Mod p Langlands program and local-global compatibility for GL

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Programme de Langlands modulo p et compatibilité local-global pour GL_2

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Titre: Programme de Langlands modulo p et compatibilité local-global pour GL_2

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Résumé: Cette thèse est consacrée au programme de Langlands modulo p pour GL_2.

Dans la première partie, j'étudie la dimension de Gelfand–Kirillov des représentations π provenant de la cohomologie modulo p des courbes de Shimura. Soit p un nombre premier et F un corps de nombres totalement réel non ramifié en des places divisant p. Soit $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ une représentation galoisienne modulaire qui satisfait l'hypothèse de Taylor–Wiles et quelques hypothèses techniques de généricité. Pour v une place fixée de F divisant p, on montre que de nombreuses représentations lisses admissibles de $\mathrm{GL}_2(F_v)$ sur $\overline{\mathbb{F}}_p$ associées à $\overline{r}$ dans les espaces propres de Hecke correspondants de la cohomologie modulo p ont une dimension de Gelfand–Kirillov $[F_v : \mathbb{Q}_p]$. Ceci s'appuie sur et étend les travaux de Breuil-Herzig-Hu-Morra-Schraen dans [BHH$^+$23] et de Hu-Wang dans [HW22], en donnant une preuve unifiée pour tous les cas ($\overline{r}$ semisimple ou non à v).

Dans la deuxième partie, j'étudie les $(\varphi, \mathcal{O}_K^\times)$-modules étales $D_A(\pi)$ associés aux représentations π provenant de la cohomologie modulo p des courbes de Shimura. Soit K une extension finie non ramifiée de $\mathbb{Q}_p$ et $\mathbb{F}$ une extension finie de $\mathbb{F}_p$. Pour π une représentation lisse admissible de $\mathrm{GL}_2(K)$ sur $\mathbb{F}$ satisfaisant certaines propriétés de multiplicité un, je calcule le rang du $(\varphi, \mathcal{O}_K^\times)$-module étale $D_A(\pi)$ associé défini dans [BHH$^+$b], ce qui étend les résultats de [BHH$^+$b] et [BHH$^+$c].

Dans la troisième partie, j'étudie les propriétés de compatibilité locale-global des $(\varphi, \mathcal{O}_K^\times)$-modules $D_A(\pi)$. Pour $\overline{\rho}$ une représentation réductible quelconque de dimension 2 de $\mathrm{Gal}(\overline{K}/K)$ sur $\mathbb{F}$, je calcule explicitement le $(\varphi, \mathcal{O}_K^\times)$-module étale $D_A^\otimes(\overline{\rho})$ défini dans [BHH$^+$c]. Ensuite, soit π une représentation lisse admissible de $\mathrm{GL}_2(K)$ sur $\mathbb{F}$ apparaissant dans certains espaces propres de Hecke de la cohomologie modulo p et $\overline{\rho}$ sa représentation sous-jacente de dimension 2 de $\mathrm{Gal}(\overline{K}/K)$ sur $\mathbb{F}$. En supposant que $\overline{\rho}$ est maximalement non-scindée, je montre sous certaines hypothèses de généricité que le $(\varphi, \mathcal{O}_K^\times)$-module étale $D_A(\pi)$ défini dans [BHH$^+$b] est isomorphe à $D_A^\otimes(\overline{\rho})$. Ceci étend les résultats de [BHH$^+$c], où $\overline{\rho}$ était supposé semisimple.

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Title: Mod p Langlands program and local-global compatibility for GL_2

Keywords: mod p Langlands program, local-global compatibility, Gelfand–Kirillov dimension, Galois deformation rings, Diamond diagrams, (φ, Γ)-modules

Abstract: This thesis is devoted to the mod p Langlands program for GL_2.

In the first part, I study the Gelfand–Kirillov dimension of the representations π coming from the mod p cohomology of Shimura curves. Let p be a prime number and F a totally real number field unramified at places above p. Let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a modular Galois representation that satisfies the Taylor–Wiles hypothesis and some technical genericity assumptions. For v a fixed place of F above p, we prove that many of the admissible smooth representations of $\mathrm{GL}_2(F_v)$ over $\overline{\mathbb{F}}_p$ associated to $\overline{r}$ in the corresponding Hecke-eigenspaces of the mod p cohomology have Gelfand–Kirillov dimension $[F_v : \mathbb{Q}_p]$. This builds on and extends the work of Breuil-Herzig-Hu-Morra-Schraen in [BHH$^+$23] and Hu-Wang in [HW22], giving a unified proof in all cases ($\overline{r}$ either semisimple or not at v).

In the second part, I study the étale $(\varphi, \mathcal{O}_K^\times)$-modules $D_A(\pi)$ associated to the representations π coming from the mod p cohomology of Shimura curves and compute there ranks. Let K be a finite unramified extension of $\mathbb{Q}_p$ and $\mathbb{F}$ a finite extension of $\mathbb{F}_p$. For π an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying certain multiplicity-one properties, we compute the rank of the associated étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ defined in [BHH$^+$b], extending the results of [BHH$^+$b] and [BHH$^+$c].

In the third part, I study the local-global compatibility properties of the étale $(\varphi, \mathcal{O}_K^\times)$-modules $D_A(\pi)$. For $\overline{\rho}$ any reducible two-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, we compute explicitly the associated étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^\otimes(\overline{\rho})$ defined in [BHH$^+$c]. Then we let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ occurring in some Hecke eigenspaces of the mod p cohomology and $\overline{\rho}$ be its underlying two-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$. Assuming that $\overline{\rho}$ is maximally non-split, we prove under some genericity assumption that the associated étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ defined in [BHH$^+$b] is isomorphic to $D_A^\otimes(\overline{\rho})$. This extends the results of [BHH$^+$c], where $\overline{\rho}$ was assumed to be semisimple.

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Contents

Chapter 1

Introduction générale

1.1 La correspondance hypothétique de Langlands modulo p pour GL_2

Soit p un nombre premier et K une extension finie de $\mathbb{Q}_p$. Le programme de Langlands modulo p pour GL_2, initié par Breuil dans [Bre03], demande une correspondance possible entre certaines représentations modulo p lisses admissibles de $\mathrm{GL}_2(K)$ et des représentations modulo p continues de dimension 2 du groupe de Galois $\mathrm{Gal}(\overline{K}/K)$.

Le cas $K = \mathbb{Q}_p$ est bien compris par les travaux de Breuil, Colmez, Emerton, Kisin, Paškūnas, etc. On liste les deux propriétés suivantes de la correspondance de Langlands modulo p pour $\mathrm{GL}_2(\mathbb{Q}_p)$.

- Colmez ([Col10]) a construit un foncteur (connu comme le foncteur de Colmez) de la catégorie des représentations modulo p admissibles de longueur finie de $\mathrm{GL}_2(\mathbb{Q}_p)$ vers la catégorie des représentations modulo p continues de dimension finie de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, en utilisant la catégorie de Fontaine des (φ, Γ)-modules ([Fon90]) comme étape intermédiaire. Cela donne une façon functorielle de réaliser la correspondance de Langlands modulo p pour $\mathrm{GL}_2(\mathbb{Q}_p)$.

- Emerton ([Eme11]) a montré que la correspondance de Langlands modulo p pour $\mathrm{GL}_2(\mathbb{Q}_p)$ satisfait la compatibilité local-global, dans le sens qu'elle peut être réalisée dans le H^1 des (tours de) courbes modulaires.

Cependant, lorsque $K \neq \mathbb{Q}_p$, la situation devient beaucoup plus compliquée. Par exemple, il y a beaucoup plus de représentations supersingulières modulo p de $\mathrm{GL}_2(K)$, et on n'a pas de classification de ces représentations ([BP12]). De plus, elles ne sont pas de présentation finie ([Sch15],[Wu21]), et il est impossible (jusqu'à présent) de décrire explicitement l'une de ces représentations. Motivés par le résultat de compatibilité local-global pour $\mathrm{GL}_2(\mathbb{Q}_p)$ ([Eme11]), on s'intéresse aux représentations modulo p de $\mathrm{GL}_2(K)$ provenant de la cohomologie des courbes de Shimura.

On présente le cadre global. Soit F un corps de nombres totalement réel qui est inerte en p (pour des raisons de simplicité dans cette introduction). Soit D une algèbre de quaternions de centre F qui est scindée en p et en exactement une place infinie. Pour chaque sous-groupe ouvert compact $U \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$ où $\mathbb{A}_F^\infty$ est l'ensemble des adèles finis de F, on note X_U la courbe de Shimura projective lisse associée sur F. Soit $\mathbb{F}$ une extension finie suffisamment grande de $\mathbb{F}_p$, qui est considérée comme le corps de coefficients à partir de maintenant. On note $K \overset{\text{déf}}{=} F_p$ la complétion de F en p et $f \overset{\text{déf}}{=} [K : \mathbb{Q}_p]$. Soit $\mathcal{O}_K$ l'anneau des entiers de K et $\mathbb{F}_q$ le

corps résiduel de K (donc $q = p^f$). On fixe un sous-groupe ouvert compact $U^p \subseteq (D \times_F \mathbb{A}_F^{\infty,p})^\times$. On considère alors la représentation lisse admissible suivante de $\mathrm{GL}_2(K)$ sur $\mathbb{F}$:

$$\pi \overset{\text{déf}}{=} \varinjlim_{U_p} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)} \left(\overline{r}, H^1_{\text{ét}}(X_{U^pU_p} \times_F \overline{F}, \mathbb{F})\right), \tag{1.1}$$

où la limite inductive est prise sur les sous-groupes ouverts compacts $U_p \subseteq (D \times_F K)^\times \cong \mathrm{GL}_2(K)$, et $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ est une représentation continue absolument irréductible telle que $\pi \neq 0$.

L'une des attentes du programme de Langlands modulo p est que la représentation π comme en (1.1) peut être utilisée pour réaliser une correspondance de Langlands modulo p pour $\mathrm{GL}_2(K)$. Plus précisément, on a l'espoir suivant.

Espoir 1.1.1. *Pour π comme en (1.1), il existe un entier $d \geq 1$ dépendant de $\overline{r}$ et U^p tel que*

$$\pi \cong \pi\left(\overline{r}|_{\mathrm{Gal}(\overline{K}/K)}\right)^{\oplus d}.$$

Ici, pour $\overline{\rho}$ une représentation modulo p continue de dimension 2 de $\mathrm{Gal}(\overline{K}/K)$, on note $\pi(\overline{\rho})$ la représentation modulo p hypothétique admissible de $\mathrm{GL}_2(K)$ lui correspondant.

Par conséquent, l'une des questions clés du programme de Langlands modulo p est de comprendre la représentation π de $\mathrm{GL}_2(K)$ en (1.1). En choisissant soigneusement le sous-groupe ouvert compact $U^p \subseteq (D \times_F \mathbb{A}_F^{\infty,p})^\times$, on suppose que l'on est dans la situation de "multiplicité un" au sens que $d = 1$ dans l'Espoir 1.1.1, ce qui est le premier cas à considérer. Pour des raisons de simplicité, on fait cette hypothèse à partir de maintenant dans cette introduction (sauf indication contraire).

Lorsque $F = \mathbb{Q}$ et $D = M_2(\mathbb{Q})$, la représentation π de $\mathrm{GL}_2(\mathbb{Q}_p)$ en (1.1) est bien comprise par [Eme11], et l'Espoir 1.1.1 est vrai. Cependant, dès que $K \neq \mathbb{Q}_p$, cette question devient particulièrement difficile. Pour l'instant, on ne connaît qu'un tout petit morceau de la représentation π. Le but de ma thèse est de mieux comprendre la représentation π en suivant la voie des résultats récents de Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$23],[BHH$^+$b],[BHH$^+$c]) et de montrer que certaines propriétés de π ne dépendent que de la restriction $\overline{r}_p \overset{\text{déf}}{=} \overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$. Le comportement de π est différent lorsque $\overline{r}_p$ est semisimple ou non. Plusieurs résultats de [BHH$^+$23], [BHH$^+$b], et [BHH$^+$c] ne traitent que du cas où $\overline{r}_p$ est semisimple, et le cas non-semisimple est beaucoup plus délicat que le cas semisimple. Le thème de ma thèse est de généraliser ces résultats à tous les $\overline{r}_p$ (y compris les $\overline{r}_p$ non-semisimples), pour lesquels on a besoin d'arguments plus élaborés en théorie des représentations.

1.2 Aperçu de quelques résultats antérieurs

On commence par un aperçu de quelques résultats antérieurs sur la représentation π comme en (1.1), qui ont pour but commun de déterminer certains sous-espaces invariants de dimension finie de la restriction de π à $\mathrm{GL}_2(\mathcal{O}_K)$ et de prouver qu'ils ne dépendent que de $\overline{r}_p$.

La première étape vers la compréhension de la représentation π est d'étudier son $\mathrm{GL}_2(\mathcal{O}_K)$-socle $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$, qui est une somme directe de poids de Serre de $\mathrm{GL}_2(\mathbb{F}_q)$, c'est-à-dire de représentations absolument irréductibles de $\mathrm{GL}_2(\mathbb{F}_q)$ sur $\mathbb{F}$. Ces poids de Serre sont prédits par [BDJ10]. Ceci est considéré comme la partie poids de la conjecture de Serre, généralisant la conjecture de Serre originale ([Ser87]) et est maintenant un théorème. A partir de maintenant, on suppose que $p > 5$ et que $\overline{r}_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ est absolument irréductible.

Théorème 1.2.1 ([EGS15],[GLS15]). *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est générique au sens de [BP12, Def. 11.7]. On a alors*

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi \cong \bigoplus_{\sigma\in W(\overline{r}_p^\vee)} \sigma,$$

où $W(\overline{r}_p^\vee)$ est l'ensemble des poids de Serre de $\overline{r}_p^\vee$ défini dans [BDJ10, §3]. En particulier, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$ ne dépend que de $\overline{r}_p$.

L'étape suivante consiste à étudier la représentation π^{K_1} de $\mathrm{GL}_2(\mathcal{O}_K)$ où $K_1 \overset{\text{déf}}{=} 1 + p\,\mathrm{M}_2(\mathcal{O}_K)$, qui contient $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$ comme sous-représentation.

Théorème 1.2.2 ([LMS22],[HW18],[Le19]). *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est générique au sens de [BP12, Def. 11.7]. On a alors*

$$\pi^{K_1} \cong D_0(\overline{r}_p^\vee),$$

où $D_0(\overline{r}_p^\vee)$ est une représentation explicite (de dimension finie) de $\mathrm{GL}_2(\mathbb{F}_q)$ sur $\mathbb{F}$ construite par Breuil-Paškūnas ([BP12, §13]). En particulier, π^{K_1} ne dépend que de $\overline{r}_p$.

Un diagramme D est une représentation D_0 de $\mathrm{GL}_2(\mathcal{O}_K)$ telle que $D_0^{K_1} = D_0$ avec un automorphisme Π sur $D_0^{I_1}$, dont le carré agit par un scalaire non nul, où $I_1 \overset{\text{déf}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ est le sous-groupe de pro-p-Iwahori. Supposons que $\overline{r}_p$ est générique au sens de [BP12, Def. 11.7]. Alors Breuil et Paškūnas ont construit une famille de diagrammes attachés à $\overline{r}_p$ telle que $D_0 = D_0(\overline{r}_p^\vee)$.

Puisque la représentation π comme en (1.1) a un caractère central, on obtient un diagramme $\mathcal{D}(\pi)$ avec $D_0 = \pi^{K_1}$ et Π donné par l'action de $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. En particulier, puisque Π normalise I_1 et I où $I \overset{\text{def}}{=} \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ est le sous-groupe d'Iwahori, il envoie un caractère χ de I à sa conjugaison χ^s par la matrice $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. Par le Théorème 1.2.2, $\mathcal{D}(\pi)$ est isomorphe à l'un des diagrammes attachés à $\overline{r}_p$ définis par Breuil et Paškūnas. On a le raffinement suivant du Théorème 1.2.2, qui distingue un diagramme unique $\mathcal{D}$ dans la famille ci-dessus, sous réserve d'une condition de compatibilité local-global lorsque $\overline{r}_p$ est suffisamment générique.

Théorème 1.2.3 ([DL21]). *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est suffisamment générique (voir [DL21, §1] pour une signification précise). Alors le diagramme $\mathcal{D}(\pi)$ ne dépend que de $\overline{r}_p$.*

L'un des principaux outils communs à la preuve du Théorème 1.2.1, du Théorème 1.2.2 et du Théorème 1.2.3 est le foncteur de patché défini par Emerton-Gee-Savitt ([EGS15]) en s'appuyant sur les travaux de Taylor et Wiles ([TW95]), et de Kisin ([Kis09]). Il s'agit d'un foncteur exact M_∞ de représentations continues de $\mathrm{GL}_2(\mathcal{O}_K)$ sur des $W(\mathbb{F})$-modules de type fini vers des R_∞-modules de type fini avec des propriétés supplémentaires, où R_∞ est l'anneau de déformation patché et est isomorphe à un anneau de séries formelles sur l'anneau de déformations cadrées universel de $\overline{r}_p^\vee$ dans notre cas (voir [EGS15] pour un énoncé précis). Ce foncteur nous permet de transférer des énoncés du côté de GL_2 vers le côté Galois. Une fois que l'on a suffisamment d'informations sur les anneaux de déformations galoisiennes, on est en mesure d'utiliser la philosophie de Breuil-Mézard ([EG14]) pour déduire des propriétés du côté GL_2. On verra des exemples de ce type d'argument au § 1.3 ci-dessous.

1.3 La dimension de Gelfand–Kirillov de π

Le premier résultat concerne la dimension de Gelfand–Kirillov de π comme en (1.1), qui mesure la croissance de la dimension des sous-espaces invariants de π sous les sous-groupes de congruence principaux. Plus précisément, On définit la dimension de Gelfand–Kirillov de π (voir [EP20]) comme l'unique entier $\dim_{\mathrm{GL}_2(K)}(\pi)$ tel qu'il existe $a \leq b$ dans $\mathbb{R}_{>0}$ satisfaisant

$$a \leq \frac{\dim_{\mathbb{F}}(\pi^{K_n})}{p^{n \dim_{\mathrm{GL}_2(K)}(\pi)}} \leq b$$

pour tout $n \geq 1$, où $K_n \overset{\text{déf}}{=} 1 + p^n \mathrm{M}_2(\mathcal{O}_K)$ pour $n \geq 1$.

Théorème 1.3.1 (Theorem 3.1.1)**.** *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est suffisamment générique (voir la condition (iv) au § 3.1 pour une signification précise). On a alors*

$$\dim_{\mathrm{GL}_2(K)}(\pi) = f.$$

Le Théorème 1.3.1 est démontré par [BHH$^+$23] lorsque $\overline{r}_p$ est semisimple et démontré par [HW22] lorsque $\overline{r}_p$ est non-semisimple en utilisant une méthode différente. D'une part, la méthode de [HW22] ne fonctionne que dans le cas non-semisimple. D'autre part, il s'avère que la méthode de [BHH$^+$23] peut être généralisée au cas non-semisimple, ce qui n'avait pas été remarqué auparavant. On adapte la méthode de [BHH$^+$23] au cas non-semisimple. Cela fournit une preuve uniforme du Théorème 1.3.1.

La preuve du Théorème 1.3.1 implique le calcul de nouveaux cas d'anneaux de déformations galoisiennes, ce qui nous permet d'utiliser davantage le foncteur de patché et donc d'aller au-delà de π^{K_1} comme dans le Théorème 1.2.2. On donne un aperçu de la preuve du Théorème 1.3.1 en suivant de près [BHH$^+$23, §1] et indique ce qui doit être changé dans le cas non-semisimple.

Soit $I \overset{\text{déf}}{=} \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ le sous-groupe d'Iwahori. Soit Z_1 le centre de K_1 et $\mathfrak{m}_{K_1}$ l'idéal maximal de l'algèbre d'Iwasawa $\mathbb{F}[\![K_1/Z_1]\!]$. On note encore $\mathfrak{m}_{K_1}$ l'idéal de $\mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$ engendré par $\mathfrak{m}_{K_1}$ sous l'inclusion naturelle $\mathbb{F}[\![K_1/Z_1]\!] \hookrightarrow \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$ lorsqu'il n'y a pas de confusion possible. Par les arguments cruciaux de Gee et Newton dans [GN20, Appendix A], on sait que $\dim_{\mathrm{GL}_2(K)}(\pi) \geq f$ pour π comme en (1.1). Afin d'obtenir la borne supérieure sur $\dim_{\mathrm{GL}_2(K)}(\pi)$, on doit appliquer le théorème de théorie des représentations suivant à π comme en (1.1). Il s'agit d'un cas particulier de [BHH$^+$23, Thm. 6.4.7] lorsque $\overline{r}_p$ est semisimple, et d'une conséquence de [HW22, Prop. 4.20] et [BHH$^+$23, Prop. 6.4.6] lorsque $\overline{r}_p$ est non-semisimple. On renvoie au Theorem 3.5.3 ci-dessous pour un énoncé plus général, qui s'applique à π pas nécessairement dans la situation de "multiplicité un".

Théorème 1.3.2 (Theorem 3.5.3)**.** *Soit π une représentation lisse admissible de $\mathrm{GL}_2(K)$ sur $\mathbb{F}$ avec un caractère central. Supposons que*

(i) *on a $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi) \cong \bigoplus_{\sigma \in W(\overline{r}_p^\vee)} \sigma$;*
(ii) *pour chaque $\sigma \in W(\overline{r}_p^\vee)$, on a $[\pi[\mathfrak{m}_{K_1}^2]|_{\mathrm{GL}_2(\mathcal{O}_K)} : \sigma] = 1$;*
(iii) *on a $\pi^{I_1} \cong D_0(\overline{r}_p^\vee)^{I_1}$ comme représentations de I.*

Alors $\dim_{\mathrm{GL}_2(K)}(\pi) \leq f$. Ici, $\pi[\mathfrak{m}_{K_1}^2]$ est l'ensemble des éléments de π (considéré comme un module sur $\mathbb{F}[\![K_1/Z_1]\!]$) annihilés par $\mathfrak{m}_{K_1}^2$, et $[\pi[\mathfrak{m}_{K_1}^2]|_{\mathrm{GL}_2(\mathcal{O}_K)} : \sigma]$ est la multiplicité de σ dans la semisimplification de $\pi[\mathfrak{m}_{K_1}^2]$ en tant que représentations de $\mathrm{GL}_2(\mathcal{O}_K)$.

On doit alors montrer que la représentation π en (1.1) satisfait les conditions du Théorème 1.3.2. Les conditions (i) et (iii) découlent du Théorème 1.2.1 et du Théorème 1.2.2, et on se concentre donc sur la condition (ii) du Théorème 1.3.2. Puisque l'on a $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma, \pi) = 1$ pour $\sigma \in W(\overline{r}_p^\vee)$ par la condition (i), il suffit de montrer que

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{Proj}_{\widetilde{\Gamma}} \sigma, \pi) \leq 1 \ \forall\, \sigma \in W(\overline{r}_p^\vee), \tag{1.2}$$

où $\widetilde{\Gamma} \overset{\text{déf}}{=} \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}^2$, et $\mathrm{Proj}_{\widetilde{\Gamma}} \sigma$ est l'enveloppe projective de σ dans la catégorie des $\widetilde{\Gamma}$-modules.

On note M_∞ le foncteur de patché comme au § 1.2. D'après la construction de M_∞, pour toute représentation V de $\mathrm{GL}_2(\mathcal{O}_K)$ de dimension finie sur $\mathbb{F}$, on a

$$\mathrm{Hom}_{\mathbb{F}}(M_\infty(V)/\mathfrak{m}_\infty, \mathbb{F}) \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(V, \pi), \tag{1.3}$$

où $\mathfrak{m}_\infty$ est l'idéal maximal de R_∞. En combinant (1.2) et (1.3), il suffit de montrer que

$$M_\infty(\mathrm{Proj}_{\widetilde{\Gamma}} \sigma) \text{ est un } R_\infty\text{-module cyclique } \forall\, \sigma \in W(\overline{r}_p^\vee). \tag{1.4}$$

On fixe $\sigma \in W(\overline{r}_p^\vee)$. Pour montrer (1.4) pour σ, on relève le $\widetilde{\Gamma}$-module $\mathrm{Proj}_{\widetilde{\Gamma}} \sigma$ sur $\mathbb{F}$ en caractéristique zéro suivant [BHH+23, §7.3] afin que l'on puisse utiliser les propriétés de compatibilité local-global de M_∞. On note $P_\sigma \overset{\text{déf}}{=} \mathrm{Proj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ l'enveloppe projective de σ dans la catégorie des $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_q)]$-modules et note $\widetilde{P}_\sigma$ le $\mathcal{O}[\mathrm{GL}_2(\mathbb{F}_q)]$-module projectif relevant P_σ. On fixe un plongement $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ et note $\sigma_j \overset{\text{déf}}{=} \sigma_0 \circ \varphi^j$ pour $j \in \mathbb{Z}$, où $\varphi : x \mapsto x^p$ est le Frobenius arithmétique sur $\mathbb{F}_q$. On note encore σ_j le plongement correspondant $\mathcal{O}_K \hookrightarrow W(\mathbb{F})$. Pour $0 \leq j \leq f-1$, on définit la représentation de $\mathrm{GL}_2(\mathcal{O}_K)$ sur $W(\mathbb{F})$

$$R_{2,j} \overset{\text{déf}}{=} \left(\mathrm{Sym}^2 W(\mathbb{F})^2 \otimes \det{}^{-1}\right)^{(j)} \otimes_{W(\mathbb{F})} \widetilde{P}_\sigma,$$

où "(j)" signifie que $\mathrm{GL}_2(\mathcal{O}_K)$ agit via le plongement $\sigma_j : \mathcal{O}_K \hookrightarrow W(\mathbb{F})$. Pour chaque j, il existe un isomorphisme

$$R_{2,j}/pR_{2,j} \cong P_\sigma \oplus P_{\sigma_j^+} \oplus P_{\sigma_j^-}$$

pour certains poids de Serre σ_j^+ et σ_j^-, ce qui induit une injection

$$\iota_j : P_\sigma \hookrightarrow R_{2,j}/pR_{2,j}.$$

On définit alors un $\mathrm{GL}_2(\mathcal{O}_K)$-réseau $R'_{2,j}$ dans $R_{2,j}[1/p]$ par

$$R'_{2,j} \overset{\text{déf}}{=} P_\sigma \times_{R_{2,j}/p} R_{2,j} = \{x \in R_{2,j} : (x \bmod pR_{2,j}) \in \iota_j(P_\sigma)\}.$$

On note $L_{-1} \overset{\text{déf}}{=} \widetilde{P}_\sigma$. Pour $0 \leq j \leq f-1$, on définit un $\mathrm{GL}_2(\mathcal{O}_K)$-réseau L_j dans $L_{-1}[1/p] \oplus \left(\bigoplus_{j'=0}^{j} R_{2,j}[1/p]\right)$ par

$$L_j \overset{\text{déf}}{=} \left\{ \begin{array}{r} \left(x, (x_{j'})_{0 \leq j' \leq j}\right) \in L_{-1} \oplus (\oplus_{j'=0}^{j} R_{2,j}) : (x_{j'} \bmod pR_{2,j'}) = (x \bmod pL_{-1}) \\ \text{via } \iota_{j'} : L_{-1}/pL_{-1} \hookrightarrow R_{2,j'}/pR_{2,j'} \ \forall\, 0 \leq j' \leq j \end{array} \right\},$$

ce qui revient à définir

$$L_j = L_{j-1} \times_{P_\sigma} R'_{2,j} \tag{1.5}$$

pour $0 \leq j \leq f-1$. On a alors $L_{f-1}/pL_{f-1} \cong \mathrm{Proj}_{\widetilde{\Gamma}} \sigma$. Par l'exactitude de M_∞, il suffit de montrer que le R_∞-module $M_\infty(L_{f-1})$ est cyclique.

Par [Le19, Thm. 4.9], on sait que le R_∞-module $M_\infty(\widetilde{P}_\sigma)$ est cyclique. Par [BHH$^+$23, Thm. 8.3.4], on sait que le R_∞-module $M_\infty(R'_{2,j})$ est cyclique pour chaque $0 \leq j \leq f-1$ lorsque $\overline{r}_p$ est semisimple. La preuve de [BHH$^+$23, Thm. 8.3.4] utilise les techniques standard de dévissage comme dans [EGS15, §10] et [Le19, Lemma 4.5], et peut être facilement généralisée à tous les $\overline{r}_p$. On peut alors utiliser (1.5) et la cyclicité de $M_\infty(\widetilde{P}_\sigma)$ et $M_\infty(R'_{2,j})$ pour montrer que $M_\infty(L_{f-1})$ est cyclique par induction.

Pour simplifier, on ne parle que de la première étape de l'induction. Par l'exactitude de M_∞, on a l'égalité de R_∞-modules

$$M_\infty(L_0) = M_\infty(\widetilde{P}_\sigma) \times_{M_\infty(P_\sigma)} M_\infty(R'_{2,0}). \tag{1.6}$$

On sait déjà que chaque terme sur la droite de (1.6) est cyclique. Pour montrer que $M_\infty(L_0)$ est cyclique, il suffit de montrer que

$$\mathrm{Ann}_{R_\infty}\big(M_\infty(P_\sigma)\big) = \mathrm{Ann}_{R_\infty}\big(M_\infty(\widetilde{P}_\sigma)\big) + \mathrm{Ann}_{R_\infty}\big(M_\infty(R'_{2,0})\big). \tag{1.7}$$

Chaque terme de (1.7) a une interprétation comme un certain anneau de déformations galoisiennes. On note $R_{\overline{r}_p^\vee}$ l'anneau de déformations cadrées universel de $\overline{r}_p^\vee$. Pour τ un type inertiel modéré, on note $R_{\overline{r}_p^\vee}^{(1,0),\tau}$ (resp. $R_{\overline{r}_p^\vee}^{(2,-1)_0,\tau}$) le quotient maximal réduit, plat sur $\mathcal{O}$ de $R_{\overline{r}_p^\vee}$ qui paramétrise les déformations potentiellement cristallines de $\overline{r}_p^\vee$ de type inertiel τ et de poids de Hodge–Tate parallèles $(1,0)$ (resp. poids de Hodge–Tate $(2,-1)$ dans le plongement $\sigma_0 : K \hookrightarrow W(\mathbb{F})[1/p]$ et $(1,0)$ ailleurs). On note

$$\mathfrak{p}_\tau^{(1,0)} \overset{\text{déf}}{=} \mathrm{Ker}(R_{\overline{r}_p^\vee} \twoheadrightarrow R_{\overline{r}_p^\vee}^{(1,0),\tau});$$
$$\mathfrak{p}_\tau^{(2,-1)_0} \overset{\text{déf}}{=} \mathrm{Ker}(R_{\overline{r}_p^\vee} \twoheadrightarrow R_{\overline{r}_p^\vee}^{(2,-1)_0,\tau}).$$

Par un certain détour (voir [BHH$^+$23, §1] pour plus de détails), pour montrer (1.7) on est réduit à la preuve de la propriété subtile de (non-)congruence suivante.

Théorème 1.3.3 (Proposition 3.4.3.3)**.** *On a*

$$p \in \cap_\tau \mathfrak{p}_\tau^{(1,0)} + \mathfrak{p}_{\tau_0}^{(2,-1)_0},$$

où τ parcourt les types inertiels modérés tels que σ est un facteur de Jordan–Hölder dans la semisimplification modulo p de $\sigma(\tau)$ (ici $\sigma(\tau)$ est la représentation lisse irréductible de $\mathrm{GL}_2(\mathcal{O}_K)$ associée par Henniart à τ dans l'appendice de [BM02]), et τ_0 est un type inertiel modéré tel que l'ensemble des constituants irréductibles de la semisimplification modulo p de $\sigma(\tau_0)$ contient l'ensemble $W(\overline{r}_p^\vee)$ (qui existe).

Lorsque $\overline{r}_p$ est semisimple, le Théorème 1.3.3 est démontré dans [BHH$^+$23, Prop. 4.3.3] par un calcul explicite des anneaux de déformations potentiellement cristallines en utilisant la machinerie des modules de Kisin, qui a d'abord été suggéré par Breuil, puis développé par [Kis06] et [LLHLM18]. Nous généralisons le calcul des anneaux de déformations potentiellement cristallines au cas non-semisimple. Ceci complète la preuve du Théorème 1.3.1.

Comme sous-produit de la preuve du Théorème 1.3.1, on peut déterminer la structure de la $\mathfrak{m}_{K_1}^2$-torsion de la représentation π en (1.1), qui est une généralisation du Théorème 1.2.2. C'est un cas particulier de [BHH$^+$23, Thm. 1.9] lorsque $\overline{r}_p$ est semisimple, et est démontré dans [HW22, Thm. 1.4] lorsque $\overline{r}_p$ n'est pas semisimple. On renvoie au Théorème 3.6.3.1(ii) ci-dessous pour un énoncé plus général, qui s'applique à π pas nécessairement dans la situation de "multiplicité un".

Théorème 1.3.4 (Theorem 3.6.3.1(ii)). *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est suffisamment générique (comme dans le Théorème 1.3.1). On a alors*

$$\pi[\mathfrak{m}_{K_1}^2] \cong \widetilde{D}_0(\overline{r}_p^\vee),$$

où $\widetilde{D}_0(\overline{r}_p^\vee)$ est une représentation explicite (de dimension finie) de $\widetilde{\Gamma}$ sur $\mathbb{F}$ définie dans [HW22, §4]. En particulier, $\pi[\mathfrak{m}_{K_1}^2]$ ne dépend que de $\overline{r}_p$.

Comme étape intermédiaire pour montrer le Théorème 1.3.1, on déduit également une propriété importante du module gradué associé du dual $\pi^\vee \overset{\text{déf}}{=} \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$ pour π comme en (1.1), ce qui généralise le résultat de [BHH+23] où $\overline{r}_p$ était supposé semisimple (cette propriété est également démontrée par [HW22] lorsque $\overline{r}_p$ n'est pas semisimple en utilisant une méthode différente). Cela conduit à une sous-catégorie abélienne de la catégorie des représentations lisses admissibles de $\mathrm{GL}_2(K)$ qui possède des propriétés de finitude agréables et qui sera introduite au § 1.4 ci-dessous. On note $\mathfrak{m}_{I_1}$ l'idéal maximal de l'algèbre d'Iwasawa $\mathbb{F}[\![I_1/Z_1]\!]$.

Théorème 1.3.5. *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est suffisamment générique (comme dans le Théorème 1.3.1). Alors le module gradué $\mathrm{gr}_{\mathfrak{m}_{I_1}} \pi^\vee$ sur l'algèbre graduée $\mathrm{gr}_{\mathfrak{m}_{I_1}} \mathbb{F}[\![I_1/Z_1]\!]$ est annihilé par un idéal à deux côtés explicite J, et l'anneau quotient $\mathrm{gr}_{\mathfrak{m}_{I_1}} \mathbb{F}[\![I_1/Z_1]\!]/J$ est commutatif et est isomorphe à*

$$\mathbb{F}[y_0, z_0, \dots, y_{f-1}, z_{f-1}]/(y_0 z_0, \dots, y_{f-1} z_{f-1}).$$

1.4 Le foncteur $\pi \mapsto D_A(\pi)$

En utilisant le Théorème 1.3.5, Breuil-Herzig-Hu-Morra-Schraen ([BHH+c]) a construit un foncteur exact D_A d'une certaine sous-catégorie de la catégorie des représentations lisses admissibles de $\mathrm{GL}_2(K)$ sur $\mathbb{F}$ à la catégorie des modules multivariables $(\varphi, \mathcal{O}_K^\times)$, qui est une généralisation du foncteur de Colmez ([Col10]). La question fondamentale est alors de déterminer la structure de $D_A(\pi)$ pour π comme en (1.1), ce qui peut être utilisée pour déduire des propriétés de π.

On rappelle tout d'abord la définition de l'anneau A, qui est un analogue en plusieurs variables de $\mathbb{F}(\!(X)\!)$. On note $N_0 \overset{\text{déf}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$. Pour $0 \le j \le f-1$, on définit

$$Y_j \overset{\text{déf}}{=} \sum_{a \in \mathbb{F}_q^\times} \sigma_0(a)^{-p^j} \begin{pmatrix} 1 & [a] \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[\![N_0]\!],$$

où $[a] \in \mathcal{O}_K^\times$ est le relève de Teichmüller de $a \in \mathbb{F}_q^\times$. On a alors $\mathbb{F}[\![N_0]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$. On définit

$$A \overset{\text{déf}}{=} \mathbb{F}[\![N_0]\!]\,[1/(Y_0 \cdots Y_{f-1})]^\wedge,$$

où la complétion se fait par rapport à la topologie $(Y_0, \dots, Y_{f-1})$-adique sur $\mathbb{F}[\![N_0]\!]$. Il existe une action $\mathbb{F}$-linéaire de $\mathcal{O}_K^\times$ sur $\mathbb{F}[\![N_0]\!]$ donnée par multiplication sur $N_0 \cong \mathcal{O}_K$, et une action $\mathbb{F}$-linéaire de Frobenius φ sur $\mathbb{F}[\![N_0]\!]$ donnée par multiplication par p sur $N_0 \cong \mathcal{O}_K$. Ces deux actions s'étendent canoniquement par continuité en des actions $\mathbb{F}$-linéaires continues de φ et $\mathcal{O}_K^\times$ qui commutent sur A. Un $(\varphi, \mathcal{O}_K^\times)$-module étale sur A est par définition un A-module libre de type fini muni d'un Frobenius semi-linéaire φ et d'une action semi-linéaire continue de $\mathcal{O}_K^\times$ commutant avec φ telle que l'image de φ engendre tout.

Pour π une représentation lisse admissible de $\mathrm{GL}_2(K)$ sur $\mathbb{F}$ avec caractère central, on considère $\pi^\vee$ comme un $\mathbb{F}[\![I_1/Z_1]\!]$-module de type fini et on le munit de la topologie $\mathfrak{m}_{I_1}$-adique. On définit

$$D_A(\pi) \overset{\text{déf}}{=} \mathbb{F}[\![N_0]\!][1/(Y_0 \cdots Y_{f-1})] \widehat{\otimes}_{\mathbb{F}[\![N_0]\!]} \pi^\vee,$$

où la complétion se fait par rapport à la topologie produit tensoriel. L'action de $\mathcal{O}_K^\times$ sur $\pi^\vee$ donnée par $f \mapsto f \circ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (pour $a \in \mathcal{O}_K^\times$) s'étend par continuité à $D_A(\pi)$, et l'action de ψ sur $\pi^\vee$ donnée par $f \mapsto f \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induit une application A-linéaire continue

$$\beta : D_A(\pi) \to A \otimes_{\varphi, A} D_A(\pi). \tag{1.8}$$

On note $\mathcal{C}$ la catégorie abélienne des représentations lisses admissibles π de $\mathrm{GL}_2(K)$ sur $\mathbb{F}$ avec des caractères centraux tels que le module gradué $\mathrm{gr}_{\mathfrak{m}_{I_1}} \pi^\vee$ est annihilé par J^n pour quelque $n \in \mathbb{N}$ (voir le Théorème 1.3.5). Pour π dans $\mathcal{C}$, $D_A(\pi)$ est un A-module libre de type fini par [BHH$^+$b, Cor. 3.1.2.9] et [BHH$^+$c, Remark. 2.6.2]. Si de plus β est un isomorphisme, alors son inverse $\beta^{-1} = \mathrm{id} \otimes \varphi$ fait de $D_A(\pi)$ un $(\varphi, \mathcal{O}_K^\times)$-module étale. En particulier, lorsque $K = \mathbb{Q}_p$, la construction ci-dessus récupère le foncteur de Colmez ([Col10]). Notre résultat principal est le suivant.

Théorème 1.4.1 (Theorem 4.1.1)**.** *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est suffisamment générique (voir la condition (v) au § 5.1 pour une signification précise). Alors π est dans $\mathcal{C}$, β dans (1.8) est un isomorphisme et*

$$\mathrm{rank}_A D_A(\pi) = 2^f.$$

Par le Théorème 1.3.5, on sait que π est dans $\mathcal{C}$. Par [BHH$^+$b, Thm. 3.3.2.3], on sait que $\mathrm{rang}_A D_A(\pi) \leq 2^f$. Le Théorème 1.4.1 est démontré par [BHH$^+$c, Thm. 3.1.3] lorsque $\overline{r}_p$ est semisimple. Nous généralisons la preuve de [BHH$^+$c] au cas non-semisimple, ce qui est sérieusement plus délicat.

La preuve du Théorème 1.4.1 se fait par une construction explicite d'une A-base du $(\varphi, \mathcal{O}_K^\times)$-module étale dual $\mathrm{Hom}_A(D_A(\pi), A)$ pour π comme en (1.1). Comme dans [BHH$^+$c, (87)], il existe une injection A-linéaire canonique

$$\mu_* : \mathrm{Hom}_A(D_A(\pi), A) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F}). \tag{1.9}$$

Pour $\underline{i} = (i_0, \dots, i_{f-1}) \in \mathbb{Z}^f_{\geq 0}$, on note $\|\underline{i}\| \overset{\text{déf}}{=} \sum_{j=0}^{f-1} i_j$ et on écrit $\underline{Y}^{\underline{i}}$ pour $\prod_{j=0}^{f-1} Y_j^{i_j} \in \mathbb{F}[\![N_0]\!]$. On a alors la proposition suivante.

Proposition 1.4.2 ([BHH$^+$c])**.** *Soit π comme en (1.1).*

(i) *L'ensemble $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ peut être identifié à l'ensemble des $(x_{\underline{i}})_{\underline{i} \in \mathbb{Z}^f}$ avec $x_{\underline{i}} \in \pi$ et*
   (a) *$\underline{Y}^{\underline{k}} x_{\underline{i}} = x_{\underline{i} - \underline{k}}$ pour tout $\underline{i} \in \mathbb{Z}^f$ et $\underline{k} \in \mathbb{Z}^f_{\geq 0}$;*
   (b) *il existe $d \in \mathbb{Z}$ tel que $x_{\underline{i}} \in \pi[\mathfrak{m}_{I_1}^{\|\underline{i}\| + d + 1}]$ pour tout $\underline{i} \in \mathbb{Z}^f$ (où $\pi[\mathfrak{m}_{I_1}^j] \overset{\text{déf}}{=} 0$ si $j \leq 0$).*

(ii) *Un élément $(x_{\underline{i}})_{\underline{i} \in \mathbb{Z}^f} \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ est dans l'image de μ_* comme en (1.9) si et seulement s'il satisfait la condition de finitude suivante :*

$$\left\{ \underline{i} \in \mathbb{Z}^f : x_{\underline{i}} \neq 0, \|\underline{i}\| = M \right\} \text{ est fini } \forall\, M \in \mathbb{Z}.$$

On va définir 2^f éléments $x_J = (x_{J,\underline{i}})_{\underline{i} \in \mathbb{Z}^f} \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ indexées par les sous-ensembles $J \subseteq \mathcal{J} \overset{\text{déf}}{=} \{0, 1, \dots, f-1\}$. On montre ensuite que les x_J pour $J \subseteq \mathcal{J}$ sont dans l'image de μ_* et forment une A-base de $\mathrm{Hom}_A(D_A(\pi), A)$. On donne un aperçu de la construction dans le cas où $\overline{r}_p$ est maximalement non-scindée (ou de façon équivalente, $|W(\overline{r}_p)| = 1$) pour plus de simplicité.

Étape 1. On définit $x_{J,\underline{0}}$ pour tout $J \subseteq \mathcal{J}$.

En prenant $d = 0$ dans la Proposition 1.4.2(i)(b), on voit que $x_{J,\underline{0}} \in \pi^{I_1}$ pour tout $J \subseteq \mathcal{J}$. D'après le Théorème 1.2.2, on sait que π^{I_1} est une somme directe de 2^f caractères distincts de I, qui peuvent être paramétrés par les sous-ensembles $J \subseteq \mathcal{J}$. On fait le choix d'un vecteur propre $x_{J,\underline{0}} \in \pi^{I_1}$ pour l'action de I pour chaque $J \subseteq \mathcal{J}$.

Etape 2. On définit $x_{J,\underline{f}}$ pour tout $J \subseteq \mathcal{J}$.

Par le Théorème 1.2.2 on a $\pi^{K_1} \cong D_0(\overline{r}_p^\vee)$, qui est explicitement connu et contient π^{I_1} comme sous-espace. Pour chaque $J \subseteq \mathcal{J}$ on définit $x_{J,\underline{f}}$ comme l'unique vecteur propre dans π^{K_1} pour l'action de $\left(\begin{smallmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{smallmatrix}\right)$ satisfaisant

$$\begin{cases} Y_j^{f+1} x_{J,\underline{f}} = 0 \ \forall\, j \in \mathcal{J} \\ (Y_0 \cdots Y_{f-1})^f x_{J,\underline{f}} = x_{J,\underline{0}}. \end{cases}$$

Etape 3. On définit $x_{J,\underline{i}}$ pour tout $J \subseteq \mathcal{J}$ et $\underline{i} \in \mathbb{Z}^f$.

Tout d'abord, en utilisant la Proposition 1.4.2(i)(a), pour $J \subseteq \mathcal{J}$ et $\underline{i} \in \mathbb{Z}^f$ tel que $i_j \leq f$ pour tout $j \in \mathcal{J}$ on définit

$$x_{J,\underline{i}} \overset{\text{déf}}{=} \underline{Y}^{\underline{f}-\underline{i}} x_{J,\underline{f}}.$$

Pour $J \subseteq \mathcal{J}$, on définit $J+1 \overset{\text{déf}}{=} \{(j+1 \bmod f) \in \mathcal{J} : j \in J\}$. On définit $\delta : \mathbb{Z}^f \to \mathbb{Z}^f$ par $\delta(\underline{i})_j \overset{\text{déf}}{=} i_{j+1}$ si $0 \leq j \leq f-2$ et $\delta(\underline{i})_{f-1} \overset{\text{déf}}{=} i_0$. En utilisant la relation entre les éléments $x_{J,\underline{f}} \in \pi^{K_1}$, on peut étendre la définition de $x_{J,\underline{i}}$ à tous les $\underline{i} \in \mathbb{Z}^f$ de manière inductive en utilisant la relation suivante

$$\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}} = \sum_{J' \subseteq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',p\delta(\underline{i})+\underline{c}^{J,J'}},$$

où $\varepsilon_J \in \{\pm 1\}$, $\mu_{J,J'} \in \mathbb{F}$ sont certaines constantes qui dépendant du choix de $x_{J,\underline{0}}$, et où $\underline{c}^{J,J'} \in \mathbb{Z}^f$ vérifie $-1 \leq c_j^{J,J'} \leq 2p$ pour tout $j \in \mathcal{J}$. On renvoie au Théorème 4.6.4 ci-dessous pour une définition précise des éléments $x_{J,\underline{i}}$ qui fonctionne pour tout $\overline{r}_p$.

Étape 4. On montre que chaque x_J satisfait la condition de finitude de la Proposition 1.4.2, qui garantit que x_J se trouve dans l'image de μ_* comme en (1.9). Une fois que l'on a montré que $x_J \in \mathrm{Hom}_A(D_A(\pi), A)$ pour tout J, il n'est pas difficile de conclure que les x_J pour $J \subseteq \mathcal{J}$ forment une A-base de $\mathrm{Hom}_A(D_A(\pi), A)$. Ceci termine la preuve du Théorème 1.4.1.

En analysant plus en détail la structure des sous-modules de π^{K_1} (pour π comme en (1.1)) et les x_J, on peut montrer la généralisation suivante du Théorème 1.4.1.

Théorème 1.4.3 (Theorem 4.1.2)**.** *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est suffisamment générique (comme dans le Théorème 1.4.1). Alors pour π_1 une sous-représentation de π, on a*

$$\mathrm{rang}_A\, D_A(\pi_1) = \left|\mathrm{JH}(\pi_1^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})\right|,$$

où $\mathrm{JH}(\pi_1^{K_1})$ est l'ensemble des facteurs de Jordan–Hölder de $\pi_1^{K_1}$ en tant que représentation de $\mathrm{GL}_2(\mathcal{O}_K)$, et $\overline{\rho}^{\mathrm{ss}}$ est la semisimplification de $\overline{\rho}$.

Comme une application du Théorème 1.4.3 à la compréhension de π en (1.1), on montre que π est engendrée par un nombre fini de vecteurs en tant que représentation de $\mathrm{GL}_2(K)$, généralisant le résultat de [BHH$^+$b] où $\overline{r}_p$ était supposé semisimple (ce résultat de génération finie est également démontré par [HW22] lorsque $\overline{r}_p$ n'est pas semisimple en utilisant une méthode complètement différente).

Théorème 1.4.4 (Corollary 4.11.3)**.** *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est suffisamment générique (comme dans le Théorème 1.4.1). Alors, en tant que représentation de $\mathrm{GL}_2(K)$, π est engendrée par $D_0(\overline{r}_p^\vee)$.*

On remarque qu l'on ignore si π est de longueur finie en tant que représentation de $\mathrm{GL}_2(K)$. Notons le point important suivant, le Théorème 1.4.3 est essentiel pour montrer que π est de longueur finie (dans le cas non-semisimple) dans un travail en cours de Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$a]).

1.5 Le foncteur $\overline{r}_p \mapsto D_A^\otimes(\overline{r}_p)$

Outre le rang de $D_A(\pi)$, Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$c]) donne une description conjecturale de $D_A(\pi)$ comme $(\varphi,\mathcal{O}_K^\times)$-module étale pour π comme en (1.1). La construction utilise le $(\varphi,\mathcal{O}_K^\times)$-module de Lubin–Tate (voir [KR09]) associé à $\overline{r}_p$ comme étape intermédiaire, ce qui peut être rendu assez explicite. On donne un exemple du $(\varphi,\mathcal{O}_K^\times)$-module de Lubin-Tate $D_K(\overline{r}_p)$ lorsque $\overline{r}_p$ est suffisamment générique. On renvoie au § 5.2 pour la description explicite de $D_K(\overline{r}_p)$ pour $\overline{r}_p$ réductible arbitraire (de dimension 2), et on renvoie à [BHH$^+$c, Lemma 2.1.6] lorsque $\overline{r}_p$ est irréductible.

Exemple 1.5.1. *Supposons que $\overline{r}_p$ est réductible et suffisamment générique comme en (5.69). Alors le $(\varphi,\mathcal{O}_K^\times)$-module de Lubin–Tate $D_K(\overline{r}_p)$ associé à $\overline{r}_p$ peut être décrit explicitement comme suit ($a\in\mathcal{O}_K^\times$):*

$$\begin{cases} D_K(\overline{r}_p) &= \prod_{j=0}^{f-1} D_{K,\sigma_j}(\overline{r}_p)\\ D_{K,\sigma_j}(\overline{r}_p) &= \mathbb{F}(\!(T_{K,\sigma_j})\!)e_0^{(j)}\oplus\mathbb{F}(\!(T_{K,\sigma_j})\!)e_1^{(j)}\\ \varphi(e_0^{(j+1)}\ e_1^{(j+1)}) &= (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(\varphi^{(j)})\\ a(e_0^{(j)}\ e_1^{(j)}) &= (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(a^{(j)}), \end{cases}$$

où T_{K,σ_j} est une variable de Lubin–Tate appropriée, les indices j étant considérés modulo f,

$$\mathrm{Mat}(\varphi^{(j)}) = \begin{pmatrix} \alpha_j T_{K,\sigma_j}^{-(q-1)h_j} & \beta_j d_j\\ 0 & \beta_j \end{pmatrix}$$

pour $\alpha_j,\beta_j\in\mathbb{F}^\times$, $d_j\in\mathbb{F}$, $0\le h_j\le p-1$, et

$$\mathrm{Mat}(a^{(j)}) \in I_2 + \mathrm{M}_2\Big(T_{K,\sigma_j}^{q-1}\mathbb{F}[\![T_{K,\sigma_j}^{q-1}]\!]\Big),$$

ce qui détermine de façon unique $\mathrm{Mat}(a^{(j)})$.

Lorsque $\overline{r}_p$ est suffisamment générique (comme dans l'Exemple 1.5.1), on définit

$$D_A^\otimes(\overline{r}_p) \overset{\text{déf}}{=} \bigotimes_{A,0\le j\le f-1} \left(A\otimes_{\mathbb{F}(\!(T_{K,\sigma_j}^{q-1})\!)} D_{K,\sigma_j}(\overline{r}_p)^{[\mathbb{F}_q^\times]}\right) \tag{1.10}$$

avec les actions canoniques de φ et $\mathcal{O}_K^\times$, où le plongement $\mathbb{F}((T_{K,\sigma_j}^{q-1})) \hookrightarrow A$ envoie T_{K,σ_j}^{q-1} sur $\varphi(Y_j)/Y_j \in A$. Il s'agit d'un $(\varphi, \mathcal{O}_K^\times)$-module étale sur A de rang 2^f. Le résultat principal est le suivant, qui généralise le Théorème 1.4.1.

Théorème 1.5.2 (Theorem 5.1.1)**.** *Soit π comme en (1.1). Supposons que $\overline{r}_p$ est maximalement non-scindée (ou de manière équivalente, $|W(\overline{r}_p)| = 1$) et suffisamment générique (voir la condition (v) au § 5.1 pour une signification précise). On a alors un isomorphisme de $(\varphi, \mathcal{O}_K^\times)$-modules étales*

$$D_A(\pi) \cong D_A^\otimes(\overline{r}_p(1)),$$

où $\overline{r}_p(1)$ est le tordu de Tate de $\overline{r}_p$. En particulier, $D_A(\pi)$ ne dépend que de $\overline{r}_p$.

Le Théorème 1.5.2 est démontré par [BHH$^+$c] lorsque $\overline{r}_p$ est semisimple. En utilisant la description explicite de $D_A^\otimes(\overline{r}_p)$ dans l'exemple 1.5.1 et (1.10), ainsi que les résultats sur $D_A(\pi)$ (voir §1.4 et §4), on est réduit au calcul de certaines constantes apparaissant sur le diagramme $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ (voir ci-dessus le Théorème 1.2.3 pour ces diagrammes). Lorsque $\overline{r}_p$ est maximalement non-scindée (et suffisamment générique), ces constantes sont calculées par [BD14] en termes du module de Fontaine–Laffaille associé à $\overline{r}_p$ ([FL82]). On remarque que notre méthode devrait s'appliquer à $W(\overline{r}_p)$ arbitraire une fois que l'on aura calculé les constantes correspondantes apparaissant sur le diagramme $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ (voir le Théorème 1.2.3).

La définition de $D_A^\otimes(\overline{r}_p)$ dans (1.10) présente l'inconvénient que le plongement $\mathbb{F}((T_{K,\sigma_j}^{q-1})) \hookrightarrow A$ ne commute pas à l'action de $\mathcal{O}_K^\times$. Par conséquent, cette définition ne fonctionne que pour $\overline{r}_p$ suffisamment générique, où le $(\varphi, \mathcal{O}_K^\times)$-module de Lubin-Tate $D_K(\overline{r}_p)$ a une forme relativement simple comme dans l'exemple 1.5.1, et il existe une action canonique de $\mathcal{O}_K^\times$ commutant avec l'action de φ. Pour $\overline{r}_p$ général, il ne peut y avoir d'action de $\mathcal{O}_K^\times$ sur $D_A^\otimes(\overline{r}_p)$ commutant avec l'action de φ si on définit encore $D_A^\otimes(\overline{r}_p)$ comme en (1.10).

Pour résoudre ce problème, Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$c]) donne une définition plus conceptuelle du $(\varphi, \mathcal{O}_K^\times)$-module étale $D_A^\otimes(\overline{r}_p)$ en utilisant des espaces perfectoïdes, que l'on rappelle brièvement. Par les résultats de [Far20] et [FF18], il existe un isomorphisme naturel $K^\times$-équivariant (où p agit comme φ sur chaque Y_j)

$$m : \mathbb{F}[\![Y_0^{1/p^\infty}, \dots, Y_{f-1}^{1/p^\infty}]\!] \xrightarrow{\sim} \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}, \dots, T_{K,\sigma_0}^{1/p^\infty}]\!]^{\Delta \rtimes \mathfrak{S}_f} \ (f \text{ copies de } T_{K,\sigma_0}),$$

où T_{K,σ_0} est la variable de Lubin–Tate, $\Delta \overset{\text{déf}}{=} \left\{(k_i) \in (K^\times)^f, \prod_i k_i = 1\right\}$ et le groupe symétrique $\mathfrak{S}_f$ permute les T_{K,σ_0}. Par conséquent, m induit un morphisme

$$m : X_{LT} \overset{\text{déf}}{=} \mathrm{Spa}\left(\mathbb{F}((T_{K,\sigma_0}^{1/p^\infty})), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!]\right)^{\times_{\mathbb{F}} f} \to X_{\mathcal{O}_K} \overset{\text{déf}}{=} \mathrm{Spa}\left(\mathbb{F}[\![Y_0^{1/p^\infty}, \dots, Y_{f-1}^{1/p^\infty}]\!]\right),$$

où on utilise le raccourci $\mathrm{Spa}(R)$ pour le spectre adique $\mathrm{Spa}(R, R)$. Il existe un sous-ensemble ouvert perfectoïde affinoïde $U \cong \mathrm{Spa}(A_\infty, A_\infty^\circ) \subseteq X_{\mathcal{O}_K}$, où A_∞ est le perfectisé-complété de A. De plus, la restriction $m : m^{-1}(U) \to U$ est un $\Delta \rtimes \mathfrak{S}_f$-torseur pro-étale. Pour $\overline{r}_p$ arbitraire, en prenant le produit tensoriel extérieur du $(\varphi^f, \mathcal{O}_K^\times)$-module de Lubin-Tate associé $D_{K,\sigma_0}(\overline{r}_p)$ avec lui-même, on obtient un fibré $(K^\times)^f \rtimes \mathfrak{S}_f$-équivariant sur X_{LT}, donc sur $m^{-1}(U)$ par restriction. En utilisant la descente pro-étale ([SW20]), on obtient un fibré $K^\times$-équivariant sur $U \cong \mathrm{Spa}(A_\infty, A_\infty^\circ)$. En prenant les sections globales et en utilisant un résultat de descente pour le Frobenius, on obtient finalement un $(\varphi, \mathcal{O}_K^\times)$-module étale sur A. On obtient donc un foncteur $\overline{r}_p \mapsto D_A^\otimes(\overline{r}_p)$. En particulier, cela donne une définition de $D_A^\otimes(\overline{r}_p)$ pour $\overline{r}_p$ arbitraire. On conjecture alors que cette définition fonctorielle de $D_A^\otimes(\overline{r}_p)$ est la bonne.

Conjecture 1.5.3 ([BHH$^+$c]). *Soit π comme en (1.1). On a un isomorphisme de $(\varphi, \mathcal{O}_K^\times)$-modules étale (pour $\overline{r}_p$ arbitraire).*

$$D_A(\pi) \cong D_A^\otimes(\overline{r}_p(1)).$$

En particulier, $D_A(\pi)$ ne dépend que de $\overline{r}_p$.

Le théorème suivant donne une description explicite de $D_A^\otimes(\overline{r}_p)$ pour $\overline{r}_p$ arbitraire (à dimension 2), généralisant le résultat de [BHH$^+$c] où $\overline{r}_p$ était supposé semisimple. En particulier, avec le Théorème 1.5.2, cela montre la conjecture 1.5.3 lorsque $\overline{r}_p$ est maximalement non-scindée et suffisamment générique (le cas semisimple étant traité par [BHH$^+$c]).

Théorème 1.5.4 (Theorem 5.5.10). (i) *Supposons que $\overline{r}_p$ est suffisamment générique (comme dans l'exemple 1.5.1). Alors la définition perfectoïde de $D_A^\otimes(\overline{r}_p)$ coïncide avec la définition dans (1.10) (donnée par la recette $T_{K,\sigma_j}^{q-1} \mapsto \varphi(Y_j)/Y_j$).*

(ii) *Pour $\overline{r}_p$ arbitraires (à dimension 2), on a une description explicite de $D_A^\otimes(\overline{r}_p)$. Voir le Théorème 5.5.10 pour plus de détails. En particulier, la recette $T_{K,\sigma_j}^{q-1} \mapsto \varphi(Y_j)/Y_j$ ne fonctionne plus, et la recette correcte implique au moins $\varphi(Y_j)/Y_j$ et $\varphi(Y_{j-1})/Y_{j-1}$.*

Enfin, on remarque que les preuves du Théorème 1.5.2 et du Théorème 1.5.4 sont très calculatoires. Il existera peut-être un jour des preuves plus conceptuelles qui éviteront les hypothèses de généricité sur $\overline{r}_p$ et les calculs techniques, prouvant ainsi complètement la conjecture 1.5.3. C'est une direction possible de recherche pour le futur.

Chapter 2

General Introduction

2.1 The hypothetical mod p Langlands correspondence for GL_2

Let p be a prime number and K be a finite extension of $\mathbb{Q}_p$. The mod p Langlands program for GL_2, initiated by Breuil in [Bre03], asks for a possible correspondence between certain admissible smooth mod p representations of $\mathrm{GL}_2(K)$ and continuous 2-dimensional mod p representations of the Galois group $\mathrm{Gal}(\overline{K}/K)$.

The case $K = \mathbb{Q}_p$ is well-understood by the work of Breuil, Colmez, Emerton, Kisin, Paškūnas, etc. We list the following two properties of the mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

- Colmez ([Col10]) constructed a functor (known as the Colmez's functor) from the category of admissible finite length mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to the category of finite-dimensional continuous mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, using Fontaine's category of (φ, Γ)-modules ([Fon90]) as an intermediate step. This gives a functorial way to realize the mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

- Emerton ([Eme11]) proved that the mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ satisfies the local-global compatibility, in the sense that it can be realized in the H^1 of (towers of) modular curves.

However, when $K \neq \mathbb{Q}_p$, the situation becomes much more complicated. For example, there are many more supersingular mod p representations of $\mathrm{GL}_2(K)$, and we don't have a classification of these representations ([BP12]). Moreover, they are not of finite presentation ([Sch15],[Wu21]), and it is impossible (so far) to write down explicitly one of these representations. Motivated by the local-global compatibility result for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([Eme11]), we are interested in the mod p representations of $\mathrm{GL}_2(K)$ coming from the cohomology of Shimura curves.

We introduce the global setup. Let F be a totally real number field that is inert at p (for simplicity in this introduction). Let D be a quaternion algebra with center F which is split at p and at exactly one infinite place. For each compact open subgroup $U \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$ where $\mathbb{A}_F^\infty$ is the set of finite adèles of F, we denote by X_U the associated smooth projective Shimura curve over F. Let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$, which is considered as the coefficient field from now on. We denote $K \overset{\text{def}}{=} F_p$ the completion of F at p and $f \overset{\text{def}}{=} [K : \mathbb{Q}_p]$. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathbb{F}_q$ be the residue field of K (hence $q = p^f$). We fix a compact open subgroup $U^p \subseteq (D \times_F \mathbb{A}_F^{\infty,p})^\times$. Then we consider the following admissible

smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$:

$$\pi \overset{\text{def}}{=} \varinjlim_{U_p} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\left(\overline{r}, H^1_{\text{ét}}(X_{U^pU_p} \times_F \overline{F}, \mathbb{F})\right), \tag{2.1}$$

where the inductive limit runs over compact open subgroups $U_p \subseteq (D \times_F K)^\times \cong \mathrm{GL}_2(K)$, and $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ is a continuous absolutely irreducible representation such that $\pi \neq 0$.

One of the expectations of the mod p Langlands program is that the representation π as in (2.1) can be used to realize a mod p Langlands correspondence for $\mathrm{GL}_2(K)$. More precisely, one has the following hope.

Hope 2.1.1. *For π as in (2.1), there exists an integer $d \geq 1$ depending on $\overline{r}$ and U^p such that*

$$\pi \cong \pi\left(\overline{r}|_{\mathrm{Gal}(\overline{K}/K)}\right)^{\oplus d}.$$

Here for $\overline{\rho}$ a continuous 2-dimensional mod p representation of $\mathrm{Gal}(\overline{K}/K)$, we denote by $\pi(\overline{\rho})$ the conjectural admissible smooth mod p representation of $\mathrm{GL}_2(K)$ corresponding to it.

Hence, one of the key questions in the mod p Langlands program is to understand the $\mathrm{GL}_2(K)$-representation π as in (2.1). By choosing the compact open subgroup $U^p \subseteq (D \times_F \mathbb{A}_F^{\infty,p})^\times$ carefully, we assume that we are in a "multiplicity one" situation in the sense that $d = 1$ in Hope 2.1.1, which is the first case to consider. For simplicity, we make this assumption from now on in this introduction (unless stated otherwise).

When $F = \mathbb{Q}$ and $D = M_2(\mathbb{Q})$, the representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ as in (2.1) is well-understood by [Eme11], and Hope 2.1.1 is true. However, as soon as $K \neq \mathbb{Q}_p$, this question becomes particularly difficult. At this moment we only know a very small piece of the representation π. The aim of my thesis is to understand more about the representation π following the path of the recent results of Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$23],[BHH$^+$b],[BHH$^+$c]) and to show that some properties of π only depend on the restriction $\overline{r}_p \overset{\text{def}}{=} \overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$. The behavior of π is different when $\overline{r}_p$ is semisimple or not. Several results of [BHH$^+$23], [BHH$^+$b], and [BHH$^+$c] only deal with the case when $\overline{r}_p$ is semisimple, and the non-semisimple case is much more delicate than the semisimple case. The theme of my thesis is to generalize these results to all $\overline{r}_p$ (including non-semisimple $\overline{r}_p$), where we need more elaborate representation-theoretic arguments.

2.2 Overview of some previous results

We begin with an overview of some previous results on the representation π as in (2.1), which have the common aim of determining certain finite-dimensional invariant subspaces of the restriction of π to $\mathrm{GL}_2(\mathcal{O}_K)$ and to prove that they only depend on $\overline{r}_p$.

The first step towards the understanding of the representation π is to study its $\mathrm{GL}_2(\mathcal{O}_K)$-socle $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$, which is a direct sum of Serre weights of $\mathrm{GL}_2(\mathbb{F}_q)$, i.e. absolutely irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{F}$. These Serre weights are predicted by [BDJ10]. This is thought of as the weight part of Serre's conjecture, generalizing the original Serre's conjecture ([Ser87]), and is now a theorem. From now on, we assume that $p > 5$ and $\overline{r}_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible.

Theorem 2.2.1 ([EGS15],[GLS15])**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is generic in the sense of [BP12, Def. 11.7]. Then we have*

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi \cong \bigoplus_{\sigma \in W(\overline{r}_p^\vee)} \sigma,$$

where $W(\overline{r}_p^\vee)$ is the set of Serre weights of $\overline{r}_p^\vee$ defined in [BDJ10, §3]. In particular, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$ *only depends on* $\overline{r}_p$.

The next step is to study the $\mathrm{GL}_2(\mathcal{O}_K)$-representation π^{K_1} where $K_1 \overset{\text{def}}{=} 1 + p\,\mathrm{M}_2(\mathcal{O}_K)$, which contains $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$ as a subrepresentation.

Theorem 2.2.2 ([LMS22],[HW18],[Le19])**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is generic in the sense of [BP12, Def. 11.7]. Then we have*

$$\pi^{K_1} \cong D_0(\overline{r}_p^\vee),$$

where $D_0(\overline{r}_p^\vee)$ is an explicit (finite-dimensional) representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{F}$ constructed by Breuil-Paškūnas ([BP12, §13]). In particular, π^{K_1} only depends on $\overline{r}_p$.

We say that a diagram $\mathcal{D}$ is a $\mathrm{GL}_2(\mathcal{O}_K)$-representation D_0 such that $D_0^{K_1} = D_0$ together with an automorphism Π on $D_0^{I_1}$ whose square acts by a nonzero scalar, where $I_1 \overset{\text{def}}{=} \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ is the pro-p-Iwahori subgroup. Assume that $\overline{r}_p$ is generic in the sense of [BP12, Def. 11.7], then Breuil and Paškūnas constructed a family of diagrams attached to $\overline{r}_p$ such that $D_0 = D_0(\overline{r}_p^\vee)$.

Since the representation π as in (2.1) has a central character, one obtains a diagram $\mathcal{D}(\pi)$ with $D_0 = \pi^{K_1}$ and Π given by the action of $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. In particular, since Π normalizes I_1 and I where $I \overset{\text{def}}{=} \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ is the Iwahori subgroup, it maps an I-character χ to its conjugation χ^s by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. By Theorem 2.2.2, $\mathcal{D}(\pi)$ is isomorphic to one of the diagrams attached to $\overline{r}_p$ defined by Breuil and Paškūnas. One has the following refinement of Theorem 2.2.2, which singles out a unique diagram $\mathcal{D}$ in the above family, subject to a local-global compatibility condition when $\overline{r}_p$ is sufficiently generic.

Theorem 2.2.3 ([DL21])**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is sufficiently generic (see [DL21, §1] for a precise meaning). Then the diagram $\mathcal{D}(\pi)$ only depends on $\overline{r}_p$.*

One of the common main tools of the proof of Theorem 2.2.1, Theorem 2.2.2 and Theorem 2.2.3 is the patching functor defined by Emerton-Gee-Savitt ([EGS15]) building on the work of Taylor and Wiles ([TW95]), and of Kisin ([Kis09]). It is an exact functor M_∞ from continuous representations of $\mathrm{GL}_2(\mathcal{O}_K)$ over finite type $W(\mathbb{F})$-modules to finite type R_∞-modules with additional properties, where R_∞ is the patched deformation ring and is isomorphic to a power series ring over the universal framed deformation ring of $\overline{r}_p^\vee$ in our case (see [EGS15] for a precise statement). This functor enables us to transfer statements from the GL_2 side to the Galois side. Once we have enough information on the Galois deformation rings, we are able to use the Breuil-Mézard philosophy ([EG14]) to deduce properties on the GL_2 side. We will see examples of this kind of argument in §2.3 below.

2.3 The Gelfand–Kirillov dimension of π

The first result is about the Gelfand–Kirillov dimension of π as in (2.1), which measures the growth of the dimension of the invariant subspaces of π under principal congruence subgroups. More precisely, we define the Gelfand–Kirillov dimension of π (see [EP20]) to be the unique integer $\dim_{\mathrm{GL}_2(K)}(\pi)$ such that there exists $a \leq b$ in $\mathbb{R}_{>0}$ satisfying

$$a \leq \frac{\dim_{\mathbb{F}}(\pi^{K_n})}{p^{n\dim_{\mathrm{GL}_2(K)}(\pi)}} \leq b$$

for all $n \geq 1$, where $K_n \overset{\text{def}}{=} 1 + p^n \mathrm{M}_2(\mathcal{O}_K)$ for $n \geq 1$.

Theorem 2.3.1 (Theorem 3.1.1). *Let π be as in (2.1). Assume that $\overline{r}_p$ is sufficiently generic (see condition (iv) in §3.1 for a precise meaning). Then we have*

$$\dim_{\mathrm{GL}_2(K)}(\pi) = f.$$

Theorem 2.3.1 is proved by [BHH$^+$23] when $\overline{r}_p$ is semisimple and proved by [HW22] when $\overline{r}_p$ is non-semisimple using a different method. On one hand, the method of [HW22] only works in the non-semisimple case. On the other hand, it turns out that the method of [BHH$^+$23] can be generalized to the non-semisimple case, and this was not noticed before. We adapt the method of [BHH$^+$23] to the non-semisimple case. This provides a uniform proof of Theorem 2.3.1.

The proof of Theorem 2.3.1 involves the computation of new cases of Galois deformation rings, which enables us to make further use of the patching functor and hence go beyond π^{K_1} as in Theorem 2.2.2. We give an overview of the proof of Theorem 2.3.1 following closely [BHH$^+$23, §1] and indicate what needs to be changed in the non-semisimple case.

Let Z_1 be the center of K_1 and $\mathfrak{m}_{K_1}$ be the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![K_1/Z_1]\!]$. We still denote by $\mathfrak{m}_{K_1}$ the ideal of $\mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$ generated by $\mathfrak{m}_{K_1}$ under the natural inclusion $\mathbb{F}[\![K_1/Z_1]\!] \hookrightarrow \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$ when there is no possible confusion. By the crucial arguments of Gee and Newton in [GN20, Appendix A], we know that $\dim_{\mathrm{GL}_2(K)}(\pi) \geq f$ for π as in (2.1). In order to get the upper bound on $\dim_{\mathrm{GL}_2(K)}(\pi)$, we need to apply the following representation-theoretic theorem to π as in (2.1). This is a special case of [BHH$^+$23, Thm. 6.4.7] when $\overline{r}_p$ is semisimple, and is a consequence of [HW22, Prop. 4.20] and [BHH$^+$23, Prop. 6.4.6] when $\overline{r}_p$ is non-semisimple. We refer to Theorem 3.5.3 below for a more general statement, which applies to π not necessarily in the "multiplicity one" situation.

Theorem 2.3.2 (Theorem 3.5.3). *Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character. Assume that*

(i) *we have $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi) \cong \bigoplus_{\sigma \in W(\overline{r}_p^\vee)} \sigma$;*
(ii) *for each $\sigma \in W(\overline{r}_p^\vee)$, we have $[\pi[\mathfrak{m}_{K_1}^2]|_{\mathrm{GL}_2(\mathcal{O}_K)} : \sigma] = 1$;*
(iii) *we have $\pi^{I_1} \cong D_0(\overline{r}_p^\vee)^{I_1}$ as I-representations.*

Then $\dim_{\mathrm{GL}_2(K)}(\pi) \leq f$. Here, $\pi[\mathfrak{m}_{K_1}^2]$ is the set of elements of π (viewed as a module over $\mathbb{F}[\![K_1/Z_1]\!]$) annihilated by $\mathfrak{m}_{K_1}^2$, and $[\pi[\mathfrak{m}_{K_1}^2]|_{\mathrm{GL}_2(\mathcal{O}_K)} : \sigma]$ is the multiplicity of σ in the semisimplification of $\pi[\mathfrak{m}_{K_1}^2]$ as $\mathrm{GL}_2(\mathcal{O}_K)$-representations.

Then we need to show that the representation π as in (2.1) satisfies the conditions of Theorem 2.3.2. The conditions (i) and (iii) follow from Theorem 2.2.1 and Theorem 2.2.2, hence we will focus on the condition (ii) of Theorem 2.3.2. Since we have $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma, \pi) = 1$ for $\sigma \in W(\overline{r}_p^\vee)$ by the condition (i), it suffices to show that

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{Proj}_{\widetilde{\Gamma}} \sigma, \pi) \leq 1 \ \forall\, \sigma \in W(\overline{r}_p^\vee), \tag{2.2}$$

where $\widetilde{\Gamma} \overset{\text{def}}{=} \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}^2$, and $\mathrm{Proj}_{\widetilde{\Gamma}} \sigma$ is the projective envelope of σ in the category of $\widetilde{\Gamma}$-modules.

We let M_∞ be the patching functor as in §2.2. From the construction of M_∞, for any finite-dimensional representation V of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$, we have

$$\mathrm{Hom}_{\mathbb{F}}(M_\infty(V)/\mathfrak{m}_\infty, \mathbb{F}) \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(V, \pi), \tag{2.3}$$

where $\mathfrak{m}_\infty$ is the maximal ideal of R_∞. Combining (2.2) and (2.3), it suffices to show that

$$M_\infty(\mathrm{Proj}_{\widetilde{\Gamma}}\,\sigma) \text{ is a cyclic } R_\infty\text{-module } \forall\,\sigma \in W(\overline{r}_p^\vee). \tag{2.4}$$

We fix $\sigma \in W(\overline{r}_p^\vee)$. To prove (2.4) for σ, we lift the $\widetilde{\Gamma}$-module $\mathrm{Proj}_{\widetilde{\Gamma}}\,\sigma$ over $\mathbb{F}$ to characteristic zero following [BHH$^+$23, §7.3] so that we can use local-global compatibility properties of M_∞. We let $P_\sigma \overset{\mathrm{def}}{=} \mathrm{Proj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$ be the projective envelope of σ in the category of $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_q)]$-modules and let $\widetilde{P}_\sigma$ be the projective $\mathcal{O}[\mathrm{GL}_2(\mathbb{F}_q)]$-module lifting P_σ. We fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and let $\sigma_j \overset{\mathrm{def}}{=} \sigma_0 \circ \varphi^j$ for $j \in \mathbb{Z}$, where $\varphi : x \mapsto x^p$ is the arithmetic Frobenius on $\mathbb{F}_q$. We still denote by σ_j the corresponding embedding $\mathcal{O}_K \hookrightarrow W(\mathbb{F})$. For $0 \le j \le f-1$, we define the $\mathrm{GL}_2(\mathcal{O}_K)$-representation over $W(\mathbb{F})$

$$R_{2,j} \overset{\mathrm{def}}{=} \left(\mathrm{Sym}^2 W(\mathbb{F})^2 \otimes \det{}^{-1}\right)^{(j)} \otimes_{W(\mathbb{F})} \widetilde{P}_\sigma,$$

where "(j)" means that $\mathrm{GL}_2(\mathcal{O}_K)$ acts via the embedding $\sigma_j : \mathcal{O}_K \hookrightarrow W(\mathbb{F})$. For each j there is an isomorphism

$$R_{2,j}/pR_{2,j} \cong P_\sigma \oplus P_{\sigma_j^+} \oplus P_{\sigma_j^-}$$

for some Serre weights σ_j^+ and σ_j^-, which induces an inclusion

$$\iota_j : P_\sigma \hookrightarrow R_{2,j}/pR_{2,j}.$$

Then we define a $\mathrm{GL}_2(\mathcal{O}_K)$-lattice $R'_{2,j}$ in $R_{2,j}[1/p]$ by

$$R'_{2,j} \overset{\mathrm{def}}{=} P_\sigma \times_{R_{2,j}/p} R_{2,j} = \{x \in R_{2,j} : (x \bmod pR_{2,j}) \in \iota_j(P_\sigma)\}.$$

We let $L_{-1} \overset{\mathrm{def}}{=} \widetilde{P}_\sigma$. For $0 \le j \le f-1$, we define a $\mathrm{GL}_2(\mathcal{O}_K)$-lattice L_j in $L_{-1}[1/p] \oplus \left(\bigoplus_{j'=0}^{j} R_{2,j}[1/p]\right)$ by

$$L_j \overset{\mathrm{def}}{=} \left\{ \begin{array}{r} \left(x, (x_{j'})_{0\le j'\le j}\right) \in L_{-1} \oplus (\oplus_{j'=0}^{j} R_{2,j}) : (x_{j'} \bmod pR_{2,j'}) = (x \bmod pL_{-1}) \\ \text{via } \iota_{j'} : L_{-1}/pL_{-1} \hookrightarrow R_{2,j'}/pR_{2,j'}\ \forall\, 0 \le j' \le j \end{array} \right\},$$

which is equivalent to defining

$$L_j = L_{j-1} \times_{P_\sigma} R'_{2,j} \tag{2.5}$$

for $0 \le j \le f-1$. Then we have $L_{f-1}/pL_{f-1} \cong \mathrm{Proj}_{\widetilde{\Gamma}}\,\sigma$. By the exactness of M_∞, it suffices to show that the R_∞-module $M_\infty(L_{f-1})$ is cyclic.

By [Le19, Thm. 4.9], we know that the R_∞-module $M_\infty(\widetilde{P}_\sigma)$ is cyclic. By [BHH$^+$23, Thm. 8.3.4], we know that the R_∞-module $M_\infty(R'_{2,j})$ is cyclic for each $0 \le j \le f-1$ when $\overline{r}_p$ is semisimple. The proof of [BHH$^+$23, Thm. 8.3.4] uses the standard dévissage techniques as in [EGS15, §10] and [Le19, Lemma 4.5], and can be easily generalized to all $\overline{r}_p$. Then one can use (2.5) and the cyclicity of $M_\infty(\widetilde{P}_\sigma)$ and $M_\infty(R'_{2,j})$ to prove that $M_\infty(L_{f-1})$ is cyclic by induction.

For simplicity, we only talk about the first step of the induction. By the exactness of M_∞, we have the equality of R_∞-modules

$$M_\infty(L_0) = M_\infty(\widetilde{P}_\sigma) \times_{M_\infty(P_\sigma)} M_\infty(R'_{2,0}). \tag{2.6}$$

We already know that each term on the RHS of (2.6) is cyclic. To prove that $M_\infty(L_0)$ is cyclic, it suffices to show that

$$\mathrm{Ann}_{R_\infty}\left(M_\infty(P_\sigma)\right) = \mathrm{Ann}_{R_\infty}\left(M_\infty(\widetilde{P}_\sigma)\right) + \mathrm{Ann}_{R_\infty}\left(M_\infty(R'_{2,0})\right). \tag{2.7}$$

Each term of (2.7) has an interpretation as a certain Galois deformation ring. We let $R_{\overline{r}_p^\vee}$ be the universal framed deformation ring of $\overline{r}_p^\vee$. For τ a tame inertial type, we let $R_{\overline{r}_p^\vee}^{(1,0),\tau}$ (resp. $R_{\overline{r}_p^\vee}^{(2,-1)_0,\tau}$) be the maximal reduced, $\mathcal{O}$-flat quotient of $R_{\overline{r}_p^\vee}$ that parametrizes potentially crystalline lifts of $\overline{r}_p^\vee$ of inertial type τ and parallel Hodge–Tate weights $(1,0)$ (resp. Hodge–Tate weights $(2,-1)$ in the embedding $\sigma_0 : K \hookrightarrow W(\mathbb{F})[1/p]$ and $(1,0)$ elsewhere). We let

$$\mathfrak{p}_\tau^{(1,0)} \overset{\text{def}}{=} \mathrm{Ker}(R_{\overline{r}_p^\vee} \twoheadrightarrow R_{\overline{r}_p^\vee}^{(1,0),\tau});$$
$$\mathfrak{p}_\tau^{(2,-1)_0} \overset{\text{def}}{=} \mathrm{Ker}(R_{\overline{r}_p^\vee} \twoheadrightarrow R_{\overline{r}_p^\vee}^{(2,-1)_0,\tau}).$$

By some detour (see [BHH+23, §1] for more details), to prove (2.7) we are reduced to the proof of the following subtle (non-)congruence property.

Theorem 2.3.3 (Proposition 3.4.3.3)**.** *We have*

$$p \in \cap_\tau \mathfrak{p}_\tau^{(1,0)} + \mathfrak{p}_{\tau_0}^{(2,-1)_0},$$

where τ runs over the tame inertial types such that σ is a Jordan–Hölder factor in the mod p semisimplification of $\sigma(\tau)$ (here $\sigma(\tau)$ is the irreducible smooth representation of $\mathrm{GL}_2(\mathcal{O}_K)$ associated by Henniart to τ in the appendix to [BM02]), and τ_0 is any tame inertial type such that the set of irreducible constituents of the mod p semisimplification of $\sigma(\tau_0)$ contains the set $W(\overline{r}_p^\vee)$ (which exists).

When $\overline{r}_p$ is semisimple, Theorem 2.3.3 is proved in [BHH+23, Prop. 4.3.3] by an explicit computation of potentially crystalline deformation rings using the machinery of Kisin modules, which was first suggested by Breuil and then developed by [Kis06] and [LLHLM18]. We generalize the computation of potentially crystalline deformation rings to the non-semisimple case. This completes the proof of Theorem 2.3.1.

As a byproduct of the proof of Theorem 2.3.1, we can determine the structure of the $\mathfrak{m}_{K_1}^2$-torsion part of the representation π in (2.1), which is a generalization of Theorem 2.2.2. This is a special case of [BHH+23, Thm. 1.9] when $\overline{r}_p$ is semisimple, and is proved in [HW22, Thm. 1.4] when $\overline{r}_p$ is non-semisimple. We refer to Theorem 3.6.3.1(ii) below for a more general statement, which applies to π not necessarily in the "multiplicity one" situation.

Theorem 2.3.4 (Theorem 3.6.3.1(ii))**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is sufficiently generic (as in Theorem 2.3.1). Then we have*

$$\pi[\mathfrak{m}_{K_1}^2] \cong \widetilde{D}_0(\overline{r}_p^\vee),$$

where $\widetilde{D}_0(\overline{r}_p^\vee)$ is an explicit (finite-dimensional) representation of $\widetilde{\Gamma}$ over $\mathbb{F}$ defined in [HW22, §4]. In particular, $\pi[\mathfrak{m}_{K_1}^2]$ only depends on $\overline{r}_p$.

As an intermediate step to prove Theorem 2.3.1, we also deduce an important property of the associated graded module of the dual $\pi^\vee \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$ for π as in (2.1), which generalizes the result of [BHH+23] where $\overline{r}_p$ was assumed to be semisimple (this property is also proved by [HW22] when $\overline{r}_p$ is non-semisimple using a different method). This leads to an abelian subcategory of the category of admissible smooth representations of $\mathrm{GL}_2(K)$ that has desirable finiteness properties and will be introduced in §2.4 below. We denote by $\mathfrak{m}_{I_1}$ the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![I_1/Z_1]\!]$.

Theorem 2.3.5. *Let π be as in (2.1). Assume that $\overline{r}_p$ is sufficiently generic (as in Theorem 2.3.1). Then the graded module $\mathrm{gr}_{\mathfrak{m}_{I_1}} \pi^\vee$ over the graded algebra $\mathrm{gr}_{\mathfrak{m}_{I_1}} \mathbb{F}[\![I_1/Z_1]\!]$ is annihilated by an explicit two-sided ideal J, and the quotient ring $\mathrm{gr}_{\mathfrak{m}_{I_1}} \mathbb{F}[\![I_1/Z_1]\!]/J$ is commutative and is isomorphic to*

$$\mathbb{F}[y_0, z_0, \dots, y_{f-1}, z_{f-1}]/(y_0 z_0, \dots, y_{f-1} z_{f-1}).$$

2.4 The functor $\pi \mapsto D_A(\pi)$

Using Theorem 2.3.5, Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$c]) constructed an exact functor D_A from a nice subcategory of the category of admissible smooth representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ to the category of multivariable $(\varphi, \mathcal{O}_K^\times)$-modules, which is a generalization of the Colmez's functor ([Col10]). Then the basic question is to determine the structure of $D_A(\pi)$ for π as in (2.1), which can be used to deduce properties of π.

First we recall the definition of the ring A, which is a multivariable analog of $\mathbb{F}(\!(X)\!)$. We let $N_0 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$. For $0 \le j \le f-1$, we define

$$Y_j \overset{\text{def}}{=} \sum_{a \in \mathbb{F}_q^\times} \sigma_0(a)^{-p^j} \begin{pmatrix} 1 & [a] \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[\![N_0]\!],$$

where $[a] \in \mathcal{O}_K^\times$ is the Teichmüller lift of $a \in \mathbb{F}_q^\times$. Then we have $\mathbb{F}[\![N_0]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$. We define

$$A \overset{\text{def}}{=} \mathbb{F}[\![N_0]\!]\,[1/(Y_0 \cdots Y_{f-1})]^\wedge,$$

where the completion is with respect to the $(Y_0, \dots, Y_{f-1})$-adic topology on $\mathbb{F}[\![N_0]\!]$. There is an $\mathbb{F}$-linear action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![N_0]\!]$ given by multiplication on $N_0 \cong \mathcal{O}_K$, and an $\mathbb{F}$-linear Frobenius φ on $\mathbb{F}[\![N_0]\!]$ given by multiplication by p on $N_0 \cong \mathcal{O}_K$. They extend canonically by continuity to commuting continuous $\mathbb{F}$-linear actions of φ and $\mathcal{O}_K^\times$ on A. Then an étale $(\varphi, \mathcal{O}_K^\times)$-module over A is by definition a finite free A-module endowed with a semi-linear Frobenius φ and a commuting continuous semi-linear action of $\mathcal{O}_K^\times$ such that the image of φ generates everything.

For π an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with central character, we view $\pi^\vee$ as a finitely generated $\mathbb{F}[\![I_1/Z_1]\!]$-module and endow it with the $\mathfrak{m}_{I_1}$-adic topology. We define

$$D_A(\pi) \overset{\text{def}}{=} \mathbb{F}[\![N_0]\!][1/(Y_0 \cdots Y_{f-1})] \widehat{\otimes}_{\mathbb{F}[\![N_0]\!]} \pi^\vee,$$

where the completion is with respect to the tensor product topology. The $\mathcal{O}_K^\times$-action on $\pi^\vee$ given by $f \mapsto f \circ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (for $a \in \mathcal{O}_K^\times$) extends by continuity to $D_A(\pi)$, and the ψ-action on $\pi^\vee$ given by $f \mapsto f \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induces a continuous A-linear map

$$\beta : D_A(\pi) \to A \otimes_{\varphi, A} D_A(\pi). \tag{2.8}$$

We let $\mathcal{C}$ be the abelian category of admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with central characters such that the graded module $\mathrm{gr}_{\mathfrak{m}_{I_1}} \pi^\vee$ is annihilated by J^n for some $n \in \mathbb{Z}_{\ge 0}$ (see Theorem 2.3.5). For π in $\mathcal{C}$, $D_A(\pi)$ is a finite free A-module by [BHH$^+$b, Cor. 3.1.2.9] and [BHH$^+$c, Remark. 2.6.2]. If moreover β is an isomorphism, then its inverse $\beta^{-1} = \mathrm{id} \otimes \varphi$ makes $D_A(\pi)$ an étale $(\varphi, \mathcal{O}_K^\times)$-module. In particular, when $K = \mathbb{Q}_p$ the above construction recovers the Colmez's functor ([Col10]). Our main result is the following.

Theorem 2.4.1 (Theorem 4.1.1)**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is sufficiently generic (see condition (v) in §5.1 for a precise meaning). Then π is in $\mathcal{C}$, β in (2.8) is an isomorphism and*

$$\mathrm{rank}_A\, D_A(\pi) = 2^f.$$

By Theorem 2.3.5, we know that π is in $\mathcal{C}$. By [BHH$^+$b, Thm. 3.3.2.3] we know that $\mathrm{rank}_A D_A(\pi) \le 2^f$. Theorem 2.4.1 is proved by [BHH$^+$c, Thm. 3.1.3] when $\overline{r}_p$ is semisimple. We generalize the proof of [BHH$^+$c] to the non-semisimple case, which is seriously more delicate.

The proof of Theorem 2.4.1 is by an explicit construction of an A-basis of the dual étale $(\varphi, \mathcal{O}_K^\times)$-module $\mathrm{Hom}_A(D_A(\pi), A)$ for π as in (2.1). As in [BHH$^+$c, (87)], there is a canonical A-linear injection

$$\mu_* : \mathrm{Hom}_A(D_A(\pi), A) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F}). \tag{2.9}$$

For $\underline{i} = (i_0, \dots, i_{f-1}) \in \mathbb{Z}^f_{\ge 0}$, we set $\|\underline{i}\| \overset{\mathrm{def}}{=} \sum_{j=0}^{f-1} i_j$ and we write $\underline{Y}^{\underline{i}}$ for $\prod_{j=0}^{f-1} Y_j^{i_j} \in \mathbb{F}[\![N_0]\!]$. Then we have the following proposition.

Proposition 2.4.2 ([BHH$^+$c])**.** *Let π be as in (2.1).*

(i) *The set $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ can be identified with the set of sequences $(x_{\underline{i}})_{\underline{i} \in \mathbb{Z}^f}$ with $x_{\underline{i}} \in \pi$ and*

  (a) *$\underline{Y}^{\underline{k}} x_{\underline{i}} = x_{\underline{i}-\underline{k}}$ for all $\underline{i} \in \mathbb{Z}^f$ and $\underline{k} \in \mathbb{Z}^f_{\ge 0}$;*

  (b) *there exists $d \in \mathbb{Z}$ such that $x_{\underline{i}} \in \pi[\mathfrak{m}_{I_1}^{\|\underline{i}\|+d+1}]$ for all $\underline{i} \in \mathbb{Z}^f$ (where $\pi[\mathfrak{m}_{I_1}^j] \overset{\mathrm{def}}{=} 0$ if $j \le 0$).*

(ii) *A sequence $(x_{\underline{i}})_{\underline{i} \in \mathbb{Z}^f} \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ lies in the image of μ_* as in (2.9) if and only if it satisfies the following finiteness condition:*

$$\left\{ \underline{i} \in \mathbb{Z}^f : x_{\underline{i}} \ne 0, \|\underline{i}\| = M \right\} \text{ is finite } \forall\, M \in \mathbb{Z}.$$

We are going to define 2^f sequences $x_J = (x_{J,\underline{i}})_{\underline{i} \in \mathbb{Z}^f} \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ indexed by the subsets $J \subseteq \mathcal{J} \overset{\mathrm{def}}{=} \{0, 1, \dots, f-1\}$. Then we prove that the sequences x_J for $J \subseteq \mathcal{J}$ lie in the image of μ_* and form an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$. We give an overview of the construction in the case $\overline{r}_p$ is maximally non-split (or equivalently, $|W(\overline{r}_p)| = 1$) for simplicity.

Step 1. We define $x_{J,\underline{0}}$ for all $J \subseteq \mathcal{J}$.

By letting $d = 0$ in Proposition 2.4.2(i)(b), we see that $x_{J,\underline{0}} \in \pi^{I_1}$ for all $J \subseteq \mathcal{J}$. From Theorem 2.2.2 we know that π^{I_1} is a direct sum of 2^f distinct I-characters, which can be parametrized by the subsets $J \subseteq \mathcal{J}$. We fix a choice of a nonzero I-eigencharacter $x_{J,\underline{0}} \in \pi^{I_1}$ for each $J \subseteq \mathcal{J}$.

Step 2. We define $x_{J,\underline{f}}$ for all $J \subseteq \mathcal{J}$.

By Theorem 2.2.2 we have $\pi^{K_1} \cong D_0(\overline{r}_p^\vee)$, which is explicitly known and contains π^{I_1} as a subset. Hence it is natural to enlarge our sequences a little bit by constructing suitable elements of π^{K_1} subject to the conditions in Proposition 2.4.2. More precisely, for each $J \subseteq \mathcal{J}$ we define $x_{J,\underline{f}}$ to be the unique $\left(\begin{smallmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{smallmatrix}\right)$-eigencharacter in π^{K_1} satisfying

$$\begin{cases} Y_j^{f+1} x_{J,\underline{f}} = 0 \ \forall\, j \in \mathcal{J} \\ (Y_0 \cdots Y_{f-1})^f x_{J,\underline{f}} = x_{J,\underline{0}}. \end{cases}$$

Step 3. We define $x_{J,\underline{i}}$ for all $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$.

First, using Proposition 2.4.2(i)(a), for $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $i_j \le f$ for all $j \in \mathcal{J}$ we define

$$x_{J,\underline{i}} \overset{\mathrm{def}}{=} \underline{Y}^{\underline{f}-\underline{i}} x_{J,\underline{f}}.$$

For $J \subseteq \mathcal{J}$, we define $J+1 \overset{\text{def}}{=} \{(j+1 \bmod f) \in \mathcal{J} : j \in J\}$. We define $\delta : \mathbb{Z}^f \to \mathbb{Z}^f$ by $\delta(\underline{i})_j \overset{\text{def}}{=} i_{j+1}$ if $0 \le j \le f-2$ and $\delta(\underline{i})_{f-1} \overset{\text{def}}{=} i_0$. Using the relation between the elements $x_{J,\underline{f}} \in \pi^{K_1}$, we are able to extend the definition of $x_{J,\underline{i}}$ to all $\underline{i} \in \mathbb{Z}^f$ inductively using the relation

$$\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}} = \sum_{J' \subseteq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',p\delta(\underline{i})+\underline{c}^{J,J'}},$$

where $\varepsilon_J \in \{\pm 1\}$, $\mu_{J,J'} \in \mathbb{F}$ are certain constants depending on the choice of $x_{J,\underline{0}}$, and $\underline{c}^{J,J'} \in \mathbb{Z}^f$ satisfies $-1 \le c_j^{J,J'} \le 2p$ for all $j \in \mathcal{J}$. We refer to Theorem 4.6.4 below for a precise definition of the sequences x_J which works for all $\overline{r}_p$.

Step 4. We prove that each sequence x_J satisfies the finiteness condition in Proposition 2.4.2, which guarantees that it lies in the image of μ_* as in (2.9). Once we prove that $x_J \in \mathrm{Hom}_A(D_A(\pi), A)$ for all J, it is not difficult to conclude that they form an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$. This completes the proof of Theorem 2.4.1.

By analyzing the submodule structure of π^{K_1} (for π as in (2.1)) and the sequences x_J in more details, we can prove the following generalization of Theorem 2.4.1.

Theorem 2.4.3 (Theorem 4.1.2)**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is sufficiently generic (as in Theorem 2.4.1). Then for π_1 a subrepresentation of π, we have*

$$\mathrm{rank}_A D_A(\pi_1) = \left|\mathrm{JH}(\pi_1^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})\right|,$$

where $\mathrm{JH}(\pi_1^{K_1})$ is the set of Jordan–Hölder factors of $\pi_1^{K_1}$ as a $\mathrm{GL}_2(\mathcal{O}_K)$-representation, and $\overline{\rho}^{\mathrm{ss}}$ is the semisimplification of $\overline{\rho}$.

As one application of Theorem 2.4.3 to the understanding of π as in (2.1), we prove that π is finitely generated as a $\mathrm{GL}_2(K)$-representation, generalizing the result of [BHH$^+$b] where $\overline{r}_p$ was assumed to be semisimple (this finite generation result is also proved by [HW22] when $\overline{r}_p$ is non-semisimple using a completely different method).

Theorem 2.4.4 (Corollary 4.11.3)**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is sufficiently generic (as in Theorem 2.4.1). Then as a $\mathrm{GL}_2(K)$-representation, π is generated by $D_0(\overline{r}_p^\vee)$.*

Moreover, we remark that a priori we do not know that π is of finite length as a $\mathrm{GL}_2(K)$-representation. Most importantly, Theorem 2.4.3 is crucially needed to prove that π is of finite length (in the non-semisimple case) in a forthcoming work of Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$a]).

2.5 The functor $\overline{r}_p \mapsto D_A^{\otimes}(\overline{r}_p)$

Besides the rank of $D_A(\pi)$, Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$c]) gives a conjectural description of $D_A(\pi)$ as an étale $(\varphi, \mathcal{O}_K^\times)$-module for π as in (2.1). The construction uses the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-module (see [KR09]) associated to $\overline{r}_p$ as an intermediate step, which can be made quite explicit. We give an example of the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-module $D_K(\overline{r}_p)$ when $\overline{r}_p$ is sufficiently generic. We refer to §5.2 for the explicit description of $D_K(\overline{r}_p)$ for arbitrary (2-dimensional) reducible $\overline{r}_p$, and refer to [BHH$^+$c, Lemma 2.1.6] when $\overline{r}_p$ is irreducible.

Example 2.5.1. *We assume that $\overline{r}_p$ is reducible and sufficiently generic as in (5.69). Then the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-module $D_K(\overline{r}_p)$ associated to $\overline{r}_p$ can be described explicitly as follows ($a \in \mathcal{O}_K^\times$):*

$$\begin{cases} D_K(\overline{r}_p) & = & \prod_{j=0}^{f-1} D_{K,\sigma_j}(\overline{r}_p) \\ D_{K,\sigma_j}(\overline{r}_p) & = & \mathbb{F}(\!(T_{K,\sigma_j})\!)e_0^{(j)} \oplus \mathbb{F}(\!(T_{K,\sigma_j})\!)e_1^{(j)} \\ \varphi(e_0^{(j+1)}\ e_1^{(j+1)}) & = & (e_0^{(j)}\ e_1^{(j)})\operatorname{Mat}(\varphi^{(j)}) \\ a(e_0^{(j)}\ e_1^{(j)}) & = & (e_0^{(j)}\ e_1^{(j)})\operatorname{Mat}(a^{(j)}), \end{cases}$$

where T_{K,σ_j} is a suitable Lubin–Tate variable, the indices j are understood modulo f,

$$\operatorname{Mat}(\varphi^{(j)}) = \begin{pmatrix} \alpha_j T_{K,\sigma_j}^{-(q-1)h_j} & \beta_j d_j \\ 0 & \beta_j \end{pmatrix}$$

for suitable $\alpha_j, \beta_j \in \mathbb{F}^\times$, $d_j \in \mathbb{F}$, $0 \le h_j \le p-1$, and

$$\operatorname{Mat}(a^{(j)}) \in I_2 + \mathrm{M}_2\Big(T_{K,\sigma_j}^{q-1}\mathbb{F}[\![T_{K,\sigma_j}^{q-1}]\!]\Big)$$

which uniquely determines $\operatorname{Mat}(a^{(j)})$.

When $\overline{r}_p$ is sufficiently generic (as in Example 2.5.1), we define

$$D_A^\otimes(\overline{r}_p) \overset{\text{def}}{=} \bigotimes_{A, 0\le j\le f-1} \left(A \otimes_{\mathbb{F}(\!(T_{K,\sigma_j}^{q-1})\!)} D_{K,\sigma_j}(\overline{r}_p)^{[\mathbb{F}_q^\times]} \right) \tag{2.10}$$

with canonical actions of φ and $\mathcal{O}_K^\times$, where the embedding $\mathbb{F}(\!(T_{K,\sigma_j}^{q-1})\!) \hookrightarrow A$ sends T_{K,σ_j}^{q-1} to $\varphi(Y_j)/Y_j \in A$. This is an étale $(\varphi, \mathcal{O}_K^\times)$-module over A of rank 2^f. Our main result is the following, which generalizes Theorem 2.4.1.

Theorem 2.5.2 (Theorem 5.1.1)**.** *Let π be as in (2.1). Assume that $\overline{r}_p$ is maximally non-split (or equivalently, $|W(\overline{r}_p)| = 1$) and sufficiently generic (see condition (v) in §5.1 for a precise meaning). Then we have an isomorphism of étale $(\varphi, \mathcal{O}_K^\times)$-modules*

$$D_A(\pi) \cong D_A^\otimes(\overline{r}_p(1)),$$

where $\overline{r}_p(1)$ is the Tate twist of $\overline{r}_p$. In particular, $D_A(\pi)$ only depends on $\overline{r}_p$.

Theorem 2.5.2 is proved by [BHH+c] when $\overline{r}_p$ is semisimple. Using the explicit description of $D_A^\otimes(\overline{r}_p)$ in Example 2.5.1 and (2.10), together with the results on $D_A(\pi)$ (see §2.4 and §4), we are reduced to the computation of some constants coming from the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ (see above Theorem 2.2.3 for diagrams). When $\overline{r}_p$ is maximally non-split (and sufficiently generic), these constants are computed by [BD14] in terms of the Fontaine–Laffaille module associated to $\overline{r}_p$ ([FL82]). We remark that our method should apply to arbitrary $W(\overline{r}_p)$ once we compute the corresponding constants coming from the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ in general (see Theorem 2.2.3).

The definition of $D_A^\otimes(\overline{r}_p)$ in (2.10) has the drawback that the embedding $\mathbb{F}(\!(T_{K,\sigma_j}^{q-1})\!) \hookrightarrow A$ does not respect the action of $\mathcal{O}_K^\times$. As a result, this definition only works for $\overline{r}_p$ sufficiently generic, where the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-module $D_K(\overline{r}_p)$ has a relatively simple form as in Example 2.5.1, and there exists a canonical action of $\mathcal{O}_K^\times$ commuting with the action of φ. For general $\overline{r}_p$, there

could be no action of $\mathcal{O}_K^\times$ on $D_A^\otimes(\overline{r}_p)$ commuting with the action of φ if we still define $D_A^\otimes(\overline{r}_p)$ as in (2.10).

To solve this problem, Breuil-Herzig-Hu-Morra-Schraen ([BHH$^+$c]) gives a more conceptual definition of the étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^\otimes(\overline{r}_p)$ using perfectoid spaces, which we briefly recall. By the results of [Far20] and [FF18], there is a natural $K^\times$-equivariant isomorphism (where p acts as φ on each Y_j)

$$m : \mathbb{F}[\![Y_0^{1/p^\infty}, \dots, Y_{f-1}^{1/p^\infty}]\!] \xrightarrow{\sim} \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}, \dots, T_{K,\sigma_0}^{1/p^\infty}]\!]^{\Delta \rtimes \mathfrak{S}_f} \ (f \text{ copies of } T_{K,\sigma_0}),$$

where T_{K,σ_0} is the Lubin–Tate variable, $\Delta \overset{\text{def}}{=} \{(k_i) \in (K^\times)^f, \prod_i k_i = 1\}$ and the symmetric group $\mathfrak{S}_f$ permutes the T_{K,σ_0}. Hence m induces a map

$$m : X_{LT} \overset{\text{def}}{=} \mathrm{Spa}\left(\mathbb{F}(\!(T_{K,\sigma_0}^{1/p^\infty})\!), \mathbb{F}[\![T_{K,\sigma_0}^{1/p^\infty}]\!]\right)^{\times_{\mathbb{F}} f} \to X_{\mathcal{O}_K} \overset{\text{def}}{=} \mathrm{Spa}\left(\mathbb{F}[\![Y_0^{1/p^\infty}, \dots, Y_{f-1}^{1/p^\infty}]\!]\right),$$

where we use the shorthand $\mathrm{Spa}(R)$ for the adic spectrum $\mathrm{Spa}(R, R)$. There is an affinoid perfectoid open subset $U \cong \mathrm{Spa}(A_\infty, A_\infty^\circ) \subseteq X_{\mathcal{O}_K}$, where A_∞ is the completed perfection of A. Moreover, the restriction $m : m^{-1}(U) \to U$ is a pro-étale $\Delta \rtimes \mathfrak{S}_f$-torsor. For arbitrary $\overline{r}_p$, by taking the self exterior tensor product of the associated Lubin–Tate $(\varphi^f, \mathcal{O}_K^\times)$-module $D_{K,\sigma_0}(\overline{r}_p)$, we get a $(K^\times)^f \rtimes \mathfrak{S}_f$-equivariant vector bundle on X_{LT}, hence on $m^{-1}(U)$ by restriction. Using pro-étale descent ([SW20]), we get a $K^\times$-equivariant vector bundle on $U \cong \mathrm{Spa}(A_\infty, A_\infty^\circ)$. Taking global sections and using Frobenius descent, we finally get an étale $(\varphi, \mathcal{O}_K^\times)$-module over A. Hence we get a functor $\overline{r}_p \mapsto D_A^\otimes(\overline{r}_p)$. In particular, this gives the definition of $D_A^\otimes(\overline{r}_p)$ for arbitrary $\overline{r}_p$. Then it is conjectured that this functorial definition of $D_A^\otimes(\overline{r}_p)$ is the correct one.

Conjecture 2.5.3 ([BHH$^+$c])**.** *Let π be as in (2.1). We have an isomorphism of étale $(\varphi, \mathcal{O}_K^\times)$-modules (for arbitrary $\overline{r}_p$)*

$$D_A(\pi) \cong D_A^\otimes(\overline{r}_p(1)).$$

In particular, $D_A(\pi)$ only depends on $\overline{r}_p$.

The following theorem gives an explicit description of $D_A^\otimes(\overline{r}_p)$ for arbitrary (2-dimensional) $\overline{r}_p$, generalizing the result of [BHH$^+$c] where $\overline{r}_p$ was assumed to be semisimple. In particular, together with Theorem 2.5.2 this proves Conjecture 2.5.3 when $\overline{r}_p$ is maximally non-split and sufficiently generic (the semisimple case being treated by [BHH$^+$c]).

Theorem 2.5.4 (Theorem 5.5.10)**.** (i) *Assume that $\overline{r}_p$ is semisimple, or non-semisimple and sufficiently generic (as in Example 2.5.1). Then the perfectoid definition of $D_A^\otimes(\overline{r}_p)$ coincides with the definition in (2.10) (given by the recipe $T_{K,\sigma_j}^{q-1} \mapsto \varphi(Y_j)/Y_j$).*

(ii) *For arbitrary (2-dimensional) $\overline{r}_p$, we have an explicit description of $D_A^\otimes(\overline{r}_p)$. See Theorem 5.5.10 for details. In particular, the recipe $T_{K,\sigma_j}^{q-1} \mapsto \varphi(Y_j)/Y_j$ no longer works, and the correct recipe involves at least $\varphi(Y_j)/Y_j$ and $\varphi(Y_{j-1})/Y_{j-1}$.*

Finally, we remark that the proof of Theorem 2.5.2 and Theorem 2.5.4 is very computational. There may exist a more conceptual proof one day, which will hopefully avoid the genericity assumptions on $\overline{r}_p$ and the technical computations, hence completely proving Conjecture 2.5.3. This is a possible direction in the future.

Chapter 3

On the mod p cohomology for GL_2

3.1 Introduction

Let p be a prime number. The mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is completely known by the work of [Bre03], [Col10], etc. However, the situation becomes much more complicated when we consider $\mathrm{GL}_2(L)$ for L a nontrivial finite extension of $\mathbb{Q}_p$. For example, there are many more irreducible admissible smooth representations of $\mathrm{GL}_2(L)$ over $\overline{\mathbb{F}}_p$ and we don't have a classification of these representations ([BP12]).

Motivated by the local-global compatibility results of Emerton ([Eme11]), we study the representations of $\mathrm{GL}_2(L)$ that come from geometry and hope that these representations can realize a mod p Langlands correspondence for $\mathrm{GL}_2(L)$.

We begin with the global setup following [BHH$^+$23]. Let F be a totally real number field which is unramified at places above p. Let D be a quaternion algebra with center F which is split at places above p and at exactly one infinite place. Let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$, which is considered as the coefficient field. For each compact open subgroup V of $(D\otimes_F \mathbb{A}_F^\infty)^\times$, we denote by X_V the associated smooth projective Shimura curve over F. Let v be a fixed place of F above p. Let F_v be the completion of F at v and $f \overset{\text{def}}{=} [F_v:\mathbb{Q}_p]$. We define an admissible smooth representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$ of the form

$$\pi \overset{\text{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\left(\overline{r}, H^1_{\text{ét}}(X_{V^vV_v}\times_F \overline{F}, \mathbb{F})\right), \tag{3.1}$$

where V^v is a fixed compact open subgroup of $(D\otimes_F \mathbb{A}_F^{\infty,v})^\times$, the inductive limit runs over compact open subgroups V_v of $\mathrm{GL}_2(F_v)$, and $\overline{r}:\mathrm{Gal}(\overline{F}/F)\to \mathrm{GL}_2(\mathbb{F})$ is a continuous absolutely irreducible representation such that $\pi\neq 0$.

One of the aims of the mod p Langlands program is to understand these representations. In the case $F=\mathbb{Q}$ and $D=\mathrm{M}_2(\mathbb{Q})$, the representations π of $\mathrm{GL}_2(\mathbb{Q}_p)$ are well-understood by [Eme11] and they realize a local mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. However as soon as $F_v\neq\mathbb{Q}_p$, the representations π of $\mathrm{GL}_2(F_v)$ are far from being known, though there have been several results on various invariant subspaces attached to these representations π ([EGS15], [Le19], etc.). In this work, we go one step further and show that under some genericity assumptions the Gelfand–Kirillov dimension of π equals to f.

To state the main theorem, we recall the Gelfand–Kirillov dimension (see [BHH$^+$23, Introduction]). Let $K_1 \overset{\text{def}}{=} 1+pM_2(\mathcal{O}_{F_v})\subseteq \mathrm{GL}_2(\mathcal{O}_{F_v})$ and Z_1 be the center of K_1. Let $\mathbb{F}[\![K_1/Z_1]\!]$ be the Iwasawa algebra of K_1/Z_1. We define the Gelfand–Kirillov dimension of π to be

$$\dim_{\mathrm{GL}_2(F_v)}(\pi) \overset{\text{def}}{=} 3f - \min\left\{i\in\mathbb{Z}_{\geq 0} : \mathrm{Ext}^i_{\mathbb{F}[\![K_1/Z_1]\!]}(\pi^\vee, \mathbb{F}[\![K_1/Z_1]\!])\neq 0\right\},$$

where $\pi^\vee \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$ is the algebraic dual of π.

Let I_{F_v} be the inertia subgroup of F_v. Let k' be the quadratic extension of the residue field of F_v and fix an embedding $k' \hookrightarrow \mathbb{F}$. Let $\omega_{f'}$ be the corresponding Serre's fundamental character of level f' for $f' \in \{f, 2f\}$. We make the following assumptions on $\overline{r}$:

(i) $\overline{r}|_{G_{F(\sqrt[p]{1})}}$ is absolutely irreducible;

(ii) for $w \nmid p$ such that either D or $\overline{r}$ ramifies, the framed deformation ring $R_{\overline{r}_w}$ of $\overline{r}_w \overset{\text{def}}{=} \overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ over the Witt vectors $W(\mathbb{F})$ is formally smooth;

(iii) for $w|p$, $\overline{r}|_{I_{F_w}}$ is generic in the sense of [BP12, §11];

(iv) $\overline{r}|_{I_{F_v}}$ is of one of the following forms up to twist:

(a) $\begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix}$ with $12 \le r_i \le p-15$,

(b) $\begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & 0 \\ 0 & \omega_{2f}^{p^f\sum_{j=0}^{f-1}(r_j+1)p^j} \end{pmatrix}$ with $13 \le r_0 \le p-14$ and $12 \le r_i \le p-15$ for $i > 0$.

Our main result is the following.

Theorem 3.1.1 (Corollary 3.6.3.1 (iv)). *Keep all the above assumptions on $F, D, \overline{r}$. Let $V^v = \prod_{w \neq v} V_w$ with $V_w = \mathrm{GL}_2(\mathcal{O}_{F_w})$ if neither D nor $\overline{r}$ ramifies at w, and $V_w \subseteq 1 + pM_2(\mathcal{O}_{F_w})$ if $w \mid p$ and $w \neq v$. Then for π as in (3.1) we have $\dim_{\mathrm{GL}_2(F_v)}(\pi) = f$.*

Theorem 3.1.1 is proved by [BHH+23, Thm. 1.1] when $\overline{r}$ is semisimple at v and is proved by [HW22, Thm. 1.1] in the "minimal" case (i.e. $\pi = \pi_v^D(\overline{r})$, see [HW22, Introduction]) when $\overline{r}$ is non-semisimple at v. So the new case of Theorem 3.1.1 is that we allow arbitrary π when $\overline{r}$ is non-semisimple at v. On one hand, the method of [HW22] only works in the non-semisimple case. On the other hand, it turns out that the method of [BHH+23] can be generalized to the non-semisimple case, and this was not noticed before. We adapt the method of [BHH+23] to the non-semisimple case and give a uniform proof of Theorem 3.1.1. As an intermediate step, [BHH+23] gives an explicit computation of some potentially crystalline deformation rings using the machinery of Kisin modules developed in [Kis06] and [LLHLM18] when $\overline{r}_v$ is semisimple. We generalize the computation of potentially crystalline deformation rings to the non-semisimple case, see Theorem 3.4.2.1 and Theorem 3.4.3.1.

Organization of the chapter

In §3.2, we recall the preliminary notions and results on algebraic groups, tame inertial types, and extension graphs. In §3.3, we recall the machinery of Kisin modules. In §3.4, we use the machinery of Kisin modules to compute explicitly some potentially crystalline deformation rings. In §3.5, we recall the notion of the Gelfand–Kirillov dimension, then recall a result that gives an upper bound for the Gelfand-Kirillov dimensions of some admissible smooth representations of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$. In §3.6 we combine all the previous results and prove Theorem 3.1.1.

Notation

If F is any field, we denote by $G_F \overset{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F, where $\overline{F}$ is a separable closure of F. If F is a local field, let $I_F \subseteq G_F$ be the inertia subgroup and $W_F \subseteq G_F$ be the Weil group. We normalize Artin's reciprocity map $\mathrm{Art}_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ so that uniformizers are sent to geometric Frobenius elements, which are elements of G_F that induce

the geometric Frobenius map on the residue field of F. If F is a number field and v is a place of F, then we write F_v for the completion of F with respect to the place v, and if v is a finite place we write $\mathcal{O}_{F_v}$ for the ring of integers of F_v, Frob_v for an arbitrary geometric Frobenius element at v and k_v for the residue field of $\mathcal{O}_{F_v}$. We also denote by $\mathbb{A}_F^\infty$ the set of finite adèles of F. If F is a perfect field in characteristic p, we denote $W(F)$ the ring of Witt vectors of F. For $x \in F$, we denote by $[x] \in W(F)$ the Techmüller lift of x.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. All number fields are considered as subfields of $\overline{\mathbb{Q}}$. For each prime number ℓ, we fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ as well as an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. All finite extensions of $\mathbb{Q}_\ell$ are considered as subfields of $\overline{\mathbb{Q}}_\ell$.

Let p be a prime number. We write $\varepsilon : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ for the cyclotomic character, ω its mod p reduction and $\widetilde{\omega}$ the Teichmüller lift of ω. We normalize the Hodge–Tate weights so that ε has Hodge–Tate weight 1.

We let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field $\mathbb{F}$. We always assume that E is large enough.

We let K be an unramified extension of $\mathbb{Q}_p$ of degree f with ring of integers $\mathcal{O}_K$ and residue field k. We fix an embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$ and we let $\sigma_j \overset{\text{def}}{=} \sigma_0 \circ \varphi^j$, where $\varphi : x \mapsto x^p$ is the arithmetic Frobenius on k. We still use σ_j to denote the corresponding embedding $K \hookrightarrow E$. We have an identification of $\mathcal{J} \overset{\text{def}}{=} \mathrm{Hom}(k, \mathbb{F})$ with $\mathrm{Hom}(K, E)$ and with $\{0, \dots, f-1\}$ given by $\sigma_j \leftrightarrow j$. We also identify $\mathcal{J}$ with the quotient $\mathbb{Z}/f\mathbb{Z}$ in an evident way so that the addition and subtraction in $\mathcal{J}$ are modulo f.

If G is a group and V is a representation of G on a finite-dimensional E-vector space, we denote by $\overline{V}$ the semisimplification of a G-stable $\mathcal{O}$-lattice in V. If V is a representation of G on a finite-dimensional vector space, we let $\mathrm{JH}(V)$ denote the set of Jordan–Hölder factors of V. If σ is an irreducible representation of G, we let $[V : \sigma]$ be the multiplicity of σ in the semisimplification of V.

For each commutative ring A and $(x_1, \dots, x_n) \in A^n$, we write $\mathrm{Diag}(x_1, \dots, x_n)$ for the diagonal matrix in $\mathrm{M}_n(A)$ whose i-th diagonal entry is x_i. If $\mu \in \mathbb{Z}^n$ and $x \in A$, then we write x^μ for the diagonal matrix $\mathrm{Diag}(x^{\mu_1}, \dots, x^{\mu_n}) \in \mathrm{M}_n(A)$. If $M \in \mathrm{GL}_n(A)$ and $N \in \mathrm{M}_n(A)$ we define $\mathrm{Ad}(M)(N) \overset{\text{def}}{=} MNM^{-1}$. If $s \in S_n$ is a permutation, we let $\dot{s}$ denote the associated permutation matrix, which we also denote by s when there is no possible confusion. We let $\mathrm{sgn}(s) \in \{\pm 1\}$ be the signature of s.

If $h : A \to B$ is a ring homomorphism and M is an A-module, we define the B-module $h^*(M) \overset{\text{def}}{=} M \otimes_A B$.

3.2 Preliminaries

In this section, we give the preliminary notions and results that we will use. We follow closely [BHH+23, §2].

3.2.1 Group theoretic preliminaries

In this subsection, we review some notions related to algebraic groups that we will use.

We consider the algebraic group GL_n defined over $\mathbb{Z}$. Let $T \subseteq \mathrm{GL}_n$ be the diagonal maximal torus and $Z \subseteq \mathrm{GL}_n$ be the center. We write R (resp. $R^\vee$) for the set of roots (resp. coroots) of (GL_n, T) and W for its Weyl group with longest element $\mathfrak{w}$. Let $R^+ \subseteq R$ be the subset of positive roots with respect to the upper triangular Borel. We identify the set of characters $X^*(T) \overset{\text{def}}{=} \mathrm{Hom}(T, \mathbb{G}_m)$ with $\mathbb{Z}^n$ in the standard way.

Example 3.2.1.1. *If $n = 2$, then $W = S_2$, $\mathfrak{w}$ is the nontrivial element of S_2 and $R^+ = \{\alpha\}$, where $\alpha \in X^*(T)$ corresponds to $(1, -1) \in \mathbb{Z}^2$.*

Let $\underline{G}$ be the algebraic group $\left(\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}_p} \mathrm{GL}_{n/\mathcal{O}_K}\right) \times_{\mathbb{Z}_p} \mathcal{O}$ with diagonal maximal torus $\underline{T}$ and center $\underline{Z}$. We write $\underline{R}$ (resp. $\underline{R}^\vee$) for the set of roots (resp. coroots) of $(\underline{G}, \underline{T})$, $\underline{W}$ for its Weyl group and $\underline{R}^+ \subseteq \underline{R}$ for the subset of positive roots with respect to the upper triangular Borel. There is a natural isomorphism $\underline{G} \cong \prod_{\mathcal{J}} \mathrm{GL}_{n/\mathcal{O}}$ induced by the ring homomorphism $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O} \cong \mathcal{O}^{\mathcal{J}}$ defined by $x \otimes y \mapsto (\sigma_j(x)y)_{j\in\mathcal{J}}$. One has similar isomorphisms for $\underline{T}$, $\underline{Z}$, $\underline{R}$, $\underline{R}^\vee$, $\underline{W}$, $\underline{R}^+$ and the character group $X^*(\underline{T})$. There is an action of $\underline{W}$ on $X^*(\underline{T})$ which is compatible with this isomorphism.

Under the identification of $X^*(\underline{T}) \cong \oplus_{\mathcal{J}} X^*(T)$ with $(\mathbb{Z}^n)^f$ as above, for each $\mu \in X^*(\underline{T})$ we can write $\mu = (\mu_j)_{0\le j\le f-1}$ with $\mu_j = (\mu_{j,1}, \dots, \mu_{j,n}) \in \mathbb{Z}^n$. Moreover, if $(a_1, \dots, a_n) \in \mathbb{Z}^n$ we write $(\underline{a_1}, \dots, \underline{a_n})$ to denote the element of $X^*(\underline{T})$ whose corresponding tuple equals $(a_1, \dots, a_n)$ at each place $j \in \mathcal{J}$. For $j \in \mathcal{J}$ we let $\eta_j \in (\mathbb{Z}^n)^f$ be $(n-1, \dots, 1, 0)$ in the j-th coordinate and 0 otherwise. We let $\eta \stackrel{\text{def}}{=} \sum_{j\in\mathcal{J}} \eta_j = (\underline{n-1}, \dots, \underline{1}, \underline{0})$. There is an automorphism π on $X^*(\underline{T})$ defined by $\pi(\mu)_j = \mu_{j-1}$.

Example 3.2.1.2. *If $n = 2$, let $\alpha_j \in X^*(\underline{T})$ be $(1, -1)$ in the j-th coordinate and 0 otherwise. Then we have $\underline{R}^+ = \{\alpha_j : j \in \mathcal{J}\}$.*

Let $\Lambda_R \subseteq X^*(\underline{T})$ be the root lattice of $\underline{G}$. Let $X^*_+(\underline{T}) \subseteq X^*(\underline{T})$ be the set of **dominant weights**, i.e. the set of weights $\lambda \in X^*(\underline{T})$ satisfying $\langle \lambda, \alpha^\vee \rangle \ge 0$ for all $\alpha \in \underline{R}^+$. Let $X_1(\underline{T}) \subseteq X^*_+(\underline{T})$ be the subset of **p-restricted weights**, i.e. the set of weights $\lambda \in X^*(\underline{T})$ satisfying $0 \le \langle \lambda, \alpha^\vee \rangle \le p-1$ for all simple roots $\alpha \in \underline{R}^+$. Let $X_{\mathrm{reg}}(\underline{T}) \subseteq X_1(\underline{T})$ be the subset of weights $\lambda \in X^*(\underline{T})$ satisfying $0 \le \langle \lambda, \alpha^\vee \rangle < p-1$ for all simple roots $\alpha \in \underline{R}^+$. Finally, let $X^0(\underline{T}) \subseteq X_{\mathrm{reg}}(\underline{T})$ be the subset of weights $\lambda \in X^*(\underline{T})$ satisfying $\langle \lambda, \alpha^\vee \rangle = 0$ for all simple roots $\alpha \in \underline{R}^+$.

The lowest alcove is defined as

$$\underline{C}_0 \stackrel{\text{def}}{=} \{\lambda \in X^*(\underline{T}) \otimes_{\mathbb{Z}} \mathbb{R} : 0 < \langle \lambda + \eta, \alpha^\vee \rangle < p \ \forall\, \alpha \in \underline{R}^+\}.$$

Given $N \ge 0$ and $\mu \in \underline{C}_0$ we say that μ is **N-deep** in $\underline{C}_0$ if $N < \langle \mu + \eta, \alpha^\vee \rangle < p - N$ for all $\alpha \in \underline{R}^+$. Thus the existence of an N-deep weight in $\underline{C}_0$ implies $p \ge 2N + 2$.

Example 3.2.1.3. *When $n = 2$, we have*

$$\begin{aligned}
\Lambda_R &= \left\{\lambda \in (\mathbb{Z}^2)^f : \lambda_{j,1} + \lambda_{j,2} = 0 \ \forall\, j \in \mathcal{J}\right\};\\
X^*_+(\underline{T}) &= \left\{\lambda \in (\mathbb{Z}^2)^f : \lambda_{j,1} \ge \lambda_{j,2} \ \forall\, j \in \mathcal{J}\right\};\\
X_1(\underline{T}) &= \left\{\lambda \in (\mathbb{Z}^2)^f : 0 \le \lambda_{j,1} - \lambda_{j,2} \le p-1 \ \forall\, j \in \mathcal{J}\right\};\\
X_{\mathrm{reg}}(\underline{T}) &= \left\{\lambda \in (\mathbb{Z}^2)^f : 0 \le \lambda_{j,1} - \lambda_{j,2} < p-1 \ \forall\, j \in \mathcal{J}\right\};\\
X^0(\underline{T}) &= \left\{\lambda \in (\mathbb{Z}^2)^f : \lambda_{j,1} = \lambda_{j,2} \ \forall\, j \in \mathcal{J}\right\};\\
\underline{C}_0 &= \left\{\lambda \in (\mathbb{R}^2)^f : -1 < \lambda_{j,1} - \lambda_{j,2} < p-1 \ \forall\, j \in \mathcal{J}\right\},
\end{aligned}$$

and $\lambda \in \underline{C}_0$ is N-deep if and only if $N \le \lambda_{j,1} - \lambda_{j,2} \le p - 2 - N$ for all $j \in \mathcal{J}$.

Given $w \in \underline{W}$, we write w_j to denote its j-th component via the identification $\underline{W} \cong \prod_{\mathcal{J}} W$. There is an automorphism π on $\underline{W}$ defined by $\pi(w)_j = w_{j-1}$.

Let $\underline{W}_a \cong \Lambda_R \rtimes \underline{W}$ (resp. $\widetilde{\underline{W}} \cong X^*(\underline{T}) \rtimes \underline{W}$) be the affine Weyl group (resp. extended affine Weyl group) of $\underline{G}$. We denote by t_λ the image of $\lambda \in X^*(\underline{T})$ in $\widetilde{\underline{W}}$. Hence an element of $\widetilde{\underline{W}}$ can be written as $(w, \lambda) \overset{\text{def}}{=} wt_\lambda$ with $w \in \underline{W}$ and $\lambda \in X^*(\underline{T})$, and the multiplication of $\widetilde{\underline{W}}$ is given by

$$(w_1, \lambda_1) \cdot (w_2, \lambda_2) = (w_1 w_2, w_2^{-1}(\lambda_1) + \lambda_2).$$

We identify $\widetilde{\underline{W}}$ with $(\mathbb{Z}^n \rtimes S_n)^f$ (the action of S_n on $\mathbb{Z}^n$ is given by $(s(a))_i = a_{s^{-1}(i)}$).

We have the **p-dot action** of $\widetilde{\underline{W}}$ on $X^*(\underline{T})$ defined as follows: if $\widetilde{w} = wt_\nu \in \widetilde{\underline{W}}$ and $\mu \in X^*(\underline{T})$ then we define

$$\widetilde{w} \cdot \mu \overset{\text{def}}{=} w(\mu + \eta + p\nu) - \eta. \tag{3.2}$$

We let Ω be the stabilizer of the lowest alcove $\underline{C}_0$ in $\widetilde{\underline{W}}$, then one easily checks that $\widetilde{\underline{W}} = \underline{W}_a \rtimes \Omega$. Concretely, when $n = 2$, Ω is the subgroup of $\widetilde{\underline{W}}$ generated by $X^0(\underline{T})$ and $\left\{1, \mathfrak{w} t_{(-1,0)}\right\}^f$.

The choice of the lowest alcove $\underline{C}_0$ endows $\underline{W}_a$ with a Bruhat order by viewing $\underline{W}_a$ as a Coxeter group generated by the walls of the alcove $\underline{C}_0$ (see [Jan03, II.6.3]). We denote this order by $\leq$. It induces a partial order $\leq$ on $\widetilde{\underline{W}}$ by defining $\widetilde{w}_a\omega \leq \widetilde{w}'_a\omega'$ in $\underline{W}_a \rtimes \Omega = \widetilde{\underline{W}}$ if and only if $\widetilde{w}_a \leq \widetilde{w}'_a$ in $\underline{W}_a$ and $\omega = \omega'$ in Ω. We denote by $\widetilde{\underline{W}}^\vee$ the group $\widetilde{\underline{W}}$, endowed with the Bruhat order induced by the choice of the antidominant base alcove, i.e.

$$\underline{C}_0^\vee \overset{\text{def}}{=} \{\lambda \in X^*(\underline{T}) \otimes \mathbb{R} : -p < \langle \lambda + \eta, \alpha^\vee \rangle < 0 \ \forall\, \alpha \in \underline{R}^+\}.$$

There is an anti-isomorphism

$$\begin{aligned} \widetilde{\underline{W}}^\vee &\xrightarrow{\sim} \widetilde{\underline{W}} \\ \widetilde{w} &\mapsto \widetilde{w}^* \end{aligned} \tag{3.3}$$

defined by $((st_\mu)^*)_j \overset{\text{def}}{=} t_{\mu_{f-1-j}} s_{f-1-j}^{-1}$ such that $\widetilde{w}_1 \leq \widetilde{w}_2$ in $\widetilde{\underline{W}}^\vee$ if and only if $\widetilde{w}_2^* \leq \widetilde{w}_1^*$ in $\widetilde{\underline{W}}$ ([LLHL19, Lemma 2.1.3]).

Given $\lambda \in X^*(\underline{T})$, we define

$$\mathrm{Adm}^\vee(t_\lambda) \overset{\text{def}}{=} \left\{\widetilde{w} \in \widetilde{\underline{W}}^\vee : \widetilde{w} \leq t_{w(\lambda)} \text{ for some } w \in \underline{W}\right\} \tag{3.4}$$

to be the **λ-admissible set** with respect to the Bruhat order defined above on $\widetilde{\underline{W}}^\vee$.

Example 3.2.1.4. *Let $n = 2$. We have*

$$\begin{aligned} \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})}) &= \left\{\widetilde{w} \in \widetilde{\underline{W}}^\vee : \widetilde{w}_j \in \{t_{(2,1)}, \mathfrak{w} t_{(2,1)}, t_{(1,2)}\} \ \forall\, j \in \mathcal{J}\right\}; \\ \mathrm{Adm}^\vee(t_{(\underline{3},\underline{0})}) &= \left\{\widetilde{w} \in \widetilde{\underline{W}}^\vee : \widetilde{w}_j \in \{t_{(3,0)}, \mathfrak{w} t_{(3,0)}, t_{(2,1)}, \mathfrak{w} t_{(2,1)}, t_{(1,2)}, \mathfrak{w} t_{(1,2)}, t_{(0,3)}\} \ \forall\, j \in \mathcal{J}\right\}. \end{aligned}$$

We will only use the Bruhat order induced by the choice of the antidominant base alcove. From now on, we use $\widetilde{\underline{W}}$ to mean $\widetilde{\underline{W}}^\vee$ for simplicity. Finally we remark that we can consider $\widetilde{\underline{W}}$ as a subgroup of $\mathrm{GL}_n\left(\mathbb{F}((v))\right)^f$ by the injective homomorphism

$$\begin{aligned} \widetilde{\underline{W}} &\hookrightarrow \mathrm{GL}_n\left(\mathbb{F}((v))\right)^f \\ st_\mu &\mapsto (\dot{s}_j v^{\mu_j})_j. \end{aligned} \tag{3.5}$$

3.2.2 Tame inertial types

In this subsection, we review the combinatorial description of tame inertial types.

An **inertial type** of K is a representation $\tau : I_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ with open kernel that can be extended to G_K. An inertial type is said **tame** if it factors through the tame inertial quotient. When $n = 2$, by a result of Henniart (see the appendix to [BM02]), given an inertial type $\tau : I_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, we can associate to it a smooth irreducible $\mathrm{GL}_2(\mathcal{O}_K)$-representation $\sigma(\tau)$ over $\overline{\mathbb{Q}}_p$, normalized as in [BM02, §2.1.1], and we may realize τ and $\sigma(\tau)$ as representations over E if E is large enough.

Now we fix a pair $(s, \mu) \in \underline{W} \times X^*(\underline{T})$. We recall how to associate to it a tame inertial type. We write $s = (s_0, \dots, s_{f-1})$ with $s_j \in S_n$ and $\mu = (\mu_0, \dots, \mu_{f-1})$ with $\mu_j \in \mathbb{Z}^n$. Let $s_\tau \overset{\text{def}}{=} s_0 s_{f-1} s_{f-2} \cdots s_1 \in S_n$. Let $r \in \mathbb{Z}$ be the order of s_τ in S_n. Let K' be the unramified extension of K of degree r and let k' be its residue field. Fix an embedding $\sigma'_0 : k' \hookrightarrow \mathbb{F}$ extending σ_0 and let $\sigma'_{j'} \overset{\text{def}}{=} \sigma_0 \circ \varphi^{j'}$. Let $f' \overset{\text{def}}{=} rf$. We have an identification of $\mathcal{J}' \overset{\text{def}}{=} \mathrm{Hom}(k', \mathbb{F})$ with $\mathrm{Hom}(K', E)$ and with $\{0, \dots, f' - 1\}$ given by $\sigma'_{j'} \leftrightarrow j'$. We also identify it with the quotient $\mathbb{Z}/f'\mathbb{Z}$ in an evident way so that the addition and subtraction in $\mathcal{J}'$ are modulo f'. Under this identification, the restriction of an embedding $k' \hookrightarrow \mathbb{F}$ to k corresponds to the natural projection $\mathbb{Z}/f'\mathbb{Z} \twoheadrightarrow \mathbb{Z}/f\mathbb{Z}$. If $j' \in \mathcal{J}'$ and $j \in \mathcal{J}$ is the image of j' under this projection, we set $s_{j'} \overset{\text{def}}{=} s_j$, $\mu_{j'} \overset{\text{def}}{=} \mu_j$ and $\eta_{j'} \overset{\text{def}}{=} \eta_j$.

We define the tame fundamental character $\omega_{f'} : I_K \to \mathbb{F}^\times$ as the composition

$$I_K = I_{K'} \xrightarrow{\mathrm{Art}_{K'}^{-1}} \mathcal{O}_{K'}^\times \twoheadrightarrow k'^\times \xhookrightarrow{\sigma'_0} \mathbb{F}^\times.$$

We also let $\widetilde{\omega}_{f'} : I_K \to \mathcal{O}^\times$ denote the Teichmüller lift of $\omega_{f'}$. We define $\boldsymbol{\alpha}'_{(s,\mu)} \in (\mathbb{Z}^n)^{f'}$ by

$$\boldsymbol{\alpha}'_{(s,\mu),j'} \overset{\text{def}}{=} s_1^{-1} s_2^{-1} \cdots s_{j'}^{-1}(\mu_{j'} + \eta_{j'}) \in \mathbb{Z}^n,\ 0 \le j' \le f' - 1.$$

In particular, we have $\boldsymbol{\alpha}'_{(s,\mu),j+kf} = s_\tau^{-k} \boldsymbol{\alpha}'_{(s,\mu),j}$ for $0 \le j \le f - 1$ and $0 \le k \le r - 1$. We also define

$$\mathbf{a}^{(0)} \overset{\text{def}}{=} \sum_{i=0}^{f-1} \boldsymbol{\alpha}'_{(s,\mu),i}\, p^i \in \mathbb{Z}^n;$$
$$\mathbf{a}'^{(j')}_{(s,\mu)} \overset{\text{def}}{=} \sum_{i'=0}^{f'-1} \boldsymbol{\alpha}'_{(s,\mu),-j'+i'}\, p^{i'} \in \mathbb{Z}^n,\ 0 \le j' \le f' - 1.$$

In particular, we have $\mathbf{a}'^{(0)}_{(s,\mu),i} = \sum_{k=0}^{r-1} \mathbf{a}^{(0)}_{s_\tau^k(i)} p^{fk}$. The following combinatorial description of tame inertial types comes from [LLHL19, Def. 2.2.1], which is based on [Her09, (6.15)].

Definition 3.2.2.1. *Let $(s, \mu) \in \underline{W} \times X^*(\underline{T})$. We define*

$$\tau(s, \mu + \eta) \overset{\text{def}}{=} \bigoplus_{i=1}^n \widetilde{\omega}_{f'}^{\mathbf{a}'^{(0)}_{(s,\mu),i}} = \bigoplus_{i=1}^n \widetilde{\omega}_{f'}^{\sum_{k=0}^{r-1} \mathbf{a}^{(0)}_{s_\tau^k(i)} p^{fk}} : I_K \to \mathrm{GL}_n(\mathcal{O}^\times). \tag{3.6}$$

This is a tame inertial type (i.e. can be extended to G_K), and we write $\overline{\tau}(s, \mu + \eta)$ for its reduction modulo ϖ. The following two definitions come from [LLHL19, Def. 2.2.5].

Definition 3.2.2.2. *Let τ be a tame inertial type and $N \in \mathbb{Z}_{\ge 0}$.*

(i) *We say that τ is N-**generic** if there is an isomorphism $\tau \cong \tau(s, \lambda+\eta)$ for some $s \in \underline{W}$ and some $\lambda \in X^*(\underline{T})$ which is N-deep in alcove $\underline{C}_0$.*

(ii) *A **lowest alcove presentation** of τ is a pair $(s,\mu) \in \underline{W} \times \underline{C}_0$ such that $\tau \cong \tau(s, \mu+\eta)$ (which by definition exists exactly when τ is 0-generic).*

Definition 3.2.2.3. *Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a Galois representation and let $N \in \mathbb{Z}_{\geq 0}$. Let $\overline{\rho}^{\mathrm{ss}}|_{I_K}$ denote the restriction to I_K of the semisimplification of $\overline{\rho}$. We say that $\overline{\rho}$ is N-**generic** if $\overline{\rho}^{\mathrm{ss}}|_{I_K} \cong \overline{\tau}(s,\mu)$ for some $s \in \underline{W}$ and $\mu - \eta \in X^*(\underline{T})$ which is N-deep in alcove $\underline{C}_0$.*

Then we introduce the **orientation** $s'_{\mathrm{or}} \in (S_n)^{f'}$ of $\boldsymbol{\alpha}'_{(s,\mu)}$ as in [LLHL19, Remark 3.2.3], which is defined by

$$s'_{\mathrm{or},j'} \overset{\mathrm{def}}{=} s_1^{-1} s_2^{-1} \cdots s_{f'-1-j'}^{-1},\ 0 \leq j' \leq f'-1. \tag{3.7}$$

In particular, we have $s'_{\mathrm{or},j+kf} = s_\tau^k s'_{\mathrm{or},j}$ for $0 \leq j \leq f-1$ and $0 \leq k \leq r-1$. By definition, we have $(s'_{\mathrm{or},j'})^{-1}\big(\boldsymbol{\alpha}'_{(s,\mu),f'-1-j'}\big) = \mu_{f'-1-j'} + \eta_{f'-1-j'}$. Since the p-adic expansion of $\mathbf{a}'^{(j')}_{(s,\mu)}$ has leading term $\boldsymbol{\alpha}'_{(s,\mu),f'-1-j'}\, p^{f'-1}$, if $\mu \in X^*(\underline{T})$ is 0-deep in alcove $\underline{C}_0$, then the element $(s'_{\mathrm{or},j'})^{-1}\big(\mathbf{a}'^{(j')}_{(s,\mu)}\big) \in X^*(T) \cong \mathbb{Z}^n$ is dominant for each $0 \leq j' \leq f'-1$. We will use the orientation in §3.3.

Example 3.2.2.4. *Let $n=2$. Let $s = (s_0, 1, \dots, 1) \in \underline{W}$ with $s_0 \in S_2$ and $\mu = (\mu_j)_j \in X^*(\underline{T})$ with $\mu_j = (r_j + m_j, m_j) \in \mathbb{Z}^2$. Then we have*

$$\boldsymbol{\alpha}'_{(s,\mu),j} = (r_j + 1 + m_j, m_j) \in \mathbb{Z}^2,\ 0 \leq j \leq f-1;$$
$$\mathbf{a}^{(0)} = \left(\sum_{j=0}^{f-1} (r_j+1+m_j)p^j, \sum_{j=0}^{f-1} m_j p^j\right) \in \mathbb{Z}^2.$$

We have the following two cases.

(i) *If $s_0 = 1$, then $s_\tau = 1$, $f' = f$, $s'_{\mathrm{or}} = (1, \dots, 1)$, and we have*

$$\tau(s, \mu+\eta) \cong \begin{pmatrix} \widetilde{\omega}_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & 0 \\ 0 & 1 \end{pmatrix} \otimes \widetilde{\omega}_f^{\sum_{j=0}^{f-1} m_j p^j};$$

(ii) *If $s_0 = \mathfrak{w}$, then $s_\tau = \mathfrak{w}$, $f' = 2f$, $s'_{\mathrm{or}} = (1, \dots, 1, \mathfrak{w}, \dots, \mathfrak{w})$, and we have*

$$\tau(s, \mu+\eta) \cong \begin{pmatrix} \widetilde{\omega}_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & 0 \\ 0 & \widetilde{\omega}_{2f}^{p^f \sum_{j=0}^{f-1}(r_j+1)p^j} \end{pmatrix} \otimes \widetilde{\omega}_f^{\sum_{j=0}^{f-1} m_j p^j}.$$

In both cases, $\tau(s, \mu+\eta)$ is N-generic if $N \leq r_j \leq p-2-N$ for all j.

3.2.3 Extension graph

In this subsection, we review the description of the extension graph for GL_2. Then we use it to describe certain sets of Serre weights.

Recall that a **Serre weight** of $\mathrm{GL}_n(k)$ is an isomorphism class of an (absolutely) irreducible representation of $\mathrm{GL}_n(k)$ over $\mathbb{F}$. By [GHS18, Lemma 9.2.4], for each $\lambda \in X_1(\underline{T})$ we can associate to it a Serre weight $F(\lambda)$ of $\mathrm{GL}_n(k)$, which induces a bijection

$$F : X_1(\underline{T})/(p-\pi)X^0(\underline{T}) \xrightarrow{\sim} \{\text{Serre weights of } \mathrm{GL}_n(k)\} \tag{3.8}$$

(see §3.2.1 for π). We say that a Serre weight σ is **regular** if $\sigma \cong F(\lambda)$ with $\lambda \in X_{\mathrm{reg}}(\underline{T})$, cf. [Her09, Def. 6.1].

Example 3.2.3.1. *When $n = 2$, let $\lambda = (\lambda_j)_j \in X_1(\underline{T})$ with $\lambda_j = (\lambda_{j,1}, \lambda_{j,2}) \in \mathbb{Z}^2$, we have*

$$F(\lambda) = \bigotimes_{j=0}^{f-1} \left(\left(\mathrm{Sym}^{\lambda_{j,1}-\lambda_{j,2}} k^2 \otimes_k \det^{\lambda_{j,2}} \right) \otimes_{k,\sigma_j} \mathbb{F} \right).$$

Moreover, $F(\lambda)$ is regular if and only if λ is 0-deep in alcove $\underline{C_0}$ if and only if $0 \leq \lambda_{j,1} - \lambda_{j,2} \leq p-2$.

In the rest of this section we let $n = 2$. Consider the algebraic group $G' \overset{\text{def}}{=} \left(\mathrm{Res}_{k/\mathbb{F}_p} \mathrm{SL}_{2/k}\right) \otimes_{\mathbb{F}_p} \mathbb{F}$. Let $\underline{T_1}$ be its maximal torus consisting of diagonal matrices of determinant 1. The corresponding root lattice is identified with Λ_R. Its weight lattice is denoted by Λ_W and we have $\Lambda_W \cong X^*(\underline{T})/X^0(\underline{T})$, which is identified with $\mathbb{Z}^f$ by the isomorphism $[(a_j, b_j)_j] \mapsto (a_j - b_j)_j$.

Fix $\omega \in X^*(\underline{T})$. Since $\widetilde{\underline{W}} = \underline{W}_a \rtimes \Omega$, there is a unique element $\widetilde{\omega} \in \Omega \cap t_{-\pi^{-1}(\omega)}\underline{W}_a$ in $\widetilde{\underline{W}}$ (see §3.2.1 for t). We also define $w_\omega \in \underline{W}$ to be the image of $\widetilde{\omega}$ in $\underline{W}$ under the projection $\widetilde{\underline{W}} \twoheadrightarrow \underline{W}$.

Now we fix $\mu \in X^*(\underline{T})$. Following [LLHLM20, §2.1], we define the **extension graph** associated to μ by

$$\Lambda_W^\mu \overset{\text{def}}{=} \{\omega \in \Lambda_W : 0 \leq \langle \overline{\mu} + \omega, \alpha^\vee \rangle < p-1, \ \forall \alpha \in \underline{R}^+\},$$

where $\overline{\mu}$ is the image of μ in Λ_W. We define a map

$$\begin{aligned} \mathfrak{t}'_\mu : X^*(\underline{T}) &\to X^*(\underline{T})/(p-\pi)X^0(\underline{T}) \\ \omega &\mapsto \widetilde{\omega} \cdot (\mu + \omega), \end{aligned}$$

where "$\cdot$" is the p-dot action defined in (3.2). This map factors through $X^*(\underline{T})/X^0(\underline{T}) = \Lambda_W$, and restricts to a map

$$\mathfrak{t}_\mu : \Lambda_W^\mu \to X_{\mathrm{reg}}(\underline{T})/(p-\pi)X^0(\underline{T}).$$

Remark 3.2.3.2. *If we compose $\mathfrak{t}_\mu$ with the bijection (3.8)*

$$F : X_1^*(\underline{T})/(p-\pi)X^0(\underline{T}) \xrightarrow{\sim} \{\textit{Serre weights of } \mathrm{GL}_2(k)\},$$

then the resulting map $\omega \mapsto F(\mathfrak{t}_\mu(\omega))$ gives a bijection between Λ_W^μ and the set of regular Serre weights of $\mathrm{GL}_2(k)$ with central character $\mu|_{\underline{Z}}$ (see [LLHLM20, Prop. 2.1.4]).

Example 3.2.3.3. *In terms of the identification $\Lambda_W \cong \mathbb{Z}^f$ the map $\mathfrak{t}_\mu$ can be described as follows: We may assume that μ has the form $(r_j, 0)_j \in X^*(\underline{T})$, because for arbitrary $\mu \in X^*(\underline{T})$ one can use the formula $\mathfrak{t}'_{\mu+\nu}(\omega) - \mathfrak{t}'_\mu(\omega) = \widetilde{\omega} \cdot (\mu + \nu) - \widetilde{\omega} \cdot \mu = w_\omega(\nu) = \nu$ (for $\nu \in X^0(\underline{T})$) to reduce to the computation for this form. Write $\omega = (\omega_j)_j = (2n_j + \delta_j)_j \in \Lambda_W^\mu$ with $n_j \in \mathbb{Z}$, $\delta_j \in \{0, 1\}$, then a representative of $\mathfrak{t}_\mu(w)$ in $X_{\mathrm{reg}}(\underline{T})$ is then given by*

$$(\mathfrak{t}_\mu(\omega))_j = \begin{cases} (r_j + n_j + \delta_j, -n_j) & \text{if } \delta_{j+1} = 0 \\ (-n_j - 1, r_j + n_j + \delta_j - p + 1) & \text{if } \delta_{j+1} = 1. \end{cases} \tag{3.9}$$

Moreover, by Example 3.2.3.1 and (3.9) one can compute that

$$F(\mathfrak{t}_\mu(\omega)) = \left(\bigotimes_{j=0}^{f-1} \left(\mathrm{Sym}^{r'_j} k^2 \otimes_{k,\sigma_j} \mathbb{F} \right) \right) \otimes_{\mathbb{F}} \det^{e(\omega)},$$

where

$$r'_j = \begin{cases} r_j + \omega_j & \text{if } 2 \mid \omega_{j+1} \\ p-2-r_j-\omega_j & \text{if } 2 \nmid \omega_{j+1}, \end{cases} \qquad e(\omega) = \frac{1}{2}\left(\delta_0(p^f-1) + \sum_{j=0}^{f-1}(r_j - r'_j)p^j\right). \tag{3.10}$$

Finally we recall the "change of origin" formula for the map t_μ.

Lemma 3.2.3.4 ([LMS22], Prop. 2.5)**.** *Let $\mu \in X^*(\underline{T})$, $\omega \in \Lambda_W^\mu$ and $\lambda \in X^*(\underline{T})$ be such that $\mathfrak{t}_\mu(\omega) \equiv \lambda \mod (p-\pi)X^0(\underline{T})$. Then $\mathfrak{t}_\lambda(\omega') = \mathfrak{t}_\mu(w_\omega^{-1}(\omega') + \omega)$ for all $\omega' \in \Lambda_W^\lambda$. Equivalently, $\mathfrak{t}_\mu(\omega') = \mathfrak{t}_\lambda(w_\omega(\omega' - \omega))$.*

Remark 3.2.3.5. *Keep the assumptions of Lemma 3.2.3.4. If we write $\omega = (a_0, \dots, a_{f-1})$ and $\omega' = (b_0, \dots, b_{f-1})$ with $a_j, b_j \in \mathbb{Z}$, then we have*

$$\mathfrak{t}_\lambda(b_0, \dots, b_{f-1}) = \mathfrak{t}_\mu\left(\sum_{j=0}^{f-1}\big(a_j + (-1)^{a_{j+1}}b_j\big)\overline{\eta}_j\right);$$
$$\mathfrak{t}_\mu(b_0, \dots, b_{f-1}) = \mathfrak{t}_\lambda\left(\sum_{j=0}^{f-1}\big((-1)^{a_{j+1}}(b_j - a_j)\big)\overline{\eta}_j\right).$$

3.3 Kisin modules

In this section, we review the machinery of Kisin modules that are used to compute the Galois deformation rings. We follow closely [LLHLM18],[LLHL19],[LLHLM20],[LLHLM23] as well as [BHH+23, §3].

3.3.1 Kisin modules with tame descent data

In this subsection, we review the notion of Kisin modules with descent data and some related objects.

Throughout this section we fix a 1-generic tame inertial type $\tau : I_K \to \mathrm{GL}_n(\mathcal{O})$ of K and a lowest alcove presentation (s, μ) for τ (hence μ is 0-deep in alcove $\underline{C_0}$ by [LLHL19, Prop. 2.2.15]). We keep the notation of §3.2.2, for example, $s_\tau = s_0 s_{f-1} s_{f-2} \cdots s_1 \in S_n$, r is the order of s_τ, K' is the unramified extension of K of degree r with residue field k' and $f' = fr$. We also define $e' \overset{\text{def}}{=} p^{f'} - 1$.

Fix an e'-th root $\pi' \overset{\text{def}}{=} (-p)^{1/e'} \in \overline{\mathbb{Q}}_p$ of $-p$. Let $L' \overset{\text{def}}{=} K'(\pi')$ and $E(u') \overset{\text{def}}{=} (u')^{e'} + p = v + p$ be the minimal polynomial of π' over K', where we define $v \overset{\text{def}}{=} (u')^{e'}$. Let $\Delta' \overset{\text{def}}{=} \mathrm{Gal}(L'/K') \subseteq \Delta \overset{\text{def}}{=} \mathrm{Gal}(L'/K)$. We define a group homomorphism $\omega_{\pi'} : \Delta' \to W(k')^\times$, $g \mapsto g(\pi')/\pi'$. It is independent of the choice of π' and satisfies $\sigma'_0 \circ \omega_{\pi'} = \widetilde{\omega}_{f'}$.

Let R be a p-adically complete Noetherian local $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}_R$. We define

$$\mathfrak{S}_{L',R} \overset{\text{def}}{=} W(k')[\![u']\!]\widehat{\otimes}_{\mathbb{Z}_p} R = \big(W(k') \otimes_{\mathbb{Z}_p} R\big)[\![u']\!].$$

The ring $\mathfrak{S}_{L',R}$ has a Δ-action defined as follows: For each $g \in \Delta'$, $g(u') = (\omega_{\pi'}(g) \otimes 1)u'$ and g acts trivially on $W(k') \otimes_{\mathbb{Z}_p} R$. Let $\sigma \in \mathrm{Gal}(L'/\mathbb{Q}_p)$ be the lift of the arithmetic Frobenius on $W(k')$ which fixes π'. Then $\sigma^f \in \Delta$ maps to a generator of $\Delta/\Delta' \cong \mathrm{Gal}(K'/K)$ and we define its action on $\mathfrak{S}_{L',R}$ by letting it act trivially on R and u' and act as the f-th power of the arithmetic Frobenius on $W(k')$. In particular, we have

$$\big(\mathfrak{S}_{L',R}\big)^{\Delta=1} = \big(W(k) \otimes_{\mathbb{Z}_p} R\big)\,[\![v]\!].$$

There is also a Frobenius endomorphism $\varphi : \mathfrak{S}_{L',R} \to \mathfrak{S}_{L',R}$, which is $W(k')$-semilinear, R-linear, and $\varphi(u') = (u')^p$. The Frobenius endomorphism and the Δ-action commute with each other. For each $\mathfrak{S}_{L',R}$-module $\mathfrak{M}$ we define the $R[\![u']\!]$-module $\mathfrak{M}^{(j')} \overset{\text{def}}{=} \mathfrak{M} \otimes_{W(k')\otimes_{\mathbb{Z}_p} R, \sigma'^{-j'}} R$, then we have a decomposition $\mathfrak{M} \cong \bigoplus_{j'\in\mathcal{J}'} \mathfrak{M}^{(j')}$.

Definition 3.3.1.1 ([LLHLM18], Def. 2.4)**.** *Let $h \in \mathbb{Z}_{\geq 0}$ and τ, R as above. A **Kisin module** over R with height in $[0,h]$ and type τ is a triple $(\mathfrak{M}, \phi_{\mathfrak{M}}, \{\widehat{g}\}_{g\in\Delta})$, such that*

(i) *$\mathfrak{M}$ is a finitely generated projective module over $\mathfrak{S}_{L',R}$;*
(ii) *the Frobenius map $\phi_{\mathfrak{M}} : \varphi^*(\mathfrak{M}) \to \mathfrak{M}$ is an injective $\mathfrak{S}_{L',R}$-linear map whose cokernel is killed by $E(u')^h$;*
(iii) *for each $g \in \Delta$, $\widehat{g} : \mathfrak{M} \to \mathfrak{M}$ is a g-semilinear map satisfying*
  (a) *the induced map $\widehat{g}^* : g^*(\mathfrak{M}) \xrightarrow{\widehat{g}\otimes\mathrm{id}} \mathfrak{M}$ is a $\mathfrak{S}_{L',R}$-linear isomorphism for all $g \in \Delta$;*
  (b) *$\widehat{g} \circ \widehat{h} = \widehat{gh}$ for all $g, h \in \Delta$;*
  (c) *$\widehat{g} \circ \phi_{\mathfrak{M}} = \phi_{\mathfrak{M}} \circ \varphi^*(\widehat{g})$ on $\varphi^*(\mathfrak{M})$ for all $g \in \Delta$;*
(iv) *for each $0 \leq j' \leq f' - 1$, we have*

$$\mathfrak{M}^{(j')}/u'\mathfrak{M}^{(j')} \cong \tau^\vee \otimes_{\mathcal{O}} R \tag{3.11}$$

as Δ'-representations over R, where $\tau^\vee \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O})$ is the algebraic dual of τ.

Morphisms of Kisin modules over R with height in $[0,h]$ and type τ are defined in the natural way and we denote by $Y^{[0,h],\tau}(R)$ the category of Kisin modules over R with height in $[0,h]$ and type τ. We often omit the additional data and just write $\mathfrak{M} \in Y^{[0,h],\tau}(R)$.

We define $\tau' : I_{K'} = I_K \xrightarrow{\tau} \mathrm{GL}_n(\mathcal{O})$. It is a tame inertial type on K'. We define the category $Y^{[0,h],\tau'}(R)$ of Kisin modules over R with height in $[0,h]$ and type τ' by replacing Δ by Δ' in Definition 3.3.1.1.

Recall that $\sigma \in \mathrm{Gal}(L'/\mathbb{Q}_p)$ is the lift of the arithmetic Frobenius on $W(k')$ which fixes π'. Let $(\tau')^{p^f}$ denote the composition of τ' (which factors through Δ') and the automorphism $g \mapsto g^{p^f}$ on Δ'. For each $\mathfrak{M} \in Y^{[0,h],\tau'}(R)$, we define its Frobenius twist $(\sigma^f)^*(\mathfrak{M}) \in Y^{[0,h],(\tau')^{p^f}}(R)$ as follows: $(\sigma^f)^*(\mathfrak{M}) \overset{\text{def}}{=} \mathfrak{M} \otimes_{W(k'),\sigma^f} W(k')$, the Frobenius map is

$$\phi_{(\sigma^f)^*(\mathfrak{M})} : \varphi^*\left((\sigma^f)^*(\mathfrak{M})\right) \xrightarrow{\sim} (\sigma^f)^*\left(\varphi^*(\mathfrak{M})\right) \xrightarrow{(\sigma^f)^*(\phi_{\mathfrak{M}})} (\sigma^f)^*(\mathfrak{M}),$$

and the Δ'-action is

$$\widehat{g^{p^f}}^*_{(\sigma^f)^*(\mathfrak{M})} : (\widehat{g^{p^f}})^*\left((\sigma^f)^*(\mathfrak{M})\right) \xrightarrow{\sim} (\sigma^f)^*\left(\widehat{g}^*(\mathfrak{M})\right) \xrightarrow{(\sigma^f)^*(\widehat{g}^*)} (\sigma^f)^*(\mathfrak{M}),$$

using that $g \mapsto g^{p^f}$ is an automorphism on Δ'. By definition, a Kisin module $\mathfrak{M} \in Y^{[0,h],\tau'}(R)$ lies in $Y^{[0,h],\tau}(R)$ if and only if its Δ'-action extends to a Δ-action. By [LLHLM18, Prop. 6.6], this is equivalent to the datum of an isomorphism $\iota_{\mathfrak{M}} : (\sigma^f)^*(\mathfrak{M}) \cong \mathfrak{M}$, such that the r-fold composite of

$$\mathfrak{M} \xrightarrow{m\mapsto m\otimes 1} (\sigma^f)^*(\mathfrak{M}) \xrightarrow[\cong]{\iota_{\mathfrak{M}}} \mathfrak{M}$$

is the identity on $\mathfrak{M}$. Note that the first map identifies $\mathfrak{M}^{(j')}$ with $\left((\sigma^f)^*(\mathfrak{M})\right)^{(j'+f)}$.

Definition 3.3.1.2 ([LLHLM20], Def. 3.1.6)**.** *Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$. An **eigenbasis** of $\mathfrak{M}$ is a collection $\beta = (\beta^{(j')})_{j'\in\mathcal{J}'}$, where $\beta^{(j')} = (f_1^{(j')}, \ldots, f_n^{(j')})$ is an $R[\![u']\!]$-basis of $\mathfrak{M}^{(j')}$ such that Δ' acts on $f_i^{(j')}$ as χ_i^{-1}, where $\chi_i \overset{\text{def}}{=} \widetilde{\omega}_{f'}^{\mathbf{a}'^{(0)}_{(s,\mu),i}}$ (see (3.6) for the notation), and satisfies $\iota_{\mathfrak{M}}\big((\sigma^f)^*(\beta^{(j')})\big) = \beta^{(j'+f)}$ for each $j' \in \mathcal{J}'$.*

Since R is assumed to be local, by (3.11) and the fact that Δ' has order prime to p, eigenbases always exist.

Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$. Under the decomposition $\mathfrak{M} \cong \bigoplus_{j' \in \mathcal{J}'} \mathfrak{M}^{(j')}$, the Frobenius map $\phi_{\mathfrak{M}}$ decomposes into $R[\![u']\!]$-linear maps $\phi_{\mathfrak{M}}^{(j')} : \varphi^*(\mathfrak{M}^{(j')}) \to \mathfrak{M}^{(j'+1)}$. The following definition combines [LLHLM18, Def. 2.11] and [LLHLM18, Prop. 2.13].

Definition 3.3.1.3. *Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ with eigenbasis β. For each $j' \in \mathcal{J}'$, the matrix $C_{\mathfrak{M},\beta}^{(j')} \in \mathrm{M}_n(R[\![u']\!])$ of $\phi_{\mathfrak{M}}$ with respect to β is defined by the formula*

$$\phi_{\mathfrak{M}}^{(j')}\left(\varphi^*(\beta^{(j')})\right) = \beta^{(j'+1)} C_{\mathfrak{M},\beta}^{(j')},$$

and the matrix $A_{\mathfrak{M},\beta}^{(j')} \in \mathrm{M}_n(R[\![v]\!])$ is defined by the formula

$$A_{\mathfrak{M},\beta}^{(j')} = \mathrm{Ad}\left((\dot{s}'_{\mathrm{or},j'+1})^{-1}(u')^{-\mathbf{a}'^{(j'+1)}_{(s,\mu)}}\right)\left(C_{\mathfrak{M},\beta}^{(j')}\right).$$

By the paragraph after [LLHLM23, Remark 5.1.7] the matrix $A_{\mathfrak{M},\beta}^{(j')}$ is upper triangular modulo v and only depends on $j' \bmod f$.

To end this subsection, we introduce the Kisin modules with more strict height conditions. These conditions are related to the Hodge–Tate weights of Galois representations. The following definition comes from [CL18, §5] (see also [LLHLM18, Prop. 4.18] for a special case).

Definition 3.3.1.4. *Let $\lambda = (\lambda_{j,1}, \dots, \lambda_{j,n})_j \in X_+^*(\underline{T})$ be a dominant weight such that $\lambda_{j,i} \in \{0, \dots, h\}$ for all $0 \le j \le f-1$, $1 \le i \le n$. We define a subcategory $Y^{\le\lambda,\tau}(R)$ of $Y^{[0,h],\tau}(R)$ whose objects consist of Kisin modules $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ such that all i by i minors of $A_{\mathfrak{M},\beta}^{(j)}$ with respect to a fixed eigenbasis β of $\mathfrak{M}$ are divisible by $(v+p)^{\sum_{k=1}^{i}\lambda_{j,n+1-k}}$ for $i \in \{1, \dots, n-1\}$ and $\det(A_{\mathfrak{M},\beta}^{(j)}) \in R[\![v]\!]^\times (v+p)^{\sum_{k=1}^{n}\lambda_{j,k}}$. In this case, we say that $\mathfrak{M}$ has **height** $\le \lambda$. This definition does not depend on the choice of the eigenbasis for $\mathfrak{M}$.*

For simplicity, we also write $Y^{\le(\lambda_1,\dots,\lambda_n),\tau}(R)$ to denote $Y^{\le(\underline{\lambda_1},\dots,\underline{\lambda_n}),\tau}(R)$ for $\lambda_i \in \mathbb{Z}$ (see §3.2.1 for the notation).

3.3.2 Gauge bases

In this subsection, we review the notion of gauge bases introduced in [LLHLM18] and $\widetilde{w}$-gauge bases introduced in [LLHLM23]. They have the property that the corresponding matrices $A_{\mathfrak{M},\beta}^{(j)}$ of the Frobenius maps defined in Definition 3.3.1.3 have standard forms.

Definition 3.3.2.1 ([LLHLM18], Def. 2.22)**.** *Let $\overline{\mathfrak{M}} \in Y^{[0,h],\tau}(\mathbb{F})$ and $\widetilde{w} = (\widetilde{w}_j)_j \in \underline{\widetilde{W}}$. Write $\mathcal{I}(\mathbb{F})$ for the Iwahori subgroup of $\mathrm{GL}_n\left(\mathbb{F}[\![v]\!]\right)$ consisting of matrices which are upper triangular modulo v. We say that $\overline{\mathfrak{M}}$ has **shape** $\widetilde{w}$ if for any choice of eigenbasis $\overline{\beta}$ the equality*

$$\mathcal{I}(\mathbb{F}) A_{\overline{\mathfrak{M}},\overline{\beta}}^{(j)} \mathcal{I}(\mathbb{F}) = \mathcal{I}(\mathbb{F}) \widetilde{w}_j \mathcal{I}(\mathbb{F})$$

holds in $\mathrm{GL}_n\left(\mathbb{F}(\!(v)\!)\right)$ for each $0 \le j \le f-1$, where we regard $\widetilde{w}_j$ as an element of $\mathrm{GL}_n\left(\mathbb{F}(\!(v)\!)\right)$ by (3.5). This notion is independent of $\overline{\beta}$. We also say that the matrix $A_{\overline{\mathfrak{M}},\overline{\beta}}^{(j)}$ has shape $\widetilde{w}_j$.

The following proposition is a restatement of [CL18, Prop. 5.4].

Proposition 3.3.2.2. *Let $\overline{\mathfrak{M}} \in Y^{[0,h],\tau}(\mathbb{F})$ and $\lambda = (\lambda_{j,1}, \dots, \lambda_{j,n})_j \in X_+^*(\underline{T})$ be a dominant weight such that $\lambda_{j,i} \in \{0, \dots, h\}$ for all j, i. Then $\overline{\mathfrak{M}}$ has shape $\widetilde{w} \in \underline{\widetilde{W}}$ for some $\widetilde{w} \in \mathrm{Adm}^\vee(t_\lambda)$ (see (3.4) for the notation), if and only if $\overline{\mathfrak{M}} \in Y^{\leq \lambda,\tau}(\mathbb{F})$.*

The property of having a fixed shape is not an open condition, as we will see in Example 3.3.2.6. Instead, we will use the notion of $\widetilde{w}$-gauge following [LLHLM23]. For simplicity of notation, the following definition is slightly different from [LLHLM23, Def. 5.2.6].

Definition 3.3.2.3. *Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ and $\widetilde{w} = (\widetilde{w}_j)_j \in \underline{\widetilde{W}}$. Write $\widetilde{w}_j = s_j t_{\nu_j}$ with $s_j \in W$ and $\nu_j \in X^*(T)$. We say that $\mathfrak{M}$ has $\widetilde{w}$-**gauge** if it has an eigenbasis β such that*

(i) $A^{(j)}_{\mathfrak{M},\beta}(v+p)^{-\nu_j}\dot{s}_j^{-1} \in \mathrm{GL}_n\left(R\left[\frac{1}{v+p}\right]\right)$ *is lower triangular modulo* $\frac{1}{v+p}$;
(ii) $A^{(j)}_{\mathfrak{M},\beta}(v+p)^{-\nu_j} \in \mathrm{GL}_n\left(R\left[\frac{1}{v+p}\right]\right)$ *is upper triangular modulo* $\frac{v}{v+p}$

*for each $0 \leq j \leq f-1$. Such a β is called a $\widetilde{w}$-**gauge basis**. We also say that the matrix $A^{(j)}_{\mathfrak{M},\beta}$ has $\widetilde{w}_j$-gauge.*

Remark 3.3.2.4. *If $A \in \mathrm{M}_n\left(R[\![v]\!]\right)$ has $\widetilde{z}$-gauge, where $\widetilde{z} = st_\nu$ with $s \in W$ and $\nu \in X^*(T)$, then its (i,j)-entry A_{ij} has the form*

$$v^{\delta_{i>j}} \left(\sum_{k=0}^{\nu_j - \delta_{i>j} - \delta_{i<s(j)}} c_{ij,k}(v+p)^k \right)$$

with $c_{ij,k} \in R$ and $c_{s(j)j,\nu_j-\delta_{s(j)>j}} \in R^\times$, and it satisfies $\det(A) \in R^\times(v+p)^3$. Here, if P is a statement, then we define $\delta_P \overset{\text{def}}{=} 1$ if P is true and $\delta_P \overset{\text{def}}{=} 0$ otherwise.

Remark 3.3.2.5. *Let $\overline{\mathfrak{M}} \in Y^{[0,h],\tau}(\mathbb{F})$. If $\overline{\mathfrak{M}}$ has shape $\widetilde{w}$, then it has $\widetilde{w}$-gauge ([LLHLM23, Remark 5.2.5]). In general, $\overline{\mathfrak{M}}$ has a unique shape, but it could have $\widetilde{w}$-gauge for many choices of $\widetilde{w}$.*

Example 3.3.2.6. *Let $n = 2$. Let $\alpha, \beta \in \mathbb{F}^\times$, and $a \in \mathbb{F}$. We list the gauges and shapes of some matrices in $\mathrm{GL}_n\left(\mathbb{F}(\!(v)\!)\right)$ that will be considered in §3.4.*

Figure 3.1: Gauges and shapes of some matrices

Matrix	One choice of gauge	Shape
$\begin{pmatrix} \alpha v^2 & 0 \\ av^2 & \beta v \end{pmatrix}$	$t_{(2,1)}$-gauge	$t_{(2,1)}$
$\begin{pmatrix} 0 & \beta v \\ \alpha v^2 & av \end{pmatrix}$	$\mathfrak{w}t_{(2,1)}$-gauge	$\mathfrak{w}t_{(2,1)}$ if $a = 0$ $t_{(2,1)}$ if $a \neq 0$
$\begin{pmatrix} \alpha v & 0 \\ 0 & \beta v^2 \end{pmatrix}$	$t_{(1,2)}$-gauge	$t_{(1,2)}$

Everything follows directly from the definitions, except the fact that the matrix $\left(\begin{smallmatrix} 0 & \beta v \\ \alpha v^2 & av \end{smallmatrix}\right)$ has shape $t_{(2,1)}$ for $a \neq 0$. To check this, we may assume that $\alpha = \beta = 1$, then it follows from the equality

$$\begin{pmatrix} 0 & v \\ v^2 & av \end{pmatrix} = \begin{pmatrix} -a^{-1} & a^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^2 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & a \end{pmatrix}.$$

The following Proposition comes from [LLHLM23, Prop. 5.2.7] which is a generalization of [LLHLM18, Thm. 4.1] and [LLHLM18, Thm. 4.16].

Proposition 3.3.2.7. *Let μ be $(h+1)$-deep in $\underline{C_0}$, $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ and $\widetilde{w} \in \widetilde{\underline{W}}$. Suppose that $\mathfrak{M}/\varpi\mathfrak{M} \in Y^{[0,h],\tau}(R/\varpi R)$ has $\widetilde{w}$-gauge, then $\mathfrak{M}$ has $\widetilde{w}$-gauge. Moreover its $\widetilde{w}$-gauge basis β is unique up to scaling by the group*

$$\left\{(t_{j'})_{j'\in\mathcal{J}'} \in T^{\mathcal{J}'}(R) \middle| t_{j'} = t_{k'} \text{ for } j' \equiv k' \bmod f\right\} \cong \underline{T}(R).$$

3.3.3 Étale φ-modules

In this subsection, we review the relation between Kisin modules and Galois representations.

We fix a compatible system $(p_n)_n$ of p-power roots of $(-p)$ in $\overline{\mathbb{Q}}_p$ and define $K_\infty \overset{\text{def}}{=} \bigcup_{n\in\mathbb{N}} K(p_n)$. Let $\mathcal{O}_{\mathcal{E},K}$ (resp. $\mathcal{O}_{\mathcal{E},L'}$) be the p-adic completion of $W(k)[\![v]\!][1/v]$ (resp. $W(k)[\![u']\!][1/u']$). It has a Frobenius endomorphism φ extending the arithmetic Frobenius on $W(k)$ (resp. $W(k')$) and such that $\varphi(v) = v^p$ (resp. $\varphi(u') = (u')^p$). Let R be a complete Noetherian local $\mathcal{O}$-algebra. The completed tensor product (see [Dee01, Def. 1.2.1]) $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$ (resp. $\mathcal{O}_{\mathcal{E},L'}\widehat{\otimes}_{\mathbb{Z}_p}R$) is naturally equipped with a Frobenius endomorphism φ. Moreover, the ring $\mathcal{O}_{\mathcal{E},L'}\widehat{\otimes}_{\mathbb{Z}_p}R$ is naturally equipped with a Δ-action such that $\left(\mathcal{O}_{\mathcal{E},L'}\widehat{\otimes}_{\mathbb{Z}_p}R\right)^{\Delta=1} = \mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$. The following definition comes from [Fon90], generalized by [Dee01] for a version with coefficients.

Definition 3.3.3.1. *An **étale φ-module** over $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$ is a pair $(\mathcal{M}, \phi_{\mathcal{M}})$, such that*

(i) *$\mathcal{M}$ is a finitely generated projective $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$-module;*
(ii) *$\phi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is φ-semilinear and the image of $\phi_{\mathcal{M}}$ generates $\mathcal{M}$ as an $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$-module.*

Morphisms of étale φ-modules over $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$ are defined in the natural way and we denote by $\Phi\,\mathrm{Mod}^{\text{ét}}(R)$ the category of étale φ-modules over $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$. We often omit $\phi_{\mathcal{M}}$ and just write $\mathcal{M} \in \Phi\,\mathrm{Mod}^{\text{ét}}(R)$.

Similar to Kisin modules, an étale φ-module $\mathcal{M}$ decomposes as $\mathcal{M} \cong \bigoplus_{j\in\mathcal{J}} \mathcal{M}^{(j)}$, where each $\mathcal{M}^{(j)}$ is a finite free module over the p-adic completion $(R[\![v]\!][1/v])^\wedge$ of the ring $R[\![v]\!][1/v]$. The Frobenius map $\phi_{\mathfrak{M}}$ decomposes into R-linear maps $\phi_{\mathfrak{M}}^{(j)} : \mathcal{M}^{(j)} \to \mathcal{M}^{(j+1)}$ which are semilinear with respect to $v \mapsto v^p$.

There is a natural inclusion $\mathfrak{S}_{L',R} \hookrightarrow \mathcal{O}_{\mathcal{E},L'}\widehat{\otimes}_{\mathbb{Z}_p}R$ and we define a functor

$$\begin{aligned} \varepsilon_\tau : Y^{[0,h],\tau}(R) &\to \Phi\,\mathrm{Mod}^{\text{ét}}(R) \\ \mathfrak{M} &\mapsto \left(\mathfrak{M} \otimes_{\mathfrak{S}_{L',R}} (\mathcal{O}_{\mathcal{E},L'}\widehat{\otimes}_{\mathbb{Z}_p}R)\right)^{\Delta=1}. \end{aligned} \tag{3.12}$$

Note that it doesn't use the lowest alcove presentation of τ.

Proposition 3.3.3.2 ([LLHLM20], Prop. 3.2.1)**.** *Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ and β an eigenbasis of $\mathfrak{M}$. Let $\mathcal{M} \overset{\text{def}}{=} \varepsilon_\tau(\mathfrak{M})$. Then there exists an $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbb{Z}_p}R$-basis $\mathfrak{f}$ of $\mathcal{M}$ such that the matrix of $\phi_{\mathcal{M}}^{(j)}$ with respect to $\mathfrak{f}$ is given by*

$$A_{\mathfrak{M},\beta}^{(j)}\, s_j^* v^{\mu_j^*+\eta_j^*} \in \mathrm{GL}_n\left((R[\![v]\!][1/v])^\wedge\right),$$

where "$*$" is defined in (3.3) and $A_{\mathfrak{M},\beta}^{(j)}$ is viewed as an element of $\mathrm{GL}_n\left((R[\![v]\!][1/v])^\wedge\right)$ under the natural inclusion $R[\![v]\!] \hookrightarrow (R[\![v]\!][1/v])^\wedge$.

Let $\mathrm{Rep}_{G_{K_\infty}}(R)$ denote the category of finite free R-modules with continuous R-linear representations of G_{K_∞}. By a result of Fontaine ([Fon90]), generalized by [Dee01] for a version with coefficients, there is an exact anti-equivalence of categories

$$\mathbb{V}_K^* : \Phi\,\mathrm{Mod}^{\text{ét}}(R) \xrightarrow{\sim} \mathrm{Rep}_{G_{K_\infty}}(R). \tag{3.13}$$

Composing it with ε_τ, we get a functor

$$T_{dd}^* : Y^{[0,h],\tau}(R) \xrightarrow{\varepsilon_\tau} \Phi\,\mathrm{Mod}^{\text{ét}}(R) \xrightarrow[\cong]{\mathbb{V}_K^*} \mathrm{Rep}_{G_{K_\infty}}(R). \tag{3.14}$$

Proposition 3.3.3.3 ([LLHL19], Prop. 3.2.18)**.** *Assume that τ is $(h+1)$-generic. Let $\overline{\mathfrak{M}}, \overline{\mathfrak{M}'} \in Y^{[0,h],\tau}(\mathbb{F})$. If $T_{dd}^*(\overline{\mathfrak{M}}) \cong T_{dd}^*(\overline{\mathfrak{M}'})$ as G_{K_∞}-representations over $\mathbb{F}$, then $\overline{\mathfrak{M}} \cong \overline{\mathfrak{M}'}$ in $Y^{[0,h],\tau}(\mathbb{F})$.*

3.4 Galois deformation rings

In this section, we compute some potentially crystalline Galois deformation rings explicitly. The semisimple case is already known by [BHH$^+$23, §4]. We combine the method of [BHH$^+$23, §4] and [Le19, §3] to deal with the non-semisimple case.

3.4.1 Setup

In this subsection, we determine the tame inertial types that we will use.

Throughout this section we fix a 2-dimensional Galois representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ such that $\overline{\rho}^{\mathrm{ss}}|_{I_K} \cong \overline{\tau}(s,\mu)$, where

(i) $s_j \neq 1$ (hence $s_j = \mathfrak{w}$) if and only if $j = 0$ and $\overline{\rho}$ is irreducible;
(ii) $\mu - \eta$ is N-deep in $\underline{C}_0$ with $N \geq 12$.

We need N to be large in order to deal with the error term coming from the monodromy condition (see (3.27) below). Twisting $\overline{\rho}$ by a power of ω_f if necessary, we furthermore assume that $\mu_j = (r_j + 2, 1) \in \mathbb{Z}^2$ with $N < r_j + 1 < p - N$ for all j so that (see Example 3.2.2.4)

$$\overline{\rho}|_{I_K} \cong \begin{cases} \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix} \otimes \omega & \text{if } \overline{\rho} \text{ is reducible,} \\ \begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & 0 \\ 0 & \omega_{2f}^{p^f\sum_{j=0}^{f-1}(r_j+1)p^j} \end{pmatrix} \otimes \omega & \text{if } \overline{\rho} \text{ is irreducible.} \end{cases} \tag{3.15}$$

Then we associate to $\overline{\rho}$ a tuple of f elements $(a_0, \dots, a_{f-1}) \in \mathbb{F}^f$ and describe the set $W(\overline{\rho})$ of Serre weights of $\overline{\rho}$ in terms of these elements. By (3.13) there exists an étale φ-module $\overline{\mathcal{M}}$ over $k((v)) \otimes_{\mathbb{F}_p} \mathbb{F}$ such that $\mathbb{V}_K^*(\overline{\mathcal{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$. Recall from §3.3.3 that we have a decomposition $\overline{\mathcal{M}} \cong \bigoplus_{j\in\mathcal{J}} \overline{\mathcal{M}}^{(j)}$ with $\overline{\mathcal{M}}^{(j)} = \mathbb{F}((v))e_1^{(j)} \oplus \mathbb{F}((v))e_2^{(j)}$. We separate the following two cases.

(i) If $\overline{\rho}$ is reducible, by [Le19, Prop. 3.1] we can take the Frobenius maps $\phi_{\overline{\mathcal{M}}}^{(j)} : \overline{\mathcal{M}}^{(j)} \to \overline{\mathcal{M}}^{(j+1)}$ to have the form

$$\begin{cases} \phi_{\overline{\mathcal{M}}}^{(j)}(e_1^{(j)}) = \alpha_{f-1-j} v^{r_{f-1-j}+2}(e_1^{(j+1)} + a_j e_2^{(j+1)}) \\ \phi_{\overline{\mathcal{M}}}^{(j)}(e_2^{(j)}) = \beta_{f-1-j} v e_2^{(j+1)} \end{cases}$$

for some $\alpha_j, \beta_j \in \mathbb{F}^\times$ and $a_j \in \mathbb{F}$. In other words, the matrices of the Frobenius maps $\phi_{\mathcal{M}}^{(j)}$ in the basis $\left\{(e_1^{(j)}, e_2^{(j)})_j\right\}$ are given by

$$\mathrm{Mat}(\phi_{\mathcal{M}}^{(f-1-j)}) = \begin{pmatrix} \alpha_j v^{r_j+2} & 0 \\ \alpha_j a_{f-1-j} v^{r_j+2} & \beta_j v \end{pmatrix}. \tag{3.16}$$

Notice that whether a_j equals 0 or not is unchanged when we rescale the basis. From now on, we fix a choice of α_j, β_j and a_j. In particular, this gives a tuple $(a_0, \dots, a_{f-1}) \in \mathbb{F}^f$.

(ii) If $\overline{\rho}$ is irreducible, by [Le19, Prop. 3.1] we can take the Frobenius maps $\phi_{\mathcal{M}}^{(j)} : \overline{\mathcal{M}}^{(j)} \to \overline{\mathcal{M}}^{(j+1)}$ to have the form

$$\mathrm{Mat}(\phi_{\mathcal{M}}^{(f-1-j)}) = \begin{cases} \begin{pmatrix} \alpha_j v^{r_j+2} & 0 \\ 0 & \beta_j v \end{pmatrix} & \text{if } j \neq 0, \\ \begin{pmatrix} 0 & -\beta_j v \\ \alpha_j v^{r_j+2} & 0 \end{pmatrix} & \text{if } j = 0 \end{cases} \tag{3.17}$$

for some $\alpha_j, \beta_j \in \mathbb{F}^\times$. In this case we define $a_j = 0$ for all j.

In particular, we see that $\overline{\rho}$ is semisimple if and only if $(a_0, \dots, a_{f-1}) = (0, \dots, 0)$.

Recall that for $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ a Galois representation, we have a set $W(\overline{\rho})$ of Serre weights of $\overline{\rho}$ defined in [BDJ10, §3] which only depends on $\overline{\rho}|_{I_K}$. In both cases, by [Le19, Prop. 3.2] it can be described by

$$W(\overline{\rho}) = \{F(\mathfrak{t}_{\mu-\eta}(b_0, \dots, b_{f-1})) : b_j \in \{0, \mathrm{sgn}(s_j)\} \text{ if } a_{f-1-j} = 0 \text{ and } b_j = 0 \text{ if } a_{f-1-j} \neq 0\}, \tag{3.18}$$

see §3.2.3 for the notation.

Then we introduce the following 3^f tame inertial types that are needed in the computation of Galois deformation rings. Given an arbitrary

$$\widetilde{w} \in \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})}) = \left\{t_{(2,1)}, \varpi t_{(2,1)}, t_{(1,2)}\right\}^f$$

(see Example 3.2.1.4), we write $\widetilde{w}^* = t_\nu w$ for some unique $(w, \nu) \in \underline{W} \times X^*(\underline{T})$ (see (3.3) for "$*$"). Then we define the tame inertial type

$$\tau_{\widetilde{w}} \overset{\text{def}}{=} \tau(sw^{-1}, \mu - sw^{-1}(\nu)) \tag{3.19}$$

with lowest alcove presentation $(s(\tau), \mu(\tau)) \overset{\text{def}}{=} \left(sw^{-1}, \mu - sw^{-1}(\nu) - \eta\right)$. In particular, $\tau_{\widetilde{w}}$ is $(N-1)$-generic.

Explicitly, $s(\tau)_j = w_j^{-1}$ except when $j = 0$ and $\overline{\rho}$ is irreducible, in which case we have $s(\tau)_0 = \varpi w_0^{-1}$. Also we have

$$\mu(\tau)_j + \eta_j = \begin{cases} (r_j, 0) & \text{if } (t_{\nu_j} w_j, s_j) \in \{(t_{(2,1)}, 1),\ (t_{(2,1)}\varpi, \varpi), (t_{(1,2)}, \varpi)\}, \\ (r_j + 1, -1) & \text{if } (t_{\nu_j} w_j, s_j) \in \{(t_{(2,1)}, \varpi),\ (t_{(2,1)}\varpi, 1), (t_{(1,2)}, 1)\}. \end{cases} \tag{3.20}$$

The following lemma tells us whether a tame inertial type contains a given Serre weight of $\overline{\rho}$.

Lemma 3.4.1.1. *There is a unique injection $\theta : W(\overline{\rho}) \hookrightarrow \left\{t_{(2,1)},\ t_{(1,2)}\right\}^f$ such that for $\sigma \in W(\overline{\rho})$ and $\widetilde{w} \in \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})})$ we have*

$$\sigma \in \mathrm{JH}\left(\overline{\sigma(\tau_{\widetilde{w}})} \otimes_{\mathbb{F}} (N_{k/\mathbb{F}_p} \circ \det)\right) \Longleftrightarrow (\widetilde{w}_j \neq \theta(\sigma)_j \ \forall\, j). \tag{3.21}$$

Proof. The proof is almost the same as [BHH$^+$23, Lemma 4.1.2] except that we replace all the bijections by injections, since a non-semisimple $\overline{\rho}$ has less Serre weights then a semisimple one. □

Remark 3.4.1.2. *By the proof of [BHH$^+$23, Lemma 4.1.2], the map θ of Lemma 3.4.1.1 is defined in the following way: If $\sigma = F(\mathfrak{t}_{\mu-\eta}(b_0,\dots,b_{f-1}))$ as in (3.18), then we define*

$$\theta(\sigma)_{f-1-j} = \begin{cases} t_{(1,2)} & \text{if } b_j = 0, \\ t_{(2,1)} & \text{if } b_j \neq 0. \end{cases}$$

Moreover, still by the proof of [BHH$^+$23, Lemma 4.1.2], the 2^f tame inertial types $\tau_{\widetilde{w}}$ appearing in (3.21) are all the tame inertial types that contain a given Serre weight σ of $\overline{\rho}$.

Let σ be a Serre weight of $\overline{\rho}$. We define the set of tame inertial types that contain σ:

$$X(\sigma) \overset{\text{def}}{=} \left\{ \widetilde{w} \in \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})}) : \sigma \in \mathrm{JH}\left(\overline{\sigma(\tau_{\widetilde{w}})} \otimes_{\mathbb{F}} (N_{k/\mathbb{F}_p} \circ \det) \right) \right\}. \tag{3.22}$$

By Lemma 3.4.1.1 we have

$$X(\sigma) = \{ \widetilde{w} \in \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})}) : \widetilde{w}_j \neq \theta(\sigma)_j \ \forall\, j \}.$$

We also define the set of tame inertial types that contain at least one Serre weight of $\overline{\rho}$:

$$X(\overline{\rho}) \overset{\text{def}}{=} \left\{ \widetilde{w} \in \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})}) : \mathrm{JH}\left(\overline{\sigma(\tau)} \otimes_{\mathbb{F}} (N_{k|\mathbb{F}_p} \circ \det) \right) \cap W(\overline{\rho}) \neq \emptyset \right\} = \bigcup_{\sigma \in W(\overline{\rho})} X(\sigma).$$

We describe this set explicitly. Let $(a_j)_j$ be the chosen elements of $\mathbb{F}$ associated to $\overline{\rho}$. If $a_{f-1-j} = 0$, then b_j can be either 0 or $\mathrm{sgn}(s_j)$ by (3.18), and $\widetilde{w}_{f-1-j}$ can be any one of the elements of $\mathrm{Adm}^\vee(t_{(2,1)}) = \{t_{(2,1)}, \mathfrak{w} t_{(2,1)}, t_{(1,2)}\}$ by Lemma 3.4.1.1 and Remark 3.4.1.2. If $a_{f-1-j} \neq 0$, then b_j has to be 0 by (3.18), and $\widetilde{w}_{f-1-j} \in \{t_{(2,1)}, \mathfrak{w} t_{(2,1)}\}$ by Lemma 3.4.1.1 and Remark 3.4.1.2. To conclude, we have

$$X(\overline{\rho}) = \left\{ \widetilde{w} \in \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})}) : \widetilde{w}_{f-1-j} \neq t_{(1,2)} \text{ if } a_{f-1-j} \neq 0 \right\}. \tag{3.23}$$

These are all the tame inertial types that we need.

Lemma 3.4.1.3. *Let $\widetilde{w} \in X(\overline{\rho})$. Up to isomorphism there exists a unique Kisin module $\overline{\mathfrak{M}} \in Y^{\leq(2,1),\tau_{\widetilde{w}}}(\mathbb{F}) \subseteq Y^{\leq(3,0),\tau_{\widetilde{w}}}(\mathbb{F})$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$ (see (3.14) for T^*_{dd}).*

Proof. We concentrate on the case that $\overline{\rho}$ is reducible. The irreducible case is similar and can also be treated as in [BHH$^+$23, Lemma 4.1.1]. Define a Kisin module $\overline{\mathfrak{M}}$ over $\mathbb{F}$ of type $\tau_{\widetilde{w}}$ by imposing the matrices of the partial Frobenius maps to be $A^{(f-1-j)} = \mathrm{Mat}(\phi_{\overline{\mathcal{M}}}^{(f-1-j)}) v^{-(\mu(\tau)_j+\eta_j)} s(\tau)_j$, where $\mathrm{Mat}(\phi_{\overline{\mathcal{M}}}^{(f-1-j)}) \in \mathrm{M}_2(\mathbb{F}[\![v]\!])$ is the matrix in (3.16), and $s(\tau)$, $\mu(\tau)$ are computed in (3.20). Explicitly, we have the three cases:

(i) If $\widetilde{w}_{f-1-j} = t_{(2,1)}$, then $s(\tau)_j = 1$ and $\mu(\tau)_j + \eta_j = (r_j, 0)$. We have

$$A^{(f-1-j)} = \begin{pmatrix} \alpha_j v^{r_j+2} & 0 \\ \alpha_j a_{f-1-j} v^{r_j+2} & \beta_j v \end{pmatrix} \begin{pmatrix} v^{-r_j} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_j v^2 & 0 \\ \alpha_j a_{f-1-j} v^2 & \beta_j v \end{pmatrix}.$$

It has shape $t_{(2,1)}$ by Example 3.3.2.6.

(ii) If $\widetilde{w}_{f-1-j} = \mathfrak{w}t_{(2,1)}$, then $s(\tau)_j = \mathfrak{w}$ and $\mu(\tau)_j + \eta_j = (r_j + 1, -1)$. We have

$$A^{(f-1-j)} = \begin{pmatrix} \alpha_j v^{r_j+2} & 0 \\ \alpha_j a_{f-1-j} v^{r_j+2} & \beta_j v \end{pmatrix} \begin{pmatrix} v^{-(r_j+1)} & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_j v \\ \beta_j v^2 & \alpha_j a_{f-1-j} v \end{pmatrix}.$$

It has shape $\mathfrak{w}t_{(2,1)}$ if $a_{f-1-j} = 0$, and shape $t_{(2,1)}$ if $a_{f-1-j} \neq 0$ by Example 3.3.2.6.

(iii) If $\widetilde{w}_{f-1-j} = t_{(1,2)}$, then $s(\tau)_j = 1$ and $\mu(\tau)_j + \eta_j = (r_j + 1, -1)$. In this case, we must have $a_{f-1-j} = 0$ by (3.23), hence

$$A^{(f-1-j)} = \begin{pmatrix} \alpha_j v^{r_j+2} & 0 \\ 0 & \beta_j v \end{pmatrix} \begin{pmatrix} v^{-(r_j+1)} & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \alpha_j v & 0 \\ 0 & \beta_j v^2 \end{pmatrix}.$$

It has shape $t_{(1,2)}$ by Example 3.3.2.6.

In all cases, we see that the matrix $A^{(f-1-j)}$ belongs to $\mathrm{M}_2\left(\mathbb{F}[\![v]\!]\right)$ and has shape contained in $\mathrm{Adm}^\vee(t_{(2,1)})$ for all j, hence $\overline{\mathfrak{M}} \in Y^{\leq(2,1),\tau_{\widetilde{w}}}(\mathbb{F}) \subseteq Y^{\leq(3,0),\tau_{\widetilde{w}}}(\mathbb{F})$ by Proposition 3.3.2.2. Moreover, we remark that $\overline{\mathfrak{M}}$ has $\widetilde{w}$-gauge by Example 3.3.2.6. Now by Proposition 3.3.3.2, the matrices of the Frobenius maps of the associated étale φ-module $\varepsilon_{\tau_{\widetilde{w}}}(\mathfrak{M})$ (see (3.12)) with respect to some basis $\mathfrak{f}$ is given by

$$\mathrm{Mat}_{\mathfrak{f}}(\phi^{(f-1-j)}) = A^{(f-1-j)} s(\tau)_j^{-1} v^{\mu(\tau)_j + \eta_j}. \tag{3.24}$$

This is the same matrix as in (3.16), hence $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$.

The uniqueness of $\overline{\mathfrak{M}}$ follows from Proposition 3.3.3.3, since $\overline{\mathfrak{M}}$ has height in $[0, 2]$, $\tau_{\widetilde{w}}$ is $(N-1)$-generic and $N - 1 \geq 3$. □

Remark 3.4.1.4. *For general $\widetilde{w} \in \mathrm{Adm}^\vee(t_{(2,1)})$, there could be some j, such that $\widetilde{w}_{f-1-j} = t_{(1,2)}$ and $a_{f-1-j} \neq 0$ (this can happen only when $\overline{\rho}$ is non-semisimple). In this case, the above construction gives a matrix $A^{(f-1-j)}$ of shape $\mathfrak{w}t_{(1,2)}$ which belongs to $\mathrm{Adm}^\vee(t_{(3,0)}) - \mathrm{Adm}^\vee(t_{(2,1)})$ (see Example 3.2.1.4), so there is still a Kisin module $\overline{\mathfrak{M}} \in Y^{\leq(3,0),\tau_{\widetilde{w}}}(\mathbb{F})$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$, which is unique up to isomorphism using Proposition 3.3.3.3 and $N - 1 \geq 4$.*

3.4.2 Single-type Galois deformation rings

In this subsection, we combine all the above preliminaries and compute the single-type deformation rings following [BHH+23, Prop. 4.2.1] (which deals with the semisimple case).

Let $\overline{\rho}$ and $(a_0, \ldots, a_{f-1}) \in \mathbb{F}^f$ an f-tuple be as in §3.4.1. Fix $\widetilde{w} \in X(\overline{\rho})$ (see (3.23)) and let $\tau_{\widetilde{w}}$ be as in (3.19). We compute the Galois deformation ring of $\overline{\rho}$ for a single-type $\tau_{\widetilde{w}}$ and Hodge–Tate weights $\leq (3, 0)$ (meaning Hodge–Tate weights $(3, 0)$ or $(2, 1)$ at each embedding).

Let $\overline{\mathfrak{M}} \in Y^{\leq(2,1),\tau_{\widetilde{w}}}(\mathbb{F})$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$ (see Lemma 3.4.1.3). By the proof of Lemma 3.4.1.3, the associated matrix $\overline{A}^{(j)}$ of $\overline{\mathfrak{M}}$ with respect to some eigenbasis $\overline{\beta}$ is of the form:

$$\overline{A}^{(f-1-j)} = \begin{cases} \begin{pmatrix} \overline{e_{11}^{*(j)}} v^2 & 0 \\ \overline{d_{21}^{(j)}} v^2 & \overline{d_{22}^{*(j)}} v \end{pmatrix} & \text{if } \widetilde{w}_{f-1-j} = t_{(2,1)}, \\ \begin{pmatrix} 0 & \overline{d_{12}^{*(j)}} v \\ \overline{d_{21}^{*(j)}} v^2 & \overline{d_{22}^{(j)}} v \end{pmatrix} & \text{if } \widetilde{w}_{f-1-j} = \mathfrak{w}t_{(2,1)}, \\ \begin{pmatrix} \overline{d_{11}^{*(j)}} v & 0 \\ 0 & \overline{e_{22}^{*(j)}} v^2 \end{pmatrix} & \text{if } \widetilde{w}_{f-1-j} = t_{(1,2)}. \end{cases} \tag{3.25}$$

Here we use the notation of Tables 3.1-3.3. Each element with " $*$ " (e.g. $\overline{e_{11}^{*(j)}}$) belongs to $\mathbb{F}^\times$. The element $\overline{d_{21}^{(j)}}$ (resp. $\overline{d_{22}^{(j)}}$) equals 0 if $a_{f-1-j}=0$, and belongs to $\mathbb{F}^\times$ if $a_{f-1-j}\neq 0$.

Let $R_{\overline{\rho}}^{\leq(3,0),\tau_{\widetilde{w}}}$ denote the maximal reduced, $\mathcal{O}$-flat quotient of the universal framed deformation ring $R_{\overline{\rho}}^{\square}$ that parametrizes potentially crystalline lifts of $\overline{\rho}$ of Hodge–Tate weights $\leq(3,0)$ in each embedding and tame inertial type $\tau_{\widetilde{w}}$ (its existence follows from [Kis08]). For each dominant character $\lambda\in X_+^*(\underline{T})$, let $R_{\overline{\rho}}^{\lambda,\tau_{\widetilde{w}}}$ denote the maximal reduced, $\mathcal{O}$-flat quotient of $R_{\overline{\rho}}^{\square}$ that parametrizes potentially crystalline lifts of $\overline{\rho}$ of Hodge–Tate weights λ_j in the j-th embedding for all j and tame inertial type $\tau_{\widetilde{w}}$.

The following result is a generalization of [BHH$^+$23, Prop. 4.2.1] (where $\overline{\rho}$ was assumed to be semisimple).

Theorem 3.4.2.1. *Let $\widetilde{w}\in X(\overline{\rho})$. We have an isomorphism*

$$R_{\overline{\rho}}^{\leq(3,0),\tau_{\widetilde{w}}}[\![X_1,\dots,X_{2f}]\!]\cong\left(R^{\tau_{\widetilde{w}}}\Big/\sum_j I^{(j)}\right)[\![Y_1,\dots,Y_4]\!],\tag{3.26}$$

where $R^{\tau_{\widetilde{w}}}\overset{\text{def}}{=}\widehat{\bigotimes}_{\mathcal{O},0\leq j\leq f-1}R^{(j)}$, and where the rings $R^{(j)}$ and the ideals $I^{(j)}$ of $R^{\tau_{\widetilde{w}}}$ are found in Tables 3.1-3.3. The irreducible components of $\operatorname{Spec}R_{\overline{\rho}}^{\leq(3,0),\tau_{\widetilde{w}}}$ *are given by* $\operatorname{Spec}R_{\overline{\rho}}^{\lambda,\tau_{\widetilde{w}}}$*, where $\lambda=(\lambda_j)\in\{(3,0),(2,1)\}^f$. More precisely, via the isomorphism (3.26), for any choice of $\lambda=(\lambda_j)\in\{(3,0),(2,1)\}^f$ the kernel of the natural surjection $R_{\overline{\rho}}^{\leq(3,0),\tau_{\widetilde{w}}}[\![X_1,\dots,X_{2f}]\!]\twoheadrightarrow R_{\overline{\rho}}^{\lambda,\tau_{\widetilde{w}}}[\![X_1,\dots,X_{2f}]\!]$ is generated by the prime ideal $\sum_{j=0}^{f-1}\mathfrak{p}^{(j),\lambda_{f-1-j}}$ of $R^{\tau_{\widetilde{w}}}$, where the ideals $\mathfrak{p}^{(j),\lambda_{f-1-j}}$ of $R^{\tau_{\widetilde{w}}}$ are found in Tables 3.1-3.3.*

Moreover, the special fiber of each $\operatorname{Spec}R_{\overline{\rho}}^{\lambda,\tau_{\widetilde{w}}}$ *is reduced.*

Proof. We follow the proof of [BHH$^+$23, Prop. 4.2.1] and use without comment the notation of *loc.cit.*

By (3.25) and Example 3.3.2.6 the eigenbasis $\overline{\beta}$ of $\overline{\mathfrak{M}}$ is a $\widetilde{w}$-gauge basis in the sense of Definition 3.3.2.3. We modify the definition of $D_{\overline{\mathfrak{M}},\overline{\beta}}^{\leq(3,0),\tau}(R)$ appearing in the proof of [BHH$^+$23, Prop. 4.2.1] by requiring β to be a $\widetilde{w}$-gauge basis instead of a gauge basis. Then by Remark 3.3.2.4, for any lift $(\mathfrak{M},\beta,\jmath)\in D_{\overline{\mathfrak{M}},\overline{\beta}}^{\leq(3,0),\tau}(R)$ the corresponding matrices $A^{(f-1-j)}$ are given in row 1 of Tables 3.1-3.3, where the entries $c_{11}^{(j)},c_{12}^{(j)},\dots$ are in R satisfying $A^{(f-1-j)}\bmod\mathfrak{m}_R$ equals $\overline{A}^{(f-1-j)}$. Here we remark that the matrix $A^{(f-1-j)}$ in row 1 of Table 3.2 has $\mathfrak{w}t_{(2,1)}$-gauge but has shape $t_{(2,1)}$ if $a_{f-1-j}\neq 0$.

Row 2 of Tables 3.1-3.3 are obtained from row 1 by applying Proposition 3.3.3.2.

Let $R^{(j)}$ be the power series ring in row 3 of Tables 3.1-3.3. Its variables $c_{11},\dots$ come from the coefficients of the matrices $A^{(f-1-j)}$ in row 1 of Tables 3.1-3.3. The condition that $\mathfrak{M}$ has height $\leq(\underline{3},\underline{0})$ (see Definition 3.3.1.4), or equivalently the determinant condition appearing in Remark 3.3.2.4, is given by $\det A^{(f-1-j)}\in R^\times(v+p)^3$. This gives the ideal $I^{(j),\leq(3,0)}$ in row 4 of Tables 3.1-3.3. Then the argument of [LLHLM18, Thm. 4.17] shows that the deformation problem $D_{\overline{\mathfrak{M}},\overline{\beta}}^{\leq(3,0),\tau}$ is represented by the maximal reduced $\mathcal{O}$-flat quotient of $\widehat{\bigotimes}_{\mathcal{O},0\leq j\leq f-1}R^{(j)}/I^{(j),\leq(3,0)}$.

As in the proof of [BHH$^+$23, Prop. 4.2.1], the monodromy condition given by [BHH$^+$23, Prop. 3.1.9] is equivalent to

$$\left(\frac{d}{dv}\right)^t\Big|_{v=-p}\left\{\left[v\frac{d}{dv}A^{(f-1-j)}-A^{(f-1-j)}\begin{pmatrix}a^{(j)}&0\\0&0\end{pmatrix}\right](v+p)^3(A^{(f-1-j)})^{-1}\right\}+O(p^{N-3-t})=0\tag{3.27}$$

for all $0 \le t \le 1$, $0 \le j \le f-1$. Here $a^{(j)} \in \mathbb{Z}_{(p)}$ is as in the proof of [BHH$^+$23, Prop. 4.2.1] and $O(p^{N-3-t})$ is certain unspecified element of $p^{N-3-t}\,\mathrm{M}_2(R)$. Then the entries of the matrices given by the left-hand side of (3.27) give the ideal $I^{(j),\nabla}$ of $R^{\tau_{\widetilde{w}}}$ which are given in row 5 of Tables 3.1-3.3. We remark that row 5 of Table 3.3 is not computed directly, but is computed using the fact that the matrix $A^{(f-1-j)}$ of Table 3.3 can be conjugated to that of Table 3.1 by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ v & 0 \end{smallmatrix}\right)$.

We have a similar diagram as (5.9) of [LLHLM18]. The vertical map labelled "f.s." in *loc.cit.* now corresponds to forgetting the $\widetilde{w}$-gauge basis on the Kisin modules, and is still formally smooth by Proposition 3.3.2.7 which is a generalization of [LLHLM18, Thm. 4.1] and [LLHLM18, Thm. 4.16]. Let $R^{\le(3,0),\tau,\nabla}_{\overline{\mathfrak{M}},\overline{\beta}}$ be the maximal reduced and $\mathcal{O}$-flat quotient of $R^{\le(3,0),\tau}_{\overline{\mathfrak{M}},\overline{\beta}}/\sum_j(I^{(j),\le(3,0)}+I^{(j),\nabla})$. Then the argument of [LLHLM18, Thm. 5.12] and [LLHLM18, Cor. 5.13] goes through and gives an isomorphism

$$R^{\le(3,0),\tau}_{\overline{\rho}}[\![X_1,\dots,X_{2f}]\!] \cong R^{\le(3,0),\tau,\nabla}_{\overline{\mathfrak{M}},\overline{\beta}}[\![Y_1,\dots,Y_4]\!].$$

The rest of the proof and the computations are completely analogous to those of [BHH$^+$23, Prop. 4.2.1] (see Remark 3.4.2.2 below). Here we notice that the equation (25) in [BHH$^+$23, Prop. 4.2.1] is guaranteed by our assumption on $\widetilde{w}$. Finally, the proof of the last statement is completely analogous to that of [BHH$^+$23, Cor. 4.2.6]. □

Remark 3.4.2.2. *The Tables 3.1-Table 3.3 are very similar to those of [BHH$^+$23, §4]. The main difference is that here the element d_{21} (resp. d_{22}) of Table 3.1 (resp. Table 3.2) is a unit in $R^{(j)}$ if and only if $a_{f-1-j} \neq 0$.*

3.4.3 Multi-type Galois deformation rings

In this subsection we compute the multi-type deformation rings following [BHH$^+$23, Prop. 4.3.1] (which deals with the semisimple case).

Let $\overline{\rho}$ and $(a_0,\dots,a_{f-1}) \in \mathbb{F}^f$ be as in §3.4.1. For $\sigma \in W(\overline{\rho})$, let $R^{\le(3,0),\sigma}_{\overline{\rho}}$ denote the maximal reduced, $\mathcal{O}$-flat quotient of $R^{\square}_{\overline{\rho}}$ that parametrizes potentially crystalline lifts of $\overline{\rho}$ of Hodge–Tate weights $\le (3,0)$ in each embedding and tame inertial type τ with $\tau \in X(\sigma)$, where $X(\sigma)$ is as in (3.22). By Lemma 3.4.1.1 and Remark 3.4.1.2, we see that $\operatorname{Spec} R^{\le(3,0),\sigma}_{\overline{\rho}}$ is the flat closure of $\bigcup_{\widetilde{w}\in X(\sigma)} \operatorname{Spec} R^{\le(3,0),\tau_{\widetilde{w}}}_{\overline{\rho}}[1/p]$ inside $\operatorname{Spec} R^{\square}_{\overline{\rho}}$. Also, we denote $\widetilde{w}_\sigma \overset{\text{def}}{=} \theta(\sigma) \in \{t_{(2,1)}, t_{(1,2)}\}^f$, where θ is defined in Lemma 3.4.1.1.

We define a bijection $i : \mathrm{Adm}^\vee(t_{(\underline{2},\underline{1})}) \to \{1,2,3\}^f$ by letting $i(\widetilde{w})$ be the f-tuple given by

$$i(\widetilde{w})_j \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \widetilde{w}_j = t_{(2,1)} \\ 2 & \text{if } \widetilde{w}_j = \mathfrak{w} t_{(2,1)} \\ 3 & \text{if } \widetilde{w}_j = t_{(1,2)}. \end{cases}$$

They will be the indices of ideals.

The following result is a generalization of [BHH$^+$23, Prop. 4.3.1] (where $\overline{\rho}$ was assumed to be semisimple).

Theorem 3.4.3.1. *We have an isomorphism*

$$R^{\le(3,0),\sigma}_{\overline{\rho}}[\![X_1,\dots,X_{2f}]\!] \cong \left(S/\bigcap_{\widetilde{w}\in X(\sigma)}\sum_j I^{(j)}_{\widetilde{w}}\right)[\![Y_1,\dots,Y_4]\!], \tag{3.28}$$

where $S \overset{\text{def}}{=} \widehat{\bigotimes}_{\mathcal{O},0\le j\le f-1} S^{(j)}$*, and where the ring* $S^{(j)}$ *and the ideals* $I_{\widetilde{w}}^{(j)}$ *of* S *are as in Table 3.4 if* $(\widetilde{w}_\sigma)_{f-1-j} = t_{(1,2)}$*, whereas the ring* $S^{(j)}$ *and the ideals* $I_{\widetilde{w}}^{(j)}$ *of* S *are as in Table 3.5 if* $(\widetilde{w}_\sigma)_{f-1-j} = t_{(2,1)}$*. The irreducible components of* $\operatorname{Spec} R_{\overline{\rho}}^{\le(3,0),\sigma}$ *are given by* $\operatorname{Spec} R_{\overline{\rho}}^{\lambda,\tau_{\widetilde{w}}}$*, where* $\lambda = (\lambda_j) \in \{(3,0),(2,1)\}^f$ *and* $\widetilde{w} \in X(\sigma)$*.*

More precisely, via the isomorphism (3.28), for any choice of $\lambda = (\lambda_j) \in \{(3,0),(2,1)\}^f$ *and* $\widetilde{w} \in X(\sigma)$ *the kernel of the natural surjection* $R_{\overline{\rho}}^{\le(3,0),\sigma}[\![X_1,\dots,X_{2f}]\!] \twoheadrightarrow R_{\overline{\rho}}^{\lambda,\tau_{\widetilde{w}}}[\![X_1,\dots,X_{2f}]\!]$ *is generated by the prime ideal* $\sum_{j=0}^{f-1} \mathfrak{p}_{\widetilde{w}}^{(j),\lambda_{f-1-j}}$ *of* S*, where the ideals* $\mathfrak{p}_{\widetilde{w}}^{(j),\lambda_{f-1-j}}$ *of* S *are found in Tables 3.4 and 3.5.*

Proof. We follow the proof of [BHH+23, Prop. 4.3.1] and use without comment the notation of *loc.cit.* The main difference in the non-semisimple case is that we need to modify the definition of the map ψ of *loc.cit.* and to prove the Claim 1 of *loc.cit.* in this case.

Recall from (3.16) and (3.17) that there exist $\delta_{12}^{(j)}, \delta_{21}^{(j)} \in \mathbb{F}^\times$ and $\delta_{22}^{(j)} \in \mathbb{F}$, such that

$$\operatorname{Mat}(\phi_{\mathcal{M}}^{(f-1-j)}) = \begin{pmatrix} \delta_{12}^{(j)} v & 0 \\ \delta_{22}^{(j)} v & \delta_{21}^{(j)} v \end{pmatrix} s_j^{-1} v^{\mu_j'}$$

where $\mu_j' \overset{\text{def}}{=} \mu_j - (1,1) = (r_j+1, 0)$. Note that $\delta_{22}^{(j)} = 0$ if and only if $a_{f-1-j} = 0$. In particular, if $\overline{\rho}$ is reducible nonsplit, there exists at least one $j \in \mathcal{J}$, such that $\delta_{22}^{(j)} \neq 0$. In this case we fix one such j and denote it j_0. Let $[\delta_{12}^{(j)}], [\delta_{21}^{(j)}] \in \mathcal{O}^\times$ and $[\delta_{22}^{(j)}] \in \mathcal{O}$ be the Teichmuller lifts of $\delta_{12}^{(j)}, \delta_{21}^{(j)} \in \mathbb{F}^\times$ and $\delta_{22}^{(j)} \in \mathbb{F}$.

Let $\overline{S} \overset{\text{def}}{=} S/\bigcap_{\widetilde{w}\in X(\sigma)} \sum_j I_{\widetilde{w}}^{(j)}$. We consider the étale φ-module $\mathcal{M}$ over $\mathcal{O}_{\mathcal{E},\overline{S}}$ given by

$$\operatorname{Mat}(\phi_{\mathcal{M}}^{(f-1-j)}) = \begin{pmatrix} (v+p)\big([\delta_{12}^{(j)}] + x_{12}^{*(j)}\big) + c_{12}^{(j)} + \frac{b_{12}^{(j)}}{v} & \frac{1}{v}\big((v+p)d_{11}^{(j)} + c_{11}^{(j)}\big) \\ (v+p)\big([\delta_{22}^{(j)}] + x_{22}^{(j)}\big) + c_{22}^{(j)} & (v+p)\big([\delta_{21}^{(j)}] + x_{21}^{*(j)}\big) + c_{21}^{(j)} + \frac{b_{21}^{(j)}}{v} \end{pmatrix} s_j^{-1} v^{\mu_j'}$$

in a suitable basis, where $b_{21}^{(j)} \overset{\text{def}}{=} 0$ if $(\widetilde{w}_\sigma)_{f-1-j} = t_{(1,2)}$ and $b_{12}^{(j)} \overset{\text{def}}{=} 0$ if $(\widetilde{w}_\sigma)_{f-1-j} = t_{(2,1)}$. In particular, we see that $\mathcal{M} \otimes_{\overline{S}} \mathbb{F} \cong \overline{\mathcal{M}}$.

Fix an $\mathbb{F}$-basis $\gamma_{\mathbb{F}}$ of $\mathbb{V}_K^*(\overline{\mathcal{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$. We demand moreover that $\gamma_{\mathbb{F},1}$, $\gamma_{\mathbb{F},2}$ span G_{K_∞}-stable lines in case $\overline{\rho}$ is reducible split. Fix an $\overline{S}$-basis γ of $\mathbb{V}_K^*(\mathcal{M})$ lifting $\gamma_{\mathbb{F}}$. Denote $\overline{S}[\![\underline{Y}]\!] \overset{\text{def}}{=} \overline{S}[\![Y_1,Y_2,Y_3,Y_4]\!]$. Then the G_{K_∞}-representation $\mathbb{V}_K^*\big(\mathcal{M}\widehat{\otimes}_{\overline{S}}\overline{S}[\![\underline{Y}]\!]\big) \cong \mathbb{V}_K^*(\mathcal{M})\widehat{\otimes}_{\overline{S}}\overline{S}[\![\underline{Y}]\!]$ over $\overline{S}[\![\underline{Y}]\!]$ together with the basis $\big(1+\left(\begin{smallmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{smallmatrix}\right)\big)(\gamma\otimes 1)$ give rise to a homomorphism $\psi_0 : R^{\square}_{\overline{\rho}|_{G_{K_\infty}}} \to \overline{S}[\![\underline{Y}]\!]$.

We extend ψ_0 to a homomorphism $\psi : R^{\square}_{\overline{\rho}|_{G_{K_\infty}}}[\![\underline{X}', \underline{X}'']\!] \to \overline{S}[\![\underline{Y}]\!]$ as follows:

$$\psi(X_j') = \begin{cases} x_{12}^{*(j)} & \text{if } 0 \le j < f-1 \text{ or } \overline{\rho} \text{ is irreducible;} \\ Y_1 & \text{if } j = f-1 \text{ and } \overline{\rho} \text{ is reducible split;} \\ x_{22}^{(j_0)} & \text{if } j = f-1 \text{ and } \overline{\rho} \text{ is reducible nonsplit;} \end{cases}$$

$$\psi(X_j'') = \begin{cases} x_{21}^{*(j)} & \text{if } 0 \le j < f-1; \\ Y_4 & \text{if } j = f-1. \end{cases}$$

Claim. The map $\psi : R^{\square}_{\overline{\rho}|_{G_{K_\infty}}}[\![\underline{X}', \underline{X}'']\!] \to \overline{S}[\![\underline{Y}]\!]$ is surjective.

Now we prove the Claim following the proof of Claim 1 in [BHH+23, Prop. 4.3.1] (which treats the semisimple case).

We check that ψ is injective on reduced tangent vectors, i.e. on $\mathbb{F}[\varepsilon]/(\varepsilon^2)$-points: Let $t_0 : \overline{S}[\![\underline{Y}]\!] \to \mathbb{F} \to \mathbb{F}[\varepsilon]/(\varepsilon^2)$ be the zero vector, where the first map maps all the variables of $\overline{S}$ and the variables Y_i to 0. Fix one continuous homomorphism $t : \overline{S}[\![\underline{Y}]\!] \to \mathbb{F}[\varepsilon]/(\varepsilon^2)$ such that $t \circ \psi = t_0 \circ \psi$. The goal is to prove that $t = t_0$.

Abusing notation, we write $t(b_{ik}^{(j)}) = \varepsilon b_{ik}^{(j)}$ for some $b_{ik}^{(j)} \in \mathbb{F}$ on the right, and similarly $t(c_{ik}^{(j)}) = \varepsilon c_{ik}^{(j)}$, $t(d_{ik}^{(j)}) = \varepsilon d_{ik}^{(j)}$, $t(x_{ik}^{*(j)}) = \varepsilon x_{ik}^{(j)}$ if $(i,k) = (2,1)$ or $(1,2)$, $t(x_{22}^{(j)}) = \varepsilon x_{22}^{(j)}$, and $t(Y_i) = \varepsilon y_i$. Since $t \circ \psi = t_0 \circ \psi$, evaluating on the variables $\underline{X}'$ and $\underline{X}''$ we deduce that

$$\begin{aligned} &x_{12}^{(j)} = x_{21}^{(j)} = 0 \text{ for } 0 \le j < f-1,\ y_4 = 0, \text{ and} \\ &\begin{cases} x_{12}^{(f-1)} = 0 & \text{if } \overline{\rho} \text{ is irreducible;} \\ y_1 = 0 & \text{if } \overline{\rho} \text{ is split reducible;} \\ x_{22}^{(j_0)} = 0 & \text{if } \overline{\rho} \text{ is nonsplit reducible.} \end{cases} \end{aligned} \tag{3.29}$$

Moreover, by the definition of ψ_0 and using $t \circ \psi = t_0 \circ \psi$, there is an isomorphism

$$\lambda : \mathcal{M}_{\overline{S}[\![\underline{Y}]\!]} \widehat{\otimes}_{\overline{S}[\![\underline{Y}]\!],t} \mathbb{F}[\varepsilon]/(\varepsilon^2) \xrightarrow{\sim} \mathcal{M}_{\overline{S}[\![\underline{Y}]\!]} \widehat{\otimes}_{\overline{S}[\![\underline{Y}]\!],t_0} \mathbb{F}[\varepsilon]/(\varepsilon^2) \tag{3.30}$$

of étale φ-modules over $\mathbb{F}[\varepsilon]/(\varepsilon^2)$ which induces the identity of $\overline{\mathcal{M}}$ modulo ε and such that $\mathbb{V}_K^*(\lambda)$ sends the basis $\left(1 + \varepsilon \left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right)\right)(\gamma \otimes 1)$ to $\gamma \otimes 1$ on the corresponding G_{K_∞}-representations over $\mathbb{F}[\varepsilon]/(\varepsilon^2)$. In particular, the isomorphism λ is realized by the change of basis (i.e. φ-conjugation) by a matrix of the form

$$1 + \varepsilon M_{f-1-j} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{E},\mathbb{F}[\varepsilon]/(\varepsilon^2)}),$$

for some $M_{f-1-j} \in \mathrm{M}_2(\mathcal{O}_{\mathcal{E},\mathbb{F}}) = \mathrm{M}_2\left(\mathbb{F}(\!(v)\!)\right)$. In other words,

$$\begin{aligned} &(1 + \varepsilon M_{j-1}) \begin{pmatrix} \delta_{12}^{(j)} & 0 \\ \delta_{22}^{(j)} & \delta_{21}^{(j)} \end{pmatrix} s_j^{-1} v^{\mu_j'} (1 - \varepsilon \varphi(M_j)) \\ &\quad = \begin{pmatrix} \delta_{12}^{(j)} + \varepsilon(x_{12}^{(j)} + c_{12}^{(j)} v^{-1} + b_{12}^{(j)} v^{-2}) & \varepsilon(d_{11}^{(j)} v^{-1} + c_{11}^{(j)} v^{-2}) \\ \delta_{22}^{(j)} + \varepsilon(x_{22}^{(j)} + c_{22}^{(j)} v^{-1}) & \delta_{21}^{(j)} + \varepsilon(x_{21}^{(j)} + c_{21}^{(j)} v^{-1} + b_{21}^{(j)} v^{-2}) \end{pmatrix} s_j^{-1} v^{\mu_j'}, \end{aligned} \tag{3.31}$$

where we have divided by v, and j is considered in $\mathbb{Z}/f\mathbb{Z}$ as usual.

First we show that the matrix $M_j \in \mathrm{M}_2\left(\mathbb{F}[\![v]\!]\right)$ for each j. Let $k_j \in \mathbb{Z}$ be minimal such that $v^{k_j} M_j \in \mathrm{M}_2\left(\mathbb{F}[\![v]\!]\right)$. The equation (3.31) is equivalent to

$$\begin{aligned} 1 - \varepsilon\varphi(M_j) = v^{-\mu_j'} s_j &\begin{pmatrix} \delta_{12}^{(j)} & 0 \\ \delta_{22}^{(j)} & \delta_{21}^{(j)} \end{pmatrix}^{-1} (1 - \varepsilon M_{j-1}) \\ \cdot &\begin{pmatrix} \delta_{12}^{(j)} + \varepsilon(x_{12}^{(j)} + c_{12}^{(j)} v^{-1} + b_{12}^{(j)} v^{-2}) & \varepsilon(d_{11}^{(j)} v^{-1} + c_{11}^{(j)} v^{-2}) \\ \delta_{22}^{(j)} + \varepsilon(x_{22}^{(j)} + c_{22}^{(j)} v^{-1}) & \delta_{21}^{(j)} + \varepsilon(x_{21}^{(j)} + c_{21}^{(j)} v^{-1} + b_{21}^{(j)} v^{-2}) \end{pmatrix} s_j^{-1} v^{\mu_j'}. \end{aligned}$$

Recall that $\mu_j' = (r_j + 1, 0)$, hence multiplying the right-hand side by $v^{r_j+1} \cdot v^{k_{j-1}} \cdot v^2$ makes it v-integral. Considering the left-hand side, it follows that $pk_j \le k_{j-1} + r_j + 3 < k_{j-1} + p - 1$ by genericity. This implies $p(\max_j k_j) < (\max_j k_j) + p - 1$, so $\max_j k_j < 1$, meaning $M_j \in \mathrm{M}_2\left(\mathbb{F}[\![v]\!]\right)$ for all j.

Comparing the coefficients of ε in (3.31) and multiplying on the right by $v^{-\mu_j'} s_j$, we get

$$\begin{aligned} M_{j-1} &\begin{pmatrix} \delta_{12}^{(j)} & 0 \\ \delta_{22}^{(j)} & \delta_{21}^{(j)} \end{pmatrix} - \begin{pmatrix} \delta_{12}^{(j)} & 0 \\ \delta_{22}^{(j)} & \delta_{21}^{(j)} \end{pmatrix} s_j^{-1} v^{\mu_j'} \varphi(M_j) v^{-\mu_j'} s_j \\ &= \begin{pmatrix} x_{12}^{(j)} + c_{12}^{(j)} v^{-1} + b_{12}^{(j)} v^{-2} & d_{11}^{(j)} v^{-1} + c_{11}^{(j)} v^{-2} \\ x_{22}^{(j)} + c_{22}^{(j)} v^{-1} & x_{21}^{(j)} + c_{21}^{(j)} v^{-1} + b_{21}^{(j)} v^{-2} \end{pmatrix}. \end{aligned} \tag{3.32}$$

From now on we assume that $\overline{\rho}$ is reducible; the case for $\overline{\rho}$ irreducible is similar and is contained in the proof of [BHH$^+$23, Prop. 4.3.1]. In particular, we have $s_j = 1$ for all j. If we write $M_j = \begin{pmatrix} m_{11}^{(j)} & m_{12}^{(j)} \\ m_{21}^{(j)} & m_{22}^{(j)} \end{pmatrix}$ with $m_{ik}^{(j)} \in \mathbb{F}[\![v]\!]$ and expand equation (3.32), we get

$$\begin{aligned} &\begin{pmatrix} m_{11}^{(j-1)}\delta_{12}^{(j)} + m_{12}^{(j-1)}\delta_{22}^{(j)} & m_{12}^{(j-1)}\delta_{21}^{(j)} \\ m_{21}^{(j-1)}\delta_{12}^{(j)} + m_{22}^{(j-1)}\delta_{22}^{(j)} & m_{22}^{(j-1)}\delta_{21}^{(j)} \end{pmatrix} \\ &- \begin{pmatrix} \delta_{12}^{(j)}\varphi(m_{11}^{(j)}) & \delta_{12}^{(j)}v^{r_j+1}\varphi(m_{12}^{(j)}) \\ \delta_{22}^{(j)}\varphi(m_{11}^{(j)}) + \delta_{21}^{(j)}v^{-(r_j+1)}\varphi(m_{21}^{(j)}) & \delta_{22}^{(j)}v^{r_j+1}\varphi(m_{12}^{(j)}) + \delta_{21}^{(j)}\varphi(m_{22}^{(j)}) \end{pmatrix} \\ &= \begin{pmatrix} x_{12}^{(j)} + c_{12}^{(j)}v^{-1} + b_{12}^{(j)}v^{-2} & d_{11}^{(j)}v^{-1} + c_{11}^{(j)}v^{-2} \\ x_{22}^{(j)} + c_{22}^{(j)}v^{-1} & x_{21}^{(j)} + c_{21}^{(j)}v^{-1} + b_{21}^{(j)}v^{-2} \end{pmatrix}. \end{aligned} \tag{3.33}$$

Compare the $(1,1)$, $(1,2)$, $(2,2)$-entries. Since M is v-integral, we deduce that $c_{12}^{(j)} = b_{12}^{(j)} = 0$, $d_{11}^{(j)} = c_{11}^{(j)} = 0$, and $c_{21}^{(j)} = b_{21}^{(j)} = 0$ for all j.

Compare the $(2,1)$-entries. Since the first term of the left-hand side is v-integral and the right-hand side times v is integral, we deduce that $v^{-r_j}\varphi(m_{21}^{(j)})$ is v-integral (recall that $\delta_{21}^{(j)} \in \mathbb{F}^\times$). Since $r_j \geq 1$, we get $v \mid m_{21}^{(j)}$ in $\mathbb{F}[\![v]\!]$. Since $p - (r_j + 1) \geq 0$ by genericity, it follows that the left-hand side of the $(2,1)$-entry of (3.33) is v-integral, hence $c_{22}^{(j)} = 0$ for all j.

Compare the $(1,2)$-entries again. Now we have the equality $m_{12}^{(j-1)}\delta_{21}^{(j)} = \delta_{12}^{(j)}v^{r_j+1}\varphi(m_{12}^{(j)})$ for all j. Comparing the order of v and using $\delta_{21}^{(j)}, \delta_{21}^{(j)} \in \mathbb{F}^\times$ for all j, we easily deduce that $m_{12}^{(j)} = 0$ for all j.

Now the equation (3.33) becomes

$$\begin{aligned} &\begin{pmatrix} \delta_{12}^{(j)}\big(m_{11}^{(j-1)} - \varphi(m_{11}^{(j)})\big) & 0 \\ \delta_{12}^{(j)}m_{21}^{(j-1)} + \delta_{22}^{(j)}\big(m_{22}^{(j-1)} - \varphi(m_{11}^{(j)})\big) - \delta_{21}^{(j)}v^{-(r_j+1)}\varphi(m_{21}^{(j)}) & \delta_{21}^{(j)}\big(m_{22}^{(j-1)} - \varphi(m_{22}^{(j)})\big) \end{pmatrix} \\ &= \begin{pmatrix} x_{12}^{(j)} & 0 \\ x_{22}^{(j)} & x_{21}^{(j)} \end{pmatrix}. \end{aligned} \tag{3.34}$$

Specializing the equation (3.34) at $v = 0$ and using that $v \mid m_{21}^{(j)}$ in $\mathbb{F}[\![v]\!]$ and $p - (r_j + 1) \geq 1$, we get

$$\begin{cases} \delta_{12}^{(j)}(m_{11}^{(j-1)} - m_{11}^{(j)})\big|_{v=0} = x_{12}^{(j)}, \\ \delta_{22}^{(j)}(m_{22}^{(j-1)} - m_{11}^{(j)})\big|_{v=0} = x_{22}^{(j)}, \\ \delta_{21}^{(j)}(m_{22}^{(j-1)} - m_{22}^{(j)})\big|_{v=0} = x_{21}^{(j)}. \end{cases} \tag{3.35}$$

By (3.29), $x_{12}^{(j)} = 0$ for $0 \leq j < f - 1$, hence $m_{11}^{(j)}\big|_{v=0}$ does not depend on j and we denote it m_{11}. By (3.35) again, it follows that $x_{12}^{(f-1)} = 0$. Similarly, $x_{21}^{(j)} = 0$ for $0 \leq j < f - 1$ by (3.29), hence $m_{22}^{(j)}\big|_{v=0}$ does not depend on j and we denote it m_{22}. As previously, it follows that $x_{21}^{(f-1)} = 0$.

If $\overline{\rho}$ is reducible split, then $\delta_{22}^{(j)} = 0$ for all j, hence by the second equation of (3.35), we get $x_{22}^{(j)} = 0$ for all j.

If $\overline{\rho}$ is reducible non-split, we know by (3.29) that $x_{22}^{(j_0)} = 0$, hence by the second equation of (3.35), we get $m_{11} = m_{22}$ since $\delta_{22}^{(j_0)} \in \mathbb{F}^\times$ by our choice of j_0. Then by the second equation of (3.35) again, we get $x_{22}^{(j)} = 0$ for all j.

As a result, the right-hand side of (3.32) vanishes and we conclude that $(M_{f-1-j})_j \in \mathrm{End}_{\varphi\text{-mod}}(\overline{\mathcal{M}})$. Denote this endomorphism by ξ. Since the isomorphism (3.30) satisfies

$$\left(1+\varepsilon\mathbb{V}_K^*(\xi)\right)\left[\left(1+\varepsilon\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4\end{pmatrix}\right)(\gamma\otimes 1)\right] = \gamma\otimes 1,$$

it follows that $\mathbb{V}_K^*(\xi) = -\left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4\end{smallmatrix}\right)$ with respect to the basis $\gamma_{\mathbb{F}}$. Moreover, we have

$$\mathrm{End}_{\varphi\text{-mod}}(\overline{\mathcal{M}}) \cong \mathrm{End}_{G_{K_\infty}}(\overline{\rho}|_{K_\infty}) \cong \mathrm{End}_{G_K}(\overline{\rho})$$

by [BHH+23, Lemma 3.2.8].

If $\overline{\rho}$ is non-split reducible, then $\mathrm{End}_{\varphi\text{-mod}}(\overline{\mathcal{M}}) = \mathbb{F}$. As $y_4 = 0$ we conclude from the formula for $\mathbb{V}_K^*(\xi)$ that $y_i = 0$ for all i.

If $\overline{\rho}$ is split reducible, then $\mathrm{End}_{\varphi\text{-mod}}(\overline{\mathcal{M}}) \cong \mathbb{F}\times\mathbb{F}$. By our condition that $\gamma_{\mathbb{F},1}$, $\gamma_{\mathbb{F},2}$ each span G_{K_∞}-stable lines, we conclude that $y_2 = y_3 = 0$. Using (3.29) we also have $y_1 = y_4 = 0$.

We have shown that $t = t_0$, completing the proof of the Claim.

The rest of the proof and the computations are completely analogous to that of [BHH+23, Prop. 4.3.1] (see Remark 3.4.3.2 below) using Theorem 3.4.2.1. As in the proof of Claim 2 of *loc.cit.*, one can identify the étale φ-modules between the tables via the change of variables given by the following.

Figure 3.2: Change of variables between the tables

Table 3.1	e_{11}^*	d_{11}	c_{11}	d_{21}	c_{12}	c_{21}	d_{22}^*	c_{22}
Table 3.4	d_{12}^*	$c_{12}-pd_{12}^*$	$b_{12}-pc_{12}$	d_{22}	d_{11}	c_{22}	d_{21}^*	c_{21}

Table 3.3	d_{11}^*	c_{11}	d_{12}	c_{12}	c_{21}	e_{22}^*	d_{22}	c_{22}
Table 3.5	d_{12}^*	c_{12}	d_{11}	c_{11}	d_{22}	d_{21}^*	$c_{21}-pd_{21}^*$	$b_{21}-pc_{21}$

□

Remark 3.4.3.2. *The Tables 3.4 and 3.5 are very similar to those of [BHH+23, §4]. The main difference is that here the element d_{22} of Table 3.4 is a unit in $R^{(j)}$ if and only if $a_{f-1-j}\neq 0$.*

The following result is a generalization of [BHH+23, Prop. 4.3.3] (where $\overline{\rho}$ was assumed to be semisimple).

Proposition 3.4.3.3. *Keep the assumption of Theorem 3.4.3.1. Fix $0\le j\le f-1$ and $\widetilde{w}\in X(\sigma)$ such that $i(\widetilde{w})_{f-1-j}=2$.*

If $(\widetilde{w}_\sigma)_{f-1-j} = t_{(1,2)}$, then we have $p\in \mathfrak{q}_1^{(j),(2,1)}\cap\mathfrak{q}_2^{(j),(2,1)}+\mathfrak{p}_{\widetilde{w}}^{(j),(3,0)}$, where

$$\mathfrak{q}_1^{(j),(2,1)} \overset{\text{def}}{=} (b_{12}-pc_{12}, c_{11}, c_{12}-pd_{12}^*, c_{21}, c_{22}, d_{11}),$$
$$\mathfrak{q}_2^{(j),(2,1)} \overset{\text{def}}{=} \left(b_{12}, c_{11}, c_{12}, c_{21}, c_{22}, \frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\right).$$

Here we omit the superscripts (j) for readability and we consider these as ideals of S.

If $(\widetilde{w}_\sigma)_{f-1-j} = t_{(2,1)}$, then we have $p\in \mathfrak{q}_2^{(j),(2,1)}\cap\mathfrak{q}_3^{(j),(2,1)}+\mathfrak{p}_{\widetilde{w}}^{(j),(3,0)}$, where

$$\mathfrak{q}_2^{(j),(2,1)} \overset{\text{def}}{=} \left(b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\right),$$
$$\mathfrak{q}_3^{(j),(2,1)} \overset{\text{def}}{=} (b_{21}-pc_{21}, c_{11}, c_{12}, c_{21}-pd_{21}^*, c_{22}, d_{22}).$$

Proof. The proof is purely computational and is completely analogous to that of [BHH$^+$23, Prop. 4.3.3] (see Remark 3.4.3.2). □

Table 3.1: $\widetilde{w}_{f-1-j} = t_{(2,1)}$ i.e. $\overline{A}^{(f-1-j)} = \begin{pmatrix} \overline{e^*_{11}}v^2 & 0 \\ \overline{d_{21}}v^2 & \overline{d^*_{22}}v \end{pmatrix}$.

$A^{(f-1-j)}$	$\begin{pmatrix} (v+p)^2e^*_{11} + (v+p)d_{11} + c_{11} & c_{12} \\ v((v+p)d_{21} + c_{21}) & (v+p)d^*_{22} + c_{22} \end{pmatrix}$
φ-module at the $(f-1-j)$-th embedding	$\begin{pmatrix} \frac{1}{v}\big((v+p)^2e^*_{11} + (v+p)d_{11} + c_{11}\big) & c_{12} \\ (v+p)d_{21} + c_{21} & (v+p)d^*_{22} + c_{22} \end{pmatrix} s_j^{-1} \begin{pmatrix} v^{r_j+1} & 0 \\ 0 & 1 \end{pmatrix}$
$R^{(j)}$	$\mathcal{O}[\![c_{11}, d_{11}, x^*_{11}, c_{12}, c_{21}, x_{21}, c_{22}, x^*_{22}]\!]$
$I^{(j),\le(3,0)}$	$c_{11}c_{22} + pc_{12}c_{21},$ $d_{11}c_{22} - c_{12}c_{21} + c_{11}d^*_{22} + pc_{12}d_{21},$ $e^*_{11}c_{22} + d_{11}d^*_{22} - c_{12}d_{21}$
$I^{(j),\nabla}$	$(a_1-1)d_{11}c_{22} + a_1c_{11}d^*_{22} + p(d_{11}d^*_{22} + 2e^*_{11}c_{22}) + O(p^{N-4}),$ $c_{22}(a_1c_{11} + pd_{11}) + O(p^{N-3}),$ $c_{12}((a_1-1)d_{11} + 2pe^*_{11}) + O(p^{N-4}),$ $c_{12}(a_1c_{11} + pd_{11}) + O(p^{N-3}),$ $(a_1-1)c_{21}c_{22} - p\big((a_1-3)d_{21}c_{22} + (a_1+1)c_{21}d^*_{22}\big) + O(p^{N-4}),$ $p\big((a_1-1)c_{21}c_{22} + p(d_{21}c_{22} - c_{21}d^*_{22})\big) + O(p^{N-3}),$ $(a_1-1)c_{12}c_{21} + c_{11}d^*_{22} - p\big((a_1-3)c_{12}d_{21} + d_{11}d^*_{22}\big) + O(p^{N-4}),$ $p\big((a_1-1)c_{12}c_{21} + c_{11}d^*_{22} + pc_{12}d_{21}\big) + O(p^{N-3})$
$I^{(j)} \stackrel{\text{def}}{=} (I^{(j),\nabla} + I^{(j),\le(3,0)})^{p\text{-sat}}$	$d_{11} + (a_1-2)\dfrac{c_{12}d_{21}}{d^*_{22}} + O(p^{N-8}),$ $c_{22} - (a_1-1)\dfrac{c_{12}d_{21}}{e^*_{11}} + O(p^{N-8}),$ $c_{21} + \dfrac{(a_1-1)(a_1-2)}{a_1}\dfrac{c_{12}(d_{21})^2}{e^*_{11}d^*_{22}} + O(p^{N-8}),$ $c_{11} - \dfrac{c_{12}d_{21}}{d^*_{22}}\Big(\dfrac{(a_1-1)^2(a_1-2)}{a_1}\dfrac{c_{12}d_{21}}{e^*_{11}d^*_{22}} - p\Big) + O(p^{N-8}),$ $\big(c_{12} + O(p^{N-8})\big)\Big((a_1-1)(a_1-2)\dfrac{c_{12}d_{21}}{e^*_{11}d^*_{22}} - 2p + O(p^{N-8})\Big)$
$\mathfrak{p}^{(j),(2,1)}$	$I^{(j)} + \big(c_{12} + O(p^{N-8})\big)$
$\mathfrak{p}^{(j),(3,0)}$	$I^{(j)} + \Big((a_1-1)(a_1-2)\dfrac{c_{12}d_{21}}{e^*_{11}d^*_{22}} - 2p + O(p^{N-8})\Big)$

Here $a_1 \in \mathbb{Z}_{(p)}$ and $a_1 \equiv -\langle s_j^{-1}(\mu_j) - (2,1), \alpha_j^\vee\rangle \equiv -\operatorname{sgn}(s_j)(r_j+1) + 1 \pmod p$. For readability we write a_1, c_{ik}, etc. instead of $a_1^{(j)}$, $c_{ik}^{(j)}$, etc. Also note that $x^*_{11} \stackrel{\text{def}}{=} e^*_{11} - [\overline{e^*_{11}}]$, $x^*_{22} \stackrel{\text{def}}{=} d^*_{22} - [\overline{d^*_{22}}]$ and $x_{21} \stackrel{\text{def}}{=} d_{21} - [\overline{d_{21}}]$.

Table 3.2: $\widetilde{w}_{f-1-j} = \mathfrak{w}t_{(2,1)}$ i.e. $\overline{A}^{(f-1-j)} = \begin{pmatrix} 0 & \overline{d^*_{12}}v \\ \overline{d^*_{21}}v^2 & \overline{d_{22}}v \end{pmatrix}$.

$A^{(f-1-j)}$	$\begin{pmatrix} (v+p)d_{11}+c_{11} & (v+p)d^*_{12}+c_{12} \\ v((v+p)d^*_{21}+c_{21}) & (v+p)d_{22}+c_{22} \end{pmatrix}$
φ-module at the $(f-1-j)$-th embedding	$\begin{pmatrix} (v+p)d^*_{12}+c_{12} & \frac{1}{v}\big((v+p)d_{11}+c_{11}\big) \\ (v+p)d_{22}+c_{22} & (v+p)d^*_{21}+c_{21} \end{pmatrix} s_j^{-1} \begin{pmatrix} v^{r_j+1} & 0 \\ 0 & 1 \end{pmatrix}$
$R^{(j)}$	$\mathcal{O}[\![c_{11}, d_{11}, c_{12}, x^*_{12}, c_{21}, x^*_{21}, c_{22}, x_{22}]\!]$
$I^{(j),\le(3,0)}$	$d_{11}d_{22} - (c_{12}d^*_{21} + d^*_{12}c_{21}) + pd^*_{12}d^*_{21}$, $c_{12}c_{21} - d_{11}c_{22} - c_{11}d_{22} - p(c_{12}d^*_{21} + d^*_{12}c_{21})$, $c_{11}c_{22} + pc_{12}c_{21}$
$I^{(j),\nabla}$	$(a_2-1)d_{11}c_{22} + a_2c_{11}d_{22} + p(d_{11}d_{22} - 2d^*_{12}c_{21} + pd^*_{12}d^*_{21}) + O(p^{N-4})$, $a_2c_{11}c_{22} + p(d_{11}c_{22} + pd^*_{12}c_{21}) + O(p^{N-3})$, $(a_2+1)c_{11}d^*_{12} + (a_2-1)d_{11}c_{12} + O(p^{N-4})$, $a_2c_{11}c_{12} + p(d_{11}c_{12} - c_{11}d^*_{12}) + O(p^{N-3})$, $(a_2-1)c_{21}c_{22} - p\big((a_2-3)d^*_{21}c_{22} + (a_2+1)c_{21}d_{22}\big) + O(p^{N-4})$, $p\big((a_2-1)c_{21}c_{22} + p(d^*_{21}c_{22} - c_{21}d_{22})\big) + O(p^{N-3})$, $(a_2-1)c_{12}c_{21} + c_{11}d_{22} - p\big((a_2-3)c_{12}d^*_{21} + (a_2-1)d^*_{12}c_{21}$ $+ d_{11}d_{22} + pd^*_{12}d^*_{21}\big) + O(p^{N-4})$, $p\big((a_2-1)c_{12}c_{21} + c_{11}d_{22} + pc_{12}d^*_{21}\big) + O(p^{N-3})$
$I^{(j)} \overset{\text{def}}{=} (I^{(j),\nabla} + I^{(j),\le(3,0)})^{p\text{-sat}}$	$c_{21} + (a_2-1)d^*_{21}\left(\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p\right) + O(p^{N-8})$, $c_{12} - a_2d^*_{12}\left(\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p\right) + O(p^{N-8})$, $c_{11} + \frac{a_2(a_2-1)}{a_2+1}d_{11}\left(\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p\right) + O(p^{N-8})$, $c_{22} - \frac{a_2(a_2-1)}{a_2-2}d_{22}\left(\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p\right) + O(p^{N-8})$, $\left(\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p + O(p^{N-8})\right)\left(\frac{a_2(a_2-1)}{(a_2-2)(a_2+1)}\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p + O(p^{N-8})\right)$
$\mathfrak{p}^{(j),(2,1)}$	$I^{(j)} + \left(\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p + O(p^{N-8})\right)$
$\mathfrak{p}^{(j),(3,0)}$	$I^{(j)} + \left(\frac{a_2(a_2-1)}{(a_2-2)(a_2+1)}\frac{d_{11}d_{22}}{d^*_{12}d^*_{21}} + p + O(p^{N-8})\right)$

Here $a_2 \in \mathbb{Z}_{(p)}$ and $a_2 \equiv -\langle \mathfrak{w}s_j^{-1}(\mu_j) - (2,1), \alpha_j^\vee \rangle \equiv \mathrm{sgn}(s_j)(r_j+1) + 1 \pmod p$. For readability we write a_2, c_{ik}, etc. instead of $a_2^{(j)}$, $c_{ik}^{(j)}$, etc. Also note that $x^*_{12} \overset{\text{def}}{=} d^*_{12} - [\overline{d^*_{12}}]$, $x^*_{21} \overset{\text{def}}{=} d^*_{21} - [\overline{d^*_{21}}]$ and $x_{22} \overset{\text{def}}{=} d_{22} - [\overline{d_{22}}]$.

Table 3.3: $\widetilde{w}_{f-1-j} = t_{(1,2)}$, i.e. $\overline{A}^{(f-1-j)} = \begin{pmatrix} \overline{d^*_{11}}v & 0 \\ 0 & \overline{e^*_{22}}v^2 \end{pmatrix}$.

$A^{(f-1-j)}$	$\begin{pmatrix} (v+p)d^*_{11}+c_{11} & (v+p)d_{12}+c_{12} \\ vc_{21} & (v+p)^2e^*_{22}+(v+p)d_{22}+c_{22} \end{pmatrix}$
φ-module at the $(f-1-j)$-th embedding	$\begin{pmatrix} (v+p)d^*_{11}+c_{11} & \frac{1}{v}\big((v+p)d_{12}+c_{12}\big) \\ vc_{21} & \frac{1}{v}\Big((v+p)^2e^*_{22}+(v+p)d_{22}+c_{22}\Big) \end{pmatrix} s_j^{-1} \begin{pmatrix} v^{r_j+1} & 0 \\ 0 & 1 \end{pmatrix}$
$R^{(j)}$	$\mathcal{O}[\![c_{11}, x^*_{11}, c_{12}, d_{12}, c_{21}, c_{22}, d_{22}, x^*_{22}]\!]$
$I^{(j),\le(3,0)}$	$c_{11}c_{22}+pc_{12}c_{21}$, $c_{11}d_{22}-c_{12}c_{21}+d^*_{11}c_{22}+pd_{12}c_{21}$, $c_{11}e^*_{22}+d^*_{11}d_{22}-d_{12}c_{21}$
$I^{(j),\nabla}$	$a_3c_{11}d_{22}+(a_3-1)d^*_{11}c_{22}-p(d^*_{11}d_{22}+2c_{11}e^*_{22})+O(p^{N-4})$, $c_{11}\big((a_3-1)c_{22}-pd_{22}\big)+O(p^{N-3})$, $c_{21}(a_3d_{22}-2pe^*_{22})+O(p^{N-4})$, $c_{21}\big((a_3-1)c_{22}-pd_{22}\big)+O(p^{N-3})$, $a_3c_{11}c_{12}-p\big((a_3+2)c_{11}d_{12}+(a_3-2)d^*_{11}c_{12}\big)+O(p^{N-4})$, $p\big(a_3c_{11}c_{12}-p(c_{11}d_{12}-d^*_{11}c_{12})\big)+O(p^{N-3})$, $a_3c_{12}c_{21}-d^*_{11}c_{22}-p\big((a_3+2)d_{12}c_{21}-d^*_{11}d_{22}\big)+O(p^{N-4})$, $p\big(a_3c_{12}c_{21}-d^*_{11}c_{22}-pd_{12}c_{21}\big)+O(p^{N-3})$
$I^{(j)} \stackrel{\text{def}}{=} (I^{(j),\nabla}+I^{(j),\le(3,0)})^{p\text{-sat}}$	$d_{22}-(a_3+1)\dfrac{d_{12}c_{21}}{d^*_{11}}+O(p^{N-8})$, $c_{11}+a_3\dfrac{d_{12}c_{21}}{e^*_{22}}+O(p^{N-8})$, $c_{12}-\dfrac{a_3(a_3+1)}{a_3-1}\dfrac{(d_{12})^2c_{21}}{d^*_{11}e^*_{22}}+O(p^{N-8})$, $c_{22}-\dfrac{d_{12}c_{21}}{d^*_{11}}\left(\dfrac{(a_3)^2(a_3+1)}{a_3-1}\dfrac{d_{12}c_{21}}{d^*_{11}e^*_{22}}-p\right)+O(p^{N-8})$, $\big(c_{21}+O(p^{N-8})\big)\left(a_3(a_3+1)\dfrac{d_{12}c_{21}}{d^*_{11}e^*_{22}}-2p+O(p^{N-8})\right)$
$\mathfrak{p}^{(j),(2,1)}$	$I^{(j)}+\big(c_{21}+O(p^{N-8})\big)$
$\mathfrak{p}^{(j),(3,0)}$	$I^{(j)}+\left(a_3(a_3+1)\dfrac{d_{12}c_{21}}{d^*_{11}e^*_{22}}-2p+O(p^{N-8})\right)$

Here $a_3 \in \mathbb{Z}_{(p)}$ and $a_3 \equiv -\langle s_j^{-1}(\mu_j)-(1,2), \alpha_j^\vee\rangle \equiv -\operatorname{sgn}(s_j)(r_j+1)-1 \pmod p$. For readability we write a_3, c_{ik}, etc. instead of $a_3^{(j)}$, $c_{ik}^{(j)}$, etc. Also, $x^*_{11} \stackrel{\text{def}}{=} d^*_{11}-[\overline{d^*_{11}}]$ and $x^*_{22} \stackrel{\text{def}}{=} e^*_{22}-[\overline{e^*_{22}}]$. Note that we necessarily have $a_{f-1-j}=0$ in this case.

Table 3.4: **Multi-type:** $(\widetilde{w}_\sigma)_{f-1-j} = t_{(1,2)}$**, so** $\widetilde{w}_{f-1-j} = t_{(2,1)}$ **and** $\mathfrak{w}t_{(2,1)}$**.**

Multi-type φ-module at the $(f-1-j)$-embedding	$\begin{pmatrix}(v+p)d_{12}^*+c_{12}+\frac{b_{12}}{v} & \frac{1}{v}\big((v+p)d_{11}+c_{11}\big)\\ (v+p)d_{22}+c_{22} & (v+p)d_{21}^*+c_{21}\end{pmatrix} s_j^{-1}\begin{pmatrix}v^{r_j+1} & 0\\ 0 & 1\end{pmatrix}$
$S^{(j)}$	$\mathcal{O}[\![c_{11}, d_{11}, b_{12}, c_{12}, x_{12}^*, c_{21}, x_{21}^*, c_{22}, x_{22}]\!]$
$I_{\widetilde{w}}^{(j)}, \quad i(\widetilde{w})_{f-1-j}=1$	$c_{11}+pd_{11},$ $c_{12}-pd_{12}^*+(a_1-2)\frac{d_{11}d_{22}}{d_{21}^*}+O(p^{N-8}),$ $c_{21}-(a_1-1)\frac{d_{11}d_{22}}{d_{12}^*}+O(p^{N-8}),$ $c_{22}+\frac{(a_1-1)(a_1-2)}{a_1}\frac{d_{11}(d_{22})^2}{d_{12}^*d_{21}^*}+O(p^{N-8}),$ $b_{12}-pc_{12}-\frac{d_{11}d_{22}}{d_{21}^*}\Big(\frac{(a_1-1)^2(a_1-2)}{a_1}\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}-p\Big)+O(p^{N-8}),$ $\big(d_{11}+O(p^{N-8})\big)\Big((a_1-1)(a_1-2)\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}-2p+O(p^{N-8})\Big)$
$I_{\widetilde{w}}^{(j)}, \quad i(\widetilde{w})_{f-1-j}=2$	$b_{12},$ $c_{21}+(a_2-1)d_{21}^*\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $c_{12}-a_2d_{12}^*\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $c_{11}+\frac{a_2(a_2-1)}{a_2+1}d_{11}\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $c_{22}-\frac{a_2(a_2-1)}{a_2-2}d_{22}\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)\Big(\frac{a_2(a_2-1)}{(a_2-2)(a_2+1)}\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(2,1)}, \quad i(\widetilde{w})_{f-1-j}=1$	$I_{\widetilde{w}}^{(j)}+\big(d_{11}+O(p^{N-8})\big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(3,0)}, \quad i(\widetilde{w})_{f-1-j}=1$	$I_{\widetilde{w}}^{(j)}+\Big(\frac{a_2(a_2-1)}{(a_2-2)(a_2+1)}\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(2,1)}, \quad i(\widetilde{w})_{f-1-j}=2$	$I_{\widetilde{w}}^{(j)}+\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(3,0)}, \quad i(\widetilde{w})_{f-1-j}=2$	$I_{\widetilde{w}}^{(j)}+\Big(\frac{a_2(a_2-1)}{(a_2-2)(a_2+1)}\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)$

For readability we write a_i, c_{ik}, etc. instead of $a_i^{(j)}$, $c_{ik}^{(j)}$, etc. Also note that $x_{12}^* \stackrel{\text{def}}{=} d_{12}^* - [\overline{d_{12}^*}]$, $x_{21}^* \stackrel{\text{def}}{=} d_{21}^* - [\overline{d_{21}^*}]$ and $x_{22} \stackrel{\text{def}}{=} d_{22} - [\overline{d_{22}}]$. Note that the constants a_1 and a_2 and the $O(p^{N-8})$ tails coming from Tables 3.1-3.2 (by the change of variables in Figure 3.2) depend on the whole f-tuple $\widetilde{w} \in X(\sigma)$.

Table 3.5: **Multi-type: $(\widetilde{w}_\sigma)_{f-1-j}=t_{(2,1)}$, so $\widetilde{w}_{f-1-j}=\mathfrak{w}t_{(2,1)}$ and $t_{(1,2)}$.**

Multi-type φ-module at the $(f-1-j)$-embedding	$\begin{pmatrix}(v+p)d_{12}^*+c_{12} & \frac{1}{v}\big((v+p)d_{11}+c_{11}\big)\\ (v+p)d_{22}+c_{22} & (v+p)d_{21}^*+c_{21}+\frac{b_{21}}{v}\end{pmatrix} s_j^{-1}\begin{pmatrix}v^{r_j+1} & 0\\ 0 & 1\end{pmatrix}$
$S^{(j)}$	$\mathcal{O}[\![c_{11},d_{11},c_{12},x_{12}^*,b_{21},c_{21},x_{21}^*,c_{22},x_{22}]\!]$
$I_{\widetilde{w}}^{(j)},\quad i(\widetilde{w})_{f-1-j}=2$	$b_{21},$ $c_{21}+(a_2-1)d_{21}^*\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $c_{12}-a_2d_{12}^*\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $c_{11}+\frac{a_2(a_2-1)}{a_2+1}d_{11}\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $c_{22}-\frac{a_2(a_2-1)}{a_2-2}d_{22}\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p\Big)+O(p^{N-8}),$ $\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)\Big(\frac{a_2(a_2-1)}{(a_2-2)(a_2+1)}\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)$
$I_{\widetilde{w}}^{(j)},\quad i(\widetilde{w})_{f-1-j}=3$	$c_{22}+pd_{22},$ $c_{21}-pd_{21}^*-(a_3+1)\frac{d_{11}d_{22}}{d_{12}^*}+O(p^{N-8}),$ $c_{12}+a_3\frac{d_{11}d_{22}}{d_{21}^*}+O(p^{N-8}),$ $c_{11}-\frac{a_3(a_3+1)}{a_3-1}\frac{(d_{11})^2d_{22}}{d_{12}^*d_{21}^*}+O(p^{N-8}),$ $b_{21}-pc_{21}-\frac{d_{11}d_{22}}{d_{12}^*}\Big(\frac{(a_3)^2(a_3+1)}{a_3-1}\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}-p\Big)+O(p^{N-8}),$ $\big(d_{22}+O(p^{N-8})\big)\Big(a_3(a_3+1)\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}-2p+O(p^{N-8})\Big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(2,1)},\quad i(\widetilde{w})_{f-1-j}=2$	$I_{\widetilde{w}}^{(j)}+\Big(\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(3,0)},\quad i(\widetilde{w})_{f-1-j}=2$	$I_{\widetilde{w}}^{(j)}+\Big(\frac{a_2(a_2-1)}{(a_2-2)(a_2+1)}\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}+p+O(p^{N-8})\Big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(2,1)},\quad i(\widetilde{w})_{f-1-j}=3$	$I_{\widetilde{w}}^{(j)}+\big(d_{22}+O(p^{N-8})\big)$
$\mathfrak{p}_{\widetilde{w}}^{(j),(3,0)},\quad i(\widetilde{w})_{f-1-j}=3$	$I_{\widetilde{w}}^{(j)}+\Big(a_3(a_3+1)\frac{d_{11}d_{22}}{d_{12}^*d_{21}^*}-2p+O(p^{N-8})\Big)$

For readability we write a_i, c_{ik}, etc. instead of $a_i^{(j)}$, $c_{ik}^{(j)}$, etc. Also, $x_{12}^*\overset{\text{def}}{=}d_{12}^*-[\overline{d_{12}^*}]$, $x_{21}^*\overset{\text{def}}{=}d_{21}^*-[\overline{d_{21}^*}]$ and $x_{22}\overset{\text{def}}{=}d_{22}$. Note that we necessarily have $a_{f-1-j}=0$ in this case. Also note that the constants a_2 and a_3 and the $O(p^{N-8})$ tails coming from Tables 3.2-3.3 (by the change of variables in Figure 3.2) depend on the whole f-tuple $\widetilde{w}\in X(\sigma)$.

3.5 Gelfand–Kirillov dimension and representations of GL_2

In this section, we recall a criterion that gives an upper bound for the Gelfand–Kirillov dimension of certain admissible smooth representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$. This criterion is given in [BHH$^+$23, §6] and is used in the semisimple case. It works in the non-semisimple case as well by the result of [HW22, §4].

We introduce some notation. Let $I \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the subgroup of upper triangular matrices modulo p. Let $I_1 \subseteq I$ be its maximal pro-p subgroup, which is the subgroup of upper unipotent matrices modulo p. Let $K_1 \overset{\text{def}}{=} 1 + pM_2(\mathcal{O}_K) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the first congruence subgroup. Let Z be the center of $\mathrm{GL}_2(K)$ and let $Z_1 \overset{\text{def}}{=} Z \cap K_1$. Let $\mathfrak{m}_{K_1}$ denote the unique maximal ideal of the Iwasawa algebra $\mathbb{F}[\![K_1/Z_1]\!]$. The ideal of $\mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$ generated by $\mathfrak{m}_{K_1}$ under the natural inclusion $\mathbb{F}[\![K_1/Z_1]\!] \hookrightarrow \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$ is also denoted by $\mathfrak{m}_{K_1}$ when there is no possible confusion. We denote $\Gamma \overset{\text{def}}{=} \mathrm{GL}_2(k)$ and $\widetilde{\Gamma} \overset{\text{def}}{=} \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}^2$.

Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be as in §3.4.1 (see (3.15)). Recall that [HW22, §4.1] constructs a finite dimensional representation $\widetilde{D}_0(\overline{\rho})$ of (the non-commutative ring) $\widetilde{\Gamma}$ over $\mathbb{F}$ (generalizing the constructions of [BP12, §13]) characterized by the following properties:

(i) $\mathrm{soc}_{\widetilde{\Gamma}} \widetilde{D}_0(\overline{\rho}) = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma$,
(ii) for each $\sigma \in W(\overline{\rho})$, we have $[\widetilde{D}_0(\overline{\rho}) : \sigma] = 1$,
(iii) $\widetilde{D}_0(\overline{\rho})$ is maximal with respect to properties (i) and (ii).

We have a decomposition of $\widetilde{\Gamma}$-representations $\widetilde{D}_0(\overline{\rho}) = \bigoplus_{\sigma \in W(\overline{\rho})} \widetilde{D}_{0,\sigma}(\overline{\rho})$, where each $\widetilde{D}_{0,\sigma}(\overline{\rho})$ satisfies $\mathrm{soc}_{\widetilde{\Gamma}} \widetilde{D}_{0,\sigma}(\overline{\rho}) = \sigma$. We have $\widetilde{D}_0(\overline{\rho})^{K_1} \cong D_0(\overline{\rho})$, which is the representation of Γ over $\mathbb{F}$ defined in [BP12, §13]. We also define the I-representation $D_1(\overline{\rho}) \overset{\text{def}}{=} D_0(\overline{\rho})^{I_1} = \widetilde{D}_0(\overline{\rho})^{I_1}$.

The following proposition comes from [HW22, Thm. 4.6].

Proposition 3.5.1. *The representation $\widetilde{D}_0(\overline{\rho})$ of $\widetilde{\Gamma}$ over $\mathbb{F}$ is multiplicity-free.*

One can describe explicitly the structure of each $\widetilde{D}_{0,\sigma}(\overline{\rho})$, where $\sigma \in W(\overline{\rho})$. We use the notation of §3.2.3. Let $\lambda \in X_1(\underline{T})$ be such that $\sigma \cong F(\lambda)$. By (3.18) and Lemma 3.2.3.4 there exist a subset $J_{\overline{\rho},\sigma} \subseteq \{0, \dots, f-1\}$ and elements $\varepsilon_j \in \{\pm 1\}$ for each $j \in J_{\overline{\rho},\sigma}$, such that

$$W(\overline{\rho}) = \{F(\mathrm{t}_\lambda(b_0, \dots, b_{f-1})) : b_j \in \{0, \varepsilon_j\} \text{ if } j \in J_{\overline{\rho},\sigma} \text{ and } b_j = 0 \text{ if } j \notin J_{\overline{\rho},\sigma}\}. \tag{3.36}$$

Then by the same argument as in [BP12, Prop. 13.4] together with [HW22, Lemma 4.8], [HW22, Cor. 2.35] and the translation formula (3.10), the multiplicity-free representation $\widetilde{D}_{0,\sigma}(\overline{\rho})$ has Jordan–Hölder factors

$$\mathrm{JH}(\widetilde{D}_{0,\sigma}(\overline{\rho})) = \left\{ \sigma_{\underline{a}} \overset{\text{def}}{=} F(t_\lambda(a_0, \dots, a_{f-1})) : a_j \in \mathbb{Z},\ \mathrm{sgn}(a_j) \neq \varepsilon_j \text{ for } j \in J_{\overline{\rho},\sigma},\ \sum_{j=0}^{f-1} \left[\frac{|a_j|}{2} \right] \leq 1 \right\} \tag{3.37}$$

and its submodule structure is determined as follows: the unique subrepresentation of $\widetilde{D}_{0,\sigma}(\overline{\rho})$ with cosocle $\sigma_{\underline{a}}$ has constituents $\sigma_{\underline{b}}$ for all $\underline{b}$ such that each b_j is between 0 and a_j. Here for $x \in \mathbb{R}$, we denote by $[x]$ the largest integer which is smaller than or equal to x.

The following proposition comes from [BHH$^+$23, Cor. 6.3.13.(i)] where the assumption that $\overline{\rho}$ is semisimple is not used in the proof of *loc.cit.*

Proposition 3.5.2. *Fix $m_\sigma \in \mathbb{Z}_{\geq 0}$ for each $\sigma \in W(\overline{\rho})$. There exists a unique (up to isomorphism) finite dimensional representation V of $\widetilde{\Gamma}$ over $\mathbb{F}$ such that*

(i) $\mathrm{soc}_{\widetilde{\Gamma}} V = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma^{m_\sigma}$,

(ii) *for each* $\sigma \in W(\overline{\rho})$*, we have* $[V : \sigma] = m_\sigma$*,*
(iii) V *is maximal with respect to properties* (i) *and* (ii)*.*

Proof. We use the same argument as in the proof of [BHH+23, Cor. 6.3.13(i)]. In particular, we can take V to be $\bigoplus_{\sigma \in W(\overline{\rho})} \widetilde{D}_{0,\sigma}(\overline{\rho})^{m_\sigma}$. □

Now we give a generalization of [BHH+23, Thm. 6.4.7] where $\overline{\rho}$ was assumed to be semisimple. This gives an upper bound for the Gelfand–Kirillov dimension $\dim_{\mathrm{GL}_2(K)}(\pi)$ of some admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$. We refer to [BHH+23, §5] for the notion of the Gelfand–Kirillov dimension.

Theorem 3.5.3. *Let* π *be an admissible smooth representation of* $\mathrm{GL}_2(K)$ *over* $\mathbb{F}$ *with a central character. Assume that:*

(i) *we have* $\mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)) = W(\overline{\rho})$ *up to multiplicity,*
(ii) *for each* $\sigma \in W(\overline{\rho})$*, we have* $[\pi[\mathfrak{m}_{K_1}^2]|_{\mathrm{GL}_2(\mathcal{O}_K)} : \sigma] = [\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi : \sigma]$*,*
(iii) *we have* $\mathrm{JH}(\pi^{I_1}) = \mathrm{JH}(D_1(\overline{\rho}))$ *up to multiplicity (as* I*-representations).*

Then $\dim_{\mathrm{GL}_2(K)}(\pi) \leq f$*.*

Proof. The proof is analogous to that of [BHH+23, Thm. 6.4.7]. For $\overline{\rho}$ not necessarily semisimple, the condition (a) of [BHH+23, §6.4] is guaranteed by Proposition 3.5.2. The condition (b) of *loc.cit.* is guaranteed by Proposition 3.5.1 and (3.37). Finally the condition (c) of *loc.cit.* is a consequence of [BHH+23, Lemma 6.4.3]. □

3.6 Global applications

In this section, we use the machinery of patching functors introduced by [EGS15] to prove the main global results: Theorem 3.6.3.1. We follow closely [BHH+23, §8] which deals with the semisimple case.

3.6.1 Patching functors

In this subsection, we recall the global setting following [BHH+23, §8.1], to which we refer the reader for more references and details.

We assume $p > 5$ and E unramified, so that $\mathcal{O} = W(\mathbb{F})$.

We fix F a totally real number field in which p is unramified. We denote by S_p the set of places of F above p. We refer to §3.1 for the notation $\mathcal{O}_F$, F_w, $\mathcal{O}_{F_w}$, Frob_w and $\mathbb{A}_F^\infty$. We fix a place $v \in S_p$.

We fix D a quaternion algebra with center F which is split at places above p and at exactly one infinite place (called the **indefinite** case) or at no infinite places (called the **definite** case). In the indefinite case we assume that $(D, F) \neq (\mathrm{GL}_2, \mathbb{Q})$ since our main theorem is already known in this case. We denote by S_D the set of finite places where D ramifies. We fix a maximal order $\mathcal{O}_D$ of D and isomorphisms $(\mathcal{O}_D)_w \xrightarrow{\sim} \mathrm{M}_2(\mathcal{O}_{F_w})$ for $w \notin S_D$, where $(\mathcal{O}_D)_w \overset{\mathrm{def}}{=} \mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_w}$.

We fix $\overline{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ a continuous representation and set $\overline{r}_w \overset{\mathrm{def}}{=} \overline{r}|_{G_{F_w}}$. Let $S_{\overline{r}}$ be the set of finite places where $\overline{r}$ ramifies. We assume that

(i) $\overline{r}|_{G_{F(\sqrt{\wp_1})}}$ is absolutely irreducible;
(ii) for each $w \in (S_D \cup S_{\overline{r}}) \setminus S_p$, the universal framed deformation ring $R_{\overline{r}_w}$ of $\overline{r}_w$ over $W(\mathbb{F})$ is formally smooth over $W(\mathbb{F})$;

(iii) for each $w \in S_p$, $\overline{r}_w$ is generic in the sense of [BP12, Def. 11.7]. In particular we have $S_p \subseteq S_{\overline{r}}$;

(iv) $\overline{r}_v$ (equivalently $\overline{r}_v^\vee$) is one of the following forms up to twist:

(a) $\overline{r}_v|_{I_{F_v}} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix}$ with $12 \leq r_i \leq p-15$,

(b) $\overline{r}_v|_{I_{F_v}} \cong \begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & 0 \\ 0 & \omega_{2f}^{p^f\sum_{j=0}^{f-1}(r_j+1)p^j} \end{pmatrix}$ with $13 \leq r_0 \leq p-14$ and $12 \leq r_i \leq p-15$ for $i>0$, where $f \overset{\text{def}}{=} [F_v : \mathbb{Q}_p]$;

Assume first that we are in the indefinite case, for each compact open subgroup $V \leq (D \otimes_F \mathbb{A}_F^\infty)^\times$ let X_V be the associated smooth projective algebraic Shimura curve over F (see e.g. [BD14, §3.1] and the references therein). We assume moreover that

(v) there exists V such that

$$\mathrm{Hom}_{G_F}\left(\overline{r}, H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})\right) \neq 0. \tag{3.38}$$

We let $\psi : G_F \to W(\mathbb{F})^\times$ be the Teichmüller lift of $\omega \det \overline{r}$ and set $\psi_w \overset{\text{def}}{=} \psi|_{G_{F_w}}$.

Exactly as in [BHH+23, §8.1], we fix a finite place w_1, a finite set of places S, a compact open subgroup $U = \prod_w U_w \subseteq \prod_w (\mathcal{O}_D)_w^\times \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$ which in particular satisfies $U_w \cong \mathrm{GL}_2(\mathcal{O}_{F_w})$ for each $w \in S_p$, a tame inertial type τ_w such that $\mathrm{JH}(\overline{\sigma(\tau_w)^\vee}) = \mathrm{JH}(\overline{\sigma(\tau_w^\vee)})$ contains exactly one Serre weight in $W(\overline{r}_w^\vee)$ for each $w \in S_p \setminus \{v\}$, and a $\mathrm{GL}_2(\mathcal{O}_{F_w})$-invariant lattice $\sigma^0(\tau_w^\vee)$ in $\sigma(\tau_w^\vee) = \sigma(\tau_w)^\vee$ for each $w \in S_p \setminus \{v\}$.

Then as in [EGS15, §6] we can define a patching functor M_∞ from the category of continuous representations σ_v of $U_v \cong \mathrm{GL}_2(\mathcal{O}_{F_v})$ on finite type $W(\mathbb{F})$-modules with central character $\psi|_{I_{F_v}}^{-1} \circ \mathrm{Art}_{F_v}|_{\mathcal{O}_{F_v}^\times}$ to the category of finite type R_∞-modules, where

$$R_\infty \cong R^{\psi_v}_{\overline{r}_v}[\![X_1, \dots, X_g]\!]$$

for some integer g. Here $R^{\psi_v}_{\overline{r}_v}$ is the framed deformation ring of $\overline{r}_v$ with fixed determinant $\varepsilon^{-1}\psi_v$.

We denote by $\mathfrak{m}_\infty$ the maximal ideal of R_∞ and for $w \in S_p \setminus \{v\}$ let σ_w be the unique Serre weight in $W(\overline{r}_w^\vee)$ that appears in $\mathrm{JH}(\overline{\sigma(\tau_w^\vee)})$. We have

$$M_\infty(\sigma_v)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{U_v/V_v}\left(\sigma_v, \mathrm{Hom}_{U^v/V^v}\left(\otimes_{w \in S_p \setminus \{v\}} \sigma_w, \mathrm{Hom}_{G_F}\left(\overline{r}, H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})\right)\right)\right)^\vee \tag{3.39}$$

for any $V = \prod V_w \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$ such that $V_w = U_w$ if $w \notin S_p$ and $V_w \subseteq 1 + p\,\mathrm{M}_2(\mathcal{O}_{F_w})$ normal in $\mathrm{GL}_2(\mathcal{O}_{F_w})$ if $w \in S_p$, and for any representation σ_v of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ over $W(\mathbb{F})$ on which V_v acts trivially. Moreover, we have $M_\infty(\sigma_v) \neq 0$ if and only if $\mathrm{JH}(\overline{\sigma_v}) \cap W(\overline{r}_v^\vee) \neq \emptyset$.

The definite case is analogous to the indefinite one. In this case, the space

$$\mathrm{Hom}_{G_F}\left(\overline{r}, H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})\right)$$

in (3.38) is replaced by $S(V, \mathbb{F})[\mathfrak{m}]$, where

$$S(V, \mathbb{F}) \overset{\text{def}}{=} \{f : D^\times \setminus (D \otimes_F \mathbb{A}_F^\infty)^\times / V \to \mathbb{F}\}$$

and $\mathfrak{m}$ is generated by $T_w - S_w \mathrm{Tr}(\overline{r}(\mathrm{Frob}_w)), \mathrm{Norm}(w) - S_w \det(\overline{r}(\mathrm{Frob}_w))$ for $w \notin S \cup \{w_1\}$ such that $V_w = (\mathcal{O}_D)_w^\times$, with T_w, S_w acting on $S(V, \mathbb{F})$ (via right translation on functions)

respectively by $V\left(\begin{smallmatrix}\varpi_w & 0\\ 0 & 1\end{smallmatrix}\right)V$, $V\left(\begin{smallmatrix}\varpi_w & 0\\ 0 & \varpi_w\end{smallmatrix}\right)V$, where ϖ_w is any uniformizer in F_w. Then, (3.39) is replaced by

$$M_\infty(\sigma_v)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{U_v/V_v}\left(\sigma_v, \mathrm{Hom}_{U^v/V^v}\left(\otimes_{w\in S_p\setminus\{v\}}\sigma_w, S(V,\mathbb{F})[\mathfrak{m}]\right)\right)^\vee. \tag{3.40}$$

3.6.2 Freeness for projective envelopes

In this subsection, we prove the freeness of some patched modules following [BHH+23, §8.2, §8.3] (which deals with the semisimple case). The main result is Proposition 3.6.2.4.

We keep the notation of §3.5 with $K \overset{\text{def}}{=} F_v$ and let k_v be the residue field of F_v. In particular we have $\Gamma = \mathrm{GL}_2(k_v)$ and $\widetilde{\Gamma} = \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_{F_v})/Z_1]\!]/\mathfrak{m}_{K_1}^2$. For σ a Serre weight, we let $P_\sigma \overset{\text{def}}{=} \mathrm{Proj}_\Gamma \sigma$ be the projective envelope of σ in the category of $\mathbb{F}[\Gamma]$-modules, $\widetilde{P}_\sigma$ be the projective $\mathcal{O}[\Gamma]$-module lifting P_σ and $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma$ be the projective envelope of σ in the category of $\widetilde{\Gamma}$-modules.

Let M_∞ be the patching functor defined in §3.6.1. If A is a ring and M is an A-module, we define the **scheme-theoretic support** of M to be the quotient $A/\operatorname{Ann}_A(M)$. For each τ a tame inertial type and $\lambda = (a_j, b_j)_j \in X^*(\underline{T})$ with $a_j > b_j$ for each j, we define

$$R_\infty^{\lambda,\tau} \overset{\text{def}}{=} R_\infty \otimes_{R_{\overline{r}_v^\vee}} R_{\overline{r}_v^\vee}^{\lambda,\tau},$$

where $R_{\overline{r}_v^\vee}^{\lambda,\tau}$ parametrizes potentially crystalline lifts of $\overline{r}_v^\vee$ of Hodge–Tate weights (a_j, b_j) in the j-th embedding $\sigma_j : F_v \hookrightarrow E$ and inertial type τ. When $a_j = a$ and $b_j = b$ for all j we write $R_\infty^{(a,b),\tau}$.

The following proposition is a generalization of [BHH+23, Prop. 8.2.3] (where $\overline{r}_v$ was assumed to be semisimple).

Proposition 3.6.2.1. *There exists an integer $r \geq 1$ such that*

(i) *for all $\sigma_v \in W(\overline{r}_v^\vee)$ the module $M_\infty(\sigma_v)$ is free of rank r over its scheme-theoretic support, which is formally smooth over $\mathbb{F}$.*
(ii) *for all tame inertial type τ such that $\mathrm{JH}(\overline{\sigma(\tau)}) \cap W(\overline{r}_v^\vee) \neq \emptyset$ and all $\mathrm{GL}_2(\mathcal{O}_{F_v})$-invariant $W(\mathbb{F})$-lattices $\sigma^0(\tau)$ in $\sigma(\tau)$ with irreducible cosocle, the module $M_\infty(\sigma^0(\tau))$ is free of rank r over its scheme-theoretic support $R_\infty^{(1,0),\tau}$, which is a domain.*

Proof. The proof is analogous to the one of [BHH+23, Thm. 8.2.3]. In the case $|\mathcal{J}| = 2$ (see the fourth paragraph of the proof of *loc.cit.*), we used the "connectedness" of $W(\overline{r}_v^\vee)$ by non-split extensions to deduce that $M_\infty(\sigma)$ has the same rank over its scheme-theoretic support for $\sigma \in W(\overline{r}_v^\vee)$. In general the Serre weights in $W(\overline{r}_v^\vee)$ can still be "connected" by non-split extensions by (3.18) and [BHH+23, Lemma 2.4.6]. □

Lemma 3.6.2.2. *Let A be a ring which is p-torsion free and M be an A-module which can be generated by d elements for some $d \geq 1$. If $M[\frac{1}{p}]$ is free of rank d over $A[\frac{1}{p}]$, then M is free of rank d over A.*

Proof. We have a surjective map of A-modules $f : A^{\oplus d} \twoheadrightarrow M$. By [Mat89, Thm. 2.4], f is an isomorphism after inverting p. Hence f is also injective since A is p-torsion free. □

The following proposition is a generalization of [BHH+23, Prop. 8.2.6] (where $\overline{r}_v$ was assumed to be semisimple).

Proposition 3.6.2.3. *Let r be the integer as in Proposition 3.6.2.1. If $\sigma_v \in W(\overline{r}_v^{\vee})$, then $M_\infty(\widetilde{P}_{\sigma_v})$ is free of rank r over $R_\infty/\cap_\tau \mathfrak{p}_\tau$, where τ runs over all tame inertial types such that $\sigma_v \in \mathrm{JH}(\overline{\sigma(\tau)})$ and $\mathfrak{p}_\tau$ is the prime ideal $\mathrm{Ker}(R_\infty \twoheadrightarrow R_\infty^{(1,0),\tau})$ of R_∞.*

Proof. The strategy of the proof is very close to the one of [Le19, Thm. 4.9] which treats the case $r = 1$. We freely use the notation from *loc.cit.*

First we show that $M_\infty(R_\mu/\operatorname{Fil}^2_\otimes R_\mu)$ is free of rank r over its scheme-theoretic support (see [Le19, Lemma 4.3]), where R_μ is the same as P_{σ_v} and $\operatorname{Fil}^2_\otimes R_\mu$ is a certain submodule of R_μ defined in [LMS22, §3]. The argument of [Le19, Lemma 4.3] gives a tame inertial type τ and a $\mathrm{GL}_2(\mathcal{O}_{F_v})$-invariant $W(\mathbb{F})$-lattice $\sigma^0(\tau)$ in $\sigma(\tau)$ such that

$$M_\infty(R_\mu/\operatorname{Fil}^2_\otimes R_\mu) \cong M_\infty\big(\overline{\sigma}^0(\tau)/\operatorname{rad}^2\overline{\sigma}^0(\tau)\big),$$

where $\overline{\sigma}^0(\tau)$ is the reduction modulo p of $\sigma^0(\tau)$ and $\operatorname{rad}^2\overline{\sigma}^0(\tau) \overset{\text{def}}{=} \operatorname{rad}\big(\operatorname{rad}\overline{\sigma}^0(\tau)\big)$ is the radical of the radical of $\overline{\sigma}^0(\tau)$ as an $\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_{F_v})]$-module. By the notation of the proof of [BHH$^+$23, Prop. 8.2.3] based on [EGS15, §10.1], the representation $\overline{\sigma}^0(\tau)/\operatorname{rad}^2\overline{\sigma}^0(\tau)$ has the form $\overline{\sigma}^{\mathcal{J}_0}$ for some capped interval $\mathcal{J}_0 \subseteq \{0,\dots,f-1\}$. Hence $M_\infty\big(\overline{\sigma}^0(\tau)/\operatorname{rad}^2\overline{\sigma}^0(\tau)\big)$ is free of rank r over its scheme-theoretic support by the proof of [BHH$^+$23, Prop. 8.2.3].

Next we show that if $I \subseteq S$ is such that $\big|I \cap \{\pm\omega^{(i)}\}\big| + \big|S_{\overline{\rho}}^{\sigma} \cap \{\pm\omega^{(i)}\}\big| = 1$ for each $0 \le i \le f-1$, then $M_\infty(\widetilde{R}_{\mu,I})$ is free of rank r over its scheme-theoretic support, which is $R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,I}}$ (see [Le19, Prop. 4.6, Prop. 4.7]). Here $S = \{\pm\omega^{(i)}\}_{0\le i\le f-1}$ whose subsets parametrize the Serre weights that appear in R_μ, $\overline{\rho}$ is the same as $\overline{r}_v^{\vee}$, $S_{\overline{\rho}}^{\sigma}$ is a subset of S satisfying $W(\overline{\rho}) = \{\sigma_J | J \subset S_{\overline{\rho}}^{\sigma}\}$, $\widetilde{R}_\mu$ is the same as $\widetilde{P}_{\sigma_v}$, $\widetilde{R}_{\mu,I}$ is a certain quotient of $\widetilde{R}_\mu$, $T_{\sigma,I}$ is the set of tame inertial types that appear as subquotients in the Γ-representation $\widetilde{R}_{\mu,I}[1/p]$ over E and $R_{\overline{\rho}}^{T_{\sigma,I}}$ is the maximal reduced $\mathcal{O}$-flat quotient of $R_{\overline{\rho}}$ parametrizing potentially crystalline framed deformations of $\overline{\rho}$ of Hodge–Tate weights $(1,0)$ at each embedding and inertial type in $T_{\sigma,I}$. In fact, the argument of [Le19, Prop. 4.6] shows that $M_\infty(\widetilde{R}_{\mu,I})$ is minimally generated by r elements, and the argument of [Le19, Prop. 4.7] shows that $M_\infty(\widetilde{R}_{\mu,I})$ has scheme-theoretic support $R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,I}}$ which is p-torsion free. Moreover, by exactness of M_∞ we have

$$M_\infty\big(\widetilde{R}_{\mu,I}\big)[1/p] \cong \bigoplus_{\tau\in T_{\sigma,I}} M_\infty(\sigma^0(\tau))[1/p]$$

for any choices of $\mathrm{GL}_2(\mathcal{O}_{F_v})$-stable $W(\mathbb{F})$-lattices $\sigma^0(\tau) \subset \sigma(\tau)$. In particular, we can take $\sigma^0(\tau)$ to have irreducible cosocle (see [EGS15, Lemma 4.1.1]). By Proposition 3.6.2.1 and the fact that the supports of $M_\infty(\sigma^0(\tau))[1/p]$ are pairwise disjoint for $\tau \in T_{\sigma,I}$, it follows that $M_\infty\big(\widetilde{R}_{\mu,I}\big)[1/p]$ is free of rank r over $\big(R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,I}}\big)[1/p]$. We deduce from Lemma 3.6.2.2 that $M_\infty(\widetilde{R}_{\mu,I})$ is free is rank r over its scheme-theoretic support, which is $R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,I}}$.

Then we show that if $I \subseteq S$ is such that $\big|I \cap \{\pm\omega^{(i)}\}\big| + \big|S_{\overline{\rho}}^{\sigma} \cap \{\pm\omega^{(i)}\}\big| \le 1$ for each $0 \le i \le f-1$, then $M_\infty(\widetilde{R}_{\mu,I})$ is free of rank r over its scheme-theoretic support, which is $R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,I}}$ (see [Le19, Thm. 4.9]). The proof is completely analogous to that of [Le19, Thm. 4.9].

In particular if we take $I = \emptyset$ so that we have $\widetilde{R}_{\mu,\emptyset} = \widetilde{R}_\mu = \widetilde{P}_{\sigma_v}$, we get that $M_\infty(\widetilde{P}_{\sigma_v})$ is free of rank r over its scheme-theoretic support, which is $R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,\emptyset}}$. Moreover, by the Chinese remainder theorem we have

$$\big(R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,\emptyset}}\big)[1/p] \cong \bigoplus_{\tau} \big(R_\infty/\mathfrak{p}_\tau\big)[1/p] \cong \big(R_\infty/\cap_\tau \mathfrak{p}_\tau\big)[1/p],$$

where τ runs over all tame inertial types such that $\sigma_v \in \mathrm{JH}(\overline{\sigma(\tau)})$. Since both $R_\infty/\mathfrak{p}_\tau$ and $R_\infty^{(1,0),\tau}$ are p-torsion free $\mathcal{O}$-algebras, we deduce that

$$R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{T_{\sigma,\emptyset}} \cong R_\infty/\cap_\tau \mathfrak{p}_\tau.$$

Finally, each $\mathfrak{p}_\tau$ is a prime ideal because $R_\infty/\mathfrak{p}_\tau = R_\infty^{(1,0),\tau}$ is a domain by [EGS15, Thm. 7.2.1]. □

Now we recall from [BHH+23, §7.3] the construction of a $\mathrm{GL}_2(\mathcal{O}_{F_v})$-representation R over $W(\mathbb{F})$ such that $R/pR \cong \mathrm{Proj}_{\widetilde{\Gamma}} \sigma_v$.

We define $L_{-1} \overset{\text{def}}{=} \widetilde{P}_{\sigma_v}$ and $R_{2,j} \overset{\text{def}}{=} \big(\mathrm{Sym}^2 W(\mathbb{F})^2 \otimes \det{}^{-1}\big)^{(j)} \otimes_{W(\mathbb{F})} \widetilde{P}_{\sigma_v}$ for each $j = 0, \dots, f-1$, where "(j)" means that the matrices of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ act via the j-th embedding $\sigma_j : \mathcal{O}_{F_v} \hookrightarrow W(\mathbb{F})$. For each j there is an isomorphism (see [BHH+23, (55)])

$$R_{2,j}/pR_{2,j} \cong P_{\sigma_v} \oplus P_{\sigma_{v_j^+}} \oplus P_{\sigma_{v_j^-}} \tag{3.41}$$

for some Serre weights $\sigma_{v_j^+}$ and $\sigma_{v_j^-}$. The isomorphism (3.41) induces an inclusion

$$\iota_j : P_{\sigma_v} \hookrightarrow R_{2,j}/pR_{2,j}.$$

As in the paragraph before [BHH+23, Lem. 8.3.2], we define a $\mathrm{GL}_2(\mathcal{O}_{F_v})$-lattice $R'_{2,j}$ in $R_{2,j}[1/p]$ by

$$R'_{2,j} \overset{\text{def}}{=} P_{\sigma_v} \times_{R_{2,j}/p} R_{2,j} = \{x \in R_{2,j} : (x \bmod pR_{2,j}) \in \iota_j(R_{\sigma_v})\}.$$

We also define a $\mathrm{GL}_2(\mathcal{O}_{F_v})$-lattice L_j in $L_{-1}[1/p] \oplus \big(\bigoplus_{j'=0}^{j} R_{2,j}[1/p]\big)$ by

$$L_j \overset{\text{def}}{=} \left\{ \begin{array}{r} (x, (x_{j'})_{0\le j'\le j}) \in L_{-1} \oplus (\oplus_{j'=0}^{j} R_{2,j}) : (x_{j'} \bmod pR_{2,j'}) = (x \bmod pL_{-1}) \\ \text{via } \iota_{j'} : L_{-1}/pL_{-1} \hookrightarrow R_{2,j'}/pR_{2,j'} \ \forall\, 0 \le j' \le j \end{array} \right\},$$

which is equivalent to defining

$$L_j = L_{j-1} \times_{P_{\sigma_v}} R'_{2,j}$$

for each $0 \le j \le f-1$ (see [BHH+23, (8)]). We define $R \overset{\text{def}}{=} L_{f-1}$. It satisfies $R/pR \cong \mathrm{Proj}_{\widetilde{\Gamma}} \sigma_v$ (see [BHH+23, Cor. 7.3.4]).

Let r be the integer as in Proposition 3.6.2.1. The following proposition is a generalization of [BHH+23, Thm. 8.3.4], [BHH+23, Thm. 8.3.9], [BHH+23, Cor. 8.3.10] and [BHH+23, Thm. 8.3.11] (where $\overline{r}_v$ was assumed to be semisimple).

Proposition 3.6.2.4. (i) *For each $0 \le j \le f-1$, the module $M_\infty(R'_{2,j})$ is free of rank r over $R_\infty/\cap_\tau \mathfrak{p}_\tau$, where τ runs over all tame inertial types such that $\sigma_v \in \mathrm{JH}(\overline{\sigma(\tau)})$ and $\mathfrak{p}_\tau$ is the prime ideal $\mathrm{Ker}(R_\infty \twoheadrightarrow R_\infty^{(2,-1)_j,\tau})$ of R_∞, where $(2,-1)_j$ is $(2,-1)$ in the j-th embedding $\sigma_j : F_v \hookrightarrow E$ and $(1,0)$ elsewhere.*

(ii) *The module $M_\infty(R)$ is free of rank r over $R_\infty/\cap_{\lambda,\tau}\mathfrak{p}_{\lambda,\tau}$, where τ runs over all tame inertial types such that $\sigma_v \in \mathrm{JH}(\overline{\sigma(\tau)})$, $\lambda = (\lambda_j)_{0\le j\le f-1}$ runs over the Hodge–Tate weights such that $\lambda_j \in \{(1,0),(2,-1)\}$ for all j and $\mathfrak{p}_{\lambda,\tau}$ is the prime ideal $\ker(R_\infty \twoheadrightarrow R_\infty^{\lambda,\tau})$ of R_∞. In particular, we have $\dim_{\mathbb{F}} M_\infty(R)/\mathfrak{m}_\infty = r$.*

(iii) *The surjection $\mathrm{Proj}_{\widetilde{\Gamma}} \sigma_v \twoheadrightarrow \sigma_v$ induces an isomorphism of nonzero finite-dimensional $\mathbb{F}$-vector spaces*

$$M_\infty\big(\mathrm{Proj}_{\widetilde{\Gamma}} \sigma_v\big)/\mathfrak{m}_\infty \xrightarrow{\sim} M_\infty(\sigma_v)/\mathfrak{m}_\infty.$$

Proof. The proof of (i) is analogous to the one of [BHH$^+$23, Thm. 8.3.4]. All the arguments concerning the set of Serre weights $W(\overline{r}_v^\vee)$ go through in the general case because we always have $W(\overline{r}_v^\vee) \subseteq W((\overline{r}_v^\vee)^{\mathrm{ss}})$ (see (3.18)). We also replace [BHH$^+$23, Prop. 8.2.6] by Proposition 3.6.2.3. Finally each $\mathfrak{p}_\tau$ is a prime ideal by Theorem 3.4.2.1.

The proof of (ii) is analogous to the one of [BHH$^+$23, Thm. 8.3.9] using the same comment on $W(\overline{r}_v^\vee)$ together with Proposition 3.6.2.3, Proposition 3.4.3.3 and (i).

Finally, (iii) is a direct consequence of Proposition 3.6.2.1, (i) and (ii). □

3.6.3 Global results

In this subsection, we state and prove the main global results following [BHH$^+$23, §8.4], (which deals with the semisimple case).

Let F, D, $\overline{r}$, ψ, S and $U = \prod_w U_w \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$ be as in §3.6.1. For each $w \in S_p \setminus \{v\}$, we fix a Serre weight $\sigma_w \in W(\overline{r}_w^\vee)$. We consider the following admissible smooth representation π' of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$ with central character $\overline{\psi}^{-1} = \omega^{-1}(\det \overline{r}_v)^{-1}$:

$$\pi' \overset{\mathrm{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{U^v/V^v}\Big(\otimes_{w\in S_p\setminus\{v\}} \sigma_w, \mathrm{Hom}_{G_F}\big(\overline{r}, H^1_{\text{ét}}(X_{V^vV_v} \times_F \overline{F}, \mathbb{F})\big)\Big) \text{ in the indefinite case,}$$

$$\pi' \overset{\mathrm{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{U^v/V^v}\big(\otimes_{w\in S_p\setminus\{v\}} \sigma_w, S(V^vV_v, \mathbb{F})[\mathfrak{m}]\big) \qquad \text{in the definite case,}$$

with $V^v = \prod_{w\neq v} V_w$ as in (3.39) or (3.40), i.e. $V_w = U_w$ if $w \notin S_p$ and $V_w \subseteq 1 + p\,\mathrm{M}_2(\mathcal{O}_{F_w})$ is normal in $\mathrm{GL}_2(\mathcal{O}_{F_w})$ if $w \in S_p$. Recall that we defined the Gelfand–Kirillov dimension $\dim_{\mathrm{GL}_2(F_v)}(\pi)$ in §3.5.

The following theorem is a generalization of [BHH$^+$23, Thm. 8.4.1], [BHH$^+$23, Thm. 8.4.2], [BHH$^+$23, Cor. 8.4.4] and [BHH$^+$23, Cor. 8.4.6], where $\overline{r}_v$ was assumed to be semisimple.

Theorem 3.6.3.1. (i) *We have* $\dim_{\mathrm{GL}_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]$.

(ii) *There is an integer* $r \geq 1$ *such that*

$$\pi[\mathfrak{m}_{K_1}^2] \cong \big(\widetilde{D}_0(\overline{r}_v^\vee)\big)^{\oplus r},$$

where $\widetilde{D}_0(\overline{r}_v^\vee)$ *is defined in §3.5. In particular, each irreducible constituent of* $\pi[\mathfrak{m}_{K_1}^2]$ *has multiplicity* r.

(iii) *Let* $x : R_\infty \to \mathcal{O}'$ *be any homomorphism of local* $W(\mathbb{F})$*-algebras, where* $\mathcal{O}'$ *is the ring of integers of a finite extension* E' *of* E*, and set*

$$V(x) \overset{\mathrm{def}}{=} \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}'}\big(\mathbb{M}_\infty \otimes_{R_\infty,x} \mathcal{O}', E'\big),$$

where $\mathbb{M}_\infty$ *is the big patched module over* R_∞ *with an* R_∞*-linear action of* $\mathrm{GL}_2(F_v)$ *defined in [CEG*$^+$*16, §2.8] and also mentioned in the proof of [BHH*$^+$*23, Thm. 8.4.1]. Then* $V(x)$ *is a* ***nonzero*** *admissible unitary Banach representation of* $\mathrm{GL}_2(F_v)$ *over* E' *with a* $\mathrm{GL}_2(F_v)$*-invariant unit ball (given by* $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}'}\big(\mathbb{M}_\infty \otimes_{R_\infty,x} \mathcal{O}', \mathcal{O}'\big)$*) lifting* $\pi \otimes_{\mathbb{F}} \mathbb{F}'$*, where* $\mathbb{F}'$ *is the residue field of* $\mathcal{O}'$.

(iv) *For any compact open subgroup*

$$V^v = \prod_{w\notin S_D\cup S_{\overline{r}}} (\mathcal{O}_D)^\times_w \prod_{w\in (S_D\cup S_{\overline{r}})\setminus\{v\}} V_w \quad \subseteq \quad \prod_{w\neq v} (\mathcal{O}_D)^\times_w$$

such that $V_w \subseteq 1 + p\,\mathrm{M}_2(\mathcal{O}_{F_w})$ *is normal in* $\mathrm{GL}_2(\mathcal{O}_{F_w})$ *for* $w \in S_p\setminus\{v\}$ *and such that* $\pi \neq 0$*, where*

$$\pi \overset{\mathrm{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{G_F}\big(\overline{r}, H^1_{\text{ét}}(X_{V^vV_v} \times_F \overline{F}, \mathbb{F})\big) \text{ in the indefinite case,}$$

$$\pi \overset{\text{def}}{=} \varinjlim_{V_v} S(V^v V_v, \mathbb{F})[\mathfrak{m}] \text{ in the definite case,}$$

we have $\dim_{\mathrm{GL}_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]$.

Proof. The proof of (i) is analogous to the one of [BHH$^+$23, Thm. 8.4.1]. We use Theorem 3.5.3 for the upper bound of the Gelfand–Kirillov dimension. The condition (ii) in Theorem 3.5.3 is guaranteed by Proposition 3.6.2.4 (iii), and the conditions (i) and (iii) in Theorem 3.5.3 are satisfied as in the proof of [BHH$^+$23, Thm. 8.4.1].

The proof of (ii) is completely analogous to the one of [BHH$^+$23, Thm 8.4.2] using Proposition 3.6.2.4 (iii) and Proposition 3.5.1.

The proof of (iii) is completely analogous to the one of [BHH$^+$23, Cor. 8.4.4] using (i).

Finally, the proof of (iv) is completely analogous to that of [BHH$^+$23, Cor 8.4.6] using (i). □

Chapter 4

On the rank of the multivariable $(\varphi, \mathcal{O}_K^\times)$-modules associated to mod p representations of $\mathrm{GL}_2(K)$

4.1 Introduction

Let p be a prime number. The mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is completely known by the work of Breuil, Colmez, Emerton, etc. In particular, Colmez ([Col10]) constructed a functor from the category of admissible finite length mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to the category of finite-dimensional continuous mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, using Fontaine's category of (φ, Γ)-modules ([Fon90]) as an intermediate step. This gives a functorial way to realize the mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

However, the situation becomes much more complicated when we consider $\mathrm{GL}_2(K)$ for K a nontrivial finite extension of $\mathbb{Q}_p$. For example, there are many more supersingular representations of $\mathrm{GL}_2(K)$ ([BP12]) and we don't have a classification of these representations. Moreover, they are not of finite presentation ([Sch15],[Wu21]), and it is impossible so far to write down explicitly one of these representations. Motivated by the local-global compatibility result of Emerton ([Eme11]) for $\mathrm{GL}_2(\mathbb{Q}_p)$, we are particularly interested in the mod p representations π of $\mathrm{GL}_2(K)$ coming from the cohomology of towers of Shimura curves.

In [BHH$^+$b], Breuil-Herzig-Hu-Morra-Schraen constructed an exact functor D_A from a nice subcategory of the category of admissible smooth mod p representations of $\mathrm{GL}_2(K)$ to the category of multivariable $(\varphi, \mathcal{O}_K^\times)$-modules. Then the key question is to determine the structure of $D_A(\pi)$ for π as above, which can be used to deduce properties of π.

We recall the construction of the functor D_A. We let K be a finite unramified extension of $\mathbb{Q}_p$ of degree $f \geq 1$ with ring of integers $\mathcal{O}_K$ and residue field $\mathbb{F}_q$ (hence $q = p^f$). Let $\mathbb{F}$ be a large enough finite extension of $\mathbb{F}_p$ and fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$. We let $N_0 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$. Then we have $\mathbb{F}[\![N_0]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$ with $Y_j \overset{\text{def}}{=} \sum_{a \in \mathbb{F}_q^\times} \sigma_0(a)^{-p^j} \left(\begin{smallmatrix} 1 & [a] \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{F}[\![N_0]\!]$ for $0 \leq j \leq f-1$, where $[a] \in \mathcal{O}_K^\times$ is the Techmüller lift of $a \in \mathbb{F}_q^\times$. We let A be the completion of $\mathbb{F}[\![N_0]\!][1/(Y_0 \cdots Y_{f-1})]$ with respect to the $(Y_0, \dots, Y_{f-1})$-adic topology on $\mathbb{F}[\![N_0]\!]$. There is an $\mathbb{F}$-linear action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![N_0]\!]$ given by multiplication on $N_0 \cong \mathcal{O}_K$, and an $\mathbb{F}$-linear Frobenius φ on $\mathbb{F}[\![N_0]\!]$ given by multiplication by p on $N_0 \cong \mathcal{O}_K$. They extend canonically by continuity to commuting continuous $\mathbb{F}$-linear actions of φ and $\mathcal{O}_K^\times$ on A. Then an étale $(\varphi, \mathcal{O}_K^\times)$-module over A is by definition a finite free A-module endowed with a semi-linear Frobenius φ and a

commuting continuous semi-linear action of $\mathcal{O}_K^\times$ such that the image of φ generates everything.

For π an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with central character, we let $\pi^\vee$ be its $\mathbb{F}$-linear dual, which is a finitely generated $\mathbb{F}[\![I_1]\!]$-module and is endowed with the $\mathfrak{m}_{I_1}$-adic topology, where $I_1 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ and $\mathfrak{m}_{I_1}$ is the maximal ideal of $\mathbb{F}[\![I_1]\!]$. We define $D_A(\pi)$ to be the completion of $\mathbb{F}[\![N_0]\!][1/(Y_0\cdots Y_{f-1})]\otimes_{\mathbb{F}[\![N_0]\!]}\pi^\vee$ with respect to the tensor product topology. The $\mathcal{O}_K^\times$-action on $\pi^\vee$ given by $f\mapsto f\circ\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (for $a\in\mathcal{O}_K^\times$) extends by continuity to $D_A(\pi)$, and the ψ-action on $\pi^\vee$ given by $f\mapsto f\circ\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induces a continuous A-linear map

$$\beta : D_A(\pi) \to A\otimes_{\varphi,A} D_A(\pi). \tag{4.1}$$

Let $\mathcal{C}$ be the abelian category of admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with central characters such that $\mathrm{gr}\,(D_A(\pi))$ is a finitely generated $\mathrm{gr}(A)$-module. Then for π in $\mathcal{C}$, $D_A(\pi)$ is a finite free A-module (see [BHH$^+$b, Cor. 3.1.2.9] and [BHH$^+$c, Remark. 2.6.2]). If moreover β is an isomorphism, then its inverse $\beta^{-1}=\mathrm{id}\otimes\varphi$ makes $D_A(\pi)$ an étale $(\varphi,\mathcal{O}_K^\times)$-module.

Let $\overline{\rho}:\mathrm{GL}_2(K)\to\mathrm{GL}_2(\mathbb{F})$ be a continuous representation of the following form up to twist:

$$\overline{\rho}|_{I_K}\cong\begin{pmatrix}\omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1\end{pmatrix} \text{ with } 2f+1\le r_j\le p-3-2f\ \forall\, 0\le j\le f-1, \tag{4.2}$$

where $\omega_f : I_K\to\mathbb{F}^\times$ is the fundamental character of level f (associated to σ_0). If $f=1$, we assume moreover that $r_0\ge 4$. In particular, we have $p\ge 4f+4$.

Let π be a smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ which satisfies

(i) $\pi^{K_1}\cong D_0(\overline{\rho})$ as $K^\times\mathrm{GL}_2(\mathcal{O}_K)$-representations, where $D_0(\overline{\rho})$ is the representation of $\mathrm{GL}_2(\mathbb{F}_q)$ defined in [BP12, §13] and is viewed as a representation of $\mathrm{GL}_2(\mathcal{O}_K)$ by inflation, and $K^\times$ acts on $D_0(\overline{\rho})$ by the character $\det(\overline{\rho})\omega^{-1}$, where ω is the mod p cyclotomic character (in particular, π is admissible and has a central character);

(ii) for any character $\chi : I\to\mathbb{F}^\times$ appearing in $\pi[\mathfrak{m}_{I_1}]=\pi^{I_1}$, we have $[\pi[\mathfrak{m}_{I_1}^3]:\chi]=1$, where $\pi[\mathfrak{m}_{I_1}^3]$ is the set of elements of π annihilated by $\mathfrak{m}_{I_1}^3$, and $[\pi[\mathfrak{m}_{I_1}^3]:\chi]$ is the multiplicity of χ in the semisimplification of $\pi[\mathfrak{m}_{I_1}^3]$ as I-representations.

In particular, (i) and (ii) are satisfied for those π coming from the cohomology of towers of Shimura curves in a "multiplicity-one" situation ([BHH$^+$23],[Wan23]). Our main result is the following:

Theorem 4.1.1. *Suppose that $\overline{\rho}$ and π are as above. Then π is in $\mathcal{C}$, β in (4.1) is an isomorphism and*

$$\mathrm{rank}_A\, D_A(\pi)=2^f.$$

By [BHH$^+$b, Remark 3.3.2.6(ii)] we know that π is in $\mathcal{C}$. By [BHH$^+$b, Thm. 3.3.2.1] and localization we know that $\mathrm{rank}_A\, D_A(\pi)\le 2^f$. Theorem 4.1.1 is proved by [BHH$^+$c, Thm. 3.1.3] when $\overline{\rho}$ is semisimple. We generalize the method of [BHH$^+$c] to the non-semisimple case, which is seriously more delicate.

The proof of Theorem 4.1.1 is by an explicit construction of an A-basis of the dual étale $(\varphi,\mathcal{O}_K^\times)$-module $\mathrm{Hom}_A(D_A(\pi),A)$. As in [BHH$^+$c, (87)], there is a canonical A-linear injection

$$\mu_* : \mathrm{Hom}_A(D_A(\pi),A)\hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi),\mathbb{F}).$$

We will construct 2^f projective systems $(x_{J,k})_{k\ge 0}$ of elements of π indexed by subsets of $J\subseteq\{0,1,\dots,f-1\}$ with $x_{J,k}\in\pi[\mathfrak{m}_{I_1}^{kf+1}]$. By [BHH$^+$c, Prop. 3.2.3], each projective system x_J can

be regarded as an element of $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi),\mathbb{F})$. Then we prove that each x_J satisfies a crucial finiteness condition (see Theorem 4.8.5), which guarantees that it lies in the image of μ_*. Once we prove that $x_J \in \mathrm{Hom}_A(D_A(\pi),A)$ for all J, it is not difficult to conclude that they form an A-basis of $\mathrm{Hom}_A(D_A(\pi),A)$.

We also prove the following generalization of Theorem 4.1.1, which is crucially needed to prove that π is of finite length (in the non-semisimple case) in [BHH$^+$a].

Theorem 4.1.2 (Theorem 4.11.2)**.** *Suppose that $\overline{\rho}$ and π are as above. Then for π_1 a subrepresentation of π, we have*

$$\mathrm{rank}_A\, D_A(\pi_1) = \left|\mathrm{JH}(\pi_1^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})\right|,$$

where $\mathrm{JH}(\pi_1^{K_1})$ is the set of Jordan–Hölder factors of $\pi_1^{K_1}$ as a $\mathrm{GL}_2(\mathcal{O}_K)$-representation, $\overline{\rho}^{\mathrm{ss}}$ is the semisimplification of $\overline{\rho}$, and $W(\overline{\rho}^{\mathrm{ss}})$ is the set of Serre weights of $\overline{\rho}^{\mathrm{ss}}$ defined in [BDJ10, §3].

Organization of the chapter

In §§4.2-4.3, we review the notion of the extension graph and recall some results of [BP12, §2] that are needed in the proof of Theorem 4.1.1 and Theorem 4.1.2. In §§4.4-4.8, we explicitly construct some projective systems of elements of π and study their basic properties. In particular, we prove the crucial finiteness condition in §4.8. In §4.9, we use these projective systems to construct an explicit basis of $D_A(\pi)$. In §4.10, we finish the proof of Theorem 4.1.1. We also compute the actions of φ and $\mathcal{O}_K^\times$ on $D_A(\pi)$. In §4.11, we finish the proof of Theorem 4.1.2. Finally, in §4.12, we give the pictures of some finite-dimensional $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentations of π.

Notation

Let p be a prime. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $K \subseteq \overline{\mathbb{Q}}_p$ be the unramified extension of $\mathbb{Q}_p$ of degree $f \geq 1$ with ring of integers $\mathcal{O}_K$ and residue field $\mathbb{F}_q$ (hence $q = p^f$). We denote by $G_K \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ the absolute Galois group of K and $I_K \subseteq G_K$ the inertia subgroup. Let $\mathbb{F}$ be a large enough finite extension of $\mathbb{F}_p$. Fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and let $\sigma_j \overset{\mathrm{def}}{=} \sigma_0 \circ \varphi^j$ for $j \in \mathbb{Z}$, where $\varphi : x \mapsto x^p$ is the arithmetic Frobenius on $\mathbb{F}_q$. We identify $\mathcal{J} \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{F}_q,\mathbb{F})$ with $\{0,1,\dots,f-1\}$, which is also identified with $\mathbb{Z}/f\mathbb{Z}$ so that the addition and subtraction in $\mathcal{J}$ are modulo f. For $a \in \mathcal{O}_K$, we denote by $\overline{a} \in \mathbb{F}_q$ its reduction modulo p. For $a \in \mathbb{F}_q$, we also view it as an element of $\mathbb{F}$ via σ_0.

For F a perfect ring of characteristic p, we denote by $W(F)$ the ring of Witt vectors of F. For $x \in F$, we denote by $[x] \in W(F)$ its Techmüller lift.

Let $I \overset{\mathrm{def}}{=} \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the Iwahori subgroup, $I_1 \overset{\mathrm{def}}{=} \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the pro-p Iwahori subgroup, $K_1 \overset{\mathrm{def}}{=} 1 + p\,\mathrm{M}_2(\mathcal{O}_K) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the first congruence subgroup, $N_0 \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)$ and $H \overset{\mathrm{def}}{=} \begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix}$.

For P a statement, we let $\delta_P \overset{\mathrm{def}}{=} 1$ if P is true and $\delta_P \overset{\mathrm{def}}{=} 0$ otherwise.

Throughout this chapter, we let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be as in (4.2) and π be a smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying the conditions (i),(ii) before Theorem 4.1.1. Since

twisting π by a character does not change the rank of $D_A(\pi)$, we assume moreover that $\overline{\rho}$ is of the following form:

$$\overline{\rho} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j}\mathrm{un}(\xi) & * \\ 0 & \mathrm{un}(\xi^{-1}) \end{pmatrix}, \tag{4.3}$$

where $\xi \in \mathbb{F}^\times$, $\mathrm{un}(\xi) : G_K \to \mathbb{F}^\times$ is the unramified character sending geometric Frobenius elements to ξ, and $\omega_f : G_K \to \mathbb{F}$ is the extension to G_K of the fundamental character of level f (associate to σ_0) such that $\omega_f(g)$ is the reduction modulo p of $g(p_f)/p_f \in \mu_{q-1}(\overline{K}^\times)$ for all $g \in G_K$ and for any choice of a $(q-1)$-th root p_f of $-p$. In particular, p acts trivially on π (by the condition (i) for π).

4.2 Combinatorics of Serre weights

In this section, we review the notion of the extension graph following [BHH+23].

We write $\underline{i}$ for an element $(i_0, \dots, i_{f-1}) \in \mathbb{Z}^f$. For $a \in \mathbb{Z}$, we denote $\underline{a} \overset{\text{def}}{=} (a, \dots, a) \in \mathbb{Z}^f$. For each $j \in \mathcal{J}$, we define $e_j \in \mathbb{Z}^f$ to be 1 in the j-th coordinate, and 0 otherwise. For $J \subseteq \mathcal{J}$, we define $\underline{e}^J \in \mathbb{Z}^f$ by $e_j^J \overset{\text{def}}{=} \delta_{j\in J}$. We say that $\underline{i} \le \underline{i}'$ if $i_j \le i_j'$ for all j. We define the left shift $\delta : \mathbb{Z}^f \to \mathbb{Z}^f$ by $\delta(\underline{i})_j \overset{\text{def}}{=} i_{j+1}$. We define

$$\begin{aligned} X_1(\underline{T}) &\overset{\text{def}}{=} \left\{(\underline{\lambda}_1, \underline{\lambda}_2) \in \mathbb{Z}^{2f} : \underline{0} \le \underline{\lambda}_1 - \underline{\lambda}_2 \le \underline{p} - \underline{1}\right\}; \\ X_{\mathrm{reg}}(\underline{T}) &\overset{\text{def}}{=} \left\{(\underline{\lambda}_1, \underline{\lambda}_2) \in \mathbb{Z}^{2f} : \underline{0} \le \underline{\lambda}_1 - \underline{\lambda}_2 \le \underline{p} - \underline{2}\right\}; \\ X^0(\underline{T}) &\overset{\text{def}}{=} \left\{(\underline{\lambda}_1, \underline{\lambda}_2) \in \mathbb{Z}^{2f} : \underline{\lambda}_1 = \underline{\lambda}_2\right\}. \end{aligned}$$

Let $\pi : \mathbb{Z}^{2f} \to \mathbb{Z}^{2f}$ be defined as $\pi(\underline{\lambda}_1, \underline{\lambda}_2) \overset{\text{def}}{=} \big(\delta(\underline{\lambda}_1), \delta(\underline{\lambda}_2)\big)$.

A **Serre weight** of $\mathrm{GL}_2(\mathbb{F}_q)$ is an absolutely irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{F}$. For $\lambda = (\underline{\lambda}_1, \underline{\lambda}_2) \in X_1(\underline{T})$, we define

$$F(\lambda) \overset{\text{def}}{=} \bigotimes_{j=0}^{f-1} \left(\left(\mathrm{Sym}^{\lambda_{1,j}-\lambda_{2,j}} \mathbb{F}_q^2 \otimes_{\mathbb{F}_q} \det^{\lambda_{2,j}}\right) \otimes_{\mathbb{F}_q,\sigma_j} \mathbb{F}\right).$$

We also denote it by $(\underline{\lambda}_1 - \underline{\lambda}_2) \otimes \det^{\underline{\lambda}_2}$. This induces a bijection

$$F : X_1(\underline{T})/(p-\pi)X^0(\underline{T}) \xrightarrow{\sim} \{\text{Serre weights of } \mathrm{GL}_2(\mathbb{F}_q)\}.$$

We say that a Serre weight σ is **regular** if $\sigma \cong F(\lambda)$ with $\lambda \in X_{\mathrm{reg}}(\underline{T})$.

For $\lambda = (\underline{\lambda}_1, \underline{\lambda}_2) \in \mathbb{Z}^{2f}$, we define the character $\chi_\lambda : I \to \mathbb{F}^\times$ by

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \mapsto (\overline{a})^{\underline{\lambda}_1} (\overline{d})^{\underline{\lambda}_2},$$

where $a, d \in \mathcal{O}_K^\times$ and $b, c \in \mathcal{O}_K$. Here, for $x \in \mathbb{F}$ and $\underline{i} \in \mathbb{Z}^f$ we define $x^{\underline{i}} \overset{\text{def}}{=} x^{\sum_{j=0}^{f-1} i_j p^j}$. In particular, if $\lambda \in X_1(\underline{T})$, then χ_λ is the I-character acting on $F(\lambda)^{I_1}$. We also denote χ_λ for its restriction to H. For each $j \in \mathcal{J}$ we define $\alpha_j \overset{\text{def}}{=} (e_j, -e_j) \in \mathbb{Z}^{2f}$, and for each $\underline{i} \in \mathbb{Z}^f$ we define $\alpha^{\underline{i}} \overset{\text{def}}{=} \sum_{j=0}^{f-1} i_j \alpha_j \in \mathbb{Z}^{2f}$. We also denote α_j and $\alpha^{\underline{i}}$ the corresponding characters χ_{α_j} and $\chi_{\alpha^{\underline{i}}}$ when there is no possible confusion. Concretely, we have

$$\alpha^{\underline{i}}\left(\begin{pmatrix} a & b \\ pc & d \end{pmatrix}\right) = \left(\overline{a}\overline{d}^{-1}\right)^{\sum_{j=0}^{f-1} i_j p^j}.$$

For $\mu = (\underline{\mu}_1, \underline{\mu}_2) \in \mathbb{Z}^{2f}$, we define the extension graph associated to μ by

$$\Lambda_W^\mu \overset{\text{def}}{=} \left\{ \underline{b} \in \mathbb{Z}^f : 0 \leq \underline{\mu}_1 - \underline{\mu}_2 + \underline{b} \leq \underline{p} - \underline{2} \right\}. \tag{4.4}$$

As in [BHH+23, p.16], there is a map

$$\mathfrak{t}_\mu : \Lambda_W^\mu \to X_{\mathrm{reg}}(\underline{T})/(p-\pi)X^0(\underline{T}),$$

such that the map $\underline{b} \mapsto F(\mathfrak{t}_\mu(\underline{b}))$ gives a bijection between Λ_W^μ and the set of regular Serre weights of $\mathrm{GL}_2(\mathbb{F}_q)$ with central character $\chi_\mu|_Z$, where $Z \cong \mathbb{F}_q^\times$ is the center of $\mathrm{GL}_2(\mathbb{F}_q)$.

We let $\mu_{\underline{r}} \overset{\text{def}}{=} (\underline{r}, \underline{0}) \in \mathbb{Z}^{2f}$ with $\underline{r} = (r_0, \dots, r_{f-1})$ and r_j as in (4.2). For $\underline{b} \in \mathbb{Z}^f$ such that $-\underline{r} \leq \underline{b} \leq \underline{p} - \underline{2} - \underline{r}$, we denote $\sigma_{\underline{b}} \overset{\text{def}}{=} F\big(\mathfrak{t}_{\mu_{\underline{r}}}(\underline{b})\big)$. For $\overline{\rho}$ as in (4.3), we let $J_{\overline{\rho}} \subseteq \mathcal{J}$ be as in [Bre14, (17)]. Then by [Bre14, Prop. A.3] and [BHH+23, (14)] we have

$$W(\overline{\rho}) = \left\{ \sigma_{\underline{b}} : \begin{array}{ll} b_j = 0 & \text{if } j \notin J_{\overline{\rho}} \\ b_j \in \{0,1\} & \text{if } j \in J_{\overline{\rho}} \end{array} \right\}. \tag{4.5}$$

In particular, $\overline{\rho}$ is semisimple if and only if $J_{\overline{\rho}} = \mathcal{J}$. For each $J \subseteq \mathcal{J}$, we define $\sigma_J \overset{\text{def}}{=} \sigma_{\underline{a}^J}$ with

$$a_j^J \overset{\text{def}}{=} \begin{cases} 0 & \text{if } j \notin J \\ 1 & \text{if } j \in J,\ j+1 \notin J \text{ or } j \in J,\ j+1 \in J,\ j \in J_{\overline{\rho}} \\ -1 & \text{if } j \in J,\ j+1 \in J,\ j \notin J_{\overline{\rho}}. \end{cases} \tag{4.6}$$

In particular, for $J \subseteq J_{\overline{\rho}}$ we have $\sigma_J = \sigma_{\underline{e}^J}$. Then as a special case of [BHH+23, (14)], we have $\sigma_J = (\underline{s}^J) \otimes \det^{\underline{t}^J}$ with

$$s_j^J \overset{\text{def}}{=} \begin{cases} r_j & \text{if } j \notin J,\ j+1 \notin J \\ r_j + 1 & \text{if } j \in J,\ j+1 \notin J \\ p-2-r_j & \text{if } j \notin J,\ j+1 \in J \\ p-1-r_j & \text{if } j \in J,\ j+1 \in J,\ j \notin J_{\overline{\rho}} \\ p-3-r_j & \text{if } j \in J,\ j+1 \in J,\ j \in J_{\overline{\rho}}; \end{cases} \tag{4.7}$$

$$t_j^J \overset{\text{def}}{=} \begin{cases} 0 & \text{if } j \notin J,\ j+1 \notin J \\ -1 & \text{if } j \in J,\ j+1 \notin J \\ r_j + 1 & \text{if } j \notin J,\ j+1 \in J \text{ or } j \in J,\ j+1 \in J,\ j \in J_{\overline{\rho}} \\ r_j & \text{if } j \in J,\ j+1 \in J,\ j \notin J_{\overline{\rho}}. \end{cases} \tag{4.8}$$

We let $\chi_J \overset{\text{def}}{=} \chi_{\lambda_J}$ with $\lambda_J \overset{\text{def}}{=} (\underline{s}^J + \underline{t}^J, \underline{t}^J)$. Then χ_J is the I-character acting on $\sigma_J^{I_1}$. For each I-character χ, we denote by χ^s its conjugation by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$.

Lemma 4.2.1. *For each $J \subseteq \mathcal{J}$, we have $\chi_J \alpha^{\underline{t}^J} = \chi_{(\underline{r},\underline{0})}$.*

Proof. By definition, we have $\chi_J \alpha^{\underline{t}^J} = \chi_\lambda$ with

$$\lambda = \lambda_J + \alpha^{\underline{t}^J} = (\underline{s}^J + \underline{t}^J, \underline{t}^J) + (\underline{t}^J, -\underline{t}^J) = (\underline{s}^J + 2\underline{t}^J, \underline{0}).$$

Since $\sigma_J = (\underline{s}^J) \otimes \det^{\underline{t}^J}$ has the same central character as $\sigma_\emptyset = (\underline{r})$, we deduce that $a^{\underline{s}^J + 2\underline{t}^J} = a^{\underline{r}}$ for all $a \in \mathbb{F}_q$, which completes the proof. □

For $J \subseteq \mathcal{J}$ and $k \in \mathbb{Z}$, we define $J + k \overset{\text{def}}{=} \{j + k : j \in J\}$. Then we define the **semisimple part** of J, the **non-semisimple part** of J and the **shifting index** of J to be respectively

$$J^{\mathrm{ss}} \overset{\text{def}}{=} J \cap J_{\overline{\rho}}, \quad J^{\mathrm{nss}} \overset{\text{def}}{=} J \setminus J_{\overline{\rho}} = J \setminus J^{\mathrm{ss}}, \quad J^{\mathrm{sh}} \overset{\text{def}}{=} J \cap (J-1) \cap J_{\overline{\rho}} \subseteq J^{\mathrm{ss}}. \tag{4.9}$$

In particular, if $\overline{\rho}$ is semisimple, then $J^{\mathrm{ss}} = J$ and $J^{\mathrm{nss}} = \emptyset$ for all J. We also remark that the set J^{sh} for each $J \subseteq \mathcal{J}$ is defined in such a way that the I-character $\chi_J \alpha^{\underline{e}^{J^{\mathrm{sh}}}}$ is independent of $J_{\overline{\rho}}$. This will play a role in Theorem 4.6.4 below. By (4.2), we have from (4.7) and (4.9)

$$2(f - \delta_{j \in J^{\mathrm{sh}}}) + 1 \leq 2(f - \delta_{j \in J^{\mathrm{sh}}}) + 1 + \delta_{f=1} \leq s_j^J \leq p - 2 - 2(f + \delta_{j \in J^{\mathrm{sh}}}) \ \forall\, j \in \mathcal{J}. \tag{4.10}$$

Lemma 4.2.2. *Let $J, J' \subseteq \mathcal{J}$ satisfying $(J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}}$. Then for each $j \in (J \Delta J') - 1$, we have*

$$2\delta_{j \in (J \cap J')^{\mathrm{nss}}} + (p - 2 - s_j^J) + \delta_{j \in J \Delta J'} = s_j^{J'}.$$

Here we recall that $J \Delta J' \overset{\text{def}}{=} (J \setminus J') \sqcup (J' \setminus J)$.

Proof. We assume that $j+1 \in J$ and $j+1 \notin J'$. Otherwise we have $j+1 \notin J$ and $j+1 \in J'$, and the proof is similar. We separate the following cases.

If $j \in J$ and $j \in J'$, then the LHS equals $2\delta_{j \notin J_{\overline{\rho}}} + (p-2-(p-3-r_j+2\delta_{j \notin J_{\overline{\rho}}})) + 0 = r_j + 1$, which equals the RHS.

If $j \in J$ and $j \notin J'$, then the LHS equals $0 + (p-2-(p-3-r_j+2\delta_{j \notin J_{\overline{\rho}}})) + 1 = r_j + 2 - 2\delta_{j \notin J_{\overline{\rho}}}$. Hence it suffices to show that $j \notin J_{\overline{\rho}}$. Indeed, if $j \in J_{\overline{\rho}}$, then $j \in (J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}} \subseteq J'$, which is a contradiction.

If $j \notin J$ and $j \in J'$, then the LHS equals $0 + (p-2-(p-2-r_j)) + 1 = r_j + 1$, which equals the RHS.

If $j \notin J$ and $j \notin J'$, then the LHS equals $0 + (p-2-(p-2-r_j)) + 0 = r_j$, which equals the RHS. □

Lemma 4.2.3. *Let $J \subseteq \mathcal{J}$ and $\underline{b} \in \mathbb{Z}^f$ such that $-\big(2(\underline{f} - \underline{e}^{J^{\mathrm{sh}}}) + \underline{1}\big) \leq \underline{b} \leq 2(\underline{f} + \underline{e}^{J^{\mathrm{sh}}})$. Then we have $F\big(\mathfrak{t}_{\lambda_J}(\underline{b})\big) = \sigma_{\underline{a}}$ with $a_j = (-1)^{\delta_{j+1 \in J}}(b_j + \delta_{j \in J}) + 2\delta_{j \in J^{\mathrm{sh}}}$ for all $j \in \mathcal{J}$. In particular,*

(i) *we have $\sigma_{J^{\mathrm{ss}}} = F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with $b_j = \delta_{j \in J^{\mathrm{nss}}}$ for all $j \in \mathcal{J}$;*
(ii) *we have $\sigma_{(J-1)^{\mathrm{ss}}} = F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with $b_j = \delta_{j \in J \Delta (J-1)^{\mathrm{ss}}}$ for all $j \in \mathcal{J}$;*
(iii) *for each $J' \subseteq \mathcal{J}$, we have $\sigma_{J'} = F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with*

$$b_j = \begin{cases} \delta_{j \in J} + \delta_{j \in J'}(-1)^{\delta_{j+1 \notin J \Delta J'}} & \text{if } j \notin J_{\overline{\rho}} \\ \big(\delta_{j \in J} - \delta_{j \notin J'}\big)(-1)^{\delta_{j+1 \in J}} & \text{if } j \in J_{\overline{\rho}}. \end{cases}$$

Proof. The assumption on $\underline{b}$ implies that $F(\mathfrak{t}_{\lambda_J}(\underline{b}))$ is well-defined. By (4.6) and a case-by-case examination we have

$$a_j^J = \delta_{j \in J}(-1)^{\delta_{j+1 \in J}} + 2\delta_{j \in J^{\mathrm{sh}}} \ \forall\, j \in \mathcal{J}.$$

Then by [BHH+23, Lemma 2.4.4] applied to $\mu = \mu_{\underline{r}}$ and $\omega = \underline{b}$, we deduce that

$$a_j = a_j^J + (-1)^{a_{j+1}^J} b_j = \big(\delta_{j \in J}(-1)^{\delta_{j+1 \in J}} + 2\delta_{j \in J^{\mathrm{sh}}}\big) + (-1)^{\delta_{j+1 \in J}} b_j = (-1)^{\delta_{j+1 \in J}}(b_j + \delta_{j \in J}) + 2\delta_{j \in J^{\mathrm{sh}}}.$$

(i). For each $j \in \mathcal{J}$, we have

$$\begin{aligned}(-1)^{\delta_{j+1 \in J}}(-\delta_{j \in J^{\mathrm{nss}}} + \delta_{j \in J}) + 2\delta_{j \in J^{\mathrm{sh}}} &= (-1)^{\delta_{j+1 \in J}} \delta_{j \in J^{\mathrm{ss}}} + 2\delta_{j+1 \in J}\delta_{j \in J^{\mathrm{ss}}} \\ &= \big((-1)^{\delta_{j+1 \in J}} + 2\delta_{j+1 \in J}\big)\delta_{j \in J^{\mathrm{ss}}} = \delta_{j \in J^{\mathrm{ss}}}.\end{aligned}$$

This proves (i) since $\underline{a}^{J^{\mathrm{ss}}} = \underline{e}^{J^{\mathrm{ss}}}$.

(ii). Since $\underline{a}^{(J-1)^{\mathrm{ss}}} = \underline{e}^{(J-1)^{\mathrm{ss}}}$, it suffices to show that

$$(-1)^{\delta_{j+1\in J}}(-\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j\in J}) + 2\delta_{j\in J^{\mathrm{sh}}} = \delta_{j\in(J-1)^{\mathrm{ss}}}.$$

If $j \in (J-1)^{\mathrm{ss}}$, then the LHS equals $-(-\delta_{j\notin J} + \delta_{j\in J}) + 2\delta_{j\in J} = \delta_{j\notin J} + \delta_{j\in J} = 1$, which equals the RHS. If $j \notin (J-1)^{\mathrm{ss}}$, then the LHS equals $(-1)^{\delta_{j+1\in J}}(-\delta_{j\in J} + \delta_{j\in J}) + 0 = 0$, which equals the RHS.

(iii). If $j \notin J_{\overline{\rho}}$, then we have

$$\begin{aligned} b_j &= \delta_{j\in J} - (-1)^{\delta_{j+1\in J}}\big(a_j^{J'} - 2\delta_{j\in J^{\mathrm{sh}}}\big) = \delta_{j\in J} - (-1)^{\delta_{j+1\in J}}\big(\delta_{j\in J'}(-1)^{\delta_{j+1\in J'}} - 0\big) \\ &= \delta_{j\in J} + \delta_{j\in J'}(-1)^{j+1\notin J\Delta J'}. \end{aligned}$$

If $j \in J_{\overline{\rho}}$, then we have

$$\begin{aligned} b_j &= \delta_{j\in J} - (-1)^{\delta_{j+1\in J}}\big(a_j^{J'} - 2\delta_{j\in J^{\mathrm{sh}}}\big) = \delta_{j\in J} - (-1)^{\delta_{j+1\in J}}\big(\delta_{j\in J'} - 2\delta_{j\in J, j+1\in J}\big) \\ &= \delta_{j\in J}\big(1 + 2\delta_{j+1\in J}(-1)^{\delta_{j+1\in J}}\big) - (-1)^{\delta_{j+1\in J}}\delta_{j\in J'} = \big(\delta_{j\in J} - \delta_{j\in J'}\big)(-1)^{\delta_{j+1\in J}}. \end{aligned}$$

This completes the proof. □

4.3 The principal series

In this section, we recall some results of [BP12, §2].

For $j \in \mathcal{J}$, we define

$$Y_j \overset{\mathrm{def}}{=} \sum_{a\in\mathbb{F}_q^\times} a^{-p^j} \begin{pmatrix} 1 & [a] \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[\![N_0]\!].$$

Then we have $\mathbb{F}[\![N_0]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$. For $\underline{i} = (i_0, \dots, i_{f-1}) \in \mathbb{Z}^f$, we set $\|\underline{i}\| \overset{\mathrm{def}}{=} \sum_{j=0}^{f-1} i_j$ and we write $\underline{Y}^{\underline{i}}$ for $\prod_{j=0}^{f-1} Y_j^{i_j}$. We recall the following results of [BHH$^+$b, Lemma 3.2.2.1].

Lemma 4.3.1. *For $j \in \mathcal{J}$ and $\mu_1, \mu_2 \in \mathbb{F}_q^\times$, we have in $\mathbb{F}[\![N_0]\!]$*

(i) $Y_j^p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) Y_{j+1}$;

(ii) $\left(\begin{smallmatrix} [\mu_1] & 0 \\ 0 & [\mu_2] \end{smallmatrix}\right) Y_j = (\mu_1\mu_2^{-1})^{p^j} Y_j \left(\begin{smallmatrix} [\mu_1] & 0 \\ 0 & [\mu_2] \end{smallmatrix}\right)$. *In particular, if V is a representation of I and $v \in V^{H=\chi}$, then for $\underline{i} \geq \underline{0}$, we have $\underline{Y}^{\underline{i}} v \in V^{H=\chi\alpha^{\underline{i}}}$.*

Let $\lambda = (\underline{\lambda}_1, \underline{\lambda}_2) \in X_1(\underline{T})$ such that $\underline{1} \leq \underline{\lambda}_1 - \underline{\lambda}_2 \leq \underline{p} - \underline{2}$. Let $f_0, \dots, f_{q-1}, \phi$ be the elements of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\lambda^s)$ defined as in [BP12, §2]. For $0 \leq \underline{i} \leq \underline{p} - \underline{1}$ we let $i \overset{\mathrm{def}}{=} \sum_{j=0}^{f-1} i_j p^j$. Then by definition and [BHH$^+$b, Lemma 3.2.2.5(ii)] we have

$$(-1)^{f-1}\left[\prod_{j=0}^{f-1} i_j!\right] \underline{Y}^{\underline{p}-\underline{1}-\underline{i}} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \phi = \begin{cases} f_i & \text{if } 0 \leq i \leq q-2 \\ f_{q-1} - f_0 & \text{if } i = q-1. \end{cases} \tag{4.11}$$

The following lemma is a restatement of some results of [BP12, §2].

Lemma 4.3.2. (i) *The $\mathrm{GL}_2(\mathcal{O}_K)$-representation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\lambda^s)$ is multiplicity-free with constituents $\{F(\mathfrak{t}_\lambda(-\underline{b})) : \underline{0} \leq \underline{b} \leq \underline{1}\}$. Moreover, the constituent $F\big(\mathfrak{t}_\lambda(-\underline{b})\big)$ corresponds to the subset $\{j : b_{j+1} = 1\}$ in the parametrization of [BP12, §2].*

(ii) *The elements* $\left\{\underline{Y}^{\underline{k}}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\phi : \underline{0}\le\underline{k}\le\underline{p}-\underline{1},\phi\right\}$ *form a basis of* $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\lambda^s)$. *Moreover,* ϕ *has* H*-eigencharacter* χ_λ^s *and* $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\phi$ *has* H*-eigencharacter* $\chi_\lambda\alpha^{-\underline{k}}=\chi_\lambda^s\alpha^{\underline{r}-\underline{k}}$.

(iii) *Let* τ *be the constituent of* $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\lambda^s)$ *corresponding to* $J\subseteq\mathcal{J}$ *as in (i) and denote by* $Q(\chi_\lambda^s,J)$ *the unique quotient of* $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\lambda^s)$ *with socle* τ *(see [BP12, Thm. 2.4(iv)]).*

(a) *If* $J=\emptyset$, *then the following* H*-eigenvectors*

$$\left\{\underline{Y}^{\underline{k}}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\phi : \underline{p}-\underline{1}-(\underline{\lambda}_1-\underline{\lambda}_2)<\underline{k}\le\underline{p}-\underline{1}, \underline{Y}^{\underline{p}-\underline{1}-(\underline{\lambda}_1-\underline{\lambda}_2)}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\phi+x\phi\right\}$$

with $x=(-1)^{\|\underline{\lambda}_1\|+(f-1)}\big(\prod_{j=0}^{f-1}(\lambda_{1,j}-\lambda_{2,j})!\big)^{-1}\in\mathbb{F}_q^\times$ *form a basis of* τ *inside* $Q(\chi_\lambda^s,\emptyset)=\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\lambda^s)$.

(b) *If* $J\neq\emptyset$, *then the following* H*-eigenvectors*

$$\left\{\underline{Y}^{\underline{k}}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\phi : \begin{matrix}0\le k_j\le p-2-(\lambda_{1,j}-\lambda_{2,j})+\delta_{j-1\in J} & \text{if } j\in J\\ p-1-(\lambda_{1,j}-\lambda_{2,j})+\delta_{j-1\in J}\le k_j\le p-1 & \text{if } j\notin J\end{matrix}\right\}$$

map to a basis of τ *inside* $Q(\chi_\lambda^s,J)$.

Proof. The first statement of (i) is [BHH+23, Lemma 6.2.1(i)], and the second statement of (i) follows from the proof of [BHH+23, Lemma 6.2.1(i)]. (ii) and (iii) are restatements of [BP12, Lemma 2.5] and [BP12, Lemma 2.7] using (4.11). □

4.4 On certain H-eigenvectors in $D_0(\overline{\rho})$

In this section, we construct some elements of $D_0(\overline{\rho})$, which is identified with π^{K_1} from now on (see condition (i) above Theorem 4.1.1). The main result is Proposition 4.4.2. They will be the first step in constructing elements of $D_A(\pi)$.

Lemma 4.4.1. (i) *The* $\mathrm{GL}_2(\mathcal{O}_K)$*-representation* $D_0(\overline{\rho})$ *is multiplicity-free with constituents*

$$\mathrm{JH}(D_0(\overline{\rho}))=\left\{\sigma_{\underline{b}}:\ \begin{matrix}b_j\in\{-1,0,1\} & \text{if } j\notin J_{\overline{\rho}}\\ b_j\in\{-1,0,1,2\} & \text{if } j\in J_{\overline{\rho}}\end{matrix}\right\}.$$

Moreover, there is a decomposition of $\mathrm{GL}_2(\mathcal{O}_K)$*-representations* $D_0(\overline{\rho})=\oplus_{J\subseteq J_{\overline{\rho}}}D_{0,\sigma_J}(\overline{\rho})$ *such that for each* $J\subseteq J_{\overline{\rho}}$, $D_{0,\sigma_J}(\overline{\rho})$ *has socle* $\sigma_J=\sigma_{\underline{e}^J}$ *and has constituents*

$$\mathrm{JH}(D_{0,\sigma_J}(\overline{\rho}))=\left\{\sigma_{\underline{b}}:\ \begin{matrix}b_j\in\{-1,0,1\} & \text{if } j\notin J_{\overline{\rho}}\\ b_j\in\{-1,0\} & \text{if } j\in J_{\overline{\rho}}\setminus J\\ b_j\in\{1,2\} & \text{if } j\in J\end{matrix}\right\}. \tag{4.12}$$

(ii) *The* I*-representation* $D_0(\overline{\rho})^{I_1}$ *is a direct sum of distinct* I*-characters. For each* $J\subseteq\mathcal{J}$, χ_J *occurs as a direct summand.*

(iii) *For each* $J\subseteq\mathcal{J}$, *the character* χ_J *appears in the component* $D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})$, *and the character* χ_J^s *appears in the component* $D_{0,\sigma_{(J-1)^{\mathrm{ss}}}}(\overline{\rho})$.

Proof. (i). For $\sigma\in W(\overline{\rho})$ and τ an arbitrary Serre weight, we let $\ell(\sigma,\tau)\in\mathbb{Z}_{\ge0}\cup\{\infty\}$ be as in [BP12, §12]. Hence, $\ell(\sigma,\tau)<\infty$ if and only if τ is a constituent of $\mathrm{Inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\sigma$, in which case it measures the distance between σ and τ in terms of the extension graph Λ_W^μ (see (4.4)). If we write $\sigma=\sigma_J=\sigma_{\underline{e}^J}$ for some $J\subseteq J_{\overline{\rho}}$ (see (4.5)), then by [BHH+23, Lemma 6.2.1(ii)] we deduce that $\ell(\sigma_J,\tau)<\infty$ if and only if $\tau=F\big(\mathfrak{t}_{\lambda_J}(\underline{b})\big)$ for some $\underline{b}\in\mathbb{Z}^f$ such that $-\underline{1}\le\underline{b}\le\underline{1}$. By

[BHH$^+$23, Lemma 2.4.4], this holds if and only if $\tau = \sigma_{\underline{b}}$ with $-\underline{1} \leq \underline{b} - \underline{e}^J \leq \underline{1}$, in which case we have $\ell(\sigma_J, \tau) = |\{j : b_j \neq \delta_{j\in J}\}|$ by [BP12, Cor. 4.11].

For τ a Serre weight, we let $\ell(\overline{\rho}, \tau) \overset{\text{def}}{=} \min_{\sigma\in W(\overline{\rho})} \ell(\sigma, \tau) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then by the previous paragraph we deduce that $\ell(\overline{\rho}, \tau) < \infty$ if and only if $\tau = F\big(\mathfrak{t}_{\lambda_J}(\underline{b})\big)$ for some $\underline{b} \in \mathbb{Z}^f$ such that $-1 \leq b_j \leq 1$ if $j \notin J_{\overline{\rho}}$ and $-1 \leq b_j \leq 2$ of $j \in J_{\overline{\rho}}$. In this case, we have $\ell(\overline{\rho}, \tau) = \ell(\sigma, \tau)$ if and only if $\sigma = \sigma_{J(\tau)}$ with $J(\tau) \overset{\text{def}}{=} \{j \in J_{\overline{\rho}} : b_j \geq 1\}$, and τ is a constituent of $D_{0,\sigma_{J(\tau)}}(\overline{\rho})$ by Proposition [BP12, Prop. 13.4]. Hence for each $J \subseteq J_{\overline{\rho}}$, $D_{0,\sigma_J}(\overline{\rho})$ has constituents τ as above such that $J(\tau) = J$, which agrees with (4.12). The other assertions then follow from [BP12, Prop. 13.4] and [BP12, Cor. 13.5].

(ii). By [BP12, Lemma 14.1], the I-representation $D_0(\overline{\rho})^{I_1}$ is a direct sum of distinct I-characters. By the proof of [BP12, Cor. 13.6], it suffices to find I-characters χ such that $\sigma_{\underline{0}} \in \mathrm{JH}\left(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi^s\right)$. Then we conclude using [Bre14, Prop. 4.2] and (4.7).

(iii). The first assertion is clear since σ_J lies in the component $D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})$ by (4.12). To prove the second assertion, we follow the notation of [BP12, §15]. In particular, we let $\mathcal{S}, \mathcal{S}^-, \mathcal{S}^+ \subseteq \mathcal{J}$ be the subsets associated to $\overline{\rho}^{\mathrm{ss}}$ and σ_J. By definition we have $\mathcal{S} = J$, $\mathcal{S}^- = \mathcal{S}^+ = \emptyset$, hence by [BP12, Lemma 15.2] applied to $\overline{\rho}^{\mathrm{ss}}$ and σ_J, we deduce that $\ell\big(\overline{\rho}^{\mathrm{ss}}, \sigma_J^{[s]}\big) = \ell\big(\sigma_{\underline{e}^{J-1}}, \sigma_J^{[s]}\big)$. Then by [BP12, Lemma 15.3] applied to $\overline{\rho}$ and $\sigma_J^{[s]}$ we deduce that $\ell\big(\overline{\rho}, \sigma_J^{[s]}\big) = \ell\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_J^{[s]}\big)$ (note that the Serre weight $\sigma^{\max}$ in the statement of [BP12, Lemma 15.3] is our $\sigma_{J_{\overline{\rho}}}$), which completes the proof using [BP12, Prop. 13.4]. □

For each $J \subseteq \mathcal{J}$ we fix a choice of $0 \neq v_J \in D_0(\overline{\rho})^{I_1}$ with I-character χ_J, which is unique up to scalar by Lemma 4.4.1(ii). The following proposition shows the existence of certain shifts of the elements v_J. We will apply $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to these elements in order to go beyond $D_0(\overline{\rho})$.

Proposition 4.4.2. *Let $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{0} \leq \underline{i} \leq \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$ (see (4.9) for J^{sh}). Then there exists a unique H-eigenvector $y \in D_0(\overline{\rho})$ satisfying*

(i) $Y_j^{i_j+1} y = 0 \ \forall\, j \in \mathcal{J}$;
(ii) $\underline{Y}^{\underline{i}} y = v_J$.

Moreover, y has H-eigencharacter $\chi_J \alpha^{-\underline{i}}$. The $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation of $D_0(\overline{\rho})$ generated by y lies in $D_{0,J^{\mathrm{ss}}}(\overline{\rho})$ and has constituents $\sigma_{\underline{b}}$ with

$$\begin{cases} b_j = \delta_{j\in J} (= \delta_{j\in J^{\mathrm{ss}}}) & \text{if } j \notin J^{\mathrm{nss}} \\ b_j \in \{0, (-1)^{\delta_{j+1\in J}}\} & \text{if } j \in J^{\mathrm{nss}},\ i_j = 0 \\ b_j \in \{-1, 0, 1\} & \text{if } j \in J^{\mathrm{nss}},\ i_j > 0. \end{cases} \tag{4.13}$$

We denote this element y by $\underline{Y}^{-\underline{i}} v_J$.

Proof. For each $y \in D_0(\overline{\rho})$ satisfying (i) and (ii), by Lemma 4.3.1(ii) the I-representation generated by y is an I/K_1-representation with socle χ_J and cosocle $\chi_J \alpha^{-\underline{i}}$, and has constituents $\chi_J \alpha^{-\underline{i}'}$ with $\underline{0} \leq \underline{i}' \leq \underline{i}$, each occurring with multiplicity 1. By [BHH$^+$23, Lemma 6.1.3], such a representation is unique up to isomorphism, and we denote it by W'. To prove the existence and uniqueness of such y, it suffices to show that there is a unique (up to scalar) I-equivariant injection $W' \hookrightarrow D_0(\overline{\rho})$. Since W' is indecomposable with I-socle χ_J, which appears in $D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})$ by Lemma 4.4.1(iii), any such injection factors through $D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})$.

Claim 1. The $\mathrm{GL}_2(\mathcal{O}_K)$-representation $V' \overset{\text{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(W')$ is multiplicity-free and $\sigma_{J^{\mathrm{ss}}} \in \mathrm{JH}(V')$.

Proof. By [BP12, Lemma 2.2], $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\left(\chi_J\alpha^{-\underline{i}'}\right)$ and $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\left(\chi_J\alpha^{-\underline{i}'}\right)^s$ have the same constituents. Since twisting χ_J by $\alpha^{-\underline{i}'}$ corresponds to shifting by $-2\underline{i}'$ in the extension graph, it follows from Lemma 4.3.2(i), [BHH$^+$23, Remark 2.4.5(ii)] and (4.10) that $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\left(\chi_J\alpha^{-\underline{i}'}\right)$ is multiplicity-free and has constituents $F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with $2\underline{i}' \le \underline{b} \le 2\underline{i}' + \underline{1}$. Hence the $\mathrm{GL}_2(\mathcal{O}_K)$-representation V' is multiplicity-free and has constituents $F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with $\underline{0} \le \underline{b} \le 2\underline{i} + \underline{1}$. By Lemma 4.2.3(i) and taking $b_j = \delta_{j\in J^{\mathrm{nss}}}$ we deduce that $\sigma_{J^{\mathrm{ss}}} \in \mathrm{JH}\left(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J)\right) \subseteq \mathrm{JH}(V')$. □

It follows from Claim 1 that there is a unique (up to scalar) $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant map $f : V' \to \mathrm{Inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\sigma_{J^{\mathrm{ss}}}$. We denote by V'' the image of f.

Claim 2. The $\mathrm{GL}_2(\mathcal{O}_K)$-representation V'' has constituents $\sigma_{\underline{b}}$ for $\underline{b}$ as in (4.13).

Proof. We let τ,τ' be constituents of V' such that $\tau = F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ and $\tau' = F\big(\mathfrak{t}_{\lambda_J}(-\underline{b} + e_{j_0})\big)$ with $\underline{0} < \underline{b} \le 2\underline{i} + \underline{1}$, $j_0 \in \mathcal{J}$ and $b_{j_0} \neq 0$. We write $\underline{b} = 2\underline{c} + \underline{\varepsilon}$ with $\underline{0} \le \underline{c} \le \underline{i}$ and $\underline{0} \le \underline{\varepsilon} \le \underline{1}$. If $\varepsilon_{j_0} = 1$, then both τ and τ' are constituents of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J\alpha^{-\underline{c}})$. We deduce from [BP12, Thm. 2.4] that V' has a length 2 subquotient with socle τ and cosocle τ'. If $\varepsilon_{j_0} = 0$, then we deduce from [HW22, Lemma 3.8] (with $j = j_0$, $\chi = \chi_J\alpha^{-\underline{c}}$, $J(\tau) = \{j : \varepsilon_{j+1} = 0\}$, $J(\tau') = J(\tau)\backslash\{j_0 - 1\}$) that V' has a length 2 subquotient with socle τ' and cosocle τ. Moreover, these are all possible non-split length 2 subquotients of V' by [BHH$^+$23, Lemma 2.4.6].

Then we use the notation of [LLHLM20, §4.1.1]. We make $\mathrm{JH}(V')$ into a directed graph by letting $\sigma \in \mathrm{JH}(V')$ point to $\sigma' \in \mathrm{JH}(V')$ if V' has a length 2 subquotient with socle σ' and cosocle σ. By construction, V'' is a quotient of V' with socle $\sigma_{J^{\mathrm{ss}}}$. It follows from the dual version of [LLHLM20, Prop. 4.1.1] that the constituents of V'' are those $\sigma \in \mathrm{JH}(V')$ which admit a path towards $\sigma_{J^{\mathrm{ss}}} = F\big(\mathfrak{t}_{\lambda_J}(-\underline{e}^{J^{\mathrm{nss}}})\big)$. From the structure of $\mathrm{JH}(V')$ we deduce that V'' has constituents $F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with

$$\begin{cases} b_j = 0 & \text{if } j \notin J^{\mathrm{nss}} \\ b_j \in \{0,1\} & \text{if } j \in J^{\mathrm{nss}},\ i_j = 0 \\ b_j \in \{0,1,2\} & \text{if } j \in J^{\mathrm{nss}},\ i_j > 0. \end{cases}$$

Then we conclude (4.13) by Lemma 4.2.3 with a case-by-case examination. □

It follows from Claim 2 and (4.12) that f factors through $D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})$. Then by Frobenius reciprocity, we have

$$\dim_{\mathbb{F}} \mathrm{Hom}_I\left(W', D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})|_I\right) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}\left(V', D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})\right) = 1.$$

To complete the proof, it remains to show that any nonzero I-equivariant map $W' \to D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})$ is injective. Since W' has I-socle χ_J, it suffices to show that the image of χ_J is nonzero. By Frobenius reciprocity, it suffices to show that the image of the subrepresentation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J)$ of V' under f is nonzero. This follows from the fact that both $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J)$ and V'' contain $\sigma_{J^{\mathrm{ss}}}$ as a constituent, and V' is multiplicity-free. □

Remark 4.4.3. *When $J_{\overline{\rho}} \neq \emptyset$, there are more I_1-invariants than these v_J for $J \subseteq \mathcal{J}$. However, Proposition 4.4.2 does not hold for the I_1-invariants other than these v_J.*

4.5 The relations between H-eigenvectors

In this section, we study various $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentations of π generated by the elements $\underline{Y}^{-\underline{i}}v_J$ defined in Proposition 4.4.2. The main results are Proposition 4.5.5 and Proposition 4.5.9. Then we study the relations between the vectors v_J for $J \subseteq \mathcal{J}$. The main results are Proposition 4.5.11 and Proposition 4.5.13. We refer to §4.12 for the pictures of some of these $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentations of π when $f = 2$.

Recall that we have defined $Q(\chi_\lambda^s, J)$ for $\lambda = (\underline{\lambda}_1, \underline{\lambda}_2) \in X_1(\underline{T})$ such that $\underline{1} \le \underline{\lambda}_1 - \underline{\lambda}_2 \le \underline{p} - \underline{2}$ and $J \subseteq \mathcal{J}$ in Lemma 4.3.2(iii). The following lemma is a generalization of [BHH$^+$b, Lemma 3.2.3.3] (where $\overline{\rho}$ was assumed to be semisimple).

Lemma 4.5.1. *Let $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{i} \le \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$.*

(i) *The $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle$ of π is multiplicity-free with socle $\sigma_{(J-1)^{\mathrm{ss}}} = \sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}}}$ and cosocle $\sigma_{\underline{c}}$ with*

$$c_j = (-1)^{\delta_{j+1 \notin J}} \big(2i_j + 1 + \delta_{j \in (J-1)^{\mathrm{ss}}} - \delta_{j \in J\Delta(J-1)^{\mathrm{ss}}}\big) \ \forall\, j \in \mathcal{J}. \tag{4.14}$$

(ii) *We have*

$$\begin{aligned} &\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle \Big/ \sum_{\underline{0} \le \underline{i}' < \underline{i}} \langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}'} v_J \rangle \\ &\qquad \cong Q\left(\chi_J^s \alpha^{\underline{i}}, \{j : j+1 \in J\Delta(J-1)^{\mathrm{ss}},\ i_{j+1} = 0\}\right). \end{aligned} \tag{4.15}$$

(iii) *Let $\underline{m} \in \mathbb{Z}^f$ with each m_j between $\delta_{j \in (J-1)^{\mathrm{ss}}}$ and c_j (as in (4.14)). Then there is a unique subrepresentation $I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big)$ of $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle$ with cosocle $\sigma_{\underline{m}}$. In particular $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle = I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{c}}\big)$. Moreover, $I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big)$ has constituents $\sigma_{\underline{b}}$ with each b_j between $\delta_{j \in (J-1)^{\mathrm{ss}}}$ and m_j, and we have*

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)} \big(I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big), \pi\big) = 1. \tag{4.16}$$

Proof. (i). We follow closely the proof of [BHH$^+$b, Lemma 3.2.3.3]. The vectors $\underline{Y}^{-\underline{i}} v_J$ and $\underline{Y}^{-\underline{i}'} v_J$ are defined in Proposition 4.4.2. We let W' (resp. W) be the I-subrepresentation of π generated by $\underline{Y}^{-\underline{i}} v_J$ (resp. $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J$) and $V \overset{\mathrm{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(W)$. In particular, W' is the same representation as in the proof of Proposition 4.4.2. By the proof of [BHH$^+$b, Lemma 3.2.3.3], we have:

(a) V is multiplicity-free as a $\mathrm{GL}_2(\mathcal{O}_K)$-representation with constituents $F\big(\mathrm{t}_{\lambda_J}(-\underline{b})\big)$ for $\underline{0} \le \underline{b} \le 2\underline{i} + 1$ (they are well-defined by (4.10));
(b) For each $\underline{0} \le \underline{b} \le 2\underline{i} + 1$, the unique subrepresentation of V with cosocle $F\big(\mathrm{t}_{\lambda_J}(-\underline{b})\big)$ has constituents $F\big(\mathrm{t}_{\lambda_J}(-\underline{a})\big)$ for $\underline{0} \le \underline{a} \le \underline{b}$;
(c) V has a filtration with subquotients $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\big(\chi_J^s \alpha^{\underline{i}'}\big)$ for $\underline{0} \le \underline{i}' \le \underline{i}$. Each subquotient $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\big(\chi_J^s \alpha^{\underline{i}'}\big)$ has constituents $F\big(\mathrm{t}_{\lambda_J}(-\underline{b})\big)$ with $2\underline{i}' \le \underline{b} \le 2\underline{i}' + 1$, and the constituent $F\big(\mathrm{t}_{\lambda_J}(-\underline{b})\big)$ of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\big(\chi_J^s \alpha^{\underline{i}'}\big)$ corresponds to the subset $\{j : b_{j+1} \text{ is odd}\} \subseteq \mathcal{J}$ (see Lemma 4.3.2(i)).

The I-equivariant inclusion $W' \hookrightarrow D_0(\overline{\rho})$ in the proof of Proposition 4.4.2 induces an I-equivariant inclusion $W \hookrightarrow \pi$ by applying $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. By Frobenius reciprocity, this induces a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant map $V \to \pi$ with image

$$\overline{V} \overset{\mathrm{def}}{=} \langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle = \langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle \subseteq \pi.$$

In particular, it follows from (b) that the cosocle of $\overline{V}$ is $F\big(\mathrm{t}_{\lambda_J}(-(2\underline{i}+1))\big)=\sigma_{\underline{c}}$ with

$$\begin{aligned} c_j &= (-1)^{\delta_{j+1\in J}}\big(-(2i_j+1)+\delta_{j\in J}\big)+2\delta_{j\in J^{\mathrm{sh}}}\\ &= (-1)^{\delta_{j+1\notin J}}\big(2i_j+1-\delta_{j\in J}+2(-1)^{\delta_{j+1\notin J}}\delta_{j\in J^{\mathrm{sh}}}\big)\\ &= (-1)^{\delta_{j+1\notin J}}\big(2i_j+1+\delta_{j\in(J-1)^{\mathrm{ss}}}-\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}}\big), \end{aligned}$$

where the first equality follows from Lemma 4.2.3 and the last equality is elementary (for example, one can separate the cases $j\in(J-1)^{\mathrm{ss}}$ and $j\notin(J-1)^{\mathrm{ss}}$).

Claim. We have $W(\overline{\rho})\cap\mathrm{JH}(V)\subseteq\mathrm{JH}\big(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)\big)$.

Proof. It suffices to show that for each $\sigma\in W(\overline{\rho})$, we have $\sigma=F\big(\mathrm{t}_{\lambda_J}(-\underline{b})\big)$ for some $\underline{b}\le\underline{1}$. We check it for $\sigma_\emptyset$, and the other cases are similar. By Lemma 4.2.3 (with $\underline{a}=\underline{0}$), we get $b_j=(-1)^{\delta_{j+1\in J}}\big(2\delta_{j\in J^{\mathrm{sh}}}\big)+\delta_{j\in J}$. If $j\notin J^{\mathrm{sh}}$, then $b_j=\delta_{j\in J}\le 1$. If $j\in J^{\mathrm{sh}}$, then $b_j=-2+1=-1$. □

Recall that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi=\bigoplus_{\sigma\in W(\overline{\rho})}\sigma$. Assume that σ is in the socle of $\overline{V}$, then we have $\sigma\in W(\overline{\rho})\cap\mathrm{JH}(V)\subseteq\mathrm{JH}\big(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)\big)$ by the claim above. Moreover, the image of the subrepresentation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)$ of V in π lies in $D_0(\overline{\rho})$, hence lies in the component $D_{0,\sigma_{(J-1)^{\mathrm{ss}}}}(\overline{\rho})$ by Lemma 4.4.1(iii) and Frobenius reciprocity, which implies that σ must be $\sigma_{(J-1)^{\mathrm{ss}}}$, the only Serre weight of $\overline{\rho}$ appearing in $D_{0,\sigma_{(J-1)^{\mathrm{ss}}}}(\overline{\rho})$. Since V is multiplicity-free by (a), we deduce that $\overline{V}$ is the unique quotient of V with socle $\sigma_{(J-1)^{\mathrm{ss}}}$.

(ii). By Lemma 4.2.3(ii), we have $\sigma_{(J-1)^{\mathrm{ss}}}=F\big(\mathrm{t}_{\lambda_J}(-\underline{e}^{J\Delta(J-1)^{\mathrm{ss}}})\big)$, hence $\overline{V}$ has constituents $F\big(\mathrm{t}_{\lambda_J}(-\underline{b})\big)$ with $\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}}\le b_j\le 2i_j+1$ for all j (or equivalently, $\sigma_{\underline{b}}$ with each b_j between $\delta_{j\in(J-1)^{\mathrm{ss}}}$ and c_j by Lemma 4.2.3). By (c), the LHS of (4.15) is the quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\big(\chi_J^s\alpha^{\underline{i}}\big)$ whose constituents are $F\big(\mathrm{t}_{\lambda_J}(-\underline{b})\big)$ with $\max(\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}},2i_j)\le b_j\le 2i_j+1$, hence it has irreducible socle $F\big(\mathrm{t}_{\lambda_J}(-\underline{a})\big)$ with $a_j=\max(\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}},2i_j)$ by (b). Since a_j is odd if and only if $i_j=0$ and $j\in J\Delta(J-1)^{\mathrm{ss}}$, it follows from (c) that the constituent $F\big(\mathrm{t}_{\lambda_J}(-\underline{a})\big)$ of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\big(\chi_J^s\alpha^{\underline{i}}\big)$ corresponds to the subset $\{j: j+1\in J\Delta(J-1)^{\mathrm{ss}},\ i_{j+1}=0\}$.

(iii). Since $\sigma_{\underline{m}}$ is a constituent of the multiplicity-free representation $\overline{V}$ by the previous paragraph, there is a unique subrepresentation of $\overline{V}$ with cosocle $\sigma_{\underline{m}}$, which moreover has constituents as in the statement by (b). We denote it by $I\big(\sigma_{(J-1)^{\mathrm{ss}}},\sigma_{\underline{m}}\big)$. By the last paragraph of the proof of (i), any constituent of $I\big(\sigma_{(J-1)^{\mathrm{ss}}},\sigma_{\underline{m}}\big)$ which is also an element of $W(\overline{\rho})$ must appear in $D_{0,\sigma_{(J-1)^{\mathrm{ss}}}}(\overline{\rho})$, hence has to be $\sigma_{(J-1)^{\mathrm{ss}}}$. Together with the fact that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi=\oplus_{\sigma\in W(\overline{\rho})}\sigma$, we deduce that

$$1\le\dim_{\mathbb{F}}\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}\big(I\big(\sigma_{(J-1)^{\mathrm{ss}}},\sigma_{\underline{m}}\big),\pi\big)\le\dim_{\mathbb{F}}\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}\big(\sigma_{(J-1)^{\mathrm{ss}}},\pi\big)=1,$$

which completes the proof. □

Remark 4.5.2. *For $\lambda=(\underline{\lambda}_1,\underline{\lambda}_2)\in X_1(\underline{T})$, $\underline{i}\in\mathbb{Z}_{\ge 0}^f$ such that $2\underline{i}+\underline{1}\le\underline{\lambda}_1-\underline{\lambda}_2\le\underline{p}-\underline{2}$ and $J'\subseteq\mathcal{J}$, we let W' be the I-representation as in the proof of Proposition 4.4.2 with χ_J replaced by χ_λ, and we denote by $Q\big(\chi_\lambda^s,\chi_\lambda^s\alpha^{\underline{i}},J'\big)$ the unique quotient of the $\mathrm{GL}_2(\mathcal{O}_K)$-representation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\big(\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)W'\big)$ whose socle is the constituent of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\lambda^s)$ corresponding to J'. Then the proof of Lemma 4.5.1 shows that $\big\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}v_J\big\rangle\cong Q\big(\chi_J^s,\chi_J^s\alpha^{\underline{i}},(J\Delta(J-1)^{\mathrm{ss}})-1\big)$.*

Corollary 4.5.3. *For each $J \subseteq J_{\overline{\rho}}$, we have (see Lemma 4.5.1(iii) for the notation)*

$$D_{0,\sigma_J}(\overline{\rho}) = \sum_{(J')^{\mathrm{ss}}=J} I\big(\sigma_J, \sigma_{\underline{e}^J+\underline{\varepsilon}^{J'}}\big) = \sum_{(J')^{\mathrm{ss}}=J} I\big(\sigma_{\underline{e}^{(J')^{\mathrm{ss}}}}, \sigma_{\underline{e}^{(J')^{\mathrm{ss}}}+\underline{\varepsilon}^{J'}}\big),$$

where $\underline{\varepsilon}^{J'} \in \{\pm 1\}^f$ with $\varepsilon_j^{J'} \overset{\mathrm{def}}{=} (-1)^{\delta_{j\notin J'}}$.

Proof. For each $J' \subseteq \mathcal{J}$ such that $(J')^{\mathrm{ss}} = J$, by applying Lemma 4.5.1(i),(iii) with $(J, \underline{i})$ there being $(J'+1, \underline{f} - \underline{e}^{(J'+1)^{\mathrm{sh}}})$, we deduce that $I\big(\sigma_J, \sigma_{\underline{e}^J+\underline{\varepsilon}^{J'}}\big)$ is well-defined. Then the result follows from Lemma 4.4.1(i), [BP12, Prop. 13.4] and (4.16). □

The following proposition is a generalization of [BHH$^+$b, Lemma 3.2.3.1] (where $\overline{\rho}$ was assumed to be semisimple), which gives a first example of the relations between the vectors $v_J \in D_0(\overline{\rho})$ and is a special case of Proposition 4.5.11 below.

Proposition 4.5.4. *For $J \subseteq \mathcal{J}$, there exists a unique element $\mu_{J,(J-1)^{\mathrm{ss}}} \in \mathbb{F}^\times$ such that*

$$\left[\prod_{j+1\in J\Delta(J-1)^{\mathrm{ss}}} Y_j^{s_j^{(J-1)^{\mathrm{ss}}}} \prod_{j+1\notin J\Delta(J-1)^{\mathrm{ss}}} Y_j^{p-1}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_J = \mu_{J,(J-1)^{\mathrm{ss}}} v_{(J-1)^{\mathrm{ss}}}. \tag{4.17}$$

Proof. By Lemma 4.5.1(ii) and its proof, we have $\big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_J \big\rangle \cong Q\big(\chi_J^s, (J\Delta(J-1)^{\mathrm{ss}}) - 1\big)$ such that $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) v_J$ corresponds to the image of $\phi \in \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)$ (see above (4.11) for ϕ) in $Q\big(\chi_J^s, (J\Delta(J-1)^{\mathrm{ss}}) - 1\big)$, and the socle is $\sigma_{(J-1)^{\mathrm{ss}}}$ which corresponds to the subset $(J\Delta(J-1)^{\mathrm{ss}}) - 1$ for $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)$ (see Lemma 4.3.2(i)). By Lemma 4.2.2 applied to J and $J' = (J-1)^{\mathrm{ss}}$, for $j \in J\Delta(J-1)^{\mathrm{ss}} - 1$ we have

$$(p - 2 - s_j^J) + \delta_{j-1\in(J\Delta(J-1)^{\mathrm{ss}})-1} = s_j^{(J-1)^{\mathrm{ss}}}.$$

Then by Lemma 4.3.2(iii) applied to $\lambda = \lambda_J$ (and recall that $\chi_J = \chi_{\lambda_J}$ with $\lambda_J = (\underline{s}^J + \underline{t}^J, \underline{t}^J)$), the LHS of (4.17) is nonzero in $\sigma_{(J-1)^{\mathrm{ss}}}$ and is the unique (up to scalar) H-eigenvector in $\sigma_{(J-1)^{\mathrm{ss}}}$ killed by all Y_j. It follows that the LHS of (4.17) is a nonzero I_1-invariant of $\sigma_{(J-1)^{\mathrm{ss}}}$, hence is a scalar multiple of $v_{(J-1)^{\mathrm{ss}}}$. □

For $J, J' \subseteq \mathcal{J}$, we define $\underline{t}^J(J') \in \mathbb{Z}^f$ by

$$t^J(J')_j \overset{\mathrm{def}}{=} p - 1 - s_j^J + \delta_{j-1\in J'}, \tag{4.18}$$

where s_j^J is defined in (4.7). In particular, by (4.10) we have

$$1 \le t^J(J')_j \le p - 1 - 2(f - \delta_{j\in J^{\mathrm{sh}}}) \ \forall\, j \in \mathcal{J}. \tag{4.19}$$

The following proposition is a generalization of [BHH$^+$b, Lemma 3.2.3.4] (where $\overline{\rho}$ was assumed to be semisimple).

Proposition 4.5.5. *Let $J \subseteq \mathcal{J}$, $j_0 \in \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{i} \le \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$ and $i_{j_0+1} = 0$. Suppose that $j_0 + 1 \in J\Delta(J-1)^{\mathrm{ss}}$. Then for each $J' \subseteq \mathcal{J}$ such that $j_0 \notin J'$, we have*

$$\begin{aligned} \left[\prod_{j\notin J'} Y_j^{2i_j + t^J(J')_j}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \big(\underline{Y}^{-\underline{i}} v_J\big) &= 0 \qquad && \text{if } J' \neq \emptyset; \\ \left[Y_{j'} \prod_{j\notin J'} Y_j^{2i_j + t^J(J')_j}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \big(\underline{Y}^{-\underline{i}} v_J\big) &= 0 \ \forall\, j' \in \mathcal{J} \qquad && \text{if } J' = \emptyset. \end{aligned} \tag{4.20}$$

Proof. The proof is analogous to the one of [BHH$^+$b, Lemma 3.2.3.4]. First we deal with the case $J' \neq \emptyset$. We denote by B the LHS of (4.20). Assume on the contrary that $B \neq 0$, then by Lemma 4.3.1(ii), B is an H-eigenvector with H-eigencharacter $\chi \overset{\text{def}}{=} \chi_J \alpha^{-\underline{i}} \big[\prod_{j \notin J'} \alpha_j^{2i_j+p-1-s_j^J+\delta_{j-1\in J'}} \big]$. To deduce a contradiction, we prove that the H-character χ does not occur in $\big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \big\rangle$. By Lemma 4.5.1(ii), it suffices to show that the H-character χ does not occur in $V_{\underline{i}'} \overset{\text{def}}{=} Q\big(\chi_J^s \alpha^{\underline{i}'}, J_{\underline{i}'}\big)$ for $\underline{0} \leq \underline{i}' \leq \underline{i}$, where $J_{\underline{i}'} \overset{\text{def}}{=} \{j : j+1 \in J \Delta (J-1)^{\mathrm{ss}}, i'_{j+1} = 0\}$. Note that $j_0 \in J_{\underline{i}'}$ for all $\underline{0} \leq \underline{i}' \leq \underline{i}$ by assumption.

We have $\chi_J \alpha^{-\underline{i}'} = \chi_{\lambda_J - \alpha^{\underline{i}'}}$ (see §4.2 for the notation). Then by Lemma 4.3.2(i),(ii),(iii)(b) applied to $\lambda = \lambda_J - \alpha^{\underline{i}'}$, the H-eigencharacters that occur in $V_{\underline{i}'}$ are $\chi_J \alpha^{-\underline{i}'} \alpha^{-\underline{k}}$ (coming from the element $\underline{Y}^{\underline{k}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \big(\underline{Y}^{-\underline{i}'} v_J\big)$), where

$$\begin{cases} 0 \leq k_j \leq p-2-(s_j^J - 2i'_j) + \delta_{j-1\in J_0} & \text{if } j \in J_0 \\ p-1-(s_j^J - 2i'_j) + \delta_{j-1\in J_0} \leq k_j \leq p-1 & \text{if } j \notin J_0 \end{cases} \tag{4.21}$$

for $J_0 \supseteq J_{\underline{i}'}$. In particular, we have $j_0 \in J_0$.

Assume $\chi = \chi_J \alpha^{-\underline{i}'} \alpha^{\underline{k}}$ for some $\underline{i}', \underline{k}$ as above, then from the definition of χ we have

$$\alpha^{-\underline{i}} \left[\prod_{j \notin J'} \alpha_j^{2i_j+p-1-s_j^J+\delta_{j-1\in J'}} \right] = \alpha^{-\underline{i}'} \alpha^{\underline{k}},$$

and thus

$$\sum_{j \notin J'} (2i_j + p - 1 - s_j^J + \delta_{j-1\in J'}) p^j - \sum_{j=0}^{f-1} (i_j - i'_j) p^j \equiv \sum_{j=0}^{f-1} k_j p^j \mod (q-1),$$

or equivalently,

$$\sum_{j \notin J'} (i_j + i'_j + p - 1 - s_j^J + \delta_{j-1\in J'}) p^j - \sum_{j \in J'} (i_j - i'_j) p^j \equiv \sum_{j=0}^{f-1} k_j p^j \mod (q-1). \tag{4.22}$$

Then we define integers $\eta_j \in \mathbb{Z}$ for all $j \in \mathcal{J}$. For $j_1 \notin J'$ (such j_1 exists since $J' \neq \emptyset$), we let $w \in \{0, \dots, f-1\}$ (depending on j_1) such that $j_1+1, \dots, j_1+w \in J'$ and $j_1+w+1 \notin J'$ (so $w = 0$ if $j_1+1 \notin J'$). We define η_j for $j = j_1+1, \dots, j_1+w+1$ as follows:

(i) If $i_{j_1+w'} = i'_{j_1+w'}$ for all $1 \leq w' \leq w$ (which is automatic if $w = 0$), then we let $\eta_j \overset{\text{def}}{=} 0$ for all $j = j_1+1, \dots, j_1+w+1$;
(ii) Otherwise, we let $w_0 \in \{1, \dots, w\}$ be minimal such that $i_{j_1+w_0} \neq i'_{j_1+w_0}$, then we let

$$\eta_j \overset{\text{def}}{=} \begin{cases} 0 & \text{if } j = j_1+1, \dots, j_1+w_0-1 \text{ (and } w_0 \neq 1) \\ p & \text{if } j = j_1+w_0 \\ p-1 & \text{if } j = j_1+w_0+1, \dots, j_1+w \text{ (and } w_0 \neq w) \\ -1 & \text{if } j = j_1+w+1. \end{cases} \tag{4.23}$$

In particular, we have

$$\sum_{j=j_1+1}^{j_1+w+1} \eta_j p^j \equiv 0 \mod (q-1). \tag{4.24}$$

When we vary $j_1 \notin J'$, we get the definition of η_j for all $j \in \mathcal{J}$.

By adding (4.24) to (4.22) for all $j_1 \notin J'$, we get

$$\sum_{j \notin J'} \big(i_j + i'_j + p - 1 - s_j^J + \delta_{j-1 \in J'} + \eta_j\big)p^j + \sum_{j \in J'} \big(\eta_j - (i_j - i'_j)\big)p^j \equiv \sum_{j=0}^{f-1} k_j p^j \mod (q-1). \quad (4.25)$$

Claim 1. Each coefficient of the LHS of (4.25) is between 0 and $p-1$, not all equal to 0 and not all equal to $p-1$.

Proof. First we prove that each coefficient of the LHS of (4.25) is between 0 and $p-1$. By (4.10) we have

$$1 \le p - 1 - s_j^J \le p - 2 - 2(f - \delta_{j \in J^{\mathrm{sh}}}). \quad (4.26)$$

We remark that the first inequality of (4.26) is weaker than (4.10), and is needed to prove Remark 4.5.6. If $j \notin J'$, then using $0 \le i_j, i'_j \le f - \delta_{j \in J^{\mathrm{sh}}}$, $\delta_{j-1 \in J'} \in \{0,1\}$ and $\eta_j \in \{-1, 0\}$ since $j \notin J'$, we deduce from (4.26) that $0 \le (i_j + i'_j + p - 1 - s_j^J + \delta_{j-1 \in J'} + \eta_j) \le p-1$. If $j \in J'$, by the definition of η_j and a case-by-case examination, we deduce that $0 \le \eta_j - (i_j - i'_j) \le p-1$.

Next we prove that the coefficients of the LHS of (4.25) are not all equal to 0. Otherwise, by the previous paragraph we must in particular have $\eta_j = -1$ for all $j \notin J'$. By the definition of η_j for $j \notin J'$ (that is, for $j = j_1 + w + 1$ in (4.23)), there exists $j' \in J'$ such that $\eta_{j'} = p$, which implies $\eta_{j'} - (i_{j'} - i'_{j'}) > 0$ since $p \ge 4f + 4$ by (4.2), a contradiction.

Finally we prove that the coefficients of the LHS of (4.25) are not all equal to $p-1$. Otherwise, by the first paragraph we must have $\eta_j = 0$ for all $j \notin J'$. By the definition of η_j for $j \notin J'$, we must have $\eta_j = 0$ for all $j \in \mathcal{J}$, hence $\eta_j - (i_j - i'_j)$ cannot be $p-1$. This implies $J' = \emptyset$, which is a contradiction. □

It follows from Claim 1 that the equation (4.25) has solution

$$k_j = \begin{cases} i_j + i'_j + p - 1 - s_j^J + \delta_{j-1 \in J'} + \eta_j & \text{if } j \notin J' \\ \eta_j - (i_j - i'_j) & \text{if } j \in J'. \end{cases} \quad (4.27)$$

Claim 2. We have $j_0 - 1 \notin J'$ and $j_0 - 1 \in J_0$.

Proof. Since $j_0 \notin J'$ and $j_0 \in J_0$, by (4.21) and (4.27) we have

$$k_{j_0} = i_{j_0} + i'_{j_0} + p - 1 - s_{j_0}^J + \delta_{j_0-1 \in J'} + \eta_{j_0} \le p - 2 - s_{j_0}^J + 2i'_{j_0} + \delta_{j_0-1 \in J_0}. \quad (4.28)$$

By the definition of η_j, if $j_0 - 1 \notin J'$, then $\eta_{j_0} = 0$ since $j_0 \notin J'$, and thus $\eta_{j_0} = -1$ implies $j_0 - 1 \in J'$. In particular, we have $\delta_{j_0-1 \in J'} + \eta_{j_0} \ge 0$. Then we deduce from (4.28) that $i_{j_0} + 1 \le i'_{j_0} + \delta_{j_0-1 \in J_0}$, which implies $i_{j_0} = i'_{j_0}$ and $j_0 - 1 \in J_0$ since $i'_{j_0} \le i_{j_0}$.

Then by (4.21) we have

$$k_{j_0-1} \le p - 2 - (s_{j_0-1}^J - 2i'_{j_0-1}) + \delta_{j_0-2 \in J_0} \le p - 1 - s_{j_0-1}^J + 2i'_{j_0-1}. \quad (4.29)$$

Suppose that $j_0 - 1 \in J'$, then by (4.28) and using $i_{j_0} = i'_{j_0}$ and $j_0 - 1 \in J_0$, we must have $\eta_{j_0} = -1$. Then by (4.23) we have $\eta_{j_0-1} \ge p - 1$, which implies $k_{j_0-1} \ge p - 1 - (i_{j_0-1} - i'_{j_0-1})$ by (4.27). Combining with (4.29) we deduce that $s_{j_0-1}^J \le i_{j_0-1} + i'_{j_0-1} \le 2(f - \delta_{j_0-1 \in J^{\mathrm{sh}}})$ since $\underline{i}' \le \underline{i} \le \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$, which contradicts (4.10). Thus we have $j_0 - 1 \notin J'$. □

Since Claim 2 proves that $j_0 - 1 \notin J'$ and $j_0 - 1 \in J_0$ assuming $j_0 \notin J'$ and $j_0 \in J_0$, we can continue this process and finally deduce that $J' = \emptyset$, which is a contradiction.

Next we deal with the case $J' = \emptyset$. As in the case $J' \neq \emptyset$, in view of the second equation in (4.20) it suffices to show that the H-eigencharacter $\chi \overset{\text{def}}{=} \chi_J \alpha_{j'} \alpha^{-\underline{i}} \big[\prod_{j \notin J'} \alpha_j^{2i_j + p - 1 - s_j^J + \delta_{j-1 \in J'}} \big]$ does not occur in $\big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \big\rangle$. Then the equation (4.22) becomes

$$\sum_{j=0}^{f-1} (i_j + i'_j + p - 1 - s_j^J + \delta_{j=j'}) p^j \equiv \sum_{j=0}^{f-1} k_j p^j \mod (q-1), \tag{4.30}$$

where the term $\delta_{j=j'}$ comes from $\alpha_{j'}$. We claim that each $(i_j + i'_j + p - 1 - s_j^J + \delta_{j=j'})$ is between 1 and $p-1$ and not all equal to $p-1$. Indeed, if $f \geq 2$, then it suffices to show that $1 \leq i_j + i'_j + p - 1 - s_j^J \leq p-2$ for all $j \in \mathcal{J}$, which follows from (4.26) and $\underline{0} \leq \underline{i}' \leq \underline{i} \leq \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$. If $f = 1$, then by assumption we have $i_0 = i'_0 = 0$ and $J^{\mathrm{sh}} = \emptyset$ (otherwise $J\Delta(J-1)^{\mathrm{ss}} = \emptyset$), hence by (4.26) we have $2 \leq i_0 + i'_0 + p - 1 - s_0^J + \delta_{j=j'} = p - s_0^J \leq p - 3$.

Then it follows from (4.30) that $k_j = i_j + i'_j + p - 1 - s_j^J + \delta_{j=j'}$ for all $j \in \mathcal{J}$. Since $j_0 \notin J'$ and $j_0 \in J_0$, a similar process as Claim 2 implies that $j_0 - 1 \in J_0, j_0 - 2 \in J_0, \dots, j' \in J_0$. Then by (4.21) using $j' \in J_0$, we have $k_{j'} = i_{j'} + i'_{j'} + p - s_{j'}^J \leq p - 2 - s_{j'}^J + 2i'_{j'} + \delta_{j'-1 \in J_0}$, which is impossible since $i'_{j'} \leq i_{j'}$. □

Remark 4.5.6. *Let $\lambda = (\underline{\lambda}_1, \underline{\lambda}_2) \in X_1(\underline{T})$, $\underline{i} \in \mathbb{Z}_{\geq 0}^f$ such that $2\underline{i} + \underline{1} \leq \underline{\lambda}_1 - \underline{\lambda}_2 \leq \underline{p} - \underline{2}$, and $J, J' \subseteq \mathcal{J}$. Assume that there exists $j_0 \in \mathcal{J}$ such that $j_0 \in J$, $j_0 \notin J'$ and $i_{j_0+1} = 0$. We consider the H-character*

$$\chi \overset{\text{def}}{=} \chi_\lambda \alpha^{-\underline{i}} \prod_{j \notin J'} \alpha_j^{2i_j + p - 1 - (\lambda_{1,j} - \lambda_{2,j}) + \delta_{j-1 \in J'}}.$$

Then the same proof as in Proposition 4.5.5 shows that (see Remark 4.5.2 for the notation):

(i) *If $J' \neq \emptyset$, then the H-character χ does not occur in $Q\big(\chi_\lambda^s, \chi_\lambda^s \alpha^{\underline{i}}, J\big)$;*
(ii) *If $J' = \emptyset$, then the H-character $\chi \alpha_{j'}$ does not occur in $Q\big(\chi_\lambda^s, \chi_\lambda^s \alpha^{\underline{i}}, J\big)$ for all $j' \in \mathcal{J}$.*

For $J, J' \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{0} \leq \underline{i} \leq \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$, we define $\underline{m} = \underline{m}(\underline{i}, J, J') \in \mathbb{Z}^f$ by

$$m_j \overset{\text{def}}{=} (-1)^{\delta_{j+1 \notin J}} (2i_j + \delta_{j \in (J-1)^{\mathrm{ss}}} - \delta_{j \in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j-1 \in J'}). \tag{4.31}$$

In particular, if $2i_j - \delta_{j \in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j-1 \in J'} \geq 0$ for all j, then by Lemma 4.5.1(iii), $\sigma_{\underline{m}}$ is a constituent of $\big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \big\rangle$.

Lemma 4.5.7. *Let $J, J' \subseteq \mathcal{J}$ satisfying $(J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}}$, and let $\underline{m} = \underline{m}\big(\underline{e}^{(J \cap J')^{\mathrm{nss}}}, J, (J \Delta J') - 1\big)$. Then we have*

$$m_j = \delta_{j \in J'} (-1)^{\delta_{j+1 \notin J}} \ \forall\, j \in \mathcal{J}.$$

Proof. For $j \in \mathcal{J}$, by definition we have

$$m_j = (-1)^{\delta_{j+1 \notin J}} \left(2\delta_{j \in (J \cap J')^{\mathrm{nss}}} + \delta_{j \in (J-1)^{\mathrm{ss}}} - \delta_{j \in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j \in J \Delta J'}\right).$$

If $j \notin J_{\overline{\rho}}$, then we have

$$\begin{aligned} m_j &= (-1)^{\delta_{j+1 \notin J}} \left(2\delta_{j \in J \cap J'} + 0 - \delta_{j \in J} + \delta_{j \in J \Delta J'}\right) \\ &= (-1)^{\delta_{j+1 \notin J}} \left(2\delta_{j \in J}\delta_{j \in J'} - \delta_{j \in J} + (\delta_{j \in J} + \delta_{j \in J'} - 2\delta_{j \in J}\delta_{j \in J'})\right) = \delta_{j \in J'} (-1)^{\delta_{j+1 \notin J}}. \end{aligned}$$

If $j \in J_{\overline{p}}$, then the assumption $(J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}}$ implies that $j \in J-1$ if and only if $j \in J'$, hence we have

$$\begin{aligned} m_j &= (-1)^{\delta_{j+1\notin J}}\big(0+\delta_{j\in J-1}-\delta_{j\in J\Delta(J-1)}+\delta_{j\in J\Delta J'}\big) \\ &= (-1)^{\delta_{j+1\notin J}}\big(\delta_{j\in J'}-\delta_{j\in J\Delta J'}+\delta_{j\in J\Delta J'}\big) = \delta_{j\in J'}(-1)^{\delta_{j+1\notin J}}. \end{aligned}$$

This completes the proof. □

The following proposition is a generalization of Proposition 4.5.5.

Proposition 4.5.8. *Let $J, J' \subseteq \mathcal{J}$, $\underline{i} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{i} \le \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$ and $\underline{m} = \underline{m}(\underline{i}, J, J')$. We denote $B \overset{\mathrm{def}}{=} \big[\prod_{j\notin J'} Y_j^{2i_j + t^J(J')_j}\big]\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\big(\underline{Y}^{-\underline{i}} v_J\big) \in \pi$.*

(i) *If $2i_j - \delta_{j\in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j-1\in J'} < 0$ for some j, then*

$$\begin{cases} B = 0 & \text{if } J' \neq \emptyset \\ Y_{j'}B = 0 \ \forall\, j' \in \mathcal{J} & \text{if } J' = \emptyset. \end{cases}$$

(ii) *If $2i_j - \delta_{j\in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j-1\in J'} \ge 0$ for all j, then (see Lemma 4.5.1(iii) for the notation)*

$$\begin{cases} B \in I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big) & \text{if } J' \neq \emptyset \\ Y_{j'}B \in I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big) \ \forall\, j' \in \mathcal{J} & \text{if } J' = \emptyset. \end{cases}$$

Proof. We assume that $J' \neq \emptyset$. The proof for the case $J' = \emptyset$ is exactly the same.

(i). Suppose that $2i_j - \delta_{j\in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j-1\in J'} < 0$ for some j, then we must have $i_j = 0$, $j \in J\Delta(J-1)^{\mathrm{ss}}$ and $j-1 \notin J'$. Hence $B = 0$ by Proposition 4.5.5 applied to $(\underline{i}, J, J')$ as above and $j_0 = j-1$.

(ii). Suppose that $2i_j - \delta_{j\in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j-1\in J'} \ge 0$ for all j. By Lemma 4.5.1(iii) and Remark 4.5.2, we have

$$\big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}} v_J\big\rangle = I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{b}}\big) \cong Q\big(\chi_J^s, \chi_J^s\alpha^{\underline{i}}, (J\Delta(J-1)^{\mathrm{ss}})-1\big)$$

with $b_j = (-1)^{\delta_{j+1\notin J}}(2i_j + \delta_{j\in(J-1)^{\mathrm{ss}}} + 1 - \delta_{j\in J\Delta(J-1)^{\mathrm{ss}}})$ for $j \in \mathcal{J}$. Since $b_j = m_j$ if and only if $j-1 \in J'$, to prove $B \in I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big)$, it suffices to show that for each $j_0 \in \mathcal{J}$ such that $j_0 - 1 \notin J'$, the image of B in the unique quotient Q of $\big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}} v_J\big\rangle$ with socle $\sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}\setminus\{j_0\}} + b_{j_0}e_{j_0}}$ is zero.

By Lemma 4.3.2(i), we have $Q \cong Q\big(\chi_J^s\alpha_{j_0}^{i_{j_0}}, \chi_J^s\alpha^{\underline{i}}, J''\big)$ with $J'' \overset{\mathrm{def}}{=} \big((J\Delta(J-1)^{\mathrm{ss}})-1\big)\cup\{j_0-1\}$. Since $(\underline{i} - i_{j_0}e_{j_0})_{j_0} = 0$, $j_0 - 1 \notin J'$ and $j_0 - 1 \in J''$, it follows from Remark 4.5.6 (with $\lambda = \lambda_J\alpha_{j_0}^{-i_{j_0}}$, $\underline{i}$ replaced with $\underline{i} - i_{j_0}e_{j_0}$ and j_0 replaced with $j_0 - 1$) that the H-eigencharacter of B does not occur in Q, hence B maps to zero in Q. □

The following proposition studies the overlaps between different $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentations $\big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}} v_J\big\rangle$ of π. This phenomenon is new in the non-semisimple case.

Proposition 4.5.9. *Let $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{i} \le \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$. Let $j_0 \in \mathcal{J}$ such that $j_0 + 1 \in (J-1)^{\mathrm{nss}}$ and $i_{j_0+1} = 0$. Let $J' \subseteq \mathcal{J}$ such that $j_0 \in J'$ if $j_0 + 1 \in J$ and $j_0 \notin J'$ if $j_0 + 1 \notin J$. We let $J'' \overset{\mathrm{def}}{=} J'\Delta\{j_0+1\}$ and let $\underline{i}' \in \mathbb{Z}^f$ be such that $i'_j = i_j$ if $j \neq j_0 + 2$ and $i'_{j_0+2} = i_{j_0+2} - \delta_{j_0+1\notin J'} + \delta_{j_0+2\in(J-1)^{\mathrm{ss}}}$. Then we have*

$$Y_{j_0+1}^{\delta_{j_0+1\notin J}}\left[\prod_{j\notin J'}Y_j^{2i_j+t^J(J')_j}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}}v_J\right)$$
$$=\frac{\mu_{J,(J-1)^{\mathrm{ss}}}}{\mu_{J\setminus\{j_0+2\},(J-1)^{\mathrm{ss}}}}Y_{j_0+1}^{\delta_{j_0+1\notin J}}\left[\prod_{j\notin J''}Y_j^{2i'_j+t^{J\setminus\{j_0+2\}}(J'')_j}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}'}v_{J\setminus\{j_0+2\}}\right),\quad(4.32)$$

where $\mu_{J,(J-1)^{\mathrm{ss}}}$ and $\mu_{J\setminus\{j_0+2\},(J-1)^{\mathrm{ss}}}$ are defined in Proposition 4.5.4, and we let $\underline{Y}^{-\underline{i}'}v_{J\setminus\{j_0+2\}}\overset{\mathrm{def}}{=}0$ if $i'_j<0$ for some $j\in\mathcal{J}$.

Note that the assumption $j_0+1\in(J-1)^{\mathrm{nss}}$ implies $\big((J\setminus\{j_0+2\})-1\big)^{\mathrm{ss}}=(J-1)^{\mathrm{ss}}$, hence $\mu_{J\setminus\{j_0+2\},(J-1)^{\mathrm{ss}}}$ is defined in Proposition 4.5.4. We claim that $\underline{i}'\le\underline{f}-\underline{e}^{(J\setminus\{j_0+2\})^{\mathrm{sh}}}$, which implies that $\underline{Y}^{-\underline{i}'}v_{J\setminus\{j_0+2\}}$ is well-defined by Proposition 4.4.2. Indeed, if $j\neq j_0+2$ or $j=j_0+2\notin(J-1)^{\mathrm{ss}}$, then we have

$$i'_j\le i_j\le f-\delta_{j\in J^{\mathrm{sh}}}\le f-\delta_{j\in(J\setminus\{j_0+2\})^{\mathrm{sh}}}.$$

If $j_0+2\in(J-1)^{\mathrm{ss}}$, then the assumption $j_0+1\in(J-1)^{\mathrm{nss}}$ implies $j_0+2\in J$ and thus $j_0+2\in J^{\mathrm{sh}}$, hence we have

$$i'_{j_0+2}\le i_{j_0+2}+1\le f-\delta_{j_0+2\in J^{\mathrm{sh}}}+1=f=f-\delta_{j_0+2\in(J\setminus\{j_0+2\})^{\mathrm{sh}}}.$$

We denote by B_1 (resp. B_2) the element on the LHS (resp. RHS) of (4.32). In order to prove Proposition 4.5.9, we need the following lemma.

Lemma 4.5.10. *Keep the assumptions of Proposition 4.5.9.*

(i) *Let $\underline{m}\overset{\mathrm{def}}{=}\underline{m}(\underline{i},J,J')$ and $\underline{m}'\overset{\mathrm{def}}{=}\underline{m}(\underline{i}',J\setminus\{j_0+2\},J'')$ (see (4.31)). Then we have $\underline{m}=\underline{m}'$ and $m_{j_0+1}=m'_{j_0+1}=0$.*

(ii) *We have (see (4.18) for $t^J(J')$)*

$$\begin{aligned}2i_j+t^J(J')_j&=2i'_j+t^{J\setminus\{j_0+2\}}(J'')_j\ \text{ if } j\neq j_0+1;\\ 2i_{j_0+1}+t^J(J')_{j_0+1}&=r_{j_0+1}+1;\\ 2i'_{j_0+1}+t^{J\setminus\{j_0+2\}}(J'')_{j_0+1}&=p-1-r_{j_0+1}.\end{aligned}\quad(4.33)$$

(iii) *We let $\underline{c},\underline{c}'\in\mathbb{Z}^f$ such that*

$$\begin{aligned}c_j&=pi_{j+1}+\delta_{j+1\in J\Delta(J-1)^{\mathrm{ss}}}s_j^{(J-1)^{\mathrm{ss}}}\\&\quad+\delta_{j+1\notin J\Delta(J-1)^{\mathrm{ss}}}(p-1)-\delta_{j\notin J'}\left(2i_j+t^J(J')_j\right)-\delta_{j=j_0+1}\delta_{j_0+1\notin J};\\ c'_j&=pi'_{j+1}+\delta_{j+1\in(J\setminus\{j_0+2\})\Delta(J-1)^{\mathrm{ss}}}s_j^{(J-1)^{\mathrm{ss}}}\\&\quad+\delta_{j+1\notin(J\setminus\{j_0+2\})\Delta(J-1)^{\mathrm{ss}}}(p-1)-\delta_{j\notin J''}\left(2i'_j+t^{J\setminus\{j_0+2\}}(J'')_j\right)-\delta_{j=j_0+1}\delta_{j_0+1\notin J}.\end{aligned}$$

Then we have $\underline{c}=\underline{c}'$.

(iv) *If moreover $2i_j-\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}}+\delta_{j-1\in J'}\ge0$ for all $j\in\mathcal{J}$, then we have $\underline{c}=\underline{c}'\ge\underline{0}$ and $\underline{Y}^{\underline{c}}B_1=\underline{Y}^{\underline{c}}B_2$. In particular, B_1 and B_2 have the same H-eigencharacter.*

Proof. (i). If $j\neq j_0+2$ or $f=1$, then by definition we have $m_j=m'_j$ and $m_{j_0+1}=m'_{j_0+1}=0$. If $j=j_0+2$ and $f\ge2$, using $j_0+1\in(J-1)^{\mathrm{nss}}$ (which implies $j_0+2\in J$) and $j_0+3\neq j_0+2$, we have

$$(-1)^{\delta_{j_0+3\notin J}}m'_{j_0+2}=(-1)^{\delta_{j_0+3\notin J\setminus\{j_0+2\}}}m'_{j_0+2}$$

$$
\begin{aligned}
&= 2i'_{j_0+2} + \delta_{j_0+2\in((J\setminus\{j_0+2\})-1)^{\mathrm{ss}}} - \delta_{j_0+2\in(J\setminus\{j_0+2\})\Delta((J\setminus\{j_0+2\})-1)^{\mathrm{ss}}} + \delta_{j_0+1\in J''}\\
&= 2\left(i_{j_0+2} - \delta_{j_0+1\notin J'} + \delta_{j_0+2\in(J-1)^{\mathrm{ss}}}\right) + \delta_{j_0+2\in(J-1)^{\mathrm{ss}}} - \delta_{j_0+2\in(J-1)^{\mathrm{ss}}} + \delta_{j_0+1\notin J'}\\
&= 2i_{j_0+2} - \delta_{j_0+1\notin J'} + 2\delta_{j_0+2\in(J-1)^{\mathrm{ss}}}\\
&= 2i_{j_0+2} - 1 + \delta_{j_0+1\in J'} + \delta_{j_0+2\in(J-1)^{\mathrm{ss}}} + 1 - \delta_{j_0+2\notin(J-1)^{\mathrm{ss}}}\\
&= 2i_{j_0+2} + \delta_{j_0+2\in(J-1)^{\mathrm{ss}}} - \delta_{j_0+2\in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j_0+1\in J'}\\
&= (-1)^{\delta_{j_0+3\notin J}} m_{j_0+2},
\end{aligned}
$$

hence $m_{j_0+2} = m'_{j_0+2}$.

(ii). We prove the case $j = j_0+2$ and $f \geq 2$, the other cases being similar and simpler. We also assume that $j_0+3 \in J$, the case $j_0+3 \notin J$ being similar. Then using (4.7), we have

$$
\begin{aligned}
2i'_{j_0+2} + t^{J\setminus\{j_0+2\}}(J'')_{j_0+2} &= 2i'_{j_0+2} + p - 1 - s_{j_0+2}^{J\setminus\{j_0+2\}} + \delta_{j_0+1\in J''}\\
&= 2\left(i_{j_0+2} - \delta_{j_0+1\notin J'} + \delta_{j_0+2\in J_{\overline{\rho}}}\right) + p - 1 - (p-2-r_{j_0+2}) + \delta_{j_0+1\notin J'}\\
&= 2i_{j_0+2} - \delta_{j_0+1\notin J'} + p - 1 - \left(p - 1 - r_{j_0+2} - 2\delta_{j_0+2\in J_{\overline{\rho}}}\right) + 1\\
&= 2i_{j_0+2} + p - 1 - s_{j_0+2}^{J} + \delta_{j_0+1\in J'}\\
&= 2i_{j_0+2} + t^J(J')_{j_0+2}.
\end{aligned}
$$

(iii). By (4.33) we have $c_j = c'_j$ for $j \neq j_0+1$, so it remains to prove that $c_{j_0+1} = c'_{j_0+1}$. We assume that $j_0+2 \in (J-1)^{\mathrm{ss}}$, the case $j_0+2 \notin (J-1)^{\mathrm{ss}}$ being similar. Then using (4.7) and (4.33) we have

$$
\begin{aligned}
c'_{j_0+1} &= p\left(i_{j_0+2} - \delta_{j_0+1\notin J'} + 1\right) + (p-2-r_{j_0+1}) + 0 - \delta_{j_0+1\in J'}(p-1-r_{j_0+1}) - \delta_{j_0+1\notin J}\\
&= p\left(i_{j_0+2} - \delta_{j_0+1\notin J'} + 1\right) - 1 + \delta_{j_0+1\notin J'}(p-1-r_{j_0+1}) - \delta_{j_0+1\notin J}\\
&= pi_{j_0+2} + 0 + (p-1) - \delta_{j_0+1\notin J'}(r_{j_0+1}+1) - \delta_{j_0+1\notin J} = c_{j_0+1}.
\end{aligned}
$$

(iv). If $j \in J'$, then by the definition of c_j and using $i_{j+1} \geq 0$, we have

$$
c_j \geq \min\left\{s_j^{(J-1)^{\mathrm{ss}}}, p-1\right\} - 1 \geq 0,
$$

where the last inequality follows from (4.10).

If $j \notin J'$, then the assumption $2i_{j+1} - \delta_{j+1\in J\Delta(J-1)^{\mathrm{ss}}} + \delta_{j\in J'} \geq 0$ implies that either $i_{j+1} \geq 1$ or $j+1 \notin J\Delta(J-1)^{\mathrm{ss}}$. By the definition of c_j and using $i_{j+1} \geq 0$, we have if $j \neq j_0+1$

$$
c_j \geq \min\left\{p, p-1\right\} - \left(2i_j + t^J(J')_j\right) \geq 0,
$$

where the last inequality follows from (4.19) and $\underline{i} \leq \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$. By the definition of c_{j_0+1} and using $i_{j_0+1} = 0$ (hence $j_0+1 \notin J\Delta(J-1)^{\mathrm{ss}}$) and (4.33), we have $c_{j_0+1} \geq (p-1) - (r_{j_0+1}+1) - 1 \geq 0$, where the last inequality follows from (4.2).

By the definition of $\underline{c}$ and since $\underline{c} \geq \underline{0}$, we have

$$
\begin{aligned}
\underline{Y}^{\underline{c}} B_1 &= \underline{Y}^{p\delta(\underline{i})}\left[\prod_{j+1\in J\Delta(J-1)^{\mathrm{ss}}} Y_j^{s_j^{(J-1)^{\mathrm{ss}}}} \prod_{j+1\notin J\Delta(J-1)^{\mathrm{ss}}} Y_j^{p-1}\right]\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}} v_J\right)\\
&= \left[\prod_{j+1\in J\Delta(J-1)^{\mathrm{ss}}} Y_j^{s_j^{(J-1)^{\mathrm{ss}}}} \prod_{j+1\notin J\Delta(J-1)^{\mathrm{ss}}} Y_j^{p-1}\right]\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right) v_J = \mu_{J,(J-1)^{\mathrm{ss}}} v_{(J-1)^{\mathrm{ss}}},
\end{aligned}
$$

where the second equality follows from Lemma 4.3.1(i) and the last equality follows from Proposition 4.5.4 applied to J. Similarly, we have (recall that $((J\setminus\{j_0+2\})-1)^{\mathrm{ss}}=(J-1)^{\mathrm{ss}}$)

$$\begin{aligned}\underline{Y}^{\underline{c}}B_2=\underline{Y}^{\underline{c}'}B_2&=\frac{\mu_{J,(J-1)^{\mathrm{ss}}}}{\mu_{J\setminus\{j_0+2\},(J-1)^{\mathrm{ss}}}}\left[\prod_{j+1\in J'''}Y_j^{s_j^{(J-1)^{\mathrm{ss}}}}\prod_{j+1\notin J'''}Y_j^{p-1}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_{J\setminus\{j_0+2\}}\\&=\mu_{J,(J-1)^{\mathrm{ss}}}v_{(J-1)^{\mathrm{ss}}},\end{aligned}$$

where $J'''\overset{\mathrm{def}}{=}(J\setminus\{j_0+2\})\Delta(J-1)^{\mathrm{ss}}$ and the last equality follows from Proposition 4.5.4 applied to $J\setminus\{j_0+2\}$. In particular, we deduce from Lemma 4.3.1(ii) that B_1 and B_2 have the same H-eigencharacter. □

Proof of Proposition 4.5.9. As in the proof of Proposition 4.5.5, the H-eigencharacters that occur in $\left\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}v_J\right\rangle$ are those in $Q\big(\chi_J^s\alpha^{\underline{i}''},J_{\underline{i}''}\big)$ for $\underline{0}\le\underline{i}''\le\underline{i}$ (where $J_{\underline{i}''}\overset{\mathrm{def}}{=}\{j:j+1\in J\Delta(J-1)^{\mathrm{ss}},\ i''_{j+1}=0\}$), which are $\chi_J\alpha^{-\underline{i}''}\alpha^{-\underline{k}}$, where

$$\begin{cases}0\le k_j\le p-2-(s_j^J-2i''_j)+\delta_{j-1\in J_0} & \text{if } j\in J_0\\ p-1-(s_j^J-2i''_j)+\delta_{j-1\in J_0}\le k_j\le p-1 & \text{if } j\notin J_0\end{cases}\tag{4.34}$$

for $J_0\supseteq J_{\underline{i}''}$. By Lemma 4.3.2(iii), unless $J_0=\emptyset$ and $k_j=p-1-(s_j^J-2i''_j)$ for all j, the H-eigencharacter in (4.34) comes from the element $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}''}v_J\right)\in\left\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}''}v_J\right\rangle$.

Suppose that $2i_j-\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}}+\delta_{j-1\in J'}<0$ for some j. By Lemma 4.5.10(i) and using $\big((J\setminus\{j_0+2\})-1\big)^{\mathrm{ss}}=(J-1)^{\mathrm{ss}}$, we have

$$2i'_j-\delta_{j\in(J\setminus\{j_0+2\})\Delta((J\setminus\{j_0+2\})-1)^{\mathrm{ss}}}+\delta_{j-1\in J''}<0$$

for the same j. Then by Proposition 4.5.8(i) applied to (i,J,J') and $(i',J\setminus\{j_0+2\},J'')$ we deduce that $B_1=B_2=0$ (if $\underline{i}'\not\ge\underline{0}$ then $B_2=0$ by definition), which proves (4.32). So in the rest of the proof we assume that $2i_j-\delta_{j\in J\Delta(J-1)^{\mathrm{ss}}}+\delta_{j-1\in J'}\ge 0$ for all j, which implies that

$$2i'_j-\delta_{j\in(J\setminus\{j_0+2\})\Delta((J\setminus\{j_0+2\})-1)^{\mathrm{ss}}}+\delta_{j-1\in J''}\ge 0$$

for all j. In particular, this implies $\underline{i}'\ge\underline{0}$. Then by Proposition 4.5.8(ii) applied to (i,J,J') and $(i',J\setminus\{j_0+2\},J'')$ we deduce that $B_1,B_2\in I\big(\sigma_{(J-1)^{\mathrm{ss}},}\sigma_{\underline{m}}\big)=I\big(\sigma_{(J-1)^{\mathrm{ss}},}\sigma_{\underline{m}'}\big)$ (see Lemma 4.5.10(i)), which is a subrepresentation of $\left\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}v_J\right\rangle$ and of $\left\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}'}v_{J\setminus\{j_0+2\}}\right\rangle$.

(i). We suppose that $j_0+1\in J$, hence $j_0\in J'$. In this case, we claim that it suffices to prove (4.32) for $J'=\mathcal{J}$, that is (using (4.33))

$$\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}}v_J\right)=\frac{\mu_{J,(J-1)^{\mathrm{ss}}}}{\mu_{J\setminus\{j_0+2\},(J-1)^{\mathrm{ss}}}}Y_{j_0+1}^{p-1-r_{j_0+1}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}'}v_{J\setminus\{j_0+2\}}\right),\tag{4.35}$$

where $i'_j=i_j$ if $j\ne j_0+2$ and $i'_{j_0+2}=i_{j_0+2}+\delta_{j_0+2\in(J-1)^{\mathrm{ss}}}$. Indeed, once (4.35) is proved, we multiply both sides of (4.35) by $\prod_{j\notin J'}Y_j^{2i_j+t^J(J')_j}$. If $j_0+1\in J'$, then using (4.33) we obtain (4.32) for J'. If $j_0+1\notin J'$, then using (4.33) together with Lemma 4.3.1(i) applied to $j=j_0+1$ we obtain (4.32) for J'.

Then we prove (4.35). Since $B_1,B_2\in I\big(\sigma_{(J-1)^{\mathrm{ss}},}\sigma_{\underline{m}}\big)\subseteq\left\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}v_J\right\rangle$ have common H-eigencharacter $\chi_J\alpha^{-\underline{i}}$ (see Lemma 4.5.10(iv)), it suffices to show that the H-eigencharacter $\chi_J\alpha^{-\underline{i}}$ only appears once in $\left\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}v_J\right\rangle$, which implies $B_1=B_2$ by Lemma 4.5.10(iv). Since $j_0+1\in J$, the assumptions $j_0+1\in(J-1)^{\mathrm{nss}}$ and $i_{j_0+1}=0$

imply that $j_0 \in J_{\underline{i}''}$ for all $\underline{i}''$ as in (4.34). In particular, we have $j_0 \in J_0$. As in the proof of Proposition 4.5.5, the equation (4.22) then becomes

$$-\sum_{j=0}^{f-1}(i_j - i_j'')p^j \equiv \sum_{j=0}^{f-1} k_j p^j \mod (q-1), \tag{4.36}$$

which is congruent to $\sum_{j=0}^{f-1}\big(p-1-(i_j-i_j'')\big)p^j$ modulo $(q-1)$. If $i_j \neq i_j''$ for some j, then we must have $k_{j_0} = p-1-(i_{j_0}-i_{j_0}'')$. Since $j_0 \in J_0$, by (4.34) we have

$$k_{j_0} = p-1-(i_{j_0}-i_{j_0}'') \leq p-2-(s_{j_0}^J - 2i_{j_0}'') + \delta_{j_0-1\in J_0} \leq p-1-(s_{j_0}^J - 2i_{j_0}'').$$

Hence $s_{j_0}^J \leq i_{j_0} + i_{j_0}'' \leq 2(f-\delta_{j_0\in J^{\mathrm{sh}}})$, which contradicts (4.10). Therefore, we must have $i_j = i_j''$ for all j and the LHS of (4.36) equals 0. Since $j_0 \in J_0$, by (4.34) and (4.10) we have $k_{j_0} < p-1$. It follows from (4.36) that $k_j = 0$ for all j.

(ii) We suppose that $j_0+1 \notin J$ (which implies $f \geq 2$), hence $j_0 \notin J'$. We prove (4.32) by the following steps.

Step 1. We prove (4.32) for $J' = \mathcal{J}\setminus\{j_0\}$.

Using (4.33), it is enough to prove that

$$Y_{j_0}^{2i_{j_0}+p-s_{j_0}^J}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\big(\underline{Y}^{-\underline{i}}v_J\big) = \frac{\mu_{J,(J-1)^{\mathrm{ss}}}}{\mu_{J\setminus\{j_0+2\},(J-1)^{\mathrm{ss}}}}\left[Y_{j_0}^{2i_{j_0}+p-s_{j_0}^J}Y_{j_0+1}^{p-1-r_{j_0+1}}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\Big(\underline{Y}^{-\underline{i}'}v_{J\setminus\{j_0+2\}}\Big), \tag{4.37}$$

where $i_j' = i_j$ if $j \neq j_0+2$ and $i_{j_0+2}' = i_{j_0+2} + \delta_{j_0+2\in(J-1)^{\mathrm{ss}}}$. Since $i_{j_0+1} = i_{j_0+1}' = 0$, by Lemma 4.3.1(i) applied to $j = j_0$ and Proposition 4.4.2, if we apply $Y_{j_0}^{s_{j_0}^J-2i_{j_0}}$ to either side of (4.37) we get zero. Moreover, $B_1, B_2 \in I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big) \subseteq \big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}v_J\big\rangle$ have common H-eigencharacter $\chi \overset{\mathrm{def}}{=} \chi_J\alpha^{-\underline{i}}\alpha_{j_0}^{2i_{j_0}+p-s_{j_0}^J}$ (see Lemma 4.5.10(iv)). Hence it suffices to show that up to scalar there exists a unique H-eigenvector $C \in \big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}v_J\big\rangle$ satisfying $Y_{j_0}^{s_{j_0}^J-2i_{j_0}}C = 0$ with H-eigencharacter χ, which implies $B_1 = B_2$ by Lemma 4.5.10(iv).

As in the proof of Proposition 4.5.5 (in the case $J' = \mathcal{J}\setminus\{j_0\}$ with the same definition of η_j), for each $\underline{i}''$ such that $\underline{0} \leq \underline{i}'' \leq \underline{i}$, the equation $\chi = \chi_J\alpha^{-\underline{i}''}\alpha^{\underline{k}}$ has at most one solution for $\underline{k}$ as in (4.34), which is given by (see (4.27) and since $j_0-1 \in J'$)

$$\begin{cases} k_{j_0} = i_{j_0} + i_{j_0}'' + p - s_{j_0}^J + \eta_{j_0} \\ k_j = \eta_j - (i_j - i_j'') & \text{if } j \neq j_0. \end{cases} \tag{4.38}$$

It follows from (4.34) that C is a linear combination of the elements $C' \overset{\mathrm{def}}{=} \underline{Y}^{\underline{k}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\big(\underline{Y}^{-\underline{i}''}v_J\big) \in \big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}''}v_J\big\rangle$ with distinct $\underline{i}''$ such that $\underline{0} \leq \underline{i}'' \leq \underline{i}$ and $\underline{k}$ as in (4.38), each of which has nonzero image in the quotient Q of $\big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}''}v_J\big\rangle$ isomorphic to $Q\big(\chi_J^s\alpha^{\underline{i}''}, J_{\underline{i}''}\big)$ (see Lemma 4.5.1(ii)).

We claim that for $\underline{i}'' \neq \underline{i}$, the element $Y_{j_0}^{s_{j_0}^J-2i_{j_0}}C' \in \big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}''}v_J\big\rangle$ also has nonzero image in $Q \cong Q\big(\chi_J^s\alpha^{\underline{i}''}, J_{\underline{i}''}\big)$. Then we deduce from Lemma 4.5.1(ii) that the coefficients of C' with $\underline{i}'' \neq \underline{i}$ in the linear combination for C must be zero, which concludes the proof of (4.37).

Then we prove the claim. We let J_0 be the subset corresponding to the H-eigencharacter of $C' = \underline{Y}^{\underline{k}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\big(\underline{Y}^{-\underline{i}''}v_J\big)$ in (4.34). Suppose that $j_0 \in J_0$. Then by Claim 2 in the proof of

Proposition 4.5.5, we deduce from $j_0 \in J_0$ and $j_0 \notin J'$ that $j_0 - 1 \notin J'$, which is a contradiction since $J' = \mathcal{J} \setminus \{j_0\}$. Hence we must have $j_0 \notin J_0$. By the definition of η_j in the case $J' = \mathcal{J} \setminus \{j_0\}$ (see (4.23)), we have either $\eta_{j_0} = -1$ or $\eta_{j_0} = 0$. Moreover, if $\eta_{j_0} = 0$, then the definition of η_j implies that $i''_j = i_j$ for all $j \neq j_0$, hence $i''_{j_0} < i_{j_0}$ since $\underline{i}'' \neq \underline{i}$. In particular, in either case we deduce from (4.38) that $k_{j_0} < 2i_{j_0} + p - s^J_{j_0}$, hence $(s^J_{j_0} - 2i_{j_0}) + k_{j_0} \leq p - 1$. Then using $j_0 \notin J_0$, the H-eigencharacter of $Y_{j_0}^{s^J_{j_0} - 2i_{j_0}} C'$ still appears in (4.34) (with the corresponding $\underline{i}''$ and J_0 unchanged), hence has nonzero image in $Q \cong Q\big(\chi^s_J \alpha^{\underline{i}''}, J_{\underline{i}''}\big)$.

Step 2. We prove (4.32) for all J' such that $j_0 \notin J'$ and $j_0 - 1 \in J'$.

We multiply both sides of (4.37) by $Y_{j_0+1}\big[\prod_{j \notin J' \cup \{j_0\}} Y_j^{2i_j + t^J(J')_j}\big]$. Since $j_0 - 1 \in J'$, we deduce that $t^J(J')_{j_0}$ is the same as in Step 1. If $j_0 + 1 \in J'$, then using (4.33) we obtain (4.32) for J'. If $j_0 + 1 \notin J'$, then using (4.33) together with Lemma 4.3.1(i) applied to $j = j_0 + 1$ we obtain (4.32) for J'.

Step 3. We prove (4.32) for all J' such that $j_0 \notin J'$ and $j_0 - 1 \notin J'$.

We multiply both sides of (4.37) by $Y_{j_0+1}\big[\prod_{j \notin J' \cup \{j_0\}} Y_j^{2i_j + t^J(J')_j}\big]$. Similarly to Step 2 but using $j_0 - 1 \notin J'$, we get $Y_{j_0} B_1 = Y_{j_0} B_2$. Moreover, $B_1, B_2 \in I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big) \subseteq \big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \big\rangle$ have H-eigencharacter $\chi \overset{\text{def}}{=} \chi_J \alpha_{j_0+1} \alpha^{-\underline{i}} \big[\prod_{j \notin J'} \alpha_j^{2i_j + p - 1 - s^J_j + \delta_{j-1 \in J'}}\big]$. Hence it suffices to show that there is no nonzero H-eigenvector $C \in I(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}})$ with H-eigencharacter χ satisfying $Y_{j_0} C = 0$, which implies $B_1 - B_2 = 0$.

As in the proof of Proposition 4.5.5, the equation $\chi = \chi_J \alpha^{-\underline{i}''} \alpha^{\underline{k}}$ is equivalent to the congruence relation (compare with (4.22))

$$p^{j_0+1} + \sum_{j \notin J'} \big(i_j + i''_j + p - 1 - s^J_j + \delta_{j-1 \in J'}\big) p^j - \sum_{j \in J'} (i_j - i''_j) p^j \equiv \sum_{j=0}^{f-1} k_j p^j \mod (q-1), \quad (4.39)$$

where the term p^{j_0+1} comes from α_{j_0+1}. Then by a similar argument as in the proof of Proposition 4.5.5, for each $\underline{i}''$ such that $\underline{0} \leq \underline{i}'' \leq \underline{i}$ the equation (4.39) has at most one solution for $\underline{k}$ as in (4.34), which satisfies (compare with (4.27) and note that $j_0 - 1 \notin J'$)

$$k_{j_0} = i_{j_0} + i''_{j_0} + p - 1 - s^J_{j_0} + \eta$$

for some $\eta \in \{0, -1\}$ and $\underline{k} \neq 2\underline{i}'' + \underline{p} - \underline{1} - \underline{s}^J$. In particular, using (4.10) we deduce that $k_{j_0} \leq p - 2$. It follows from (4.34) that C is a linear combination of the elements $C' \overset{\text{def}}{=} \underline{Y}^{\underline{k}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \big(\underline{Y}^{-\underline{i}''} v_J\big) \in \big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}''} v_J \big\rangle$ with distinct $\underline{i}''$ such that $\underline{0} \leq \underline{i}'' \leq \underline{i}$ and $\underline{k}$ such that $k_{j_0} \leq p-2$, each of which has nonzero image in the quotient Q of $\big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}''} v_J \big\rangle$ isomorphic to $Q\big(\chi^s_J \alpha^{\underline{i}''}, J_{\underline{i}''}\big)$ (see Lemma 4.5.1(ii)).

We let J_0 be the subset corresponding to the H-eigencharacter of $C' = \underline{Y}^{\underline{k}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \big(\underline{Y}^{-\underline{i}''} v_J\big)$ in (4.34) and claim that $j_0 \notin J_0$. Since $m_{j_0+1} = 0$, any constituent $\sigma_{\underline{a}}$ of $I\big(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}}\big)$ satisfies $a_{j_0+1} = 0$. If we write $\sigma_{\underline{a}} = F\big(\mathrm{t}_{\lambda_J}(\underline{b})\big)$, then by Lemma 4.2.3 we have $0 = a_{j_0+1} = (-1)^{\delta_{j_0+2 \in J}} (b_{j_0+1} + \delta_{j_0+1 \in J}) + 2\delta_{j_0+1 \in J^{\mathrm{sh}}}$. Since $j_0 + 1 \notin J$, we deduce that $b_{j_0+1} = 0$, which is even. Since $C \in I(\sigma_{(J-1)^{\mathrm{ss}}}, \sigma_{\underline{m}})$ we deduce from Lemma 4.3.2(i) applied to $\lambda = \lambda_J - \alpha^{\underline{i}''}$ that $j_0 \notin J_0$.

Since $j_0 \notin J_0$ and $k_{j_0} \leq p - 2$, the H-eigencharacter of $Y_{j_0} C'$ still appears in (4.34) (with the corresponding $\underline{i}''$ and J_0 unchanged), hence has nonzero image in $Q \cong Q\big(\chi^s_J \alpha^{\underline{i}''}, J_{\underline{i}''}\big)$. Since we ask $Y_{j_0} C = 0$, it follows that the coefficient of each C' in the linear combination for C must be zero, which completes the proof. $\square$

The following Proposition is a generalization of Proposition 4.5.4 and gives more relations between the vectors $v_J \in D_0(\overline{\rho})$.

Proposition 4.5.11. *Let $J, J' \subseteq \mathcal{J}$ such that $(J')^{\mathrm{nss}} \neq \mathcal{J}$ (i.e. $(J', J_{\overline{\rho}}) \neq (\mathcal{J}, \emptyset)$) and satisfying*

$$\begin{cases} (J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}} \\ (J')^{\mathrm{nss}} \subseteq (J-1)^{\mathrm{nss}} \Delta (J'-1)^{\mathrm{nss}}. \end{cases} \tag{4.40}$$

Then there exists a unique element $\mu_{J,J'} \in \mathbb{F}^\times$, such that

$$\left[\prod_{j+1 \in J \Delta J'} Y_j^{s_j^{J'}} \prod_{j+1 \notin J \Delta J'} Y_j^{p-1} \right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \left(\underline{Y}^{-\underline{e}^{(J \cap J')^{\mathrm{nss}}}} v_J \right) = \mu_{J,J'} v_{J'}, \tag{4.41}$$

where $\underline{Y}^{-\underline{e}^{(J \cap J')^{\mathrm{nss}}}} v_J$ is defined in Proposition 4.4.2.

Proof. If $f = 1$, then the assumption implies $J' = (J-1)^{\mathrm{ss}}$ and $(J \cap J')^{\mathrm{nss}} = \emptyset$, and the proposition is already proved in Proposition 4.5.4. Hence in the rest of the proof we assume that $f \geq 2$. We denote by B the LHS of (4.41).

Claim 1. The element B is nonzero and has H-eigencharacter $\chi_{J'}$.

Proof. By Lemma 4.5.1(ii), the representation $\left\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \underline{Y}^{-\underline{e}^{(J \cap J')^{\mathrm{nss}}}} v_J \right\rangle$ has a quotient Q isomorphic to $Q\big(\chi_J^s \alpha^{\underline{e}^{(J \cap J')^{\mathrm{nss}}}}, J'''\big)$ with $J''' \overset{\mathrm{def}}{=} \big((J \Delta (J-1)^{\mathrm{ss}}) \setminus (J \cap J')^{\mathrm{nss}}\big) - 1$. By the proof of Lemma 4.5.1, Q has constituents $F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with

$$\max\big(\delta_{j \in J \Delta (J-1)^{\mathrm{ss}}}, 2\delta_{j \in (J \cap J')^{\mathrm{nss}}}\big) \leq b_j \leq 2\delta_{j \in (J \cap J')^{\mathrm{nss}}} + 1. \tag{4.42}$$

We claim that $\sigma_{J'}$ is a constituent of Q and corresponds to the subset $J'' \overset{\mathrm{def}}{=} (J \Delta J') - 1$ for $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \big(\chi_J^s \alpha^{\underline{e}^{(J \cap J')^{\mathrm{nss}}}}\big)$ (see Lemma 4.3.2(i)). Indeed, by Lemma 4.2.3(iii), we have $\sigma_{J'} = F\big(\mathfrak{t}_{\lambda_J}(-\underline{b})\big)$ with

$$b_j = \begin{cases} \delta_{j \in J} + \delta_{j \in J'} (-1)^{\delta_{j+1 \notin J \Delta J'}} & \text{if } j \notin J_{\overline{\rho}} \\ \big(\delta_{j \in J} - \delta_{j \notin J'}\big) (-1)^{\delta_{j+1 \in J}} & \text{if } j \in J_{\overline{\rho}}. \end{cases} \tag{4.43}$$

We need to check that b_j satisfies (4.42). We assume that $j \notin J_{\overline{\rho}}$, the case $j \in J_{\overline{\rho}}$ being similar. By (4.40), we have $j \in J'$ implies $j+1 \in J \Delta J'$. Hence we have $\delta_{j \in J'} (-1)^{\delta_{j+1 \notin J \Delta J'}} = \delta_{j \in J'}$, and from (4.42) it suffices to show that

$$\max\big(\delta_{j \in J}, 2\delta_{j \in J \cap J'}\big) \leq \delta_{j \in J} + \delta_{j \in J'} \leq 2\delta_{j \in J \cap J'} + 1,$$

which is easy. Then we prove the second assertion. By Lemma 4.3.2(i) applied to $\lambda = \lambda_J - \alpha^{\underline{e}^{(J \cap J')^{\mathrm{nss}}}}$, it suffices to show that $b_j = 2\delta_{j \in (J \cap J')^{\mathrm{nss}}} + 1$ if and only if $j \in J \Delta J'$. Once again we assume that $j \notin J_{\overline{\rho}}$, and the case $j \in J_{\overline{\rho}}$ is similar. Then it suffices to show that $\delta_{j \in J} + \delta_{j \in J'} = 2\delta_{j \in J \cap J'} + 1$ if and only if $j \in J \Delta J'$, which is easy.

By Lemma 4.2.2, for $j \in J''$ we have $\big(p - 2 - (s_j^J - 2\delta_{j \in (J \cap J')^{\mathrm{nss}}})\big) + \delta_{j-1 \in J''} = s_j^{J'}$. Then by Lemma 4.3.2(iii) applied to $\lambda = \lambda_J - \alpha^{\underline{e}^{(J \cap J')^{\mathrm{nss}}}}$, we deduce that the image of B in $Q \cong Q\big(\chi_J^s \alpha^{\underline{e}^{(J \cap J')^{\mathrm{nss}}}}, J'''\big)$ is a nonzero I_1-invariant of $\sigma_{J'}$. In particular, B is nonzero and has H-eigencharacter $\chi_{J'}$. □

Claim 2. The element B is K_1-invariant.

Proof. First we claim that $\underline{m}\big(\underline{e}^{(J\cap J')^{\mathrm{nss}}}, J, J''\big) = \underline{a}^{J'}$ (see (4.31) for $\underline{m}$ and (4.6) for $\underline{a}^{J'}$). Indeed, using Lemma 4.5.7 it suffices to show that $\delta_{j\in J'}(-1)^{\delta_{j+1\notin J}} = a_j^{J'}$ for $j \in \mathcal{J}$. If $j \in J_{\overline{\rho}}$, then the assumption $(J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}}$ implies that $j \in J-1$ if and only if $j \in J'$, hence we have $\delta_{j\in J'}(-1)^{\delta_{j+1\notin J}} = \delta_{j\in J'}(-1)^{\delta_{j\notin J'}} = \delta_{j\in J'}$, which equals $a_j^{J'}$ by (4.6). If $j \notin J_{\overline{\rho}}$, then $j \in J'$ implies $j+1 \in J\Delta J'$ by the second formula of (4.40), hence we have $\delta_{j\in J'}(-1)^{\delta_{j+1\notin J}} = \delta_{j\in J'}(-1)^{\delta_{j+1\in J'}}$, which equals $a_j^{J'}$ by (4.6).

By Proposition 4.5.8(ii) applied to $\underline{i} = \underline{e}^{(J\cap J')^{\mathrm{nss}}}$ and with J' there being J'', we deduce that

$$Y_{j'}\left[\prod_{j+1\notin J\Delta J'} Y_j^{2\delta_{j\in(J\cap J')^{\mathrm{nss}}}+t^J(J'')_j}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{e}^{(J\cap J')^{\mathrm{nss}}}}v_J\right) \in I(\sigma_{(J-1)^{\mathrm{ss}}},\sigma_{J'}) \subseteq D_{0,\sigma_{(J-1)^{\mathrm{ss}}}}(\overline{\rho}) \tag{4.44}$$

for all $j' \in \mathcal{J}$, where the last inclusion follows from the fact that $\sigma_{J'}$ is a constituent of $D_{0,\sigma_{(J-1)^{\mathrm{ss}}}}(\overline{\rho})$ (which follows from Lemma 4.4.1(iii) and (4.40)). Since $s_j^{J'} \geq 1$ and $2\delta_{j\in(J\cap J')^{\mathrm{nss}}} + t^J(J'')_j \leq p-2$ for all j by (4.10), (4.19) and $f \geq 2$, multiplying (4.44) by a suitable power of $\underline{Y}$, we deduce that $B \in D_{0,\sigma_{(J-1)^{\mathrm{ss}}}}(\overline{\rho})$, hence is K_1-invariant. □

Claim 3. We have $Y_{j_0}B = 0$ for all $j_0 \in \mathcal{J}$.

Proof. (i). Suppose that $j_0+1 \notin J\Delta J'$ and $j_0+1 \notin (J\cap J')^{\mathrm{nss}}$. By Proposition 4.4.2, we have $Y_{j_0+1}\big(\underline{Y}^{-\underline{e}^{(J\cap J')^{\mathrm{nss}}}}v_J\big) = 0$. Hence it follows from Lemma 4.3.1(i) applied to $j = j_0$ that $Y_{j_0}B = 0$.

(ii). Suppose that $j_0+1 \in J\Delta J'$, which equals $J\Delta\big((J')^{\mathrm{ss}}\Delta(J')^{\mathrm{nss}}\big) = \big(J\Delta(J')^{\mathrm{ss}}\big)\Delta(J')^{\mathrm{nss}}$. Hence for each $j \in J\Delta J'$, we have either $j \in J\Delta(J')^{\mathrm{ss}}$, $j \notin (J')^{\mathrm{nss}}$ or $j \notin J\Delta(J')^{\mathrm{ss}}$, $j \in (J')^{\mathrm{nss}}$, and in the latter case we have $j+1 \in J\Delta J'$ by (4.40). In particular, since $(J')^{\mathrm{nss}} \neq \mathcal{J}$, there exists $0 \leq w \leq f-1$ such that $j \notin J\Delta(J')^{\mathrm{ss}}$, $j \in (J')^{\mathrm{nss}}$ for $j = j_0+1,\dots,j_0+w$ and $j_0+w+1 \in J\Delta(J')^{\mathrm{ss}}$, $j_0+w+1 \notin (J')^{\mathrm{nss}}$.

By (4.40) we have $j_0+w+1 \in J\Delta(J')^{\mathrm{ss}} = J\Delta(J-1)^{\mathrm{ss}}$. Then by proposition 4.5.5 applied to $\underline{i} = \underline{e}^{(J\cap J')^{\mathrm{nss}}}$, j_0 replaced by j_0+w and J' replaced by $J_1''-1$ with $J_1'' \overset{\mathrm{def}}{=} (J\Delta J')\setminus\{j_0+1,\dots,j_0+w+1\}$, and possibly multiplying (4.20) by Y_{j_0+w+1}, we have

$$Y_{j_0+w+1}\left[\prod_{j+1\notin J_1''} Y_j^{2\delta_{j\in(J\cap J')^{\mathrm{nss}}}+p-1-s_j^J+\delta_{j\in J_1''}}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{e}^{(J\cap J')^{\mathrm{nss}}}}v_J\right) = 0.$$

Since $2\delta_{j\in(J\cap J')^{\mathrm{nss}}} + p - 1 - s_j^J + \delta_{j\in J_1''} \leq p-1$ for all j by (4.10), to prove that $Y_{j_0}B = 0$, it suffices (from the formula of B) to show that

$$s_j^{J'} + \delta_{j=j_0} = 2\delta_{j\in(J\cap J')^{\mathrm{nss}}} + p - 1 - s_j^J + \delta_{j\in(J\Delta J')\setminus\{j_0+1,\dots,j_0+w+1\}} + \delta_{j=j_0+w+1}$$

for $j+1 \in (J\Delta J')\setminus J_1''$, that is $j = j_0,\dots,j_0+w$. This follows from Lemma 4.2.2 with J, J' as above noting that $j = j_0+w+1$ and $j \in \{j_0,\dots,j_0+w\}$ imply that $j = j_0$ and $w = f-1$.

(iii). Suppose that $j_0+1 \in (J\cap J')^{\mathrm{nss}}$, then by Lemma 4.3.1(i) applied to $j = j_0$ and using $j_0+1 \notin J\Delta J'$, we have

$$Y_{j_0}B = \left[\prod_{j+1\in J\Delta J'} Y_j^{s_j^{J'}} \prod_{j+1\notin(J\Delta J')\cup\{j_0+1\}} Y_j^{p-1}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{e}^{(J\cap J')^{\mathrm{nss}}\setminus\{j_0+1\}}}v_J\right). \tag{4.45}$$

As in (ii) (with the difference that $j_0+1 \notin J\Delta J'$), there exists $1 \leq w \leq f-1$ such that $j \notin J\Delta(J')^{\mathrm{ss}}$, $j \in (J')^{\mathrm{nss}}$ for $j = j_0+2,\dots,j_0+w$ and $j_0+w+1 \in J\Delta(J')^{\mathrm{ss}}$, $j_0+w+1 \notin (J')^{\mathrm{nss}}$.

By (4.40) we have $j_0+w+1 \in J\Delta(J')^{\mathrm{ss}} = J\Delta(J-1)^{\mathrm{ss}}$. Then by proposition 4.5.5 applied to $\underline{i} = \underline{e}^{(J\cap J')^{\mathrm{nss}}\setminus\{j_0+1\}}$, j_0 replaced by j_0+w and J' replaced by $J_2''-1$ with $J_2'' \overset{\mathrm{def}}{=} \big((J\Delta J')\setminus\{j_0+2,\dots,j_0+w+1\}\big)\cup\{j_0+1\}$, we have

$$\left[\prod_{j+1\notin J_2''} Y_j^{2\delta_{j\in(J\cap J')^{\mathrm{nss}}\setminus\{j_0+1\}}+p-1-s_j^J+\delta_{j\in J_2''}}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{e}^{(J\cap J')^{\mathrm{nss}}\setminus\{j_0+1\}}}v_J\right)=0.$$

Since $2\delta_{j\in(J\cap J')^{\mathrm{nss}}\setminus\{j_0+1\}}+p-1-s_j^J+\delta_{j\in J_2''}\le p-1$ for all j by (4.10), to prove that $Y_{j_0}B=0$, comparing with (4.45) it suffices to show that

$$s_j^{J'} = 2\delta_{j\in(J\cap J')^{\mathrm{nss}}\setminus\{j_0+1\}}+p-1-s_j^J+\delta_{j\in((J\Delta J')\setminus\{j_0+2,\dots,j_0+w+1\})\cup\{j_0+1\}}$$

for $j+1\in(J\Delta J')\setminus J_2''$, that is $j=j_0+1,\dots,j_0+w$. This follows from Lemma 4.2.2 with J,J' as above. □

From Claim 2 and Claim 3, we deduce that B is I_1-invariant. Since $B\neq 0$ has H-eigencharacter $\chi_{J'}$ by Claim 1 and since $D_0(\overline{\rho})^{I_1}$ is multiplicity-free by Lemma 4.4.1(ii), we conclude that B is a scalar multiple of $v_{J'}$, which completes the proof. □

Remark 4.5.12. *For $J\subseteq\mathcal{J}$, we define the **right boundary** of J by $\partial J \overset{\mathrm{def}}{=} \{j\in J : j+1\notin J\}$. Then the second formula in (4.40) is equivalent to*

$$(\partial J')^{\mathrm{nss}} \subseteq (J-1)^{\mathrm{nss}} \subseteq \big((J'\setminus\partial J')^c\big)^{\mathrm{nss}}.$$

If $J_{\overline{\rho}}=\emptyset$, then we define $x_{\emptyset,\underline{r}} \overset{\mathrm{def}}{=} \mu_{\emptyset,\emptyset}^{-1}\underline{Y}^{\underline{p-1-r}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_\emptyset$ so that $\underline{Y}^{\underline{r}}x_{\emptyset,\underline{r}}=v_\emptyset$ by (4.17) applied to $J=\emptyset$. This agrees with the definition of $x_{\emptyset,\underline{r}}$ given in Theorem 4.6.4 below, see (4.123) below. Then we have the following complement of Proposition 4.5.11 which together with Proposition 4.5.11 gives all possible relations between the vectors $v_J\in D_0(\overline{\rho})$.

Proposition 4.5.13. *Assume that $J_{\overline{\rho}}=\emptyset$. Then for $\emptyset\neq J\subseteq\mathcal{J}$, there exists a unique element $\mu_{J,\mathcal{J}}\in\mathbb{F}^\times$ such that*

$$\left[\prod_{j+1\notin J} Y_j^{p-1-r_j}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_J = \mu_{J,\mathcal{J}}v_{\mathcal{J}}+\mu_{J,\emptyset}x_{\emptyset,\underline{r}},$$

where $\mu_{J,\emptyset}$ is defined in Proposition 4.5.4.

Proof. By Lemma 4.5.1(ii) and its proof, the isomorphism $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\emptyset^s)\cong\big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_\emptyset\big\rangle$ identifies the element ϕ in §4.3 with $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)v_\emptyset$, which is a scalar multiple of $v_{\mathcal{J}}$ since $\chi_{\mathcal{J}}=\chi_\emptyset^s$ when $J_{\overline{\rho}}=\emptyset$. Hence by Lemma 4.3.2(iii)(a) applied to $\lambda=\lambda_\emptyset$, any nonzero element in the I-cosocle of $\sigma_\emptyset$ is a linear combination of $v_{\mathcal{J}}$ and $x_{\emptyset,\underline{r}}$ with nonzero coefficients.

By Lemma 4.5.1(i),(ii) and its proof, the representation $\big\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_J\big\rangle\cong Q\big(\chi_J^s,J-1\big)$ has socle $\sigma_\emptyset$, and identifies $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)v_J$ with the element ϕ in §4.3. Since $J\neq\emptyset$, we deduce from Lemma 4.3.2(iii)(b) applied to $\lambda=\lambda_J$ that the element $B\overset{\mathrm{def}}{=}\big[\prod_{j+1\notin J}Y_j^{p-1-r_j}\big]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_J$ is nonzero and lies in the I-cosocle of $\sigma_\emptyset$, hence $B=\mu_{J,\mathcal{J}}v_{\mathcal{J}}+\mu'_{J,\emptyset}x_{\emptyset,\underline{r}}$ for some $\mu_{J,\mathcal{J}},\mu'_{J,\emptyset}\in\mathbb{F}^\times$ by the previous paragraph. Finally, by applying $\underline{Y}^{\underline{r}}$ to B and recalling that $\underline{Y}^{\underline{r}}v_{\mathcal{J}}=0$ since $v_{\mathcal{J}}$ is I_1-invariant, we deduce from Proposition 4.5.4 (with $J_{\overline{\rho}}=J=\emptyset$) that $\mu'_{J,\emptyset}=\mu_{J,\emptyset}$. □

Example 4.5.14. *Some examples for $f = 3$ and $J_{\overline{\rho}} = \emptyset$:*

$$\begin{aligned}
Y_0^{r_0} Y_1^{p-1} Y_2^{p-1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\{1\}} &= \mu_{\{1\},\emptyset} v_\emptyset; \\
Y_0^{r_0+1} Y_1^{p-1} Y_2^{p-2-r_2} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\{1\}} &= \mu_{\{1\},\{0\}} v_{\{0\}}; \\
Y_0^{r_0+1} Y_1^{p-2-r_1} Y_2^{p-1-r_2} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\{1\}} &= \mu_{\{1\},\{0,2\}} v_{\{0,2\}}; \\
Y_1^{p-1-r_1} Y_2^{p-1-r_2} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\{1\}} &= \mu_{\{1\},\{0,1,2\}} v_{\{0,1,2\}} + \mu_{\{1\},\emptyset} x_{\emptyset,(r_0,r_1,r_2)}.
\end{aligned}$$

$$\begin{aligned}
Y_0^{r_0} Y_1^{p-1} Y_2^{r_2} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\{0,1\}} &= \mu_{\{0,1\},\emptyset} v_\emptyset; \\
Y_0^{r_0+1} Y_1^{p-1} Y_2^{p-1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(Y_0^{-1} v_{\{0,1\}}\right) &= \mu_{\{0,1\},\{0\}} v_{\{0\}}; \\
Y_0^{r_0} Y_1^{p-2-r_1} Y_2^{r_2+1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\{0,1\}} &= \mu_{\{0,1\},\{2\}} v_{\{2\}}; \\
Y_0^{p-1} Y_1^{p-1-r_1} Y_2^{r_2+1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(Y_1^{-1} v_{\{0,1\}}\right) &= \mu_{\{0,1\},\{1,2\}} v_{\{1,2\}}; \\
Y_1^{p-1-r_1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\{0,1\}} &= \mu_{\{0,1\},\{0,1,2\}} v_{\{0,1,2\}} + \mu_{\{0,1\},\emptyset} x_{\emptyset,(r_0,r_1,r_2)}.
\end{aligned}$$

Lemma 4.5.15. *Let $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$ such that the pairs $(J_1, J_3), (J_1, J_4), (J_2, J_3), (J_2, J_4)$ satisfy the assumptions of either Proposition 4.5.11 or Proposition 4.5.13 (here we say that (J, J') satisfies the assumption of Proposition 4.5.13 if $J_{\overline{\rho}} = \emptyset$, $J \neq \emptyset$ and $J' = \mathcal{J}$). Then we have*

$$\frac{\mu_{J_1,J_3}}{\mu_{J_1,J_4}} = \frac{\mu_{J_2,J_3}}{\mu_{J_2,J_4}}, \tag{4.46}$$

where each term of (4.46) is defined in either Proposition 4.5.11 or Proposition 4.5.13.

Proof. First we suppose that $J_{\overline{\rho}} = \emptyset$ and $J_4 = \mathcal{J}$. If $J_3 = \mathcal{J}$, then (4.46) is clear. If $J_3 \neq \mathcal{J}$, then by the proof of Proposition 4.5.13 and using that the I-cosocle of $\sigma_\emptyset$ has dimension 1 over $\mathbb{F}$, the ratio $\mu_{J,\mathcal{J}}/\mu_{J,\emptyset}$ does not depend on J, hence we can replace $J_4 = \mathcal{J}$ by $J_4 = \emptyset$.

From now on, we assume that all the pairs $(J_1, J_3), (J_1, J_4), (J_2, J_3), (J_2, J_4)$ satisfy the assumptions of Proposition 4.5.11. In particular, we have $(J_1 - 1)^{\mathrm{ss}} = (J_2 - 1)^{\mathrm{ss}} = J_3^{\mathrm{ss}} = J_4^{\mathrm{ss}}$. Using that

$$\frac{\mu_{J_i,J_3}}{\mu_{J_i,J_4}} = \frac{\mu_{J_1,J_3}}{\mu_{J_i,J_4^{\mathrm{ss}}}} \left(\frac{\mu_{J_i,J_4}}{\mu_{J_i,J_4^{\mathrm{ss}}}}\right)^{-1}$$

for $i = 1, 2$ with each term defined in Proposition 4.5.11, we may assume that $J_4 = J_4^{\mathrm{ss}} \subseteq J_{\overline{\rho}}$.

Then using Remark 4.5.12, the assumption (4.40) for the pairs $(J_1, J_3), (J_1, J_4), (J_2, J_3), (J_2, J_4)$ is equivalent to

$$(\partial J_3)^{\mathrm{nss}} \sqcup J_4 \subseteq J_i - 1 \subseteq \left((J_3 \setminus \partial J_3)^c\right)^{\mathrm{nss}} \sqcup J_4 \tag{4.47}$$

for $i = 1, 2$. By choosing a sequence $J^0, J^1, \ldots, J^r \subseteq \mathcal{J}$ for some $r \geq 0$ such that $J^0 = J_1$, $J^r = J_2$, $|J_i \Delta J_{i-1}| = 1$ for $1 \leq i \leq r$ and

$$(\partial J_3)^{\mathrm{nss}} \sqcup J_4 \subseteq J^i - 1 \subseteq \left((J_3 \setminus \partial J_3)^c\right)^{\mathrm{nss}} \sqcup J_4$$

for $0 \leq i \leq r$, it suffices to prove the proposition with J_1, J_2 as in (4.47) such that $(J_1 - 1) = (J_2 - 1) \sqcup \{j_0 + 1\}$ for some $j_0 \in \mathcal{J}$. In particular, we have $j_0 + 1 \in (J_1 - 1)^{\mathrm{nss}}$ (since $(J_1 - 1)^{\mathrm{ss}} = (J_2 - 1)^{\mathrm{ss}} = J_4$) and $j_0 + 1 \notin J_3$.

(i). Suppose that $j_0 \in J_1 - 1$. By Proposition 4.5.11 applied to (J_1, J_3), we have

$$\left[\prod_{j+1 \in J_1 \Delta J_3} Y_j^{s_j^{J_3}} \prod_{j+1 \notin J_1 \Delta J_3} Y_j^{p-1}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{e}^{(J_1 \cap J_3)^{\mathrm{nss}}}} v_{J_1}\right) = \mu_{J_1,J_3} v_{J_3}. \tag{4.48}$$

Since $j_0+1 \in J_1$ and $j_0+1 \notin (J_1 \cap J_3)^{\mathrm{nss}}$ (since $j_0+1 \notin J_3$), by Proposition 4.5.9 applied to $(\underline{i}, J, J') = \bigl(\underline{e}^{(J_1 \cap J_3)^{\mathrm{nss}}}, J_1, \mathcal{J}\bigr)$ with j_0 as above together with (4.33) (and note that $(J_1-1)^{\mathrm{ss}} = J_4$), we have

$$\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{e}^{(J_1 \cap J_3)^{\mathrm{nss}}}} v_{J_1}\right) = \frac{\mu_{J_1,J_4}}{\mu_{J_2,J_4}} Y_{j_0+1}^{p-1-r_{j_0+1}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}'} v_{J_2}\right) \tag{4.49}$$

with

$$\begin{aligned} \underline{i}' &\overset{\mathrm{def}}{=} \underline{e}^{(J_1 \cap J_3)^{\mathrm{nss}}} + \delta_{j_0+2 \in J_4} e_{j_0+2} \\ &= \bigl(\underline{e}^{(J_2 \cap J_3)^{\mathrm{nss}}} + \delta_{j_0+2 \in J_3^{\mathrm{nss}}} e_{j_0+2}\bigr) + \delta_{j_0+2 \in J_3^{\mathrm{ss}}} e_{j_0+2} = \underline{e}^{(J_2 \cap J_3)^{\mathrm{nss}}} + \delta_{j_0+2 \in J_3} e_{j_0+2}, \end{aligned} \tag{4.50}$$

where the second equality uses $J_1 \setminus J_2 = \{j_0+2\}$ and $J_3^{\mathrm{ss}} = J_4$. We assume that $j_0+2 \in J_3$, the case $j_0+2 \notin J_3$ being similar. Since $j_0+1 \notin J_3$, we have $s_{j_0+1}^{J_3} = p-2-r_{j_0+1}$ by (4.7). Combining (4.48) and (4.49), we have

$$\begin{aligned} \mu_{J_1,J_3} v_{J_3} &= \frac{\mu_{J_1,J_4}}{\mu_{J_2,J_4}} \left[\prod_{j+1 \in J_1 \Delta J_3} Y_j^{s_j^{J_3}} \prod_{j+1 \notin J_1 \Delta J_3} Y_j^{p-1}\right] Y_{j_0+1}^{p-1-r_{j_0+1}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}'} v_{J_2}\right) \\ &= \frac{\mu_{J_1,J_4}}{\mu_{J_2,J_4}} \left[\prod_{j+1 \in J_2 \Delta J_3} Y_j^{s_j^{J_3}} \prod_{j+1 \notin J_2 \Delta J_3} Y_j^{p-1}\right] Y_{j_0+1}^{r_{j_0+1}} Y_{j_0+1}^{p-1-r_{j_0+1}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}'} v_{J_2}\right) \\ &= \frac{\mu_{J_1,J_4}}{\mu_{J_2,J_4}} \left[\prod_{j+1 \in J_2 \Delta J_3} Y_j^{s_j^{J_3}} \prod_{j+1 \notin J_2 \Delta J_3} Y_j^{p-1}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{e}^{(J_2 \cap J_3)^{\mathrm{nss}}}} v_{J_2}\right) \\ &= \frac{\mu_{J_1,J_4}}{\mu_{J_2,J_4}} \mu_{J_2,J_3} v_{J_3}, \end{aligned}$$

where the second equality uses $j_0+2 \in J_1$, $j_0+2 \notin J_2$, $j_0+2 \in J_3$ (hence $j_0+2 \notin J_1 \Delta J_3$ and $j_0+2 \in J_2 \Delta J_3$) and $s_{j_0+1}^{J_3} = p-2-r_{j_0+1}$, the third equality follows from Lemma 4.3.1(i) applied to $j = j_0+1$ and (4.50) using $j_0+2 \in J_3$, and the last equality follows from Proposition 4.5.11 applied to (J_2, J_3). Therefore, we have $\mu_{J_1,J_3} = (\mu_{J_1,J_4}/\mu_{J_2,J_4})\mu_{J_2,J_3}$, which completes the proof.

(ii). Suppose that $j_0 \notin J_1 - 1$ (which implies $f \geq 2$). Similar to (i), by Proposition 4.5.9 applied to $(\underline{i}, J, J') = \bigl(\underline{e}^{(J_1 \cap J_3)^{\mathrm{nss}}}, J_1, \mathcal{J} \setminus \{j_0\}\bigr)$ with j_0 as above together with (4.33), we have

$$\begin{aligned} &\left[Y_{j_0+1} Y_{j_0}^{2\delta_{j_0 \in (J_1 \cap J_3)^{\mathrm{nss}}} + p - s_{j_0}^{J_1}}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}} v_J\right) \\ &\qquad = \frac{\mu_{J_1,J_4}}{\mu_{J_2,J_4}} \left[Y_{j_0+1} Y_{j_0}^{2\delta_{j_0 \in (J_1 \cap J_3)^{\mathrm{nss}}} + p - s_{j_0}^{J_1}}\right] Y_{j_0+1}^{p-1-r_{j_0+1}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}'} v_{J \setminus \{j_0+2\}}\right), \end{aligned}$$

where $\underline{i}' = \underline{e}^{(J_2 \cap J_3)^{\mathrm{nss}}} + \delta_{j_0+2 \in J_3} e_{j_0+2}$. We claim that $\bigl[\prod_{j+1 \in J_1 \Delta J_3} Y_j^{s_j^{J_3}} \prod_{j+1 \notin J_1 \Delta J_3} Y_j^{p-1}\bigr]$ is a multiple of $Y_{j_0+1} Y_{j_0}^{2\delta_{j_0 \in (J_1 \cap J_3)^{\mathrm{nss}}} + p - s_{j_0}^{J_1}}$. Indeed, since $s_{j_0+1}^{J_3} \geq 1$ and $2\delta_{j_0 \in (J_1 \cap J_3)^{\mathrm{nss}}} + p - s_{j_0}^{J_1} \leq p-1$ by (4.10), the claim follows from the fact that $j_0+1 \notin J_1$, $j_0+1 \notin J_3$ (hence $j_0+1 \notin J_1 \Delta J_3$) and $j_0+1 \neq j_0$. Once we have the claim, we can argue exactly as in (i) to conclude the proof. □

To end this section, we extend the definition of $\mu_{J,J'}$ to all $J, J' \subseteq \mathcal{J}$ such that $(J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}}$ as follows:

$$\begin{cases} \mu_{J,J'} \overset{\mathrm{def}}{=} \dfrac{\mu_{((J')^{\mathrm{ss}} \sqcup (\partial J')^{\mathrm{nss}})+1, J'} \mu_{J,(J')^{\mathrm{ss}}}}{\mu_{((J')^{\mathrm{ss}} \sqcup (\partial J')^{\mathrm{nss}})+1,(J')^{\mathrm{ss}}}} & \text{if } (J_{\overline{\rho}}, J') \neq (\emptyset, \mathcal{J}) \\ \mu_{\emptyset,\mathcal{J}} \overset{\mathrm{def}}{=} \dfrac{\mu_{\emptyset,\emptyset} \mu_{\mathcal{J},\mathcal{J}}}{\mu_{J,\emptyset}} & \text{if } J_{\overline{\rho}} = \emptyset \end{cases} \tag{4.51}$$

(and $\mu_{J,\mathcal{J}}$ as in Proposition 4.5.13 if $J_{\overline{\rho}} = \emptyset$ and $J \neq \emptyset$), where each term on the RHS of (4.51) are defined in either Proposition 4.5.11 or Proposition 4.5.13. Then the equation (4.46) holds for arbitrary $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$ such that $(J_1 - 1)^{\mathrm{ss}} = (J_2 - 1)^{\mathrm{ss}} = J_3^{\mathrm{ss}} = J_4^{\mathrm{ss}}$. In particular, for J, J' such that $(J-1)^{\mathrm{ss}} = (J'-1)^{\mathrm{ss}}$, the quantity $\mu_{J,J''}/\mu_{J',J''}$ does not depend on J'' for $(J'')^{\mathrm{ss}} = (J-1)^{\mathrm{ss}}$ and we denote it by $\mu_{J,*}/\mu_{J',*}$. Similarly, for J, J' such that $J^{\mathrm{ss}} = (J')^{\mathrm{ss}}$, the quantity $\mu_{J'',J}/\mu_{J'',J'}$ does not depend on J'' for $(J''-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$ and we denote it by $\mu_{*,J}/\mu_{*,J'}$.

4.6 Projective systems in π

In this section, we define certain projective systems $x_{J,\underline{i}}$ of elements of π indexed by $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$, see Theorem 4.6.4. They will give rise to a basis of the A-module $D_A(\pi)$. The definition of these elements is much more involved than in the semisimple case (compared with [BHH$^+$c, (104)]).

The following quantities will appear in the definition of these elements in Theorem 4.6.4.

Definition 4.6.1. *Let $J \subseteq \mathcal{J}$.*

(i) *We define $\underline{r}^J \in \mathbb{Z}^f$ by*

$$r_j^J \overset{\text{def}}{=} \begin{cases} 0 & \text{if } j \notin J,\ j+1 \notin J \\ -1 & \text{if } j \in J,\ j+1 \notin J \\ r_j + 1 & \text{if } j \notin J,\ j+1 \in J \\ r_j & \text{if } j \in J,\ j+1 \in J. \end{cases} \tag{4.52}$$

(ii) *We define $\underline{c}^J \in \mathbb{Z}^f$ by*

$$c_j^J \overset{\text{def}}{=} \begin{cases} p-1 & \text{if } j \notin J,\ j+1 \notin J \\ r_j + 1 & \text{if } j \in J,\ j+1 \notin J \\ p-2-r_j & \text{if } j \notin J,\ j+1 \in J \\ 0 & \text{if } j \in J,\ j+1 \in J. \end{cases} \tag{4.53}$$

(iii) *We define $\varepsilon_J \in \{\pm 1\}$ by (see Remark 4.5.12 for ∂J)*

$$\varepsilon_J \overset{\text{def}}{=} \begin{cases} (-1)^{f-1} & \text{if } J_{\overline{\rho}} = \emptyset,\ J = \mathcal{J} \\ (-1)^{|(J \setminus \partial J)^{\mathrm{nss}}|} & \text{otherwise.} \end{cases} \tag{4.54}$$

Remark 4.6.2. (i) *By definition, for all $J \subseteq \mathcal{J}$ and $j \in \mathcal{J}$ we have*

$$r_j^J = \delta_{j+1 \in J}(r_j + 1) - \delta_{j \in J}; \tag{4.55}$$

$$c_j^J = \delta_{j \notin J}(p - 2 - r_j) + \delta_{j+1 \notin J}(r_j + 1). \tag{4.56}$$

(ii) *The definition of $\underline{c}^J$ is a variant of [BHH$^+$c, (95)] (where $\overline{\rho}$ was assumed to be semisimple). Also, by (4.2) we have $\underline{0} \leq \underline{c}^J \leq \underline{p} - \underline{1}$.*

Lemma 4.6.3. (i) *For $J \subseteq \mathcal{J}$, we have $\underline{t}^J = \underline{r}^J + \underline{e}^{J^{\mathrm{sh}}}$ (see (4.8) for $\underline{t}^J$ and (4.9) for J^{sh}).*

(ii) *For $J_1, J_2 \subseteq \mathcal{J}$ such that $J_1 \cap J_2 = \emptyset$, we have $\underline{r}^{J_1 \cup J_2} = \underline{r}^{J_1} + \underline{r}^{J_2}$.*

(iii) *For $J \subseteq \mathcal{J}$, we have $\alpha^{\underline{c}^J} = \alpha^{\underline{r}^{J+1} - \underline{r}^J}$.*

(iv) *Let $J' \subseteq J \subseteq \mathcal{J}$ and $J'' \overset{\text{def}}{=} J'\Delta(J-1)$. Let $\underline{c} \overset{\text{def}}{=} p\underline{e}^{J' \cap (J-1)} + \underline{c}^{J'} - \underline{f} - \underline{r}^{J \setminus J'}$. Let $\underline{\delta} \in \{0,1\}^f$. Then for all $j \in \mathcal{J}$ we have (see (4.18) for $\underline{t}^J(J')$)*

$$c_j - \delta_j \geq \delta_{j \notin J''} \left[2\big(\delta_{j \in (J'+1) \cap J} - \delta_{j \in (J'+1)^{\mathrm{sh}}}\big) + t^{J'+1}(J'')_j \right] + \delta_{J''=\emptyset}.$$

(v) *Let $J' \subseteq J \subseteq \mathcal{J}$. Then for all $j \in \mathcal{J}$ we have*

$$c_j^{J'} - r_j^{J \setminus J'} = c_j^J + \delta_{j \in J \setminus J'}(p - 1 - r_j).$$

Proof. (i). This follows directly from the definition.

(ii). This follows immediately from (4.55).

(iii). By (4.55) and (4.56) we have

$$\begin{aligned} c_j^J + r_j^J - r_j^{J+1} &= \big(\delta_{j \notin J}(p - 1 - r_j) + \delta_{j+1 \notin J}(r_j + 1) - \delta_{j \notin J}\big) \\ &\qquad + \big(\delta_{j+1 \in J}(r_j + 1) - \delta_{j \in J}\big) - \big(\delta_{j \in J}(r_j + 1) - \delta_{j-1 \in J}\big) \\ &= p\delta_{j \notin J} - \big(\delta_{j \notin J} + \delta_{j \in J}\big)(r_j + 1) \\ &\qquad + \big(\delta_{j+1 \notin J} + \delta_{j+1 \in J}\big)(r_j + 1) - \big(\delta_{j \notin J} + \delta_{j \in J}\big) + \delta_{j-1 \in J} \\ &= p\delta_{j \notin J} - 1 + \delta_{j-1 \in J} = p\delta_{j \notin J} - \delta_{j-1 \notin J}. \end{aligned}$$

Hence we have

$$\alpha^{\underline{c}^J + \underline{r}^J - \underline{r}^{J+1}} = \prod_{j \notin J} \alpha_j^p \prod_{j-1 \notin J} \alpha_j^{-1} = \prod_{j \notin J} \alpha_{j+1} \prod_{j-1 \notin J} \alpha_j^{-1} = 1,$$

which proves (iii).

(iv). We assume that $j \notin J''$, the case $j \in J''$ being similar and simpler. By definition we have

$$\begin{aligned} &2\big(\delta_{j \in (J'+1) \cap J} - \delta_{j \in (J'+1)^{\mathrm{sh}}}\big) + \big(t^{J'+1}(J'')_j + \delta_{J''=\emptyset}\big) + \delta_j \\ &\qquad \le 2\big(\delta_{j \in (J'+1) \cap J} - \delta_{j \in (J'+1)^{\mathrm{sh}}}\big) + \big(p - 1 - s_j^{J'+1} + 1\big) + 1 \\ &\qquad \le 2\big(\delta_{j \in (J'+1) \cap J} - \delta_{j \in (J'+1)^{\mathrm{sh}}}\big) + p - 1 - \big(2(f - \delta_{j \in (J'+1)^{\mathrm{sh}}}) + 1 + \delta_{f=1}\big) + 1 + 1 \\ &\qquad = p - 2f + 2\delta_{j \in (J'+1) \cap J} - \delta_{f=1} \le p - 2f + 2 - \delta_{f=1} \le p - f, \end{aligned}$$

where the second inequality follows from (4.10). Since $j \notin J''$, we have either $j \in J' \cap (J - 1)$, or $j \notin J'$ and $j \notin J - 1$. We give the proof when $j \in J' \cap (J - 1)$, the other case being similar. By the definition of $\underline{c}$, (4.53) and (4.52) we have

$$\begin{aligned} c_j &= \begin{cases} p + 0 - f - 0 & \text{if } j + 1 \in J' \\ p + (r_j + 1) - f - (r_j + 1) & \text{if } j + 1 \notin J' \end{cases} \\ &= p - f, \end{aligned}$$

which proves (iv).

(v). By (4.55) and (4.56) we have

$$\begin{aligned} c_j^{J'} - r_j^{J \setminus J'} - c_j^J &= \big(\delta_{j \notin J'}(p - 1 - r_j) + \delta_{j+1 \notin J'}(r_j + 1) - \delta_{j \notin J'}\big) - \big(\delta_{j+1 \in J \setminus J'}(r_j + 1) - \delta_{j \in J \setminus J'}\big) \\ &\qquad - \big(\delta_{j \notin J}(p - 1 - r_j) + \delta_{j+1 \notin J}(r_j + 1) - \delta_{j \notin J}\big) \\ &= \big(\delta_{j \notin J'} - \delta_{j \notin J}\big)(p - 1 - r_j) + \big(\delta_{j+1 \notin J'} - \delta_{j+1 \in J \setminus J'} - \delta_{j+1 \notin J}\big)(r_j + 1) \\ &\qquad - \big(\delta_{j \notin J'} - \delta_{j \in J \setminus J'} - \delta_{j \notin J}\big) \\ &= \delta_{j \in J \setminus J'}(p - 1 - r_j). \end{aligned}$$

This proves (v). □

Theorem 4.6.4. *There exists a unique family of elements $\big\{x_{J,\underline{i}} : J \subseteq \mathcal{J}, \underline{i} \in \mathbb{Z}^f\big\}$ of π satisfying the following properties:*

(i) *For each $J \subseteq \mathcal{J}$, we have $x_{J,\underline{f}} = \underline{Y}^{-(\underline{f}-\underline{e}^{J^{\mathrm{sh}}})} v_J$ (defined in Proposition 4.4.2).*

(ii) *For each $J \subseteq \mathcal{J}$, $\underline{i} \in \mathbb{Z}^f$ and $\underline{k} \in \mathbb{Z}^f_{\geq 0}$, we have $\underline{Y}^{\underline{k}} x_{J,\underline{i}} = x_{J,\underline{i}-\underline{k}}$.*

(iii) *For each $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$, we have (see Proposition 4.5.11 and (4.51) for $\mu_{J+1,J'}$)*

$$\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}} = \sum_{J^{\mathrm{ss}} \subseteq J' \subseteq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',p\delta(\underline{i})+\underline{c}^J+\underline{r}^{J\setminus J'}}.$$

Remark 4.6.5. *The extra term $\underline{e}^{J^{\mathrm{sh}}}$ in 4.6.4(i) has the advantage that the constants $\underline{c}^J$ and $\underline{r}^J$ in Theorem 4.6.4(iii) work for arbitrary $J_{\overline{\rho}}$.*

Remark 4.6.6. *Let $0 \leq k \leq f$. Assume that Theorem 4.6.4 is true for $|J| \leq k$. Then by Theorem 4.6.4(ii),(iii), for all $|J| \leq k$, $\underline{i} \in \mathbb{Z}^f$ and $\underline{\ell} \in \mathbb{Z}^f_{\geq 0}$ we have*

$$\underline{Y}^{\underline{\ell}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}} = \sum_{J^{\mathrm{ss}} \subseteq J' \subseteq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',p\delta(\underline{i})+\underline{c}^J+\underline{r}^{J\setminus J'}-\underline{\ell}}. \tag{4.57}$$

Moreover, the LHS and each term of the summation in (4.57) are H-eigenvectors with common H-eigencharacter $\chi_J \alpha^{\underline{e}^{J^{\mathrm{sh}}}+\underline{\ell}-\underline{i}}$, see the proof of Corollary 4.6.10 below.

Example 4.6.7. *Some examples for $f = 3$ and $J_{\overline{\rho}} = \emptyset$:*

$$\begin{aligned}
\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{\emptyset,(i_0,i_1,i_2)} &= \mu_{\emptyset,\emptyset} x_{\emptyset,(pi_1+p-1,pi_2+p-1,pi_0+p-1)};\\
\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{\{1\},(i_0,i_1,i_2)} &= \mu_{\{1\},\{0\}} x_{\{0\},(pi_1+r_0+1,pi_2+p-1,pi_0+p-2-r_2)}\\
&\quad + \mu_{\{1\},\emptyset} x_{\emptyset,(pi_1+r_0,pi_2+p-1,pi_0+p-1)};\\
\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{\{1,2\},(i_0,i_1,i_2)} &= -\mu_{\{1,2\},\{0,1\}} x_{\{0,1\},(pi_1,pi_2+r_1+1,pi_0+p-2-r_2)}\\
&\quad + \mu_{\{1,2\},\{1\}} x_{\{1\},(pi_1-1,pi_2+r_1+1,pi_0+p-1)}\\
&\quad + \mu_{\{1,2\},\{0\}} x_{\{0\},(pi_1+r_0+1,pi_2+r_1,pi_0+p-2-r_2)}\\
&\quad + \mu_{\{1,2\},\emptyset} x_{\emptyset,(pi_1+r_0,pi_2+r_1,pi_0+p-1)}.
\end{aligned}$$

The proof of Theorem 4.6.4 will occupy the current and the next sections.

Proof of Theorem 4.6.4. We define the elements $x_{J,\underline{i}} \in \pi$ by increasing induction on $|J|$ and on $\max_j i_j$. For each $J \subseteq \mathcal{J}$, we let $x_{J,\underline{f}} \overset{\text{def}}{=} \underline{Y}^{-(\underline{f}-\underline{e}^{J^{\mathrm{sh}}})} v_J$, which is defined in Proposition 4.4.2. Then for each $\underline{i} \leq \underline{f}$, we define $x_{J,\underline{i}} \overset{\text{def}}{=} \underline{Y}^{\underline{f}-\underline{i}} x_{J,\underline{f}}$. By Proposition 4.4.2, for $\underline{i} \leq \underline{f}$ we have

$$x_{J,\underline{i}} = \begin{cases} \underline{Y}^{-(\underline{i}-\underline{e}^{J^{\mathrm{sh}}})} v_J & \text{if } \underline{i} \geq \underline{e}^{J^{\mathrm{sh}}} \\ 0 & \text{otherwise.} \end{cases} \tag{4.58}$$

Then we let $|J| = k$ for $0 \leq k \leq f$ and $\max_j i_j = m > f$. Assume that $x_{J,\underline{i}}$ is defined for $|J| \leq k-1$ and all $\underline{i} \in \mathbb{Z}^f$, and for $|J| = k$ and $\max_j i_j \leq m-1$. We write $\underline{i} = p\delta(\underline{i}') + \underline{c}^J - \underline{\ell}$ for the unique $\underline{i}', \underline{\ell} \in \mathbb{Z}^f$ such that $\underline{0} \leq \underline{\ell} \leq \underline{p} - \underline{1}$. Then we claim that $\max_j i'_j < \max_j i_j = m$. Indeed, for each j we have

$$i'_{j+1} = \left(i_j - c^J_j + \ell_j\right)/p \leq \left(m - 0 + (p-1)\right)/p < m/p + 1 < m, \tag{4.59}$$

where the last inequality uses $m > f \geq 1$. Then we define $x_{J,\underline{i}}$ by the formula

$$\varepsilon_J \mu_{J+1,J} x_{J,\underline{i}} \overset{\text{def}}{=} \underline{Y}^{\underline{\ell}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}'} - \sum_{J^{\mathrm{ss}} \subseteq J' \subsetneq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',\underline{i}+\underline{r}^{J\setminus J'}}, \tag{4.60}$$

where each term on the RHS of (4.60) is defined by the induction hypothesis (hence a priori (4.60) holds for all $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\max_j i_j > f$).

Lemma 4.6.8. *Let $0 \le k \le f$. Assume that Theorem 4.6.4 is true for $|J| \le k-1$. If (4.60) holds for $|J| = k$ and $\underline{i} = \underline{f}$, then Theorem 4.6.4 is true for $|J| = k$.*

Proof. By (4.58), Theorem 4.6.4(ii) is true for all $J \subseteq \mathcal{J}$ and $\underline{i} \le \underline{f}$. Then we let $J \subseteq \mathcal{J}$ such that $|J| = k$. We define $\underline{c}'^J \in \mathbb{Z}^f$ by

$$c'^J_j \overset{\text{def}}{=} \begin{cases} p-1-f & \text{if } j \notin J,\ j+1 \notin J \\ r_j+1-f & \text{if } j \in J,\ j+1 \notin J \\ p-2-r_j-f & \text{if } j \notin J,\ j+1 \in J \\ p-f & \text{if } j \in J,\ j+1 \in J. \end{cases} \tag{4.61}$$

In particular, by (4.2) we have $\underline{0} \le \underline{c}'^J \le \underline{p}-\underline{1}$, and by (4.61) and (4.53) we have

$$\underline{f} = p\delta\big(\underline{e}^{J\cap(J+1)}\big) + \underline{c}^J - \underline{c}'^J. \tag{4.62}$$

Since (4.60) holds for J as above and $\underline{i} = \underline{f}$ by assumption, using (4.62) we have

$$\varepsilon_J \mu_{J+1,J} x_{J,\underline{f}} = \underline{Y}^{\underline{c}'^J} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{e}^{J\cap(J+1)}} - \sum_{J^{\mathrm{ss}} \subseteq J' \subsetneq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',\underline{f}+\underline{r}^{J\setminus J'}}. \tag{4.63}$$

For each $\underline{i} \le \underline{f}$, we write $\underline{i} = p\delta(\underline{i}') + \underline{c}^J - \underline{\ell}$ as in (4.60). In particular, comparing with (4.62) we have $\underline{i}' \le \underline{e}^{J\cap(J+1)}$. By Lemma 4.3.1(i) and Theorem 4.6.4(ii) (applied with $(J, \underline{i}, \underline{k})$ there replaced by $\big(J+1, \underline{e}^{J\cap(J+1)}, \underline{e}^{J\cap(J+1)} - \underline{i}'\big)$ and using $\underline{e}^{J\cap(J+1)} \le \underline{f}$) we deduce that

$$\underline{Y}^{\underline{f}-\underline{i}} \left[\underline{Y}^{\underline{c}'^J} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{e}^{J\cap(J+1)}}\right] = \underline{Y}^{\underline{\ell}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}'}. \tag{4.64}$$

Since Theorem 4.6.4(ii) is true for $|J| \le k-1$, by applying $\underline{Y}^{\underline{f}-\underline{i}}$ to (4.63) and using (4.64) we deduce that (4.60) is true for J as above and $\underline{i} \le \underline{f}$, hence for all $\underline{i} \in \mathbb{Z}^f$ by definition.

Then we use increasing induction on $\max_j i_j$ to prove that Theorem 4.6.4(ii) is true (for J as above, which satisfies $|J| = k$). We already know that Theorem 4.6.4(ii) is true for $\underline{i} \le \underline{f}$. Then for each $\underline{i} \in \mathbb{Z}^f$ and $\underline{k} \in \mathbb{Z}^f_{\ge 0}$, if we write $\underline{i} - \underline{k} = p\delta(\underline{i}'') + \underline{c}^J - \underline{\ell}'$ for the unique $\underline{i}'', \underline{\ell}' \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{\ell}' \le \underline{p}-\underline{1}$, then we have

$$\begin{aligned} \underline{Y}^{\underline{k}} \varepsilon_J \mu_{J+1,J} x_{J,\underline{i}} &= \underline{Y}^{\underline{k}} \left[\underline{Y}^{\underline{\ell}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}'} - \sum_{J^{\mathrm{ss}} \subseteq J' \subsetneq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',\underline{i}+\underline{r}^{J\setminus J'}}\right] \\ &= \underline{Y}^{\underline{\ell}'} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}''} - \sum_{J^{\mathrm{ss}} \subseteq J' \subsetneq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',\underline{i}-\underline{k}+\underline{r}^{J\setminus J'}} \\ &= \varepsilon_J \mu_{J+1,J} x_{J,\underline{i}-\underline{k}}, \end{aligned}$$

where the first and the third equality follow from (4.60), and the second equality follows in a similar way as (4.64) using the induction hypothesis. This shows that $\underline{Y}^{\underline{k}} x_{J,\underline{i}} = x_{J,\underline{i}-\underline{k}}$.

Finally, by taking $\underline{i}$ such that $\underline{\ell} = \underline{0}$ in (4.60), we conclude that Theorem 4.6.4(iii) is true (for J as above). □

Assume that Theorem 4.6.4 is true for $|J| \le k-1$. By Lemma 4.6.8, it suffices to prove that (4.60) is true for $|J| = k$ and $\underline{i} = \underline{f}$. For $J \subseteq \mathcal{J}$, we denote (see (4.61) for $\underline{c}'^J$)

$$\begin{aligned} z_J &\overset{\text{def}}{=} \underline{Y}^{\underline{c}'^J} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{e}^{J\cap(J+1)}} = \underline{Y}^{\underline{c}'^J} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{e}^{(J\cap(J+1))^{\mathrm{nss}}}} v_{J+1}\right) \in \pi; \\ w_J &\overset{\text{def}}{=} \sum_{J^{\mathrm{ss}} \subseteq J' \subsetneq J} \varepsilon_{J'} \mu_{J+1,J'} x_{J',\underline{f}+\underline{r}^{J\setminus J'}} \in \pi. \end{aligned} \tag{4.65}$$

Then by (4.63), it is equivalent to proving that for $|J| = k$ we have

$$z_J - w_J = \varepsilon_J \mu_{J+1,J} x_{J,\underline{f}} = \varepsilon_J \mu_{J+1,J} \underline{Y}^{-(\underline{f}-\underline{e}^{J^{\mathrm{sh}}})} v_J. \tag{4.66}$$

Lemma 4.6.9. *Let $0 \leq k \leq f$. Assume that Theorem 4.6.4 is true for $|J| \leq k-1$. Then for $|J| = k$, we have $z_J, w_J \in D_0(\overline{\rho})$.*

Proof. (i). First we prove that $z_J \in D_0(\overline{\rho})$. Let $\underline{i} \overset{\mathrm{def}}{=} \underline{e}^{(J\cap(J+1))^{\mathrm{nss}}}$, $J_1'' \overset{\mathrm{def}}{=} J\Delta(J-1)$ and $\underline{m} \overset{\mathrm{def}}{=} \underline{m}(\underline{i}, J+1, J_1'')$ (see (4.31)). By Lemma 4.5.7 applied with (J, J') there replaced by $(J+1, J)$, we have $m_j = \delta_{j\in J}(-1)^{\delta_{j\notin J}} = \delta_{j\in J}$ for all $j \in \mathcal{J}$. Then by Proposition 4.5.8(ii) applied to $(\underline{i}, J+1, J_1'')$, we have for all $j' \in \mathcal{J}$

$$Y_{j'}^{\delta_{J_1''=\emptyset}} \left[\prod_{j\notin J_1''} Y_j^{2i_j + t^{J+1}(J_1'')_j}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}} v_{J+1}\right) \in I\left(\sigma_{J^{\mathrm{ss}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}}}\right) = I\left(\sigma_{J^{\mathrm{ss}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}} + \underline{e}^{J^{\mathrm{nss}}}}\right) \subseteq D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho}),$$

where the last inclusion follows from Corollary 4.5.3. To prove that $z_J \in D_0(\overline{\rho})$, it suffices to show that for all $j \in \mathcal{J}$ we have

$$c_j^{\prime J} \geq \delta_{j\notin J_1''}\left(2i_j + t^{J+1}(J_1'')_j\right) + \delta_{J_1''=\emptyset}.$$

This is a consequence of Lemma 4.6.3(iv) applied with $J' = J$ and $\underline{\delta} = \underline{0}$, and (4.62).

(ii). Next we prove that $w_J \in D_0(\overline{\rho})$. To do this, we prove by increasing induction on $|J'|$ that $x_{J',\underline{f}+\underline{r}^{J\setminus J'}} \in D_0(\overline{\rho})$ for each J' such that $J^{\mathrm{ss}} \subseteq J' \subsetneqq J$ (which implies $(J')^{\mathrm{ss}} = J^{\mathrm{ss}}$). Let $\underline{i} \overset{\mathrm{def}}{=} \underline{e}^{(J'+1)\cap J} - \underline{e}^{(J'+1)^{\mathrm{sh}}} = \underline{e}^{((J'+1)\cap J)^{\mathrm{nss}}}$ (using $(J')^{\mathrm{ss}} = J^{\mathrm{ss}}$), $J_2'' \overset{\mathrm{def}}{=} J'\Delta(J-1)$ and $\underline{m} \overset{\mathrm{def}}{=} \underline{m}(\underline{i}, J'+1, J_2'')$ (see (4.31)). By Proposition 4.5.8(ii) applied to $(\underline{i}, J'+1, J_2'')$, we have for all $j' \in \mathcal{J}$

$$Y_{j'}^{\delta_{J_2''=\emptyset}} \left[\prod_{j\notin J_2''} Y_j^{2i_j + t^{J'+1}(J_2'')_j}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}} v_{J'+1}\right) \in I\left(\sigma_{(J')^{\mathrm{ss}}}, \sigma_{\underline{m}}\right). \tag{4.67}$$

By Lemma 4.5.7 applied with (J, J') there replaced by $(J'+1, J)$, we have $m_j = \delta_{j\in J}(-1)^{\delta_{j\notin J'}}$ for all $j \in \mathcal{J}$. Then a case-by-case examination using $J^{\mathrm{ss}} \subseteq J' \subseteq J$ shows that $m_j - \delta_{j\in(J')^{\mathrm{ss}}}$ equals 0 if $j \in J_{\overline{\rho}}$ and equals $\delta_{j\in J}(-1)^{\delta_{j\notin J'}}$ if $j \notin J_{\overline{\rho}}$. Hence by Corollary 4.5.3 we deduce that $I\left(\sigma_{(J')^{\mathrm{ss}}}, \sigma_{\underline{m}}\right) \subseteq D_{0,\sigma_{(J')^{\mathrm{ss}}}}(\overline{\rho})$.

We let $\underline{c} \in \mathbb{Z}^f$ be as in Lemma 4.6.3(iv). On one hand, since $\underline{Y}^{-\underline{i}} v_{J'+1} = x_{J'+1,\underline{e}^{(J'+1)\cap J}}$ by (4.58), multiplying (4.67) by a suitable power of $\underline{Y}$ and using Lemma 4.6.3(iv) (with $\underline{\delta} = \underline{0}$) we deduce that

$$\underline{Y}^{\underline{c}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J'+1,\underline{e}^{(J'+1)\cap J}} \in D_0(\overline{\rho}).$$

On the other hand, since $|J'| \leq k-1$ by assumption, by (4.57) applied to J' and using $(J')^{\mathrm{ss}} = J^{\mathrm{ss}}$ we have

$$\begin{aligned}
\underline{Y}^{\underline{c}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J'+1,\underline{e}^{(J'+1)\cap J}} &= \sum_{J^{\mathrm{ss}}\subseteq J''\subseteq J'} \varepsilon_{J''}\mu_{J'+1,J''} x_{J'',\left(p\delta(\underline{e}^{(J'+1)\cap J}) + \underline{c}^{J'} + \underline{r}^{J'\setminus J''} - \underline{c}\right)} \\
&= \sum_{J^{\mathrm{ss}}\subseteq J''\subseteq J'} \varepsilon_{J''}\mu_{J'+1,J''} x_{J'',\left(\underline{r}^{J'\setminus J''} + \underline{f} + \underline{r}^{J\setminus J'}\right)} \\
&= \sum_{J^{\mathrm{ss}}\subseteq J''\subseteq J'} \varepsilon_{J''}\mu_{J'+1,J''} x_{J'',\underline{f}+\underline{r}^{J\setminus J''}},
\end{aligned}$$

where the second equality follows from the definition of $\underline{c}$ and the last equality follows from Lemma 4.6.3(ii). By the induction hypothesis, we have $x_{J'',\underline{f}+\underline{r}^{J\setminus J''}} \in D_0(\overline{\rho})$ for all $J^{\mathrm{ss}} \subseteq J'' \subsetneqq J'$. It follows that $x_{J',\underline{f}+\underline{r}^{J\setminus J'}} \in D_0(\overline{\rho})$, which completes the proof. □

By Lemma 4.7.9 and Lemma 4.7.10 below with a case-by-case examination, we have (see (4.54) for ε_J)

$$Y_j^{f+1-\delta_{j\in J^{\mathrm{sh}}}}(z_J - w_J) = 0 \ \forall\, j \in \mathcal{J};$$
$$\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}}(z_J - w_J) = \varepsilon_J v_J.$$

Then (4.66) is a consequence of Lemma 4.6.9 and Proposition 4.4.2. This proves the existence of the family $\{x_{J,\underline{i}} : J \subseteq \mathcal{J}, \underline{i} \in \mathbb{Z}^f\}$. Finally, the uniqueness is clear from the construction. This completes the proof of Theorem 4.6.4 (assuming Lemma 4.7.9 and Lemma 4.7.10). □

Corollary 4.6.10. *Let $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$. Then (assuming Theorem 4.6.4) H acts on $x_{J,\underline{i}}$ (possibly zero) by the character $\chi'_J\alpha^{-\underline{i}}$, where $\chi'_J \overset{\mathrm{def}}{=} \chi_J\alpha^{\underline{e}^{J^{\mathrm{sh}}}}$ (see §4.2 for χ_J).*

Proof. By Lemma 4.2.1 and Lemma 4.6.3(i), for each $J \subseteq \mathcal{J}$ we have

$$\chi'_J\alpha^{\underline{r}^J} = \chi_{(\underline{r},\underline{0})}. \tag{4.68}$$

We prove the result by increasing induction on $|J|$ and $\max_j i_j$. If $\underline{i} \le \underline{f}$, then the claim follows from (4.58) and Proposition 4.4.2. Next we assume that $\max_j i_j > f$ and write $\underline{i} = p\delta(\underline{i}') + \underline{c}^J - \underline{\ell}$ for the unique $\underline{i}', \underline{\ell} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{\ell} \le \underline{p} - \underline{1}$. In particular, we have $\max_j i'_j < \max_j i_j$ (see 4.59). By the induction hypothesis and Lemma 4.3.1(ii), H acts on $\underline{Y}^{\underline{\ell}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}'}$ by the character

$$\chi'_{J+1}\alpha^{-\underline{i}'+\underline{\ell}} = \chi'_{J+1}\alpha^{-p\delta(\underline{i}')+\underline{\ell}} = \chi'_J\alpha^{\underline{r}^J-\underline{r}^{J+1}-p\delta(\underline{i}')+\underline{\ell}} = \chi'_J\alpha^{-\underline{c}^J-p\delta(\underline{i}')+\underline{\ell}} = \chi'_J\alpha^{-\underline{i}},$$

where the second equality follows from (4.68) and the third equality follows from Lemma 4.6.3(iii). By the induction hypothesis, for each J' such that $J^{\mathrm{ss}} \subseteq J' \subsetneqq J$, H acts on $x_{J',\underline{i}+\underline{r}^{J\setminus J'}}$ by the character

$$\chi'_{J'}\alpha^{-\underline{i}-\underline{r}^{J\setminus J'}} = \chi'_J\alpha^{\underline{r}^J-\underline{r}^{J'}-\underline{i}-\underline{r}^{J\setminus J'}} = \chi'_J\alpha^{-\underline{i}},$$

where the first equality follows from (4.68) and the second equality follows from Lemma 4.6.3(ii). Hence we deduce from (4.60) that H acts on $x_{J,\underline{i}}$ by the character $\chi'_J\alpha^{-\underline{i}}$. □

4.7 Some vanishing results

In this section, we prove that certain elements in the projective systems $\{x_{J,\underline{i}} : J \subseteq \mathcal{J}, \underline{i} \in \mathbb{Z}^f\}$ are zero, see Lemma 4.7.1, Proposition 4.7.4 and Corollary 4.7.6. We then use these vanishing results to deduce Lemma 4.7.9 and Lemma 4.7.10, which finish the proof of Theorem 4.6.4.

Lemma 4.7.1. *Let $0 \le k \le f$. Assume that Theorem 4.6.4 is true for $|J| \le k$. Let $J \subseteq \mathcal{J}$ with $|J| \le k$, $j_0 \in \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$. Suppose that $j_0 \notin J$, $j_0 + 1 \notin J$ and $i_{j_0} < 0$. Then we have $x_{J,\underline{i}} = 0$.*

Proof. We prove the result by increasing induction on $|J| \le k$ and on $\max_j i_j$ (the base case being $|J| = -1$, which is automatic). We let $J \subseteq \mathcal{J}$ such that $|J| \le k$ and $\underline{i} \in \mathbb{Z}^f$. If $\underline{i} \le \underline{f}$, then the lemma follows directly from (4.58). If $\max_j i_j > f$, then we write $\underline{i} = p\delta(\underline{i}') + \underline{c}^{J+1} - \underline{\ell}$ for the

unique $\underline{i}', \underline{\ell} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{\ell} \le \underline{p}-\underline{1}$. In particular, we have $\max_j i'_j < \max_j i_j$ (see (4.59)). Since $i_{j_0} < 0$ and $c_{j_0}^{J+1} = p-1$ by (4.53), we have $i'_{j_0+1} < 0$, hence by the induction hypothesis on $\max_j i_j$ we deduce that $x_{J+1,\underline{i}'} = 0$. For each $J' \subsetneqq J$, by (4.52) we have $i_{j_0} + r_{j_0}^{J\setminus J'} = i_{j_0} < 0$, hence by the induction hypothesis on $|J|$ we deduce that $x_{J',\underline{i}+\underline{r}^{J\setminus J'}} = 0$ for $J' \subsetneqq J$. Then by (4.60) we conclude that $x_{J,\underline{i}} = 0$. □

To prove more vanishing results, we need some variants of [BHH$^+$c, Lemma 3.4.2] and [BHH$^+$c, Lemma 3.4.4] (where $\overline{\rho}$ was assumed to be semisimple). Since there could be overlaps between different $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentations $\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle$ of π (see Proposition 4.5.9), we need to be more precise about the region where the elements $x_{J,\underline{i}}$ vanish. This motivates the following somewhat technical definition.

Definition 4.7.2. *Let $J \subseteq \mathcal{J}$.*

(i) *Let $j \in \mathcal{J}$ and $x \in \mathbb{Z}$. We write $x = 2n+\delta$ with $n \in \mathbb{Z}$ and $\delta \in \{0,1\}$. Then we define $t_j^J(x) \overset{\text{def}}{=} np + \delta\left(\delta_{j+1\notin J}(r_j+1) + \delta_{j+1\in J}(p-1-r_j)\right)$.*

(ii) *Let $\underline{n} \in \mathbb{Z}^f$. Suppose that there exists $j_0 \in \mathcal{J}$ such that*

(a) *$n_{j_0+1} = 0$;*

(b) *$1 \le n_j \le 2f - \delta_{j\in J}$ if $j \ne j_0+1$,*

then we define $\underline{a}^J(\underline{n}) \in \mathbb{Z}^f$ by

$$a^J(\underline{n})_j \overset{\text{def}}{=} \begin{cases} t_{j_0}^J(n_{j_0+1}) = 0 & \text{if } j = j_0 \text{ and } j_0 \in J^{\mathrm{sh}} \\ t_j^J(n_{j+1}) - n_j & \text{otherwise.} \end{cases}$$

Lemma 4.7.3. *Let $J \subseteq \mathcal{J}$, $\underline{n} \in \mathbb{Z}^f$ and $j_0 \in \mathcal{J}$ as in Definition 4.7.2(ii). Let $J' \subseteq J$ such that $j_0+1 \notin J \setminus J'$. Suppose that either $j_0 \notin J^{\mathrm{sh}}$ or $J^{\mathrm{ss}} \cup \{j_0+1\} \subseteq J'$, then we have (see §4.2 for $\underline{e}^{J\setminus J'}$)*

$$\underline{a}^J(\underline{n}) + \underline{r}^{J\setminus J'} = \underline{a}^{J'}\left(\underline{n} + \underline{e}^{J\setminus J'}\right).$$

Proof. Since $j_0+1 \notin J\setminus J'$, we have $\left(\underline{n} + \underline{e}^{J\setminus J'}\right)_{j_0+1} = n_{j_0+1} + \delta_{j_0+1\in J\setminus J'} = 0$. By definition, we also have for $j \ne j_0+1$

$$1 \le n_j \le n_j + \delta_{j\in J\setminus J'} \le (2f - \delta_{j\in J}) + \delta_{j\in J\setminus J'} = 2f - \delta_{j\in J'}.$$

Hence $\underline{a}^{J'}\left(\underline{n} + \underline{e}^{J\setminus J'}\right)$ is well-defined.

First we suppose that $j_0 \notin J^{\mathrm{sh}}$. We need to prove that for each $j \in \mathcal{J}$ we have

$$t_j^J(n_{j+1}) - n_j + r_j^{J\setminus J'} = t_j^{J'}\left(n_{j+1} + \delta_{j+1\in J\setminus J'}\right) - (n_j + \delta_{j\in J\setminus J'}).$$

Since $r_j^{J\setminus J'} = \delta_{j+1\in J\setminus J'}(r_j+1) - \delta_{j\in J\setminus J'}$ by (4.55), it suffices to show that for each $j \in \mathcal{J}$ we have

$$t_j^J(n_{j+1}) + \delta_{j+1\in J\setminus J'}(r_j+1) = t_j^{J'}\left(n_{j+1} + \delta_{j+1\in J\setminus J'}\right). \tag{4.69}$$

We fix $j \in \mathcal{J}$ and write $n_{j+1} = 2n+\delta$ with $n \in \mathbb{Z}$ and $\delta \in \{0,1\}$. If $\delta = 0$, then we have (as $\delta_{j+1\in J\setminus J'}\delta_{j+1\in J'} = 0$)

$$\begin{aligned} t_j^{J'}\left(n_{j+1} + \delta_{j+1\in J\setminus J'}\right) &= np + \delta_{j+1\in J\setminus J'}\left(\delta_{j+1\notin J'}(r_j+1) + \delta_{j+1\in J'}(p-1-r_j)\right) \\ &= np + \delta_{j+1\in J\setminus J'}(r_j+1) \\ &= t_j^J(n_{j+1}) + \delta_{j+1\in J\setminus J'}(r_j+1). \end{aligned}$$

If $\delta = 1$, then we have

$$\begin{aligned}
t_j^J(n_{j+1}) + \delta_{j+1\in J\setminus J'}(r_j+1) &= np + \delta_{j+1\notin J}(r_j+1) + \delta_{j+1\in J}(p-1-r_j) + \delta_{j+1\in J\setminus J'}(r_j+1)\\
&= np + \delta_{j+1\notin J'}(r_j+1) + \delta_{j+1\in J}(p-1-r_j);\\
t_j^{J'}\big(n_{j+1}+\delta_{j+1\in J\setminus J'}\big) &= np + \delta_{j+1\notin J'}(r_j+1) + \delta_{j+1\in J'}(p-1-r_j)\\
&\qquad + \delta_{j+1\in J\setminus J'}\big(\delta_{j+1\notin J'}(p-1-r_j) + \delta_{j+1\in J'}(r_j+1)\big)\\
&= np + \delta_{j+1\notin J'}(r_j+1) + \delta_{j+1\in J'}(p-1-r_j) + \delta_{j+1\in J\setminus J'}(p-1-r_j)\\
&= np + \delta_{j+1\notin J'}(r_j+1) + \delta_{j+1\in J}(p-1-r_j).
\end{aligned}$$

Then it remains to show that $a^J(\underline{n})_{j_0} + r_{j_0}^{J\setminus J'} = a^{J'}\big(\underline{n}+\underline{e}^{J\setminus J'}\big)_{j_0}$ when $j_0 \in J^{\mathrm{sh}}$ and $J^{\mathrm{ss}} \cup \{j_0+1\} \subseteq J' \subseteq J$. By assumption we have $j_0 \in (J')^{\mathrm{sh}}$, hence by definition we have $a^J(\underline{n})_{j_0} = \underline{a}^{J'}\big(\underline{n}+\underline{e}^{J\setminus J'}\big)_{j_0} = 0$. By assumption we also have $j_0, j_0+1 \in J'$, hence $j_0, j_0+1 \notin J\setminus J'$ and $r_{j_0}^{J\setminus J'} = 0$ by (4.52). This completes the proof. □

Proposition 4.7.4. *Let $0 \le k \le f$. Suppose that Theorem 4.6.4 is true for $|J| \le k$. Let $J \subseteq \mathcal{J}$ with $|J| \le k$. Let $\underline{n} \in \mathbb{Z}^f$ and $j_0 \in \mathcal{J}$ be as in Definition 4.7.2(ii). Then we have $x_{J,\underline{a}^J(\underline{n})-e_{j_0+1}} = 0$.*

Proof. If $f = 1$, then we have $a^J(0) - 1 = -1$, and the proposition follows directly from (4.58). Hence in the rest of the proof we assume that $f \ge 2$, and we prove the result by increasing induction on $|J|$. We let $\underline{i} \in \mathbb{Z}^f$ and $J' \subseteq \mathcal{J}$ be the unique pair such that

$$n_j = 2i_j - \delta_{j\in J+1} + \delta_{j\notin J} + \delta_{j-1\in J'} \tag{4.70}$$

for all $j \in \mathcal{J}$. In particular, we have $\underline{i} \le \underline{f}$ since $n_j \le 2f - \delta_{j\in J}$ for all $j \in \mathcal{J}$.

Claim 1. We let $\underline{i}' \overset{\mathrm{def}}{=} \underline{i} - \underline{e}^{(J+1)^{\mathrm{sh}}}$. Then we have for all $j \in \mathcal{J}$

$$n_j = 2i'_j - \delta_{j\in(J+1)\Delta J^{\mathrm{ss}}} + \delta_{j\notin J^{\mathrm{nss}}} + \delta_{j-1\in J'}. \tag{4.71}$$

Indeed, this follows from (4.70) and the following computation:

$$\begin{aligned}
&-2\delta_{j\in(J+1)^{\mathrm{sh}}} - \delta_{j\in(J+1)\Delta J^{\mathrm{ss}}} + \delta_{j\notin J^{\mathrm{nss}}}\\
&\qquad = -2\delta_{j\in J+1}\delta_{j\in J^{\mathrm{ss}}} - \big(\delta_{j\in J+1} + \delta_{j\in J^{\mathrm{ss}}} - 2\delta_{j\in J+1}\delta_{j\in J^{\mathrm{ss}}}\big) + \big(\delta_{j\notin J} - \delta_{j\in J^{\mathrm{ss}}}\big)\\
&\qquad = -\delta_{j\in J+1} + \delta_{j\notin J}.
\end{aligned}$$

Claim 2: We let $\underline{c} \in \mathbb{Z}^f$ such that (see (4.18) for $t^{J+1}(J')_j$)

$$c_j = pi_{j+1} + c_j^J - \delta_{j\notin J'}\big(2i'_j + t^{J+1}(J')_j\big). \tag{4.72}$$

If either $j_0 \notin J^{\mathrm{sh}}$ or $j_0 \in J'$, then we have

$$\begin{cases} c_j \ge a^J(\underline{n})_j - 1 & \text{if } j = j_0+1,\ j_0+1 \in J' \text{ and } j_0+1 \notin J\\ c_j \ge a^J(\underline{n})_j & \text{otherwise.}\end{cases} \tag{4.73}$$

Proof. Indeed, by (4.7) and a case-by-case examination we have

$$\begin{aligned}
2i'_j + t^{J+1}(J')_j &= 2i_j + p - 1 - \big(s_j^{J+1} + 2\delta_{j\in(J+1)^{\mathrm{sh}}}\big) + \delta_{j-1\in J'}\\
&= 2i_j + \delta_{j\notin J}(p-1-r_j) + \delta_{j\in J}(r_j+1) - \delta_{j\in J+1} + \delta_{j-1\in J'}.
\end{aligned} \tag{4.74}$$

If $j \in J'$, then by definition and a case-by-case examination we have

$$\begin{aligned} t_j^J(n_{j+1}) &= t_j^J\big(2i_{j+1} - \delta_{j\in J} + \delta_{j+1\notin J} + 1\big) = t_j^J\big(2i_{j+1} + \delta_{j\notin J} + \delta_{j+1\notin J}\big) \\ &= pi_{j+1} + \delta_{j\notin J}(p-1-r_j) + \delta_{j+1\notin J}(r_j+1). \end{aligned} \tag{4.75}$$

Combining (4.56), (4.74) and (4.75) we deduce that

$$c_j - a^J(\underline{n})_j = n_j - \delta_{j\notin J} \geq -\delta_{j=j_0+1, j_0+1\notin J},$$

unless when $j = j_0$ and $j_0 \in J^{\mathrm{sh}}$, in which case we have $c_{j_0} - a^J(\underline{n})_{j_0} = -\delta_{j_0\notin J} = 0$. If $j \notin J'$, then by assumption we have either $j \neq j_0$ or $j_0 \notin J^{\mathrm{sh}}$. By definition and a case-by-case examination we have

$$t_j^J(n_{j+1}) = t_j^J\big(2i_{j+1} - \delta_{j\in J} + \delta_{j+1\notin J}\big) = pi_{j+1} - \delta_{j\in J}(r_j+1) + \delta_{j+1\notin J}(r_j+1). \tag{4.76}$$

Combining (4.56), (4.70), (4.74) and (4.76) we deduce that $c_j = a^J(\underline{n})_j$. □

Using the decomposition

$$\mathcal{J} = \big(J^c \cap (J+1)^c\big) \sqcup \big(\big((J+1)\Delta J^{\mathrm{ss}}\big) \cup J^{\mathrm{nss}}\big) \sqcup (J+1)^{\mathrm{sh}}, \tag{4.77}$$

we separate the proof into the following four cases.

(a). Suppose that $j_0 \notin J$ and $j_0 + 1 \notin J$ (which implies $j_0 \notin J^{\mathrm{sh}}$). Since $n_{j_0+1} = 0$ and $n_{j_0} > 0$ by assumption (recall that $f \geq 2$), we have $\big(\underline{a}^J(\underline{n}) - e_{j_0+1}\big)_{j_0} = -n_{j_0} - 0 < 0$. Then we deduce from Lemma 4.7.1 that $x_{J,\underline{a}^J(\underline{n})-e_{j_0+1}} = 0$.

(b). Suppose that $j_0 + 1 \in (J+1)\Delta J^{\mathrm{ss}}$ and $j_0 + 1 \notin J^{\mathrm{nss}}$ (which implies $j_0 \notin J^{\mathrm{sh}}$). Using (4.71), we deduce from $n_{j_0+1} = 0$ that $i'_{j_0+1} = 0$ and $j_0 \notin J'$, and deduce from $\underline{i} \leq \underline{f}$ that $\underline{0} \leq \underline{i}' \leq \underline{f} - \underline{e}^{(J+1)^{\mathrm{sh}}}$. By Proposition 4.5.5 applied to $\big(\underline{i}', J+1, J'\big)$ with j_0 as above, we have

$$Y_{j_0+1}^{\delta_{J'=\emptyset}} \left[\prod_{j\notin J'} Y_j^{2i'_j + t^{J+1}(J')_j}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{i}'} v_{J+1}\right) = 0. \tag{4.78}$$

Multiplying (4.78) by $Y_{j_0+1}^{\delta_{j_0+1\notin J'}}$ when $J' \neq \emptyset$ and using (4.58), we deduce that

$$Y_{j_0+1}^{\delta_{j_0+1\notin J'}} \left[\prod_{j\notin J'} Y_j^{2i'_j + t^{J+1}(J')_j}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}} = 0.$$

Then by (4.57) we have (see (4.72) for $\underline{c}$)

$$\sum_{J^{\mathrm{ss}}\subseteq J_1\subseteq J} \varepsilon_{J_1}\mu_{J+1,J_1} x_{J_1,\left(\underline{c}+\underline{r}^{J\setminus J_1} - \delta_{j_0+1\notin J'} e_{j_0+1}\right)} = 0. \tag{4.79}$$

By (4.73) we have $\underline{c} \geq \underline{a}^J(\underline{n}) - \delta_{j_0+1\in J'\setminus J} e_{j_0+1} \geq \underline{a}^J(\underline{n}) - \delta_{j_0+1\in J'} e_{j_0+1}$. Moreover, for each $J_1 \subseteq \mathcal{J}$ such that $J^{\mathrm{ss}} \subseteq J_1 \subseteq J$, we have $j_0 + 1 \notin J \setminus J_1$ (since $j_0 + 1 \notin J^{\mathrm{nss}}$), hence by Lemma 4.7.3 (recall that $j_0 \notin J^{\mathrm{sh}}$) we have $\underline{a}^J(\underline{n}) + \underline{r}^{J\setminus J_1} = \underline{a}^{J_1}\big(\underline{n} + \underline{e}^{J\setminus J_1}\big)$. In particular, multiplying (4.79) by a suitable power of $\underline{Y}$ and using Theorem 4.6.4(ii) (applied to J_1 such that $J^{\mathrm{ss}} \subseteq J_1 \subseteq J$) we deduce that

$$\sum_{J^{\mathrm{ss}}\subseteq J_1\subseteq J} \varepsilon_{J_1}\mu_{J+1,J_1} x_{J_1,\underline{a}^{J_1}(\underline{n}+\underline{e}^{J\setminus J_1}) - e_{j_0+1}} = 0.$$

By the induction hypothesis, we have $x_{J_1,\underline{a}^{J_1}(\underline{n}+\underline{e}^{J\setminus J_1})-e_{j_0+1}}=0$ for all $J^{\mathrm{ss}}\subseteq J_1\subsetneqq J$, hence we conclude that $x_{J,\underline{a}^J(\underline{n})-e_{j_0+1}}=0$.

(c). Suppose that $j_0+1\in J^{\mathrm{nss}}$. Using (4.71), we deduce from $n_{j_0+1}=0$ that $i'_{j_0+1}=0$, and $j_0\in J'$ if and only if $j_0+1\in J+1$. Then we deduce from (4.71) and $\underline{i}\le\underline{f}$ that $\underline{0}\le\underline{i}'\le\underline{f}-\underline{e}^{(J+1)^{\mathrm{sh}}}$. By Proposition 4.5.9 applied to $\big(\underline{i}',J+1,J'\big)$ with j_0 as above, we have (see the end of §4.5 for $\mu_{J_1,*}/\mu_{J_2,*}$)

$$Y_{j_0+1}^{\delta_{j_0\notin J}}\left[\prod_{j\notin J'}Y_j^{2i'_j+t^{J+1}(J')_j}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\Big(\underline{Y}^{-\underline{i}'}v_{J+1}\Big)$$
$$=\frac{\mu_{J+1,*}}{\mu_{(J+1)\setminus\{j_0+2\},*}}Y_{j_0+1}^{\delta_{j_0\notin J}}\left[\prod_{j\notin J''}Y_j^{2i''_j+t^{(J+1)\setminus\{j_0+2\}}(J'')_j}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\Big(\underline{Y}^{-\underline{i}''}v_{(J+1)\setminus\{j_0+2\}}\Big),$$

where $\underline{i}''\overset{\mathrm{def}}{=}\underline{i}'-\delta_{j_0+1\in J'}e_{j_0+2}+\delta_{j_0+2\in J^{\mathrm{ss}}}e_{j_0+2}$ and $J''\overset{\mathrm{def}}{=}J'\Delta\{j_0+1\}$. As in (b), using (4.58), (4.57) and $j_0+1\notin J^{\mathrm{ss}}$, we deduce that

$$\sum_{J^{\mathrm{ss}}\subseteq J_1\subseteq J}\varepsilon_{J_1}\mu_{J+1,J_1}x_{J_1,\left(\underline{c}+\underline{r}^{J\setminus J_1}-\delta_{j_0\notin J}e_{j_0+1}\right)}$$
$$=\sum_{J^{\mathrm{ss}}\subseteq J_2\subseteq J\setminus\{j_0+1\}}\varepsilon_{J_2}\mu_{J+1,J_2}x_{J_2,\left(\underline{c}'+\underline{r}^{(J\setminus\{j_0+1\})\setminus J_2}-\delta_{j_0\notin J}e_{j_0+1}\right)},\quad(4.80)$$

where $\underline{c}$ is defined in (4.72) and $c'_j\overset{\mathrm{def}}{=}pi'''_{j+1}+c_j^{J\setminus\{j_0+1\}}-\delta_{j\notin J''}\big(2i''_j+t^{(J+1)\setminus\{j_0+2\}}(J'')_j\big)$ with

$$\underline{i}'''\overset{\mathrm{def}}{=}\underline{i}''+\underline{e}^{((J+1)\setminus\{j_0+2\})^{\mathrm{sh}}}=\underline{i}''+\underline{e}^{(J+1)^{\mathrm{sh}}}-\delta_{j_0+2\in J^{\mathrm{ss}}}e_{j_0+2}$$
$$=\underline{i}'+\underline{e}^{(J+1)^{\mathrm{sh}}}-\delta_{j_0+1\in J'}e_{j_0+2}=\underline{i}-\delta_{j_0+1\in J'}e_{j_0+2}.$$

Claim 3: We have $\underline{c}'=\underline{c}+\underline{r}^{\{j_0+1\}}$.

Proof. By (4.33) we have $c_j=c'_j$ for $j\ne j_0$ and $j\ne j_0+1$. If $j=j_0$, then by (4.33) and (4.53) we deduce that $c'_{j_0}-c_{j_0}=c_{j_0}^{J\setminus\{j_0+1\}}-c_{j_0}^J=r_{j_0}+1$. If $j=j_0+1$, we assume that $j_0+2\in J$, the case $j_0+2\notin J$ being similar. Then by (4.33) and (4.53) we deduce that

$$\begin{aligned}c'_{j_0+1}&=p\big(i_{j_0+2}-\delta_{j_0+1\notin J'}\big)+(p-2-r_{j_0+1})-\delta_{j_0+1\in J'}(p-1-r_{j_0+1})\\&=pi_{j_0+2}-\delta_{j_0+1\notin J'}(r_{j_0+1}+1)-(\delta_{j_0+1\notin J'}-1+\delta_{j_0+1\in J'})(p-1-r_{j_0+1})-1\\&=pi_{j_0+2}-\delta_{j_0+1\notin J'}(r_{j_0+1}+1)-1=c_{j_0+1}-1.\end{aligned}$$

The claim then follows from (4.52). □

By Lemma 4.6.3(ii), for each $J_2\subseteq J\setminus\{j_0+1\}$ we have $\underline{r}^{\{j_0+1\}}+\underline{r}^{(J\setminus\{j_0+1\})\setminus J_2}=\underline{r}^{J\setminus J_2}$, hence the RHS of (4.80) cancels with the terms in the LHS of (4.80) for the J_1 such that $j_0+1\notin J_1$. Since $j_0+1\in J$, and $j_0\in J'$ whenever $j_0\in J^{\mathrm{sh}}$, by (4.73) we have $\underline{c}\ge\underline{a}^J(\underline{n})$. Moreover, for each $J_1\subseteq\mathcal{J}$ such that $J^{\mathrm{ss}}\cup\{j_0+1\}\subseteq J_1\subseteq J$, by Lemma 4.7.3 we have $\underline{a}^J(\underline{n})+\underline{r}^{J\setminus J_1}=\underline{a}^{J_1}\big(\underline{n}+\underline{e}^{J\setminus J_1}\big)$. Then multiplying (4.80) by a suitable power of $\underline{Y}$ and using Theorem 4.6.4(ii) we deduce that

$$\sum_{J^{\mathrm{ss}}\cup\{j_0+1\}\subseteq J_1\subseteq J}\varepsilon_{J_1}\mu_{J+1,J_1}x_{J_1,\underline{a}^{J_1}(\underline{n}+\underline{e}^{J\setminus J_1})-e_{j_0+1}}=0.$$

By the induction hypothesis, we have $x_{J_1,\underline{a}^{J_1}(\underline{n}+\underline{e}^{J\setminus J_1})-e_{j_0+1}}=0$ for all $J^{\mathrm{ss}}\cup\{j_0+1\}\subseteq J_1\subsetneqq J$, hence we conclude that $x_{J,\underline{a}^J(\underline{n})-e_{j_0+1}}=0$.

(d). Suppose that $j_0+1\in(J+1)^{\mathrm{sh}}$. By (4.71) we have $i'_{j_0+1}=-1$ and $j_0\in J'$, hence $x_{J+1,\underline{i}}=0$ by (4.58). Then by (4.57), we have

$$0=\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)x_{J+1,\underline{i}}=\sum_{J^{\mathrm{ss}}\subseteq J_1\subseteq J}\varepsilon_{J_1}\mu_{J+1,J_1}x_{J_1,p\delta(\underline{i})+\underline{c}^J+\underline{r}^{J\setminus J_1}}. \tag{4.81}$$

By (4.73) we have $p\delta(\underline{i})+\underline{c}^J\geq\underline{c}\geq\underline{a}^J(\underline{n})-e_{j_0+1}$. Moreover, for each $J_1\subseteq\mathcal{J}$ such that $J^{\mathrm{ss}}\subseteq J_1\subseteq J$, by assumption we have $j_0+1\in(J+1)^{\mathrm{sh}}\subseteq J^{\mathrm{ss}}\subseteq J_1$. Then as in (c), we deduce from (4.81), Lemma 4.7.3 and Theorem 4.6.4(ii) that

$$\sum_{J^{\mathrm{ss}}\subseteq J_1\subseteq J}\varepsilon_{J_1}\mu_{J+1,J_1}x_{J_1,\underline{a}^{J_1}(\underline{n}+\underline{e}^{J\setminus J_1})-e_{j_0+1}}=0.$$

By the induction hypothesis, we have $x_{J_1,\underline{a}^{J_1}(\underline{n}+\underline{e}^{J\setminus J_1})-e_{j_0+1}}=0$ for all $J^{\mathrm{ss}}\subseteq J_1\subsetneqq J$, hence we conclude that $x_{J,\underline{a}^J(\underline{n})-e_{j_0+1}}=0$. □

Example 4.7.5. *Some examples for $f=3$, $J_{\overline{\rho}}=\emptyset$ and $J=\{0,1\}$:*

$$\begin{aligned}
&x_{J,(-1,r_1,p-2-r_2)}=x_{J,(-1,p-1,p-3-r_2)}=x_{J,(-2,r_1,p-1)}=x_{J,(-2,p-1,p-2)}=0;\\
&x_{J,(p-2-r_0,-1,p-2-r_2)}=x_{J,(p-3-r_0,-1,p-1)}=x_{J,(p-1,-2,p-2-r_2)}=x_{J,(p-2,-2,p-1)}=0;\\
&x_{J,(p-2-r_0,r_1,-1)}=x_{J,(p-1,r_1-1,-1)}=x_{J,(p-2-r_0,p-1,-2)}=x_{J,(p-1,p-2,-2)}=0.
\end{aligned}$$

Corollary 4.7.6. *Let $0\leq k\leq f$. Suppose that Theorem 4.6.4 is true for $|J|\leq k-1$. Let $J,J'\subseteq\mathcal{J}$ with $|J|\leq k$ and $J^{\mathrm{ss}}\subseteq J'\subsetneqq J\subseteq\mathcal{J}$. Let $j_0\in\mathcal{J}$ such that $j_0+1\notin J\setminus J'$. Then we have $x_{J',\left(\underline{r}^{J\setminus J'}+\underline{f}-(f+1-\delta_{j_0\in J^{\mathrm{sh}}})e_{j_0}\right)}=0$.*

Proof. If $f=1$, then the assumption is never satisfied. Hence in the rest of the proof we assume that $f\geq 2$. We let $\underline{n}\in\mathbb{Z}^f$ such that $n_{j_0+1}=0$, $n_{j_0}=1+\delta_{j_0\in J\setminus J'}$ and $n_j=2$ for $j\neq j_0,j_0+1$. In particular, $\underline{n}$ satisfies the conditions in Definition 4.7.2(ii) for J' and j_0. Since $|J'|\leq k-1$ by assumption, we deduce from Proposition 4.7.4 applied to J' and j_0 that $x_{J',\underline{a}^{J'}(\underline{n})-e_{j_0+1}}=0$. Then the result follows Theorem 4.6.4(ii) (applied to J') and the Claim below.

Claim. We have

$$\underline{a}^{J'}(\underline{n})-e_{j_0+1}\geq\underline{r}^{J\setminus J'}+\underline{f}-\left(f+1-\delta_{j_0\in J^{\mathrm{sh}}}\right)e_{j_0}.$$

Proof. If either $j\neq j_0,j_0-1$, or $j=j_0-1$ and $j_0\in J\setminus J'$, then we have $n_{j+1}=2$. Hence

$$\begin{aligned}
a^{J'}(\underline{n})_j-\delta_{j=j_0+1}&=t_j^{J'}(2)-n_j-\delta_{j=j_0+1}\\
&\geq p-2-1\geq(p-2-2f)+f\geq r_j+1+f\geq r_j^{J\setminus J'}+f,
\end{aligned}$$

where the third inequality follows from (4.2) and the last inequality follows from (4.52).

If $j=j_0-1$ and $j_0\notin J\setminus J'$, then we have $n_{j_0}=1$, and $r_{j_0}^{J\setminus J'}\leq 0$ by (4.52). Hence

$$\begin{aligned}
a^{J'}(\underline{n})_{j_0-1}-\delta_{j_0-1=j_0+1}&=t_{j_0-1}^{J'}(1)-n_{j_0-1}-\delta_{j_0-1=j_0+1}\\
&\geq\left(\delta_{j_0\notin J'}(r_{j_0-1}+1)+\delta_{j_0\in J'}(p-1-r_{j_0-1})\right)-2-1\\
&\geq(2f+2)-3\geq f\geq r_{j_0-1}^{J\setminus J'}+f,
\end{aligned}$$

where the second inequality follows from (4.2).

Finally, we let $j = j_0$. Since $j_0+1 \notin J \setminus J'$, by (4.52) we have $r_{j_0}^{J\setminus J'} = -\delta_{j_0\in J\setminus J'}$. If $j_0 \notin J^{\mathrm{sh}}$, then we have (using $n_{j_0+1}=0$)

$$a^{J'}(\underline{n})_{j_0} - \delta_{j_0=j_0+1} = t_{j_0}^{J'}(0) - n_{j_0} = -n_{j_0} = -\delta_{j_0\in J\setminus J'} - 1 = r_{j_0}^{J\setminus J'} + f - (f+1).$$

If $j_0 \in J^{\mathrm{sh}}$, then $J^{\mathrm{sh}} \subseteq J^{\mathrm{ss}} \subseteq J'$ implies $j_0 \in J'$, hence $j_0 \notin J\setminus J'$, which implies $r_{j_0}^{J\setminus J'} = 0$. Then we have

$$a^{J'}(\underline{n})_{j_0} - \delta_{j_0=j_0+1} = 0 = r_{j_0}^{J\setminus J'} + f - f,$$

which completes the proof. □

Then we list some consequences of Lemma 4.7.1 and Corollary 4.7.6. In particular, we prove Lemma 4.7.9 and Lemma 4.7.10 which finish the proof of Theorem 4.6.4. The following lemma will be needed in the proof of Lemma 4.7.8 and Lemma 4.7.10.

Lemma 4.7.7. *Let $J, J' \subseteq \mathcal{J}$ such that $J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}} \subsetneqq J' \subseteq J$ (see Remark 4.5.12 for ∂J) and $J^{\mathrm{nss}} \neq \mathcal{J}$. Then we have*

$$\underline{Y}^{\underline{c}^{J'} + (p-1)\underline{e}^{J^{\mathrm{sh}}} - \underline{r}^{J\setminus J'}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J'+1, \underline{e}^{(J^{\mathrm{sh}}+1)}} = 0. \tag{4.82}$$

Proof. Our assumption implies $f \geq 2$. By Lemma 4.6.3(v), the LHS of (4.82) is well-defined (since $\underline{c}^J \geq \underline{0}$) and it suffices to show that

$$\left[\underline{Y}^{\underline{c}^J + (p-1)\underline{e}^{J^{\mathrm{sh}}}} \prod_{j\in J\setminus J'} Y_j^{p-1-r_j}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J'+1, \underline{e}^{(J^{\mathrm{sh}}+1)}} = 0. \tag{4.83}$$

We let $\underline{i} \overset{\mathrm{def}}{=} \underline{e}^{J^{\mathrm{sh}}+1} - \underline{e}^{(J'+1)^{\mathrm{sh}}}$. If $\underline{i} \not\geq \underline{0}$, then by (4.58) we have $x_{J'+1,\underline{e}^{J^{\mathrm{sh}}+1}} = 0$, which proves (4.83). From now on we assume that $\underline{i} \geq \underline{0}$, which implies $(J'+1)^{\mathrm{sh}} \subseteq J^{\mathrm{sh}} + 1$. Then we claim that

$$(J')^{\mathrm{nss}} \cap \left(J^{\mathrm{ss}} - 1\right) = \emptyset. \tag{4.84}$$

Otherwise, there exists $j_1 \in (J')^{\mathrm{nss}}$ such that $j_1 + 1 \in J^{\mathrm{ss}} = (J')^{\mathrm{ss}}$, which implies $j_1 + 1 \in (J'+1)^{\mathrm{sh}} \subseteq J^{\mathrm{sh}} + 1$. Hence $j_1 \in J^{\mathrm{sh}}$, which is a contradiction since $j_1 \notin J_{\overline{\rho}}$.

Since $J^{\mathrm{nss}} \neq \mathcal{J}$ by assumption, we divide J^{nss} into a disjoint union of intervals not adjacent to each other. Since $(\partial J)^{\mathrm{nss}} \subsetneqq (J')^{\mathrm{nss}} \subseteq J^{\mathrm{nss}}$ by assumption, we choose an interval I as above such that $(J'\setminus \partial J)^{\mathrm{nss}} \cap I \neq \emptyset$ and denote by j_0 the right boundary of I. Since $j_0+1 \notin J^{\mathrm{nss}}$ by construction, we have either $j_0+1 \in J^{\mathrm{ss}} = (J')^{\mathrm{ss}}$, which implies $j_0 \notin J'$ by (4.84), or $j_0+1 \notin J$, which implies $j_0 \in (\partial J)^{\mathrm{nss}} \subseteq J'$. In particular, in both cases we have $j_0 \notin (J'\setminus \partial J)^{\mathrm{nss}}$ and $j_0 + 1 \in (J'+1)\Delta(J')^{\mathrm{ss}}$.

(a). First we suppose that $j_0 + 1 \notin J$, which implies $j_0 \in (\partial J)^{\mathrm{nss}} \subseteq J'$. We let $1 \leq w \leq f-1$ be minimal such that $j_0 - w \in (J'\setminus \partial J)^{\mathrm{nss}}$. By the construction of I we have $j_0 - w, j_0 - w + 1, \dots, j_0 \in J^{\mathrm{nss}}$ and $j_0 - w + 1, \dots, j_0 - 1 \notin J'$. Then by (4.7) we have if $w \geq 2$

$$s_j^{J'+1} = \begin{cases} p-2-r_j & \text{if } j = j_0 \\ r_j & \text{if } j = j_0 - w + 2, \dots, j_0 - 1 \text{ (and } w \geq 3) \\ r_j + 1 & \text{if } j = j_0 - w + 1, \end{cases} \tag{4.85}$$

and $s_{j_0}^{J'+1} = p - 1 - r_{j_0}$ if $w = 1$.

Then we let $J'' \overset{\text{def}}{=} \mathcal{J} \setminus \{j_0 - w + 1, \dots, j_0\}$. For each $j = j_0 - w + 1, \dots, j_0 + 1$ we have $j - 1 \notin J_{\overline{\rho}}$ by construction, hence $j \notin J^{\mathrm{sh}} + 1$, which implies $i_j \leq 0$ and hence $i_j = 0$ (since $\underline{i} \geq \underline{0}$). Then by (4.18) and (4.85) we have

$$\begin{aligned} 2i_j + t^{J'+1}(J'')_j &= p - 1 - s_j^{J'+1} + \delta_{j-1 \in J''} \\ &= \begin{cases} r_j + 1 & \text{if } j = j_0 \\ p - 1 - r_j & \text{if } j = j_0 - w + 1, \dots, j_0 - 1 \text{ (and } w \geq 2). \end{cases} \end{aligned}$$

By Proposition 4.5.5 applied to $(\underline{i}, J' + 1, J'')$ with j_0 as above and using (4.58), we have

$$\left[Y_{j_0}^{r_{j_0}+1} \prod_{j=j_0-w+1}^{j_0-1} Y_j^{p-1-r_j} \right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J'+1, \underline{e}^{(J^{\mathrm{sh}}+1)}} = 0.$$

Since $j_0 - w + 1, \dots, j_0 - 1 \in J \setminus J'$, to prove (4.83) it is enough to show that $c_{j_0}^J + (p-1)\delta_{j_0 \in J^{\mathrm{sh}}} \geq r_{j_0} + 1$, which follows from (4.53) since $j_0 \in J$ and $j_0 + 1 \notin J$.

(b). Then we suppose that $j_0 + 1 \in J^{\mathrm{ss}} = (J')^{\mathrm{ss}}$, which implies $j_0 \notin J'$. We use the same definition of w, J'' as in (a). In particular, we still have $j_0 - w, j_0 - w + 1, \dots, j_0 \in J^{\mathrm{nss}}$ and $j_0 - w + 1, \dots, j_0 - 1 \notin J'$. Then by (4.7) and (4.18) we have

$$s_j^{J'+1} = \begin{cases} r_j & \text{if } j = j_0 - w + 2, \dots, j_0 \\ r_j + 1 & \text{if } j = j_0 - w + 1 \end{cases}$$

and $2i_j + t^{J'+1}(J'')_j = p - 1 - r_j$ for $j = j_0 - w + 1, \dots, j_0$. By Proposition 4.5.5 applied to $(\underline{i}, J' + 1, J'')$ with j_0 as above and using (4.58), we have

$$\left[\prod_{j=j_0-w+1}^{j_0} Y_j^{p-1-r_j} \right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J'+1, \underline{e}^{(J^{\mathrm{sh}}+1)}} = 0.$$

Since $j_0 - w + 1, \dots, j_0 \in J \setminus J'$, this completes the proof of (4.83). □

The following lemma gives some examples of elements in the projective systems defined in Theorem 4.6.4 and will be needed in the proof of Lemma 4.7.9.

Lemma 4.7.8. *Let $0 \leq k \leq f$. Suppose that Theorem 4.6.4 is true for $|J| \leq k - 1$. Let $J, J' \subseteq \mathcal{J}$ such that $|J| \leq k$ and $J^{\mathrm{ss}} \subseteq J' \subsetneqq J$.*

(i) *If $J^{\mathrm{nss}} \neq \mathcal{J}$, then we have (see the end of §4.5 for $\mu_{*,J}/\mu_{*,J'}$)*

$$\varepsilon_{J'} x_{J', \underline{r}^{J \setminus J'} + \underline{e}^{J^{\mathrm{sh}}}} = \begin{cases} 0, & \text{if } J' \not\supseteq J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}} \\ (-1)^{|(J' \setminus \partial J)^{\mathrm{nss}}|} \frac{\mu_{*,J}}{\mu_{*,J'}} v_J, & \text{if } J' \supseteq J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}}. \end{cases}$$

(ii) *If $J^{\mathrm{nss}} = \mathcal{J}$ (i.e. $(J, J_{\overline{\rho}}) = (\mathcal{J}, \emptyset)$, which implies $k = f$) and $J' \neq \emptyset$, then we have*

$$\varepsilon_{J'} x_{J', \underline{r}^{J \setminus J'}} = (-1)^{|J'|+1} \frac{\mu_{*,J}}{\mu_{*,J'}} v_J.$$

Proof. (i). We let $J^{\mathrm{nss}} \neq \mathcal{J}$ and separate the following cases:

(a). Suppose that $J' \not\supseteq J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}}$. Since $J' \supseteq J^{\mathrm{ss}}$, we have $J' \not\supseteq \partial J$. Then we let $j_0 \in \partial J$ (i.e. $j_0 \in J$ and $j_0 + 1 \notin J$) and $j_0 \notin J'$. This implies $j_0 \in J \setminus J'$ and $j_0 + 1 \notin J \setminus J'$, hence $r_{j_0}^{J \setminus J'} = -1$ by (4.52). We also have $j_0 \notin J^{\mathrm{sh}}$. Then we deduce from Lemma 4.7.1 that $x_{J', \underline{r}^{J \setminus J'} + \underline{e}^{J^{\mathrm{sh}}}} = 0$.

(b). Suppose that $J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}} \subseteq J' \subsetneqq J$. We use increasing induction on $\big|J' \setminus \big(J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}}\big)\big|$, which equals $|(J' \setminus \partial J)^{\mathrm{nss}}|$.

First we assume that $J' = J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}}$. By Proposition 4.5.11 applied to $(J' + 1, J)$, we have

$$\begin{aligned} \mu_{J'+1,J} v_J &= \left[\prod_{j \in J_0} Y_j^{s_j^J} \prod_{j \notin J_0} Y_j^{p-1} \right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \left(\underline{Y}^{-\underline{e}^{((J'+1) \cap J)^{\mathrm{nss}}}} v_{J'+1} \right) \\ &= \left[\prod_{j \in J_0} Y_j^{s_j^J} \prod_{j \notin J_0} Y_j^{p-1} \right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) x_{J'+1, \underline{e}^{J_1}}, \end{aligned} \tag{4.86}$$

where the second equality follows from (4.58) and

$$\begin{aligned} J_0 &\overset{\mathrm{def}}{=} \big((J'+1)\Delta J\big) - 1 = J'\Delta(J-1) = \big(J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}}\big)\Delta\big((J-1)^{\mathrm{ss}} \sqcup (J-1)^{\mathrm{nss}}\big) \\ &= \big(J\Delta(J-1)\big)^{\mathrm{ss}} \sqcup \big((\partial J)\Delta(J-1)\big)^{\mathrm{nss}} = \big(J\Delta(J-1)\big)^{\mathrm{ss}} \sqcup \big(J \cup (J-1)\big)^{\mathrm{nss}}; \\ J_1 &\overset{\mathrm{def}}{=} \big((J'+1) \cap J\big)^{\mathrm{nss}} \sqcup (J'+1)^{\mathrm{sh}} = \big((J'+1) \cap J^{\mathrm{nss}}\big) \sqcup \big((J'+1) \cap (J')^{\mathrm{ss}}\big) \\ &= \big((J'+1) \cap J^{\mathrm{nss}}\big) \sqcup \big((J'+1) \cap J^{\mathrm{ss}}\big) = (J'+1) \cap J = \big(J' \cap (J-1)\big) + 1. \end{aligned}$$

We write $\underline{s} \in \mathbb{Z}^f$ with $s_j \overset{\mathrm{def}}{=} s_j^J$ if $j \in J_0$ and $s_j \overset{\mathrm{def}}{=} p - 1$ if $j \notin J_0$.

Claim. We have

$$p\underline{e}^{J_1 - 1} + \underline{c}^{J'} - \underline{s} = \underline{r}^{J \setminus J'} + \underline{e}^{J^{\mathrm{sh}}}. \tag{4.87}$$

Proof. Fix $j \in \mathcal{J}$. We assume that $j \in J_{\overline{\rho}}$, the case $j \notin J_{\overline{\rho}}$ being similar. In particular, we have $j \in J'$ if and only if $j \in J$, which implies

$$p\delta_{j \in J_1 - 1} - \delta_{j \in J^{\mathrm{sh}}} = (p-1)\delta_{j \in J \cap (J-1)}. \tag{4.88}$$

Since $j \in J'$ if and only if $j \in J$, and $j \in J_0$ if and only if $j \in J\Delta(J-1)$, by (4.52), (4.53) and (4.7) with a case-by-case examination we have

$$\begin{aligned} r_j^{J \setminus J'} &= \delta_{j+1 \in J \setminus J'}(r_j + 1); \\ c_j^{J'} &= \delta_{j \notin J'}(p - 2 - r_j) + \delta_{j+1 \notin J'}(r_j + 1) = \delta_{j \notin J}(p - 2 - r_j) + \delta_{j+1 \notin J'}(r_j + 1); \\ s_j &= \delta_{j \notin J}(p - 2 - r_j) + \delta_{j+1 \notin J}(r_j + 1) + (p-1)\delta_{j \in J \cap (J-1)}. \end{aligned} \tag{4.89}$$

Combining (4.88) and (4.89) we get (4.87). □

By (4.57) applied to $J' + 1$ and using $(J')^{\mathrm{ss}} = J^{\mathrm{ss}}$, we deduce from (4.86) that

$$\begin{aligned} \mu_{J'+1,J} v_J &= \sum_{J^{\mathrm{ss}} \subseteq J'' \subseteq J'} \varepsilon_{J''} \mu_{J'+1,J''} x_{J'', \left(p\underline{e}^{J_1 - 1} + \underline{c}^{J'} + \underline{r}^{J' \setminus J''} - \underline{s}\right)} \\ &= \sum_{J^{\mathrm{ss}} \subseteq J'' \subseteq J'} \varepsilon_{J''} \mu_{J'+1,J''} x_{J'', \left(\underline{r}^{J \setminus J'} + \underline{r}^{J' \setminus J''} + \underline{e}^{J^{\mathrm{sh}}}\right)} \end{aligned}$$

$$= \sum_{J^{\mathrm{ss}} \subseteq J'' \subseteq J'} \varepsilon_{J''} \mu_{J'+1,J''} x_{J'', \underline{r}^{J \setminus J''} + \underline{e}^{J^{\mathrm{sh}}}},$$

where the second equality follows from (4.87) and the last equality follows from Lemma 4.6.3(ii). We know from (a) that $x_{J'', \underline{r}^{J \setminus J''} + \underline{e}^{J^{\mathrm{sh}}}} = 0$ for $J^{\mathrm{ss}} \subseteq J'' \subsetneqq J'$, hence we conclude that

$$\varepsilon_{J'} \mu_{J'+1,J'} x_{J', \underline{r}^{J \setminus J'} + \underline{e}^{J^{\mathrm{sh}}}} = \mu_{J'+1,J} v_J,$$

which proves (i) when $J' = J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}}$.

Next we assume that $J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}} \subsetneqq J' \subsetneqq J$. By Lemma 4.7.7, (4.57) applied to $J'+1$ and using $(J')^{\mathrm{ss}} = J^{\mathrm{ss}}$, we have

$$\begin{aligned} 0 &= \underline{Y}^{\underline{c}^{J'} + (p-1)\underline{e}^{J^{\mathrm{sh}}} - \underline{r}^{J \setminus J'}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J'+1, \underline{e}^{(J^{\mathrm{sh}}+1)}} \\ &= \sum_{J^{\mathrm{ss}} \subseteq J'' \subseteq J'} \varepsilon_{J''} \mu_{J'+1,J''} x_{J'', \left(p\underline{e}^{J^{\mathrm{sh}}} + \underline{c}^{J'} + \underline{r}^{J' \setminus J''} - \left(\underline{c}^{J'} + (p-1)\underline{e}^{J^{\mathrm{sh}}} - \underline{r}^{J \setminus J'}\right)\right)} \\ &= \sum_{J^{\mathrm{ss}} \subseteq J'' \subseteq J'} \varepsilon_{J''} \mu_{J'+1,J''} x_{J'', \underline{r}^{J \setminus J''} + \underline{e}^{J^{\mathrm{sh}}}}, \end{aligned}$$

where the last equality follows from Lemma 4.6.3(ii). We know from (a) that $x_{J'', \underline{r}^{J \setminus J''} + \underline{e}^{J^{\mathrm{sh}}}} = 0$ for $J'' \not\supseteq J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}}$, hence we have

$$\sum_{J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}} \subseteq J'' \subseteq J'} \varepsilon_{J''} \mu_{J'+1,J''} x_{J'', \underline{r}^{J \setminus J''} + \underline{e}^{J^{\mathrm{sh}}}} = 0. \tag{4.90}$$

By the induction hypothesis, we have for $J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}} \subseteq J'' \subsetneqq J'$

$$\varepsilon_{J''} x_{J'', \underline{r}^{J \setminus J''} + \underline{e}^{J^{\mathrm{sh}}}} = (-1)^{|(J'' \setminus \partial J)^{\mathrm{nss}}|} \frac{\mu_{*,J}}{\mu_{*,J''}} v_J. \tag{4.91}$$

Moreover, if we denote $m \overset{\mathrm{def}}{=} |(J' \setminus \partial J)^{\mathrm{nss}}|$, then (by the definition of $\mu_{*,J}/\mu_{*,J''}$) we have

$$\begin{aligned} \sum_{J^{\mathrm{ss}} \sqcup (\partial J)^{\mathrm{nss}} \subseteq J'' \subsetneqq J'} (-1)^{|(J'' \setminus \partial J)^{\mathrm{nss}}|} \frac{\mu_{*,J}}{\mu_{*,J''}} \mu_{J'+1,J''} &= \left[\sum_{(J'' \setminus \partial J)^{\mathrm{nss}} \subsetneqq (J' \setminus \partial J)^{\mathrm{nss}}} (-1)^{|(J'' \setminus \partial J)^{\mathrm{nss}}|} \right] \mu_{J'+1,J} \\ &= \left[\sum_{i=0}^{m-1} (-1)^i \binom{m}{i} \right] \mu_{J'+1,J} = (-1)^{m+1} \mu_{J'+1,J}. \end{aligned} \tag{4.92}$$

Combining (4.90), (4.91) and (4.92), we conclude that

$$\varepsilon_{J'} x_{J', \underline{r}^{J \setminus J'} + \underline{e}^{J^{\mathrm{sh}}}} = (-1)^{|(J' \setminus \partial J)^{\mathrm{nss}}|} \frac{\mu_{*,J}}{\mu_{*,J'}} v_J,$$

which proves (i).

(ii). Let $(J, J_{\overline{p}}) = (\mathcal{J}, \emptyset)$ and $\emptyset \neq J' \neq \mathcal{J}$, which implies $J^{\mathrm{ss}} = (J')^{\mathrm{ss}} = \emptyset$. We prove the result by increasing induction on $|J'|$. By Proposition 4.5.13 applied to $J'+1$, we have

$$\mu_{J'+1,\mathcal{J}} v_{\mathcal{J}} + \mu_{J'+1,\emptyset} x_{\emptyset,\underline{r}} = \left[\prod_{j \notin J'} Y_j^{p-1-r_j} \right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{J'+1}$$

$$
\begin{aligned}
&= \sum_{J''\subseteq J'} \varepsilon_{J''}\mu_{J'+1,J''}x_{J'',\left(c_j^{J'+1}+r_j^{J'\setminus J''}-\delta_{j\notin J'}(p-1-r_j)\right)_j}\\
&= \sum_{J''\subseteq J'} \varepsilon_{J''}\mu_{J'+1,J''}x_{J'',\left(\underline{c}^{\mathcal{J}}+\underline{r}^{\mathcal{J}\setminus J'}+\underline{r}^{J'\setminus J''}\right)}\\
&= \sum_{J''\subseteq J'} \varepsilon_{J''}\mu_{J'+1,J''}x_{J'',\underline{r}^{\mathcal{J}\setminus J''}},
\end{aligned}
$$

where the second equality follows from (4.57) applied to $J'+1$ and $\underline{i}=\underline{0}$, the third equality follows from Lemma 4.6.3(v), and the last equality uses Lemma 4.6.3(ii) and $\underline{c}^{\mathcal{J}}=\underline{0}$ (see (4.53)). Since $\underline{r}=\underline{r}^{\mathcal{J}}$ by (4.52), we deduce that

$$
\mu_{J'+1,\mathcal{J}}v_{\mathcal{J}} = \sum_{\emptyset\neq J''\subseteq J'} \varepsilon_{J''}\mu_{J'+1,J''}x_{J'',\underline{r}^{\mathcal{J}\setminus J''}}. \tag{4.93}
$$

By the induction hypothesis, we have for $\emptyset\neq J''\subsetneqq J'$

$$
\varepsilon_{J''}x_{J'',\underline{r}^{\mathcal{J}\setminus J''}} = (-1)^{|J''|+1}\frac{\mu_{*,\mathcal{J}}}{\mu_{*,J''}}v_{\mathcal{J}}. \tag{4.94}
$$

Moreover, we have (by the definition of $\mu_{*,J}/\mu_{*,J''}$)

$$
\begin{aligned}
\sum_{\emptyset\neq J''\subsetneqq J'} (-1)^{|J''|+1}\frac{\mu_{*,\mathcal{J}}}{\mu_{*,J''}}\mu_{J'+1,J''} &= -\left[\sum_{\emptyset\neq J''\subsetneqq J'}(-1)^{|J''|}\right]\mu_{J'+1,\mathcal{J}}\\
&= -\left[\sum_{i=1}^{|J'|-1}(-1)^i\binom{|J'|}{i}\right]\mu_{J'+1,\mathcal{J}} = \left(1+(-1)^{|J'|}\right)\mu_{J'+1,\mathcal{J}}.
\end{aligned} \tag{4.95}
$$

Combining (4.93),(4.94) and (4.95), we conclude that

$$
\varepsilon_{J'}x_{J',\underline{r}^{\mathcal{J}\setminus J'}} = (-1)^{|J'|+1}\frac{\mu_{*,\mathcal{J}}}{\mu_{*,J'}}v_{\mathcal{J}},
$$

which proves (ii). □

The following two lemmas complete the proof of Theorem 4.6.4.

Lemma 4.7.9. *Let $0\le k\le f$. Suppose that Theorem 4.6.4 is true for $|J|\le k-1$. Then for $|J|\le k$, we have (see (4.65) for w_J)*

$$
Y_{j_0}^{f+1-\delta_{j_0\in J^{\mathrm{sh}}}}w_J = \begin{cases} \displaystyle\sum_{J^{\mathrm{ss}}\subseteq J'\subseteq J\setminus\{j_0+1\}} \varepsilon_{J'}\mu_{J+1,J'}x_{J',\left(\underline{r}^{J\setminus J'}+\underline{f}-(f+1-\delta_{j_0\in J^{\mathrm{sh}}})e_{j_0}\right)} & \text{if } j_0+1\in J^{\mathrm{nss}}\\ 0 & \text{if } j_0+1\notin J^{\mathrm{nss}}; \end{cases}
$$

$$
\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}}w_J = \begin{cases} 0 & \text{if } J^{\mathrm{nss}}=(\partial J)^{\mathrm{nss}}\\ (-1)^{|(J\setminus\partial J)^{\mathrm{nss}}|+1}\mu_{J+1,J}v_J & \text{if } J^{\mathrm{nss}}\neq(\partial J)^{\mathrm{nss}} \text{ and } J^{\mathrm{nss}}\neq\mathcal{J}\\ \left(1+(-1)^f\right)\mu_{J,J}v_J+\mu_{J,\emptyset}x_{\emptyset,\underline{r}} & \text{if } J^{\mathrm{nss}}=\mathcal{J}. \end{cases}
$$

Proof. (i). We prove the first equality. By definition and Theorem 4.6.4(ii) (applied to J' such that $J^{\mathrm{ss}}\subseteq J'\subsetneqq J$, which implies $|J'|\le k-1$), we have

$$
Y_{j_0}^{f+1-\delta_{j_0\in J^{\mathrm{sh}}}}w_J = \sum_{J^{\mathrm{ss}}\subseteq J'\subsetneqq J} \varepsilon_{J'}\mu_{J+1,J'}x_{J',\left(\underline{r}^{J\setminus J'}+\underline{f}-(f+1-\delta_{j_0\in J^{\mathrm{sh}}})e_{j_0}\right)}.
$$

By Corollary 4.7.6, we have $x_{J',\left(\underline{r}^{J\setminus J'}+\underline{f}-(f+1-\delta_{j_0\in J^{\mathrm{sh}}})e_{j_0}\right)}=0$ if $J^{\mathrm{ss}}\subseteq J'\subseteq J$ and $j_0+1\notin J\setminus J'$. Then we easily conclude.

(ii). We prove the second equality. First we suppose that $J^{\mathrm{nss}}\neq\mathcal{J}$. Then by Theorem 4.6.4(ii) (applied to J' such that $J^{\mathrm{ss}}\subseteq J'\subsetneqq J$) and Lemma 4.7.8(i) we have

$$\begin{aligned}\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}}w_J&=\sum_{J^{\mathrm{ss}}\subseteq J'\subsetneqq J}\varepsilon_{J'}\mu_{J+1,J'}x_{J',\underline{r}^{J\setminus J'}+\underline{e}^{J^{\mathrm{sh}}}}=\left[\sum_{J^{\mathrm{ss}}\sqcup(\partial J)^{\mathrm{nss}}\subseteq J'\subsetneqq J}(-1)^{|(J'\setminus\partial J)^{\mathrm{nss}}|}\right]\mu_{J+1,J}v_J\\&=\begin{cases}0&\text{if } J^{\mathrm{nss}}=(\partial J)^{\mathrm{nss}}\\(-1)^{|(J\setminus\partial J)^{\mathrm{nss}}|+1}\mu_{J+1,J}v_J&\text{if } J^{\mathrm{nss}}\neq(\partial J)^{\mathrm{nss}},\end{cases}\end{aligned}$$

where the last equality follows as in (4.92). Next we suppose that $J^{\mathrm{nss}}=\mathcal{J}$, or equivalently $(J,J_{\overline{\rho}})=(\mathcal{J},\emptyset)$, which implies $J^{\mathrm{ss}}=J^{\mathrm{sh}}=\emptyset$ and $J+1=J$. Then by Theorem 4.6.4(ii) (applied to $J'\neq\mathcal{J}$) and Lemma 4.7.8(ii) we have

$$\begin{aligned}\underline{Y}^{\underline{f}}w_{\mathcal{J}}&=\sum_{J'\neq\mathcal{J}}\varepsilon_{J'}\mu_{\mathcal{J},J'}x_{J',\underline{r}^{\mathcal{J}\setminus J'}}=\mu_{\mathcal{J},\emptyset}x_{\emptyset,\underline{r}}+\left[\sum_{\emptyset\neq J'\neq\mathcal{J}}(-1)^{|J'|+1}\right]\mu_{\mathcal{J},\mathcal{J}}v_{\mathcal{J}}\\&=\left(1+(-1)^f\right)\mu_{\mathcal{J},\mathcal{J}}v_{\mathcal{J}}+\mu_{\mathcal{J},\emptyset}x_{\emptyset,\underline{r}},\end{aligned}$$

where the last equality follows as in (4.95). □

Lemma 4.7.10. *Let $0\leq k\leq f$. Suppose that Theorem 4.6.4 is true for $|J|\leq k-1$. Then for $|J|\leq k$, we have*

$$Y_{j_0}^{f+1-\delta_{j_0\in J^{\mathrm{sh}}}}z_J=\begin{cases}\displaystyle\sum_{J^{\mathrm{ss}}\subseteq J'\subseteq J\setminus\{j_0+1\}}\varepsilon_{J'}\mu_{J+1,J'}x_{J',\left(\underline{r}^{J\setminus J'}+\underline{f}-(f+1-\delta_{j_0\in J^{\mathrm{sh}}})e_{j_0}\right)}&\text{if } j_0+1\in J^{\mathrm{nss}}\\0&\text{if } j_0+1\notin J^{\mathrm{nss}};\end{cases}$$

$$\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}}z_J=\begin{cases}\mu_{J+1,J}v_J,&\text{if } J^{\mathrm{nss}}=(\partial J)^{\mathrm{nss}}\\0,&\text{if } J^{\mathrm{nss}}\neq(\partial J)^{\mathrm{nss}} \text{ and } J^{\mathrm{nss}}\neq\mathcal{J}\\\mu_{J,J}v_J+\mu_{J,\emptyset}x_{\emptyset,\underline{r}},&\text{if } J^{\mathrm{nss}}=\mathcal{J},\end{cases}$$

where z_J is defined in (4.65).

Proof. (i) We prove the first equality. We recall from (4.65) that

$$z_J=\underline{Y}^{\underline{c}'^J}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)x_{J+1,\underline{e}^{J\cap(J+1)}}=\underline{Y}^{\underline{c}'^J}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{e}^{(J\cap(J+1))^{\mathrm{nss}}}}v_{J+1}\right).\tag{4.96}$$

Using the decomposition (4.77), we separate the proof into the following five cases.

(a). Suppose that $j_0\notin J$ and $j_0+1\notin J$, which implies $j_0\notin J^{\mathrm{sh}}$. Since $j_0+1\notin\left(J\cap(J+1)\right)^{\mathrm{nss}}$, by Lemma 4.3.1(i) applied to j_0 and Proposition 4.4.2 we have

$$Y_{j_0}^{p}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{e}^{(J\cap(J+1))^{\mathrm{nss}}}}v_{J+1}\right)=0.\tag{4.97}$$

Since $c'^J_{j_0}=p-1-f$ by (4.61), we deduce from (4.96) and (4.97) that $Y_{j_0}^{f+1}z_J=0$.

(b). Suppose that $j_0+1 \in (J+1)^{\mathrm{sh}}$, which implies $j_0+1 \notin \big(J \cap (J+1)\big)^{\mathrm{nss}}$. In particular, the equality (4.97) still holds as in (a). Since $c'^{J}_{j_0} = p-f$ by (4.61), we deduce from (4.96) and (4.97) that $Y_{j_0}^{f+1-\delta_{j_0\in J^{\mathrm{sh}}}} z_J = 0$.

(c). Suppose that $j_0+1 \in (J+1)\Delta J^{\mathrm{ss}}$ and $j_0+1 \notin J^{\mathrm{nss}}$, which implies $f \geq 2$ and $j_0 \in J\Delta(J-1)$. In particular, we have $j_0 \notin J^{\mathrm{sh}}$. By (4.7) we have

$$\begin{aligned} s_j^{J+1} - 2\delta_{j\in(J\cap(J+1))^{\mathrm{nss}}} &= \begin{cases} \big(r_j + \delta_{j-1\in J}\big) - 0 & \text{if } j \notin J \\ \big(p-1-r_j-\delta_{j-1\notin J} - 2\delta_{j\in(J+1)^{\mathrm{ss}}}\big) - 2\delta_{j\in(J+1)^{\mathrm{nss}}} & \text{if } j \in J \end{cases} \\ &= \begin{cases} r_j + \delta_{j-1\in J\Delta(J-1)} & \text{if } j \notin J \\ p-3-r_j+\delta_{j-1\in J\Delta(J-1)} & \text{if } j \in J. \end{cases} \end{aligned}$$

We let $\underline{i} \overset{\mathrm{def}}{=} \underline{e}^{(J\cap(J+1))^{\mathrm{nss}}}$ and $J'' \overset{\mathrm{def}}{=} \big(J\Delta(J-1)\big)\setminus\{j_0\}$. In particular, we have $i_{j_0+1}=0$. Then by (4.18), for $j \neq j_0$ we have

$$\begin{aligned} \delta_{j\notin J''}\big(2i_j + t^{J+1}(J'')_j\big) &= \delta_{j\notin J''}\big(2\delta_{j\in(J\cap(J+1))^{\mathrm{nss}}} + p - 1 - s_j^{J+1} + \delta_{j-1\in J''}\big) \\ &= \begin{cases} p-1-r_j-\delta_{j=j_0+1} & \text{if } j\notin J,\ j+1\notin J \\ r_j+2-\delta_{j=j_0+1} & \text{if } j\in J,\ j+1\in J \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{4.98}$$

and $2i_{j_0} + t^{J+1}(J'')_{j_0}$ equals $p-1-r_j$ if $j_0 \notin J$ (which implies $j_0+1 \in J$), and equals r_j+2 if $j_0 \in J$ (which implies $j_0+1 \notin J$). In particular, by (4.61) we have $2i_{j_0}+t^{J+1}(J'')_{j_0} = f+1+c'^{J}_{j_0}$.

By Proposition 4.5.5 applied to $(\underline{i}, J+1, J'')$ with j_0 as above, taking $j' = j_0+1$ in (4.20) when $J''=\emptyset$ and multiplying $Y_{j_0+1}^{\delta_{J''\neq\emptyset}}$ when $j_0+1 \notin J'$, we deduce that

$$\left[Y_{j_0}^{f+1+c'^{J}_{j_0}} \prod_{j\notin J, j+1\notin J} Y_j^{p-1-r_j} \prod_{j\in J, j+1\in J} Y_j^{r_j+2}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \big(\underline{Y}^{-\underline{i}} v_{J+1}\big) = 0. \tag{4.99}$$

Comparing (4.96) and (4.99), to prove $Y_{j_0}^{f+1} z_J = 0$, it is enough to show that

$$c'^{J}_j \geq \begin{cases} p-1-r_j & \text{if } j\notin J,\ j+1\notin J \\ r_j+2 & \text{if } j\in J,\ j+1\in J, \end{cases} \tag{4.100}$$

which follows directly from (4.61) and (4.2).

(d). Suppose that $j_0 \notin J$ and $j_0+1 \in J^{\mathrm{nss}}$, which implies $f \geq 2$ and $j_0 \in J\Delta(J-1)$. In particular, we have $j_0 \notin J^{\mathrm{sh}}$. We let $\underline{i} \overset{\mathrm{def}}{=} \underline{e}^{(J\cap(J+1))^{\mathrm{nss}}}$ and $J'' \overset{\mathrm{def}}{=} \big(J\Delta(J-1)\big)\setminus\{j_0\}$. In particular, we have $i_{j_0+1}=0$. As in (c), the equality (4.98) still holds and we have $2i_{j_0} + t^{J+1}(J'')_{j_0} = f+1+c'^{J}_{j_0}$. We denote

$$Z \overset{\mathrm{def}}{=} \prod_{j+1\notin J, j+2\notin J} Y_j^{p-1-r_j} \prod_{j+1\in J, j+2\in J} Y_j^{r_j+2} \in \mathbb{F}[\![N_0]\!].$$

Then by Proposition 4.5.9 applied to $(\underline{i}, J+1, J'')$ with j_0 as above, using (4.33) and together with Lemma 4.3.1(i) applied to j_0+1 if moreover $j_0+1 \notin J''$, we have

$$\left[Y_{j_0+1}Y_{j_0}^{f+1+c'^{J}_{j_0}}Z\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}}v_{J+1}\right)$$
$$=\frac{\mu_{J+1,*}}{\mu_{(J+1)\backslash\{j_0+2\},*}}Y_{j_0+1}^{p-1-r_{j_0+1}}\left[Y_{j_0+1}Y_{j_0}^{f+1+c'^{J}_{j_0}}Z\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}'}v_{(J+1)\backslash\{j_0+2\}}\right), \quad (4.101)$$

where $\underline{i}' \overset{\text{def}}{=} \underline{i}+\delta_{j_0+2\in J^{\mathrm{ss}}}e_{j_0+2}$. Then using (4.100) together with $c'^{J}_{j_0+1}\geq 1$ by (4.61) and (4.2), we deduce from (4.96) and (4.101) that

$$Y_{j_0}^{f+1}z_J=\frac{\mu_{J+1,*}}{\mu_{(J+1)\backslash\{j_0+2\},*}}\left[Y_{j_0}^{f+1}Y_{j_0+1}^{p-1-r_{j_0+1}}\underline{Y}^{\underline{c}'^{J}}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}'}v_{(J+1)\backslash\{j_0+2\}}\right). \quad (4.102)$$

Since we have

$$\underline{i}'+\underline{e}^{((J+1)\backslash\{j_0+2\})^{\mathrm{sh}}}=\underline{i}+\underline{e}^{(J+1)^{\mathrm{sh}}}=\underline{e}^{J\cap(J+1)},$$

by (4.58) we have $\underline{Y}^{-\underline{i}'}v_{(J+1)\backslash\{j_0+2\}}=x_{(J+1)\backslash\{j_0+2\},\underline{e}^{J\cap(J+1)}}$. Then by (4.57) applied to $J\backslash\{j_0+1\}$ and $\underline{i}=\underline{e}^{J\cap(J+1)}$, and using $j_0+1\notin J^{\mathrm{ss}}$, we deduce from (4.102) that

$$Y_{j_0}^{f+1}z_J=\frac{\mu_{J+1,*}}{\mu_{(J+1)\backslash\{j_0+2\},*}}\left[\sum_{J^{\mathrm{ss}}\subseteq J'\subseteq J\backslash\{j_0+1\}}\mu_{(J+1)\backslash\{j_0+2\},J'}x_{J',\underline{c}(J')}\right]=\sum_{J^{\mathrm{ss}}\subseteq J'\subseteq J\backslash\{j_0+1\}}\mu_{J+1,J'}x_{J',\underline{c}(J')}$$

with $\underline{c}(J')\in\mathbb{Z}^f$ such that

$$\begin{aligned}c(J')_j&=p\delta_{j+1\in J\cap(J+1)}+c_j^{J\backslash\{j_0+1\}}+r_j^{(J\backslash\{j_0+1\})\backslash J'}-c'^{J}_j-(p-1-r_{j_0+1})\delta_{j=j_0+1}-(f+1)\delta_{j=j_0}\\&=c_j^{J\backslash\{j_0+1\}}+r_j^{(J\backslash\{j_0+1\})\backslash J'}-c_j^{J}-(p-1-r_{j_0+1})\delta_{j=j_0+1}+f-(f+1)\delta_{j=j_0}\\&=r_j^{(J\backslash\{j_0+1\})\backslash J'}+r_j^{\{j_0+1\}}+f-(f+1)\delta_{j=j_0}\\&=r_j^{J\backslash J'}+f-(f+1)\delta_{j=j_0},\end{aligned}$$

where the second equality follows from (4.62), the third equality follows from Lemma 4.6.3(v) applied to $J'=J\setminus\{j_0+1\}$, and the last equality follows from Lemma 4.6.3(ii). This proves the desired formula.

(e). Suppose that $j_0\in J$ and $j_0+1\in J^{\mathrm{nss}}$. We let $\underline{i}\overset{\text{def}}{=}\underline{e}^{(J\cap(J+1))^{\mathrm{nss}}\backslash\{j_0+1\}}$. In particular, we have $i_{j_0+1}=0$. Then by Proposition 4.5.9 applied to $(\underline{i},J+1,\mathcal{J})$ with j_0 as above together with (4.33), we have

$$\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}}v_{J+1}\right)=\frac{\mu_{J+1,*}}{\mu_{(J+1)\backslash\{j_0+2\},*}}Y_{j_0+1}^{p-1-r_{j_0+1}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}'}v_{(J+1)\backslash\{j_0+2\}}\right), \quad (4.103)$$

where $\underline{i}'\overset{\text{def}}{=}\underline{i}+\delta_{j_0+2\in J^{\mathrm{ss}}}e_{j_0+2}$. Since $c'^{J}_{j_0}=p-f$ by (4.61), we deduce from (4.96) that

$$\begin{aligned}Y_{j_0}^{f+1-\delta_{j_0\in J^{\mathrm{sh}}}}z_J&=\underline{Y}^{\underline{c}'^{J}+(f+1-\delta_{j_0\in J^{\mathrm{sh}}}-p)e_{j_0}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}}v_{J+1}\right)\\&=\frac{\mu_{J+1,*}}{\mu_{(J+1)\backslash\{j_0+2\},*}}\left[Y_{j_0+1}^{p-1-r_{j_0+1}}\underline{Y}^{\underline{c}'^{J}+(f+1-\delta_{j_0\in J^{\mathrm{sh}}}-p)e_{j_0}}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}'}v_{(J+1)\backslash\{j_0+2\}}\right)\\&=\frac{\mu_{J+1,*}}{\mu_{(J+1)\backslash\{j_0+2\},*}}\left[Y_{j_0}^{f+1-\delta_{j_0\in J^{\mathrm{sh}}}}Y_{j_0+1}^{p-1-r_{j_0+1}}\underline{Y}^{\underline{c}'^{J}}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\left(\underline{Y}^{-\underline{i}''}v_{(J+1)\backslash\{j_0+2\}}\right),\end{aligned}$$

where the first and the third equalities follow from Lemma 4.3.1(i) applied to j_0, the second equality follows from (4.103), and $\underline{i}''\overset{\text{def}}{=}\underline{i}'+e_{j_0+1}$ is the same as the $\underline{i}'$ in (4.102) by definition.

Then the rest of the proof is completely analogous to the one of (d), replacing $f+1$ by $f+1-\delta_{j_0\in J^{\mathrm{sh}}}$.

(ii). We prove the second equality. By (4.96) we have

$$\begin{aligned}\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}} z_J &= \underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}+\underline{c}'^J} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{e}^{J\cap(J+1)}} \\ &= \underline{Y}^{\underline{c}^J+(p-1)\underline{e}^{J^{\mathrm{sh}}}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{\underline{e}^{(J\cap(J+1))^{\mathrm{nss}}+1}} x_{J+1,\underline{e}^{J\cap(J+1)}}\right) \qquad (4.104) \\ &= \underline{Y}^{\underline{c}^J+(p-1)\underline{e}^{J^{\mathrm{sh}}}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{e}^{(J^{\mathrm{sh}}+1)}}, \qquad (4.105)\end{aligned}$$

where the second equality follows from (4.62) and Lemma 4.3.1(i), and the last equality follows from (4.58) and the equality $\big(J\cap(J+1)\big)\setminus\big((J\cap(J-1))^{\mathrm{nss}}+1\big)=J^{\mathrm{sh}}+1$.

Suppose that $J^{\mathrm{nss}}=(\partial J)^{\mathrm{nss}}$, or equivalently $\big(J\cap(J-1)\big)^{\mathrm{nss}}=\emptyset$. Then using (4.7) and (4.53), a case-by-case examination shows that

$$\underline{c}^J+(p-1)\underline{e}^{J^{\mathrm{sh}}} = \begin{cases} s_j^J & \text{if } j\in J\Delta(J-1) \\ p-1 & \text{if } j\notin J\Delta(J-1). \end{cases}$$

We also have $x_{J+1,\underline{e}^{J\cap(J+1)}}=\underline{Y}^{-\underline{e}^{(J\cap(J+1))^{\mathrm{nss}}}} v_{J+1}$ by (4.58). Then by Proposition 4.5.11 applied to $(J+1,J)$, we deduce from (4.104) that $\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}} z_J=\mu_{J+1,J}v_J$.

Suppose that $J^{\mathrm{nss}}\neq(\partial J)^{\mathrm{nss}}$ and $J^{\mathrm{nss}}\neq\mathcal{J}$, which implies $J^{\mathrm{ss}}\sqcup(\partial J)^{\mathrm{nss}}\subsetneq J$. Then by Lemma 4.7.7 applied to (J,J) and using $\underline{r}^{\emptyset}=\underline{0}$ by (4.52), we deduce from (4.105) that $\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}} z_J=0$.

Suppose that $J^{\mathrm{nss}}=\mathcal{J}$, or equivalently $(J,J_{\overline{\rho}})=(\mathcal{J},\emptyset)$, which implies $J+1=J$ and $J^{\mathrm{sh}}=\emptyset$. Then by Proposition 4.5.13 applied to $\mathcal{J}$ and using $\underline{c}^{\mathcal{J}}=\underline{0}$ by (4.53), we deduce from (4.105) that $\underline{Y}^{\underline{f}-\underline{e}^{J^{\mathrm{sh}}}} z_J=\mu_{J,J}v_J+\mu_{J,\emptyset}x_{\emptyset,\underline{r}}$. □

4.8 The finiteness condition

In this section, we prove the crucial finiteness condition for the family of elements $(x_{J,\underline{i}})_{J,\underline{i}}$ of Theorem 4.6.4 to give rise to a basis of $D_A(\pi)$. The main result is Theorem 4.8.5.

The following lemma will be crucial for the induction arguments in Proposition 4.8.2 and Proposition 4.8.4.

Lemma 4.8.1. *Let $J\subseteq\mathcal{J}$ and $\underline{i}\in\mathbb{Z}^f$. Suppose that $x_{J',\underline{i}}=0$ for all $|J'|\leq|J|$. Then we have $x_{J,p\delta(\underline{i})+\underline{c}^J}=0$.*

Proof. We use increasing induction on $|J|$. By Theorem 4.6.4(iii) we have

$$0=\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J+1,\underline{i}} = \sum_{J^{\mathrm{ss}}\subseteq J'\subseteq J} \varepsilon_{J'}\mu_{J+1,J'}x_{J',p\delta(\underline{i})+\underline{c}^J+\underline{r}^{J\setminus J'}}.$$

By the induction hypothesis, for all $J'\subsetneq J$ we have $x_{J',p\delta(\underline{i})+\underline{c}^{J'}}=0$. Since $\underline{c}^{J'}\geq\underline{c}^J+\underline{r}^{J\setminus J'}$ by Lemma 4.6.3(v), we deduce from Theorem 4.6.4(ii) that $x_{J',p\delta(\underline{i})+\underline{c}^J+\underline{r}^{J\setminus J'}}=0$ for all $J'\subsetneq J$. Hence we conclude that $x_{J,p\delta(\underline{i})+\underline{c}^J}=0$. □

Proposition 4.8.2. *Let $\underline{i}\in\mathbb{Z}^f$ satisfying*

(i) $\|\underline{i}\|\leq f$;
(ii) $i_j\leq -1$ *for some* $j\in\mathcal{J}$.

Then we have $x_{J,\underline{i}} = 0$ for all $J \subseteq \mathcal{J}$.

Proof. If $f = 1$, then we conclude using (4.58). From now on we assume that $f \geq 2$. We use increasing induction on $\max_j i_j$. If $\underline{i} \leq \underline{f}$, then the lemma follows from (4.58). Then we assume that $\max_j i_j > f$ and let $J_0 \subseteq \mathcal{J}$. We write $\underline{i} = p\delta(\underline{i}') + \underline{c}^{J_0} - \underline{\ell}$ for the unique $\underline{i}', \underline{\ell} \in \mathbb{Z}^f$ such that $\underline{0} \leq \underline{\ell} \leq \underline{p} - \underline{1}$. In particular, we have $\max_j i'_j < \max_j i_j$ (see (4.59)). Since $\underline{c}^{J_0} \geq \underline{0}$ by (4.53), we also have

$$\|\underline{i}'\| = \left(\|\underline{i}\| - \|\underline{c}^{J_0}\| + \|\underline{\ell}\|\right)/p \leq (f - 0 + (p-1)f)/p = f.$$

Suppose that $i'_j < 0$ for some j. Then by the induction hypothesis, we have $x_{J,\underline{i}'} = 0$ for all $J \subseteq \mathcal{J}$ (in particular, for all $|J| \leq |J_0|$). Since $p\delta(\underline{i}') + \underline{c}^{J_0} \geq \underline{i}$, we deduce from Lemma 4.8.1 and Theorem 4.6.4(ii) that $x_{J_0,\underline{i}} = 0$.

Suppose that $i'_j \geq 0$ for all j, which implies $i_j \geq -(p-1)$ for all j. Since $\|i\| \leq f$, we deduce that $i_j \leq (f-1)(p-1) + f$ for all j. We write $\min_j i_j = -m'$ with $1 \leq m' \leq p-1$ and fix $j_0 \in \mathcal{J}$ such that $i_{j_0} = -m'$. Then we let $\underline{n} \in \mathbb{Z}^f$ with $n_{j_0+1} = 0$, $n_j = m'$ for $j \neq j_0 + 1$ if $1 \leq m' \leq 2f-2$, and $n_j = 2f-1$ for $j \neq j_0+1$ if $2f-1 \leq m' \leq p-1$. In particular, $\underline{n}$ satisfies the conditions in Definition 4.7.2(ii) for J_0 and j_0. By Proposition 4.7.4 applied to J_0 and j_0 we have $x_{J_0,\underline{a}^{J_0}(\underline{n}) - e_{j_0+1}} = 0$. Then the result follows from Theorem 4.6.4(ii) and the Claim below.

Claim. We have $\underline{a}^{J_0}(\underline{n}) - e_{j_0+1} \geq \underline{i}$.

Proof. We assume that $1 \leq m' \leq 2f-2$, the case $2f-1 \leq m' \leq p-1$ being similar. Since $\|\underline{i}\| \leq f$ and $i_j \geq -m'$ for all j, we deduce that $i_j \leq (f-1)m' + f$ for all j. Hence it suffices to show that

$$\begin{cases} a^{J_0}(\underline{n})_{j_0} \geq -m' \\ a^{J_0}(\underline{n})_j - \delta_{j=j_0+1} \geq (f-1)m' + f \quad \text{if } j \neq j_0. \end{cases}$$

By Definition 4.7.2(ii), we have $a^{J_0}(\underline{n})_{j_0} = 0 \geq -m'$ if $j_0 \in J_0^{\mathrm{sh}}$, and $a^{J_0}(\underline{n})_{j_0} = t_{j_0}^{J_0}(0) - n_{j_0} = -m'$ (since $n_{j_0+1} = 0$) if $j_0 \notin J_0^{\mathrm{sh}}$. For $j \neq j_0$, by definition we have

$$\begin{aligned} a^{J_0}(\underline{n})_j - \delta_{j=j_0+1} &= t_j^{J_0}(m') - n_j - \delta_{j=j_0+1} \\ &= p[m'/2] + \delta_{2\nmid m'}\left(\delta_{j+1\notin J}(r_j+1) + \delta_{j+1\in J}(p-1-r_j)\right) - (n_j + \delta_{j=j_0+1}) \\ &\geq (4f+4)[m'/2] + \delta_{2\nmid m'}(2f+2) - 1 \\ &= (2f+2)m' - 1 = fm' + fm' + (2m'-1) > (f-1)m' + f, \end{aligned}$$

where the first inequality uses (4.2) and $p \geq 4f+4$, and the last inequality uses $m' \geq 1$. Here for $x \in \mathbb{R}$, we denote by $[x]$ the largest integer which is smaller than or equal to x. This proves the claim. □

Corollary 4.8.3. *Let $J \subseteq \mathcal{J}$, $\underline{i} \in \mathbb{Z}^f$ and $\underline{k} \in \mathbb{Z}^f_{\geq 0}$.*

(i) *If $\|\underline{k}\| > \|\underline{i}\| - |J^{\mathrm{sh}}|$, then we have $\underline{Y}^{\underline{k}} x_{J,\underline{i}} = 0$.*
(ii) *If $\|\underline{k}\| = \|\underline{i}\| - |J^{\mathrm{sh}}|$ and $\underline{Y}^{\underline{k}} x_{J,\underline{i}} \neq 0$, then we have $\underline{k} = \underline{i} - \underline{e}^{J^{\mathrm{sh}}}$.*

Proof. By Theorem 4.6.4(ii) we have $\underline{Y}^{\underline{k}} x_{J,\underline{i}} = x_{J,\underline{\ell}}$ with $\underline{\ell} \overset{\text{def}}{=} \underline{i} - \underline{k}$. In both cases, we have $\|\underline{\ell}\| \leq f$ since $|J^{\mathrm{sh}}| \leq f$. If $\underline{\ell} \geq \underline{0}$, then we have $\underline{\ell} \leq \underline{f}$, and the result follows from (4.58). If $\underline{\ell} \not\geq \underline{0}$, then the result follows from Proposition 4.8.2. □

Proposition 4.8.4. *Let $m \in \mathbb{Z}_{\geq 0}$ and $\underline{i} \in \mathbb{Z}^f$ satisfying*

(i) $\|\underline{i}\| \le p^m + f - 1$;
(ii) $i_j \le -p^m$ *for some* $j \in \mathcal{J}$.

Then we have $x_{J,\underline{i}} = 0$ *for all* $J \subseteq \mathcal{J}$.

Proof. We prove the result by increasing induction on m. For $m = 0$, this is exactly Proposition 4.8.2. Then we let $m \ge 1$ and fix $J \subseteq \mathcal{J}$. We write $\underline{i} = p\delta(\underline{i}') + \underline{c}^J - \underline{\ell}$ for the unique $\underline{i}', \underline{\ell} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{\ell} \le \underline{p} - \underline{1}$ and fix $j_0 \in \mathcal{J}$ such that $i_{j_0} \le -p^m$. Since $\underline{c}^J \ge \underline{0}$ by (4.53), we have

$$\|\underline{i}'\| = \big(\|\underline{i}\| - \|\underline{c}^{J+1}\| + \|\underline{\ell}\|\big)/p \le \big((p^m + f - 1) - 0 + (p-1)f\big)/p = p^{m-1} + f - 1/p,$$

which implies $\|\underline{i}'\| \le p^{m-1} + f - 1$. We also have

$$i'_{j_0+1} = \Big(i_{j_0} - c_{j_0}^{J+1} + \ell_{j_0}\Big)/p \le \big(-p^m - 0 + (p-1)\big)/p = -p^{m-1} + (p-1)/p,$$

which implies $i'_{j_0+1} \le -p^{m-1}$. By the induction hypothesis, we have $x_{J',\underline{i}'} = 0$ for all $J' \subseteq \mathcal{J}$. Since $p\delta(\underline{i}') + \underline{c}^J \ge \underline{i}$, we conclude from Lemma 4.8.1 and Theorem 4.6.4(ii) that $x_{J,\underline{i}} = 0$. □

Theorem 4.8.5 (Finiteness condition)**.** *For* $J \subseteq \mathcal{J}$ *and* $M \in \mathbb{Z}$, *the set* $\big\{\underline{i} \in \mathbb{Z}^f : x_{J,\underline{i}} \ne 0, \|\underline{i}\| = M\big\}$ *is finite.*

Proof. We choose m large enough such that $p^m + f - 1 \ge M$. If $i_{j_0} \le -p^m$ for some j_0, then by Proposition 4.8.4 we have $x_{J,\underline{i}} = 0$. Otherwise, we have $i_j > -p^m$ for all j. Together with the restriction $\|\underline{i}\| = M$, this set is finite. □

4.9 An explicit basis of $\mathrm{Hom}_A(D_A(\pi), A)$

In this section, we construct an explicit basis of $\mathrm{Hom}_A(D_A(\pi), A)$. In particular, we prove that $D_A(\pi)$ has rank 2^f, see Theorem 4.9.5.

First we recall the definition of the ring A and the A-module $D_A(\pi)$. We let $\mathfrak{m}_{N_0}$ be the maximal ideal of $\mathbb{F}[\![N_0]\!]$. Then we have $\mathbb{F}[\![N_0]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$ and $\mathfrak{m}_{N_0} = (Y_0, \dots, Y_{f-1})$. Consider the multiplicative subset $S \overset{\text{def}}{=} \{(Y_0 \cdots Y_{f-1})^n : n \ge 0\}$ of $\mathbb{F}[\![N_0]\!]$. Then $A \overset{\text{def}}{=} \widehat{\mathbb{F}[\![N_0]\!]_S}$ is the completion of the localization $\mathbb{F}[\![N_0]\!]_S$ with respect to the $\mathfrak{m}_{N_0}$-adic filtration

$$F_n\left(\mathbb{F}[\![N_0]\!]_S\right) = \bigcup_{k \ge 0} \frac{1}{(Y_0 \cdots Y_{f-1})^k} \mathfrak{m}_{N_0}^{kf-n},$$

where $\mathfrak{m}_{N_0}^m \overset{\text{def}}{=} \mathbb{F}[\![N_0]\!]$ if $m \le 0$. We denote by $F_n A$ ($n \in \mathbb{Z}$) the induced filtration on A and endow A with the associated topology ([LvO96, §1.3]). There is an $\mathbb{F}$-linear action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![N_0]\!]$ given by multiplication on $N_0 \cong \mathcal{O}_K$, and an $\mathbb{F}$-linear Frobenius φ on $\mathbb{F}[\![N_0]\!]$ given by multiplication by p on $N_0 \cong \mathcal{O}_K$. They extend canonically by continuity to commuting continuous $\mathbb{F}$-linear actions of φ and $\mathcal{O}_K^\times$ on A which satisfies (for each $j \in \mathcal{J}$)

$$\begin{aligned} \varphi(Y_j) &= Y_{j-1}^p; \\ [a](Y_j) &= a^{p^j} Y_j \ \forall\, a \in \mathbb{F}_q^\times. \end{aligned} \tag{4.106}$$

We let $\pi^\vee$ be the $\mathbb{F}$-linear dual of π, which is a finitely generated $\mathbb{F}[\![I_1]\!]$-module and is endowed with the $\mathfrak{m}_{I_1}$-adic topology, where $\mathfrak{m}_{I_1}$ is the maximal ideal of $\mathbb{F}[\![I_1]\!]$. We define $D_A(\pi)$ to be the completion of $\mathbb{F}[\![N_0]\!]_S \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$ with respect to the tensor product topology. The $\mathcal{O}_K^\times$-action on $\pi^\vee$ given by $f \mapsto f \circ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (for $a \in \mathcal{O}_K^\times$) extends by continuity to $D_A(\pi)$, and the ψ-action on $\pi^\vee$

given by $f \mapsto f \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induces a continuous A-linear map $\beta : D_A(\pi) \to A \otimes_{\varphi,A} D_A(\pi)$. Moreover, $D_A(\pi)$ is a finite free A-module by [BHH$^+$b, Remark 3.3.2.6(ii)], [BHH$^+$b, Cor. 3.1.2.9] and [BHH$^+$c, Remark. 2.6.2].

Then we recall some constructions in [BHH$^+$c, §3]. As in [BHH$^+$c, (87)], there exists an injective A-linear map

$$\mu_* : \mathrm{Hom}_A(D_A(\pi), A) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F}). \tag{4.107}$$

By [BHH$^+$c, Prop. 3.2.3], $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ is identified with the set of sequences $(x_k)_{k\geq 0}$ with $x_k \in \pi$ and

(i) $\underline{Y}^{\underline{1}} x_k = x_{k-1}$ for all $k \geq 1$;
(ii) there exists $d \in \mathbb{Z}$ such that $x_k \in \pi[\mathfrak{m}_{I_1}^{fk+d+1}]$ for all $k \geq 0$ (where $\pi[\mathfrak{m}_{I_1}^{j}] \overset{\mathrm{def}}{=} 0$ if $j \leq 0$),

and the A-module structure on $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ is given as follows: for $a \in A$ and $x = (x_k)_{k\geq 0} \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$, we have $a(x) = (y_k)_{k\geq 0}$ with

$$y_k = (\underline{Y}^{\underline{\ell-k}} a) x_\ell \tag{4.108}$$

for $\ell \gg_k 0$. See [BHH$^+$c, Remark 3.8.2] for the explanation of (4.108). We are going to construct 2^f elements of $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ using the elements $\left\{x_{J,\underline{i}} : J \subseteq \mathcal{J}, \underline{i} \in \mathbb{Z}^f\right\}$ in π (see Theorem 4.6.4). Then we prove that they lie in the image of μ_*, and give rise to an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$.

Let $Z_1 \cong 1 + p\mathcal{O}_K$ be the center of I_1. Since π has a central character, Z_1 acts trivially on π. We still denote by $\mathfrak{m}_{I_1}$ the maximal ideal of $\mathbb{F}[\![I_1/Z_1]\!]$ when there is no possible confusion. For $0 \leq j \leq f-1$, we view Y_j as an element of $\mathbb{F}[\![I_1/Z_1]\!]$ and we define

$$Z_j \overset{\mathrm{def}}{=} \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \begin{pmatrix} 1 & 0 \\ p[\lambda] & 1 \end{pmatrix} \in \mathbb{F}[\![I_1/Z_1]\!].$$

Since Z_j commutes with each other, for $\underline{i} \in \mathbb{Z}_{\geq 0}^f$ we write $\underline{Z}^{\underline{i}}$ for $\prod_{j=0}^{f-1} Z_j^{i_j}$. For $0 \leq j \leq f-1$, we denote by $y_j, z_j \in \mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$ (the graded ring for the $\mathfrak{m}_{I_1}$-adic filtration) the associated elements of $Y_j, Z_j \in \mathbb{F}[\![I_1/Z_1]\!]$. We define the $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module

$$\mathrm{gr}(\pi) \overset{\mathrm{def}}{=} \bigoplus_{n\geq 0} \pi[\mathfrak{m}_{I_1}^{n+1}]/\pi[\mathfrak{m}_{I_1}^{n}].$$

By the proof of [BHH$^+$23, Cor. 5.3.5] and the assumptions on π (see above Theorem 4.1.1) and taking $\mathbb{F}$-linear dual, the $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module $\mathrm{gr}(\pi)$ is annihilated by the ideal $(y_j z_j, z_j y_j;\ 0 \leq j \leq f-1)$, hence becomes a graded module over $R \overset{\mathrm{def}}{=} \mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/(y_j z_j, z_j y_j;\ 0 \leq j \leq f-1)$, which is a commutative ring, isomorphic to $\mathbb{F}[y_j, z_j]/(y_j z_j;\ 0 \leq j \leq f-1)$ with y_j, z_j of degree -1 (see [BHH$^+$23, Thm. 5.3.4]). For $v \in \pi$, as in [BHH$^+$c, §3.5] we define

$$\deg(v) \overset{\mathrm{def}}{=} \min\{n \geq -1 : v \in \pi[\mathfrak{m}_{I_1}^{n+1}]\} \in \mathbb{Z}_{\geq -1}.$$

We denote $\mathrm{gr}(v) \in \pi[\mathfrak{m}_{I_1}^{\deg(v)+1}]/\pi[\mathfrak{m}_{I_1}^{\deg(v)}] \subseteq \mathrm{gr}(\pi)$ if $v \neq 0$ and $\mathrm{gr}(v) = 0$ if $v = 0$ the associated graded element of v.

Lemma 4.9.1. *Let $v \in \pi$ with $\deg(v) = d \geq 0$.*

(i) *For $j \in \mathcal{J}$, we have* $\deg(Y_j v) \le \deg(v) - 1$. *If moreover $d > 0$, then the equality holds if and only if* $y_j \operatorname{gr}(v) = \operatorname{gr}(Y_j v) \neq 0$ *in* $\operatorname{gr}(\pi)$. *Similar statements hold for Z_j.*

(ii) *There exists $\underline{a}, \underline{b} \in \mathbb{Z}^f_{\ge 0}$ satisfying $\|\underline{a}\| + \|\underline{b}\| = d$ such that $0 \neq \underline{Y}^{\underline{a}} \underline{Z}^{\underline{b}} v \in \pi^{I_1}$.*

(iii) *We have* $\deg\left(\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v\right) \le pd + (p-1)f$.

Proof. (i). This follows from the fact that $y_j, z_j \in R$ has degree -1.

(ii). If $d = 0$, then the statement is trivial, so we let $d > 0$. Since $y_0, \dots, y_{f-1}, z_0, \dots, z_{f-1}$ form an $\mathbb{F}$-basis of the degree -1 part of R, which equals $\mathfrak{m}_{I_1}/\mathfrak{m}_{I_1}^2$, there exists one of them, say y_j, such that $y_j \operatorname{gr}(v) \neq 0$ (otherwise, $\operatorname{gr}(v)$ is annihilated by $\mathfrak{m}_{I_1}/\mathfrak{m}_{I_1}^2$ in $\operatorname{gr}(\pi)$, so $\mathfrak{m}_{I_1} v \subseteq \pi[\mathfrak{m}_{I_1}^{d-1}]$, i.e. $v \in \pi[\mathfrak{m}_{I_1}^{d}]$, a contradiction). By (i), we have $\deg(Y_j v) = d - 1$. If $d - 1 > 0$, continue this process to $Y_j v \in \pi$ and so on.

In particular, there exist $W_1, \dots, W_d \in \{Y_0, \dots, Y_{f-1}, Z_0, \dots, Z_{f-1}\}$ such that $W_1 \cdots W_d\, v \in \pi$ has degree 0 and $w_1 \cdots w_d \operatorname{gr}(v) \neq 0$ in $\operatorname{gr}(\pi)$, where $w_i \in R$ is the associated graded element of W_i for $1 \le i \le d$. We let $W_1', \dots, W_d'$ be a permutation of $W_1, \dots, W_d$ such that $W_1' \cdots W_d'$ is of the form $\underline{Y}^{\underline{a}} \underline{Z}^{\underline{b}}$ as in the statement. Since R is commutative, we have $w_1' \cdots w_d' \operatorname{gr}(v) \neq 0$ in $\operatorname{gr}(\pi)$. As a consequence, $W_1' \cdots W_d' v \neq 0$ and has degree zero by (i), hence belongs to π^{I_1}.

(iii). By (ii), it suffices to show that $\underline{Y}^{\underline{a}} \underline{Z}^{\underline{b}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v = 0$ for all $\underline{a}, \underline{b} \in \mathbb{Z}^f_{\ge 0}$ such that $\|\underline{a}\| + \|\underline{b}\| \ge pd + (p-1)f + 1$. We write $\underline{a} = p\underline{c} + \underline{\ell}$ for the unique $\underline{c} \ge \underline{0}$ and $\underline{0} \le \underline{\ell} \le \underline{p} - \underline{1}$. One easily checks that $Z_j \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) Z_{j-1}^p$ for all $j \in \mathcal{J}$. Together with Lemma 4.3.1(i), we have

$$\underline{Y}^{\underline{a}} \underline{Z}^{\underline{b}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v = \underline{Y}^{\underline{\ell}} \underline{Y}^{p\underline{c}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Z}^{p\delta(\underline{b})} v = \underline{Y}^{\underline{\ell}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{\delta^{-1}(\underline{c})} \underline{Z}^{p\delta(\underline{b})} v.$$

Since $\deg(v) = d$, using (i) it suffices to show that $\|\delta^{-1}(\underline{c})\| + \|p\delta(\underline{b})\| > d$. Indeed, we have

$$\begin{aligned} \|\delta^{-1}(\underline{c})\| + \|p\delta(\underline{b})\| &= \|\underline{c}\| + p\|\underline{b}\| = (\|a\| - \|\ell\|)/p + p\|b\| \\ &\ge (\|a\| - (p-1)f)/p + p\|b\| \ge (\|a\| + \|b\| - (p-1)f)/p \ge (pd+1)/p > d, \end{aligned}$$

which completes the proof. □

Recall that we have constructed $x_{J,\underline{i}} \in \pi$ for $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ in Theorem 4.6.4.

Lemma 4.9.2. *Let $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{i} \ge \underline{e}^{J^{\mathrm{sh}}}$ (see §4.2 for $\underline{e}^{J^{\mathrm{sh}}}$ and note that $\|\underline{e}^{J^{\mathrm{sh}}}\| = |J^{\mathrm{sh}}|$).*

(i) *If $z_j \operatorname{gr}(x_{J,\underline{i}}) = 0$ for all $j \in \mathcal{J}$, then we have* $\deg(x_{J,\underline{i}}) = \|\underline{i}\| - |J^{\mathrm{sh}}|$.

(ii) *If* $\deg(x_{J,\underline{i}}) > \|\underline{i}\| - |J^{\mathrm{sh}}|$, *then there exists $j_0 \in \mathcal{J}$ such that* $\deg(x_{J,\underline{i}+e_{j_0}}) \ge \deg(x_{J,\underline{i}}) + 2$.

Proof. (i). By the second paragraph of the proof of [BHH+c, Prop. 3.5.1], there exists $\underline{a} \in \mathbb{Z}^f_{\ge 0}$ such that $0 \neq \underline{Y}^{\underline{a}} x_{J,\underline{i}} \in \pi^{I_1}$ and $\deg(x_{J,\underline{i}}) = \|\underline{a}\|$. By Theorem 4.6.4(ii) and (4.60), we have $\underline{Y}^{\underline{i}-\underline{e}^{J^{\mathrm{sh}}}} x_{J,\underline{i}} = x_{J,\underline{e}^{J^{\mathrm{sh}}}} = v_J \neq 0$, hence $\|\underline{a}\| = \deg(x_{J,\underline{i}}) \ge \|\underline{i}\| - |J^{\mathrm{sh}}|$ by Lemma 4.9.1(i). Then by Corollary 4.8.3(i),(ii), we must have $\underline{a} = \underline{i} - \underline{e}^{J^{\mathrm{sh}}}$, hence $\deg(x_{J,\underline{i}}) = \|\underline{i}\| - |J^{\mathrm{sh}}|$.

(ii). By (i), there exists $j_0 \in \mathcal{J}$ such that $z_{j_0} \operatorname{gr}(x_{J,\underline{i}}) \neq 0$. Since $Y_{j_0} x_{J,\underline{i}+e_{j_0}} = x_{J,\underline{i}}$ by Theorem 4.6.4(ii), we have $\deg(x_{J,\underline{i}+e_{j_0}}) \ge \deg(x_{J,\underline{i}}) + 1$ by Lemma 4.9.1(i). Assume on the contrary that $\deg(x_{J,\underline{i}+e_{j_0}}) = \deg(x_{J,\underline{i}}) + 1$, then by Lemma 4.9.1(i) we have $y_{j_0} \operatorname{gr}(x_{J,\underline{i}+e_{j_0}}) = \operatorname{gr}(x_{J,\underline{i}})$, hence $z_{j_0} y_{j_0} \operatorname{gr}(x_{J,\underline{i}+e_{j_0}}) = z_{j_0} \operatorname{gr}(x_{J,\underline{i}}) \neq 0$. This is a contradiction since $z_{j_0} y_{j_0} = 0$ in R. □

Lemma 4.9.3. *For $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{i} \le \underline{f} + \underline{1}$, we have $x_{J,\underline{i}} \in \pi^{K_1}$.*

Proof. By Theorem 4.6.4(ii), it suffices to show that $x_{J,\underline{f}+\underline{1}} \in \pi^{K_1}$.

Recall that $\underline{c}'^J \in \mathbb{Z}^f$ is defined in (4.61), which satisfies $\underline{1} \le \underline{c}'^J \le \underline{p} - \underline{1}$ by (4.2). By the proof of Lemma 4.6.9 except that we apply Lemma 4.6.3(iv) with $\underline{\delta} = \underline{1}$ instead of $\underline{\delta} = \underline{0}$, we have

(i) $\underline{Y}^{\underline{c}'^J-\underline{1}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)x_{J+1,\underline{e}^{J\cap(J+1)}}\in\pi^{K_1}$;
(ii) $x_{J',\underline{f}+\underline{1}+\underline{r}^{J\setminus J'}}\in\pi^{K_1}$ for each $J'\subseteq\mathcal{J}$ such that $J^{\mathrm{ss}}\subseteq J'\subsetneqq J$ (see (4.52) for $\underline{r}^{J\setminus J'}$).

Moreover, by (4.62) we have (see (4.53) for $\underline{c}^J$)

$$\underline{f}+\underline{1}=p\delta\big(\underline{e}^{J\cap(J+1)}\big)+\underline{c}^J-(\underline{c}'^J-\underline{1}).$$

Hence we deduce from (4.60) (with $\underline{i}=\underline{f}+\underline{1}$) that $x_{J,\underline{f}+\underline{1}}\in\pi^{K_1}$. □

The following proposition is a generalization of [BHH$^+$c, Prop. 3.5.1] (where $\overline{\rho}$ was assumed to be semisimple).

Proposition 4.9.4. *For $J\subseteq\mathcal{J}$ and $\underline{i}\in\mathbb{Z}^f$, we have* $\deg\big(x_{J,\underline{i}}\big)\le\|\underline{i}\|-|J^{\mathrm{sh}}|$. *If moreover $\underline{i}\ge\underline{e}^{J^{\mathrm{sh}}}$, then we have* $\deg\big(x_{J,\underline{i}}\big)=\|\underline{i}\|-|J^{\mathrm{sh}}|$.

Proof. First we make the following observation. Let $J\subseteq\mathcal{J}$ and $m\ge1$. Assume that $\deg(x_{J,\underline{i}})=\|\underline{i}\|-|J^{\mathrm{sh}}|$ for all $\underline{e}^{J^{\mathrm{sh}}}\le\underline{i}\le\underline{m}$, then we have $\deg(x_{J,\underline{i}})\le\|\underline{i}\|-|J^{\mathrm{sh}}|$ for all $\underline{i}\le\underline{m}$. Indeed, by Theorem 4.6.4(ii) we have $x_{J,\underline{i}}=\underline{Y}^{\underline{m}-\underline{i}}x_{J,\underline{m}}$, hence $\deg(x_{J,\underline{i}})\le(\|\underline{m}\|-|J^{\mathrm{sh}}|)-\|\underline{m}-\underline{i}\|=\|\underline{i}\|-|J^{\mathrm{sh}}|$ by Lemma 4.9.1(i). In particular, we only need to prove the result for $\underline{i}\ge\underline{e}^{J^{\mathrm{sh}}}$.

We prove the result by increasing induction on $|J|$ and on $\max_j i_j$. For $J\subseteq\mathcal{J}$ and $\underline{i}\in\mathbb{Z}^f$ such that $\underline{e}^{J^{\mathrm{sh}}}\le\underline{i}\le\underline{f}+\underline{1}$, by Lemma 4.9.3 we have $Z_jx_{J,\underline{i}}=0$ for all $j\in\mathcal{J}$, hence $z_j\,\mathrm{gr}(x_{J,\underline{i}})=0$ for all $j\in\mathcal{J}$. By Lemma 4.9.2(i) we deduce that $\deg(x_{J,\underline{i}})=\|\underline{i}\|-|J^{\mathrm{sh}}|$.

Then we let $0\le k\le f-1$ and $m\ge f+1$. Assume that the result is true for

(a) $|J|\le k-1$ and $\underline{i}\in\mathbb{Z}^f$;
(b) $|J|=k$ and $\max_j i_j\le m$,

we prove the result for $|J|=k$ and $\underline{i}\ge\underline{e}^{J^{\mathrm{sh}}}$ such that $\max_j i_j=m+1$.

Claim. For $J\subseteq\mathcal{J}$ such that $|J|=k$ and $\underline{i}\in\mathbb{Z}^f$ such that $\max_j i_j\le pm$, we have

$$\deg(x_{J,\underline{i}})\le\|\underline{i}\|+(p-1)f.$$

Proof. We write $\underline{i}=p\delta(\underline{i}')+\underline{c}^J-\underline{\ell}$ for the unique $\underline{i}',\underline{\ell}\in\mathbb{Z}^f$ such that $\underline{0}\le\underline{\ell}\le\underline{p}-\underline{1}$ (see (4.53) for $\underline{c}^J$). Then we claim that $\max_j i'_j\le m$. Indeed, for each j we have

$$i'_{j+1}=\big(i_j-c_j^J+\ell_j\big)/p\le\big(pm-0+(p-1)\big)/p<m+1,$$

hence $i'_{j+1}\le m$. Since $|J+1|=|J|=k$, by (b) we have $\deg(x_{J+1,\underline{i}'})\le\|\underline{i}'\|-|(J+1)^{\mathrm{sh}}|\le\|\underline{i}'\|$. Then by Lemma 4.9.1(i),(iii) we have

$$\deg\Big(\underline{Y}^{\underline{\ell}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)x_{J+1,\underline{i}'}\Big)\le p\|\underline{i}'\|+(p-1)f-\|\underline{\ell}\|=\|\underline{i}\|-\|\underline{c}^J\|+(p-1)f\le\|\underline{i}\|+(p-1)f,\quad(4.109)$$

where the last inequality uses $\underline{c}^J\ge\underline{0}$. For $J'\subseteq\mathcal{J}$ such that $J^{\mathrm{ss}}\subseteq J'\subsetneqq J$, by (a) we have (see (4.52) for $\underline{r}^{J\setminus J'}$)

$$\deg\big(x_{J',\underline{i}+\underline{r}^{J\setminus J'}}\big)\le\|\underline{i}\|+\|\underline{r}^{J\setminus J'}\|-|(J')^{\mathrm{sh}}|\le\|\underline{i}\|+(p-1)f,\qquad(4.110)$$

where the last inequality uses $r_j^{J\setminus J'}\le p-1$ for all $j\in\mathcal{J}$ by (4.2). Combining (4.60), (4.109) and (4.110), we deduce that $\deg(x_{J,\underline{i}})\le\|\underline{i}\|+(p-1)f$. □

Assume on the contrary that $\deg(x_{J_0,\underline{i}^{(1)}}) \geq \|\underline{i}^{(1)}\| - |J_0^{\mathrm{sh}}| + 1$ for some $|J_0| = k$ and $\underline{i}^{(1)} \geq \underline{e}^{J^{\mathrm{sh}}}$ such that $\max_j i_j^{(1)} = m+1$. By Lemma 4.9.2(ii), there exists $\underline{i}^{(2)} = \underline{i}^{(1)} + e_{j_1}$ for some $j_1 \in \mathcal{J}$ such that $\deg\left(x_{J_0,\underline{i}^{(2)}}\right) \geq \|\underline{i}^{(1)}\| - |J_0^{\mathrm{sh}}| + 3 = \|\underline{i}^{(2)}\| - |J_0^{\mathrm{sh}}| + 2$. Moreover, we have $\max_j i_j^{(2)} \leq m+2$. Similarly, there exists $\underline{i}^{(3)} = \underline{i}^{(2)} + e_{j_2}$ for some $j_2 \in \mathcal{J}$ such that $\deg\left(x_{J_0,\underline{i}^{(3)}}\right) \geq \|\underline{i}^{(3)}\| - |J_0^{\mathrm{sh}}| + 3$, which moreover satisfies $\max_j i_j^{(3)} \leq m+3$. Continue this process, there exists $\underline{i}^{((p-1)m)} \in \mathbb{Z}^f$ such that $\max_j i_j^{((p-1)m)} \leq pm$ and

$$\deg\left(x_{J_0,\underline{i}^{((p-1)m)}}\right) \geq \|\underline{i}^{((p-1)m)}\| - |J_0^{\mathrm{sh}}| + (p-1)m.$$

By the Claim above, we also have

$$\deg\left(x_{J_0,\underline{i}^{((p-1)m)}}\right) \leq \|\underline{i}^{((p-1)m)}\| + (p-1)f.$$

This is a contradiction since $m \geq f+1$ and $|J_0^{\mathrm{sh}}| \leq f \leq p-2$ by (4.2). □

For $J \subseteq \mathcal{J}$, we define the sequence $x_J = (x_{J,k})_{k\geq 0}$ by $x_{J,k} \overset{\mathrm{def}}{=} x_{J,\underline{k}}$, which is defined in Theorem 4.6.4. Since $x_{J,k} \in \pi\left[\mathfrak{m}_{I_1}^{kf-|J^{\mathrm{sh}}|+1}\right]$ for all $k \geq 0$ by Proposition 4.9.4, we have $x_J \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$. Then we have the following generalization of [BHH$^+$c, Thm. 3.7.1] (where $\overline{\rho}$ was assumed to be semisimple).

Theorem 4.9.5. *The sequences $\{x_J : J \subseteq \mathcal{J}\}$ are contained in the image of the injection*

$$\mu_* : \mathrm{Hom}_A(D_A(\pi), A) \to \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$$

and form an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$. In particular, $D_A(\pi)$ is a free A-module of rank 2^f.

Proof. We follow closely the proof of [BHH$^+$c, Thm. 3.7.1] and use without comment the notation of *loc.cit.*.

First, the proof of *loc.cit.* using Theorem 4.8.5 shows that each $x_J \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ comes from an element of $\mathrm{Hom}_A(D_A(\pi), A)$, and we still denote it by x_J.

For each $J \subseteq \mathcal{J}$, we define another sequence $x'_J = \left(x'_{J,k}\right)_{k\geq 0}$ by $x'_{J,k} \overset{\mathrm{def}}{=} x_{J,\underline{k}+\underline{e}^{J^{\mathrm{sh}}}}$. In particular, we have $x'_{J,0} = v_J$ by (4.58). By (4.108) we have $x'_J = \underline{Y}^{-\underline{e}^{J^{\mathrm{sh}}}} x_J$, which implies that $x'_J \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi), \mathbb{F})$ and comes from an element of $\mathrm{Hom}_A(D_A(\pi), A)$ (recall $\underline{Y}^{-\underline{e}^{J^{\mathrm{sh}}}} \in A$), and we still denote it by x'_J.

Since $\underline{Y}^{-\underline{e}^{J^{\mathrm{sh}}}}$ is invertible in A, it suffices to show that $\{x'_J, J \subseteq \mathcal{J}\}$ form an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)$. As in the proof of *loc.cit.*, it suffices to show that the elements $\{\mathrm{gr}(x'_J) : J \subseteq \mathcal{J}\}$ form a $\mathrm{gr}\, A$-basis of $\mathrm{Hom}_{\mathrm{gr}\, A}(\mathrm{gr}\, D_A(\pi), \mathrm{gr}\, A)$.

Since $\pi^{I_1} = D_0(\overline{\rho})^{I_1}$ is multiplicity-free by Lemma 4.4.1(ii) (see the assumptions on π above Theorem 4.1.1), there exist unique I-eigenvectors $v_J^* \in (\pi^{I_1})^\vee = \mathrm{gr}_0(\pi^\vee)$ for $J \subseteq \mathcal{J}$ such that $\langle v_J, v_{J'}^* \rangle = \delta_{J=J'}$. As in the proof of [BHH$^+$c, Lemma 3.7.2], we know that $\mathrm{gr}\, D_A(\pi)$ is a free $\mathrm{gr}\, A$-module, and that there exists a surjection of $\mathrm{gr}\, A$-modules

$$\bigoplus_{J\subseteq\mathcal{J}} \mathrm{gr}\, A \twoheadrightarrow \mathrm{gr}\, D_A(\pi), \tag{4.111}$$

sending the standard basis element indexed by J on the left to the image of v_J^* in $\mathrm{gr}\, D_A(\pi)$ (still denoted v_J^*). To complete the proof, it is enough to show that $\langle \mathrm{gr}(x'_J), v_{J'}^* \rangle = \delta_{J=J'} \underline{y}^{-\underline{1}}$ in $\mathrm{gr}\, A$ for all $J, J' \subseteq \mathcal{J}$, which implies that the surjection (4.111) is an isomorphism. The argument here is completely analogous to that of [BHH$^+$c, Thm. 3.7.1], using Corollary 4.8.3 and Proposition 4.9.4. □

4.10 The actions of φ and $\mathcal{O}_K^\times$ on $\mathrm{Hom}_A(D_A(\pi),A)(1)$

In this section, we finish the proof of Theorem 4.1.1. We also compute the actions of φ and $\mathcal{O}_K^\times$ on $\mathrm{Hom}_A(D_A(\pi),A)(1)$. The main results are Proposition 4.10.4 and Corollary 4.10.5. Here the $\mathcal{O}_K^\times$-action is much more technical to compute explicitly than in the semisimple case (see [BHH$^+$c, Prop. 3.8.3]). Instead, we give a congruence relation which uniquely determines the $\mathcal{O}_K^\times$-action.

By definition, $D_A(\pi)$ is a finite free $(\psi,\mathcal{O}_K^\times)$-module over A in the sense of [BHH$^+$b, Def. 3.1.2.1]. Then the construction of [BHH$^+$c, §3.2] makes $\mathrm{Hom}_A(D_A(\pi),A)(1)$ a $(\varphi,\mathcal{O}_K^\times)$-module, which is étale if and only if β as in (4.1) is an isomorphism. Here for D a $(\varphi,\mathcal{O}_K^\times)$-module over A, we write $D(1)$ to be D with the action of φ unchanged and the action of $a\in\mathcal{O}_K^\times$ multiplied by $N_{\mathbb{F}_q/\mathbb{F}_p}(\overline{a})$. Then by [BHH$^+$c, Lemma 3.8.1(ii)] and [BHH$^+$c, (114)], under the injection (4.107) the actions of φ and $\mathcal{O}_K^\times$ on $\mathrm{Hom}_A(D_A(\pi),A)(1)$ can be expressed in terms of sequences as follows:

(i) for $k\geq 0$ and $p\ell\geq k$, we have

$$(\varphi(x_J))_k=(-1)^{f-1}\underline{Y}^{p\underline{\ell}-\underline{k}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)x_{J,\ell};\tag{4.112}$$

(ii) for $a\in\mathcal{O}_K^\times$, $k\geq 0$ and $\ell\gg_k 0$, we have

$$(a(x_J))_k=\frac{a(\underline{Y}^{\underline{\ell}})}{\underline{Y}^{\underline{k}}}\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)x_{J,\ell}.\tag{4.113}$$

We denote by $\mathrm{Mat}(\varphi)$ and $\mathrm{Mat}(a)$ ($a\in\mathcal{O}_K^\times$) the matrices of the actions of φ and $\mathcal{O}_K^\times$ on $\mathrm{Hom}_A(D_A(\pi),A)(1)$ with respect to the basis $\{x_J: J\subseteq\mathcal{J}\}$ of Theorem 4.9.5, whose rows and columns are indexed by the subsets of $\mathcal{J}$. For $J,J'\subseteq\mathcal{J}$ such that $(J-1)^{\mathrm{ss}}=(J')^{\mathrm{ss}}$, we let

$$\gamma_{J,J'}\overset{\mathrm{def}}{=}(-1)^{f-1}\varepsilon_{J'}\mu_{J,J'},\tag{4.114}$$

where $\varepsilon_{J'}$ is defined in (4.54) and $\mu_{J,J'}$ is defined in (4.51). Then by definition and the sentence after (4.51), for $J_1,J_2,J_3,J_4\subseteq\mathcal{J}$ such that $(J_1-1)^{\mathrm{ss}}=(J_2-1)^{\mathrm{ss}}=J_3^{\mathrm{ss}}=J_4^{\mathrm{ss}}$ we have

$$\frac{\gamma_{J_1,J_3}}{\gamma_{J_1,J_4}}=\frac{\gamma_{J_2,J_3}}{\gamma_{J_2,J_4}}.$$

We define $\frac{\gamma_{*,J}}{\gamma_{*,J'}}$ for $J^{\mathrm{ss}}=(J')^{\mathrm{ss}}$ in a similar way as $\frac{\mu_{*,J}}{\mu_{*,J'}}$.

Proposition 4.10.1. (i) *We have (see (4.53) for $\underline{c}^J$ and (4.52) for $\underline{r}^{J\setminus J'}$)*

$$\mathrm{Mat}(\varphi)_{J',J+1}=\begin{cases}\gamma_{J+1,J'}\underline{Y}^{-(\underline{c}^J+\underline{r}^{J\setminus J'})} & \text{if } J^{\mathrm{ss}}\subseteq J'\subseteq J\\ 0 & \text{otherwise.}\end{cases}$$

(ii) *For $a\in[\mathbb{F}_q^\times]$, $\mathrm{Mat}(a)$ is a diagonal matrix with $\mathrm{Mat}(a)_{J,J}=\overline{a}^{\underline{r}^{J^c}}$.*

Proof. (i). Let $J\subseteq\mathcal{J}$. For $k\geq 0$ and $p\ell\geq k$, by (4.112) and (4.57) we have

$$\begin{aligned}(\varphi(x_{J+1}))_k&=(-1)^{f-1}\underline{Y}^{p\underline{\ell}-\underline{k}}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)x_{J+1,\underline{\ell}}=\sum_{J^{\mathrm{ss}}\subseteq J'\subseteq J}(-1)^{f-1}\varepsilon_{J'}\mu_{J+1,J'}x_{J',\left(p\underline{\ell}+\underline{c}^J+\underline{r}^{J\setminus J'}-(p\underline{\ell}-\underline{k})\right)}\\&=\sum_{J^{\mathrm{ss}}\subseteq J'\subseteq J}\gamma_{J+1,J'}x_{J',\underline{c}^J+\underline{r}^{J\setminus J'}+\underline{k}}.\end{aligned}$$

Then using (4.108) one easily checks that

$$\varphi(x_{J+1}) = \sum_{J^{\mathrm{ss}} \subseteq J' \subseteq J} \gamma_{J+1,J'} \underline{Y}^{-(\underline{c}^J + \underline{r}^{J \setminus J'})}(x_{J'}),$$

which proves (i).

(ii). Let $J \subseteq \mathcal{J}$ and $a \in [\mathbb{F}_q^\times]$. By Corollary 4.6.10 we have

$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J,\underline{i}} = \chi'_J\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \overline{a}^{-\underline{i}} x_{J,\underline{i}} = \overline{a}^{\underline{r}-\underline{r}^J-\underline{i}} x_{J,\underline{i}} = \overline{a}^{\underline{r}^{\mathcal{J}}-\underline{r}^J-\underline{i}} x_{J,\underline{i}} = \overline{a}^{\underline{r}^{J^c}-\underline{i}} x_{J,\underline{i}},$$

where the second equality follows from (4.68), the third equality follows from (4.52), and the last equality follows from Lemma 4.6.3(ii). Then for $k \geq 0$, by (4.113) we have

$$(a(x_J))_k = \frac{a(\underline{Y}^{\underline{k}})}{\underline{Y}^{\underline{k}}} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) x_{J,k} = \overline{a}^{\underline{k}} \left(\overline{a}^{\underline{r}^{J^c}-\underline{k}} x_{J,k}\right) = \overline{a}^{\underline{r}^{J^c}} x_{J,k},$$

where the second equality follows from (4.106). Using (4.108), we conclude that $a(x_J) = \overline{a}^{\underline{r}^{J^c}}(x_J)$, which proves (ii). □

Proof of Theorem 4.1.1. By Theorem 4.9.5, π is in $\mathcal{C}$ and $D_A(\pi)$ has rank 2^f. Moreover, by Proposition 4.10.1(i) we have $\mathrm{Mat}(\varphi) \in \mathrm{GL}_{2^f}(A)$, hence $\mathrm{Hom}_A(D_A(\pi), A)$ is an étale $(\varphi, \mathcal{O}_K^\times)$-module over A, which implies that β as in (4.1) is an isomorphism. □

Then we are going to describe the $\mathcal{O}_K^\times$-action. The following two lemmas are needed in the proof of Proposition 4.10.4 below.

Lemma 4.10.2. *Let $a \in A$, $\lambda \in \mathbb{F}^\times$ and $\underline{s} \in \mathbb{Z}^f$ such that $a = \lambda \underline{Y}^{\underline{s}} \varphi_q(a)$. If $\underline{s} = (q-1)\underline{t}$ for some $\underline{t} \in \mathbb{Z}^f$ and $\lambda = 1$, then we have $a \in \mathbb{F}\underline{Y}^{-\underline{t}}$. Otherwise, we have $a = 0$.*

Proof. Let $m > 0$ be large enough such that q^m is a multiple of $|\mathbb{F}|$ and $\lambda^m = 1$. In particular, φ_q^m acts as $x \mapsto x^{q^m}$ on A. By iteration, we have $a^{q^m} = \underline{Y}^{-((q^m-1)/(q-1))\underline{s}} a$. Suppose that $a \neq 0$. Since A is an integral domain, we have $a^{q^m-1} = \underline{Y}^{-((q^m-1)/(q-1))\underline{s}}$. In particular, we have $a \in A^\times$, hence we can write $a = c\underline{Y}^{-\underline{t}} a_1$ with $c \in \mathbb{F}^\times$, $\underline{t} \in \mathbb{Z}^f$ and $a_1 \in 1 + F_{-1}A$. Then we deduce that $a_1 = 1$ and $(q^m-1)\underline{t} = ((q^m-1)/(q-1))\underline{s}$, which implies $\underline{s} = (q-1)\underline{t}$, and we necessarily have $\lambda = 1$. □

For $a \in A^\times$ and $k = \sum_{i=0}^m k_i \varphi^i \in \mathbb{Z}[\varphi]$ with $m \in \mathbb{Z}_{\geq 0}$ and $k_i \in \mathbb{Z}$ for all $0 \leq i \leq m$, we define

$$a^k \overset{\mathrm{def}}{=} \prod_{i=0}^m \varphi^i(a^{k_i}) \in A^\times.$$

This makes $A^\times$ a $\mathbb{Z}[\varphi]$-module. By completeness, $1 + F_{-1}A$ is a $\mathbb{Z}_{(p)}[\varphi]$-module, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ with respect to the prime ideal (p).

Lemma 4.10.3. *Let $J, J' \subseteq \mathcal{J}$, $\lambda_j \in \mathbb{F}^\times$ and $1 \leq h_j \leq p-2$ for all $j \in \mathcal{J}$. Consider the map*

$$\begin{aligned} \theta : \left(A^{[\mathbb{F}_q^\times]}\right)^f &\to \left(A^{[\mathbb{F}_q^\times]}\right)^f \\ (a_i)_{i \in \mathcal{J}} &\mapsto \left(a_i - \lambda_i \left[\prod_{j - i \in J \setminus J'} Y_j^{h_j(1-\varphi)} \prod_{j - i \in J' \setminus J} Y_j^{-h_j(1-\varphi)} \right] \varphi(a_{i+1}) \right)_{i \in \mathcal{J}}. \end{aligned}$$

(i) *If $J' \neq J$, then $\theta(\underline{a}) = \underline{0}$ implies $\underline{a} = \underline{0}$.*

(ii) *If* $J' = J$ *and* $\lambda_i = 1$ *for all* i*, then* $\theta(\underline{a}) = \underline{0}$ *implies* $\underline{a} = \underline{\mu}$ *for some* $\mu \in \mathbb{F}$*.*

(iii) *If* $J' = J \setminus \{j_0\}$ *for some* $j_0 \in J$ *and* $\underline{0} \neq \underline{b} \in \big((F_0A \setminus F_{-1}A) \cap A^{[\mathbb{F}_q^\times]}\big)^f$*, then the equation* $\theta(\underline{a}) = \underline{b}$ *has no solution.*

(iv) *If* $J' \subsetneqq J$ *and* $b_i \in F_{|J\setminus J'|(1-p)}A \cap A^{[\mathbb{F}_q^\times]}$ *for all* i*, then the equation* $\theta(\underline{a}) = \underline{b}$ *has at most one solution. If moreover* $1 \leq h_j \leq p-1-f$ *for all* j*, then there is a unique solution which moreover satisfies* $a_i \equiv b_i \mod F_{(f+1)(1-p)}A$ *for all* i*.*

Proof. (i). We write $\lambda \overset{\text{def}}{=} \prod_{i=0}^{f-1} \lambda_i$ and $h^{(j)} \overset{\text{def}}{=} \sum_{i=0}^{f-1} h_{j+i}p^i$ for all $j \in \mathcal{J}$. If $\theta(\underline{a}) = 0$, then by iteration we have $a_0 = \lambda \underline{Y}^{\underline{s}} \varphi_q(a_0)$, where

$$\underline{s} \overset{\text{def}}{=} \sum_{j \in J\setminus J'} h^{(j)}(e_j - pe_{j-1}) - \sum_{j \in J'\setminus J} h^{(j)}(e_j - pe_{j-1}) \neq \underline{0}. \tag{4.115}$$

Claim. Suppose that $\underline{s} = (q-1)\underline{t}$ for some $\underline{t} \in \mathbb{Z}^f$. Then we have $|t_j| \leq p-1$ for all $j \in \mathcal{J}$ and $\underline{0} \neq \underline{t} \neq \pm(\underline{p} - \underline{1})$.

Proof. Since $0 \leq h^{(j)} \leq (p-2)(1+p+\cdots+p^{f-1})$ for all $j \in \mathcal{J}$, we deduce from (4.115) that

$$|t_j| \leq \frac{(p-2)(1+p+\cdots+p^{f-1})(p+1)}{q-1} = \frac{(p-2)(p+1)}{p-1} < p,$$

hence $|t_j| \leq p-1$ for all $j \in \mathcal{J}$. We also deduce from (4.115) that

$$\big|\|\underline{t}\|\big| \leq \frac{(p-2)(1+p+\cdots+p^{f-1})(p-1)f}{q-1} = (p-2)f,$$

which implies $\underline{t} \neq \pm(\underline{p} - \underline{1})$. □

By Lemma 4.10.2, the only possible nonzero solution for a_0 in A is a scalar multiple of $\underline{Y}^{-\underline{t}}$, which is not fixed by $[\mathbb{F}_q^\times]$ since $-(\underline{p}-\underline{1}) < \underline{t} < \underline{p}-\underline{1}$ and $\underline{t} \neq \underline{0}$. Hence $a_0 = 0$, and we conclude that $a_i = 0$ for all i.

(ii). If $\theta(\underline{a}) = \underline{0}$, then by iteration we have $a_0 = \varphi_q(a_0)$. By Lemma 4.10.2, we deduce that $a_0 = \mu \in \mathbb{F}$, hence $a_i = \mu$ for all i.

(iii). In this case, the equation $\theta(\underline{a}) = \underline{b}$ becomes

$$a_i = \lambda_i Y_{j_0+i}^{h_{j_0+i}(1-\varphi)} \varphi(a_{i+1}) + b_i \ \forall\, i \in \mathcal{J}. \tag{4.116}$$

For $0 \neq a \in A$, we say that a has degree m if $a \in F_{-m}A \setminus F_{-(m+1)}A$. We also define $\deg(0) \overset{\text{def}}{=} \infty$. In particular, a nonzero scalar has degree zero, and φ multiplies the degree by p (see (4.106)). We choose $i_0 \in \mathcal{J}$ such that $b_{i_0} \neq 0$ (hence $\deg(b_{i_0}) = 0$) and let $i = i_0$ in (4.116). Since the degree of $Y_{j_0+i_0}^{h_{j_0+i_0}(1-\varphi)}$ is not a multiple of p, the two terms of the RHS of (4.116) have different degrees. Comparing the degrees of both sides of (4.116), we deduce that $\deg(a_{i_0}) \leq 0$.

Then we let $i = i_0 - 1$ in (4.116). Since $\deg(a_{i_0}) \leq 0$, we have

$$\deg\Big(Y_{j_0+i_0-1}^{h_{j_0+i_0-1}(1-\varphi)} \varphi(a_{i_0})\Big) = p\deg(a_{i_0}) - (p-1)h_{j_0+i_0-1} < \min\{\deg(a_{i_0}), 0\}.$$

Comparing the degrees of both sides of (4.116), we deduce that $\deg(a_{i_0-1}) < \deg(a_{i_0})$.

Then we let $i = i_0 - 2$ in (4.116) and continue this process. We finally deduce that

$$\deg(a_{i_0}) = \deg(a_{i_0-f}) < \deg(a_{i_0-f+1}) < \cdots < \deg(a_{i_0-1}) < \deg(a_{i_0}),$$

which is a contradiction.

(iv). By (i), the equation $\theta(\underline{a}) = \underline{b}$ has at most one solution. If moreover $1 \leq h_j \leq p-1-f$ for all $j \in \mathcal{J}$, since $b_i \in F_{|J\setminus J'|(1-p)}A$ for all $i \in \mathcal{J}$, we have for all $i \in \mathcal{J}$

$$\begin{aligned}\deg\Big(\big((\mathrm{id}-\theta)(\underline{b})\big)_i\Big) &= \deg\left(\lambda_i\left[\prod_{j-i\in J\setminus J'} Y_j^{h_j(1-\varphi)}\right]\varphi(b_{i+1})\right)\\ &\geq p\deg(b_{i+1}) - |J\setminus J'|(p-1)(p-1-f)\\ &\geq \max\{(f+1)(p-1), \deg(b_{i+1})+1\}.\end{aligned}$$

Hence the series

$$\underline{a} \overset{\mathrm{def}}{=} \underline{b} + \sum_{k=1}^{\infty}(\mathrm{id}-\theta)^k(\underline{b})$$

converges in $\big(A^{[\mathbb{F}_q^\times]}\big)^f$, gives a solution of the equation and satisfies the required congruence relation. □

Let $Q \in \mathrm{GL}_{2^f}(A)$ be the diagonal matrix with $Q_{J,J} = \underline{Y}^{\underline{r}^{J^c}}$ for $J \subseteq \mathcal{J}$. Then the matrices $\mathrm{Mat}(\varphi)'$, $\mathrm{Mat}(a)'$ ($a \in \mathcal{O}_K^\times$) with respect to the new basis $\{x''_J \overset{\mathrm{def}}{=} \underline{Y}^{-\underline{r}^{J^c}}(x_J) : J \subseteq \mathcal{J}\}$ of $\mathrm{Hom}_A(D_A(\pi), A)(1)$ are given by $\mathrm{Mat}(\varphi)' = Q\,\mathrm{Mat}(\varphi)\varphi(Q)^{-1}$ and $\mathrm{Mat}(a)' = Q\,\mathrm{Mat}(a)a(Q)^{-1}$.

Proposition 4.10.4. (i) *We have*

$$\mathrm{Mat}(\varphi)'_{J',J+1} = \begin{cases}\gamma_{J+1,J'}\prod_{j\notin J} Y_j^{(r_j+1)(1-\varphi)} & \text{if } J^{\mathrm{ss}} \subseteq J' \subseteq J\\ 0 & \text{otherwise.}\end{cases}$$

(ii) *For $a \in [\mathbb{F}_q^\times]$, we have $\mathrm{Mat}(a)' = I$.*

(iii) *Assume that $J_{\overline{\rho}} \neq \mathcal{J}$. Up to twist by a continuous character $\mathcal{O}_K^\times \to \mathbb{F}^\times$, there exists a unique $\mathcal{O}_K^\times$-action on $\mathrm{Hom}_A(D_A(\pi), A)(1)$ which satisfies (ii) and commutes with φ as in (i). Moreover, the matrix $\mathrm{Mat}(a)'$ ($a \in \mathcal{O}_K^\times$) satisfies for $J, J' \subseteq \mathcal{J}$*
   (a) *$\mathrm{Mat}(a)'_{J',J} = 0$ if $J' \not\subseteq J$;*
   (b) *$\mathrm{Mat}(a)'_{J',J} \in F_{|J\setminus J'|(1-p)}A$ if $J' \subseteq J$.*

(iv) *Assume that $J_{\overline{\rho}} = \mathcal{J}$. Up to diagonal matrices $B \in \mathrm{GL}_{2^f}(\mathbb{F})$ such that $B_{J,J} = B_{J+1,J+1}$ for all $J \subseteq \mathcal{J}$, there exists a unique $\mathcal{O}_K^\times$-action on $\mathrm{Hom}_A(D_A(\pi), A)(1)$ which satisfies (ii) and commutes with φ as in (i). Moreover, the matrix $\mathrm{Mat}(a)'$ is diagonal for all $a \in \mathcal{O}_K^\times$.*

Proof. (i). We have $\mathrm{Mat}(\varphi)'_{J',J+1} = Q_{J',J'}\,\mathrm{Mat}(\varphi)_{J',J+1}\varphi(Q_{J+1,J+1})^{-1}$. Hence we deduce from Proposition 4.10.1(i) that $\mathrm{Mat}(\varphi)'_{J',J+1} \neq 0$ if and only if $J^{\mathrm{ss}} \subseteq J' \subseteq J$, in which case we have

$$\mathrm{Mat}(\varphi)'_{J',J+1} = \gamma_{J+1,J'}\underline{Y}^{\underline{r}^{(J')^c}}\underline{Y}^{-(\underline{c}^J+\underline{r}^{J\setminus J'})}\varphi\big(\underline{Y}^{\underline{r}^{(J+1)^c}}\big)^{-1} = \gamma_{J+1,J'}\underline{Y}^{\underline{r}^{J^c}-\underline{c}^J-p\delta(\underline{r}^{(J+1)^c})},$$

where the last equality follows from Lemma 4.6.3(ii) and (4.106). By (4.55) and (4.56) we have

$$\begin{aligned}r_j^{J^c} - c_j^J - pr_{j+1}^{(J+1)^c} &= \big(\delta_{j+1\notin J}(r_j+1) - \delta_{j\notin J}\big) - \big(\delta_{j\notin J}(p-2-r_j) + \delta_{j+1\notin J}(r_j+1)\big)\\ &\quad - p\big(\delta_{j+1\notin J}(r_{j+1}+1) - \delta_{j\notin J}\big)\\ &= \delta_{j\notin J}(r_j+1) - \delta_{j+1\notin J}\big(p(r_{j+1}+1)\big),\end{aligned}$$

which proves the required formula for $\mathrm{Mat}(\varphi)'_{J',J+1}$ using (4.106).

(ii). Let $a \in [\mathbb{F}_q^\times]$. We deduce from Proposition 4.10.1(ii) and (4.106) that $\mathrm{Mat}(a)'$ is a diagonal matrix with

$$\mathrm{Mat}(a)'_{J,J} = Q_{J,J}\,\mathrm{Mat}(a)_{J,J}a(Q_{J,J})^{-1} = \underline{Y}^{r_{J^c}}\overline{a}^{r_{J^c}}a\big(\underline{Y}^{r_{J^c}}\big)^{-1} = 1.$$

(iii) and (iv). For simplicity, we denote by P_φ the matrix $\mathrm{Mat}(\varphi)'$ and we let $P_a \in \mathrm{GL}_{2^f}(A)$ ($a \in \mathcal{O}_K^\times$) be the matrices for the $\mathcal{O}_K^\times$-action. Since $[\mathbb{F}_q^\times]$ fixes the basis $\{x''_J : J \subseteq \mathcal{J}\}$ by (ii), it also fixes the matrices P_a. By the commutativity of the actions of φ and $\mathcal{O}_K^\times$, we have

$$P_a\,a(P_\varphi) = P_\varphi\,\varphi(P_a). \tag{4.117}$$

Since $(P_\varphi)_{J',J+1} \neq 0$ if and only if $J^{\mathrm{ss}} \subseteq J' \subseteq J$ by (i), comparing the $(J', J+1)$-entries of (4.117) we get

$$\sum_{J^{\mathrm{ss}}\subseteq J''\subseteq J}(P_a)_{J',J''}\,a(P_\varphi)_{J'',J+1} = \sum_{J'':(J'')^{\mathrm{ss}}\subseteq J'\subseteq J''}(P_\varphi)_{J',J''+1}\varphi(P_a)_{J''+1,J+1}. \tag{4.118}$$

Claim 1. For $j \in \mathcal{J}$ we let $P_{a,j} \overset{\mathrm{def}}{=} f_{a,j}^{h^{(j)}(1-\varphi)/(1-q)} \in 1 + F_{1-p}A$, where $f_{a,j} \overset{\mathrm{def}}{=} \overline{a}^{p^j}Y_j/a(Y_j) \in 1 + F_{1-p}A$ and $h^{(j)} = \sum_{i=0}^{f-1} h_{j+i}p^i$ as in the proof of Lemma 4.10.3 with $h_j \overset{\mathrm{def}}{=} r_j + 1$. For $J \subseteq \mathcal{J}$ we let $P_{a,J} \overset{\mathrm{def}}{=} \prod_{j\notin J} P_{a,j} \in 1 + F_{1-p}A$. In particular, $P_{a,J}$ is fixed by $[\mathbb{F}_q^\times]$. Then for all $J \subseteq \mathcal{J}$, we have

$$P_{a,J}\,a(P_\varphi)_{J,J+1} = (P_\varphi)_{J,J+1}\varphi(P_{a,J+1}).$$

Proof. By (i) and by definition, it suffices to show that for all $j \in \mathcal{J}$ we have

$$P_{a,j}\ a\left(Y_j^{(r_j+1)(1-\varphi)}\right) = Y_j^{(r_j+1)(1-\varphi)}\varphi(P_{a,j+1}).$$

Since $\varphi(Y_{j+1}) = Y_j^p$ by (4.106), it suffices to show that for all $j \in \mathcal{J}$ we have

$$f_{a,j}^{h^{(j)}/(1-q)}\,a\big(Y_j^{r_j+1}\big) = Y_j^{r_j+1}f_{a,j}^{ph^{(j+1)}/(1-q)},$$

which follows from the equality $ph^{(j+1)} - h^{(j)} = (q-1)(r_j+1)$. □

We define $Q_a \in \mathrm{GL}_{2^f}(A)$ by $(Q_a)_{J',J} = (P_a)_{J',J}P_{a,J}^{-1}$, which is fixed by $[\mathbb{F}_q^\times]$. Then it suffices to prove the uniqueness for Q_a. Dividing the LHS of (4.118) by $P_{a,J}\,a(P_\varphi)_{J,J+1} \in A^\times$ and the RHS of (4.118) by $(P_\varphi)_{J,J+1}\varphi(P_{a,J+1}) \in A^\times$ using Claim 1 and (i), we get

$$\begin{aligned}&\sum_{J^{\mathrm{ss}}\subseteq J''\subseteq J}\left[\frac{\gamma_{*,J''}}{\gamma_{*,J}}(Q_a)_{J',J''}\prod_{j\in J\setminus J''}P_{a,j}\right]\\ &\quad=\sum_{J'':(J'')^{\mathrm{ss}}\subseteq J'\subseteq J''}\left[\frac{\gamma_{J''+1,J'}}{\gamma_{J+1,J}}\left(\prod_{j\in J\setminus J''}Y_j^{h_j(1-\varphi)}\prod_{j\in J''\setminus J}Y_j^{-h_j(1-\varphi)}\right)\varphi(Q_a)_{J''+1,J+1}\right].\end{aligned} \tag{4.119}$$

(a). We assume that $J' \not\subseteq J$. We use increasing induction on $|J| - |J'|$ (which ranges from $-f$ to f) to show that $(Q_a)_{J',J} = 0$. By the induction hypothesis, we have $(Q_a)_{J',J''} = 0$ if $J'' \subsetneqq J$, and $(Q_a)_{J''+1,J+1} = 0$ if $J'' \supsetneqq J'$. Hence it follows from (4.119) that

$$(Q_a)_{J',J} = \frac{\gamma_{J'+1,J'}}{\gamma_{J+1,J}} \left[\prod_{j \in J \setminus J'} Y_j^{h_j(1-\varphi)} \prod_{j \in J' \setminus J} Y_j^{-h_j(1-\varphi)} \right] \varphi(Q_a)_{J'+1,J+1}. \tag{4.120}$$

A similar equality holds replacing (J', J) with $(J' + i, J + i)$ (for all $i \in \mathcal{J}$), hence it follows from Lemma 4.10.3(i) (with $\lambda_i = \gamma_{J'+i+1,J'+i}/\gamma_{J+i+1,J+i}$) that $(Q_a)_{J',J} = 0$.

In the case $J_{\overline{\rho}} = \mathcal{J}$, which implies $J^{\mathrm{ss}} = J$ for all $J \subseteq \mathcal{J}$, the equation (4.119) is the same as (4.120). Then as in the previous paragraph, we deduce from Lemma 4.10.3(i) that $(Q_a)_{J',J} = 0$ for all $J' \neq J$.

(b). We assume that $J' = J$. Then by a similar argument, the equation (4.120) still holds and becomes $(Q_a)_{J,J} = \varphi(Q_a)_{J+1,J+1}$. By Lemma 4.10.3(ii), we deduce that $(Q_a)_{J,J} = \xi_{a,J}$ for some $\xi_{a,J} \in \mathbb{F}^\times$ (nonzero since Q_a is invertible), and we have $\xi_{a,J} = \xi_{a,J+1}$. In particular, this completes the proof of (iv).

Claim 2. If $J_{\overline{\rho}} \neq \mathcal{J}$, then $\xi_{a,J}$ does not depend on J.

Proof. It suffices to show that $\xi_{a,J} = \xi_{a,J'}$ for all J, J' such that $J' = J \setminus \{j_0\}$ for some $j_0 \in J$. Since $(Q_a)_{J',J} = 0$ for $J' \not\subseteq J$, we deduce from (4.119) that

$$(Q_a)_{J',J} + \delta_{j_0 \notin J_{\overline{\rho}}} \frac{\gamma_{*,J'}}{\gamma_{*,J}} \xi_{a,J'} P_{a,j_0} = \frac{\gamma_{J'+1,J'}}{\gamma_{J+1,J}} Y_{j_0}^{h_{j_0}(1-\varphi)} \varphi(Q_a)_{J'+1,J+1} + \delta_{j_0 \notin J_{\overline{\rho}}} \frac{\gamma_{*,J'}}{\gamma_{*,J}} \xi_{a,J}.$$

A similar equality holds replacing (J', J) with $(J' + i, J + i)$ (hence j_0 is replaced with $j_0 + i$). For each $i \in \mathcal{J}$, we let

$$b_i \overset{\mathrm{def}}{=} \delta_{j_0+i \notin J_{\overline{\rho}}} \frac{\gamma_{*,J'+i}}{\gamma_{*,J+i}} \left(\xi_{a,J+i} - \xi_{a,J'+i} P_{a,j_0+i} \right) = \delta_{j_0+i \notin J_{\overline{\rho}}} \frac{\gamma_{*,J'+i}}{\gamma_{*,J+i}} \left(\xi_{a,J} - \xi_{a,J'} P_{a,j_0+i} \right).$$

Suppose on the contrary that $\xi_{a,J} \neq \xi_{a,J'}$. Since $P_{a,j} \in 1 + F_{1-p}A$ for all j, we deduce that $b_i \in (F_0 A \setminus F_{-1}A) \cap A^{[\mathbb{F}_q^\times]}$ for all i, and not all equal to 0 since $J_{\overline{\rho}} \neq \mathcal{J}$. Then by Lemma 4.10.3(iii) (with $\lambda_i = \gamma_{J'+i+1,J'+i}/\gamma_{J+i+1,J+i}$) we deduce a contradiction. □

(c). In the rest of the proof we assume that $J_{\overline{\rho}} \neq \mathcal{J}$. Since $\xi_{a,J}$ does not depend on J by Claim 2, we denote it by ξ_a. Since $(Q_a)_{J',J} = 0$ for all $J' \not\subseteq J$ by (a), the assignment $a \mapsto \xi_a$ defines a continuous character of $\mathcal{O}_K^\times$ with values in $\mathbb{F}^\times$. By considering $\xi_a^{-1} P_a$, we may assume that $\xi_a = 1$ for all $a \in \mathcal{O}_K^\times$. To finish the proof of (iii), we use increasing induction on $|J \setminus J'|$ to show that for $J' \subseteq J$ there is a unique choice of $(Q_a)_{J',J}$, which moreover satisfies

$$(Q_a)_{J',J} \equiv \begin{cases} \frac{\gamma_{*,J'}}{\gamma_{*,J}} \prod_{j \in J \setminus J'} (1 - P_{a,j}) & \mod F_{(f+1)(1-p)}A \quad \text{if } J' \supseteq J^{\mathrm{ss}} \\ 0 & \mod F_{(f+1)(1-p)}A \quad \text{if } J' \not\supseteq J^{\mathrm{ss}}. \end{cases}$$

Since $(Q_a)_{J,J} = \xi_a = 1$ by (b) and assumption, the case $J' = J$ is true. Then we assume that $J' \subsetneqq J$. Since $(Q_a)_{J',J} = 0$ for $J' \not\subseteq J$ by (a), (4.119) gives

$$\sum_{J' \cup J^{\mathrm{ss}} \subseteq J'' \subseteq J} \left[\frac{\gamma_{*,J''}}{\gamma_{*,J}} (Q_a)_{J',J''} \prod_{j \in J \setminus J''} P_{a,j} \right]$$

$$= \sum_{J'':(J'')^{\mathrm{ss}}\subseteq J'\subseteq J''\subseteq J} \left[\frac{\gamma_{J''+1,J'}}{\gamma_{J+1,J}} \left(\prod_{j\in J\setminus J''} Y_j^{h_j(1-\varphi)} \right) \varphi(Q_a)_{J''+1,J+1} \right], \quad (4.121)$$

which implies that

$$(Q_a)_{J',J} - \frac{\gamma_{J'+1,J'}}{\gamma_{J+1,J}} \left[\prod_{j\in J\setminus J'} Y_j^{h_j(1-\varphi)} \right] \varphi(Q_a)_{J'+1,J+1} = b_0, \quad (4.122)$$

where $b_0 \overset{\mathrm{def}}{=} b_{0,1} - b_{0,2}$ with

$$b_{0,1} \overset{\mathrm{def}}{=} \sum_{J'':(J'')^{\mathrm{ss}}\subseteq J'\subsetneq J''\subseteq J} \left[\frac{\gamma_{J''+1,J'}}{\gamma_{J+1,J}} \left(\prod_{j\in J\setminus J''} Y_j^{h_j(1-\varphi)} \right) \varphi(Q_a)_{J''+1,J+1} \right];$$

$$b_{0,2} \overset{\mathrm{def}}{=} \sum_{J'\cup J^{\mathrm{ss}}\subseteq J''\subsetneq J} \left[\frac{\gamma_{*,J''}}{\gamma_{*,J}} (Q_a)_{J',J''} \prod_{j\in J\setminus J''} P_{a,j} \right].$$

By the induction hypothesis together with $1-P_{a,j}\in F_{1-p}A$ and $h_j\le p-1-f$ (by (4.2)), each term in the summation of $b_{0,1}$ lies in $F_{(f+1)(1-p)}A$ unless the term for $J''=J$, which appears if and only if $J'\supseteq J^{\mathrm{ss}}$. If $J'\not\supseteq J^{\mathrm{ss}}$, then we have $b_{0,1}\in F_{(f+1)(1-p)}A$. Moreover, for each J'' such that $J'\cup J^{\mathrm{ss}}\subseteq J''\subsetneq J$, we have $J'\not\supseteq J^{\mathrm{ss}}=(J'')^{\mathrm{ss}}$. Hence by the induction hypothesis, we deduce that $b_{0,2}\in F_{(f+1)(1-p)}A$, hence $b_0\in F_{(f+1)(1-p)}A$. If $J'\supseteq J^{\mathrm{ss}}$, then by the induction hypothesis we have

$$\begin{aligned} b_0 = b_{0,1} - b_{0,2} &\equiv \frac{\gamma_{*,J'}}{\gamma_{*,J}} - \sum_{J'\subseteq J''\subsetneq J} \left[\frac{\gamma_{*,J''}}{\gamma_{*,J}} \frac{\gamma_{*,J'}}{\gamma_{*,J''}} \prod_{j\in J''\setminus J'} (1-P_{a,j}) \prod_{j\in J\setminus J''} P_{a,j} \right] \\ &= \frac{\gamma_{*,J'}}{\gamma_{*,J}} \left[\prod_{j\in J\setminus J'} \big((1-P_{a,j})+P_{a,j}\big) - \sum_{J'\subseteq J''\subsetneq J} \left(\prod_{j\in J''\setminus J'} (1-P_{a,j}) \prod_{j\in J\setminus J''} P_{a,j} \right) \right] \\ &= \frac{\gamma_{*,J'}}{\gamma_{*,J}} \prod_{j\in J\setminus J'} (1-P_{a,j}) \quad \big(\mathrm{mod}\ \ F_{(f+1)(1-p)}A\big). \end{aligned}$$

In particular, we have $b_0\in F_{|J\setminus J'|(1-p)}A$ since $1-P_{a,j}\in F_{1-p}A$ for all j. For $i\in\mathcal{J}$, we define b_i in a similar way as b_0 replacing (J',J) with $(J'+i,J+i)$, and a similar equality as (4.122) holds replacing (J',J) with $(J'+i,J+i)$ and b_0 with b_i. Then we deduce from Lemma 4.10.3(iv) (with $\lambda_i=\gamma_{J'+i+1,J'+i}/\gamma_{J+i+1,J+i}$) that there is a unique solution of $(Q_a)_{J',J}$, which satisfies

$$(Q_a)_{J',J} \equiv b_0 \mod F_{(f+1)(1-p)}A.$$

This completes the proof. □

Finally, we can determine the $\mathcal{O}_K^\times$-action on $\mathrm{Hom}_A(D_A(\pi),A)(1)$. In the semisimple case, this is computed explicitly in [BHH$^+$c, Prop. 3.8.3].

Corollary 4.10.5. *If $J_{\overline{\rho}}\neq\mathcal{J}$, then the $\mathcal{O}_K^\times$-action on $\mathrm{Hom}_A(D_A(\pi),A)(1)$ is the unique one in Proposition 4.10.4(iii) which satisfies $\mathrm{Mat}(a)'_{J,J}\in 1+F_{1-p}A$ for all $a\in\mathcal{O}_K^\times$ and $J\subseteq\mathcal{J}$.*

Proof. By the proof of Proposition 4.10.4(iii), there exists a continuous character $\xi : \mathcal{O}_K^\times \to \mathbb{F}^\times$ such that for all $a \in \mathcal{O}_K^\times$ and $J \subseteq \mathcal{J}$ we have $\mathrm{Mat}(a)'_{J,J} = \xi(a)P_{a,J}$ with $P_{a,J} \in 1 + F_{1-p}A$. To prove that ξ is trivial, it suffices to show that $\mathrm{Mat}(a)'_{\emptyset,\emptyset} \in 1 + F_{1-p}A$. Using the change of basis matrix Q which is diagonal, it suffices to show that $\mathrm{Mat}(a)_{\emptyset,\emptyset} \in \underline{a}^{\underline{r}}(1 + F_{1-p}A)$. Hence it is enough to prove that $a(x_\emptyset) \in \underline{a}^{\underline{r}}(1 + F_{1-p}A)x_\emptyset$.

We claim that for all $\underline{i} \in \mathbb{Z}^f$, we have

$$x_{\emptyset,\underline{i}} = \mu_{\emptyset,\emptyset}^{-n} \underline{Y}^{\underline{p^n - 1 - i}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^n v_\emptyset \tag{4.123}$$

for any $n \geq 0$ such that $\underline{p^n - 1 - i} \geq \underline{0}$. Indeed, by Proposition 4.5.4 with $J = \emptyset$, we have $\underline{Y}^{\underline{p-1}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_\emptyset = \mu_{\emptyset,\emptyset} v_\emptyset$, hence using Lemma 4.3.1(i) the RHS of (4.123) does not depend on n. By (4.58) and (4.66) with $J = \emptyset$, we deduce that (4.123) is true for $\underline{i} = \underline{f}$. Moreover, using Lemma 4.3.1(i) one easily checks that the RHS of (4.123) satisfies Theorem 4.6.4(ii),(iii) for $J = \emptyset$. Hence by the uniqueness of $x_{\emptyset,\underline{i}}$ (see Theorem 4.6.4 and its proof) we deduce that (4.123) is true for all $\underline{i} \in \mathbb{Z}^f$.

In particular, $x_{\emptyset,\underline{i}}$ has the same expression as in the semisimple case, see [BHH$^+$c, (103)]. Then we conclude by the explicit computation for the semisimple case, see [BHH$^+$c, Prop. 3.8.3]. □

Remark 4.10.6. *If $J_{\overline{\rho}} = \mathcal{J}$, then similar to the proof of Corollary 4.10.5 and using the explicit computation in [BHH$^+$c, Prop. 3.8.3] for all J, one can show that the $\mathcal{O}_K^\times$-action on $\mathrm{Hom}_A(D_A(\pi), A)(1)$ is the unique one in Proposition 4.10.4(iv) which satisfies $\mathrm{Mat}(a)'_{J,J} \in 1 + F_{1-p}A$ for all $a \in \mathcal{O}_K^\times$ and $J \subseteq \mathcal{J}$.*

4.11 On the subrepresentations of π

In this section, we finish the proof of Theorem 4.1.2, see Theorem 4.11.2. This theorem is crucially needed to prove that π is of finite length in the non-semisimple case in [BHH$^+$a]. As a corollary, we prove that π is generated by $D_0(\overline{\rho})$ under the assumption that $\pi^\vee$ is essentially self-dual of grade $2f$ in the sense of [BHH$^+$b, (176)], see Corollary 4.11.3, which gives another proof of [HW22, Thm. 1.6] (but under a stronger genericity condition).

Lemma 4.11.1. *Let π_1 be a subrepresentation of π. Then there exists a set S of subsets of $\mathcal{J}$ which is stable under $J \mapsto J - 1$, and is moreover stable under taking subsets if $J_{\overline{\rho}} \neq \mathcal{J}$, such that*

$$\begin{aligned} \mathrm{JH}(\pi_1^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}}) &= \left\{\sigma_{\underline{e}^J} : J \in S\right\}; \\ \mathrm{JH}(\pi_1^{K_1}) &= \left\{\sigma_{\underline{b}} \in \mathrm{JH}\left(D_0(\overline{\rho})\right) : \{j : b_j \geq 1\} \in S\right\}, \end{aligned} \tag{4.124}$$

where $\overline{\rho}^{\mathrm{ss}}$ is the semisimplification of $\overline{\rho}$, $\sigma_{\underline{b}}$ and $\underline{e}^J$ are defined in §4.2, and see Lemma 4.4.1(i) for $\mathrm{JH}\left(D_0(\overline{\rho})\right)$.

Proof. We recall from Corollary 4.5.3 that for each $J \subseteq \mathcal{J}$ we have $\underline{\varepsilon}^J \in \{\pm 1\}^f$ with $\varepsilon_j^J = (-1)^{\delta_{j \notin J}}$. We also recall from (4.6) that $\sigma_J = \sigma_{\underline{e}^J}$ for $J \subseteq J_{\overline{\rho}}$.

Claim 1. If $\sigma_{\underline{e}^J} \in \mathrm{JH}(\pi_1^{K_1})$ for some $J \subseteq \mathcal{J}$, then $\pi_1^{K_1}$ contains $I\left(\sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}}}, \sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}} + \underline{\varepsilon}^{J-1}}\right)$ and $I\left(\sigma_{\underline{e}^{J^{\mathrm{ss}}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}} + \underline{\varepsilon}^J}\right)$ (see Lemma 4.5.1(iii) for the notation).

Proof. We prove the claim by increasing induction on $|J|$. Fix $J \subseteq \mathcal{J}$ and assume that $\sigma_{\underline{e}^J} \in \mathrm{JH}(\pi_1^{K_1})$. Since $\pi_1^{K_1}$ is a $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation of $\pi^{K_1} = D_0(\overline{\rho})$ and $\sigma_{\underline{e}^J} \in \mathrm{JH}\left(D_{0,\sigma_{J^{\mathrm{ss}}}}(\overline{\rho})\right)$ by Lemma 4.4.1(i), we deduce from Corollary 4.5.3 that $\pi_1^{K_1}$ contains $I\left(\sigma_{\underline{e}^{J^{\mathrm{ss}}}}, \sigma_{\underline{e}^J}\right)$, which is

a subrepresentation of $I\big(\sigma_{\underline{e}^{J^{\mathrm{ss}}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}}+\underline{\varepsilon}^{J}}\big)$ by Lemma 4.5.1(iii). In particular, for each $J' \subseteq \mathcal{J}$ such that $J^{\mathrm{ss}} \subseteq J' \subsetneqq J$, we have $\sigma_{\underline{e}^{J'}} \in \mathrm{JH}(\pi_1^{K_1})$. Then by the induction hypothesis and using $(J')^{\mathrm{ss}} = J^{\mathrm{ss}}$, we deduce that $\pi_1^{K_1}$ contains $I\big(\sigma_{\underline{e}^{J^{\mathrm{ss}}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}}+\varepsilon^{J'}}\big)$. In particular, by Lemma 4.5.1(iii), $\mathrm{JH}(\pi_1^{K_1})$ contains all $\sigma_{\underline{b}}$ with

$$\begin{cases} b_j = \delta_{j\in J^{\mathrm{ss}}} & \text{if } j \notin J^{\mathrm{nss}} \\ b_j \in \{0,-1\} & \text{if } j \in J^{\mathrm{nss}},\ j \notin J' \\ b_j \in \{0,1\} & \text{if } j \in J^{\mathrm{nss}},\ j \in J'. \end{cases}$$

By varying J' such that $J^{\mathrm{ss}} \subseteq J' \subsetneqq J$ and using $\sigma_{\underline{e}^{J}} \in \mathrm{JH}(\pi_1^{K_1})$, we deduce that $\mathrm{JH}(\pi_1^{K_1})$ contains $\sigma_{\underline{b}}$ with

$$\begin{cases} b_j = \delta_{j\in J^{\mathrm{ss}}} & \text{if } j \notin J^{\mathrm{nss}} \\ b_j \in \{-1,0,1\} & \text{if } j \in J^{\mathrm{nss}}. \end{cases}$$

Hence we have $\underline{Y}^{-\underline{1}} v_J \in \pi_1^{K_1}$ by Proposition 4.4.2.

By Lemma 4.5.1(i),(iii), we have

$$\big\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{1}} v_J \big\rangle = I\big(\sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}}}, \sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}}+\underline{c}}\big) \subseteq \pi_1 \tag{4.125}$$

with $c_j = (-1)^{\delta_{j+1\notin J}}\big(3 - \delta_{j\in J\Delta(J-1)^{\mathrm{ss}}}\big)$. Since $\varepsilon_j^{J-1} = (-1)^{\delta_{j+1\notin J}}$ and $3 - \delta_{j\in J\Delta(J-1)^{\mathrm{ss}}} \geq 1$, we deduce from (4.125) and Corollary 4.5.3 that

$$I\big(\sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}}}, \sigma_{\underline{e}^{(J-1)^{\mathrm{ss}}}+\underline{\varepsilon}^{J-1}}\big) \subseteq \pi_1 \cap \pi^{K_1} = \pi_1^{K_1}, \tag{4.126}$$

which proves the first part of the claim.

By Lemma 4.5.1(iii) and (4.126), we have $\sigma_{\underline{e}^{J-1}} \in \mathrm{JH}(\pi_1^{K_1})$. Continuing the above process with J replaced with $J-1$ and so on, we deduce that $\pi_1^{K_1}$ contains $I\big(\sigma_{\underline{e}^{(J-i)^{\mathrm{ss}}}}, \sigma_{\underline{e}^{(J-i)^{\mathrm{ss}}}+\underline{\varepsilon}^{J-i}}\big)$ for all $i \geq 0$. In particular, taking $i = f$, the second part of the claim follows from the fact that $J - f = J$. □

Claim 2. Suppose that $J_{\overline{\rho}} \neq \mathcal{J}$. If $\sigma_{\underline{e}^{J}} \in \mathrm{JH}(\pi_1^{K_1})$ for some $J \subseteq \mathcal{J}$, then $\sigma_{\underline{e}^{J'}} \in \mathrm{JH}(\pi_1^{K_1})$ for all $J' \subseteq J$.

Proof. Without loss of generality, we may assume that $|J \setminus J'| = 1$ and write $J \setminus J' = \{j_0\}$ for some $j_0 \in J$. Since $J_{\overline{\rho}} \neq \mathcal{J}$, by replacing (J, J', j_0) with $(J+i, J'+i, j_0+i)$ for some $0 \leq i \leq f-1$ using Claim 1, we may assume that $j_0 \notin J_{\overline{\rho}}$, which implies $J^{\mathrm{ss}} \subseteq J' \subsetneqq J$. Then we have $\sigma_{\underline{e}^{J'}} \in \mathrm{JH}(\pi_1^{K_1})$ by the first paragraph of the proof of Claim 1. □

We let $S \overset{\mathrm{def}}{=} \big\{J \subseteq \mathcal{J} : \sigma_{\underline{e}^{J}} \in \mathrm{JH}(\pi_1^{K_1})\big\}$. Then by Claim 1 and Claim 2, S is stable under $J \mapsto J-1$, and is moreover stable under taking subsets if $J_{\overline{\rho}} \neq \mathcal{J}$. By (4.5), we have $W(\overline{\rho}^{\mathrm{ss}}) = \big\{\sigma_{\underline{e}^{J}} : J \subseteq \mathcal{J}\big\}$, hence the first formula of (4.124) follows from the definition of S.

Then by Claim 1, we have

$$\pi' \overset{\mathrm{def}}{=} \sum_{J\in S} I\big(\sigma_{\underline{e}^{J^{\mathrm{ss}}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}}+\underline{\varepsilon}^{J}}\big) \subseteq \pi_1^{K_1} \subseteq \pi^{K_1} = D_0(\overline{\rho}). \tag{4.127}$$

Since $\mathrm{JH}(\pi') = \big\{\sigma_{\underline{b}} \in \mathrm{JH}\left(D_0(\overline{\rho})\right) : \{j : b_j \geq 1\} \in S\big\}$ by Lemma 4.5.1(iii), to prove the second formula of (4.124), it suffices to show that the first inclusion in (4.127) is an equality.

Suppose on the contrary that the first inclusion in (4.127) is strict, then there exists $\sigma_{\underline{b}} \in \pi_1^{K_1} \subseteq D_0(\overline{\rho})$ such that $J_0 \overset{\mathrm{def}}{=} \{j : b_j \geq 1\} \notin S$. By Corollary 4.5.3, $\pi_1^{K_1}$ must contain $I\big(\sigma_{\underline{e}^{J_0^{\mathrm{ss}}}}, \sigma_{\underline{b}}\big)$, which contains $\sigma_{\underline{e}^{J_0}}$ as a constituent by Lemma 4.5.1(iii). This contradicts the definition of S. □

Theorem 4.11.2. *Let π_1 be a subrepresentation of π. Then we have*

$$\operatorname{rank}_A D_A(\pi_1) = \left|\mathrm{JH}(\pi_1^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})\right|.$$

Proof. Recall from Theorem 4.9.5 that $\mathrm{Hom}_A(D_A(\pi), A)$ has rank 2^f with A-basis $\{x_J : J \subseteq \mathcal{J}\}$, where x_J is defined before Theorem 4.9.5. Let S be the set of subsets of $\mathcal{J}$ in Lemma 4.11.1. It suffices to show that $\{x_J : J \in S\}$ form an A-basis of $\mathrm{Hom}_A(D_A(\pi_1), A) \hookrightarrow \mathrm{Hom}_A(D_A(\pi), A)$.

First we prove that x_J is an element of $\mathrm{Hom}_A(D_A(\pi_1), A)$ for all $J \in S$. By Proposition 4.4.2, for all $J \in S$ and $\underline{0} \le \underline{i} \le \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$, the element $\underline{Y}^{-\underline{i}} v_J \in D_0(\overline{\rho})$ lies in the subrepresentation of $D_0(\overline{\rho})$ with constituents $\sigma_{\underline{b}}$ for $\underline{b}$ as in (4.13). Hence we have $\underline{Y}^{-\underline{i}} v_J \in \pi_1^{K_1}$ for all $J \in S$ and $\underline{0} \le \underline{i} \le \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$ by the second equality in (4.124), which implies that $x_{J,\underline{i}} \in \pi_1$ for all $J \in S$ and $\underline{i} \le \underline{f}$ by (4.58). Since S is stable under $J \mapsto J - 1$, and is moreover stable under taking subsets if $J_{\overline{\rho}} \neq \mathcal{J}$, using (4.60), an increasing induction on $|J|$ and on $\max_j i_j$ shows that $x_{J,i} \in \pi_1$ for all $J \in S$ and $\underline{i} \in \mathbb{Z}^f$ (if $J_{\overline{\rho}} = \mathcal{J}$ then we have $J^{\mathrm{ss}} = J$ for all $J \subseteq \mathcal{J}$ and we only use increasing induction on $\max_j i_j$). By the definition of x_J and Proposition 4.9.4, we have $x_J \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi_1), \mathbb{F})$ for all $J \in S$. Then as in the proof of 4.9.5, we deduce from Theorem 4.8.5 that $x_J \in \mathrm{Hom}_A(D_A(\pi_1), A)$ for all $J \in S$.

Next we prove that any element of $\mathrm{Hom}_A(D_A(\pi_1), A)$ is an A-linear combination of x_J for $J \in S$. Suppose on the contrary that $\sum_{i=1}^m a_{J_i} x_{J_i} \in \mathrm{Hom}_A(D_A(\pi_1), A)$ for $J_i \notin S$ distinct and $a_{J_i} \in A \setminus \{0\}$. We let J_0 be a maximal (under inclusion) element among those J_i such that $\deg(a_{J_i}) - |\partial J_i|$ is minimal for $1 \le i \le m$ (see the proof of Lemma 4.10.3(iii) for the definition of the degree and see Remark 4.5.12 for ∂J_i). Up to rescaling $\sum_{i=1}^m a_{J_i} x_{J_i}$ by a suitable $\lambda \underline{Y}^{\underline{s}} \in A^\times$ with $\lambda \in \mathbb{F}^\times$ and $\underline{s} \in \mathbb{Z}^f$, we may assume that

$$a_{J_0} = \underline{Y}^{-\underline{e}^{J_0 \setminus \partial J_0}} + \big(\text{terms of degree} \ge -|J_0 \setminus \partial J_0| \text{ and not in } \mathbb{F}\underline{Y}^{-\underline{e}^{J_0 \setminus \partial J_0}}\big), \tag{4.128}$$

which has degree $-|J_0 \setminus \partial J_0|$, hence we have $\deg(a_{J_0}) - |\partial J_0| = -|J_0|$. Then by the assumption on J_0, we have

$$\begin{cases} \deg(a_{J_i}) \ge -|J_0| + |\partial J_i| + 1 = -\big(|J_i \setminus \partial J_i| - |J_i \setminus J_0| - 1\big) & \text{if } J_i \supsetneq J_0 \\ \deg(a_{J_i}) \ge -|J_0| + |\partial J_i| \ge -f & \text{if } J_i \not\supseteq J_0. \end{cases} \tag{4.129}$$

We define the following $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation of $D_0(\overline{\rho})$ (see Corollary 4.5.3):

$$V \overset{\text{def}}{=} \begin{cases} \sum\limits_{J \not\supseteq J_0} I\big(\sigma_{\underline{e}^{J^{\mathrm{ss}}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}} + \underline{e}^J}\big) & \text{if } J_{\overline{\rho}} \neq \mathcal{J} \\ \sum\limits_{J \neq J_0} I\big(\sigma_{\underline{e}^{J^{\mathrm{ss}}}}, \sigma_{\underline{e}^{J^{\mathrm{ss}}} + \underline{e}^J}\big) = \bigoplus\limits_{J \neq J_0} D_{0,\sigma_J}(\overline{\rho}) & \text{if } J_{\overline{\rho}} = \mathcal{J}, \end{cases}$$

By Lemma 4.5.1(iii), V has constituents $\sigma_{\underline{b}}$ with $\{j : b_j \ge 1\} \not\supseteq J_0$ if $J_{\overline{\rho}} \neq \mathcal{J}$, and $\{j : b_j \ge 1\} \neq J_0$ if $J_{\overline{\rho}} = \mathcal{J}$. In particular, since $J_0 \notin S$ and S is stable under taking subsets if $J_{\overline{\rho}} \neq \mathcal{J}$, we deduce from the second equality in (4.124) that $\pi_1^{K_1} \subseteq V$.

Claim. We have the following properties:

(i) If $J \not\supseteq J_0$ and $\|\underline{i}\| \le f$, then $x_{J,\underline{i}} \in V$.
(ii) If $J \supsetneq J_0$ and $\|\underline{i}\| \le |J \setminus \partial J| - |J \setminus J_0| - 1$, then $x_{J,\underline{i}} \in V$.
(iii) If $J = J_0$ and $\|\underline{i}\| \le |J_0 \setminus \partial J_0|$, then we have $x_{J,\underline{i}} \in V$ if $\underline{i} \neq \underline{e}^{J_0 \setminus \partial J_0}$, and $x_{J,\underline{e}^{J_0 \setminus \partial J_0}} \in D_0(\overline{\rho}) \setminus V$.

Proof. (i). By (4.58), we may assume that $\underline{i} \geq \underline{e}^{J^{\mathrm{sh}}}$, in which case we have $x_{J,i} = \underline{Y}^{-\underline{i}'} v_J \in D_0(\overline{\rho})$ with $\underline{i}' \overset{\mathrm{def}}{=} \underline{i} - \underline{e}^{J^{\mathrm{sh}}}$, which is defined in Proposition 4.4.2 since $\underline{0} \leq \underline{i}' \leq \underline{f} - \underline{e}^{J^{\mathrm{sh}}}$. We let $\sigma_{\underline{b}} \in \mathrm{JH}\big(D_0(\overline{\rho})\big)$ be an arbitrary constituent of the $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation of $D_0(\overline{\rho})$ generated by $x_{J,\underline{i}}$. Then by (4.13) we have

$$\begin{aligned} \{j : b_j \geq 1\} &\subseteq J^{\mathrm{ss}} \sqcup \big\{j \in J^{\mathrm{nss}} : i'_j > 0\big\} \sqcup \big\{j : j \in (\partial J)^{\mathrm{nss}} : i'_j = 0\big\} \\ &= J \setminus \big\{j \in (J \setminus \partial J)^{\mathrm{nss}} : i'_j = 0\big\} \subseteq J. \end{aligned} \tag{4.130}$$

In particular, if $J \not\supseteq J_0$, then we deduce from (4.130) that $\{j : b_j \geq 1\} \not\supseteq J_0$, hence $\sigma_{\underline{b}}$ is a constituent of V, which proves that $x_{J,\underline{i}} \in V$.

(ii). By (4.130), we also have

$$\big|\{j : b_j \geq 1\}\big| \leq |J| - \big|\big\{j \in (J \setminus \partial J)^{\mathrm{nss}} : i'_j = 0\big\}\big| \leq |J| - \big(|(J \setminus \partial J)^{\mathrm{nss}}| - \|\underline{i}'\|\big), \tag{4.131}$$

where the second inequality becomes an equality if and only if $\underline{i}' = \underline{e}^{J_1}$ for some $J_1 \subseteq (J \setminus \partial J)^{\mathrm{nss}}$, which is equivalent to $\underline{i} = \underline{e}^{J_2}$ for some $J^{\mathrm{sh}} \subseteq J_2 \subseteq J \setminus \partial J$. If $\|\underline{i}\| \leq |J \setminus \partial J| - |J \setminus J_0| - 1$, which implies $\|\underline{i}'\| \leq \big|(J \setminus \partial J)^{\mathrm{nss}}\big| - |J \setminus J_0| - 1$, then we deduce from (4.131) that $\big|\{j : b_j \geq 1\}\big| \leq |J_0| - 1$, which implies $\{j : b_j \geq 1\} \not\supseteq J_0$, hence $\sigma_{\underline{b}}$ is a constituent of V, which proves that $x_{J,\underline{i}} \in V$.

(iii). Suppose that $J = J_0$ and $\|\underline{i}\| \leq |J_0 \setminus \partial J_0|$. Then by (4.131) we have

$$\big|\{j : b_j \geq 1\}\big| \leq |J_0| - \big(|(J_0 \setminus \partial J_0)^{\mathrm{nss}}| - \|\underline{i}'\|\big) = |J_0| - \big(|J_0 \setminus \partial J_0| - \|\underline{i}\|\big) \leq |J_0|,$$

and at least one inequality is strict if $\underline{i} \neq \underline{e}^{J_0 \setminus \partial J_0}$, in which case we have $\{j : b_j \geq 1\} \not\supseteq J_0$, hence $\sigma_{\underline{b}}$ is a constituent of V. This proves $x_{J_0,\underline{i}} \in V$ if $\|\underline{i}\| \leq |J_0 \setminus \partial J_0|$ and $\underline{i} \neq \underline{e}^{J_0 \setminus \partial J_0}$. Finally, the $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation of $D_0(\overline{\rho})$ generated by $\underline{Y}^{-\underline{e}^{(J_0 \setminus \partial J_0)^{\mathrm{nss}}}} v_{J_0}$ has $\sigma_{\underline{e}^{J_0}}$ as a constituent by (4.13), hence it follows from (4.58) and the description of the constituents of V that $x_{J_0,\underline{e}^{J_0 \setminus \partial J_0}} = \underline{Y}^{-\underline{e}^{(J_0 \setminus \partial J_0)^{\mathrm{nss}}}} v_{J_0} \in D_0(\overline{\rho}) \setminus V$. □

Using (4.107), we identify $\sum_{i=1}^m a_{J_i} x_{J_i} \in \mathrm{Hom}_A(D_A(\pi_1), A)$ as $(z_k)_{k \geq 0} \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(D_A(\pi_1), \mathbb{F})$, which is a sequence of elements of π_1. By writing each $a_{J_i} \in A$ as an infinite sum of monomials in $\underline{Y}$ together with (4.128) and (4.129), we deduce from (4.108), [BHH+c, Remark 3.8.2] and the definition of x_{J_i} that the zeroth term z_0 of the sequence $\sum_{i=1}^m a_{J_i} x_{J_i}$ is a linear combination of $x_{J,\underline{i}}$ satisfying the assumptions of (i),(ii),(iii) of the claim above and with exactly one of the terms equals $x_{J_0,\underline{e}^{J_0 \setminus \partial J_0}}$, hence is an element of $D_0(\overline{\rho}) \setminus V$. By definition, we also have $z_0 \in \pi_1$, hence $z_0 \in \pi_1 \cap D_0(\overline{\rho}) = \pi_1^{K_1} \subseteq V$, which is a contradiction. □

Corollary 4.11.3. *Assume moreover that $\pi^\vee$ is essentially self-dual of grade $2f$ in the sense of [BHH+b, (176)]. Then as a $\mathrm{GL}_2(K)$-representation, π is generated by $D_0(\overline{\rho})$.*

Proof. We use the notation of [BHH+b, Prop. 3.3.5.3]. By Theorem 4.1.1 and the proof of [BHH+b, Prop. 3.3.5.3(i)], we deduce that $\dim_{\mathbb{F}((X))} D_\xi^\vee(\pi') = \mathfrak{m}_{\mathfrak{p}_0}(\pi'^\vee)$ for any subquotient π' of π. By Theorem 4.11.2 and [BHH+b, Remark 3.3.5.4(ii)], we deduce that $D_\xi^\vee(\pi') \neq 0$ for π' a subrepresentation of π. Then the proof of [BHH+b, Prop. 3.3.5.3(iii)] shows that $D_\xi^\vee(\pi') \neq 0$ for π' a quotient of π, see [BHH+b, Remark 3.3.5.4(i)]. Then we can conclude as in the proof of [BHH+b, Thm. 3.3.5.5]. □

4.12 Some pictures for f=2

We give some pictures of the representations $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle \cong Q(\chi_J^s, \chi_J^s \alpha^{\underline{i}}, J - 1)$ (see §4.5 for the notation) when $f = 2$, $J_{\overline{\rho}} = \emptyset$, $p = 17$, $r_0 = 6$, $r_1 = 9$ and $\underline{i} = \underline{1}$.

Let J be $\emptyset$,$\{0\}$,$\{1\}$,$\{0,1\}$ in figures 4.1,4.2,4.3,4.4 respectively. In each picture, the whole representation is $Q(\chi_J^s, \chi_J^s\alpha^{\underline{i}}, \emptyset)$, the shadow part is the kernel of the surjection $Q(\chi_J^s, \chi_J^s\alpha^{\underline{i}}, \emptyset) \twoheadrightarrow \langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle$ and the remaining part is the representation $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle \subseteq \pi$. Each square is a principal series $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s\alpha^{\underline{i}'})$ for some $\underline{0} \leq \underline{i}' \leq \underline{1}$. Each rectangle is a Serre weight which is a constituent of the corresponding principal series, and the rectangle on the right bottom is the socle. We list all the Serre weights in $D_0(\overline{\rho})$. Each lattice point is an H-eigencharacter, except that at each intersection point of two rectangles, it is a double point, each belongs to one of the rectangles. See Lemma 4.3.2(ii),(iii) for a justification of the picture. The upper left corner of each square is the element $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}'} v_J$ for some $\underline{0} \leq \underline{i}' \leq \underline{1}$. The operator Y_0 acts as going one step to the right, and the operator Y_1 acts as going one step to the bottom. Hence, each lattice point is of the form $\underline{Y}^{\underline{k}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}'} v_J$ for some $\underline{0} \leq \underline{k} \leq \underline{p}-\underline{1}$ and $\underline{0} \leq \underline{i}' \leq \underline{1}$, except at the double point, where one needs to make a modification, see 4.3.2(iii)(a). By Proposition 4.5.5, each lattice point in the shadow equals zero (a priori, it is only zero in the subquotient of $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle$ which is the principal series containing it quotiented by the shadow part). Moreover, the overlaps of different representations $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-\underline{i}} v_J \rangle$ are studied in Proposition 4.5.9.

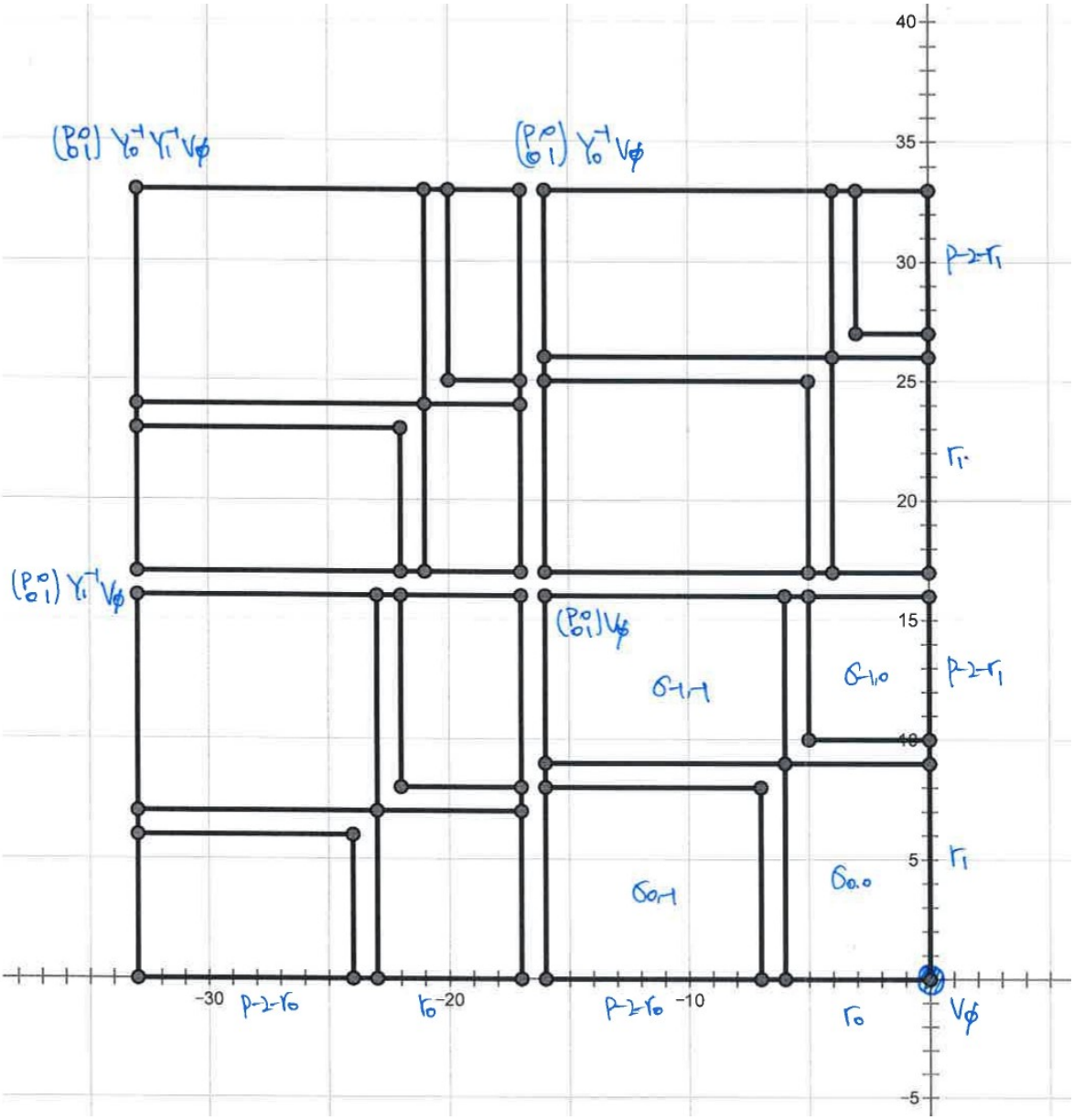

Figure 4.1: $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-1} v_\emptyset \rangle$

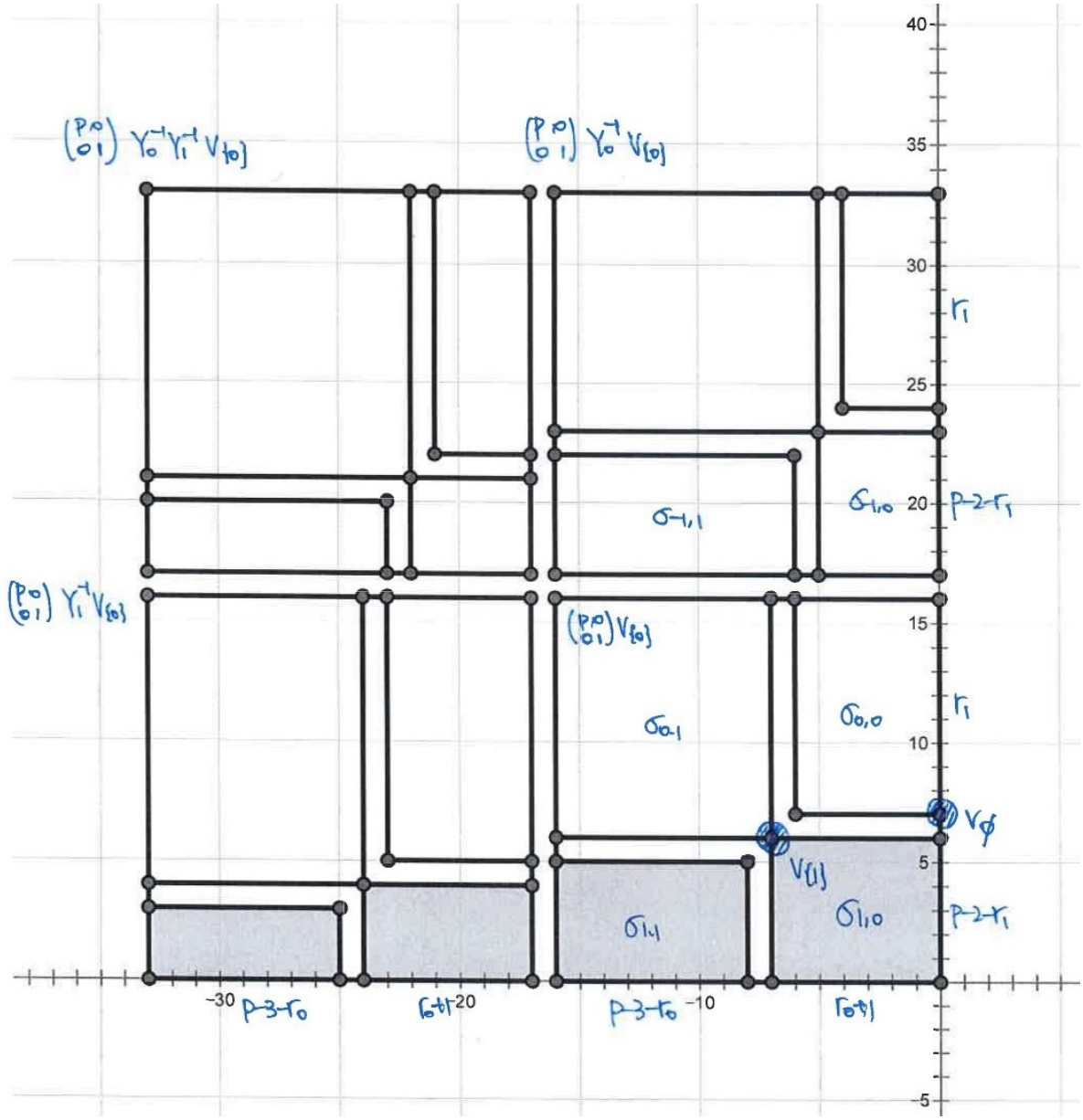

Figure 4.2: $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-1} v_{\{0\}} \rangle$

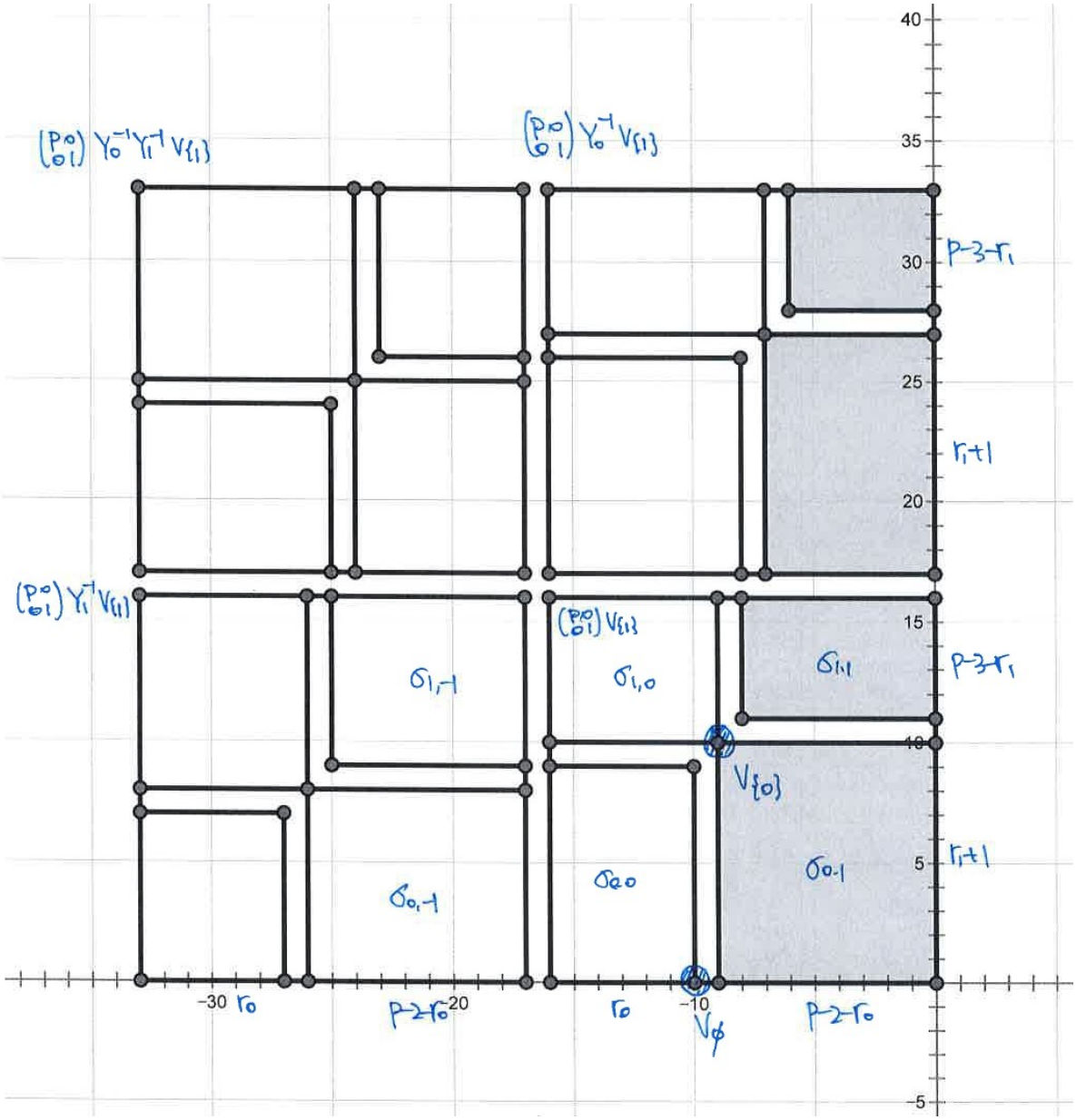

Figure 4.3: $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-1} v_{\{1\}} \rangle$

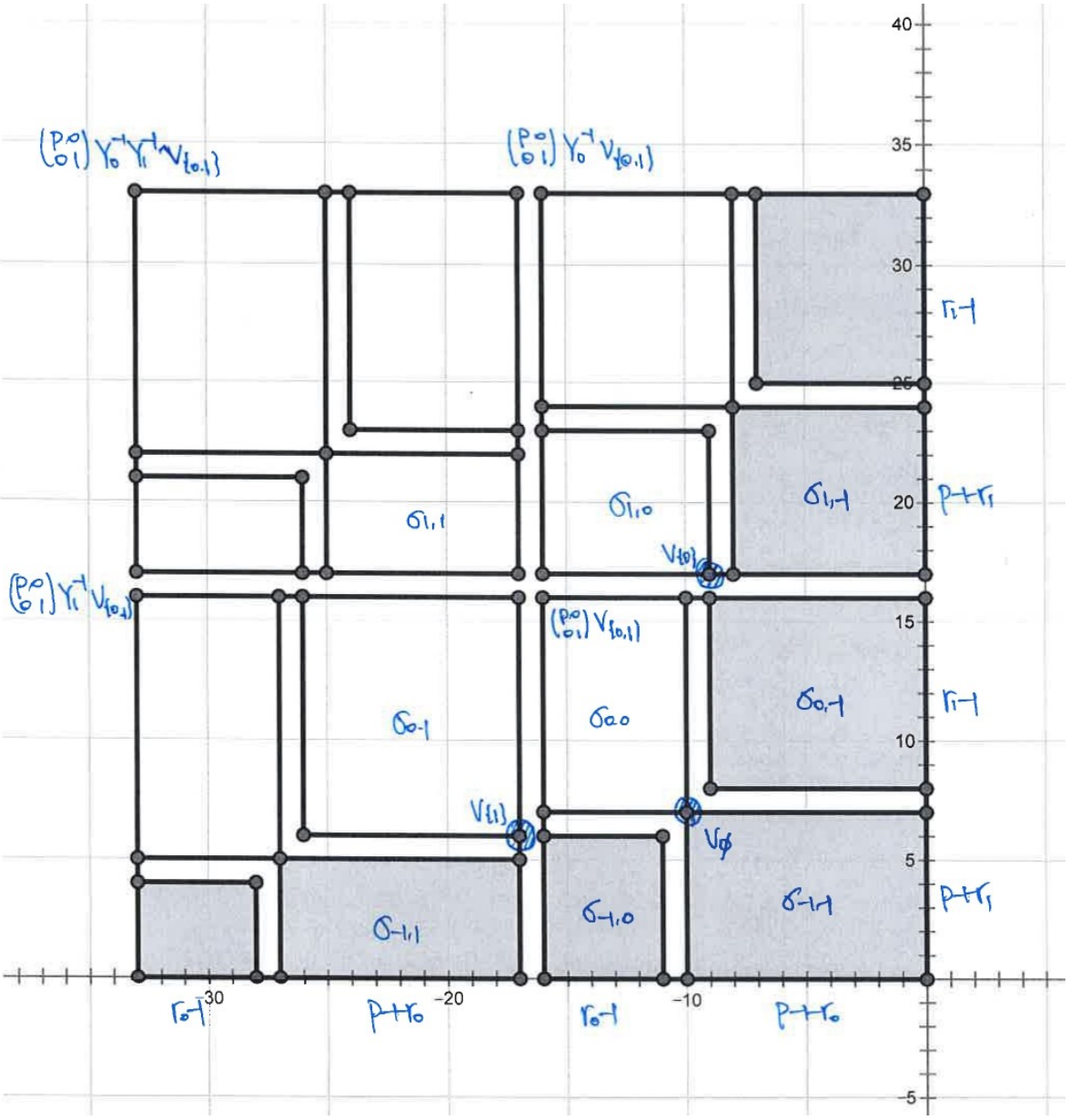

Figure 4.4: $\langle \mathrm{GL}_2(\mathcal{O}_K) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \underline{Y}^{-1} v_{\{0,1\}} \rangle$

Chapter 5

Lubin–Tate and multivariable $(\varphi, \mathcal{O}_K^\times)$-modules in dimension 2

5.1 Introduction

Let p be a prime number. The mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is completely known by the work of Breuil, Colmez, Emerton, etc. In particular, Colmez ([Col10]) constructed a functor from the category of admissible finite length mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to the category of finite-dimensional continuous mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, using Fontaine's category of (φ,Γ)-modules ([Fon90]) as an intermediate step. This gives a functorial way to realize the mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

However, the situation becomes much more complicated when we consider $\mathrm{GL}_2(K)$ for K a nontrivial finite extension of $\mathbb{Q}_p$. For example, there are many more supersingular representations of $\mathrm{GL}_2(K)$ ([BP12]) and we don't have a classification of these representations. Motivated by the local-global compatibility result of Emerton ([Eme11]) for $\mathrm{GL}_2(\mathbb{Q}_p)$, we are particularly interested in the mod p representations π of $\mathrm{GL}_2(K)$ coming from the cohomology of towers of Shimura curves.

We introduce the global setup following [BHH$^+$c]. Let F be a totally real number field that is unramified at places above p. Let D be a quaternion algebra with center F which is split at places above p and at exactly one infinite place. For each compact open subgroup $U \subseteq (D \otimes_F \mathbb{A}_F^\infty)$ where $\mathbb{A}_F^\infty$ is the set of finite adèles of F, we denote by X_U the associated smooth projective algebraic Shimura curve over F.

Let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$. We fix an absolutely irreducible continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$. For w a finite place of F, we write $\overline{r}_w \overset{\mathrm{def}}{=} \overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$. We let S_D be the set of finite places where D ramifies, $S_{\overline{r}}$ be the set of finite places where $\overline{r}$ ramifies, and S_p the set of places above p. We fix a place $v \in S_p$ and write $K \overset{\mathrm{def}}{=} F_v$. We assume that

(i) $p \geq 5$, $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible and the image of $\overline{r}\big(\mathrm{Gal}(\overline{F}/F(\sqrt[5]{1}))\big)$ in $\mathrm{PGL}_2(\mathbb{F})$ is not isomorphic to A_5;
(ii) $\overline{r}_w$ is generic in the sense of [BP12, Def. 11.7] for $w \in S_p$;
(iii) $\overline{r}_w$ is non-scalar for $w \in S_D$.

Then there is a so-called "local factor" defined in [BD14, §3.3] and [EGS15, §6.5] as follows:

$$\pi \overset{\mathrm{def}}{=} \mathrm{Hom}_{U^v}\left(\overline{M}^v, \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\left(\overline{r}, \varinjlim_V H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})\right)\right)[\mathfrak{m}'], \tag{5.1}$$

where the inductive limit runs over the compact open subgroups $V \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$, and we refer to [BD14, §3.3] and [EGS15, §6.5] for the definitions of the compact open subgroup $U^v \subseteq (D \otimes_F \mathbb{A}_F^{\infty,v})^\times$, the (finite-dimensional) irreducible smooth representation $\overline{M}^v$ of U^v over $\mathbb{F}$, and the maximal ideal $\mathfrak{m}'$ in a certain Hecke algebra.

In [BHH+b], Breuil-Herzig-Hu-Morra-Schraen attached to π an étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ over A, which we briefly recall as follows. We write $f \overset{\text{def}}{=} [F_v : \mathbb{Q}_p]$. Let $\mathbb{F}_q$ be the residue field of F_v (hence $q = p^f$) and fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$. Then we have $\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$ with $Y_j \overset{\text{def}}{=} \sum_{a \in \mathbb{F}_q^\times} \sigma_0(a)^{-p^j} \delta_{[a]} \in \mathbb{F}[\![\mathcal{O}_K]\!]$ for $0 \le j \le f-1$, where $[a] \in \mathcal{O}_K^\times$ is the Techmüller lift of $a \in \mathbb{F}_q^\times$ and $\delta_{[a]}$ is the corresponding element in $\mathbb{F}[\![\mathcal{O}_K]\!]$. We let A be the completion of $\mathbb{F}[\![\mathcal{O}_K]\!][1/(Y_0 \cdots Y_{f-1})]$ with respect to the $(Y_0, \dots, Y_{f-1})$-adic topology. There is an $\mathbb{F}$-linear action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![\mathcal{O}_K]\!]$ given by multiplication on $\mathcal{O}_K$, and an $\mathbb{F}$-linear Frobenius φ on $\mathbb{F}[\![\mathcal{O}_K]\!]$ given by multiplication by p on $\mathcal{O}_K$. They extend canonically by continuity to commuting continuous $\mathbb{F}$-linear actions of φ and $\mathcal{O}_K^\times$ on A. Then an étale $(\varphi, \mathcal{O}_K^\times)$-module over A is by definition a finite free A-module endowed with a semi-linear Frobenius φ and a commuting continuous semi-linear action of $\mathcal{O}_K^\times$ such that the image of φ generates everything.

For π as in (5.1), we let $\pi^\vee$ be its $\mathbb{F}$-linear dual, which is a finitely generated $\mathbb{F}[\![I_1]\!]$-module and is endowed with the $\mathfrak{m}_{I_1}$-adic topology, where $I_1 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ and $\mathfrak{m}_{I_1}$ is the maximal ideal of $\mathbb{F}[\![I_1]\!]$. We define $D_A(\pi)$ to be the completion of $\mathbb{F}[\![\mathcal{O}_K]\!][1/(Y_0 \cdots Y_{f-1})] \otimes_{\mathbb{F}[\![\mathcal{O}_K]\!]} \pi^\vee$ with respect to the tensor product topology, where we view $\pi^\vee$ as an $\mathbb{F}[\![\mathcal{O}_K]\!]$-module via $\mathbb{F}[\![\mathcal{O}_K]\!] \cong \mathbb{F}[\![\left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)]\!] \subseteq \mathbb{F}[\![I_1]\!]$. The $\mathcal{O}_K^\times$-action on $\pi^\vee$ given by $f \mapsto f \circ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (for $a \in \mathcal{O}_K^\times$) extends by continuity to $D_A(\pi)$, and the ψ-action on $\pi^\vee$ given by $f \mapsto f \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induces a continuous A-linear isomorphism $\beta : D_A(\pi) \xrightarrow{\sim} A \otimes_{\varphi, A} D_A(\pi)$ (Theorem 4.1.1). In particular, the inverse $\beta^{-1} = \mathrm{id} \otimes \varphi$ makes $D_A(\pi)$ an étale $(\varphi, \mathcal{O}_K^\times)$-module ([BHH+b, Cor. 3.1.2.9] and [BHH+c, Remark. 2.6.2]).

In [BHH+c], Breuil-Herzig-Hu-Morra-Schraen also gave a conjectural description of $D_A(\pi)$ in terms of $\overline{r}_v$. They constructed a functor $D_A^\otimes$ from the category of finite-dimensional continuous representations of $\mathrm{Gal}(\overline{F}_v/F_v)$ over $\mathbb{F}$ to the category of étale $(\varphi, \mathcal{O}_K^\times)$-modules over A, using the category of Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-modules as an intermediate step. We refer to §5.5 for the precise definition. Then they conjectured that $D_A(\pi)$ is isomorphic to $D_A^\otimes(\overline{r}_v(1))$ as étale $(\varphi, \mathcal{O}_K^\times)$-modules over A, where $\overline{r}_v(1)$ is the Tate twist of $\overline{r}_v$. We compute explicitly the structure of the étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^\otimes(\overline{r}_v(1))$ in Theorem 5.5.10, extending the results of [BHH+c] where $\overline{r}_v$ was assumed to be semisimple.

We assume moreover that

(iv) the framed deformation ring $R_{\overline{r}_w}$ of $\overline{r}_w$ over the Witt vectors $W(\mathbb{F})$ is formally smooth for $w \in (S_D \cup S_{\overline{r}}) \setminus S_p$;

(v) $\overline{r}_v$ is of the following form up to twist:

$$\overline{r}_v|_{I_{F_v}} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix} \text{ with } \max\{12, 2f+1\} \le r_j \le p - \max\{15, 2f+3\} \ \forall\, j,$$

where $I_{F_v} \subseteq \mathrm{Gal}(\overline{F}_v/F_v)$ is the decomposition group.

Our main result is the following:

Theorem 5.1.1. *Let π be as in* (5.1) *and keep all the assumptions on $\overline{r}$. Assume moreover that $|W(\overline{r}_v)| = 1$, where $W(\overline{r}_v)$ is the set of Serre weights of $\overline{r}_v$ defined in [BDJ10, §3]. Then we have an isomorphism of étale $(\varphi, \mathcal{O}_K^\times)$-modules*

$$D_A(\pi) \cong D_A^\otimes(\overline{r}_v(1)).$$

Theorem 5.1.1 is proved by [BHH$^+$c, Thm. 3.1.3] when $\overline{r}_v$ is semisimple. Using the explicit computation of $D_A^{\otimes}(\overline{r}_v(1))$ in Theorem 5.5.10 and the results of §4 on $D_A(\pi)$, we are reduced to the computation of some constants coming from the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ in the sense of [DL21]. When $|W(\overline{r}_v)| = 1$ (i.e. $\overline{r}_v$ is maximally non-split), these constants are computed by [BD14] in terms of the Fontaine–Laffaille module associated to $\overline{r}_v$ ([FL82]). We remark that our method should apply to arbitrary $W(\overline{r}_v)$ once we compute the corresponding constants coming from the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ in general.

The proof of Theorem 5.1.1 is very computational. There may exist a more conceptual proof one day, which will hopefully avoid the genericity assumptions on $\overline{r}_v$ and the technical computations, but such proof is not known so far.

Organization of the chapter

In §5.2 and §5.3, we give an explicit parametrization of the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-modules and the cyclotomic $(\varphi, \mathbb{Z}_p^\times)$-modules associated to reducible two-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$. In §5.4, we construct explicitly some étale $(\varphi, \mathcal{O}_K^\times)$-modules over A that will be needed in §5.5, where we compute explicitly the associated étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^{\otimes}(\overline{\rho})$ for $\overline{\rho}$ an arbitrary reducible two-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ in Theorem 5.5.10. Finally, in §5.6, we combine all the previous results and the results of §4 and [BD14] to finish the proof of Theorem 5.1.1.

Notation

Let p be an odd prime. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $K \subseteq \overline{\mathbb{Q}}_p$ be the unramified extension of $\mathbb{Q}_p$ of degree $f \geq 1$ with ring of integers $\mathcal{O}_K$ and residue field $\mathbb{F}_q$ (hence $q = p^f$). We denote by $G_K \overset{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ the absolute Galois group of K and $I_K \subseteq G_K$ the inertia subgroup. Let $\mathbb{F}$ be a large enough finite extension of $\mathbb{F}_p$. Fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and let $\sigma_j \overset{\text{def}}{=} \sigma_0 \circ \varphi^j$ for $j \in \mathbb{Z}$, where $\varphi : x \mapsto x^p$ is the arithmetic Frobenius on $\mathbb{F}_q$. We identify $\mathcal{J} \overset{\text{def}}{=} \mathrm{Hom}(\mathbb{F}_q, \mathbb{F})$ with $\{0, 1, \dots, f-1\}$, which is also identified with $\mathbb{Z}/f\mathbb{Z}$ so that the addition and subtraction in $\mathcal{J}$ are modulo f. For $a \in \mathcal{O}_K$, we denote by $\overline{a} \in \mathbb{F}_q$ its reduction modulo p. For $a \in \mathbb{F}_q$, we also view it as an element of $\mathbb{F}$ via σ_0.

For F a perfect ring of characteristic p, we denote by $W(F)$ the ring of Witt vectors of F. For $x \in F$, we denote by $[x] \in W(F)$ its Techmüller lift.

Let $I \overset{\text{def}}{=} \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the Iwahori subgroup, $I_1 \overset{\text{def}}{=} \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the pro-p Iwahori subgroup, $K_1 \overset{\text{def}}{=} 1 + p\,\mathrm{M}_2(\mathcal{O}_K) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ be the first congruence subgroup, $N_0 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)$ and $H \overset{\text{def}}{=} \begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix}$.

For P a statement, we let $\delta_P \overset{\text{def}}{=} 1$ if P is true and $\delta_P \overset{\text{def}}{=} 0$ otherwise.

Throughout this chapter, we let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be of the following form:

$$\overline{\rho} \cong \begin{pmatrix} \omega_f^h \,\mathrm{un}(\lambda_0) & * \\ 0 & \mathrm{un}(\lambda_1) \end{pmatrix}, \tag{5.2}$$

where $0 \leq h \leq q-2$, $\lambda_0, \lambda_1 \in \mathbb{F}^\times$, for $\xi \in \mathbb{F}^\times$ we denote by $\mathrm{un}(\xi) : G_K \to \mathbb{F}^\times$ the unramified character sending geometric Frobenius elements to ξ, and $\omega_f : G_K \to \mathbb{F}$ is the extension to G_K of the fundamental character of level f (associate to σ_0) such that $\omega_f(g)$ is the reduction

modulo p of $g(p_f)/p_f \in \mu_{q-1}(\overline{K}^\times)$ for all $g \in G_K$ and for any choice of a $(q-1)$-th root p_f of $-p$.

Then we can write $h = \sum_{i=0}^{f-1} p^j h_j$ with $0 \leq h_j \leq p-1$ for $0 \leq j \leq f-1$ in a unique way. We extend the definition of h_j to all $j \in \mathbb{Z}$ by the relation $h_{j+f} = h_j$ for all $j \in \mathbb{Z}$. For $j \geq 0$, we set

$$[h]_j \overset{\text{def}}{=} h_0 + ph_1 + \cdots + p^j h_j.$$

In particular, we have $[h]_{f-1} = h$. We also define $[h]_{-1} \overset{\text{def}}{=} 0$ and $[h]_{-2} \overset{\text{def}}{=} -h_{f-1}/p$, hence $[h]_{j+f} = h + q[h]_j$ for all $j \geq -2$.

We define $h'_j \overset{\text{def}}{=} h_{j+1}$ for $j \in \mathbb{Z}$ and $h' \overset{\text{def}}{=} \sum_{i=0}^{f-1} p^j h'_j = \sum_{i=0}^{f-1} p^j h_{j+1}$. Then we define $[h']_j$ for $j \geq -2$ in a similar way.

5.2 Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-modules

In this section, we give an explicit parametrization of the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-modules corresponding to $\overline{\rho}$ as in (5.2). The main result is Theorem 5.2.10.

Let G_{LT} be the unique (up to isomorphism) Lubin–Tate formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ associated to the uniformizer p. We choose the formal variable T_K of G_{LT} so that the logarithm ([Lan90, §8.6]) is given by the power series $\sum_{n=0}^{\infty} p^{-n} T_K^{q^n}$. For $a \in \mathcal{O}_K$ we have power series $a_{\mathrm{LT}}(T_K) \in aT_K + T_K^2 \mathcal{O}_K[\![T_K]\!]$.

As in [BHH$^+$c, §2.1], there is a continuous $\mathbb{F}$-linear endomorphism φ of $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T_K)\!)$ which is the p-th power map on $\mathbb{F}_q$ and satisfies $\varphi(T_K) = T_K^p$, and a continuous $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q$-linear action (commuting with φ) of $\mathcal{O}_K^\times$ on $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T_K)\!)$ satisfying $a(T_K) = a_{\mathrm{LT}}(T_K)$ for $a \in \mathcal{O}_K^\times$, where we still denote by $a_{\mathrm{LT}}(T_K) \in \mathbb{F}_q[\![T_K]\!]$ the reduction modulo p of $a_{\mathrm{LT}}(T_K) \in \mathcal{O}_K[\![T_K]\!]$. Then there is a covariant exact equivalence of categories compatible with tensor products between the category of finite-dimensional continuous representations of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ and the category of étale $(\varphi, \mathcal{O}_K^\times)$-modules over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T_K)\!)$.

For D_K an étale φ-module over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T_K)\!)$, the isomorphism

$$\begin{aligned} \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T_K)\!) &\xrightarrow{\sim} \mathbb{F}(\!(T_{K,\sigma_0})\!) \times \mathbb{F}(\!(T_{K,\sigma_1})\!) \times \cdots \times \mathbb{F}(\!(T_{K,\sigma_{f-1}})\!) \\ \lambda \otimes (\textstyle\sum_{n \gg -\infty} c_n T_K^n) &\mapsto \big((\textstyle\sum_{n \gg -\infty} \lambda \sigma_0(c_n) T_{K,\sigma_0}^n), \ldots, (\textstyle\sum_{n \gg -\infty} \lambda \sigma_{f-1}(c_n) T_{K,\sigma_{f-1}}^n) \big) \end{aligned} \tag{5.3}$$

induces a decomposition

$$D_K \xrightarrow{\sim} D_{K,\sigma_0} \times \cdots \times D_{K,\sigma_{f-1}}.$$

For each $0 \leq i \leq f-1$, the functor $D_K \mapsto D_{K,\sigma_i}$ induces an equivalence of categories between the category of étale $(\varphi, \mathcal{O}_K^\times)$-modules over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T_K)\!)$ and the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over $\mathbb{F}(\!(T_{K,\sigma_i})\!)$. Here $\varphi_q \overset{\text{def}}{=} \varphi^f$, and $\mathbb{F}(\!(T_{K,\sigma_i})\!)$ is endowed with an $\mathbb{F}$-linear endomorphism φ_q such that $\varphi_q(T_{K,\sigma_i}) = T_{K,\sigma_i}^q$, and a continuous $\mathbb{F}$-linear action (commuting with φ_q) of $\mathcal{O}_K^\times$ such that $a(T_{K,\sigma_i}) = a_{\mathrm{LT}}(T_{K,\sigma_i})$ for $a \in \mathcal{O}_K^\times$, where $a_{\mathrm{LT}}(T_{K,\sigma_i}) \in \mathbb{F}[\![T_{K,\sigma_i}]\!]$ is the image of $a_{\mathrm{LT}}(T_K) \in \mathbb{F}_q[\![T_K]\!]$ in $\mathbb{F}[\![T_{K,\sigma_i}]\!]$ via the embedding $\sigma_i : \mathbb{F}_q \hookrightarrow \mathbb{F}$.

For $\overline{\rho}$ a finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, we denote by $D_K(\overline{\rho})$ the associated étale $(\varphi, \mathcal{O}_K^\times)$-module over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T_K)\!)$, and for each $0 \leq i \leq f-1$ we denote by $D_{K,\sigma_i}(\overline{\rho})$ the associated étale $(\varphi_q, \mathcal{O}_K^\times)$-module over $\mathbb{F}(\!(T_{K,\sigma_i})\!)$.

For $a \in \mathcal{O}_K^\times$, we set

$$f_a^{\mathrm{LT}} \overset{\text{def}}{=} \overline{a} T_K / a(T_K) \in 1 + T_K \mathbb{F}[\![T_K]\!].$$

We still denote by f_a^{LT} its image in $\mathbb{F}(\!(T_{K,\sigma_0})\!)$ via σ_0 when there is no possible confusion.

Any (continuous) character of G_K over $\mathbb{F}$ is of the form $\omega_f^h \operatorname{un}(\lambda)$ for $0 \leq h \leq q-2$ and $\lambda \in \mathbb{F}^\times$. By [BHH$^+$c, Lemma 2.1.8], the corresponding étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{K,\sigma_0}\big(\omega_f^h \operatorname{un}(\lambda)\big)$ can be described as follows ($a \in \mathcal{O}_K^\times$):

$$\begin{cases} D_{K,\sigma_0}\big(\omega_f^h \operatorname{un}(\lambda)\big) &= \mathbb{F}(\!(T_{K,\sigma_0})\!)e \\ \varphi_q(e) &= \lambda T_{K,\sigma_0}^{-(q-1)h} e \\ a(e) &= \big(f_a^{\mathrm{LT}}\big)^h e. \end{cases} \tag{5.4}$$

Lemma 5.2.1. *We have $f_a^{\mathrm{LT}} = 1$ for $a \in [\mathbb{F}_q^\times]$. More generally, we have for $a \in \mathcal{O}_K^\times$*

$$\big(f_a^{\mathrm{LT}}\big)^{-1} \in 1 + c_a T_K^{q-1} - c_a^{p^{f-1}} T_K^{(q-1)(p^{f-1}+1)} + T_K^{(q-1)(2p^{f-1}+1)} \mathbb{F}_q[\![T_K^{q-1}]\!],$$

where $c_a \in \mathbb{F}_q$ is the reduction modulo p of $(1-a^{q-1})/p \in \mathcal{O}_K$.

Proof. By [Lan90, Lemma 8.6.2] we have equality in $\mathcal{O}_K[\![T_K]\!]$

$$\sum_{n=0}^{\infty} \frac{a_{\mathrm{LT}}(T_K)^{q^n}}{p^n} = a \sum_{n=0}^{\infty} \frac{T_K^{q^n}}{p^n}. \tag{5.5}$$

In particular, for $a \in [\mathbb{F}_q^\times]$ we have $a_{\mathrm{LT}}(T_K) = aT_K$, which implies $f_a^{\mathrm{LT}} = 1$. Then the commutativity of the actions of $\mathcal{O}_K^\times$ and $[\mathbb{F}_q^\times]$ implies that $a_{\mathrm{LT}}(T_K) \in aT_K\big(1 + T_K^{q-1}\mathcal{O}_K[\![T_K^{q-1}]\!]\big)$ for $a \in \mathcal{O}_K^\times$, and we write in $\mathcal{O}_K[\![T_K^{q-1}]\!]$

$$a_{\mathrm{LT}}(T_K) = aT_K \left(1 + \sum_{i=1}^{\infty} x_a(i) T_K^{(q-1)i}\right) \tag{5.6}$$

for $x_a(i) \in \mathcal{O}_K$. Then by (5.5) we have

$$1 + \sum_{i=1}^{2p^{f-1}} x_a(i) T_K^{(q-1)i} + \frac{a^{q-1}T_K^{q-1}}{p} \left(1 + \sum_{i=1}^{2p^{f-1}} x_a(i) T_K^{(q-1)i}\right)^q \equiv 1 + \frac{T_K^{q-1}}{p} \mod T_K^{(q-1)(2p^{f-1}+1)}. \tag{5.7}$$

Comparing the coefficients of T_K^{q-1}, we get $x_a(1) = (1-a^{q-1})/p$. Also, each term of the expansion $\big(1 + \sum_{i=1}^{2p^{f-1}} x_a(i) T_K^{(q-1)i}\big)^q$ has the form

$$\frac{q!}{n_0! \cdots n_{2p^{f-1}}!} \prod_{i=1}^{2p^{f-1}} x_a(i)^{n_i} T_K^{(q-1)\sum_{i=1}^{2p^{f-1}} i n_i} \tag{5.8}$$

with $0 \leq n_i \leq q$ and $\sum_{i=0}^{2p^{f-1}} n_i = q$.

Claim. For the terms in (5.8) such that $\sum_{i=1}^{2p^{f-1}} i n_i \leq 2p^{f-1}-1$, we have $v_p\big(q!/(n_0! \cdots n_{2p^{f-1}}!)\big) \geq 2$ except in the following two cases:

(a) $n_0 = q$ and $n_i = 0$ for $i \neq 0$, in which case the term in (5.8) is 1;
(b) $n_0 = (p-1)p^{f-1}$, $n_1 = p^{f-1}$ and $n_i = 0$ for $i > 1$, in which case the term in (5.8) is congruent to $p x_a(1)^{p^{f-1}} T_K^{(q-1)p^{f-1}}$ modulo p^2.

Proof. Recall that $v_p(n!) = (n - S_p(n))/(p-1)$, where $S_p(n)$ is the sum of the digits in the p-adic expansion of n. Hence we have

$$v_p\left(\frac{q!}{n_0!\cdots n_{2p^{f-1}}!}\right) = \frac{1}{p-1}\left[\left(\sum_{i=0}^{2p^{f-1}} S_p(n_i)\right) - 1\right].$$

If $v_p\big(q!/(n_0!\cdots n_{2p^{f-1}}!)\big) \leq 1$, then we have $\sum_{i=0}^{2p^{f-1}} S_p(n_i) \leq p$, which implies that each n_i must be a multiple of p^{f-1}, hence (a) and (b) are the only possibilities since $\sum_{i=1}^{2p^{f-1}} i n_i \leq 2p^{f-1} - 1$. Moreover, we have by Lucas theorem

$$\frac{1}{p}\cdot\frac{q!}{((p-1)p^{f-1})!\,(p^{f-1})!} = \binom{p^f-1}{p^{f-1}-1} \equiv 1 \mod p,$$

hence the term in (5.8) in case (b) is congruent to $p x_a(1)^{p^{f-1}} T_K^{(q-1)p^{f-1}}$ modulo p^2. □

By the claim above and (5.7), for $1 \leq i \leq 2p^{f-1}$ we have $x_a(i) \in p\mathcal{O}_K$ except possibly in the following two cases:

(i) $x_a(1) = (1 - a^{q-1})/p$;
(ii) $x_a(p^{f-1}+1) \equiv -a^{q-1} x_a(1)^{p^{f-1}} \equiv -x_a(1)^{p^{f-1}} \mod p$.

Then by reducing (5.6) modulo p we have

$$\big(f_a^{\mathrm{LT}}\big)^{-1} = a_{\mathrm{LT}}(T_K)/(\overline{a}T_K) \in 1 + c_a T_K^{q-1} - c_a^{p^{f-1}} T_K^{(q-1)(p^{f-1}+1)} + T_K^{(q-1)(2p^{f-1}+1)}\mathbb{F}_q[\![T_K^{q-1}]\!],$$

which completes the proof. □

Remark 5.2.2. *The map $\mathcal{O}_K^\times \to \mathbb{F}_q$, $a \mapsto c_a$ is a group homomorphism and satisfies:*

(i) *If $a \in [\mathbb{F}_q^\times]$, then $c_a = 0$.*
(ii) *If $a = 1 + pb$ for some $b \in \mathcal{O}_K$, then $c_a = \overline{b}$.*

Since $a(T_{K,\sigma_0}) = \overline{a}T_{K,\sigma_0}$ for $a \in [\mathbb{F}_q^\times]$ by Lemma 5.2.1, we have $\mathbb{F}((T_{K,\sigma_0}))^{[\mathbb{F}_q^\times]} = \mathbb{F}((T_{K,\sigma_0}^{q-1}))$. Then for $\overline{\rho}$ as in (5.2), we have $D_{K,\sigma_0}(\overline{\rho}) \cong \mathbb{F}((T_{K,\sigma_0}))\otimes_{\mathbb{F}((T_{K,\sigma_0}^{q-1}))} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$, where $D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]}$ has the following form (using (5.4), and $a \in \mathcal{O}_K^\times$):

$$\begin{cases} D_{K,\sigma_0}(\overline{\rho})^{[\mathbb{F}_q^\times]} &= \mathbb{F}((T_{K,\sigma_0}^{q-1}))e_0 \oplus \mathbb{F}((T_{K,\sigma_0}^{q-1}))e_1 \\ \varphi_q(e_0\ e_1) &= (e_0\ e_1)\,\mathrm{Mat}(\varphi_q) \\ a(e_0\ e_1) &= (e_0\ e_1)\,\mathrm{Mat}(a) \end{cases}$$

with

$$\begin{cases} \mathrm{Mat}(\varphi_q) &= \begin{pmatrix} \lambda_0 T_{K,\sigma_0}^{-(q-1)h} & \lambda_1 D \\ 0 & \lambda_1 \end{pmatrix} \\ \mathrm{Mat}(a) &= \begin{pmatrix} \big(f_a^{\mathrm{LT}}\big)^h & E_a \\ 0 & 1 \end{pmatrix} \end{cases}$$

for some $D \in \mathbb{F}((T_{K,\sigma_0}^{q-1}))$ and $E_a \in \mathbb{F}((T_{K,\sigma_0}^{q-1}))$.

Definition 5.2.3. *Let $0 \leq h \leq q-2$ and $\lambda_0, \lambda_1 \in \mathbb{F}^\times$. We define W^{LT} to be the set of equivalence classes of tuples $[B] = \big(D, (E_a)_{a\in\mathcal{O}_K^\times}\big)$ such that*

(i) $D \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$, $E_a \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$ *for all* $a \in \mathcal{O}_K^\times$, *and the map* $\mathcal{O}_K^\times \to \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$, $a \mapsto E_a$ *is continuous;*

(ii) $E_{ab} = E_a + \big(f_a^{\mathrm{LT}}\big)^h a(E_b)$ *for all* $a, b \in \mathcal{O}_K^\times$;

(iii) $\big(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\big)(E_a) = \big(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\big)(D)$ *for all* $a \in \mathcal{O}_K^\times$;

(iv) *two tuples* $\big(D, (E_a)_{a\in\mathcal{O}_K^\times}\big)$ *and* $\big(D', (E'_a)_{a\in\mathcal{O}_K^\times}\big)$ *are equivalent if and only if there exists* $b \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$ *such that*

$$\begin{cases} D' = D + \left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)(b) \\ E'_a = E_a + \left(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\right)(b) \quad \forall\, a \in \mathcal{O}_K^\times. \end{cases}$$

It has a natural structure of an $\mathbb{F}$*-vector space.*

By the definition of W^{LT} and the equivalence of categories $\overline{\rho} \mapsto D_{K,\sigma_0}(\overline{\rho})$, there is an isomorphism of $\mathbb{F}$-vector spaces

$$W^{\mathrm{LT}} \cong \mathrm{Ext}^1\left(D_{K,\sigma_0}\big(\mathrm{un}(\lambda_1)\big), D_{K,\sigma_0}\big(\omega_f^h\,\mathrm{un}(\lambda_0)\big)\right) \cong H^1\left(G_K, \mathbb{F}\big(\omega_f^h\,\mathrm{un}(\lambda_0\lambda_1^{-1})\big)\right), \tag{5.9}$$

where Ext^1 is defined in the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over $\mathbb{F}(\!(T_{K,\sigma_0})\!)$. For each $[B] \in W^{\mathrm{LT}}$, we denote by $D([B])$ the corresponding étale $(\varphi_q, \mathcal{O}_K^\times)$-module over $\mathbb{F}(\!(T_{K,\sigma_0})\!)$. Note that $D([B]) \cong D(\lambda[B])$ as étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over $\mathbb{F}(\!(T_{K,\sigma_0})\!)$ for $\lambda \in \mathbb{F}^\times$.

Lemma 5.2.4. *Let* $0 \le h \le q-2$.

(i) *For* $i \ge -1$ *and* $a \in \mathcal{O}_K^\times$, *we have*

$$\left(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\right)\left(T_{K,\sigma_0}^{-(q-1)[h]_i}\right) \in T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].$$

(ii) *For* $i \ge -1$ *and* $a \in \mathcal{O}_K^\times$, *we have*

$$\left(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\right)\left(T_{K,\sigma_0}^{-(q-1)([h]_i+p^{i+1})}\right) \in (h_{i+1}-1)c_a^{p^{i+1}}T_{K,\sigma_0}^{-(q-1)[h]_i} + T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].$$

(iii) *For* $i \ge f-1$ *such that* $h_i = 1$ *and* $a \in \mathcal{O}_K^\times$, *we have*

$$\left(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\right)\left(T_{K,\sigma_0}^{-(q-1)([h]_i+p^{i+1-f})}\right) \in -c_a^{p^{i+1}}T_{K,\sigma_0}^{-(q-1)[h]_i} + c_a^{p^i}T_{K,\sigma_0}^{-(q-1)[h]_{i-1}} + T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].$$

Proof. For $s \in \mathbb{Z}$ and $a \in \mathcal{O}_K^\times$, by definition we have

$$\left(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\right)\left(T_{K,\sigma_0}^{-(q-1)s}\right) = T_{K,\sigma_0}^{-(q-1)s}\left(1 - \big(f_a^{\mathrm{LT}}\big)^{h+(q-1)s}\right). \tag{5.10}$$

(i). Take $s = [h]_i$. Since $h+(q-1)[h]_i = [h]_{i+f} - [h]_i$ is a multiple of p^{i+1} and $p^{i+1} \ge [h]_i + 1$, we deduce from (5.10) and Lemma 5.2.1 that

$$\left(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\right)\left(T_{K,\sigma_0}^{-(q-1)[h]_i}\right) \in T_{K,\sigma_0}^{-(q-1)[h]_i}\left(T_{K,\sigma_0}^{(q-1)p^{i+1}}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!]\right) \subseteq T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].$$

(ii). Take $s = [h]_i + p^{i+1}$. We have

$$h + (q-1)([h]_i + p^{i+1}) = [h]_{i+f} - [h]_i + qp^{i+1} - p^{i+1} \in (h_{i+1}-1)p^{i+1} + p^{i+2}\mathbb{Z}.$$

Then using $p^{i+1} \geq [h]_i + 1$, we deduce from (5.10) and Lemma 5.2.1 that

$$\begin{aligned}
&\Big(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\Big)\Big(T_{K,\sigma_0}^{-(q-1)([h]_i+p^{i+1})}\Big) \\
&\qquad \in T_{K,\sigma_0}^{-(q-1)([h]_i+p^{i+1})}\Big((h_{i+1}-1)c_a^{p^{i+1}} T_{K,\sigma_0}^{(q-1)p^{i+1}} + T_{K,\sigma_0}^{2(q-1)p^{i+1}} \mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!]\Big) \\
&\qquad \subseteq (h_{i+1}-1)c_a^{p^{i+1}} T_{K,\sigma_0}^{-(q-1)[h]_i} + T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].
\end{aligned}$$

(iii). Take $s = [h]_i + p^{i+1-f}$. We have

$$h + (q-1)([h]_i + p^{i+1-f}) = [h]_{i+f} - [h]_i + p^{i+1} - p^{i+1-f} \in -p^{i+1-f} + p^{i+1}\mathbb{Z}.$$

Then we deduce from (5.10) and Lemma 5.2.1 that

$$\begin{aligned}
&\Big(\mathrm{id} - \big(f_a^{\mathrm{LT}}\big)^h a\Big)\Big(T_{K,\sigma_0}^{-(q-1)([h]_i+p^{i+1-f})}\Big) \\
&\qquad \in T_{K,\sigma_0}^{-(q-1)([h]_i+p^{i+1-f})}\Big(- c_a^{p^{i+1}} T_{K,\sigma_0}^{(q-1)p^{i+1-f}} + c_a^{p^i} T_{K,\sigma_0}^{(q-1)(p^{f-1}+1)p^{i+1-f}} \\
&\qquad\qquad\qquad + T_{K,\sigma_0}^{(q-1)(2p^{f-1}+1)p^{i+1-f}} \mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!]\Big) \\
&\qquad \subseteq -c_a^{p^{i+1}} T_{K,\sigma_0}^{-(q-1)[h]_i} + c_a^{p^i} T_{K,\sigma_0}^{-(q-1)[h]_{i-1}} + T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!],
\end{aligned}$$

where the first inclusion uses $p \geq 3$ (hence $p^f \geq 2p^{f-1}+1$), and the second inclusion uses $h_i = 1$ (hence $[h]_i = [h]_{i-1} + p^i < 2p^i$). □

Definition 5.2.5. *Let $0 \leq h \leq q-2$, $\lambda_0, \lambda_1 \in \mathbb{F}^\times$ and $0 \leq j \leq f-1$. We define $D_j^{\mathrm{LT}}, D_{\mathrm{tr}}^{\mathrm{LT}}, D_{\mathrm{un}}^{\mathrm{LT}} \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$ as follows:*

(i) *If $h_j \neq 0$, we define*

$$D_j^{\mathrm{LT}} \overset{\mathrm{def}}{=} T_{K,\sigma_0}^{-(q-1)[h]_{j-1}}.$$

If $h_j = 0$, we let $0 \leq r \leq f-1$ such that $h_{j+1} = \cdots = h_{j+r} = 1$ and $h_{j+r+1} \neq 1$, then we define

$$\begin{aligned}
D_j^{\mathrm{LT}} &\overset{\mathrm{def}}{=} \lambda_0\lambda_1^{-1}\left[T_{K,\sigma_0}^{-(q-1)([h]_{f+j+r}+p^{f+j+r+1})} + (h_{j+r+1}-1)\sum_{i=0}^{r} T_{K,\sigma_0}^{-(q-1)([h]_{f+j+i}+p^{j+i+1})}\right] \\
&= \lambda_0\lambda_1^{-1}\Bigg[T_{K,\sigma_0}^{-(q-1)(h+q([h]_{j-1}+p^j(p+p^2+\cdots+p^{r+1})))} \\
&\qquad\qquad + (h_{j+r+1}-1)\sum_{i=0}^{r} T_{K,\sigma_0}^{-(q-1)(h+q([h]_{j-1}+p^j((p+p^2+\cdots+p^i))+p^{j+i+1}))}\Bigg].
\end{aligned}$$

(ii) *If $h = 1 + p + \cdots + p^{f-1}$ and $\lambda_0\lambda_1^{-1} = 1$, we define*

$$D_{\mathrm{tr}}^{\mathrm{LT}} \overset{\mathrm{def}}{=} \sum_{i=0}^{f-1} T_{K,\sigma_0}^{-(q-1)([h]_{f+i-1}+p^i)} = \sum_{i=0}^{f-1} T_{K,\sigma_0}^{-(q-1)(1+p+\cdots+p^{i-1}+2p^i+p^{i+1}+\cdots+p^{f+i-1})}.$$

Otherwise (i.e. either $h \neq 1 + p + \cdots + p^{f-1}$ or $\lambda_0\lambda_1^{-1} \neq 1$), we define $D_{\mathrm{tr}}^{\mathrm{LT}} \overset{\mathrm{def}}{=} 0$.

(iii) *If $h = 0$ and $\lambda_0\lambda_1^{-1} = 1$, we define $D_{\mathrm{un}}^{\mathrm{LT}} \overset{\mathrm{def}}{=} 1$. Otherwise, we define $D_{\mathrm{un}}^{\mathrm{LT}} \overset{\mathrm{def}}{=} 0$.*

Corollary 5.2.6. *Let $0 \leq h \leq q-2$ and $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.*

(i) *For all* $0 \le j \le f-1$ *and* $a \in \mathcal{O}_K^\times$, *we have*

$$\left(\mathrm{id} - \left(f_a^{\mathrm{LT}}\right)^h a\right)\left(D_j^{\mathrm{LT}}\right) \in T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].$$

(ii) *If* $h = 1 + p + \cdots + p^{f-1}$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then for all* $a \in \mathcal{O}_K^\times$, *we have*

$$\left(\mathrm{id} - \left(f_a^{\mathrm{LT}}\right)^h a\right)\left(D_{\mathrm{tr}}^{\mathrm{LT}}\right) \in \left(\mathrm{id} - T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)\left(c_a^{p^{f-1}} T_{K,\sigma_0}^{-(q-1)(1+p+\cdots+p^{f-2})}\right) + T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!].$$

Proof. This follows from Lemma 5.2.4. Note that for i such that $h_i = 0$ we have $[h]_i = [h]_{i-1}$. □

Lemma 5.2.7. *Let* $0 \le h \le q-2$ *and* $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.

(i) *For any* $y \in T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!]$, *the equation* $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)(x) = y$ *has a unique solution in* $T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!]$, *given by the convergent series* $x = \sum_{n=0}^{\infty}\left(\lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)^n(y)$.

(ii) *For any* $y \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$, *the equation* $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)(x) = y$ *has at most one solution in* $\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$ *unless* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$.

(iii) *We let*

$$y = \sum_{i=0}^{m} a_i T_{K,\sigma_0}^{-(q-1)(h+qh+q^2i))} + \sum_{j=0}^{n} b_j T_{K,\sigma_0}^{-(q-1)(h+qj)} + \sum_{k=0}^{t} c_k T_{K,\sigma_0}^{-(q-1)k} \tag{5.11}$$

with $m, n \ge -1$, $t \ge 0$, $a_i, b_j, c_k \in \mathbb{F}$, $a_m \ne 0$, $b_n \ne 0$, $c_t \ne 0$ *and* $t \notin h + q\mathbb{Z}$. *If* $m, n < t$, *then the equation* $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)(x) = y$ *has no solution in* $\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$.

Proof. (i). The proof is similar to that of Lemma 5.5.6 below using $h < q-1$. We omit the details.

(ii). It suffices to show that the equality

$$\varphi_q(x) = \lambda_0^{-1}\lambda_1 T_{K,\sigma_0}^{(q-1)h} x \tag{5.12}$$

for $x \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$ implies $x = 0$ unless $h = 0$ and $\lambda_0\lambda_1^{-1} = 1$.

First we assume that $h \ne 0$. If $x \ne 0$, we assume that the lowest degree term of x has degree $(q-1)s$ for $s \in \mathbb{Z}$, then the lowest degree on both sides of (5.12) are $(q-1)qs$ and $(q-1)(s+h)$, which cannot be equal since $0 < h < q-1$. Hence we must have $x = 0$.

Next we assume that $h = 0$ and $\lambda_0\lambda_1^{-1} \ne 1$. We let $m \ge 0$ be large enough so that $(\lambda_0\lambda_1^{-1})^m = 1$ and $q^m \ge |\mathbb{F}|$, then φ_q^m acts as $x \mapsto x^{q^m}$ on $\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$, and by (5.12) we have $x^{q^m} = \varphi_q^m(x) = x$, hence $x \in \mathbb{F}$. Since $\lambda_0\lambda_1^{-1} \ne 1$, by (5.12) again we conclude that $x = 0$.

(iii). Suppose that $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)(x) = y$ for $x \in \mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$. Then we have

$$\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)(z) = \sum_{k=0}^{t} c_k' T_{K,\sigma_0}^{-(q-1)k}, \tag{5.13}$$

where

$$z \stackrel{\mathrm{def}}{=} x + \left(\lambda_0^{-1}\lambda_1\right)^2 \sum_{i=0}^{m} a_i T_{K,\sigma_0}^{-(q-1)i} + \lambda_0^{-1}\lambda_1 \sum_{i=0}^{m} a_i T_{K,\sigma_0}^{-(q-1)(h+qi)} + \lambda_0^{-1}\lambda_1 \sum_{j=0}^{n} b_j T_{K,\sigma_0}^{-(q-1)j}$$

and $c_k' \in \mathbb{F}$, and we have $c_t' = c_t \ne 0$ since $m, n < t$.

We write $z = c_s T_{K,\sigma_0}^{-(q-1)s} + (\text{terms with degree} > -(q-1)s)$. Since the RHS of (5.13) does not lie in $T_{K,\sigma_0}^{q-1}\mathbb{F}[\![T_{K,\sigma_0}^{q-1}]\!]$, we must have $s \geq 0$ (since $h < q-1$), hence the lowest degree term of the LHS of (5.13) has degree $-(q-1)(h+qs)$. However, the lowest degree term of the RHS of (5.13) has degree $-(q-1)t$, which does not lie in $-(q-1)(h+q\mathbb{Z})$ by assumption. This is a contradiction. □

Proposition 5.2.8. *Let $0 \leq h \leq q-2$ and $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.*

(i) *For all $0 \leq j \leq f-1$, the tuple $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ with*

$$\begin{cases} D &= D_j^{\mathrm{LT}} \\ E_a &= E_{j,a}^{\mathrm{LT}} \overset{\mathrm{def}}{=} \left(\mathrm{id} - \lambda_0\lambda_1^{-1} T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)^{-1} \left[\left(\mathrm{id} - \left(f_a^{\mathrm{LT}}\right)^h a\right)\left(D_j^{\mathrm{LT}}\right)\right] \\ &= \sum_{n=0}^{\infty} \left(\lambda_0\lambda_1^{-1} T_{K,\sigma_0}^{-(q-1)h}\varphi_q\right)^n \left[\left(\mathrm{id} - \left(f_a^{\mathrm{LT}}\right)^h a\right)\left(D_j^{\mathrm{LT}}\right)\right] \end{cases}$$

defines an element of W^{LT}. We denote it by $[B_j^{\mathrm{LT}}]$.

(ii) *If $h = 1 + p + \cdots + p^{f-1}$ and $\lambda_0\lambda_1^{-1} = 1$, then the tuple $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ with*

$$\begin{cases} D &= D_{\mathrm{tr}}^{\mathrm{LT}} \\ E_a &= E_{\mathrm{tr},a}^{\mathrm{LT}} \overset{\mathrm{def}}{=} \left(\mathrm{id} - T_{K,\sigma_0}^{-(q-1)(1+p+\cdots+p^{f-1})}\varphi_q\right)^{-1} \left[\left(\mathrm{id} - \left(f_a^{\mathrm{LT}}\right) a\right)\left(D_{\mathrm{tr}}^{\mathrm{LT}}\right)\right] \\ &= c_a^{p^{f-1}} T_{K,\sigma_0}^{-(q-1)(1+p+\cdots+p^{f-2})} + \sum_{n=0}^{\infty} \left(T_{K,\sigma_0}^{-(q-1)(1+p+\cdots+p^{f-1})}\varphi_q\right)^n \Big[\left(\mathrm{id} - \left(f_a^{\mathrm{LT}}\right) a\right)\left(D_{\mathrm{tr}}^{\mathrm{LT}}\right) \\ &\qquad - \left(\mathrm{id} - T_{K,\sigma_0}^{-(q-1)(1+p+\cdots+p^{f-1})}\varphi_q\right)\left(c_a^{p^{f-1}} T_{K,\sigma_0}^{-(q-1)(1+p+\cdots+p^{f-2})}\right) \Big] \end{cases}$$

defines an element of W^{LT}. We denote it by $[B_{\mathrm{tr}}^{\mathrm{LT}}]$. Otherwise, we define $E_{\mathrm{tr},a}^{\mathrm{LT}} \overset{\mathrm{def}}{=} 0$ for all $a \in \mathcal{O}_K^\times$ and $[B_{\mathrm{tr}}^{\mathrm{LT}}] \overset{\mathrm{def}}{=} [0]$ in W^{LT}.

(iii) *If $h = 0$ and $\lambda_0\lambda_1^{-1} = 1$, then the tuple $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ with*

$$\begin{cases} D &= D_{\mathrm{un}}^{\mathrm{LT}} = 1 \\ E_a &= E_{\mathrm{un},a}^{\mathrm{LT}} \overset{\mathrm{def}}{=} 0 \end{cases}$$

defines an element of W^{LT}. We denote it by $[B_{\mathrm{un}}^{\mathrm{LT}}]$. Otherwise, we define $E_{\mathrm{un},a}^{\mathrm{LT}} \overset{\mathrm{def}}{=} 0$ for all $a \in \mathcal{O}_K^\times$ and $[B_{\mathrm{un}}^{\mathrm{LT}}] \overset{\mathrm{def}}{=} [0]$ in W^{LT}.

Proof. (iii) is direct. For (i) and (ii), each E_a is well-defined by Corollary 5.2.6 and Lemma 5.2.7(i), and condition (ii) in Definition 5.2.3 is guaranteed by the uniqueness of solution in Lemma 5.2.7(i),(ii). □

Remark 5.2.9. *Suppose that $h = 0$ and $\lambda_0\lambda_1^{-1} = 1$. For $0 \leq j \leq f-1$, we let $[B_j]$ be the element of W^{LT} defined by the tuple $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ with $D = 0$ and $E_a = c_a^{p^j}$. Then we have $[B_j] = -[B_{j+1}^{\mathrm{LT}}]$ for $0 \leq j \leq f-2$ and $[B_{f-1}] = -[B_0^{\mathrm{LT}}]$ in W^{LT}.*

Theorem 5.2.10. *Let $0 \leq h \leq q-2$ and $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.*

(i) *If $h = 0$ and $\lambda_0\lambda_1^{-1} = 1$, then $\left\{[B_0^{\mathrm{LT}}], \ldots, [B_{f-1}^{\mathrm{LT}}], [B_{\mathrm{un}}^{\mathrm{LT}}]\right\}$ form a basis of W^{LT}.*

(ii) *If $h = 1 + p + \cdots + p^{f-1}$ and $\lambda_0\lambda_1^{-1} = 1$, then $\left\{[B_0^{\mathrm{LT}}], \ldots, [B_{f-1}^{\mathrm{LT}}], [B_{\mathrm{tr}}^{\mathrm{LT}}]\right\}$ form a basis of W^{LT}.*

(iii) *In the remaining cases,* $\{[B_0^{\mathrm{LT}}],\dots,[B_{f-1}^{\mathrm{LT}}]\}$ *form a basis of* W^{LT}.

Remark 5.2.11. *If* $h=1+p+\cdots+p^{f-1}$ *and* $\lambda_0\lambda_1^{-1}=1$, *then* $\{[B_0^{\mathrm{LT}}],\dots,[B_{f-1}^{\mathrm{LT}}]\}$ *form a basis of the subspace of* W^{LT} *which corresponds to peu ramifiées representations under* (5.9).

Proof of Theorem 5.2.10. By (5.9), we have $\dim_{\mathbb{F}} W^{\mathrm{LT}}=\dim_{\mathbb{F}} H^1\big(G_K,\mathbb{F}\big(\omega_f^h\,\mathrm{un}(\lambda_0\lambda_1^{-1})\big)\big)=f$ except the cases ($h=0$, $\lambda_0\lambda_1^{-1}=1$) and ($h=1+p+\cdots+p^{f-1}$, $\lambda_0\lambda_1^{-1}=1$), in which case the dimension is $f+1$. So it is enough to show that the elements of W^{LT} as in the statements are $\mathbb{F}$-linearly independent (using Definition 5.2.3(iv)).

(iii). Suppose that $\sum_{j=0}^{f-1}c_j[B_j^{\mathrm{LT}}]=[0]$ in W^{LT}. By definition, there exists $b\in\mathbb{F}(\!(T_{K,\sigma_0}^{q-1})\!)$ such that

$$\Big(\mathrm{id}-\lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)h}\varphi_q\Big)(b)=\sum_{j=0}^{f-1}c_jD_j^{\mathrm{LT}}. \tag{5.14}$$

Step 1. Assuming $h\neq 0$, we prove that $c_j=0$ for all j such that $h_j=0$.

By symmetry (since one can replace D_{K,σ_0} with D_{K,σ_i} if necessary, see the proof of Corollary 5.5.12 below), it is enough to prove that $c_{f-2}=0$ assuming $h_{f-2}=0$ (which implies $f\geq 2$ since $h\neq 0$). Suppose on the contrary that $c_{f-2}\neq 0$.

For each $0\leq j\leq f-1$ such that $h_j=0$, we let $0\leq r\leq f-1$ be the corresponding integer in Definition 5.2.5(i). Since $h_{f-2}=0$, we have $r\leq f-2$ if $j=f-1$ and $r+j\leq f-3$ if $0\leq j\leq f-3$.

- If $j+r\geq f-1$, then we have
$$[h]_{f+j+r}+p^{f+j+r+1}=h+qh+q^2\big([h]_{j+r-f}+p^{j+r+1-f}\big)\leq h+qh+q^2\big([h]_{f-2}+p^{f-1}\big).$$
- If $j+r\leq f-2$, then we have
$$[h]_{f+j+r}+p^{f+j+r+1}=h+q\big([h]_{j+r}+p^{j+r+1}\big)\leq h+q\big([h]_{f-2}+p^{f-1}\big).$$
- If $0\leq i\leq r$ such that $j+i\geq f-1$, then we have (since $r\neq f-1$ if $j=f-1$)
$$[h]_{f+j+i}+p^{j+i+1}=h+q\big([h]_{j+i}+p^{j+i+1-f}\big)<h+q\big([h]_{2f-2}+p^{f-1}\big).$$
- If $0\leq i\leq r$ such that $j+i\leq f-2$, then we have $[h]_{f+j+i}+p^{j+i+1}\leq[h]_{2f-2}+p^{f-1}$, with equality holds if and only if $j+i=f-2$, which implies $j=f-2$ and $i=0$ since $r+j\leq f-3$ if $0\leq j\leq f-3$.

In particular, by the definition of D_j^{LT} together with $c_{f-2}\neq 0$ and $[h]_{f-2}<[h]_{2f-2}$ (since $h\neq 0$), the RHS of (5.14) has the form (5.11) with $t=[h]_{2f-2}+p^{f-1}$ and $m,n<t$. Then we deduce a contradiction by Lemma 5.2.7(iii).

Step 2. Assuming $h\neq 0$, we prove that $c_j=0$ for all j.

By Step 1, we already know that $c_j=0$ for all $0\leq j\leq f-1$ such that $h_j=0$. Suppose on the contrary that $c_j\neq 0$ for some j. We let j_0 be the largest integer in $\{0,1,\dots,f-1\}$ such that $h_{j_0}\neq 0$. Then we have $[h]_{j_0-1}\notin h+q\mathbb{Z}$. By the definition of D_j^{LT} (in the case $h_j\neq 0$) the RHS of (5.14) has the form (5.11) with $m=n=-1$ and $t=[h]_{j_0-1}$. Then we deduce a contradiction by Lemma 5.2.7(iii).

Step 3. Assuming $h=0$ (hence $\lambda_0\lambda_1^{-1}\neq 1$ by assumption), we prove that $c_j=0$ for all j.

By definition we have $D_j^{\mathrm{LT}} = \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)p^{f+j+1}} - \lambda_0\lambda_1^{-1}T_{K,\sigma_0}^{-(q-1)p^{j+1}}$ for all $0 \le j \le f-1$. Then by replacing b with $b + \big(\lambda_0^{-1}\lambda_1 - 1\big)c_{f-1} + \sum_{j=0}^{f-1} c_j T_{K,\sigma_0}^{-(q-1)p^{j+1}}$ in (5.14), the RHS of (5.14) becomes $\sum_{j=0}^{f-1} c'_j T_{K,\sigma_0}^{-(q-1)p^j}$ with $c'_0 = \big(\lambda_0^{-1}\lambda_1 - 1\big)c_{f-1}$ and $c'_j = \big(1 - \lambda_0\lambda_1^{-1}\big)c_{j-1}$ for $1 \le j \le f-1$. Suppose on the contrary that $c_j \neq 0$ for some j. We let j_0 be the largest integer in $\{0, 1, \dots, f-1\}$ such that $c'_{j_0} \neq 0$ (which exists since $\lambda_0\lambda_1^{-1} \neq 1$). Then we deduce a contradiction by Lemma 5.2.7(iii) with $m = n = -1$ and $t = p^{j_0}$.

(i). Let $h = 0$ and $\lambda_0\lambda_1^{-1} = 1$. Suppose that $c_{\mathrm{un}}[B_{\mathrm{un}}^{\mathrm{LT}}] + \sum_{j=0}^{f-1} c_j[B_j^{\mathrm{LT}}] = [0]$ in W^{LT}. By Proposition 5.2.8(iii) and Remark 5.2.9, the element $c_{\mathrm{un}}[B_{\mathrm{un}}^{\mathrm{LT}}] + \sum_{j=0}^{f-1} c_j[B_j^{\mathrm{LT}}] \in W^{\mathrm{LT}}$ is represented by the tuple $\big(D, (E_a)_{a\in\mathcal{O}_K^\times}\big)$ with

$$\begin{cases} D & = c_{\mathrm{un}} \\ E_a & = -c_0 c_a^{p^{f-1}} - \sum_{j=1}^{f-1} c_j c_a^{p^{j-1}}. \end{cases}$$

Since $\mathrm{Im}(\mathrm{id} - \varphi_q) \cap \mathbb{F} = \{0\}$, we deduce from Definition 5.2.3(iv) that $c_{\mathrm{un}} = 0$. Since the characters $c_a, c_a^p, \dots, c_a^{p^{f-1}}$ are linearly independent (using for example Remark 5.2.2(ii)) and since $\mathrm{Ker}(\mathrm{id} - \varphi_q) = \mathbb{F}$, we deduce from Definition 5.2.3(iv) that $c_j = 0$ for all j.

(ii). Let $h = 1 + p + \cdots + p^{f-1}$ and $\lambda_0\lambda_1^{-1} = 1$. Suppose that $c_{\mathrm{tr}}[B_{\mathrm{tr}}^{\mathrm{LT}}] + \sum_{j=0}^{f-1} c_j[B_j^{\mathrm{LT}}] = [0]$ in W^{LT}. If $c_{\mathrm{tr}} = 0$, then the proof of (iii) shows that $c_j = 0$ for all j, which proves (ii). If $c_{\mathrm{tr}} \neq 0$, then by the definition of $D_{\mathrm{tr}}^{\mathrm{LT}}$ and D_j^{LT} (in the case $h_j \neq 0$), and since $[h]_{f+i-1} + p^i \notin h + q\mathbb{Z}$ for all $0 \le i \le f-1$, the sum $c_{\mathrm{tr}}D_{\mathrm{tr}}^{\mathrm{LT}} + \sum_{j=0}^{f-1} c_j D_j^{\mathrm{LT}}$ has the form (5.11) with $m = n = -1$ and $t = [h]_{2f-2} + p^{f-1}$. Then we deduce a contradiction by Lemma 5.2.7(iii). □

5.3 Cyclotomic $(\varphi, \mathbb{Z}_p^\times)$-modules

In this section, we give an explicit parametrization of the cyclotomic $(\varphi, \mathbb{Z}_p^\times)$-modules corresponding to $\overline{\rho}$ as in (5.2) (compare with the parametrization of [CD11] where we use a different variable). The main result is Theorem 5.3.9.

We choose the formal variable T of the formal group $\mathbb{G}_m$ so that the logarithm [Lan90, §8.6] is given by the power series $\sum_{n=0}^{\infty} p^{-n}T^{p^n}$. For $a \in \mathbb{Z}_p$ we have power series $a_{\mathrm{cyc}}(T) \in aT + T^2\mathbb{Z}_p[\![T]\!]$.

As in §5.2, there is a continuous $\mathbb{F}$-linear endomorphism φ of $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T)\!)$ which is the p-th power map on $\mathbb{F}_q$ and satisfies $\varphi(T) = T^p$, and a continuous $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q$-linear action (commuting with φ) of $\mathbb{Z}_p^\times$ on $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T)\!)$ satisfying $a(T) = a_{\mathrm{cyc}}(T)$ for $a \in \mathbb{Z}_p^\times$, where we still denote by $a_{\mathrm{cyc}}(T) \in \mathbb{F}_q[\![T]\!]$ the reduction modulo p of $a_{\mathrm{cyc}}(T) \in \mathbb{Z}_p[\![T]\!]$. Then there is a covariant exact equivalence of categories compatible with tensor products between the category of finite-dimensional continuous representations of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ and the category of étale $(\varphi, \mathbb{Z}_p^\times)$-modules over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T)\!)$.

Using a decomposition analogous to (5.3), for each $0 \le i \le f-1$ there is an equivalence $D \mapsto D_{\sigma_i}$ between the category of étale $(\varphi, \mathbb{Z}_p^\times)$-modules over $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_q(\!(T)\!)$ and the category of étale $(\varphi_q, \mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(T_{\sigma_i})\!)$. Here $\varphi_q \overset{\mathrm{def}}{=} \varphi^f$, and $\mathbb{F}(\!(T_{\sigma_i})\!)$ is endowed with an $\mathbb{F}$-linear endomorphism φ_q such that $\varphi_q(T_{\sigma_i}) = T_{\sigma_i}^q$, and a continuous $\mathbb{F}$-linear action commuting with φ_q such that $a(T_{\sigma_i}) = a_{\mathrm{cyc}}(T_{\sigma_i})$ for $a \in \mathbb{Z}_p^\times$, where $a_{\mathrm{cyc}}(T_{\sigma_i}) \in \mathbb{F}[\![T_{\sigma_i}]\!]$ is the image of $a_{\mathrm{cyc}}(T) \in \mathbb{F}_q[\![T]\!]$ in $\mathbb{F}[\![T_{\sigma_i}]\!]$ via the embedding $\sigma_i : \mathbb{F}_q \hookrightarrow \mathbb{F}$.

For $\overline{\rho}$ a finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$, we denote by $D(\overline{\rho})$ the associated étale $(\varphi,\mathbb{Z}_p^\times)$-modules over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T)\!)$, and for each $0\le i\le f-1$ we denote by $D_{\sigma_i}(\overline{\rho})$ the associated étale $(\varphi_q,\mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(T_{\sigma_i})\!)$.

For $a\in\mathbb{Z}_p^\times$, we set

$$f_a^{\mathrm{cyc}}\overset{\mathrm{def}}{=}\overline{a}T/a(T)\in 1+T\mathbb{F}[\![T]\!].$$

We still denote by f_a^{cyc} its image in $\mathbb{F}(\!(T_{\sigma_0})\!)$ via σ_0 when there is no possible confusion. Then for $0\le h\le q-2$ and $\lambda\in\mathbb{F}^\times$, the étale $(\varphi_q,\mathbb{Z}_p^\times)$-module $D_{\sigma_0}\big(\omega_f^h\,\mathrm{un}(\lambda)\big)$ can be described as follows (see for example [Bre11, Prop. 3.5] where we use a different variable, $a\in\mathbb{Z}_p^\times$ and see below (5.2) for h'):

$$\begin{cases} D_{\sigma_0}(\omega_f^h\,\mathrm{un}(\lambda)) &= \mathbb{F}(\!(T_{\sigma_0})\!)e\\ \varphi_q(e) &= \lambda T_{\sigma_0}^{-(p-1)h'}e\\ a(e) &= (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,e,\end{cases}\tag{5.15}$$

where $(f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}$ is well-defined since $f_a^{\mathrm{cyc}}\in 1+T\mathbb{F}[\![T]\!]$ and $\frac{p-1}{q-1}h'\in\mathbb{Z}_p$.

Lemma 5.3.1. *We have $f_a^{\mathrm{cyc}}=1$ for all $a\in[\mathbb{F}_p^\times]$. More generally, we have for $a\in\mathbb{Z}_p^\times$*

$$(f_a^{\mathrm{cyc}})^{-1}\in 1+c_a^{\mathrm{cyc}}T^{p-1}-c_a^{\mathrm{cyc}}T^{2(p-1)}+T^{3(p-1)}\mathbb{F}[\![T^{p-1}]\!],$$

where $c_a^{\mathrm{cyc}}\in\mathbb{F}_p$ is the reduction modulo p of $(1-a^{p-1})/p\in\mathbb{Z}_p$.

Proof. This is a special case of Lemma 5.2.1 by taking $f=1$. □

Since $a(T_{\sigma_0})=\overline{a}T_{\sigma_0}$ for $a\in[\mathbb{F}_p^\times]$ by Lemma 5.3.1, we have $\mathbb{F}(\!(T_{\sigma_0})\!)^{[\mathbb{F}_p^\times]}=\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$. Then for $\overline{\rho}$ as in (5.2), we have $D_{\sigma_0}(\overline{\rho})\cong\mathbb{F}(\!(T_{\sigma_0})\!)\otimes_{\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)}D_{\sigma_0}(\overline{\rho})^{[\mathbb{F}_p^\times]}$, where $D_{\sigma_0}(\overline{\rho})^{[\mathbb{F}_p^\times]}$ has the following form (using (5.15), and $a\in\mathbb{Z}_p^\times$):

$$\begin{cases} D_{\sigma_0}(\overline{\rho})^{[\mathbb{F}_p^\times]} &= \mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)e_0\oplus\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)e_1\\ \varphi_q(e_0\ e_1) &= (e_0\ e_1)\,\mathrm{Mat}(\varphi_q)\\ a(e_0\ e_1) &= (e_0\ e_1)\,\mathrm{Mat}(a)\end{cases}$$

with

$$\begin{cases} \mathrm{Mat}(\varphi_q) &= \begin{pmatrix}\lambda_0T_{\sigma_0}^{-(p-1)h'} & \lambda_1D\\ 0 & \lambda_1\end{pmatrix}\\ \mathrm{Mat}(a) &= \begin{pmatrix}(f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'} & E_a\\ 0 & 1\end{pmatrix}\end{cases}$$

for some $D\in\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$ and $E_a\in\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$.

Definition 5.3.2. *Let $0\le h\le q-2$, $\lambda_0,\lambda_1\in\mathbb{F}^\times$ and see below (5.2) for h'. We define W^{cyc} to be the set of equivalence classes of tuples $[B]=\big(D,(E_a)_{a\in\mathbb{Z}_p^\times}\big)$ such that*

(i) *$D\in\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$, $E_a\in\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$ for all $a\in\mathbb{Z}_p^\times$, and the map $\mathbb{Z}_p^\times\to\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$, $a\mapsto E_a$ is continuous;*

(ii) *$E_{ab}=E_a+(f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a(E_b)$ for all $a,b\in\mathbb{Z}_p^\times$;*

(iii) *$\big(\mathrm{id}-\lambda_0\lambda_1^{-1}T_{\sigma_0}^{-(p-1)h'}\varphi_q\big)(E_a)=\big(\mathrm{id}-(f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\big)(D)$ for all $a\in\mathbb{Z}_p^\times$;*

(iv) *two tuples* $\big(D,(E_a)_{a\in\mathbb{Z}_p^\times}\big)$ *and* $\big(D',(E'_a)_{a\in\mathbb{Z}_p^\times}\big)$ *are equivalent if and only if there exists* $b\in\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$ *such that*

$$\begin{cases} D' = D + \Big(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{\sigma_0}^{-(p-1)h'}\varphi_q\Big)(b) \\ E'_a = E_a + \Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)(b) \quad \forall\, a\in\mathbb{Z}_p^\times. \end{cases}$$

It has a natural structure of an $\mathbb{F}$*-vector space.*

As in §5.2, there is an isomorphism of $\mathbb{F}$-vector spaces

$$W^{\mathrm{cyc}} \cong \mathrm{Ext}^1\Big(D_{\sigma_0}\big(\mathrm{un}(\lambda_1)\big), D_{\sigma_0}\big(\omega_f^h\,\mathrm{un}(\lambda_0)\big)\Big) \cong H^1\Big(G_K, \mathbb{F}\big(\omega_f^h\,\mathrm{un}(\lambda_0\lambda_1^{-1})\big)\Big), \tag{5.16}$$

where Ext^1 is defined in the category of étale $(\varphi_q,\mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(T_{\sigma_0})\!)$. For each $[B]\in W^{\mathrm{cyc}}$, we denote by $D([B])$ the corresponding étale $(\varphi_q,\mathbb{Z}_p^\times)$-module over $\mathbb{F}(\!(T_{\sigma_0})\!)$. Note that $D([B])\cong D(\lambda[B])$ as étale $(\varphi_q,\mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(T_{\sigma_0})\!)$ for $\lambda\in\mathbb{F}^\times$.

Lemma 5.3.3. *Let* $0\le h\le q-2$ *and see below (5.2) for* h'.

(i) *For* $i\ge -1$ *and* $a\in\mathbb{Z}_p^\times$*, we have*

$$\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\Big(T_{\sigma_0}^{-(p-1)[h']_i}\Big) \in T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!].$$

(ii) *For* $i\ge -1$ *and* $a\in\mathbb{Z}_p^\times$*, we have*

$$\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\Big(T_{\sigma_0}^{-(p-1)([h']_i+p^{i+1})}\Big) \in (h'_{i+1}-1)c_a^{\mathrm{cyc}}T_{\sigma_0}^{-(p-1)[h']_i} + T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!].$$

(iii) *For* $i\ge 0$ *such that* $h'_i=1$ *and* $a\in\mathbb{Z}_p^\times$*, we have*

$$\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\Big(T_{\sigma_0}^{-(p-1)([h']_i+p^i)}\Big) \in -c_a^{\mathrm{cyc}}T_{\sigma_0}^{-(p-1)[h']_i} + c_a^{\mathrm{cyc}}T_{\sigma_0}^{-(p-1)[h']_{i-1}} + T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!].$$

Proof. For $s\in\mathbb{Z}$ and $a\in\mathbb{Z}_p^\times$, by definition we have

$$\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\Big(T_{\sigma_0}^{-(p-1)s}\Big) = T_{\sigma_0}^{-(p-1)s}\Big(1 - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}(h'+(q-1)s)}\Big). \tag{5.17}$$

(i). Take $s=[h']_i$. Since $\frac{p-1}{q-1}\,(h'+(q-1)[h']_i) = \frac{p-1}{q-1}\,([h']_{i+f}-[h']_i)\in p^{i+1}\mathbb{Z}_p$ and $p^{i+1}\ge [h']_i+1$, we deduce from (5.17) and Lemma 5.3.1 that

$$\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\Big(T_{\sigma_0}^{-(p-1)[h']_i}\Big) \in T_{\sigma_0}^{-(p-1)[h']_i}\Big(T_{\sigma_0}^{(p-1)p^{i+1}}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!]\Big) \subseteq T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!].$$

(ii). Take $s=[h']_i+p^{i+1}$. We have

$$\tfrac{p-1}{q-1}\big(h'+(q-1)([h']_i+p^{i+1})\big) = \tfrac{p-1}{q-1}\big([h']_{i+f}-[h']_i+qp^{i+1}-p^{i+1}\big) \in (h'_{i+1}-1)p^{i+1}+p^{i+2}\mathbb{Z}_p.$$

Using $p^{i+1}\ge [h']_i+1$, we deduce from (5.17) and Lemma 5.3.1 that

$$\begin{aligned} &\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\Big(T_{\sigma_0}^{-(p-1)([h']_i+p^{i+1})}\Big) \\ &\qquad \in T_{\sigma_0}^{-(p-1)([h']_i+p^{i+1})}\Big((h'_{i+1}-1)c_a^{\mathrm{cyc}}T_{\sigma_0}^{(p-1)p^{i+1}} + T_{\sigma_0}^{2(p-1)p^{i+1}}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!]\Big) \\ &\qquad \subseteq (h'_{i+1}-1)c_a^{\mathrm{cyc}}T_{\sigma_0}^{-(p-1)[h']_i} + T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!]. \end{aligned}$$

(iii). Take $s = [h']_i + p^i$. We have

$$\tfrac{p-1}{q-1}\left(h' + (q-1)([h']_i + p^i)\right) = \tfrac{p-1}{q-1}\left([h']_{i+f} - [h']_i + qp^i - p^i\right) \in -p^i + p^{i+1}\mathbb{Z}_p.$$

Then we deduce from (5.17) and Lemma 5.3.1 that

$$\begin{aligned}\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\, a\Big)\Big(T_{\sigma_0}^{-(p-1)([h']_i+p^i)}\Big)&\\ \in T_{\sigma_0}^{-(p-1)([h']_i+p^i)}\left(-c_a^{\mathrm{cyc}}T_{\sigma_0}^{(p-1)p^i} + c_a^{\mathrm{cyc}}T_{\sigma_0}^{2(p-1)p^i} + T_{\sigma_0}^{3(p-1)p^i}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!]\right)&\\ \subseteq -c_a^{\mathrm{cyc}}T_{\sigma_0}^{-(p-1)[h']_i} + c_a^{\mathrm{cyc}}T_{\sigma_0}^{-(p-1)[h']_{i-1}} + T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!],&\end{aligned}$$

where the first inclusion use $p \geq 3$, and the second inclusion uses $h'_i = 1$. □

Definition 5.3.4. *Let $0 \leq h \leq q-2$, $\lambda_0, \lambda_1 \in \mathbb{F}^\times$, $0 \leq j \leq f-1$ and see below (5.2) for h'. We define $D_j^{\mathrm{cyc}}, D_{\mathrm{tr}}^{\mathrm{cyc}}, D_{\mathrm{un}}^{\mathrm{cyc}} \in \mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$ as follows:*

(i) *If $h'_j \neq 0$, we define*

$$D_j^{\mathrm{cyc}} \overset{\mathrm{def}}{=} T_{\sigma_0}^{-(p-1)[h']_{j-1}}.$$

If $h'_j = 0$, we let $0 \leq r \leq f-1$ such that $h'_{j+1} = \cdots = h'_{j+r} = 1$ and $h'_{j+r+1} \neq 1$, then we define

$$\begin{aligned}D_j^{\mathrm{cyc}} &\overset{\mathrm{def}}{=} T_{\sigma_0}^{-(p-1)([h']_{j+r}+p^{j+r+1})} + (h'_{j+r+1}-1)\sum_{i=0}^{r} T_{\sigma_0}^{-(p-1)([h']_{j+i}+p^{j+i})}\\ &= T_{\sigma_0}^{-(p-1)([h']_{j-1}+p^j(p+p^2+\cdots+p^{r+1}))} + (h'_{j+r+1}-1)\sum_{i=0}^{r} T_{\sigma_0}^{-(p-1)([h']_{j-1}+p^j((p+p^2+\cdots+p^i)+p^i))}.\end{aligned}$$

(ii) *If $h = 1+p+\cdots+p^{f-1}$ and $\lambda_0\lambda_1^{-1} = 1$, we define*

$$D_{\mathrm{tr}}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} \sum_{i=0}^{f-1} T_{\sigma_0}^{-(p-1)([h']_i+p^i)} = \sum_{i=0}^{f-1} T_{\sigma_0}^{-(p-1)(1+p+\cdots+p^{i-1}+2p^i)}.$$

Otherwise (i.e. either $h \neq 1+p+\cdots+p^{f-1}$ or $\lambda_0\lambda_1^{-1} \neq 1$), we define $D_{\mathrm{tr}}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} 0$.

(iii) *If $h = 0$ and $\lambda_0\lambda_1^{-1} = 1$, we define $D_{\mathrm{un}}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} 1$. Otherwise, we define $D_{\mathrm{un}}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} 0$.*

Corollary 5.3.5. *Let $0 \leq h \leq q-2$, $\lambda_0, \lambda_1 \in \mathbb{F}^\times$ and see below (5.2) for h'.*

(i) *For all $0 \leq j \leq f-1$ and $a \in \mathbb{Z}_p^\times$, we have*

$$\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\, a\Big)\big(D_j^{\mathrm{cyc}}\big) \in T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!].$$

(ii) *If $h = 1+p+\cdots+p^{f-1}$ and $\lambda_0\lambda_1^{-1} = 1$, then for all $a \in \mathbb{Z}_p^\times$, we have*

$$\big(\mathrm{id} - (f_a^{\mathrm{cyc}})\, a\big)\big(D_{\mathrm{tr}}^{\mathrm{cyc}}\big) \in c_a^{\mathrm{cyc}}\big(1 - T_{\sigma_0}^{-(q-1)}\big) + T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!].$$

Proof. This follows from Lemma 5.3.3. Note that for i such that $h'_i = 0$ we have $[h]_i = [h]_{i-1}$, and in (ii) we have $\frac{p-1}{q-1}h' = 1$. □

Lemma 5.3.6. *Let $0 \leq h \leq q-2$, $\lambda_0, \lambda_1 \in \mathbb{F}^\times$ and see below (5.2) for h'.*

(i) *For any* $y \in T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!]$, *the equation* $\big(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{\sigma_0}^{-(p-1)h'}\varphi_q\big)(x) = y$ *has a unique solution in* $T_{\sigma_0}^{p-1}\mathbb{F}[\![T_{\sigma_0}^{p-1}]\!]$, *given by the convergent series* $x = \sum_{n=0}^{\infty}\big(\lambda_0\lambda_1^{-1}T_{\sigma_0}^{-(p-1)h'}\varphi_q\big)^n(y)$.

(ii) *For any* $y \in \mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$, *the equation* $\big(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{\sigma_0}^{-(p-1)h'}\varphi_q\big)(x) = y$ *has at most one solution in* $\mathbb{F}(\!(T_{\sigma_0}^{p-1})\!)$ *unless* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$.

Proof. The proof is similar to that of Lemma 5.2.7. We omit the details. □

Proposition 5.3.7. *Let* $0 \le h \le q-2$, $\lambda_0, \lambda_1 \in \mathbb{F}^\times$ *and see below (5.2) for* h'.

(i) *For all* $0 \le j \le f-1$, *the tuple* $\big(D, (E_a)_{a\in\mathbb{Z}_p^\times}\big)$ *with*

$$\begin{cases} D &= D_j^{\mathrm{cyc}} \\ E_a &= E_{j,a}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} \Big(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{\sigma_0}^{-(p-1)h'}\varphi_q\Big)^{-1}\Big[\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\big(D_j^{\mathrm{cyc}}\big)\Big] \\ &= \sum_{n=0}^{\infty}\Big(\lambda_0\lambda_1^{-1}T_{\sigma_0}^{-(p-1)h'}\varphi_q\Big)^n\Big[\Big(\mathrm{id} - (f_a^{\mathrm{cyc}})^{\frac{p-1}{q-1}h'}\,a\Big)\big(D_j^{\mathrm{cyc}}\big)\Big] \end{cases}$$

defines an element of W^{cyc}. *We denote it by* $[B_j^{\mathrm{cyc}}]$.

(ii) *If* $h = 1 + p + \cdots + p^{f-1}$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then the tuple* $\big(D, (E_a)_{a\in\mathbb{Z}_p^\times}\big)$ *with*

$$\begin{cases} D &= D_{\mathrm{tr}}^{\mathrm{cyc}} \\ E_a &= E_{\mathrm{tr},a}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} \Big(\mathrm{id} - T_{\sigma_0}^{-(q-1)}\varphi_q\Big)^{-1}\big[\big(\mathrm{id} - (f_a^{\mathrm{cyc}})a\big)\big(D_{\mathrm{tr}}^{\mathrm{cyc}}\big)\big] \\ &= c_a^{\mathrm{cyc}} + \sum_{n=0}^{\infty}\Big(T_{\sigma_0}^{-(q-1)}\varphi_q\Big)^n\Big[\big(\mathrm{id} - (f_a^{\mathrm{cyc}})a\big)\big(D_{\mathrm{tr}}^{\mathrm{cyc}}\big) - c_a^{\mathrm{cyc}}\big(1 - T_{\sigma_0}^{-(q-1)}\big)\Big] \end{cases}$$

defines an element of W^{cyc}. *We denote it by* $[B_{\mathrm{tr}}^{\mathrm{cyc}}]$. *Otherwise, we define* $E_{\mathrm{tr},a}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} 0$ *for all* $a \in \mathbb{Z}_p^\times$ *and* $[B_{\mathrm{tr}}^{\mathrm{cyc}}] \overset{\mathrm{def}}{=} [0]$ *in* W^{cyc}.

(iii) *If* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then the tuple* $\big(D, (E_a)_{a\in\mathbb{Z}_p^\times}\big)$ *with*

$$\begin{cases} D &= D_{\mathrm{un}}^{\mathrm{cyc}} = 1 \\ E_a &= E_{\mathrm{un},a}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} 0 \quad \forall\, a \in \mathbb{Z}_p^\times \end{cases}$$

defines an element of W^{cyc}. *We denote it by* $[B_{\mathrm{un}}^{\mathrm{cyc}}]$. *Otherwise, we define* $E_{\mathrm{un},a}^{\mathrm{cyc}} \overset{\mathrm{def}}{=} 0$ *for all* $a \in \mathbb{Z}_p^\times$ *and* $[B_{\mathrm{un}}^{\mathrm{cyc}}] \overset{\mathrm{def}}{=} [0]$ *in* W^{cyc}.

Proof. (iii) is direct. For (i) and (ii), each E_a is well-defined by Corollary 5.3.5 and Lemma 5.3.6(i), and condition (ii) of Definition 5.3.2 is guaranteed by the uniqueness of solution in Lemma 5.3.6(i),(ii). □

Remark 5.3.8. *Suppose that* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$. *We let* $[B]$ *be the element of* W^{cyc} *defined by the tuple* $\big(D, (E_a)_{a\in\mathbb{Z}_p^\times}\big)$ *with* $D = 0$ *and* $E_a = c_a^{\mathrm{cyc}}$. *Then we have* $[B] = -\big([B_0^{\mathrm{cyc}}] + \cdots + [B_{f-1}^{\mathrm{cyc}}]\big)$ *in* W^{cyc}.

Theorem 5.3.9. *Let* $0 \le h \le q-2$ *and* $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.

(i) *If* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then* $\big\{[B_0^{\mathrm{cyc}}], \ldots, [B_{f-1}^{\mathrm{cyc}}], [B_{\mathrm{un}}^{\mathrm{cyc}}]\big\}$ *form a basis of* W^{cyc}.

(ii) *If* $h = 1 + p + \cdots + p^{f-1}$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then* $\big\{[B_0^{\mathrm{cyc}}], \ldots, [B_{f-1}^{\mathrm{cyc}}], [B_{\mathrm{tr}}^{\mathrm{cyc}}]\big\}$ *form a basis of* W^{cyc}.

(iii) *In the remaining cases,* $\big\{[B_0^{\mathrm{cyc}}], \ldots, [B_{f-1}^{\mathrm{cyc}}]\big\}$ *form a basis of* W^{cyc}.

Proof. The proof is similar to that of Theorem 5.2.10. We omit the details. If $p \geq 5$, one can also conclude from Theorem 5.2.10 using Corollary 5.5.12 below. □

Remark 5.3.10. *If* $h = 1 + p + \cdots + p^{f-1}$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then* $\{[B_0^{\mathrm{cyc}}], \ldots, [B_{f-1}^{\mathrm{cyc}}]\}$ *form a basis of the subspace of* W^{cyc} *which corresponds to peu ramifiées representations under* (5.16).

5.4 Étale $(\varphi, \mathcal{O}_K^\times)$-modules over A

In this section, we give an explicit construction of some étale $(\varphi, \mathcal{O}_K^\times)$-modules over A of rank 2 that will be needed in §5.5. The main construction is Proposition 5.4.7. We also give a comparison between some of these étale $(\varphi, \mathcal{O}_K^\times)$-modules that are constructed using different systems of variables, see Proposition 5.4.8.

First we recall the definition of the ring A. Let $\mathfrak{m}_{\mathcal{O}_K}$ be the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![\mathcal{O}_K]\!]$. For $j \in \mathcal{J}$, we define

$$Y_j \overset{\text{def}}{=} \sum_{a \in \mathbb{F}_q^\times} a^{-p^j} \delta_{[a]} \in \mathfrak{m}_{\mathcal{O}_K} \setminus \mathfrak{m}_{\mathcal{O}_K}^2,$$

where $\delta_{[a]} \in \mathbb{F}[\![\mathcal{O}_K]\!]$ corresponds to $[a] \in \mathcal{O}_K$. Then we have $\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![Y_0, \ldots, Y_{f-1}]\!]$ and $\mathfrak{m}_{\mathcal{O}_K} = (Y_0, \ldots, Y_{f-1})$. Consider the multiplicative subset $S \overset{\text{def}}{=} \{(Y_0 \cdots Y_{f-1})^n : n \geq 0\}$ of $\mathbb{F}[\![\mathcal{O}_K]\!]$. Then $A \overset{\text{def}}{=} \widehat{\mathbb{F}[\![\mathcal{O}_K]\!]_S}$ is the completion of the localization $\mathbb{F}[\![\mathcal{O}_K]\!]_S$ with respect to the $\mathfrak{m}_{\mathcal{O}_K}$-adic filtration

$$F_n\left(\mathbb{F}[\![\mathcal{O}_K]\!]_S\right) = \bigcup_{k \geq 0} \frac{1}{(Y_0 \cdots Y_{f-1})^k} \mathfrak{m}_{\mathcal{O}_K}^{kf-n},$$

where $\mathfrak{m}_{\mathcal{O}_K}^m \overset{\text{def}}{=} \mathbb{F}[\![\mathcal{O}_K]\!]$ if $m \leq 0$. We denote by F_nA ($n \in \mathbb{Z}$) the induced filtration on A and endow A with the associated topology ([LvO96, §1.3]). There is an $\mathbb{F}$-linear action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![\mathcal{O}_K]\!]$ given by multiplication, and an $\mathbb{F}$-linear Frobenius φ on $\mathbb{F}[\![\mathcal{O}_K]\!]$ given by multiplication by p. They extend canonically by continuity to commuting continuous $\mathbb{F}$-linear actions of φ and $\mathcal{O}_K^\times$ on A which satisfies (for each $j \in \mathcal{J}$)

$$\begin{aligned} \varphi(Y_j) &= Y_{j-1}^p; \\ [a](Y_j) &= a^{p^j} Y_j \ \forall\, a \in \mathbb{F}_q^\times. \end{aligned} \tag{5.18}$$

Then we introduce another system of variables for $\mathbb{F}[\![\mathcal{O}_K]\!]$ following [BHH$^+$c]. For R a perfectoid $\mathbb{F}$-algebra, we denote by R° the subring of power-bounded elements in R and by $R^{\circ\circ} \subseteq R^\circ$ the subset of topologically nilpotent elements. We let $\mathbf{B}^+(R)$ be the Fréchet K-algebra defined as the completion of $W(R^\circ)[1/p]$ for the family of norms $|\cdot|_\rho$ for $0 \leq \rho \leq 1$ given by $\left|\sum_{n \gg -\infty} [x_n] p^n\right|_\rho \overset{\text{def}}{=} \sup_{n \in \mathbb{Z}} \{|x_n| \rho^n\}$. Then as in [BHH$^+$c, p.27], there exist elements $X_0, \ldots, X_{f-1} \in \mathbb{F}[\![\mathcal{O}_K]\!]$ satisfying $\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![X_0, \ldots, X_{f-1}]\!]$ and such that for any perfectoid $\mathbb{F}$-algebra R we have an isomorphism of K-vector spaces

$$\begin{aligned} \mathrm{Hom}_{\mathbb{F}\text{- alg}}^{\mathrm{cont}}\left(\mathbb{F}[\![K]\!], R\right) = \mathrm{Hom}_{\mathbb{F}\text{- alg}}^{\mathrm{cont}}\left(\mathbb{F}[\![\mathcal{O}_K]\!], R\right) &\cong \mathbf{B}^+(R)^{\varphi_q = p^f} \\ \left(X_i \mapsto x_i \in R^{\circ\circ}\right)_{0 \leq i \leq f-1} &\mapsto \sum_{i=0}^{f-1} \sum_{n \in \mathbb{Z}} [x_i^{p^{-i-nf}}] p^{i+nf}, \end{aligned} \tag{5.19}$$

where $\mathbb{F}[\![K]\!]$ is the $\mathfrak{m}_{\mathcal{O}_K}$-adic completion of $\mathbb{F}[K] \otimes_{\mathbb{F}[\mathcal{O}_K]} \mathbb{F}[\![\mathcal{O}_K]\!]$ and K acts on $\mathbb{F}[\![K]\!]$ by multiplication. By [BHH$^+$c, (41)] we have (for $0 \leq i \leq f-1$)

$$\begin{aligned} \varphi(X_i) &= X_{i-1}^p; \\ [a](X_i) &= a^{p^i} X_i \ \forall\, a \in \mathbb{F}_q^\times, \end{aligned} \tag{5.20}$$

where we extend the definition of X_i to all $i \in \mathbb{Z}$ by the relation $X_{i+f} = X_i$.

By considering the $[\mathbb{F}_q^\times]$-action in (5.18) and (5.20) (see [BHH$^+$c, (55)]), for each $0 \leq i \leq f-1$ there exists $\mu_i \in \mathbb{F}^\times$ such that

$$Y_i = \mu_i X_i + (\text{degree} \geq 2 \text{ in the variables } X_i) \text{ and } Y_i \in \mu_i X_i (1 + F_{1-p}A). \tag{5.21}$$

In particular, for each i we have $Y_i^{1-\varphi}/X_i^{1-\varphi} \in 1+F_{1-p}A$. Here, for $a \in A^\times$ and $k = \sum_{i=0}^m k_i \varphi^i \in \mathbb{Z}[\varphi]$ with $m \in \mathbb{Z}_{\geq 0}$ and $k_i \in \mathbb{Z}$ for all $0 \leq i \leq m$, we write $a^k \overset{\text{def}}{=} \prod_{i=0}^m \varphi^i(a^{k_i}) \in A^\times$. This makes $A^\times$ a $\mathbb{Z}[\varphi]$-module. Moreover, $1 + F_{-1}A$ is a $\mathbb{Z}_p[\varphi]$-module by completeness.

For $a \in \mathcal{O}_K^\times$ and $0 \leq j \leq f-1$, we set:

$$\begin{aligned} f_{a,j} &\overset{\text{def}}{=} \overline{a}^{p^j} X_j / a(X_j) \in 1 + F_{1-p}A; \\ f_{a,\sigma_j} &\overset{\text{def}}{=} \overline{a}^{p^j} Y_j / a(Y_j) \in 1 + F_{1-p}A. \end{aligned}$$

As in [BHH$^+$c, (25)], for $0 \leq h \leq q-2$ and $\lambda \in \mathbb{F}^\times$ we define the étale $(\varphi_q, \mathcal{O}_K^\times)$-module $D_{A,\sigma_0}\big(\omega_f^h \operatorname{un}(\lambda)\big)$ over A as follows ($a \in \mathcal{O}_K^\times$):

$$\begin{cases} D_{A,\sigma_0}(\omega_f^h \operatorname{un}(\lambda)) &= Ae \\ \varphi_q(e) &= \lambda X_0^{h(1-\varphi)} e \\ a(e) &= f_{a,0}^{h(1-\varphi)/(1-q)} e. \end{cases} \tag{5.22}$$

Using (5.21), we get an isomorphic étale $(\varphi_q, \mathcal{O}_K^\times)$-module over A if we replace X_0 by Y_0 (and thus $f_{a,0}$ by f_{a,σ_0}).

Definition 5.4.1. *Let $0 \leq h \leq q-2$ and $\lambda_0, \lambda_1 \in \mathbb{F}^\times$. We define W^X to be the set of equivalence classes of tuples $[B] = \big(D, (E_a)_{a \in \mathcal{O}_K^\times}\big)$ such that*

(i) *$D \in A$, $E_a \in A$ for all $a \in \mathcal{O}_K^\times$, and the map $\mathcal{O}_K^\times \to A$, $a \mapsto E_a$ is continuous;*
(ii) *$E_{ab} = E_a + f_{a,0}^{h(1-\varphi)/(1-q)} a(E_b)$ for all $a, b \in \mathcal{O}_K^\times$;*
(iii) *$\big(\mathrm{id} - \lambda_0 \lambda_1^{-1} X_0^{h(1-\varphi)} \varphi_q\big)(E_a) = \big(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\big)(D)$ for all $a \in \mathcal{O}_K^\times$;*
(iv) *two tuples $\big(D, (E_a)_{a \in \mathcal{O}_K^\times}\big)$ and $\big(D', (E'_a)_{a \in \mathcal{O}_K^\times}\big)$ are equivalent if and only if there exists $b \in A$ such that*

$$\begin{cases} D' = D + \left(\mathrm{id} - \lambda_0 \lambda_1^{-1} X_0^{h(1-\varphi)} \varphi_q\right)(b) \\ E'_a = E_a + \left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(b) \quad \forall\, a \in \mathcal{O}_K^\times. \end{cases}$$

It has a natural structure of an $\mathbb{F}$-vector space.

We define W^Y in a similar way replacing X_0 by Y_0.

By the definition of W^X, there is an isomorphism of $\mathbb{F}$-vector spaces

$$W^X \cong \operatorname{Ext}^1\left(D_{A,\sigma_0}\big(\operatorname{un}(\lambda_1)\big), D_{A,\sigma_0}\big(\omega_f^h \operatorname{un}(\lambda_0)\big)\right),$$

where Ext^1 is defined in the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A. For $[B] = \big(D, (E_a)_{a\in\mathcal{O}_K^\times}\big) \in W^X$, we denote by $D([B])$ the corresponding étale $(\varphi_q, \mathcal{O}_K^\times)$-module over A. It has an A-basis with respect to which the matrices of the actions of φ_q and $\mathcal{O}_K^\times$ have the form (using (5.22))

$$\begin{cases} \mathrm{Mat}(\varphi_q) &= \begin{pmatrix} \lambda_0 X_0^{h(1-\varphi)} & \lambda_1 D \\ 0 & \lambda_1 \end{pmatrix} \\ \mathrm{Mat}(a) &= \begin{pmatrix} f_{a,0}^{h(1-\varphi)/(1-q)} & E_a \\ 0 & 1 \end{pmatrix} \quad \forall\, a \in \mathcal{O}_K^\times. \end{cases}$$

Note that $D([B]) \cong D(\lambda[B])$ as étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A for $\lambda \in \mathbb{F}^\times$.

We denote by A_∞ the completed perfection of A (see [BHH$^+$c, Lemma 2.4.2(i)]).

Lemma 5.4.2. *Let $0 \le j \le f-1$. We have $f_{a,j} = f_{a,\sigma_j} = 1$ for all $a \in [\mathbb{F}_q^\times]$. More generally we have for $a \in \mathcal{O}_K^\times$*

$$\begin{aligned} f_{a,j}^{-1} &\in 1 + c_a^{p^j} X_j^{\varphi-1} - c_a^{p^{j-1}} X_j^{\varphi-1} X_{j-1}^{\varphi-1} + F_{3-3p}A; \\ f_{a,\sigma_j}^{-1} &\in 1 + c_a^{p^j} Y_j^{\varphi-1} - c_a^{p^{j-1}} Y_j^{\varphi-1} Y_{j-1}^{\varphi-1} + F_{3-3p}A, \end{aligned} \tag{5.23}$$

where c_a is as in Lemma 5.2.1.

Proof. Recall that we have $\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![X_0, \dots, X_{f-1}]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$ with maximal ideal $\mathfrak{m}_{\mathcal{O}_K} = (X_0, \dots, X_{f-1}) = (Y_0, \dots, Y_{f-1})$.

If $a \in [\mathbb{F}_q^\times]$, then we have $f_{a,j} = f_{a,\sigma_j} = 1$ for all $0 \le j \le f-1$ by (5.18) and (5.20).

If $a = 1 + p^2 b$ for some $b \in \mathcal{O}_K$. Then for each $x \in \mathcal{O}_K$, we have (recall that $\delta_x \in \mathbb{F}[\![\mathcal{O}_K]\!]$ corresponds to x)

$$a(\delta_x) = \delta_{(1+p^2b)x} = \delta_x + (\delta_{p^2b} - 1)\delta_x = \delta_x + \big(1 + (\delta_b - 1)^{p^2}\big)\delta_x \in \delta_x + \mathfrak{m}_{\mathcal{O}_K}^{p^2}.$$

From this we deduce that (for all $0 \le j \le f-1$)

$$\begin{aligned} a(X_j) &\in X_j(1 + F_{1-p^2}A); \\ a(Y_j) &\in Y_j(1 + F_{1-p^2}A). \end{aligned}$$

Hence the lemma holds (since $p^2 - 1 \ge 3p - 3$ and $c_a = 0$ for $a = 1 + p^2 b$).

It remains to prove the lemma for $a = 1 + p[\mu]$ with $\mu \in \mathbb{F}_q^\times$. We refer to [FF18, §1.10.2] for the definition of the ring of Witt bi-vectors $BW(A_\infty)$. Since the isomorphism (5.19) respects the $\mathcal{O}_K^\times$-actions, we have equality in $\mathbf{B}^+(A_\infty)^{\varphi_q = p^f}$ (which equals $BW(A_\infty)^{\varphi_q = p^f}$ by [FF18, Prop. 4.2.1]):

$$\begin{aligned} \sum_{i=0}^{f-1} \sum_{n\in\mathbb{Z}} [a(X_i)^{p^{-i-nf}}] p^{i+nf} &= a \sum_{i=0}^{f-1} \sum_{n\in\mathbb{Z}} [X_i^{p^{-i-nf}}] p^{i+nf} \\ &= \sum_{i=0}^{f-1} \sum_{n\in\mathbb{Z}} [X_i^{p^{-i-nf}}] p^{i+nf} + \sum_{i=0}^{f-1} \sum_{n\in\mathbb{Z}} [\mu X_i^{p^{-i-nf}}] p^{i+nf+1} \\ &= \sum_{i=0}^{f-1} \sum_{n\in\mathbb{Z}} \Big([X_i^{p^{-i-nf}}] + [(\mu^{p^i} X_{i-1}^p)^{p^{-i-nf}}] \Big) p^{i+nf}, \end{aligned} \tag{5.24}$$

where the last equality follows from a reindexation.

For $n \in \mathbb{Z}_{\geq 0}$, we let $S_n \in \mathbb{Z}[a_0, \dots, a_n, b_0, \dots, b_n]$ be the additional law of the Witt vectors, given inductively by the equalities in $\mathbb{Z}[a_0, \dots, a_n, b_0, \dots, b_n]$

$$\sum_{i=0}^{n} p^i a_i^{p^{n-i}} + \sum_{i=0}^{n} p^i b_i^{p^{n-i}} = \sum_{i=0}^{n} p^i S_i^{p^{n-i}}. \tag{5.25}$$

By [FF18, §1.10.2], the additional law in the ring of Witt bi-vectors BW is given by

$$\sum_{i\in\mathbb{Z}} [a_i^{p^{-i}}]p^i + \sum_{i\in\mathbb{Z}} [b_i^{p^{-i}}]p^i = \sum_{i\in\mathbb{Z}} [c_i^{p^{-i}}]p^i,$$

where $c_i \overset{\text{def}}{=} \lim_{n\to\infty} c_{i,n} \in \mathbb{Z}[\![\dots, a_i, \ \dots, b_i]\!]$ with

$$c_{i,n} \overset{\text{def}}{=} S_n(a_{i-n}, a_{i-n+1}, \dots, a_i, b_{i-n}, b_{i-n+1}, \dots, b_i) \in \mathbb{Z}[a_{i-n}, \dots, a_i, b_{i-n}, \dots, b_i].$$

In particular, for $i \in \mathbb{Z}$ we have

$$\begin{aligned} c_{i,0} &= a_i + b_i \in \mathbb{Z}[a_i, b_i]; \\ c_{i,1} &= a_i + b_i - \sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} a_{i-1}^{p-s} b_{i-1}^{s} \in \mathbb{Z}[a_{i-1}, a_i, b_{i-1}, b_i]. \end{aligned} \tag{5.26}$$

Moreover, for $i \in \mathbb{Z}$ and $n \geq 0$, we have in $\mathbb{Z}[a_{i-n-1}, \dots, a_i, b_{i-n-1}, \dots, b_i]$

$$\sum_{\ell=0}^{n} p^\ell a_{i-n+\ell}^{p^{n-\ell}} + \sum_{\ell=0}^{n} p^\ell b_{i-n+\ell}^{p^{n-\ell}} = \sum_{\ell=0}^{n} p^\ell c_{i-n+\ell,\ell}^{p^{n-\ell}}; \tag{5.27}$$

$$\sum_{\ell=0}^{n+1} p^\ell a_{i-(n+1)+\ell}^{p^{n+1-\ell}} + \sum_{\ell=0}^{n+1} p^\ell b_{i-(n+1)+\ell}^{p^{n+1-\ell}} = \sum_{\ell=0}^{n+1} p^\ell c_{i-(n+1)+\ell,\ell}^{p^{n+1-\ell}}. \tag{5.28}$$

Considering (5.28) $- p \cdot$ (5.27) and using $c_{i-(n+1)} = a_{i-(n+1)} + b_{i-(n+1)}$, we get

$$a_{i-(n+1)}^{p^{n+1}} + b_{i-(n+1)}^{p^{n+1}} = \left(a_{i-(n+1)} + b_{i-(n+1)}\right)^{p^{n+1}} + \sum_{\ell=1}^{n+1} p^\ell \left(c_{i-(n+1)+\ell,\ell}^{p^{n+1-\ell}} - c_{i-(n+1)+\ell,\ell-1}^{p^{n+1-\ell}} \right).$$

Hence we have

$$\begin{aligned} c_{i,n+1} - c_{i,n} = \frac{1}{p^{n+1}} \Bigg[& a_{i-(n+1)}^{p^{n+1}} + b_{i-(n+1)}^{p^{n+1}} - \left(a_{i-(n+1)} + b_{i-(n+1)}\right)^{p^{n+1}} \\ & - \sum_{\ell=1}^{n} p^\ell \left(c_{i-(n+1)+\ell,\ell}^{p^{n+1-\ell}} - c_{i-(n+1)+\ell,\ell-1}^{p^{n+1-\ell}} \right) \Bigg]. \end{aligned} \tag{5.29}$$

From (5.29) and using induction on n, we deduce that for $i \in \mathbb{Z}$ and $n \geq 1$,

- each term of $c_{i,n+1} - c_{i,n}$ involves both the variable a_k for some $k \leq i$ and the variable b_ℓ for some $\ell \leq i$;
- the minimal degree (in the variables a_k, b_k for $k \leq i$) of each term of $c_{i,n+1} - c_{i,n}$ is at least $2p-1$, and tends to infinity as n tends to ∞.

In particular, using (5.26) we have

$$c_i = a_i + b_i - \sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} a_{i-1}^{p-s} b_{i-1}^{s} + (\deg \geq 2p-1)\,, \tag{5.30}$$

where each term of $(\deg \geq 2p-1)$ involves both the variable a_k for some $k \leq i$ and the variable b_ℓ for some $\ell \leq i$, and has degree at least $2p-1$. Then combining (5.24), (5.25) and (5.30), we conclude that (for $a = 1 + p[\mu]$ and $0 \leq j \leq f-1$)

$$\begin{aligned} a(X_j) &\in X_j + \mu^{p^j} X_{j-1}^p - \sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} X_{j-1}^{p-s} \left(\mu^{p^{j-1}} X_{j-2}^p\right)^s + (\deg \geq 3p-2) \\ &= X_j + \mu^{p^j} X_{j-1}^p - \mu^{p^{j-1}} X_{j-1}^{p-1} X_{j-2}^p + (\deg \geq 3p-2) \\ &\subseteq X_j \left(1 + \mu^{p^j} X_j^{\varphi-1} - \mu^{p^{j-1}} X_j^{\varphi-1} X_{j-1}^{\varphi-1} + F_{3-3p} A\right), \end{aligned}$$

which proves the first formula in (5.23).

Next we turn to the variables Y_j, still with $a = 1 + p[\mu]$ for some $\mu \in [\mathbb{F}_q^\times]$.

Claim. We have $\delta_1 \in 1 - Y_0 - \cdots - Y_{f-1} + \mathfrak{m}_{\mathcal{O}_K}^2$ in $\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![Y_0, \ldots, Y_{f-1}]\!]$.

Proof. Recall that $Y_j \overset{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \delta_{[\lambda]} \in \mathbb{F}[\![\mathcal{O}_K]\!]$ for $0 \leq j \leq f-1$. On one hand, we have

$$\sum_{i=0}^{q-2} \sum_{\lambda \in \mathbb{F}_q} \lambda^i \delta_{[\lambda]} = \sum_{\lambda \in \mathbb{F}_q} \left[\sum_{i=0}^{q-2} \lambda^i\right] \delta_{[\lambda]} = 1 - \delta_1 \in \mathfrak{m}_{\mathcal{O}_K}, \tag{5.31}$$

where we use the convention that $0^0 \overset{\text{def}}{=} 1$. On the other hand, for each $0 \leq i \leq q-2$, if we write $i = \sum_{j=0}^{f-1} i_j p^j$ with $0 \leq i_j \leq p-1$, then by [BHH+b, Lemma 3.2.2.5(i)] we have in $\mathbb{F}[\![\mathcal{O}_K]\!]$

$$\sum_{\lambda \in \mathbb{F}_q} \lambda^i \delta_{[\lambda]} \equiv (-1)^{f-1} \left[\prod_{j=0}^{f-1} i_j!\right] \prod_{j=0}^{f-1} Y_j^{p-1-i_j} \mod \mathfrak{m}_{\mathcal{O}_K}^p. \tag{5.32}$$

Combining (5.31) and (5.32), we deduce that $\delta_1 \in 1 - a_0 Y_0 - \cdots - a_{f-1} Y_{f-1} + \mathfrak{m}_{\mathcal{O}_K}^2$ in $\mathbb{F}[\![\mathcal{O}_K]\!]$ with $a_j = (-1)^{f-1}(p-2)!\,((p-1)!)^{f-1} = 1$ in $\mathbb{F}$ for all $0 \leq j \leq f-1$. □

For each $0 \leq j \leq f-1$, by the claim above we have (for $a = 1 + p[\mu]$)

$$\begin{aligned} a(Y_j) &= \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \delta_{(1+p[\mu])[\lambda]} = \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \delta_{[\lambda]} \cdot [\mu\lambda](\delta_p) \\ &\in \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \delta_{[\lambda]} \cdot [\mu\lambda] \left((1 - Y_0 - \cdots - Y_{f-1} + \mathfrak{m}_{\mathcal{O}_K}^2)^p\right) \\ &= \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \delta_{[\lambda]} \cdot [\mu\lambda] \left(1 - Y_0^p - \cdots - Y_{f-1}^p\right) + \mathfrak{m}_{\mathcal{O}_K}^{2p} \\ &= \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \delta_{[\lambda]} - \sum_{i=0}^{f-1} \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-p^j} \delta_{[\lambda]} (\mu\lambda)^{p^{i+1}} Y_i^p + \mathfrak{m}_{\mathcal{O}_K}^{2p} \quad \text{(by (5.18))} \end{aligned}$$

$$= Y_j - \sum_{i=0}^{f-1} \mu^{p^{i+1}} \left[\sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{p^{i+1}-p^j} \delta_{[\lambda]} \right] Y_i^p + \mathfrak{m}_{\mathcal{O}_K}^{2p}.$$

If $f = 1$, then by (5.32) we have

$$\sum_{\lambda \in \mathbb{F}_p^\times} \lambda^{p-1} \delta_{[\lambda]} = \sum_{\lambda \in \mathbb{F}_p^\times} \delta_{[\lambda]} \equiv Y_0^{p-1} - 1 \mod \mathfrak{m}_{\mathcal{O}_K}^p.$$

If $f > 1$, then by (5.32), we deduce that

$$\sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{p^{i+1}-p^j} \delta_{[\lambda]} \in \begin{cases} -1 + \mathfrak{m}_{\mathcal{O}_K}^p & \text{if } i \equiv j-1 \mod f \\ Y_{j-1}^{p-1} + \mathfrak{m}_{\mathcal{O}_K}^p & \text{if } i \equiv j-2 \mod f \\ \mathfrak{m}_{\mathcal{O}_K}^p & \text{otherwise.} \end{cases}$$

In both cases, we conclude that

$$a(Y_j) \in Y_j + \mu^{p^j} Y_{j-1}^p - \mu^{p^{j-1}} Y_{j-1}^{p-1} Y_{j-2}^p + \mathfrak{m}_{\mathcal{O}_K}^{2p}. \tag{5.33}$$

Using (5.18) and the commutativity of the actions of a and $[\mathbb{F}_q^\times]$ on A, we deduce that each term in $\mathfrak{m}_{\mathcal{O}_K}^{2p}$ of (5.33) has degree congruent to 1 modulo $p-1$, hence we have (for $a = 1 + p[\mu]$)

$$a(Y_j) \in Y_j \left(1 + \mu^{p^j} Y_j^{\varphi-1} - \mu^{p^{j-1}} Y_j^{\varphi-1} Y_{j-1}^{\varphi-1} + F_{3-3p} A\right),$$

which proves the second formula in (5.23). □

Lemma 5.4.3. *Let $0 \le h \le q-2$.*

(i) *For $i \ge -1$ and $a \in \mathcal{O}_K^\times$, we have*

$$\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right) \left(X_0^{[h]_i(1-\varphi)}\right) \in F_{1-p} A.$$

(ii) *For $i \ge -1$ and $a \in \mathcal{O}_K^\times$, we have*

$$\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right) \left(X_0^{([h]_i - p^{i+1})(1-\varphi)} X_1^{p^{i+1}(1-\varphi)}\right) \in F_{1-p} A.$$

(iii) *For $i \ge -1$ and $a \in \mathcal{O}_K^\times$, we have*

$$\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right) \left(X_0^{([h]_i + p^{i+1})(1-\varphi)}\right) \in (h_{i+1} - 1) c_a^{p^{i+1}} X_0^{[h]_i(1-\varphi)} + F_{1-p} A.$$

(iv) *For $i \ge 0$ such that $h_i = 1$ and $a \in \mathcal{O}_K^\times$, we have*

$$\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right) \left(X_0^{[h]_i(1-\varphi)} X_1^{p^i(1-\varphi)}\right) \in -c_a^{p^{i+1}} X_0^{[h]_i(1-\varphi)} + c_a^{p^i} X_0^{[h]_{i-1}(1-\varphi)} + F_{1-p} A.$$

Proof. (i). By definition we have

$$\begin{aligned} &\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right) \left(X_0^{[h]_i(1-\varphi)}\right) \\ &\qquad = X_0^{[h]_i(1-\varphi)} \left(1 - f_{a,0}^{(h+(q-1)[h]_i)(1-\varphi)/(1-q)}\right) \\ &\qquad \in X_0^{[h]_i(1-\varphi)} \left(1 - f_{a,0}^{p^{i+1}\mathbb{Z}_p(1-\varphi)}\right) \end{aligned}$$

$$\subseteq X_0^{[h]_i(1-\varphi)} F_{(1-p)p^{i+1}} A \subseteq F_{1-p} A,$$

where the second inclusion follows from Lemma 5.4.2.

(ii). By definition we have

$$\begin{aligned}
&\Big(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\Big)\Big(X_0^{([h]_i - p^{i+1})(1-\varphi)} X_1^{p^{i+1}(1-\varphi)}\Big) \\
&\quad = X_0^{([h]_i - p^{i+1})(1-\varphi)} X_1^{p^{i+1}(1-\varphi)} \Big(1 - f_{a,0}^{(h+(q-1)([h]_i - p^{i+1}))(1-\varphi)/(1-q)} f_{a,1}^{-p^{i+1}(1-\varphi)}\Big) \\
&\quad \in X_0^{([h]_i - p^{i+1})(1-\varphi)} X_1^{p^{i+1}(1-\varphi)} \Big(1 - f_{a,0}^{p^{i+1}\mathbb{Z}_p(1-\varphi)} f_{a,1}^{-p^{i+1}(1-\varphi)}\Big) \\
&\quad \subseteq X_0^{([h]_i - p^{i+1})(1-\varphi)} X_1^{p^{i+1}(1-\varphi)} F_{(1-p)p^{i+1}} A \subseteq F_{1-p} A,
\end{aligned}$$

where the second inclusion follows from Lemma 5.4.2.

(iii). By definition we have

$$\begin{aligned}
&\Big(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\Big)\Big(X_0^{([h]_i + p^{i+1})(1-\varphi)}\Big) \\
&\quad = X_0^{([h]_i + p^{i+1})(1-\varphi)} \Big(1 - f_{a,0}^{(h+(q-1)([h]_i + p^{i+1}))(1-\varphi)/(1-q)}\Big) \\
&\quad \in X_0^{([h]_i + p^{i+1})(1-\varphi)} \Big(1 - f_{a,0}^{(p^{i+1}(h_{i+1}-1) + p^{i+2}\mathbb{Z}_p)(1-\varphi)}\Big) \\
&\quad \subseteq X_0^{([h]_i + p^{i+1})(1-\varphi)} \Big((h_{i+1}-1) c_a^{p^{i+1}} X_0^{p^{i+1}(\varphi-1)} + F_{2(1-p)p^{i+1}} A\Big) \\
&\quad \subseteq (h_{i+1}-1) c_a^{p^{i+1}} X_0^{[h]_i(1-\varphi)} + F_{1-p} A,
\end{aligned}$$

where the second inclusion follows from Lemma 5.4.2.

(iv). By definition we have

$$\begin{aligned}
&\Big(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\Big)\Big(X_0^{[h]_i(1-\varphi)} X_1^{p^i(1-\varphi)}\Big) \\
&\quad = X_0^{[h]_i(1-\varphi)} X_1^{p^i(1-\varphi)} \Big(1 - f_{a,0}^{(h+(q-1)[h]_i)(1-\varphi)/(1-q)} f_{a,1}^{-p^i(1-\varphi)}\Big) \\
&\quad \in X_0^{[h]_i(1-\varphi)} X_1^{p^i(1-\varphi)} \Big(1 - f_{a,0}^{p^{i+1}\mathbb{Z}_p(1-\varphi)} f_{a,1}^{-p^i(1-\varphi)}\Big) \\
&\quad \subseteq X_0^{[h]_i(1-\varphi)} X_1^{p^i(1-\varphi)} \Big(-c_a^{p^{i+1}} X_1^{p^i(\varphi-1)} + c_a^{p^i} X_1^{p^i(\varphi-1)} X_0^{p^i(\varphi-1)} + F_{3(1-p)p^i} A\Big) \\
&\quad \subseteq -c_a^{p^{i+1}} X_0^{[h]_i(1-\varphi)} + c_a^{p^i} X_0^{[h]_{i-1}(1-\varphi)} + F_{1-p} A,
\end{aligned}$$

where the second inclusion follows from Lemma 5.4.2 and uses $p \geq 3$ (hence $p^{i+1}(p-1) \geq 3p^i(p-1)$), and the last inclusion uses $h_i = 1$. □

Definition 5.4.4. *Let $0 \leq h \leq q-2$, $\lambda_0, \lambda_1 \in \mathbb{F}^\times$ and $0 \leq j \leq f-1$. We define $D_j^X, D_j'^X, D_{\mathrm{tr}}^X, D_{\mathrm{un}}^X \in A$ as follows:*

(i) *If $h_j \neq 0$, we define*

$$D_j^X \overset{\mathrm{def}}{=} X_0^{[h]_{j-1}(1-\varphi)};$$

If $h_j = 0$, we let $0 \leq r \leq f-1$ such that $h_{j+1} = \cdots = h_{j+r} = 1$ and $h_{j+r+1} \neq 1$, then we define

$$\begin{aligned}
D_j^X &\overset{\mathrm{def}}{=} X_0^{([h]_{j+r} + p^{j+r+1})(1-\varphi)} + (h_{j+r+1}-1) \sum_{i=0}^{r} X_0^{[h]_{j+i}(1-\varphi)} X_1^{p^{j+i}(1-\varphi)} \\
&= X_0^{([h]_{j-1} + p^j(p+p^2+\cdots+p^{r+1}))(1-\varphi)} + (h_{j+r+1}-1) \sum_{i=0}^{r} X_0^{([h]_{j-1} + p^j(p+p^2+\cdots+p^i))(1-\varphi)} X_1^{p^{j+i}(1-\varphi)}.
\end{aligned}$$

(ii) *We define*

$$D_j'^X \stackrel{\text{def}}{=} X_0^{([h]_{j-1}-p^j)(1-\varphi)} X_1^{p^j(1-\varphi)}.$$

(iii) *If* $h = 1 + p + \cdots + p^{f-1}$ *and* $\lambda_0\lambda_1^{-1} = 1$, *we define*

$$D_{\mathrm{tr}}^X \stackrel{\text{def}}{=} \sum_{i=0}^{f-1} X_0^{[h]_i(1-\varphi)} X_1^{p^i(1-\varphi)} = \sum_{i=0}^{f-1} X_0^{(1+p+\cdots+p^i)(1-\varphi)} X_1^{p^i(1-\varphi)}.$$

Otherwise (i.e. either $h \neq 1 + p + \cdots + p^{f-1}$ *or* $\lambda_0\lambda_1^{-1} \neq 1$*), we define* $D_{\mathrm{tr}}^X \stackrel{\text{def}}{=} 0$.

(iv) *If* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$, *we define* $D_{\mathrm{un}}^X \stackrel{\text{def}}{=} 1$. *Otherwise, we define* $D_{\mathrm{un}}^X \stackrel{\text{def}}{=} 0$.

Corollary 5.4.5. *Let* $0 \leq h \leq q-2$ *and* $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.

(i) *For all* $0 \leq j \leq f-1$ *and* $a \in \mathcal{O}_K^\times$, *we have*

$$\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(D_j^X) \in F_{1-p}A;$$
$$\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(D_j'^X) \in F_{1-p}A.$$

(ii) *If* $h = 1 + p + \cdots + p^{f-1}$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then for all* $a \in \mathcal{O}_K^\times$, *we have*

$$\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(D_{\mathrm{tr}}^X) \in c_a\left(1 - X_0^{h(1-\varphi)}\right) + F_{1-p}A.$$

Proof. This follows from Lemma 5.4.3. Note that for i such that $h_i = 0$ we have $[h]_i = [h]_{i-1}$. □

Lemma 5.4.6. *Let* $0 \leq h \leq q-2$ *and* $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.

(i) *For any* $y \in F_{1-p}A$, *the equation* $\left(\mathrm{id} - \lambda_0\lambda_1^{-1} X_0^{h(1-\varphi)}\varphi_q\right)(x) = y$ *has a unique solution in* $F_{1-p}A$, *given by the convergent series* $x = \sum_{n=0}^{\infty}\left(\lambda_0\lambda_1^{-1} X_0^{h(1-\varphi)}\varphi_q\right)^n(y)$.

(ii) *For any* $y \in A$, *the equation* $\left(\mathrm{id} - \lambda_0\lambda_1^{-1} X_0^{h(1-\varphi)}\varphi_q\right)(x) = y$ *has at most one solution in* A *unless* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$.

Proof. The proof is similar to that of Lemma 5.2.7. We omit the details. □

Proposition 5.4.7. *Let* $0 \leq h \leq q-2$ *and* $\lambda_0, \lambda_1 \in \mathbb{F}^\times$.

(i) *For all* $0 \leq j \leq f-1$, *the tuple* $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ *with*

$$\begin{cases} D &= D_j^X \\ E_a &= E_{j,a}^X \stackrel{\text{def}}{=} \left(\mathrm{id} - \lambda_0\lambda_1^{-1} X_0^{h(1-\varphi)}\varphi_q\right)^{-1}\left[\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(D_j^X)\right] \\ &= \sum_{n=0}^{\infty}\left(\lambda_0\lambda_1^{-1} X_0^{h(1-\varphi)}\varphi_q\right)^n\left[\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(D_j^X)\right] \end{cases}$$

defines an element of W^X. *We denote it by* $[B_j^X]$.

(ii) *For all* $0 \leq j \leq f-1$, *the tuple* $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ *with*

$$\begin{cases} D &= D_j'^X \\ E_a &= E_{j,a}'^X \stackrel{\text{def}}{=} \left(\mathrm{id} - \lambda_0\lambda_1^{-1} X_0^{h(1-\varphi)}\varphi_q\right)^{-1}\left[\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(D_j'^X)\right] \\ &= \sum_{n=0}^{\infty}\left(\lambda_0\lambda_1^{-1} X_0^{h(1-\varphi)}\varphi_q\right)^n\left[\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)} a\right)(D_j'^X)\right] \end{cases}$$

defines an element of W^X. *We denote it by* $[B_j'^X]$.

(iii) *If* $h = 1 + p + \cdots + p^{f-1}$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then the tuple* $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ *with*

$$\begin{cases} D & = D^X_{\mathrm{tr}} \\ E_a & = E^X_{\mathrm{tr},a} \overset{\mathrm{def}}{=} \left(\mathrm{id} - X_0^{h(1-\varphi)}\varphi_q\right)^{-1}\left[\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)}a\right)(D^X_{\mathrm{tr}})\right] \\ & = c_a + \sum_{n=0}^{\infty}\left(X_0^{h(1-\varphi)}\varphi_q\right)^n\left[\left(\mathrm{id} - f_{a,0}^{h(1-\varphi)/(1-q)}a\right)(D^X_{\mathrm{tr}}) - c_a\left(1 - X_0^{h(1-\varphi)}\right)\right] \end{cases}$$

defines an element of W^X. *We denote it by* $[B^X_{\mathrm{tr}}]$. *Otherwise, we define* $E^X_{\mathrm{tr},a} \overset{\mathrm{def}}{=} 0$ *for all* $a \in \mathcal{O}_K^\times$ *and* $[B^X_{\mathrm{tr}}] \overset{\mathrm{def}}{=} [0]$ *in* W^X.

(iv) *If* $h = 0$ *and* $\lambda_0\lambda_1^{-1} = 1$, *then the tuple* $\left(D, (E_a)_{a\in\mathcal{O}_K^\times}\right)$ *with*

$$\begin{cases} D & = D^X_{\mathrm{un}} = 1 \\ E_a & = E^X_{\mathrm{un},a} \overset{\mathrm{def}}{=} 0 \end{cases}$$

defines an element of W^X. *We denote it by* $[B^X_{\mathrm{un}}]$. *Otherwise, we define* $E^X_{\mathrm{un},a} \overset{\mathrm{def}}{=} 0$ *for all* $a \in \mathcal{O}_K^\times$ *and* $[B^X_{\mathrm{un}}] \overset{\mathrm{def}}{=} [0]$ *in* W^X.

Proof. (iv) is direct. For (i), (ii) and (iii), each E_a is well-defined by Corollary 5.4.5 and Lemma 5.4.6(i), and condition (ii) in Definition 5.4.1 is guaranteed by the uniqueness of solution in Lemma 5.4.6(i),(ii). □

By Lemma 5.4.2, we can give similar definitions for the variables Y_i instead of X_i. We have the following partial comparison result:

Proposition 5.4.8. *Suppose that* $c_0, \ldots, c_{f-1}, c'_0, \ldots, c'_{f-1}, c_{\mathrm{un}} \in \mathbb{F}$ *such that* $c_j = 0$ *if* $h_j = 0$, *then we have an isomorphism of étale* $(\varphi_q, \mathcal{O}_K^\times)$*-modules over* A*:*

$$D\left(\sum_{j=0}^{f-1} c_j[B_j^X] + \sum_{j=0}^{f-1} c'_j[B_j'^X] + c_{\mathrm{un}}[B^X_{\mathrm{un}}]\right) \cong D\left(\sum_{j=0}^{f-1} c_j[B_j^Y] + \sum_{j=0}^{f-1} c'_j[B_j'^Y] + c_{\mathrm{un}}[B^Y_{\mathrm{un}}]\right).$$

Proof. Let e_0^X, e_1^X be an A-basis of $D\big(\sum_{j=0}^{f-1} c_j[B_j^X] + \sum_{j=0}^{f-1} c'_j[B_j'^X] + c_{\mathrm{un}}[B^X_{\mathrm{un}}]\big)$ with respect to which the matrices of the actions of φ_q and $\mathcal{O}_K^\times$ have the form

$$\begin{cases} \mathrm{Mat}_A^X(\varphi_q) & = \begin{pmatrix} \lambda_0 X_0^{h(1-\varphi)} & \lambda_1 D^X \\ 0 & \lambda_1 \end{pmatrix} \\ \mathrm{Mat}_A^X(a) & = \begin{pmatrix} f_{a,0}^{h(1-\varphi)/(1-q)} & E_a^X \\ 0 & 1 \end{pmatrix} \quad \forall\, a \in \mathcal{O}_K^\times, \end{cases}$$

where

$$\begin{cases} D^X & \overset{\mathrm{def}}{=} \sum_{j=0}^{f-1} c_j D_j^X + \sum_{j=0}^{f-1} c'_j D_j'^X + c_{\mathrm{un}} D^X_{\mathrm{un}} \\ E_a^X & \overset{\mathrm{def}}{=} \sum_{j=0}^{f-1} c_j E_{j,a}^X + \sum_{j=0}^{f-1} c'_j E_{j,a}'^X + c_{\mathrm{un}} E^X_{\mathrm{un},a} \quad \forall\, a \in \mathcal{O}_K^\times. \end{cases}$$

Let e_0^Y, e_1^Y be an A-basis of $D\big(\sum_{j=0}^{f-1} c_j[B_j^Y] + \sum_{j=0}^{f-1} c'_j[B_j'^Y] + c_{\mathrm{un}}[B^Y_{\mathrm{un}}]\big)$ with respect to which the matrices of the actions of φ_q and $\mathcal{O}_K^\times$ have the form

$$\begin{cases} \mathrm{Mat}_A^Y(\varphi_q) & = \begin{pmatrix} \lambda_0 Y_0^{h(1-\varphi)} & \lambda_1 D^Y \\ 0 & \lambda_1 \end{pmatrix} \\ \mathrm{Mat}_A^Y(a) & = \begin{pmatrix} f_{a,\sigma_0}^{h(1-\varphi)/(1-q)} & E_a^Y \\ 0 & 1 \end{pmatrix} \quad \forall\, a \in \mathcal{O}_K^\times, \end{cases}$$

where

$$\begin{cases} D^Y & \stackrel{\text{def}}{=} \sum_{j=0}^{f-1} c_j D_j^Y + \sum_{j=0}^{f-1} c'_j D_j'^Y + c_{\text{un}} D_{\text{un}}^Y \\ E_a^Y & \stackrel{\text{def}}{=} \sum_{j=0}^{f-1} c_j E_{j,a}^Y + \sum_{j=0}^{f-1} c'_j E_{j,a}'^Y + c_{\text{un}} E_{\text{un},a}^Y \quad \forall\, a \in \mathcal{O}_K^\times. \end{cases}$$

To prove the proposition, it is enough to find a change of basis formula $(e_0^Y \ e_1^Y) = (e_0^X \ e_1^X)Q$ for some $Q = \left(\begin{smallmatrix} b_{00} & b_{01} \\ 0 & b_{11} \end{smallmatrix}\right) \in I_2 + M_2(F_{1-p}A)$ such that $Q^{-1}\,\mathrm{Mat}_A^X(\varphi_q)\varphi_q(Q) = \mathrm{Mat}_A^Y(\varphi_q)$, or equivalently

$$\begin{pmatrix} b_{00}^{-1} & -b_{00}^{-1}b_{01}b_{11}^{-1} \\ 0 & b_{11}^{-1} \end{pmatrix} \begin{pmatrix} \lambda_0 X_0^{h(1-\varphi)} & \lambda_1 D^X \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} \varphi_q(b_{00}) & \varphi_q(b_{01}) \\ 0 & \varphi_q(b_{11}) \end{pmatrix} = \begin{pmatrix} \lambda_0 Y_0^{h(1-\varphi)} & \lambda_1 D^Y \\ 0 & \lambda_1 \end{pmatrix}. \tag{5.34}$$

Then the $\mathcal{O}_K^\times$-actions also agree by Lemma 5.4.6(i) using $E_a^X, E_a^Y \in F_{1-p}A$.

Comparing the (2,2)-entries of (5.34), we have $b_{11} = 1$.

Comparing the (1,1)-entries of (5.34), we need to solve $\varphi_q(b_{00})b_{00}^{-1} = \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^h$. So we can take $b_{00} = \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)}$, which makes sense since $Y_0^{1-\varphi}/X_0^{1-\varphi} \in 1 + F_{1-p}A$ by (5.21).

Comparing the (1,2)-entries of (5.34), we need to solve

$$b_{00}^{-1}\lambda_0 X_0^{h(1-\varphi)}\varphi_q(b_{01}) + b_{00}^{-1}\lambda_1 D^X \varphi_q(b_{11}) - b_{00}^{-1}b_{01}b_{11}^{-1}\lambda_1\varphi_q(b_{11}) = \lambda_1 D^Y.$$

Replacing b_{00}, b_{11} by their previous values, we get

$$\left(\mathrm{id} - \lambda_0\lambda_1^{-1}X_0^{h(1-\varphi)}\varphi_q\right)(b_{01}) = D^X - D^Y\left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)}. \tag{5.35}$$

Then we deduce from Lemma 5.4.6(i) and the claim below that there is a unique solution of $b_{01} \in F_{1-p}A$, which completes the proof.

Claim. Then RHS of (5.35) is in $F_{1-p}A$.

Proof. For each $0 \le j \le f-1$ such that $h_j \ne 0$, we have

$$\begin{aligned} &D_j^X - D_j^Y\left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)} \\ &\qquad = X_0^{[h]_{j-1}(1-\varphi)} - Y_0^{[h]_{j-1}(1-\varphi)}\left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)} \\ &\qquad = X_0^{[h]_{j-1}(1-\varphi)}\left[1 - \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{[h]_{j-1}+h/(q-1)}\right] \\ &\qquad \in X_0^{[h]_{j-1}(1-\varphi)}\left[1 - \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{p^j\mathbb{Z}_p}\right] \\ &\qquad \subseteq X_0^{[h]_{j-1}(1-\varphi)}F_{(1-p)p^j}A \subseteq F_{1-p}A. \end{aligned}$$

For each $0 \le j \le f-1$, we have

$$\begin{aligned} &D_j'^X - D_j'^Y\left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)} \\ &\qquad = X_0^{([h]_{j-1}-p^j)(1-\varphi)}X_1^{p^j(1-\varphi)} - Y_0^{([h]_{j-1}-p^j)(1-\varphi)}Y_{\sigma_1}^{p^j(1-\varphi)}\left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)} \end{aligned}$$

$$= X_0^{([h]_{j-1}-p^j)(1-\varphi)} X_1^{p^j(1-\varphi)} \left[1 - \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{[h]_{j-1}-p^j+h/(q-1)} \left(Y_{\sigma_1}^{1-\varphi}/X_1^{1-\varphi}\right)^{p^j}\right]$$
$$\in X_0^{([h]_{j-1}-p^j)(1-\varphi)} X_1^{p^j(1-\varphi)} \left[1 - \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{p^j\mathbb{Z}_p} \left(Y_{\sigma_1}^{1-\varphi}/X_1^{1-\varphi}\right)^{p^j}\right]$$
$$\subseteq X_0^{([h]_{j-1}-p^j)(1-\varphi)} X_1^{p^j(1-\varphi)} F_{(1-p)p^j}A \subseteq F_{1-p}A.$$

Moreover, we have

$$D_{\mathrm{un}}^X - D_{\mathrm{un}}^Y \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)} = D_{\mathrm{un}}^X \left[1 - \left(Y_0^{1-\varphi}/X_0^{1-\varphi}\right)^{h/(q-1)}\right] \in F_{1-p}A.$$

Then the claim follows since $c_j = 0$ if $h_j = 0$. □

Remark 5.4.9. *In general, we do not know how to write $D([B_j^X])$ (in the case $h_j = 0$) and $D([B_{\mathrm{tr}}^X])$ in terms of elements of W^Y.*

5.5 The étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^\otimes(\overline{\rho})$

In this section, we recall the definition of the functor $\overline{\rho} \mapsto D_A^\otimes(\overline{\rho})$ defined in [BHH$^+$c] and give an explicit computation of $D_A^\otimes(\overline{\rho})$ for all reducible two-dimensional $\overline{\rho}$ when $p \geq 5$. As a corollary, we give the comparison between the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-modules and the cyclotomic $(\varphi, \mathbb{Z}_p^\times)$-modules corresponding to $\overline{\rho}$ as in (5.2) when $p \geq 5$, see Corollary 5.5.12.

Recall that A_∞ is the completed perfection of A. The actions of φ and $\mathcal{O}_K^\times$ on A extends naturally to A_∞, and $A_\infty^\times$ becomes a $\mathbb{Q}_p[\varphi]$-module.

Proposition 5.5.1 ([BHH$^+$c], Cor. 2.6.6)**.** *The functor $D \mapsto A_\infty \otimes_A D$ induces an equivalence of categories between the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A and the category of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A_∞, which is rank-preserving and compatible with tensor products.*

As in [BHH$^+$c], we let

$$A'_\infty \overset{\mathrm{def}}{=} \mathbb{F}((T_{K,0}^{1/p^\infty})) \left\langle \left(\frac{T_{K,i}}{T_{K,0}^{p^i}}\right)^{\pm 1/p^\infty}, 1 \leq i \leq f-1 \right\rangle.$$

There is an $\mathbb{F}$-linear Frobenius φ on A'_∞ given by (for each $0 \leq i \leq f-1$)

$$\varphi(T_{K,i}) = T_{K,i+1}, \tag{5.36}$$

where we use the convention that $T_{K,f} \overset{\mathrm{def}}{=} T_{K,0}^q$. There is also an $(\mathcal{O}_K^\times)^f$-action on A'_∞ commuting with $\varphi_q (\overset{\mathrm{def}}{=} \varphi^f)$ given by $(a_i \in \mathcal{O}_K^\times)$

$$(a_0, \ldots, a_{f-1})(T_{K,i}) = a_i(T_{K,i}),$$

where $\mathcal{O}_K^\times$ acts on each variable $T_{K,i}$ in the same way as they act on T_{K,σ_0} in §5.2.

For $0 \leq i \leq f-1$ and $a \in \mathcal{O}_K^\times$, we define $j_i(a) \in (K^\times)^f$ to be a in the i-th coordinate and 1 otherwise. There is an inclusion $\iota_i : \mathbb{F}((T_{K,\sigma_0})) \hookrightarrow A'_\infty$ defined by $T_{K,\sigma_0} \mapsto T_{K,i}$, which commutes with φ_q, and the action of $a \in \mathcal{O}_K^\times$ on $\mathbb{F}((T_{K,\sigma_0}))$ is identified with the action of $j_i(a)$ on A'_∞. In particular, we regard $\mathbb{F}((T_{K,\sigma_0}))$ as a subfield of A'_∞ via the inclusion ι_0. By [BHH$^+$c, Prop. 2.4.4], we can also regard A_∞ as a subring of A'_∞, which is compatible with φ, and the action of $a \in \mathcal{O}_K^\times$ on A_∞ is identified with the action of $(a, 1, \ldots, 1)$ on A'_∞. Moreover,

if we denote $\Delta_1 \overset{\text{def}}{=} \mathrm{Ker}\big((\mathcal{O}_K^\times)^f \to \mathcal{O}_K^\times\big)$ the kernel of the multiplication map, then we have $A_\infty = (A'_\infty)^{\Delta_1}$ (see the paragraph before [BHH$^+$c, Thm. 2.5.1]).

For $\overline{\rho}$ a finite-dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ and $0 \le i \le f-1$, we define

$$D^{(i)}_{A_\infty}(\overline{\rho}) \overset{\text{def}}{=} \Big(A'_\infty \otimes_{\iota_i, \mathbb{F}((T_{K,\sigma_0}))} D_{K,\sigma_0}(\overline{\rho})\Big)^{\Delta_1}.$$

We endow it with a φ_q-action given by $\varphi_q = \varphi_q \otimes \varphi_q$, and an $\mathcal{O}_K^\times$-action such that $a \in \mathcal{O}_K^\times$ acts by $j_i(a) \otimes a$. By the result of [BHH$^+$c], these actions are well-defined and make $D^{(i)}_{A_\infty}(\overline{\rho})$ an étale $(\varphi_q, \mathcal{O}_K^\times)$-module over A_∞. Moreover, there is an isomorphism

$$\phi_i : D^{(i)}_{A_\infty}(\overline{\rho}) \xrightarrow{\sim} D^{(i+1)}_{A_\infty}(\overline{\rho})$$

given by $\phi_i(x \otimes v) \overset{\text{def}}{=} \varphi(x) \otimes v$ if $i < f-1$, and $\phi_i(x \otimes v) \overset{\text{def}}{=} \varphi(x) \otimes \varphi_q(v)$ if $i = f-1$. Finally, we define the étale $(\varphi, \mathcal{O}_K^\times)$-module over A_∞:

$$D^{\otimes}_{A_\infty}(\overline{\rho}) \overset{\text{def}}{=} \bigotimes_{i=0}^{f-1} D^{(i)}_{A_\infty}(\overline{\rho}),$$

where the φ-action is given by $\varphi(v_0 \otimes \cdots \otimes v_{f-1}) \overset{\text{def}}{=} \phi_{f-1}(v_{f-1}) \otimes \phi_0(v_0) \otimes \cdots \otimes \phi_{f-2}(v_{f-2})$, and the $\mathcal{O}_K^\times$-action is the diagonal action.

By the equivalence of categories in Proposition 5.5.1, up to isomorphism there are unique étale $(\varphi_q, \mathcal{O}_K^\times)$-modules $D^{(i)}_A(\overline{\rho})$ for $0 \le i \le f-1$ and $D^{\otimes}_A(\overline{\rho})$ over A such that

$$A_\infty \otimes_A D^{(i)}_A(\overline{\rho}) \cong D^{(i)}_{A_\infty}(\overline{\rho});$$
$$A_\infty \otimes_A D^{\otimes}_A(\overline{\rho}) \cong D^{\otimes}_{A_\infty}(\overline{\rho}).$$

Lemma 5.5.2. *There exists a unique element $u \in T_{K,0}(1 + (A'_\infty)^{\circ\circ}) \subseteq A'_\infty$ such that:*

(i) $u^{q-1} = X_0^{\varphi-1} \in A \subseteq A_\infty \subseteq A'_\infty$;
(ii) *for any $(a_0, \dots, a_{f-1}) \in \Delta_1$, we have $(a_0, \dots, a_{f-1})(u) = \overline{a_0} u$, hence*

$$(a_0, \dots, a_{f-1})\big(uT_{K,0}^{-1}\big) = f_{a_0}^{\mathrm{LT}} uT_{K,0}^{-1};$$

(iii) *for any $a \in \mathcal{O}_K^\times$, we have $(a, 1, \dots, 1)(u) = \overline{a} f_{a,0}^{(1-\varphi)/(q-1)} u$, hence*

$$(a, 1, \dots, 1)\big(uT_{K,0}^{-1}\big) = f_a^{\mathrm{LT}} f_{a,0}^{(1-\varphi)/(q-1)} uT_{K,0}^{-1};$$

(iv) $\varphi_q(u) = u^q$.

Proof. (i),(ii),(iii) follow from [BHHMS3, Lemma 2.9.2] and (iv) follows from [BHHMS3, Remark 2.9.4]. □

Lemma 5.5.3. *There is a unique multiplicative norm $|\cdot|$ on A'_∞ inducing the topology of A'_∞ such that $|T_{K,0}| = p^{-1}$. It also satisfies:*

(i) $|T_{K,i}| = p^{-p^i}$ *for all* $0 \le i \le f-1$;
(ii) $|\varphi(x)| = |x|^p \ \forall\, x \in A'_\infty$;
(iii) *for any $(a_0, \dots, a_{f-1}) \in (\mathcal{O}_K^\times)^f$, we have $|(a_0, \dots, a_{f-1})(x)| = |x| \ \forall\, x \in A'_\infty$;*
(iv) $|X_i| = |Y_i| = p^{-(1+p+\cdots+p^{f-1})}$ *for all* $0 \le i \le f-1$. *In particular, for any $x \in F_{1-p}A$, we have $|x| \le p^{-(q-1)}$.*

Proof. Recall that the desired norm on A'_∞ is the unique multiplicative extension to A'_∞ of the Gauss norm on the ring $\mathbb{F}((T_{K,0}))\langle T_{K,i}/T_{K,0}^{p^i}, 1 \leq i \leq f-1\rangle$ with $T_{K,0}$-adic topology such that $|T_{K,0}| = p^{-1}$ (see [BHH$^+$c, Lemma 2.4.7(iii)] and the proof of [BHH$^+$c, Lemma 2.4.2(iii)]). In particular, for $0 \leq i \leq f-1$ we have $|T_{K,i}| = |T_{K,i}/T_{K,0}^{p^i}| \cdot |T_{K,0}|^{p^i} = p^{-p^i}$, which proves (i).

The assignment $\|x\| \overset{\text{def}}{=} |\varphi(x)|$ is a multiplicative norm on A'_∞ inducing the topology of A'_∞ such that $\|T_{K,0}\| = p^{-p}$. By uniqueness we get $|\varphi(x)| = |x|^p \ \forall\, x \in A'_\infty$, which proves (ii). Similarly, for any $(a_0, \ldots, a_{f-1}) \in (\mathcal{O}_K^\times)^f$, the assignment $\|x\|' \overset{\text{def}}{=} |(a_0, \ldots, a_{f-1})(x)|$ is a multiplicative norm on A'_∞ inducing the topology of A'_∞ such that $\|T_{K,0}\| = p^{-1}$. By uniqueness we get $|(a_0, \ldots, a_{f-1})(x)| = |x| \ \forall\, x \in A'_\infty$, which proves (iii).

Then we prove (iv). Recall from [BHH$^+$c, (63)] that we have $X_0 = T_{K,0} \cdots T_{K,f-1}(1 + w_0)$ for some $|w_0| < 1$. Then we deduce from (i) that $|X_0| = |T_{K,0} \cdots T_{K,f-1}| = p^{-(1+p+\cdots+p^{f-1})}$. By the proof of [BHH$^+$c, Lemma 2.4.2(iii)], we have $|X_i| = |X_0| = p^{-(1+p+\cdots+p^{f-1})}$ for $1 \leq i \leq f-1$. Finally, we deduce from (5.21) that $|Y_i| = |X_i| = p^{-(1+p+\cdots+p^{f-1})}$ for $0 \leq i \leq f-1$. □

For $r \in \mathbb{R}_{>0}$, we denote $B(r) \overset{\text{def}}{=} \{x \in A'_\infty : |x| \leq p^{-r}\}$ and $B^\circ(r) \overset{\text{def}}{=} \{x \in A'_\infty : |x| < p^{-r}\}$.

Lemma 5.5.4. *We have the following relations in A'_∞.*

(i) *We have*

$$\begin{aligned} X_1^{1-\varphi} &\in \sum_{i=0}^{f-1} T_{K,i}^{-p(1-q^{-1})} - T_{K,0}^{q-1}\left(\sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})}\right) + B\Big(\tfrac{(q-1)(2p-2)}{p}\Big) \\ &\subseteq \sum_{i=0}^{f-1} T_{K,i}^{-p(1-q^{-1})} + B\Big(\tfrac{(q-1)(p-1)}{p}\Big) \subseteq T_{K,f-1}^{-p(1-q^{-1})}\left[1 + B\Big(\tfrac{(q-1)(p-1)}{p}\Big)\right]. \end{aligned}$$

(ii) *Let $u \in A'_\infty$ be as in Lemma 5.5.2, then we have*

$$uT_{K,0}^{-1} \in 1 + T_{K,0}^{q-1}\left(\sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})}\right) + B\Big(\tfrac{(q-1)(2p-1)}{p}\Big) \subseteq 1 + B\Big(\tfrac{(q-1)(p-1)}{p}\Big).$$

Proof. Recall from the proof of [BHH$^+$c, Lemma 2.9.2] (especially the second formula before [BHH$^+$c, (63)]) that the element

$$\sum_{n=0}^{\infty}[x_n]p^n \overset{\text{def}}{=} \prod_{i=0}^{f-1}\sum_{n\geq 0}[T_{K,i}^{q^{-n}}]p^n - \sum_{n\geq 0}\sum_{i=0}^{f-1}[X_i^{-nf-i}]p^{nf+i} \in W((A'_\infty)^\circ) \tag{5.37}$$

satisfies $|x_i| < p^{-c}$ for all $i \geq 0$, and the proof of *loc.cit.* shows that we can take $c = q-1$. In particular, we have

$$|x_0| = |T_{K,0} \cdots T_{K,f-1} - X_0| < p^{-c},$$

hence

$$X_0 \in T_{K,0} \cdots T_{K,f-1}\left[1 + B^\circ\Big(c - (1+p+\cdots+p^{f-1})\Big)\right]. \tag{5.38}$$

By a direct computation in the ring of Witt vectors, we have from (5.37)

$$|x_1| = \left|\sum_{i=0}^{f-1} T_{K,0} \cdots T_{K,i}^{q^{-1}} \cdots T_{K,f-1} - X_1^{p^{-1}} - \sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}(T_{K,0} \cdots T_{K,f-1})^{(p-s)/p}(-X_0)^{s/p}\right| < p^{-c},$$

hence

$$X_1^{p^{-1}} \in T_{K,0}\cdots T_{K,f-1}\left[\sum_{i=0}^{f-1} T_{K,i}^{-(1-q^{-1})} - \sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}(-1)^s\left[1+\left(\tfrac{X_0}{T_{K,0}\cdots T_{K,f-1}}-1\right)\right]^{s/p}\right] + B^\circ(c) \tag{5.39}$$

$$\subseteq T_{K,0}\cdots T_{K,f-1}\left[\sum_{i=0}^{f-1} T_{K,i}^{-(1-q^{-1})} - \sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}(-1)^s\left[1+B^\circ(c')\right]^s\right]$$

$$\subseteq T_{K,0}\cdots T_{K,f-1}\left[\sum_{i=0}^{f-1} T_{K,i}^{-(1-q^{-1})} + B^\circ(c')\right] \tag{5.40}$$

with $c' \stackrel{\text{def}}{=} \big(c-(1+p+\cdots+p^{f-1})\big)/p$, where the second inclusion follows from (5.38), and the last inclusion uses $\sum_{s=1}^{p-1} p^{-1}\binom{p}{s}(-1)^s = 0$ (since $p\geq 3$ is odd). Applying φ to (5.40) using (5.36) and Lemma 5.5.3(i),(ii), we get

$$X_0 \in T_{K,0}\cdots T_{K,f-1}\left[1+T_{K,0}^{q-1}\left(\sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})}\right) + B^\circ(pc'+q-1)\right]. \tag{5.41}$$

Then we put (5.41) into (5.39). Since $c > 1+p+\cdots+p^{f-1}+c'$, we get

$$X_1^{p^{-1}} \in T_{K,0}\cdots T_{K,f-1}\left[\sum_{i=0}^{f-1} T_{K,i}^{-(1-q^{-1})} - \sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}(-1)^s\left(1+\sum_{i=1}^{f-1}\frac{T_{K,0}^{(q-1)/p}}{T_{K,i}^{(1-q^{-1})/p}} + B^\circ\Big(c'+\tfrac{q-1}{p}\Big)\right)^s\right]$$

$$\subseteq T_{K,0}\cdots T_{K,f-1}\left[\sum_{i=0}^{f-1} T_{K,i}^{-(1-q^{-1})} - \sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}(-1)^s\left(1+s\sum_{i=1}^{f-1}\frac{T_{K,0}^{(q-1)/p}}{T_{K,i}^{(1-q^{-1})/p}} + B\Big(\tfrac{(q-1)(2p-2)}{p^2}\Big)\right)\right],$$

where the last inclusion uses $(q-1)(2p-2)/p^2 < c' + (q-1)/p$. Using $\sum_{s=1}^{p-1} p^{-1}\binom{p}{s}(-1)^s = 0$ and $\sum_{s=1}^{p-1} p^{-1}\binom{p}{s}(-1)^s s = 1$ (since $p\geq 3$ is odd), we get

$$X_1^{p^{-1}} \in T_{K,0}\cdots T_{K,f-1}\left[\sum_{i=0}^{f-1} T_{K,i}^{-(1-q^{-1})} - T_{K,0}^{(q-1)/p}\left(\sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})/p}\right) + B\Big(\tfrac{(q-1)(2p-2)}{p^2}\Big)\right] \tag{5.42}$$

$$\subseteq T_{K,0}\cdots T_{K,f-1}\left[\sum_{i=0}^{f-1} T_{K,i}^{-(1-q^{-1})} + B\Big(\tfrac{(q-1)(p-1)}{p^2}\Big)\right] \tag{5.43}$$

$$\subseteq T_{K,0}\cdots T_{K,f-2}T_{K,f-1}^{q^{-1}}\left[1+B\Big(\tfrac{(q-1)(p-1)}{p^2}\Big)\right]. \tag{5.44}$$

Applying φ to (5.44) using (5.36) and Lemma 5.5.3(ii), we get

$$X_0 = \varphi(X_1^{p^{-1}}) \in T_{K,0}\cdots T_{K,f-1}\left[1+B\Big(\tfrac{(q-1)(p-1)}{p}\Big)\right]. \tag{5.45}$$

Dividing (5.42) by (5.45) and then raising to the p-th power, we get

$$X_1^{1-\varphi} \in \sum_{i=0}^{f-1} T_{K,i}^{-p(1-q^{-1})} - T_{K,0}^{q-1}\left(\sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})}\right) + B\Big(\tfrac{(q-1)(2p-2)}{p}\Big),$$

which proves (i).

Dividing (5.43) by (5.45) and then applying φ, we get

$$X_0^{1-\varphi} \in T_{K,0}^{-(q-1)} + \sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})} + B\Big(\tfrac{(q-1)(2p-1)}{p}\Big). \tag{5.46}$$

By the definition of u (see the lines below [BHH$^+$c, (64)]) and using (5.46), we get

$$\begin{aligned} uT_{K,0}^{-1} &\stackrel{\text{def}}{=} \left(X_0^{\varphi-1}/T_{K,0}^{q-1}\right)^{1/(q-1)} \in \left(X_0^{1-\varphi}T_{K,0}^{q-1}\right)^{1+q\mathbb{Z}_p} \\ &\subseteq 1 + T_{K,0}^{q-1}\left(\sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})}\right) + B\Big(\tfrac{(q-1)(2p-1)}{p}\Big), \end{aligned}$$

which proves (ii). □

Lemma 5.5.5. *We have the following equalities of operators on A'_∞:*

(i) *for $a \in \mathcal{O}_K^\times$ and $h \in \mathbb{Z}$, we have*

$$\left(T_{K,0}^{-(q-1)h}\varphi_q\right) \circ \left(\left(f_a^{\mathrm{LT}}\right)^h (a, 1, \dots, 1)\right) = \left(\left(f_a^{\mathrm{LT}}\right)^h (a, 1, \dots, 1)\right) \circ \left(T_{K,0}^{-(q-1)h}\varphi_q\right);$$

(ii) *for $(a_0, \dots, a_{f-1}) \in \Delta_1$ and $h \in \mathbb{Z}$, we have*

$$\left(T_{K,0}^{-(q-1)h}\varphi_q\right) \circ \left(\left(f_{a_0}^{\mathrm{LT}}\right)^h (a_0, \dots, a_{f-1})\right) = \left(\left(f_{a_0}^{\mathrm{LT}}\right)^h (a_0, \dots, a_{f-1})\right) \circ \left(T_{K,0}^{-(q-1)h}\varphi_q\right);$$

(iii) *for $h \in \mathbb{Z}$, we have*

$$\left(T_{K,0}^{-(q-1)h}\varphi_q\right) \circ \left(\left(uT_{K,0}^{-1}\right)^{-h}\right) = \left(uT_{K,0}^{-1}\right)^{-h} X_0^{h(1-\varphi)}\varphi_q.$$

Proof. All the equalities are direct calculations, (i) and (ii) using the definition of f_a^{LT}, and (iii) using Lemma 5.5.2(i),(iv). We omit the details. Here we recall that we identify $T_{K,\sigma_0} \in \mathbb{F}(\!(T_{K,\sigma_0})\!)$ with $T_{K,0} \in A'_\infty$ via the inclusion ι_0. □

Lemma 5.5.6. *Let $0 \le h \le q-2$ and $\lambda_0, \lambda_1 \in \mathbb{F}^\times$. Then for any $y \in A'_\infty$ with $|y| < p^{-h}$, the equation $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)(x) = y$ has a unique solution $x \in A'_\infty$ with $|x| < p^{-h}$, given by the convergent series $x = \sum_{n=0}^\infty \left(\lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)^n(y)$.*

Proof. For any $x \in A'_\infty$, we have (by Lemma 5.5.3(i),(ii)) $\left|\lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q(x)\right| = |x|^q p^{(q-1)h}$. In particular, if $|x| < p^{-h}$ and $x \ne 0$, then we have $\left|\lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q(x)\right| < |x|$. If $x_1, x_2 \in A'_\infty$ such that $|x_1|, |x_2| < p^{-h}$ and $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)(x_1) = \left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)(x_2)$, then we have $|x_1 - x_2| = \left|\lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q(x_1 - x_2)\right|$, which implies $x_1 = x_2$. This proves uniqueness. Then given $|y| < p^{-h}$, one easily checks that the element $x \stackrel{\text{def}}{=} \sum_{n=0}^\infty \left(\lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)^n(y)$ converges, and satisfies $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)(x) = y$ and $|x| = |y| < p^{-h}$. □

Definition 5.5.7. *Let $0 \le h \le q-2$ and $0 \le j \le f-1$. We define $H_j \in \mathbb{Z}$ as follows:*

(i) *If $h_{j-1} \ne p-1$, we define $H_j \stackrel{\text{def}}{=} 0$.*
(ii) *If $h_{j-1} = p-1$ and $h_j \ne 0$, we define $H_j \stackrel{\text{def}}{=} h_j$.*
(iii) *If $h_{j-1} = p-1$ and $h_j = 0$, we let $0 \le r \le f-1$ such that $h_{j+1} = \cdots = h_{j+r} = 1$ and $h_{j+r+1} \ne 1$, then we define $H_j \stackrel{\text{def}}{=} h_{j+r+1} - 1$.*

Definition 5.5.8. *Let $\overline{\rho}$ be as in (5.2). Suppose that (see Theorem 5.2.10)*

$$D_{K,\sigma_0}(\overline{\rho}) \cong D\left(\sum_{j=0}^{f-1} c_j[B_j^{\mathrm{LT}}] + c_{\mathrm{tr}}[B_{\mathrm{tr}}^{\mathrm{LT}}] + c_{\mathrm{un}}[B_{\mathrm{un}}^{\mathrm{LT}}]\right)$$

for some $c_0, \dots, c_{f-1}, c_{\mathrm{tr}}, c_{\mathrm{un}} \in \mathbb{F}$, then we define (see Proposition 5.4.7 for the notation)

$$D_{A,\sigma_0}(\overline{\rho}) \stackrel{\mathrm{def}}{=} D\left(\sum_{j=0}^{f-1} c_j\left([B_j^X] + H_j[B_{j-1}'^X]\right) + c_{\mathrm{tr}}[B_{\mathrm{tr}}^X] + c_{\mathrm{un}}[B_{\mathrm{un}}^X]\right),$$

where we use the convention that $[B_{-1}'^X] \stackrel{\mathrm{def}}{=} \lambda_0\lambda_1^{-1}[B_{f-1}'^X]$ in W^X. This is an étale $(\varphi_q, \mathcal{O}_K^\times)$-module of rank 2 over A and is well-defined up to isomorphism.

Lemma 5.5.9. *Let $0 \le h \le q-2$ and $0 \le j \le f-1$.*

(i) *If $h_{j-1} \ne p-1$, then we have $(q-1)\big((p-1)p^{j-1} - [h]_{j-1}\big) > h$.*
(ii) *We have $p^j - [h]_{j-1} - p^{j-f} > h$.*

Proof. (i). If $j \ge 1$, then using $h_{j-1} \ne p-1$ we have

$$(q-1)\big((p-1)p^{j-1} - [h]_{j-1}\big) \ge q-1 > h. \tag{5.47}$$

If $j = 0$, then using $h_{f-1} \ne p-1$ we have (since $[h]_{-1} = 0$)

$$(q-1)\big((p-1)p^{j-1} - [h]_{j-1}\big) = (q-1)(p-1)/p > (p-1)p^{f-1} - 1 \ge h. \tag{5.48}$$

(ii). If $[h]_{j-1} \ne (p-1)(1+p+\cdots+p^{j-1})$, then we have

$$(q-1)(p^j - [h]_{j-1} - p^{j-f}) > q-1 > h;$$

If $[h]_{j-1} = (p-1)(1+p+\cdots+p^{j-1})$, then we can't have $h_j = h_{j+1} = \cdots = h_{f-1} = p-1$ (otherwise $h = q-1$), so we get

$$(q-1)(p^j - [h]_{j-1} - p^{j-f}) \ge (q-1)(1-p^{j-f}) > q-1-p^j \ge h.$$

This completes the proof. □

Theorem 5.5.10. *Suppose that $p \ge 5$, then for $\overline{\rho}$ as in (5.2), we have an isomorphism of étale $(\varphi_q, \mathcal{O}_K^\times)$-modules over A:*

$$D_A^{(0)}(\overline{\rho}) \cong D_{A,\sigma_0}(\overline{\rho}).$$

Proof. By Proposition 5.5.1, it suffices to show that

$$A_\infty \otimes_A D_{A,\sigma_0}(\overline{\rho}) = \Big(A'_\infty \otimes_{\mathbb{F}((T_{K,\sigma_0}))} D_{K,\sigma_0}(\overline{\rho})\Big)^{\Delta_1}. \tag{5.49}$$

Let $e_0^{\mathrm{LT}}, e_1^{\mathrm{LT}}$ be an $\mathbb{F}((T_{K,\sigma_0}))$-basis of $D_{K,\sigma_0}(\overline{\rho})$ with respect to which the matrices of the actions of φ_q and $\mathcal{O}_K^\times$ have the form

$$\begin{cases} \mathrm{Mat}_K(\varphi_q) & = \begin{pmatrix} \lambda_0 T_{K,\sigma_0}^{-(q-1)h} & \lambda_1 D^{\mathrm{LT}} \\ 0 & \lambda_1 \end{pmatrix} \\ \mathrm{Mat}_K(a) & = \begin{pmatrix} \big(f_a^{\mathrm{LT}}\big)^h & E_a^{\mathrm{LT}} \\ 0 & 1 \end{pmatrix} \quad \forall\, a \in \mathcal{O}_K^\times, \end{cases}$$

where

$$\begin{cases} D^{\mathrm{LT}} & \overset{\mathrm{def}}{=} \sum\limits_{j=0}^{f-1} c_j D_j^{\mathrm{LT}} + c_{\mathrm{tr}} D_{\mathrm{tr}}^{\mathrm{LT}} + c_{\mathrm{un}} D_{\mathrm{un}}^{\mathrm{LT}} \\ E_a^{\mathrm{LT}} & \overset{\mathrm{def}}{=} \sum\limits_{j=0}^{f-1} c_j E_{j,a}^{\mathrm{LT}} + c_{\mathrm{tr}} E_{\mathrm{tr},a}^{\mathrm{LT}} + c_{\mathrm{un}} E_{\mathrm{un},a}^{\mathrm{LT}} \quad \forall\, a \in \mathcal{O}_K^{\times}. \end{cases}$$

Let e_0^X, e_1^X be an A-basis of $D_{A,\sigma_0}(\overline{\rho})$ with respect to which the matrices of the actions of φ_q and $\mathcal{O}_K^{\times}$ have the form

$$\begin{cases} \mathrm{Mat}_A(\varphi_q) & = \begin{pmatrix} \lambda_0 X_0^{h(1-\varphi)} & \lambda_1 D^X \\ 0 & \lambda_1 \end{pmatrix} \\ \mathrm{Mat}_A(a) & = \begin{pmatrix} f_{a,0}^{h(1-\varphi)/(1-q)} & E_a^X \\ 0 & 1 \end{pmatrix} \quad \forall\, a \in \mathcal{O}_K^{\times} \end{cases}$$

where

$$\begin{cases} D^X & \overset{\mathrm{def}}{=} \sum\limits_{j=0}^{f-1} c_j \left(D_j^X + H_j D_{j-1}'^X \right) + c_{\mathrm{tr}} D_{\mathrm{tr}}^X + c_{\mathrm{un}} D_{\mathrm{un}}^X \\ E_a^X & \overset{\mathrm{def}}{=} \sum\limits_{j=0}^{f-1} c_j \left(E_{j,a}^X + H_j E_{j-1,a}'^X \right) + c_{\mathrm{tr}} E_{\mathrm{tr},a}^X + c_{\mathrm{un}} E_{\mathrm{un},a}^X \quad \forall\, a \in \mathcal{O}_K^{\times}. \end{cases}$$

To prove (5.49), it is enough to find a change of basis formula $(e_0^X \ e_1^X) = (e_0^{\mathrm{LT}} \ e_1^{\mathrm{LT}})Q$ for some $Q = \left(\begin{smallmatrix} b_{00} & b_{01} \\ 0 & b_{11} \end{smallmatrix}\right) \in \mathrm{GL}_2(A'_\infty)$, such that

(i) $Q^{-1}\,\mathrm{Mat}_K(\varphi_q)\varphi_q(Q) = \mathrm{Mat}_A(\varphi_q)$;
(ii) $Q^{-1}\,\mathrm{Mat}_K(a)a(Q) = \mathrm{Mat}_A(a)\ \forall\, a \in \mathcal{O}_K^{\times}$;
(iii) the basis $(e_0^X \ e_1^X) = (e_0^{\mathrm{LT}} \ e_1^{\mathrm{LT}})Q$ is fixed by $(a_0, \ldots, a_{f-1})\ \forall\, (a_0, \ldots, a_{f-1}) \in \Delta_1$.

More concretely, we are going to solve the equation

$$\begin{pmatrix} b_{00}^{-1} & -b_{00}^{-1} b_{01} b_{11}^{-1} \\ 0 & b_{11}^{-1} \end{pmatrix} \begin{pmatrix} \lambda_0 T_{K,0}^{-(q-1)h} & \lambda_1 D^{\mathrm{LT}} \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} \varphi_q(b_{00}) & \varphi_q(b_{01}) \\ 0 & \varphi_q(b_{11}) \end{pmatrix} = \begin{pmatrix} \lambda_0 X_0^{h(1-\varphi)} & \lambda_1 D^X \\ 0 & \lambda_1 \end{pmatrix}, \tag{5.50}$$

and then check that the following equalities hold:

$$\begin{aligned} &\begin{pmatrix} b_{00}^{-1} & -b_{00}^{-1} b_{01} b_{11}^{-1} \\ 0 & b_{11}^{-1} \end{pmatrix} \begin{pmatrix} \left(f_a^{\mathrm{LT}}\right)^h & E_a^{\mathrm{LT}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (a,1,\ldots,1)(b_{00}) & (a,1,\ldots,1)(b_{01}) \\ 0 & (a,1,\ldots,1)(b_{11}) \end{pmatrix} \\ &\qquad = \begin{pmatrix} f_{a,0}^{h(1-\varphi)/(1-q)} & E_a^X \\ 0 & 1 \end{pmatrix} \ \forall\, a \in \mathcal{O}_K^{\times}; \end{aligned} \tag{5.51}$$

$$\begin{aligned} &\begin{pmatrix} b_{00}^{-1} & -b_{00}^{-1} b_{01} b_{11}^{-1} \\ 0 & b_{11}^{-1} \end{pmatrix} \begin{pmatrix} \left(f_{a_0}^{\mathrm{LT}}\right)^h & E_{a_0}^{\mathrm{LT}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (a_0,\ldots,a_{f-1})(b_{00}) & (a_0,\ldots,a_{f-1})(b_{01}) \\ 0 & (a_0,\ldots,a_{f-1})(b_{11}) \end{pmatrix} \\ &\qquad = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \forall\, (a_0,\ldots,a_{f-1}) \in \Delta_1. \end{aligned} \tag{5.52}$$

Comparing the (2,2)-entries of (5.50), we can take $b_{11} = 1$. Then the equalities of the (2,2)-entries of (5.51) and (5.52) are clear.

Comparing the (1,1)-entries of (5.50), we need to solve $\varphi_q(b_{00})b_{00}^{-1} = T_{K,0}^{(q-1)h} X_0^{h(1-\varphi)}$. By Lemma 5.5.2(i),(iv) we can take $b_{00} = \left(uT_{K,0}^{-1}\right)^{-h}$. Then the equalities of the (1,1)-entries of (5.51) and (5.52) follow directly from Lemma 5.5.2(ii),(iii).

Comparing the (1,2)-entries of (5.50), we need to solve

$$b_{00}^{-1}\lambda_0 T_{K,0}^{-(q-1)h}\varphi_q(b_{01}) + b_{00}^{-1}\lambda_1 D^{\mathrm{LT}}\varphi_q(b_{11}) - b_{00}^{-1}b_{01}b_{11}^{-1}\lambda_1\varphi_q(b_{11}) = \lambda_1 D^X.$$

Replacing b_{00}, b_{11} by their previous values, we get:

$$\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)(b_{01}) = D_{01} \stackrel{\mathrm{def}}{=} D^{\mathrm{LT}} - \left(uT_{K,0}^{-1}\right)^{-h} D^X. \tag{5.53}$$

Without loss of generality, we may assume that one of $c_0, \dots, c_{f-1}, c_{\mathrm{tr}}, c_{\mathrm{un}}$ is 1 and the others are 0. We separate the following cases:

Case 1: $c_j = 1$ for some $0 \le j \le f-1$, $h_j \ne 0$ and $h_{j-1} \ne p-1$.

By definition, we have

$$\begin{aligned} D_{01} &= D_j^{\mathrm{LT}} - \left(uT_{K,0}^{-1}\right)^{-h} D_j^X \\ &= T_{K,0}^{-(q-1)[h]_{j-1}} - \left(uT_{K,0}^{-1}\right)^{-h} X_0^{[h]_{j-1}(1-\varphi)} \\ &= T_{K,0}^{-(q-1)[h]_{j-1}} \left[1 - \left(uT_{K,0}^{-1}\right)^{-(h+(q-1)[h]_{j-1})}\right] \\ &\in T_{K,0}^{-(q-1)[h]_{j-1}} \left[1 - \left[1 + B\Big(\tfrac{(q-1)(p-1)}{p}\Big)\right]^{p^j\mathbb{Z}}\right] \\ &\subseteq T_{K,0}^{-(q-1)[h]_{j-1}} B\big((q-1)(p-1)p^{j-1}\big) \subseteq B^\circ(h), \end{aligned} \tag{5.54}$$

where the third equality uses Lemma 5.5.2(i), the first inclusion follows from Lemma 5.5.4(ii), and the last inclusion follows from Lemma 5.5.9(i). By Lemma 5.5.6, we take $b_{01} \in A'_\infty$ to be the unique solution of (5.53) satisfying $|b_{01}| < p^{-h}$.

Then we check the equality of the (1,2)-entries of (5.51) for the previous values of b_{00}, b_{01}, b_{11}, or equivalently (for $a \in \mathcal{O}_K^\times$)

$$\left(f_a^{\mathrm{LT}}\right)^h (a,1,\dots,1)(b_{01}) + E_a^{\mathrm{LT}} - b_{01} = \left(uT_{K,0}^{-1}\right)^{-h} E_a^X. \tag{5.55}$$

By Lemma 5.5.3(i),(iii),(iv) and $q-1 > h$, each term of (5.55) has norm $< p^{-h}$, hence by Lemma 5.5.6 it suffices to check the equality after applying the operator $\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)$. We have

$$\begin{aligned} &\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)\left(\left(f_a^{\mathrm{LT}}\right)^h (a,1,\dots,1)(b_{01})\right) \\ &\qquad = \left(f_a^{\mathrm{LT}}\right)^h (a,1,\dots,1)\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)(b_{01}) \qquad \text{(by Lemma 5.5.5(i))} \\ &\qquad = \left(f_a^{\mathrm{LT}}\right)^h (a,1,\dots,1)\left(D^{\mathrm{LT}} - \left(uT_{K,0}^{-1}\right)^{-h} D^X\right) \qquad \text{(by (5.53))} \\ &\qquad = \left(f_a^{\mathrm{LT}}\right)^h a(D^{\mathrm{LT}}) - f_{a,0}^{h(1-\varphi)/(1-q)}\left(uT_{K,0}^{-1}\right)^{-h} a(D^X); \qquad \text{(by Lemma 5.5.2(iii))} \\ &\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)\left(E_a^{\mathrm{LT}}\right) = D^{\mathrm{LT}} - \left(f_a^{\mathrm{LT}}\right)^h a(D^{\mathrm{LT}}); \qquad \text{(by Proposition 5.2.8(i))} \\ &\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)(b_{01}) = D^{\mathrm{LT}} - \left(uT_{K,0}^{-1}\right)^{-h} D^X; \qquad \text{(by (5.53))} \\ &\left(\mathrm{id} - \lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)\left(\left(uT_{K,0}^{-1}\right)^{-h} E_a^X\right) \\ &\qquad = \left(uT_{K,0}^{-1}\right)^{-h}\left(\mathrm{id} - \lambda_0\lambda_1^{-1}X_0^{h(1-\varphi)}\varphi_q\right)\left(E_a^X\right) \qquad \text{(by Lemma 5.5.5(iii))} \\ &\qquad = \left(uT_{K,0}^{-1}\right)^{-h}\left(D^X - f_{a,0}^{h(1-\varphi)/(1-q)} a(D^X)\right). \qquad \text{(by Proposition 5.4.7(i))} \end{aligned}$$

Hence the equality (5.55) holds.

Finally, we check the equality of the (1,2)-entries of (5.52) for the previous values of b_{00}, b_{01}, b_{11}, or equivalently (for $(a_0, \dots, a_{f-1}) \in \Delta_1$)

$$\left(f_{a_0}^{\mathrm{LT}}\right)^h (a_0, \dots, a_{f-1})(b_{01}) + E_{a_0} - b_{01} = 0. \tag{5.56}$$

By Lemma 5.5.3(i),(iii) and $q-1 > h$, each term of (5.56) has norm $< p^{-h}$, hence by Lemma 5.5.6 it suffices to check the equality after applying the operator $\left(\mathrm{id} - \lambda_0 \lambda_1^{-1} T_{K,0}^{-(q-1)h} \varphi_q\right)$. We have

$$\begin{aligned}
&\left(\mathrm{id} - \lambda_0 \lambda_1^{-1} T_{K,0}^{-(q-1)h} \varphi_q\right)\left(\left(f_{a_0}^{\mathrm{LT}}\right)^h (a_0, \dots, a_{f-1})(b_{01})\right) \\
&\quad = \left(f_{a_0}^{\mathrm{LT}}\right)^h (a_0, \dots, a_{f-1}) \left(D^{\mathrm{LT}} - \left(u T_{K,0}^{-1}\right)^{-h} D^X\right) \qquad \text{(by Lemma 5.5.5(ii))} \\
&\quad = \left(f_{a_0}^{\mathrm{LT}}\right)^h a_0(D^{\mathrm{LT}}) - \left(u T_{K,0}^{-1}\right)^{-h} D^X. \qquad \text{(by Lemma 5.5.2(ii))}
\end{aligned}$$

Here we recall that $D^X \in A$, hence is invariant under Δ_1. We also have

$$\begin{aligned}
&\left(\mathrm{id} - \lambda_0 \lambda_1^{-1} T_{K,0}^{-(q-1)h} \varphi_q\right)\left(E_{a_0}^{\mathrm{LT}}\right) = D^{\mathrm{LT}} - \left(f_{a_0}^{\mathrm{LT}}\right)^h a_0(D^{\mathrm{LT}}); \qquad \text{(by Proposition 5.2.8(i))} \\
&\left(\mathrm{id} - \lambda_0 \lambda_1^{-1} T_{K,0}^{-(q-1)h} \varphi_q\right)(b_{01}) = D^{\mathrm{LT}} - \left(u T_{K,0}^{-1}\right)^{-h} D^X. \qquad \text{(by (5.53))}
\end{aligned}$$

Hence the equality (5.56) holds.

In the remaining cases, we will prove that

$$D_{01} \in \left(\mathrm{id} - \lambda_0 \lambda_1^{-1} T_{K,0}^{-(q-1)h} \varphi_q\right)(b) + B^\circ(h) \tag{5.57}$$

for certain $b \in A'_\infty$. By Lemma 5.5.6 there is a unique choice of $b_{01} \in b + B^\circ(h) \subseteq A'_\infty$ satisfying (5.53). Then one can check the equalities of the (1,2)-entries of (5.51) and (5.52) as in Case 1.

Case 2: $c_j = 1$ for some $0 \le j \le f-1$, $h_j \ne 0$ and $h_{j-1} = p-1$.

We have

$$\begin{aligned}
\left(u T_{K,0}^{-1}\right)^{-h} X_0^{[h]_{j-1}(1-\varphi)} &= T_{K,0}^{-(q-1)[h]_{j-1}} \left(u T_{K,0}^{-1}\right)^{-(h+(q-1)[h]_{j-1})} \\
&\in T_{K,0}^{-(q-1)[h]_{j-1}} \left[1 + T_{K,0}^{q-1} \left(\sum_{i=1}^{f-1} T_{K,i}^{-(1-q^{-1})}\right) + B\left(\tfrac{(q-1)(2p-1)}{p}\right)\right]^{-p^j h_j + p^{j+1}\mathbb{Z}} \\
&\subseteq T_{K,0}^{-(q-1)[h]_{j-1}} \left[1 - h_j T_{K,0}^{(q-1)p^j} \left(\sum_{i=1}^{f-1} T_{K,i}^{-p^j(1-q^{-1})}\right) + B\big((q-1)(2p^j - 2p^{j-1})\big)\right] \\
&\subseteq T_{K,0}^{-(q-1)[h]_{j-1}} - h_j T_{K,0}^{(q-1)(p^j - [h]_{j-1})} \left(\sum_{i=1}^{f-1} T_{K,i}^{-p^j(1-q^{-1})}\right) + B^\circ(q-1),
\end{aligned} \tag{5.58}$$

where the first equality uses Lemma 5.5.2(i), the first inclusion follows from Lemma 5.5.4(ii), and the last inclusion uses $2p^j - 2p^{j-1} - [h]_{j-1} > 1$. We also have

$$\begin{aligned}
&\left(u T_{K,0}^{-1}\right)^{-h} X_0^{([h]_{j-2} - p^{j-1})(1-\varphi)} X_1^{p^{j-1}(1-\varphi)} \\
&\quad = T_{K,0}^{-(q-1)([h]_{j-2} - p^{j-1})} \left(u T_{K,0}^{-1}\right)^{-\left(h + (q-1)([h]_{j-2} - p^{j-1})\right)} X_1^{p^{j-1}(1-\varphi)}
\end{aligned}$$

$$\begin{aligned}
&\in T_{K,0}^{(q-1)(p^{j-1}-[h]_{j-2})}\Big[1+B\Big(\tfrac{(q-1)(p-1)}{p}\Big)\Big]^{p^j\mathbb{Z}}\left[\sum_{i=0}^{f-1}T_{K,i}^{-p(1-q^{-1})}+B\Big(\tfrac{(q-1)(p-1)}{p}\Big)\right]^{p^{j-1}}\\
&\subseteq T_{K,0}^{(q-1)(p^{j-1}-[h]_{j-2})}\left[\sum_{i=0}^{f-1}T_{K,i}^{-p^j(1-q^{-1})}+B\big((q-1)(p^{j-1}-p^{j-2})\big)\right]\\
&\subseteq T_{K,0}^{(q-1)(p^j-[h]_{j-1})}\left(\sum_{i=0}^{f-1}T_{K,i}^{-p^j(1-q^{-1})}\right)+B^\circ(q-1),
\end{aligned}\tag{5.59}$$

where the first equality uses Lemma 5.5.2(i), the first inclusion follows from Lemma 5.5.4(i),(ii), and the last inclusion uses $h_{j-1}=p-1$ (hence $p^{j-1}-[h]_{j-2}=p^j-[h]_{j-1}$, and $p^{j-1}-p^{j-2}+(p^{j-1}-[h]_{j-2})=(p^j-[h]_{j-1})+(p^{j-1}-p^{j-2})>1$). Combining (5.58) and (5.59), we get

$$\begin{aligned}
T_{K,0}^{-(q-1)[h]_{j-1}}-\big(uT_{K,0}^{-1}\big)^{-h}\left(X_0^{[h]_{j-1}}+h_jX_0^{([h]_{j-2}-p^{j-1})(1-\varphi)}X_1^{p^{j-1}(1-\varphi)}\right)\\
\in -h_jT_{K,0}^{(q-1)(p^j-[h]_{j-1}-p^{j-f})}+B^\circ(q-1)\subseteq B^\circ(h),
\end{aligned}\tag{5.60}$$

where the last inclusion follows from Lemma 5.5.9(ii) and $h<q-1$. In particular, for $j\geq 1$ we have $|D_{01}|<p^{-h}$, which proves (5.57) (with $b=0$).

Next we assume that $j=0$, so that $h_{f-1}=p-1$. Recall that $[B'^X_{-1}]\overset{\text{def}}{=}\lambda_0\lambda_1^{-1}[B'^X_{f-1}]$ in W^X. Then the difference of D_{01} and the LHS of (5.60) is

$$\begin{aligned}
&h_0\big(uT_{K,0}^{-1}\big)^{-h}\left[\lambda_0\lambda_1^{-1}X_0^{([h]_{f-2}-p^{f-1})(1-\varphi)}X_1^{p^{f-1}(1-\varphi)}-X_0^{-(1-\varphi)}X_1^{p^{-1}(1-\varphi)}\right]\\
&\quad=-h_0\big(uT_{K,0}^{-1}\big)^{-h}\left(\mathrm{id}-\lambda_0\lambda_1^{-1}X_0^{h(1-\varphi)}\varphi_q\right)\left[X_0^{-(1-\varphi)}X_1^{p^{-1}(1-\varphi)}\right]\\
&\quad=\left(\mathrm{id}-\lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\right)\left[-h_0\big(uT_{K,0}^{-1}\big)^{-h}X_0^{-(1-\varphi)}X_1^{p^{-1}(1-\varphi)}\right],
\end{aligned}$$

where the first equality uses $h_{f-1}=p-1$ (hence $[h]_{f-2}-p^{f-1}=h-q$), and the second equality uses Lemma 5.5.5(iii). This proves (5.57) (with $b=-h_0\big(uT_{K,0}^{-1}\big)^{-h}X_0^{-(1-\varphi)}X_1^{p^{-1}(1-\varphi)}$).

Case 3: $c_j=1$ for some $0\leq j\leq f-1$, $h_j=0$ and $h_{j-1}\neq p-1$.

Let $0\leq r\leq f-1$ such that $h_{j+1}=\cdots=h_{j+r}=1$ and $h_{j+r+1}\neq 1$. We have

$$\begin{aligned}
&\big(uT_{K,0}^{-1}\big)^{-h}X_0^{([h]_{j+r}+p^{j+r+1})(1-\varphi)}=T_{K,0}^{-(q-1)([h]_{j+r}+p^{j+r+1})}\big(uT_{K,0}^{-1}\big)^{-\left(h+(q-1)([h]_{j+r}+p^{j+r+1})\right)}\\
&\in T_{K,0}^{-(q-1)([h]_{j+r}+p^{j+r+1})}\left[1+T_{K,0}^{q-1}\left(\sum_{\ell=1}^{f-1}T_{K,\ell}^{-(1-q^{-1})}\right)+B\Big(\tfrac{(q-1)(2p-1)}{p}\Big)\right]^{p^{j+r+1}(1-h_{j+r+1})+p^{j+r+2}\mathbb{Z}}\\
&\subseteq T_{K,0}^{-(q-1)([h]_{j+r}+p^{j+r+1})}\left[1-(h_{j+r+1}-1)\sum_{\ell=1}^{f-1}\frac{T_{K,0}^{(q-1)p^{j+r+1}}}{T_{K,\ell}^{p^{j+r+1}(1-q^{-1})}}+B\big((q-1)(2p^{j+r+1}-2p^{j+r})\big)\right]\\
&\subseteq T_{K,0}^{-(q-1)([h]_{j+r}+p^{j+r+1})}-(h_{j+r+1}-1)T_{K,0}^{-(q-1)[h]_{j+r}}\left(\sum_{\ell=1}^{f-1}T_{K,\ell}^{-p^{j+r+1}(1-q^{-1})}\right)+B^\circ(q-1),
\end{aligned}\tag{5.61}$$

where the first equality uses Lemma 5.5.2(i), the first inclusion follows from Lemma 5.5.4(ii), and the last inclusion uses $h_{j+r}=1$ and $p\geq 5$ (hence $2p^{j+r+1}-2p^{j+r}-([h]_{j+r}+p^{j+r+1})>1$). For $0\leq i\leq r$, we have

$$\big(uT_{K,0}^{-1}\big)^{-h}X_0^{[h]_{j+i}(1-\varphi)}X_1^{p^{j+i}(1-\varphi)}=T_{K,0}^{-(q-1)[h]_{j+i}}\big(uT_{K,0}^{-1}\big)^{-(h+(q-1)[h]_{j+i})}X_1^{p^{j+i}(1-\varphi)}$$

$$\in T_{K,0}^{-(q-1)[h]_{j+i}}\Big[1+B\Big(\tfrac{(q-1)(p-1)}{p}\Big)\Big]^{p^{j+i+1}\mathbb{Z}}\Bigg[\sum_{\ell=0}^{f-1}T_{K,\ell}^{-p(1-q^{-1})}-\sum_{\ell=1}^{f-1}\tfrac{T_{K,0}^{q-1}}{T_{K,\ell}^{1-q^{-1}}}+B\Big(\tfrac{(q-1)(2p-2)}{p}\Big)\Bigg]^{p^{j+i}}$$
$$\subseteq T_{K,0}^{-(q-1)[h]_{j+i}}\Bigg[\sum_{\ell=0}^{f-1}T_{K,\ell}^{-p^{j+i+1}(1-q^{-1})}-\sum_{\ell=1}^{f-1}\tfrac{T_{K,0}^{(q-1)p^{j+i}}}{T_{K,\ell}^{p^{j+i}(1-q^{-1})}}+B\big((q-1)(2p^{j+i}-2p^{j+i-1})\big)\Bigg], \tag{5.62}$$

where the first equality uses Lemma 5.5.2(i), and the first inclusion follows from Lemma 5.5.4(i),(ii).

If $1\le i\le r$, then using $h_{j+i}=1$, $h_{j+i-1}\in\{0,1\}$ and $p\ge 5$ (hence $[h]_{j+i}-p^{j+i}=[h]_{j+i-1}$ and $2p^{j+i}-2p^{j+i-1}-[h]_{j+i}>1$) we deduce from (5.62) that

$$\big(uT_{K,0}^{-1}\big)^{-h}X_0^{[h]_{j+i}(1-\varphi)}X_1^{p^{j+i}(1-\varphi)}$$
$$\in T_{K,0}^{-(q-1)[h]_{j+i}}\Bigg(\sum_{\ell=0}^{f-1}T_{K,\ell}^{-p^{j+i+1}(1-q^{-1})}\Bigg)-T_{K,0}^{-(q-1)[h]_{j+i-1}}\Bigg(\sum_{\ell=1}^{f-1}T_{K,\ell}^{-p^{j+i}(1-q^{-1})}\Bigg)+B^\circ(q-1). \tag{5.63}$$

If $i=0$, then using $h_j=0$ (hence $[h]_j=[h]_{j-1}$ and $2p^j-2p^{j-1}-[h]_j>1$) we deduce from (5.62) that

$$\big(uT_{K,0}^{-1}\big)^{-h}X_0^{[h]_j(1-\varphi)}X_1^{p^j(1-\varphi)}$$
$$\in T_{K,0}^{-(q-1)[h]_j}\Bigg(\sum_{\ell=0}^{f-1}T_{K,\ell}^{-p^{j+1}(1-q^{-1})}\Bigg)-T_{K,0}^{(q-1)(p^j-[h]_{j-1})}\Bigg(\sum_{\ell=1}^{f-1}T_{K,\ell}^{-p^{j}(1-q^{-1})}\Bigg)+B^\circ(q-1). \tag{5.64}$$

Since $h_{j-1}\ne p-1$ by assumption, we deduce from (5.64), Lemma 5.5.3(i) and Lemma 5.5.9(i) that

$$\big(uT_{K,0}^{-1}\big)^{-h}X_0^{[h]_j(1-\varphi)}X_1^{p^j(1-\varphi)}\in T_{K,0}^{-(q-1)[h]_j}\Bigg(\sum_{\ell=0}^{f-1}T_{K,\ell}^{-p^{j+1}(1-q^{-1})}\Bigg)+B^\circ(h). \tag{5.65}$$

Combining (5.62), (5.63) (with $1\le i\le r$) and (5.65), we get

$$\big(uT_{K,0}^{-1}\big)^{-h}D_j^X\in D'+B^\circ(h) \tag{5.66}$$

with

$$D'\overset{\text{def}}{=}T_{K,0}^{-(q-1)([h]_{j+r}+p^{j+r+1})}+(h_{j+r+1}-1)\sum_{i=0}^{r}T_{K,0}^{-(q-1)([h]_{j+i}+p^{j+i+1-f})}. \tag{5.67}$$

By the definition of D_j^{LT}, we deduce from (5.66) that $D_{01}\in\big(\mathrm{id}-\lambda_0\lambda_1^{-1}T_{K,0}^{-(q-1)h}\varphi_q\big)(-D')+B^\circ(h)$, which proves (5.57).

Case 4: $c_j=1$ for some $0\le j\le f-1$, $h_j=0$ and $h_{j-1}=p-1$.

Let $0\le r\le f-1$ such that $h_{j+1}=\cdots=h_{j+r}=1$ and $h_{j+r+1}\ne 1$. For simplicity, we assume that $j\ge 1$. The case $j=0$ can be treated as in Case 2. Combining (5.62), (5.63) (with $1\le i\le r$), (5.64) and (5.59), we get (for D' as in (5.67))

$$\big(uT_{K,0}^{-1}\big)^{-h}\big(D_j^X+(h_{j+r+1}-1)D_j'^X\big)\in D'+T_{K,0}^{(q-1)(p^j-[h]_{j-1}-p^{j-f})}+B^\circ(q-1)\subseteq D'+B^\circ(h),$$

where the last inclusion follows from Lemma 5.5.9(ii) and $h<q-1$. This proves (5.57) (with $b=-D'$) as in Case 3.

Case 5: $h = 1 + p + \cdots + p^{f-1}$, $\lambda_0\lambda_1^{-1} = 1$ and $c_{\mathrm{tr}} = 1$.

Since $h_j = 1$ for all j, the relation (5.63) still holds for $j = 0$ and $0 \le i \le f-1$, from which we deduce that

$$\begin{aligned}
\big(uT_{K,0}^{-1}\big)^{-h} D_{\mathrm{tr}}^X &= \big(uT_{K,0}^{-1}\big)^{-h} \left(\sum_{i=0}^{f-1} X_0^{[h]_i(1-\varphi)} X_1^{p^i(1-\varphi)} \right) \\
&\in \sum_{i=0}^{f-1} \left[T_{K,0}^{-(q-1)[h]_i} \left(\sum_{\ell=0}^{f-1} T_{K,\ell}^{-p^{i+1}(1-q^{-1})} \right) - T_{K,0}^{-(q-1)[h]_{i-1}} \left(\sum_{\ell=1}^{f-1} T_{K,\ell}^{-p^i(1-q^{-1})} \right) \right] + B^\circ(h) \\
&= -\sum_{\ell=1}^{f-1} T_{K,\ell}^{-(1-q^{-1})} + \sum_{i=0}^{f-2} T_{K,0}^{-(q-1)([h]_i + p^{i+1-f})} + T_{K,0}^{-(q-1)h} \left(\sum_{\ell=0}^{f-1} T_{K,\ell}^{-p^f(1-q^{-1})} \right) + B^\circ(h) \\
&= \Big(\mathrm{id} - T_{K,0}^{-(q-1)h} \varphi_q \Big) \left[-\sum_{\ell=1}^{f-1} T_{K,\ell}^{-(1-q^{-1})} + \sum_{\ell=0}^{f-2} T_{K,0}^{-(q-1)([h]_i + p^{i+1-f})} \right] + D_{\mathrm{tr}}^{\mathrm{LT}} + B^\circ(h),
\end{aligned}$$

which proves (5.57).

Case 6: $h = 0$, $\lambda_0\lambda_1^{-1} = 1$ and $c_{\mathrm{un}} = 1$.

This case is easy, because we can simply take $Q = \left(\begin{smallmatrix} b_{00} & b_{01} \\ 0 & b_{11} \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. □

Remark 5.5.11. *By [BHH+c, Cor. 2.6.7], the functor $\overline{\rho} \mapsto D_A^{(0)}(\overline{\rho})$ is compatible with tensor products. Since we have $D_A^{(0)}\big(\omega_f^h \mathrm{un}(\lambda)\big) \cong D_{A,\sigma_0}\big(\omega_f^h \mathrm{un}(\lambda)\big)$ for all $h \in \mathbb{Z}$ and $\lambda \in \mathbb{F}^\times$ by [BHH+c, Thm. 2.9.5] and since any reducible 2-dimensional mod p representation of G_K is isomorphic to $\overline{\rho}$ as in (5.2) up to twist, we know $D_A^{(0)}(\overline{\rho})$ for all 2-dimensional mod p representations $\overline{\rho}$ of G_K (the irreducible case being treated in [BHH+c, Thm. 2.9.5]) when $p \ge 5$.*

As a corollary of Theorem 5.5.10, we give the comparison between the Lubin–Tate $(\varphi_q, \mathcal{O}_K^\times)$-modules and the cyclotomic $(\varphi_q, \mathbb{Z}_p^\times)$-modules corresponding to $\overline{\rho}$ as in (5.2).

Corollary 5.5.12. *Suppose that $p \ge 5$. Let $\overline{\rho}$ be as in (5.2). Suppose that (see Theorem 5.2.10)*

$$D_{K,\sigma_0}(\overline{\rho}) \cong D\left(\sum_{j=0}^{f-1} c_j [B_j^{\mathrm{LT}}] + c_{\mathrm{tr}}[B_{\mathrm{tr}}^{\mathrm{LT}}] + c_{\mathrm{un}}[B_{\mathrm{un}}^{\mathrm{LT}}] \right)$$

for some $c_0, \dots, c_{f-1}, c_{\mathrm{tr}}, c_{\mathrm{un}} \in \mathbb{F}$, then we have (see Definition 5.5.7 for H_j and see Theorem 5.3.9)

$$D_{\sigma_0}(\overline{\rho}) \cong D\left(\sum_{j=0}^{f-1} c_j \Big([B_{j-1}^{\mathrm{cyc}}] + H_j [B_{j-2}^{\mathrm{cyc}}] \Big) + c_{\mathrm{tr}}[B_{\mathrm{tr}}^{\mathrm{cyc}}] + c_{\mathrm{un}}[B_{\mathrm{un}}^{\mathrm{cyc}}] \right).$$

Here we use the convention that $[B_{-1}^{\mathrm{cyc}}] \overset{\mathrm{def}}{=} \lambda_0\lambda_1^{-1}[B_{f-1}^{\mathrm{cyc}}]$ and $[B_{-2}^{\mathrm{cyc}}] \overset{\mathrm{def}}{=} \lambda_0\lambda_1^{-1}[B_{f-2}^{\mathrm{cyc}}]$.

Proof. Recall from the proof of [BHH+c, Prop. 2.8.1] that the canonical inclusion

$$\mathbf{B}^+(R)^{\varphi=p} \hookrightarrow \mathbf{B}^+(R)^{\varphi_q=p^f}$$

for any perfectoid $\mathbb{F}$-algebra R induces a map $Z_{\mathbb{Z}_p} \to Z_{\mathcal{O}_K}$ of perfectoid spaces over $\mathbb{F}$, which is induced by the map

$$\mathrm{tr} : A_\infty \twoheadrightarrow \mathbb{F}(\!(T^{p^{-\infty}})\!)$$

coming from the trace map $\mathbb{F}[\![K]\!] \overset{\mathrm{tr}}{\twoheadrightarrow} \mathbb{F}[\![\mathbb{Q}_p]\!] \cong \mathbb{F}[\![T^{p^\infty}]\!]$, where T is the variable in §5.3. By the definition of T and X_i $(0 \le i \le f-1)$, we have the relation in $\mathbf{B}^+\big(\mathbb{F}(\!(T^{p^{-\infty}})\!)\big)$ which is analogous to [BHH$^+$c, (62)]:

$$\sum_{n\in\mathbb{Z}}[T^{p^{-n}}]p^n = \sum_{n\in\mathbb{Z}}\sum_{i=0}^{f-1}[\mathrm{tr}(X_i)^{p^{-nf-i}}]p^{nf+i}.$$

Hence we deduce that

$$\mathrm{tr}(X_i) = T \quad \forall\ 0 \le i \le f-1. \tag{5.68}$$

Suppose that

$$D_{K,\sigma_0}(\overline{\rho}) \cong D\left(\sum_{j=0}^{f-1} c_j[B_j^{\mathrm{LT}}] + c_{\mathrm{tr}}[B_{\mathrm{tr}}^{\mathrm{LT}}] + c_{\mathrm{un}}[B_{\mathrm{un}}^{\mathrm{LT}}]\right)$$

for some $c_0,\dots,c_{f-1},c_{\mathrm{tr}},c_{\mathrm{un}} \in \mathbb{F}$. By Theorem 5.5.10, we have

$$D_A^{(0)}(\overline{\rho}) \cong D\left(\sum_{j=0}^{f-1} c_j\left([B_j^X] + H_j[B_{j-1}'^X]\right) + c_{\mathrm{tr}}[B_{\mathrm{tr}}^X] + c_{\mathrm{un}}[B_{\mathrm{un}}^X]\right).$$

Then by [BHH$^+$c, Prop. 2.8.1], [BHH$^+$c, Remark 2.8.2] (comparing (5.4) and (5.15)) and (5.68), we deduce that

$$D_{\sigma_{f-1}}(\overline{\rho}) \cong D\left(\sum_{j=0}^{f-1} c_j\left([B_j^{\mathrm{cyc},h}] + H_j[B_{j-1}^{\mathrm{cyc},h}]\right) + c_{\mathrm{tr}}[B_{\mathrm{tr}}^{\mathrm{cyc}}] + c_{\mathrm{un}}[B_{\mathrm{un}}^{\mathrm{cyc}}]\right),$$

where $[B_j^{\mathrm{cyc},h}]$ is defined in the same way as $[B_j^{\mathrm{cyc}}]$, replacing h' by h for $-1 \le j \le f-1$. In particular, the corollary is true for $f=1$.

To prove the corollary for $f \ge 2$, without loss of generality we may assume that one of $c_0,\dots,c_{f-1},c_{\mathrm{tr}},c_{\mathrm{un}}$ is 1 and the others are 0. We separate the following cases:

Case 1: $c_j = 1$ for some $0 \le j \le f-1$.

Consider the following étale $(\varphi,\mathbb{Z}_p^\times)$-module over $\mathbb{F}\otimes_{\mathbb{F}_p}\mathbb{F}_q(\!(T)\!)$ $(a \in \mathbb{Z}_p^\times)$:

$$\begin{cases} D & = & \prod_{i=0}^{f-1} D_{\sigma_i} = \prod_{i=0}^{f-1}\left(\mathbb{F}(\!(T)\!)e_0^{(i)} \oplus \mathbb{F}(\!(T)\!)e_1^{(i)}\right) \\ \varphi(e_0^{(i+1)},e_1^{(i+1)}) & = & (e_0^{(i)},e_1^{(i)})\,\mathrm{Mat}(\varphi)^{(i)} \\ a(e_0^{(i)},e_1^{(i)}) & = & (e_0^{(i)},e_1^{(i)})\,\mathrm{Mat}(a)^{(i)} \end{cases}$$

(here we view i as an element of $\mathbb{Z}/f\mathbb{Z}$) with

$$\mathrm{Mat}(\varphi)^{(i)} = \begin{pmatrix} \alpha_i T^{-(p-1)h_{i+1}} & \beta_i d_i \\ 0 & \beta_i \end{pmatrix},$$

where

$$\alpha_i = \begin{cases} 1 & \text{if } i \ne f-2 \\ \lambda_0 & \text{if } i = f-2; \end{cases} \qquad \beta_i = \begin{cases} 1 & \text{if } i \ne f-2 \\ \lambda_1 & \text{if } i = f-2; \end{cases}$$

$$d_i = \begin{cases} 1 & \text{if } i \equiv j-1 \bmod f \text{ and } h_j \neq 0 \\ T^{-(p-1)(p+p^2+\cdots+p^{r+1})} + (h_{j+r+1}-1)\sum_{i=0}^{r} T^{-(p-1)((p+p^2+\cdots+p^i)+p^i)} & \\ & \text{if } i \equiv j-1 \bmod f,\ h_j = 0,\ h_{j+1} = \cdots = h_{j+r} = 1 \text{ and } h_{j+r+1} \neq 1 \\ H_j & \text{if } i \equiv j-2 \bmod f \\ 0 & \text{if } i \not\equiv j-1, j-2 \bmod f, \end{cases}$$

and the $\mathbb{Z}_p^\times$-action is the unique one which commutes with φ and satisfies $\mathrm{Mat}(a)^{(i)} \in I_2 + \mathrm{Mat}\left(T^{p-1}\mathbb{F}[\![T^{p-1}]\!]\right)$ for all i. Then by computing the actions $\varphi_q = \varphi \circ \cdots \circ \varphi$ (f-fold) on $D_{\sigma_{f-1}}$ and D_{σ_0} one can check that

$$D_{\sigma_{f-1}} \cong D\left([B_{j-1}^{\mathrm{cyc},h}] + H_j[B_{j-2}^{\mathrm{cyc},h}]\right) \cong D_{\sigma_{f-1}}(\overline{\rho});$$
$$D_{\sigma_0} \cong D\left([B_{j-1}^{\mathrm{cyc}}] + H_j[B_{j-2}^{\mathrm{cyc}}]\right).$$

as étale $(\varphi_q, \mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(T)\!)$, which completes the proof.

Case 2: $h = 1 + p + \cdots + p^{f-1}$, $\lambda_0\lambda_1^{-1} = 1$ and $c_{\mathrm{tr}} = 1$.
The proof is similar to Case 1 by taking $d_i = T^{-2(p-1)}$ for all $0 \leq i \leq f-1$.

Case 3: $h = 0$, $\lambda_0\lambda_1^{-1} = 1$ and $c_{\mathrm{un}} = 1$.
The proof is similar to Case 1 by taking $d_0 = 1$ and $d_i = 0$ for $1 \leq i \leq f-1$. □

5.6 The main theorem on $D_A(\pi)$

In this section, we recall the results of §4 on $D_A(\pi)$ and finish the proof of Theorem 5.1.1. To do this, we need to prove that certain constants appearing on $D_A(\pi)$ and on $D_A^\otimes(\overline{\rho})$ match, see Proposition 5.6.3.

We let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be of the following form:

$$\overline{\rho} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j}\mathrm{un}(\xi) & * \\ 0 & \mathrm{un}(\xi^{-1}) \end{pmatrix} \tag{5.69}$$

with $\xi \in \mathbb{F}^\times$, $0 \leq r_j \leq p-3$ for $0 \leq j \leq f-1$ and $r_j \neq 0$ for some j. Up to enlarging $\mathbb{F}$, we fix an f-th root $\sqrt[f]{\xi} \in \mathbb{F}^\times$ of ξ. By Theorem 5.2.10(iii) (with $h_j = r_j + 1$, $\lambda_0 = \xi$ and $\lambda_1 = \xi^{-1}$), the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-module $D_K(\overline{\rho})$ associated to $\overline{\rho}$ has the following form ($a \in \mathcal{O}_K^\times$):

$$\begin{cases} D_K(\overline{\rho}) & = \prod_{j=0}^{f-1} D_{K,\sigma_j}(\overline{\rho}) = \prod_{j=0}^{f-1}\left(\mathbb{F}(\!(T_{K,\sigma_j})\!)e_0^{(j)} \oplus \mathbb{F}(\!(T_{K,\sigma_j})\!)e_1^{(j)}\right) \\ \varphi(e_0^{(j+1)}\ e_1^{(j+1)}) & = (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(\varphi^{(j)}) \\ a(e_0^{(j)}\ e_1^{(j)}) & = (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(a^{(j)}), \end{cases} \tag{5.70}$$

where

$$\mathrm{Mat}(\varphi^{(j)}) = \begin{pmatrix} \sqrt[f]{\xi}\,T_{K,\sigma_j}^{-(q-1)(r_j+1)} & \sqrt[f]{\xi}^{-1}d_j \\ 0 & \sqrt[f]{\xi}^{-1} \end{pmatrix} \tag{5.71}$$

for some $d_j \in \mathbb{F}$ and $\mathrm{Mat}(a^{(j)}) \in I_2 + \mathrm{M}_2\left(T_{K,\sigma_j}^{q-1}\mathbb{F}[\![T_{K,\sigma_j}^{q-1}]\!]\right)$ which uniquely determines $\mathrm{Mat}(a^{(j)})$. By Theorem 5.5.10, Proposition 5.4.8 and the assumption on $\overline{\rho}$, the étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^\otimes(\overline{\rho})$

is obtained from $\bigotimes_{i=0}^{f-1} D_{K,\sigma_j}(\overline{\rho})$ by the recipe $T_{K,\sigma_j}^{q-1} \mapsto \varphi(Y_j)/Y_j$. Hence, if we consider the A-basis $\{e_J \overset{\text{def}}{=} \bigotimes_{j=0}^{f-1} e_{\delta_{j\in J}}^{(j)}\}_{J\subseteq\mathcal{J}}$ for $D_A^{\otimes}(\overline{\rho})$, the corresponding matrix $\mathrm{Mat}(\varphi) \in \mathrm{GL}_{2^f}(A)$ (with its rows and columns indexed by the subsets of $\mathcal{J}$) for the φ-action is given by

$$\mathrm{Mat}(\varphi)_{J',J+1} = \begin{cases} \nu_{J+1,J'} \prod_{j\notin J} Y_j^{(r_j+1)(1-\varphi)} & \text{if } J' \subseteq J \\ 0 & \text{if } J' \not\subseteq J, \end{cases} \tag{5.72}$$

where $\nu_{J,J'} \overset{\text{def}}{=} \sqrt[f]{\xi}^{|J^c|-|J|} \prod_{j\in(J-1)\setminus J'} d_j$ for $J' \subseteq J-1$. Also, the corresponding matrices for the $\mathcal{O}_K^\times$-action satisfy $\mathrm{Mat}(a) \in I_{2^f} + \mathrm{M}_{2^f}(F_{1-p}A)$ for all $a \in \mathcal{O}_K^\times$.

We also describe the Fontaine–Laffaille module associated to $\overline{\rho}$ (see [FL82]).

Lemma 5.6.1. *The Fontaine–Laffaille module $FL(\overline{\rho})$ associated to $\overline{\rho}$ has the following form:*

$$\begin{cases} FL(\overline{\rho}) & = & \prod_{j=0}^{f-1} FL_{\sigma_j}(\overline{\rho}) = \prod_{j=0}^{f-1} \left(\mathbb{F}e_0^{(j)} \oplus \mathbb{F}e_1^{(j)}\right) \\ \mathrm{Fil}^{r_j+1} FL_{\sigma_j}(\overline{\rho}) & = & \mathbb{F}e_0^{(j)} \\ \varphi_{r_{j+1}+1}(e_0^{(j+1)}) & = & \sqrt[f]{\xi}^{-1}(e_0^{(j)} - d_{j+1}e_1^{(j)}) \\ \varphi(e_1^{(j+1)}) & = & \sqrt[f]{\xi}\, e_1^{(j)}, \end{cases} \tag{5.73}$$

where $d_j \in \mathbb{F}^\times$ is as in (5.71).

Proof. Let T be the variable in §5.3. In particular, the uniformizer is p, hence $\mathbb{Z}_p[\![T]\!] = \mathbb{Z}_p[\![X]\!]$ where X is the usual variable corresponding to the formal group law $(1+X)^p - 1$. By Corollary 5.5.12 and (5.70), the cyclotomic $(\varphi, \mathbb{Z}_p^\times)$-module $D(\overline{\rho})$ associated to $\overline{\rho}$ has the following form ($a \in \mathbb{Z}_p^\times$):

$$\begin{cases} D(\overline{\rho}) & = & \prod_{j=0}^{f-1} \left(\mathbb{F}(\!(T_{\sigma_j})\!)e_0^{(j)} \oplus \mathbb{F}(\!(T_{\sigma_j})\!)e_1^{(j)}\right) \\ \varphi(e_0^{(j+1)}\ e_1^{(j+1)}) & = & (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(\varphi^{(j)}) \\ a(e_0^{(j)}\ e_1^{(j)}) & = & (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(a^{(j)}), \end{cases}$$

where

$$\mathrm{Mat}(\varphi^{(j)}) = \begin{pmatrix} \sqrt[f]{\xi}\, T_{\sigma_j}^{-(p-1)(r_{j+1}+1)} & \sqrt[f]{\xi}^{-1} d_{j+1} \\ 0 & \sqrt[f]{\xi}^{-1} \end{pmatrix}$$

for the same d_j as in (5.71) and $\mathrm{Mat}(a^{(j)}) \in I_2 + \mathrm{M}_2(T_{\sigma_j}^{p-1}\mathbb{F}[\![T_{\sigma_j}^{p-1}]\!])$ which uniquely determines $\mathrm{Mat}(a^{(j)})$.

Let $Q \overset{\text{def}}{=} \varphi(T)/T \in T^{p-1} + p(1+T\mathbb{Z}_p[\![T]\!])$, where φ acts on $\mathbb{Z}_p[\![T]\!]$ as p_{cyc}. Since $a_{\mathrm{cyc}}(T) = aT$ for $a \in [\mathbb{F}_p^\times]$, the commutativity of the action of $a \in \mathbb{Z}_p^\times$ with $[\mathbb{F}_p^\times]$ implies that $a_{\mathrm{cyc}}(T) \in aT\left(1 + T^{p-1}\mathbb{Z}_p[\![T^{p-1}]\!]\right)$. Then we let

$$\Lambda_a \overset{\text{def}}{=} \prod_{i\geq 0} \varphi^{1+if}(Q/a_{\mathrm{cyc}}(Q)) \in 1 + T^{p-1}\mathbb{Z}_p[\![T^{p-1}]\!].$$

We construct a Wach module (see e.g. [CD11, §2.4]) over $W(\mathbb{F})\otimes_{\mathbb{Z}_p}\mathcal{O}_K[\![T]\!]$ of the form ($a \in \mathbb{Z}_p^\times$):

$$\begin{cases} M & = & \prod_{j=0}^{f-1} M^{(j)} = \prod_{j=0}^{f-1} \left(W(\mathbb{F})[\![T]\!]e_0^{(j)} \oplus W(\mathbb{F})[\![T]\!]e_1^{(j)}\right) \\ \varphi(e_0^{(j+1)}\ e_1^{(j+1)}) & = & (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(\varphi^{(j)}) \\ a(e_0^{(j)}\ e_1^{(j)}) & = & (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(a^{(j)}) \end{cases}$$

with

$$\begin{cases} \mathrm{Mat}(\varphi^{(j)}) = \begin{pmatrix} [\sqrt[f]{\xi}]^{-1}Q^{r_{j+1}+1} & 0 \\ [\sqrt[f]{\xi}]^{-1}[d_{j+1}]Q^{r_{j+1}+1} & [\sqrt[f]{\xi}] \end{pmatrix} \\ \mathrm{Mat}(a^{(j)}) = \begin{pmatrix} P_a^{(j)} & 0 \\ P_a^{(j)}E_a^{(j)} & 1 \end{pmatrix}, \end{cases}$$

where $P_a^{(j)} \overset{\text{def}}{=} \prod_{i=0}^{f-1} \varphi^i(\Lambda_a)^{r_{i+j+1}+1} \in 1+T^{p-1}\mathbb{Z}_p[\![T^{p-1}]\!]$, and $E_a^{(j)} \in T^{p-1}\mathbb{Z}_p[\![T^{p-1}]\!]$ is the unique solution for the system of equations ($j \in \mathcal{J}$)

$$E_a^{(j)} - [\sqrt[f]{\xi}]^2 Q^{-(r_{j+1}+1)}\varphi(E_a^{(j+1)}) = [d_{j+1}]\left((P_a^{(j)})^{-1} - 1\right).$$

To prove uniqueness, up to dividing p we may assume that $p \nmid (E_a^{(j)} - E_a^{\prime(j)})$ for some j, then we reduce modulo p and compare the degrees in T. The existence of the solution follows as in the proof of Lemma 4.10.3(iv). Then one can check that M is a Wach module over $W(\mathbb{F})\otimes_{\mathbb{Z}_p}\mathcal{O}_K[\![T]\!]$ such that $M \otimes_{\mathbb{Z}_p[\![T]\!]} \mathbb{F}((T))$ is the dual étale $(\varphi, \mathcal{O}_K^\times)$-module of $D(\overline{\rho})$.

We give M a filtration defined by

$$\mathrm{Fil}^i M \overset{\text{def}}{=} \{x \in M : \varphi(x) \in Q^i M\}.$$

Then for $f(T), g(T) \in \mathbb{Z}_p[\![T]\!]$, we have

$$\begin{aligned} &f(T)e_0^{(j)} + g(T)e_1^{(j)} \in \mathrm{Fil}^i M^{(j)} \iff \\ &\varphi(f(T))\left([\sqrt[f]{\xi}]^{-1}Q^{r_j+1}e_0^{(j-1)} - [\sqrt[f]{\xi}]^{-1}[d_{j+1}]Q^{r_j+1}e_1^{(j-1)}\right) + \varphi(g(T))[\sqrt[f]{\xi}]e_1^{(j-1)} \in Q^i M^{(j-1)}. \end{aligned}$$

If $i \le 0$, this is automatic. If $1 \le i \le r_j + 1$, then we need $Q^i | \varphi(g(T))$, which is equivalent to $T^i | g(T)$. If $i > r_j + 1$, then we need $Q^{i-(r_j+1)} | \varphi(f(T))$ and $Q^i | \varphi(g(T))$, which is equivalent to $T^{i-(r_j+1)} | f(T)$ and $T^i | g(T)$. To summarize, we have

$$\mathrm{Fil}^i M^{(j)} = \begin{cases} W(\mathbb{F})[\![T]\!]e_0^{(j)} \oplus W(\mathbb{F})[\![T]\!]e_1^{(j)} & \text{if } i \ge 0 \\ W(\mathbb{F})[\![T]\!]e_0^{(j)} \oplus T^i W(\mathbb{F})[\![T]\!]e_1^{(j)} & \text{if } 1 \le i \le r_j + 1 \\ T^{i-(r_j+1)}W(\mathbb{F})[\![T]\!]e_0^{(j)} \oplus T^i W(\mathbb{F})[\![T]\!]e_1^{(j)} & \text{if } i > r_j + 1. \end{cases}$$

Then the "module filtré" over $W(\mathbb{F})$ associated to M in [Wac97, Thm. 3] is of the form:

$$\begin{cases} M/TM &= \prod_{j=0}^{f-1}\left(W(\mathbb{F})e_0^{(j)} \oplus W(\mathbb{F})e_1^{(j)}\right) \\ \mathrm{Fil}^{r_j+1}(M^{(j)}/TM^{(j)}) &= \mathbb{F}e_0^{(j)} \\ \varphi_{r_{j+1}+1}(e_0^{(j+1)}) &= [\sqrt[f]{\xi}]^{-1}(e_0^{(j)} - [d_{j+1}]e_1^{(j)}) \\ \varphi(e_1^{(j+1)}) &= [\sqrt[f]{\xi}]\, e_1^{(j)}. \end{cases}$$

Its reduction modulo p is the Fontaine–Laffaille module in (5.73), which is also the Fontaine–Laffaille module of $\overline{\rho}$ by [Wac97, Thm. 1']. This completes the proof. □

Then we recall some results on $D_A(\pi)$ following §4. Keep the notation of §5.1. We let π be as in (5.1) with $\overline{r}$ satisfying the assumptions (i)-(v) above Theorem 5.1.1. By [DL21, Thm. 1.1] we have $\pi^{K_1} = D_0(\overline{r}_v^\vee)$ as $K^\times \mathrm{GL}_2(\mathcal{O}_K)$-representations, where $D_0(\overline{r}_v^\vee)$ is the representation of $\mathrm{GL}_2(\mathbb{F}_q)$ defined in [BP12, §13] and is viewed as a representation of $\mathrm{GL}_2(\mathcal{O}_K)$ by inflation, and

$K^\times$ acts on $D_0(\overline{r}_v^\vee)$ by the character $\det(\overline{r}_v^\vee)\omega^{-1}$, where ω is the mod p cyclotomic character. Since $12 \le r_j \le p-15$ for all j, the proof of Theorem 3.6.3.1(i) shows that π satisfies (i),(ii),(iii) of Theorem 3.5.3, hence satisfies the conditions (a),(b),(c) of [BHH$^+$23, §6.4]. By [BHH$^+$23, Prop. 6.4.6] we deduce that $[\pi[\mathfrak{m}_{I_1}^3] : \chi] = 1$ for any character $\chi : I \to \mathbb{F}^\times$ appearing in π^{I_1}, where $\mathfrak{m}_{I_1}$ is the maximal ideal of $\mathbb{F}[\![I_1]\!]$, $\pi[\mathfrak{m}_{I_1}^3]$ is the set of elements of π annihilated by $\mathfrak{m}_{I_1}^3$, and $[\pi[\mathfrak{m}_{I_1}^3] : \chi]$ is the multiplicity of χ in the semisimplification of $\pi[\mathfrak{m}_{I_1}^3]$ as I-representations. In particular, π satisfies the conditions (i),(ii) above Theorem 4.1.1 with $\overline{\rho} = \overline{r}_v^\vee$. Twisting $\overline{\rho}$ and π using [BHH$^+$c, Lemma 2.9.7] and [BHH$^+$c, Lemma 3.1.1], we may assume that $\overline{\rho}$ is as in (5.69) with $\max\{12, 2f+1\} \le r_j \le p - \max\{15, 2f+3\}$ for all j. In particular, p acts trivially on π.

From now on, we assume that $|W(\overline{\rho})| = 1$, which is equivalent to $J_{\overline{\rho}} = \emptyset$ by [Bre14, Prop. A.3], where $J_{\overline{\rho}} \subseteq \mathcal{J}$ is the subset defined in [Bre14, (17)]. In particular, by [Bre14, (18)] with $e^j = e_1^{(f-j)}$, $f^j = e_0^{(f-j)}$, $\alpha_j = \sqrt[f]{\xi}$, $\beta_j = \sqrt[f]{\xi}^{-1}$ and $\mu_j = d_{f+1-j}$ for all $j \in \mathcal{J}$ in [Bre14, (16)], we deduce that $d_j \in \mathbb{F}^\times$ for all $j \in \mathcal{J}$ (see (5.71) for d_j). We denote $\sigma_\emptyset \overset{\text{def}}{=} \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$.

We write $\underline{i}$ for an element $(i_0, \dots, i_{f-1}) \in \mathbb{Z}^f$, and we write $\underline{Y}^{\underline{i}}$ for $\prod_{j=0}^{f-1} Y_j^{i_j} \in A$. For $J \subseteq \mathcal{J}$, we define $\underline{e}^J \in \mathbb{Z}^f$ by $e_j^J \overset{\text{def}}{=} \delta_{j\in J}$. We say that $\underline{i} \le \underline{i}'$ if $i_j \le i_j'$ for all j. For each $J \subseteq \mathcal{J}$, we define $\underline{s}^J, \underline{r}^J \in \mathbb{Z}^f$ by

$$s_j^J \overset{\text{def}}{=} \begin{cases} r_j, & \text{if } j \notin J,\ j+1 \notin J \\ r_j + 1, & \text{if } j \in J,\ j+1 \notin J \\ p-2-r_j, & \text{if } j \notin J,\ j+1 \in J \\ p-1-r_j, & \text{if } j \in J,\ j+1 \in J; \end{cases} \tag{5.74}$$

$$r_j^J \overset{\text{def}}{=} \begin{cases} 0, & \text{if } j \notin J,\ j+1 \notin J \\ -1, & \text{if } j \in J,\ j+1 \notin J \\ r_j + 1, & \text{if } j \notin J,\ j+1 \in J \\ r_j, & \text{if } j \in J,\ j+1 \in J. \end{cases} \tag{5.75}$$

We define the character $\chi_J : I \to \mathbb{F}^\times$ by $\left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \mapsto (\overline{a})^{\underline{s}^J + \underline{r}^J}(\overline{d})^{\underline{r}^J}$. Here, for $x \in \mathbb{F}$ and $\underline{i} \in \mathbb{Z}^f$ we define $x^{\underline{i}} \overset{\text{def}}{=} x^{\sum_{j=0}^{f-1} i_j p^j}$. We identify π^{K_1} with $D_0(\overline{\rho})$. Then by the proof of Lemma 4.4.1(ii) we have $\pi^{I_1} = D_0(\overline{\rho})^{I_1} = \bigoplus_{J\subseteq\mathcal{J}} \chi_J$ as I-representations. For each $J \subseteq \mathcal{J}$ we fix a choice of $0 \ne v_J \in D_0(\overline{\rho})^{I_1}$ with I-character χ_J, which is unique up to scalar. We recall the following results of §4 in the case $J_{\overline{\rho}} = \emptyset$.

Proposition 5.6.2. (i) *(Proposition 4.4.2) Let $J \subseteq \mathcal{J}$ and $\underline{i} \in \mathbb{Z}^f$ such that $\underline{0} \le \underline{i} \le \underline{f}$. Then there exists a unique H-eigenvector $\underline{Y}^{-\underline{i}} v_J \in D_0(\overline{\rho})$ satisfying*

(a) $Y_j^{i_j+1}\left(\underline{Y}^{-\underline{i}} v_J\right) = 0\ \forall\, j \in \mathcal{J}$;
(b) $\underline{Y}^{\underline{i}}\left(\underline{Y}^{-\underline{i}} v_J\right) = v_J$.

(ii) *(Proposition 4.5.11) Let $J, J' \subseteq \mathcal{J}$ such that $J' \ne \mathcal{J}$ and $J'+1 \subseteq J\Delta J' \overset{\text{def}}{=} (J\backslash J') \sqcup (J'\backslash J)$. Then there exists a unique element $\mu_{J,J'} \in \mathbb{F}^\times$, such that*

$$\left[\prod_{j+1\in J\Delta J'} Y_j^{s_j^{J'}} \prod_{j+1\notin J\Delta J'} Y_j^{p-1}\right] \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\underline{Y}^{-\underline{e}^{J\cap J'}} v_J\right) = \mu_{J,J'} v_{J'}.$$

(iii) *(Proposition 4.5.13) We write $x_{\emptyset,\underline{r}} \overset{\text{def}}{=} \mu_{\emptyset,\emptyset}^{-1} \underline{Y}^{\underline{p-1-r}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_\emptyset$ so that $\underline{Y}^{\underline{r}} x_{\emptyset,\underline{r}} = v_\emptyset$ by (ii).*

Then for $\emptyset \neq J \subseteq \mathcal{J}$, there exists a unique element $\mu_{J,\mathcal{J}} \in \mathbb{F}^\times$ such that

$$\prod_{j+1 \notin J} Y_j^{p-1-r_j} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_J = \mu_{J,\mathcal{J}} v_{\mathcal{J}} + \mu_{J,\emptyset} x_{\emptyset,\underline{r}},$$

where $\mu_{J,\emptyset}$ is defined in (ii).

(iv) *(Lemma 4.5.15) Let $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$. Then we have*

$$\frac{\mu_{J_1,J_3}}{\mu_{J_1,J_4}} = \frac{\mu_{J_2,J_3}}{\mu_{J_2,J_4}} \tag{5.76}$$

whenever all of them are defined in either (ii) or (iii).

We extend the definition of $\mu_{J,J'}$ to arbitrary $J, J' \subseteq \mathcal{J}$ by the formula

$$\begin{cases} \mu_{J,J'} \overset{\text{def}}{=} \mu_{(J')^c,J'} \mu_{J,\emptyset} / \mu_{(J')^c,\emptyset} \text{ if } J' \neq \mathcal{J}; \\ \mu_{\emptyset,\mathcal{J}} \overset{\text{def}}{=} \mu_{\emptyset,\emptyset} \mu_{\mathcal{J},\mathcal{J}} / \mu_{J,\emptyset} \end{cases}$$

(and $\mu_{J,\mathcal{J}}$ as in Proposition 5.6.2(iii) for $J \neq \emptyset$). Then the equation (5.76) holds for arbitrary $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$. By Theorem 4.1.1 and the construction of [BHH$^+$c, §3.2], $\mathrm{Hom}_A(D_A(\pi), A)(1)$ is an étale $(\varphi, \mathcal{O}_K^\times)$-module over A of rank 2^f. Here for D a $(\varphi, \mathcal{O}_K^\times)$-module over A, we write $D(1)$ to be D with the action of φ unchanged and the action of $a \in \mathcal{O}_K^\times$ multiplied by $N_{\mathbb{F}_q/\mathbb{F}_p}(a)$. Moreover, by Proposition 4.10.4(i),(iii) and Corollary 4.10.5 there is an A-basis of $\mathrm{Hom}_A(D_A(\pi), A)(1)$ such that

(i) the corresponding matrix $\mathrm{Mat}(\varphi)' \in \mathrm{GL}_{2^f}(A)$ for the φ-action is given by

$$\mathrm{Mat}(\varphi)'_{J',J+1} = \begin{cases} \gamma_{J+1,J'} \prod_{j \notin J} Y_j^{(r_j+1)(1-\varphi)} & \text{if } J' \subseteq J \\ 0 & \text{if } J' \not\subseteq J, \end{cases} \tag{5.77}$$

where $\gamma_{J,J'} \overset{\text{def}}{=} (-1)^{f-1} \varepsilon_{J'} \mu_{J,J'}$ with $\varepsilon_J \overset{\text{def}}{=} (-1)^{|J \cap (J-1)|}$ if $J \neq \mathcal{J}$ and $\varepsilon_{\mathcal{J}} \overset{\text{def}}{=} (-1)^{f-1}$.

(ii) the corresponding matrices $\mathrm{Mat}(a)'$ for the $\mathcal{O}_K^\times$-action satisfy $\mathrm{Mat}(a)'_{J,J} \in 1 + F_{1-p}A$ for all $a \in \mathcal{O}_K^\times$ and $J \subseteq \mathcal{J}$, which uniquely determines $\mathrm{Mat}(a)'$.

We also extend the definition of $\nu_{J,J'}$ (see (5.71)) to all $J, J' \subseteq \mathcal{J}$ by the formula

$$\nu_{J,J'} \overset{\text{def}}{=} \sqrt[f]{\xi}^{|J^c|-|J|} \frac{\prod_{j \notin J'} d_j}{\prod_{j+1 \notin J} d_j},$$

where $d_j \in \mathbb{F}^\times$ is as in (5.71). Then it is easy to check that (5.76) holds for $\nu_{J,J'}$, and that

$$\frac{\nu_{J,\emptyset}}{\nu_{J^c,\emptyset} \nu_{J,J^c}} = \sqrt[f]{\xi}^{|J^c|-|J|} \frac{\prod_{j \notin J, j+1 \in J} d_j}{\prod_{j \in J, j+1 \notin J} d_j}. \tag{5.78}$$

Proposition 5.6.3. *Keep the assumptions of π and assume that $|W(\overline{\rho})| = 1$. Then for $J \subseteq \mathcal{J}$ we have (see (5.77) for $\gamma_{J,J'}$ and ε_J)*

$$\frac{\gamma_{J,\emptyset}}{\gamma_{J^c,\emptyset} \gamma_{J,J^c}} = (-1)^{f-1} \varepsilon_{J^c} \frac{\mu_{J,\emptyset}}{\mu_{J^c,\emptyset} \mu_{J,J^c}} = \sqrt[f]{\xi}^{|J^c|-|J|} \frac{\prod_{j \notin J, j+1 \in J} d_j}{\prod_{j \in J, j+1 \notin J} d_j}. \tag{5.79}$$

Proof. The first equality follows directly from the definition. Then we prove the second equality. Since the LHS of (5.79) is unchanged when we rescale the basis $(v_J)_{J\in\mathcal{J}}$ and since χ_{J^c} is the conjugation of χ_J by the matrix $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$, we may assume that $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)v_J = v_{J^c}$ for all J (note that p acts trivially on π).

First we compute $\mu_{J,\emptyset}/\mu_{J^c,\emptyset}$. We apply [BD14, Thm. 1.1] with J replaced by $J-1$ and $v = v_J$. Together with [BHH$^+$b, Lemma 3.2.2.5(i)], we get

$$\begin{aligned}(-1)^{f-1}&\left[\prod_{j+1\in J}(p-1-r_j)!\prod_{j+1\in J}Y_j^{r_j}\prod_{j+1\notin J}Y_j^{p-1}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_J\\ &= x(J-1)(-1)^{f-1}\left[\prod_{j+1\notin J}(p-1-r_j)!\prod_{j+1\notin J}Y_j^{r_j}\prod_{j+1\in J}Y_j^{p-1}\right]\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_{J^c},\quad (5.80)\end{aligned}$$

where $x(J-1)$ is computed by [BD14, Thm. 1.2] with $\alpha_{v,\sigma_j} = \sqrt[f]{\xi}$, $\beta_{v,\sigma_j} = \sqrt[f]{\xi}^{-1}$ and $x_{v,\sigma_j} = -d_j$ by Lemma 5.6.1. By Proposition 5.6.2(ii) applied to $(J,\emptyset)$ and $(J^c,\emptyset)$, we deduce from (5.80) that

$$\begin{aligned}\frac{\mu_{J,\emptyset}}{\mu_{J^c,\emptyset}} &= x(J-1)\frac{\prod_{j+1\notin J}(p-1-r_j)!}{\prod_{j+1\in J}(p-1-r_j)!}\\ &= \left[-\sqrt[f]{\xi}^{|J^c|-|J|}\frac{\prod_{j+1\in J,j\notin J}(-d_j)(r_j+1)}{\prod_{j+1\notin J,j\in J}(-d_j)(r_j+1)}\right]\frac{\prod_{j+1\in J}(-1)^{r_j+1}r_j!}{\prod_{j+1\notin J}(-1)^{r_j+1}r_j!}\\ &= -\sqrt[f]{\xi}^{|J^c|-|J|}\frac{\left[\prod_{j\in J,j+1\in J}(-1)^{r_j+1}r_j!\right]\left[\prod_{j\notin J,j+1\in J}(-1)^{r_j}(r_j+1)!d_j\right]}{\left[\prod_{j\notin J,j+1\notin J}(-1)^{r_j+1}r_j!\right]\left[\prod_{j\in J,j+1\notin J}(-1)^{r_j}(r_j+1)!d_j\right]},\quad (5.81)\end{aligned}$$

where the second equality follows from [BD14, Thm. 1.2] and

$$\big((p-1-r)!\big)^{-1}\equiv(-1)^{r+1}r!\ \bmod p\ \forall\, 0\le r\le p-1. \quad (5.82)$$

Next we compute μ_{J,J^c} for $J\neq\emptyset$. By Lemma 4.5.1(ii) and its proof (with $J_{\overline{\rho}}=\emptyset$), there is a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant surjection (see §4.3 for the element $\phi\in\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)$)

$$\begin{aligned}\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)&\twoheadrightarrow\left\langle\mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_J\right\rangle\\ \phi&\mapsto\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)v_J = v_{J^c}\end{aligned}$$

which is not an isomorphism when $J\neq\emptyset$, hence it maps the socle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J^s)$ to zero. By definition, it is elementary to check that $(-1)^{\underline{s}^J+\underline{r}^J} = (-1)^{\underline{r}^{J^c}}$ (see (5.74) for $\underline{s}^J$ and (5.75) for $\underline{r}^J$). Then we deduce from Lemma 4.3.2(iii)(a) that

$$\underline{Y}^{\underline{p-1-s}^J}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_J+(-1)^{f-1}(-1)^{\underline{r}^{J^c}}\left[\prod_{j=0}^{f-1}(s_j^J)!\right]v_{J^c}=0. \quad (5.83)$$

By Proposition 5.6.2(ii) applied to (J, J^c), we deduce from (5.83) that

$$\mu_{J,J^c} = (-1)^{\underline{r}^{J^c}+\underline{1}} / \left[\prod_{j=0}^{f-1}(s_j^J)!\right] = \frac{\left[\prod_{j\in J, j+1\in J}(-1)^{r_j} r_j!\right]\left[\prod_{j\notin J, j+1\in J}(-1)^{r_j}(r_j+1)!\right]}{\left[\prod_{j\notin J, j+1\notin J}(-1)^{r_j+1} r_j!\right]\left[\prod_{j\in J, j+1\notin J}(-1)^{r_j}(r_j+1)!\right]}, \quad (5.84)$$

where the second equality follows from (5.74), (5.75) and (5.82). Combining (5.81) and (5.84), we get

$$\frac{\mu_{J,\emptyset}}{\mu_{J^c,\emptyset}\mu_{J,J^c}} = (-1)^{|J\cap(J-1)|+1}\, \sqrt[f]{\xi}^{|J^c|-|J|} \frac{\prod_{j\notin J, j+1\in J} d_j}{\prod_{j\in J, j+1\notin J} d_j}.$$

By definition, it is elementary to check that $(-1)^{f-1}\varepsilon_{J^c} = (-1)^{|J\cap(J-1)|+1}$ for $J \neq \emptyset$. This proves the proposition for $J \neq \emptyset$.

It remains to prove the proposition for $J = \emptyset$. By (5.76) we have $\mu_{\emptyset,\emptyset}/(\mu_{\mathcal{J},\emptyset}\mu_{\emptyset,\mathcal{J}}) = \mu_{\mathcal{J},\mathcal{J}}^{-1}$, hence it suffices to show that $\mu_{\mathcal{J},\mathcal{J}} = \xi^{-1}$. We let

$$y \overset{\text{def}}{=} \underline{Y}^{\underline{p}-\underline{1}-\underline{r}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_\emptyset + (-1)^{f-1}(-1)^{\underline{r}} \left[\prod_{j=0}^{f-1} r_j!\right]^{-1} v_{\mathcal{J}} \in \pi. \quad (5.85)$$

By Lemma 4.3.2(iii)(a), both the elements y and $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathcal{J}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) v_\emptyset$ are nonzero and lie in the I-cosocle of $\sigma_\emptyset = \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$, hence they are equal up to a scalar. By Proposition 5.6.2(ii) applied to $(\emptyset,\emptyset)$ and since $v_{\mathcal{J}} \in \pi^{I_1}$, we have $\underline{Y}^{\underline{r}} y = \mu_{\emptyset,\emptyset} v_\emptyset$. By Proposition 5.6.2(iii) applied to $J = \mathcal{J}$, we have (see Proposition 5.6.2(iii) for $x_{\emptyset,\underline{r}}$)

$$\underline{Y}^{\underline{r}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathcal{J}} = \mu_{\mathcal{J},\mathcal{J}} \underline{Y}^{\underline{r}} v_{\mathcal{J}} + \mu_{\mathcal{J},\emptyset} \underline{Y}^{\underline{r}} x_{\emptyset,\underline{r}} = \mu_{\mathcal{J},\emptyset} v_\emptyset, \quad (5.86)$$

where the second equality uses $v_{\mathcal{J}} \in \pi^{I_1}$. Then we deduce from $\underline{Y}^{\underline{r}} y = \mu_{\emptyset,\emptyset} v_\emptyset$ and (5.86) that $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathcal{J}} = (\mu_{\mathcal{J},\emptyset}/\mu_{\emptyset,\emptyset}) y$, hence we have

$$\mu_{\mathcal{J},\mathcal{J}} v_{\mathcal{J}} + \mu_{\mathcal{J},\emptyset} x_{\emptyset,\underline{r}} = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) v_{\mathcal{J}} = \frac{\mu_{\mathcal{J},\emptyset}}{\mu_{\emptyset,\emptyset}} y = \mu_{\mathcal{J},\emptyset} x_{\emptyset,\underline{r}} + \frac{\mu_{\mathcal{J},\emptyset}}{\mu_{\emptyset,\emptyset}} (-1)^{f-1}(-1)^{\underline{r}} \left[\prod_{j=0}^{f-1} r_j!\right]^{-1} v_{\mathcal{J}},$$

where the first equality follows from Proposition 5.6.2(iii) applied to $J = \mathcal{J}$ and the last equality follows from (5.85), which implies that

$$\mu_{\mathcal{J},\mathcal{J}} = \frac{\mu_{\mathcal{J},\emptyset}}{\mu_{\emptyset,\emptyset}} (-1)^{f-1}(-1)^{\underline{r}} \left[\prod_{j=0}^{f-1} r_j!\right]^{-1} = \xi^{-1},$$

where the last equality follows from (5.81) applied to $J = \mathcal{J}$. This completes the proof. □

Finally, we need the following lemma.

Lemma 5.6.4. *Let $B \in \mathrm{M}_{2^f}(\mathbb{F})$ with nonzero entries whose rows and columns are indexed by the subsets of $\mathcal{J}$ and satisfies $B_{J_1,J_3}/B_{J_1,J_4} = B_{J_2,J_3}/B_{J_2,J_4}$ for all $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$. Then up to conjugation by diagonal matrices, B is uniquely determined by the quantities*

$$\left\{\frac{B_{J,\emptyset}}{B_{J^c,\emptyset} B_{J,J^c}}\right\}_{J\subseteq\mathcal{J}}. \quad (5.87)$$

Proof. First, it is easy to check that conjugation by a diagonal matrix does not change these quantities. Next, given such a matrix B, after conjugation we may assume that $B_{J,\emptyset} = 1$ for all $J \neq \emptyset$. Then $B_{\emptyset,\emptyset}$ is determined by letting $J = \mathcal{J}$ in (5.87), and the rest of the entries of B are determined by the formula (for $J' \neq \emptyset$)

$$B_{J,J'} = B_{(J')^c,J'} \frac{B_{J,\emptyset}}{B_{(J')^c,\emptyset}} = \left(\frac{B_{(J')^c,\emptyset}}{B_{J',\emptyset} B_{(J')^c,J'}} \right)^{-1} \frac{B_{J,\emptyset}}{B_{J',\emptyset}}.$$

This completes the proof. □

Suppose that the matrices $(\gamma_{J,J'})$ and $(\nu_{J,J'})$ are conjugated by the diagonal matrix Q, then the matrices $\left(\gamma_{J,J'}\delta_{J'\subseteq J-1}\right)$ and $\left(\nu_{J,J'}\delta_{J'\subseteq J-1}\right)$ are also conjugated by Q.

Proof of Theorem 5.1.1. We prove that $D_A(\pi) \cong D_A^{\otimes}(\overline{\rho}^\vee(1))$ as étale $(\varphi, \mathcal{O}_K^\times)$-modules over A. Since $D_K(\overline{\rho}^\vee)$ is dual to $D_K(\overline{\rho})$ as étale $(\varphi, \mathcal{O}_K^\times)$-modules, by definition and the equivalence of categories [BHH$^+$c, Thm. 2.5.1] and Proposition 5.5.1, there is a perfect pairing $D_A^{\otimes}(\overline{\rho}) \times D_A^{\otimes}(\overline{\rho}^\vee) \to A$ which is equivariant for the actions of φ and $\mathcal{O}_K^\times$. Hence it suffices to show that $\mathrm{Hom}_A(D_A(\pi), A) \cong D_A^{\otimes}(\overline{\rho}(-1)) \cong D_A^{\otimes}(\overline{\rho})(-1)$, or equivalently, $\mathrm{Hom}_A(D_A(\pi), A)(1) \cong D_A^{\otimes}(\overline{\rho})$. By Proposition 4.10.4(iii) and Corollary 4.10.5, it suffices to compare the matrices $\mathrm{Mat}(\varphi)$ (see (5.72)) and $\mathrm{Mat}(\varphi)'$ (see (5.77)). Then by Lemma 5.6.4 it suffices to show that $\gamma_{J,\emptyset}/(\gamma_{J^c,\emptyset}\gamma_{J,J^c}) = \nu_{J,\emptyset}/(\nu_{J^c,\emptyset}\nu_{J,J^c})$ for all $J \subseteq \mathcal{J}$. This is a consequence of (5.78) and Proposition 5.6.3. □

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