The Littlewood problem and non-harmonic Fourier series

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Le problème de Littlewood et les séries de Fourier non-harmoniques

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Title: The Littlewood problem and non-harmonic Fourier series.

Abstract: We investigate trigonometric polynomials both in the harmonic and the non-harmonic (non-periodic) case. More precisely, we are interested in lower bounds for L^1-norms or $\mathcal{B}_1$-norms (Besicovitch-norms) of such polynomials.

We study harmonic trigonometric polynomials with quadratic frequencies. We extend previous results for polynomials having only zero or one as coefficients to polynomials with complex coefficients. Furthermore, we obtain that up to a multiplicative constant, the square root is a lower bound for the L^1-norms of polynomials with monotone and uniformly bounded coefficients.

Next, we give explicit lower bounds for L^1 and $\mathcal{B}_1$-norms of non-harmonic trigonometric polynomials. If the polynomials have only zero or one as coefficients, we deduce a logarithmic lower bound for its L^1 or $\mathcal{B}_1$-norms with respect to the number of terms.

Afterwards, we look at non-harmonic trigonometric polynomials with gaps (between the frequencies) going to infinity. We give lower bounds for the L^1-norm of such polynomials.

Finally, we investigate two particular trigonometric polynomials: Lacunary and polynomials with frequencies having multidimensional structure. We extend multiple results from the integer to non-integer case. We obtain lower bounds for $\mathcal{B}_1$-norms of such polynomials.

Keywords: Littlewood conjecture, Besicovitch norm, non-harmonic Fourier series, lacunary series.

Titre: Le problème de Littlewood et les séries de Fourier non-harmoniques.

Résumé: Nous étudions les polynômes trigonométriques à la fois dans le cadre harmonique et non-harmonique (non-périodique). Plus précisément, nous nous intéressons aux bornes inférieures au sens de la norme L^1 et de la norme $\mathcal{B}_1$ (de Besicovitch) de ces polynômes.

D'abord, nous considérons les polynômes trigonométriques avec des fréquences quadratiques et des coefficients complexes. Nous étendons ainsi les résultats précédents sur les polynômes ayant seulement zéro ou un comme coefficients. Pour les polynômes à coefficients monotones et uniformément bornés, nous obtenons une minoration de la norme L^1 par la racine du nombre de termes.

Ensuite, nous donnons aussi des bornes inférieures explicites pour les normes L^1 et $\mathcal{B}_1$ de tels polynômes. Lorsque les polynômes n'ont que zéro ou un comme coefficients, nous en déduisons une minoration de la norme L^1 ou $\mathcal{B}_1$ par le logarithme du nombre de termes.

Nous nous intéressons aussi aux polynômes trigonométriques non-harmoniques tels que les différences successives entre les fréquences tendent vers l'infini. Nous donnons des minorations de la norme L^1 pour de tels polynômes.

Enfin, nous étudions deux types de polynômes trigonométriques. D'une part ceux dont les fréquences sont lacunaires et d'autre part, ceux dont les fréquences ayant une structure multidimensionnelle. Nous étendons plusieurs résultats du cas entier au cas non-entier. Nous obtenons des bornes inférieures pour la norme $\mathcal{B}_1$ pour de tels polynômes.

Mots-clés: Conjecture de Littlewood, norme de Besicovitch, series lacunaires, séries de Fourier non-harmoniques.

Contents

Chapter 1

Introduction

English version

In this thesis, we consider trigonometric polynomials of the form

$$\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}, \tag{1.0.1}$$

where $(a_k)_{k=0,\dots,N}$ is a sequence of complex numbers and $(\lambda_k)_{k=0,\dots,N}$ is a sequence of real numbers commonly known as frequencies. We focus on lower bounds of the L^1-norm or the Besicovitch $\mathcal{B}^1$-norm of non-harmonic (non-periodic) trigonometric polynomials of the form (1.0.1). Recall that, for $1 \leq p < +\infty$, the Besicovitch $\mathcal{B}_p$-norms are defined by

$$\|\Phi\|_{\mathcal{B}_p}^p = \lim_{T\to+\infty} \frac{1}{T} \int_{[-T/2,T/2]} |\Phi(x)|^p \,\mathrm{d}x.$$

Those norms can be seen as a substitute to $L^p([-1/2,1/2])$-norms to investigate non-harmonic trigonometric polynomials.

The starting point of this thesis is the investigation by Littlewood [26] of the properties of the trigonometric polynomials having only 0 or 1 as coefficients

$$\sum_{k=0}^{N} e^{2i\pi n_k t},$$

where the n_k's are distinct integers. In particular, Littlewood conjectured that

$$L_N := \inf_{n_0<n_1,<\cdots<n_N} \int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq C \ln(N+1) \tag{1.0.2}$$

for some constant $C \leq \dfrac{4}{\pi^2}$.

The first non-trivial estimate was obtained by Cohen [4] who proved that

$$L_N \geq C(\ln(N+1)/\ln\ln(N+1))^{1/8}$$

for $N \geq 3$. Subsequent improvements are due to Davenport [5], Fournier [7] and crucial contributions by Pichorides [33, 34, 35, 36] leading to

$$L_N \geq C \ln(N+1)/(\ln\ln(N+1))^2.$$

Finally, Littlewood's conjecture was proved independently by Konyagin [24] and McGehee, Pigno, Smith [27] in 1981. In both papers, Littlewood's conjecture is actually obtained as a corollary of a stronger result (and they are not consequences of one another). Here, we are particularly interested in the result of McGehee, Pigno and Smith [27, 20]:

Theorem 1.0.1 (McGehee, Pigno & Smith)**.** *For $n_0 < n_1 < \cdots < n_N$ integers and $a_0, \ldots, a_N$ complex numbers,*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq C_{MPS} \sum_{k=0}^{N} \frac{|a_k|}{k+1}$$

where C_{MPS} is a universal constant ($C_{MPS} = 1/30$ would do).

Taking the a_k's to have modulus 1, one thus obtains a lower bound

$$L_N \geq C \ln(N+1).$$

The year after, Stegeman [38] and Yabuta [41] independently suggested some modifications of the argument in [27] that lead to a better bound of L_N, namely:

Theorem 1.0.2 (Stegeman, Yabuta)**.** *Let $N \geq 3$. For $n_0 < n_1 < \cdots < n_N$ integers*

$$\int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln(N+1).$$

Our first results concern the study of finite trigonometric polynomials with quadratic frequencies $\lambda_k = k^2$, which appear for example, in the solution of the Schrödinger's equation. Let $a = (a_k)_{k=0,\ldots,N}$ be a sequence of complex numbers, we write

$$\mathbb{E}\left[|a|^2\right] = \frac{1}{N+1} \sum_{k=0}^{N} |a_k|^2.$$

Thus for the constant sequence $a_k = 1,\ k = 0, \ldots, N$, Zalcwasser's result [43] can be written as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} e^{2i\pi k^2 x} \right| \mathrm{d}x \geq C\sqrt{N} \left(\mathbb{E}[|a|^2]\right)^{1/2}, \tag{1.0.3}$$

for some positive constant C. Our first result is an extension of this estimate to trigonometric polynomial with complex coefficients a_k. It is written in terms of the l^1-norm of the increments of the sequence a:

$$\|\partial a\|_{1,N} = |a_0| + \sum_{k=1}^{N} |a_k - a_{k-1}|,$$

and can be stated as follows

Theorem 1.0.3. *For every $\varepsilon > 0$ there exists a constant C_ε such that if $(a_k)_{k=0,\dots,N}$ is a sequence of complex numbers, then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi k^2 x} \right| \mathrm{d}x \geq C_\varepsilon \sqrt{N} \left(\frac{\left(\mathbb{E}[|a|^2]\right)^{\frac{1}{2}}}{\|\partial a\|_{1,N}} \right)^{2+\varepsilon} (\mathbb{E}[|a|^2])^{\frac{1}{2}}. \tag{1.0.4}$$

The proof is an adaptation of that of Zalcwasser and is based on an approximate functional equation obtained with the residue Theorem, combined with continued fractions decomposition of irrational numbers.

We now compare our result, Theorem 1.0.3, with Zalcwasser's and McGehee, Pigno, Smith's results respectively by giving two examples:

1. Let $\alpha, \beta > 0$ and $(a_k)_{k=0,\dots,N}$ be an increasing sequence such that $\alpha < a_k < \beta$. For N large, we have
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi k^2 x} \right| \mathrm{d}x \geq \beta \Big(\frac{\alpha}{\beta}\Big)^{(3+\varepsilon)} \sqrt{N}.$$

2. let $\alpha \in \mathbb{R}$ and $a_k = (1+k)^\alpha$ for $k = 0, \dots, N$. When looking at the asymptotic behavior, if $-1/6 < \alpha < 0$, the lower bound in Theorem 1.0.1 behaves as constant while the lower bound in Theorem 1.0.3 grows as $N^{\frac{1}{2}+\alpha(3+\varepsilon)}$.

For non-integer frequencies, the first result has been obtained by Hudson and Leckband [14] who used a clever perturbation argument to prove the following:

Theorem 1.0.4 (Hudson & Leckband)**.** *For $\lambda_0 < \lambda_1 < \ldots < \lambda_N$ real numbers and $a_0, \dots, a_N$ complex numbers,*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C_{MPS} \sum_{k=0}^{N} \frac{|a_k|}{k+1}$$

where C_{MPS} is the same constant as in Theorem 1.0.1.

A further extension is due to Nazarov [30] who showed that such a result holds not only when $T \to +\infty$ but as soon as $T > 1$:

Theorem 1.0.5 (Nazarov)**.** *For $T > 1$, there exists a constant C_T such that, for $0 < \lambda_0 < \cdots < \lambda_N$ real numbers verifying $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ (or equivalently $\lambda_{k+1} - \lambda_k \geq k$) and $a_0, \dots, a_N$ complex numbers,*

$$\int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C_T \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{1.0.5}$$

Note that the constant in Nazarov's proof is not explicit. We will improve on Nazarov's proof to obtain a more precise and explicit estimate of the constant C_T. This also allows us to directly obtain the result of Hudson and Leckband:

Theorem 1.0.6. *Let $\lambda_0 < \lambda_1 < \cdots < \lambda_N$ be real numbers and $a_0, \dots, a_N$ be complex numbers. Then*

i. we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{26} \sum_{k=0}^{N} \frac{|a_k|}{k+1}.$$

ii. If further $a_0, \dots, a_N$ all have modulus larger or equal to 1, $|a_k| \geq 1$ then

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln(N+1).$$

iii. Assume further that for $k = 0, \dots, N-1$, $\lambda_{k+1} - \lambda_k \geq 1$, then, for every $T > 1$, there exists a constant $C(T)$ such that, for every $a_0, \dots, a_N \in \mathbb{C}$,

$$\frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C(T) \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{1.0.6}$$

Moreover,

(a) for $T \geq 72$ we can take $C(T) = \dfrac{1}{122}$;

(b) for $1 < T \leq 2$, $C(T) = O\big((T-1)^{15/2}\big)$.

Remark. *For $2 < T < 72$, (1.0.6) follows from the case (b) with $T = 2$, but the constant is not explicit.*

The proof is related to the one implemented by McGehee, Pigno and Smith as extended by Nazarov to prove Theorem 1.0.5. Here we follow constants more closely by introducing and optimizing various parameters throughout the proof.

Theorem 1.0.6 can be used to lower bound a curve length of trigonometric polynomials. Let $\lambda_{k+1} \geq \lambda_k + 1$ and consider a curve in the complex plane of the form

$$\Gamma = \{z = P(t),\ t \in [0,T]\} \qquad \text{with} \quad P(t) = \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}.$$

Figure 1.1 shows two such curves.

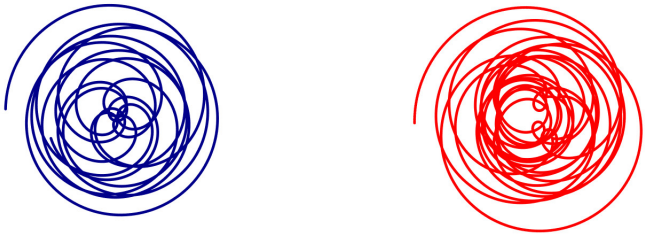


Fig. 1.1 – Left: $1 + e^{4i\pi t} + e^{20it}$ and right $1 + e^{16it} + e^{24it} + e^{34it}$ both for $t = 0$ to 5.

Since

$$P'(t) = 2i\pi \sum_{k=0}^{N} a_k \lambda_k e^{2i\pi\lambda_k t},$$

it follows from Theorem 1.0.6 that, when $T \geq 72$, the length of Γ is lower bounded by

$$\ell(\Gamma) = \int_0^T |P'(t)|\,\mathrm{d}t \geq \frac{T}{20} \sum_{k=0}^{N} \frac{|\lambda_k||a_k|}{k+1}.$$

Next, we investigate the L^1-norms of non-harmonic trigonometric polynomials with frequencies forming a sequence with gaps going to infinity. Those polynomials were studied in the L^2-case by Kahane [22] who improved on a result by Ingham [15]. Our main result here is an L^1-analogue of Kahane's result.

We first recall several well-known results in the L^2-setting:

Theorem 1.0.7 (Ingham)**.** *Let $\gamma > 0$ and $T > \dfrac{1}{\gamma}$. Then there exist constants $0 < A_2(T,\gamma) \leq B_2(T,\gamma)$ such that*

– for every sequence of real numbers $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \geq \gamma$,

– for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$,

$$A_2(T,\gamma) \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 dt \leq B_2(T,\gamma) \sum_{k\in\mathbb{Z}} |a_k|^2$$

Then Kahane showed that the condition $T > \dfrac{1}{\gamma}$ can be lifted if $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$:

Theorem 1.0.8 (Kahane)**.** *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$. Then, for every $T > 0$, there exist constants $0 < A_2(T,\Lambda) \leq B_2(T,\Lambda)$ such that*

$$A_2(T,\Lambda) \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 dt \leq B_2(T,\Lambda) \sum_{k\in\mathbb{Z}} |a_k|^2$$

holds for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$.

Here, our main result concern the L^1-case and states as follows

Theorem 1.0.9. *Let $\Lambda = (\lambda_k)_{k\in\mathbb{Z}}$ be an increasing sequence with $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$. Then, for every $T > 0$, there exists a constant $\tilde{A}_1(T,\Lambda) > 0$ such that, if $(a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ is a sequence of complex numbers, and $N \geq 1$, then*

$$\tilde{A}_1(T,\Lambda) \sum_{k=0}^{N} \frac{|a_k|}{1+k} \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t. \tag{1.0.7}$$

If further $\sum_{k\in\mathbb{Z}} \frac{1}{1+|\lambda_k|}$ *converges, then there also exists a constant* $A_1(T,\Lambda)$ *such that, for every* $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ *and every* $N \geq 1$,

$$A_1(T,\Lambda) \max_{k=-N,\dots,N} |a_k| \leq \frac{1}{T}\int_{-T/2}^{T/2} \left| \sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t. \tag{1.0.8}$$

The main difficulty in the proof of this result is that Kahane's argument cannot be adapted directly. Indeed, Kahane used in a crucial way that in Ingham's Inequality the L^2-norm of a trigonometric polynomial is both lower and upper bounded by the ℓ^2-norm of its coefficients. In the L^1-case, the upper bound is in terms of the ℓ^1-norm of the coefficients and does not match the lower bound, which is given in terms of a weighted l^1 norm. Instead, our proof uses Nazarov's Theorem, a compactness argument and a trick allowing us to benefit from Kahane's result.

In the end, we investigate the Besicovitch $\mathcal{B}_1$-norms of lacunary trigonometric polynomials and polynomials with frequencies having multidimensional structure. Indeed, non-harmonic lacunary polynomials are trigonometric polynomials

$$\sum_{k=0}^{\infty} a_k e^{2i\pi\lambda_k t},$$

with

$$\frac{\lambda_{k+1}}{\lambda_k} > q > 1.$$

Our main result is an extension to real setting of a result in the integer case done by Zygmund [44] and states as follows

Theorem 1.0.10. *Let* $q > 1$ *and* $(\lambda_k)_{k\geq 0}$ *be a sequence of real numbers verifying*

$$\lambda_0 > 1 \quad \textit{and} \quad \lambda_{k+1} \geq q\lambda_k$$

and $a_0, \dots, a_N$ *be a sequence of complex numbers. Then for* $1 \leq p < \infty$*, there exists positive constant* $A_{p,q}$*,* $B_{p,q}$ *such that*

$$A_{p,q}\left(\sum_{k=0}^{N}|a_k|^2\right)^{1/2} \leq \lim_{T\to+\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,q}\left(\sum_{k=0}^{N}|a_k|^2\right)^{1/2}.$$

Next let A be a finite subset of $\mathbb{R}$, we say that A is $(\delta; m, n)$ strongly 2-dimensional in $\mathbb{R}$ if there exists two real numbers d and D with $D > (2+\delta)d$ such that

$$A = \bigcup_{k\in I}(A_k + kD) \tag{1.0.9}$$

for some set I containing m integers and real subsets $A_k \subseteq [-d, d]$ verifying $|A_k| \geq n$.

We extend a result by Hanson [10] from the integer to the non-integer case, we obtain the following

Theorem 1.0.11. *Let $\delta > 0$ and m, n be two positive integers satisfying*

$$m \geq \pi^3 2^{21} C_{MPS}^3 \ln(m)^3 \ln(n)^3 \quad \textit{and} \quad n \geq \pi^3 2^{21} C_{MPS}^3 \ln(n)^3,$$

where C_{MPS} is the constant in Theorem 1.0.1. Suppose A is $(\delta; m, n)$ strongly 2-dimensional subset of $\mathbb{R}$. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{a\in A} e^{2i\pi at} \right| \mathrm{d}t \geq \frac{C_{MPS}^2}{(2^9\pi)^2\big(2 + \ln(1 + \frac{2}{\delta})\big)} \ln(m) \ln(n).$$

Both proofs rely on an argument of Hudson and Leckband [14] used to extend the solution to the Littlewood conjecture from integer to real setting. Indeed, the idea is to approximate real numbers by rational numbers via a Lemma of Dirichlet.

Manuscript organization

In Chapter 2, we recall some classical results such as Ingham's inequalities, asymptotic estimate of the L^1-norm of the Dirichlet kernel and Trigub's Theorem which is indeed a solution to Littlewood's Conjecture 1.0.2.

In Chapter 3, we focus on the particular case of harmonic trigonometric polynomials whose frequencies are quadratic. We extend Zalcwasser's result (1.0.3) to trigonometric polynomials with complex coefficients.

In Chapter 4, we are interested in non-harmonic trigonometric polynomials. Depending on the integration interval, we give quantitative versions of Nazarov's Theorem 1.0.5.

In Chapter 5, we study non-harmonic trigonometric polynomials whose increment between frequencies tends towards infinity and we give an L^1-version of Kahane's Theorem 1.0.8

In Chapter 6, we extend results by Zygmund and Hanson concerning respectively bounds for the L^1-norm of lacunary harmonic polynomials and harmonic polynomials whose frequencies have multidimensional structure to the non-harmonic framework.

Version française

Dans cette thèse, nous considérons des polynômes trigonométriques non-harmoniques (c'est à dire non-périodiques) de la forme

$$\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}, \tag{1.0.10}$$

où $(a_k)_{k=0,\dots,N}$ est une suite de nombres complexes et les λ_k sont des nombres réels appelés fréquences. Nous nous intéressons aux bornes inférieures de la norme L^1 ou $\mathcal{B}^1$ (de Besicovitch) de ces polynômes trigonométriques.

Rappelons que lorsque $1 \leq p < +\infty$, les normes de Besicovitch $\mathcal{B}_p$ sont définies par

$$\|\Phi\|_{\mathcal{B}_p}^p = \lim_{T\to+\infty} \frac{1}{T} \int_{[-T/2,T/2]} |\Phi(x)|^p \,\mathrm{d}x.$$

Ces normes peuvent être considérées comme un substitut aux normes $L^p([-1/2,1/2])$ pour étudier les polynômes trigonométriques non-harmoniques.

Le point de départ de cette thèse est l'étude par Littlewood [26] des propriétés des polynômes trigonométriques n'ayant que, 0 ou 1 comme coefficients

$$\sum_{k=0}^{N} e^{2i\pi n_k t},$$

où les n_k sont des entiers distincts. En particulier, Littlewood a conjecturé que

$$L_N := \inf_{n_0<n_1,<\cdots<n_N} \int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq C \ln(N+1) \tag{1.0.11}$$

pour une constante $C \leq \dfrac{4}{\pi^2}$.

La première minoration non triviale a été obtenue par Cohen [4] qui a démontré que

$$L_N \geq C(\ln(N+1)/\ln\ln(N+1))^{1/8}$$

pour $N \geq 3$. Des améliorations ultérieures sont dues à Davenport [5], Fournier [7] et aux contributions de Pichorides [33, 34, 35, 36] menant à

$$L_N \geq C\ln(N+1)/(\ln\ln(N+1))^2.$$

Enfin, la conjecture de Littlewood a été démontrée indépendamment par Konyagin [24] et McGehee, Pigno, Smith [27] en 1981. Dans ces deux articles, la conjecture de Littlewood est obtenue comme corollaire d'un résultat plus fort (et ils ne sont pas des conséquences l'un de l'autre). Ici, nous nous intéressons particulièrement au résultat de McGehee, Pigno et Smith [27, 20] :

Théorème 1.0.1 (McGehee, Pigno et Smith)**.** *Pour toute suite finie d'entiers $n_0 < n_1 < \cdots < n_N$, et toute suite finie de nombres complexes $a_0, \ldots, a_N$,*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq C_{MPS} \sum_{k=0}^{N} \frac{|a_k|}{k+1}$$

où C_{MPS} est une constante universelle ($C_{MPS} = 1/30$ convient).

Si pour $0 \leq k \leq N$, a_k est de module 1, on obtient

$$L_N \geq C \ln(N+1).$$

Stegeman [38] et Yabuta [41] ont indépendamment modifié l'argument de [27] et ils ont obtenu une meilleure borne de L_N:

Théorème 1.0.2 (Stegeman, Yabuta)**.** *Soient $N \geq 3$ et $n_0 < n_1 < \cdots < n_N$ une suite d'entiers, alors*

$$\int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln(N+1).$$

Notre premier résultat concerne l'étude des polynômes trigonométriques avec fréquences quadratiques $\lambda_k = k^2$, qui apparaissent par exemple dans la solution de l'équation de Schrödinger. Soit $a = (a_k)_{k=0,\ldots,N}$ une suite de nombres complexes, on pose

$$\mathbb{E}\left[|a|^2\right] = \frac{1}{N+1} \sum_{k=0}^{N} |a_k|^2.$$

Lorsque la suite $a_k = 1,\ k = 0, \ldots, N$, le résultat de Zalcwasser [43] peut s'écrire de la manière suivante

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} e^{2i\pi k^2 x} \right| \mathrm{d}x \geq C\sqrt{N} \left(\mathbb{E}[|a|^2]\right)^{1/2}, \tag{1.0.12}$$

où C est une constante positive. Notre premier résultat est une extension de cette inégalité à des polynômes avec des coefficients a_k complexes. On pose

$$\|\partial a\|_{1,N} = |a_0| + \sum_{k=1}^{N} |a_k - a_{k-1}|,$$

nous obtenons le théorème suivant

Théorème 1.0.3. *Pour tout $\varepsilon > 0$, il existe une constante $C_\varepsilon > 0$ telle que pour toute suite $a = (a_k)_{k=0,\ldots,N}$ de nombres complexes, on a*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi k^2 x} \right| \mathrm{d}x \geq C_\varepsilon \sqrt{N} \left(\frac{\left(\mathbb{E}[|a|^2]\right)^{\frac{1}{2}}}{\|\partial a\|_{1,N}} \right)^{2+\varepsilon} \left(\mathbb{E}[|a|^2]\right)^{\frac{1}{2}}.$$

La démonstration est une adaptation de celle de Zalcwasser et est basée sur une équation fonctionnelle approchée obtenue par le théorème des résidus et sur la décomposition en fractions continues de nombres irrationnels.

Comparons maintenant notre résultat, le théorème 1.0.3, avec les résultats de Zalcwasser et McGehee, Pigno, Smith respectivement en donnant deux exemples:

1. Soient $\alpha, \beta > 0$ et $(a_k)_{k=0,\dots,N}$ une suite croissante telle que $\alpha < a_k < \beta$. Pour N grand, on a
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi k^2 x} \right| \mathrm{d}x \geq \beta \Big(\frac{\alpha}{\beta}\Big)^{(3+\varepsilon)} \sqrt{N}.$$

2. Soient $\alpha \in \mathbb{R}$ et $a_k = (1+k)^\alpha$ pour $k = 0, \dots, N$. Pour N grand et $-1/6 < \alpha < 0$, la borne inférieure du théorème 1.0.1 se comporte comme une constante tandis que la borne inférieure du théorème 1.0.3 croît comme $N^{\frac{1}{2}+\alpha(3+\varepsilon)}$.

Pour les fréquences non-entières, le premier résultat a été obtenu par Hudson et Leckband [14] qui ont utilisé un argument de perturbation pour démontrer le théorème suivant:

Théorème 1.0.4 (Hudson & Leckband)**.** *Pour $\lambda_0 < \lambda_1 < \ldots < \lambda_N$ nombres réels et $a_0, \dots, a_N$ nombres complexes,*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C_{MPS} \sum_{k=0}^{N} \frac{|a_k|}{k+1}$$

où C_{MPS} est la même constante que dans le théorème 1.0.1.

Nazarov [30] a montré qu'un tel résultat est valable non seulement lorsque $T \to +\infty$ mais aussi dès que $T > 1$:

Théorème 1.0.5 (Nazarov)**.** *Pour $T > 1$, il existe une constante C_T telle que, pour toute suite $0 < \lambda_0 < \cdots < \lambda_N$ de réels avec $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ et toute suite $a_0, \dots, a_N$ de complexes, on a*

$$\int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C_T \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{1.0.13}$$

La constante C_T dans le théorème de Nazarov n'est pas explicite. Nous améliorons la démonstration de Nazarov pour obtenir une estimation explicite de la constante C_T. Cela nous permet également d'obtenir directement le résultat de Hudson et Leckband:

Théorème 1.0.6. *Pour toute suite $\lambda_0 < \lambda_1 < \cdots < \lambda_N$ de réels et toute suite $a_0, \dots, a_N$ de complexes,*

1. *on a*
$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{26} \sum_{k=0}^{N} \frac{|a_k|}{k+1}.$$

2. *Si $a_0, \dots, a_N$ sont tous de module supérieur à 1, $|a_k| \geq 1$, alors*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln(N+1).$$

3. *Si de plus, pour $k = 0, \dots, N-1$, $\lambda_{k+1} - \lambda_k \geq 1$, alors, pour tout $T > 1$, il existe une constante $C(T)$ telle que, pour chaque $a_0, \dots, a_N \in \mathbb{C}$,*

$$\frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C(T) \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{1.0.14}$$

De plus,

   (a) *Pour $T \geq 72$, on a $C(T) = \dfrac{1}{122}$;*

   (b) *Pour $1 < T \leq 2$, $C(T) = O\big((T-1)^{15/2}\big)$.*

Remarque. *Pour $2 < T < 72$, l'inégalité* (1.0.14) *reste vraie et se déduit du cas (b) avec $T = 2$, mais la constante n'est pas explicite.*

La démonstration est liée à celle introduite par McGehee, Pigno, Smith et étendue par Nazarov pour démontrer le théorème 1.0.5. Ici, nous nous focalisons les constantes en introduisant et en optimisant divers paramètres tout au long de la démonstration.

Comme application, considérons une courbe dans le plan complexe de la forme

$$\Gamma = \left\{ z = P(t) = \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t},\ t \in [0,T] \right\} \quad \text{avec} \quad \lambda_{k+1} \geq \lambda_k + 1.$$

La Figure 1.2 montre deux de ces courbes.

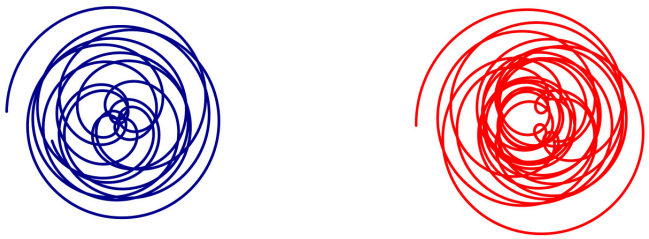


Fig. 1.2 – Gauche: $1 + e^{4i\pi t} + e^{20it}$ et droite $1 + e^{16it} + e^{24it} + e^{34it}$ pour $t = 0$ à 5.

Par le théorème 1.0.6, et lorsque $T \geq 72$, la longueur de Γ est bornée par

$$\ell(\Gamma) = \int_0^T |P'(t)|\, \mathrm{d}t \geq \frac{T}{20} \sum_{k=0}^{N} \frac{|\lambda_k||a_k|}{k+1}.$$

Nous étudions ensuite les normes L^1 des polynômes trigonométriques non-harmoniques dont les fréquences forment une suite avec des écarts tendant vers l'infini. Ces polynômes ont été étudiés dans le cas L^2 par Kahane [22] qui a amélioré un résultat d'Ingham [15]. Notre résultat principal est un analogue L^1 du résultat de Kahane.

Nous rappelons d'abord les résultats d'Ingham et de Kahane;

Théorème 1.0.7 (Ingham)**.** *Soient $\gamma > 0$ et $T > \dfrac{1}{\gamma}$. Alors il existe deux constantes positives $0 < A_2(T,\gamma) \leq B_2(T,\gamma)$ telles que*

– *pour toute suite $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ de réels vérifiant $\lambda_{k+1} - \lambda_k \geq \gamma$,*

– *pour toute suite $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$,*

$$A_2(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \leq B_2(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2$$

Kahane a ensuite montré que la condition $T > \dfrac{1}{\gamma}$ peut être levée si $\lambda_{k+1} - \lambda_k \to +\infty$ lorsque $k \to \pm\infty$:

Théorème 1.0.8 (Kahane)**.** *Soit $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ tel que $\lambda_{k+1} - \lambda_k \to +\infty$ quand $k \to \pm\infty$. Alors, pour tout $T > 0$, il existe deux constantes $0 < A_2(T,\Lambda) \leq B_2(T,\Lambda)$ telles que*

$$A_2(T,\Lambda)\sum_{k\in\mathbb{Z}}|a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \leq B_2(T,\Lambda)\sum_{k\in\mathbb{Z}}|a_k|^2$$

est valable pour toute suite $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$.

Nous nous intéressons ici au cas L^1. Notre résultat principal est le suivant

Théorème 1.0.9. *Soit $\Lambda = (\lambda_k)_{k\in\mathbb{Z}}$ une suite croissante qui vérifie $\lambda_{k+1} - \lambda_k \to +\infty$ lorsque $k \to \pm\infty$. Alors, pour tout $T > 0$, il existe une constante $\tilde{A}_1(T,\Lambda) > 0$ telle que, si $(a_k)_{k\in\mathbb{N}}$ est une suite de nombres complexes, et $N \geq 1$, on a*

$$\tilde{A}_1(T,\Lambda)\sum_{k=0}^{N}\frac{|a_k|}{1+k} \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t. \tag{1.0.15}$$

Si de plus $\displaystyle\sum_{k\in\mathbb{Z}}\frac{1}{1+|\lambda_k|}$ converge, alors il existe aussi une constante $A_1(T,\Lambda)$ telle que, pour toute suite $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ et tout $N \geq 1$,

$$A_1(T,\Lambda)\max_{k=-N,\dots,N}|a_k| \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t. \tag{1.0.16}$$

La difficulté principale de la démonstration est que l'argument de Kahane ne peut pas être adapté directement. En effet, Kahane a utilisé le fait que dans l'inégalité d'Ingham, la norme L^2 d'un polynôme trigonométrique est bornée inférieurement et supérieurement par la norme ℓ^2 de ses coefficients. Dans le cas L^1, la borne supérieure est exprimée en termes de norme ℓ^1 des coefficients et ne correspond pas à la borne inférieure, qui elle, est donnée en terme d'une norme l^1 pondérée. Notre démonstration utilise le théorème de Nazarov, un argument de compacité, ainsi qu'une astuce permettant de nous ramener au théorème de Kahane.

Nous conclurons par l'étude des normes $\mathcal{B}_1$ de Besicovitch des polynômes trigonométriques lacunaires et des polynômes avec des fréquences ayant une structure multidimensionnelle. En effet, un polynôme trigonométrique non-harmonique est lacunaire s'il est de la forme suivante

$$\sum_{k=0}^{\infty} a_k e^{2i\pi\lambda_k t}, \qquad \text{avec} \qquad \frac{\lambda_{k+1}}{\lambda_k} > q > 1.$$

Notre résultat principal est une extension au cas réel d'un résultat dans le cas entier obtenu par Zygmund [44], nous obtenons

Théorème 1.0.10. *Soient $q > 1$ et $(\lambda_k)_{k\geq 0}$ une suite de nombres réels vérifiant*

$$\lambda_0 > 1 \quad et \quad \lambda_{k+1} \geq q\lambda_k$$

et $a_0, \ldots, a_N$ une suite de nombre complexes. Alors pour $1 \leq p < \infty$, il existe deux constantes positives $A_{p,q}$, $B_{p,q}$ telles que

$$A_{p,q}\left(\sum_{k=0}^{N}|a_k|^2\right)^{1/2} \leq \lim_{T\to+\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\lambda_k t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,q}\left(\sum_{k=0}^{N}|a_k|^2\right)^{1/2}.$$

Ensuite, concernant les polynômes trigonométriques avec fréquences dont la structure est multidimensionnelle. Soit A un sous ensemble fini de $\mathbb{R}$, on dit que A est $(\delta; m, n)$ fortement 2-dimensionnel dans $\mathbb{R}$, s'il existe deux nombres réels d et $D > (2+\delta)d$ tels que

$$A = \bigcup_{k\in I}(A_k + kD) \tag{1.0.17}$$

où I est un ensemble d'entiers de cardinal m et les $A_k \subset [-d, d]$ sont des sous-ensembles de $\mathbb{R}$ et de cardinal supérieur ou égal à n.

On étend un résultat de Hanson [10] du cas entier au cas non entier et on obtient le théorème suivant

Théorème 1.0.11. *Soient $\delta > 0$ et m, n deux entiers positifs tels que*

$$m \geq \pi^3 2^{21} C_{MPS}^3 \ln(m)^3 \ln(n)^3 \quad et \quad n \geq \pi^3 2^{21} C_{MPS}^3 \ln(n)^3,$$

où C_{MPS} est la constante du théorème 1.0.1. Soit A un ensemble $(\delta; m, n)$ fortement 2-dimensionnel de $\mathbb{R}$. On a

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{a\in A}e^{2i\pi at}\right|\mathrm{d}t \geq \frac{C_{MPS}^2}{(2^9\pi)^2\big(2+\ln(1+\frac{2}{\delta})\big)}\ln(m)\ln(n).$$

Les deux démonstrations sont basées sur un argument de Hudson et Leckband [14] utilisé pour étendre la solution de la conjecture de Littlewood du cadre entier au cadre réel. En effet, l'idée est d'approcher des nombres réels par des nombres rationnels via un lemme de Dirichlet.

Organisation du manuscrit

Dans le Chapitre 2, on rappelle certains résultats classiques comme les inégalités d'Ingham, une estimation asymptotique de la norme L^1 du noyau de Dirichlet et le théorème de Trigub qui propose une solution à la Conjecture de Littlewood 1.0.2.

Dans le Chapitre 3, on se focalise sur le cas particulier des polynômes trigonométriques harmoniques dont les fréquences sont quadratiques. On étend le résultat de Zalcwasser (1.0.12) à des polynômes trigonométriques avec coefficients complexes.

Dans le Chapitre 4, on s'intéresse aux polynômes trigonométriques non-harmoniques. Selon l'intervalle d'intégration, on donne des versions quantitatives du théorème de Nazarov 1.0.5.

Dans le Chapitre 5, on étudie les polynômes trigonométriques non-harmoniques dont l'incrément entre les fréquences tend vers l'infini et on donne une version L^1 du théorème de Kahane 1.0.8

Dans le Chapitre 6, on étend les résultats de Zygmund et de Hanson au cadre non-harmonique. Ces résultats concernent respectivement des bornes pour la norme L^1 des polynômes harmoniques lacunaires et les polynômes harmoniques dont les fréquences ont une structure multidimensionnelle.

Chapter 2

Some classical results

In this chapter, we recall some known results which will be used repeatedly in this thesis. It is divided into two main parts. We first study the L^2-estimates of exponential sums with real frequencies. In this setting, the results are due to Ingham who, under conditions on the length of the interval of integration, have extended Parseval's identity to non-integer setting.

In the second part, we look at the L^1-case. We start by recalling the L^1-norm of Dirichlet kernel and end with Trigub's solution to the Littlewood problem.

2.1 Ingham's inequalities

A non harmonic Fourier series is an expression of the type $\sum_k a_k e^{2i\pi\lambda_k t}$, in which the frequencies λ_k are real and not all integers. Paley and Wiener [31] and Levison [25] were among the first to study such series. Their main interest was to characterize the sets of frequencies for which, for each real function $f(t)$ of a given class, one can find an expression (a sum of exponentials) as defined above, summable to f for almost every $-1/2 \leq t \leq 1/2$.

Not long after Paley and Wiener, Ingham [15] showed that, under a uniform gap condition on the frequencies, and up to a constant, the L^2-norm of a non harmonic Fourier series is lower bounded by the l^2-norm of its coefficients. In the same paper, he showed an inverse inequality and hence extended Parseval's identity to the real frequencies setting. Later, Ingham's inequalities were also extended to complex valued sequences by Haraux [11] and others [1],[42].

The aim of this section is to show the following: let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a 1-separated sequence; $|\lambda_k - \lambda_\ell| \geq 1$ if $k \neq \ell$. Let $\mathcal{P}(\Lambda)$ be the set of (non-harmonic) trigonometric polynomials

$$\mathcal{P}(\Lambda) = \left\{ P(t) := \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \ : (a_k)_{k\in\mathbb{Z}} \subset \mathbb{C} \text{ with finite support} \right\}.$$

Since the set of function $\{t \to e^{2i\pi\lambda t}\}_{\lambda\in\mathbb{R}}$ is linearly independent over any interval, we can

then define two natural norms on $\mathcal{P}(\Lambda)$, namely

$$\|P\|_{L^2([-T/2,T/2])} := \left(\frac{1}{T}\int_{-T/2}^{T/2} |P(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \quad \text{and} \quad \|P\|_{\ell^2} := \left(\sum_{k\in\mathbb{Z}} |a_k|^2\right)^{\frac{1}{2}}.$$

We will show that, when $T > 1$, these two norms are equivalent. This is done by proving two inequalities. The first one is the direct inequality:

2.1.1 Ingham's direct inequality

Proposition 2.1.1. *Let $\gamma > 0$. Let $(a_k)_{k\in\mathbb{Z}}$ be a finitely supported sequence of complex numbers and $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers with $\lambda_{k+1} - \lambda_k \geq \gamma$. For every $T > 0$,*

$$\frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \leq 2\frac{\gamma T + 1}{\gamma T}\sum_{k\in\mathbb{Z}} |a_k|^2. \tag{2.1.1}$$

Proof. By the change of variable $t = s/\gamma$ we may assume that $\gamma = 1$. We consider the function h on $\mathbb{R}$ defined by

$$h(x) = \begin{cases} \cos \pi x & \text{when } |x| \leq \dfrac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

As h is real and even, its Fourier transform is given by

$$\widehat{h}(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\pi x) e^{-2i\pi t x} \,\mathrm{d}x = \frac{2}{\pi}\frac{\cos(\pi t)}{1 - 4t^2}.$$

with the understanding that $\widehat{h}(1/2) = \dfrac{1}{2}$. From this, one shows that $\widehat{h}(t) \geq \dfrac{1}{2}$ for $|t| \leq \dfrac{1}{2}$. Finally, let

$$g(x) = h * h(x) = \begin{cases} \dfrac{\sin \pi|x| - \pi(|x| - 1)\cos \pi x}{2\pi} & \text{when } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

One easily shows that g is even, non-negative, supported in $[-1, 1]$ and that $g(0) = \dfrac{1}{2}$. Further its Fourier transform is $\widehat{g}(t) = \widehat{h}(t)^2$. In particular, $\widehat{g}(t) \geq 0$ and $\widehat{g}(t) \geq \dfrac{1}{4}$ for $|t| \leq \dfrac{1}{2}$.

But then, if $(a_k)_{k\in\mathbb{Z}}$ is finitely supported and $P(t) = \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}$,

$$\begin{aligned}\int_{-1/2}^{1/2} |P(t)|^2 \,\mathrm{d}t &\leq 4\int_{-1/2}^{1/2} \widehat{g}(t)|P(t)|^2 \,\mathrm{d}t \\ &\leq 4\int_{\mathbb{R}} \widehat{g}(t)\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \\ &= 4\sum_{j,k\in\mathbb{Z}} a_j\overline{a_k}\int_{\mathbb{R}} \widehat{g}(t)e^{2i\pi(\lambda_j-\lambda_k)t}\,\mathrm{d}t \\ &= 4\sum_{j,k\in\mathbb{Z}} a_j\overline{a_k}g(\lambda_j-\lambda_k).\end{aligned}$$

Note that the sums are actually finite. Further, if $j \neq k$ then $|\lambda_j - \lambda_k| \geq 1$ and, as g is supported in $[-1,1]$, we then have $g(\lambda_j - \lambda_k) = 0$. This implies that

$$\int_{-1/2}^{1/2} |P(t)|^2 \,\mathrm{d}t \leq 4g(0)\sum_{k\in\mathbb{Z}} |a_k|^2$$

so that the inequality is proven for $T = 1$ since $4g(0) = 2$.

For $T < 1$ we simply write

$$\frac{1}{T}\int_{-T/2}^{T/2} |P(t)|^2 \,\mathrm{d}t \leq \frac{1}{T}\int_{-1/2}^{1/2} |P(t)|^2 \,\mathrm{d}t \leq \frac{2}{T}\sum_{k\in\mathbb{Z}} |a_k|^2.$$

To conclude, notice first that, if $I = [a - 1/2, a + 1/2]$ and $P(t) = \sum_{k\in\mathbb{Z}} a_j e^{2i\pi\lambda_k t}$ then

$$\begin{aligned}\int_I |P(t)|^2 \,\mathrm{d}t &= \int_{-1/2}^{1/2} |P(a+t)|^2 \,\mathrm{d}t = \int_{-1/2}^{1/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k a}e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \\ &\leq 2\sum_{k\in\mathbb{Z}} |a_k e^{2i\pi\lambda_k a}|^2 = 2\sum_{k\in\mathbb{Z}} |a_k|^2\end{aligned}$$

from the case $T = 1$.

Now let $T > 1$ and cover the interval $[-T/2, T/2]$ by $K = \lceil T\rceil \leq T + 1$ intervals $I_1, \ldots, I_K$ of length 1. Then

$$\frac{1}{T}\int_{-T/2}^{T/2} |P(t)|^2 \,\mathrm{d}t \leq \frac{1}{T}\sum_{k=1}^{K}\int_{I_k} |P(t)|^2 \,\mathrm{d}t \leq 2\frac{T+1}{T}\sum_{k\in\mathbb{Z}} |a_k|^2.$$

This completes the proof. □

We now show that a converse inequality also holds, but for $T > \dfrac{1}{\gamma}$:

2.1.2 Ingham's converse inequality

Proposition 2.1.2. *Let $\gamma > 0$. Let $(a_k)_{k\in\mathbb{Z}}$ be a finitely supported sequence of complex numbers and $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers with $\lambda_{k+1} - \lambda_k \geq \gamma$. For every $T > \dfrac{1}{\gamma}$,*

$$C(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \tag{2.1.2}$$

with

$$C(T,\gamma) = \begin{cases} \dfrac{\pi^2}{8}\dfrac{(\gamma T)^2-1}{(\gamma T)^3} & \text{for } \dfrac{1}{\gamma} < T \leq \dfrac{2}{\gamma} \\ \dfrac{\pi^2}{64} & \text{for } T \geq \dfrac{2}{\gamma} \end{cases}. \tag{2.1.3}$$

Proof. Changing variable $t = s/\gamma$ we find that $C(T,\gamma) = C(\gamma T, 1)$ so that we may assume that $\gamma = 1$. We will prove this inequality in three steps. We first establish this inequality for $1 < T \leq 2$.

As in the previous proof, let h be again defined by $h(x) = \mathbb{1}_{[-1/2,1/2]}(x)\cos\pi x$. Notice that, as h is non-negative, even, continuous with support $[-1/2, 1/2]$, then $h * h$ is non-negative, even, continuous with support $[-1, 1]$.

Next $h \in H^1(\mathbb{R})$ with $h' = -\pi\mathbb{1}_{[-1/2,1/2]}\sin\pi x$ and

$$\widehat{h'}(t) = 4it\frac{\cos\pi t}{1-4t^2}$$

thus

$$\widehat{h'*h'}(t) = -(2\pi t)^2\widehat{h}^2(t)$$

We now consider $k_T = \pi^2T^2h*h + h'*h'$ so that k_T is continuous, real valued, even and supported in $[-1, 1]$.

$$\widehat{k_T}(t) = \pi^2\left(T^2 - 4t^2\right)\widehat{h}^2(t)$$

is even (so k_T is the Fourier transform of $\widehat{k_T}$) and in L^1. Further $\widehat{k_T}$ is non-negative on $[-T/2, T/2]$ and negative on $\mathbb{R}\setminus[-T/2, T/2]$.

This implies that

$$\begin{aligned}\int_{-T/2}^{T/2}\widehat{k_T}(t)\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t &\geq \int_{\mathbb{R}}\widehat{k_T}(t)\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \\ &= \sum_{k,\ell\in\mathbb{Z}} a_k\overline{a_\ell}\int_{\mathbb{R}}\widehat{k_T}(t)e^{2i\pi(\lambda_k-\lambda_\ell)t}\,\mathrm{d}t \\ &= \sum_{k,\ell\in\mathbb{Z}} a_k\overline{a_\ell}k_T(\lambda_k-\lambda_\ell) = \sum_{k\in\mathbb{Z}}|a_k|^2k_T(0).\end{aligned}$$

In the last line, we use that $|\lambda_k - \lambda_\ell| \geq 1$ when $k \neq \ell$ thus $k_T(\lambda_k - \lambda_\ell) = 0$.

Now, for $\xi \in [-T/2, T/2]$,

$$\widehat{k_T}(\xi) = \pi^2(T^2 - 4\xi^2)\widehat{h}^2(\xi) \leq \pi^2(T^2 - 4\xi^2)\widehat{h}^2(0) \leq 4T^2$$

while

$$k_T(0) = \pi^2 \int_{-1/2}^{1/2} T^2 \cos^2 \pi t - \sin^2 \pi t \, \mathrm{d}t = \frac{\pi^2}{2}(T^2 - 1)$$

which leads to

$$\int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t \geq \frac{\pi^2}{8} \frac{T^2 - 1}{T^2} \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{2.1.4}$$

For $2 \leq T \leq 6$, we simply write

$$\int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t \geq \int_{-1}^{1} \left| \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t \geq \frac{3\pi^2}{32} \sum_{k\in\mathbb{Z}} |a_k|^2$$

where the second inequality is (2.1.4) with $T = 2$, establishing (2.1.2) with $C = \dfrac{3\pi^2}{32T} \geq \dfrac{\pi^2}{64}$.

Now let $T \geq 6$ and $M_T = [T/2]$ so that $M_T \geq \dfrac{T}{2} - 1 \geq \dfrac{T}{3}$. For $j = 0, \ldots, M_T - 1$, let $t_j = -T/2 + 2j + 1$ so that the intervals $[t_j - 1, t_j + 1[$ are disjoint and $\bigcup_{j=0}^{M_T-1} [t_j - 1, t_j + 1[\subseteq [-T/2, T/2]$ thus

$$\begin{aligned} \int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} b_k e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t &\geq \sum_{j=0}^{M_T-1} \int_{t_j-1}^{t_j+1} \left| \sum_{k\in\mathbb{Z}} b_k e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t \\ &= \sum_{j=0}^{M_T-1} \int_{-1}^{1} \left| \sum_{k\in\mathbb{Z}} b_k e^{2i\pi\lambda_k t_j} e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t. \end{aligned}$$

Now, apply (2.1.4) with $a_k = b_k e^{2i\pi\lambda_k t_j}$ and $T = 2$ to get

$$\frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} b_k e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t \geq \frac{3\pi^2}{32} \frac{M_T}{T} \sum_{k\in\mathbb{Z}} |a_k|^2 \geq \frac{\pi^2}{32} \sum_{k\in\mathbb{Z}} |a_k|^2,$$

establishing (2.1.2) with $C = \dfrac{\pi^2}{32}$. □

Finally, we notice that, with a change of variable, and a simple limiting argument to remove the condition on the support of (a_k), we have just proved the following;

Theorem 2.1.3 (Ingham)**.** *Let $\gamma > 0$ and $T > \dfrac{1}{\gamma}$ and let $C(T, \gamma)$ be given by* (2.1.3)*. Then*

– for every sequence of real numbers $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \geq \gamma$;

– *for every sequence* $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$,

$$C(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2 \le \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \le 2\frac{\gamma T+1}{\gamma T}\sum_{k\in\mathbb{Z}}|a_k|^2. \tag{2.1.5}$$

We now show that the condition $T > \dfrac{1}{\gamma}$ can not be fully removed for (2.1.2) to hold for every Λ and every $P \in \mathcal{P}(\Lambda)$.

2.1.3 Ingham's counterexample

Proposition 2.1.4 (Ingham)**.** *Let $\gamma > 0$. There exists a real sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ with $\lambda_{k+1} - \lambda_k \ge \gamma$ and a family of sequences $\big(a_k(\alpha)\big)_{k\in\mathbb{Z},0<\alpha<1/2}$ such that, if*

$$C\sum_{k=-N}^{N}|a_k|^2 \le \int_{-1/2\gamma}^{1/2\gamma}\left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right|^2 dt \tag{2.1.6}$$

holds for every $N \ge 1$ and every $c_{-N},\dots,c_N \in \mathbb{C}$, then $C = 0$.

In other words, the condition $T > \dfrac{1}{\gamma}$ is necessary in Ingham's inequality to obtain a meaningful lower bound.

Proof. After scaling we again assume that $\gamma = 1$. Let $0 < \alpha < 1/2$ and define, for $|z| < 1$,

$$g_\alpha(z) = (1+z)^{-\alpha} = \frac{\exp\left(-i\alpha \arctan\frac{\Im(z)}{1+\Re(z)}\right)}{|1+z|^\alpha} \quad , \quad |z| < 1.$$

Of course, we may also write g_α as a power series

$$g_\alpha(z) = \sum_{n=0}^{+\infty}\frac{(-\alpha)_n}{n!}z^n$$

where $(\alpha)_0 = 1$, $(-\alpha)_n = -\alpha(-\alpha-1)\cdots(-\alpha-n+1)$.

Next define

$$\begin{aligned} f_\alpha(r,t) &= 2\Re\big(e^{i\pi(\alpha+1)t}g_\alpha(re^{2i\pi t})\big) \\ &= e^{i\pi(\alpha+1)t}g_\alpha(re^{2i\pi t}) + e^{-i\pi(\alpha+1)t}g_\alpha(re^{-2i\pi t}) \\ &= \sum_{n=0}^{+\infty}\frac{(-\alpha)_n}{n!}r^n e^{2i\pi\left(n+\frac{\alpha+1}{2}\right)t} + \sum_{n=0}^{+\infty}\frac{(-\alpha)_n}{n!}r^n e^{-2i\pi\left(n+\frac{\alpha+1}{2}\right)t}. \end{aligned}$$

Now set $\Lambda = \{\lambda_j\}_{j\in\mathbb{Z}}$ with $\lambda_j = j + \dfrac{\alpha+1}{2}$ when $j \ge 0$ and $\lambda_j = j + 1 - \dfrac{\alpha+1}{2}$ for $j \le -1$, then $\lambda_{j+1} - \lambda_j \ge 1$ (and even $= 1$ excepted for $|\lambda_0 - \lambda_{-1}| = 1+\alpha$). In particular,

if we set

$$\begin{aligned} P_{m,r}(t) &= \sum_{n=0}^{m} \frac{(-\alpha)_n}{n!} r^n e^{2i\pi\left(n+\frac{\alpha+1}{2}\right)t} + \sum_{n=0}^{m} \frac{(-\alpha)_n}{n!} r^n e^{-2i\pi\left(n+\frac{\alpha+1}{2}\right)t} \\ &:= \sum_{k\in\mathbb{Z}} a_{m,r}(k) e^{2i\pi\lambda_k t} \in \mathcal{P}(\Lambda) \end{aligned}$$

and $P_{m,r} \to f_\alpha$ when $m \to +\infty$, uniformely over $t \in [-1/2, 1/2]$.

Further, Parseval's relation reads

$$\sum_{n=0}^{+\infty} \left| \frac{(-\alpha)_n}{n!} r^n \right|^2 = \int_{-1/2}^{1/2} \left| \sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} r^n e^{2i\pi nt} \right|^2 \mathrm{d}t = \int_{-1/2}^{1/2} |g_\alpha(re^{2i\pi t})|^2 \,\mathrm{d}t$$

thus

$$\lim_{m\to+\infty} \sum_{k\in\mathbb{Z}} |a_{m,r}(k)|^2 = \lim_{m\to+\infty} 2\sum_{n=0}^{+\infty} \left| \frac{(-\alpha)_n}{n!} r^n \right|^2 = 2\int_{-1/2}^{1/2} |g_\alpha(re^{2i\pi t})|^2 \,\mathrm{d}t.$$

It follows that, if we had

$$\int_{-1/2}^{1/2} |P_{m,r}(t)|^2 \,\mathrm{d}t \geq C \sum_{k\in\mathbb{Z}} |a_{m,r}(k)|^2 \tag{2.1.7}$$

then, letting $m \to +\infty$, we would also have

$$\int_{-1/2}^{1/2} |f_\alpha(r,t)|^2 \,\mathrm{d}t \geq 2C \int_{-1/2}^{1/2} |g_\alpha(re^{2i\pi t})|^2 \,\mathrm{d}t \tag{2.1.8}$$

for every $0 < r < 1$ and every $0 < \alpha < \dfrac{1}{2}$.

But, if we fix $t \in]-1/2, 1/2[$ then, when $r \to 1$,

$$g_\alpha(re^{\pm 2i\pi t}) = \frac{1}{\left(1 + re^{\pm 2i\pi t}\right)^\alpha} \to \frac{1}{\left(1 + e^{\pm 2i\pi t}\right)^\alpha} = \frac{e^{\mp i\alpha\pi t}}{2^\alpha \cos^\alpha \pi t}$$

(this is where we use that $T \leq 1$) while

$$|g_\alpha(re^{\pm 2i\pi t})|^2 = \frac{1}{\left((1-r)^2 + 4r\cos^2 \pi t\right)^\alpha} \leq \frac{1}{4\cos^{2\alpha} \pi t}$$

for $\frac{1}{2} < r < 1$. Similar bounds follow for $f_\alpha(r,t)$:

$$f_\alpha(r,t) = e^{i\pi(\alpha+1)t} g_\alpha(re^{2i\pi t}) + e^{-i\pi(\alpha+1)t} g_\alpha(re^{-2i\pi t}) \to \frac{e^{i\pi t} + e^{-i\pi t}}{2^\alpha \cos^\alpha \pi t} = \frac{1}{2^{\alpha-1}\cos^{\alpha-1} \pi t}$$

while

$$|f_\alpha(r,t)|^2 \leq \frac{1}{\cos^{2\alpha} \pi t}.$$

When $2\alpha < 1$ the majorants are integrable so that we can let $r \to 1$ in (2.1.8). This leads to

$$2^{2-2\alpha} \int_{-1/2}^{1/2} \frac{\mathrm{d}t}{\cos^{2\alpha-2} \pi t} \geq 2^{1-2\alpha} C \int_{-1/2}^{1/2} \frac{\mathrm{d}t}{\cos^{2\alpha} \pi t}. \tag{2.1.9}$$

Letting $\alpha \to \dfrac{1}{2}$, the left hand side stays bounded while the right hand side goes to $+\infty$ unless $C = 0$. □

2.1.4 Ingham's estimate in L^1

In the same paper [15], Ingham also proved an L^1-estimate for trigonometric sums:

Theorem 2.1.5 (Ingham)**.** *Let $\gamma > 0$. Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that $\lambda_{k+1} - \lambda_k \geq \gamma$. Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers. Then, for $T > \dfrac{1}{\gamma}$ and every N,*

$$\frac{2(T^2\gamma^2 - 1)}{\pi T^2 \gamma^2} \max_{k=-N,\dots,N} |a_k| \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t.$$

Ingham's estimate is a bit weaker than Nazarov's estimate (Theorem 4.1.5) when the sum is one sided, *i.e.* if $a_k = 0$ for $k = -N, \dots, -1$. On the other hand, the estimate by McGehee, Pigno and Smith and thus also the one in Nazarov's inequality are not valid for every two sided trigonometric polynomial (*see* Subsection 4.2.5).

Proof. We first prove the result for $\gamma > 1$ and $T = 1$.

Thus, take $\gamma_1 > 1$ and a sequence $(\lambda_k)_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \geq \gamma_1$ thus for $k \neq \ell$,

$$|\lambda_k - \lambda_\ell| \geq \gamma_1 |k - \ell| > 1. \tag{2.1.10}$$

We then fix $N \geq 1$ and a finite sequence $(a_k)_{k=-N,\dots,N}$. We take ℓ so that $|a_\ell| = \max_{k=-N,\dots,N} |a_k|$.

Note that if $h \in L^1(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathbb{R}} h(t) \sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t} e^{-2i\pi\lambda_\ell t} \,\mathrm{d}t &= \int_{\mathbb{R}} \sum_{k=-N}^{N} a_k h(t) e^{2i\pi(\lambda_k - \lambda_\ell)t} \,\mathrm{d}t \\
&= \sum_{k=-N}^{N} a_k \widehat{h}(\lambda_k - \lambda_\ell) \\
&= a_\ell \widehat{h}(0) + \sum_{k\in\{-N,\dots,N\}\setminus\{\ell\}} a_k \widehat{h}(\lambda_k - \lambda_\ell).
\end{aligned}$$

As for Ingham's L^2-estimate, we consider $h(t) = \mathbb{1}_{[-1/2,1/2]}(t) \cos \pi t$. As h is supported in $[-1/2, 1/2]$ and $|h| \leq 1$, and as $|a_k| \leq |a_\ell|$ we obtain

$$|a_\ell| \left(|\widehat{h}(0)| - \sum_{k\in\{-N,\dots,N\}\setminus\{\ell\}} |\widehat{h}(\lambda_k - \lambda_\ell)| \right) \leq \int_{-1/2}^{1/2} \left| \sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t.$$

On the other hand, by definition of h, $\widehat{h}(0) = \dfrac{2}{\pi}$ and, for $|t| \geq 1$, $|\widehat{h}(t)| \leq \dfrac{\widehat{h}(0)}{4t^2-1}$. With (2.1.10) we thus get

$$\begin{aligned}\sum_{k\in\{-N,\dots,N\}\setminus\{\ell\}} |\widehat{h}(\lambda_k-\lambda_\ell)| &\leq \widehat{h}(0) \sum_{k\in\{-N,\dots,N\}\setminus\{\ell\}} \frac{1}{4\gamma_1^2(k-\ell)^2-1} \\ &\leq \frac{2\widehat{h}(0)}{\gamma_1^2}\sum_{j=1}^{+\infty}\frac{1}{4j^2-1} = \frac{\widehat{h}(0)}{\gamma_1^2}.\end{aligned}$$

We thus obtain

$$\frac{2(\gamma_1^2-1)}{\pi\gamma_1^2}\max_{k\in\{-N,\dots,N\}}|a_k| \leq \int_{-1/2}^{1/2}\left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

For the general case, $\gamma > 0$ and $T > \dfrac{1}{\gamma}$, $(\lambda_k)_{k\in\mathbb{Z}}$ a sequence such that $\lambda_{k+1}-\lambda_k \geq \gamma$, $N \geq 1$ and a finite sequence $(a_k)_{k=-N,\dots,N}$, by the change of variable $t = Ts$

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t &= \int_{-1/2}^{1/2}\left|\sum_{k=-N}^{N} a_k e^{2i\pi T\lambda_k s}\right|\mathrm{d}s \\ &\geq \frac{2(T^2\gamma^2-1)}{\pi T^2\gamma^2}\max_{k\in\{-N,\dots,N\}}|a_k|\end{aligned}$$

since $T\lambda_{k+1} - T\lambda_k \geq \gamma_1 := T\gamma > 1$. □

Letting $T \to +\infty$ we obtain:

Corollary 2.1.6. *Let $\gamma > 0$. Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that $\lambda_{k+1}-\lambda_k \geq \gamma$. Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers. Then, for every N,*

$$\frac{2}{\pi}\max_{k\in\{-N,\dots,N\}}|a_k| \leq \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

In particular, for every N,

$$\frac{2}{\pi}\max_{k\in\{-N,\dots,N\}}|a_k| \leq \int_{-1/2}^{1/2}\left|\sum_{k=-N}^{N} a_k e^{2i\pi k t}\right|\mathrm{d}t.$$

Another corollary of Theorem 2.1.5 is the following

Corollary 2.1.7. *Let $\gamma > 0$. Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that $\lambda_{k+1}-\lambda_k \geq \gamma$. Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers with finite support. Then for $T > \dfrac{1}{\gamma}$, every N and every $\eta > 1$, there a positive constant $C = C_{\eta,\gamma,T}$ such that*

$$\sum_{k=-N}^{N}\frac{|a_k|}{(1+|k|)^\eta} \leq \frac{C}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t \tag{2.1.11}$$

Remark 2.1.8. *Fejer kernel shows that the condition $\eta > 1$ is mandatory. However if $a_k = 0$ for $k < 0$ then 2.1.7 holds for $\eta = 1$.*

The next part is dedicated to the L^1-case, we start by the L^1-norm of Dirichlet kernel.

2.2 Littlewood conjecture in the periodic case

2.2.1 The L^1 norm of the Dirichlet kernel

Recall that the Dirichlet kernel is given by

$$D_N(t) = \sum_{k=-N}^{N} e^{2i\pi kt} = \frac{\sin(2N+1)\pi t}{\sin \pi t}.$$

The following is a classical estimate of the L^1-norm of this kernel, which is called the *Lebesgue constant*.

Theorem 2.2.1. *When $N \to +\infty$,*

$$\|D_N\|_1 := \int_{-1/2}^{1/2} |D_N(t)| \, \mathrm{d}t = \frac{4}{\pi^2} \ln N + O(1).$$

Proof. The proof is based on the following inequality which is easy to establish: for $-\frac{\pi}{2} \leq s \leq \frac{\pi}{2}$, $|s| \left(1 - \frac{s^2}{3!}\right) \leq |\sin s| \leq |s|$. In particular, for $|t| \leq \frac{1}{2}$,

$$\frac{1}{\pi|t|} \leq \frac{1}{|\sin \pi t|} \leq \frac{1}{\pi|t|} \frac{1}{1 - \pi^2 t^2/6} \leq \frac{1}{|\pi t|} + \frac{\pi|t|}{3}$$

since $\frac{1}{1-u} \leq 1 + 2u$ for $u \leq \frac{1}{2}$. It follows that

$$\begin{aligned} \int_{-1/2}^{1/2} \frac{|\sin(2N+1)\pi t|}{|\pi t|} \, \mathrm{d}t &\leq \int_{-1/2}^{1/2} \frac{|\sin(2N+1)\pi t|}{\sin \pi t} \, \mathrm{d}t \\ &\leq \int_{-1/2}^{1/2} \frac{|\sin(2N+1)\pi t|}{|\pi t|} \, \mathrm{d}t + \int_{-1/2}^{1/2} |\sin(2N+1)\pi t| \frac{\pi|t|}{3} \, \mathrm{d}t. \end{aligned}$$

As

$$\int_{-1/2}^{1/2} |\sin(2N+1)\pi t| \frac{\pi|t|}{3} \, \mathrm{d}t \leq \int_{-1/2}^{1/2} \frac{\pi|t|}{3} \, \mathrm{d}t = \frac{\pi}{12}$$

we get

$$\|D_N\|_1 = \int_{-1/2}^{1/2} \frac{|\sin(2N+1)\pi t|}{|\pi t|} \, \mathrm{d}t + O(1).$$

Using parity and the change of variable $s = (2N+1)\pi t$ we obtain

$$\int_{-1/2}^{1/2} \frac{|\sin(2N+1)\pi t|}{|\pi t|}\,\mathrm{d}t = 2\int_0^{1/2} \frac{|\sin(2N+1)\pi t|}{|\pi t|}\,\mathrm{d}t = \frac{2}{\pi}\int_0^{N\pi+\pi/2} \frac{|\sin s|}{s}\,\mathrm{d}s.$$

Note that

$$\int_{N\pi}^{N\pi+\pi/2} \frac{|\sin s|}{s}\,\mathrm{d}s \le \int_{N\pi}^{N\pi+\pi/2} \frac{\mathrm{d}s}{s} = \ln\frac{N\pi+\pi/2}{N\pi} = O(1/N)$$

while

$$\int_0^{\pi} \frac{|\sin s|}{s}\,\mathrm{d}s \le \int_0^{\pi} 1\,\mathrm{d}s = \pi.$$

It remains to estimate

$$\begin{aligned}\frac{2}{\pi}\int_{\pi}^{N\pi} \frac{|\sin s|}{s}\,\mathrm{d}s &= \frac{2}{\pi}\sum_{j=1}^{N-1}\int_{j\pi}^{(j+1)\pi} \frac{|\sin s|}{s}\,\mathrm{d}s \\ &= \frac{2}{\pi}\sum_{j=1}^{N-1}\int_0^{\pi} \frac{|\sin s|}{s+j\pi}\,\mathrm{d}s \\ &= \frac{2}{\pi}\sum_{j=1}^{N-1}\int_0^{\pi} \frac{\sin s}{s+j\pi}\,\mathrm{d}s\end{aligned}$$

as $\sin s \ge 0$ on $[0,\pi]$. But then

$$\frac{2}{(j+1)\pi} = \int_0^{\pi} \frac{\sin s}{\pi+j\pi}\,\mathrm{d}s \le \int_0^{\pi} \frac{\sin s}{s+j\pi}\,\mathrm{d}s \le \int_0^{\pi} \frac{\sin s}{j\pi}\,\mathrm{d}s = \frac{2}{j\pi}.$$

Writing

$$\sum_{j=1}^{N-1}\frac{1}{j+1} = \sum_{j=1}^{N}\frac{1}{j} - 1 = \sum_{j=1}^{N-1}\frac{1}{j} - 1 + \frac{1}{N},$$

it follows that

$$\frac{2}{\pi}\int_{\pi}^{N\pi} \frac{|\sin s|}{s}\,\mathrm{d}s = \frac{4}{\pi^2}\sum_{j=1}^{N}\frac{1}{j} + O(1) = \ln N + O(1),$$

as stated. □

2.2.2 Solution to the Littlewood conjecture

In this section, we will present Trigub's solution [39] to the Littlewood problem (1.0.2). The proof given below will give $C = \dfrac{1}{112}$ which is not the best possible. For simplicity, we take $a_0 = 0$.

Theorem 2.2.2 (Trigub)**.** *For any increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ and any complex sequence $(a_k)_{k=1}^{\infty}$, there exists an absolute positive constant C such that*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=1}^{\infty} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq C \sum_{k=1}^{\infty} \frac{|a_k|}{k} \tag{2.2.1}$$

Proof. Let $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\|f\|_p$ be the norm of a function f in $L^p(I)$, $p \geq 1$. We denote by $a_k(f)$ the kth Fourier coefficient of f. For any sequence $(c_k)_{k=1}^{\infty}$, we define the following (finite) norm

$$\|c\| \coloneqq \sup_k \left(2^k \sum_{2^{k-1} \leq s < 2^k} |c_s|^2 \right)^{1/2}.$$

Let $b = (b_k)_1^{\infty}$ and $b_k = a_{n_k}(f)$. For any $\varepsilon > 0$ and any sequence (c_k), we will construct a function $f = f_{c,\varepsilon}$ with the following properties

1. $\|f\|_{\infty} \leq \frac{1}{\varepsilon}$
2. $\|b - c\| \leq 14\varepsilon \|c\|^2$.

Suppose that such f is given, then we can conclude as follows

$$\varepsilon \left| \sum_{k=1}^{\infty} \overline{b_k} a_k \right| = \left| \varepsilon \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{f(t)} \sum_{k=1}^{\infty} a_k e^{2i\pi n_k t} \, \mathrm{d}t \right| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=1}^{\infty} a_k e^{2i\pi n_k t} \right| \mathrm{d}t,$$

then, if

$$a = (a_k)_1^{\infty} \quad \text{and} \quad \|a\|' = \sup_{\|c\| \leq 1} \left| \sum_{1}^{\infty} c_k a_k \right| < \infty,$$

by choosing c in a special way for $\|c\| = 1$, we obtain

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=1}^{\infty} a_k e^{2i\pi n_k t} \right| \mathrm{d}t &\geq \varepsilon \left| \sum \overline{c_k} a_k \right| - \varepsilon \sum |b_k - c_k| \, |a_k| \\ &\geq \varepsilon \|a\|' - \varepsilon \|a\|' \|b - c\| \geq \|a\|' \varepsilon (1 - 14\varepsilon). \end{aligned}$$

By taking $c_k = 2^{-s+1/2}$ for $2^{s-1} \leq k < 2^s - 1$, we get

$$\|a\|' \geq \sup_{\|c\| \leq 1} \sum_{s=1}^{\infty} \sum_{k=2^{s-1}}^{2^s - 1} |c_k| \, |a_k| \geq \frac{1}{2} \sum_{k=1}^{\infty} \frac{|a_k|}{k}.$$

Setting $\varepsilon = 1/28$, we obtain the required inequality with $C = \dfrac{1}{112}$.
Let us now construct f with the desired properties. For any sequence $c = (c_k)_k$, we set, for $k \geq 1$

$$f_k(t) = \sum_{2^{k-1} \leq s < 2^k} c_s e^{2i\pi n_s t}.$$

We then write the Fourier series of each $|f_k| \in L^2(I)$ as

$$|f_k(t)| = \sum_{s\in\mathbb{Z}} a_s(|f_k|)e^{2i\pi st}$$

to which we associate $h_k \in L^2(I)$ defined via its Fourier series as

$$h_k(t) = a_0(|f_k|) + 2\sum_{s=-\infty}^{-1} a_s(|f_k|)e^{2i\pi st} = a_{0,k} + 2\sum_{s=-\infty}^{-1} a_{s,k}e^{2i\pi st}.$$

Since $|f_k|$ is real, $a_{0,k}$ is also real and $\overline{a_{s,k}} = -a_{s,k}$. We also have $\Re(h_k) = |f_k|$.

For any $\varepsilon > 0$, we set

$$f = f_{c,\varepsilon} = \sum_{k=1}^{\infty} f_k e^{-\varepsilon\sum_{j\geq k} h_j}.$$

By Parseval's identity, we have

$$\begin{aligned}
\sum_{j\geq 1}\|h_j\|_2 &= \sum_{j\geq 1}\left(a_{0,j}^2 + 4\sum_{s=-\infty}^{-1}|a_{s,j}|^2\right)^{1/2} \\
&\leq \sum_{j\geq 1}\left(2a_{0,j}^2 + 2\sum_{s=1}^{\infty}|a_{s,j}|^2 + 2\sum_{s=-\infty}^{-1}|a_{s,j}|^2\right)^{1/2} \\
&= 2\sum_j \|f_j\|_2 = 2\sum_j\left(\sum_{2^{j-1}\leq s<2^j}|c_s|^2\right)^{1/2} \\
&\leq 2\|c\|\sum_j 2^{-j/2} = \frac{2}{\sqrt{2}-1}\|c\|. \qquad (2.2.2)
\end{aligned}$$

Let $x_k = -\sum_{j\geq k}|f_j|$, we have

$$\begin{aligned}
|f| &\leq \sum_{k=1}^{\infty}|f_k||e^{-\varepsilon\sum_{j\geq k}h_j}| \leq \sum_{k=1}^{\infty}|f_k|e^{-\varepsilon\sum_{j\geq k}|f_j|} \\
&= \sum_{k=1}^{\infty} e^{\varepsilon x_k}(x_{k+1}-x_k) \leq \int_{-\infty}^{0} e^{\varepsilon x}\,\mathrm{d}x = \frac{1}{\varepsilon}.
\end{aligned}$$

then $f \in L^\infty$. It remains to prove that

$$\|b-c\| \leq 14\varepsilon\|c\|^2$$

where $b = (b_k)$ and $b_k = a_{n_k}(f)$. Since for $\Re(z_1) \leq 0$ and $\Re(z_2) \leq 0$ the inequality

$$|e^{z_1} - e^{z_2}| \leq |z_1 - z_2|$$

holds, and by (2.2.2), it follows that

$$\left\| e^{-\varepsilon \sum_s^\infty h_j} - e^{-\varepsilon \sum_s^N h_j} \right\|_1 \leq \varepsilon \left\| \sum_{j=N+1}^{\infty} h_j \right\|_1 \leq \varepsilon \sum_{j=N+1}^{\infty} \|h_j\|_1 \leq \varepsilon \sum_{j=N+1}^{\infty} \|h_j\|_2 \to 0,\ N \to \infty.$$

Next we define the spectrum of a function F by $\mathrm{spec}(F) = \{k \in \mathbb{Z};\ a_k(F) \neq 0\}$. Then

$$\mathrm{spec}(e^{-\varepsilon \sum h_j}) \subset]-\infty, 0]] \qquad \text{and} \qquad \mathrm{spec}(f_k) \subset [\![2^{k-1}, 2^k[\![.$$

Therefore, if $p \in [\![2^{s-1}; 2^s[\![$, then

$$b_p = a_{n_p}(f) = \sum_{k=1}^{\infty} a_{n_p}\left(f_k e^{-\varepsilon \sum h_j}\right) = \sum_{k=s}^{\infty} a_{n_p}\left(f_k e^{-\varepsilon \sum h_j}\right). \tag{2.2.3}$$

Indeed, let us show that $a_{n_p}\left(f_k e^{-\varepsilon \sum h_j}\right) = 0$ for $p \in [\![2^{s-1}, 2^s[\![$ and $k \leq s-1$. If it is not the case, then

$$n_p \in spec\left(f_k e^{-\varepsilon \sum h_j}\right) \subset [\![2^{k-1}, 2^k[\![+]-\infty, 0]],$$

hence $n_p < 2^k \leq 2^{s-1}$. Since $(n_p)_p$ is an increasing sequence of integers, $n_p \geq p \geq 2^{s-1}$ implying the result.
Furthermore, since $b_p = a_{n_p}(f)$ and $a_{n_p}(f_k) = c_p$, then using the inequality $|a_k(g)| \leq \|g\|_1$ and Cauchy Schwarz inequality

$$\begin{aligned}
|b_p - c_p| &= \left| \sum_{k=s}^{\infty} a_{n_p}(f_k e^{-\varepsilon \sum_{j\geq k} h_j} - f_k) \right| \leq \sum_{k=s}^{\infty} \|f_k(e^{-\varepsilon \sum_{j\geq k} h_j} - 1)\|_1 \\
&\leq \sum_{k=s}^{\infty} \left\| f_k \varepsilon \sum_{j\geq k} h_j \right\|_1 \leq 2\varepsilon \sum_{k\geq s} \|f_k\|_2 \sum_{j\geq k} \|f_j\|_2 \\
&= 2\varepsilon \sum_{k\geq s} \left(\sum_{2^{k-1}\leq s<2^k} |c_s|^2 \right)^{1/2} \sum_{j\geq k} \left(\sum_{2^{j-1}\leq q<2^j} |c_q|^2 \right)^{1/2} \\
&\leq 2\varepsilon \|c\|^2 \sum_{k\geq s} 2^{-k/2} \sum_{j\geq k} 2^{-j/2} \leq \frac{4\sqrt{2}}{\sqrt{2}-1} \varepsilon \|c\|^2 2^{-s}.
\end{aligned}$$

Hence

$$\|b - c\| = \sup_s \left(2^s \sum_{2^{s-1}\leq p<2^s} |b_p - c_p|^2 \right)^{1/2} \leq \frac{4\sqrt{2}}{\sqrt{2}-1} \varepsilon \|c\|^2 < 14\varepsilon \|c\|^2,$$

and the Theorem is proved. □

Chapter 3

Quadratic frequencies

3.1 Introduction

In this chapter, we estimate

$$s_n(x) = \sum_{k=0}^{n} a_k e^{i\pi k^2 x},$$

where the a_k's are complex numbers. We give a lower bound to the L^1-norm of this sums and then we give results similar to the one proved by Hardy and Littlewood concerning the asymptotic behavior of this sum for x irrational.

For any sequence $a = (a_k)_{k=0,\dots,n}$ of complex numbers, we define

1. The l^1-norm of the increment of a

$$\|\partial a\|_{1,n} = |a_0| + \sum_{k=1}^{n} |a_k - a_{k-1}|.$$

   Note that $\|a\|_\infty \leq \|\partial a\|_{1,n}$.

2. The arithmetic mean of $|a| = (|a_0|, |a_1|, \dots, |a_n|)$

$$\mathbb{E}\left[|a|\right] = \frac{1}{n+1} \sum_{k=0}^{n} |a_k|.$$

Zalcwasser [43] gave a lower bound of L^1-norm of s_n when $a_k = 1$ for all k. Here we are interested in the case where a is not necessarily constant. More precisely, we will prove the following result

Theorem 3.1.1. *For every $\varepsilon > 0$ there exists a constant C_ε such that if $a = (a_k)_{k=0,\dots,n}$ is a sequence of complex numbers, then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{n} a_k e^{2i\pi k^2 x} \right| \mathrm{d}x \geq C_\varepsilon \sqrt{n} \left(\frac{\left(\mathbb{E}[|a|^2]\right)^{\frac{1}{2}}}{\|\partial a\|_{1,n}} \right)^{2+\varepsilon} (\mathbb{E}[|a|^2])^{\frac{1}{2}}. \tag{3.1.1}$$

Remarks

1. When $a_k = 1$ for all k, we obtain Zalcwasser's result [43]:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{n} e^{2i\pi k^2 x} \right| \mathrm{d}x \geq C\sqrt{n} \left(\mathbb{E}[|a|^2] \right)^{1/2}. \tag{3.1.2}$$

2. By a change of variable, the inequality (3.1.1) reads

$$\int_0^2 \left| \sum_{k=0}^{n} a_k e^{i\pi k^2 x} \right| \mathrm{d}x \geq C_\varepsilon \sqrt{n} \left(\frac{\left(\mathbb{E}[|a|^2] \right)^{\frac{1}{2}}}{\|\partial a\|_{1,n}} \right)^{2+\varepsilon} (\mathbb{E}[|a|^2])^{\frac{1}{2}}. \tag{3.1.3}$$

   Hence, as one can reduce the problem to only x between 0 and 1 instead of 0 and 2 (see remark before Lemma 3.5.3), it is simpler to prove (3.1.3).

Examples

Let us now compare our result, Theorem 3.1.1, with McGehee, Pigno and Smith's Theorem 1.0.1, by giving two examples. More precisely we will investigate the behavior of the lower bounds in both theorems when n is large. That is

$$A_n \coloneqq \sum_{k=0}^{n} \frac{|a_k|}{k+1} \qquad \text{and} \qquad B_n \coloneqq \frac{\sqrt{n}}{\|\partial a\|_{1,n}^{2+\varepsilon}} \left(\frac{1}{n+1} \sum_{k=0}^{n} |a_k|^2 \right)^{\frac{3+\varepsilon}{2}},$$

where $\varepsilon > 0$.

1. Let $\alpha, \beta > 0$ and let $(a_k)_{k=0,\dots,n}$ be an increasing sequence such that $\alpha < a_k < \beta$. For n large, we have

$$B_n \geq \beta \Big(\frac{\alpha}{\beta} \Big)^{(3+\varepsilon)} \sqrt{n} \geq \beta \ln(n) \geq A_n.$$

   In other words, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{n} a_k e^{2i\pi k^2 x} \right| \mathrm{d}x \geq \beta \Big(\frac{\alpha}{\beta} \Big)^{(3+\varepsilon)} \sqrt{n},$$

   hence Theorem 3.1.1 extend (3.1.2) to more general trigonometric polynomials.

2. Let $\alpha \in \mathbb{R}$ and $a_k = (k+1)^\alpha,\ k = 0, \dots n$.

   (a) If $\alpha \geq 0$, then $B_n \sim n^{\alpha+\frac{1}{2}}$, $A_n \sim \ln(n)$ if $\alpha = 0$ and $A_n \sim n^\alpha$ if $\alpha > 0$, thus our result is better.

(b) If $\alpha < 0$, then A_n is only a constant. For B_n, we distinguish different cases. If $-\frac{1}{6} < \alpha < 0$ then $B_n \sim n^{\frac{1}{2}+\alpha(3+\varepsilon)}$, which is better than A_n. In the other case, B_n goes to 0. Indeed

$$B_n \sim \begin{cases} \ln(n)^{\frac{3+\varepsilon}{2}} n^{-\frac{2+\varepsilon}{2}} & \text{if } \alpha = -\dfrac{1}{2}, \\ n^{\frac{1}{2}+\alpha(3+\varepsilon)} & \text{if } -\dfrac{1}{2} < \alpha < -\dfrac{1}{6}, \\ n^{-\frac{2+\varepsilon}{2}} & \text{if } \alpha < -\dfrac{1}{2}. \end{cases}$$

Thus, we see that our result is better when $\alpha > -\frac{1}{6}$.

3.2 Strategy of the proof

The starting point is to establish an approximate functional equation (A.F.E). Since the Lebesgue measure of the set of all rational numbers is zero, without loss of generality, one can reduce the study to the case of only irrational x's. Moreover, we will see that one can also reduce the proof to irrationals between 0 and 1 instead of 0 and 2. Hence, subsection 3.4 is dedicated to the study of irrational numbers in $]0,1[$, more precisely we recall some known properties regarding continued fraction to be used in order to deliver a more practical form of our approximate functional equation.

The second step (subsection 3.5) consists in iterating this new A.F.E (that is why we say it is more practical) to obtain the following; for $x \in]0,1[$,

$$\left|\sum_{k=0}^{n} a_k e^{i\pi k^2 x}\right| \leq |a_0|\frac{n}{\sqrt{q_s}} + C\|\partial a\|_{1,n}\sqrt{q_s},$$

where C is an absolute positive constant and q_s is the $s-$th convergent of the continued fraction expansion of x.

We then use the previous upper bound with some properties of the continued fraction expansion of x to find C, such that for $t > 0$

$$\lambda\left(\{x \in [0,1] \ : \ \left|\sum_{k=0}^{n} a_k e^{i\pi k^2 x}\right| > t\sqrt{n}\|\partial a\|_{1,n}\}\right) \leq ct^{-4}, \tag{3.2.1}$$

where λ is the Lebesgue measure on $[0,1]$. In order to benefit from (3.2.1), we use the layer cake representation to prove that, for $p \in]2,4[$

$$\left\|\sum_{k=0}^{n} a_k e^{i\pi k^2 x}\right\|_{L^p([0,1])} \leq C_p\|\partial a\|_{1,n}\sqrt{n}$$

Then by log-convexity of L^p-norms, we interpolate 2 between 1 and p and using Parseval's identity, one can conclude the proof.

3.3 The approximate functional equation

In the following, we will investigate more general sums and henceforth we set

$$s_n(x,t)=\sum_{k=0}^{n}a_k e^{i\pi k^2 x+2i\pi kt} \quad \text{and} \quad s_n^1(x,t)=\sum_{k=0}^{n}e^{i\pi k^2 x+2i\pi kt},$$

where n is a non-negative integer, a_k are complex numbers, x and t are real numbers with

$$0<x<2 \qquad \text{and} \quad 0\le t\le 1.$$

Let $[x]$ be the integral part of x. When nx is not an integer, we define the sum s_{nx} by $s_{nx}:=s_{[nx]}$.

The aim of this section is to establish the following theorem.

Theorem 3.3.1 (Approximate Functional Equation)**.** *For $0<x<2$ and $0\le t\le 1$, we have*

$$s_n(x,t)=a_0\frac{e^{i\pi/4}}{\sqrt{x}}e^{-i\frac{\pi t^2}{x}}s_{nx}^1\left(-\frac{1}{x},\frac{t}{x}\right)+R(x,t), \tag{3.3.1}$$

where $|R(x,t)|\le\dfrac{C}{\sqrt{x}}\|\partial a\|_{1,n}$ and C is an absolute constant.

To prove this theorem, we will follow the same steps used by Mordell [28] to demonstrate the approximate functional inequality in the case $a_n=1$ for all n. This proof is simpler than that of Hardy and Littlewood [13]. We will now fix the sequence (a_n) and without loss of generality, we assume that $\|\partial a\|_{1,n}=1$ so that $|a_n|\le 1$ for all n. We also fix x,t with $0<x<2$ and $0\le t\le 1$.

We will need some preliminary notions and results. Let

$$g_{x,t}(z)=e^{i\pi z^2x+2i\pi zt} \quad \text{and} \quad f_{x,t}(z)=\frac{1}{e^{2i\pi z}-1}\sum_{k=0}^{n-1}a_k g_{x,t}(z+k).$$

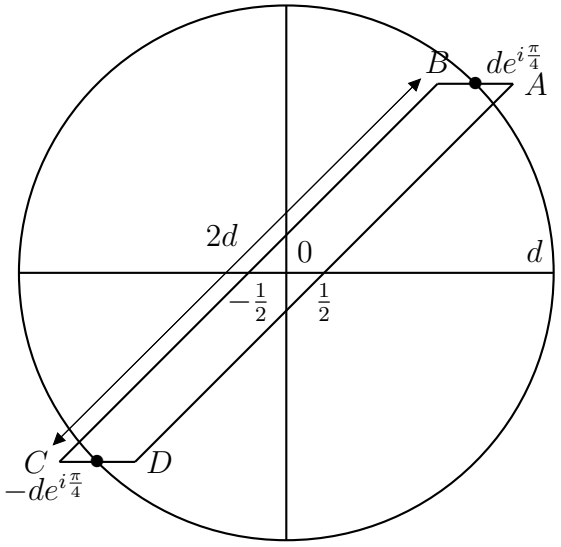
When it is unnecessary to indicate the dependency on the variables x,t, we will simply write $g=g_{x,t}$ and $f=f_{x,t}$. Thus, we have

$$s_{n-1}^1(x,t)=\sum_{k=0}^{n-1}g_{x,t}(k) \quad \text{and} \quad s_{n-1}(x,t)=\sum_{k=0}^{n-1}a_k g_{x,t}(k).$$

By the Residue Theorem, if γ is a closed contour such that the index of 0 with respect to γ is 1, the other poles of f (*i.e.*, the non-zero integers) being of index zero, we have

$$s_{n-1}(x,t)=\sum_{k=0}^{n-1}a_k g_{x,t}(k)=\int_\gamma f_{x,t}(z)\,\mathrm{d}z.$$

Let γ be the following parallelogram $ABCD$



More precisely:

- AB is parametrised by $z = \gamma_{AB}(u) = e^{i\pi/4}d + u$, u from $\dfrac{1}{2}$ to $-\dfrac{1}{2}$,
- BC is parametrised by $z = \gamma_{BC}(u) = -\dfrac{1}{2} + e^{i\pi/4}u$, u from d to $-d$
- CD is parametrised by $z = \gamma_{CD}(u) = -e^{i\pi/4}d + u$, $-\dfrac{1}{2} \leq u \leq \dfrac{1}{2}$,
- DA is parametrised by $z = \gamma_{DA}(u) = \dfrac{1}{2} + e^{i\pi/4}u, -d \leq u \leq d$.

As a result

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_{AB} f(z)\,\mathrm{d}z + \int_{BC} f(z)\,\mathrm{d}z + \int_{CD} f(z)\,\mathrm{d}z + \int_{DA} f(z)\,\mathrm{d}z \\ &= I(AB) + I(BC) + I(CD) + I(DA).\end{aligned}$$

Lemma 3.3.2. *For all $n \geq 1$, $s_{n-1}(x,t) = \lim\limits_{d\to+\infty} \int_{CB} (f_{x,t}(z+1) - f_{x,t}(z))\,\mathrm{d}z$.*

Proof. We start by proving that

$$I(AB) \underset{d\to+\infty}{\longrightarrow} 0 \qquad \text{and} \qquad I(CD) \underset{d\to+\infty}{\longrightarrow} 0.$$

Since

$$f(z) = \frac{1}{e^{2i\pi z} - 1} \sum_{k=0}^{n-1} a_k g(z+k),$$

and $|a_k| \leq \|a\|_\infty \leq \|\partial a\|_{1,n} \leq 1$, we have

$$|f(z)| \leq \frac{1}{|e^{2i\pi z} - 1|} \sum_{k=0}^{n-1} |g(z+k)|.$$

then

$$\left|\int_{AB} f(z)\,\mathrm{d}z\right| = \left|\int_{1/2}^{-1/2} f(e^{i\pi/4}d+u)\,\mathrm{d}u\right| \leq \sum_{k=0}^{n-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{g(de^{i\pi/4}+u+k)}{e^{2i\pi(de^{i\pi/4}+u)}-1}\right| \mathrm{d}u.$$

It is therefore enough to show that

$$\left|\frac{g(de^{i\pi/4}+u+k)}{e^{2i\pi(de^{i\pi/4}+u)}-1}\right| \underset{d\to+\infty}{\longrightarrow} 0$$

uniformly in $u \in \left[-\frac{1}{2},\frac{1}{2}\right]$. But

$$\begin{aligned} |g(de^{i\pi/4}+u+k)| &= |e^{i\pi(de^{i\pi/4}+u+k)^2x+2i\pi(de^{i\pi/4}+u+k)t}| \\ &= e^{-\pi d^2x-\sin(\pi/4)2(u+k)d\pi x-2\pi d\sin(\pi/4)t}. \end{aligned}$$

Using the inequality

$$|e^{\zeta}-1| \geq \big||e^{\zeta}|-1\big| = \big|e^{\Re(\zeta)}-1\big|,$$

we then obtain that

$$\left|\frac{g(de^{i\pi/4}+u+k)}{e^{2i\pi(de^{i\pi/4}+u)}-1}\right| \leq \frac{e^{-\pi d^2x+\sqrt{2}\pi((k+1/2)x+t)d}}{|e^{-\pi\sqrt{2}d}-1|} \underset{d\to+\infty}{\longrightarrow} 0$$

which allows us to conclude for the integral over the segment AB. The integral over CD is obtained in the same way. Thus

$$s_{n-1}(x,t) = \lim_{d\to+\infty}\left(\int_{DA} f_{x,t}(z)\,\mathrm{d}z - \int_{CB} f_{x,t}(z)\,\mathrm{d}z\right).$$

Since $DA = CB+1$, we have

$$\begin{aligned} \int_{DA} f(z)\,\mathrm{d}z &= \int_{-d}^{d} f(\gamma_{DA}(u))\gamma'_{DA}(u)\,\mathrm{d}u \\ &= \int_{-d}^{d} f(\gamma_{BC}(u)+1)\gamma'_{BC}(u)\,\mathrm{d}u \\ &= \int_{CB} f(z+1)\,\mathrm{d}z. \end{aligned}$$

As a result,

$$s_{n-1}(x,t) = \lim_{d\to+\infty}\int_{CB}(f_{x,t}(z+1)-f_{x,t}(z))\,\mathrm{d}z,$$

as stated. □

Lemma 3.3.3. *We can decompose*

$$s_{n-1}(x,t) = a_{n-1}J_n(x,t) - a_0J_0(x,t) + I_n(x,t)$$

where

- $J_k(x,t) = \lim\limits_{d\to+\infty} \displaystyle\int_{CB} \frac{g_{x,t}(z+k)}{e^{2i\pi z}-1}\,\mathrm{d}z,$
- $I_n(x,t) = \lim\limits_{d\to+\infty} \displaystyle\sum_{k=1}^{n-1} \left((a_{k-1}-a_k) \int_{CB} \frac{g_{x,t}(z+k)}{e^{2i\pi z}-1}\,\mathrm{d}z \right).$

Proof. We want to decompose

$$\lim_{d\to+\infty} \int_{CB} (f(z+1)-f(z))\,\mathrm{d}z.$$

But, Abel's summation shows that

$$\sum_{k=0}^{n-1} a_k\big(g(z+1+k)-g(z+k)\big) = a_{n-1}g(z+n) - a_0 g(z) + \sum_{k=1}^{n-1}(a_{k-1}-a_k)g(z+k).$$

Since $e^{2i\pi(z+1)} = e^{2i\pi z}$ and by definition of f, we deduce that

$$f(z+1)-f(z) = a_{n-1}\frac{g(z+n)}{e^{2i\pi z}-1} - a_0\frac{g(z)}{e^{2i\pi z}-1} + \sum_{k=1}^{n-1}(a_{k-1}-a_k)\frac{g(z+k)}{e^{2i\pi z}-1}.$$

By integrating over CB and passing to the limit when $d \longrightarrow +\infty$, we obtain the desired decomposition. □

Lemma 3.3.4. *Suppose that* $\mu \coloneqq \left(n-\frac{1}{2}\right) x+t \in [0,1]$ *then there exists a uniform constant* $C>0$ *such that* $|J_n(x,t)| \leq \dfrac{C}{\sqrt{x}}$.

Remark. *Until further notice, we will look at* $\mu \in [0,1]$ *and we will see at the end of the proof how to overcome this condition.*

Proof. In the definition of J the integral is over z of the form $z = \gamma_{BC}(u) = -\dfrac{1}{2} + ue^{i\pi/4}$ with $u \in [-d,d]$. But, for these z,

$$\begin{aligned} g(z+n) &= e^{i\pi(z+n)^2 x + 2i\pi(z+n)t} = e^{i\pi(-\frac{1}{2}+ue^{i\pi/4}+n)^2 x + 2i\pi(-\frac{1}{2}+ue^{i\pi/4})t} \\ &= e^{i\pi[(n-1)^2 + u^2 e^{i\pi/2} + 2(n-\frac{1}{2})ue^{i\pi/4}]x - i\pi t + 2iu\pi e^{i\pi/4}t} \\ &= e^{\pi[i(n-1)^2 - u^2 + \sqrt{2}(n-\frac{1}{2})u(-1+i)]x - i\pi t + \sqrt{2}(-1+i)u\pi}. \end{aligned}$$

Subsequently,

$$\begin{aligned} |g(z+n)| &= e^{-\pi x u^2 - \pi\sqrt{2}(n-\frac{1}{2})ux - u\pi\sqrt{2}t} = e^{-\pi x u^2 - u\pi\sqrt{2}[(n-\frac{1}{2})x+t]} \\ &= e^{-\pi x u^2 - u\pi\sqrt{2}\mu}. \end{aligned}$$

Since $0 \leq \mu \leq 1$, we deduce the estimate

$$\begin{aligned} \left|g\big(\gamma_{BC}(u)+n\big)\right| &\leq \begin{cases} e^{-\pi x u^2} & \text{if } u \geq 0 \\ e^{-\pi x u^2}(e^{-u\pi\sqrt{2}}) & \text{if } u < 0 \end{cases} \\ &\leq \begin{cases} e^{-\pi x u^2} & \text{if } u \geq 0 \\ e^{-\pi x u^2}\left|e^{2i\pi\gamma_{BC}(u)}\right| & \text{if } u < 0 \end{cases} \end{aligned}$$

since $|e^{2i\pi\gamma_{BC}(u)}| = |e^{-2\pi\Im\left(\gamma_{BC}(u)\right)}| = e^{-\sqrt{2}\pi u}$. We deduce that, when $z = \gamma_{BC}(u)$

$$\frac{\left|g\big(\gamma_{BC}(u)+n\big)\right|}{|e^{2i\pi\gamma_{BC}(u)}-1|} \leq e^{-\pi x u^2} M(u)$$

where $M(u) = \begin{cases} \dfrac{1}{|e^{2i\pi\gamma_{BC}(u)}-1|} & \text{if } u \geq 0 \\ \dfrac{|e^{2i\pi\gamma_{BC}(u)}|}{|e^{2i\pi\gamma_{BC}(u)}-1|} & \text{if } u \leq 0 \end{cases}$.

Note that

$$\begin{aligned} |e^{2i\pi z}-1|^2 &= e^{2i\pi(z-\bar{z})} - (e^{2i\pi z} + e^{-2i\pi\bar{z}}) + 1 \\ &= e^{-4\pi\Im(z)} + 2e^{-2\pi\Im(z)}\cos 2\pi\Re(z) + 1 \end{aligned}$$

and hence

$$|e^{2i\pi\gamma_{BC}(u)}-1|^2 = e^{-2\sqrt{2}\pi u} - 2e^{-\sqrt{2}\pi u}\sin\sqrt{2}\pi u + 1.$$

Since

$$e^{-\sqrt{2}\pi u}\sin\sqrt{2}\pi u \to 0 \quad \text{when} \quad u \to 0,$$

there exists u_0 such that, if $|u| < u_0$,

$$2e^{-\sqrt{2}\pi u}\sin\sqrt{2}\pi u < 1,$$

then

$$|e^{2i\pi\gamma_{BC}(u)}-1|^2 \geq e^{-2\sqrt{2}\pi u_0}.$$

For $u > u_0$,

$$|e^{2i\pi\gamma_{BC}(u)}-1|^2 \geq e^{-2\sqrt{2}\pi u} - 2e^{-\sqrt{2}\pi u} + 1 = (1-e^{-\sqrt{2}\pi u})^2 \geq (1-e^{-\sqrt{2}\pi u_0})^2$$

while, for $u < -u_0$,

$$|e^{2i\pi\gamma_{BC}(u)}-1|^2 \geq e^{-2\sqrt{2}\pi u} - 2e^{-\sqrt{2}\pi u} + 1 = (e^{-\sqrt{2}\pi u}-1)^2 \geq (e^{\sqrt{2}\pi u_0}-1)^2.$$

The denominator in the definition of M is therefore lower bounded by a strictly positive quantity and M is continuous. Moreover, M is bounded on $[0,+\infty)$. For $u \to -\infty$,

$|e^{2i\pi\gamma_{BC}(u)}| \to +\infty$ so $M(u) = \dfrac{|e^{2i\pi\gamma_{BC}(u)}|}{|e^{2i\pi\gamma_{BC}(u)} - 1|} \to 1$ when $u \to +\infty$. Thus M is bounded on $\mathbb{R}$ and

$$\frac{\big|g\big(\gamma_{BC}(u) + n\big)\big|}{|e^{2i\pi\gamma_{BC}(u)} - 1|} \leq \|M\|_\infty e^{-\pi x u^2}.$$

But then $J_n(x,t) = \displaystyle\int_{-\infty}^{+\infty} \frac{g_{x,t}\big(\gamma_{BC}(u) + k\big)}{e^{2i\pi\gamma_{BC}(u)} - 1} \gamma'_{BC}(u)\,\mathrm{d}u$ converges and is bounded by

$$\int_{-\infty}^{+\infty} \|M\|_\infty e^{-\pi x u^2}\,\mathrm{d}u = \frac{\|M\|_\infty}{\sqrt{x}},$$

as announced. □

Lemma 3.3.5. *Let $\mu = \left(n - \dfrac{1}{2}\right) x + t$, and suppose that $\mu \in [0,1]$. For at least one λ among $[nx]$ and $[nx]+1$, we have*

$$-J_0(x,t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{k=0}^{\lambda-1} e^{-i\pi(t+k)^2/x} + \tilde{J}_0(x,t) \text{ with } |\tilde{J}_0(x,t)| \leq \frac{C}{\sqrt{x}},$$

where C is a uniform constant.

Proof. We fix a non-negative integer λ, to be adjusted later on. Since

$$\sum_{k=0}^{\lambda-1} e^{2i\pi zk} = \frac{1}{1 - e^{2i\pi z}} - \frac{e^{2i\pi z\lambda}}{1 - e^{2i\pi z}},$$

we can write

$$-\int_{CB} \frac{g(z)}{e^{2i\pi z} - 1}\,\mathrm{d}z = \sum_{k=0}^{\lambda-1} \int_{CB} g(z) e^{2i\pi zk}\,\mathrm{d}z + \int_{CB} \frac{g(z) e^{2i\pi z\lambda}}{1 - e^{2i\pi z}}\,\mathrm{d}z.$$

We are going to show that each term in the sum does indeed converge to a term of the desired form and that the last integral does indeed converge to a remainder $\tilde{J}_0$.

First, we start with the terms in the sum. For $z = \gamma_{BC}(u)$,

$$\begin{aligned} g(z)e^{2i\pi zk} &= e^{i\pi z^2 x + 2i\pi z(t+k)} = e^{i\pi x\left(z + \frac{t+k}{x}\right)^2 - \frac{i\pi(t+k)^2}{x}} \\ &= e^{-\frac{i\pi(t+k)^2}{x}} e^{i\pi x\left(ue^{i\frac{\pi}{4}} - \frac{1}{2} + \frac{t+k}{x}\right)^2} \\ &= e^{-\frac{i\pi(t+k)^2}{x}} e^{-\pi x(u+v)^2} \end{aligned}$$

with $v = -ie^{i\frac{\pi}{4}}\left(\dfrac{t+k}{x} - \dfrac{1}{2}\right)$. It follows immediately that

$$\int_{CB} g(z) e^{2i\pi zk}\,\mathrm{d}z \underset{d\to+\infty}{\longrightarrow} e^{i\frac{\pi}{4}} . e^{-\frac{i\pi(t+k)^2}{x}} \int_{\mathbb{R}} e^{-\pi x(u+v)^2}\,\mathrm{d}u$$

since this integral is convergent. It is easy to show that the function $\phi : w \to \int_{\mathbb{R}} e^{-\pi x(u+w)^2}\,\mathrm{d}u$ is an entire function, when w is real $\phi(w) = \int_{\mathbb{R}} e^{-\pi x u^2}\,\mathrm{d}u = \frac{1}{\sqrt{x}}$ using a change of variable. By uniqueness of the analytic continuation, we conclude that

$$\phi(w) = \frac{1}{\sqrt{x}} \text{ for all w} \in \mathbb{C}$$

and so

$$\lim_{d\to+\infty} \int_{CB} g(z)e^{2i\pi kz}\,\mathrm{d}z = \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} e^{-i\pi(t+k)^2/x}.$$

Let us now show that the last term is indeed a remainder term when λ is well chosen. First, as in the proof of Lemma 3.3.4, when $z = \gamma_{BC}(u)$,

$$|g(z)e^{2i\pi z\lambda}| = e^{-\pi x u^2 - \sqrt{2}\pi u\left(-\frac{x}{2}+t\right)} e^{-\pi\lambda\sqrt{2}u} = e^{-\pi x u^2 - \sqrt{2}\pi u\left(-\frac{x}{2}+t+\lambda\right)}.$$

Let $\lambda \in \mathbb{N}$ such that $0 \le t + \lambda - \frac{x}{2} \le 1$. Equivalently,

$$-t + \frac{x}{2} \le \lambda \le 1 - t + \frac{x}{2}. \tag{3.3.2}$$

Note that one of the 2 integers $[nx]$ and $[nx]+1$ satisfies (3.3.2). Otherwise we would have only one of the following cases

- $1 - t + \frac{x}{2} \le [nx] \le nx$ which contradicts $\mu \le 1$,
- $[nx] + 1 \le -t + \frac{x}{2}$ which contradicts $0 \le \mu$.

We then choose among $[nx]$ and $[nx]+1$ an integer λ which satisfies (3.3.2) and the same reasoning as in the proof of Lemma 3.3.4 shows that, when $d \to +\infty$,

$$\int_{CB} \frac{g(z)e^{2i\pi z\lambda}}{1 - e^{2i\pi z}}\,\mathrm{d}z$$

converges to a term $J_1(x,t)$ satisfying $|J_1(x,t)| \le \frac{C}{\sqrt{x}}$ with C a uniform constant. □

Lemma 3.3.6. *Let $n \in \mathbb{N}$ and $\mu = \left(n - \frac{1}{2}\right)x + t$. If $0 \le \mu \le 1$, then*

$$|I_n(x,t)| \le \frac{C}{\sqrt{x}} \|\partial a\|_{1,n},$$

with C a uniform constant.

Proof. Again, when $z = \gamma_{BC}(u) = -\dfrac{1}{2} + e^{i\pi/4}u$,

$$\left|g\big(\gamma_{BC}(u)+k\big)\right| = e^{-\pi x u^2 - 2\pi \sin(\frac{\pi}{4})u[(k-\frac{1}{2})x+t]}.$$

For $k = 1, \ldots, n-1$, let $\mu_k = \left(k - \dfrac{1}{2}\right)x + t$, and notice that $0 \leq \mu_k \leq \mu \leq 1$. As in the proof of Lemma 3.3.4, we have

$$\left|\frac{g\big(\gamma_{BC}(u)+k\big)}{e^{2i\pi\gamma_{BC}(u)}-1}\right| = e^{-\pi x u^2} M(u)$$

and the function M is bounded on $\mathbb{R}$. Finally,

$$\begin{aligned} |I_n| &= \left|\lim_{d\to+\infty} \sum_{k=1}^{n-1} \left((a_{k-1}-a_k)\int_{CB} \frac{g(z+k)}{e^{2i\pi z}-1}\,\mathrm{d}z\right)\right| \\ &\leq \sum_{k=1}^{n-1} |a_{k-1}-a_k| \int_{\mathbb{R}} \left|\frac{g(z+k)}{e^{2i\pi z}-1}\right| \mathrm{d}z \\ &\leq \frac{C}{\sqrt{x}}\|\partial a\|_{1,n}, \end{aligned}$$

since $\displaystyle\sum_{k=0}^{n} |a_k - a_{k-1}| \leq \|\partial a\|_{1,n}$. □

Lemma 3.3.7. *If $0 \leq \mu \leq 1$, then*

$$s_n(x,t) = a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{k=0}^{[nx]} e^{-\frac{i\pi(t+k)^2}{x}} + R(x,t) \quad \textit{where} \quad |R(x,t)| \leq \frac{C}{\sqrt{x}}\|\partial a\|_{1,n}.$$

Proof. Recall that $|a_k| \leq \|\partial a\|_{1,n}$ for all $k \geq 0$. According to the Lemmas 3.3.3, 3.3.4, 3.3.5 and 3.3.6,

$$s_{n-1}(x,t) = a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{k=0}^{\lambda-1} e^{-\frac{i\pi(t+k)^2}{x}} + R_0(x,t) \quad \text{where} \quad |R_0(x,t)| \leq \frac{C}{\sqrt{x}}\|\partial a\|_{1,n},$$

and λ is either $[nx]$ or $[nx]+1$. First, we write $s_n(x,t) = s_{n-1}(x,t) + a_n e^{i\pi n^2 x + 2i\pi nt}$. Since $0 < x < 2$ and $|a_n| \leq \|\partial a\|_{1,n}$, we have $\left|a_n e^{i\pi n^2 x + 2i\pi nt}\right| \leq \dfrac{\sqrt{2}}{\sqrt{x}}\|\partial a\|_{1,n}$. We can therefore write s_n instead of s_{n-1} by replacing $R_0(x,t)$ by $R_1(x,t) = R_0(x,t) + a_n e^{i\pi n^2 x + 2i\pi nt}$.

When $\lambda = [nx]+1$, the lemma is established. When $\lambda = [nx]$, we write

$$a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{k=0}^{[nx]-1} e^{-\frac{i\pi(t+k)^2}{x}} = a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{k=0}^{[nx]} e^{-\frac{i\pi(t+k)^2}{x}} - a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} e^{-\frac{i\pi(t+[nx])^2}{x}}.$$

Since $|a_0| \leq \|\partial a\|_{1,n}$, the last term is absorbed into R_1. □

It remains to prove the theorem without the restriction on μ.

Proof of Theorem 3.3.1. Let $p = -\left[(n-\frac{1}{2})x+t\right] = -\left[nx-\frac{x}{2}+t\right]$ such that

$$0 \leq (n-\frac{1}{2})x+t+p \leq 1. \tag{3.3.3}$$

By applying the Lemma 3.3.7 for $s = t+p$ instead of t, we obtain

$$\sum_{k=0}^{n} a_k e^{i\pi k^2 x+2i\pi ks} = a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{k=0}^{nx} e^{-i\pi(s+k)^2/x} + R(x,s) \quad \text{where} \quad |R(x,s)| \leq \frac{C}{\sqrt{x}} \|\partial a\|_{1,n}.$$

In the first sum, we notice that $e^{2i\pi ks} = e^{2i\pi kt}$ and in the second, we make a change of summation index $j \to p+k$ and we get

$$\sum_{k=0}^{n} a_k e^{i\pi k^2 x+2i\pi kt} = a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{j=p}^{p+[nx]} e^{-i\pi(t+j)^2/x} + R(x,t-p). \tag{3.3.4}$$

where $|R(x,t-p)| \leq \frac{C}{\sqrt{x}} \|\partial a\|_{1,n}$.

We will now replace the sum over j going from p to $p+[nx]$ by a sum going from $-[nx]$ to 0. Since $-1 \leq -\frac{x}{2}+t \leq 1$ and $p = -\left[(n-\frac{1}{2})x+t\right] = -\left[nx-\frac{x}{2}+t\right]$, there are at most 2 terms to remove to go from p to $-[nx]$ and 2 terms to add to go from $p+[nx]$ to 0. Each of these terms has modulus $\frac{|a_0|}{\sqrt{x}} \leq \frac{\|\partial a\|_{1,n}}{\sqrt{x}}$ and can therefore be absorbed in the remainder term R.

Thus (3.3.4) writes

$$\begin{aligned}
\sum_{k=0}^{n} e^{i\pi k^2 x+2i\pi kt} &= a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{j=-[nx]}^{0} e^{-i\pi(t+j)^2/x} + R(x,t-p) \\
&= a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} \sum_{j=0}^{[nx]} e^{-i\pi(t-j)^2/x} + R(x,t-p) \\
&= a_0 \frac{e^{i\frac{\pi}{4}} e^{-i\pi t^2/x}}{\sqrt{x}} \sum_{k=0}^{[nx]} e^{-i\pi k^2/x+2i\pi kt/x} + R(x,t-p) \\
&= a_0 \frac{e^{i\frac{\pi}{4}}}{\sqrt{x}} e^{-i\pi t^2/x} s^1_{[nx]}\left(-\frac{1}{x}, \frac{t}{x}\right) + R(x,t-p)
\end{aligned}$$

with $|R(x,t-p)| \leq \frac{C}{\sqrt{x}} \|\partial a\|_{1,n}$ and C a constant independent of x,t,p. □

3.4 Continued fraction

The following method essentially consists of extending Euclid's algorithm to the case of irrational numbers. For more details on this part, we refer to the book by Choimet and Queffélec [3]. Suppose that x irrational of $]0,1[$ and t an element of [0.1],

$$\begin{cases} x = \omega_0, \\ \frac{1}{\omega_0} = v_1' = v_1 + \omega_1, & \text{with } v_1 \in \mathbb{N}^*, 0 < \omega_1 < 1, \\ \frac{1}{\omega_1} = v_2' = v_2 + \omega_2, & \text{with } v_2 \in \mathbb{N}^*, 0 < \omega_2 < 1, \\ \vdots & \\ \frac{1}{\omega_n} = v_{n+1}' = v_{n+1} + \omega_{n+1}, & \text{with } v_{n+1} \in \mathbb{N}^*, 0 < \omega_{n+1} < 1, \\ \vdots & \end{cases} . \tag{3.4.1}$$

As x is irrational, the process never stops. The v_n are called the partial quotients of the continued fraction expansion of x. The convergents of this expansion are the rationals $\frac{p_n}{q_n}$ $(n \geq 0)$ defined by the following sequences of integers $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$:

$$\begin{cases} p_{-1} = 1, & p_0 = 0 \quad \text{and} \quad p_{n+1} = v_{n+1}p_n + p_{n-1} \text{ for } n \geq 0, \\ q_{-1} = 0, & q_0 = 1 \quad \text{and} \quad q_{n+1} = v_{n+1}q_n + q_{n-1} \quad \text{for } n \geq 0. \end{cases}$$

For $n \geq 1$, we have $\dfrac{p_n}{q_n} = \cfrac{1}{v_1 + \cfrac{1}{v_2 + \cfrac{1}{v_3 + \ddots \; v_{n-1} + \cfrac{1}{v_n}}}}$.

Finally let $h_n = p_{n-1}q_n - p_nq_{n-1}$ for $n \geq 0$.

Remark. *We have:*

1. *Every irrational number (more generally positive real number x [6]) can be represented in precisely one way as an infinite continued fraction. The continued fraction is finite if and only if x is rational.*

2. *ω_j satisfies the following inequality:*

$$\omega_j\omega_{j+1} \leq \frac{1}{2} \qquad \textit{for all } j. \tag{3.4.2}$$

Indeed, since $v_j \geq 1$ for all j,

$$\omega_j = \frac{1}{v_{j+1} + \omega_{j+1}} \leq \frac{1}{1 + \omega_{j+1}},$$

and then

$$\omega_j\omega_{j+1} \leq \frac{\omega_{j+1}}{1+\omega_{j+1}} \leq \frac{1}{2},$$

since $0 \leq \omega_{j+1} < 1$ *and the function* $x \longrightarrow \dfrac{x}{1+x}$ *is increasing on* $[0,1]$.

The following result relates the ω_n to the convergents.

Lemma 3.4.1. *For all* $n \geq 0$, *we have*

$$(-1)^n\omega_0\omega_1 \ldots \omega_n = q_n x - p_n.$$

Proof. Let $\beta_n = (-1)^n\omega_0\omega_1 \ldots \omega_n$ for $n \geq 0$. We proceed by induction on n. $\beta_0 = \omega_0 = x = q_0x + p_0$ and $\beta_1 = -\omega_0\omega_1 = -\omega_0(\dfrac{1}{\omega_0} - v_1) = v_1\omega_0 - 1 = v_1x - 1 = q_1x - p_1$. Suppose the lemma is true up to order n.

$$\begin{aligned}
\beta_{n+1} &= -\beta_n\omega_{n+1} = -\beta_n(\frac{1}{\omega_n} - v_{n+1}) = v_{n+1}\beta_n - \frac{\beta_n}{\omega_n} \\
&= v_{n+1}\beta_n + \beta_{n-1} \\
&= v_{n+1}(q_nx - p_n) + q_{n-1}x - p_{n-1} \\
&= x(v_{n+1}q_n + q_{n-1}) - (v_{n+1}p_n + p_{n-1}) \\
&= q_{n+1}x - p_{n+1}
\end{aligned}$$

by construction of the sequence (p_n, q_n). □

Lemma 3.4.2. *For all* $n \geq 0$, *on a* $h_n = (-1)^n$.

Proof. By induction on n, $h_0 = p_{-1}q_0 - p_0q_{-1} = 1$ and if $h_n = (-1)^n$ then

$$\begin{aligned}
h_{n+1} &= p_nq_{n+1} - p_{n+1}q_n = p_n(v_{n+1}q_n + q_{n-1}) - q_n(v_{n+1}p_n + p_{n-1}) \\
&= v_{n+1}p_nq_n + p_nq_{n-1} - v_{n+1}p_nq_n - q_np_{n-1} \\
&= p_nq_{n-1} - q_np_{n-1} \\
&= -h_n
\end{aligned}$$

which is indeed $(-1)^{n+1}$ with the induction hypothesis. □

Lemma 3.4.3. *For all* $n \geq 0$, *we have* $x = \dfrac{v'_{n+1}p_n + p_{n-1}}{v'_{n+1}q_n + q_{n-1}}$.

Proof. For $n = 0$,

$$\frac{v'_1p_0 + p_{-1}}{v'_1q_0 + q_{-1}} = \frac{1}{v'_1} = \omega_0 = x.$$

Moreover,

$$\begin{aligned}\frac{v'_{n+2}p_{n+1}+p_n}{v'_{n+2}q_{n+1}+q_n} &= \frac{\frac{p_{n+1}}{\omega_{n+1}}+p_n}{\frac{q_{n+1}}{\omega_{n+1}}+q_n} = \frac{\frac{v_{n+1}p_n+p_{n-1}}{\omega_{n+1}}+p_n}{\frac{v_{n+1}q_n+q_{n-1}}{\omega_{n+1}}+q_n} \\ &= \frac{p_n(v_{n+1}+\omega_{n+1})+p_{n-1}}{q_n(v_{n+1}+\omega_{n+1})+q_{n-1}} \\ &= \frac{v'_{n+1}p_n+p_{n-1}}{v'_{n+1}q_n+q_{n-1}} = x.\end{aligned}$$

The result is then established by induction. □

Lemma 3.4.4. *Let $q'_{n+1} = v'_{n+1}q_n + q_{n-1}$. For all $n \geq 0$,*

$$|q_n x - p_n| = \frac{1}{q'_{n+1}}.$$

Proof. By Lemma 3.4.3,

$$\begin{aligned}q_n x - p_n &= q_n\left(\frac{v'_{n+1}p_n+p_{n-1}}{v'_{n+1}q_n+q_{n-1}} - p_n\right) \\ &= \frac{q_n(v'_{n+1}p_n+p_{n-1}) - p_n(v'_{n+1}q_n+q_{n-1})}{q'_{n+1}} \\ &= \frac{q_n p_{n-1} - p_n q_{n-1}}{q'_{n+1}} \\ &= \frac{h_n}{q'_{n+1}}.\end{aligned}$$

hence $|q_n x - p_n| = \frac{1}{q'_{n+1}}$. □

Corollary 3.4.5. *For all $n \geq 0$, we have*

$$q_{n+1} \leq q'_{n+1} \leq 2q_{n+1}$$

and

$$\frac{1}{2q_{n+1}} \leq \omega_0\omega_1\ldots\omega_n \leq \frac{1}{q_{n+1}}.$$

Proof. By Lemmas 3.4.1 and 3.4.4, we have

$$\omega_0\ldots\omega_n = |(-1)^n\omega_0\ldots\omega_n| = |q_n x - p_n| = \frac{1}{q'_{n+1}}.$$

So, the second inequality follows from the first one. However,

$$v'_{n+1} = v_{n+1} + \omega_{n+1} \quad \text{with} \quad 0 < \omega_{n+1}$$

hence

$$v'_{n+1} \geq v_n \quad \text{and} \quad q'_{n+1} \geq q_{n+1}.$$

Moreover, $\omega_{n+1} < 1$ then $v'_{n+1} \leq v_{n+1} + 1$, and we have

$$\begin{aligned} q'_{n+1} &= v'_{n+1}q_n + q_{n-1} \\ &\leq (1 + v_{n+1})q_n + q_{n-1} \\ &= q_n + q_{n+1} \\ &\leq 2q_{n+1}, \end{aligned}$$

since the sequence $(q_n)_{n\geq 0}$ increases. This implies the result. □

3.5 Intermediate results

We will proceed by combining two essential ingredients: the approximate functional equation (3.3.1) that we will give in a more practical form, well suited to an iteration and the continued fraction expansion of irrational numbers as described in section 3.5 (see [12]).

Lemma 3.5.1. *Let $t \in [0,1]$ and $0 < x < 2$. We define $\omega_1 = \dfrac{1}{x} - \left[\dfrac{1}{x}\right]$, then there exists $u = u(x,t) \in \mathbb{C}$ of modulus 1 and $\tilde{t} = \tilde{t}(x,t) \in [0,1]$ such that*

$$s_n(x,t) = a_0 \frac{u}{\sqrt{x}} \overline{s^1_{[nx]}(\omega_1, \tilde{t}\,)} + R(x,t).$$

where $|R(x,t)| \leq \dfrac{C}{\sqrt{x}} \|\partial a\|_{1,n}$ with C a uniform constant.

Proof. Let $\theta = -\dfrac{t}{x}$ and $\dfrac{1}{x} = v_1 + \omega_1$ with $v_1 = \left[\dfrac{1}{x}\right]$ and $\omega_1 = \dfrac{1}{x} - \left[\dfrac{1}{x}\right]$. By Theorem 3.3.1

$$\begin{aligned} s_n(x,t) &= a_0 \frac{e^{i\pi/4}}{\sqrt{x}} e^{-i\pi t^2/x} s^1_{nx}\left(-\frac{1}{x}, \frac{t}{x}\right) + R \\ &= a_0 \frac{u}{\sqrt{x}} s^1_{[nx]}(-v_1 - \omega_1, -\theta) + R, \end{aligned}$$

with $u = e^{i\frac{\pi}{4}}.e^{-i\frac{\pi t^2}{x}}$ and $|R| = |R(x,t)| \leq \dfrac{C}{\sqrt{x}} \|\partial a\|_{1,n}$where C is a uniform constant.

Since $(-1)^{k^2} = (-1)^k = e^{i\pi k}$ when k is an integer, then, for any integer m, s^1_m satisfies another (non approximate) functional equation;

$$\begin{aligned} s^1_m(x-1,t) &= \sum_{k=0}^{m} e^{i\pi k^2(x-1)+2i\pi kt} = \sum_{k=0}^{m} (-1)^{k^2} e^{i\pi k^2 x + 2i\pi kt} \\ &= \sum_{k=0}^{m} e^{i\pi k^2 x + 2i\pi kt + i\pi k} = s^1_m\left(x, t + \frac{1}{2}\right). \end{aligned}$$

By iterating, we obtain more generally that, for all integers ℓ, m, $s^1_m(x-\ell,t) = s^1_m\left(x, t+\frac{\ell}{2}\right)$. In particular, we have

$$s_n(x,t) = a_0\frac{u}{\sqrt{x}}s^1_{[nx]}(-\omega_1, -\theta+\frac{v_1}{2}) + R.$$

If we define $\tilde{t} = \theta - \frac{v_1}{2} - \left[\theta - \frac{v_1}{2}\right] \in [0,1]$, then by 1-periodicity of $s^1_{[nx]}$ in the second variable

$$s_n(x,t) = a_0\frac{u}{\sqrt{x}}s^1_{[nx]}(-\omega_1, -\tilde{t}\,) + R.$$

We conclude by noticing that $s^1_{[nx]}(-\omega_1, -\tilde{t}\,) = \overline{s^1_{[nx]}(\omega_1, \tilde{t}\,)}$. □

Theorem 3.5.2. *Let x be an irrational in the interval $]0,1[$ and $t \in [0,1]$. Then, for $n,\ s \geq 1$, we have*

$$|s_n(x,t)| \leq |a_0|\frac{n}{\sqrt{q_s}} + C\|\partial a\|_{1,n}\sqrt{q_s},$$

where $C > 0$ is an absolute constant, q_s denotes the denominator of the s-th convergent of the continued fraction expansion of x.

Proof. This proof is similar to the one by Zalcwasser [43] and consists in iterating Lemma 3.5.1. Here, C is the constant of the Lemma 3.5.1 and the ω_j have been defined in (3.4.1).

Since $x = \omega_0$, by Lemma 3.5.1, if we take $n_1 = [n\omega_0]$, $t_1 = \tilde{t}(x,t)$ and $u_1 = u(x,t)$, we have

$$s_n(x,t) = a_0\frac{u_1}{\sqrt{x}}\overline{s^1_{n_1}(\omega_1,t_1)} + R_1 = a_0\frac{u_1}{\sqrt{\omega_0}}\overline{s^1_{n_1}(\omega_1,t_1)} + R_1$$

where $|R_1| = |R(x,t)| \leq \frac{C}{\sqrt{x}}\|\partial a\|_{1,n}$.

Again, by Lemma 3.5.1, writing $t_2 = \tilde{t}(\omega_1, t_1)$,

$$\begin{aligned} s_n(x,t) &= a_0\frac{u_1}{\sqrt{\omega_0}}\left[\frac{u_2}{\sqrt{\omega_1}}s^1_{[n_1\omega_1]}(\omega_2,t_2) + R(\omega_1,t_1)\right] + R_1 \\ &= a_0\frac{u_1u_2}{\sqrt{\omega_0\omega_1}}s^1_{[n_1\omega_1]}(\omega_2,t_2) + a_0\frac{u_1}{\sqrt{\omega_0}}R(\omega_1,t_1) + R_1 \\ &= a_0\frac{u_1u_2}{\sqrt{\omega_0\omega_1}}s^1_{[n_1\omega_1]}(\omega_2,t_2) + R_2. \end{aligned}$$

But since $|R(\omega_1,t_1)| \leq \frac{C}{\sqrt{\omega_1}}$, (note that here, we apply the Lemma 3.5.1 to the constant sequence $\mathbf{1} = (1,\ldots)$ and that $\|\partial\mathbf{1}\|_{1,n} = 1$), we have

$$\begin{aligned} |R_2| = \left|a_0\frac{u_1}{\sqrt{\omega_0}}R(\omega_1,t_1) + R_1\right| &\leq \frac{C}{\sqrt{\omega_0\omega_1}}|a_0| + \frac{C}{\sqrt{\omega_0}}\|\partial a\|_{1,n} \\ &\leq C\left(\frac{1}{\sqrt{\omega_0}} + \frac{1}{\sqrt{\omega_0\omega_1}}\right)\|\partial a\|_{1,n}. \end{aligned}$$

Then, by induction, we define the sequence n_j by $n_{j+1} = [n_j\omega_j]$ and since $n_{j+1} \leq n_j\omega_j$ then $n_p\omega_p \leq n\omega_0 \cdots \omega_p$.

By applying $p+1$ times the Lemma 3.5.1, we deduce that

$$s_n(x,t) = a_0 \frac{u_1 \cdots u_{p+1}}{\sqrt{\omega_0 \dots \omega_p}} s^1_{[n_p\omega_p]}(\omega_{p+1}, t_{p+1}) + R_{p+1}$$

where

– $t_{p+1} = \tilde{t}(\omega_p, t_p)$;

– the u_j are complex numbers of modulus 1;

– $R_{p+1} \leq C\left(\frac{1}{\sqrt{\omega_0}} + \cdots + \frac{1}{\sqrt{\omega_0 \dots \omega_p}}\right)\|\partial a\|_{1,n}$.

Since $|s^1_{[n_p\omega_p]}| \leq [n_p\omega_p] \leq n\omega_0 \cdots \omega_p$, we obtain

$$\begin{aligned} |s_n(x,t)| &\leq |a_0| \frac{n\omega_0 \dots \omega_p}{\sqrt{\omega_0 \dots \omega_p}} + C\left(\frac{1}{\sqrt{\omega_0}} + \dots \frac{1}{\sqrt{\omega_0 \dots \omega_p}}\right)\|\partial a\|_{1,n} \\ &\leq |a_0| n\sqrt{\omega_0 \dots \omega_p} + \frac{C}{\sqrt{\omega_0 \dots \omega_p}}(1 + \sqrt{\omega_p} + \sqrt{\omega_p\omega_{p-1}} + \dots \sqrt{\omega_p \dots \omega_1})\|\partial a\|_{1,n}. \end{aligned}$$

By Inequality (3.4.2), we have

$$\begin{aligned} |s_n(x,t)| &\leq |a_0| n\sqrt{\omega_0 \dots \omega_p} + \frac{C}{\sqrt{\omega_0 \dots \omega_p}}\left(1 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2} + \dots\right)\|\partial a\|_{1,n} \\ &\leq |a_0| n\sqrt{\omega_0 \dots \omega_p} + \left(\frac{7C}{\sqrt{\omega_0 \dots \omega_p}}\right)\|\partial a\|_{1,n}, \end{aligned}$$

since $2\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k = \frac{2\sqrt{2}}{\sqrt{2}-1} \leq 7$. We now select $p = s-1$ and use Corollary 3.4.5 to obtain the bound

$$|s_n(x,t)| \leq |a_0|\frac{n}{\sqrt{q_s}} + 7\sqrt{2}C\sqrt{q_s}\|\partial a\|_{1,n}$$

as stated. □

Let

$$S_n(x) = s_n(x,0) = \sum_{k=0}^{n} a_k e^{i\pi k^2 x}$$

Since $|S_n(2-x)| = |\sum_{k=0}^{n} \overline{a_k} e^{i\pi k^2 x}|$ and $\|\partial\overline{a}\|_{1,n} = \|\partial a\|_{1,n}$, the study of $S_n(x)$, $x \in]0,2[$ is reduced to the case $x \in]0,1[$.

Lemma 3.5.3. *There exists an absolute constant c such that, for $t > 0$ we have*

$$|\{x \in [0,1] \,:\, |S_n(x)| > t\sqrt{n}\|\partial a\|_{1,n}\}| \leq ct^{-4}.$$

Proof. Let C be the constant appearing in Theorem 3.5.2. We fix $t > 0$ and $n \in \mathbb{N}^*$, and denote E the set of irrationals numbers x in $]0,1[$ such that $|S_n(x)| > t\sqrt{n}\|\partial a\|_{1,n}$. Let x in E and q_j, as previously, the denominator of the convergents in the continued fraction expansion of x. We distinguish 2 cases:

- **First case,** $q_1 > \dfrac{t^2 n}{4C^2}$**.** Then,

$$x < \frac{1}{v_1} = \frac{1}{q_1} \leq \frac{4C^2}{t^2 n},$$

in other words $x \in \left]0, \frac{4C^2}{t^2 n}\right]$.

- **Second case,** $q_1 \leq \dfrac{t^2 n}{4C^2}$**.** By the hypothesis and Theorem 3.5.2, we have for $s \geq 1$:

$$t\sqrt{n}\|\partial a\|_{1,n} < |S_n(x)| \leq |a_0|\frac{n}{\sqrt{q_s}} + C\|\partial a\|_{1,n}\sqrt{q_s}.$$

Subsequently

$$\text{either} \quad |a_0|\frac{n}{\sqrt{q_s}} > \frac{t\sqrt{n}\|\partial a\|_{1,n}}{2} \quad \text{or} \quad C\sqrt{q_s} > \frac{t\sqrt{n}}{2}.$$

Since $|a_0| \leq \|\partial a\|_{1,n}$, then for all $s \geq 1$, we have the following alternative:

$$\text{either} \quad q_s < \frac{4|a_0|^2 n}{t^2\|\partial a\|_{1,n}^2} \leq \frac{4n}{t^2} \quad \text{or} \quad q_s > \frac{t^2 n}{4C^2}. \tag{3.5.1}$$

Since we assumed that $q_1 \leq \dfrac{t^2 n}{4C^2}$ and $q_s \underset{s\to+\infty}{\longrightarrow} \infty$, there exists an integer $s_0 \geq 0$ such that

$$q_{s_0} \leq \frac{t^2 n}{4C^2} \leq q_{s_0+1}.$$

In this case, by Lemma 3.4.4 and Corollary 3.4.5,

$$|q_{s_0}x - p_{s_0}| = \frac{1}{q'_{s_0+1}} < \frac{1}{q_{s_0+1}},$$

thus

$$\left|x - \frac{p_{s_0}}{q_{s_0}}\right| \leq \frac{1}{q_{s_0}q_{s_0+1}} \leq \frac{1}{q_{s_0}}\frac{4C^2}{t^2 n},$$

so that

$$x \in \left[\frac{p_{s_0}}{q_{s_0}} - \frac{1}{q_{s_0}}\frac{4C^2}{t^2 n}, \frac{p_{s_0}}{q_{s_0}} + \frac{1}{q_{s_0}}\frac{4C^2}{t^2 n}\right].$$

In summary if $x \in E$ we have

$$x \in \left]0, \frac{4C^2}{t^2 n}\right] \cup \left[\frac{p_{s_0}}{q_{s_0}} - \frac{1}{q_{s_0}}\frac{4C^2}{t^2 n}, \frac{p_{s_0}}{q_{s_0}} + \frac{1}{q_{s_0}}\frac{4C^2}{t^2 n}\right].$$

Moreover,

$$\begin{aligned} |\{x \in [0,1] \,:\, |S_n(x)| > t\sqrt{n}\|\partial a\|_{1,n}\}| &\leq \frac{4C^2}{t^2 n} + \frac{2}{q_{s_0}}\frac{4C^2}{t^2 n} \\ &\leq \frac{4C^2}{t^2 n} + \frac{8C^2}{t^2 n} = \frac{12C^2}{t^2 n}, \end{aligned}$$

since $q_{s_0} \geq 1$.

Finally, since $t\sqrt{n}\|\partial a\|_{1,n} \geq 2n\|\partial a\|_{1,n} > |S_n|$, if $t > 2\sqrt{n}$,

$$|\{x \in [0,1] \,:\, |S_n(x)| > t\sqrt{n}\|\partial a\|_{1,n}\}| \leq |\{x \in [0,1] \,:\, |S_n(x)| > 2n\|\partial a\|_{1,n}\}| = 0.$$

Furthermore, if $t \leq 2\sqrt{n}$ *i.e* if $\dfrac{1}{n} \leq \dfrac{4}{t^2}$ we find

$$|\{x \in [0,1] \,:\, |S_n(x)| > t\sqrt{n}\|\partial a\|_{1,n}\}| \leq \frac{48C^2}{t^4}$$

as stated. □

Lemma 3.5.4. *Let p be a real number in $]2,4[$. There exists a constant C_p depending on p such that*

$$\left\| \frac{S_n}{\sqrt{n}} \right\|_{L^p([0,1])} \leq C_p \|\partial a\|_{1,n}.$$

Proof. Let $X = \dfrac{|S_n|}{\sqrt{n}\|\partial a\|_{1,n}}$. By Lemma 3.5.3, $\lambda(|X| > t) \leq \min(1, ct^{-4})$ then

$$\begin{aligned} \|X\|_p^p &= \int_0^\infty p t^{p-1} \lambda(|X| > t)\, \mathrm{d}t \\ &\leq \int_0^1 p t^{p-1}\, \mathrm{d}t + \int_1^\infty p t^{p-1} c t^{-4}\, \mathrm{d}t \\ &\leq 1 + c\frac{p}{4-p} := C_p^p \end{aligned}$$

as stated. □

3.6 Last step

We can now now prove the main theorem.

Proof of Theorem 3.1.1. Let $p \in]2,4[$. The end of the proof consists in applying Lemma 3.5.4 and Holder's Inequality. By Lemma 3.5.4,

$$\begin{aligned}\|S_n\|^p_{L^p([0,2])} = \int_0^2 \left|\sum_{k=0}^n a_k e^{i\pi k^2 x}\right|^p \mathrm{d}x &= \int_0^1 \left|\sum_{k=0}^n a_k e^{i\pi k^2 x}\right|^p \mathrm{d}x + \int_0^1 \left|\sum_{k=0}^n \overline{a_k} e^{i\pi k^2 x}\right|^p \mathrm{d}x \\ &\leq C_p \|\partial a\|_{1,n}\sqrt{n},\end{aligned}$$

since both terms in the right hand side satisfy Lemma 3.5.4. Next, we interpolate 2 between 1 and p we thus write

$$\frac{1}{2} = \frac{\eta}{1} + \frac{1-\eta}{p} \tag{3.6.1}$$

By Holder's Inequality, or equivalently log convexity of L^p norms

$$\|S_n\|_2 \leq \|S_n\|_p^{1-\eta}\|S_n\|_1^{\eta} \leq C_p^{1-\eta} n^{\frac{1-\eta}{2}} \|\partial a\|_{1,n}^{1-\eta}\|S_n\|_1^{\eta},$$

but

$$\|S_n\|_{L^2([0,2])} = \sqrt{2}\|a\|_{\ell^2},$$

then

$$\|a\|_{\ell^2} \leq C_p^{1-\eta} n^{\frac{1-\eta}{2}} \|\partial a\|_{1,n}^{1-\eta}\|S_n\|^{\eta}_{L^1([0,2])}$$

and we get

$$\|S_n\|_{L^1([0,2])} \geq \frac{C_{p,\eta}}{n^{\frac{1-\eta}{2\eta}}\|\partial a\|_{1,n}^{\frac{1-\eta}{\eta}}}\|a\|_{\ell^2}^{\frac{1}{\eta}}.$$

From (3.6.1), one can write p as a function of η. Let $\varepsilon > 0$, since $0 < \eta < \frac{1}{3}$, one can write $\eta = \dfrac{1}{3+\varepsilon}$, yielding the result. □

3.7 Estimations of the uniform norm

We finish this chapter with two upper bounds for

$$\sup_{0\leq t\leq 1} |s_n(x,t)|$$

when $x \in]0,1[$ is a fixed irrational number. The proof depends heavily on the Diophantine properties x, and is a direct application of Theorem 3.5.2.

Corollary 3.7.1. *Let x be an irrational in $]0,1[$ and a be a sequence of complex numbers verifying $\sum_{k=0}^{\infty} |a_k - a_{k-1}| < \infty$. Then*

1. *We have*

$$\sup_{0\leq t\leq 1} |s_n(x,t)| = o(n) \text{ as } n \to \infty.$$

2. *If further x has bounded partial quotients, that is, if the sequence $(v_k)_{k\geq 1}$ is bounded, then*
$$\sup_{0\leq t\leq 1} |s_n(x,t)| = \mathcal{O}(\sqrt{n}) \text{ as } n \to \infty.$$

Proof. Let $M_n = \sup_{0\leq t\leq 1} |s_n(x,t)|$.

1. Theorem 3.5.2 gives, for $n, s \geq 1$:
$$\frac{M_n}{n} \leq \frac{|a_0|}{\sqrt{q_s}} + \frac{C\|\partial a\|_{1,n}\sqrt{q_s}}{n}$$
but then, for s fixed, by letting n goes to infinity and then s goes to infinity we obtain the result.

2. Let n be an integer greater than 1. As $n \geq 1 = q_0$, and since $(q_k)_k$ is an increasing sequence, there exists an integer $s \geq 1$ such that $q_{s-1} \leq n \leq q_s$. Since $v_s \geq 1$ and by definition of q_s
$$q_s = v_s q_{s-1} + q_{s-2} \leq 2v_s q_{s-1} \leq 2v_s n.$$
Now the upper bound of the Theorem 3.5.2 gives
$$|s_n(x,t)| \leq |a_0|\sqrt{n} + C\|\partial a\|_{1,n}\sqrt{2v_s n}.$$
But since x has bounded partial quotients v_s, say $|v_s| \leq M$, we have
$$|s_n(x,t)| \leq \sqrt{n}\|\partial a\|_{1,n}(1 + C\sqrt{2M}),$$
which is indeed
$$\sup_{0\leq t\leq 1} |s_n(x,t)| = \mathcal{O}(\sqrt{n}),$$

yielding the corollary. □

Chapter 4

Littlewood problem and non-harmonic Fourier series

4.1 Introduction

In the fifties, Littlewood [26] suggested that, up to a constant, the log function is a nice lower bound for L^1-norm of trigonometric polynomials having only 0 or 1 as coefficients and integer frequencies (a question referred to as the Littlewood problem):

$$\int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq C \ln(N+1),$$

where C is independent of N.

Nearly 40 years later, Konyagin [24], and independently McGehee, Pigno and Smith [27] affirmed that Littlewood's thoughts were correct. Both obtained the solution as a corollary of a stronger result. Here we are interested in the latter result [27] which states as follows

Theorem 4.1.1 (McGehee, Pigno, Smith)**.** *For $n_0 < n_1 < \cdots < n_N$ integers and $a_0, \ldots, a_N$ complex numbers,*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi n_k t} \right| \mathrm{d}t \geq C_{MPS} \sum_{k=0}^{N} \frac{|a_k|}{k+1},$$

where C_{MPS} is a universal constant ($C_{MPS} = 1/30$ would do it).

The year after, Stegeman and Yabuta independently improved on the result of McGehee, Pigno and Smith using some modified version of their proof:

Theorem 4.1.2 (Stegeman [38], Yabuta [41])**.** *Let $N \geq 3$. For $n_0 < n_1 < \cdots < n_N$ integers*

$$\int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln(N+1).$$

Next, one can ask the same question as Littlewood but for real frequencies. The problem is quite different since we lose periodicity of the exponential sums as well as the orthogonality. The first result in this direction was by Hudson and Leckband who used a clever perturbation argument based on a lemma by Dirichlet ([44, p 235], [8]) to prove the following

Theorem 4.1.3 (Hudson & Leckband [14]). *For $\lambda_0 < \lambda_1 < \ldots < \lambda_N$ real numbers and $a_0, \ldots, a_N$ complex numbers,*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \, \mathrm{d}t \geq C_{MPS} \sum_{k=0}^{N} \frac{|a_k|}{k+1}$$

where C_{MPS} is the same constant as in Theorem 4.1.1.

Not long after, and using the same method, Nazarov [30] showed that one can extend the estimate to the case of non-integer frequencies. However, and to be able to do so, Nazarov slightly enlarged the interval of integration; he replaced $[-\frac{1}{2}, \frac{1}{2}]$ by $[-\frac{T}{2}, \frac{T}{2}]$ for $T > 1$.

Theorem 4.1.4 (Nazarov [30]). *For $T > 1$, there exists a constant C_T such that, for $0 < \lambda_0 < \cdots < \lambda_N$ real numbers such that $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ and $a_0, \ldots, a_N$ complex numbers,*

$$\int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \, \mathrm{d}t \geq C_T \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{4.1.1}$$

It is worth mentioning that the constants in Nazarov's proof are not explicit and that the problem is still open for $T = 1$.

Furthermore, its not clear that Ingham's counter-example [15] for the L^2-case can be used to prove that the theorem does not hold for $T = 1$. From here, one can see that, when tackling the problem of finding explicit constant, one should deal with multiple cases.

The first part concerns Nazarov's theorem for large enough intervals [17], the aim is to improve on Nazarov's proof to obtain a quantitative version of his result, that is an estimate of the constant C_T. By doing so, one can recover previous results for both integer, and real frequencies. Moreover, we obtain the best constants known today.

The second part is dedicated to the case where T is sufficiently small *i.e* T is near 1 [20]. Our aim remains the same, to obtain an estimate of the constant C_T. Combining the two cases, our main result states as follows

Theorem 4.1.5. *Let $\lambda_0 < \lambda_1 < \cdots < \lambda_N$ be distinct real numbers and $a_0, \ldots, a_N$ be complex numbers. Then*

i. we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \, \mathrm{d}t \geq \frac{1}{26} \sum_{k=0}^{N} \frac{|a_k|}{k+1}.$$

ii. If further $a_0, \ldots, a_N$ all have modulus larger than 1, $|a_k| \geq 1$ then

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \, \mathrm{d}t \geq \frac{4}{\pi^3} \ln(N+1).$$

iii. Assume further that for $k = 0, \dots, N-1$, $\lambda_{k+1} - \lambda_k \geq 1$, then, for every $T > 1$, there exists a constant $C(T)$ such that, for every $a_0, \dots, a_N \in \mathbb{C}$,

$$\frac{1}{T}\int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq C(T) \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{4.1.2}$$

Moreover,

(a) for $T \geq 72$ we can take $C(T) = \dfrac{1}{122}$;

(b) for $1 < T \leq 2$, $C(T) = O\big((T-1)^{15/2}\big)$.

Remark. *For $2 < T < 72$, (4.1.2) follows from the case (b) with $T = 2$, but the constant is not totally explicit.*

Let us make a few comments on the result. First, the limit in the statement of the result are well-known to exist and are the Besicovitch norms of $\displaystyle\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}$. Further, when the λ_k's are all integers, then

$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t = \int_{-1/2}^{1/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t$$

so that we recover Theorems 4.1.1 and 4.1.2 (with the best constants as of today). As the constants in Theorem 4.1.5 are the best known for C_{MPS} we also recover Theorem 4.1.3 while at this stage, we only recover Theorem 4.1.4 for large enough T which is due to the strategy of proof (see below).

Further, note that the left hand side in Theorem 4.1.5*i)* and *ii)* is unchanged if one replaces the λ_k's by $\alpha\lambda_k + \beta$. In the proof we will thus assume that $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ (or equivalently that $\lambda_{k+1} - \lambda_k \geq 1$). In Theorem 4.1.5*iii)* this restriction only affects the critical T for which our proof works.

4.2 Quantitative version for large T

We are going to prove the first part of Theorem 4.1.5, that is i, ii and iii a).

4.2.1 Strategy of the proof

Without loss of generality one can suppose $a_0 = 0$ *i.e.* the sums start from one. This proof is closely related to the one implemented by McGehee, Pigno and Smith as extended by Nazarov to prove Theorem 4.1.4, but we here follow constants more closely. Further, we introduce various parameters which will be optimized in the last step. We fix a (non-harmonic) trigonometric polynomial

$$\phi(t) = \sum_{k=1}^{N} a_k e^{2\pi i\lambda_k t} \quad \text{and} \quad S = \sum_{k=1}^{N} \frac{|a_k|}{k}. \tag{4.2.1}$$

We then write $|a_k| = a_k u_k$ with u_k complex numbers of modulus 1 and introduce

$$U(t) = \sum_{k=1}^{N} \frac{u_k}{k} e^{-2\pi i \lambda_k t}.$$

Using the orthogonality relation

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{2i\pi\lambda t} e^{-2i\pi\mu t}\, \mathrm{d}t = \delta_{\lambda,\mu} \tag{4.2.2}$$

we see that

$$S = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi(t) U(t)\, \mathrm{d}t. \tag{4.2.3}$$

The second step will consist in correcting U into V in such a way that $\|V\|_\infty \le A$ where A is a numerical constant (that does not depend on N or T) and so that, for each k,

$$\lim_{T\to+\infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} \big(U(t) - V(t)\big) e^{2i\pi\lambda_k t}\, \mathrm{d}t \right| \le \frac{\alpha}{k}$$

with $\alpha < 1$. In particular, if we multiply by a_k and sum over k, we get

$$\lim_{T\to+\infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} \big(U(t) - V(t)\big) \phi(t)\, \mathrm{d}t \right| \le \alpha S.$$

Then, writing

$$S = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi(t) V(t)\, \mathrm{d}t + \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi(t) \big(U(t) - V(t)\big)\, \mathrm{d}t$$

we would obtain

$$S \le \|V\|_\infty \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\phi(t)|\, \mathrm{d}t + \alpha S$$

that is

$$S \le \frac{A}{1-\alpha} \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\phi(t)|\, \mathrm{d}t$$

as desired.

The difficulty in implementing this strategy lies in the fact that one must control ϕ, U, V over the entire real line. We will instead fix a large T and use an auxiliary function adapted to $[-T/2, T/2]$ so as to only do the computations over this interval while controlling errors. Here we will exploit the fact that T is large that allows us to change Nazarov's auxiliary function into a better behaved one. The first task is then to estimate the error made when replacing the limit in (4.2.3) with the mean over $[-T/2, T/2]$. The second step is then the correction of U into a bounded V. This correction is only made over the interval $[-T/2, T/2]$ and is roughly done the same way as was originally done by McGehee, Pigno, Smith, but implementing the improvements made by Stegeman and Yabuta and again controlling errors.

The remaining of the section is devoted to the proof that is divided into three steps, a subsection being devoted to each of them.

4.2.2 An auxiliary function and the estimate of U

We now introduce several notations and parameters that will be fixed later:

– a parameter $\delta \geq 2$ and the sequence $(\beta_j)_{j\geq 0}$ given by $\beta_0 = 1$, $\beta_{j+1} = \beta_j + \delta^j$ that is $\beta_j = \sum_{k=0}^{j-1} \delta^k = \frac{\delta^j - 1}{\delta - 1}$. Up to enlarging N, we can assume that $N = \beta_{n+1} - 1$ for some $n \geq 2$. We then define

$$\mathcal{D}_j = \{k \in \mathbb{Z} \,:\, \beta_j \leq k < \beta_{j+1}\}$$

so that $|\mathcal{D}_j| = \delta^j$. Note that for every $\ell \in \{1, \dots, N\}$ there is a unique $j_\ell \in \{1, \dots, n\}$ such that $\ell \in \mathcal{D}_{j_\ell}$. Further, this allows to write $\sum_{k=1}^N$ in the form $\sum_{j=0}^n \sum_{k\in\mathcal{D}_j}$. Note also that if $k \in \mathcal{D}_j$,

$$\frac{1}{k+1} \leq \frac{1}{k + \frac{1}{\delta - 1}} \leq (\delta - 1)\delta^{-j} \tag{4.2.4}$$

– a sequence of real numbers $(\lambda_k)_{k=1,\dots,N}$ such that for every k, ℓ $|\lambda_k - \lambda_\ell| \geq |k - \ell|$ (or equivalently $\lambda_{k+1} \geq \lambda_k + 1$ for $k = 1, \dots, N-1$);

– a sequence of complex numbers $(a_k)_{k=1,\dots,N}$ and we write $|a_k| = a_k u_k$ with $(u_k)_{k=1,\dots,N}$ a sequence of complex numbers of modulus 1;

– an integer $p \geq 4$ and an interval $I_p = \left[-\frac{p^2+p}{2}, \frac{p^2+p}{2}\right]$ of length $|I_p| = p^2 + p$.

We then define inductively $\varphi_1 = \mathbf{1}_{[-p^2/2, p^2/2]} * \mathbf{1}_{[-1/2,1/2]}$ and $\varphi_{j+1} = \varphi_j * \mathbf{1}_{[-1/2,1/2]}$. Note that, φ_j is even, non-negative and $\|\varphi_j\|_\infty \leq 1$ while $\|\varphi_j\|_1 = p^2$. We then define $\varphi = \frac{p^2+p}{p^2}\varphi_p$ so that φ is supported in I_p, is bounded by 2 and has Fourier transform

$$\mathcal{F}[\varphi](\lambda) := \int_{\mathbb{R}} \varphi(t) e^{-2i\pi\lambda t}\, \mathrm{d}t = (p^2+p)\frac{\sin p^2\pi\lambda}{p^2\pi\lambda}\left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^p.$$

We will mainly need that,

$$\|\varphi\|_\infty = \frac{p^2+p}{p^2} \quad , \quad \mathcal{F}[\varphi](0) = |I_p| \quad \text{and} \quad |\mathcal{F}[\varphi](\lambda)| \leq \frac{|I_p|}{(\pi\lambda)^p}. \tag{4.2.5}$$

Finally, we will write

$$\begin{aligned} \phi(t) &= \sum_{k=1}^N a_k e^{2i\pi\lambda_k t}, \\ U(t) &= \sum_{j=0}^n \frac{1}{|\mathcal{D}_j|} \sum_{k\in\mathcal{D}_j} u_k e^{-2i\pi\lambda_k t}, \\ S &= \sum_{j=0}^n \frac{1}{|\mathcal{D}_j|} \sum_{k\in\mathcal{D}_j} |a_k|. \end{aligned}$$

Note that in view of (4.2.4),

$$\sum_{k=1}^N \frac{|a_k|}{k+1} \leq (\delta - 1)S \tag{4.2.6}$$

so that it is enough to bound S.

The following is the key estimate in this section:

Lemma 4.2.1. *With the previous notation, there is a $p(\delta) \geq 2$ such that, when $p \geq p(\delta)$ then for $1 \leq \ell \leq N$ and j_ℓ be the unique index for which $\ell \in \mathcal{D}_{j_\ell}$, we have*

$$\sum_{j=0}^{n} \frac{1}{|\mathcal{D}_j|} \sum_{k \in \mathcal{D}_j \setminus \{\ell\}} |\mathcal{F}[\varphi](\lambda_k - \lambda_\ell)| \leq \frac{1}{2}\, \delta^{-j_\ell}. \tag{4.2.7}$$

Proof. Write

$$\begin{aligned} E &:= \sum_{j=0}^{n} \frac{1}{|\mathcal{D}_j|} \sum_{k \in \mathcal{D}_j \setminus \{\ell\}} |\mathcal{F}[\varphi](\lambda_k - \lambda_\ell)| \\ &= \sum_{j=j_\ell-1}^{n} \frac{1}{|\mathcal{D}_j|} \sum_{k \in \mathcal{D}_j \setminus \{\ell\}} |\mathcal{F}[\varphi](\lambda_k - \lambda_\ell)| + \sum_{j=0}^{j_\ell-2} \frac{1}{|\mathcal{D}_j|} \sum_{k \in \mathcal{D}_j} |\mathcal{F}[\varphi](\lambda_k - \lambda_\ell)| \\ &= E_+ + E_-. \end{aligned}$$

For the estimate of E_+, as $|\lambda_k - \lambda_\ell| \geq |k - \ell|$, $|\mathcal{F}[\varphi](\lambda_k - \lambda_\ell)| \leq \dfrac{|I_p|}{\pi^p} \dfrac{1}{|k-\ell|^p}$ and

$$E_+ \leq \frac{|I_p|}{\pi^p} \delta^{-(j_\ell-1)} \sum_{j=j_\ell-1}^{n} \sum_{k \in \mathcal{D}_j \setminus \{\ell\}} \frac{1}{|k-\ell|^p} \leq \frac{2|I_p|}{\pi^p} \delta^{-(j_\ell-1)} \sum_{m=1}^{\infty} \frac{1}{m^p} \leq \frac{4|I_p|}{\pi^p} \delta^{-(j_\ell-1)}$$

since $\delta > 1$ and the last series is bounded by $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \leq 2$. It remains to notice that $\dfrac{4\delta|I_p|}{\pi^p} = \dfrac{4\delta(p^2+p)}{\pi^p} \leq \dfrac{1}{4}$ when p is large enough to get

$$E_+ \leq \frac{\delta^{-j_\ell}}{4}. \tag{4.2.8}$$

For the second sum, note that it is only present when $j_\ell \geq 2$. So, if $k \in \mathcal{D}_j$ with $j \leq j_\ell - 2$ then

$$\ell - k \geq \beta_{j_\ell} - \beta_{j_\ell - 1} = \delta^{j_\ell - 1}$$

thus

$$|\mathcal{F}[\varphi](\lambda_k - \lambda_j)| \leq \frac{|I_p|}{\pi^p \delta^{(j_\ell-1)p}}.$$

It follows that

$$E_- \leq \sum_{j=0}^{j_\ell-2} \sum_{k \in \mathcal{D}_j} \frac{|I_p|}{\pi^p \delta^{(j_\ell-1)p}} \leq \beta_{j_\ell - 1} \frac{|I_p|}{\pi^p \delta^{(j_\ell-1)p}}$$

since there are at most $\beta_{j_\ell-1}$ terms in this sum. But $\beta_{j_\ell-1} \leq \dfrac{\delta^{j_\ell-1}}{\delta-1}$ so that

$$E_- \leq \frac{|I_p|\delta}{\pi^p(\delta-1)\delta^{(j_\ell-1)(p-2)}}\delta^{-j_\ell} \leq \frac{1}{4}\delta^{-j_\ell} \tag{4.2.9}$$

when p is large enough to have $\dfrac{|I_p|\delta}{\pi^p(\delta-1)} \leq \dfrac{1}{4}$ since $\delta^{(j_\ell-1)(p-2)} \geq 1$.

It remains to put (4.2.8)-(4.2.9) into (4.2.8) to obtain the result. □

Remark 4.2.2. *The proof shows that $p(\delta)$ is the smallest integer such that*

$$\frac{4\delta(p^2+p)}{\min(1,\delta-1)\pi^p} \leq \frac{1}{4}.$$

For instance, if we choose $\delta = 4$, we will obtain $p(\delta) = 8$.

From this, we deduce the following:

Corollary 4.2.3. *With the notations above, for $\ell = 1, \ldots, N$*

$$\left|\frac{1}{|I_p|}\int_{I_p} U(t)e^{2i\pi\lambda_\ell t}\varphi(t)\,\mathrm{d}t - \frac{u_\ell}{\delta^{j_\ell}}\right| \leq \frac{1}{2|I_p|}\frac{1}{\delta^{j_\ell}} \tag{4.2.10}$$

Proof. Indeed

$$\begin{aligned}
\frac{1}{|I_p|}\int_{I_p} U(t)e^{2i\pi\lambda_\ell t}\varphi(t)\,\mathrm{d}t &= \sum_{j=0}^{n}\frac{1}{|\mathcal{D}_j|}\sum_{k\in\mathcal{D}_j} u_k \frac{1}{|I_p|}\int_{I_p} e^{-2i\pi\lambda_k t}e^{2i\pi\lambda_\ell t}\varphi(t)\,\mathrm{d}t \\
&= \frac{1}{|I_p|}\sum_{j=0}^{n}\frac{1}{|\mathcal{D}_j|}\sum_{k\in\mathcal{D}_j} u_k\mathcal{F}[\varphi](\lambda_k-\lambda_\ell) \\
&= u_\ell\delta^{-j_\ell}\frac{\mathcal{F}[\varphi](0)}{|I_p|} + \frac{1}{|I_p|}\sum_{j=0}^{n}\frac{1}{|\mathcal{D}_j|}\sum_{k\in\mathcal{D}_j\setminus\{\ell\}} u_k\mathcal{F}[\varphi](\lambda_k-\lambda_\ell).
\end{aligned}$$

As $\mathcal{F}[\varphi](0) = |I_p|$, we get

$$\left|\frac{1}{|I_p|}\int_{I_p} U(t)e^{2i\pi\lambda_\ell t}\varphi(t)\,\mathrm{d}t - \frac{u_\ell}{\delta^{j_\ell}}\right| \leq \frac{1}{|I_p|}\sum_{j=0}^{n}\frac{1}{|\mathcal{D}_j|}\sum_{k\in\mathcal{D}_j\setminus\{\ell\}} |\mathcal{F}[\varphi](\lambda_k-\lambda_\ell)|$$

and Lemma 4.2.1 gives the result. □

This allows us to obtain the approximation of S by an integral of $\phi(t)U(t)$.

Proposition 4.2.4. *Under the previous notation*

$$S \leq \frac{2|I_p|}{2|I_p|-1}\left|\frac{1}{|I_p|}\int_{I_p} U(t)\phi(t)\varphi(t)\,\mathrm{d}t\right|.$$

Proof. According to (4.2.10),

$$\left|\frac{1}{|I_p|}\int_{I_p} U(t)e^{2i\pi\lambda_\ell t}\varphi(t)\,\mathrm{d}t - u_\ell\delta^{-j_\ell}\right| \leq \frac{1}{2|I_p|}\delta^{-j_\ell}.$$

Multiplying the expression in the absolute value by a_ℓ, we get

$$\left|\frac{1}{|I_p|}\int_{I_p} U(t)a_\ell e^{2i\pi\lambda_\ell t}\varphi(t)\,\mathrm{d}t - \frac{|a_\ell|}{\delta^{j_\ell}}\right| \leq \frac{|a_\ell|}{2|I_p|}\delta^{-j_\ell}.$$

The triangle inequality then gives

$$\left|\frac{1}{|I_p|}\int_{I_p} U(t)\phi(t)\varphi(t)\,\mathrm{d}t - S\right| \leq \frac{1}{2|I_p|}S.$$

The result follows with the reverse triangular inequality. □

4.2.3 Construction of V

Before we start this section, let us recall Hilbert's inequality (*see e.g.* [3, Chapter 10]).

Lemma 4.2.5 (Hilbert's inequality)**.** *Let $\lambda_1,\dots,\lambda_N$ be real numbers with $|\lambda_k-\lambda_\ell|\geq 1$ when $k\neq\ell$, and let $z_1,\dots,z_N$ be complex numbers. We have*

$$\left|\sum_{\substack{1\leq k,\ell\leq N\\ k\neq\ell}}\frac{z_k\overline{z_\ell}}{\lambda_k-\lambda_\ell}\right| \leq \pi\sum_{k=1}^{N}|z_k|^2.$$

We will now decompose U into $\mathcal{D}_j$-blocs f_j. More precisely, we set

$$\tilde{f}_j(t)=\sum_{k\in\mathcal{D}_j}u_k e^{-2i\pi\lambda_k t},\quad f_j=\frac{1}{|\mathcal{D}_j|}\tilde{f}_j$$

so that

$$U(t)=\sum_{j=0}^{n}\frac{1}{|\mathcal{D}_j|}\sum_{k\in\mathcal{D}_j}u_k e^{-2i\pi\lambda_k t}=\sum_{j=0}^{n}f_j(t).$$

Our aim in this section is to modify U in such a way that we obtain a trigonometric polynomial V that is in a sense still similar to U but satisfies an L^∞ bound that is uniform in N.

We start by estimating the norms of the f_j's:

Lemma 4.2.6. *With the above notation, we have*

1. $\|f_j\|_{L^2(I_p)}\leq\delta^{-j/2}\sqrt{|I_p|+1}$;

2. $\|f_j\|_\infty \leq 1$.

Proof. For the second bound, obviously $\|\tilde{f}_j\|_\infty \leq |\mathcal{D}_j| = \delta^j$. For the first bound, we have

$$\begin{aligned}
\|\tilde{f}_j\|^2_{L^2(I_p)} &= \int_{I_p} \tilde{f}_j(t)\overline{\tilde{f}_j(t)}\,\mathrm{d}t = \int_{I_p} \sum_{k,\ell\in\mathcal{D}_j} u_k\overline{u_\ell} e^{-2i\pi(\lambda_k-\lambda_\ell)t}\,\mathrm{d}t \\
&= |I_p| \sum_{k\in\mathcal{D}_j} |u_k|^2 + \sum_{\substack{k,\ell\in\mathcal{D}_j \\ k\neq\ell}} u_k\overline{u_\ell} \int_{-|I_p|/2}^{|I_p|/2} e^{-2i\pi(\lambda_k-\lambda_\ell)t}\,\mathrm{d}t \\
&= |I_p|\,|\mathcal{D}_j| + \sum_{\substack{k,\ell\in\mathcal{D}_j \\ k\neq\ell}} u_k\overline{u_\ell} \left(\frac{e^{i|I_p|\pi(\lambda_k-\lambda_\ell)} - e^{-i|I_p|\pi(\lambda_k-\lambda_\ell)}}{2i\pi(\lambda_k-\lambda_\ell)}\right).
\end{aligned}$$

Now, set $z_k^\pm = u_k e^{\pm i|I_p|\pi\lambda_k}$ so that $|z_k^\pm| = 1$. We have just shown that

$$\|\tilde{f}_j\|^2_{L^2(I_p)} = |I_p|\,|\mathcal{D}_j| + \frac{1}{2i\pi} \sum_{\substack{k,\ell\in\mathcal{D}_j \\ k\neq\ell}} \frac{z_k^+\overline{z_\ell^+}}{\lambda_k-\lambda_\ell} - \frac{1}{2i\pi} \sum_{\substack{k,\ell\in\mathcal{D}_j \\ k\neq\ell}} \frac{z_k^-\overline{z_\ell^-}}{\lambda_k-\lambda_\ell}.$$

Applying Hilbert's Inequality to the last two sums, we get

$$\|\tilde{f}_j\|^2_{L^2(I_p)} \leq |I_p|\,|\mathcal{D}_j| + \frac{1}{2}\sum_{k\in\mathcal{D}_j} |z_k^+|^2 + \frac{1}{2}\sum_{k\in\mathcal{D}_j} |z_k^-|^2 = (|I_p|+1)|\mathcal{D}_j|.$$

The bound for $\|f_j\|^2_{L^2(I_p)}$ follows. □

Note that the proof also shows that $\|f_j\|_{L^2(I_p)} \geq \delta^{-j/2}\sqrt{|I_p|-1}$.

Notation 4.2.7. *For a function $F \in L^2(I_p)$ and $s \in \mathbb{Z}$, we write*

$$c_s^p(F) = \frac{1}{|I_p|}\int_{I_p} F(t)e^{-2i\pi\frac{st}{|I_p|}}\,\mathrm{d}t$$

for the Fourier coefficients of F. Its Fourier series is then

$$F(t) = \sum_{s\in\mathbb{Z}} c_s^p(F)e^{2i\pi\frac{st}{|I_p|}}$$

and Parseval's relation reads

$$\frac{1}{|I_p|}\int_{I_p} |F(t)|^2\,\mathrm{d}t = \sum_{s\in\mathbb{Z}} |c_s^p(F)|^2.$$

We then write the Fourier series of each $|f_j| \in L^2(I_p)$ as

$$|f_j|(t) = \sum_{s\in\mathbb{Z}} c_s^p(|f_j|)e^{2i\pi\frac{st}{|I_p|}}$$

to which we associate $h_j \in L^2(I_p)$ defined via its Fourier series as

$$h_j(t) = c_0^p(|f_j|) + 2\sum_{s=1}^{\infty} c_s^p(|f_j|)e^{2i\pi\frac{st}{|I_p|}}.$$

Lemma 4.2.8. *For $0 \leq j \leq n$, the following properties hold*

1. $\mathrm{Re}(h_j) = |f_j| \leq 1$;
2. $\|h_j\|_{L^2(I_p)} \leq \sqrt{2}\|f_j\|_{L^2(I_p)}$.

Proof. First, as $|f_j|$ is real valued, $c_0^p(|f_j|)$ is also real, and $\overline{c_s^p(|f_j|)} = c_{-s}^p(|f_j|)$ for every $s \geq 1$. A direct computation then shows that $\mathrm{Re}(h_j) = |f_j|$ which is less than 1 by Lemma 4.2.6 while Parseval shows that $\|h_j\|_2 \leq \sqrt{2}\|f_j\|_2$. □

We now define a sequence $(F_j)_{j=0,\ldots,n}$ inductively through

$$F_0 = f_0 \quad \text{and} \quad F_{j+1} = F_j e^{-\eta h_{j+1}} + f_{j+1}$$

where $0 < \eta \leq 1$ is a real number that we will fix later. Further set

$$E_\eta := \sup_{0<x\leq 1} \frac{x}{1-e^{-\eta x}} = \frac{1}{\eta} \sup_{0<x\leq \eta} \frac{x}{1-e^{-x}} = \frac{1}{1-e^{-\eta}}.$$

Lemma 4.2.9. *For $0 \leq j \leq n$, $\|F_j\|_\infty \leq E_\eta$.*

Proof. By definition of E_η, if $C \leq E_\eta$ and $0 \leq x \leq 1$, then $Ce^{-\eta x} + x \leq E_\eta e^{-\eta x} + x \leq E_\eta$.

We can now prove by induction over j that $|F_j| \leq E_\eta$ from which the lemma follows. First, when $j = 0$, from Lemma 4.2.6 we get

$$\|F_0\|_\infty = \|f_0\|_\infty \leq 1 \leq E_\eta.$$

Assume now that $\|F_j\|_\infty \leq E_\eta$, then

$$\begin{aligned} |F_{j+1}(t)| &= |F_j(t)e^{-\eta h_{j+1}(t)} + f_{j+1}(t)| \leq |F_j(t)|e^{-\eta \Re\left(h_{j+1}(t)\right)} + |f_{j+1}(t)| \\ &= |F_j(t)|e^{-\eta|f_{j+1}(t)|} + |f_{j+1}(t)|. \end{aligned}$$

As $|f_{j+1}(t)| \leq 1$ and $|F_j(t)| \leq E_\eta$, we get $|F_{j+1}(t)| \leq E_\eta$ as claimed. □

Lemma 4.2.10. *For $0 \leq \ell \leq n$ and $j = 0, \ldots, k$, let $g_{j,k} = e^{-\eta H_{j,k}}$ with*

$$H_{j,k} = \begin{cases} h_{j+1} + \ldots + h_k & \text{when } j < k \\ 0 & \text{when } j = k \end{cases}.$$

Then $F_k = \sum_{j=0}^{k} f_j g_{j,k}$. Moreover

$$\|H_{j,k}\|_{L^2(I_p)} \leq \frac{\sqrt{2(|I_p|+1)}}{\sqrt{\delta}-1}\delta^{-j/2}.$$

Proof. For the first part, we use induction on k. First, when $k = 0$, $H_{0,0} = 0$ thus $g_{0,0} = 1$ and, indeed, we have

$$F_0 = f_0 = f_0 g_{0,0}.$$

Assume now that the formula has been established at rank $k-1$ and let us show that $F_k = \sum_{j=0}^{k} f_j g_{j,k}$. By construction, we have

$$F_k = F_{k-1} e^{-\eta h_k} + f_k = \left(\sum_{j=0}^{k-1} f_j g_{j,k-1} \right) e^{-\eta h_k} + f_k.$$

with the induction hypothesis. It remains to notice that $g_{k,k} = e^{-\eta H_{k,k}} = 1$ and that, for $j = 0, \dots, k-1$, $H_{j,k} = H_{j,k-1} + h_k$ thus $g_{j,k} = g_{j,k-1} e^{-\eta h_k}$ so that, indeed, we have $F_k = \sum_{j=0}^{k} f_j g_{j,k}$.

Next, it is enough to estimate $H_{j,k}$ when $j < k$ in which case

$$\|H_{j,k}\|_{L^2(I_p)} \leq \sum_{r=j+1}^{k} \|h_r\|_{L^2(I_p)} \leq \sqrt{2} \sum_{r=j+1}^{k} \|f_r\|_{L^2(I_p)}$$

with Lemma 4.2.8. But then, from Lemma 4.2.6 we get

$$\|H_{j,k}\|_{L^2(I_p)} \leq \sqrt{2(|I_p|+1)} \sum_{r=j+1}^{k} \delta^{-\frac{r}{2}} \leq \sqrt{2(|I_p|+1)} \frac{\delta^{-\frac{j+1}{2}}}{1-\delta^{-1/2}} = \frac{\sqrt{2(|I_p|+1)}}{\sqrt{\delta}-1} \delta^{-j/2}$$

as claimed. □

Next, we will need the following well-known lemma:

Lemma 4.2.11. *If $H \in H^\infty$ and $Re(H) \geq 0$, then $e^{-H} \in H^\infty$ and*

$$\|e^{-H} - 1\|_2 \leq \|H\|_2.$$

Proof. Since H^∞ is a Banach algebra, the partial sums $\sum_{k=0}^{n} (-1)^k \frac{H^k}{k!}$ of e^{-H} are elements of H^∞. Moreover, since H is bounded, these sums converge uniformly toward e^{-H}, with $e^{-H} \in H^\infty$. Finally, if $z \in \mathbb{C}$ and $\Re(z) \geq 0$,

$$\left| e^{-z} - 1 \right| = \left| \int_0^1 z e^{-tz} dt \right| \leq \int_0^1 |z| e^{-t\Re(z)} dt \leq |z|.$$

In our case $z = H(t)$, and we have

$$\left| e^{-H(t)} - 1 \right| \leq |H(t)|$$

and by integration we have the desired inequality. □

Lemma 4.2.12. *Let $0 \le k \le n$ and $0 \le j \le k$, then*

1. *the negative Fourier coefficients of $g_{j,k}(t) - 1$ vanish so that its Fourier series writes*
$$g_{j,k}(t) - 1 = \sum_{s \ge 0} c_s^p (g_{j,k} - 1) e^{i\pi \frac{st}{|I_p|}};$$
2. $\|g_{j,k} - 1\|_{L^2(I_p)} \le \eta \dfrac{\sqrt{2(|I_p|+1)}}{\sqrt{\delta} - 1} \delta^{-j/2}$.

Proof. When $j = k$, $g_{k,k}(t) - 1 = 0$ and there is nothing to do. When $j < k$, $\Re(H_{j,k}) = \sum_{r=j+1}^{k} \Re(h_r) \ge 0$ so from the previous lemma, we obtain the first statement and that
$$\|g_{j,k} - 1\|_{L^2(I_p)} := \|e^{-\eta H_{j,k}} - 1\|_{L^2(I_p)} \le \eta \|H_{j,k}\|_{L^2(I_p)}.$$
We then conclude with Lemma 4.2.10. □

Recall that
$$U = \sum_{j=0}^{n} f_j$$
and we set
$$V^\eta = F_n = \sum_{j=0}^{n} f_j g_{j,n}$$
where the dependence on η comes from the definition of the $g_{j,n}$'s. In particular,
$$\|V^\eta\|_\infty \le E_\eta. \tag{4.2.11}$$

The key estimate here is the following:

Proposition 4.2.13. *Let $0 < \varepsilon \le 1$, $N \ge 1$ and $\delta \ge e$ then there exists P such that, if $p \ge P$, there exists $\eta = \eta(p) \in (0,1)$ such that, for $1 \le \ell \le N$ and j_ℓ the unique index for which $\ell \in \mathcal{D}_{j_\ell}$*
$$\left| \frac{1}{|I_p|} \int_{I_p} \big(U(t) - V^\eta(t)\big) e^{2i\pi\lambda_\ell t} \varphi(t)\, \mathrm{d}t \right| \le \varepsilon \delta^{-j_\ell}. \tag{4.2.12}$$
Moreover, when $p \to +\infty$, $\eta(p) \to \eta_\infty = \dfrac{(\delta - 1)(\sqrt{\delta} - 1)}{\sqrt{2}\delta} \varepsilon$.

Proof. To simplify notation, we write $g_j = g_{j,n}$ and $V = V^\eta$. Then
$$\begin{aligned} R &:= \frac{1}{|I_p|} \int_{I_p} \big(U(t) - V(t)\big) e^{2i\pi\lambda_\ell t} \varphi(t)\, \mathrm{d}t \\ &= \frac{1}{|I_p|} \int_{I_p} \sum_{0 \le j \le j_\ell - 2} f_j (g_j - 1) e^{2i\pi\lambda_\ell t} \varphi(t)\, \mathrm{d}t + \frac{1}{|I_p|} \int_{I_p} f_{j_\ell - 1} (g_{j_\ell - 1} - 1) e^{2i\pi\lambda_\ell t} \varphi(t)\, \mathrm{d}t \\ &\quad + \frac{1}{|I_p|} \int_{I_p} \sum_{j_\ell \le j \le n} f_j (g_j - 1) e^{2i\pi\lambda_\ell t} \varphi(t)\, \mathrm{d}t \\ &:= R_- + R_0 + R_+. \end{aligned}$$

Let us first bound R_- (which is only present when $j_\ell \geq 2$). For this, notice that if $s \in \mathbb{Z}$,

$$\begin{aligned}\int_{I_p} \tilde{f}_j(t)\varphi(t)e^{2i\pi\lambda_\ell t}e^{-2i\pi\frac{s}{|I_p|}t}\,\mathrm{d}t &= \int_{I_p}\sum_{r\in\mathcal{D}_j} u_r\varphi(t)e^{-2i\pi(\lambda_r-\lambda_\ell+\frac{s}{|I_p|})t}\,\mathrm{d}t \\ &= \sum_{r\in\mathcal{D}_j} u_r\mathcal{F}[\varphi]\left(\lambda_r-\lambda_\ell+\frac{s}{|I_p|}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\int_{I_p} \tilde{f}_j(g_j-1)e^{2i\pi\lambda_\ell t}\varphi(t)\,\mathrm{d}t &= \int_{I_p}\tilde{f}_j(t)\varphi(t)e^{2i\pi\lambda_\ell t}\sum_{s\geq 0}c_s^p(g_j-1)e^{2i\pi\frac{st}{|I_p|}}\,\mathrm{d}t \\ &= \sum_{s=0}^{+\infty}c_s^p(g_j-1)\int_{I_p}\tilde{f}_j(t)\varphi(t)e^{2i\pi\lambda_\ell t+\frac{2i\pi st}{|I_p|}}\,\mathrm{d}t \\ &= \sum_{s=0}^{+\infty}c_s^p(g_j-1)\sum_{r\in\mathcal{D}_j}u_r\mathcal{F}[\varphi]\left(\lambda_r-\lambda_\ell-\frac{s}{|I_p|}\right) \\ &= \sum_{r\in\mathcal{D}_j}u_r\sum_{s=0}^{\infty}c_s^p(g_j-1)\mathcal{F}[\varphi]\left(\lambda_r-\lambda_\ell-\frac{s}{|I_p|}\right).\end{aligned}$$

Finally, we get

$$R_- = \frac{1}{|I_p|}\sum_{0\leq j\leq j_\ell-2}\frac{1}{|\mathcal{D}_j|}\sum_{r\in\mathcal{D}_j}u_r\sum_{s=0}^{\infty}c_s^p(g_j-1)\mathcal{F}[\varphi]\left(\lambda_r-\lambda_\ell-\frac{s}{|I_p|}\right).$$

As

$$\left(\sum_{s\geq 0}|c_s^p(g_j-1)|^2\right)^{1/2} = \frac{1}{\sqrt{|I_p|}}\|g_j-1\|_{L^2(I_p)} \leq \eta\frac{\sqrt{2(|I_p|+1)/|I_p|}}{\sqrt{\delta}-1}$$

we get with Cauchy-Schwarz

$$|R_-| \leq \eta\frac{\sqrt{2(|I_p|+1)/|I_p|}}{\sqrt{\delta}-1}\sum_{0\leq j\leq j_\ell-2}\frac{1}{|\mathcal{D}_j|}\sum_{r\in\mathcal{D}_j}\left(\sum_{s=0}^{\infty}\frac{1}{|I_p|^2}\left|\mathcal{F}[\varphi]\left(\lambda_r-\lambda_\ell-\frac{s}{|I_p|}\right)\right|^2\right)^{1/2}.$$

We will first estimate the innermost sum. We will use the following simple estimate valid for $u \geq 1$:

$$\begin{aligned}\sum_{s=0}^{+\infty}\frac{1}{(s+u)^{2p}} = \frac{1}{u^{2p}}+\sum_{s=1}^{+\infty}\frac{1}{(s+u)^{2p}} &\leq \frac{1}{u^{2p}}+\int_{s=0}^{+\infty}\frac{\mathrm{d}s}{(s+u)^{2p}} \\ &= \frac{1}{u^{2p}}+\frac{1}{(2p-1)u^{2p-1}} \leq \frac{2}{u^{2p-1}}.\end{aligned}$$

Here $u = |I_p|(\lambda_\ell-\lambda_r)$.

First, for $0 \leq j \leq j_\ell - 2$, if $r \in \mathcal{D}_j$ then $r < \beta_{j+1}$ and $\beta_\ell \leq \ell < \beta_{\ell+1}$ we first get

$$|I_p|(\lambda_\ell - \lambda_r) \geq |I_p|(\beta_\ell - \beta_{j+1}) = |I_p|\frac{\delta^{j_\ell} - \delta^{j+1}}{\delta - 1} \geq \frac{|I_p|}{2}\delta^{j_\ell - 1}$$

for all $0 \leq j \leq j_\ell - 2$. This then implies that

$$\begin{aligned} E_r := \sum_{s=0}^{\infty} \frac{1}{|I_p|^2}\left|\mathcal{F}[\varphi]\left(\lambda_k - \lambda_r + \frac{s}{|I_p|}\right)\right|^2 &\leq \sum_{s=0}^{\infty} \frac{1}{\left[\pi\left(\lambda_k - \lambda_r + \frac{s}{|I_p|}\right]\right)^{2p}} \\ &= \left(\frac{|I_p|}{\pi}\right)^{2p} \sum_{s=0}^{\infty} \frac{1}{\left(|I_p|(\lambda_k - \lambda_r) + s\right)^{2p}} \\ &\leq \frac{2^{2p}|I_p|}{\pi^{2p}}\delta^{-(j_\ell - 1)(2p-1)}. \end{aligned}$$

It follows that

$$\begin{aligned} |R_-| &\leq \eta\frac{\sqrt{2(|I_p|+1)}}{\sqrt{\delta}-1} \sum_{0 \leq j \leq j_\ell - 2} \frac{\delta^{-(j_\ell-1)(p-1/2)}}{(\pi/2)^p} \\ &= \eta\frac{\delta\sqrt{2(|I_p|+1)}}{(\pi/2)^p(\sqrt{\delta}-1)}(j_\ell - 1)\delta^{-(j_\ell-1)(p-3/2)}\delta^{-j_\ell}. \end{aligned}$$

Now, simple calculus shows that, when $a \geq \dfrac{1}{2}$, $x \geq 1$,

$$(x-1)e^{-a(x-1)} \leq xe^a e^{-ax} \quad \text{which is decreasing (in } x\text{)}.$$

Thus, as $x \geq 2$ in our case

$$(j_\ell - 1)\delta^{-(j_\ell-1)(p-3/2)} \leq \frac{2}{e^{(p-\frac{3}{2})}}$$

leading to the bound

$$|R_-| \leq \eta\frac{2\delta\sqrt{2(|I_p|+1)}}{e^{(p-\frac{3}{2})}(\pi/2)^p(\sqrt{\delta}-1)}\delta^{-j_\ell}. \tag{4.2.13}$$

It is crucial for the sequel to note that we can write this as

$$|R_-| \leq \eta\mu_p^-\delta^{-j_\ell} \qquad \text{with } \mu_p^- = \mu_p^-(\delta) \to 0 \text{ when } p \to +\infty.$$

On the other hand, if $j = j_\ell - 1$, $r \in \mathcal{D}_j$, then $|I_p|(\lambda_\ell - \lambda_r) \geq |I_p|$ and the same computation gives $E_r \leq \dfrac{2|I_p|}{\pi^{2p}}$. Repeating the computation of R_- gives $|R_0| \leq \eta\dfrac{2\sqrt{|I_p|+1}}{\pi^p}\dfrac{1}{\sqrt{\delta}-1}$. We write this in the form

$$|R_0| \leq \eta\delta^{-j_\ell}\mu_p^0(\delta, n)$$

where $\mu_p^0(\delta, n) \to 0$ when $p \to +\infty$.

We can now estimate R_+.

$$\begin{aligned} |R_+| &\leq \sum_{j_\ell \leq j \leq n} \frac{1}{|I_p|} \int_{I_p} |f_j(t)|\,|g_j(t)-1|\,|\varphi(t)|\,\mathrm{d}t \qquad (4.2.14)\\ &\leq \|\varphi\|_\infty \sum_{j_\ell \leq j \leq n} \frac{1}{|I_p|} \|f_j\|_{L^2(I_p)} \|g_j - 1\|_{L^2(I_p)} \\ &\leq \eta \frac{\sqrt{2}}{\sqrt{\delta}-1} \frac{p^2+p}{p^2} \frac{|I_p|+1}{|I_p|} \sum_{j_\ell \leq j \leq n} \delta^{-j} \\ &\leq \eta \frac{\sqrt{2}\delta}{(\sqrt{\delta}-1)(\delta-1)} \frac{p^2+p}{p^2} \frac{|I_p|+1}{|I_p|} \delta^{-j_\ell} \\ &= \eta \frac{\sqrt{2}\delta}{(\sqrt{\delta}-1)(\delta-1)} \frac{p^2+p+1}{p^2} \delta^{-j_\ell}. \end{aligned}$$

We write this in the form

$$|R_+| \leq \eta \left(\frac{\sqrt{2}\delta}{(\sqrt{\delta}-1)(\delta-1)} + \mu_p^+ \right) \delta^{-j_\ell} \quad \text{with } \lim_{p\to+\infty} \mu_p^+ = 0$$

and conclude that $|R| \leq \eta \left(\frac{\sqrt{2}\delta}{(\sqrt{\delta}-1)(\delta-1)} + \mu_p \right) \delta^{-j_\ell}$ with $\mu_p = \mu_p^-(\delta) + \mu_p^0(\delta, n) + \mu_p^+ \to 0$ (depending on δ and n thus N). □

Remark 4.2.14. *An inspection of the proof shows that the dependence of P on N only comes from R_0. This is harmless when we let $p \to +\infty$ which then implies $|I_p| \to +\infty$ i.e. when we prove Theorem 4.1.5 i) & ii) but is not possible when proving iii). To avoid that issue, one can then bound R_- and $R_0 + R_+$ instead of $R_- + R_0$ and R_+. The same proof works but the price to pay are slightly worse constants:*

$$|R_-| \leq \eta \frac{2\delta\sqrt{2(|I_p|+1)}}{e^{(p-\frac{3}{2})}(\pi/2)^p(\sqrt{\delta}-1)} \delta^{-j_\ell}.$$

and

$$|R_0 + R_+| \leq \eta \frac{\sqrt{2}\delta^2}{(\sqrt{\delta}-1)(\delta-1)} \frac{p^2+p}{p^2} \frac{|I_p|+1}{|I_p|} \delta^{-j_\ell}$$

since we can include R_0 into the sum (4.2.14) defining R_+ by starting it at $j_\ell - 1$ instead of j_ℓ. The consequence is that the δ on the numerator of the bound of R_+ becomes δ^2.

But then

$$|R| \leq \eta \left(\frac{2\delta\sqrt{2(|I_p|+1)}}{e^{(p-\frac{3}{2})}(\pi/2)^p(\sqrt{\delta}-1)} + \frac{\sqrt{2}\delta^2}{(\sqrt{\delta}-1)(\delta-1)} \frac{p^2+p+1}{p^2} \right) \delta^{-j_\ell}.$$

Taking any

$$\begin{aligned}\eta &\leq \frac{\varepsilon}{\frac{2\delta\sqrt{2(|I_p|+1)}}{e^{(p-\frac{3}{2})}(\pi/2)^p(\sqrt{\delta}-1)}+\frac{\sqrt{2}\delta^2}{(\sqrt{\delta}-1)(\delta-1)}\frac{p^2+p+1}{p^2}}\\ &= \frac{\varepsilon(\sqrt{\delta}-1)(\delta-1)}{\frac{2\delta\sqrt{2(|I_p|+1)}(\delta-1)}{e^{(p-\frac{3}{2})}(\pi/2)^p}+\sqrt{2}\delta^2\frac{p^2+p+1}{p^2}}\end{aligned}$$

will then still give (4.2.12)*, provided this* η *satisifies* $0<\eta<1$.

However, this quantity is too complicated to hope to be able to handle it in an optimisation process. Instead, we will determine the condition on p *for the parameter that almost optimise the case* $p\to+\infty$*, namely* $\varepsilon=\frac{1}{2}$ *and* $\delta=4$*. In this case, the smallest possible* p *in the first part is* $p=8$*. One can then do a computer check to see that, when* $\varepsilon=1/2$ *and* $p\geq 8$*,* (4.2.12) *is valid for* $\eta=0.058$.

Corollary 4.2.15. *Under the conditions of Proposition 4.2.13*

$$\left|\frac{1}{|I_p|}\int_{I_p}\big(U(t)-V^\eta(t)\big)\phi(t)\varphi(t)\,\mathrm{d}t\right|\leq\varepsilon S. \tag{4.2.15}$$

Further, if $\eta=0.058$ *and* $\delta=4$*, then for* $p\geq 8$*,* (4.2.15) *holds for* $\varepsilon=1/2$.

Proof. As $\phi(t)=\sum_{k=1}^N a_k e^{2i\pi\lambda_k t}$, it suffices to use the triangular inequality and (4.2.12). □

4.2.4 End of the proof

The end of the proof consists in applying first Proposition 4.2.4

$$S\leq\frac{2|I_p|}{2|I_p|-1}\left|\frac{1}{|I_p|}\int_{I_p}U(t)\phi(t)\varphi(t)\,\mathrm{d}t\right|.$$

Then, we fix an $0<\varepsilon<\frac{2|I_p|-1}{2|I_p|}$ and take $\eta<\eta(p)$ as in Proposition 4.2.13 and apply (4.2.15) to get

$$\begin{aligned}S &\leq \frac{2|I_p|}{2|I_p|-1}\left|\frac{1}{|I_p|}\int_{I_p}\big(U(t)-V^\eta(t)\big)\phi(t)\varphi(t)\,\mathrm{d}t\right|+\frac{2|I_p|}{2|I_p|-1}\left|\frac{1}{|I_p|}\int_{I_p}V^\eta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right|\\ &\leq \frac{2|I_p|}{2|I_p|-1}\varepsilon S+\frac{2|I_p|}{2|I_p|-1}\|V^\eta\|_\infty\|\varphi\|_\infty\frac{1}{2|I_p|}\int_{I_p}|\Phi(t)|\,\mathrm{d}t\\ &\leq \frac{2|I_p|}{2|I_p|-1}\varepsilon S+\frac{p^2+p}{p^2}\frac{2|I_p|}{2|I_p|-1}E_\eta\frac{1}{|I_p|}\int_{I_p}|\Phi(t)|\,\mathrm{d}t.\end{aligned}$$

We thus obtain from (4.2.16) that

$$\left(1-\frac{2|I_p|}{2|I_p|-1}\varepsilon\right)S\leq\frac{p^2+p}{p^2}\frac{2|I_p|}{2|I_p|-1}E_\eta\frac{1}{|I_p|}\int_{I_p}|\Phi(t)|\,\mathrm{d}t. \tag{4.2.16}$$

Taking $\delta = 4$, $\varepsilon = 1/2$, $p \geq 8$ so that $T = |I_p| \geq 72$ and $\eta = 0.058$, (4.2.16) reads

$$S \leq \frac{2 \times (72)^2}{71 \times 64} \frac{1}{1 - e^{-0.058}} \frac{1}{T} \int_{-T/2}^{T/2} |\Phi(t)| \, \mathrm{d}t.$$

From (4.2.4), we finally get

$$\sum_{k=1}^{N} \frac{|a_k|}{k+1} \leq 3 \times \frac{(144)^2}{64 \times 142} \frac{1}{1 - e^{-0.058}} \frac{1}{T} \int_{-T/2}^{T/2} |\Phi(t)| \, \mathrm{d}t \leq \frac{122}{T} \int_{-T/2}^{T/2} |\Phi(t)| \, \mathrm{d}t$$

establishing Theorem 4.1.5*iii)*.

We will now establish Theorem 4.1.5*i)*. To do so, let $p \to +\infty$ in (4.2.16) to get

$$(1 - \varepsilon) S \leq \frac{1}{1 - e^{-\eta}} \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\Phi(t)| \, \mathrm{d}t. \tag{4.2.17}$$

This relation is valid for every $\delta > 1$, every $0 < \varepsilon < 1$ and every

$$\eta < \eta_\infty = \frac{(\delta - 1)(\sqrt{\delta} - 1)}{\sqrt{2}\delta} \varepsilon.$$

By continuity, we can thus replace η by η_∞ in (4.2.17). Further, we may use again (4.2.4) to get

$$\sum_{k=1}^{N} \frac{|a_k|}{k+1} \leq \frac{\delta - 1}{(1 - \varepsilon)\big(1 - e^{-\eta_\infty}\big)} \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{j=0}^{N} a_j e^{2i\pi\lambda_j t} \right| \mathrm{d}t. \tag{4.2.18}$$

It remains to choose the parameters δ and ε so as to minimize the factor of the Besicovitch norm. A computer search shows that

$$\frac{\delta - 1}{(1 - \varepsilon)\big(1 - e^{-\eta_\infty}\big)} = \frac{\delta - 1}{(1 - \varepsilon)\big(1 - e^{-\frac{(\delta-1)(\sqrt{\delta}-1)}{\sqrt{2}\delta}\varepsilon}\big)}$$

takes its minimal value ~ 25.1624 for some ε, δ with $0.4768 \leq \varepsilon \leq 0.4772$ and $3.70 \leq \delta \leq 3.75$. This gives the claimed inequality

$$\sum_{k=1}^{N} \frac{|a_k|}{k+1} \leq 25.2 \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t.$$

A somewhat better estimate is possible when $|a_k| \geq 1$. Indeed, in this case we proved in (4.2.17) that

$$n + 1 \leq \frac{1}{(1 - \varepsilon)\big(1 - e^{-\frac{(\delta-1)(\sqrt{\delta}-1)}{\sqrt{2}\delta}\varepsilon}\big)} \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t.$$

But n was defined as $N = \beta_{n+1} - 1 = \dfrac{\delta^{n+1}-1}{\delta-1} - 1$ that is

$$n+1 = \frac{\ln\big(1+(\delta-1)(N+1)\big)}{\ln\delta}$$

since $\delta \geq 2$. We thus have

$$\ln[1+(\delta-1)N] \leq \frac{\ln\delta}{(1-\varepsilon)\left(1-e^{-\frac{(\delta-1)(\sqrt{\delta}-1)}{\sqrt{2}\delta}\varepsilon}\right)} \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t.$$

and we are looking for ε, δ that minimize

$$\frac{\ln\delta}{(1-\varepsilon)\big(1-e^{-\frac{(\delta-1)(\sqrt{\delta}-1)}{\sqrt{2}\delta}\varepsilon}\big)}$$

and for the value of this minimum. The best value we obtain is 7.714 for $\varepsilon = 0.28$ and $\delta = 89.254$ which is essentially the same value as in [38] (note that $1/7.714 = 0.1296$ is the value given there).

We then obtain the following: if $|a_k| \geq 1$, then

$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq 0.1296\ln(1+88N).$$

4.2.5 Further comments

First, the inequality for a fixed T implies the inequality for the Besicovitch norm. This follows from a simple trick already used for Ingham's inequality. Indeed, once we establish that

$$C_0\int_{-T_0/2}^{T_0/2}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq \sum_{k=1}^{N}\frac{|a_k|}{k+1}$$

for some $T_0 > 0$ then, changing variable $t = t_0 + s$ we also have

$$C_0\int_{t_0-T_0/2}^{t_0+T_0/2}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t = C_0\int_{-T_0/2}^{T_0/2}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t_0} e^{2i\pi\lambda_k s}\right| \mathrm{d}s \geq \sum_{k=1}^{N}\frac{|a_k|}{k+1}.$$

Next, for any integer M, covering $[-MT_0, MT_0]$ by $2M$ intervals of length T_0, we get

$$C_0\int_{-MT_0}^{MT_0}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq 2M\sum_{k=1}^{N}\frac{|a_k|}{k+1}.$$

Dividing by $2M$ and letting $M \to +\infty$ we get

$$C_0 T_0 \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t \geq \sum_{k=1}^{N}\frac{|a_k|}{k+1}.$$

In particular, Nazarov's result also implies the result of Hudson and Leckband (but with worse constants).

As said, the proof is an adaptation of the proof by Nazarov. The main difference is that we use an auxiliary function φ_p that depends on $T = p^2 + p$ and exploit the fact that $p \to +\infty$ to obtain the numerical constants.

There is a major difference between the sums that appear in Ingham's Theorem 2.1.3 and those that appear in McGehee, Pigno, Smith 4.1.1 and Nazarov's Theorems 4.1.4. In the L^2-case, the sums can be two sided and not in the L^1-case. Indeed let $T > 1$ and consider the Fejer kernel

$$F_N(t) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) e^{2i\pi kt},$$

then, as F_N is a positive function

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F_N(t)\,\mathrm{d}t \leq \frac{1}{T}\int_{-\frac{\lfloor T\rfloor+1}{2}}^{\frac{\lfloor T\rfloor+1}{2}} F_N(t)\,\mathrm{d}t \leq \frac{\lfloor T\rfloor+1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)\,\mathrm{d}t \leq 2 \tag{4.2.19}$$

since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)\,\mathrm{d}t = 1.$$

But

$$\sum_{k=-N}^{N}\left(\frac{1-\frac{|k|}{N+1}}{1+|k|}\right) \geq 2\sum_{k=1}^{\frac{N+1}{2}}\left(\frac{1-\frac{k}{N+1}}{1+k}\right) \geq \sum_{k=1}^{\frac{N+1}{2}}\frac{1}{k+1} \to \infty \quad \text{as} \quad N \to \infty,$$

since

$$1 - \frac{k}{N+1} \geq \frac{1}{2} \quad \text{for} \quad k \leq \frac{N+1}{2}.$$

In other words, one cannot find a constant C, independent of N such that Nazarov's (or M.P.S) theorem holds. However we have seen in Corollary 2.1.7 that a slightly weaker result holds

Remark. *In the proof of Corollary 2.1.7, the constant depends also on T. However since we are applying the corollary for $T \geq 72$ and the function $T \longrightarrow C_T$ is decreasing (when T is large C_T is bounded uniformly), we get a constant which depends only on η.*

4.3 Quantitative version for T near 1.

In the previous section, we investigated Nazarov's theorem when the length of the interval is large enough. In this section, we will look at the opposite case. Let $\delta > 0$ and $T = 1 + \delta$ (this δ has nothing to do with the parameter δ used in the proof of the remaining part of Theorem 4.1.5). We are going to prove the second part of Theorem 4.1.5 *iii i.e.* a quantitative estimate of Nazarov's theorem.

To do so, we will follow the same steps used by Nazarov to prove Theorem 4.1.4, however, we will explicitly define a slightly different auxiliary function φ which will allow us to follow the constants more closely. Some proofs are similar to the previous section and hence we will omit some details and where appropriate, refer to previous proofs.

4.3.1 An auxiliary function and the estimation of T

First, we start by some preliminary notations and results. Let $\delta > 0$ and $I = I_\delta := [-\frac{1+\delta}{2}, \frac{1+\delta}{2}]$.

Let us fix $(\lambda_k)_{k=0,\dots,N} \subset \mathbb{R}$ with $\lambda_{k+1} - \lambda_k \geq 1$ for every k, $(a_k)_{k=0,\dots,N}$ a sequence of complex numbers and write $|a_k| = a_k u_k$ with $|u_k| = 1$. Let

$$S = \sum_{k=0}^{N} \frac{|a_k|}{k+1} \qquad \text{and} \qquad \phi(t) = \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t},$$

so that we must find C_δ such that:

$$\|\phi\|_{L^1(I)} \geq C_\delta S. \tag{4.3.1}$$

We define

$$S_\delta = \sum_{k=0}^{N} \frac{|a_k|}{k+N_\delta} \quad \text{and} \quad T_\delta(t) = \sum_{k=0}^{N} \frac{u_k}{k+N_\delta} e^{-2i\pi\lambda_k t}$$

where N_δ is a large integer that we will adjust through the proof. This integer will be of the form $N_\delta = 2^{m_\delta}$. We will prove that

$$\|\phi\|_{L^1(I)} \geq B_\delta S_\delta \tag{4.3.2}$$

and, as $k + N_\delta \leq (k+1)N_\delta$, $S_\delta \geq \dfrac{S}{N_\delta}$ so that we obtain the desired inequality (4.3.1) with a constant $C_\delta = \dfrac{B_\delta}{N_\delta}$.

The meaning of N_δ is the following. Consider:

– a new sequence of frequencies $(\tilde{\lambda}_j)_{j\in\mathbb{Z}}$ such that $\tilde{\lambda}_j = \lambda_{j-N_\delta}$ for $j = N_\delta, \dots, N_\delta + N$. and then $\tilde{\lambda}_j = \lambda_0 + j - N_\delta$ for $j < N_\delta$ and $\tilde{\lambda}_j = \lambda_N + j - N$ for $j > N + N_\delta$. In particular, we still have $|\tilde{\lambda}_{j+1} - \tilde{\lambda}_j| \geq 1$. In other words, the sequence $(\lambda_j)_{j=0,\dots,N}$ is completed into a sequence $(\lambda_j)_{j\in\mathbb{Z}}$ that is still 1-separated and then shifted by N_δ.

– A new sequence of complex numbers $(\tilde{a}_j)_{j\in\mathbb{Z}}$ with $\tilde{a}_j = a_{j-N_\delta}$ for $j = N_\delta, \dots, N_\delta + N$ and $\tilde{a}_j = 0$ for other j's. In other words, the sequence $(a_j)_{j=0,\dots,N}$ is completed into a sequence $(a_j)_{j\in\mathbb{Z}}$ by 0-padding it and then shifted by N_δ.

Then (4.3.2) reads

$$\int_{I_\delta} \left| \sum_{j\in\mathbb{Z}} \tilde{a}_j e^{2i\pi\tilde{\lambda}_j t} \right| \mathrm{d}t \geq B_\delta \sum_{j\in\mathbb{Z}} \frac{|\tilde{a}_j|}{j+N_\delta}$$

with the convention that $0/0 = 0$.

Note that, up to adding 0 terms at the end of the sequence a_j, we may assume that $N + N_\delta$ is of the form $2^{n_\delta} - 1$ for some integer n_δ. This will allow us to write

$$\sum_{k=0}^{N} = \sum_{j=m_\delta}^{n_\delta} \sum_{2^j \leq r+N_\delta < 2^{j+1}}.$$

Next, as in Ingham's proof, we will introduce an auxiliary function. Again we consider

$$h(t) = \begin{cases} \cos(\pi t) & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

whose Fourier transform is given by

$$\widehat{h}(\lambda) = \frac{2}{\pi} \frac{\cos(\pi\lambda)}{1 - 4\lambda^2}.$$

We will need to smooth a bit this function to obtain a better decay of the Fourier transform and thereby slightly enlarge its support. More precisely, let $p = 10$, $q = 8$ and let

$$f_\delta(t) = \frac{p+q}{\delta} \mathbb{1}_{[-\frac{\delta}{2(p+q)}, \frac{\delta}{2(p+q)}]}(t)$$

and define

$$g_\delta = *_{p+q} f_\delta,$$

where $*_k \psi$ denotes the k-fold convolution of ψ by itself. More precisely,

$$*_2 \psi(x) = \psi(x) * \psi(x) = \int_{\mathbb{R}} \psi(t)\psi(x-t)\,\mathrm{d}t$$

and, for $k \geq 2$,

$$*_{(k+1)} \psi = *_k \psi * \psi.$$

Clearly g_δ is non-negative, even and with support $[-\frac{\delta}{2}, \frac{\delta}{2}]$. Finally, we define φ_δ as

$$\varphi_\delta = \frac{\pi}{2} h * g_\delta. \tag{4.3.3}$$

In the following lemma, we list the properties needed on φ_δ. They are all established via easy calculus and straight forward Fourier analysis.

Lemma 4.3.1. *There is a $c_0 > 0$ and a $\delta_0 > 0$ such that, if $0 < \delta < \delta_0$ then,*

1. $\widehat{\varphi_\delta}(\lambda) = \dfrac{\cos(\pi\lambda)}{1 - 4\lambda^2} \operatorname{sinc}^{p+q}\left(\dfrac{\pi\delta\lambda}{p+q}\right)$,
2. $\|\varphi_\delta\|_\infty \leq \dfrac{\pi}{2}$.

3. Let $D_\delta = c_0 \delta^{-\frac{p+q}{p+2}} \geq 1$ *and* $\nu > 0$ *then, for* $|\lambda| \geq \max(1, D_\delta \nu^{\frac{1}{p+2}})$,

$$|\widehat{\varphi}(\lambda)| \leq \frac{\nu}{|\lambda|^q}$$

4. Let $\gamma_\delta = \left(\operatorname{sinc} \frac{\pi\delta}{p+q}\right)^{p+q}$ *then, for* $|\lambda| \geq 1$,

$$|\widehat{\varphi_\delta}(\lambda)| \leq \frac{\gamma_\delta}{4\lambda^2 - 1}.$$

From now on, we will assume that $0 < \delta < \frac{7}{3\pi} < 1$ so that $D_\delta \geq 1$. In particular, when $|\lambda| \geq D_\delta$, we have $|\widehat{\varphi}(\lambda)| \leq |\lambda|^{-3}$.

Proof of Lemma 4.3.1. The first one is simple Fourier analysis.

For the second one, we write

$$\|\varphi_\delta\|_\infty \leq \frac{\pi}{2} \|g_\delta\|_1 \|h\|_\infty \leq \frac{\pi}{2} \|f_\delta\|_1^{p+q} \|h\|_\infty$$

and use simple calculus to conclude.

For the third one, we notice that $4\lambda^2 - 1 \geq 3\lambda^2$ when $\lambda \geq 1$ thus

$$|\widehat{\varphi_\delta}(\lambda)| = \left| \frac{\cos(\pi\lambda)}{4\lambda^2 - 1} \left(\frac{\sin(\frac{\pi\delta\lambda}{p+q})}{\frac{\pi\delta\lambda}{p+q}} \right)^{2p} \right| \leq \frac{1}{3} \left(\frac{p+q}{\pi} \right)^{p+q} \delta^{-(p+q)} \frac{1}{|\lambda|^{p+2}} \frac{1}{|\lambda|^q} \leq \frac{\nu}{|\lambda|^q},$$

since $|\lambda| \geq \dfrac{1}{3^{\frac{1}{p+2}}} \left(\dfrac{p+q}{\pi} \right)^{\frac{p+q}{p+2}} \delta^{-\frac{p+q}{p+2}} \nu^{\frac{1}{p+2}}$ thus we get the result with $c_0 = \dfrac{1}{3^{\frac{1}{p+2}}} \left(\dfrac{p+q}{\pi} \right)^{\frac{p+q}{p+2}}$.

For the last one, we take δ_0 small enough to have $\operatorname{sinc} \dfrac{\pi\delta_0}{p+q} = \sup_{t \geq \delta_0} |\operatorname{sinc} t|$ and then $|\widehat{g_\delta}(\lambda)| \leq \gamma_\delta$ when $|\lambda| \geq 1$. The first identity allows to conclude. □

We can now state the first crucial result in this proof.

Lemma 4.3.2. *There exist* $\delta_1 > 0$, $c_1 > 0$ *such that, if* $0 < \delta < \delta_1$, $0 < c_1\delta^2 < 1 - \sqrt{\gamma_\delta}$.

Moreover, let

$$\alpha_\delta = 1 - \left(c_1\delta^2 + \frac{\gamma_\delta}{1 - c_1\delta^2} \right)$$

and let m_δ *be such that*

$$N_\delta = 2^{m_\delta} \geq \delta^{-7/2}.$$

Then, for $0 \leq k \leq N$,

$$\sum_{\substack{0 \leq j \leq N \\ j \neq k}} \frac{|\widehat{\varphi}(\lambda_j - \lambda_k)|}{j + N_\delta} \leq \frac{1 - \alpha_\delta}{k + N_\delta}.$$

Proof. The Taylor expansion when $\delta \to 0$ of $1 - \sqrt{\gamma_\delta}$ is of the form

$$1 - \sqrt{\gamma_\delta} = A\delta^2 + O(\delta^4),$$

with $A > 0$. Thus, if $c_1 < A$, for δ small enough, $0 < c_1\delta^2 < 1 - \sqrt{\gamma_\delta}$. Next, notice that $0 < 1 - \left(\beta + \dfrac{\gamma_\delta}{1-\beta}\right) < 1$ if $0 < \beta < 1 - \sqrt{\gamma_\delta}$ which shows that $0 < \alpha_\delta < 1$. For future use, note that there is a $\kappa > 0$ such that

$$\alpha_\delta = \kappa\delta^2 + O(\delta^4). \tag{4.3.4}$$

We will further assume that δ is small enough for

$$\delta^{-7/2} \geq \max\left(c_1^{-\frac{q+1}{q-2}}\delta^{-2\frac{q+1}{q-2}}, c_2\delta^{-2-\frac{p+q}{p+2}}\right) = \max\left(c_1^{-\frac{q+1}{q-2}}\delta^{-3}, \frac{c_0}{c_1}\delta^{-7/2}\right)$$

since we have chosen $q = 8$ and $p = 10$.

Note that, for every $\varepsilon > 0$, the power $7/2$ could be reduced to $3 + \varepsilon$ by taking p large enough, but could not be reduced below 3 with this construction.

We can now turn to the estimate itself. Set $\beta = c_1\delta^2$ and split the sum in the left hand side of the main inequality into two sums

$$E := \sum_{\substack{0\leq j\leq N\\ j\neq k}} \frac{|\widehat{\varphi}(\lambda_j - \lambda_k)|}{j + N_\delta} = E_1 + E_2$$

where

$$E_1 = \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} \frac{|\widehat{\varphi}(\lambda_j - \lambda_k)|}{j + N_\delta}$$

and

$$E_2 = \sum_{\substack{j+N_\delta\geq(1-\beta)(k+N_\delta)\\ j\neq k}} \frac{|\widehat{\varphi}(\lambda_j - \lambda_k)|}{j + N_\delta}.$$

The result is obtained if we prove the estimates

$$E_1 \leq \frac{\beta}{k + N_\delta} \quad \text{et} \quad E_2 \leq \frac{\gamma_\delta/(1-\beta)}{k + N_\delta}.$$

Now, as $N_\delta \geq \dfrac{c_0}{c_1}\delta^{-2-\frac{p+q}{p+2}}$, $\beta N_\delta \geq D_\delta$. Then, if j is an index corresponding to E_1,

$$|\lambda_k - \lambda_j| \geq |k - j| = (k + N_\delta) - (j + N_\delta) \geq \beta(k + N_\delta) \geq \beta N_\delta \geq D_\delta$$

hence, from part 3 of Lemma 4.3.1 (with $\nu = 1$),

$$\begin{aligned} E_1 &\leq \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} |\widehat{\varphi}(\lambda_j - \lambda_k)| \leq \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} \frac{1}{|\lambda_j - \lambda_k|^q} \\ &\leq \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} \frac{1}{\big(\beta(k + N_\delta)\big)^q}. \end{aligned}$$

But, E_1 contains less than k terms so

$$E_1 \leq \frac{k}{k+N_\delta}\frac{\beta^{-(q+1)}}{(k+N_\delta)^{q-2}}\frac{\beta}{k+N_\delta} \leq \frac{\beta}{k+N_\delta} \tag{4.3.5}$$

since $N_\delta \geq \beta^{-\frac{q+1}{q-2}} = c_1^{-\frac{q+1}{q-2}}\delta^{-2\frac{q+1}{q-2}}$.

We shall now bound E_2. In this sum,

$$j+N_\delta \geq (1-\beta)(k+N_\delta) \quad \text{and} \quad |\lambda_j-\lambda_k| \geq |j-k|.$$

Then

$$\begin{aligned} E_2 &\leq \frac{1}{(1-\beta)(k+N_\delta)} \sum_{\substack{1\leq j\leq N\\ j\neq k}} \frac{\gamma_\delta}{4(\lambda_j-\lambda_k)^2-1} \\ &\leq \frac{\gamma_\delta}{(1-\beta)(k+N_\delta)} \sum_{\substack{1\leq j\leq N\\ j\neq k}} \frac{1}{4(j-k)^2-1} \\ &\leq \frac{\gamma_\delta}{(1-\beta)(k+N_\delta)} \sum_{\ell=1}^{\infty} \frac{2}{4\ell^2-1}. \end{aligned}$$

Since $\frac{2}{4\ell^2-1} = \frac{1}{2\ell-1} - \frac{1}{2\ell+1}$, we obtain the expected bound $E_2 \leq \frac{\gamma_\delta/(1-\beta)}{k+N_\delta}$. □

The following lemma is a first step toward proving the last part of Theorem 4.1.5 *iii* and is a consequence of Lemma 4.3.2.

Lemma 4.3.3. *Let us use the notation of the Lemma 4.3.2. Then,*

$$\left| \int_{I_\delta} T_\delta(t) \left(\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right) \varphi(t)\,\mathrm{d}t \right| \geq \alpha_\delta \sum_{k=0}^{N} \frac{|a_k|}{k+N_\delta}. \tag{4.3.6}$$

Proof. By definition of T_δ,

$$\begin{aligned} \int_{I_\delta} T_\delta(t) e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t &= \sum_{j=0}^{N} \frac{u_j}{j+N_\delta} \int_{I_\delta} e^{-2i\pi\lambda_j t} e^{2i\pi\lambda_k t}\varphi(t)dt \\ &= \sum_{j=0}^{N} \frac{u_j}{j+N_\delta}\widehat{\varphi}(\lambda_j-\lambda_k) \\ &= \frac{u_k}{k+N_\delta} + \sum_{\substack{0\leq j\leq N\\ j\neq k}} \frac{u_j}{j+N_\delta}\widehat{\varphi}(\lambda_j-\lambda_k) \end{aligned}$$

thus

$$\left| \int_{I_\delta} T_\delta(t) e^{2i\pi\lambda_k t}\varphi(t)dt - \frac{u_k}{k+N_\delta} \right| \leq \sum_{\substack{0\leq j\leq N\\ j\neq k}} \frac{1}{j+N_\delta}\widehat{\varphi}(\lambda_j-\lambda_k).$$

By applying the Lemma 4.3.2, we obtain that

$$\left| \int_{I_\delta} T_\delta(t) e^{2i\pi\lambda_k t} \varphi(t) dt - \frac{u_k}{k+N_\delta} \right| \leq \frac{1-\alpha_\delta}{k+N_\delta}.$$

It follows that

$$\left| \int_{I_\delta} T_\delta(t) a_k e^{2i\pi\lambda_k t} \varphi(t) dt - \frac{u_k a_k}{k+N_\delta} \right| \leq \frac{1-\alpha_\delta}{k+N_\delta} |a_k|.$$

Using the fact that $u_k a_k = |a_k|$ and the triangular inequality, we obtain

$$\left| \int_{I_\delta} T_\delta(t) \left(\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right) \varphi(t) dt - \sum_{k=0}^{N} \frac{|a_k|}{k+N_\delta} \right| \leq (1-\alpha_\delta) \sum_{k=0}^{N} \frac{|a_k|}{k+N_\delta}$$

from which the lemma follows immediately. □

4.3.2 Construction of $\tilde{T}_\delta$

Recall that we defined $\mathcal{D}_j = \{k \in \mathbb{N} : 2^j \leq k < 2^{j+1}\}$. From our assumption on N_δ and N, we can write

$$T_\delta(t) = \sum_{k=0}^{N} \frac{u_k}{k+N_\delta} e^{-2i\pi\lambda_k t} = \sum_{j=m_\delta}^{n_\delta} f_j(t)$$

with

$$f_j(t) = \sum_{r+N_\delta \in \mathcal{D}_j} \frac{u_r}{r+N_\delta} e^{-2i\pi\lambda_r t}.$$

Next we estimate the norms of the f_j's. Lemma 4.2.8 (with $\delta = 2$) also shows that

Lemma 4.3.4. *For $m_\delta \leq j \leq n_\delta$,*

1. $\|f_j\|_{L^2(I_\delta)} \leq 2^{-\frac{j}{2}} \sqrt{|I_\delta| + 1}$.
2. $\|f_j\|_\infty \leq 1$.

We then write again the Fourier series expansion of $|f_j| \in L^2(I_\delta)$ as

$$|f_j| = \sum_{s \in \mathbb{Z}} a_{s,j} e^{\frac{2i\pi}{|I|} t},$$

and define $h_j \in L^2(I_\delta)$ via its Fourier series expansion

$$h_j(t) = a_{0,j} + 2 \sum_{s=1}^{\infty} a_{s,j} e^{\frac{2is\pi}{|I|} t}.$$

Lemma 4.3.5. *For $m_\delta \leq j \leq n_\delta$, the following properties hold*

1. $\mathrm{Re}(h_j) = |f_j|$;
2. $\|h_j\|_2 \leq \sqrt{2}\|f_j\|_2$;
3. $h_j \in H^\infty(I_\delta)$.

This is the analogue of Lemma 4.2.8, the proof is quite similar, hence we will skip it. We now define a sequence $(F_j)_{j\geq m_\delta}$ inductively through

$$F_{m_\delta} = f_{m_\delta} \text{ et } F_{j+1} = F_j e^{-\varepsilon h_{j+1}} + f_{j+1}$$

where $0 < \varepsilon < 1$ is a small parameter that we will adjust later. Further set

$$E_\varepsilon := \sup_{0<x\leq 1} \frac{x}{1-e^{-\varepsilon x}} = \frac{1}{\varepsilon} \sup_{0<x\leq \varepsilon} \frac{x}{1-e^{-x}} = \frac{1}{1-e^{-\varepsilon}}.$$

From the next to last identity, it is easy to obtain the following simple bound:

$$\frac{1}{\varepsilon} \leq E_\varepsilon \leq \frac{2}{\varepsilon}.$$

Lemma 4.3.6. *Let $m_\delta \leq n \leq n_\delta$. For $j = m_\delta, \ldots, n$ we define $g_{j,n} = e^{-\varepsilon H_{j,n}}$ with*

$$H_{j,n} = \begin{cases} h_{j+1} + \ldots + h_n & \text{if } j < n \\ 0 & \text{if } j = n \end{cases}.$$

Then $F_n = \sum_{j=m_\delta}^{n} f_j g_{j,n}$ with $\|F_n\|_\infty \leq \frac{2}{\varepsilon}$. Moreover $\|H_{j,n}\|_2 \leq \frac{\sqrt{2(|I_\delta|+1)}}{\sqrt{2}-1} 2^{-\frac{j}{2}}$.

Lemma 4.3.7. *Assume that $0 < \varepsilon \leq \frac{\sqrt{|I_\delta|}(\sqrt{2}-1)}{\sqrt{2(|I_\delta|+1)}}$. Then, for $m_\delta \leq j \leq n \leq n_\delta$,*

1. $\|g_{j,n} - 1\|_2 \leq \varepsilon \|H_{j,n}\|_2 \leq \frac{\sqrt{2(|I_\delta|+1)}}{\sqrt{2}-1} \varepsilon 2^{-\frac{j}{2}}$
2. *The Fourier series of $g_{j,n}(t) - 1$ writes $g_{j,n}(t) - 1 = \sum_{s\geq 0} c_{s,j} e^{\frac{2is\pi}{|I_\delta|}t}$, with $\sum_{s=0}^{+\infty} |c_{s,j}|^2 \leq 1$.*

Proof. By Lemma 4.2.11, $\|g_j - 1\|_2 \leq \varepsilon \|H_j\|_2$. Then, since $g_{j,n}$ is analytic, its Fourier series writes

$$g_j(t) - 1 = \sum_{s\geq 0} c_{s,j} e^{\frac{2i\pi}{|I_\delta|}st}.$$

But then, with Parseval

$$\left(\sum_{s\geq 0} |c_{s,j}|^2\right)^{\frac{1}{2}} = \frac{1}{\sqrt{|I_\delta|}} \|g_j - 1\|_2 \leq \frac{\varepsilon}{\sqrt{|I_\delta|}} \|H_j\|_2 \leq \frac{\varepsilon}{\sqrt{|I_\delta|}} \frac{\sqrt{2(|I_\delta|+1)}}{\sqrt{2}-1} 2^{-\frac{j}{2}} \leq 2^{-\frac{j}{2}} < 1$$

which implies the claimed bound. □

Now recall that

$$T_\delta = \sum_{m_\delta \le j \le n_\delta} f_j$$

and define

$$\tilde{T}_\delta = F_{n_\delta}.$$

In particular, from Lemma 4.3.6, we have

$$\|\tilde{T}_\delta\|_\infty \le \frac{2}{\varepsilon}. \tag{4.3.7}$$

The key estimates here is the following;

Lemma 4.3.8. *Once again, we use the notations of the lemma 4.3.2. There exists $\delta_2 > 0$ such that, if $0 < \delta < \delta_2$ and $N_\delta \ge \delta^{-7/2}$ then*

$$\left| \int_{I_\delta} (\tilde{T}_\delta - T_\delta)(t) \left(\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right) \varphi(t)\, \mathrm{d}t \right| \le \frac{2}{3} \alpha_\delta \sum_{k=0}^{N} \frac{|a_k|}{k + N_\delta}. \tag{4.3.8}$$

where φ is the function defined in (4.3.3).

Proof. It is enough to prove that, under the conditions of the lemma, for $0 \le k \le N$ we have

$$\left| \int_{I_\delta} (\tilde{T}_\delta - T_\delta)(t) e^{2i\pi\lambda_k t} \varphi(t)\, \mathrm{d}t \right| \le \frac{2}{3} \frac{\alpha_\delta}{k + N_\delta}. \tag{4.3.9}$$

Once (4.3.9) is established, it will then be enough to multiply the left hand side by a_k and to use the triangular inequality.

We fix $k \in [0, N]$ and let ℓ be the index such that $k + N_\delta \in \mathcal{D}_\ell$. We define R, R_1 and R_2 as follows

$$\begin{aligned} R &= \int_{I_\delta} (\tilde{T}_{\delta,\beta} - T_{\delta,\beta})(t) e^{2i\pi\lambda_k t} \varphi(t)\, \mathrm{d}t \\ &= \int_I \sum_{m_\delta \le j \le \ell - 2} f_j(t) \big(g_j(t) - 1 \big) e^{2i\pi\lambda_k t} \varphi(t\, \mathrm{d}t + \int_{I_\delta} \sum_{\ell - 1 \le j \le n_\delta} f_j(t) \big(g_j(t) - 1 \big) e^{2i\pi\lambda_k t} \varphi(t)\, \mathrm{d}t \\ &:= R_1 + R_2. \end{aligned}$$

We will first bound R_1. Note that if $s \in \mathbb{Z}$,

$$\begin{aligned} \int_{I_\delta} f_j(t) \varphi(t) e^{2i\pi\lambda_k t} e^{\frac{2i\pi}{|I_\delta|} st}\, \mathrm{d}t &= \int_{I_\delta} \sum_{r + N_\delta \in \mathcal{D}_j} \frac{u_r}{r + N_\delta} \varphi(t) e^{2i\pi(-\lambda_r + \lambda_k + \frac{s}{|I_\delta|})}\, \mathrm{d}t \\ &= \sum_{r + N_\delta \in \mathcal{D}_j} \frac{u_r}{r + N_\delta} \widehat{\varphi}(\lambda_r - \lambda_k - \frac{s}{|I_\delta|}). \end{aligned}$$

From this, we obtain

$$\begin{aligned}\int_{I_\delta} f_j(t)\big(g_j(t)-1\big)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t &= \int_{I_\delta} f_j(t)\varphi(t)e^{2i\pi\lambda_k t}\sum_{s\geq 0} c_{s,j}e^{\frac{2is\pi}{|I_\delta|}t}\,\mathrm{d}t\\ &= \sum_{s=0}^{+\infty} c_{s,j}\int_{I_\delta} f_j(t)\varphi(t)e^{2i\pi\lambda_k t}e^{\frac{2i\pi}{|I_\delta|}st}\,\mathrm{d}t\\ &= \sum_{s=0}^{+\infty} c_{s,j}\sum_{r+N_\delta\in\mathcal{D}_j}\frac{u_r}{r+N_\delta}\widehat{\varphi}(\lambda_r-\lambda_k-\frac{s}{|I_\delta|})\\ &= \sum_{r+N_\delta\in\mathcal{D}_j}\frac{u_r}{r+N_\delta}\sum_{s=0}^{\infty} c_{s,j}\widehat{\varphi}(\lambda_r-\lambda_k-\frac{s}{|I_\delta|}).\end{aligned}$$

So finally we get

$$R_1=\sum_{m_\delta\leq j\leq \ell-2}\ \sum_{r+N_\delta\in\mathcal{D}_j}\frac{u_r}{r+N_\delta}\sum_{s=0}^{\infty} c_{s,j}\widehat{\varphi}(\lambda_r-\lambda_k-\frac{s}{|I_\delta|}).$$

Let $c_s(r)=c_{s,j}$ if $r+N_\delta\in I_j$. Since $\widehat{\varphi}$ is an even function, we can write

$$R_1=\sum_{2^{m_\delta}\leq r+N_\delta<2^{\ell-1}}\frac{u_r}{r+N_\delta}\sum_{s=0}^{\infty} c_s(r)\widehat{\varphi}(\lambda_k-\lambda_r+\frac{s}{|I_\delta|}):=\sum_{2^{m_\delta}\leq r+N_\delta<2^{\ell-1}}\frac{u_r}{r+N_\delta}E_r.$$

From Lemma 4.3.1, recall that, with $D_\delta=c_0\delta^{-\frac{p+q}{p+2}}$ and $\nu=\dfrac{\alpha_\delta}{|I_\delta|^{1/q}}$, for

$$|\lambda|\geq\max(1,D_\delta\nu^{-\frac{1}{p+2}}),$$

we have

$$|\widehat{\varphi}(\lambda)|\leq\frac{\alpha_\delta}{(|I_\delta||\lambda|)^q}. \tag{4.3.10}$$

Now, $s\geq 0$, $|I_\delta|\geq 1$ and, as $r+N_\delta<2^{\ell-1}$, $2^\ell\leq k+N_\delta<2^{\ell+1}$, $\lambda_k>\lambda_r$ thus

$$\begin{aligned}|I_\delta|\left|\lambda_k-\lambda_r+\frac{s}{|I_\delta|}\right| &= |I_\delta|(\lambda_k-\lambda_r)+s\geq\lambda_k-\lambda_r+s\geq k-r+s \qquad (4.3.11)\\ &= (k+N_\delta)-(r+N_\delta)>2^\ell-2^{\ell-1}=2^{\ell-1}\geq 2^{m_\delta-1}.\end{aligned}$$

Further, from (4.3.4) and $1\leq|I_\delta|\leq 2$, $D_\delta\nu^{-\frac{1}{p+2}}\leq c_3\delta^{-\frac{p+q+2}{p+2}}=c_3\delta^{-\frac{20}{12}}$. Thus, choosing m_δ sufficiently large for $N_\delta=2^{m_\delta}\delta^{-7/2}$ and $\delta\leq\delta_2$ for some $\delta_2>0$ small enough, we are able to apply (4.3.10) and obtain, with (4.3.11) that

$$|\widehat{\varphi}(\lambda_k-\lambda_r+\frac{s}{|I_\delta|})|\leq\frac{\alpha_\delta}{(k-r+s)^q}.$$

We can now bound E_r. Since $\sum_{s=0}^{\infty} |c_s(r)|^2 \leq 1$, the previous bound and the Cauchy-Schwarz inequality give us

$$\begin{aligned} |E_r| &\leq \sum_{s=0}^{\infty} |c_s(r)| \left| \widehat{\varphi}\left(\lambda_k - \lambda_r + \frac{s}{1+\delta}\right)\right| \leq \alpha_\delta \left(\sum_{s=0}^{\infty} \frac{1}{(k-r+s)^{2q}}\right)^{1/2} \\ &\leq \alpha_\delta \left(\sum_{n=k-r}^{\infty} \frac{1}{n^{2q}}\right)^{1/2} \\ &\leq \alpha_\delta \left(\int_{k-r-1}^{\infty} \frac{dt}{t^{2q}}\right)^{1/2} = \frac{\alpha_\delta}{\sqrt{2q-1}(k-r-1)^{q-1/2}}. \end{aligned}$$

But $k - r > 2^{\ell-1}$ then $k - r - 1 \geq 2^{\ell-1}$. Since $k + N_\delta \in \mathcal{D}_\ell$ *i.e* $2^\ell \leq k + N_\delta \leq 2^{\ell+1}$ we get $\dfrac{1}{k-r-1} \leq \dfrac{4}{k+N_\delta}$ and then

$$|E_r| \leq \frac{4^{q-1/2}\alpha_\delta}{\sqrt{2q-1}(k+N_\delta)^{q-1/2}}.$$

Finally, we deduce that

$$\begin{aligned} |R_1| &= \left| \sum_{2^{m_\delta} \leq r+N_\delta < 2^{\ell-1}} \frac{u_r}{r+N_\delta} E_r \right| \leq \sum_{2^{m_\delta} \leq r+N_\delta < 2^{\ell-1}} \frac{|E_r|}{r+N_\delta} \leq \sum_{2^{m_\delta} \leq r+N_\delta < 2^{\ell-1}} |E_r| \\ &\leq \frac{4^{q-1/2}\alpha_\delta}{\sqrt{2q-1}(k+N_\delta)^{q-3/2}} \leq \frac{\alpha_\delta}{3}\frac{1}{k+N_\delta}. \end{aligned}$$

since the last sum has at most $2^{\ell-1} \leq k + N_\delta$ terms.

We will now bound R_2.

$$\begin{aligned} |R_2| &\leq \sum_{\ell-1 \leq j \leq n_\delta} \int_I |f_j(t)|\,|g_j(t)-1|\,|\varphi(t)|\,\mathrm{d}t \\ &\leq \|\varphi\|_\infty \sum_{\ell-1 \leq j \leq n_\delta} \|f_j\|_2 \|g_j - 1\|_2. \end{aligned}$$

According to the Lemmas 4.3.4 (1) and 4.3.7 (1) we get

$$|R_2| \leq \|\varphi\|_\infty \varepsilon \frac{\sqrt{2}(2+\delta)}{\sqrt{2}-1} \sum_{l-1 \leq j \leq m} 2^{-j}.$$

since

$$\sum_{\ell-1 \leq j \leq n_\delta} 2^{-j} \leq \sum_{j=\ell-1}^{\infty} 2^{-j} = 2^{-\ell+2} = 8.2^{-(\ell+1)}$$

and $k+N_\delta \in I_l$ then $\dfrac{1}{2^{\ell+1}} \leq \dfrac{1}{k+N_\delta}$. Consequently

$$\sum_{\ell-1\leq j\leq n_\delta} 2^{-j} \leq \frac{8}{k+N_\delta}.$$

Finally, we deduce that

$$R_2 \leq \varepsilon\|\varphi\|_\infty \frac{\sqrt{2}(|I_\delta|+1)}{\sqrt{2}-1}\frac{8}{k+N_\delta}$$

and we obtain $R_2 \leq \dfrac{\alpha_\delta}{3}\dfrac{1}{k+N_\delta}$ when $\varepsilon \leq \dfrac{(\sqrt{2}-1)}{24\|\varphi\|_\infty(|I_\delta|+1)\sqrt{2}}\alpha_\delta$. □

Note that, from (4.3.4) and $|I_\delta| \leq 2$, we can take $\varepsilon = c_4\delta^2$ for some $c_4 > 0$.

It is now easy to deduce part *iii* of Theorem 4.1.5 using the 2 inequalities (4.3.6) and (4.3.8).

4.3.3 End of the proof

Proof of Theorem 4.1.5 iii (b). Let $S_\delta = \displaystyle\sum_{k=0}^{N} \frac{|a_k|}{k+N_\delta}$ and $\Phi(t) = \displaystyle\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}$ as previously defined. Recall that in (4.3.6) , we have shown that

$$\begin{aligned}
\alpha_\delta S_\delta &\leq \left|\int_{I_\delta} T_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| \\
&\leq \left|\int_{I_\delta} \tilde{T}_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| + \left|\int_{I_\delta} \big(\tilde{T}_\delta(t)-T_\delta(t)\big)\phi(t)\varphi(t)\,\mathrm{d}t\right| \\
&\leq \left|\int_{I_\delta} \tilde{T}_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| + \frac{2}{3}\alpha_\delta S_\delta
\end{aligned}$$

with (4.3.8).

It follows that

$$\begin{aligned}
S_\delta &\leq \frac{3}{\alpha_\delta}\left|\int_I \tilde{T}_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| \\
&\leq \frac{3\|\tilde{T}\|_\infty\|\varphi\|_\infty}{\alpha_\delta}\int_{I_\delta}|\phi(t)|\,\mathrm{d}t.
\end{aligned}$$

But, if δ is small enough and $N_\delta \geq \delta^{-7/2}$,
– from Lemma 4.3.1, $\|\varphi\|_\infty \leq \dfrac{\pi}{2}$;
– from (4.3.4), $\alpha_\delta = \kappa\delta^2 + O(\delta^4)$
– as $\varepsilon = c_4\delta^2$, from (4.3.7), $\|\tilde{T}\|_\infty = \dfrac{2}{c_4\delta^2}$.

Therefore, there are two absolute constants δ_* and c_* such that, if $\delta \leq \delta_*$, then

$$S_\delta \leq \frac{c_*}{\delta^4} \int_{I_\delta} |\phi(t)| \, \mathrm{d}t.$$

As noticed at the start of the proof, this implies that

$$\sum_{k=0}^{N} \frac{|a_k|}{k+1} \leq \frac{c_*}{\delta^{\frac{15}{2}}} \int_{-\frac{1+\delta}{2}}^{\frac{1+\delta}{2}} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t$$

for every N, every sequence of real numbers $(\lambda_k)_{k\geq 0}$ with $\lambda_{k+1} - \lambda_k \geq 1$ and every complex sequence $(a_k)_{k\geq 0}$. □

4.3.4 Further comments

Let us make a few comments.

- We have not fully optimized the proof, by taking q sufficiently large and p/q sufficiently large, one can replace $\delta^{15/2}$ by $\delta^{7+\eta}$ for any fixed η.
- One cannot seek any gain by changing the choice of g_δ since the function should be even and verifies $\widehat{g}(0) = 1$.
- Changing the dyadic decomposition i.e the intervals I_j will only change the constant A in (4.3.6). Thus it will not give any better result.
- The result depends heavily on the choice of α_δ. But from the (4.3.4) we deduce that the best choice is of order 2 (we cannot go below the power 2).

Hence, using this method, this is the best result we can aim to.

Chapter 5

On L^1-norms of non-harmonic trigonometric polynomials with sparse frequencies

5.1 Introduction

The aim of this chapter is to establish a lower bound of L^1-norms of non-harmonic trigonometric polynomials with sparse frequencies [18]. The results are then applied to obtain L^1-observability estimate of certain PDEs, including the free Schrödinger equation. We thus obtain L^1-analogues of a result of Kahane [22] and Haraux [11] on the L^2-norm of sparse trigonometric polynomials while the L^2-observability result was previously obtained by Jaming and Komornik in [19].

Let us now be more precise. We first recall the well-known results in the L^2-setting. The celebrated Ingham Inequality gives a lower and upper bound of $L^2([-T,T])$-norms of (non-harmonic) trigonometric polynomials and is stated as follows:

Theorem 5.1.1 (Ingham [15]). *Let $\gamma > 0$ and $T > \dfrac{1}{\gamma}$. Then there exist constants $0 < A_2(T,\gamma) \leq B_2(T,\gamma)$ such that*

– *for every sequence of real numbers $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \geq \gamma$;*

– *for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$,*

$$A_2(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \leq B_2(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2.$$

Note that $A_2(T,\gamma), B_2(T,\gamma)$ are explicit constants (*see* [23, 20]). Ingham has also shown that the upper bound is valid for any $T > 0$ while the lower bound may not be true for $T \leq \dfrac{1}{\gamma}$. In his seminal work on almost periodic functions [22], Kahane has shown that this condition can be lifted if $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$:

Theorem 5.1.2 (Kahane). *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$. Then, for every $T > 0$, there exist constants $0 < A_2(T,\Lambda) \leq B_2(T,\Lambda)$ such that*

$$A_2(T,\Lambda)\sum_{k\in\mathbb{Z}} |a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \leq B_2(T,\Lambda)\sum_{k\in\mathbb{Z}} |a_k|^2$$

holds for every sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$.

The constants are not explicit in [22], they were later obtained by Haraux [11] (but with constants that are difficult to compute explicitly, *see e.g.* [23, 20]).

Those inequalities have found many applications in control theory. Among the numerous results (*see* the book [23] for a good introduction to the subject), our starting point is a result of Jaming and Komornik [19]. To state it, let us introduce some notation. We write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \sum_{k\in\mathbb{Z}}(1+|k|^2)^2|c_k(f)|^2 < \infty\}$, where the $c_k(f)$'s are the Fourier coefficients of f. Then the following holds:

Theorem 5.1.3 (Jaming and Komornik). *Fix $(t_1, x_1) \in \mathbb{R}^2$, $a \in \mathbb{R}$ and $T > 0$. For $u_0 \in H^2(\mathbb{T})$, let u be the solution of*

$$\begin{cases} u_t = \dfrac{i}{2\pi}u_{xx} & \text{in } \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{T}. \end{cases}$$

i. There exists $D_2(T,a)$ such that, for every $u_0 \in H^2(\mathbb{T})$,

$$\int_0^T |u(t_1+t, x_1+at)|^2\, dt \leq D_2(T,a)\|u_0\|_{L^2}^2$$

ii. If $a \notin \mathbb{Z}$, then there exists $C_2(T,a)$ such that, for every $u_0 \in H^2(\mathbb{T})$,

$$C_2(T,a)\|u_0\|_{L^2}^2 \leq \int_0^T |u(t_1+t, x_1+at)|^2\, dt \tag{5.1.1}$$

also holds.

iii. If $a \in \mathbb{Z}$, then there exists $u_0 \neq 0$ such that $u(t_0+t, x_0+at) = 0$ so that (5.1.1) fails.

Let us sketch the proof. If we write $u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx}$ then the solution of the Schrödinger equation can be written as a Fourier series $u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi(k^2t+kx)}$, and the fact that $u_0 \in H^2(\mathbb{T})$ implies that this series is uniformly convergent. One can thus restrict it to a segment:

$$u(t_0+t, x_0+2at) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi k^2(t_0+t)+2i\pi k(x_0+2at)} := \sum_{k\in\mathbb{Z}} d_k e^{2i\pi\lambda_k t}.$$

Then one shows that the λ_k's are such that Kahane's theorem applies (provided a is not an integer). Our aim is to extend this argument to the L^1-setting.

The first task is thus to obtain an L^1-version of Ingham's inequality. An $L^1-\ell^\infty$ estimate was obtained by Ingham [15] (and is an easy adaptation of the L^2-proof) and a much more evolved L^1 to weighted ℓ^1-inequality was obtained by Nazarov, inspired by the proof of Littlewood's conjecture by McGehee-Pigno-Smith. The results are as follows

Theorem 5.1.4. *Let $(\lambda_k)_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers such that there exists $\gamma > 0$ for which $\lambda_{k+1} - \lambda_k \geq \gamma$ for every k. Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers.*

- Ingham [16]: *For $T \geq \dfrac{1}{\gamma}$, there exists a constant $A_1(T,\gamma) > 0$ such that, for every $N \geq 1$,*
$$A_1(T,\gamma) \max_{k=-N,\dots,N} |a_k| \leq \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t.$$

- Nazarov [30]: *For $T > \dfrac{1}{\gamma}$, there exists a constant $\tilde{A}_1(T,\gamma) > 0$ such that, for every $N \geq 1$,*
$$\tilde{A}_1(T,\gamma) \sum_{k=0}^{N} \frac{|a_k|}{1+k} \leq \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t.$$

Ingham established the first inequality for $T > \dfrac{1}{\gamma}$ in [15] and improved his result in [16] showing that it also holds for $T = \dfrac{1}{\gamma}$, and that one may take $A_1(T,\gamma) = \dfrac{1}{2}$. This was further improved by Mordell [29]. There is a major difference between the two inequalities: the right hand side in Ingham's inequality is generally much smaller than in Nazarov's inequality (*e.g.* take $|a_k| = 1$ for all k then Ingham provides a constant lower bound while Nazarov provides a logarithmic one). On the other hand, in Nazarov's inequality the sum starts at 0 and may fail for symmetric sums. Also its validity for $T = \dfrac{1}{\gamma}$ is an open question.

This result is sufficient to partially extend Theorem 5.1.3 to the L^1-setting. The only thing that would be missing is that in Theorem 5.1.3, there is no minimal time needed thanks to Kahane's extension of Ingham's inequality. However, so far this is unknown in the L^1-case and our first result is precisely to prove this:

Theorem 5.1.5. *Let $\Lambda = (\lambda_k)_{k\in\mathbb{Z}}$ be an increasing sequence with $\lambda_{k+1} - \lambda_k \to +\infty$ when $k \to \pm\infty$. Then, for every $T > 0$, there exists a constant $\tilde{A}_1(T,\Lambda) > 0$ such that, if $(a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ is a sequence of complex numbers, and $N \geq 1$, then*

$$\tilde{A}_1(T,\Lambda) \sum_{k=0}^{N} \frac{|a_k|}{1+k} \leq \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t. \tag{5.1.2}$$

If further $\sum_{k\in\mathbb{Z}} \frac{1}{1+|\lambda_k|}$ *converges, then there also exists a constant* $A_1(T,\Lambda)$ *such that, for every* $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ *and every* $N \geq 1$,

$$A_1(T,\Lambda) \max_{k=-N,\dots,N} |a_k| \leq \frac{1}{T}\int_{-T/2}^{T/2} \left| \sum_{k=-N}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t. \tag{5.1.3}$$

The main difficulty in the proof of this result is that both Kahane's and Haraux's argument cannot directly be adapted. Indeed, both use in a crucial way that in Ingham's inequality the L^2-norm of a trigonometric polynomial is both lower and upper bounded by the ℓ^2-norm of its coefficients. In the L^1-case, the upper bound is in terms of the ℓ^1-norm of the coefficients and does not match the lower bound. Instead, our proof uses a compactness argument so that we do not obtain an estimate of $A_1(\gamma,T), \tilde{A}_1(\gamma,T)$ in this case. It would be interesting to obtain such an estimate.

Finally, we apply this result to an observability inequality for the Schrödinger equation. We show the following: take $u_0 \in H^2(\mathbb{T})$ and write its Fourier series $u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kt}$. Let u be the solution of

$$\begin{cases} u_t = \dfrac{i}{2\pi} u_{xx} & \text{in } \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{T}. \end{cases}$$

then, for every $a \notin \mathbb{Z}$ and every $T > 0$, there exists a constant $C(a,T) > 0$ such that

$$\frac{1}{T}\int_0^T |u(t_0+t, x_0+at)|\,\mathrm{d}t \geq C \sum_{k\in\mathbb{Z}} \frac{|c_k|}{1+|k|}.$$

Similar results are then obtained for higher order Schrödinger equations.

5.2 Proof of Theorem 5.1.5

First note that replacing the sequence $(\lambda_k)_{k\in\mathbb{Z}}$ by a translate $\lambda_k + \lambda$, leaves (5.1.2)-(5.1.3) unchanged. So there is no loss of generality in assuming that $\lambda_0 > 0 > \lambda_{-1}$. We now fix $T > 0$.

Define K to be an integer such that, if $|k| \geq K$, $\lambda_{k+1} - \lambda_k \geq \frac{2}{T}$. As a consequence, from Nazarov's inequality, the following holds for every sequence $(b_k)_{k\in\mathbb{Z}}$ and every $N \geq 0$:

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} \left| \sum_{k=K}^{K+N} b_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} b_{k+K} e^{2i\pi\lambda_{k+\mathcal{N}} t} \right| \mathrm{d}t \\ &\geq \tilde{A}_1\left(T, \frac{2}{T}\right) \sum_{k=0}^{N} \frac{|b_{k+K}|}{k+1} \\ &\geq \tilde{A}_1\left(T, \frac{2}{T}\right) \sum_{k=K}^{K+N} \frac{|b_k|}{k+1}. \end{aligned} \tag{5.2.1}$$

While Ingham's inequality shows that

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{K\leq|k|\leq K+N} b_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t \geq A_1\left(T,\frac{2}{T}\right)\frac{1}{2}\max_{K\leq|k|\leq K+N}|b_k|. \tag{5.2.2}$$

We first prove (5.1.2). To do so, we will adopt the following convention. An element of $\mathbb{C}^N$ will be indexed starting at 0, $z=(z_0,\dots,z_{N-1})$. It will also be considered as an element of $\mathbb{C}^M$, for $M\geq N$ as a sequence $(z_k)_{k\geq 0}$ by setting $z_k=0$ for $k\geq N$. An element of $\mathbb{C}^N$ is thus called a vector or a sequence. On $\mathbb{C}^N$, we introduce two norms through

$$\|(a_0,\dots,a_{N-1})\|_{\ell^{1,-1}_N}=\sum_{k=0}^{N-1}\frac{|a_k|}{1+k}$$

and

$$\|(a_0,\dots,a_{N-1})\|_{L^1_N}=\left\|\sum_{k=0}^{N-1}a_k e^{2i\pi\lambda_k t}\right\|_{L^1([-T/2,T/2])}:=\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N-1}a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

The first one is clearly a norm while for the second one, it is enough to notice that the set $\{t\to e^{2i\pi\lambda t}\}_{\lambda\in\mathbb{R}}$ is linearly independent in $L^1([-T/2,T/2])$.

As $\|\cdot\|_{\ell^{1,-1}_N}$ and $\|\cdot\|_{L^1_N}$ are both norms on the finite dimensional space $\mathbb{C}^N$, they are equivalent. Thus there are $\kappa_N\leq 1\leq\Lambda_N$ such that, for every $a\in\mathbb{C}^N$,

$$\kappa_N\|a\|_{\ell^{1,-1}_N}\leq\|a\|_{L^1_N}\leq\Lambda_N\|a\|_{\ell^{1,-1}_N}. \tag{5.2.3}$$

Nazarov's theorem asserts that one may choose κ_N independently of N provided T is large enough. Our aim is to show that this is possible for every T under our additional condition on $(\lambda_k)_{k\geq 0}$.

Assume towards a contradiction that this is not the case. Then, for every integer $n\geq 1$, there exist an integer K_n and $a^{(n)}=(a_0^{(n)},\dots,a_{K_n-1}^{(n)})\in\mathbb{C}^{K_n}$ such that $\|a^{(n)}\|_{\ell^{1,-1}_{K_n}}=1$ while $\|a^{(n)}\|_{L^1_{K_n}}\leq\dfrac{1}{n}$. The first observation is that $K_n\to+\infty$ otherwise, we would contradict (5.2.3) when n is large enough. Hence, without loss of generality, we will assume that $K_{n+1}>K_n\geq K$ for every n, where K was defined such that if $|k|\geq K$, $\lambda_{k+1}-\lambda_k\geq\dfrac{2}{T}$.

Next, we split $a^{(n)}$ into two vectors

$$a_-^{(n)}=(a_0^{(n)},\dots,a_{K-1}^{(n)},0,\dots)\quad\text{and}\quad a_+^{(n)}=a^{(n)}-a_-^{(n)}.$$

With an obvious abuse of notation, we consider that $a_-^{(n)}\in\mathbb{C}^K$. In particular $\|a_-^{(n)}\|_{\ell^{1,-1}_K}\leq\|a_-^{(n)}\|_{\ell^{1,-1}_{K_n}}\leq 1$. Thus, up to taking a subsequence, we may assume that $a_-^{(n)}\to(a_0,\dots,a_{K-1})$.

Next, define the following functions

1. The functions $\varphi^{(n)}$ given by

$$\varphi^{(n)}(t) = \sum_{k=0}^{K_n} a_k^{(n)} e^{2i\pi\lambda_k t}$$

so that $\|\varphi^{(n)}\|_{L^1([-T/2,T/2])} \leq \dfrac{1}{n} \to 0$ *i.e.* $\varphi^{(n)} \to 0$ in $L^1([-T/2,T/2])$.

2. The functions $\varphi_-^{(n)}, \varphi_-$ given by

$$\varphi_-^{(n)}(t) = \sum_{k=0}^{K-1} a_k^{(n)} e^{2i\pi\lambda_k t} \quad \text{and} \quad \varphi_-(t) = \sum_{k=0}^{K-1} a_k e^{2i\pi\lambda_k t}.$$

This functions are in a finite dimensional subspace of $L^1([-T/2,T/2])$ so that the convergence $a_k^{(n)} \to a_k$ for $k = 0, \ldots, K-1$ implies that $\varphi_-^{(n)} \to \varphi_-$ in $L^1([-T/2,T/2])$.

3. The functions

$$\varphi_+^{(n)} = \varphi^{(n)} - \varphi_-^{(n)} = \sum_{k=K}^{K_n} a_k^{(n)} e^{2i\pi\lambda_k t}.$$

Note that $\varphi_+^{(n)} \to -\varphi_-$ in $L^1([-T/2,T/2])$. On the other hand, for $n \geq m$ we can apply (5.2.1) to $\varphi_+^{(n)} - \varphi_+^{(m)}$ leading to

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} |\varphi_+^{(n)}(t) - \varphi_+^{(m)}(t)|\,\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2} \left| \sum_{k=K}^{K_n} \big(a_k^{(n)} - a_k^{(m)}\big) e^{2i\pi\lambda_k t} \right| \mathrm{d}t \\ &\geq \tilde{A}_1\left(T, \frac{2}{T}\right) \sum_{k=K}^{K_n} \frac{\big|a_k^{(n)} - a_k^{(m)}\big|}{k+1}. \end{aligned}$$

Using also that $a_k^{(n)} \to a_k$ for $k = 0, \ldots, K-1$ this shows that $\big(a_k^{(n)}\big)_{k\geq 0}$ is a Cauchy sequence in the Banach space

$$\ell^{1,-1} = \left\{ (b_k)_{k\geq 0} \,:\, \|(b_k)\|_{\ell^{1,-1}} := \sum_{k=0}^{+\infty} \frac{|b_k|}{k+1} \right\}.$$

In particular, $\big(a_k^{(n)}\big)_{k\geq 0} \to (a_k)_{k\geq 0}$ in $\ell^{1,-1}$. This implies that, for all k, $a_k^{(n)} \to a_k$ and that

$$1 = \|a^{(n)}\|_{\ell^{1,-1}_{K_n}} = \|a^{(n)}\|_{\ell^{1,-1}} \to \|a\|_{\ell^{1,-1}}.$$

We will thus reach a contradiction if we show that $a_k = 0$ for all k.

To do so, we introduce further functions via

$$\Phi_\pm^{(n)}(x) = \int_0^x \varphi_\pm^{(n)}(t)\,\mathrm{d}t \quad \text{and} \quad \Phi_-(x) = \int_0^x \varphi_-(t)\,\mathrm{d}t = \frac{1}{2i\pi}\sum_{k=0}^{K-1} \frac{a_k}{\lambda_k}\big(e^{2i\pi\lambda_k x} - 1\big).$$

Note that as $\varphi_\pm^{(n)} \to \pm\varphi_-$ in $L^1([-T/2,T/2])$, $\Phi_\pm^{(n)} \to \pm\Phi_-$ uniformly over $[-T/2,T/2]$ thus also in $L^2([-T/2,T/2])$.

Next, as $(\lambda_n)_{n\in\mathbb{N}}$ is increasing with $\lambda_0 > 0$ and $\lambda_{n+1} - \lambda_n \to +\infty$, there exists $\alpha > 0$ such that $\lambda_n \geq \alpha(n+1)$. It follows that

$$\sum_{k=0}^{+\infty} \frac{|a_k|}{\lambda_k} \leq \frac{1}{\alpha}\sum_{k=0}^{+\infty} \frac{|a_k|}{k+1} < +\infty \quad \text{and} \quad \sum_{k=0}^{+\infty} \frac{|a_k^{(n)} - a_k|}{\lambda_k} \leq \frac{1}{\alpha}\sum_{k=0}^{+\infty} \frac{|a_k^{(n)} - a_k|}{k+1} \to 0.$$

As $|e^{2i\pi\lambda_k x} - 1| \leq 2$, it follows that

$$\Phi_+^{(n)} = \frac{1}{2i\pi}\sum_{k=K}^{K_n} \frac{a_k^{(n)}}{\lambda_k}\big(e^{2i\pi\lambda_k x} - 1\big) \to \Phi_+ = \frac{1}{2i\pi}\sum_{k=K}^{\infty} \frac{a_k}{\lambda_k}\big(e^{2i\pi\lambda_k x} - 1\big)$$

where the series defining Φ_+ is uniformly convergent over $[-T/2,T/2]$ and the convergence $\Phi_+^{(n)} \to \Phi_+$ is uniform over $[-T/2,T/2]$, thus also in $L^2([-T/2,T/2])$. But we also know that $\Phi_+^{(n)} \to -\Phi_-$ in $L^2([-T/2,T/2])$ thus $\Phi_+ + \Phi_- = 0$.

It remains to apply Kahane's extension of Ingham's Inequality to obtain that

$$\begin{aligned} 0 = \frac{1}{T}\int_{-T/2}^{T/2} |\Phi_+(t) + \Phi_-(t)|^2 \,\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2} \left| -\frac{1}{2i\pi}\sum_{k=0}^{+\infty} \frac{a_k}{\lambda_k} + \sum_{k=0}^{+\infty} \frac{a_k}{2i\pi\lambda_k} e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \\ &\geq A_2(T,\Lambda)\left(\left|\frac{1}{2i\pi}\sum_{k=0}^{+\infty}\frac{a_k}{\lambda_k}\right|^2 + \sum_{k=0}^{+\infty}\left|\frac{a_k}{2i\pi\lambda_k}\right|^2\right) \end{aligned}$$

thus $a_k = 0$ for all k and we obtain the desired contradiction.

The proof of (5.1.3) is similar, so we give less detail. Elements of $\mathbb{C}^{2N+1}$ will be indexed from $-N$ to N, *i.e.* $z = (z_{-N},\dots,z_N)$ and will be considered as an element of $\mathbb{C}^{2M+1}$, $M \geq N$ and also as a sequence $(z_k)_{k\in\mathbb{Z}}$ by setting $z_k = 0$ when $|k| > N$. We again consider two norms on $\mathbb{C}^{2K+1}$, the ℓ^∞ norm and (with a small abuse of notation)

$$\|(a_{-K},\dots,a_K)\|_{L^1_N} = \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-K}^{K} a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

For every K there exists $\tilde{\kappa}_K$ such that, for every $a \in \mathbb{C}^{2K+1}$,

$$\tilde{\kappa}_K\|a\|_\infty \leq \|a\|_{L^1_N}.$$

Ingham's theorem asserts that one may choose κ_K independently of K provided T is large enough. Our aim is again to show that this is possible for every T under our additional condition on λ_k.

Assume towards a contradiction that this is not possible.

Then, for every integer $n \geq 1$, there exist an integer $K_n \to +\infty$ with $K_{n+1} > K_n \geq K$ and $a^{(n)} = (a_{-K_n}^{(n)},\dots,a_{K_n}^{(n)}) \in \mathbb{C}^{2K_n+1}$ such that $\|a^{(n)}\|_\infty = 1$ while $\|a^{(n)}\|_{L^1_{K_n}} \leq \dfrac{1}{n}$. So,

without loss of generality, we will assume that $K_{n+1} > K_n \geq K$ for every n. Recall that we defined K so that if $|k| \geq K$, $\lambda_{k+1} - \lambda_k \geq \dfrac{2}{T}$.

We split $a^{(n)}$ into two vectors

$$a_-^{(n)} = (\dots, 0, a_{-K+1}^{(n)}, \dots, a_{K-1}^{(n)}, 0, \dots) \in \mathbb{C}^{2K-1} \quad \text{and} \quad a_+^{(n)} = a^{(n)} - a_-^{(n)}.$$

As $\|a_-^{(n)}\|_\infty \leq \|a^{(n)}\|_\infty = 1$, there is no loss of generality in assuming that

$$a_-^{(n)} \to (a_{-K+1}, \dots, a_{K-1}).$$

We again consider

$$\varphi^{(n)}(t) = \sum_{k=-K_n}^{K_n} a_k^{(n)} e^{2i\pi\lambda_k t} \to 0$$

in $L^1([-T/2, T/2])$,

$$\varphi_-^{(n)}(t) = \sum_{k=-K+1}^{K-1} a_k^{(n)} e^{2i\pi\lambda_k t} \to \varphi_-(t) = \sum_{k=-K+1}^{K-1} a_k e^{2i\pi\lambda_k t}$$

in $L^1([-T/2, T/2])$ and

$$\varphi_+^{(n)} = \varphi^{(n)} - \varphi_-^{(n)} = \sum_{K \leq |k| \leq K_n} a_k^{(n)} e^{2i\pi\lambda_k t} \to -\varphi_-$$

in $L^1([-T/2, T/2])$.

Using (5.2.2) instead of (5.2.1) we get, for $n \geq m$

$$\frac{1}{T} \int_{-T/2}^{T/2} |\varphi_+^{(n)}(t) - \varphi_+^{(m)}(t)| \, \mathrm{d}t \geq A_1\left(T, \frac{2}{T}\right) \max_{K \leq |k| \leq K_n} \left|a_k^{(n)} - a_k^{(m)}\right|$$

so that $\left(a_k^{(n)}\right)_{k\in\mathbb{Z}}$ is a Cauchy sequence in ℓ^∞ and we call $a = (a_k)_{k\in\mathbb{Z}}$ its limit. Of course $\|a\|_\infty = 1$ so that we will again reach a contradiction if we show that $a_k = 0$ for all k.

To do so, we introduce again

$$\Phi_\pm^{(n)}(x) = \int_0^x \varphi_\pm^{(n)}(t) \, \mathrm{d}t \quad \text{and} \quad \Phi_-(x) = \int_0^x \varphi_-(t) \, \mathrm{d}t = \frac{1}{2i\pi} \sum_{k=0}^{K-1} \frac{a_k}{\lambda_k} \left(e^{2i\pi\lambda_k x} - 1\right)$$

so that $\Phi_\pm^{(n)} \to \pm\Phi_-$ uniformly over $[-T/2, T/2]$ thus also in $L^2([-T/2, T/2])$.

Next, as $\lambda_k \neq 0$ and $\displaystyle\sum_{k\in\mathbb{Z}} \frac{1}{1 + |\lambda_n|}$ converges so is $\displaystyle\sum_{k\in\mathbb{Z}} \frac{1}{|\lambda_n|}$. As $(a_k) \in \ell^\infty$ and $|a_k^{(n)} - a_k| \to 0$ in ℓ^∞ we get

$$\sum_{k\in\mathbb{Z}} \frac{|a_k|}{\lambda_k} < +\infty \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \frac{|a_k^{(n)} - a_k|}{\lambda_k} \to 0.$$

As $|e^{2i\pi\lambda_k x} - 1| \leq 2$, it follows that

$$\Phi_+^{(n)} = \frac{1}{2i\pi} \sum_{K \leq |k| \leq K_n} \frac{a_k^{(n)}}{\lambda_k} \big(e^{2i\pi\lambda_k x} - 1\big) \to \Phi_+ = \frac{1}{2i\pi} \sum_{|k| \geq K} \frac{a_k}{\lambda_k} \big(e^{2i\pi\lambda_k x} - 1\big)$$

where the series defining Φ_+ is uniformly convergent over $[-T/2, T/2]$ and the convergence $\Phi_+^{(n)} \to \Phi_+$ is uniform over $[-T/2, T/2]$, thus also in $L^2([-T/2, T/2])$. But we also know that $\Phi_+^{(n)} \to -\Phi_-$ in $L^2([-T/2, T/2])$ thus $\Phi_+ + \Phi_- = 0$. Applying again Kahane's extension of Ingham's inequality we obtain $a_k = 0$ for all k which is the desired contradiction.

5.3 Application to 1-periodic Schrödinger equation

Recall that the Wiener algebra is defined as

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) \,:\, \|f\|_{A(\mathbb{T})} = \sum_{k \in \mathbb{Z}} |c_k(f)| < +\infty\}.$$

Theorem 5.3.1. *Let u be a weak solution of the Schrödinger equation*

$$\begin{cases} i\partial_t u(t,x) = \dfrac{1}{2\pi} \partial_x^2 u(t,x) & t \in \mathbb{R}, x \in \mathbb{T} \\ u_0 = u(0,x) & x \in \mathbb{R} \end{cases}, \tag{5.3.1}$$

with initial value $u_0(x) = \sum_{k \in \mathbb{Z}} c_k e^{2i\pi kt} \in A(\mathbb{T})$. Let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{T}$. Then

1. *For $a \in \mathbb{R} \setminus \mathbb{Z}$, for every $T > 0$ there exists a constant $C(a,T) > 0$ such that*

$$\frac{1}{T} \int_0^T |u(t_0 + t, x_0 + at)| \, \mathrm{d}t \geq C \sum_{k \in \mathbb{Z}} \frac{|c_k|}{1 + |k|}. \tag{5.3.2}$$

2. *If $a \in \mathbb{Z}$ then there exists $u_0 \neq 0$ such that $u(t_0 + t, x_0 + at) = 0$ for all t. In particular,* (5.3.2) *fails.*

Remark 5.3.2. *Recall also that if $u_0 \in H^s(\mathbb{T})$ with $s > \dfrac{1}{2}$ then, with Cauchy-Schwarz,*

$$\|u_0\|_{A(\mathbb{T})} \leq \left(\sum_{k \in \mathbb{Z}} \frac{1}{(1 + |k|^2)^s} \right)^{1/2} \left(\sum_{k \in \mathbb{Z}} (1 + |k|^2)^s |c_k|^2 \right)^{1/2} < +\infty.$$

One may thus replace the condition $u_0 \in A(\mathbb{T})$ with a more familiar condition like $u_0 \in H^1(\mathbb{T})$.

Proof. Write $u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx}$ so that $u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi(k^2t+kx)}$. This series is uniformly convergent over $\mathbb{R}\times\mathbb{T}$ since $\sum|c_k|$ converges thus u is continuous. Further

$$v(t) = u(t_0+t, x_0+at) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi k^2(t_0+t)+2i\pi k(x_0+at)} = \sum_{k\in\mathbb{Z}} d_k e^{2i\pi\lambda_k t}$$

with

$$d_k = c_k e^{2i\pi(k^2t_0+kx_0)} \quad \text{and} \quad \lambda_k = k^2+2bk.$$

Note that $|d_k| = |c_k|$. On the other hand

$$\begin{aligned} \lambda_k - \lambda_m &= k^2+2ak-(m^2+am) = k^2-m^2+a(k-m) \\ &= (k-m)(k+m+a). \end{aligned} \tag{5.3.3}$$

Assume first that $a\in\mathbb{Z}$. This case was already treated in [19] but let us reproduce the proof here for completeness. In this case, the frequencies (λ_k) satisfy the symmetry property $\lambda_k = \lambda_{-a-k}$. Now fix $k\neq -a$ and notice that $-a-k\neq 0$ so that, if we fix $c_k\neq 0$ we can choose c_{-a-k} so that $d_{-a-k} = -d_k$ that is

$$c_{-a-k} = -c_k e^{2i\pi\left((k^2t_0+kx_0)-((-a-k)^2t_0+(-a-k)x_0)\right)} = -c_k e^{-2i\pi\left(a(a+2k)t_0+x_0)\right)}.$$

Setting

$$u_0(x) = c_k\left(e^{2i\pi kt} - e^{-2i\pi\left(a(a+2k)t_0+2ax_0)\right)}e^{-2i\pi(a+k)t}\right)$$

we obtain $u(t_0+t, x_0+at) = 0$.

From now on, we assume that $a\notin\mathbb{Z}$ so that from (5.3.3) we deduce that $\lambda_k\neq\lambda_m$ when $k\neq m$. It will be convenient to write $a = 2b$. We can then further split the sequence $(\lambda_k)_{k\in\mathbb{Z}}$ into a disjoint union, $(\lambda_k)_{k\in\mathbb{Z}} = (\lambda_k^+)_{k\geq 0}\cup(\lambda_k^-)_{k\geq 1}$ with

$$\lambda_k^+ := \lambda_{-[b]+k} = (-[b]+k)^2+2b(-[b]+k) \quad \text{for} \quad k\geq 0$$

and

$$\lambda_k^- := \lambda_{-[b]-k} = (-[b]-k)^2+2b(-[b]-k) \quad \text{for} \quad k\geq 1.$$

By definition

$$\lambda_0^+ = [b]^2-2b[b] \quad \text{and} \quad \lambda_1^- = \lambda_0^+ + 1 - 2(b-[b]).$$

We will now distinguish two cases:

First case: Assume that $\dfrac{1}{2} < b-[b] < 1$ so that $\lambda_1^- < \lambda_0^+$.

In this case, the frequencies interlace as follows:

$$\lambda_{k+1}^- < \lambda_k^+ < \lambda_{k+2}^- \quad \text{for all} \quad k\geq 0.$$

Indeed, for all $k\geq 0$

$$\lambda_k^+ - \lambda_{k+1}^- = 2(2k+1)\left(b-[b]-\frac{1}{2}\right) > 0 \quad \text{and} \quad \to\infty \quad \text{as} \quad k\to\infty$$

and

$$\lambda^-_{k+2} - \lambda^+_k = 4(k+1)\big(1 - (b - [b]))\big) > 0 \quad \text{and} \quad \to \infty \quad \text{as} \quad k \to \infty$$

with our hypothesis on $b - [b]$. In particular, if we set $\mu_{2k} = \lambda^-_{k+1} = \lambda_{-[b]-k-1}$ and $\mu_{2k+1} = \lambda^+_k = \lambda_{[-b]+k}$ for $k \geq 0$ then $0 < \mu_{2k+1} - \mu_{2k} \to +\infty$. From Theorem 5.1.5, we thus get that

$$\int_0^T |v(t)|\,\mathrm{d}t \geq C(T) \sum_{k=0}^{\infty} \left(\frac{|c_{-[a]-k-1}|}{2k+1} + \frac{|c_{-[a]+k}|}{2k+2} \right).$$

Finally, for $k \geq 0$, $2k+1 \leq \alpha_a(|-[a]-k-1|+1)$ and $2k+2 \leq \alpha_a(|-[a]+k|+1)$ with a constant α_a depending on a only, so that

$$\int_0^T |v(t)|\,\mathrm{d}t \geq \frac{C(T)}{\alpha_a} \sum_{k\in\mathbb{Z}} \frac{|c_k|}{|k|+1}$$

as claimed.

Second case: $0 < b - [b] < \dfrac{1}{2}$. Note that $b - [b] \neq 0, \dfrac{1}{2}, 1$ since $a = 2b \notin \mathbb{Z}$ all cases are covered. In this case, similar computations show that the frequencies interlace as

$$\lambda^+_k < \lambda^-_{k+1} < \lambda^+_{k+1} \quad \text{for all} \quad k \geq 0$$

with $\lambda^-_{k+1} - \lambda^+_k, \lambda^+_{k+1} - \lambda^-_{k+1} \to +\infty$. The remaining of the proof is the same and is thus omitted. □

5.4 General Case

Let x in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}^+$, we consider the following equations

$$\begin{cases} i\partial_t u(t,x) = 2\pi P\left(\dfrac{\partial_x}{2i\pi}\right) u \\ u_0 = u(0,x) = \displaystyle\sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx} \in \mathbb{A}(\mathbb{T}) \end{cases} . \tag{5.4.1}$$

where

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0$$

with $n \geq 2$ and $a_n \neq 0$. Note that up to replacing u by $\bar{u}$, there is no loss of generality in assuming that $a_n > 0$.

If $u_0(x) = \displaystyle\sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx} \in A(\mathbb{T})$, then the solution to this system is given by

$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{-2i\pi P(k)t} e^{2i\pi kx}.$$

Again, this is a continuous function.

Let $a \in \mathbb{R}$ to be chosen later. For any $(t_0, x_0) \in \mathbb{R} \times \mathbb{T}$, we define

$$u(t_0 + t, x_0 + at) = \sum_{k\in\mathbb{Z}} c_k e^{-2i\pi P(k)(t_0+t)+2i\pi k(x_0+at)} = \sum_{k\in\mathbb{Z}} d_k e^{-2i\pi\lambda_k t}$$

with

$$d_k = c_k e^{-2i\pi(P(k)t_0 - kx_0)} \quad \text{and} \quad \lambda_k = P(k) - ak.$$

Note that $\lambda_k - \lambda_m = (k-m)\big(Q(k,m) - a\big)$ with

$$\begin{aligned} Q(k,m) &= a_n(k^{n-1} + k^{n-2}m + \ldots + m^{n-1}) + a_{n-1}(k^{n-2} + \ldots + m^{n-2}) + \ldots + a_1 \\ &= \sum_{\ell=1}^{n} a_\ell \sum_{j=0}^{\ell-1} k^{\ell-1-j} m^j. \end{aligned}$$

Define

$$E = \{Q(k,m) : \ k, m \in \mathbb{Z} \text{ such that } k \neq m\}$$

which is countable (thus of measure 0).

Theorem 5.4.1. *Let u be any solution of the Schrödinger equation* (5.4.1) *with initial value $u_0 = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi kx} \in A(\mathbb{T})$. Then*

1. *If $a \notin E$, for all $T > 0$ there exists a constant $C(a,T) > 0$ such that*

$$\frac{1}{T}\int_0^T |u(t_0 + t, x_0 + at)|\,\mathrm{d}t \geq C \max_{k\in\mathbb{Z}} |c_k|. \tag{5.4.2}$$

If n is even, there also exists a constant $C(a,T) > 0$ such that

$$\frac{1}{T}\int_0^T |u(t_0 + t, x_0 + at)|\,\mathrm{d}t \geq C \sum_{k\in\mathbb{Z}} \frac{|c_k|}{1+|k|}. \tag{5.4.3}$$

2. *If $a \in E$ then both* (5.4.2)-(5.4.3) *fail.*

An L^2-analogue of this result can be found in [40].

Proof. The last part of the theorem is the same as for the Schrödinger equation in the previous section. Indeed, if $a \in E$, we can choose two indexes $k \neq m$ such that $\lambda_k = \lambda_m$ and then choose c_k, c_m such that $d_k = -d_m$. Taking $u_0 = c_k e^{2i\pi kt} + c_m e^{2i\pi mt}$, the corresponding solution u satisfies $u(t_0 + t, x_0 + at) = 0$.

We now assume that $a \notin E$ so that $\lambda_k \neq \lambda_m$ for all $k, m \in \mathbb{Z}$. We will further show that the $(\lambda_k)'s$ can be ordered as a sequence with gaps going to infinity. Here we need to distinguish between n even or odd. We start with the odd case.

If n is odd, then $\lambda_k = P(k) - ak \to \pm\infty$ when $k \to \pm\infty$. Note also that, as P has degree at least 3, $\sum \dfrac{1}{1+|\lambda_k|}$ converges.

Further $\lambda_{k+1}-\lambda_k = Q(k+1,k)-a = a_n k^{n-1}+o(k^{n-1}) \to +\infty$ when $k \to \pm\infty$. so that, there exists K such that, for $k \geq K$, λ_k is increasing as well as for $k \leq -K$. There further exists $K' \geq K$ such that, if $k, \ell \geq K'$, then

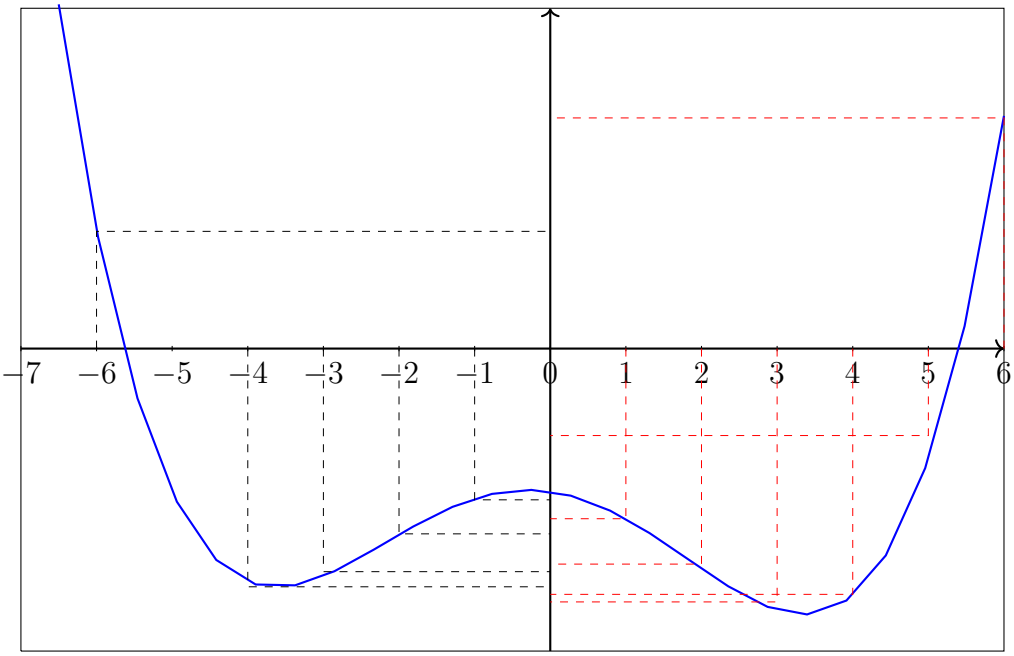
$$\lambda_{-\ell} \leq \min_{|j|\leq K} \lambda_j \leq \max_{|j|\leq K} \lambda_j \leq \lambda_k.$$

We then define $(\mu_k)_{|k|\leq K'}$ as an ordering of $(\lambda_k)_{|k|\leq K'}$ and $\mu_k = \lambda_k$ for $|k| > K'$. Note that those λ_k's are not one of the $(\mu_k)_{|k|\leq K'}$'s. Then $(\mu_k)_{k\in\mathbb{Z}}$ is an increasing sequence with gaps $\mu_{k+1}-\mu_k \to +\infty$ when $k \to \pm\infty$. We can then apply (5.1.3) to conclude.

We now assume that $n = 2p$ is even. In this case $\lambda_k = P(k) - ak \to +\infty$ when $k \to \pm\infty$ and $\lambda_{k+1}-\lambda_k \to \pm\infty$ when $k \to \pm\infty$. In this case, the ordering needs to be made differently.

The idea is rather simple, there is an oscillating part and we are going to show that, for k, ℓ large, the λ_k's and $\lambda_{-\ell}$'s interlace. In the generic case we actually have $\lambda_{k+q_0} < \lambda_{-k} < \lambda_{k+q_0+1}$ for some fixed q_0 and large enough k. This shows that, for some K_0, $(\lambda_k)_{k\notin\{-K_0,\dots,K_0+q_0\}}$ can be rearranged in an increasing way as:

$\lambda_{K_0+q_0+1}, \lambda_{-K_0-1}, \lambda_{K_0+q_0+2}, \lambda_{-K_0-2}, \dots$. The finite number of remaining λ_k's are rearranged separately and, provided K_0 is large enough, they can be put at the start and the resulting sequence $(\mu_k)_{k\geq 0}$ is then increasing with gaps going to infinity. A key aspect of this construction is that each μ_k is a $\lambda_{k'}$ with $\big|k-|k'|\big| \leq C_\Lambda$ depending only on Λ. The idea is the same in the exceptional case.



The picture shows the case of a polynomial P of degree 4. The reordering here is $\mu_0 = \lambda_3$, $\mu_1 = \lambda_4$, $\mu_2 = \lambda_{-4}$, $\mu_3 = \lambda_{-3}$ $\mu_4 = \lambda_2$, $\mu_5 = \lambda{-}2$, $\mu_6 = \lambda_1$, $\mu_7 = \lambda_1$, $\mu_8 = \lambda_{-5}$ (not represented to keep the picture readable), $\mu_9 = \lambda_{-1}$, $\mu_{10} = \lambda_0$, $\mu_{11} = \lambda_5$, $\mu_{12} = \lambda_{-6}$, $\mu_{13} = \lambda_6$ and more generally $\mu_{12+2k} = \lambda_{-6-k}$ while $\mu_{13+2k} = \lambda_{6+k}$.

Let us now be more precise.

We again take K such that from $-\infty$ to K, λ_k is decreasing while from K to $+\infty$, λ_k

is increasing and define K' such that if $k, \ell \geq K'$, then

$$\max_{|j|\leq K} \lambda_j \leq \lambda_{-\ell}, \lambda_k.$$

Next, an easy computation shows that

$$\sum_{j=0}^{\ell}(-1)^j = \begin{cases} 1 & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd} \end{cases} \quad \text{and} \quad \sum_{j=0}^{\ell}(-1)^j j = \begin{cases} \ell/2 & \text{if } \ell \text{ is even} \\ -(\ell+1)/2 & \text{if } \ell \text{ is odd} \end{cases}$$

so that

$$\lambda_{k+q} - \lambda_{-k} = (2k+q)\left(\sum_{\ell=1}^{2p} a_\ell \sum_{j=0}^{\ell-1}(-1)^j (k+q)^{\ell-1-j} k^j - a\right).$$

But

$$\begin{aligned} a_{2p}\sum_{j=0}^{2p-1}(-1)^j(k+q)^{\ell-1-j}k^j &= a_{2p}k^{2p-1}\sum_{j=0}^{2p-1}(-1)^j + a_{2p}qk^{2p-2}\sum_{j=0}^{2p-2}(-1)^j j + o(k^{2p-2}) \\ &= (p-1)a_{2p}qk^{2p-2} + o(k^{2p-2}) \end{aligned}$$

and

$$a_{2p-1}\sum_{j=0}^{2p-2}(-1)^j(k+q)^{2p-2-j}k^j = a_{2p-1}k^{2p-2} + o(k^{2p-2})$$

so that

$$\lambda_{k+q} - \lambda_{-k} = \begin{cases} \big((p-1)a_{2p}q + a_{2p-1}\big)k^{2p-1} + o(k^{2p-1}) & \text{if } p \geq 2 \\ \big(a_2 q + a_1 - a\big)k + o(k) & \text{if } p = 1 \end{cases}. \tag{5.4.4}$$

Set $\alpha_q = \begin{cases} a_2 q + a_1 - a & \text{if } p = 1 \\ (p-1)a_{2p}q + a_{2p-1} & \text{if } p \geq 2 \end{cases}$ so that $\lambda_{k+q} - \lambda_{-k} = \alpha_q k^{2p-1} + o(k^{2p-1})$.

There are now two cases:

Case 2.1 *For every q, $\alpha_q \neq 0$*

Then there exists q_0 such that $\alpha_{q_0} > 0$ and $\alpha_{q_0-1} < 0$. But then, $\lambda_{k+q_0} - \lambda_{-k} \to +\infty$ while $\lambda_{k+q_0-1} - \lambda_{-k} \to -\infty$.

We now take $K'' > \max(K' - q_0, K')$ such that, for $k \geq K''$, $\lambda_{k+q_0} - \lambda_{-k} > 0$ and $\lambda_{k+q_0-1} - \lambda_{-k} < 0$, that is $\lambda_{k+q_0-1} < \lambda_{-k} < \lambda_{k+q_0}$. The choice of K'' also implies that $\lambda_{-K''+1}, \ldots, \lambda_{K''+q_0-1}$ are all $< \min(\lambda_{-K''}, \lambda_{K''+q_0})$. We can thus reorder those terms as an increasing sequence $(\mu_k)_{k=0,\cdots,\hat{K}}$ with $\hat{K} = 2K'' + q_0 - 2$, that we then complete into a sequence $(\mu_k)_{k\in\mathbb{N}}$ by adding successively a term $\lambda_{K''+k+q_0}$ and a term $\lambda_{-K''-k}$ and the resulting sequence is an increasing rearrangement of (λ_k) such that $\mu_k \to +\infty$ and $\mu_{k+1} - \mu_k \to +\infty$. Note that if we define σ the mapping $\mathbb{Z} \to \mathbb{N}$ defined by $\mu_k = \lambda_{\sigma(k)}$ then there is a constant C_Λ such that $\big||k| - \sigma(k)\big| \leq C_\Lambda$.

It follows from (5.1.2) that

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2}|u(t_0+t,x_0+at)|\,\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}} d_k e^{-2i\pi\lambda_k t}\right|\mathrm{d}t\\ &= \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{+\infty} d_{\sigma^{-1}(k)} e^{-2i\pi\mu_k t}\right|\mathrm{d}t\\ &\geq \tilde{A}_1(T,\Lambda)\sum_{k=0}^{+\infty}\frac{|d_{\sigma^{-1}(k)}|}{1+k} = \tilde{A}_1(T,\Lambda)\sum_{j\in\mathbb{Z}}\frac{|c_j|}{1+\sigma(j)}\\ &\geq \frac{\tilde{A}_1(T,\Lambda)}{1+C_\Lambda}\sum_{k\in\mathbb{Z}}\frac{|c_k|}{1+|k|}.\end{aligned}$$

Note that the series $\sum_{k\in\mathbb{Z}} d_k e^{-2i\pi\lambda_k t}$ is uniformly convergent so that we can re-order terms.

Case 2.2 *There exists q_0 such that $\alpha_{q_0}=0$.*

The proof is essentially the same, but the interlacing of the λ_k and $\lambda_{-\ell}$ for k,ℓ large may be different. This comes from the fact that the leading term in (5.4.4) is 0. Nevertheless, $\alpha_{q_0+1}>0$ and $\alpha_{q_0-1}<0$ so that, for k large enough $\lambda_{k+q_0+1}-\lambda_{-k}>0$ while $\lambda_{k+q_0-1}-\lambda_{-k}<0$. So, for each k, either $\lambda_{k+q_0-1}<\lambda_{k+q_0}<\lambda_{-k}<\lambda_{k+q_0+1}$ or $\lambda_{k+q_0-1}<\lambda_{-k}<\lambda_{k+q_0}<\lambda_{k+q_0+1}$ (actually only one can occur for k large enough) and we define the rearrangement μ_k accordingly. □

Chapter 6

On Besicovitch norms of non-harmonic trigonometric polynomials with sparse frequencies

6.1 Introduction

As we already mentioned, Hudson and Leckband were the first to generalize results on the Littlewood conjecture to the real setting, they showed that the coefficients of a non harmonic polynomials are controlled by its Besicovitch norm;

Theorem 6.1.1 (Hudson & Leckband [14]). *For $\lambda_0 < \lambda_1 < \ldots < \lambda_N$ real numbers and $a_0, \ldots, a_N$ complex numbers*

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{30} \sum_{k=0}^{N} \frac{|a_k|}{k+1}.$$

To prove this result, they mainly used a perturbation argument based on a lemma by Dirichlet ([44, p 235], [8]) which allows them to benefit from McGehee, Pigno and Smith's result in the integer setting (Mc-Gehee, Pigno and Smith's Theorem 1.0.1).

In this chapter, we give lower bounds (occasionally upper bounds too) of the Besicovitch norm of the sums

$$\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t},$$

where the frequencies are real. Depending on the properties of the frequencies, we are primarily focused on two different cases: Lacunary sums and the case where the set of frequencies has a multidimensional structure. In each case, we will define properly the framework, present some already established results (mainly in the integer setting) and then we will use the technique of Hudson and Leckband to extend existing results to the case of real frequencies.

6.1.1 Hadamard Lacunary trigonometric series

Hadamard lacunary trigonometric series or simply lacunary series have the following form

$$\sum_{k=0}^{\infty} a_k e^{2i\pi n_k t}, \tag{6.1.1}$$

where the $n_k's$ are q-lacunary, meaning that, for all k, there exists a $q > 1$ such that

$$\frac{n_{k+1}}{n_k} > q. \tag{6.1.2}$$

Here t is a real variable in $[0, 2\pi]$, $n_k's$ are non-zero integers, and the coefficients are complex.

In the integer setting, Zygmund showed that if

$$\|a\|_{l^2(\mathbb{N})} = \sum_{k=0}^{\infty} |a_k|^2 < \infty,$$

then the series (6.1.1) defines a function $f \in \bigcap_{1\le p<\infty} L^p(0,1)$, and there are positive constants $A_{p,q}$ and $B_{p,q}$ such that

$$A_{p,q}\|a\|_{l^2(\mathbb{N})} \le \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p \,\mathrm{d}t\right)^{1/p} \le B_{p,q}\|a\|_{l^2(\mathbb{N})}. \tag{6.1.3}$$

In Section 6.2, We will present the proof of (6.1.3) and then we will see how we can get a similar result for the Besicovitch norms;

Theorem 6.1.2. *Let $q > 1$ and $(\lambda_k)_{k\ge 0}$ be real numbers verifying*

$$\lambda_0 > 1 \quad \textit{and} \quad \lambda_{k+1} \ge q\lambda_k$$

and $a_0, \dots, a_N$ be a sequence of complex numbers. Then for $1 \le p < \infty$, there exists positive constant $A_{p,q}$, $B_{p,q}$ such that

$$A_{p,q}\left(\sum_{k=0}^{N} |a_k|^2\right)^{1/2} \le \lim_{T\to+\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \le B_{p,q}\left(\sum_{k=0}^{N} |a_k|^2\right)^{1/2}.$$

6.1.2 Sets with multidimensional structure

There are multiple notions of sets with multidimensional structure. The simplest one is sets of dimension one which are projections of higher dimensional sets [21]. A classic example of such sets is a finite generalized arithmetic progression of rank 2, which is a set of the following form; let $a, b \in \mathbb{Z}$,

$$GAP = \{am + bn : 1 \le m \le M, 1 \le n \le N\}.$$

These sets can be thought of as projection of boxes in $\mathbb{Z}^2$: If a and b are mutually prime, then

$$GAP = P(B),$$

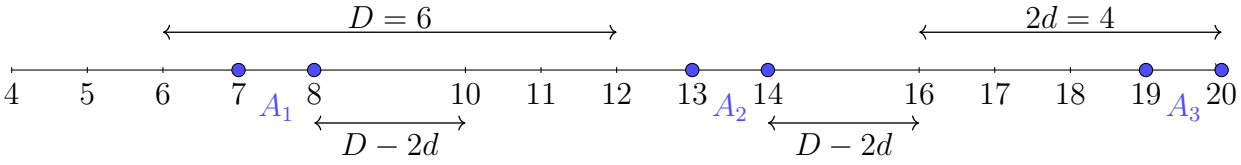
where P is the surjection defined by $P(m;n) = am + bn$.

Here we will adopt a more general notion; let $\delta > 0$ and $(m,n) \in \mathbb{N}^2$. A subset A of $\mathbb{Z}$ is $(\delta; m, n)$-strongly 2-dimensional if there exists numbers d and D with $D > (2+\delta)d$ such that

$$A = \bigcup_{k \in I} (A_k + kD) \tag{6.1.4}$$

for some set I containing at least m integers and subsets $A_k \subseteq \{-d, \dots, d\}$ verifying $|A_k| \geq n$. The simplest example of sets in $\mathbb{Z}$ with multidimensional structure are sets which are unions of intervals separated by a gap.

To see the relation between the two definitions, let us give an example where a generalized arithmetic progression of rank 2, verifies the strongly multidimensional conditions. Let $A = \{7, 8, 13, 14, 19, 20\}$. It is easy to check that, for $a = 1$, $b = 6$, the set A is a generalized arithmetic progression of rank 2 (with $M = 2$ and $N = 3$). Furthermore, A is $(1/2; 3, 2)$ strongly multidimensional, since it can be seen as union of 3 interval, each containing at least 2 integers.



Hanson's motivation came from the following question; Can we find a relation between the additive structure of a set A and the L^1-norm of exponential sums over the set A. A known question in this direction is the inverse Littlewood problem; Given a positive constant C, can we characterize the sets A for which

$$\int_0^1 \left| \sum_{a \in A} e^{2i\pi at} \right| \mathrm{d}t \leq C \ln(|A|).$$

Green [9] suggested that such A might be very close to being unions of a few arithmetic progressions and if so, he pointed out to a relation with sum-free sets established by Bourgain [2]. More discussion about multidimensional sets and relation with Littlewood problem can be found in [32]. Before stating our main result, let us first recall Hanson's Theorem

Theorem 6.1.3 (Hanson [10]). *Let $\delta > 0$ and m, n be two positive integers satisfying*

$$m \geq \pi^3 2^{21} C_{MPS}^3 \ln(m)^3 \ln(n)^3 \quad \text{and} \quad n \geq \pi^3 2^{21} C_{MPS}^3 \ln(n)^3, \tag{6.1.5}$$

where C_{MPS} is the constant in Theorem 4.1.1. Suppose A is $(\delta; m, n)$ strongly 2-dimensional subset of $\mathbb{Z}$. Then

$$\int_0^1 \left| \sum_{a \in A} e^{2i\pi at} \right| \mathrm{d}t \geq \frac{C_{MPS}^2}{(2^9\pi)^2\big(2 + \ln(1 + \frac{2}{\delta})\big)} \ln(m) \ln(n)$$

By induction, this theorem can be generalized in a straightforward manner to $r-$dimensional sets ($r > 2$). Combining this result with Theorem 3.3 in [37], we see that this theorem is also best possible up to the constant.

A simple interpretation is that any set with multidimensional structure cannot verify the Inverse Littlewood problem for small C.

Next let $A \subset \mathbb{R}$. We say that A is $(\delta; m, n)$ strongly 2-dimensional in $\mathbb{R}$ if there exist numbers d and D with $D > (2 + \delta)d$ such that

$$A = \bigcup_{k \in I} (A_k + kD) \tag{6.1.6}$$

for some set I containing m integers and real subsets $A_k \subseteq [-d, \ldots, d]$ verifying $|A_k| \geq n$. The decomposition is hence the same as before, the main difference is that the sets A_k's can now be real. We can now state our main result:

Theorem 6.1.4. *Let $\delta > 0$ and m, n be two positive integers satisfying*

$$m \geq \pi^3 2^{21} C_{MPS}^3 \ln(m)^3 \ln(n)^3 \quad \textit{and} \quad n \geq \pi^3 2^{21} C_{MPS}^3 \ln(n)^3,$$

where $C_{MPS}(= 1/30)$. Suppose A is $(\delta; m, n)$ strongly 2-dimensional subset of $\mathbb{R}$. Then

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{a \in A} e^{2i\pi at} \right| \mathrm{d}t \geq \frac{C_{MPS}^2}{(2^9\pi)^2\big(2 + \ln(1 + \frac{2}{\delta})\big)} \ln(m) \ln(n).$$

6.2 Lacunary series

We take a lacunary trigonometric series of the form

$$\sum_{k=0}^{\infty} a_k e^{2i\pi n_k t} \tag{6.2.1}$$

where the a_k's are complex numbers, and the n_k's verifies Hadamard's condition;

$$\frac{n_{k+1}}{n_k} > q > 1.$$

6.2.1 Integer case: A result by Zygmund

First we will work in the integer setting and we aim to prove the following result [44];

Theorem 6.2.1 (Zygmund)**.** *Let $q > 1$ and $(n_k)_{k\geq 0}$ be a q-lacunary sequence of integers, $n_0 \geq 1$ and $n_{k+1} \geq qn_k$. Let $1 \leq p < +\infty$. There are two constants $A_{p,q}, B_{p,q}$ such that, if $(a_k)_{k\geq 0} \in \ell^2$, then $g(t) = \sum_{k\geq 0} a_k e^{2i\pi n_k t}$ is in $L^p([0,1])$ with*

$$A_{p,q}\left(\sum_{k=0}^{+\infty}|a_k|^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left|\sum_{k=0}^{\infty} a_k e^{2i\pi n_k t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,q}\left(\sum_{k=0}^{+\infty}|a_k|^2\right)^{\frac{1}{2}}. \tag{6.2.2}$$

Remark 6.2.2. *Note that a simple change of variable also shows that, for every integer M,*

$$A_{p,q}\left(\sum_{k=0}^{+\infty}|a_k|^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{M}\int_{-M/2}^{M/2} \left|\sum_{k=0}^{\infty} a_k e^{2i\pi \frac{n_k}{M} t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,q}\left(\sum_{k=0}^{+\infty}|a_k|^2\right)^{\frac{1}{2}}. \tag{6.2.3}$$

Also, we may assume that $q \to A_{p,q}, B_{p,q}$ are continuous.

To prove Theorem 6.2.1, we need some auxiliary lemmas;

Lemma 6.2.3. *A q-lacunary sequence is a finite union of q'-lacunary sequences with $q' \geq 3$*

Proof. We consider a sequence of integers $(n_k)_k$ such that

$$\frac{n_{k+1}}{n_k} \geq q > 1.$$

If $q \geq 3$ there is nothing to prove. For $1 < q < 3$, take N an integer such that $q^N \geq 3$ and write

$$n_k^{(\ell)} = n_{\ell+kN},$$

then $\left(n_k^{(\ell)}\right)_k$ is q^N-Lacunary and

$$\{n_k\} = \bigcup_{\ell=0}^{N-1} \{n_k^{(\ell)}\}.$$

as announced. □

Next, for $q \geq 3$, q-lacunary sequences have a particular arithmetic property:

Lemma 6.2.4. *Let $q \geq 3$ and $(n_k)_{k\geq 0}$ a sequence such that $n_0 \geq 1$ and $n_{k+1} \geq qn_k$. Consider two finite sequences $\varepsilon_\ell, \eta_\ell \in \{-1, 0, 1\}$ for $\ell = 0, \ldots, m$ and assume that*

$$\sum_{\ell=0}^{m} \varepsilon_\ell n_\ell = \sum_{\ell=0}^{m} \eta_\ell n_\ell \tag{6.2.4}$$

then $\varepsilon_\ell = \eta_\ell$ for every ℓ.

In other words, an integer can be represented in at most one way as $\sum \pm n_\ell$. Such a sequence is called *quasi-independent*. Note that this result is valid when the n_k's are real, not only for integers.

Proof. First observe that $n_j \leq \dfrac{1}{q^{m-j}} n_m = \dfrac{q^j}{q^m} n_m$ for $j = 0, \dots, m$.

Assume that (6.2.4) holds and define $\nu_\ell = \varepsilon_\ell - \eta_\ell$ so that

$$\sum_{\ell=0}^{m} \nu_\ell n_\ell = 0.$$

Assume towards a contradiction that there is an ℓ such that $\nu_\ell \neq 0$. Without loss of generality, we may assume that the largest such ℓ is m and, up to exchanging ε_ℓ and η_ℓ, that $\nu_m \geq 1$.

Observe that $\nu_\ell \in \{-2, -1, 0, 1, 2\}$ so that we obtain the desired contradiction writing

$$\begin{aligned} 0 = \sum_{\ell=0}^{m} \nu_\ell n_\ell &= \nu_m n_m + \sum_{\ell=0}^{m-1} \nu_\ell n_\ell \geq n_m - 2\sum_{\ell=0}^{m-1} n_\ell \geq n_m - 2\sum_{\ell=0}^{m-1} \frac{q^\ell}{q^m} n_m \\ &= \left(1 - \frac{2}{q^m}\frac{q^m - 1}{q-1}\right) n_m = \frac{q^{m+1} - 3q^m + 2}{(q-1)q^m} n_m > 0 \end{aligned}$$

since $q \geq 3$ and $n_m > 0$. □

Now in order to prove Theorem 6.2.1, we follow closely [44, Chapter V.8] which goes through Rademacher series. First let us introduce those series.

To start, let us denote by $\mathcal{D}_k = \{[j2^{-k-1}, (j+1)2^{-k-1}[,\ j = 0, \dots, 2^{k+1} - 1\}$ the dyadic intervals of generation k and $\mathcal{D} = \bigcup_{k\geq 0} \mathcal{D}_k$ the set of all dyadic intervals. Also, if $I, J \in \mathcal{D}$ then either $I \cap J = \emptyset$ or $I \subset J$ or $J \subset I$. The Rademacher functions of generation k are then functions that take alternative values $+1$ and -1 on successive intervals in $\mathcal{D}_k$, that is

$$r_k(t) = \sum_{j=0}^{2^{k+1}-1} (-1)^j \mathbb{1}_{]j2^{-k-1}, (j+1)2^{-k-1}[}(t) = \operatorname{sign}\big(\sin(2\pi 2^k t)\big).$$

The first observation is that, if $I \in \mathcal{D}_\ell$ and $k > \ell$ then r_k takes the value $+1$ on half of I and -1 on the other half so that $\int_I r_k = 0$. A first consequence is that r_k is orthogonal to r_ℓ in $L^2([0,1])$ since r_ℓ is constant on each $I \in \mathcal{D}_\ell$ so that $\int_I r_k r_\ell = 0$ and $\mathcal{D}_\ell$ is a covering of $[0,1]$. Moreover, as $|r_k| = 1$, the family $(r_k)_{k\geq 0}$ is an orthonormal sequence in $L^2([0,1])$.

In particular, we now fix a sequence $(c_k)_{k\geq 0}$ such that $(c_k) \in l^2(\mathbb{N})$, we can define

$$f = \sum_{k=0}^{+\infty} c_k r_k$$

and this series converges in $L^2([0,1])$ thus $f \in L^2([0,1])$ with $\|f\|_{L^2} = \|c_k\|_{l^2(\mathbb{N})}$. We actually have a bit better:

Theorem 6.2.5. *Let $(c_k) \in l^2(\mathbb{N})$ and f defined by*

$$f = \sum_{k=0}^{+\infty} c_k r_k.$$

Then this series converges both in $L^2([0,1])$ and almost everywhere.

Proof. The L^2-convergence has already been established. Further, let $F = \int f$ be the indefinite integral of f and let $E \subset [0,1]$ be the set of Lebesgue points of f so that $|E| = 1$ and on E, F' exists and is finite.

Now let, $S_n[f]$ be the n-th partial sum of this series

$$\mathcal{S}_n[f](x) = \sum_{k=0}^{n} c_k r_k(x).$$

As $\mathcal{S}_n[f] \to f$ in $L^2([0,1])$, for every $0 \leq a < b \leq 1$,

$$\left| \int_a^b \big(f(x) - \mathcal{S}_n[f](x)\big)\,\mathrm{d}x \right| \leq \int_0^1 |f(x) - \mathcal{S}_n[f](x)|\,\mathrm{d}x \leq \left(\int_0^1 |f(x) - \mathcal{S}[f](x)|^2\,\mathrm{d}x \right)^{1/2} \to 0.$$

We have just shown that, if I is an interval, then $\int_I \mathcal{S}_n[f] \to \int_I f$ thus also, if we fix $\ell \geq 1$,

$$\int_I (\mathcal{S}_n[f] - \mathcal{S}_{\ell-1}[f]) \to \int_I (f - \mathcal{S}_{\ell-1}[f]).$$

On the other hand, if $I \in \mathcal{D}_{\ell-1}$ and $k \geq \ell$, then $\int_I r_k = 0$ so that $\int_I \mathcal{S}_n[f] = \int_I \mathcal{S}_{\ell-1}[f]$. Letting $n \to +\infty$ we obtain that

$$\int_I f(x)\,\mathrm{d}x = \int_I \mathcal{S}_{\ell-1}[f](x)\,\mathrm{d}x \quad \text{for every } I \in \mathcal{D}_{\ell-1}.$$

Next, let $x_0 \in E$ not a dyadic rational ($x_0 \neq \dfrac{p}{2^q}$, $p, q \in \mathbb{N}$) and let $I_k =]j2^{-k}, (j+1)2^{-k}[$ be such that $x_0 \in E \cap I_k$. Then, as $\mathcal{S}_{k-1}[f]$ is constant over I_k

$$\mathcal{S}_{k-1}[f](x_0) = \frac{1}{|I_k|} \int_{I_k} \mathcal{S}_{k-1}[f](x)\,\mathrm{d}x = \frac{1}{|I_k|} \int_{I_k} f(x)\,\mathrm{d}x \to F'(x_0)$$

when $k \to +\infty$. □

The second result is that f is actually in every L^p space:

Theorem 6.2.6. *Let $(c_k)|k \geq 0 \in \ell^2$ and $f = \sum_{k=0}^{+\infty} c_k r_k$. Then, for $1 \leq p < +\infty$, $f \in L^p([0,1])$. Moreover, there exist A_p, B_p, depending on p only, such that*

$$A_p \left(\sum_{k=0}^{+\infty} |c_k|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 |f(x)|^p\,\mathrm{d}x \right)^{\frac{1}{p}} \leq B_p \left(\sum_{k=0}^{+\infty} |c_k|^2 \right)^{\frac{1}{2}}.$$

Proof. Let us first notice that the theorem holds for $p = 2$ since

$$\gamma := \left(\int_0^1 |f(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = \left(\sum_{k=0}^{+\infty} |c_k|^2 \right)^{\frac{1}{2}}$$

i.e. the inequalities are equalities with $A_2 = B_2 = 1$.

Next, let us notice that this implies the lower bound when $p > 2$ with $A_p = 1$ since then, with Hölder

$$\left(\int_0^1 |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \geq \left(\int_0^1 |f(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = \gamma.$$

It also implies the upper bound with $B_p = 1$ for $p < 2$ since now Hölder implies that

$$\left(\int_0^1 |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = \gamma.$$

Further, take $2(m-1) < p \leq 2m$ for some integer $m \geq 2$, and assume that that the upper bound

$$\left(\int_0^1 |f(x)|^{2m} \, \mathrm{d}x \right)^{\frac{1}{2m}} \leq B_{2m} \gamma \tag{6.2.5}$$

holds. Then Hölder implies that

$$\left(\int_0^1 |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x)|^{2m} \, \mathrm{d}x \right)^{\frac{1}{2m}} \leq B_{2m} \gamma$$

that is, $B_p \leq B_{2m}$ for $2(m-1) < p \leq 2m$.

Next, let us show that the upper bound for $p = 4$ implies the lower bound for $p < 2$. Assume for the moment that we are able to prove that

$$\left(\int_0^1 |f(x)|^4 \, \mathrm{d}x \right)^{\frac{1}{4}} \leq B_4 \gamma.$$

Let $1 \leq q < 2$ and write $2 = qt + 4(1-t)$, that is, take $t = \dfrac{2}{4-q}$. Then, from Hölder

$$\begin{aligned} \gamma^2 &= \int_0^1 |f(x)|^2 \, \mathrm{d}x = \int_0^1 |f(x)|^{qt} |f(x)|^{4(1-t)} \, \mathrm{d}x \leq \left(\int_0^1 |f(x)|^q \, \mathrm{d}x \right)^t \left(\int_0^1 |f(x)|^4 \, \mathrm{d}x \right)^{1-t} \\ &\leq (B_4 \gamma)^{4(1-t)} \left(\int_0^1 |f(x)|^q \, \mathrm{d}x \right)^t = (B_4 \gamma)^{2-qt} \left(\int_0^1 |f(x)|^q \, \mathrm{d}x \right)^t \end{aligned}$$

thus

$$\left(\int_0^1 |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \geq B_4^{1-\frac{4-q}{q}} \gamma.$$

So it remains to prove (6.2.5) for every $m \geq 2$. Notice also that it is enough to prove this inequality with real c_k's. The constant in the complex case is then multiplied by 2: write $f = f_r + if_i$ where $f_r = \sum \Re(c_k) r_k$ and $f_i = \sum \Im(c_k) r_k$. Then

$$\|f\|_{2m} \leq \|f_r\|_{2m} + \|f_i\|_{2m} \leq B_{2m}^{\mathbb{R}} \left[\left(\sum_{k=0}^{+\infty} |\Re(c_k)|^2 \right)^{\frac{1}{2}} + \left(\sum_{k=0}^{+\infty} |\Im(c_k)|^2 \right)^{\frac{1}{2}} \right] \leq 2B_{2m}^{\mathbb{R}} \gamma$$

since $|\Re(c_k)|, |\Im(c_k)| \leq |c_k|$.

To conclude, we write

$$\int_0^1 S_n[f](x)^{2m} \, \mathrm{d}x = \sum_{\ell_0 + \cdots + \ell_n = 2m} A_{\ell_0, \ldots, \ell_n} c_0^{\ell_0} \cdots c_n^{\ell_n} \int_0^1 r_0^{\ell_0}(x) \cdots r_n^{\ell_n}(x) \, \mathrm{d}x$$

where $\ell_j \geq 0$ for every j and

$$A_{\ell_0, \ldots, \ell_n} = \frac{(\ell_0 + \cdots + \ell_n)!}{\ell_0! \cdots \ell_j!}.$$

Now observe that

$$\int_0^1 r_0^{\ell_0}(x) \cdots r_n^{\ell_n}(x) \, \mathrm{d}x = \begin{cases} 1 & \text{if all the } \ell_j\text{'s are even} \\ 0 & \text{otherwise} \end{cases}$$

and that

$$\left(\sum_{k=0}^{n} c_k^2 \right)^m = \sum_{\ell_0 + \cdots + \ell_n = m} A_{\ell_0, \ldots, \ell_n} (c_0^2)^{\ell_0} \cdots (c_n^2)^{\ell_n}.$$

Further, when $\ell_0 + \cdots + \ell_n = m$,

$$\frac{A_{2\ell_0, \ldots, 2\ell_n}}{A_{\ell_0, \ldots, \ell_n}} = \frac{(m+1)(m+2) \cdots 2m}{\prod_{j=0}^n (\ell_j + 1)(\ell_j + 2) \cdots 2\ell_j} \leq \frac{(m+1)(m+2) \cdots 2m}{2^m} \leq m^m$$

(with the convention that the denominator is $(\ell_j + 1)(\ell_j + 2) \cdots 2\ell_j = 1$ when $\ell_j = 0$). It follows that

$$\int_0^1 S_n[f](x)^{2m} \, \mathrm{d}x \leq m^m \left(\sum_{k=0}^{n} |c_k|^2 \right)^m.$$

As $S_n[f] \to f$ a.e., we conclude that

$$\left(\int_0^1 |f(x)|^{2m} \, \mathrm{d}x \right)^{\frac{1}{2m}} \leq m^{1/2} \left(\sum_{k=0}^{+\infty} |c_k|^2 \right)^{\frac{1}{2}}$$

that is $B_{2m} = 2m^{1/2}$. □

The estimate $B_{2m} = 2m^{1/2}$ allows to improve a bit the result:

Corollary 6.2.7. *Let* $(c_k) \in \ell^2$ *and* $f = \sum_{k=0}^{+\infty} c_k r_k$. *Then, for every* $\mu > 0$, $\exp(\mu|f|^2) \in L^1([0,1])$.

Proof. Let us fix $\mu > 0$. We first show that if $\gamma := \|c_k\|_2$ is small enough, then $\exp(\mu|f|^2) \in L^1([0,1])$. Indeed

$$\int_0^1 \exp(\mu|f(x)|^2)\,\mathrm{d}x = \sum_{m=0}^{+\infty} \frac{\mu^m}{m!} \int_0^1 |f(x)|^{2m}\,\mathrm{d}x \leq \sum_{m=0}^{+\infty} \frac{m^m}{m!}(4\mu\gamma^2)^m. \tag{6.2.6}$$

But $\dfrac{m^m}{m!} \leq \sum_{n=0}^{+\infty} \dfrac{m^n}{n!} = e^m$ so that

$$\int_0^1 \exp(\mu|f(x)|^2)\,\mathrm{d}x \leq \sum_{m=0}^{+\infty} (4e\mu\gamma^2)^m = \frac{1}{1-4e\mu\gamma^2} < +\infty$$

provided $\gamma^2 < \dfrac{1}{4e\mu}$.

Next, take any $f \in L^1(0,1)$, and apply the first part to $f - \mathcal{S}_n[f] = \sum_{k=n+1}^{+\infty} c_k r_k$. As $\gamma_n^2 := \sum_{k=n+1}^{+\infty} |c_k|^2 \to 0$, for n large enough $\gamma_n^2 < \dfrac{1}{8e\mu}$ thus $\exp(2\mu|f - \mathcal{S}_n[f]|^2) \in L^1([0,1])$.

Finally, as $|f|^2 \leq 2|f - \mathcal{S}_n[f]|^2 + 2|\mathcal{S}_n[f]|^2$, we have

$$\exp(\mu|f|^2) \leq \exp(2\mu|f - \mathcal{S}_n[f]|^2)\exp(2\mu|\mathcal{S}_n[f]|^2) \in L^1$$

since $|\mathcal{S}_n[f]| \in L^\infty$ thus also $\exp(2\mu|\mathcal{S}_n[f]|^2) \in L^\infty$. □

Next, we consider series of the form

$$\sum_{k=0}^{+\infty} c_k e^{2i\pi kt} r_k(x).$$

The idea is that such series are of the form $\sum \pm c_j e^{2i\pi jt}$, that is, choosing $x \in (0,1)$ at random, we randomly change the sign of c_j. Our first result is the following:

Theorem 6.2.8. *Let* $(c_k) \in \ell^2$ *and* $f_x(t) = \sum_{k=0}^{+\infty} c_k r_k(x) e^{2i\pi kt}$. *Then, for almost every* $x \in (0,1)$, *the series converges almost everywhere in* $t \in (0,1)$ *and* $f_x \in L^p([0,1])$ *for every* $1 \leq p < +\infty$.

Proof. Let E be the set of $(x,t) \in [0,1]^2$ where the series defining f converges.

According to Theorem 6.2.5, for every $t \in [0,1]$, the set $E_t^2 = \{(x,t) \in E\}$ has measure $|E_t^2| = 1$. It follows that $|E| = 1$ but then, for almost every $x \in [0,1]$, $E_x^1 = \{(x,t) \in E\}$ has also measure $|E_x^1| = 1$.

Next, set $\gamma = \|c_k\|_2$ and fix $n \geq 1$. As in (6.2.6),

$$\frac{\mu^n}{n!} \int_0^1 |f_x(t)|^{2n}\,\mathrm{d}x \leq \sum_{m=0}^{+\infty} \frac{\mu^m}{m!} \int_0^1 |f_x(t)|^{2m}\,\mathrm{d}x = \int_0^1 \exp(\mu |f_x(t)|^2)\,\mathrm{d}x \leq \frac{1}{1-4e\mu\gamma^2} \quad (6.2.7)$$

provided $\mu < \dfrac{1}{4e\gamma^2}$. It follows that

$$\int_0^1 \int_0^1 |f_x(t)|^{2n}\,\mathrm{d}t\,\mathrm{d}x = \int_0^1 \int_0^1 |f_x(t)|^{2n}\,\mathrm{d}x\,\mathrm{d}t \leq \frac{n!}{(1-4e\mu\gamma^2)\mu^n} < +\infty.$$

But then, for every n, there is a set $F_n \subset [0,1]$ with $|F_n| = 0$ such that, if $x \in [0,1] \setminus F_n$,

$$\int_0^1 |f_x(t)|^{2n}\,\mathrm{d}t < +\infty.$$

Setting $F = \bigcup F_n$, $|F| = 0$ and, for every $x \in [0,1] \setminus F$, for every n, $f_x \in L^{2n}$. Using the inclusion of $L^{2n}([0,1]) \subset L^p([0,1])$ when $p \leq 2n$, we obtain that, for almost every x, $f_x \in L^p([0,1])$ for every $p \geq 1$, as claimed. □

We can now prove Zygmund's Theorem

Proof. The beginning of the proof is the same as for Theorem 6.2.6. Parseval's identity shows that (6.2.2) is satisfied when $p = 2$ with $A_{2,q} = B_{2,q} = 1$. The lower bound is then automatically satisfied for $p \geq 2$ with $A_{p,q} = 1$ while the upper bound is satisfied for $p \leq 2$ with $B_{2,q} = 1$. Finally, if we establish the upper bound for $p > 2$, using Hölder's inequality in the same way as in the proof of Theorem 6.2.6, the lower bound follows for $p < 2$ with $A_{2,q} = B_{4,q}^{1-\frac{4-p}{p}}$. Also, it is enough to prove the upper bound when $p = 2m$, $m \geq 2$ and then, if $2(m-1) < p \leq 2m$, $B_{p,q} \leq B_{2m,q}$. Another reduction is that, by homogeneity, it is enough to prove the theorem when $\displaystyle\sum_{k=0}^{+\infty} |c_k|^2 = 1$.

A further restriction is that it is enough to prove the theorem for $q \geq 3$. Indeed, for $1 < q < 3$, we introduce an integer N_q such that $q^{N_q} \geq 3$ and write $n_k^{(\ell)} = n_{kN_q+\ell}$ for $\ell = 0, \ldots, N_q - 1$. Then $n_{k+1}^{(\ell)} \geq q^{N_q} n_k^{(\ell)}$. If the theorem is established when $q \geq 3$ then, for each ℓ, the upper bound in (6.2.2) reads

$$\left(\int_0^1 \left| \sum_{k \geq 0} c_{kN_q+\ell} e^{2i\pi n_k^{(\ell)} t} \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \leq B_{p,q^{N_q}} \left(\sum_{k=0}^{+\infty} |c_{kN_q+\ell}|^2 \right)^{\frac{1}{2}}.$$

But then, with the triangular inequality in L^p,

$$\begin{aligned}
\left(\int_0^1 \left|\sum_{j\geq 0} c_j e^{2i\pi n_j t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} &= \left(\int_0^1 \left|\sum_{\ell=0}^{N_q-1}\sum_{k\geq 0} c_{kN_q+\ell} e^{2i\pi n_k^{(\ell)} t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq \sum_{\ell=0}^{N_q-1}\left(\int_0^1 \left|\sum_{k\geq 0} c_{kN_q+\ell} e^{2i\pi n_k^{(\ell)} t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq B_{p,q^{N_q}} \sum_{\ell=0}^{N_q-1}\left(\sum_{k=0}^{+\infty} |c_{kN_q+\ell}|^2\right)^{\frac{1}{2}} \\
&\leq N_q^{1/2} B_{p,q^{N_q}} \left(\sum_{\ell=0}^{N_q-1}\sum_{k=0}^{+\infty} |c_{kN_q+\ell}|^2\right)^{\frac{1}{2}} \\
&= N_q^{1/2} B_{p,q^{N_q}} \left(\sum_{j\geq 0} |c_j|^2\right)^{\frac{1}{2}}
\end{aligned}$$

where we have used Cauchy-Schwarz in $\mathbb{R}^{N_q}$ in the next to last line.

A last reduction comes from the observation that, for every k

$$\int_0^1 \exp(\mu|g(t)|^2)\,\mathrm{d}t = \sum_{n=0}^{+\infty} \frac{\mu^n}{n!}\int_0^1 |g(t)|^{2n}\,\mathrm{d}t \geq \frac{\mu^m}{m!}\int_0^1 |g(t)|^{2m}\,\mathrm{d}t$$

It is therefore enough to prove that there is a $\mu(q)$ and a $C>0$ such that, if $\mu<\mu(q)$

$$\int_0^1 \exp(\mu|g(t)|^2)\,\mathrm{d}t \leq C \tag{6.2.8}$$

which would then imply that

$$\int_0^1 |g(t)|^{2m}\,\mathrm{d}t \leq C\frac{m!}{\mu^m}$$

as desired.

In order to prove (6.2.8), let us introduce

$$f_x(t) = \sum_{j\geq 0} c_j r_{n_j}(x) e^{2i\pi n_j t}.$$

Integrating (6.2.7) with respect to t and using Fubini, we deduce that

$$\int_0^1\int_0^1 \exp(\mu|f_x(t)|^2)\,\mathrm{d}t\,\mathrm{d}x \leq K := \frac{1}{1-4e\mu\gamma^2}.$$

But then, there is an x_0 (that we can assume not to be a dyadic rational $x_0 \neq 2^j/k$) such that

$$\int_0^1 \exp(\mu|f_{x_0}(t)|^2)\,\mathrm{d}t \leq K.$$

Next, we consider the Riesz product

$$P_k(x) = \prod_{j=0}^{k}\big(1 + r_{n_j}(x_0)\cos 2\pi n_j t\big) = \prod_{j=0}^{k}\left(1 + r_{n_j}(x_0)\frac{e^{2\pi n_j t} + e^{-2\pi n_j t}}{2}\right) = \sum_{j\in\mathbb{Z}} \gamma_j e^{2i\pi jt}$$

where the Fourier coefficients have the following property:

– $\gamma_0 = 1$;

– $\gamma_j = 0$ if j is an integer that is not of the form $\sum \pm n_\ell$, in particular when $|j| > \sum_{\ell=0}^{n} n_\ell$;

– if $j = \sum \varepsilon_\ell n_\ell$ with $\varepsilon_\ell \in \{-1, 0, 1\}$. As $q > 3$, this ε_ℓ's are unique. Then $\gamma_j = \prod_{\varepsilon_\ell \neq 0} \frac{r_{n_\ell}(x_0)}{2}$. In particular, $\gamma_{n_j} = \frac{r_{n_j}(x_0)}{2}$ for $j = 0, \dots, k$ and $\gamma_{n_j} = 0$ for $j > k$.

As a consequence, the partial sums of the Fourier series of g are given by

$$S_{n_k}[g](t) := \sum_{j=0}^{k} c_j e^{2i\pi n_j t} = \sum_{j=0}^{k} c_j r_j(x_0)^2 e^{2i\pi n_j t} = 2\int_0^1 f_{x_0}(s) P_k(t-s)\,\mathrm{d}s.$$

Note that $P_k \geq 0$ and $\displaystyle\int_0^1 P_k(t)\,\mathrm{d}t = \gamma_0 = 1$ so that $\nu_k = P_k(t)\,\mathrm{d}t$ is a probability measure. As $\varphi(s) = \exp(\mu s^2)$ is increasing and convex, we apply Jensen's inequality (with the measure ν_k) to obtain

$$\varphi\left(\frac{1}{2}|S_{n_k}[g](t)|\right) \leq \varphi\left(\int_0^1 |f_{x_0}(s)| P_k(t-s)\,\mathrm{d}s\right) \leq \int_0^1 \varphi\big(|f_{x_0}(s)|\big) P_k(t-s)\,\mathrm{d}s.$$

Integrating over $[0, 1]$ and using Fubini, we get

$$\int_0^1 \varphi\left(\frac{1}{2}|S_{n_k}[g](t)|\right)\,\mathrm{d}t \leq \int_0^1 \varphi\big(|f_{x_0}(s)|\big)\int_0^1 P_k(t-s)\,\mathrm{d}t\,\mathrm{d}s = \int_0^1 \varphi\big(|f_{x_0}(s)|\big)\,\mathrm{d}s \leq K.$$

Letting $k \to +\infty$, we obtain

$$\int_0^1 \exp\left(\frac{\mu}{4}|g(t)|^2\right)\,\mathrm{d}t \leq K$$

as claimed (up to $\mu/4$ instead of μ). □

6.2.2 Extension to real frequencies

We end this section with a generalization to the real (non harmonic) setting on sufficiently large intervals. The proof is based on a lemma by Dirichlet which mainly allow us to approximate real numbers with quotients of integers.

Theorem 6.2.9. *Let $q > 1$ and $(\lambda_k)_{k\geq 0}$ be real numbers verifying*

$$\lambda_{k+1} - \lambda_k > 0, \qquad \lambda_0 > 1 \quad \textit{and} \quad \lambda_{k+1} \geq q\lambda_k$$

and $a_0, \dots, a_N$ be a sequence of complex numbers. Then for $1 \le p < \infty$,

$$A_{p,q}\left(\sum_{k=0}^{N}|a_k|^2\right)^{1/2} \le \lim_{T\to+\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\lambda_k t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \le B_{p,q}\left(\sum_{k=0}^{N}|a_k|^2\right)^{1/2}$$

with $A_{p,q}$, $B_{p,q}$ the constants in Theorem 6.2.1.

Proof. Let $a_0, \dots, a_N$ be complex numbers, $\lambda_0 < \lambda_1 < \cdots < \lambda_N$ be real numbers with $\lambda_{k+1} \ge q\lambda_k$ and

$$\Phi(t) = \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}.$$

Let $\varepsilon > 0$. By a lemma of Dirichlet ([44, p 235], [9]), there is an increasing sequence of integers $(M_n)_{n\ge1}$ and, for each $n \ge 1$ a finite family of integers $(N_{k,n})_{k=0,\dots,N}$ such that

$$\left|\lambda_k - \frac{N_{k,n}}{M_n}\right| < \frac{\varepsilon}{M_n} \qquad \text{for } k = 0, \dots, N$$

which implies that

$$\left|e^{2i\pi\lambda_k t} - e^{2i\pi\frac{N_{k,n}}{M_n}t}\right| \le 2\pi\left|\lambda_k - \frac{N_{k,n}}{M_n}\right||t| \le 2\pi\frac{\varepsilon}{M_n}|t| \qquad \text{for } k = 0, \dots, N.$$

Define the M_n-periodic function

$$\Psi_n(t) = \sum_{k=0}^{N} a_k e^{2i\pi N_{k,n}t/M_n},$$

and note that, for $t \in [-M_n/2, M_n/2]$,

$$|\Phi(t) - \Psi_n(t)| \le \sum_{k=0}^{N}|a_k|\left|e^{2i\pi\lambda_k t} - e^{2i\pi\frac{N_{k,n}}{M_n}t}\right| \le 2\pi\varepsilon\sum_{k=0}^{N}|a_k|.$$

But then

$$\begin{aligned}\left|\left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Phi(t)|^p\,\mathrm{d}t\right)^{1/p} - \left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Psi_n(t)|^p\,\mathrm{d}t\right)^{1/p}\right| &\\ \le \left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Phi(t)-\Psi_n(t)|^p\,\mathrm{d}t\right)^{1/p} &\\ \le \left(2\pi\sum_{k=0}^{N}|a_k|\right)^{1/p}\varepsilon^{1/p}. \quad (6.2.9)&\end{aligned}$$

We can now conclude as follows. Let $\tilde{q} = \tilde{q}(\varepsilon) = q - (1+q)\varepsilon < q$. Observe that

$$\begin{aligned}\frac{N_{k+1,n}}{M_n} &\geq \lambda_{k+1} - \left|\lambda_{k+1} - \frac{N_{k+1,n}}{M_n}\right| \geq q\lambda_k - \frac{\varepsilon}{M_n} \\ &\geq q\frac{N_{k,n}}{M_n} - q\left|\lambda_k - \frac{N_{k,n}}{M_n}\right| - \frac{\varepsilon}{M_n} \\ &\geq q\frac{N_{k,n}}{M_n} - (1+q)\frac{\varepsilon}{M_n},\end{aligned}$$

that is $N_{k+1,n} \geq qN_{k,n} - (1+q)\varepsilon \geq \tilde{q}N_{k,n}$.

Applying (6.2.3) to $n_k = N_{k,n}$ and $M = M_n$ we obtain

$$A_{p,\tilde{q}}\left(\sum_{k=0}^{N}|a_k|^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\frac{N_{k,n}}{M_n}t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,\tilde{q}}\left(\sum_{k=0}^{N}|a_k|^2\right)^{\frac{1}{2}}.$$

From (6.2.9) we conclude that

$$\begin{aligned}A_{p,\tilde{q}}\left(\sum_{k=0}^{N}|a_k|^2\right)^{\frac{1}{2}} - \left(2\pi\sum_{k=0}^{N}|a_k|\right)^{1/p}\varepsilon^{1/p} &\leq \left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\lambda_k t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq B_{p,\tilde{q}}\left(\sum_{k=0}^{N}|a_k|^2\right)^{\frac{1}{2}} + \left(2\pi\sum_{k=0}^{N}|a_k|\right)^{1/p}\varepsilon^{1/p}.\end{aligned}$$

The result follows by letting $\varepsilon \to 0$ so that $\tilde{q} \to q$, $M_n \to M$, $A_{p,\tilde{q}} \to A_{p,q}$, $B_{p,\tilde{q}} \to B_{p,q}$ since those constants are continuous in their parameters. □

6.3 2-dimensional sets

Given a set $I \subseteq \mathbb{Z}$, a positive integer q, and an arbitrary integer s, we define

$$I(q;s) = \{k \in I : k = s \pmod{\mathrm{q}}\}.$$

The proof of Theorem 6.1.3 is a direct consequence of the following two lemmas

Lemma 6.3.1. *Let I be a set of integers with $|I| \geq 8$. Then there are positive integers q and s such that*

$$\frac{|I|^{\frac{1}{3}}}{8} \leq |I(q;s)| \leq q^{1/2}.$$

Proof. For each $j \geq 1$, we choose any s_j such that $|I(4^j;s_j)|$ is maximal. But, on one hand,

$$I = \bigcup_{s=0}^{4^j-1} I(4^j,s)$$

and on the other hand, for j fixed, the sets $I(4^j; s)$ are disjoints, so at least one of them has cardinality larger than $4^{-j}|I|$. In particular,

$$|I(4^j, s_j)| \geq 4^{-j}|I|. \tag{6.3.1}$$

For $j = 1$, we thus have $|I(4; s_1)| \geq \dfrac{|I|}{4} \geq 2$. On the other hand, if $j = s \mod kp$ then $j = s \mod p$ so that, for any s

$$I(4^m; s) \subset I(4^\ell; s) \quad \text{for} \quad \ell < m,$$

and, for sufficiently large j we have $|I(4^j; s_j)| = 1 \leq 2^j$. Therefore, there exists a minimal j_0 such that $|I(4^{j_0}; s_{j_0})| \leq 2^{j_0}$. Let $q = 4^{j_0}$, and $s = s_{j_0}$ then using (6.3.1) and the definition of j_0

$$\frac{|I|}{q} \leq |I(q; s)| = |I(4^{j_0}; s_{j_0})| \leq 2^{j_0} = q^{\frac{1}{2}}.$$

In particular

$$|I|^{\frac{1}{3}} \leq q^{\frac{1}{2}}. \tag{6.3.2}$$

By minimality of j_0

$$\frac{q^{\frac{1}{2}}}{2} = 2^{j_0-1} \leq \left|I\left(4^{j_0-1}; s_{j_0-1}\right)\right| \leq \sum_{r=0}^{3} \left|I\left(4^{j_0}; s_{j_0-1} + r4^{j_0-1}\right)\right| \leq 4|I(4^{j_0}; s_{j_0})| = 4|I(q; s)|$$

by definition of s_{j_0}. We thus get

$$|I(q; s)| \geq \frac{q^{\frac{1}{2}}}{8} \geq \frac{|I|^{\frac{1}{3}}}{8},$$

with (6.3.2). □

Lemma 6.3.2. *Let $\delta > 0$ and let d and D be positive integers with $(2 + \delta)d < D$. Suppose I is a finite set of integers, and let*

$$F(t) = \sum_{k \in I} f_k(t) e^{2i\pi Dkt}$$

where

$$f_k(t) = \sum_{|n| \leq d} a_{n,k} e^{2i\pi nt}$$

Let q and s with $q > 4\pi$ and suppose $I(q; s) = \{k_1, \ldots, k_J\}$ then we have

$$\|F\|_{L^1([0,1]} \geq \frac{1}{32\pi(2 + \ln(1 + \frac{2}{\delta}))} \sum_{j=1}^{J} \|f_{k_j}\|_{L^1([0,1])} \left(\frac{C_{MPS}}{2j} - \frac{2\pi d}{qD}\right).$$

We split the remaining of this section into two parts. In the first one, we show that Lemmas 6.3.1 and 6.3.2 imply Theorem 6.1.5. In the second one, we prove the Lemma 6.3.2.

6.3.1 Strongly 2 dimensional subsets of $\mathbb{Z}$: A proof by Hanson

First we recall the proof of Hanson's result

Proof of Theorem 6.1.3. Let $\delta > 0$ m, n be two integers satisfying the conditions of the theorem and let A be strongly (δ, m, n)-regular. Thus, there are two integers d, D with $D > (2+\delta)d$, such that we can write

$$A = \bigcup_{k\in I}(A_k + kD),$$

with $|I| \geq m$ and $A_k \subset \{-d, \ldots, d\}$ with $|A_k| \geq n$. We can then write

$$F(t) := \sum_{a\in A} e^{2i\pi at} = \sum_{k\in I} f_k(t)e^{2i\pi Dkt}$$

with

$$f_k(t) = \sum_{a\in A_k} e^{2i\pi at} = \sum_{n=-d}^{d} a_{n,k}e^{2i\pi nt}$$

with $a_{n,k} = 1$ if $n \in A_k$ and $a_{n,k} = 0$ otherwise.

Assume first that there exists $k_1 \in I$ such that

$$\|f_{k_1}\|_{L^1([0,1])} \geq \frac{C_{MPS}}{2^9\pi}\ln(m)\ln(n). \tag{6.3.3}$$

We then choose $q \geq \dfrac{16\pi}{7C_{MPS}}$ in such a way that there is an s such that $I(q,s) = \{k_1\}$. Hence by Lemma 6.3.2,

$$\begin{aligned}\|F\|_{L^1([0,1])} &\geq \frac{\|f_{k_1}\|_{L^1([0,1])}}{32\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}\left(\frac{C_{MPS}}{2} - \frac{2\pi d}{qD}\right)\\ &\geq \frac{C_{MPS}\|f_{k_1}\|_{L^1([0,1])}}{2^6\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}\left(1-\frac{7d}{8D}\right).\end{aligned}$$

As $D > 2d$, using (6.3.3), we conclude that, in this case

$$\|F\|_{L^1([0,1])} \geq \left(\frac{C_{MPS}}{2^9\pi}\right)^2 \frac{\ln(n)\ln(m)}{2+\ln\left(1+\frac{2}{\delta}\right)}$$

which establishes the theorem.

We will thus assume that, for each $k \in I$,

$$\|f_k\|_{L^1([0,1])} \leq \frac{C_{MPS}}{2^9\pi}\ln(m)\ln(n). \tag{6.3.4}$$

Note that, from Theorem 1.0.1,

$$\|f_k\|_{L^1([0,1])} \geq C_{MPS}\ln(n)$$

so that $2^9\pi \leq \ln(m)$, in particular, $m \geq 8$. We then take q and s given by Lemma 6.3.1 applied to the set I so that $J = |I(q;s)|$ satisfies

$$\frac{m^{\frac{1}{3}}}{8} \leq J \leq q^{\frac{1}{2}}. \tag{6.3.5}$$

We write

$$I(q;s) = \{k_1 < \ldots < k_J\}$$

From Lemma 6.3.2, we get that

$$\begin{aligned} \|F\|_{L^1([0,1])} &\geq \frac{1}{2^5\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)} \sum_{j=1}^{J} \|f_{k_j}\|_{L^1} \left(\frac{C_{MPS}}{2j} - \frac{2\pi d}{qD}\right) \\ &= \frac{1}{2^5\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}(T_1 - T_2) \end{aligned}$$

with

$$T_1 = \frac{C_{MPS}}{2} \sum_{j=1}^{J} \frac{\|f_{k_j}\|_{L^1}}{j} \quad \text{and} \quad T_2 = \frac{2\pi d}{qD} \sum_{j=1}^{J} \|f_{k_j}\|_{L^1}.$$

Next, as $\|f_{k_j}\|_{L^1} \geq C_{MPS}\ln(n)$,

$$T_1 \geq \frac{C_{MPS}^2 \ln(n)}{2} \sum_{j=1}^{J} \frac{1}{j} \geq \frac{C_{MPS}^2 \ln(n)}{2} \ln J \geq \frac{C_{MPS}^2 \ln(n)\ln(m)}{8}$$

with (6.3.5).

On the other hand, from (6.3.4), we get

$$T_2 \leq \frac{2\pi J d}{qD} \frac{C_{MPS}}{2^9\pi} \ln(m)\ln(n) \leq \frac{2\pi}{q^{\frac{1}{2}}} \frac{C_{MPS}}{2^9\pi} \ln(m)\ln(n),$$

since $d \leq \dfrac{D}{2}$ and $J \leq q^{\frac{1}{2}}$ with (6.3.4). Further, (6.3.4) also implies that

$$q^{\frac{1}{2}} \geq \frac{m^{\frac{1}{3}}}{8} \geq 2^4\pi C_{MPS}\ln(m)\ln(n)$$

with (6.1.5), leading to $T_2 \leq \dfrac{1}{2^{13}\pi}$.

We have established that

$$\begin{aligned} \|F\|_{L^1([0,1])} &\geq \frac{1}{2^5\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)} \left(\frac{C_{MPS}^2 \ln(m)\ln(n)}{2^3} - \frac{1}{2^{13}\pi}\right) \\ &= \frac{C_{MPS}^2}{(2^9)^2\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)} \left(2^{10}\ln(m)\ln(n) - \frac{1}{C_{MPS}^2}\right) \\ &\geq \frac{C_{MPS}^2}{(2^9\pi)^2\left(2+\ln(1+\frac{2}{\delta})\right)} \ln(m)\ln(n) \end{aligned}$$

since $\ln(m)\ln(n)C_{MPS}^2 \geq \dfrac{1}{2^{10}\pi - 1}$. □

Auxiliary Lemmas

The rest of this chapter consists in proving Lemma 6.3.2. The proof is quite long and is divided into several lemmas. Throughout, we say that f is a trigonometric polynomial of degree d if

$$f(t) = \sum_{|k|\leq d} a_k e^{2i\pi kt}.$$

Lemma 6.3.3 (Bernstein's inequality)**.** *Let $f :\longrightarrow \mathbb{C}$ be a trigonometric polynomial of degree d. Then*

$$\|f'\|_{L^1([0,1])} \leq 2\pi d \|f\|_{L^1([0,1])}.$$

Lemma 6.3.4. *Let N be a positive integer. Then for any trigonometric polynomial f of degree d*

$$\left| \|f\|_{L^1([0,1])} - \frac{1}{N}\sum_{j=0}^{N-1} \left| f\left(\frac{j}{N}\right)\right| \right| \leq \frac{2\pi d}{N}\|f\|_{L^1([0,1])}.$$

Note that, as f is 1-periodic, writing $N = R + S$,

$$\frac{1}{N}\sum_{j=0}^{N-1} \left| f\left(\frac{j}{N}\right)\right| = \frac{1}{R+S}\sum_{j=-R}^{S-1} \left| f\left(\frac{j}{R+S}\right)\right|,$$

since

$$\begin{aligned}
\sum_{j=0}^{N-1} \left| f\left(\frac{j}{R+S}\right)\right| &= \sum_{j=0}^{S-1} \left| f\left(\frac{j}{R+S}\right)\right| + \sum_{j=S}^{R+S-1} \left| f\left(\frac{j}{R+S}\right)\right| \\
&= \sum_{j=0}^{S-1} \left| f\left(\frac{j}{R+S}\right)\right| + \sum_{j=-R}^{-1} \left| f\left(\frac{j}{R+S}\right)\right| \\
&= \sum_{j=-R}^{S-1} \left| f\left(\frac{j}{R+S}\right)\right|
\end{aligned}$$

where we used the change of index $J = j - N$ and the 1-periodicity of f in the next to last line.

Proof. We write, using the triangular and reverse triangular inequalities

$$\begin{aligned}\left|\int_0^1 |f(t)|\,\mathrm{d}t - \frac{1}{N}\sum_{j=0}^{N-1}\left|f\left(\frac{j}{N}\right)\right|\right| &= \left|\sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}\left(|f(t)| - \left|f\left(\frac{j}{N}\right)\right|\right)\mathrm{d}t\right| \\ &\leq \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}\left|f(t) - f\left(\frac{j}{N}\right)\right|\mathrm{d}t = \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}\left|\int_{\frac{j}{N}}^{t} f'(s)\,\mathrm{d}s\right|\mathrm{d}t \\ &\leq \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j}{N}}^{t}|f'(s)|\,\mathrm{d}s\,\mathrm{d}t = \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}|f'(s)|\int_{s}^{\frac{j+1}{N}}\mathrm{d}t\,\mathrm{d}s \\ &\leq \frac{1}{N}\sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}|f'(s)|\,\mathrm{d}s = \int_0^1 |f'(s)|\,\mathrm{d}s,\end{aligned}$$

where we have used Fubini and the bound $\displaystyle\int_s^{\frac{j+1}{N}}\mathrm{d}t \leq \frac{1}{N}$ when $\displaystyle\frac{j}{N} \leq s \leq \frac{j+1}{N}$. We conclude with Bernstein's inequality. □

For a finitely supported sequence $\big(A(k)\big)_{k\in\mathbb{Z}}$, we define its discrete Fourier transform (or **Z**-Fourier transform) as

$$\mathcal{F}_d[A](t) = \sum_{k\in\mathbb{Z}} A(k)e^{-2i\pi kt}.$$

If A, B are two finitely supported sequences, their convolution is the sequence $A*B$ defined by

$$A*B(k) = B*A(k) = \sum_{n\in\mathbb{Z}} A(k-n)B(n).$$

The Convolution Theorem is also valid here: $\mathcal{F}_d[A*B](t) = \mathcal{F}_d[A](t)\mathcal{F}_d[B](t)$. Two classical examples are

– the Dirichlet kernel: set $\mathbb{d}_L(k) = \mathbb{1}_{-L,\dots,L}(k)$ so that

$$D_L(t) = \mathcal{F}_d[\mathbb{d}_L](t) = \sum_{|k|\leq L} e^{2i\pi kt} = \frac{\sin(\pi(2L+1)t)}{\sin(\pi t)};$$

– the Fejér kernel: set $\mathbb{f}_L(k) = \left(1 - \frac{|k|}{L+1}\right)\mathbb{1}_{-L,\dots,L}(k)$ so that

$$F_L(t) = \mathcal{F}_d[\mathbb{f}_L](t) = \sum_{|k|\leq L}\left(1 - \frac{|k|}{L+1}\right)e^{2i\pi kt} = \frac{1}{L+1}\frac{(\sin(\pi(L+1)t))^2}{(\sin(\pi t))^2}.$$

Lemma 6.3.5. *Let M, N, R, S be integers with $2 \leq M < N$. Then there exists a function $K_{M,N}$ with the following properties:*

1. *$K_{M,N}(k) = 1$ for $|k| \leq N$,*
2. *$K_{M,N}(k) = 0$ for $|k| \geq N + 2M$*

3. If further $R+S \geq 2N+4M$, then

$$\frac{1}{R+S}\sum_{j=-R}^{S-1}\left|\mathcal{F}_d[K_{M,N}]\left(\frac{j}{R+S}\right)\right| \leq 16\pi(2+\ln(1+N/M)).$$

Proof. Define

$$\begin{aligned} K_{M,N}(k) &= \frac{1}{M}\mathrm{d}_{N+M} * \mathbb{f}_{M-1}(k) \\ &= \frac{1}{M}\sum_{n\in\mathbb{Z}}\left(1-\frac{|n|}{M}\right)\mathbb{1}_{\{-M-N,\ldots,M+N\}}(k-n)\mathbb{1}_{\{-M+1,\ldots,M-1\}}(n) \\ &= \frac{1}{M}\sum_{\substack{|n|\leq M-1 \\ |n-k|\leq N+M}}\left(1-\frac{|n|}{M}\right). \end{aligned}$$

First, for $|k| \leq N$, if $|n| \leq M-1$, then $|n-k| \leq |n|+|k| \leq N+M-1$, so that

$$K_{M,N}(k) = \frac{1}{M}\sum_{|n|\leq M-1}\left(1-\frac{|n|}{M}\right) = 1,$$

since

$$\begin{aligned} \sum_{|n|\leq M-1}\left(1-\frac{|n|}{M}\right) &= \sum_{n=1-M}^{0}\left(1+\frac{n}{M}\right)+\sum_{n=1}^{M-1}\left(1-\frac{n}{M}\right) \\ &= \frac{M+1}{2}+\frac{M-1}{2} = M. \end{aligned}$$

On the other hand, if $|k| \geq N+2M$ and $|n| \leq M-1$ then $|k-n| \geq |k|-|n| \geq N+M+1$ so that the sum defining $K_{M,N}$ is empty and $K_{M,N}=0$.

To prove the last item, the Convolution Theorem shows that

$$\mathcal{F}_d[K_{M,N}](t) = \frac{1}{M}D_{N+M}(t)F_{M-1}(t).$$

As D_{N+M} and F_{M-1} are both even, so is $K_{M,N}$ thus

$$\int_0^1 |\mathcal{F}_d[K_{M,N}](t)|\,\mathrm{d}t = 2\int_0^{\frac{1}{2}} |\mathcal{F}_d[K_{M,N}](t)|\,\mathrm{d}t = 2(I_1+I_2+I_3)$$

where

$$\begin{aligned} I_1 &= \frac{1}{M}\int_0^{\frac{1}{N+M}} |D_{N+M}(t)F_{M-1}(t)|\,\mathrm{d}t \\ I_2 &= \frac{1}{M}\int_{\frac{1}{N+M}}^{\frac{1}{M}} |D_{N+M}(t)F_{M-1}(t)|\,\mathrm{d}t, \\ I_3 &= \frac{1}{M}\int_{\frac{1}{M}}^{\frac{1}{2}} |D_{N+M}(t)F_{M-1}(t)|\,\mathrm{d}t. \end{aligned}$$

We have

$$|D_{N+M}(t)| \leq 2(N+M)+1,\ 0 \leq F_{M-1}(t) \leq M,\ \int_0^1 F_{M-1}(t)\,\mathrm{d}t = 1.$$

It follows that

$$I_1 \leq \frac{1}{M}\int_0^{\frac{1}{N+M}} (2(M+N)+1)M\,\mathrm{d}t = 2 + \frac{1}{M+N} \leq 3.$$

since M, N are positive integers.

Using the explicit expressions of D_{N+M} and F_{M-1}, we have

$$\begin{aligned} I_2 &= \frac{1}{M^2}\int_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{|\sin(\pi(2(N+M)+1)t)|\sin^2(\pi Mt)}{\sin^3(\pi t)}\,\mathrm{d}t \\ &\leq \frac{1}{M^2}\int_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{\sin^2(\pi Mt)}{\sin^3(\pi t)}\,\mathrm{d}t \\ &\leq \frac{\pi^2}{8}\int_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{\mathrm{d}t}{t} = \frac{\pi^2}{8}\ln\left(1+\frac{N}{M}\right). \end{aligned}$$

using that $\sin \pi t \leq \pi t$ for $t \geq 0$ and that $\sin \pi t \geq 2t$ for $0 \leq t \leq \frac{1}{2}$.

Finally, for I_3, we do the same computation to bound

$$I_3 \leq \frac{1}{M^2}\int_{\frac{1}{M}}^{\frac{1}{2}} \frac{\sin^2 \pi Mt}{t^3}\,\mathrm{d}t = \int_1^{\frac{M}{2}} \frac{\sin^2 \pi s}{s^3}\,\mathrm{d}s \leq \int_1^{\frac{M}{2}} \frac{\mathrm{d}s}{s^3} \leq \frac{1}{2}.$$

Grouping all terms and slightly upper bounding the numerical constants, we obtain

$$\int_0^1 |\mathcal{F}_d[K_{M,N}](t)|\,\mathrm{d}t \leq 8\left(2+\ln\left(1+\frac{N}{M}\right)\right).$$

By Lemma 6.3.4 (and the 1-periodicity of $\mathcal{F}_d[K_{M,N}]$), we obtain

$$\frac{1}{R+S}\sum_{j=-R}^{S-1}\left|\mathcal{F}_d[K_{M,N}]\left(\frac{j}{R+S}\right)\right| \leq \left(1+\frac{2\pi d}{R+S}\right)\|\mathcal{F}_d[K_{M,N}]\|_{L^1([0,1])}.$$

But $d = N+2M-1$ and $R+S \geq 2N+4M$ so

$$\frac{2\pi d}{R+S} \leq \frac{\pi(2N+4M-2)}{2N+4M} \leq \pi.$$

Hence we get

$$\begin{aligned} \frac{1}{R+S}\sum_{j=-R}^{S-1}\left|\mathcal{F}_d[K_{M,N}]\left(\frac{j}{R+S}\right)\right| &\leq (1+\pi)\|\mathcal{F}_d[K_{M,N}]\|_{L^1([0,1])} \\ &\leq 16\pi\left(2+\ln\left(1+\frac{N}{M}\right)\right), \end{aligned}$$

concluding the proof. □

Lemma 6.3.6. *Let R, S be positive integers and $K = \big(K(-R), \dots, K(S-1)\big) \in \mathbb{C}^{R+S}$. We extend K into*

– an $R+S$-periodic sequence $K^{(p)}\big(j(R+S)+\ell\big) = K(\ell)$, for $\ell = -R, \dots, S-1$ and $j \in \mathbb{Z}$;

– a finitely supported sequence $K^{(0)}$ by setting $K^{(0)}(\ell) = K(\ell)$ for $\ell = -R, \dots, S-1$ and $K^{(0)}(\ell) = 0$ for $\ell \geq S$ and for $\ell \leq -R-1$.

Then

$$\int_0^1 \left| \sum_{m\in\mathbb{Z}} a_m K^{(p)}(m) e^{2i\pi mt} \right| \mathrm{d}t \leq \frac{1}{R+S} \sum_{\ell=-R}^{S-1} \left| \mathcal{F}_d[K^{(0)}]\left(\frac{\ell}{R+S}\right)\right| \int_0^1 \left| \sum_{m\in\mathbb{Z}} a_m e^{2i\pi mt} \right| \mathrm{d}t$$

Proof. Write elements of $\mathbb{C}^{R+S}$ as $(a_{-R}, \dots, a_{S-1})$ and the scalar product $\langle a, b\rangle = \sum_{\ell=-R}^{S-1} a_\ell \overline{b_\ell}$.

For $j = -R, \dots, S-1$, denote by $e_k := \left[\frac{1}{\sqrt{R+S}} e^{2i\pi \frac{k\ell}{R+S}}\right]_{\ell=-R}^{S-1}$ so that $(e_k)_{k=-R,\dots,S-1}$ is an orthonormal basis of $\mathbb{C}^{R+S}$.

Write $A_\ell = \sum_{n\in\mathbb{Z}} \overline{a_{n(R+S)+\ell}} e^{-2i\pi(n(R+S)+\ell)t}$ for $\ell = -R, \dots, S-1$ and $A = (A_\ell) \in \mathbb{C}^{R+S}$. Then, by periodicity of $K^{(p)}$

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} a_m K^{(p)}(m) e^{2i\pi mt} &= \sum_{\ell=-R}^{S-1} K(\ell) \sum_{j\in\mathbb{Z}} a_{j(R+S)+\ell} e^{2i\pi(j(R+S)+\ell)t} \\
&= \langle K, A\rangle = \sum_{k=-R}^{S-1} \langle K, e_k\rangle \overline{\langle A, e_k\rangle} \\
&= \frac{1}{R+S} \sum_{k=-R}^{S-1} \left(\sum_{\ell=-R}^{S-1} K(\ell) e^{-2i\pi k \frac{\ell}{R+S}} \right) \left(\sum_{\ell=-R}^{S-1} \sum_{n\in\mathbb{Z}} a_{n(R+S)+\ell} e^{2i\pi(n(R+S)+\ell)t} e^{2i\pi k \frac{\ell}{R+S}} \right) \\
&= \frac{1}{R+S} \sum_{k=-R}^{S-1} \mathcal{F}_d[K^0]\left(\frac{k}{R+S}\right) \left(\sum_{\ell=-R}^{S-1} \sum_{n\in\mathbb{Z}} a_{n(R+S)+\ell} e^{2i\pi[(n(R+S)+\ell)t + \ell\frac{k}{R+S}]} \right).
\end{aligned}$$

Noticing that $e^{2i\pi\ell\frac{k}{R+S}} = e^{2i\pi[(n(R+S)+\ell)\frac{k}{R+S}]}$, we may write

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} a_m K^{(p)}(m) e^{2i\pi mt} &= \frac{1}{R+S} \sum_{k=-R}^{S-1} \left(\mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right) \right. \\
&\qquad \left. \times \sum_{\ell=-R}^{S-1} \sum_{n\in\mathbb{Z}} a_{n(R+S)+\ell} e^{2i\pi(n(R+S)+\ell)(t+\frac{k}{R+S})} \right) \\
&= \frac{1}{R+S} \sum_{k=-R}^{S-1} \mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right) \sum_{m\in\mathbb{Z}} a_m e^{2i\pi m(t+\frac{k}{R+S})}.
\end{aligned}$$

From this, we deduce that

$$\int_0^1 \left|\sum_{m\in\mathbb{Z}} a_m K^{(p)}(m) e^{2i\pi mt}\right| \mathrm{d}t$$
$$\leq \frac{1}{R+S}\sum_{k=-R}^{S-1}\left|\mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right)\right| \int_0^1 \left|\sum_{m\in\mathbb{Z}} a_m e^{2i\pi m(t+\frac{k}{R+S})}\right| \mathrm{d}t$$
$$= \frac{1}{R+S}\sum_{k=-R}^{S-1}\left|\mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right)\right| \int_0^1 \left|\sum_{m\in\mathbb{Z}} a_m e^{2i\pi mu}\right| \mathrm{d}u,$$

with the change of variable $u = t + \dfrac{j}{R+S}$ and periodicity of $u \longrightarrow \displaystyle\sum_m a_m e^{2i\pi mu}$. □

Lemma 6.3.7. *Let d, D and q be positive integers with $(2+2\delta)d + 4 \leq D$ for some $\delta > 0$ and $q \geq 4\pi$.*

Suppose I is a finite set of integers and, for each $k \in I$ let f_k be a trigonometric polynomial of degree at most d. Then, for any integer s, we have:

$$\int_0^1 \left|\sum_{k\in I(q;s)} f_k(t) e^{2i\pi Dkt}\right| \mathrm{d}t \leq 32\pi(2+\ln(1+2/\delta)) \int_0^1 \left|\sum_{k\in I} f_k(t) e^{2i\pi Dkt}\right| \mathrm{d}t.$$

Proof. Assume we can prove the lemma for $s = 0$, that is, for any sequence $(f_k)_{k\in\mathbb{Z}}$ of trigonometric polynomials of degree at most d and any finite set I,

$$\int_0^1 \left|\sum_{q\ell\in I} f_{\ell q}(t) e^{2i\pi D\ell qt}\right| \mathrm{d}t \leq 32\pi(2+\ln(1+2/\delta)) \int_0^1 \left|\sum_{k\in I} f_k(t) e^{2i\pi Dkt}\right| \mathrm{d}t.$$

Replacing I with $I - s$ and replacing (f_k) with (f_{k+s}) we get

$$\begin{aligned}
\int_0^1 \left|\sum_{k\in I(q;s)} f_k(t) e^{2i\pi D\ell qt}\right| \mathrm{d}t &= \int_0^1 \left|\sum_{s+\ell q\in I} f_{s+\ell q}(t) e^{2i\pi D\ell qt}\right| \mathrm{d}t \\
&\leq 32\pi(2+\ln(1+2/\delta)) \int_0^1 \left|\sum_{k\in I-s} f_{s+k}(t) e^{2i\pi Dkt}\right| \mathrm{d}t \\
&= 32\pi(2+\ln(1+2/\delta)) \int_0^1 \left|e^{-2i\pi Dst}\sum_{\ell\in I} f_\ell(t) e^{2i\pi Dlt}\right| \mathrm{d}t
\end{aligned}$$

which is the desired estimate since $|e^{-2i\pi Dst}| = 1$. So there is no loss of generality is assuming $s = 0$.

Next, we write $f_k = \displaystyle\sum_{-d\leq\ell\leq d} a_\ell^k e^{2i\pi(Dk+\ell)t}$ and

$$F(t) = \sum_{k\in I}\sum_{-d\leq\ell\leq d} a_\ell^k e^{2i\pi(Dk+\ell)t} = \sum_m a_m e^{2i\pi mt}$$

where

$$\begin{cases} a_m = a_\ell^k & \text{when } m = Dk + \ell, \quad k \in I \quad \text{and} \quad |\ell| \le d \\ 0 & \text{otherwise} \end{cases}. \tag{6.3.6}$$

Let $N = d$, $M = \left\lceil \frac{\delta d}{2} \right\rceil$ the smallest integer larger than $\frac{\delta d}{2}$ and let $K_{M,N}$ the sequence from Lemma 6.3.5. Since

$$N + 2M \le d + 2\left(\frac{\delta d}{2} + 1\right) = d + \delta d + 2 \le \frac{D}{2}$$

then we have

$$\operatorname{supp}(K_{M,N}) = [-N - 2M, N + 2M] \subseteq \left[-\frac{D}{2}, \frac{D}{2}\right].$$

Further $K_{M,N}(m) = 1$ for $m \in [-d, d]$. Next, let $K_{M,N}^{(p)}$ be the qD-periodic sequence defined by $K_{M,N}^{(p)}(jqD + \ell) = K_{M,N}(\ell)$ for $\ell = -N - 2M, \ldots, N + 2M$ and $K_{M,N}^{(p)}(k) = 0$ for all other k's. We take $R, S > N + 2M$ such that $R + S = qD$ and then $K_{M,N}^{(p)}(k) = 0$ for $k = -R, \ldots, -N - 2M - 1$ and for $k = N + 2M + 1, \ldots, S - 1$.

From Lemma 6.3.6, we get

$$\begin{aligned} \int_0^1 \left| \sum_m a_m K_{M,N}^{(p)}(m) e^{2i\pi mt} \right| \mathrm{d}t &\le \frac{1}{R+S} \sum_{j=-R}^{S-1} \left| \mathcal{F}_d[K_{M,N}]\left(\frac{j}{R+S}\right) \right| \int_0^1 \left| \sum_{m \in \mathbb{Z}} a_m e^{2i\pi mu} \right| \mathrm{d}u \\ &\le 16\pi \left(2 + \ln\left(1 + \frac{N}{M}\right)\right) \|F\|_{L^1([0,1])} \end{aligned}$$

with Lemma 6.3.5.

As $\operatorname{supp}(K_{M,N}) \subseteq \left[-\frac{D}{2}, \frac{D}{2}\right]$ and $K_{M,N}^{(p)}$ is periodic of period $R + S = qD$ then

$$K_{M,N}^{(p)}(m) \neq 0 \quad \text{if} \quad m = jqD + \ell' \quad \text{for} \quad j \in \mathbb{Z} \quad \text{and} \quad |\ell'| \le \frac{D}{2}. \tag{6.3.7}$$

Combining (6.3.6) and (6.3.7) we have that $a_m K_{M,N}^{(p)}(m) \neq 0$ only when

$$m = Dk + \ell = jqD + \ell'.$$

Hence $|jqD - Dk| = |\ell - \ell'| \le d + \frac{D}{2} < D$. But this can only happen when $jq = k$, which then also implies $\ell = \ell'$. In particular, $m = jqD + \ell$ with $|\ell| \le d$ and then

$$a_m K_{M,N}^{(p)}(m) = a_{jqD+\ell} K_{M,N}^{(p)}(jqD + \ell) = a_{jqD+\ell} K_{M,N}^{(p)}(\ell) = a_\ell^{jqD}.$$

It follows that

$$\begin{aligned}\sum_{m\in\mathbb{Z}} a_m K_{M,N}(m) e^{2i\pi mt} &= \sum_{j\in\mathbb{Z}} \sum_{-d\le\ell\le d} a_{jqD+\ell} e^{2i\pi[(jqD+\ell)t]} \\ &= \sum_{\substack{k=0 \bmod q \\ k\in I}} \sum_{-d\le\ell\le d} a_{kD+\ell} e^{2i\pi kDt} e^{2i\pi\ell t} \\ &= \sum_{k\in I(q;0)} e^{2i\pi kDt} \sum_{-d\le\ell\le d} a_{kD+\ell} e^{2i\pi\ell t} \\ &= \sum_{k\in I(q;0)} f_k(t) e^{2i\pi kDt}.\end{aligned}$$

Finally we get

$$\begin{aligned}\int_0^1 \left| \sum_{k\in I(q;0)} f_k(t) e^{2i\pi Kdt} \right| \mathrm{d}t &= \int_0^1 \left| \sum_m a_m K_{M,N}(m) e^{2i\pi mt} \right| \mathrm{d}t \\ &\le 32\pi \left(2 + \ln(1 + N/M)\right) \|F\|_{L^1([0,1])} \\ &\le 32\pi(2 + \ln(1 + 2/\delta))\|F\|_{L^1([0,1])}\end{aligned}$$

since $\dfrac{N}{M} \le \dfrac{2d}{\delta D} \le \dfrac{2}{\delta}$. □

We can now prove the lemma.

Proof of Lemma 6.3.2. Write $I(q;s) = \{k_1, \ldots, k_J\}$ and write each k_j in the form $k_j = r_j q + s$. Applying Lemma 6.3.7 yields

$$\begin{aligned}\|F\|_{L^1([0,1])} &\ge \frac{1}{32\pi(2 + \ln(1 + 2/\delta))} \int_0^1 \left| \sum_{j=1}^J f_{k_j}(t) e^{2i\pi k_j Dt} \right| \mathrm{d}t \\ &= \int_0^1 \left| e^{2i\pi Dst} \sum_{j=1}^J f_{k_j}(t) e^{2i\pi r_j qDt} \right| \mathrm{d}t \\ &= \frac{1}{qD} \int_0^{qD} \left| \sum_{j=1}^J f_{k_j}\left(\frac{s}{qD}\right) e^{2i\pi r_j s} \right| \mathrm{d}s \\ &= \frac{1}{qD} \sum_{m=0}^{qD-1} \int_m^{m+1} \left| \sum_{j=1}^J f_{k_j}\left(\frac{s}{qD}\right) e^{2i\pi r_j s} \right| \mathrm{d}s.\end{aligned}$$

But

$$\sum_{j=1}^J f_{k_j}\left(\frac{s}{qD}\right) e^{2i\pi r_j s} = \sum_{j=1}^J f_{k_j}\left(\frac{m}{qD}\right) e^{2i\pi r_j s} + \sum_{j=1}^J \left[f_{k_j}\left(\frac{s}{qD}\right) - f_{k_j}\left(\frac{m}{qD}\right) \right] e^{2i\pi r_j s}$$

so that

$$\|F\|_{L^1([0,1])} \geq \frac{1}{32(2+\ln(1+2/\delta))}(T_1 - T_2)$$

with

$$T_1 = \frac{1}{qD}\sum_{m=0}^{qD-1}\int_m^{m+1}\left|\sum_{j=1}^{J} f_{k_j}\left(\frac{m}{qD}\right)e^{2i\pi r_j s}\right|\,\mathrm{d}s,$$

and

$$T_2 = \frac{1}{qD}\sum_{m=0}^{qD-1}\int_m^{m+1}\left|\sum_{j=1}^{J}\left(f_{k_j}\left(\frac{s}{qD}\right) - f_{k_j}\left(\frac{m}{qD}\right)\right)e^{2i\pi r_j s}\right|\,\mathrm{d}s.$$

It remains to show that

$$T_1 \geq \frac{C_{MPS}}{2}\sum_{j=1}^{J}\frac{\|f_{k_j}\|_{L^1}}{j} \quad \text{and} \quad T_2 \leq \frac{2\pi d}{qD}\sum_{j=1}^{J}\|f_{k_j}\|_{L^1}.$$

Let us start with T_1: using the 1-periodicity in s,

$$\begin{aligned} T_1 &= \frac{1}{qD}\sum_{m=0}^{qD-1}\int_0^{1}\left|\sum_{j=1}^{J} f_{k_j}\left(\frac{m}{qD}\right)e^{2i\pi r_j s}\right|\,\mathrm{d}s, \\ &\geq \frac{C_{MPS}}{qD}\sum_{m=0}^{qD-1}\sum_{j=1}^{J}\frac{\left|f_{k_j}\left(\frac{m}{qD}\right)\right|}{j} \\ &= \frac{C_{MPS}}{qD}\sum_{j=1}^{J}\frac{1}{j}\sum_{m=0}^{qD-1}\left|f_{k_j}\left(\frac{m}{qD}\right)\right| \end{aligned}$$

with Theorem 4.1.1. Applying Lemma 6.3.4 to f_{k_j} and using that $\dfrac{2\pi d}{qD} < \dfrac{1}{2}$ with our hypothesis on q and D, we get

$$\frac{1}{qD}\sum_{m=0}^{qD-1}\left|f_{k_j}\left(\frac{m}{qD}\right)\right| \geq \frac{\|f_{k_j}\|_{L^1}}{2}$$

and the desired estimate of T_1 follows immediately.

Let us now estimate T_2. For $s \in [m, m+1]$, we have

$$\begin{aligned} \left|\sum_{j=1}^{J}\left(f_{k_j}\left(\frac{s}{qD}\right) - f_{k_j}\left(\frac{m}{qD}\right)\right)e^{2i\pi r_j s}\right| &= \left|\sum_{j=1}^{J}\int_{\frac{m}{qD}}^{\frac{s}{qD}} f'_{k_j}(t)e^{2i\pi r_j s}\,\mathrm{d}t\right| \\ &\leq \int_{\frac{m}{qD}}^{\frac{m+1}{qD}}\left|\sum_{j=1}^{J} f'_{k_j}(t)e^{2i\pi r_j s}\right|\mathrm{d}t. \end{aligned}$$

From the 1-periodicity in s, the integral of this quantity over $[m, m+1]$ is the same as the integral over $[0, 1]$. Thus

$$\begin{aligned} T_2 &\leq \frac{1}{qD} \sum_{m=0}^{qD-1} \int_m^{m+1} \int_{\frac{m}{qD}}^{\frac{m+1}{qD}} \left| \sum_{j=1}^{J} f'_{k_j}(t) e^{2i\pi r_j s} \right| \mathrm{d}t \, \mathrm{d}s \\ &= \frac{1}{qD} \sum_{m=0}^{qD-1} \int_{\frac{m}{qD}}^{\frac{m+1}{qD}} \int_0^1 \left| \sum_{j=1}^{J} f'_{k_j}(t) e^{2i\pi r_j s} \right| \mathrm{d}s \, \mathrm{d}t \\ &= \frac{1}{qD} \int_0^1 \int_0^1 \left| \sum_{j=1}^{J} f'_{k_j}(t) e^{2i\pi r_j s} \right| \mathrm{d}s \, \mathrm{d}t \\ &\leq \frac{1}{qD} \sum_{j=1}^{J} \| f'_{k_j} \|_{L^1([0,1])} \\ &\leq \frac{2\pi d}{qD} \sum_{j=1}^{J} \| f_{k_j} \|_{L^1([0,1])} \end{aligned}$$

with Bernstein's inequality. □

6.3.2 Strongly 2-dimensional subsets of $\mathbb{R}$

We start by extending the definition of strongly dimensional structure to subsets of $\mathbb{R}$. We will use the same decomposition as in the case of strongly subset of $\mathbb{Z}$. The main difference with strongly subset of $\mathbb{Z}$ is that the sets A_k are now subsets of $\mathbb{R}$ instead of $\mathbb{Z}$. Let $\delta > 0$ and $(m, n) \in \mathbb{N}^2$. A finite subset A of $\mathbb{R}$ is $(\delta; m, n)$-strongly 2-dimensional if there exist two real numbers d and D with $D > (2+\delta)d$ such that

$$A = \bigcup_{k \in I} (A_k + kD)$$

for some set I containing m integers and subsets $A_k \subseteq [-d, d]$ verifying $|A_k| \geq n$.

Theorem 6.3.8. *Let $\delta > 0$ and m, n be two positive integers satisfying*

$$m \geq \pi^3 2^{21} C_{MPS}^3 \ln(m)^3 \ln(n)^3 \quad \text{and} \quad n \geq \pi^3 2^{21} C_{MPS}^3 \ln(n)^3,$$

where $C_{MPS}(= 1/30)$ is the constant in Theorem 4.1.1. Suppose A is $(\delta; m, n)$ strongly 2-dimensional subset of $\mathbb{R}$. Then

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{a \in A} e^{2i\pi a t} \right| \mathrm{d}t \geq \frac{C_{MPS}^2}{(2^9\pi)^2 \left(2 + \ln(1 + \frac{2}{\delta})\right)} \ln(m) \ln(n).$$

Proof. The idea is similar to the proof by Hudson and Leckband 6.1.1. We will use Dirichlet's Lemma to approximate the real frequencies by integers and then apply Hanson's Theorem for strongly multidimensional subsets of $\mathbb{Z}$.

Let A be a strongly multidimensional subset of $\mathbb{R}$:

$$A = \bigcup_{k\in I}(A_k + kD), \qquad A_k = \{a_{k,1}, \ldots, a_{k,n_k}\},$$

and

$$\Phi(t) = \sum_{a\in A} e^{2i\pi at} = \sum_{k=1}^{m}\sum_{j=1}^{n_k} e^{2i\pi(a_{k,j}+kD)t}.$$

Let $0 < \varepsilon < 1$. By a lemma of Dirichlet ([44, p 235], [8]), there exists an increasing sequence of integers $(m_l)_{l\geq 1}$ and, for each $l \geq 1$ finite families of integers $(a_{k,j}^{(l)})_{k=1,\ldots,m}$ and D_l such that, for $k = 1, \ldots, m$ and $j = 1, \ldots, n_k$

$$\left|a_{k,j} - \frac{a_{k,j}^{(l)}}{m_l}\right| < \frac{\varepsilon}{m_l} \quad \text{and} \quad \left|D - \frac{D_l}{m_l}\right| < \frac{\varepsilon}{m_l} \tag{6.3.8}$$

which implies that for $k = 1, \ldots, m$ and $j = 1, \ldots, n_k$

$$\left|e^{2i\pi(a_{k,j}+kD)t} - e^{2i\pi\left(\frac{a_{k,j}^{(l)}}{m_l}+\frac{kD}{m_l}\right)t}\right| \leq 2\pi\left(\left|a_{k,j} - \frac{a_{k,j}^{(l)}}{m_l}\right| + k\left|D - \frac{D_l}{m_l}\right|\right)|t| \leq \frac{2(1+m)\pi\varepsilon}{m_l}|t|.$$

Define the m_l-periodic function

$$\Psi_l(t) = \sum_{k=1}^{m}\sum_{j=1}^{n_k} e^{2i\pi(a_{k,j}^{(l)}+kD_l)t/m_l}$$

and note that, for $t \in [-m_l/2, m_l/2]$,

$$|\Phi(t) - \Psi_l(t)| \leq \sum_{k=1}^{m}\sum_{j=1}^{n}\left|e^{2i\pi(a_{k,j}+kD)t} - e^{2i\pi\left(\frac{a_{k,j}^{(l)}}{m_l}+\frac{kD_l}{m_l}\right)t}\right| \leq \pi(1+m)|A|\varepsilon.$$

but then

$$\begin{aligned}\left|\left(\frac{1}{m_l}\int_{-m_l/2}^{m_l/2}|\Phi(t)|\,\mathrm{d}t\right) - \left(\frac{1}{m_l}\int_{-m_l/2}^{m_l/2}|\Psi_l(t)|\,\mathrm{d}t\right)\right| & \\ \leq \left(\frac{1}{m_l}\int_{-m_l/2}^{m_l/2}|\Phi(t) - \Psi_l(t)|\,\mathrm{d}t\right) & \\ \leq \pi(1+m)|A|\varepsilon & \end{aligned}$$

But

$$|a_{k,j}^{(l)} - a_{k,j}m_l| \leq \varepsilon < 1 \qquad \text{and} \qquad -d \leq a_{k,j} \leq d$$

imply that

$$-dm_l - 1 \leq a_{k,j}^{(l)} \leq dm_l + 1$$

i.e

$$a_{k,j}^{(l)} \in \{-d_l, \ldots, d_l\} \qquad \text{with} \qquad d_l = \lceil dm_l \rceil + 1.$$

Note also that, for each k, the $a_{k,j}^{(l)}$'s are distinct, provided l is large enough. On the other hand

$$\frac{D}{d} > 2 + \delta$$

hence there exists δ' such that

$$\frac{D}{d} > 2 + \delta' > 2 + \delta.$$

Since

$$|D_l - Dm_l| \leq \varepsilon \qquad \text{and} \qquad \varepsilon < 1$$

we get

$$\begin{aligned} D_l &> Dm_l - 1 \geq (2+\delta')dm_l - 1 \\ &\geq (2+\delta')(\lceil dm_l \rceil - 1) - 1 \\ &= (2+\delta')d_l - 5 - 2\delta' \end{aligned}$$

then

$$\frac{D_l}{d_l} > 2 + \delta,$$

provided l is large enough.

It follows that

$$\bigcup_{k=1}^{n} (A_l + kD_l) \qquad \text{with} \qquad A_l = \{a_{k,j}^l\}$$

is strongly $(\delta; m, n)$ strongly 2-dimensional. Then if

$$C = \frac{C_{MPS}^2}{(2^9\pi)^2\big(2 + \ln(1 + \frac{2}{\delta})\big)},$$

from Hanson's Theorem 6.1.3, we get that

$$\begin{aligned} C\ln(m)\ln(n) &\leq \int_0^1 \left| \sum_{k=1}^{m} \sum_{j=1}^{n_k} e^{2i\pi(a_{k,j}^{(l)} + KD_l)t} \right| \mathrm{d}t \\ &= \frac{1}{m_l} \int_{-m_l/2}^{m_l/2} \left| \sum_{k=1}^{m} \sum_{j=1}^{n_k} e^{2i\pi(a_{k,j}^{(l)} + KD_l)t/m_l} \right| \mathrm{d}t \\ &\leq \frac{1}{m_l} \int_{-m_l/2}^{m_l/2} |\Phi(t)| \,\mathrm{d}t + \frac{1}{m_l} \int_{-m_l/2}^{m_l/2} |\Phi(t) - \Psi_l(t)| \,\mathrm{d}t \\ &\leq \frac{1}{m_l} \int_{-m_l/2}^{m_l/2} |\Phi(t)| \,\mathrm{d}t + \pi(1+m)|A|\varepsilon \end{aligned}$$

Letting $l \to +\infty$ and then $\varepsilon \to 0$ we obtain Theorem 6.3.8:

$$\lim_{T\to\infty} \int_{-T/2}^{T/2} \left| \sum_{a\in A} e^{2i\pi at} \right| \, \mathrm{d}t \geq \frac{C_{MPS}^2}{(2^9\pi)^2\big(2+\ln(1+\frac{2}{\delta})\big)} \ln(m)\ln(n).$$

□

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