Diverse aspects of hyperbolic geometry and group dynamics

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Divers aspects de la géométrie hyperbolique et de la dynamique de groupes

Thèse de doctorat de l'Université Paris-Saclay

École doctorale n°574 Mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques Fondamentales
Graduate school : Mathématiques. Référent : Faculté des sciences d'Orsay.

Thèse préparée dans l'unité de recherche **Laboratoire de mathématiques d'Orsay (Université Paris-Saclay, CNRS)**, sous la direction de **Bruno DUCHESNE**, Professeur.

Thèse soutenue à Paris-Saclay, le 21 juin 2024, par

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Thèse de doctorat

NNT : 2024UPASM016

獻之皇祖，君諱耀坤。既敦既純，且嚴復慈。
享其壽祉，遺芳後昆。幼承庭訓，烝烝孺慕。
鸞停鵠峙，感思舊君。紀之以文，天靈永慰。

Remerciements

Das wundert mich nicht. Für die Mathematik hatte der zu wenig Phantasie, aber zum Dichten reicht es.

David Hilbert

De dons très chers, ma thèse a récolté des vies.
Bruno Duchesne, t'as éclairé mes envies :
Plus qu'un guide, tel phare imposant qui se livre,
Ami, modèle, sagesse que l'on délivre.

Aucune doute, nos chemins se croiseront,
Des théories du futur, nous en bâtissons.
À cher Bruno, tout mon respect, et mon honneur,
En ces rimes, célébrons-nous cette lueur.

Je prenais la lanterne à midi en la main,
Dans le dédale de math, cherchant un chemin.
Vers moi, Nicolas Monod, il s'est avancé
Pour me montrer ce sujet de toute beauté.

Avec Chénmín Sūn, nos discours de plénitude
Produisaient de belles idées mathématiques.
À Rick Vitale, j'exprime ma gratitude
Pour m'avoir appris l'analyse stochastique.

Lorsque j'errais dans le désert de l'inconnu,
trompé par mirage, j'étais presque perdu.
Jesus Hernandez-Hernandez m'a expliqué
Comment mon petit faux pas peut être éffacé.

Mes pairs d'étude, sources d'inspirations,
J'ai puisé le savoir de nos discussions :
Antonio Lopez-Neumann, mon bon ami,
Il m'a présenté des exemples très jolis.

J'étais tout ignorant comme un petit bébé,
H. Lajoinie m'apprends la propriété (T).
De nombreuses idées ont été partagées
Avec David Xu, pendant des années passées.

Si jamais je flottais sur la mer de l'ennui,
Mes amis seraient mes sauveteurs courageux.
Mille fois merci à Jocelyne et Louis
Pour le parcours dans le Sud, toujours très joyeux.

Lorsque j'avais des doutes existentiels,
Ma chère famille m'a toujours soutenu,
Et ils restent mon appui émotionnel
Quand je rencontre des accidents imprévus.

À tous tout également, mon cœur vous enlace,
Et *prima inter pares*, Míngwèi, je confesse :
Ta compagne tendre, chaque geste caresse,
Dans les joies partagées, ou le temps de tritesse.

前川如激箭，幾度濁還清。
問道經三歲，今朝又啓程。
從來多聚散，無用訴愁情。
若憶求師處，中庭滿落英。

Chapitre 0

Panorama (Fr)

兩句三年得，一吟雙淚流。
知音如不賞，歸臥故山秋。[1]

賈島（779–843）

Au cours des dernières décennies, la géométrie hyperbolique s'est imposée comme un point central de la recherche en géométrie. Depuis le XIXe siècle, lorsque des visionnaires comme János Bolyai et Nikolai Lobachevsky ont introduit pour la première fois le concept de géométrie non euclidienne, l'exploration des subtilités de la géométrie hyperbolique a suscité un intérêt soutenu et fervent. Pendant la seconde moitié du XXe siècle, à mesure que la topologie en basses dimensions était explorée de manière approfondie, la géométrie hyperbolique a inspiré une série de résultats profonds et d'une beauté remarquable : il suffit de mentionner l'héritage de William P. Thuston et la contribution de Mikhael Gromov, qui constituent la plupart du fondement théorique de cette thèse.

Simultanément, l'étude des groupes infinis a également émergé dans la seconde moitié du XIXe siècle, établissant des liens étroits avec des considérations géométriques, telles que le programme d'Erlangen (Klein, 1872) et les contributions de Lie et Poincaré. Dans le domaine de la géométrie, les groupes apparaissent naturellement lors de l'étude des symétries et des transformations géométriques. Un exemple notable de l'interaction entre la géométrie et les groupes infinis est le résultat de Dehn sur le groupe fondamental des surfaces, où la courbure négative est un élément géométrique important. Ce résultat marque le point de départ de la relation inhérente entre les groupes infinis et la géométrie

[1] Traduction en français :
Deux vers forgés sur trois années résonnent,
Lus une fois, et deux lignes de larmes coulent.
Si mes chers amis ne peuvent mon travail apprécier,
Aux montagnes d'automne natales, je reviendrai m'allonger.

hyperbolique.

J'espère que les lecteurs pourront supporter ma verbosité et me permettre de prendre un peu d'espace pour discuter quelques réflexions philosophiques sur la géométrie et la dynamique de groupes. Cela aidera les lecteurs non mathématiciens à mieux saisir la saveur de mon travail. En premier lieu, je partage le même avis que Pierre de la Harpe [Har00] :

> (Traduction en français) *L'une de mes convictions personnelles est que la fascination pour les symétries et les groupes est un moyen de faire face aux frustrations liées aux limites de la vie : nous aimons reconnaître les symétries qui nous permettent d'aller au-delà de ce que nous pouvons voir.*

Mais les ambitions des mathématiciens vont plus loin. Nous aspirons à reconnaître, à travers les symétries qui nous sont facilement visibles, les symétries plus obscures qui leur sont proches mais qui ne sont pas en général intuitivement apparentes. Lorsque la complexité des objets géométriques d'intérêt devient infinie, comme dans le cas des structures fractales, des graphes infinis ou des surfaces de type infini discutées dans cette thèse, cette perspective devient particulièrement cruciale. Dans de tels cas, les symétries deviennent non dénombrables, et l'étude isolée de symétries individuelles ne contribue souvent pas de manière significative à nous donner une vision globale de la structure des groupes associés. Dans ces cas, il devient nécessaire de percevoir le groupe, constitué de symétries, non seulement comme une entité algébrique abstraite, mais comme un objet géométrique lui-même. La notion d'un groupe topologique sert cet objectif ; il nous permet de traduire les similitudes entre ces symétries en proximité entre les éléments du groupe (ou, dans le langage de Bourbaki, en *uniformité*), ou même en distance, comme les groupes polonais dont nous parlerons plus tard dans cette thèse. En laissant ces groupes agir continûment sur des espaces topologiques, nous pouvons comprendre la structure du groupe en tant qu'un objet géométrique en examinant les régularités de ces actions. Cela nous amène au deuxième élément du titre de cette thèse : la dynamique de groupes.

La beauté de la géométrie hyperbolique et de la dynamique de groupes ne réside pas seulement dans leur profondeur, mais aussi dans leur diversité très riche : objets, points de vue, méthodes, outils ... Comparée à l'immense gamme d'objets et de possibilités dans le domaine de la géométrie hyperbolique et de la dynamique de groupes, cette thèse ne représente qu'une petite partie de l'ensemble. Néanmoins, elle traite déjà des corps convexes, des surfaces, des graphes infinis et de la géométrie grossière des espaces métriques. De plus, les outils et les méthodes utilisés pour résoudre les questions dans cette thèse proviennent d'un large éventail de disciplines, notamment la topologie, l'analyse fonctionnelle, la théorie des probabilité, la géométrie métrique, logique mathématique, et cætera.

Cette thèse sera divisée en trois parties : « *Shoulders of Giants* » (*les épaules des géants*), « *Farther Sight of Dwarfs* » (*la vue plus lointaine des nains*), et « *Neither Fish, Flesh, nor Fowl* » (idiom. *ni chair ni poisson*).

La première partie, « *Shoulders of Giants* », comme le suggère le titre, présentera le contexte de la recherche et des résultats importants déjà connus. Dans le Chapitre 2, on commencera par introduire la notion d'espace hyperbolique algébrique (de dimension infinie) en utilisant le modèle de l'hyperboloïde. Il s'agit d'espace le plus fondamental en géométrie hyperbolique. On citera aussi le résultat de Duchesne [Duc23] sur la topologie polonaise du groupe des isométries de l'espace hyperbolique algébrique de dimension infinie, comme un premier exemple (dans cette thèse) de mis en distance des similitudes entre les transformations. On expliquera également les noyaux de type hyperbolique mise en avant par [MP14; MP19; Mon20]. Ils permettent de construire un plongement d'un ensemble dans un espace hyperbolique algébrique. La Section 2.2 sera consacrée à l'exposition d'une généralisation de l'espace hyperbolique algébrique, appelée *espace hyperbolique au sens de Gromov*. Il s'agit de capturer l'hyperbolicité à *grande échelle*. Ces espaces jouissent de nombreuses propriétés similaires à celles de l'espace hyperbolique algébrique : la trichotomie des isométries, la dynamique des isométries sur le bord à l'infini, et cétéra. Pour des raisons de limitation d'espace, on est malheureusement contraint de mettre de côté la riche théorie des espaces $\mathrm{CAT}(0)$, mais cela ne signifie nullement que cette généralisation infinitésimal est négligéable.

Dans le Chapitre 3, on introduira et expliquera plusieurs concepts et propriétés fondamentales de la dynamique des actions continues des groupes topologiques sur des espaces compacts. Dans la Section 3.1, on détaillera la construction du *flot minimal universel*. Parmi ses nombreuses constructions équivalentes, on a choisi une approche topologique qui est légèrement plus compliquée par rapport aux autres. Ce choix est motivé principalement par plusieurs considérations : premièrement, il s'inscrit dans la tradition de *Topologie Générale* de Bourbaki [Bou07], car cette construction repose sur le concept d'*espace uniforme* introduit par Bourbaki (en particulier, André Weil), abstrayant mathématiquement la notion de proximité tout en restant conforme à l'intuition géométrique ; deuxièmement, cette construction nous offre une compréhension plus approfondie du flot minimal universel, en nous permettant d'établir des liens avec certaines propriétés de points fixes pour des groupes topologiques, telles que la *moyennabilité*, et la *moyennabilité extrême*. Par exemple, cela nous permet de le comprendre comme un sous-espace de fonctionnelles linéaires pour les fonctions uniformément continues à droite bornées définies sur le groupe, ou plus spécifiquement, comme des *moyennes* de ces fonctions ; de plus, cette approche évite certaines discussions sur la théorie des ensembles en logique mathématique, bien que ces discussions ne soient pas particulièrement difficiles. On s'intéressera en particulier aux

groupes moyennables, proposés par von Neumann à la suite de son étude du paradoxe de Banach-Tarski [Neu29]. Ce paradoxe constate que l'on peut découper en un nombre fini de morceaux une boule solide de dimension 3 et les recoller pour former deux boules identiques à la première, en utilisant uniquement des translations et des rotations. On énumérera plusieurs définitions équivalentes d'un groupe topologique moyennable, qui interagissent avec une diversité de disciplines mathématiques. On abordera également ses propriétés héréditaires et traitera de manière similaire la moyennabilité extrême, un renforcement de la moyennabilité.

Le Chapitre 4 se penchera principalement sur les surfaces orientables (de type fini ou infini) et leurs groupes de difféotopies. En termes simples, les groupes de difféotopies contiennent les symétries les plus fondamentales des objets géométriques : au sein des ces groupes, deux symétries sont identifiées si l'une peut se transformer continûment en l'autre (en temps fini) et *vice versa*. Cette transformation continue est appelée une *homotopie*. Il existe différentes méthodes pour étudier les groupes de difféotopies, notamment des approches algébriques (via des actions sur l'homologie, sur le groupe des automorphismes extérieurs du groupe fondamental de la surface, sur le groupe de Torelli par automorphismes intérieurs, et cétéra), des approches géométriques ou holomorphiques (via des actions sur la courbe universelle de Teichmüller et sur les espaces de Teichmüller, et cétéra), ainsi que des approches topologiques (via des actions sur l'espace projectif des feuilletages mesurés, sur l'espace des laminations géodésiques, ou encore sur l'espace des feuilletages non mesurées, et cétéra), voir par exemple un survol dans [Pap15]. Malheureusement, il n'est pas possible de présenter toutes ces méthodes une à une, car cela transformerait cette thèse en une longue documentation bibliographique sur les vastes contributions de William P. Thurston et de ses successeurs. Par conséquent, nous nous concentrerons uniquement sur les approches combinatoires ou de la théorie géométrique des groupes, à savoir les actions simpliciales (ou isométriques) des groupes de difféotopies sur les graphes de courbes et d'autres complexes dérivés ou similaires. L'avantage réside dans deux aspects : ces complexes ou graphes sont souvent hyperboliques au sens de Gromov, offrant un meilleur accès aux outils de la géométrie hyperbolique ; en outre, cette perspective nous permet de reconnaître les groupes de difféotopies comme des groupes d'automorphismes de structures dénombrables, ou encore, des groupes polonais non archimédiens, et nous disposons alors des outils provenant de la logique mathématique et de la dynamique topologique, en particulier de la théorie des modèles et de la théorie descriptive des ensembles. Tous ces outils peuvent nous aider à déterminer la moyennabilité ou la moyennabilité extrême des groupes de difféotopies, comme nous le verrons plus loin dans la thèse.

La deuxième partie de la thèse, intitulée « *Farther Sight of Dwarfs* », se compose de trois textes académiques rédigés pendant la période de doctorat, respectivement [Lon23b],

[Lon23a] et [LT24].

Le Chapitre 5 portera sur le plongement des classes homothétiques de certains corps convexes dans l'espace de Hilbert dans un espace hyperbolique de dimension infinie sur les nombres réels, en utilisant les noyaux de type hyperbolique. L'idée remonte à la caractérisation due à William P. Thurston des métriques plates sur la sphère [Thu98]. Dans un papier récent, Debin et Fillastre [DF22] donnent un plongement des classes homothétiques de corps convexes de dimension finie. Ils ont adopté une stratégie connue sous le nom de « *géométrie intégrale* ». Cette stratégie vise à associer un corps convexe à une unique fonction continue sur la sphère unité, puis à étudier l'espace de ces fonctions via l'analyse harmonique ou l'analyse fonctionnelle, pour en déduire des conclusions sur les corps convexes dans l'espace euclidien. L'analyse ou le calcul dans les espaces euclidiens classiques a longtemps été perfectionné depuis l'époque de Newton et de Leibniz. Cependant, lorsque la dimension augmente jusqu'à l'infini, comme dans les espaces de Hilbert séparables, il y a des points de vue diverse. Les traditions déterministes nous conduisent souvent vers l'analyse fonctionnelle. Or, dans la seconde moitié du XXe siècle, une analyse stochastique dans les espaces de dimension infinie, telle que le calcul de Malliavin, a été développée (voir [Da 06]). Dans le Chapitre 5, outre la poursuite du travail laissé inachevé en dimensions infinies par Debin et Fillastre, on tente d'étendre la stratégie de la « géométrie intégrale » aux dimensions infinies à l'aide du calcul de Malliavin. Cela implique d'associer de manière unique chaque corps convexe à une variable aléatoire, *i.e.* la fonction d'appui stochastique, et d'utiliser ensuite les technologies de l'analyse stochastique pour déduire des conclusions sur les corps convexes de dimension infinie. Un résultat important du Chapitre 5 consiste à fournir des formules pour les volumes intrinsèques du premier et du second degré des corps convexes à dimension infinie en utilisant les fonctions d'appui.

Pour une surface de type fini, le groupe de difféotopies sera un groupe discret de type fini qui, dans la plupart des cas, contient un sous-groupe libre sur deux générateurs. Il en résulte la non-moyennabilité de cette classe de groupes. Cependant, lorsque la complexité de la surface sous-jacente devient infinie, les groupes de difféotopies ne sont plus discrets et le fait d'avoir des sous-groupes libres n'exclut pas la possibilité d'être moyennable. Bien que l'on ne sache toujours pas si les groupes de difféotopies des surfaces de type infini (ou *gros groupes de difféotopies*) sont moyennables, nous montrerons au Chapitre 6, en utilisant les technologies de la théorie descriptive des ensembles [KPT05], que les groupes de difféotopies ne peuvent jamais être extrêmement moyennables, à l'exception des cas triviaux.

Récemment, de manière analogue au graphe de courbes, la notion de graphe de courbes fin a été introduite dans le but d'étudier les propriétés combinatoires et géométriques du groupe d'homéomorphismes des surfaces [BHW22]. Néanmoins, ces graphes hyperboliques non localement dénombrables montrent une différence significative par rapport

aux graphes de courbes en termes de leurs bords à l'infini [BHW22; Bow+22; GM23b; GM23a]. Malgré toutes ces différences, dans le Chapitre 7, nous montrerons que pour les surfaces de genre au moins 2, le bord de Gromov des graphes fins de courbes restent connexe (par arcs) comme les graphes de courbes classiques, répondant positivement un analogue de la question de Peter Storm dans [KL08].

La troisième partie est appelée « *Neither Fish, Fleish, Nor Fowl* », car elle s'efforce de présenter des recherches qui n'avaient pas encore pleinement abouti au moment de la rédaction de cette thèse.

Pendant une longue période, un théorème folklorique, *une action par isométries d'un groupe moyennable sur un espace hyperbolique au sens de Gromov ne peut pas être de type général*, a été vérifié dans diverses situations, notamment lorsque l'espace hyperbolique est propre, ou lorsque le groupe moyennable est localement compact et que l'action est métriquement propre. Mais les arguments ne sont plus valables lorsque nous ne supposons aucune condition de propreté ou de compacité, par exemple action sur le complexe de projection d'un gros groupe de difféotopies. Afin de faire face à ces situations, nous utiliserons dans le Chapitre 9 une autre compactification (ou bordification) pour l'espace hyperbolique au sens de Gromov que la bordification traditionnelle de Gromov. Cette compactification témoignera de la rigidité des groupes topologiques moyennables agissant par isométries sur les espaces hyperboliques géodésiques et séparable, ce qui peut nous aider à prouver la non-moyennabilité de certains gros groupes de difféotopies.

Dans son travail célèbre [Gro87], Mikhael Gromov a énoncé sans preuve le suivant : *chaque espace hyperbolique géodésique peut se plonger isométriquement dans un espace ultracomplet d'une manière essentiellement surjective*. Dans le Chapitre 8, nous démonterons la première partie de son énoncé en passant à une construction par ultralimite, mais la deuxième partie de cet énoncé reste encore inconnue. À la fin du Chapitre 8, nous étudierons aussi quelques familles d'espaces hyperboliques au sens de Gromov dont le bord à l'infini est préservé sous quasi-isométries.

Les lecteurs auront peut-être déjà remarqué qu'au long de cette thèse, il s'agit plutôt d'un *parcours en largeur* qu'*en profondeur* d'inconnues. Cela indique que le processus de recherche est loin d'être complet. Pour chaque sujet abordé dans cette thèse, il y a plus d'un problème ouvert et beaucoup d'entre eux n'ont même pas encore été attaqués. Voici une liste incomplète de ces questions :

- Comment calculer les volumes intrinsèques de degré supérieur des corps convexes de dimension infinie en utilisant uniquement leurs fonctions d'appui ?
- Existe-t-il des interprétations géométriques pour la projection orthogonale des fonctions d'appui dans le n-ième chaos de Wiener pour $n \geq 2$?
- Comment peut-on déterminer si un graphe infini dénombrable est un graphe de courbes

d'une surface orientable ?

- Les gros groupes de difféotopies sont-ils également non moyennables ?
- Quels gros groupes de difféotopies ont un flot minimal universel non métrisable ?
- Quelle est une condition nécessaire raisonnable pour que le bord à l'infini des espaces hyperboliques au sens de Gromov soit préservée par les quasi-isométries ?
- Peut-on plonger par une quasi-isométrie un graphe de courbes d'une surface de type fini dans un espace hyperbolique algébrique tel que l'action du groupe de difféotopies soit équivariante ?
- Les groupes de difféotopies des surfaces de type fini ont-ils de la propriété (T) de Kazhdan ? Quid de la propriété de Haagerup ?
- Existe-t-il une interprétation géométrique ou dynamique des points sur le bord de Gromov des graphes fins de courbes ? Si oui, quid de l'interprétation géométrique ou dynamique de la topologie sur ce bord ?
- Le graphe de courbes fin est-il ultracomplet ?

Ces problèmes ouverts posent des défis passionnants et offrent des opportunités pour la recherche future, invitant à l'exploration et à l'innovation dans la quête de solutions.

C'est le point final de ma recherche et formation doctorale, mais sans aucun doute le tout début de ma « *Wissenschaft als Beruf* ». Comme il reste encore beaucoup à découvrir, permettez-moi de finir ce chapitre introductif par le dicton suivant adapté du célèbre paradoxe socratique :

Dixi, sed scio me adhuc nihil scire.

Chapter 1

Panorama (En)

兩句三年得，一吟雙淚流。
知音如不賞，歸臥故山秋。[1]

賈島（779–843）

Over the past few decades, hyperbolic geometry has emerged as a key focal point in geometric research. Since the 19th century, when visionaries like János Bolyai and Nikolai Lobachevsky first introduced the concept of non-Euclidean geometry, there has been a sustained and fervent interest in exploring the intricacies of hyperbolic geometry. As the second half of the 20th century unfolded, with the deepening exploration of low-dimensional topology, hyperbolic geometry gave rise to a series of profound and remarkably beautiful results: it suffices to mention the legacy of William P. Thuston and the contribution of Mikhael Gromov, which form the most of theoretic foundation of this thesis.

Simultaneously, the study on infinite groups also emerged in the latter half of the 19th century, establishing close connections with geometric considerations, such as the Erlangen program (Klein, 1872) and the contributions of Lie and Poincaré. In the realm of geometry, groups naturally appear when investigating symmetries and geometric transformations. A notable instance of the interplay between geometry and infinite groups is Dehn's result on the fundamental group of surfaces, where negative curvature serves as a significant geometric element in Dehn's work. This marks the starting point of the inherent relationship between infinite groups and hyperbolic geometry.

I hope readers can bear with my verbosity and allow me to take some space to dis-

[1] Translation in English:
Two verses crafted across three years resound,
Read once, yet two lines of tears abound.
If my dear friends fail to grasp their worth,
To the autumn mountains of home, I'll return forth.

cuss some philosophical reflections on geometry and group dynamic. This will help non-mathematician readers to better capture the flavour of my work. Firstly, I share the same opinion with Pierre de la Harpe [Har00]:

> *One of my personal beliefs is that fascination with symmetries and groups is one way of coping with frustrations of life's limitations: we like to recognise symmetries which allow us to recognise more than what we can see.*

But mathematicians' ambitions go beyond that. We aspire to recognise, through the symmetries readily visible to us, the more obscure symmetries that are close to them but not in general intuitively apparent. When the complexity of the geometric objects of interest becomes infinite, such as in the case of fractal structures, infinite graphs, or the infinite-type surfaces discussed in this paper, this perspective becomes particularly crucial. In such instances, the symmetries become even uncountable, and studying individual symmetries in isolation often does not contribute significantly to our overall view of the structure of the associated groups. In these cases, it becomes necessary to perceive the group, consisting of symmetries, not merely as an abstract algebraic entity but as a geometric object itself. The concept of a topological group serves this purpose; it allows us to translate the similarities between those symmetries into closeness between group elements (or, in the language of Bourbaki, *uniformity*), or even into distance, such as Polish groups that we will talk about later in this thesis. By letting these groups act continuously on topological spaces, we can understand the structure of the group as a geometric object by examining the regularities of these actions. This leads to the second element in the title of this thesis: the group dynamic.

The beauty of hyperbolic geometry and group dynamics is not just in their depth, but also in their very rich diversity: objects, viewpoints, methods, tools... When compared to the immense range of objects and possibilities within the realms of hyperbolic geometry and group dynamics, what this thesis represents is only a small part of the whole picture. Yet, it already treats convex bodies, surfaces, infinite graphs, and the coarse geometry of metric spaces. Not to mention, the tools and methods employed in the solutions to questions appeared in this thesis come from a broad spectrum of disciplines, including topology, functional analysis, probability theory, metric geometry, mathematical logic, *etc.*

The thesis is divided into three parts: "*Shoulders of Giants*", "*Farther Sight of Dwarfs*", and "*Neither Fish, Flesh, nor Fowl*".

The first part, "Shoulders of Giants", as suggested by the title, will present the context of the research and significant results already known. In Chapter 2, we will begin by introducing the concept of algebraic hyperbolic space (of infinite dimension) using the hyperboloid model. This is the most fundamental space in hyperbolic geometry. Duchesne's result [Duc23] on the Polish topology of the group of isometries of the algebraic hyperbolic

space of infinite dimension will also be mentioned as a first example (in this thesis) of measuring the similarities between transformations by distance. The kernels of hyperbolic type highlighted by [MP14; MP19; Mon20], which allow us to construct an embedding of a set into an algebraic hyperbolic space, will also be discussed. Section 2.2 will be devoted to presenting a generalisation of algebraic hyperbolic space, called *Gromov hyperbolic space*. It aims to capture hyperbolicity on a large scale. This space shares many properties with algebraic hyperbolic space: the trichotomy of isometries, dynamics of isometries on the boundary at infinity, and so on. Unfortunately, due to space limitations, we are compelled to set aside the rich theory of $\mathrm{CAT}(0)$ spaces, but this by no means implies that this infinitesimal generalisation is negligible.

In Chapter 3, we will introduce and explain several fundamental concepts and properties of the dynamics of continuous actions of topological groups on compact spaces. In Section 3.1, we will unravel the construction of the *universal minimal flow*. Among its many equivalent constructions, we have chosen a slightly more complicated topological approach. This choice is motivated mainly by several considerations: firstly, it follows the tradition of Bourbaki's *General Topology* [Bou07], since this construction is based on the concept of *uniform space* introduced by Bourbaki (in particular, André Weil), abstracting mathematically the notion of proximity while remaining consistent with geometric intuition; secondly, this construction offers us a deeper understanding of the universal minimal flow, by allowing us to link it to certain fixed point properties for topological groups, such as *amenability* and *extreme amenability*. For example, it allows us to see it as a subspace of linear functionals for bounded right uniformly continuous functions defined on the group, or more specifically, as *means* of such functions; moreover, this approach avoids some discussions of set theory in mathematical logic, although these discussions are not particularly difficult. In particular, we will be interested in amenable groups, proposed by von Neumann following his study of Banach-Tarski paradox [Neu29]. This paradox states that a 3-dimensional solid ball can be cut into a finite number of pieces and glued back together to form two balls identical to the first one, using only translations and rotations. We will list several equivalent definitions of an amenable topological group, which interact with a diversity of mathematical disciplines. We will also discuss its hereditary properties and give a similar treatment to extreme amenability, a reinforcement of amenability.

Chapter 4 will primarily delve into orientable surfaces (of finite or infinite type) and their mapping class groups. In simple terms, mapping class groups contain the most fundamental symmetries of geometric objects: within these groups, two symmetries are considered equivalent if one can continuously transform into the other (in finite time) and *vice versa*. We refer to this continuous transformation as *homotopy*. There are various methods to study mapping class groups, including algebraic approaches (via actions on

homology, on the outer automorphism group of the fundamental group of the surface, on the Torelli group by inner automorphisms, *etc.*), geometric or holomorphic approaches (via actions on the universal Teichmüller curve and on Teichmüller spaces, *etc.*), as well as topological approaches (via actions on the projectivized space of measured foliations, on the space of geodesic laminations, on the space of unmeasured foliations, *etc.*), see for example a survey in [Pap15]. Unfortunately, introducing all these methods one by one is impractical, as it would turn this thesis into extensive records on the vast contributions of William P. Thurston and his successors. Hence, we will only focus on geometric group theoretic or combinatorial approaches, *viz.* the simplicial (or isometric) actions of mapping class groups on curve graphs and other derived or similar complexes. The advantage lies in two aspects: these complexes or graphs are often Gromov hyperbolic, offering a better access to tools from hyperbolic geometry; also, this perspective allows us to view the mapping class groups as the automorphism groups of countable structures, or in other words, non-Archimedean Polish groups, which further enables us to use tools from mathematical logic and topological dynamics, especially from model theory and descriptive set theory. These can all help us to determine the amenability or extreme amenability of mapping class groups as shown later in the thesis.

The second part of the thesis, "*Farther Sight of Dwarfs*", consists of three pieces of academic writing prepared during the doctoral period, respectively [Lon23b], [Lon23a] and [LT24].

Chapter 5 will revolve around the embedding of homothetic classes of certain convex bodies in Hilbert space into an infinite-dimensional hyperbolic space over real numbers, by using the kernels of hyperbolic type. The idea can be traced back to William P. Thurston's characterisation of flat metrics on the sphere [Thu98]. In a recent paper, Debin and Fillastre [DF22] give an embedding homothetic classes of finite-dimensional convex bodies. They adopted a strategy known as "*integral geometry*". This strategy aims to associate a convex body to a unique continuous function on the unit sphere, then study the space of these functions via harmonic analysis or functional analysis, deducing conclusions about convex bodies in Euclidean space. The analysis or calculus in classical Euclidean spaces has long been polished ever since the time of Newton and Leibniz. However, as the dimension increases to infinity as in separable Hilbert spaces, diverse perspectives emerge. Deterministic traditions often lead us toward functional analysis. Yet, in the latter half of the 20th century, a stochastic analysis of infinite-dimensional spaces, such as Malliavin calculus, has been developed (see [Da 06]). In Chapter 5, aside from continuing the work left incomplete in infinite dimensions by Debin and Fillastre, an attempt is made to extend the strategy of "integral geometry" to infinite dimensions using Malliavin calculus. This involves uniquely associating each convex body with a random variable, *i.e.* stochastic sup-

port function, and then using technologies from stochastic analysis to deduce conclusions about infinite-dimensional convex bodies. One significant result in Chapter 5 is providing formulas for the intrinsic volumes of first and second degree of infinite-dimensional convex bodies using the support functions.

For a surface of finite type, the mapping class group will be a finitely generated discrete group that in most cases, contains a free group on two generators. This results in the non-amenability of this class of groups. However, as the complexity of the underlying surface goes to infinity, the mapping class groups are no longer discrete and having free subgroups does not exclude the possibility of being amenable. Although it is still unknown if mapping class groups of infinite-type surfaces (or *big mapping class groups*) are amenable, in Chapter 6, using the techniques from descriptive set theory [KPT05], we will show that mapping class groups can never be extremely amenable except for the trivial cases.

Recently, analogous to the curve graph, the notion of fine curve graph has been introduced to study the combinatorial and geometric properties of the homeomorphism group of surfaces [BHW22]. Nevertheless, these non-locally countable hyperbolic graphs show a significant difference from curve graphs in terms of their Gromov boundaries [BHW22; Bow+22; GM23b; GM23a]. Despite all these differences, in Chapter 7, we will show that for surfaces of genus at least 2, the Gromov boundary of fine curve graphs remains (path) connected like the classical curve graphs, providing a positive answer to an analogue of Peter Storm's question in [KL08].

The third part is called "*Neither Fish, Fleish, Nor Fowl*" because it attempts to present research that had not yet been fully completed at the time of writing this thesis.

In his famous work [Gro87], Mikhael Gromov stated without proof the following: *every geodesic hyperbolic space can be embedded isometrically into an ultracomplete space in an essentially surjective way*. In Chapter 8, we will prove the first part of this statement by passing to an ultralimit construction, but the second part of this statement is still unknown. At the end of Chapter 8, we will also study some families of Gromov hyperbolic spaces whose boundary at infinity is preserved under quasi-isometries.

Over the long term, a folklore theorem, *isometric actions of amenable groups on Gromov hyperbolic space cannot be of general type*, has been tested true in various situations, notably when the Gromov hyperbolic space is proper, or when the amenable group is locally compact and the action is metrically proper. But the arguments no longer hold when we do not assume any properness or compactness condition, *e.g.* big mapping class groups acting on projection complexes. In order to cope with these situations, in Chapter 9, we will use some other compactification (or bordification) for Gromov hyperbolic space than the traditional Gromov bordification. This compactification will witness the rigidity of topological amenable groups acting on Gromov hyperbolic spaces by isometries, which can further

help us to prove the non-amenability of some big mapping class groups.

Readers might have already noticed that this thesis is more about *breadth-first search* than *depth-first search* for unknowns. This indicates that the research process is far from complete. For each topic covered in this thesis, there is more than one open problem and many of them remain even unattacked. Here is an incomplete list of these questions:

- How to compute the higher degree intrinsic volumes of infinite-dimensional GB convex bodies using only their support functions?
- Are there any geometric interpretations for the orthogonal projection of support functions in n-th Wiener chaos for $n \geq 2$?
- How can one determine whether a countably infinite graph is a curve graph of an orientable surface?
- Are big mapping class groups also non-amenable?
- Which big mapping class groups admit a non-metrisable universal minimal flow?
- What is a reasonable necessary condition for the boundary of Gromov hyperbolic spaces being preserved by quasi-isometries?
- Can one embed curve graph of finite type surfaces quasi-isometrically into an algebraic hyperbolic space so that the action of mapping class group is equivariant?
- Does mapping class groups of finite type surfaces enjoy Kazhdan's property (T)? What about Haagerup property?
- Is there a geometric or dynamical interpretation of points on the Gromov boundary of fine curve graphs? If so, what is the geometric or dynamical interpretation of the topology on this boundary?
- Is the fine curve graph ultracomplete?

These open problems present exciting challenges and opportunities for future research, inviting exploration and innovation in the quest for solutions.

This is the end point of my doctoral research and education, but no doubt the very starting point of my "*Wissenschaft als Beruf*". As there is still much to be discovered, let me finish this introductory chapter with the following saying adapted from the reknown Socratic paradox:

> *Dixi, sed scio me adhuc nihil scire.*

Contents

Part I

Shoulders of Giants

Chapter 2

Hyperbolic Spaces and Their Geometry

– Elle est donc belle ?
– Belle hyperboliquement.

Thomas Corneille (1625–1709),
Le Berger Extravagant.

2.1 Algebraic hyperbolic spaces

For a number field, say $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$, *viz.* real numbers, complex numbers or quaternion numbers[1], one can construct hyperbolic space via projectivitising the Minkowski space similarly to the real cases. As Gromov commented in [Gro93, p. 121]:

> *These spaces look as cute and sexy to me as their finite dimensional siblings but they have been neglected by geometers and algebraists alike.*

Fortunately, this is no longer the case. Since last decades, the study of these infinite dimensional hyperbolic spaces becomes popular. In this section, we will present some selected results about these spaces.

2.1.1 Hyperboloid model

Let $\mathbf{K}$ be a normed number field. Let J be an index set. We define the associated ℓ^2-space by

$$\mathcal{H}^J := \left\{ (x_j)_{j\in J} \in \mathbf{K}^J : \sum_{j\in J} |x_j|^2 < \infty \right\}.$$

[1] In the sequel, the boldface symbols will mean that we tend to treat them under the context of number fields, whereas the usage of blackboard bold, such as $\mathbb{R}$ or $\mathbb{C}$, means that they are considered more from a metric space perspective.

We emphasise on the fact that J is not necessarily countable, and when it is uncountable, the summation $\sum_{j\in J}|x_j|^2$ makes sense when every but countably many x_j's are null. As usual, we also define for every $\mathbf{x}=(x_j)\in\mathcal{H}^J$ its norm by

$$\|\mathbf{x}\| := \left(\sum_{j\in J}|x_j|^2\right)^{1/2}$$

and impose for each pair $\mathbf{x},\mathbf{y}\in\mathcal{H}^J$ the inner product

$$(\mathbf{x},\mathbf{y})_{\mathcal{H}^J} := \sum_{j\in J} x_j\overline{y_j}$$

to make $\mathcal{H}^J$ a Hilbert space.

Now let us consider the vector space $\mathcal{L} := \mathbf{K}\oplus\mathcal{H}^J$ carrying the bilinear form

$$B_0\big((x_0,\mathbf{x}),(y_0,\mathbf{y})\big) = x_0y_0 - (\mathbf{x},\mathbf{y})_{\mathcal{H}^J}.$$

The associated quadratic form $B_0(x,x)$ is called a *Lorentzian quadratic form* and the space $\mathcal{L}$ is a *Minkowski space*.

Let $\mathbb{P}\mathcal{L}$ be the projective space of the Minkowski space $\mathcal{L}$, *i.e.* the quotient space of $\mathcal{L}\setminus\{0\}$ under the equivalent relation $x\sim tx$ for every $t\in\mathbf{K}\setminus\{0\}$ and every $x\in\mathcal{L}\setminus\{0\}$. The *hyperboloid model* for $|J|$*-dimensional* $\mathbf{K}$*-hyperbolic space* is given by

$$\mathbb{H}^J_{\mathbf{K}} := \{[x]\in\mathbb{P}\mathcal{L} : B_0(x,x)>0\}$$

equipped with a distance function $d_{\mathbb{H}}\colon \mathbb{H}^J_{\mathbf{K}}\times\mathbb{H}^J_{\mathbf{K}}\to[0,\infty)$ such that

$$d_{\mathbb{H}}([x],[y]) = \cosh^{-1}\left(\frac{|B_0(x,y)|}{\sqrt{|B_0(x,x)|\cdot|B_0(y,y)|}}\right) \tag{2.1}$$

for every $[x],[y]\in\mathbb{H}^J_{\mathbf{K}}$. The metric $d_{\mathbb{H}}$ from (2.1) will also be called the *hyperbolic metric*. Also, to make a distinction from Gromov hyperbolic spaces that will be introduced below, the construction $\mathbb{H}^J_{\mathbf{K}}$ will be called an *algebraic hyperbolic space*.

In the following, we will also use the notation $\mathbb{H}^\alpha_{\mathbf{K}}$ to denote an α-dimensional $\mathbf{K}$-hyperbolic space, *i.e.* α is the cardinal of J.

Now let us recall some basic notions from metric geometry. Let (X,d_X) and (Y,d_Y) be two metric spaces. A map $f\colon(X,d_X)\to(Y,d_Y)$ is called an *isometric embedding* if $d_Y\big(f(x),f(x')\big)=d_X(x,x')$ for any $x,x'\in X$. Let $I\subset\mathbb{R}$ be an interval, carrying the usual metric. An isometric embedding $\gamma: I\to(X,d_X)$ is then called a *geodesic*. Moreover,

if the interval I is bounded, then γ becomes a *geodesic segment*; if the interval I can be chosen as $[0,\infty)$, then γ is a *geodesic ray*; and if $I=\mathbb{R}$, then γ is a *geodesic line*. A metric space where every two points can be connected by a geodesic is called a *geodesic metric space*. If two points $x,y\in X$ can be joined by a geodesic segment, then we will denote by $[x,y]$ the (or anyone, if there are many of them) geodesic segment between these two points.

In fact, this metric make $\mathbb{P}\mathcal{L}$ a metric space, *i.e.* $(\mathbb{H}^I_{\mathbf{K}},d_{\mathbb{H}})$ as mentioned above. Moreover, it enjoys a stronger property than being geodesic:

Proposition 2.1.1 ([DSU17], Proposition 2.2.2)**.** *The hyperbolic metric $d_{\mathbb{H}}$ is compatible with the quotient topology induced by the projectivisation $\mathbb{P}\mathcal{L}$. Also, for any $[x],[y]\in\mathbb{H}^I_{\mathbf{K}}$, there is a unique geodesic connecting $[x]$ to $[y]$.*

Remark 2.1.2. The second half of Proposition 2.1.1 states that $(\mathbb{H}^I_{\mathbf{K}},d_{\mathbb{H}})$ is *uniquely geodesic*.

When the dimension is finite and $\mathbf{K}\in\{\mathbf{R},\mathbf{C},\mathbf{H}\}$, the algebraic hyperbolic space is a Riemannian manifold where the hyperbolic metric $d_{\mathbb{H}}$ becomes the length metric associated to a certain Riemannian metric, see [Jos02, §4.4] for the discussion when $\mathbf{K}=\mathbf{R}$. For other number fields, the construction of such Riemannian metric is of similar flavour. The same approach also works for infinite dimensional algebraic hyperbolic spaces $\mathbb{H}^\alpha_{\mathbf{K}}$, *i.e.* $\alpha\geq\aleph_0$. Reader can find information about infinite dimensional Riemannian manifolds from [Lan99]. Moreover, one should notice that the arguments from [Jos02, §4.4] do not involve dimensionality, it can soon be seen that $\mathbb{H}^\alpha_{\mathbf{R}}$ has constant sectional curvature -1.

Recall that a *proper map* between topological spaces is such that preimages of compact sets are compact. A *proper metric space* (X,d_X) if for any point $o\in X$ the function $x\mapsto d_X(o,x)$ is a proper map, or equivalently every closed ball is compact.

We wish to record the following corollary immediate from Proposition 2.1.1 to emphasize the difference between the infinite dimensional algebraic hyperbolic spaces and the finite dimensional ones:

Corollary 2.1.3. *The metric space $(\mathbb{H}^\alpha_{\mathbf{K}},d_{\mathbb{H}})$ is proper if and only if $\alpha<\aleph_0$.*

2.1.2 Isometries on algebraic hyperbolic spaces

An *isometry* is a bijective isometric embedding. For a metric space (X,d_X), the collection of all isometries $X\to X$, denoted $\mathrm{Isom}(X)$, carries a natural binary operation $(f,g)\mapsto f\circ g$ by composition that makes it into a group. This group is called the *group of isometries* of (X,d_X).

Fix $\mathbf{K}\in\{\mathbf{R},\mathbf{C},\mathbf{H}\}$ and a cardinality $\alpha>0$, let $\mathbb{H}^\alpha_{\mathbf{K}}$ be the algebraic hyperbolic space defined as above, $B_0(x,x)$ be the corresponding Lorentzian quadratic form, and $\mathcal{L}$ be the

associated Minkowski space. We define the *orthogonal group* $\mathrm{O}_{\mathbf{K}}(\mathcal{L}; B_0)$ to be the group of $\mathbf{K}$-linear automorphisms T of $\mathcal{L}$ such that $B_0(Tx, Tx) = B_0(x, x)$ for all $x \in \mathcal{L}$.

The projectivisation of $\mathrm{O}_{\mathbf{K}}(\mathcal{L}; B_0)$ will also yield a group, *i.e.* the *projective orthogonal group*, denoted $\mathrm{PO}_{\mathbf{K}}(\mathcal{L}; B_0)$, of which the elements act on the proectivisation of the Minkowski space $\mathbb{P}\mathcal{L} \simeq \mathbb{H}^{\alpha}_{\mathbf{K}}$. It is not hard to see that the action $\mathrm{PO}_{\mathbf{K}}(\mathcal{L}; B_0) \curvearrowright \mathbb{H}^{\alpha}_{\mathbf{K}}$ is by isometries. But actually we can deduce a stronger result:

Theorem 2.1.4 ([DSU17], Theorem 2.3.3)**.** *If* $\mathbf{K} \in \{\mathbf{R}, \mathbf{H}\}$, *then* $\mathrm{Isom}(\mathbb{H}^{\alpha}_{\mathbf{K}}) = \mathrm{PO}_{\mathbf{K}}(\mathcal{L}; B_0)$. *If* $\mathbf{K} = \mathbf{C}$, *then* $\mathrm{PO}_{\mathbf{K}}(\mathcal{L}; B_0) < \mathrm{Isom}(\mathbb{H}^{\alpha}_{\mathbf{K}})$ *is a subgroup of index* 2.

In the sequel, we will mainly focus on the discussion of the cases where $\mathbf{K} = \mathbf{R}$ and we write $\mathbb{H}^{\alpha}$ instead of $\mathbb{H}^{\alpha}_{\mathbf{R}}$. Also, in the stead of $\mathrm{PO}_{\mathbf{K}}(\mathcal{L}; B_0)$, we will write $\mathrm{PO}_{\mathbf{K}}(1, \alpha)$ and even $\mathrm{PO}(1, \alpha)$ whenever $\mathbf{K} = \mathbf{R}$.

Either from an algebraic perspective (matrix groups) or from the geometric perspective (groups of isometries), for any strictly positive integer n, there is a standard group embedding $\mathrm{PO}(1, n) \hookrightarrow \mathrm{PO}(1, \infty)$.

Recall that for $\mathrm{Isom}(X)$, the *pointwise convergence topology* of $\mathrm{Isom}(X)$ is the coarsest topology on the group such that the orbit map $\mathrm{Isom}(X) \to X$ given by $g \mapsto gx$ is continuous for every $x \in X$.

Proposition 2.1.5 ([Duc23], Lemma 3.2)**.** *Endowed with the pointwise convergence topology, the group* $\mathrm{PO}_{\mathbf{R}}(1, \infty)$ *is the completion of the union of standard embeddings of* $\mathrm{PO}_{\mathbf{R}}(1, \infty)$.

But the standard embedding is not the only way that $\mathrm{PO}_{\mathbf{R}}(1, n)$ for $n \in \mathbb{N}_{>0}$ can be put into $\mathrm{PO}_{\mathbf{R}}(1, \infty)$. This reveals that hyperbolicity in infinite dimensions is genuinely different from that in finite dimensions.

Theorem 2.1.6 ([MP14; MP19])**.** *Let* $n \in \mathbb{N}_{>0} \cup \{\infty\}$. *For any* $0 < t \leq 1$, *there exists a continuous irreducible representation* $\varrho_t \colon \mathrm{PO}_{\mathbf{R}}(1, n) \to \mathrm{PO}_{\mathbf{R}}(1, \infty)$ *with a* ϱ_t*-equivariant embedding* $f_t \colon \mathbb{H}^n_{\mathbf{R}} \to \mathbb{H}^{\infty}_{\mathbf{R}}$ *with* $\cosh d\big(f_t(x), f_t(y)\big) = \big(\cosh d(x, y)\big)^t$ *for all* $x, y \in \mathbb{H}^{\infty}_{\mathbf{R}}$. *Moreover, the representation* ϱ_t *is unique up to conjugacy and is standard if and only if* $t = 1$.

Remark 2.1.7. In view of the standard embedding $\mathbb{H}^n_{\mathbf{R}} \hookrightarrow \mathbb{H}^{\infty}_{\mathbf{R}}$, Theorem 2.1.6 can be restated by only considering the irreducible self-representations of $\mathrm{PO}_{\mathbf{R}}(1, \infty)$: for each parametre $0 < t \leq 1$ and each positive integer $n < \infty$, the representation $\varrho_t \colon \mathrm{PO}_{\mathbf{R}}(1, n) \to \mathrm{PO}_{\mathbf{R}}(1, \infty)$ is essentially the composition of $\varrho_t \colon \mathrm{PO}_{\mathbf{R}}(1, \infty) \to \mathrm{PO}_{\mathbf{R}}(1, \infty)$ with the standard embedding $\mathrm{PO}(1, n) \hookrightarrow \mathrm{PO}(1, \infty)$.

Remark 2.1.8. We first remark that as a quotient group of the orthogonal group $\mathrm{O}_{\mathbf{R}}(1, \infty)$, the projective orthogonal group $\mathrm{PO}_{\mathbf{R}}(1, \infty)$ also acts linearly on the Hilbert space $\mathcal{H}^J < \mathcal{L}$. Here by irreducibility of a group representation $G \to \mathrm{PO}_{\mathbf{R}}(1, \infty)$, we mean that the associated group representation $G \to \mathrm{PO}_{\mathbf{R}}(1, \infty) \to \mathrm{GL}(\mathcal{H}^J)$ is irreducible.

Remark 2.1.9. When $t \neq 1$, the representation ϱ_t is called an *exotic representation*. The nomenclature "*exotic*" is due to the fact that, unlike Karpelevich-Mostow theorem that holds for continuous isometric action on *finite dimensional* spaces, the irreducible representations ϱ_t with $t \neq 1$ does not preserve a proper totally geodesic subset of $\mathbb{H}^\infty$. Similar phenomena were also observed earlier for $\mathrm{PSL}(2, \mathbf{R})$ in [DP12] and for automorphism groups of trees in [BIM05].

Complex projective orthogonal groups $\mathrm{PO}_{\mathbf{C}}(1, \infty)$ also admits exotic representations in the following sense:

Theorem 2.1.10 ([Mon20])**.** *For any $0 < t \leq 1$, there exists a continuous irreducible representation $\varrho_t \colon \mathrm{Isom}(\mathbb{H}^\infty_{\mathbf{C}}) \to \mathrm{Isom}(\mathbb{H}^\infty_{\mathbf{C}})$ with a ϱ_t-equivariant embedding $f_t \colon \mathbb{H}^\infty_{\mathbf{C}} \to \mathbb{H}^\infty_{\mathbf{C}}$ such that $\cosh d\big(f_t(x), f_t(y)\big) = \big(\cosh d(x, y)\big)^t$ for all $x, y \in \mathbb{H}^\infty_{\mathbf{C}}$.*

Remark 2.1.11. The choice of the number field is essential and Theorem 2.1.10 no longer holds if the target group is changed into $\mathrm{PO}_{\mathbf{R}}(1, \infty)$, as observed in [Sto22].

2.1.3 Kernels of hyperbolic type

The construction of exotic representations appeared in Theorem 2.1.6 and Theorem 2.1.10 relies on the notion of *kernel of hyperbolic type*, which allows one to put a space into some hyperbolic space and further yields a canonical hyperbolic representation of the automorphism group of this kernel.

Definition 2.1.12 (Kernel of (real) hyperbolic type)**.** Given a set X, a *kernel of hyperbolic type (over $\mathbf{R}$)* on X is a function $\beta : X \times X \to \mathbf{R}$ that is symmetric, non-negative, equal to 1 on the diagonal with

$$\sum_{i,j=1}^{n} c_i c_j \beta(x_i, x_j) \leq \left(\sum_{k=1}^{n} c_k \beta(x_k, x_0) \right)^2 \tag{2.2}$$

for all $n \geq 1$, any $x_0, x_1, \ldots, x_n \in X$ and any $c_1, \ldots, c_n \in \mathbf{R}$.

Remark 2.1.13. If $n = 1$, then (2.2) implies that $\beta(x, y) \geq 1$ for every $x, y \in X$. In particular, it is easy to see that $\beta \equiv 1$ gives a trivial example of a kernel of real hyperbolic type. Moreover, it is direct from the definition that the set of kernels of hyperbolic type is closed under taking pointwise limits.

Remark 2.1.14. For the reason of conciseness, here we only present the kernel of *real* hyperbolic type. A complex version of kernel of hyperbolic type can also be designed, wherein an extra parametre, *Cartan argument*, is needed, and we refer reader to [Mon20] for details.

Kernels of positive type and of conditionally negative type are classical tools for the study of embeddings into spherical and Euclidean spaces respectively (see for example [BLV08, Appendix C] for brief introduction). Kernels of hyperbolic type is closely connected to these two notions:

Proposition 2.1.15. *The function $\beta(\cdot,\cdot)\colon X\times X\to \mathbf{R}$ satisfies (2.2) if and only if for every $z\in X$, the function $N(x,y):=\beta(x,z)\beta(y,z)-\beta(x,y)$ is a kernel of positive type, i.e. $\sum_{i,j=1}^{n} c_i c_j N(x_i,x_j)\geq 0$ for all $n\geq 1$, any $x_0,x_1,\dots,x_n\in X$ and any $c_1,\dots,c_n\in\mathbf{R}$.*

Proof. It is merely a rearrangement of terms. □

Similarly, kernels of hyperbolic type are designed for embedding sets into hyperbolic spaces.

Theorem 2.1.16 ([MP19])**.** *Let X be a non-empty set with a kernel of hyperbolic type β. Then there exists a real hyperbolic space $\mathbb{H}^{\alpha}$ for some cardinal α and a map $f\colon X\to \mathbb{H}^{\alpha}$ such that*

$$d_{\mathbb{H}}\big(f(x),f(y)\big)=\cosh^{-1}\big(\beta(x,y)\big). \tag{2.3}$$

Moreover, the space $\mathbb{H}^{\alpha}$ and the map f are unique up to a unique isometry of hyperbolic spaces.

Therefore, denoting by $\mathrm{Aut}(X,\beta)$ the group of bijections of X that preserve the kernel β, there is a canonical representation $\mathrm{Aut}(X,\beta)\to \mathrm{Isom}(\mathbb{H}^{\alpha})$ for which f is equivariant.

Remark 2.1.17. One way to understand Theorem 2.1.16 is the following. Proposition 2.1.15 implies that each kernel of hyperbolic type can be uniquely associated to a positive type kernel. By *Gelfand–Naimark–Segal construction* or simply *GNS construction*, one can build a Hilbert space via the associated kernel of positive type (we refer to [BLV08, Appendix C]), which can be further used to construct an algebraic hyperbolic space where the set carrying the hyperbolic type kernel is embedded.

2.2 Gromov hyperbolic spaces

In this section, we will present the general theory of Gromov hyperbolic spaces. These spaces can be viewed as generalisations of algebraic hyperbolic spaces. Soon after its birth, Gromov hyperbolic space has been profoundly studied by many mathematicians from different aspects. Whereas here, we will only cover a small part of this topic and will mainly focus on demonstrating the differences between proper settings and non-proper settings. We refer the reader, if interested, to classic mathematical literature such as [HG90], [BH13, §III. H], and [DSU17].

2.2.1 Definitions

Recall that the *Hausdorff distance*, denoted d_{Haus}, between two closed subsets A, B in a metric space (X, d_X) is given by

$$d_{\mathrm{Haus}}(A, B) := \sup \{r \geq 0 : A \subset \mathcal{N}_r(B) \text{ and } B \subset \mathcal{N}_r(A)\},$$

where $\mathcal{N}_r$ stands for the closed r-neighbourhood in X.

Definition 2.2.1 (*Gromov hyperbolic space* (version 1))**.** A geodesic metric space (X, d_X) is *Gromov hyperbolic*, if there exists some $\delta \geq 0$ such that for any three points $x, y, z \in X$, the geodesic segment $[x, y]$ is contained in $\mathcal{N}_\delta([x, z] \cup [y, z])$.

In other words, the *geodesic triangles* in Gromov hyperbolic space are *δ-slim* for a uniform $\delta \geq 0$. This condition is attributed to Rips by Gromov in [Gro87] and is thus also often called *Rips condition*, see for example [HG90, §1.4 Définition 27] and [DSU17, §4.3].

Although Definition 2.2.1 also serves well as a geometric intuition for Gromov hyperbolicity, it only works for geodesic spaces. If one wishes to go beyond geodesic spaces, *e.g.* discrete metric spaces, this definition will no longer be valid. For this reason, we are introducing another definition of Gromov hyperbolicity.

Let $o \in X$ be a base point. Define the function $\langle \cdot, \cdot \rangle_o \colon X \times X \to [0, \infty)$ by

$$\langle x, y \rangle_o := \frac{1}{2}\big(d_X(x, o) + d_X(y, o) - d_X(x, y)\big) \tag{2.4}$$

for every $x, y \in X$. The quantity $\langle x, y \rangle_o$ in (2.4) is called the *Gromov product of x and y based at o*, or simply the *Gromov product of x and y*. It is clear that Gromov product is symmetric and $\langle x, x \rangle_o = d_X(x, o)$.

Definition 2.2.2 (*Gromov hyperbolic space* (version 2))**.** A metric space (X, d_X) is *Gromov hyperbolic*, if there exists some $\delta \geq 0$ such that

$$\langle x, y \rangle_o \geq \min \{\langle x, z \rangle_o, \langle z, y \rangle_o\} - \delta \tag{2.5}$$

for all $x, y, z, o \in X$.

It is not hard to see that (2.5) is equivalent to the following relaxed inequality, *i.e.* a metric space (X, d_X) is Gromov hyperbolic if and only if there exists a $\delta \geq 0$ such that

$$d_X(x, y) + d_X(z, o) \leq \max \{d_X(x, z) + d_X(y, o), d_X(x, o) + d_X(y, z)\} + 2\delta \tag{2.6}$$

for all $x, y, z, o \in X$.

The equivalent conditions (2.5) and (2.6) are called the *Gromov four points condition*. When we wish to emphasise that the metric space (X, d_X) is Gromov hyperbolic for a specific $\delta \geq 0$, then we will call this space a *δ-hyperbolic space*.

For a geodesic space, this the Gromov four points condition is equivalent to the condition that every geodesic triangles are δ'-slim for some other $\delta' \geq 0$, see for example [HG90, §2.3 Proposition 21] and [BH13, §III.H, Proposition 1.22].

Remark 2.2.3. We mention in the interest of curiosity the following notion. As an analogue to the Gromov four points condition, one can generalise hyperbolicity for a metric space to *combinatorial higher-rank hyperbolicity* in the sense of [JL23]. Similar to Rips condition, for a coarsely injective metric space, having combinatorial higher-rank hyperbolicity is equivalent to enjoying a slim simplex property [JL23].

Remark 2.2.4. Although Gromov hyperbolicity can be defined for non-geodesic metric space, many interesting or well-known properties of Gromov hyperbolic spaces are shown under the context of geodesic metric spaces. But this will not cause us too much difficulty. Because we can always conveniently embed the original Gromov hyperbolic space isometrically into its *injective hull*, and this injective hull is always geodesic and Gromov hyperbolic for the same δ [Lan13, Proposition 1.3]. Therefore, we can easily extend many properties that hold in geodesic hyperbolic spaces to more general non-geodesic hyperbolic spaces.

Gromov product $\langle x, y\rangle_o$ can be interpreted geometrically as "*coarsely the distance between the base point o to any geodesic segment $[x, y]$*":

Proposition 2.2.5 ([BH13], §III.H.1)**.** *If (X, d_X) is a geodesic δ-hyperbolic space, then*

$$d(o, [x, y]) - \delta \leq \langle x, y\rangle_o \leq d(o, [x, y]), \tag{2.7}$$

for all $x, y \in X$. In particular, if $[x, y] \subset [o, x]$ or $[x, y] \subset [o, y]$, then $\langle x, y\rangle_o = d(o, [x, y])$.

Here we provide several basic examples and non-examples of Gromov hyperbolic space.

Example 2.2.6 (Bounded spaces)**.** The trivial examples of Gromov hyperbolic space are *bounded* metric spaces. Indeed, if the metric space is bounded, then the Gromov product will always be bounded. By taking a sufficiently large δ, the inequality (2.4) will always be verified.

Example 2.2.7 (Trees)**.** A *simplicial tree* is an unidirected graph where any pair of distinct vertices is connected by a unique path. If each edge in this graph is considered to have length 1, then it becomes a geodesic metric space. Moreover, any geodesic triangles in a simplicial tree is 0-slim. A simplicial tree is special case of an *$\mathbb{R}$-tree*, *i.e.* a geodesic 0-hyperbolic space.

Example 2.2.8 (Algebraic hyperbolic spaces)**.** Every algebraic hyperbolic space (over the number field **R**, **C** or **H**) is $\log(2)$-hyperbolic. In fact, as Riemannian manifold, the sectional curvature of an algebraic hyperbolic space is bounded between -4 and -1 and as a result, an algebraic hyperbolic space is $\mathrm{CAT}(-1)$, which further implies that it is $\log(2)$-hyperbolic. See [DSU17, §3.2 & §3.3] for more detailed discussion.

Example 2.2.9 (Euclidean spaces)**.** The metric space $\mathbb{R}$ is 0-hyperbolic. But $\mathbb{R}^d$ with Euclidean metric is not Gromov hyperbolic whenever $d \geq 2$: for any $\delta > 0$, one can always find a (large enough) equilateral triangle that is not δ-slim.

The following lemma provides an abundance of Gromov hyperbolic spaces and it can be shown directly from the definition:

Lemma 2.2.10 (Hierarchical property of Gromov hyperbolic space)**.** *Any subspace of a Gromov hyperbolic space is also Gromov hyperbolic. Equivalently, any non Gromov hyperbolic space cannot be isometrically embedded into a Gromov hyperbolic space.*

2.2.2 Gromov boundary

Let (X, d_X) be a Gromov hyperbolic space and let $o \in X$ be an arbitrary base point. A sequence $(x_n)_{n\geq 0}$ in δ-hyperbolic space is *Cauchy-Gromov* if $\langle x_n, x_m\rangle_o \to \infty$ as $n, m \to \infty$. We remark that being Cauchy-Gromov of a sequence does not depend on the choice of the base point $o \in X$. A Cauchy-Gromov sequence $(x_n)_{n\geq 0}$ is *equivalent* to another one $(y_m)_{m\geq 0}$, if $\langle x_n, y_m\rangle_o \to \infty$ as $n, m \to \infty$.

Definition 2.2.11 (*Gromov boundary*)**.** Let (X, d_X) be a Gromov hyperbolic space. Then we define its *Gromov boundary* by the equivalent classes of Cauchy-Gromov sequences in X and we denote it by ∂X. We say that a sequence $(x_n)_{n\geq 0}$ converges to a boundary point $\xi \in \partial X$ if it is a representative of ξ. The disjoint union $\overline{X} := X \sqcup \partial X$ is called the *Gromov bordification* of X.

Remark 2.2.12. It is easy to check that the concepts of Cauchy-Gromov sequences and their equivalence do not depend on the base point o. In particular, the Gromov boundary ∂X is independent of the choice of o.

Remark 2.2.13. In the literature, when the Gromov hyperbolic space is proper and geodesic, its Gromov boundary will also be taken as the classes of geodesic rays issued from o, equivalent up to bounded Hausdorff distance. This definition via geodesic rays is sometimes also called as "*visual boundary*" of Gromov hyperbolic space. By the virtue of Proposition 2.2.5, visual boundary is a subspace of Gromov boundary. But if the Gromov hyperbolic space is geodesic and proper, these two notions are equivalent [BH13, §III.H, Lemma 3.13].

The Gromov product can be extended to the boundary in the following way. For all $x \in X$ and $\xi \in \partial X$, we define

$$\langle x, \xi \rangle_o := \sup \liminf_{n \to \infty} \langle x, x_n \rangle_o,$$

where the supremum is taken among all sequences $(x_n)_{n \geq 0}$ converging to $\xi \in \partial X$, and also by setting for all $\eta, \xi \in \partial X$

$$\langle \eta, \xi \rangle_o := \sup \liminf_{n,m \to \infty} \langle y_m, x_n \rangle_o,$$

where the supremum is taken among all sequences $x_n \to \xi$ and $y_m \to \eta$. In particular, note that $\langle \xi, \xi \rangle_o = \infty$ for all $\xi \in \partial X$. One also remarks that for any $\xi, \eta \in \partial X$ and any two sequences $x_n \to \xi$ and $y_m \to \eta$, we have

$$\langle \xi, \eta \rangle_o - 2\delta \leq \liminf_{n,m \to \infty} \langle x_n, y_m \rangle_o \leq \langle \xi, \eta \rangle_o. \tag{2.8}$$

For further detail on this estimation, we refer to [HG90, §7.2, Remarque 8]. In particular, we remark that $\langle \xi, \xi \rangle_o = \infty$ for every $\xi \in \partial X$.

With Gromov product, the Gromov boundary can be endowed with the topology generated by the neighbourhood basis for $\xi \in \partial X$ in the form of $\{\eta \in \partial X : \langle \xi, \eta \rangle_o \geq R\}$, where $R \geq 0$. Similarly, we can define shadow of $x \in \overline{X}$ based at $o \in X$ of radius $R > 0$ by

$$\overline{\mathcal{S}}_o(x, R) := \{y \in \overline{X} : \langle y, x \rangle_o \geq R\}, \tag{2.9}$$

of which the family generates a topology for the Gromov bordification $\overline{X}$.

The following hierarchical property is deduced immediately from the definition.

Proposition 2.2.14 (Hierarchical property of Gromov boundary)**.** *Let (X, d_X) and (Y, d_Y) be two Gromov hyperbolic spaces. For any isometric embedding $f : Y \hookrightarrow X$, it can be extended continuously to the entire Gromov bordification $\overline{f} : \overline{Y} \to \overline{X}$ and $\overline{f}|_{\partial Y} : \partial Y \hookrightarrow \partial X$ is a topological embedding.*

The Gromov bordification has a quite nice topology:

Proposition 2.2.15 ([DSU17], Proposition 3.4.18)**.** *Let (X, d_X) be a Gromov hyperbolic space. Then with the topology above, the Gromov bordification $\overline{X}$ is completely metrisable. If X is in addition proper and geodesic, then the Gromov bordification $\overline{X}$ is compact and the Gromov boundary $\partial X \subset \overline{X}$ is a closed subspace and thus is also compact. If X is separable, then $\overline{X}$ and ∂X will both be separable.*

Remark 2.2.16. The compactness result of Proposition 2.2.15 will fail if the properness assumption is dropped. In the literature, when the space is proper and geodesic, the Gromov

bordification is sometimes also called "*Gromov compactification*".

Example 2.2.17 (Bounded spaces)**.** The Gromov boundary of any bounded metric space is empty: any Cauchy-Gromov sequence must be unbounded. But there exist unbounded Gromov hyperbolic space of which the Gromov boundary is empty: the space consisting of one point and issued from this point, infinitely many segments of increasing length til infinity, see Figure 2.1.

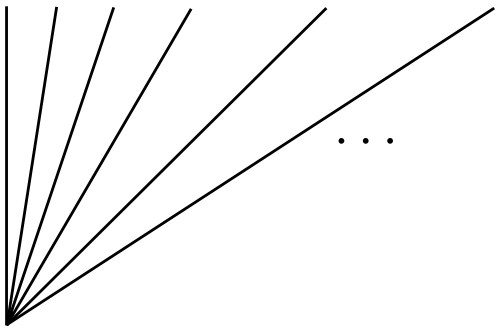

Figure 2.1: An unbounded 0-hyperbolic space with empty Gromov boundary.

Example 2.2.18 (Boundary of regular trees)**.** The Gromov boundary of $\mathbb{R}$ consists of two points, namely $\{\pm\infty\}$. The Gromov boundary of T_n $(n \geq 3)$, the regular tree of valency n, *i.e.* the tree where each vertex has exactly valency n, is homeomorphic to the Cantor set. This uses the fact that all T_n $(n \geq 3)$ are quasi-isometric and the Gromov boundary is invariant under quasi-isometries for geodesic Gromov hyperbolic spaces, see §2.2.3. Also, the Cantor set is compact.

Example 2.2.19 (Boundary of algebraic hyperbolic spaces)**.** The Gromov boundary of an algebraic hyperbolic space $\mathbb{H}^{\alpha}_{\mathbf{K}}$ is homeomorphic to the space $\{[x] \in \mathbb{P}\mathcal{L} : B_0(x,x) = 0\}$, *i.e.* the natural boundary of hyperboloid [DSU17, §3.5.1]. The same arguments as Corollary 2.1.3 show that $\partial\mathbb{H}^{\alpha}_{\mathbf{K}}$ is compact if and only if it is proper, or equivalently if and only if $\alpha < \aleph_0$.

The following property is a strengthening of being geodesic:

Definition 2.2.20 (*ultracompleteness*)**.** A Gromov hyperbolic space (X, d_X) is *ultracomplete* if for every distinct two points $x, y \in \overline{X}$, there exists a geodesic $[x, y]$.

Remark 2.2.21. A geodesic line $\gamma\colon \mathbb{R} \to X$ is said to connect $\eta^- \in \partial X$ to $\eta^+ \in \partial X$ if $\gamma(\pm n)$ converges to $\eta^{\pm}$ as Cauchy-Gromov sequences.

Remark 2.2.22. Any ultracomplete Gromov hyperbolic space is *a fortiori* a geodesic metric space, but the converse is not true: in the space $X := \{(x,y) \in \mathbb{R}^2 : 0 \leq y < 1 \text{ or } (x,y) = (0,1)\}$, the point $(0,1)$ cannot be connected to any point on the Gromov boundary by a geodesic ray, since such a geodesic must be horizontal. See Figure 2.2.

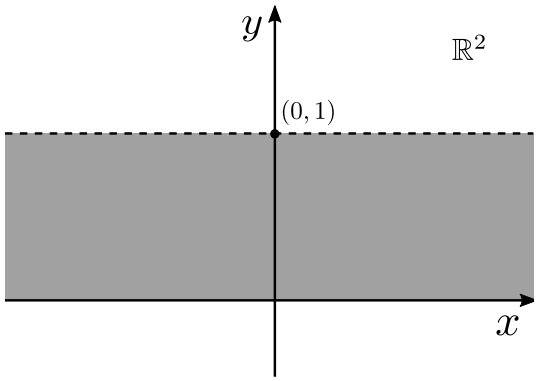

Figure 2.2: A geodesic non-proper Gromov hyperbolic space that is not ultracomplete.

Remark 2.2.23. There exists geodesic Gromov hyperbolic space that is ultracomplete but not proper: the curve graph of a hyperbolic surface [Min10, Lemma 5.14]. For the definition of such a graph, see §4.1.2.

Remark 2.2.24. Ultracompleteness property is sometimes confused with "*visibility property*", *viz.* each pair of distinct points on the Gromov boundary can be connected by a geodesic line, but ultracompleteness is *a priori* strictly stronger than visibility. Figure 2.2 also gives an example of Gromov hyperbolic space that has visibility property but is not ultracomplete. In particular, when the space is not geodesic, it can never be ultracomplete while it can still have visibility property. See in Figure 2.3 such a metric subspace in $\mathbb{R}^2$ made of infinitely many triangles and it has visibility property since the geodesic $y = 1$ connects two boundary points at infinity.

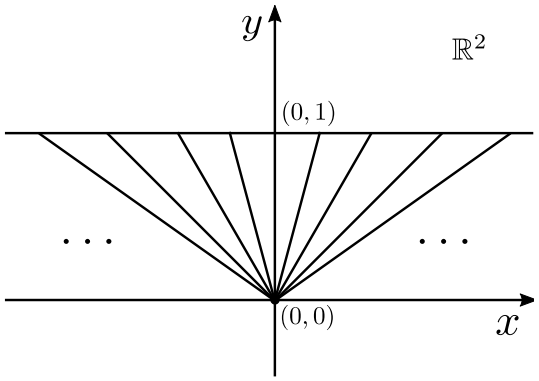

Figure 2.3: A proper (non-geodesic) Gromov hyperbolic space that is not ultracomplete.

The metric space from Figure 2.2 is not proper while that from Figure 2.3 is not geodesic. But if a Gromov hyperbolic space is both geodesic and proper, then necessarily it is ultracomplete:

Proposition 2.2.25. *If (X, d_X) is proper geodesic space that is Gromov hyperbolic, then (X, d_X) is ultracomplete.*

Proof. Under the assumptions above, for any point $x \in X$ and any boundary point $\xi \in \partial X$, there is a geodesic ray connecting x to ξ [BH13, §III.H, Lemma 3.1]. Also, the metric space (X, d_X) enjoys visibility property [BH13, §III.H, Lemma 3.2]. Moreover, it is geodesic. □

The following statement appears in [Gro87, §7.5], where Gromov claims it without proof. The correctness of this statement remains unknown; it is still in the status of conjecture:

Conjecture 2.2.26 (Gromov)**.** *Every (geodesic) Gromov hyperbolic metric space (X, d_X) can be isometrically embedded via ι into some ultracomplete space (Y, d_Y) such that $\sup_{y \in Y} d_Y(y, \iota(X)) < C < \infty$.*

If Conjecture 2.2.26 holds, then it means that for any Gromov hyperbolic space (X, d_X), one can find a larger ultracomplete space (Y, d_Y) with $\partial Y \simeq \partial X$ so that one can always assume that it is ultracomplete when discussing the dynamics of $\mathrm{Isom}(X)$.

We will show a partial affirmation to the above Gromov ultracompleteness conjecture in Chapter 8.

Finally, we consider the dynamic of isometries on a Gromov hyperbolic space.

Let X be a δ-hyperbolic geodesic space and let G acts on X by isometries. The *limit set* of G in bord$X$, denoted $\partial_X G$, is the collection of elements on ∂X that can be represented by Cauchy-Gromov sequences of the form $(g_n x)_{n \geq 0}$ for some $x \in X$ and $g_n \in G$. It is worth noticing that if $(g_n x)_{n \geq 0}$ defines an element ξ on ∂X for some $x \in X$, then one observes that for any other $y \in X$, the sequence $(g_n y)_{n \geq 0}$ is also Cauchy-Gromov and equivalent to $(g_n x)_{n \geq 0}$ as g acts by isometries. Hence the definition of the limit set $\partial_X G$ does not depend on the choice of x.

The following trichotomy for isometries on a Gromov hyperbolic space is very famous. Although some proofs of the classification suppose that the Gromov hyperbolic space is proper, see [Cap+15] and [CDP90, Chapitre 9] for example. To be precise, an isometry g on X is:

- *elliptic* if and only if $\partial_X \langle g \rangle$ is empty, or if $\langle g \rangle$ has bounded orbits;
- *parabolic* if and only if $|\partial_X \langle g \rangle| = 1$, or if $d(x, g^n x)/n \to 0$ as $n \to \infty$;
- *hyperbolic* if and only if $|\partial_X \langle g \rangle| = 2$, or if $d(x, g^n x)/n \to c > 0$ as $n \to \infty$.

Several facts about the type of isometries are listed in [CDP90, §9.1]. One basic result is that an isometry shares the same type with its powers.

Recall that in [Gro87, §3.1] Gromov gives a classification for an action of an arbitrary group Γ on a Gromov hyperbolic geodesic proper space X by isometries. Such a group action is called:

- *elementary* and
  - *bounded* if it has bounded orbits;
  - *horocyclic* if it is unbounded and has no hyperbolic elements;
  - *lineal* if it has hyperbolic elements but any two hyperbolic elements have the same endpoints;
- *non-elementary* and
  - *focal* if it has hyperbolic elements, is not lineal, and any two of its hyperbolic elements have one common endpoint;
  - *general type* if it has hyperbolic elements with no common endpoint.

But in Chapter 9, we will see how these classifications can be generalised to any Gromov hyperbolic space.

2.2.3 Quasi-isometry

The notion of *quasi-isometry* is pivotal in coarse geometry, the study of metric spaces at its large scale, which is one of the central foci of this thesis.

Definition 2.2.27 (*quasi-isometry*)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces. A (not necessarily continuous) map $f\colon X \to Y$ is a (λ, k)-*quasi-isometry embedding* for some $\lambda \geq 1$ and $k \geq 0$ if for every $x, x' \in X$,

$$\frac{1}{\lambda} d_X(x, x') - k \leq d_Y\big(f(x), f(x')\big) \leq \lambda d_X(x, x') + k.$$

If, in addition, there exists a constant $r \geq 0$ such that the r-neighbourhood $\mathcal{N}_r\big(f(X)\big) = Y$, then f is called a (λ, k)-*quasi-isometry*. Moreover, two spaces X, Y are *quasi-isometric* whenever such a map exists.

Remark 2.2.28. The condition "there exists a constant $r \geq 0$ such that the r-neighbourhood $\mathcal{N}_r\big(f(X)\big) = Y$" is to say that the image of X via f is *cobounded* in Y, or f is *essentially surjective*.

Remark 2.2.29. As a trivial remark, each isometric embedding $\iota\colon X \to Y$ is a quasi-isometric embedding. Each bi-Lipschitz map $f\colon X \to Y$, *i.e.* there exists $0 < a < A$ such that $a d_X(x, y) \leq d_Y\big(f(x), f(y)\big) \leq A d_X(x, y)$ for all $x, y \in X$, is also a quasi-isometric embedding.

Remark 2.2.30. It is not hard to check that being quasi-isometric forms an equivalent relation among metric spaces. Indeed, for any (λ, k)-quasi-isometry $f\colon X \to Y$, one can construct a (λ', k')-quasi-isometry $f'\colon Y \to X$ and a constant $C > 0$ such that $d_Y\big(f \circ f'(y), y\big) \leq C$ and $d_X\big(f' \circ f(x), x\big) \leq C$ for all $x \in X$ and all $y \in Y$. Such f' will be called a *quasi-inverse* of f. Also, the composition of a (λ, k)-quasi-isometry and a (λ', k')-quasi-isometry is a $(\lambda\lambda', \lambda' k + k')$-quasi-isometry.

Remark 2.2.31. For a metric space (X, d_X), the *quasi-isometry group*, denoted $\mathcal{QI}(X)$, consists of equivalent classes of quasi-isometries $X \to X$ up to bounded distance of images, *i.e.* two quasi-isometries f and g are equivalent if $\sup_{x\in X} d_X\big(f(x), g(x)\big) < \infty$. This group only depends on the quasi-isometry type of metric spaces, *i.e.* a quasi-isometry $X \to Y$ will induce a group isomorphism between $\mathcal{QI}(X) \to \mathcal{QI}(Y)$. Since each isometry is a quasi-isometry, there is a natural group homomorphism $\mathrm{Isom}(X) \to \mathcal{QI}(X)$, but this homomorphism is in general *not* an isomorphism: isometry groups are not invariant under quasi-isometries.

Example 2.2.32. It is an trivial example that all bounded metric spaces are quasi-isometric to each other and in particular, they are all quasi-isometric to a singleton. Conversely, any metric space that is quasi-isometric to a point must be of finite diameter.

Example 2.2.33. Let T_n be the regular tree of valency n. For any distinct $n, m \geq 3$, the trees T_n and T_m are quasi-isometric. To show this, it suffices to prove that T_3 is quasi-isometric to T_n for any $n \geq 4$. Denote by d_3 and d_n the metric on T_3 and T_n respectively. Label the edges of T_3 by $\{1, 2, 3\}$ so that no edges of the same label are adjacent. By collapsing to a point every path in T_3 of length $n-3$ consisting of edges sequentially labelled $1, 2, 3, 1, 2, \ldots$, we define a map $f\colon T_3 \to T_n$ that is a $(n-2, 1)$-quasi-isometry. See Figure 2.4 for an example.

Example 2.2.34 (Cayley graph)**.** Let G be a finitely generated group. Let S be a finite generating sets in G. The *Cayley graph* of G with respect to S, denoted $\mathrm{Cay}(G, S)$, is a graph where vertices are elements in G and g is connected to h by an edge if and only if $g^{-1}h \in S$. It is a geodesic proper space if one endows it with the simplicial metric d_S. We note that d_S is exactly the *word metric* associated to S, *i.e.* the distance $d_S(g, h)$ is the shortest word over alphabet S representing $g^{-1}h$. If S and S' are two distinct finite generating sets in G, then there exists a constant $\lambda := \max\big(\max_{s'\in S'} d_S(s'), \max_{x\in S} d_{S'}(s)\big)$, *i.e.* the length of the longest word in the dictionary of translation, such that the map $\mathrm{Id}\colon \mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S')$ by $g \mapsto g$ for every $g \in G$ is a $(\lambda, 0)$-quasi-isometry. This means that the Cayley graphs of a finitely generated group has a well-defined quasi-isometric type.

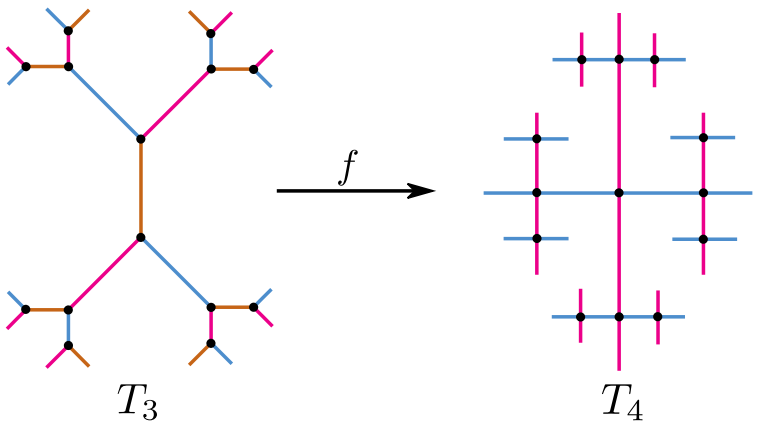

Figure 2.4: A $(2,1)$-quasi-isometry from T_3 to T_4.

Definition 2.2.35 (*quasi-geodesic*)**.** A *quasi-geodesic* in a metric space (X, d_X) is a quasi-isometric embedding $I \to X$, where $I \subset \mathbb{R}$ is an interval.

Quasi-geodesics can also be used to characterise Gromov hyperbolicity for geodesic metric spaces: a geodesic δ-hyperbolic space is where all quasi-geodesic triangles are M-slim ([BH13, §III.H, Corollary 1.8]). This relies on the following fact, referred as *Morse Lemma*:

Proposition 2.2.36. *Let X be a δ-hyperbolic space. There exists a function $M : [1,\infty) \times [0,\infty) \to [0,\infty)$ depending on δ so that given any pair $(\lambda, k) \in [1,\infty) \times [0,\infty)$ and any two points $x, y \in X$, all (λ, k)-quasi-geodesics connecting x to y are within Hausdorff distance $M(\lambda, k)$ of each other.*

Remark 2.2.37. Different from [BH13, §III.H, Theorem 1.7], we do *not* assume that X is geodesic. In fact, by the virtue of [Lan99], one can isometrically embed X into its injective hull, denoted $\mathrm{Inj}(X)$, which is a geodesic metric space that is hyperbolic with the same δ. Moreover, any quasi-geodesic in X remains quasi-geodesic in $\mathrm{Inj}(X)$. So we can conclude Proposition 2.2.36 by applying the geodesic version of Morse Lemma [BH13, §III.H, Theorem 1.7] to $\mathrm{Inj}(X)$.

Remark 2.2.38. Since a geodesic is always a (λ, k)-quasi-geodesic for any λ and k, Morse Lemma also means that in (geodesic) Gromov hyperbolic spaces, quasi-geodesics stay closed to geodesics, whenever they exist. But this is utterly wrong in general cases: the spiral in $\mathbb{R}^2$ parametrised as $t \to (t, \log(1+t))$ in *polar coordinates* is a quasi-geodesic ray but it stays arbitrarily far away from any geodesic ray in $\mathbb{R}^2$.

Remark 2.2.39. The function $M : [1,\infty) \times [0,\infty) \to [0,\infty)$ is called the *Morse function* and for fixed constant λ and k, the value $M(\lambda, k)$ is called the *Morse constant*. In particular, the

Hausdorff distance between the two sides of a geodesic bigon in a δ-hyperbolic space is at most a multiple of δ (see [HG90, §7.1, Corollaire 3]).

Combining Proposition 2.2.36 and the fact that quasi-geodesic triangles are slim in Gromov hyperbolic spaces, we can soon conclude the following important fact:

Theorem 2.2.40 ([BH13], §III.H, Theorem 1.9)**.** *Let (X, d_X) and (Y, d_Y) be geodesic metric spaces and let $f : Y \to X$ be a quasi-isometric embedding. If X is Gromov hyperbolic, then so is Y (with a different $\delta \geq 0$).*

Remark 2.2.41. Theorem 2.2.40 states that among *geodesic metric spaces*, Gromov hyperbolicity is invariant under quasi-isometries. But the geodesic assumption cannot be dropped. The graph $\{(x, y) \in \mathbb{R}^2 : y = |x|\}$ as a subspace of $\mathbb{R}^2$, is not a geodesic metric space but it is quasi-isometric to a 0-hyperbolic space $\mathbb{R}$, whereas it is not Gromov hyperbolic as $\mathbb{R}$ is.

Moreover, we have the following quantitative estimation called *quasi-invariance of Gromov product*

Proposition 2.2.42 ([HG90], §5.2, Proposition 15)**.** *Let (X, d_X) and (Y, d_Y) be δ-hyperbolic geodesic spaces and let $f : Y \to X$ be a (λ, K)-quasi-isometric embedding. Then there exists a constant C, depending only on δ, λ and K, such that*

$$\frac{1}{\lambda}\langle x, y\rangle_z - C \leq \langle f(x), f(y)\rangle_{f(z)} \leq \lambda\langle x, y\rangle_z + C$$

We have seen in Remark 2.2.31 that the quasi-isometry group is invariant under quasi-isometries and in Theorem 2.2.40 that Gromov hyperbolocity is also invariant when the space is geodesic. But quasi-isometries have the following much stronger property:

Theorem 2.2.43 ([BH13], §III.H, Theorem 3.9)**.** *Let $f : X \to Y$ be a quasi-isometric embedding between proper geodesic metric spaces. Assume that X and Y are Gromov hyperbolic. Then it induces a topological embedding $f_\partial : \partial X \to \partial Y$. If f is a quasi-isometry, then $\partial X \simeq \partial Y$.*

Remark 2.2.44. Theorem 2.2.43 is the classical case of the folklore theorem: quasi-isometric Gromov hyperbolic spaces can be basically treated as the same space. In Chapter 9, we will give a proof for non proper length spaces as an application of ultralimit. Another (more quantitative) proof using visual metric can be found in [Väi05, Theorem 5.35]. The use of Arzelá-Ascoli Theorem is not a necessity and similar geometric arguments also prove the same result in the setting of geodesic non proper spaces, see [CDP90, Chapitre 3, Théorème 2.2].

Remark 2.2.45. If the space is not Gromov hyperbolic but only CAT(0), then one can define *contracting boundary*, which behaves similarly to Gromov boundary, and endows it with topology generated by shadows, but this boundary is not invariant under quasi-isometries [Cas16].

Chapter 3

Analysis on Topological Groups

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3.1 Universal minimal flow

The *universal minimal flow* of a topological group G is a canonical compact topological space on which G acts continuously. It is of great interest in topological dynamics and is closely connected to various analytic properties of topological groups. For some important non-locally compact topological groups that appear in geometry, combinatorics and logic, this canonical flow can be computed explicitly and has meaningful interpretation under the corresponding contexts. In this section, we will give a brief survey on this notion.

We remark that this section aims at treating a general theory for topological groups that are not necessarily Hausdorff, which might be handy for the study of topological dynamics of algebraic groups.

3.1.1 Uniform spaces

Uniform spaces provide a natural setting for studying topological dynamics and analysis on groups. It can be viewed as a generalisation of metric spaces and in particular, every metric space is a uniform space defined in the following:

Definition 3.1.1 (uniform space)**.** A *uniform space* is a pair $(X,\mathcal{U})$, consisting of a set X and a *uniform structure* $\mathcal{U} \subset \mathcal{P}(X \times X)$, where the elements $E \in \mathcal{U}$ is called an *entourage* satisfying the following conditions:

[1] Translation from classical Chinese: *By analysing the zigzags, everything is evidently clear.*

(U1) the family $\mathcal{U}$ is closed under finite intersection and supsets, *i.e.* a *filter*;
(U2) any entourage contains the *diagonal* $\Delta = \{(x,x) : x \in X\}$;
(U3) if $E \in \mathcal{U}$, then there exists $F \in \mathcal{U}$ such that $F \subset E^{-1} := \{(x,y) \in X \times X : (y,x) \in E\}$;
(U4) if $E \in \mathcal{U}$, then there exists $F \in \mathcal{U}$ such that $F \circ F := \{(x,z) \in X \times X : \exists y \in X \text{ such that } (x,y),(y,z) \in F\} \subset E$.

Moreover, the uniform structure $\mathcal{U}$ is *separated* if

(U5) the intersection of all entourages is the diagonal.

In the sequel, we will write $\mathcal{U}(X)$ for the uniform structure on a given uniform space X whenever it is understood.

Definition 3.1.2 (*uniform continuity*)**.** A map $f\colon X \to Y$ between uniform spaces is *uniformly continuous* if for any entourage $V \in \mathcal{U}(Y)$, there is an entourage $E \in \mathcal{U}(X)$ such that $(f \times f)(E) := \{(f(x), f(y)) \in Y \times Y : (x,y) \in E\} \subset V$.

The simplest examples of uniformly continuous maps could be the uniformly continuous functions on metric spaces. Moreover, we remark that the uniform spaces and uniformly continuous mappings form a category in a very natural way.

Let X be a uniform space. We say that a point y is *E-closed to* x if the pair (x,y) is contained in the entourage $E \in \mathcal{U}(X)$. Each uniform space is naturally a topological space: the *uniform topology* in which a neighbourhood base of a point $x \in X$ consists of the sets of points that are E-closed to x as E runs though the entourages of X. In what follows, we denote by $E[x]$ the points in X that are E-closed to x. Note that a separated uniform structure induces necessarily a Hausdorff uniform topology.

It is worth remarking that not all topological spaces admit a uniform structure. In fact, the topological spaces that can carry an associated uniform structure described as above are exactly the completely regular ones (see for example [Jam13, Proposition 11.5]). Such a topological space is called a *uniformisable space*. If we embed the category of uniformisable spaces into the category of topological spaces, then the category of the uniform spaces will become a sub-category of uniformisable spaces.

Here we give several important examples of uniformisable spaces:

Example 3.1.3 (Discrete countable space)**.** For a uniformisable space, there could exist several uniform structure on it that can induce the same uniform topology. For example, the space $\mathbb{Z}$ with discrete topology can be induced by the discrete uniform structure, *i.e.* the uniform structure consisting of all subsets of $X \times X$ containing the diagonal Δ, and also by the uniform structure generated by sets of the form $\{(x,y) : x = y \text{ or } x,y \geq n\}$. These uniform structures are said *compatible* to the topology.

Example 3.1.4 (Compact spaces)**.** An important example of uniformisable space is the compact Hausdorff spaces. The result is known as *uniformisation theorem of compact space* (see

[Jam13, Proposition 8.20] or [Bou07, TG II §4, Théorème 1]) and it states that there is a unique compatible uniform structure on every compact Hausdorff space consisting of all neighbourhoods of the diagonal as entourages.

Example 3.1.5 (Topological groups)**.** Let G be a topological group. There are two standard uniform structures on G. The *right uniform structure* (respectively *left uniform structure*) on a topological group G consists of entourages E_U containing all the pairs (g, h) such that $gh^{-1} \in U$ (respectively $g^{-1}h \in U$) for a neighbourhood U of the identity element in G. We denote the above uniform structure by $\mathcal{U}_{\mathrm{r}}(G)$ (respectively $\mathcal{U}_{\mathrm{l}}(G)$). If in addition, the topology on G is Hausdorff, then both $\mathcal{U}_{\mathrm{r}}(G)$ and $\mathcal{U}_{\mathrm{l}}(G)$ are separated. But in general, the left uniform structure and the right uniform structure of a topological group do not coincide.

At the categorical level, we can define the *sum* (or *coproduct*), *product*, *quotient* and *subobject* for objects. For uniformisable spaces, the notions of sum, product and subobject are consistent in the larger category of topological spaces. Namely, the uniform topology induced by the uniform structure defined on the disjoint union, Cartesian product and subspace of uniform spaces are respectively the disjoint union topology, Tychonoff topology and the subspace topology. See [Isb64] or [Jam13] for example. We will define the product of uniform spaces in the next paragraph.

Nevertheless, believing that quotient uniform structure will induced quotient topology is "*horribly false*", as stated in [Isb64]. In fact, for every non-normal uniformisable spaces X equipped with the finest compatible uniform structure, one can always construct a quotient map $f : X \to Y$ of uniform spaces so that the quotient topology on Y induced by f is not uniformisable (see [Isb64]), *e.g.* the Moore plane.

3.1.2 Completion and compactification

Recall that a complete metric space is a compact topological space if and only if it is totally bounded. We have similar characterisation of compactness for uniform spaces.

A uniform space X is *compact* if the uniform topology on X induced by $\mathcal{U}(X)$ is compact, *i.e.* every open covering of X admits a finite sub-covering.

Similarly, we can define the *completeness* of a uniform space. First, let us introduce the following notion:

Definition 3.1.6 (*Cauchy filter*)**.** A *Cauchy filter* over a uniform space X is a filter $\mathcal{F} \subset \mathcal{P}(X)$ (see condition (U1)) if for every $E \in \mathcal{U}(X)$, there exists $A \in \mathcal{F}$ such that $A \times A \in E$.

Recall that a filter $\mathcal{F}$ over a uniform space X *converges* to a point $x \in X$ if for every $E \in \mathcal{U}(X)$, there exists $A \in \mathcal{F}$ such that $A \subset E[x]$, or more general, a filter converges to a

point in a topological space if every neighbourhood of this point is contained in the filter. We can immediately conclude the following proposition from the definitions, of which we exhibit the proof as an example:

Proposition 3.1.7. *Let $\mathcal{F}$ be a filter over a uniform space X. If $\mathcal{F}$ converges to some point $x \in X$, then $\mathcal{F}$ is a Cauchy filter.*

Proof. For any entourage $E \in \mathcal{U}(X)$, one can seek a symmetric entourage by E' such that $E' \circ E' \subset E$. Since $\mathcal{F}$ converges to x, there exists $A \in \mathcal{F}$ such that $A \times A \subset E'[x] \times E'[x] \subset E' \circ E' \subset E$. This shows that $\mathcal{F}$ is Cauchy. □

But the converse is not true: any non-complete metric space admits non-convergent Cauchy sequence and thus a non-convergent Cauchy filter. So we define the following:

Definition 3.1.8 (*completeness*)**.** A uniform space X is *complete* if every Cauchy filter converges to some point in X.

There is a useful connection between completeness and compactness in the class of uniform spaces:

Definition 3.1.9 (*total boundedness*)**.** A uniform structure $\mathcal{U}$ over a set X is *totally bounded* if for any entourage $E \in \mathcal{U}$, there exists a finite cover $\{U_1, \ldots, U_n\}$ such that $U_k \times U_k \subset E$ for all $1 \leq k \leq n$. A uniform space X is totally bounded if $\mathcal{U}(X)$ is totally bounded.

The following result is a generalisation of Heine-Borel Theorem in the uniform space settings and the proof is a straightforward adaption of the proof for metric space cases. A proof in categorical language is available in [Isb64, §II.28].

Theorem 3.1.10 (Heine-Borel)**.** *A uniform space is compact if and only if it is complete and totally bounded.*

It somehow responses to the principle proposed by Isbell [Isb64, §I.14]:

> *As far as a single entourage is concerned*[2]*, all uniform spaces are like metric spaces.*

Further pursue of this principle allows us to associate the *completion* of a uniform space to *means* and finally leads to analytic properties on topological groups.

Let us first take a glimpse at the following result, which completely establishes the connection between metric spaces and uniform spaces. The proof is constructive but not difficult to follow and is given in [Isb64, §I.14]. Here we only state the theorem by reformulating it in the language of entourages:

[2] In the original text, it says: "*As far as a single coverings are concerned, all uniform spaces are like metric spaces.*" Since the definition of uniform spaces via entourages is used here instead of that via uniform coverings, we make a small modification.

Theorem 3.1.11. *Let X be a uniform space. Then for every entourage $E \in \mathcal{U}(X)$, there exists a metric space (M, d_M) and a uniformly continuous map $f\colon X \to M$ such that for every bounded subset $B \subset M$ with diameter $\operatorname{diam}(B) < 1$, we have $f^{-1}(B) \times f^{-1}(B) \subset E$.*

Given a uniform space X, we say that an entourage $E \in \mathcal{U}(X)$ is *realised* by a (uniformly continuous) map $f\colon X \to Y$ if $f^{-1}(F[y]) \times f^{-1}(F[y]) \subset E$ for some $F \in \mathcal{U}(Y)$ and every $y \in Y$. Hence Theorem 3.1.11 says that for any uniform space X, every entourage $E \in \mathcal{U}(X)$ is realised by a uniformly continuous map into a metric space.

Recall that the product of uniform spaces Y_α, also denoted $\prod_\alpha Y_\alpha$, is a uniform space where the uniform structure is generated by all entourages in the form of

$$\left\{ \big((x_\alpha),(y_\alpha)\big) \in \left(\prod_\alpha Y_\alpha\right) \times \left(\prod_\alpha Y_\alpha\right) : (x_{\alpha_i}, y_{\alpha_i}) \in V_i \in \mathcal{U}\big(Y_{\alpha_i}\big),\ i = 1, \dots, n \right\},$$

where $n \in \mathbb{N}$. The product uniform structure is the coarsest uniform structure such that the canonical projection $\iota_\alpha\colon \prod_\alpha Y_\alpha \twoheadrightarrow Y_\alpha$ via $(y_\alpha) \mapsto y_\alpha$ is uniformly continuous for each α. This definition is similar to the definition of Tychonoff topology on the product of topological spaces.

Proposition 3.1.12 (cf. §II.18, [Isb64])**.** *Let X be a separated uniform space and let $f_\alpha\colon X \to Y_\alpha$ be a family of uniformly continuous maps that realises every entourage in $\mathcal{U}(X)$. Let $\iota_\alpha\colon \prod_\alpha Y_\alpha \twoheadrightarrow Y_\alpha$ be the canonical projection. Then there is a uniform embedding $f\colon X \to \prod_\alpha Y_\alpha$ such that $\iota_\alpha \circ f = f_\alpha$ for every α.*

Proof. Define $f(x) \coloneqq (f_\alpha(x))_\alpha \in \prod_\alpha Y_\alpha$. It is not hard to verify that f is uniformly continuous. Moreover, since X is separated, for any distinct $x \neq x' \in X$, there is a symmetric entourage $E \in \mathcal{U}(X)$ such that $(x, x') \notin E$. As the family f_α realises every entourage in $\mathcal{U}(X)$, there exists α such that $x' \notin E[x] \supset f_\alpha^{-1}\big(F[f_\alpha(x)]\big)$ for some $F \in \mathcal{U}(Y_\alpha)$, which implies that $f_\alpha(x) \neq f_\alpha(x')$. Hence f is a monomorphism. Again, using the fact that f_α realises every entourage in $\mathcal{U}(X)$, one can easily show that if $f(X) \subset \prod_\alpha Y_\alpha$ is equipped with subspace uniform structure, the map f^{-1} is uniformly continuous and hence f is an isomorphism. □

Remark 3.1.13. The condition of realising entourages is not necessary for f to be a uniform embedding. Moreover, the product is always defined in the set-theoretic settings: the *class* of indices can indeed be taken as a *set* which is of cardinal at most $|\mathcal{U}(X)| \leq |\mathcal{P}(X \times X)|$.

Remark 3.1.14. In general, if the uniform space is not separated, then the uniformly continuous map $f\colon X \to \prod_\alpha Y_\alpha$ above is no longer necessarily a uniform embedding: $f(X)$ can be isomorphic to a uniform space between X and its *separated replica*, *i.e.* the quotient uniform space X/R, where $R = \bigcap\{E : E \in \mathcal{U}(X)\}$. To be precisely, if X_s is the separated replica of X, then we can show that $X \twoheadrightarrow f(X) \twoheadrightarrow X_s$ for some uniformly continuous maps.

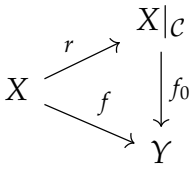
A *completion* of a uniform space X is the complete uniform space in which X is densely embedded via uniformly continuous map.

The following theorem is deduced from Theorem 3.1.11 and Proposition 3.1.12:

Theorem 3.1.15 (cf. §II, [Isb64])**.** *Every separated uniform space can be embedded into a product of complete metric spaces and thus up to isomorphism, it has a unique completion.*

Proof. Theorem 3.1.11 yields a family of uniformly continuous map $f_\alpha \colon X \to M_\alpha$ that realises every entourage $E \in \mathcal{U}(X)$. We may assume M_α to be complete by taking its completion. Proposition 3.1.12 tells us that there exists a uniform embedding $f \colon X \hookrightarrow \prod_\alpha M_\alpha$. Note that $\prod_\alpha M_\alpha$ is also complete. The closure of $f(X)$ in $\prod_\alpha M_\alpha$ is a completion of X. Moreover, as $f(X) \simeq X$ and $f(X)$ is a subspace of the complete uniform space $\overline{f(X)}$, it is not hard to see that any uniformly continuous map $X \to Y$ into a complete uniform space Y has a unique uniformly continuous extension over $\overline{f(X)}$. Hence any completion of X must be isomorphic to $\overline{f(X)}$. □

Definition 3.1.16 (*reflection*)**.** Let X be a uniform space and $\mathcal{C}$ be a subcategory of uniform spaces. The *reflection* of X in $\mathcal{C}$, denoted $X|_{\mathcal{C}}$, is a uniform space in $\mathcal{C}$ together with a uniformly continuous map $r \colon X \to X|_{\mathcal{C}}$ such that for every uniformly continuous map $f \colon X \to Y$ with $Y \in \mathcal{C}$, there is a unique $f_0 \colon X|_{\mathcal{C}} \to Y$ such that $f = f_0 \circ r$, *i.e.* the following diagram commutes:



Remark 3.1.17. The second half of the proof of Theorem 3.1.15 actually shows that for separated uniform spaces, the completion is a reflection in complete separated uniform spaces. In general, a reflection if exists, is always unique. Moreover, the uniformly continuous reflection map $r \colon X|_{\mathcal{C}} \to (X|_{\mathcal{C}})\big|_{\mathcal{C}}$ is equivalent to the identity map of $X|_{\mathcal{C}}$. Moreover, the reflection can be considered as a universal object for X in the subcategory $\mathcal{C}$.

Remark 3.1.18. The *separated replica* of a uniform space X is the reflection of X in the category of separated uniform spaces, because every equivalent classes in the separated replica must be sent to the same point under a uniformly continuous map into a separated uniform space.

Remark 3.1.19. It is easy to see that the reflection of a reflection is still a reflection.

Let $\mathrm{UCB}(X)$ be the space of *bounded uniformly continuous functions* on X. Equipped with the norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$, it is a Banach space. By Banach-Alaoglu theorem, carrying the weak-* topology, the unit ball in $\mathrm{UCB}(X)^*$ is compact and thus totally bounded

in view of Heine-Borel theorem 3.1.10. It is not difficult to verify that the map $x \mapsto \delta_x$, where $\delta_x(f) \coloneqq f(x)$ for all $f \in \mathrm{UCB}(X)$, is a uniformly continuous map. Hence for every uniform space X, there is always at least one uniformly continuous map from X to a totally bounded separated uniform space.

Now adopting a similar idea of Stone-Čech compactification, we can constructing totally bounded reflection of uniform spaces:

Theorem 3.1.20. *Every separated uniform space X admits a reflection in the totally bounded separated uniform spaces.*

Proof. Let X be a separated uniform space and let f_α be the class of all uniformly continuous map $X \to Y_\alpha$ to some totally bounded separated uniform space Y_α. So

$$\left\{E \in \mathcal{U}(X) : E = \left(f^{-1} \times f^{-1}\right)(F),\ \exists F \in \mathcal{U}(Y_\alpha)\right\}$$

is non-empty and forms a coarser uniform structure $\mathcal{U}^*(X)$ on X than the original $\mathcal{U}(X)$. Moreover, by definition there is a family with cardinal at most $|\mathcal{U}(X)|$ of uniformly continuous maps that realises $\mathcal{U}^*(X)$. Hence by Proposition 3.1.12, $(X,\mathcal{U}^*(X))$ is isomorphic to a subspace of a product of totally bounded spaces, which is again totally bounded. The uniform space $(X,\mathcal{U}^*(X))$ is the desired reflection. □

Now we are ready to introduce the following notion:

Definition 3.1.21 (*Samuel compactification*)**.** For every uniform space X, the *Samuel compactification* of X is denoted $\mathcal{S}(X)$ and is defined by the completion of the totally bounded separated reflection of its separated replica.

Remark 3.1.22. In the light of Remark 3.1.19, the Samuel compactification of a uniform space is is a reflection in compact separated uniform spaces and thus is unique up to isomorphism. Moreover, by its construction, there is a uniformly continuous monomorphism $X \hookrightarrow \mathcal{S}(X)$, of which the image is dense.

Remark 3.1.23. If X is a discrete uniform space, *i.e.* the every the supset in $X \times X$ of the diagonal is an entourage, then its uniform topology is also discrete. As a discrete topological space X, the *Stone-Čech compactification* βX can be identified with the compact space of all ultrafilters on X (see for example [HS11, Theorem 3.27]). Moreover, the canonical identification $\iota\colon X \to \beta X$ has dense image. As a result, both βX and $\mathcal{S}(X)$ are a completion of its totally bounded reflection $(X,\mathcal{U}^*(X))$, so by Theorem 3.1.15, they are isomorphic.

Finally, we shall establish the connection between Samuel compactification and means.

Definition 3.1.24 (*mean*)**.** Let V be a Banach space of functions. A *mean* μ over V is linear functional $\mu \in V^*$ such that $\mu(f) \geq 0$ for all $f \geq 0$ and $\mu(1) = 1$. The collection of means

over V is denoted $\mathfrak{M}(V)$. If in addition $\mu \in \mathfrak{M}(V)$ is such that $\mu(fh) = \mu(f)\mu(h)$ for all $f, h \in V$, then μ is said *multiplicative*.

Remark 3.1.25. It can be easily deduce from the definition that for all $f \in V$ and $\mu \in \mathfrak{M}(V)$, we have $\inf(f) \leq \mu(f) \leq \sup(f)$. If V is in addition a Banach function space carrying with the norm $\|\cdot\|_\infty$, then $\mathfrak{M}(V)$ is a closed subset in the unit ball of V^* and is compact under the weak-* topology by Banach-Alaoglu theorem.

If we write $\mathfrak{M}_m(V)$ the space of all multiplicative means on V, then for a compact space, we have the following identification:

Lemma 3.1.26. *If X is a separated compact uniform space (or equivalently, a compact Hausdorff space), then $\mathfrak{M}_m\big(\mathrm{UCB}(X)\big) \simeq X$.*

Proof. By Riesz-Markov-Kakutani representation theorem, a mean $\mu \in \mathfrak{M}_m\big(\mathrm{UCB}(X)\big)$ can be viewed as a probability measure on X with the multiplicative condition

$$\int_X fh \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu \int_X h \,\mathrm{d}\mu,$$

for all $f, h \in \mathrm{UCB}(X)$. We claim that the support of μ is necessarily a singleton and thus a Dirac measure. If x, y are two distinct points in the support of μ, by Urysohn's lemma, we can find two strictly positive functions $f, h \in \mathrm{UCB}(X)$ with disjoint supports and $f(x) = h(y) = 1$. But this will imply that

$$0 = \int_X fh \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu \int_X h \,\mathrm{d}\mu > 0,$$

which is a contradiction. Hence $\delta \colon X \to \mathfrak{M}_m(\mathrm{UCB}(X))$ via $x \mapsto \delta_x$ is a continuous bijection and thus an isomorphism between separated compact spaces. □

The following characterisation is adopted as one definition of Samuel compactification in [Pes06], while the original construction in [Sam48] uses ultrafilters and does not involve means.

Theorem 3.1.27. *Let X be a uniform space. Then its Samuel compactification $\mathcal{S}(X)$ is isomorphic to $\mathfrak{M}_m(\mathrm{UCB}(X))$, i.e. the space of all multiplicative means on* $\mathrm{UCB}(X)$.

Proof. Since $\mathcal{S}(X)$ is a reflection of X in separated compact uniform spaces, every function $f \in \mathrm{UCB}(X)$ will induce a unique function $\phi_f \in \mathrm{UCB}(\mathcal{S}(X))$. Conversely, any function in $\mathrm{UCB}(\mathcal{S}(X))$ will yield a function in $\mathrm{UCB}(X)$ by restricting it onto the image of the reflection map. Hence $\mathrm{UCB}(X) = \mathrm{UCB}(\mathcal{S}(X))$. By Lemma 3.1.26, we have $\mathcal{S}(X) \simeq \mathfrak{M}_m(\mathrm{UCB}(\mathcal{S}(X))) = \mathfrak{M}_m(\mathrm{UCB}(X))$. □

3.1.3 Minimal flows

The classical references of this topic are [Ell69; Pes06] and what we are presenting here is slightly different from their descriptions.

Let us start with a definition:

Definition 3.1.28 (*minimal flow*)**.** Let G be a topological group. A compact Hausdorff topological space X is a *G-flow* if G acts on X continuously. A G-flow is *minimal* if it is its only subflow.

Remark 3.1.29. Here we insist that being Hausdorff is necessary for a proper definition of G-flows, for the purpose that it has a well-defined uniform structure (see [Jam13, Proposition 8.20] or [Bou07, Théorème 1, TG II §4]). However, we do not require that G is Hausdorff.

It is clear from Zorn's Lemma that every G-flow contains at least one minimal G-flow, but there might be several of them. It is easy to verify that a G-flow X is minimal if and only if for any $x \in X$, the orbit Gx is dense in X.

Let G be a topological group, we define the following notion:

Definition 3.1.30 (*bounded right uniformly continuous function*)**.** A *bounded right uniformly continuous function* $f\colon G \to \mathbb{R}$ is a uniformly continuous map with respect to the right uniform structure on the group G such that $\sup_{g\in G} |f(g)| < \infty$. The space of all right uniformly continuous bounded function is denoted $\mathrm{RUCB}(G)$.

By Theorem 3.1.27, we can see that the Samuel compactification $\mathcal{S}(G)$ for the right uniform structure $\mathcal{U}_r(G)$ of a topological group G is isomorphic to $\mathfrak{M}_m(\mathrm{RUCB}(G))$. Hence there is a natural continuous G-action on $\mathcal{S}(G)$ by setting $\lambda_g(f)\colon h \mapsto f(g^{-1}h)$ and

$$(g\mu)(f) := \mu(\lambda_g(f))$$

for every $\mu \in \mathfrak{M}_m(\mathrm{RUCB}(G)) \simeq \mathcal{S}(G)$, every $f \in \mathrm{RUCB}(G)$ and every $g, h \in G$.

Let X be a G-flow such that the orbit of $o \in X$ is dense in X. Then the orbit map $\rho_o\colon g \mapsto go$ is a uniformly continuous map. Since $\mathcal{S}(G)$ is a reflection, it will yield a unique uniformly continuous G-equivariant map $\mathcal{S}(G) \to X$ such that $\delta_e \mapsto o$, where $e \in G$ is the identity element. As a result, for every point $a \in \mathcal{S}(G)$, there is a G-equivariant uniformly continuous map $R_a\colon \mathcal{S}(G) \to \overline{Ga} \subset \mathcal{S}(G)$ with $\delta_e \mapsto a$. Now define the continuous *right translation* on $\mathcal{S}(G)$ by

$$\mathcal{S}(G) \times \mathcal{S}(G) \ni (x, y) \mapsto xy := R_y(x) \in \mathcal{S}(G),$$

which makes $\mathcal{S}(G)$ into a compact *left monoid*. In fact, we have $R_{ab} = R_b R_a$, $R_{\delta_e} = \mathrm{Id}$,

$ga = R_a(\delta_g)$ for all $a, b \in \mathcal{S}(G)$ and all $g \in G$. In other words, the uniformly continuous map R_a is G-equivariant for all $a \in \mathcal{S}(G)$.

Moreover, every minimal subflow of $\mathcal{S}(G)$ is also a compact *left semitopological semigroup*: since G leaves a subflow M invariant, so will all right translations by elements in $M \subset \mathcal{S}(G)$.

Theorem 3.1.31 (Ellis)**.** *Every non-empty compact left semitopological semigroup contains a idempotent.*

Proof. Let S be such a semigroup. By Zorn's Lemma, there is at least a minimal closed non-empty subsemigroup, denoted T. We claim that this semigroup T only contains idempotents (and thus is trivial). Let $a \in T$. Then Ta is also a closed subsemigroup of T and by minimality, $Ta = T$. But this implies that the closed subsemigroup $\{t \in T : ta = a\}$ is not empty and has to be the entire T, which implies that $a^2 = a$. □

Now we prove the following result:

Theorem 3.1.32. *Let G be a topological group. Then for any pair of minimal G-subflows $M, M' \subset \mathcal{S}(G)$, there is a G-equivariant homeomorphism $M \to M'$.*

Proof. Since M and M' are both compact left semitopological semigroup, they contain idempotents $p \in M$ and $p' \in M'$ respectively. We claim that $R_{p'}R_p$ is the desired homeomorphism.

First, we remark that by definition $R_p \colon \mathcal{S}(G) \to M$ and $R_{p'} \colon \mathcal{S}(G) \to M'$. By minimality of M', we can conclude that $Mpp' = M'$. To show that it is a homeomorphism, it suffices to prove that it is injective, or it admits an inverse map $M' \to M$.

Note that by the minimality of M, we have $Mp = M$, so for any $x \in M$, there exists some $y \in M$ such that $x = yp$. Hence $xp = yp^2 = yp = x$.

By the same arguments, we have $M'p = M$. Let $b := pp'p \in M$. Similarly, by the minimality of M, we can conclude by minimality that $Mb = M$. Hence there exists $c \in M$ such that $cb = p$. It is not hard to see that R_pR_c is a (right-)inverse of $R_{p'}R_p$. Indeed, for any $x \in M$, we compute $R_pR_{p'}R_pR_c(x) = xcpp'p = xcb = xp = x$. This completes the proof. □

Remark 3.1.33. If M is a minimal G-subflow in $\mathcal{S}(G)$ and $f \colon M \to M$ is G-equivariant continuous map, then f must be a G-isomorphism. In fact, the last sentences of Theorem 3.1.32 means that R_pR_c is a right-inverse of f, where $c \in M$ is such that $cf(p) = p$.

A minimal G-subflow in $\mathcal{S}(G)$ is universal in the following sense:

Proposition 3.1.34. *A minimal G-flow M is G-isomorphic to a minimal flow $M \subset \mathcal{S}(G)$ if and only if for any G-flow X, there is a continuous G-equivariant map $f \colon M' \to X$. If X is in addition minimal, then f is onto.*

Proof. If X is minimal, then by minimality, f must be surjective whenever it exists. The necessity of the statement is clear since $\mathcal{S}(G)$ is a reflection of G in the category of compact uniform spaces. For the sufficiency, let $f\colon M' \to M$ by assumption and $h\colon \mathcal{S}(G) \to M'$, then $f \circ h|_M\colon M \to M$ is G-equivariant and thus a G-isomorphism. This forces f and h G-isomorphisms. □

Theorem 3.1.32 hence leads to the following notion:

Definition 3.1.35 (*universal minimal flow*)**.** A G-flow is called *universal* if it is G-isomorphic to a minimal G-subflow in $\mathcal{S}(G)$. Such a flow is denoted $\mathcal{M}(G)$.

Remark 3.1.36. The universal minimal flow of G can also be viewed as a product space of minimal G-flow. Indeed, the totally bounded separated reflection can be embedded into a product of totally bounded separated uniform spaces Y_α with a uniformly continuous map $G \to Y_\alpha$ (see the proof of Theorem 3.1.20). By the uniqueness of completion, the Samuel compfactification $\mathcal{S}(G)$ must be embedded into a product of compact spaces $\overline{Y_\alpha}$ and the map $G \to Y_\alpha$ will then yield a G-action on $\overline{Y_\alpha}$. So the G-action on $\mathcal{S}(G)$ can be factored into diagonal G-action on the components.

For non-Hausdorff topological groups, this definition remains valid. It suffices to notice the following fact:

Proposition 3.1.37. *The separated replica $\widehat{G}$ of a topological group G is still a topological group and $\mathcal{M}(\widehat{G}) \simeq \mathcal{M}(G)$.*

Proof. Let $e \in G$ be the identity elements and $\mathcal{N}(e)$ be the collection of all neighbourhoods of e. By definition, $\widehat{G} = G/\sim$, where $g' \sim g$ if and only if $g'g^{-1} \in \bigcap\{V : V \in \mathcal{N}(e)\}$. To show the result, it suffices to show that $\widehat{G}$ is a group, which can be reduced to proving that $[g][h] = [gh]$ is well-defined. Let $g' \sim g$ and $h' \sim h$. We need to show that $g'h'h^{-1}g^{-1} \in \bigcap\{V : V \in \mathcal{N}(e)\}$. Indeed, for any $V \in \mathcal{N}(e)$, by continuity of the multiplications, there is a $U \in \mathcal{N}(e)$ such that $U^2 \subset V$ and a $W \in \mathcal{N}(e)$ with $gWg^{-1} \subset U$. Since $g' \sim g$, $g'g^{-1} \in U$, which implies that $g' \in Ug$. Also, we have $hh^{-1} \in W$. So

$$g'h'h^{-1}g^{-1} \in Ugh'h^{-1}g \in UgWg^{-1} \in U^2 \subset V.$$

This proves that $\widehat{G}$ is a group. By definition, $\mathcal{S}(G)$ is $\mathcal{S}(\widehat{G})$. Note that any uniformly continuous map $f\colon G \to X$ into a separated uniform space must satisfy $f(g) = f(h)$ whenever $g \sim h$. Hence each G-flow is a well-defined $\widehat{G}$-flow and *vice versa*. So their minimal subflows must be the same in $\mathcal{S}(G) = \mathcal{S}(\widehat{G})$. □

The universal minimal flow of a topological group is usually difficult to compute, but here we give several examples.

Example 3.1.38 (trivial topology)**.** If G is carrying the trivial topology, then Proposition says that $\mathcal{M}(G)$ is actually a singleton since its separated replica $\widehat{G}$ is the trivial group.

Example 3.1.39 (compact groups)**.** For a compact Hausdorff group G, its universal minimal flow is itself. Indeed, in this case $\mathcal{S}(G) = G$ and G acts on itself transitively. If it is not Hausdorff, then its universal minimal flow is its separated replica $\widehat{G}$.

Example 3.1.40 (Homeomorphism group of the circle)**.** Let $\mathrm{Homeo}^+(S^1)$ be the group consisting of all orientation-preserving homeomorphismsof the circle, equipped with the compact-open topology. Then its universal minimal flow is S^1 itself [Pes98].

Example 3.1.41 (Infinite symmetric group)**.** The infinite symmetric group S_∞ over a countable set ω can be equipped with pointwise convergence topology. Its universal minimal flow is the compact space of all linear orderings on ω [GW02].

3.2 Amenability

There is a nice survey paper on this topic [GH17] and one may also refer to [Pes06] for connections between amenability and minimal flows.

3.2.1 Amenable topological groups

Let G be a topological group that is not necessarily locally compact nor Hausdorff. Recall that a probability measure on a topological space is a positive regular Borel measure with total measure 1. If G acts on X continuously and μ is a probability measure on X, then μ is *G-invariant* if the *pushforward measure* $g_*\mu := \mu \circ g^{-1}$ coincides with μ for every $g \in G$.

Definition 3.2.1 (*amenability*)**.** A topological group G is *amenable* if it admits a G-invariant probability measure on every G-flow.

Remark 3.2.2. This definition depends on the topology and we remark that for a fixed group G, if it is amenable with a given group topology (G, τ), then for any coarser group topology $\tau' > \tau$, the topological group (G, τ') will remain amenable. On one hand, every group is amenable if it is equipped with the trivial topology: any continuous action on a G-flow will be the trivial action and every Dirac measure on the G-flow is G-invariant. On the other hand, if a group G is amenable for its discrete topology, then it is amenable for every group topology. Such a group will be called a *discretely amenable group*.

Before proceeding the other characterisations of amenability of a topological group, we need to recall some notions from functional analysis.

Let S be a locally convex topological vector space and let $K \subset S$ be a convex compact subset. We say that G acts on K continuously and *affinely* if the G-action on K is continuous and satisfies

$$g\big(\lambda x + (1-\lambda)y\big) = \lambda gx + (1-\lambda)gy$$

for all $\lambda \in [0,1]$, $x, y \in K$ and $g \in G$. Such K will be called an *affine convex compact G-space*.

Let S be a locally convex topological vector space and $K \subset S$ be a convex compact subset. For each probability measure μ on K, there exists a unique point b_μ in K such that $f(b_\mu) = \int_K f \,\mathrm{d}\mu$ for all linear forms $f \in S^*$. This point is called the *barycenter* of μ. See [Phe01, Proposition 1.1].

Finally, for a real Banach space E, we denote by E^* its dual and by E_1^* the unit ball in E^* with respect to the operator norm. By Banach-Alaoglu theorem, the unit ball E_1^* is compact if it is equipped with the weak-$*$-topology. Moreover, given a strongly continuous representation $G \to \mathrm{Isom}(E)$, it will naturally induces a continuous affine action on E_1^*.

Theorem 3.2.3. *Let G be a topological group. Then the following are equivalent:*
(A1) G is amenable;
(A2) G admits a G-invariant probability measure on its universal minimal flow $\mathcal{M}(G)$;
(A3) G admits a G-invariant probability measure on its Samuel compactification $\mathcal{S}(G)$;
(A4) Every affine convex compact G-space has a G-fixed point;
(A5) G admits a G-left-invariant mean on $\mathrm{RUCB}(G)$.

Proof. The implications (A1)⇒(A2)⇒(A3) are trivial. For (A3)⇒(A4), since a affine convex compact G-space K is a G-flow, there will be a G-equivariant continuous map $f\colon \mathcal{S}(G) \to K$. If μ is a G-invariant probability measure on $\mathcal{S}(G)$, then the pushforward measure $f_*\mu$ will be a G-invariant probability measure on K, which makes its barycenter G-invariant. For (A4)⇒(A5), it suffices to notice that the means $\mathfrak{M}\big(\mathrm{RUCB}(G)\big) \subset \mathrm{RUCB}(G)^*$ is an affine convex compact G-space. For (A5)⇒(A1), if X is a G-flow and $m \in \mathfrak{M}\big(\mathrm{RUCB}(G)\big)$ is G-invariant, then one can define $F_f \in \mathrm{RUCB}(G)$ by $g \mapsto \int_X f(gx)\,\mathrm{d}\nu$ for some fixed probability measure ν on X so that the linear form $P_m\colon \mathcal{C}(X) \to \mathbb{R}$ given by $f \mapsto m(F_f)$ will yield a G-invariant probability measure on X by Riesz theorem. □

Remark 3.2.4. If the topological group G is locally compact, then being amenable is equivalent to having an invariant mean on $L^\infty(G)$, the essentially bounded functions with respect to the Haar measure on G.

Recall that a group G is a *directed union* of a family of subgroups $(H_\alpha)_{\alpha\in A}$ if $G = \bigcup_{\alpha\in A} H_\alpha$ and for each $\alpha_1, \alpha_2 \in A$, there is $\alpha_3 \in A$ such that $H_{\alpha_1} \cup H_{\alpha_2} \subset H_{\alpha_3}$. We remark that H_α will be carrying the subspace topology inherited from the group topology on G.

The class of amenable groups enjoys the following hereditary properties.

Proposition 3.2.5. *Let G be a topological group.*

(HA1) If G is amenable, then very open subgroup $H < G$ is amenable;

(HA2) If G is a directed union of a family of amenable subgroups $(H_\alpha)_{\alpha\in A}$, then G is amenable;

(HA3) If G has an amenable closed normal subgroup N such that the quotient G/N is also amenable, then G is amenable;

(HA4) If $H \to G$ is a continuous homomorphism with dense image and if H is amenable, then G is amenable;

(HA5) A dense subgroup $H < G$ is amenable if and only if G is amenable;

(HA6) G is amenable if and only if its separated replica $\widehat{G}$ is amenable.

Suppose that G is in addition locally compact.

(HA7) If G is amenable, then every subgroup of G is amenable.

On the proof. The proof of (HA1)-(HA4) is classical and can be found in [Ric67, §4]. For (HA5) and (HA6), it suffices to notice that a dense subgroup H and the separated replica $\widehat{G}$ share the same space of bounded right uniformly continuous functions as G. For (HA7), it is also classical that any closed subgroup in a locally compact amenable group remains amenable (see for example [Zim13, §4.2]), so every subgroup of a locally compact amenable group is amenable since its closure is amenable. □

Example 3.2.6 (Compact groups)**.** Every compact group is amenable because the normalised Haar measure serves as an invariant mean on the right uniformly continuous functions. In particular, every finite group is discretely amenable.

Example 3.2.7 (Abelian groups)**.** The group $\mathbb{Z}$ is discretely amenable: the accumulation point in $\mathfrak{M}\big(\mathrm{RUCB}(\mathbb{Z})\big)$ of $\mu_n := \sum_{i=1}^{n} \delta_i/n$ is a $\mathbb{Z}$-invariant mean. Every finitely generated abelian group is also discretely amenable because it is a product of finite copies of $\mathbb{Z}$ and finite groups. Every abelian group is discretely amenable because it is the directed union of finitely generated subgroups in it. This result is also called Markov-Kakutani theorem.

Example 3.2.8 (Free groups)**.** Let $n \geq 2$ and F_n be the free group generated by n elements. Then F_n are not discretely amenable since it admits no invariant finitely additive probability measure on itself, which further implies that it has no invariant mean $\mathrm{RUCB}(F_n)$ for the discrete topology. But as subgroups of compact group $\mathrm{SO}(3)$, the free subgroups F_n are amenable with the induced (Hausdorff) topology.

Example 3.2.9 (Countable symmetric group)**.** Let $\mathcal{S}_\infty$ be the group of all permutation on $\mathbb{N}$. Endow with the pointwise convergence topology, it becomes a non-locally compact topological group. It is amenable: the group of finitely supported permutations is a dense subgroup in $\mathcal{S}_\infty$ and is the directed union of $\mathcal{S}_n$ for $n \geq 1$. But $\mathcal{S}_\infty$ is not a discretely amenable group because by identifying $\mathbb{N}$ with vertices in the Cayley graph $\mathrm{Cay}(F_2)$, it

will yield a surjective group homomorphism $\mathcal{S}_\infty \to F_2$. The discrete amenability of closed subgroups in $\mathcal{S}_\infty$ is related to structural Ramsey theory [Moo13].

Example 3.2.10 (Unitary group)**.** Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. Let $\mathrm{U}(\mathcal{H})$ be its unitary group. Equipped with the strong operator topology, the group $\mathrm{U}(\mathcal{H})$ is amenable as it contains a dense subgroup being a directed union of compact groups $\mathrm{U}(n)$. But it contains every countable group as a discrete subgroup [GH17, Lemma 5.1], so (HA1) cannot be replaced by *closed* subgroups.

3.2.2 Extremely amenable groups

Now we pass to a stronger property than amenability:

Definition 3.2.11 (*extreme amenability*)**.** A topological group G is *extremely amenable* if it admits a fixed point on every G-flow.

Remark 3.2.12. Similarly, extreme amenability depends on the topology and any extremely amenable topological group remains extremely amenable for any coarser group topology. In particular, if G is extremely amenable, then it is amenable.

Theorem 3.2.13. *Let G be a topological group. Then the following are equivalent:*
(E1) G is extremely amenable;
(E2) the universal minimal flow $\mathcal{M}(G)$ reduces to a singleton;
(E3) G has a fixed point on its Samuel compactificatin $\mathcal{S}(G)$;
(E4) G admits a G-left-invariant multiplicative mean on $\mathrm{RUCB}(G)$.

Proof. In light of Theorem 3.1.27, these conditions are all equivalent. □

Remark 3.2.14. However, it is worth noticing that, other than the trivial group, no locally compact groups are extremely amenable [Vee77], thus *a fortiori* no discrete ones [Ell60].

The class of extremely amenable groups enjoys the following hereditary properties.

Proposition 3.2.15. *Let G be a topological group.*
(HE1) If G is extremely amenable, then very open subgroup $H < G$ is extremely amenable;
(HE2) If G is a directed union of a family of extremely amenable subgroups $(H_\alpha)_{\alpha\in A}$, then G is extremely amenable;
(HE3) If G has an extremely amenable closed normal subgroup N such that the quotient G/N is also extremely amenable, then G is amenable;
(HE4) If $H \to G$ is a continuous homomorphism with dense image and if H is extremely amenable, then G is extremely amenable;
(HE5) A dense subgroup $H < G$ is extremely amenable if and only if G is extremely amenable;

(HE6) G is extremely amenable if and only if its separated replica $\widehat{G}$ is extremely amenable.

Proof. The proofs of (HA1) and (HA3) use the existence of an invariant mean, which also works if the mean is multiplicative, so one can conclude (HE1) and (HE3). The proofs of (HA2) and (HA4) use the fixed point criterion for convex compact spaces, which can be adapted to any compact spaces, hence we can also conclude (HE2) and (HE4). (HE5) and (HE6) can easily deduced as (HA5) and (HA6). □

Example 3.2.16 (Unitary group)**.** Let $\mathcal{H}$ be a separable complex Hilbert space of infinite dimension and let $\mathrm{U}(\mathcal{H})$ be its unitary group. Equipped with the strong operator topology, the group $\mathrm{U}(\mathcal{H})$ is extremely amenable. See for example [Pes06, Section 2.2] for detail.

Example 3.2.17. The group $\mathrm{Homeo}^+([0,1])$ of orientation-preserving homeomorphisms is extremely amenable if it is equipped with compact-open topology. See [Pes98] again for example.

Example 3.2.18. The group $\mathrm{Aut}(\mathbb{Q},\leq)$ of order-preserving automorphisms on $\mathbb{Q}$ is a closed subgroup of $\mathcal{S}_\infty$ with pointwise convergence topology and is extremely amenable. See [Pes98] for example. But $\mathcal{S}_\infty$ itself is not extremely amenable because it does not preserve an order on $\mathbb{N}$, see below.

Example 3.2.19 (Non-archimedean Polish groups)**.** More general, with the pointwise convergence topology, a closed subgroup $G < \mathcal{S}_\infty$ (or equivalently a *non-archimedean Polish groups*) is extremely amenable if and only if G preserves an ordering and has Ramsey property [KPT05].

Chapter 4

Mapping Classes on Surfaces

Semper in finem determinatur res.[1]

Christine de Pizan (1364– circa 1430),
Le Livre de Paix.

4.1 Orientable surfaces

A *surface* is a locally compact secondly countable 2-dimensional orientable real manifold and up to homeomorphism, it is uniquely determined by its genus and the space of its planar and non-planar ends.

4.1.1 Classification of surfaces

Let S be a surface above. Unless otherwise mentioned, all surfaces here below are assumed to be connected.

The following notion is introduced under more general settings by [Fre31] and here we give a simplified version. Let S be a surface. If it contains non-empty boundary component, then we identify it temporarily with its interior, *i.e.* $S \setminus \partial S$. Then we define the following notion:

Definition 4.1.1 (*end*)**.** The *ends* of surface S are the equivalent classe of proper rays, denoted by $\mathrm{End}(S)$, where two proper rays α and β are equivalent if for any compact subset $K \subset S$, there exists $R > 0$ such that $\alpha\big([R, \infty)\big)$ and $\beta\big([R, \infty)\big)$ are contained in the same connected component in $S \setminus K$.

[1] Translation from Latin: *A thing is always determined in its end.*

The space $S \sqcup \mathrm{End}(S)$ is a topological space and the induced topology on S is its original topology. If $p \in \mathrm{End}(S)$ and a proper ray α is a representative of p, then a basis of neighbourhood of p consists of the connected component of $C \subset S \setminus K$ for compact subset $K \subset S$, together with $\mathrm{End}(C)$, that contains $\alpha\big([R, \infty)\big)$ for some large $R > 0$.

An end $p \in \mathrm{End}(S)$ is *planar* if there exists an open neighbourhood U of p in $S \sqcup \mathrm{End}(S)$ such that $S \cap U$ as a subsurface is of genus 0. Otherwise, an end will be *non-planar*, of which the collection is denoted by $\mathrm{End}_\infty(S)$. Note that $\mathrm{End}_\infty(S)$ is a subset of $\mathrm{End}(S)$ and both $\mathrm{End}_\infty(S)$ and $\mathrm{End}(S)$ are carrying the induced topology from the compactification $S \sqcup \mathrm{End}(S)$ described as above.

Remark 4.1.2. We remark that an accumulation point $p \in \mathrm{End}(S)$ in the end space is not necessarily a non-planar end, even though each neighbourhood of p in S has infinitely many ends as subsurface. There exist zero-genus surfaces where all ends are planar but of which the end space has accumulation point. See Example 4.1.7 and Example 4.1.9.

For a planar end, there are several types. A *puncture* in particular is an end such that it admits a neighbourhood in S that is homeomorphic to a once-punctured disc, *i.e.* $\{z \in \mathbb{C} : 0 < |z| < 1\}$. An end with *boundary component* is an end such that it admits a neighbourhood in S that is homeomorphic to an annulus. There are also other type of planar end, see Example 4.1.9 for instance.

The following result gives a complete classifications for orientable surfaces as we mention just above:

Theorem 4.1.3 (Richards [Ric63])**.** *Let S and S' be two orientable surfaces. There exists a homeomorphism $S \to S'$ if and only if the genus $g(S) = g(S')$, the end spaces $\mathrm{End}(S) \simeq \mathrm{End}(S')$ and $\mathrm{End}_\infty(S) \simeq \mathrm{End}_\infty(S')$ are homeomorphic respectively.*

We define the *complexity* of an orientable surface S by $\xi(S) := 3g(S) + |\mathrm{End}(S)| - 3$. An orientable surface is of *finite type* if $\xi(S) < \infty$, otherwise it will be an *infinite-type surface*.

Moreover, there is a complete description of end space of an orientable surface, see [AS15, Chapter 1] and [Ric63] for example. In brief, any nested pair of closed subsets of a Cantor set can be realised as the end spaces of an orientable surface and *vice versa*:

Theorem 4.1.4. *Let X, Y be two closed subset of a Cantor set with $Y \subset X$. Then there exists a unique (up to homeomorphism) orientable surface S such that $\mathrm{End}(S) \simeq X$ and $\mathrm{End}_\infty(S) \simeq Y$. Conversely, for any orientable surface S, the spaces $\mathrm{End}_\infty(S) \subset \mathrm{End}(S)$ are homeomorphic to closed subsets of a Cantor set.*

The following are some important examples of infinite-type surfaces:

Example 4.1.5 (The Loch Ness monster surface)**.** It is the infinite-genus surface with exactly one end (which is necessarily non-planar). See Figure 4.1. The unique non-planar end is shown on Figure 4.1 as the infinity at right-hand side.

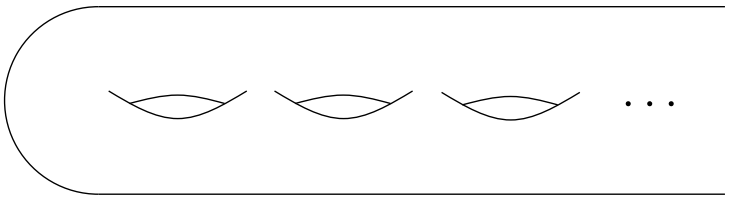


Figure 4.1: The Loch Ness monster surface.

Example 4.1.6 (Jacob's ladder surface)**.** It is the infinite-genus surface with exactly two ends and both ends are non-planar. See Figure 4.2. The two non-planar ends are visualised in Figure 4.2 respectively as the infinity at left-hand side and right-hand side.

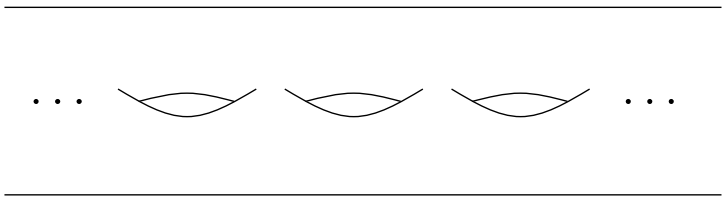


Figure 4.2: Jacob's ladder surface.

Example 4.1.7 (The Cantor tree surface)**.** It is the zero-genus surface with end space homeomorphic to a Cantor set. This surface is homeomorphic to a sphere with a Cantor set removed. See Figure 4.3. This surface is also called an *arboreal surface* as per [GLU22]: equipped with a hyperbolic metric, it is quasi-isometric to a simplicial tree and there is a cobounded isometric embedding of simplicial tree in the surface. By Richards' classification Theorem 4.1.3, every arboreal surface associated to an regular n-valence tree with

$n \geq 3$ are homeomorphic. In general, the arboreal surfaces associated to a tree whose ends is homeomorphic to a Cantor set are all homeomorphic.

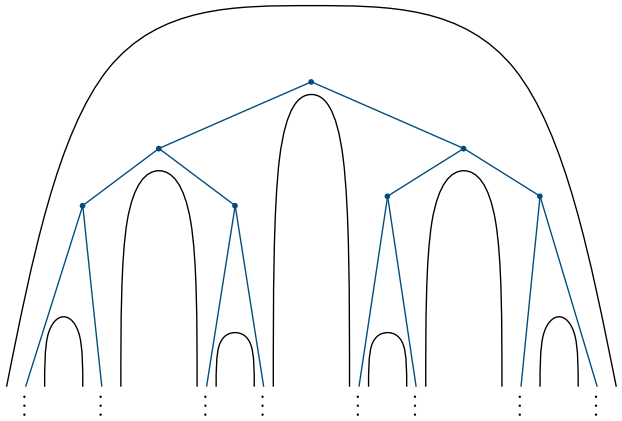


Figure 4.3: The Cantor tree surface and a cobounded isometrically embedded Cantor tree.

Example 4.1.8 (The blooming Cantor tree surface)**.** It is the infinite-genus surface whose end space is homeomorphic to a Cantor set and such that every end is non-planar. See Figure 4.4.

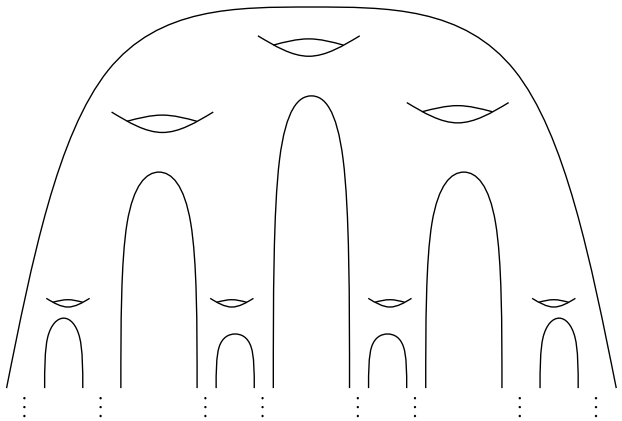


Figure 4.4: The blooming Cantor tree surface.

Example 4.1.9 (The flute surface)**.** The zero-genus surface whose end space has a unique accumulation point, *i.e.* is homeomorphic to $\{0\} \cup \{1/n : n \in \mathbb{N}_{>0}\}$ viewed as subset of $\mathbb{R}$. This surface is homeomorphic to $\mathbb{R}^2 \setminus \mathbb{Z}$. See Figure 4.5. The unique accumulation point in the end space is visualised as the infinity at right-hand side in Figure 4.5.

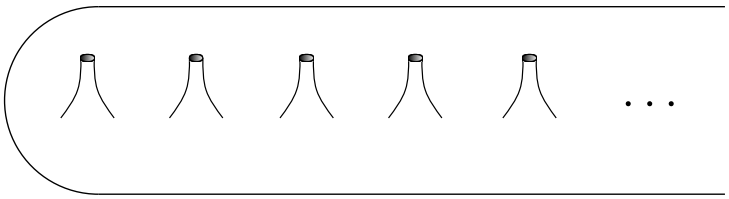


Figure 4.5: The flute surface.

4.1.2 Simple closed curves and their graph

A *simple closed curve* α is a proper topological embedding of the circle S^1 into an orientable surface S. In most of time, we identify a simple closed curve with its image. A simple closed curve $\alpha \subset S$ is *separating* if $S \setminus \alpha$ is not connected, otherwise the curve will be *non-separating*. Moreover, a simple closed curve α is *non-essential* if it is separating and one connected component of $S \setminus \alpha$ is a topological disc, annulus or a once-punctured topological disc, otherwise α is an *essential* simple closed curve.

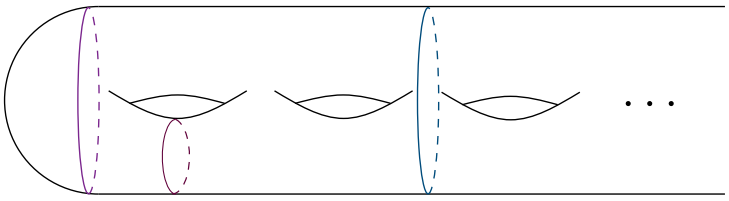


Figure 4.6: Example of non-essential, non-separating and separating curves (from left to right) on the Loch Ness monster surface.

Recall that an *isotopy* is a continuous map $\Phi\colon [0,1] \times Y \to X$ such that for every $t \in [0,1]$,

the map $\Phi(t,\cdot)\colon Y \to X$ is a topological embedding. An isotopy is a special case of a *homotopy*, where we only require Φ to be continuous. In particular, a simple closed curve on a surface S is not essential if and only if it is homotopic to a singleton, a boundary component of S or a puncture.

We say that two simple closed curves α and β on a surface S are *isotopic* if there exists an isotopy $\Phi\colon [0,1]\times S^1 \to S$ such that $\Phi(0,\cdot)=\alpha$ and $\Phi(1,\cdot)=\beta$. We remark that being isotopic is an equivalence relation among essential simple closed curves on a surface S.

The collection of isotopic classes of essential simple closed curves on an orientable surface S is denoted by $\mathcal{C}(S)$. We say that two isotopic classes of essential simple closed curves are *disjoint* if they admit representatives that are disjoint. We can also define a simplicial metric for isotopic classes $\mathcal{C}(S)$. Indeed, we can construct a graph where the vertices are isotopic classes $\mathcal{C}(S)$ of essential simple closed curves on S and two vertices is connected by an edge if they are disjoint. The distance between two isotopic classes $[\alpha]$ and $[\beta]$ is defined as the number of edges of the shortest path connecting $[\alpha]$ to $[\beta]$. This metric graph is called *the curve graph* of S.

One important geometric property of curve graph is the following:

Theorem 4.1.10 ([MM99])**.** *Let S be an orientable surface of finite type. If $\xi(S) > 1$, then the curve graph $\mathcal{C}(S)$ is δ-hyperbolic.*

Remark 4.1.11. If $g(S)=0$ and $|\operatorname{End}(S)|\leq 2$, then there is no essential simple closed curve on S, *i.e.* $\mathcal{C}(S)$ is empty. If $g(S)=0$ and $|\operatorname{End}(S)|=3,4$, or if $g(S)=1$ and $|\operatorname{End}(S)|\leq 1$, then every pair of non-isotopic essential simple closed curves cannot be disjoint, so the curve graph will not be connected. If the surface is of infinite type, then the graph $\mathcal{C}(S)$ is bounded.

In general, the curve graph is not proper (even if the surface is of finite type), but it is still possible to connect an point on its Gromov boundary by a geodesic ray to a vertex in it. More general, it is ultracomplete [Min10, Lemma 5.14].

Another way to classify essential simple closed curves is to look at their *topological type*: for an essential simple closed curve α on surface S, the topological type of α is the homeomorphic classes of $\S\setminus\alpha$, *i.e.* another curve $\beta\subset S$ has the same topological type as α if and only if for any connected component of $S\setminus\alpha$, there exists a homeomorphic connected component of $S\setminus\beta$, and *vice versa*.

Recall that a *foliation* is *minimal* if no trajectory is a simple closed curve. The space of minimal foliations is exactly the space of ending laminations (see for example [Ohs90]). There is a geometric interpretation of the Gromov boundary of curve graph:

Theorem 4.1.12 ([Kla22])**.** *Let S be an orientable surface of finite type. If $\xi(S) > 1$, then the Gromov boundary $\partial\mathcal{C}(S)$ of its curve graph is homeomorphic to the space of minimal foliations.*

There is a wealth of literature available on this topic and the classical ones are [MM99; MM00; Min10]. Additionally, by using curves and arcs on orientable surfaces, we can further construct various hyperbolic spaces. For example, the graph consisting of only isotopy classes of non-separating essential simple closed curves is also δ-hyperbolic [Ham14]. When the surface is with boundaries, then an arc on the surface is a proper topological embedding of a real interval. The *arc graph*, where the vertices are isotopy classes of curves and arcs, is also a hyperbolic graph [KP10]. Also, a famous technique of Bestvina-Bromberg-Fujiwara allows us to construct a quasi-tree, which is also δ-hyperbolic, using subsurface projection [BBF15]. This technique is later on applied to infinite-type surfaces, for example in [HQR22].

4.2 Mapping Class Group

In this section, we will describe some generalities of mapping class groups of both finite-type and infinite-type surfaces and focus more on infinite-type surfaces. As before, we only discuss a very limited part of this topic. For mapping class groups of finite-type surfaces, the classical reference is [FM11]. For infinite-type surfaces, the mapping class groups are discussed in a survey paper [AV20].

4.2.1 Definition and basic properties

Let S be an orientable surface of finite or infinite type. Let us consider the group of all homeomorphisms $S \to S$ that fix pointwisely the (possibly empty) boundary component ∂S. We denote this group by $\mathrm{Homeo}(S)$ and endow it with compact-open topology for the canonical action of $\mathrm{Homeo}(S)$ on S. In particular, the *orientation preserving homeomorphism group* $\mathrm{Homeo}^+(S)$ is a closed subgroup in $\mathrm{Homeo}(S)$ of index 2.

We say that two homeomorphisms $\phi, \psi \in \mathrm{Homeo}^+(S)$ are *isotopic* if they are connected by a continuous path in $\mathrm{Homeo}^+(S)$. We remark that for any simple closed curves $\alpha \subset S$, the image $\phi(\alpha)$ is isotopic to $\psi(\alpha)$ as curves. We remark that the (path-)connected component in $\mathrm{Homeo}^+(S)$ of identity element contains exactly the orientation preserving homeomorphisms that are isotopic to the identity element. It is a closed normal subgroup in $\mathrm{Homeo}^+(S)$.

The *mapping class group* of an orientable surface S is denoted by $\mathcal{MCG}(S)$ and is defined by the group of all isotopic classes of orientation preserving homeomorphisms $S \to S$ preserving pointwisely the boundary components. Following our discussion above, the mapping class group is also defined by the quotient group

$$\mathcal{MCG}(S) := \mathrm{Homeo}^+(S)/\{\text{isotopy}\},$$

where {isotopy} is the closed normal subgroup consisting of homeomorphisms isotopic to the identity element. Similarly, we can define the *extended mapping class group*, denoted by $\mathcal{MCG}^{\pm}(S)$, by the quotient group $\mathrm{Homeo}(S)/\{\text{isotopy}\}$. Moreover, the mapping class group $\mathcal{MCG}(S)$ is a closed subgroup of index 2 in $\mathcal{MCG}^{\pm}(S)$.

If one is interested in the combinatoric or structural properties of mapping class groups, a good way is to study its action on the curve graph. We remark that the natural action of mapping classes on the isotopic classes of essential simple closed curves respect the disjointness, hence the action of $\mathcal{MCG}(S)$ on $\mathcal{C}(S)$ is simplicial and thus by isometries. We also remark that this action is in general not transitive, but the action of $\mathcal{MCG}(S)$ is transitive among isotopic classes of curves with the same topological type.

The following fundamental fact about the action of mapping class group on the curve graph is proven in series of articles [Iva97; Kor99; Luo00; HMV18; BDR20] and we summarise it in below:

Theorem 4.2.1. *Let S be an orientable surface that is not a twice-punctured torus. Let $\mathcal{C}(S)$ be the curve graph of S and $\mathrm{Aut}(\mathcal{C}(S))$ be the automorphism group of $\mathcal{C}(S)$, equipped with the pointwise convergence topology. Then the extended mapping class group $\mathcal{MCG}^{\pm}(S)$ is isomorphic to $\mathrm{Aut}(\mathcal{C}(S))$ as topological groups.*

Theorem 4.2.1 allows us to describe explicitly the topology defined on $\mathcal{MCG}(S)$: it is the permutation topology and a basis of neighbourhoods at the identity element $\mathrm{Id} \in \mathcal{MCG}(S)$ is given by

$$U(F) := \{g \in \mathcal{MCG}(S) : ga = a \ (\forall a \in F)\}, \tag{4.1}$$

where F is a finite collection of isotopic classes of essential simple closed curves.

A weaker result than Theorem 4.2.1 is the following consequence that is usually referred to as *Alexander's method*:

Proposition 4.2.2 ([FM11], §2.3 and [HMV19])**.** *Let S be an orientable surface (of finite or infinite type) and let $\phi \in \mathrm{Homeo}^{+}(S)$. If for any essential simple closed curve $\alpha \subset S$, the image $\phi(\alpha)$ is isotopic to α, then ϕ is isotopic to the identity.*

Let S be an orientable surface of finite type. A pair of essential simple closed curves α, β are called *filling* if $S \setminus (\alpha \cup \beta)$ are disjoint union of annuli, topological disc or once punctured disc, *i.e.* any other essential simple closed curve must intersect at least one of α and β. We remark that such a pair of curves always exists whenever $\xi(S) > 1$, *viz.* when the curve graph is of infinite diameter. In particular, if a mapping class fixes a filling pair of curves, then it necessarily leaves every isotopic class invariant, which by Alexander's method implies that this mapping class is trivial. By treating the finite cases when $\xi(S) \leq 1$, we can conclude the following result:

Proposition 4.2.3. *Let S be an orientable surface of finite type. Then $\mathcal{MCG}(S)$ is a discrete group.*

One particular class of mapping class allows us to further study the mapping class group of an orientable surface. Let us consider an orientable surface S and an essential simple closed curve $\alpha \subset S$. Let us take a small neighbourhood N of α that is homeomorphic to annulus: by endowing S a Riemannian metric, we may assume (up to isotopy) that α is geodesic and such neighbourhood can be taken as a small normal neighbourhood of α. Let $A := S^1 \times [0,1]$ be the standard annulus. The orientation on A is given by the embedding $A \hookrightarrow \mathbb{R}^2$ with $(\theta, t) \mapsto (\theta, t+1)$ under the polar coordinates of $\mathbb{R}^2$. A "twist map" $T\colon A \to A$ is a homeomorphism given by the formula $T(\theta, t) := (\theta + 2\pi t, t)$. Let $\psi\colon A \to N \subset S$ be an orientation preserving homeomorphism such that $\psi(S^1 \times \{1/2\}) = \alpha$. Now we define the *Dehn twist* along α, denoted by T_α, to be an orientation preserving homeomorphism $S \to S$ given by

$$T_\alpha(x) := \begin{cases} \psi \circ T \circ \psi^{-1}(x) & \text{if } x \in N \\ x & \text{otherwise.} \end{cases}$$

We remark that if α and β are isotopic, then so will be T_α and T_β. Hence we are able to define Dehn twist for isotopic class of simple closed curves, namely $T_a \in \mathcal{MCG}(S)$ the mapping class consisting the Dehn twist along α and $\alpha \in a \in \mathcal{C}(S)$.

These Dehn twists are very useful for the study of mapping class groups. Let $\mathcal{PMCG}(S)$ be the subgroup of $\mathcal{MCG}(S)$ that induces trivial action on $\mathrm{End}(S)$. This group is called the *pure mapping class group*. We note that if S is finite-type and if $\mathrm{End}(S) \neq$, then $\mathcal{MCG}(S)$ is generated by $\mathcal{PMCG}(S)$ and permutation of ends. Here permutation of ends can be realised by products of *half twists* along two ends.

Theorem 4.2.4 (Chapter 4, [FM11])**.** *Let S be an orientable surface of finite type. Then $\mathcal{MCG}(S)$ is finitely generated. In particular, the pure mapping class group $\mathcal{PMCG}(S)$ is generated by finitely many Dehn twists along non-separating simple closed curves.*

But the case where S is infinite-type is much more complicated. We first remark that $\mathcal{MCG}(S)$ is not locally compact, *a fortiori* not discrete. Indeed, let S be an infinite-type surface and let $F \subset \mathcal{S}$ be any finite subset. Due to the infinite type of surface S, there exists $a \in \mathcal{C}(S)$ that is disjoint from every element in F. Then the cyclic subgroup generated by the Dehn twist T_a is contained in $U(F)$ as per (4.1), but it admits no accumulation point. Moreover, it turns out that compact sets in $\mathcal{MCG}(S)$ are nowhere dense. Since $\mathcal{MCG}(S)$ is Baire space, it cannot be algebraically generated by compact sets. See discussion in Chapter 6 or [AV20, Theorem 4.2]. We record these in the following:

Theorem 4.2.5. *Let S be an orientable surface of infinite type. Then $\mathcal{MCG}(S)$ is not locally compact, nor is it algebraically generated by a compact set.*

4.2.2 Isometric actions on metric spaces

For a finite-type orientable surface S, the curve graph $\mathcal{C}(S)$ is a very useful tool for the study of mapping class group. But when the surface becomes infinite-type, this graph becomes bounded and is no longer convenient for geometric group theoretic purposes.

Moreover, there are big mapping class groups that do not admit unbounded continuous action by isometries on a metric space:

Theorem 4.2.6 ([MR23])**.** *Let S be an infinite-type surface so that either every end of S is planar or every end of S is non-planar. If the end space of S is countable and homeomorphic to $\omega^\alpha + 1$, then $\mathcal{MCG}(S)$ cannot act continuously and unboundedly by isometries on a metric space.*

But for another class of infinite-type surfaces, there exist interesting continuous actions by isometries on a Gromov hyperbolic space. As an example, we will mainly treat in below the *projection complex*.

Let S be a connected orientable surface of infinite type. Let $\mathcal{MCG}(S)$ be its mapping class group. As in [MR23, Definition 1.8], a connected compact subsurface $K \subseteq S$ of finite type is a *non-displaceable subsurface* if $K \cap gK \neq \emptyset$ for every $g \in \mathcal{MCG}(S)$. More rigorously, the element g should be acting on the isotopic classes of K instead of K itself, but one may replace it by any of its representatives $\phi_g \in \mathrm{Homeo}^+(S)$ in the group of orientation-preserving homeomorphism of S, and this won't hinder the definition given above. In the sequel, one will omit the nuances between g and its representative ϕ_g, and by abuse of symbols, one allows $\mathcal{MCG}(S)$ to act on S and its subset whenever the action is understood.

For our purpose here, let us consider a non-displaceable subsurface $K \subset S$ such that

$$\overline{S} \setminus K = \bigsqcup_\beta \overline{P}_\beta,$$

where each $\overline{P}_\beta$ is a connected neighborhood of a clopen subset of $\mathrm{End}(S)$ in $\overline{S}$. In fact, these neighborhoods can be indexed by components $\beta \subseteq \partial K$ of its boundary. This larger subsurface is again, after the same arguments, non-displaceable in S.

Denote by $[K]$ the isotopic class of K and it is well-defined since the isotopy between two sets on a surface clearly yields an equivalence relation. Note that $[A] \sqcup [B] := [A \sqcup B]$ is meaningful when A and B are essential subsurfaces such that up to isotopy, the intersection $\partial A \cap \partial B$ is a disjoint union of simple closed curves, and $[A \sqcup B]$ is defined to be the isotopic class of a larger subsurface obtained by gluing together the common part of boundaries.

A *filling system* is a filling pair of simple closed curves on subsurface K. Let us consider the family $\mathcal{I} = \{g[K] : g \in \mathcal{MCG}(S)\}$ consisting of isotopic classes of non-displaceable subsurfaces interchangeable with K. Let $\mathcal{F}_K$ be a filling system of K. Without loss of generality, one assumes that $[\partial K] \subseteq [\mathcal{F}_K]$. Then it is clear that $g\mathcal{F}_K$ yields a filling system in

$K' := gK$. Pick one $g_{K'} \in \mathcal{MCG}(S)$ such that $K' = g_{K'}K$ and define $\mathcal{F}_{K'} = g_{K'}F_K$. It is clear that $\mathcal{F}_{K'}$ also lies in a minimal position and that $\partial K' = g_{K'}\partial K$. There is a corresponding between the class of filling systems and the class of non-displaceable subsurfaces:

Proposition 4.2.7. *Let $g \in \mathcal{MCG}(S)$. If $g[\mathcal{F}_K] = [\mathcal{F}_K]$, then $g[K] = [K]$.*

Proof. By definition, one sees that

$$\overline{S} \setminus \mathcal{F}_K = \left(\bigsqcup_{\alpha \in A} D_\alpha \right) \sqcup \left(\bigsqcup_{\beta \subseteq \partial K} \overline{P}_\beta \right), \tag{4.2}$$

where A is a finite index set, D_α is a topological disk indexed by $\alpha \in A$, β is a component of ∂K, and $\overline{P}_\beta$ is as above. So alternatively, one can write

$$K = \mathcal{F}_K \sqcup \left(\bigsqcup_{\alpha \in A} D_\alpha \right). \tag{4.3}$$

By applying g on both sides of (4.2) and (4.3), and taking the isotopic classes, one gets

$$\overline{S} \setminus [\mathcal{F}_K] = \overline{S} \setminus g[\mathcal{F}_K] = \left(\bigsqcup_{\alpha \in A} g\left[D_\alpha\right] \right) \sqcup \left(\bigsqcup_{\beta \subseteq \partial K} g\left[\overline{P}_\beta\right] \right) \tag{4.4}$$

and

$$g[K] = g[\mathcal{F}_K] \sqcup \left(\bigsqcup_{\alpha \in A} g\left[D_\alpha\right] \right) = [\mathcal{F}_K] \sqcup \left(\bigsqcup_{\alpha \in A} g\left[D_\alpha\right] \right), \tag{4.5}$$

As $g \in \mathcal{MCG}(S)$ can only send topological disks to disks and neighborhoods of clopen sets in $\mathrm{End}(S)$ to other neighborhoods, the action of g on $([D_\alpha])_{\alpha \in A}$ reduces to a permutation of A after comparing (4.2) and (4.4). This further render (4.5) identical to the class form of (4.3). Hence $g[K] = [K]$. □

In particular, one can conclude the following result:

Corollary 4.2.8. *Let $[K'] \in \mathcal{I}$. If $[K'] \neq [K]$, then $[\mathcal{F}_{K'}] \neq [\mathcal{F}_K]$.*

Remark 4.2.9. It should be noticed that the converse is not necessarily true. For example, the Dehn twist τ along an essential simple closed curve in K can yield a filling system that is not isotopic to the original one, but τ is nevertheless supported on K and thus leave the class $[K]$ invariant. This also demonstrates that this corollary will fail when one considers the filling systems and subsurfaces but not their isotopic classes.

Moreover, by the virtue of Corollary 4.2.8, one can deduce:

Corollary 4.2.10. *There are only countably many distinct classes in $\mathcal{I}$.*

Proof. For each $[K'] \in \mathcal{I}$, the filling system is *a priori* $[\mathcal{F}_{K'}]$ fixed. Two distinct classes $[K]$ and $[K']$ in $\mathcal{I}$ would yield two corresponding finite subsets $[\mathcal{F}_K]$ and $[\mathcal{F}'_K]$ in $\mathcal{C}(S)$, the collection of isotopic classes of simple closed curves on S. Since $\mathcal{C}(S)$ is countable, there is only countably many possibilities for the finite collection $[\mathcal{F}_{K'}]$'s and thus for $[K']$'s. □

Now one reviews the celebrated construction of Bestvina, Bromberg and Fujiwara from [BBF15].

Let $\mathbb{Y}$ be a collection of metric spaces. Let G be a group acting on $\mathbb{Y}$ as well as on each $Y \in \mathbb{Y}$. The action of G is *metric-preserving* if for every $g \in G$ and every $Y \in \mathbb{Y}$, there exists an isometry $\iota_g^Y : Y \to gY$ such that $\iota_{gh}^Z = \iota_g^{hZ} \circ \iota_g^Z$ for every $Z \in \mathbb{Y}$ and all $g, h \in G$.

A collection $\left(\mathbb{Y}, (\pi_Y(Z))_{Y \neq Z}\right)$ with a G action is a *G-equivariant projection family* if the following are satisfied:

- $\mathbb{Y}$ is a collection of metric spaces equipped with a metric-preserving G-action,
- $\pi_Y(Z)$ is a nonempty subset of Y for any two distinct members $Y \neq Z$ of $\mathbb{Y}$, and
- for every $g \in G$ and any two distinct $Y, Z \in \mathbb{Y}$, one has $\pi_{gY}(gZ) = \iota_g^Y(\pi_Y(Z))$.

Here $\pi_Y : \mathbb{Y} \to \mathcal{P}(Y)$ is called a *projection function*.

Let $\mathcal{Y} := \bigsqcup_{Y \in \mathbb{Y}} Y$ be the collection of all points in the member metric spaces in $\mathbb{Y}$. The projection function π_Y can be extended to $\mathcal{Y}$ by setting

$$\pi_Y(x) = \begin{cases} \{x\} & \text{if } x \in Y \\ \pi_Y(X) & \text{if } x \in X \neq Y \end{cases}$$

for every $x \in \mathcal{Y}$.

The *projection distance* is a positive function given by

$$d_Y^\pi(X, Z) := \operatorname{diam}\left(\pi_Y(X) \cup \pi_Y(Z)\right) = \sup\left\{d_Y(a, b) : a, b \in \pi_Y(X) \cup \pi_Y(Z)\right\}$$

for any $X, Z \neq Y$, where d_Y is the metric in Y. The definition naturally passes to the level of $\mathcal{Y}$ by setting $d_Y^\pi(x, z) = \operatorname{diam}\left(\pi_Y(x) \cup \pi_Y(z)\right)$.

Definition 4.2.11 (BBF family)**.** Let G be a group and $\left(\mathbb{Y}, (\pi_Y(Z))_{Y \neq Z}\right)$ be a G-equivariant projection family. The family is a BBF family if or every $X, Y, Z \in \mathbb{Y}$ with $Y \neq X, Z$, there exists a $\theta > 0$ such that

(P0) For all distinct $X, Y \in \mathbb{Y}$, one has $d_Y^\pi(X, X) \leq \theta$.

(P1) For three distinct $X, Y, Z \in \mathbb{Y}$, if $d_Y^\pi(X, Z) < \theta$, then $d_X^\pi(Y, Z) \geq \theta$.

(P2) For all $X, Z \in \mathbb{Y}$, there are only finitely many $Y \neq X, Z$ such that $d_Y^\pi(X, Z) > \theta$.

Given a BBF family $\left(\mathbb{Y}, (\pi_Y(Z))_{Y \neq Z}\right)$, one can construct a quasi-tree and a hyperbolic space from this family as the following statement:

Theorem 4.2.12 (Bestvina-Bromberg-Fujiwara [BBF15]). *Let G be a group. Assume that there exists a BBF family $\left(\mathbb{Y}, (\pi_Y(Z))_{Y \neq Z}\right)$ for G and that there exists a $\delta \geq 0$ such that every $Y \in \mathbb{Y}$ is δ-hyperbolic. Then for large enough $M > 0$, there exists a projection complex $\mathcal{P}_M(\mathbb{Y})$ and a hyperbolic space $\mathbb{X}$ where every $Y \in \mathbb{Y}$ is embedded as geodesically convex subspace and G acts on $\mathbb{X}$ by isometries. In addition, the projection complex $\mathcal{P}_M(\mathbb{Y})$ is a quasi-tree.*

Using this theorem above, we are able to construct a Gromov hyperbolic metric space on which certain big mapping class groups can act continuously and unboundedly by isometries.

Let S be an infinite-type surface and $K \subset S$ be a finite-type non-displaceable subsurface. Let $\mathcal{C}_S(K)$ be the curve graph whose vertices are the isotopic classes of simple closed curves that have a representative contained in K which is essential in K. This graph only depends on the isotopic class $[K]$ (see [HQR22, Lemma 2.1]), so one can write $\mathcal{C}_S([K])$ instead. Moreover, one can regard $\mathcal{C}_S([K])$ as a subgraph of $\mathcal{C}(S)$ (see [HQR22, Lemma 2.2]).

Equip each $\mathcal{C}_S([K])$ with the simplicial metric. Then the action of $\mathcal{MCG}(S)$ on $\mathcal{C}_S([K])$ preserves the metric. For any $[K_1], [K_2] \in \mathcal{I}$, when $[K_1] \neq [K_2]$, one can find K_1 and K_2 respective representatives such that there is at least one component of ∂K_2 intersects K_1 in an essential curve or arc. One defines the subsurface projection

$$\pi_{\mathcal{C}_S([K_1])}\left(\mathcal{C}_S([K_2])\right) \subseteq \mathcal{C}_S([K_1]),$$

by collection of the curves on $[K_1]$ that are essentially disjoint from the boundary of $[K_2]$.

Proposition 4.2.13 ([HQR22]). *Let S be an infinite-type surface and K be a non-displaceable subsurface of S. Then the family*

$$\mathbb{Y}_K = \left\{\mathcal{C}_S([K']) : [K'] \in \mathcal{I}\right\}$$

together with subsurface projections given above is a BBF family, and for large enough $M > 0$, one can construct a quasi-tree $\mathcal{P}_M(\mathbb{Y}_K)$ and the corresponding blown-up projection complex $\mathbb{X}$ from $\mathbb{Y}_K$ as in Theorem 4.2.12. Moreover, the group $\mathcal{MCG}(S)$ acts continuously on $\mathcal{P}_M(\mathbb{Y}_K)$ and $\mathbb{X}$ by isometries and the action on $\mathcal{P}_M(\mathbb{Y}_K)$ is of general type.

Moreover, by looking carefully into the construction of the Gromov hyperbolic space $\mathbb{X}_K$ in [BBF15], we can conclude that the vertices of $\mathbb{X}_K$ are the union of $\mathcal{C}_S([K])$. By Corollary 4.2.10, we can reorganise the result above in more favourable terms:

Proposition 4.2.14. *Let S be an infinite-type surface and K be a non-displaceable subsurface of S. Then there exists a separable geodesic Gromov hyperbolic metric space $\mathbb{X}_K$ on which $\mathcal{MCG}(S)$ act continuously by isometries. Moreover, this action is of general type.*

As an application, we mention one interesting result about the normal subgroup structure of mapping class group of infinite-type surfaces admitting compact non-displaceable subsurfaces, which can also be viewed as a characterisation of this class of surfaces:

Theorem 4.2.15 ([HQR22])**.** *Let S be an orientable surface of infinite type. Then $\mathcal{MCG}(S)$ contains a non-trivial normal free subgroup if and only if S contains a non-displaceable subsurface of finite type.*

As discrete group, it follows immediately that $\mathcal{MCG}(S)$ is not amenable, since it contains non-abelian free subgroups. In fact, almost every mapping class groups of an orientable surface is not discretely amenable: if $a, b \in \mathcal{C}(S)$ are two curves that intersect more than twice, then the Dehn twists T_a and T_b generate a free subgroup in $\mathcal{MCG}(S)$. But the topological (non)-amenability of big mapping class groups are more difficult to deduce. Combining Proposition 4.2.14 and Theorem 9.1.1, we can give a partial answer to this question:

Corollary 4.2.16. *Let S be an infinite-type surface. If S contains a non-displaceable subsurface of finite type, then $\mathcal{MCG}(S)$ is not amenable.*

Part II

Farther Sight of Dwarfs

Chapter 5

Hyperbolic Embedding of Convex Bodies

Les questions les plus importantes de la vie ne sont en effet, pour la plupart, que des problèmes de probabilité.

Pierre-Simon Laplace (1749–1827),
Théorie Analytique des Probabilités.

5.1 Introduction

The idea of endowing the collection of flat figures with a hyperbolic structure dates back to W. P. Thurston's work [Thu98], where he provides a complex hyperbolic description of "triangulations" of a 2-sphere using flat metrics. A notable generalization of this concept is the study of subspaces of P endowed with an isometric involution, as explored in [BG92]. To begin, it is important to recognize that the homothety classes of ellipses in $\mathbb{R}^2$ naturally correspond to the real hyperbolic plane. These subspaces are isometric to spaces of homothety classes of plane convex polygons with fixed edge directions and real hyperbolic distances. This approach has been extended to higher dimension by using mixed volumes to define hyperbolic metrics on spaces of convex polytopes in $\mathbb{R}^n$. For $n = 3$, some of these spaces, which are isometric to real hyperbolic polyhedra, can be isometrically embedded into P [FI16; FI17]. Further, Debin and Fillastre advance this methodology by using intrinsic volumes to hyperbolise the homothety classes of Euclidean convex bodies in higher dimension [DF22].

Given $2 \leq d < \infty$, a *convex body* in $\mathbb{R}^d$ is a non-void convex compact subset of $\mathbb{R}^d$. Between any two convex bodies, it is possible to define an "area distance" by using the intrinsic volumes of convex bodies and mimicking the definition of the distance on the Klein model of real hyperbolic spaces. The "area distance" then becomes a metric on the

space of homothety classes of convex bodies K in $\mathbb{R}^d$ with $2 \leq \dim(K) \leq d$. In [DF22], Debin and Fillastre show that this metric space can be isometrically embedded into a real hyperbolic space of dimension at most $\aleph_0$. Their idea is to identify convex bodies with their support functions restricted to the unit sphere S^{d-1}, which reside in the Sobolev space of functions defined on S^{d-1}, and then compare the Sobolev subspace with the Klein model. Moreover, they show that if one normalises the convex bodies $K \subset \mathbb{R}^d$ so that $\operatorname{diam}(K) = 1$ and positions them in the way that their Steiner points lie at 0, then the hyperbolic embedding will be homeomorphic to the space of normalised convex bodies equipped with the Hausdorff distance.

At the end [DF22, §4], they consider the canonical isometric embedding of $\mathbb{R}^d$ into $\mathbb{R}^{d+k}$. By identifying two convex bodies $K \subset \mathbb{R}^d$ and $K' \subset \mathbb{R}^\delta$ if K only differs K' from a homothety in $\mathbb{R}^{\max(d,\delta)}$, they are able to send all convex bodies of finite dimension into an infinite-dimensional real hyperbolic space. But this space is not complete. Examples are the sequence of n-dimensional unit balls (Example 5.3.13) and the increasing sequence of non-GB rectangles (Example 5.3.14). So they ask the following question: *can one give a description on the completion of the hyperbolic embedding of finite-dimensional convex bodies?*

In answering this question, the present article first gives an elementary proof for embedding homothety classes of Euclidean convex bodies into a real hyperbolic space. This proof relies on the result about kernels of (real) hyperbolic type given in [MP19]. These kernels can be viewed as the hyperbolic analogue to kernels of positive and of conditionally negative type (compare to, for example, [BLV08, Appendix C]). The definition of the herein involved kernel of hyperbolic type only uses the *intrinsic volumes* of convex bodies.

For finite-dimensional convex bodies, the volume of $K + rB^d \subset \mathbb{R}^d$, where B^d is d-dimensional unit ball, is polynomial in $r > 0$ (Steiner formula) and the intrinsic volumes of K are defined as the normalised coefficients of this polynomial. When it comes to infinite dimension, the intrinsic volume of a convex body K will be defined as the supremum of intrinsic volumes of finite-dimensional convex bodies contained in K. So a natural extension of the aforementioned hyperbolisation process to infinite-dimension is to consider the class of infinite-dimensional convex bodies with finite intrinsic volumes.

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$. Dudley introduces the notion of *Gaußian bounded* (abbv. *GB*) subsets [Dud67]. This family of subsets in $\mathcal{H}$ has been profoundly studied in the context of geometric probability and also finds its applications in ergodic theory [Dud73; Bou88; Web94]. In [Che76], Chevet first defines the i-th intrinsic volume V_i, for $i \geq 1$, of an infinite-dimensional convex compact subset K in $\mathcal{H}$ (called an *infinite-dimensional convex body in* $\mathcal{H}$) by the supremum of $V_i(K')$ for all finite-dimensional convex bodies $K' \subset K$ and then shows that $K \subset \mathcal{H}$ is GB if and only if its intrinsic volumes $V_i(K)$ are finite for $i \geq 1$ (see Proposition 5.2.4). So the hyperbolisation process is naturally

applied to the homothety classes of GB convex bodies in $\mathcal{H}$, wherein are contained the finite-dimensional ones.

Recall that a *real hyperbolic space* $\mathbb{H}^{\alpha}_{\mathbb{R}}$ is defined by the hyperboloid model constructed from a Hilbert space $\mathcal{H}'$ on $\mathbb{R}$ via a Lorentzian quadratic form, where $\alpha = \dim(\mathcal{H}')$ is a cardinal and is called the *dimension* of $\mathbb{H}^{\alpha}_{\mathbb{R}}$. It is a Gromov-hyperbolic space. A real hyperbolic space is of *infinite dimension* if $\alpha \geq \aleph_0$, and if $\alpha = \aleph_0$, one simply writes $\mathbb{H}^{\infty}_{\mathbb{R}}$ for convenience.

Theorem 5.1.1. *Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$. Let $\mathbb{K}_2$ be the collection of homothety classes of GB convex bodies in $\mathcal{H}$ with dimension at least 2. Then there exists an embedding $\iota : \mathbb{K}_2 \hookrightarrow \mathbb{H}^{\infty}_{\mathbb{R}}$ of $\mathbb{K}_2$ into the $\aleph_0$-dimensional real hyperbolic space, thus defining a metric on $\mathbb{K}_2$, and its image $\iota(\mathbb{K}_2)$ forms a convex subset.*

Moreover, the map ι extends continuously to the homothety classes of segments in $\mathcal{H}$ and sends them to the Gromov boundary $\partial\big(\iota(\mathbb{K}_2)\big)$.

To understand GB convex bodies of infinite dimension, the techniques of infinite-dimensional analysis become indispensable.

Sudakov discovers that a GB convex body K can be associated to a random variable $h_K(X)$ in $L^1(\Omega)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which the isonormal Gaußian process is defined, and the first intrinsic volume is the expectation of this random variable [Sud71]. It is after decades that this random variable is recognised as the support function of the convex bodies in a separable real Hilbert space $\mathcal{H}$ and notions such as the Steiner point (or *"centre"*) find their generalisations in the context of infinite-dimensional GB convex bodies [Vit01].

To treat these random variables, we turn to Malliavin calculus, which allows us to compute the Malliavin derivatives $Dh_K(X)$ of support function of a GB convex body, an $\mathcal{H}$-valued random variable representing an extremal point in K where the isonormal Gaußian process is maximised (see Proposition 5.4.18), so that K can be recovered by taking the closed convex hull of the essential range of $Dh_K(X)$ (see Corollary 5.4.19). Moreover, the Steiner point of K is exactly the barycenter in K with respect to the pushforward probability measure induced by $\omega \mapsto Dh_K(X)(\omega)$ (see Proposition 5.4.20).

The support function $h_K(X)$ of any GB convex body $K \subset \mathcal{H}$ lies in the Sobolev space $\mathbb{D}^{1,2}$ and in particular, if K is of finite dimension, then it recovers the Sobolev space introduced in [Sch14; DF22]. Moreover, one can also generalise the formula from [Sch14, pp.298] or [Che76, Théorème 3.10]. If we set the polarisation

$$V_2(K_1, K_2) := \frac{1}{2}\left(V_2(K_1 + K_2) - V_2(K_1) - V_2(K_2)\right),$$

then we have the following result:

Theorem 5.1.2. *Let $K, K' \subset \mathcal{H}$ be GB convex bodies and X be an isonormal Gaußian process on $\mathcal{H}$. Then*

$$V_2(K) = \pi \mathbb{E}\left[|h_K(X)|^2 - \|Dh_K(X)\|_{\mathcal{H}}^2\right] \tag{5.1}$$

and

$$V_2(K, K') = \pi \mathbb{E}\left[h_K(X)h_{K'}(X) - (Dh_K(X), Dh_{K'}(X))_{\mathcal{H}}\right], \tag{5.2}$$

where $h_K, h_{K'}$ are the support functions of K and K'.

With formulae (5.1) and (5.2) above, we are able to characterise the equality cases of Alexandrov-Fenchel inequality for the second intrinsic volumes of GB convex bodies:

Theorem 5.1.3. *Let $K, K' \subset \mathcal{H}$ be GB convex bodies in $\mathcal{H}$ of dimension at least 2. Then $V_2(K, K')^2 = V_2(K)V_2(K')$ if and only if $K = tK' + v$ for some $t > 0$ and $v \in \mathcal{H}$.*

For polytopes, Chevet also gives general formulae to compute their intrinsic volumes of higher degree using the support function $h_P(X)$ and some other quantities [Che76, Proposition 3.6′]. The technique of associating an infinite-dimensional GB convex body to its support function will allow us to work on function spaces instead of geometric objects while we try to understand GB convex bodies. But the answer to the following question remains unclear: *is it possible to generalise the formulae for GB convex bodies, e.g. the notions from Malliavin calculus?*

Recall that $L^2(\Omega)$ admits an orthogonal decomposition $\overline{\bigoplus}_{n\geq 0}\mathfrak{H}_n$ by the *n-th Wiener chaos*. It turns out that the orthogonal projections of the support function $h_K(X)$ in $\mathfrak{H}_n$'s can completely determine the size, position and shape of K. The support function $h_K(X)$ of K is approximated by the support functions $h_P(X)$ of polytopes contained in it. In a more general context, it is the estimation of the suprema for infinite Gaußian processes by its finite sub-processes [BLM13, Chapter 13].

It is worth remarking that in [DF22], Debin and Fillastre also discuss the orthogonal projection of $h_K(X)$ to $\mathfrak{H}_n$. The Malliavin calculus generalises their discussion and further gives to these projections geometric significations: the projection of $h_K(X)$ to $\mathfrak{H}_0$ is the constant function of $V_1(K)/\sqrt{2\pi}$, the projection to $\mathfrak{H}_1$ is the Steiner point of K and its projection to $\overline{\bigoplus}_{n\geq 2}\mathfrak{H}_n$ stands for its shape (see Section 5.4.2). Nevertheless, *is it possible to tell the geometric signification of the projections of $h_K(X)$ to $\mathfrak{H}_n$ for each $n \geq 2$?*

Debin and Fillastre [DF22] show that the homothety classes of the n-dimensional unit balls $[B^n]$ converge to a point $O \in \mathbb{H}^\infty_{\mathbb{R}}$ (see also Example 5.3.13), but they do not converge to any point in $\iota(\mathbb{K}_2)$. Using the tools from Malliavin derivative, it is possible to adapt integral geometry into infinite dimension. Particularly, we are able to answer the question asked

by Debin and Fillastre: the completion for the hyperbolic embedding of finite-dimensional convex bodies is but the convex hull in $\mathbb{H}^\infty_{\mathbb{R}}$ of homothetic GB convex bodies and the point O, *i.e.*

Theorem 5.1.4. *Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$ and $\iota : \mathbb{K}_2 \to \mathbb{H}^\infty_{\mathbb{R}}$ be the embedding in Theorem 5.1.1. Then $\overline{\iota(\mathbb{K}_2)} = \mathrm{co}\,(\iota(\mathbb{K}_2) \cup \{O\})$ is the geodesic convex hull, or equivalently each point in $\overline{\iota(\mathbb{K}_2)}$ is uniquely associated to a function $h_K(X) + c \in \mathbb{D}^{1,2}$, where $c \geq 0$ is a constant, $K \subset \mathcal{H}$ is a GB convex body with $\mathrm{Stein}(K) = 0$ and $h_K(X)$ is its support function.*

5.2 Gaußian bounded convex bodies

In the sequel, unless otherwise indicated, the space $\mathcal{H}$ will be referred to as a separable infinite-dimensional real Hilbert space carrying an inner product $(\cdot,\cdot)_{\mathcal{H}}$. Let $(e_i)_{i\geq 1}$ be an orthonormal basis of $\mathcal{H}$.

5.2.1 GB and GC sets

Recall that a *centred Gaußian process on T* is a collection of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ indexed by $t \in T$ such that for any finite subset $\{v_1, \ldots, v_n\}$ of T, the random vector $(X_{v_1}, \ldots, X_{v_n}) \sim \mathcal{N}(0, \Sigma)$ is Gaußian.

For the Hilbert space $\mathcal{H}$, a centered *isonormal Gaußian process* is a Gaussian process $(X_v)_{v\in\mathcal{H}}$ on $\mathcal{H}$ such that $\mathbb{E}[X_v] = 0$, $\mathbb{E}[X_v X_u] = (v,u)_{\mathcal{H}}$ for every $v, u \in \mathcal{H}$ and $X_{av+bu} = aX_v + bX_u$ for any $a, b \in \mathbb{R}$ and any $v, u \in \mathcal{H}$.

The term "isonormal" is due to Segal and this process is introduced in [Seg54] where it bears the name of "canonical normal distribution".

Such a process can be constructed explicitly as the following. Let $(X_i)_{i\geq 1}$ be a sequence of *orthogaußian* random variables, *i.e.* independent and identically distributed random variables following $\mathcal{N}(0,1)$. Then an isonormal Gaußian process on $\mathcal{H}$ can be defined by

$$X_v(\omega) := \sum_{i=1}^{\infty} (v, e_i)_{\mathcal{H}} X_i(\omega)$$

for all $v \in \mathcal{H}$ and all $\omega \in \Omega$.

Conversely, for any orthonormal basis $(e_i)_{i\geq 1}$ in $\mathcal{H}$, an isonormal Gaußian process $(X_v)_{v\in\mathcal{H}}$ on $\mathcal{H}$ must be such that $(X_{e_i})_{i\geq 1}$ are orthogaußian. So the isonormal Gaußian process on $\mathcal{H}$ is essentially unique.

Alternatively, it is also possible to regard the isonormal Gaußian process as a random variable $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ and identify $\mathcal{H}$ with ℓ^2, so that the Gaußian variable $X_v = (v, X) = \sum_{i=1}^{\infty} v_i X_i \sim \mathcal{N}(0, \|v\|^2)$.

It is a consequence of Weil's converse to Haar theorem [Wei65, Appendice] that in infinite-dimensional real Hilbert space $\mathcal{H}$, there is no complete analogue to Lebesgue or Haar measure. But there is still a need for measuring subsets in $\mathcal{H}$. In view of this, Dudley introduced the following class of subsets that are suitable for measuring [Dud67]:

Definition 5.2.1 (Gaußian bounded sets)**.** Let K be a subset of $\mathcal{H}$ and $(X_v)_{v\in\mathcal{H}}$ be an isonormal Gaußian process over $\mathcal{H}$. Then K is *Gaußian bounded*, or *GB* for abbreviation, if for any countable (dense) subset $C \subset K$,

$$\mathbb{P}\left(\left\{\omega \in \Omega : \exists M \in (0,\infty) \text{ such that } \sup_{v\in C} X_v(\omega) < M\right\}\right) = 1.$$

Remark 5.2.2. For a countable set C, the supremum $\sup_{v\in C} X_v$ clearly defines a random variable. Since the isonormal Gaußian process X is linear, if $K \subset \mathcal{H}$ is in addition convex, then for any dense subset $C \subset K$, $\sup_{v\in C} X_v$ is actually $\sup_{v\in K} X_v$. So being GB means that the sample function $X(\cdot,\omega)$ of the isonormal Gaußian process is uniformly bounded on K for almost all $\omega \in \Omega$. We also remark that for every countable index set C, the random variable $\sup_{v\in C} |X_v|$ is almost surely bounded if and only if it has a finite expectation [LS70]. Moreover, for a GB convex body K, the random variable $\sup_{v\in C} X_v(\omega)$ does not depend on the choice of countable dense subsets in K [Vit01]. Thus, we will simply write this random variable as $\sup_{v\in K} X_v$ in the sequel.

Let us focus on some properties of GB sets in a separable Hilbert space for a while. For two sets A, B in the vector space $\mathcal{H}$, one can define the *Minkowski sum* by

$$A + B \coloneqq \{x + y \in \mathcal{H} : x \in A \text{ and } y \in B\}.$$

Moreover, subsets of $\mathcal{H}$ are also carrying the scalar multiplication

$$tA \coloneqq \{tx \in \mathcal{H} : x \in A\}$$

for any t, and when $t > 0$, it is called a *dilation*. As usual, a *translation* is a map $A \mapsto A + p$ for some vector $p \in \mathcal{H}$. A finite combination of dilations and translations will then be called a *homothety*. It is obvious from the definition that the class of GB sets are stable under Minkowski additions and homotheties, as it is also remarked in [DFL71].

Moreover, if K is a GB set in $\mathcal{H}$, then so will be its convex hull, which is the collection of all convex combinations of points in K. A subset of the GB set K in $\mathcal{H}$ is also GB. These follow directly from the definition. Since being GB is a closed condition, this implies that the closure of a GB set K is also GB.

Finally, let us mention the following compactness result about GB set: *every GB set in* $\mathcal{H}$

is totally bounded and thus is relatively compact [Dud67, Proposition 3.4].

By taking the closed convex hull of a GB set, we are allowed to only focus on convex, compact, GB subsets in $\mathcal{H}$, which will be called *GB convex bodies* in $\mathcal{H}$. It is clear that a GB convex body in $\mathcal{H}$ cannot have a non-empty interior, otherwise it would contain an open ball and would not be totally bounded.

Another similar notion to GB sets is the following:

Definition 5.2.3 (Gaußian continuous set)**.** Let K be a subset of $\mathcal{H}$ and $(X_v)_{v\in\mathcal{H}}$ be an isonormal Gaußian process over $\mathcal{H}$. Then K is *Gaußian continuous*, or *GC* for abbreviation, if for almost all $\omega \in \Omega$, the sample function $X(\cdot, \omega)$ is continuous on K.

It is clear that every compact GC set is GB. But the non-GC compacta amongst the GB sets are quite narrow.

Let K be a subset of $\mathcal{H}$ that is convex and symmetric. For each $v \in \mathcal{H}$, define $\|v\|_K := \sup\{|(u, v)_{\mathcal{H}}| : u \in K\}$.

For any two bounded convex subsets K, K' in $\mathcal{H}$, note $K \prec K'$ if $K \subset \operatorname{span}(K')$ and K is compact for $\|\cdot\|_{s(K')}$, where $s(K')$ is the symmetric closed convex hull of K'. Then a *maximal* GB set is such that K' will never be GB whenever $K \prec K'$. As a result, *every GB set is either maximal or GC* [Dud67, Theorem 4.7].

5.2.2 Intrinsic volumes

Let K be a subset of $\mathcal{H}$. The dimension of K will be defined by the dimension of the subspaces in $\mathcal{H}$ generated by K. If K is a convex body in $\mathcal{H}$ of dimension $d < \infty$, then it can be identified with a convex body in $\mathbb{R}^d$ and its *k-th intrinsic volume*, denoted by $V_k(K)$, is a positive function that can be characterised by the *Steiner formula*

$$\operatorname{vol}_d\left(K + rB^d\right) = \sum_{k=0}^{d} r^{d-k}\kappa_{d-k}V_k(K), \tag{5.3}$$

where vol_d is the Lebesgue measure in $\mathbb{R}^d$, B^d is the unit ball in $\mathbb{R}^d$ and κ_k is the Lebesgue measure of the unit k-ball. Although Steiner formula depends on the dimension of the ambient Euclidean space, we emphasise that the intrinsic volumes *are independent of the dimension of the ambient Euclidean space*. We remark that if K is d-dimensional, then $V_d(K)$ is its Lebesgue measure in $\mathbb{R}^d$.

The polarisation of V_2 by setting

$$V_2(K_1, K_2) := \frac{1}{2}\left(V_2(K_1 + K_2) - V_2(K_1) - V_2(K_2)\right)$$

provides a positively bilinear form on finite-dimensional convex bodies. More precisely, $V_2(\cdot,\cdot)$ enjoys the following properties for finite-dimensional convex bodies:

(C1) $V_2(K,K) = V_2(K)$.

(C2) $V_2(K_1,K_2) = V_2(K_2,K_1)$

(C3) For any $t \geq 0$, $V_2(tK_1,K_2) = tV_2(K_1,K_2)$.

(C4) $V_2(K_1+K_2,K_3) = V_2(K_1,K_3) + V_2(K_2,K_3)$.

(C5) $K_1 \subseteq K_2$ implies $V_2(K_1,K_3) \leq V_2(K_2,K_3)$.

(C6) $V_2(K_1,K_2) \geq 0$ and the equality holds if and only if K_1 or K_2 is a point, or both are segments in the same direction.

(C7) (*Alexandrov-Fenchel inequality*) for convex bodies K_1 and K_2 in $\mathbb{R}^d$ of dimension at least 2, $V_2(K_1,K_2)^2 \geq V_2(K_1)V_2(K_2)$, and equality holds if and only if $K_1 = tK_2 + v$ for some $t > 0$ and $v \in \mathbb{R}^d$.

These properties are classical and can be shown in geometrical means, see for example [Sch14, §5]. In terms of *mixed volume* or *quermaßintegrals*, $V_2(K,K') = V_2(K,K',B^d,\ldots,B^d)$ if $K,K' \subset \mathbb{R}^d$, where B^d is the unit ball in $\mathbb{R}^d$ [Sch14, §5 & §6].

If K is a convex body in $\mathcal{H}$ with infinite dimension, then following [Che76], its k-th intrinsic volume will be defined by

$$V_k(K) \coloneqq \sup\{V_k(C) : C \subset K \text{ convex body with } \dim(C) < \infty\}.$$

With this definition, the following properties give a full characterisation of GB convex bodies in $\mathcal{H}$ [Che76, Proposition 4.1]:

Proposition 5.2.4. *Let K be a convex body in $\mathcal{H}$. Then the following assertions are equivalent:*

(i) K is GB.

(ii) For all $k \geq 1$, the intrinsic volume $V_k(K)$ is finite.

(iii) There exists a $k \geq 1$ such that the intrinsic volume $V_k(K)$ is finite.

Remark 5.2.5. The definition of the intrinsic volumes of a finite-dimensional convex body K in $\mathcal{H}$ does not rely on the choice of the ambient finite-dimensional subspace. So the intrinsic volumes are invariant under the action of the orthogonal group $O(\mathcal{H})$. The same holds for the infinite-dimensional GB convex bodies.

In particular, the first intrinsic volume of a GB convex body K is given by

$$V_1(K) = \sqrt{2\pi}\mathbb{E}\left[\sup_{v\in K} X_v\right] < \infty, \tag{5.4}$$

see [Sud71; BC74]. More generally, the following formula due to Tsirelson (called *Kubota-Tsirelson formula*) provides a way to define or to compute the intrinsic volumes of a GB

convex body without approximating it by finite-dimensional convex bodies [Tsi86]:

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!\kappa_k}\mathbb{E}\left[\lambda_k\left(\left\{(X_v^1,\dots,X_v^k)\in\mathbb{R}^k : v\in K\right\}\right)\right], \tag{5.5}$$

where $X^1,\cdots,X^k$ are k independent isonormal Gaußian process on $\mathcal{H}$ and λ_k is the Lebesgue measure on $\mathbb{R}^k$. In particular, we remark that it is direct from Kubota-Tsirelson formula that V_1 is additive.

By passing to the limit, properties (C1)-(C7) also hold for all GB convex bodies (but not the equality condition in (C7)).

5.2.3 Examples

In this section, we record three examples of infinite-dimensional GB convex bodies given in [Dud67]. Again, let $(e_i)_{i\geq 1}$ be an orthonormal basis of the separable Hilbert space $\mathcal{H}$.

Let $(b_i)_{i\geq 1}$ be a sequence of non-negative real numbers. We define the associated *ellipsoid* by

$$E\big((b_i),(e_i)\big) = \left\{x=\sum_{i\geq 1}x_ie_i\in\mathcal{H} : \sum_{b_i>0}\frac{x_i^2}{b_i^2}\leq 1\right\}.$$

We remark that such an ellipsoid is compact if and only if b_i converges to 0. It is shown that $E\big((b_i),(e_i)\big)$ is GB if and only if $(b_i)_{i\geq 1}\in\ell^2$, or if E is a *Schmidt ellipsoid* [Dud67, Proposition 6.3]. Again, the closed unit ball in $\mathcal{H}$ is not GB.

Also, for a sequence $(\ell_i)_{i\geq 1}$ of positive real numbers, we define the *rectangle* by

$$R\big((\ell_i),(e_i)\big) = \left\{x=\sum_{i\geq 1}x_ie_i\in\mathcal{H} : |x_i|\leq \ell_i/2\right\}.$$

Similarly, the rectangle $R\big((\ell_i),(e_i)\big)$ is GB if and only if $(\ell_i)_{i\geq 1}\in\ell^1$ [Dud67, Proposition 6.6]. Moreover, we can explicitly compute out the intrinsic volumes of a GB rectangle:

Lemma 5.2.6. *Let $R = R\big((\ell_i),(e_i)\big)$ be a GB rectangle as above. Then*

$$V_k(R) = \sum_{i_1<i_2<\cdots<i_k}\ell_{i_1}\ell_{i_2}\cdots\ell_{i_k}.$$

Proof. Recall the formula of intrinsic volumes for orthogonal product (see [Che76, (4.4.2)] or [KR97, Proposition 4.2.3, Theorem 9.7.1] for example): for every GB convex bodies $A, B\subset$

$\mathcal{H}$ with $(A, B)_{\mathcal{H}} = 0$,

$$V_k(A + B) = \sum_{i+j=k} V_i(A) V_j(B), \tag{5.6}$$

where $V_0(A) = V_0(B) = 1$. The result follows from an induction on $k \geq 1$. □

It is worth noticing that both GB ellipsoids and GB rectangles are not maximal, *i.e.* they are all GC sets.

Another important example are infinite-dimensional hyperoctahedra. Let $(a_i)_{i\geq 1}$ be a sequence of positive numbers. Define

$$\mathrm{Oc}\big((a_i), (e_i)\big) = \left\{ v = \sum_{i\geq 1} a_i x_i e_i \in \mathcal{H} : \sum_{i\geq 1} |x_i| = 1 \right\}$$

to be the symmetric closed convex hull of $\{0\} \cup \{a_i e_i \in \mathcal{H} : i \geq 1\}$. Then $\mathrm{Oc}\big((a_i), (e_i)\big)$ is GB if and only if $a_i = O\big((\log i)^{-1/2}\big)$, and it is GC if and only if $a_i = o\big((\log i)^{-1/2}\big)$.

5.2.4 Vitale distance

Recall that the *support function* of a convex body K is defined by $h_K(x) = \sup_{v\in K}(v, x)_{\mathcal{H}}$. The support function can also be formally extended to $\mathbb{R}^{\mathbb{N}}$ by

$$h_K(\vec{x}) = \sup_{v\in K} \sum_{i\geq 1} (v, e_i)_{\mathcal{H}} x_i \tag{5.7}$$

for every $\vec{x} = (x_i)_{i\geq 1} \in \mathbb{R}^{\mathbb{N}}$.

The random variable $h_K(X)$ makes sense for an isonormal Gaußian process X and coincides with $\sup_{v\in K} X_v$, so $V_1(K) = \sqrt{2\pi}\mathbb{E}[h_K(X)]$.

Recall that for a GB convex body K, the *Steiner point* of K is defined (formally) by

$$\mathrm{Stein}(K) := \mathbb{E}\left[h_K(X) X\right], \tag{5.8}$$

where X is the isonormal Gaußian process on $\mathcal{H}$ (see [Vit01]). This definition of Steiner point is understood in the sense that it is uniquely determined by

$$\big(\mathrm{Stein}(K), v\big)_{\mathcal{H}} = \mathbb{E}[h_K(X) X_v] \in \mathbb{R}$$

for every vector $v \in V$ (see Section 5.4.1 and Section 5.4.2 for discussions on the convergence of this expectation). For finite-dimensional convex bodies in $\mathcal{H}$, this definition is the same as the original definition introduced in [Grü03]. Moreover, we will later see that the

Steiner point $\mathrm{Stein}(K)$ is exactly the barycenter of K with respect to the probability supported on the extremal points $\mathrm{Ext}(K)$ and inherited from the isonormal Gaußian process $(X_v)_{v\in\mathcal{H}}$ (see Proposition 5.4.20).

In the context of a Hilbert space $\mathcal{H}$, the Hausdorff distance between two convex bodies $K, K' \subset \mathcal{H}$ is given by $d_{\mathrm{Haus}}(K,K') = \|h_K - h_{K'}\|_{L^\infty(B_\mathcal{H})}$, where $B_\mathcal{H}$ is the closed unit ball in $\mathcal{H}$. But this distance function is insufficient for describing the behaviours of GB convex bodies: the Steiner point is not continuous with respect to the Hausdorff distance [Vit85] and neither are intrinsic volumes (a unit ball of radius r always has infinite intrinsic volumes even when $r \to 0$, while $\{0\}$ is GB with $V_k(\{0\}) = 0$ for all $k \geq 1$).

Let K, K' be two convex bodies in $\mathcal{H}$. A GB convex body $L \subset \mathcal{H}$ is said to be *equalising* K and K' if $K \subset K' + L$ and $K' \subset K + L$. So mimicking the definition of Hausdorff distance, Vitale defines in [Vit01] the *Vitale distance* for GB convex bodies by

$$d_{\mathrm{Vit}}(K,K') := \inf\left\{V_1(L) : L \subset \mathcal{H} \text{ is GB equalising } K \text{ and } K'\right\}.$$

The function $d_{\mathrm{Vit}}(\cdot,\cdot)$ yields a distance on GB convex bodies and an écart on compact convex subsets of $\mathcal{H}$.

On one hand, for any convex body $K, K' \subset \mathcal{H}$, $d_{\mathrm{Hauss}}(K,K') \leq d_{\mathrm{Vit}}(K,K')$: the diameter of L must be less than $V_1(L)$ by monotonicity of V_1, thus must be contained in a Hilbert ball with radius $V_1(L)$, and if in addition L equalises K and K', then $d_{\mathrm{Hauss}}(K,K') \leq V_1(L)$.

On the other hand, if $K, K' \subset \mathbb{R}^d$, then the GB set L that equalises K and K' in $\mathbb{R}^d$ is at most a Euclidean ball, so $d_{\mathrm{Vit}}(K,K') \leq V_1(B^d) d_{\mathrm{Haus}}(K,K')$, where B^d is the d-dimensional Euclidean unit ball.

Equipped with d_{Vit}, both the space of all GB convex bodies and the space of all GC convex bodies are complete, and the completion of finite-dimensional convex bodies under d_{Vit} is GC convex bodies [Vit01, Theorem 5]. Also, there exists a constant $C > 0$ such that $\|\mathrm{Stein}(K) - \mathrm{Stein}(K')\|_\mathcal{H} \leq C d_{\mathrm{Vit}}(K,K')$ [Vit01, Theorem 8], and this also demonstrates that the Steiner point is well-defined for all GB convex bodies.

Readers can refer to [Vit01] and [Le08] for further discussion on Vitale distance and its relation with oscillation of GB convex bodies.

5.3 Embedding into hyperbolic space

In this section, we will construct a distance function for the family of GB convex bodies in $\mathcal{H}$ and isometrically embed it into the infinite-dimensional real hyperbolic space and its boundary.

5.3.1 Embedding via hyperbolic kernel

Let us first recall the following basic fact about hyperbolic kernel: by rearranging the terms, it is not hard to see that $\beta(\cdot,\cdot)$ is a kernel of hyperbolic type if and only if for every $z \in X$, the function

$$N(x,y) := \beta(x,z)\beta(y,z) - \beta(x,y)$$

is a kernel of positive type, *i.e.* $\sum_{i,j=1}^{n} c_i c_j N(x_i,x_j) \geq 0$ for all $n \geq 1$, any $x_0, x_1, \ldots, x_n \in X$ and any $c_1, \ldots, c_n \in \mathbb{R}$.

Let $\mathcal{K}_2$ be the family of translation classes of GB convex bodies in $\mathcal{H}$ with $\dim(K) \geq 2$, *i.e.* K and K' are identified in $\mathcal{K}_2$ if there exists a $p \in \mathcal{H}$ such that $K = K' + p$.

Two main inconveniences of treating $(\mathcal{K}_2, V_2)$ is that (A3) only holds for positive numbers and that $K + (-K) \neq \{0\}$, say $B^n + (-B^n) = 2B^n$. Due to these observations, the space $\mathcal{K}_2$ looks more like the positive cone of a vector space instead of the entire space. To rule out the difficulties, let $\widetilde{\mathcal{K}_2}$ be the *real vector space* spanned by elements $\widetilde{K}$ for all $K \in \mathcal{K}_2$, with the identification $t\widetilde{K} = \widetilde{tK}$ for every $t \geq 0$ and $\widetilde{K_1} + \widetilde{K_2} = \widetilde{K_1 + K_2}$ for every $K_1, K_2 \in \mathcal{K}_2$. Under this convention, we have formally $\widetilde{K} + (-\widetilde{K}) = 0$. As a result, for any $v \in \widetilde{\mathcal{K}_2}$, it can be decomposed into $\widetilde{K_1} - \widetilde{K_2}$ for some $K_1, K_2 \in \mathcal{K}_2$. Furthermore, we can also linearly extend V_2 to $\widetilde{\mathcal{K}_2}$ by setting $V_2(-\widetilde{K_1}, \widetilde{K_2}) = -V_2(\widetilde{K_1}, \widetilde{K_2})$.

Choose an $M \in \mathcal{K}_2$ and define

$$\rho_M(v,w) := V_2\left(v, \widetilde{M}\right) V_2\left(w, \widetilde{M}\right) - V_2(M)V_2(v,w) \tag{5.9}$$

for every pair $v, w \in \widetilde{\mathcal{K}_2}$.

Proposition 5.3.1. *For any $M \in \mathcal{K}_2$, the bilinear form ρ_M given as (5.9) defines a positive semi-definite scalar product on the vector space $\widetilde{\mathcal{K}_2}$.*

Proof. It is clear from the definition that ρ_M is symmetric and bilinear. So it suffices to show the positive semi-definiteness. Since $\rho_{tM} = t^2\rho_M$ for every $t > 0$, we may assume that $V_2(M) = 1$. Take any $t > 0$ and $K_1, K_2 \in \mathcal{K}_2$, by Alexandrov-Fenchel inequality, we have

$$\begin{aligned} 0 &\leq V_2(K_1 + tM, K_2 + tM)^2 - V_2(K_1 + tM)V_2(K_2 + tM) \\ &= t^2\big[2V_2(K_1,K_2) + V_2(K_1,M)^2 + V_2(K_2,M)^2 - 2V_2(K_1,M)V_2(K_2,M) \\ &\qquad - V_2(K_1) - V_2(K_2)\big] + C_1(K_1,K_2)t + C_0(K_1,K_2), \end{aligned}$$

where $C_1(K_1,K_2)$ and $C_0(K_1,K_2)$ are constants depending only on K_1 and K_2. Because the above polynomial in t always stays positive, its leading coefficient must be positive as well,

i.e.

$$\begin{aligned} 2V_2(K_1,K_2) + V_2(K_1,M)^2 + V_2(K_2,M)^2 \geq \\ 2V_2(K_1,M)V_2(K_2,M) + V_2(K_1) + V_2(K_2). \end{aligned} \tag{5.10}$$

Now taking any $v \in \widetilde{\mathcal{K}_2}$, we may write $v = \widetilde{K_1} - \widetilde{K_2}$ and

$$\begin{aligned} \rho_M(v,v) &= \rho_M\left(\widetilde{K_1} - \widetilde{K_2}, \widetilde{K_1} - \widetilde{K_2}\right) \\ &= \rho_M\left(\widetilde{K_1}, \widetilde{K_1}\right)^2 + \rho_M\left(\widetilde{K_2}, \widetilde{K_2}\right) - 2\rho_M\left(\widetilde{K_1}, \widetilde{K_2}\right) \\ &= 2V_2(K_1,K_2) + V_2(K_1,M)^2 + V_2(K_2,M)^2 \\ &\quad - \left[2V_2(K_1,M)V_2(K_2,M) + V_2(K_1) + V_2(K_2)\right] \\ &\geq 0. \end{aligned}$$

This completes the proof. □

Let us consider the projective space of $\widetilde{\mathcal{K}_2}$, denoted by $\mathbb{P}\widetilde{\mathcal{K}_2}$.

Recall that two sets $K, K' \subset \mathcal{H}$ are homothetic if they K' is the image of K under a homothety, which is a finite combination of translations and dilations. By starting with $\widetilde{\mathcal{K}_2}$, we are taking the quotient by translations; by taking the projective space, we rule out the dilations. So the projective space $\mathbb{P}\widetilde{\mathcal{K}_2}$ contains all homothety class of GB convex bodies of dimension at least 2. We denote by $\mathbb{K}_2 \subset \mathbb{P}\widetilde{\mathcal{K}_2}$ the space of homothety classes of GB convex bodies of dimension at least 2 and the elements in it by $[K]$ for some $K \in \mathcal{K}_2$.

Proposition 5.3.2. *There is an embedding $\iota : \mathbb{K}_2 \hookrightarrow \mathbb{H}^{\alpha}_{\mathbb{R}}$ for some cardinal α.*

Proof. By [MP19, Proposition 3.3] and the discussion above, it suffices to design a kernel of hyperbolic type for $\mathbb{K}_2$. We claim that

$$\beta([K_1],[K_2]) := \frac{V_2(K_1,K_2)}{\sqrt{V_2(K_1)V_2(K_2)}}$$

is of hyperbolic type. By Proposition 5.3.1, this is equivalent to saying that for any $M \in \mathcal{K}_2$, the kernel

$$N_M([K_1],[K_2]) := \beta([K_1],[M])\beta([K_2],[M]) - \beta([K_1],[K_2]) = \frac{\rho_M\left(\widetilde{K_1}, \widetilde{K_2}\right)}{V_2(M)\sqrt{V_2(K_1)V_2(K_2)}}$$

is of positive type. Since $V_2(M) > 0$, we have

$$\sum_{i,j=1}^{n} c_i c_j N_M([K_i],[K_j]) = \rho_M\left(\sum_{i=1}^{n} \frac{c_i \widetilde{K}_i}{\sqrt{V_2(K_i)}}, \sum_{i=1}^{n} \frac{c_i \widetilde{K}_i}{\sqrt{V_2(K_i)}}\right) \Big/ V_2(M) \geq 0,$$

for any $c_1, \ldots, c_n \in \mathbb{R}$ and $K_1, \ldots, K_2 \in \mathcal{K}_2$. This finishes the proof. □

Remark 5.3.3. We notice that the orthogonal group $O(\mathcal{H})$ also acts on the homothety classes of GB convex bodies in $\mathcal{H}$ and, as mentioned in Remark 5.2.5, this action preserves the intrinsic volumes, so it induces an isometric action of $O(\mathcal{H})$ on $\iota(\mathbb{K}_2) \subset \mathbb{H}^{\alpha}_{\mathbb{R}}$.

Moreover, as per (2.3), there is an explicit formula for computing the distance of homothety classes of two GB convex bodies, namely

$$d_{\mathbb{H}}\big(\iota([K]), \iota([K'])\big) = \cosh^{-1}\left(\frac{V_2(K,K')}{\sqrt{V_2(K)V_2(K')}}\right). \tag{5.11}$$

It is immediate that $d_{\mathbb{H}}$ is continuous with respect to d_{Vit} for convex bodies with $V_2(K) = 1$: if L is a GB set body K and K', then $d_{\mathrm{Vit}}(K,K') \to 0$ implies that we can make $V_1(L) \to 0$, which further indicates that $V_2(L) \to 0$ as per [Che76, (4.4.1)] or (5.25), hence $V_2(K,K') \to 1$ and $d_{\mathbb{H}}\big(\iota([K]), \iota([K'])\big) \to 0$ after property (C4) and (C5).

For the embedding granted by Proposition 5.3.2, there is a minimal cardinal ([MP19, §3]). Nevertheless, to conclude the minimal dimension α, we still need some more information on the hyperbolic geometry of convex bodies.

5.3.2 Hyperbolic geometry of GB convex bodies

Debin and Fillastre show that the image of ι restricted to homothety classes of finite-dimensional convex bodies is geodesic [DF22]. The same result also holds for infinite-dimensional GB convex bodies:

Proposition 5.3.4. *Let $K_0, K_1 \subset \mathcal{H}$ be two GB convex bodies with $\dim(K_0), \dim(K_1) \geq 2$. Then there is a unique geodesic in $\mathbb{H}^{\alpha}_{\mathbb{R}}$ connecting $\iota([K_0])$ and $\iota([K_1])$ given by $\iota([(1-t)K_0 + tK_1])$ for $t \in [0,1]$.*

Proof. Without loss of generality, we may assume that $V_2(K_0) = V_2(K_1) = 1$. Let $a = V_2(K_0, K_1)$. By Alexandrov-Fenchel inequality, we have $a \geq 1$. Define $K_t = (1-t)K_0 + tK_1$

for $t \in [0,1]$. Then we will have

$$\begin{aligned}
&d_{\mathbb{H}}\big(\iota([K_0]),\iota([K_t])\big) + d_{\mathbb{H}}\big(\iota([K_1]),\iota([K_t])\big) \\
&= \cosh^{-1}\left(\frac{V_2(K_t,K_0)}{\sqrt{V_2(K_t)}}\right) + \cosh^{-1}\left(\frac{V_2(K_t,K_1)}{\sqrt{V_2(K_t)}}\right) \\
&= \cosh^{-1}\left(\frac{ta+(1-t)}{\sqrt{t^2+(1-t)^2+2t(1-t)a}}\right) + \cosh^{-1}\left(\frac{(1-t)a+t}{\sqrt{t^2+(1-t)^2+2t(1-t)a}}\right) \\
&= \cosh^{-1}\left(\frac{(1-t+t^2)a+(t-t^2)a^2}{t^2+(1-t)^2+2t(1-t)a}\right) =: \cosh^{-1}\big(\phi(t,a)\big).
\end{aligned}$$

If $a = 1$, then it implies that $[K_0] = [K_1] = [K_t]$ and the existence of a geodesic is automatic. So suppose that $a > 1$. Since

$$\frac{\partial}{\partial t}\phi(t,a) = \frac{(a-1)a(2t-1)}{\big(t^2+(1-t)^2+2t(1-t)a\big)^2},$$

the function has maxima $\phi(1,a) = \phi(0,a) = a$. Also, we notice that $\phi(t,a) \geq 0$. It soon follows that

$$d_{\mathbb{H}}\big(\iota([K_0]),\iota([K_t])\big) + d_{\mathbb{H}}\big(\iota([K_1]),\iota([K_t])\big) \leq \cosh^{-1}(a) = d_{\mathbb{H}}\big(\iota([K_0]),\iota([K_1])\big).$$

By triangle inequality, it forces the path $\big(\iota([K_t])\big)_{t\in[0,1]}$ to be the geodesic between $\iota([K_0])$ and $\iota([K_1])$ after a suitable parametrisation. □

A quick computation allows us to give an isometric parametrisation for the geodesic segments in $\iota(\mathbb{K}_2)$:

Corollary 5.3.5. *Let $K_0, K_1 \subset \mathcal{H}$ be two GB convex bodies with $\dim(K_0), \dim(K_1) \geq 2$. Then*

$$\iota\left(\left[\frac{K_0}{V_2(K_0)} + \frac{K_1}{V_2(K_1)}\right]\right) \in \mathbb{H}^{\alpha}_{\mathbb{R}}$$

is the midpoint on the geodesic segment between $\iota([K_0])$ and $\iota([K_1])$.

Proof. By assuming $V_2(K_0) = V_2(K_1) = 1$, we will have

$$d_{\mathbb{H}}\big(\iota([K_0]),\iota([K_0+K_1])\big) = d_{\mathbb{H}}\big(\iota([K_1]),\iota([K_0+K_1])\big) = \cosh^{-1}\left(\frac{V_2(K_0,K_1)+1}{\sqrt{V_2(K_0+K_1)}}\right),$$

which completes the proof. □

Let $\mathbb{K}$ be the homothety classes of GB convex bodies with *non-zero dimension* in $\mathcal{H}$, *i.e.* GB convex bodies that do not reduce to a singleton. Then we have the following result:

Corollary 5.3.6. *The embedding ι can be extended to $\mathbb{K} \hookrightarrow \overline{\mathbb{H}^{\alpha}_{\mathbb{R}}}$ and there is a bijection between the Gromov boundary $\partial\big(\iota(\mathbb{K})\big)$ and the projective space $\mathbb{P}\mathcal{H}$.*

Proof. It suffices to show that homothety classes of segments are embedded in $\partial\mathbb{H}^{\alpha}_{\mathbb{R}}$. Take a GB convex body $K \subset \mathcal{H}$ with $\dim(K) \geq 2$ and $V_2(K) = 1$. Let P be any segment. Then for every $t \in [0, \infty)$, we have $\dim(K + tP) \geq 2$. It is clear that the path $\big(\iota([K + tP])\big)_{t\in[0,\infty)}$ is a geodesic ray, since any finite segment of it is geodesic by Proposition 5.3.4 and

$$\lim_{t\to\infty} d_{\mathbb{H}}(\iota([K]), \iota([K + tP])) = \lim_{t\to\infty} \cosh^{-1}\left(\frac{tV_2(K, P) + 1}{\sqrt{2tV_2(K, P) + 1}}\right) = \infty.$$

In particular, the sequence $\big(\iota([K + nP])\big)_{n\in\mathbb{N}}$ is a Cauchy-Gromov sequence converging to a point $\partial\mathbb{H}^{\alpha}_{\mathbb{R}}$, denoted by $\iota([P])$.

We remark that $\iota([P])$ does not depend on the choice of K. Indeed, for two distinct GB convex bodies $K, K' \subset \mathcal{H}$ with $\dim(K), \dim(K') \geq 2$ and $V_2(K) = V_2(K') = 1$, we have

$$d_{\mathbb{H}}\big(\iota([K + tP], \iota([K' + tP]\big) = \cosh^{-1}\left(\frac{tV_2(K + K', P) + 1}{\sqrt{(1 + 2tV_2(K, P))(1 + 2tV_2(K', P)}}\right)$$

is bounded uniformly in $t \in [0, \infty)$, thus both $\iota([K + tP])$ and $\iota([K' + tP])$ converge to the same point on $\partial\mathbb{H}^{\alpha}_{\mathbb{R}}$.

We shall show that ι is injective. Given two segments $P, P' \subset \mathcal{H}$ that are not in the same direction, then we have

$$\begin{aligned} &d_{\mathbb{H}}\big(\iota([K + nP], \iota([K + mP']\big) \\ &= \cosh^{-1}\left(\frac{1 + nV_2(K, P) + mV_2(K, P) + nmV_2(P, P')}{\sqrt{(1 + 2nV_2(K, P))(1 + 2mV_2(K, P'))}}\right) \to \infty \end{aligned}$$

as $n, m \to \infty$. This means that the geodesic between $\iota([K])$ and $\iota([P])$ is not fellow travelling with that between $\iota([K])$ and $\iota([P'])$ and forces $\iota([P]) \neq \iota([P'])$.

Finally, we remark that the homothetic classes of 1-dimensional convex bodies are in bijection with $\mathbb{P}\mathcal{H} \simeq \mathrm{Gr}(1, \mathcal{H})$ by sending a segment passing through the origin $0 \in \mathcal{H}$ to the subspace generated by it. Hence $\mathbb{P}\mathcal{H}$ is in bijection with $\partial\big(\iota(\mathbb{K})\big)$ via ι. □

Remark 5.3.7. The fact that $\mathbb{P}\mathcal{H} = \partial\iota(\mathbb{K})$ is not trivial. The Hilbert space that we use to construct $\mathbb{H}^{\infty}_{\mathbb{R}}$ is an abstract Hilbert space $\widetilde{\mathcal{K}_2}$ obtained via GNS construction and the projective space $\mathbb{P}\widetilde{\mathcal{K}_2}$, which is in bijection with $\partial\mathbb{H}^{\infty}_{\mathbb{R}}$, does not *a priori* have a relation with $\mathbb{P}\mathcal{H}$. It is still unclear, for example, whether ι restricted to the segments, $\iota : \mathbb{K} \setminus \mathbb{K}_2 \to \partial\mathbb{H}^{\infty}_{\mathbb{R}}$ is surjective or not. The same remark implies that the action of $O(\mathcal{H})$ by isometries on $\iota(\mathbb{K}_2)$, as mentioned in Remark 5.3.3, is also non-trivial.

Since the space $\mathbb{H}^\alpha_{\mathbb{R}}$ is regularly geodesic, we can further generalise the result of Proposition 5.3.4 to the boundary as follows:

Corollary 5.3.8. *For any distinct $[K_1], [K_2] \in \mathbb{K}$, the geodesic connecting $\iota([K_1])$ to $\iota([K_2])$ is the path $\iota([tK_1 + (1-t)K_2])$ for $t \in (0,1)$ under a suitable parametrisation.*

In the hyperbolic structure, any GB convex bodies can be approximated by the finite-dimensional convex bodies contained in it.

Proposition 5.3.9. *Let K be a GB convex body in $\mathcal{H}$ and $(K_n)_{n\geq 1}$ be a sequence of finite-dimensional convex bodies contained in K such that $V_2(K_n) \to V_2(K)$ as $n \to \infty$. Then $\iota([K_n]$ converges to $\iota([K])$.*

Proof. Indeed, the Alexandrov-Fenchel's inequality and the monotonicity of V_2 gives the following estimation

$$1 \leq \frac{V_2(K_n + K) - V_2(K_n) - V_2(K)}{2\sqrt{V_2(K_n)V_2(K)}} \leq \frac{4V_2(K) - V_2(K_n) - V_2(K)}{2\sqrt{V_2(K_n)V_2(K)}} \to 1$$

as $n \to \infty$. □

Let $\mathbb{K}_{2,f}$ be the homothety classes of finite-dimensional convex bodies that do not reduce to a singleton or a segment. In view of Proposition 5.3.9, if $\overline{\iota(\mathbb{K}_{2,f})}$ is the completion of $\iota(\mathbb{K}_{2,f})$ in $\mathbb{H}^\alpha_{\mathbb{R}}$, then $\iota(\mathbb{K}_2) \subset \overline{\iota(\mathbb{K}_{2,f})}$.

Corollary 5.3.10. *The space $\iota(\mathbb{K}_2)$ is separable and the minimal dimension $\alpha \leq \aleph_0$.*

Proof. For any $d \geq 2$, the image in $\mathbb{H}^\alpha_{\mathbb{R}}$ of the homothety classes of the convex bodies in $\mathbb{R}^d$ is homeomorphic to the space of all convex bodies K in $\mathbb{R}^d$ with $\mathrm{Stein(K)} = 0$ and $V_2(K) = 1$, endowed with Hausdorff distance [DF22]. So it is separable. It soon follows that $\iota(\mathbb{K}_{2,f})$ is a countable union of separable spaces and is thus separable. As a subspace of $\overline{\iota(\mathbb{K}_{2,f})}$, $\iota(\mathbb{K}_2)$ is also separable. □

Proposition 5.3.11. *The minimal dimension α for the embedding $\mathbb{K} \to \overline{\mathbb{H}^\alpha_{\mathbb{R}}}$ is $\aleph_0$.*

Proof. Corollary 5.3.10 proves one side, so it remains to show that $\alpha \geq \aleph_0$. Suppose *ab absurdo* that $\alpha < \aleph_0$. Then $\partial\mathbb{H}^\alpha_{\mathbb{R}}$ is homeomorphic to $S^{\alpha-1}$. Let $(e_i)_{i\geq 1}$ be an orthonormal system and let $\sigma_i = \{x \in \mathcal{H} : x = te_i,\ t \in [0,1]\}$ be the corresponding unit segments. Suppose that σ_i are sent to $\eta_i \in \partial\mathbb{H}^\alpha_{\mathbb{R}}$ via ι. Passing to a subsequence, we may assume by compactness of $S^{\alpha-1}$ that $(\eta_i)_{i\geq 1}$ converges to some η in $\partial\mathbb{H}^\alpha_{\mathbb{R}}$.

For each $i \geq 1$, we choose a Cauchy-Gromov sequence $\left(\iota([K_i^{(n)}])\right)_{n\geq 1}$ along the geodesic $[\eta_i, \eta_{i+1}]$ that converges to η_i. By Corollary 5.3.8, the convex bodies $K_i^{(n)}$ are rectangles of

the form $t\sigma_i + (1-t)\sigma_{i+1}$. For convenience reasons, fix a rectangle

$$R = \left\{ x_1 e_1 + x_2 e_2 \in \mathcal{H} : -\frac{1}{2} \leq x_1, x_2 \leq \frac{1}{2} \right\}.$$

By choosing $K_i^{(i)}$ so that $\langle \eta_i, \iota([K_i^{(i)}]) \rangle_{\iota([R])} \geq 2^i$, the δ-hyperbolicity of $\mathbb{H}^\alpha_{\mathbb{R}}$ yields that

$$\langle \iota([K_i^{(i)}]), \eta \rangle_{\iota([R])} \geq \min \left(\langle \eta_i, \eta \rangle_{\iota([R])}, \langle \eta_i, \iota([K_i^{(i)}]) \rangle_{\iota([R])} \right) - 2\delta \to \infty$$

as $i \to \infty$. Hence $\left(\iota([K_i^{(i)}])\right)_{i \geq 1}$ is a Cauchy-Gromov sequence converging to η.

Let $t_i \in (0,1)$ be such that

$$K_i^{(i)} = t_i \sigma_i + (1-t_i)\sigma_{i+1}.$$

We may also assume *a posteriori* that $t_i \to 0$ as $i \to \infty$. For any $\lambda \in [0,1]$, we pose

$$K_\lambda^i = \lambda K_i^{(i)} + (1-\lambda) K_{i+1}^{(i+1)} = \lambda t_i \sigma_i + (\lambda - \lambda t_i + t_{i+1} - \lambda t_{i+1})\sigma_{i+1} + (1-\lambda) t_{i+1} \sigma_{i+2}.$$

Applying Lemma 5.2.6, we have for sufficiently large i

$$\begin{aligned} d_{\mathbb{H}}\left(\iota([R]), \iota([K_\lambda^i])\right) &= \cosh^{-1}\left(\frac{\frac{1}{2}\left(V_2(R+K_\lambda^i) - V_2(R) - V_2(K_\lambda^i)\right)}{\sqrt{V_2(R)V_2(K_\lambda^i)}} \right) \\ &\sim \cosh^{-1}\left(\frac{1}{\sqrt{\lambda(1-\lambda)}} \right) \geq \cosh^{-1}(2), \end{aligned}$$

where we have used the asymptotic identification $t_i, t_{i+1} \sim 0$ as i becomes large enough. Hence by (2.7), we have

$$\langle \iota([K_i^{(i)}]), \iota([K_{i+1}^{(i+1)}]) \rangle_{\iota([R])} \leq \min_{\lambda \in [0,1]} d_{\mathbb{H}}\left(\iota([R]), \iota([K_\lambda^i])\right) \sim \cosh^{-1}(2) < \infty,$$

which contradicts to $\left(\iota([K_i^{(i)}])\right)_{i \geq 1}$ being Cauchy-Gromov. □

For convenience, we will simply denote in the sequel by $\mathbb{H}^\infty_{\mathbb{R}}$ the $\aleph_0$-dimensional real hyperbolic space, in which $\mathbb{K}_2$ is embedded.

Moreover, it is worth remarking that the homothety classes of polytopes are dense in $\iota(\mathbb{K}_2)$.

Proposition 5.3.12. *The image of homothety classes of polytopes with dimension at least two is dense in* $\iota(\mathbb{K}_2) \subset \mathbb{H}^\infty_{\mathbb{R}}$.

Proof. By the remark [Che76, (3.9.1)] (this is an important fact and will be used frequently in the sequel), for any GB convex body $K \subset \mathcal{H}$ with dimension at least 2, $V_2(K)$ is the supremum amongst $V_2(P)$ of polytopes P contained in K. Then it is possible to choose a sequence $(P_n)_{n\geq 1}$ of polytopes contained in K such that $V_2(P_n) \to V_2(K)$ as $n \to \infty$. Then

$$\begin{aligned} d_{\mathbb{H}}\big(\iota([P_n]), \iota([K])\big) &= \cosh^{-1}\left(\frac{V_2(P_n+K) - V_2(P_n) - V_2(K)}{2\sqrt{V_2(P_n)V_2(K)}}\right) \\ &\leq \cosh^{-1}\left(\frac{V_2(2K) - V_2(P_n) - V_2(K)}{2\sqrt{V_2(P_n)V_2(K)}}\right) \\ &\to \cosh^{-1}(1) = 0 \end{aligned}$$

as $n \to \infty$. □

5.3.3 Examples and non-examples of Cauchy sequences

At the end of [DF22], Debin and Fillastre ask about the completion of $\iota(\mathbb{K}_{2,f})$ inside of $\mathbb{H}^\infty_{\mathbb{R}}$. One pathological phenomenon observed by Debin and Fillastre is that there are Cauchy sequences in $\iota(\mathbb{K}_{2,f})$ that do not converge to any GB convex body. In this section, some more examples will be presented. In fact, these examples suggest that this is the only ill-behaved case.

As we remark, $\iota(\mathbb{K}_2) \subset \overline{\iota(\mathbb{K}_{2,f})}$. But the converse is not true, *i.e.* $\iota(\mathbb{K}_2)$ is not the completion of $\iota(\mathbb{K}_{2,f})$ in $\mathbb{H}^\infty_{\mathbb{R}}$. Here are several examples:

Example 5.3.13 (Unit balls)**.** Let $B^n \subset \operatorname{span}(e_1, \dots, e_n)$ be the unit ball of dimension n in $\mathcal{H}$. Then by the Steiner formula (5.3), one can compute that $V_1(B^n) = n\kappa_n/\kappa_{n-1}$ and $V_2(B^n) = (n-1)\pi$. Using the Steiner formula (5.3), for $m \geq n$

$$\operatorname{vol}_m(B^n + rB^m) = \sum_{k=0}^{m} r^{m-k}\kappa_{m-k}V_k(B^n + B^m) = \sum_{k=0}^{m}(r-1)^{m-k}\kappa_{m-k}V_k(B^n),$$

and comparing the terms for $k = 0, 1, 2$ while $r \to \infty$, we are able to compute $V_2(B^n + B^m)$ in terms of κ_k's, so hence $V_2(B^n, B^m)/\sqrt{V_2(B^n)V_2(B^m)}$. By Stirling's approximation, we can deduce that $\big(\iota([B^n])\big)_{n\geq 2}$ is a Cauchy sequence in $\mathbb{H}^\infty_{\mathbb{R}}$. This is already known in [DF22, §4]. We remark that $V_1(B^n)/\sqrt{2V_2(B^n)} \to 1$ as $n \to \infty$.

Example 5.3.14 (Non GB rectangles)**.** Let $(\ell_i)_{i\geq 1}$ be a sequence of strictly positive numbers. Suppose that $(\ell_i)_{i\geq 1}$ is not in ℓ^1 so that $R\big((\ell_i), (e_i)\big)$ is not GB. But its n-dimensional sections

$R_n \coloneqq \prod_{i=1}^n [-\ell_i/2, \ell_i/2]$ still define a Cauchy sequence in $\mathbb{H}$ if, and only if,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \ell_i^2}{\left(\sum_{i=1}^n \ell_i\right)^2} = 0,$$

or equivalently if

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \ell_i}{\sqrt{2\sum_{1\le i<j\le n} \ell_i\ell_j}} = \lim_{n\to\infty} \frac{V_1(R^n)}{\sqrt{2V_2(R^n)}} = 1.$$

The proof of this claim is an asymptotic analysis exercise: it suffices to use

$$\sum_{1\le i<j\le n} \ell_i\ell_j \sim \frac{1}{2}\left(\sum_{i=1}^n \ell_i\right)^2$$

to prove the necessity and for the sufficiency, we can deduce a contradiction by assuming

$$\sum_{i=1}^n \ell_i^2 \sim k\left(\sum_{i=1}^n \ell_i\right)^2$$

for some $k \in (0,1]$. In particular, if ℓ_i is of at most polynomial growth, then the corresponding sections converge; but they diverge when ℓ_i is exponentially increasing. Moreover, given any two sequences $(a_i)_{i\ge1}$ and $(b_i)_{i\ge1}$ as in the claim above, we likewise define the n-dimensional sections R_n and R'_n. By Cauchy-Schwarz inequality,

$$0 \le \frac{\sum_{i=1}^n a_ib_i}{\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)} \le \frac{\left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}}{\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)} \to 0$$

as $n, m \to \infty$. Hence

$$\frac{V_2(R_n, R'_n)}{\sqrt{V_2(R_n)V_2(R'_n)}} \sim \frac{\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) - \sum_{i=1}^n a_ib_i}{\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)} \to 1$$

as $n \to \infty$. This shows that all convergent n-dimensional sections that are not converging to a GB rectangle converge to the same point in $\mathbb{H}^\infty_{\mathbb{R}}$.

Example 5.3.15 (Comparison between rectangles and balls)**.** Let $I_n = \prod_{i=1}^n [-1,1]$ and B^n be the unit ball as in Example 5.3.13. Both $(I_n)_{n\ge2}$ and $(B^n)_{n\ge2}$ define *a priori* convergent sequences in $\mathbb{H}^\infty_{\mathbb{R}}$. Since I_n and B^n both lie in $\operatorname{span}(e_1, \dots, e_n) \simeq \mathbb{R}^n$, the identity (see [Sch14,

pp.298] for example)

$$V_2(K,K') = \frac{n-1}{2\kappa_{n-2}}\left((h_K,h'_K)_{L^2(S^{n-1})} - \frac{1}{n-1}\langle \nabla h_K, \nabla h_{K'}\rangle_{L^2(S^{n-1})}\right), \tag{5.12}$$

where ∇ is the gradient on S^{n-1}, yields

$$V_2(I_n,B^n) = \frac{n-1}{2\kappa_{n-2}}\int_{S^{n-1}} h_{I_n}(v)\,\mathrm{d}v. \tag{5.13}$$

Notice that $h_{I_n}(v) = \sum_{k=1}^n |v_k|$ for every $v \in S^{n-1}$, where $v_k = (v,e_k)_{\mathcal{H}}$. Applying orthogonal decomposition and Fubini's theorem to (5.13), we get

$$V_2(I_n,B^n) = \sum_{k=1}^n \frac{(n-1)\pi\kappa_{n-3}}{\kappa_{n-2}}\int_{-1}^1 |t|\left(1-t^2\right)^{\frac{n-3}{2}}\,\mathrm{d}t = \frac{2n\pi\kappa_{n-3}}{\kappa_{n-2}}.$$

Hence by Stirling's approximation

$$d_{\mathbb{H}}\big(\iota([I_n]),\iota([B^n])\big) = \cosh^{-1}\left(\frac{2n\pi\kappa_{n-3}}{\kappa_{n-2}}\frac{1}{\sqrt{2\pi n(n-1)^2}}\right) \to 0$$

as $n \to \infty$. As a result, those two sequences converge to the same point in $\mathbb{H}^\infty_{\mathbb{R}}$.

Now, let $O \in \mathbb{H}^\infty_{\mathbb{R}}$ be the limit of homothety classes of n-dimensional unit balls as in Example 5.3.13.

Example 5.3.16 (Other non-GB limits)**.** It is possible to construct some more Cauchy sequences of $\iota(\mathbb{K}_2)$ in $\mathbb{H}^\infty_{\mathbb{R}}$ that do not converge to the image of any homothety class of a GB convex body in $\mathcal{H}$. Let $I = \{te_1 : t \in [0,1]\}$ be the unit interval. Then for every $n \geq 1$, $K_n := I + c_n B^{n+1}$ is a GB convex body in $\mathcal{H}$ of dimension at least 2 for some $c_n > 0$. Corollary 5.3.6 indicates that $\iota([K_n])$ is on the geodesic between $\iota([B^{n+1}])$ and $\iota([I]) \in \partial\mathbb{H}^\infty_{\mathbb{R}}$. As $\mathbb{H}^\infty_{\mathbb{R}}$ is regularly geodesic, by choosing a suitable parameter $c_n > 0$ for every $n \geq 1$ according to Corollary 5.3.5, we can make sure it converges in $\mathbb{H}^\infty_{\mathbb{R}}$ to a point on the geodesic between O and $\iota([I])$.

5.4 Malliavin calculus and intrinsic volumes

Debin and Fillastre show that the hyperbolisation process can be realised by treating the Sobolev space via spherical harmonics [DF22]. When it comes to infinite dimension, spherical harmonics will no longer be available since the unit sphere in the Hilbert space $\mathcal{H}$ does not admit a Haar measure (due to the converse Haar's theorem of Weil [Wei65]). In infinite dimension, Malliavin calculus becomes indispensable.

5.4.1 Wiener-Itô decomposition and Malliavin derivative

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $L^2(\Omega)$ be the space of real L^2-functions on Ω with respect to the probability measure $\mathbb{P}$. Let $\mathcal{H}$ be a separable Hilbert space on $\mathbb{R}$ and X be an $\mathcal{F}$-measurable isonormal Gaußian process on $\mathcal{H}$.

Recall that the *Hermite polynomials* are polynomials $(H_n)_{n\geq 0}$ determined by the recurrence relation $H_0 = 1$ and $H_n'(x) = nH_{n-1}(x)$ for all $n \geq 1$ and that $\mathbb{E}[H_n(Z)] = 0$ for all $n \geq 1$, where $Z \sim \mathcal{N}(0,1)$ is a normal Gaußian random variable with variance 1. For each $n \geq 0$, the *n-th Wiener chaos* $\mathfrak{H}_n$ is defined as the closure in $L^2(\Omega)$ of the linear span of the set $\{H_n(X_v) : v \in \mathcal{H}\}$, where X is the concerned isonormal Gaußian process on the Hilbert space $\mathcal{H}$. In particular, the space $\mathfrak{H}_0$ consists of all constant functions and $\mathfrak{H}_1 = \{X_v : v \in \mathcal{H}\}$.

The terminology follows from the original article [Wie38] where the construction is similar to the one introduced here, yet the equivalent definition appears decades later in [Seg56]. The following decomposition is considered due to [Itô51] and reader can refer to [Nua06, Theorem 1.1.1] or [Wie33, pp.64] for a detailed proof:

Theorem 5.4.1 (Wiener-Itô decomposition)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$, $L^2(\Omega)$ and $\mathfrak{H}_n$ be as above. Then one has an orthogonal decomposition $L^2(\Omega) = \overline{\bigoplus}_{n\geq 0}\mathfrak{H}_n$.*

Remark 5.4.2. Here the symbol $\overline{\bigoplus}$ refers to the Hilbert space direct sum, which is the closure of algebraic direct sum inside of a Hilbert space.

Let $d > 0$ be an integer and $C_P^\infty(\mathbb{R}^d)$ be the space of smooth functions on $\mathbb{R}^d$ which, together with all their partial derivatives, have at most polynomial growth. By $\mathcal{S}$ one denotes the class of random variables $\varphi : \Omega \to \mathbb{R}$ such that there exists an $n \in \mathbb{N}$, vectors $v_1, \ldots, v_n \in \mathcal{H}$ and a function $f \in C_P^\infty(\mathbb{R}^d)$ verifying

$$\varphi(\omega) = f\big(X_{v_1}(\omega), \cdots, X_{v_n}(\omega)\big)$$

for almost every $\omega \in \Omega$. The random variables $\varphi \in \mathcal{S}$ are then called *smooth random variables* and the function $f \in C_P^\infty(\mathbb{R}^d)$ appearing in the definition for $\varphi \in \mathcal{S}$ is then called a *(smooth) representation of φ*.

By using the derivatives of its smooth representation, the definition of Malliavin derivative for smooth random variables soon follows:

Definition 5.4.3 (Malliavin derivative)**.** Let $\varphi(\omega) = f\big(X_{v_1}(\omega), \cdots, X_{v_n}(\omega)\big)$ be a smooth random variable as above and f be its smooth representation. Then the *Malliavin derivative $D\varphi$* of φ is defined by the $\mathcal{H}$-valued random variable

$$D\varphi := \sum_{j=1}^{n} \partial_j f\big(X_{v_1}(\omega), \cdots, X_{v_n}(\omega)\big) v_j : \Omega \to \mathcal{H},$$

where $\partial_j f$ is the j-th partial derivative of f.

Following from Cameron-Martin theorem (see for example [Bog98]), this definition does not depend on the smooth representations of random variables.

Moreover, Malliavin derivative enjoys the integral-by-parts property, namely

$$\mathbb{E}\left[(D\varphi, v)_{\mathcal{H}}\right] = \mathbb{E}\left[\varphi X_v\right] \tag{5.14}$$

for every $v \in \mathcal{H}$ and every $\varphi \in \mathcal{S}$ [Nua06, Lemma 1.2.1].

For each $N \geq 0$, the space $\bigoplus_{n=0}^{N} \mathfrak{H}_n$ consists of smooth random variables with polynomial representations of degree at most N. By Stone-Weierstraß Theorem, the set $\mathcal{S} \cap \bigoplus_{n=0}^{N} \mathfrak{H}_n$ is dense in $\bigoplus_{n=0}^{N} \mathfrak{H}_n$. It soon follows from Wiener-Itô decomposition that $\mathcal{S}$ is dense in $L^2(\Omega)$. So it is possible to define the Malliavin derivative of an L^2-random variable via approximating by smooth ones.

Let E and F be two Banach spaces. Let $A : \mathrm{Dom}(A) \to F$ be an (unbounded) operator from E to F, where $\mathrm{Dom}(A)$ is a subspace of E on which A is defined. Such an operator is called *closed*, if its graph $\Gamma(A) := \{(x, Ax) \in E \times F : x \in \mathrm{Dom}(A)\}$ is closed. An operator is called *closable*, if the closure of its graph is again the graph of an operator, called the closure of A.

By virtue of the following result, the Malliavin derivative can be extended to the entire $L^2(\Omega)$ [Nua06, Proposition 1.2.1]:

Proposition 5.4.4. *The Malliavin derivative $D : \mathcal{S} \subset L^2(\Omega) \to L^2(\Omega; \mathcal{H})$ is closable.*

Abusing notation, let D be again the closure of Malliavin derivative defined on $\mathcal{S}$ and by density $D : L^2(\Omega) \to L^2(\Omega; \mathcal{H})$ gives the Malliavin derivative for all L^2-random variables. By passing to limits, the formula (5.14) also holds for every $v \in \mathcal{H}$ and every $\varphi \in L^2(\Omega)$.

The *Sobolev space* $\mathbb{D}^{1,2}$ is the Hilbert space defined by the random variables $\varphi \in L^2(\Omega)$ such that the norm

$$\|\varphi\|_{1,2} := \left(\mathbb{E}\left[|\varphi|^2\right] + \mathbb{E}\left[\|D\varphi\|_{\mathcal{H}}^2\right]\right)^{1/2}$$

is finite, where the inner product is given by $\langle \psi, \varphi \rangle_{1,2} = \mathbb{E}[\psi\varphi] + \mathbb{E}\left[(D\psi, D\varphi)_{\mathcal{H}}\right]$. In particular, for each $n \geq 0$, $\mathfrak{H}_n \subset \mathbb{D}^{1,2}$.

Let us introduce the adjoint of the Malliavin derivative. One defines the adjoint operator δ, called the *divergence operator* or *Skorohod integral*, by setting taking a random variable δV for an $\mathcal{H}$-valued random variable $V \in L^2(\Omega; \mathcal{H})$ such that

$$\mathbb{E}[X(\delta V)] = \mathbb{E}[(V, DX)_{\mathcal{H}}] \tag{5.15}$$

for every $X \in L^2(\Omega)$. Moreover, Meyer's inequality implies that $\delta : L^2(\Omega; \mathcal{H}) \to L^2(\Omega)$ is well-defined and is a continuous operator [Nua06, Proposition 1.5.4].

One also defines the *divergence operator* by $\triangle = \delta D : L^2(\Omega) \to L^2(\Omega)$. The following result seems to be folklore. The proof is not difficult, but it is crucial for our purpose, so a brief proof is provided:

Proposition 5.4.5. *For every $n \geq 0$, the n-th Wiener chaos $\mathfrak{H}_n$ is the eigenspace of Ornstein-Uhlenbeck operator $\triangle$ for the eigenvalue n.*

Proof. Let $(e_i)_{i\geq 1}$ be an orthonormal basis of $\mathcal{H}$. Let $\vec{k}$ be a multi-index on $\mathbb{N}$ and define

$$\Phi_{\vec{k}} := \prod_{i \in \mathrm{supp}(\vec{k})} H_{k_i}(X_{e_i}), \tag{5.16}$$

where the H_j are the Hermite polynomials and X is an isonormal Gaußian process on $\mathcal{H}$. By definition, the collection of $\Phi_{\vec{k}}$ with $|\vec{k}| = n$ is dense in $\mathfrak{H}_n$. By the recurrence relation $DH_n(X_v) = nH_{n-1}(X_v)v$, one has

$$D\Phi_{\vec{k}} = \sum_i k_i \Phi_{\vec{k}-\delta_i} e_i,$$

where $\delta_i(j) = \delta_{i,j}$ is the Kronecker multi-index. Recall that by [Nua06, (1.46)], one has $\delta(\varphi v) + (D\varphi, v)_{\mathcal{H}} = \varphi X_v$ for every $\varphi \in \mathcal{S}$ and every $v \in \mathcal{H}$. Hence

$$\begin{aligned}\triangle\Phi_{\vec{k}} &= \sum_i k_i \Phi_{\vec{k}-\delta_i} X_{e_i} - \sum_{i,j} k_i\big(k_j - \delta_i(j)\big)\Phi_{\vec{k}-\delta_i-\delta_j}(e_i, e_j)_{\mathcal{H}} \\ &= \sum_i k_i \left(\Phi_{\vec{k}-\delta_i} X_{e_i} - (k_i - 1)\Phi_{\vec{k}-2\delta_i}\right).\end{aligned}$$

Since the Hermite polynomials also enjoy the relations

$$H_{k_i-1}(x)x - (k_i - 1)H_{k_i-2}(x) = H_{k_i}(x),$$

it soon follows that $\triangle\Phi_{\vec{k}} = n\Phi_{\vec{k}}$, which extends to the entire $\mathfrak{H}_n$ by density. This proves the claimed. □

5.4.2 Support functions as random variables

In conformity with Section 5.2.4, let K be a GB convex body in $\mathcal{H}$ and let h_K be its support function. Then $h_K(X)$ becomes a random variable over Ω. In particular, 5.4 implies that K is a GB convex body if and only if the random variable $h_K \in L^1(\Omega)$.

Let $P = \overline{\mathrm{co}}(v_1, \ldots, v_n)$ be the closed convex hull of n points $v_1, \ldots, v_n \in \mathcal{H}$, *i.e.* the

polytope generated these points. By [Che76, Théorème 3.10],

$$V_2(P) = \pi \mathbb{E}\left[h_P(X)^2 - \|\sigma_P(X)\|_{\mathcal{H}}^2\right] \tag{5.17}$$

where $\sigma_P(X)$ is the $\mathcal{H}$-valued random variable such that $(\sigma_P(X), X) = h_P(X)$, and for any two polytopes $P, P' \subset \mathcal{H}$,

$$V_2(P, P') = \pi \mathbb{E}\left[h_P(X) h_{P'}(X) - (\sigma_P(X), \sigma_{P'}(X))_{\mathcal{H}}\right]. \tag{5.18}$$

However, since $P, P' \subset \mathbb{R}^d = \operatorname{span}(e_1, \dots, e_d)$ for some $d > 0$, after [Sch14, §1.7], it is clear that $\sigma_P(X) = \sigma_P((X_{e_1}, \dots, X_{e_d})) = \nabla h_P((X_{e_1}, \dots, X_{e_d}))$. As a result, (5.18) is exactly the same as (5.12).

Remark 5.4.6. More precisely, in [Sch14, §1.7], it is seen that $\sigma_P(X)$ is the *Fréchet derivative* of the support function $h_K(X)$. For more information about the connection between Malliavin derivative and Fréchet derivative of functions in Wanatabe-Sobolev space, one can refer to for example [CKL06; Kru14]. But the situation here is much simpler.

Since a polytope is always bounded, $\sigma_P(X) \in L^2(\Omega; \mathcal{H})$. It soon follows from (5.17) that $h_P(X) \in L^2(\Omega)$. Moreover, the random variable $h_P(X)$ has a almost-everywhere differentiable representation, namely $h_K : \mathbb{R}^d \to \mathbb{R}$, it soon yields $\sigma_P(X) = Dh_K(X)$.

Although the Malliavin derivative is closable, it is in general not continuous. To generalise the formulae (5.17) and (5.18), the following lemmata will be needed:

Lemma 5.4.7. *Let $\varphi_n \in \mathbb{D}^{1,2}$ be a sequence of random variables such that $\varphi_n \to \varphi$ in $L^2(\Omega)$. Suppose that $\sup_{n\in\mathbb{N}} \mathbb{E}[\|D\varphi_n\|_{\mathcal{H}}^2] < \infty$. Then $\varphi \in \mathbb{D}^{1,2}$ and $D\varphi_n$ converges weakly in $L^2(\Omega; \mathcal{H})$ to $D\varphi$, or equivalently, φ_n converges weakly to φ in $\mathbb{D}^{1,2}$.*

Proof. Since φ_n converges to φ in $L^2(\Omega)$ and $\sup_{n\in\mathbb{N}} \mathbb{E}[\|D\varphi_n\|_{\mathcal{H}}^2] < \infty$, $(\varphi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{D}^{1,2}$. As $\mathbb{D}^{1,2}$ is a Hilbert space, by Banach-Alaoglu theorem, one can extract a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ that converges weakly to a function $\phi \in \mathbb{D}^{1,2}$. In particular, the sequence φ_{n_k} converges to ϕ in $L^2(\Omega)$, which indicates that $\phi = \varphi$ and $\varphi \in \mathbb{D}^{1,2}$. These arguments can be applied to any subsequence of φ_n. It follows that φ_n converges weakly to φ in $\mathbb{D}^{1,2}$. This proves the claimed result. □

Lemma 5.4.8. *Let $K \subset \mathcal{H}$ be a GB convex body and let $h_K(X) \in \mathbb{D}^{1,2}$ be its support function. If there exists a sequence of polytopes $(P_m)_{m\geq 1}$ such that $h_{P_m}(X)$ converges weakly to $h_K(X)$ in $\mathbb{D}^{1,2}$, then there exists another $(\widetilde{P}_m)_{m\geq 1}$ such that $h_{\widetilde{P}_m}(X) \to h_K(X)$ in $\mathbb{D}^{1,2}$.*

Proof. By Banach-Saks theorem, one can extract a subsequence P_{n_k} such that the Cesàro sums

$$\widetilde{P}_m = \frac{P_{n_1} + P_{n_2} + \cdots + P_{n_m}}{m}$$

have support functions $h_{\widetilde{P}_m}(X)$ converging to $h_K(X)$ in $\mathbb{D}^{1,2}$. □

Proposition 5.4.9. *Let $K \subset \mathcal{H}$ be GB convex bodies and X be an isonormal Gaußian process on $\mathcal{H}$. Then there exists a sequence of polytopes $(\widetilde{P}_N)_{N\geq 1}$ such that $V_2(\widetilde{P}_N) \to V_2(K)$ and $h_{\widetilde{P}_N}(X)$ converges to $h_K(X)$ in $\mathbb{D}^{1,2}$ as $N \to \infty$.*

Proof. Let $(P_n)_{n\in\mathbb{N}}$ be an increasing sequence of polytopes included in a GB convex body K such that $V_i(P_n) \to V_i(K)$ as $n \to \infty$ for $i = 1, 2$. Since $h_{P_n}(X) \leq h_{P_{n+1}}(X) \leq h_K(X)$, it turns out that $h_{P_n}(X)$ converges to $h_K(X)$ in $L^1(\Omega)$. By passing to a subsequence, we may assume that $h_{P_n}(X)$ converges to $h_K(X)$ almost surely. By monotone convergence theorem, $h_{P_n}(X)$ also converges to $h_K(X)$ in $L^2(\Omega)$. Since K is bounded, there exists an $R > 0$ such that the random variable taking values amongst the extremal points of P_n satisfies $\|Dh_{P_n}(X)\|_{\mathcal{H}} < R$ for every $n \in \mathbb{N}$. By Lemma 5.4.7, $h_K(X) \in \mathbb{D}^{1,2}$ and $h_{P_n}(X)$ converges weakly to $h_K(X)$ in $\mathbb{D}^{1,2}$. By Lemma 5.4.8, it is possible to construct polytopes $\widetilde{P}_N$ contained in K with $V_1(\widetilde{P}_N) \to V_1(K)$ as $N \to \infty$ and that $h_{\widetilde{P}_N}(X)$ converges to $h_K(X)$ in $\mathbb{D}^{1,2}$. Note that Proposition 5.3.12 asserts that $V_2(P_i, P_j) \to V_2(K)$ as $i, j \to \infty$. Moreover, by (C1) and (C4), we have

$$V_2(\widetilde{P}_N) = \frac{\sum_{i,j=1}^{N} V_2(P_{n_i}, P_{n_j})}{N^2} \to V_2(K)$$

as $N \to \infty$. □

Remark 5.4.10. In [Che76, Proposition 3.6'], a more general form for $V_k(K)$ with $k \geq 1$ is given. But it is yet unclear whether $V_k(K)$ can also be rewritten in a similar way as (5.1).

Corollary 5.4.11. *Let K be a GB convex body in $\mathcal{H}$ and X be an isonormal Gaußian process over $\mathcal{H}$. Then there exists $M > 0$ such that the Malliavin derivative of its support function satisfies $\|Dh_K(X)\|^2_{\mathcal{H}} \leq M$ almost surely.*

Proof. Since K is compact, there exists $M > 0$ such that $K \subset B_{\mathcal{H}}(0, M)$. Let $\widetilde{P}_N$ be as in the proof of Proposition 5.4.9. Then $Dh_{\widetilde{P}_N}(X)$ converges to $Dh_K(X)$ in $L^2(\Omega; \mathcal{H})$. By passing to a subsequence, the convergence is with probability 1. But $\widetilde{P}_N \subset K \subset B_{\mathcal{H}}(0, M)$, so $\|Dh_{\widetilde{P}_N}(X)\|^2_{\mathcal{H}} \leq M$. The desired result follows from letting N tend to ∞. □

Proof of Theorem 5.1.2. Formulae (5.1) and (5.2) follows directly from (5.18) and (5.17) by passing to the limits. □

Debin and Fillastre in [DF22] show that the support functions of convex bodies in $\mathbb{R}^d$ of dimension at least 2 is embedded into a Sobolev space and if in addition that their Steiner points are positioned at 0, then the support functions restricted to S^{d-1} are L^2-orthogonal to the eigenspace of spherical Laplacian for the minimal positive eigenvalue. This result also holds for infinite-dimensional GB convex bodies in the context of Malliavin calculus.

Lemma 5.4.12. *Let $K \subset \mathcal{H}$ be a GB convex bodies such that* $\mathrm{Stein}(K) = 0$*, X be an isonormal Gaußian process over $\mathcal{H}$ and h_K be the support function of K. Then $h_K(X) \in \overline{\bigoplus}_{n\neq 1}\mathfrak{H}_n$.*

Proof. The formula (5.1) shows that $h_K \in \mathbb{D}^{1,2}$. By definition (5.8), $\mathbb{E}[h_K(X)X] = 0$, in particular, for every $v \in \mathcal{H}$, $(\mathbb{E}[h_K(X)X], v)_{\mathcal{H}} = \mathbb{E}[h_K(X)X_v] = 0$. But X_v are exactly the elements in $\mathfrak{H}_1$. This proves the desired result. □

Let $J_n : L^2(\Omega) \to \mathfrak{H}_n$ be the orthogonal projection onto the n-th Wiener chaos. Then more generally than Lemma 5.4.12, for a GB convex body $K \subset \mathcal{H}$ and an isonormal Gaußian process X over $\mathcal{H}$, the image $J_0\big(h_K(X)\big) = \mathbb{E}[h_K(X)] = V_1(K)/\sqrt{2\pi}$ and $J_1\big(h_K(X)\big) = X_{\mathrm{Stein}(K)}$.

Moreover, by Proposition 5.4.5, the Ornstein-Uhlenbeck operator $\triangle$ is commutative with J_n by linearity of the operator and orthogonality of $\mathfrak{H}_n$. At this point, it is possible to deduce the following Rayleigh's eigenvalue theorem for random variables in $\overline{\bigoplus}_{n\neq 1}\mathfrak{H}_n$:

Proposition 5.4.13. *Let $\varphi \in \overline{\bigoplus}_{n\neq 1}\mathfrak{H}_n$. Then $\mathbb{E}[\|D\varphi\|^2_{\mathcal{H}}] \geq 2\|\varphi\|^2_{L^2(\Omega)} - 2\left(\mathbb{E}[\varphi]\right)^2 \geq 0$.*

Proof. By orthogonality, $\varphi = \sum_{n\neq 1} J_n\varphi$. Using (5.15), one can compute

$$\begin{aligned}
\mathbb{E}[\|D\varphi\|^2_{\mathcal{H}}] + 2\left(\mathbb{E}[\varphi]\right)^2 &= \sum_{n\neq 1}\mathbb{E}[J_n\varphi(\triangle J_n\varphi)] + 2\left(\mathbb{E}[\varphi]\right)^2 \\
&= \sum_{n\neq 1} n\mathbb{E}[|J_n\varphi|^2] + 2\left(J_0\varphi\right)^2 \\
&= \sum_{n\geq 1} n\mathbb{E}[|J_n\varphi|^2] + 2\left(J_0\varphi\right)^2 \\
&\geq 2\sum_{n\neq 1}\|J_n\varphi\|^2_{L^2(\Omega)} = 2\|\varphi\|^2_{L^2(\Omega)}.
\end{aligned}$$

The inequality $2\|\varphi\|^2_{L^2(\Omega)} - 2\left(\mathbb{E}[\varphi]\right)^2 \geq 0$ is a direct application of Cauchy-Schwarz inequality. This completes the proof. □

Remark 5.4.14. Let $K \subset \mathcal{H}$ be a GB convex body with dimension at least 2 and positioned so that $\mathrm{Stein}(K) = 0$. Let $J_n : L^2(\Omega) \to \mathfrak{H}_n$ be the orthogonal projection. Then it is clear that $h_K(X) \in \overline{\bigoplus}_{n\neq 1}\mathfrak{H}_n$ and $J_0(h_K(X)) = \mathbb{E}[h_K(X)] = V_1(K)/\sqrt{2\pi} > 0$. Similarly to the discussion above, if $h \in \mathbb{D}^{1,2} \cap \overline{\bigoplus}_{n\geq 2}\mathfrak{H}_n$, we shall have $\mathbb{E}[\|Dh\|^2_{\mathcal{H}}] \geq 2\mathbb{E}[h^2]$, hence $-V_2(\cdot,\cdot)$ defines an inner product on the space $\mathbb{D}^{1,2} \cap \overline{\bigoplus}_{n\geq 2}\mathfrak{H}_n$. This implies that the bilinear form $V_2(\cdot,\cdot)$ defined on the space of support functions of GB convex bodies with dimension at least 2 and $\mathrm{Stein}(K) = 0$ is of Lorentzian signature, from which one can construct the infinite-dimensional hyperbolic space $\mathbb{H}^\infty_{\mathbb{R}}$. This recovers the discussion in [DF22, Proposition 2.4], in which $\mathbb{D}^{1,2} \cap \overline{\bigoplus}_{n\geq 2}\mathfrak{H}_n$ is identified with $H^1(\mathbb{S}^{n-1})_{01}$.

5.4.3 Convex bodies and support functions

This section mainly concerns the following question: *given a function in the Sobolev space $\mathbb{D}^{1,2}$, how can one tell whether it is the support function of a GB convex body in $\mathcal{H}$?*

In finite dimension, the support function of a convex body is a function $\mathbb{R}^d \to \mathbb{R}$ that is convex, positively homogeneous and semi-lower continuous. Moreover, each of such functions uniquely define a convex body in $\mathbb{R}^d$ by

$$K = \bigcap_{x \in \mathbb{R}^d} \left\{ y \in \mathbb{R}^d : x \cdot y \leq h_K(x) \right\}.$$

It turns out that $K' \subset K \subset \mathbb{R}^d$ if and only if $h_{K'}(x) \leq h_K(x)$ for all $x \in \mathbb{R}^d$.

This property can be further generalised to infinite dimension:

Proposition 5.4.15. *Let $K', K \subset \mathcal{H}$ be two GB convex bodies. Then $K' \subset K$ if and only if almost surely $h_{K'}(X) \leq h_K(X)$, where X is an isonormal Gaußian process on $\mathcal{H}$.*

Proof. It is clear that $K' \subset K$ implies $h_{K'}(X) \leq h_K(X)$ by definition. So it remains to show the converse. Note that by taking the contraposition, the converse is equivalent to $\mathbb{P}\{h_K(X) < 0\} > 0$ whenever $0 \notin K$. So suppose now that $K \subset \mathcal{H}$ is a GB convex body and $0 \notin K$. Then there exists a unique point $v \in K$ such that $\|v\|_{\mathcal{H}} = \inf\{\|v'\|_{\mathcal{H}} : v' \in K\}$. Indeed, v is the nearest point projection to K and if two distinct points $v, v' \in K$ have both the least distance to 0 amongst points in K, then the midpoint $v'' = v/2 + v'/2 \in K$ would have a strictly lesser distance to 0. By Gram-Schmidt process, let $(e_i)_{i \geq 1}$ be an orthonormal basis of $\mathcal{H}$ such that $v = te_1$ with $t < 0$. Hence

$$t = \sup_{w \in K} (w, e_1)_{\mathcal{H}}. \tag{5.19}$$

For simplicity, we write $w_i = (w, e_i)_{\mathcal{H}}$. So

$$h_K(X) = \sup_{w \in K} \left(w_1 X_{e_1} + \sum_{i=2}^{\infty} w_i X_{e_i} \right) \leq \sup_{w \in K} w_1 X_{e_1} + \sup_{w' \in K} \sum_{i=2}^{\infty} w_i' X_{e_i}. \tag{5.20}$$

When $X_{e_1} > 0$, one can use (5.19) to rewrite the right-hand side of (5.20) into

$$t X_{e_1} + \sup_{w' \in K} \sum_{i=2}^{\infty} w_i' X_{e_i} \tag{5.21}$$

Since K is GB, the last term of (5.21) $\sup_{w' \in K} \sum_{i=2}^{\infty} w_i' X_{e_i}$ is almost surely finite. By independence of X_{e_i} for $i \geq 1$, the two terms in (5.21) are also independent. By making $X_{e_1} \gg 0$ large enough, (5.21) will be negative for a positive probability, and so will be $h_K(X)$. □

We denote $\overline{\mathrm{co}}(C)$ be the convex hull of a set C. It immediately follows from Proposition 5.4.15 that for any index set I, we have

$$\sup_{i\in I} h_{K_i}(X) = h_{\widetilde{K}}(X), \tag{5.22}$$

where $\widetilde{K} := \overline{\mathrm{co}}\Big(\bigcup_{i\in I} K_i\Big)$ is the closed convex hull of the union of K_i's.

Let $\phi \in \mathbb{D}^{1,2}$. Define

$$\mathcal{F}_\phi := \{K \subset \mathcal{H} : K \text{ is a convex body with } h_K(X) \le \phi\}.$$

Then the following result can be deduced immediately from Proposition 5.4.15:

Corollary 5.4.16. *Let $K \subset \mathcal{H}$ be a GB convex body. Then*

$$K = \overline{\mathrm{co}}\Big(\bigcup_{P\in\mathcal{F}_{h_K(X)}} P\Big). \tag{5.23}$$

Proof. Let K' be the right-hand side of (5.23). It is clear that $K \subset K'$ since $K \in \mathcal{F}_{h_K(X)}$. Conversely, by (5.22),

$$h_{K'}(X) = \sup_{P\in\mathcal{F}_{h_K(X)}} h_P(X) \le h_K(X),$$

which implies $K' \subset K$ by Proposition 5.4.15. Hence the equality is obtained. □

Remark 5.4.17. It is somehow tautological but Corollary 5.4.16 also means that K is the maximal convex body so that $h_K(X)$ is bounded by $\phi = h_K(X) \in \mathbb{D}^{1,2}$.

For GC convex body, the isonormal Gaußian process has almost surely a continuous sample function, so the maxima of the sample function are well-defined and attained. More generally, for GB convex body $K \subset \mathcal{H}$, we can similarly define a point $v \in K$ to be the *maximal point of* K at state $\omega \in \Omega$ if for every $\varepsilon > 0$,

$$\sup\{X_w(\omega) : w \in K \cap B_{\mathcal{H}}(v,\varepsilon)\} = h_K(X)(\omega).$$

Since $\mathcal{H}$ is separable and $h_K(X) < \infty$ almost surely for any $K \subset \mathcal{H}$ GB convex body, for almost every state $\omega \in \Omega$, there exists a maximal point of K. Moreover, for each $v \in K$, the asymptotic error

$$\limsup_{\varepsilon\to 0} \{X_w(\omega) - X'_w(\omega) : w, w' \in K \cap B_{\mathcal{H}}(v,\varepsilon)\}$$

is uniformly bounded in $L^1(\Omega)$ sense [Vit01].

The Malliavin derivative of the supremum of a Gaußian process is almost surely the maximal point:

Proposition 5.4.18. *Let $K \subset \mathcal{H}$ be a GB convex body and $h_K(X)$ be its support function. Then for almost every $\omega \in \Omega$, $Dh_K(X)(\omega)$ is the unique maximal point of K.*

Proof. Note that the proof of [KP90, Lemma 2.6] is available for any non-degenerate Gaußian process on a compact subset of a separable Banach space, with continuous covariance kernel and having at least a maximal point for almost every $\omega \in \Omega$. It soon follows that almost surely there exists a unique maximal point for the isonormal Gaußian process $(X_v)_{v\in K}$. The fact $Dh_K(X)$ being the unique maximal point of K can be deduced via approximation arguments [DN08, Lemma 3.1]. □

Recall that for $\mathbf{f} \in L^2(\Omega;\mathcal{H})$, one can define its *essential range* by

$$\overline{\mathrm{ess}}(\mathbf{f}) := \bigcap_{\mathbf{u}=\mathbf{f}\ \text{a.s.}} \overline{\mathbf{u}(\Omega)}.$$

This intersection is non-void since $\mathcal{H}$ is supposed to be separable. Note that by definition the essential range remains the same for functions that only differ on a negligible set.

Corollary 5.4.19. *Let $K \subset \mathcal{H}$ be a GB convex body and $h_K(X)$ be its support function. Then the closed convex hull of the essential range of the Malliavin derivative $\overline{\mathrm{co}}\big(\mathrm{ess}(Dh_K(X))\big) = K$.*

Proof. Let $K' = \overline{\mathrm{co}}\big(\mathrm{ess}(Dh_K(X))\big)$. *A priori* $K' \subset K$, as $Dh_K(X) \in K$ almost surely by Proposition 5.4.18. Conversely, note that

$$K' = \bigcap_{n\geq 1} \left\{ \left[\mathrm{co}\big(\mathrm{ess}(Dh_K(X))\big) + B_{\mathcal{H}}\left(0, 1/n\right)\right] \cap K \right\} =: \bigcap_{n\geq 1} K_n.$$

Moreover, the support function $h_{K_n}(X)$ is a decreasing sequence convergent to $h_{K'}(X)$. But for every $n \geq 1$, by the definition of maximal points $h_{K_n}(X) \geq h_K(X)$. Letting $n \to \infty$ yields $h_{K'}(X) \geq h_K(X)$, which indicates that $K \subset K'$ after Proposition 5.4.15. □

A Maximal point of a GB convex body K, if it is well-defined, is necessarily contained in the **extremal points** $\mathrm{Ext}(K)$ of K. Hence we can define a measurable map $\Omega \to \mathrm{Ext}(K)$ by $\omega \mapsto Dh_K(X)(\omega)$, which will further yield a probability measure μ_K on $\mathrm{Ext}(K)$ by pushing-forward.

From a functional analysis point of view, for a GB convex body in $\mathcal{H}$, *Choquet's theory* also provides a natural center called the **barycenter** when a probability measure μ is defined on $\mathrm{Ext}(K)$. The barycenter is the unique point $b \in K$ such that $\int_{\mathrm{Ext}(K)} \phi(v)\,\mathrm{d}\mu(v) = \phi(b)$ for every affine function ϕ defined on $\mathcal{H}$. See [Phe01] for more details.

The following proposition allows us to connect the stochastic point of view to this functional analysis point of view:

Proposition 5.4.20. *Let $K \subset \mathcal{H}$ be a GB convex body and let μ_K be the pushforawrd measure as above. Then* $\mathrm{Stein}(K)$ *is the barycenter for μ_K.*

Proof. By Riesz representation theorem, it suffices to show that

$$\big(\mathrm{Stein}(K), w\big)_{\mathcal{H}} = \int_{\mathrm{Ext}(K)} (v, w)_{\mathcal{H}} \, \mathrm{d}\mu_K(v)$$

for any $w \in \mathcal{H}$. Since we have $\delta(\varphi v) + (D\varphi, v)_{\mathcal{H}} = \varphi X_v$ for every $\varphi \in \mathbb{D}^{1,2}$ and every $v \in \mathcal{H}$ [Nua06, (1.46)], it follows that

$$\begin{aligned}
\mathbb{E}\big[(Dh_K(X), w)_{\mathcal{H}}\big] &= \mathbb{E}\big[h_K(X) X_w - \delta\big(h_K(X) w\big)\big] \\
&= \big(\mathrm{Stein}(K), w\big)_{\mathcal{H}} - \mathbb{E}\big[\delta\big(h_K(X) w\big)\big] \\
&= \big(\mathrm{Stein}(K), w\big)_{\mathcal{H}} - \mathbb{E}\big[\big(h_K(X) w, D1\big)_{\mathcal{H}}\big] \\
&= \big(\mathrm{Stein}(K), w\big)_{\mathcal{H}},
\end{aligned}$$

where we have applied the definition of the Steiner point (5.8) and the definition of the divergence operator δ. As

$$\mathbb{E}\big[(Dh_K(X), w)_{\mathcal{H}}\big] = \int_{\mathrm{Ext}(K)} (v, w)_{\mathcal{H}} \, \mathrm{d}\mu_K(v)$$

following from the definition of the pushforward measure, we thus complete the proof. □

Now we can show Theorem 5.1.3 with the descriptions above.

Proof of Theorem 5.1.3. In classical spectral theory, it is well-known that we can find an orthonormal basis $(\phi_n)_{n \geq 1}$ in $L^2(\Omega)$ such that $\triangle \phi_n = \lambda_n \phi_n$ for every $n \in \mathbb{N}$. This basis can be constructed using the Gram-Schmidt process from the random variables $\Phi_{\vec{k}}$ as described in (5.16). By Proposition 5.4.5, we may assume that $\lambda_0 = 0$, $\phi_0 = 1$ and $\lambda_n \geq 1$ for all $n > 0$.

Let $K, K' \subset \mathcal{H}$ be two GB convex bodies such that $V_2(K, K')^2 = V_2(K) V_2(K')$. Without loss of generality, we may assume that $\mathrm{Stein}(K) = \mathrm{Stein}(K') = 0$. It suffices to show that $K = tK'$ for some $t > 0$.

By eigendecomposition, we write $h_K(X) = \sum_{n \geq 1} a_n \phi_n$ and $h_{K'}(X) = \sum_{n \geq 1} b_n \phi_n$ with $a_n, b_n \in \mathbb{R}$. By Remark 5.4.14, we have $\lambda_n > 1$ for all $n \geq 1$. Using the formulae (5.1) and (5.2), as well as substituting $h_K(X), h_{K'}(X)$ by the eigendecompositions, we can equivalently

write the assumption $V_2(K,K')^2 = V_2(K)V_2(K')$ as

$$\left(a_0b_0 + \sum_{n\geq 1}(1-\lambda_n)a_nb_n\right)^2 = \left(a_0^2 + \sum_{n\geq 1}(1-\lambda_n)a_n^2\right)\left(b_0^2 + \sum_{n\geq 1}(1-\lambda_n)b_n^2\right),$$

where we have also used the definition $\triangle = \delta D$. Rearranging this formula by grouping terms into sums of squares, we then get

$$\sum_{n\geq 1}(1-\lambda_n)(a_nb_0 - a_0b_n)^2 = \sum_{i<j}(1-\lambda_i)(1-\lambda_j)(a_ib_j - a_jb_i)^2. \tag{5.24}$$

Since $\lambda_n > 1$ for all $n \geq 1$, the left-hand side of (5.24) is non-positive, while the right-hand side of (5.24) is non-negative. This forces each square in (5.24) to vanish, *i.e.* $a_ib_j = a_jb_i$ for all $i,j \geq 0$. Consequently, we have $h_K(X) = th_{K'}(X)$ for some $t \in \mathbb{R}$. Applying (5.2) again, we then have $tV_2(K') = V_2(K,K') > 0$, which shows that $t > 0$. Finally, Corollary 5.4.19 allows us to deduce $K = tK'$ by taking the closed convex hull of the essential range of the Malliavin derivatives. □

5.4.4 Completion of hyperbolic embedding

Let us consider any sequence $(K_n)_{n\geq 1}$ of GB convex bodies in $\mathcal{H}$ of dimension at least 2 such that $V_2(K_n) = 1$ and $\mathrm{Stein}(K_n) = 0$ for all $n \geq 1$. In this section, the criteria for $\iota([K_n])$ defines a Cauchy sequence in $\mathbb{H}^\infty_{\mathbb{R}}$ will be treated.

Recall the following inequality from [Che76, (4.4.1)]:

$$V_2(K) \leq \frac{V_1(K)^2}{2} \leq 2\pi\mathrm{diam}(K) + V_2(K). \tag{5.25}$$

Since there is a segment of length $\mathrm{diam}(K)$ contained in K, the monotonicity of the intrinsic volume implies $V_1(K) \geq \mathrm{diam}(K)$. Hence we can conclude the following result:

Proposition 5.4.21. *Let $(K_n)_{n\geq 1}$ be a sequence of GB convex bodies of dimension at least 2 with $V_2(K_n) = 1$ and $\mathrm{Stein}(K_n) = 0$ for all $n \geq 1$. If $\mathrm{diam}(K_n) \to \infty$ as $n \to \infty$, then $\iota([K_n])$ will eventually leave every bounded subset of $\mathbb{H}^\infty_{\mathbb{R}}$.*

Proof. Let $(K_n)_{\geq 1}$ be a sequence as above. Suppose that K is also a GB convex body with $\dim(K) \geq 2$, $V_2(K) = 1$ and $\mathrm{Stein}(K) = 0$. Then one has by (5.25) and $V_1(K) \geq \mathrm{diam}(K)$

the inequality

$$\begin{aligned}\cosh d_{\mathbb{H}}\big(\iota([K]),\iota([K_n])\big) &= \frac{V_2(K_n+K)-2}{2}\\ &\geq \frac{(V_1(K)+V_1(K_n))^2-4-4\pi\mathrm{diam}(K)-4\pi\mathrm{diam}(K_n)}{4}\\ &\geq \frac{(V_1(K)+\mathrm{diam}(K_n))^2-4-4\pi\mathrm{diam}(K)-4\pi\mathrm{diam}(K_n)}{4},\end{aligned}$$

which diverges to ∞ as $n\to\infty$. □

Let $O\in\mathbb{H}^\infty_{\mathbb{R}}$ be the limit of image under ι of the homothety classes of n-dimensional unit balls. If K_n are GB convex bodies as above with $\mathrm{diam}(K_n)\to 0$ as $n\to\infty$, then they forcibly converge to $O\in\mathbb{H}^\infty_{\mathbb{R}}$. This result is already suggested by Example 5.3.13 and Example 5.3.14.

Proposition 5.4.22. *Let $(K_n)_{n\geq 1}$ be a sequence of GB convex bodies of dimension at least 2 such that $V_2(K_n)=1$ and $\mathrm{Stein}(K_n)=0$ for all $n\geq 1$. Let $O\in\mathbb{H}^\infty_{\mathbb{R}}$ be the limit of the image under ι of homothety classes of n-dimensional unit balls. If $\mathrm{diam}(K_n)\to 0$ as $n\to\infty$, then $\iota([K_n])$ converges to O in $d_{\mathbb{H}}$.*

Proof. Let $(K_n)_{n\geq 1}$ be such that $\mathrm{diam}(K_n)\to\infty$ as $n\to\infty$. Taking limits on every terms in (5.25) by letting n tend to ∞ yields $V_1(K_n)\to\sqrt{2}$. Hence

$$\begin{aligned}1\leq\cosh\Big(d_{\mathbb{H}}\big(\iota([K_n]),\iota([K_m])\big)\Big) &= \frac{V_2(K_n+K_m)-2}{2}\\ &\leq \frac{(V_1(K_n)+V_1(K_m))^2-4}{4}\to 1\end{aligned}$$

as $n,m\to\infty$. So $\iota([K_n])$ defines a Cauchy sequence in $\mathbb{H}^\infty_{\mathbb{R}}$ converging to a point. If we take the sequence $K_1, B^2/V_2(B^2), K_2, B^3/V_2(B^3),\ldots$, then their images under ι will become a new Cauchy sequence, which implies that $\iota([K_n])$ converges to $O\in\mathbb{H}^\infty_{\mathbb{R}}$. □

Remark 5.4.23. Although this proof is geometric, the point O has a very specific meaning in the Sobolev space $\mathbb{D}^{1,2}$. Under the setting above, $\iota([K_n])$ yields a Cauchy sequence in $\mathbb{H}^\infty_{\mathbb{R}}$ if and only if $V_2(K_n,K_m)\to 1$ as $n,m\to\infty$, which is again by (5.2) equivalent to

$$\mathbb{E}[\|Dh_{K_n}(X)-Dh_{K_m}(X)\|^2_{\mathcal{H}}-|h_{K_n}(X)-h_{K_m}(X)|^2]\to 0 \tag{5.26}$$

as $n,m\to\infty$. If $\mathrm{diam}(K_n)\to 0$, it turns out that $\|Dh_{K_n}(X)\|_{L^\infty(\Omega;\mathcal{H})}\to 0$. Hence in view of Proposition 5.4.13 and (5.1), $h_{K_n}(X)$ converges in $\mathbb{D}^{1,2}$ to the constant function $1/\sqrt{\pi}$. This function corresponds to the limit O of unit balls in $\mathbb{H}^\infty_{\mathbb{R}}$ but it is not the support function of any GB convex body in view of Corollary 5.4.16 and Corollary 5.4.19: it only bounds the singleton $\{0\}$ but differs from the support function of $\{0\}$.

We are now able to conclude Theorem 5.1.4.

Proof of Theorem 5.1.4. Let $(K_n)_{n\geq 1}$ be a sequence of GB convex bodies in $\mathcal{H}$. Suppose that $\dim(K_n) \geq 2$, $\mathrm{Stein}(K_n) = 0$ and $V_2(K_n) = 1$, such that $\iota([K_n])$ defines a Cauchy sequence in $\mathbb{H}^{\infty}_{\mathbb{R}}$. By Proposition 5.3.12, we may assume that K_n's are all polytopes.

Now suppose that there exists $< A < \infty$ such that $\mathrm{diam}(K_n) \leq A$ for every $n \geq 1$. Hence $V_1(K_n)$ is bounded after (5.25). By passing to a subsequence, we may assume that $V_1(K_n)$ converges. Together with Proposition 5.4.13, one gets

$$\begin{aligned}&\|h_{K_n}(X) - h_{K_m}(X)\|^2_{L^2(\Omega)} - (V_1(K_n) - V_1(K_m))^2/2\pi\\ \leq&\mathbb{E}[\|Dh_{K_n}(X) - Dh_{K_m}(X)\|^2_{\mathcal{H}} - |h_{K_n}(X) - h_{K_m}(X)|^2].\end{aligned}$$

Letting $n, m \to \infty$, (5.26) forces $\|h_{K_n}(X) - h_{K_m}(X)\|^2_{L^2(\Omega)} \to 0$ as $n, m \to \infty$. Hence $h_{K_n}(X)$ converges to a random variable $\phi \in L^2(\Omega)$. Moreover, by assumption and Corollary 5.4.11, we have $\|Dh_{K_n}\|_{\mathcal{H}} \leq \mathrm{diam}(K_n) < A$ almost surely, by Lemma 5.4.7, $h_{K_n}(X)$ converges to ϕ weakly in $\mathbb{D}^{1,2}$.

By Lemma 5.4.8, it is possible to construct another sequence $(P_m)_{m\geq 1}$ of polytopes with $\mathrm{Stein}(P_m) = 0$ and $h_{P_m}(X) \to \phi$ in $\mathbb{D}^{1,2}$ as $m \to \infty$; indeed, P_m is the m-th Cesàro sum of a subsequence of $(K_n)_{n\geq 1}$. Moreover, by passing to a subsequence, we may assume that $h_{P_m}(X)$ (resp. $Dh_{P_m}(X)$) converges to ϕ (resp. $D\phi$) almost surely.

Let $\mathcal{F}_\phi$ be the collection of GB convex bodies such that $h_K(X) \leq \phi$ almost surely. Define

$$K_\phi := \overline{\mathrm{co}}\left(\bigcup_{P\in\mathcal{F}_\phi} P\right)$$

to be the largest GB convex body such that $h_{K_\phi}(X) \leq \phi$ and define $\psi = \phi - h_{K_\phi}(X)$.

We claim that $\overline{\mathrm{co}}\big(\mathrm{ess}(D\phi)\big) \subset K_\phi$. To this end, it suffices to show that for every $v \in \mathrm{ess}(D\phi)$, the random variables satisfies $X_v \leq \phi$ almost surely. Let $S \subset \Omega$ be a subset such that $\mathbb{P}(S) = 1$ with $h_{P_m}(X) \to \phi$ and $Dh_{P_m}(X) \to D\phi$ pointwisely on S. Then for each $v \in \mathrm{ess}(D\phi)$, there exists $\omega' \in S$ with $Dh_{P_m}(X)(\omega') \to v$ as $m \to \infty$. Moreover,

$$\mathbb{E}\left[\left|X_v - X_{Dh_{P_m}(X)(\omega')}\right|^2\right] = \|v - Dh_{P_m}(X)(\omega')\|^2_{\mathcal{H}} \to 0$$

as $m \to \infty$, *i.e.* the convergence is in $L^2(\Omega)$. By passing to a subsequence, there exists $S_v \subset S \subset \Omega$ with $\mathbb{P}(S_v) = 1$ such that $X_{Dh_{P_m}(X)(\omega')}(\omega)$ converges to $X_v(\omega)$ for every $\omega \in S_v$. By taking the limit along the subsequence (and abusing the notations since the

limits are the same), it soon follows that

$$X_v(\omega) = \lim_{k\to\infty} X_{Dh_{P_m}(X)(\omega')}(\omega) \leq \lim_{m\to\infty} X_{Dh_{P_m}(X)(\omega)}(\omega) = \lim_{m\to\infty} h_{P_m}(X)(\omega) = \phi(\omega)$$

for every $\omega \in S_v$. Hence $\overline{\mathrm{co}}\big(\mathrm{ess}(D\phi)\big) \subset K_\phi$. As $h_{K_\phi}(X) + \psi = \phi$, by taking the Malliavin derivative and their closed convex hull on both sides, we have by Corollary 5.4.19 that

$$K_\phi \subset \overline{\mathrm{co}}\big(\mathrm{ess}(Dh_{K_\phi}(X))\big) \subset \overline{\mathrm{co}}\big(\mathrm{ess}(Dh_{K_\phi}(X)) + \mathrm{ess}(D\psi)\big) = \overline{\mathrm{co}}\big(\mathrm{ess}(D\phi)\big) \subset K_\phi.$$

But $\overline{\mathrm{co}}\big(\mathrm{ess}(Dh_{K_\phi}(X)) + \mathrm{ess}(D\psi)\big) = \overline{\mathrm{co}}\big(\mathrm{ess}(Dh_{K_\phi}(X)) + \overline{\mathrm{co}}\big(\mathrm{ess}(D\psi)\big)$. As a result, $K_\phi + \overline{\mathrm{co}}\big(\mathrm{ess}(D\psi)\big) \subset K_\phi$, which forces $D\psi = 0$ almost surely, *i.e.* ψ is almost surely a constant function. Also, because P_m is the Cesàro sum of K_n's, it is clear that $V_2(P_m)$ converges to 1 as $m \to \infty$. By convergence of $h_{P_m}(X)$ to ϕ in $\mathbb{D}^{1,2}$, it soon yields that

$$\pi\mathbb{E}\left[\phi^2 - \|D\phi\|_{\mathcal{H}}^2\right] = \lim_{m\to\infty} V_2(P_m) = 1.$$

Alternatively, the function ϕ is also the limit point of the support function of GB convex body $K_\phi + b_n B^n$, where $b_n > 0$ and B^n is the n-dimensional unit ball, with $b_n h_{B^n}(X) \to \psi$ in $\mathbb{D}^{1,2}$. But $\iota([K_\phi + b_n B^n])$ is a point on the geodesic between $\iota([K_\phi])$ and $\iota([B^n])$ in view of Proposition 5.3.4. As $\mathbb{H}^\infty_{\mathbb{R}}$ is regularly geodesic, ϕ also represents a point on the geodesic $\iota([K_\phi])$ and O. In particular, one remarks that $\dim(K_\phi) < 2$ is possible. If $\dim(K_\phi) = 1$, then $\iota([K_\phi]) \in \partial\mathbb{H}^\infty_{\mathbb{R}}$; or if $\dim(K_\phi) = 0$, *i.e.* $K_\phi = \{0\}$, in which case $h_{K_\phi}(X) = 0$ and the sequence $(K_n)_{n\geq 1}$ converges to $O \in \mathbb{H}^\infty_{\mathbb{R}}$. □

Remark 5.4.24. As mentioned in Remark 5.3.3, the orthogonal group $O(\mathcal{H})$ acts on $\iota(\mathbb{K}_2)$ by isometries. This action soon extends to the completion $\overline{\iota(\mathbb{K}_2)}$ and the point O is the unique $O(\mathcal{H})$-invariant point.

Chapter 6

Dynamics of Big Mapping Class Groups

Celui qui s'arrête fait remarquer l'emportement des autres, comme un point fixe.

Blaise Pascal (1623–1662), *Pensées*.

6.1 Introduction

From a model theoretic aspect, a non-archimedean Polish group can always be realised as the automorphism group of some countable first-order relational structure. In particular, one may ask the following question:

Question 6.1.1. Given a countable first-order relational structure $\mathcal{F}$, how can one detect if its automorphism group $\mathrm{Aut}(\mathcal{F})$ can be realised as the (extended) mapping class group of an orientable surface?

This chapter tends to give a partial answer to Question 6.1.1: the mapping class groups of all but finitely many orientable surfaces can never be the automorphism group of a countable first-order relational structure $\mathcal{F}$ such that $\mathrm{Age}(\mathcal{F})$ has Ramsey property.

Recall that a topological group G is said to have *fixed point on compacta property* or *extremely amenable* if every continuous G-action on a compact Hausdorff space admits a fixed point, or equivalently the universal minimal G-flow $M(G)$ reduces to a singleton. However, it is worth noticing that, other than the trivial group, no locally compact groups are extremely amenable [Vee77], thus *a fortiori* no discrete ones [Ell60].

In a celebrated paper [KPT05], Kechris, Pestov and Todorčević develop a surprising correspondence (*abbrv.* KPT correspondence) between model theory, combinatorics and topological dynamics: if $\mathcal{F}$ is a structure with universe $\mathbb{N}$, then the non-archimedean Polish group $\mathrm{Aut}(\mathcal{F})$ is extremely amenable if and only if the *age* $\mathrm{Age}(\mathcal{F})$ has Ramsey property.

So it is a natural question to ask for a non-archimedean Polish group arising in the study of a geometry object whether or not it is extremely amenable. In particular, it is worth knowing whether big mapping class groups are extremely amenable or not. One shall notice that big mapping class groups are not locally compact [AV20, Theorem 4.2].

Theorem 6.1.2. *Let Σ be an orientable surface of finite or infinite topological type. Then $\mathcal{MCG}(\Sigma)$ is not extremely amenable unless Σ is a sphere or a once-punctured sphere, in which cases the mapping class groups are trivial.*

Denote by $\mathrm{End}(\Sigma)$ the end space of a surface Σ. It is a compact space with a natural continuous $\mathcal{MCG}(\Sigma)$-action. In many cases, this action is fixed-point free, which will witness the non extreme amenability of the mapping class group. But this is not always the case. For example, the *Loch Ness monster* surface has infinite genus but only one end and the action of its mapping class group on the end space is trivial. Other non trivial examples are (non self-similar) surfaces with a unique maximal end [MR22].

The proof of Theorem 6.1.2 mainly relies on the description of extremely amenable non-archimedean Polish groups provided in [KPT05] and can be divided into the discussion of two cases, depending on whether the surface is zero-genus or not. When the surface Σ has zero genus, the Mann-Rafi ordering [MR23] yields two disjoint curves of the same topological type unless Σ is a sphere or a once-punctured sphere. In this case, the existence of these two curves further indicates that $\mathcal{MCG}(\Sigma)$ is not extremely amenable. If $g(\Sigma) > 0$, then one may also find another pair of curves with the same topological type, not necessarily disjoint, of which the existence also implies the non extreme amenability of $\mathcal{MCG}(\Sigma)$.

Moreover, the proof of Theorem 6.1.2 remains valid for closed subgroup of $\mathcal{MCG}(\Sigma)$ containing a non-trivial mapping class with finite orbit, including pure mapping class groups $\mathcal{PMCG}(\Sigma)$, *i.e.* the subgroup in $\mathcal{MCG}(\Sigma)$ that fixes pointwise every ends, and the closure of compactly supported mapping class group $\mathcal{MCG}_c(\Sigma)$.

Theorem 6.1.3. *Let Σ be an orientable surface of finite or infinite type. If $G < \mathcal{MCG}(\Sigma)$ is a subgroup containing a mapping class $\phi \in G$ such that for some simple closed curve c on Σ, the orbit $\{\phi^n(c) : n \in \mathbb{Z}\}$ is finite, then G is not extremely amenable. In particular, the groups $\mathcal{PMCG}(\Sigma)$ and $\mathcal{MCG}_c(\Sigma)$ of a surface Σ with $g(\Sigma) \geq 1$ are not extremely amenable.*

Note that the non extreme amenability of $\mathcal{PMCG}(\Sigma)$ can also be shown in a way that one constructs a $\mathcal{PMCG}(\Sigma)$-action on the circle S^1 without fixed-point from the homomorphisms built in [APV20]. But this method cannot be adapted to $\mathcal{MCG}_c(\Sigma)$.

6.2 Non extreme amenability

Recall that the *extended mapping class group* $\mathcal{MCG}(\Sigma)^{\pm}$ is defined similarly as $MCG(\Sigma)$ but using the group of all automorphisms $\mathrm{Homeo}(\Sigma)$ in the stead of the orientation preserving one. The result below is already known following several basic facts about mapping class groups:

Proposition 6.2.1. *If the surface Σ is of complexity at least two, then the mapping class group $\mathcal{MCG}(\Sigma)$ and the extended mapping class group $\mathcal{MCG}^{\pm}(\Sigma)$ are non-archimedean Polish groups.*

Proof. Let $\mathcal{C}(\Sigma)$ be the curve graph of the surface Σ, where vertices are isotopic classes of essential simple closed curves and an edge is attached to two vertices if the two classes have disjoint representatives. It is well-known that for a surface Σ with complexity at least two, of finite or infinite type, we have the isomorphism $\mathcal{MCG}^{\pm}(\Sigma) \simeq \mathrm{Aut}(\mathcal{C}(\Sigma))$ between topological groups [Iva97; Luo00; HMV18; BDR20]. Note that the vertices in $\mathcal{C}(\Sigma)$ are countable. The graph $\mathcal{C}(\Sigma)$ is actually a countable relational first-order structure. Such an automorphism group is closed subgroup of S_∞ and thus a non-archimedean Polish group (see for example [Kec12, Part I, §9.B(7)]). Hence $\mathcal{MCG}^{\pm}(\Sigma)$ is a non-archimedean Polish group. To show that $\mathcal{MCG}(\Sigma)$ is also one, it suffices to demonstrate that $\mathcal{MCG}(\Sigma)$ is closed in $\mathcal{MCG}^{\pm}(\Sigma)$. Indeed, given a convergent sequence of homeomorphisms $\Sigma \to \Sigma$ in $\mathrm{Homeo}(\Sigma)$ for compact-open topology, if they are all orientation preserving, then so will be their limit[1]: when $g_n \to g$, for a fixed curve $\alpha \subset \Sigma$, on one hand the Dehn twist $T_{g_n\alpha}$ shall converge to $T_{g\alpha}$ as $g_n\alpha$ converges to $g\alpha$ uniformly; but if g is orientation reversing, then on the other hand $T_{g_n\alpha} = g_n T_\alpha g_n^{-1}$ converges to $gT_\alpha g^{-1}$, which is a reversing Dehn twist along $g\alpha$ but not $T_{g\alpha}$. □

To further discuss the extreme amenability of a non-archimedean Polish group, we need to introduce the following notions. Let G be a group acting on a space X and Y be a subspace of X. We call the subgroup $G_{(Y)} = \{g \in G : gy = y,\ \forall y \in Y\}$ of G the *pointwise stabiliser* of Y. Similarly, the subgroup $G_Y = \{g \in G : gY = Y\}$ is called the *setwise stabiliser* of Y in G.

The following observation is a natural consequence after KPT correspondence (see [KPT05, Proposition 4.3]), which can find its root in [GW02].

Lemma 6.2.2. *If G is an extremely amenable non-archimedean Polish group acting continuously on the discrete space $\mathbb{N}$, then $G_{(F)} = G_F$ for any finite subset $F \subset \mathbb{N}$.*

[1] This fact holds more generally for orientable manifolds and can be seen via using the homologous definition of orientation, yet here we do not wish to introduce the entire voluminousity of homological theory.

Sketch of Proof. If G is extremely amenable, then its continuous action on the compact space $\mathrm{LO}(\mathbb{N})$, the space of linear orders on $\mathbb{N}$, has a fixed point, *i.e.* G fixes a linear order on $\mathbb{N}$. Let $F \subset \mathbb{N}$ be a finite set. Since G fixes a linear order on F, the only way that an element in G leave F invariant is to fix every elements in F. □

Remark 6.2.3. For mapping class group $\mathcal{MCG}(\Sigma)$ viewed as a non-archimedean Polish group, the discrete space $\mathbb{N}$ on which it acts on is identified with isotopy classes of curves on Σ, *i.e.* vertices in $\mathcal{C}(\Sigma)$.

The end space $\mathrm{End}(\Sigma)$ is a totally disconnected, separable, metrisable topological space (and thus a closed subset of Cantor set). Among the ends, there are *non planary ends*, of which the collection is denoted $\mathrm{End}_\infty(\Sigma)$, in the sense that every neighbourhood in Σ of a non planary end has at least genus 1. The non planary ends $\mathrm{End}_\infty(\Sigma)$ form a closed subset of $\mathrm{End}(\Sigma)$.

Remark 6.2.4. There is a slight *abus de langage* here. The neighbourhood mentioned above resides in the end compactification of Σ instead of $\mathrm{End}(\Sigma)$, but its intersection with Σ becomes a subsurface in Σ and can still be regarded as a "neighbourhood" of an end.

Let D, D' be two subsets of $\mathrm{End}(\Sigma)$. We say that D and D' are *isomorphic* if D is homeomorphic to D' and $D \cap \mathrm{End}_\infty(\Sigma)$ is homeomorphic $D' \cap \mathrm{End}_\infty(\Sigma)$ simultaneously.

Mann and Rafi give a way to order the ends of a surface by their similarity [MR23]. Let $x, y \in \mathrm{End}(\Sigma)$ be two ends. Then we write $x \preccurlyeq y$ if every clopen neighbourhood of y contains a clopen subset that is isomorphic to a neighbourhood of x. Two ends are said *equivalent* if both $x \preccurlyeq y$ and $y \preccurlyeq x$. They are said *non-comparable* if neither case happens.

It is worth noticing that each compact neighbourhood of an end of Σ corresponds to one of its clopen neighbourhoods D in $\mathrm{End}(\Sigma)$ and this compact neighbourhood can be chosen to be the end compactification of a subsurface in Σ whose ends are the union of D with an additional isolated point.

Since the group $\mathrm{Homeo}^+(\Sigma)$ acts on the surface Σ continuously, this action has a natural continuous extension onto the end compactification of Σ. By taking the quotient, it is not hard to see that $\mathcal{MCG}(\Sigma)$ acts continuously by homeomorphisms on $\mathrm{End}(\Sigma)$ and $\mathrm{End}_\infty(\Sigma)$ is an invariant subspace of this group action. Moreover, from the definition, this action preserves the ordering given above, namely $gx \preccurlyeq gy$ for any $g \in \mathcal{MCG}(\Sigma)$ if and only if $x \preccurlyeq y$. In particular, if there exists $g \in \mathcal{MCG}(\Sigma)$ such that $gx = y$, then necessarily $x \preccurlyeq y$.

Adopting the terminology from [FM11], the *topological type* of a simple closed curve c on Σ is $\Sigma \setminus c$. Two curves c and c' on Σ are said to *have the same topological type* if there is a homeomorphism between $\Sigma \setminus c$ and $\Sigma \setminus c'$.

Remark 6.2.5. Since no distinction will be needed here, in the sequel, a curve on the surface can mean either a topological embedding of S^1 or its isotopy class, depending on the

context.

An argument of the renowned *Alexander method* (see for example [FM11, §2.3] and [HMV19]) is that given two distinct simple closed curves (up to isotopy) of the same topological type, one can always find a non-trivial mapping class sending one curve to the other. Moreover, this mapping class g can be chosen to have finite order if the surface Σ is of finite type. This is not true in general for infinite-type surfaces, but only some special cases will be needed for our purpose here and similar arguments yield the same result for these special cases:

Lemma 6.2.6. *Let Σ be a surface with genus 0. Suppose that there exists two disjoint simple closed curves c, c' on Σ with the same topological type. Assume that c cuts $\mathrm{End}(\Sigma) = E \sqcup N$ and c' cuts $\mathrm{End}(\Sigma) = E' \sqcup N'$ in a way that $E \simeq E'$ and that E, E' are disjoint. Then there exists a non-trivial mapping class $\phi \in \mathcal{MCG}(\Sigma)$ such that $\phi(c) = c'$ and $\phi^2 = \mathrm{Id}$.*

Proof. Figure 6.1 depicts how such a surface Σ looks like. The simplest case is when E and E' reduces to singleton and Σ is a twice-punctured sphere. Now the desired $\phi \in \mathcal{MCG}(\Sigma)$ is just the symmetry sending E to E'. More precisely, cut Σ into three parts $S \sqcup S' \sqcup \big(\Sigma \setminus (S \cup S')\big)$ along c and c' so that $\partial S = c$ and $\partial S' = c'$. By Richards' theorem, there is a homeomorphism $\varphi \colon S \to S'$ sending ∂S to $\partial S'$. Take a symmetry $\sigma \in \mathcal{MCG}(\Sigma \setminus (S \cup S'))$ that interchanges the position of c and c': σ is the continuous extension by identity of the symmetry $\overline{\sigma} \in \mathcal{MCG}(S_{0,3})$, where $S_{0,3} \subset \Sigma \setminus (S \cup S')$ is a subsurface of finite type with two of its ends being $\partial S = c$ and $\partial S' = c'$. Now ϕ is piecewise defined by σ on $\Sigma \setminus (S \cup S')$, φ on S and φ^{-1} on S' (up to isotopy). □

Let us first deal with the (non) extreme amenability of $\mathcal{MCG}(\Sigma)$ for surfaces Σ with genus 0 by using the results above.

Lemma 6.2.7. *Let Σ be an orientable surface with genus 0 and complexity at least 2. If $\mathcal{MCG}(\Sigma)$ is extremely amenable, then any two distinct ends in $\mathrm{End}(\Sigma)$ are non-comparable.*

Proof. Suppose for contradiction that $\mathcal{MCG}(\Sigma)$ is extremely amenable but there exist two distinct $x, y \in \mathrm{End}(\Sigma)$ such that $x \preccurlyeq y$. Since $\mathrm{End}(\Sigma)$ is metrisable and *a fortiori* Hausdorff, we can take a clopen neighbourhood N of y so that $x \notin N$. By definition, inside of N, there exists a homeomorphic copy of a clopen neighbourhood of x but excludes x. This implies the existence of $x' \in \mathrm{End}(\Sigma)$ which is distinct from x but equivalent to x. Moreover, one can take a clopen neighbourhood D of x and a clopen neighbourhood D' of x' in the way that $D \simeq D'$ but $D \cap D' = \emptyset$. We should note that $\mathrm{End}(\Sigma) \setminus D$ is also homeomorphic to $\mathrm{End}(\Sigma) \setminus D'$. Now associate D to a subsurface with boundary $S \subset \Sigma$ as described above so that $\mathrm{End}(S) = D \sqcup \{*\}$ and find a subsurface $S' \subset \Sigma$ likewise for D'. By Richards' theorem, the curves ∂S and $\partial S'$ have the same topological type and satisfies the hypothesis

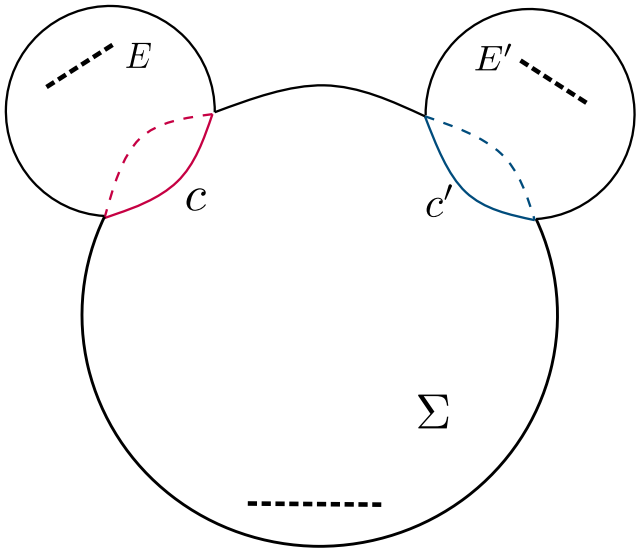


Figure 6.1: Surface of Lemma 6.2.6

of Lemma 6.2.6. But Lemma 6.2.6 indicates that there exists a mapping class $\phi \in \mathcal{MCG}(\Sigma)$ such that if $F = \{\partial S, \partial S'\}$, then $\phi \in \mathcal{MCG}(\Sigma)_F \setminus \mathcal{MCG}(\Sigma)_{(F)}$. This yields a contradiction to Lemma 6.2.2. □

Now one can show the following proposition:

Proposition 6.2.8. *Let Σ be an orientable surface with genus 0. Then $\mathcal{MCG}(\Sigma)$ is extremely amenable if and only if Σ is a sphere or a once-punctured sphere, in which case $\mathcal{MCG}(\Sigma)$ is the trivial group.*

Proof. If the surface has complexity less than 2, then it has non-trivial discrete group as mapping class group unless it is a sphere or a once-punctured sphere. So by the virtue of Lemma 6.2.7, it remains to show that there is no surface with genus 0, complexity at least 2 and pairwise non-comparable ends unless it is a sphere or a once-punctured sphere, where the statement is satisfied vacuously. Indeed, let Σ be a such surface. Assume that there exist distinct $x, y \in \mathrm{End}(\Sigma)$ and an element $g \in \mathcal{MCG}(\Sigma)$ so that $gx = y$. Then necessarily $x \preccurlyeq y$ as remarked above, which contradicts to the non-comparing assumption. Hence the action of $\mathcal{MCG}(\Sigma)$ on $\mathrm{End}(\Sigma)$ must be trivial. This implies that $\mathcal{MCG}(\Sigma) = \mathcal{PMCG}(\Sigma)$. By [PV18, Theorem 3], this means that $\mathcal{MCG}(\Sigma) = \mathcal{PMCG}(\Sigma)$ are simultaneously residually finite and as a result, the surface Σ must be of finite type. But as the ends of Σ must be all non-comparable, it only happens when Σ is a sphere or a once punctured sphere, in which cases $\mathcal{MCG}(\Sigma)$ is trivial. □

Let S_1^1 be the surface of genus 1 with one boundary component and let $S_{1,1}$ be once-punctured torus. Then there is an element $g \in \mathcal{MCG}(S_1^1)$ and a simple closed curve c on S_1^1 such that $\{g^n(c) : n \in \mathbb{Z}\}$ is finite. Indeed, by the inclusion homomorphism [FM11, Theorem 3.18], we have the short sequence

$$1 \to \mathbb{Z} \to \mathcal{MCG}(S_1^1) \to \mathrm{SL}_2(\mathbb{Z}) \simeq \mathcal{MCG}(S_{1,1}) \to 1.$$

Note that

$$\phi := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \simeq \mathcal{MCG}(S_{1,1})$$

is a torsion and that there is a simple closed curve c on $S_{1,1}$, which can also be regarded as a curve in S_1^1, such that $\{\phi^n(c) : n \in \mathbb{Z}\}$ is finite. As a result, the orbit of c under the action of any pre-image $g \in \mathcal{MCG}(S_1^1)$ of $\phi \in \mathcal{MCG}(S_{1,1})$ is finite.

With the observation above, one can now prove Theorem 6.1.2:

Proof of Theorem 6.1.2. If the surface Σ has genus 0, then the extreme amenability of $\mathcal{MCG}(\Sigma)$ is determined by Proposition 6.2.8. For the torus, the mapping class group is not extremely amenable since it is a non-trivial discrete group. Suppose now that the surface Σ has complexity at least 2 and non-zero genus. If Σ contains an essential (sub)surface that is homeomorphic to S_1^1. Then we take a $g \in \mathcal{MCG}(S_1^1)$ and a curve c on $S_1^1 \subset \Sigma$ so that $\{g^n(c) : n \in \mathbb{Z}\}$ is finite. By the virtue of the inclusion homomorphism [FM11, Theorem 3.18], one can extend g by identity to an element in $\mathcal{MCG}(\Sigma)$. Now the pointwise stabiliser of $F := \{g^n(c) : n \in \mathbb{Z}\}$ in $\mathcal{MCG}(\Sigma)$ is different from its setwise stabiliser. Hence by Lemma 6.2.2, $\mathcal{MCG}(\Sigma)$ is not extremely amenable. □

Finally, there are still some details in Theorem 6.1.3 that need further clarifications.

Lemma 6.2.9. *Let G be a topological group and H be a dense subgroup of G. Then G is extremely amenable if and only if H is.*

Proof. Note that a topological group G is extremely amenable if and only if it admits a G-invariant multiplicative mean over $\mathrm{RUCB}(G)$, the space of right-uniformly continuous functions on G, or equivalently the G-action on its *Samuel compactification* σG has a fixed point (see for example [Pes06, §1.1]). But if H is a dense subgroup of G, then $\mathrm{RUCB}(H) \simeq \mathrm{RUCB}(G)$ and any continuous H-action on the multiplicative means of $\mathrm{RUCB}(G)$ can be extended continuously to a G-action on it. Conversely, the restriction of a continuous G-action on H yields an H-action. Hence H and G can only be simultaneously extremely amenable. □

Proof of Theorem 6.1.3. For any subgroup $G < \mathcal{MCG}(\Sigma)$ with complexity $\xi(\Sigma) > 2$ that contains a mapping class ϕ such that for some simple closed curve c on the surface, the

orbit $F := \{\phi^n(c) : n \in \mathbb{Z}\}$ is finite, it is clear that ϕ belongs to $\overline{G}_F$ but not $\overline{G}_{(F)}$. Hence the closure of such subgroup G in $\mathcal{MCG}(\Sigma)$ is not extremely amenable. It follows from Lemma 6.2.9 that G itself is neither extremely amenable. □

6.3 Perspectives

In a recent paper, Disarlo, Koberda and González [DKN23] establish a model theoretic connection between the mapping class groups and the curve graph of non-sporadic finite-type surfaces, which is motivated by Ivanov's metaconjecture [Iva06]. Following their ideas, another way to ask Question 6.1.1 is the following:

Question 6.3.1. Given a graph on countable vertices, how can one detect if it is the curve graph of an orientable surface?

Another notion that is closely related to the extreme amenability is the amenability of a topological group, *i.e.* every continuous group action on a compact Hausdorff space admits an invariant probability measure over the space. Although the non extreme amenability is already clear, it remains unknown if there are amenable big mapping class groups. For every but finitely many finite-type orientable surfaces, the mapping class group is non-amenable because as a discrete group, it contains a non-abelian free subgroup on two generators. If the surface is of infinite type, the amenability of its mapping class group is less clear. For the surfaces Σ of infinite type having a non-displaceable subsurface S of finite type, one can construct a *blown-up projection complex* from the curve graphs of the $\mathcal{MCG}(\Sigma)$-orbit of S, see for example [HQR22; DD22]. Equipped with the combinatorial metric, the blown-up projection complex is a separable geodesic Gromov-hyperbolic space on which the mapping class group $\mathcal{MCG}(\Sigma)$ acts continuously by isometries and the $\mathcal{MCG}(\Sigma)$-action is of general type. However, an amenable group can never have a continuous action of general type on a separable geodesic Gromov-hyperbolic space by isometries (see Chapter 9). This implies that these big mapping class groups are not amenable.

Chapter 7

Connectedness of the Gromov boundary of fine curve graphs

Joint work with Dong Tan.

> Everything is connected to everything else.
>
> Oliver Wendell Holmes Jr. (1841-1935)
> *The Bar as a Profession.*

7.1 Introduction

Let S be an orientable finite-type surface of genus $g \geq 2$. In order to study the group of diffeomorphisms of surface G, Bowden, Hensel and Webb design for S its *fine curve graph* $\mathcal{C}^\dagger(S)$ [BHW22], where the vertices are smooth curves instead of their isotopy classes, and the edges are defined by disjointness.

Indeed, the fine curve graph $\mathcal{C}^\dagger(S)$ shares a lot of similarity with the curve graph of S and can be considered as an analogue of the curve graph for the mapping class group: the graph is Gromov hyperbolic [BHW22] and the group of homeomorphisms of surface S is isomorphic to the automorphism group of the topological version of the fine curve graph [Lon+21].

But compared to the classical curve graph, the fine curve graph $\mathcal{C}^\dagger(S)$ is much more complicated and the dynamics of homeomorphism group on $\mathcal{C}^\dagger(S)$ is much richer than the mapping class group on the curve graph: $\mathcal{C}^\dagger(S)$ is a locally uncountable; there exist homeomorphisms on the surface S that induce parabolic isometries on $\mathcal{C}^\dagger(S)$ and by consequences there are points on the Gromov boundary that are not represented by ending laminations [BHW22; Bow+22], whereas it is the case for curve graphs [Kla22].

The study of geometric or topological interpretation of points at infinity in the fine curve graph is still an ongoing project. Efforts are made in understanding the stabilisers of these points in terms of (homological) rotation sets [Bow+22; GM23a; GM23b].

In this paper, we are more interested in the topological properties of the Gromov boundary of fine curve graphs. This paper is written as one of the initiatives in this direction.

For the curve graph, Peter Storm asked an interesting question about the (path)-connectedness of its the Gromov boundary, which is recorded in [KL08, Question 10]. This question is answered positively by [Gab09; LS09; LMS11]. Recently, [Wri23] provides a new proof and proves that the boundary of curve graph is in addition linearly connected (see Definition 7.2.6).

Let $\mathcal{NC}^\dagger(S)$ be the subgraph of $\mathcal{C}^\dagger(S)$ consisting of non-separating curves.

In this paper, we conclude the following results for the topology of Gromov boundary of the fine curve graph:

Theorem 7.1.1. *Let S be an orientable finite-type surface of genus $g \geq 2$. Then the Gromov boundary $\partial\mathcal{C}^\dagger(S)$ of the fine curve graph is path connected.*

Moreover, for any $o \in \mathcal{NC}^\dagger(S)$ and any small enough $\varepsilon > 0$, if $b \in (1, 1+\varepsilon)$ and $\rho_{o,b}$ is the visual metric defined on the Gromov boundary, then it is linearly connected.

We remark that Theorem 7.1.1 cannot be deduced from the path-connectedness of the Gromov boundary of curve graphs, since there are boundary points in the fine curve graph that are cannot be represented by ending laminations and that path-connectedness is usually not preserved under taking closures.

We say that a subgraph G of a graph Γ is *connected* if for any two points $a, b \in G$, there exists a finite sequence $a = v_0, v_1, \ldots, v_n = b$ such that v_i is connected to v_{i+1} by an edge for all $0 \leq i \leq n-1$ and $v_i \in G$ for all $0 \leq i \leq n$.

Theorem 7.1.2. *Let S be an orientable finite-type surface of genus $g \geq 2$. For any integer $r > 0$ and any $o \in \mathcal{NC}^\dagger(S)$, the r-sphere $\{\alpha \in \mathcal{NC}^\dagger(S) \colon d^\dagger(\alpha, o) = r\}$ is connected.*

The main difficulties of dealing with the fine curve graph is the non-transversality between topological curves and the lack of analogue in the setting of fine curve graphs of *tight geodesics* as in the curve graphs. In order to deduce the results, we first generalise the criterion introduced by Wright in [Wri23] to *any* hyperbolic graphs with no "*dead ends*" (see Proposition 7.2.7). In order to apply Proposition 7.2.7, it suffices to check that the assumptions are satisfied by a subgraph quasi-isometric to $\mathcal{C}^\dagger(S)$. In principle, under the *non-separating* and *transverse* settings, as long as there are only finitely many topological curves involved, *the fine curve graph looks like the curve graph of a punctured subsurface* (see Proposition 7.3.4), which allows us to make use of [Wri23, Proposition 5.4]. The technical part of the proof lies in passing non-transverse pairs of curves into transverse ones. At

the end of this paper, we also gave several interesting observations on the group action of $\mathrm{Homeo}(S)$ on the Gromov bordification of $\mathcal{C}^{\dagger}(S)$.

7.2 Sufficient condition for linear connecitivty

The following result is pivotal in large scale geometry *à la Gromov* and is well-known in proper cases. But we remark that it actually holds for non-proper spaces [Has22; Väi05]. For convenience reason, we state it here below. There are several different proofs from different approaches and the one we present here might have already been known to [Cap+15]:

Proposition 7.2.1. *Let (X, d_X) be a δ-hyperbolic geodesic space and $o \in X$ be an arbitrary base point. Then ∂X is in bijection with the equivalent classes of quasi-geodesic rays issued from o and two quasi-geodesic rays are equivalent if they have bounded Hausdorff distance. Moreover, a quasi-isometric embedding $f\colon X \to Y$ induces a topological embedding $f_\partial\colon \partial X \to \partial Y$ and f_∂ is a homeomorphism if f is essentially surjective.*

Proof. See Chapter 8 and [Väi05]. One idea is to take a ultracomplete supspace and deduce similarly as the classical cases, where taking ultralimits play the rôle of Arzelà-Ascoli theorem. □

Another powerful tool to study the Gromov boundary is the following notion of visual metric that is carefully treated in [DSU17, §3.6] and [Väi05] under general settings:

Theorem 7.2.2 (Väsäilä)**.** *Let (X, d_X) be a Gromov hyperbolic space. Then ∂X is completely metrizable. For any based point $o \in X$, the topology is compatible to the visual metric $\rho_{o,b}$ from $o \in X$ with parameter $b \in (1, 1+\varepsilon)$ for some small enough $\varepsilon > 0$. Moreover, we have the estimation*

$$b^{-\langle \xi,\eta\rangle_o}/4 \leq \rho_{o,b}(\xi,\eta) \leq b^{-\langle \xi,\eta\rangle_o} \tag{7.1}$$

for every $\xi, \eta \in \partial X$.

Let X be a δ-hyperbolic geodesic space and let $\gamma \subset X$ be a geodesic. We define for a point $x \in X$ its *nearest points projection* $\pi_\gamma(x)$ by the points $y \in \gamma$ satisfying $d_X(x,y) = d_X(x,\gamma)$. We remark that $\pi_\gamma(x)$ is not empty in a geodesic space: it consists of the minima of the continuous proper function $d_X(x,\cdot)\colon \gamma \to [0,\infty)$.

The first result about this notion is referred as the *reverse triangle inequality*, see (8.1) from Proposition 8.2.1.

Using this inequality, we can deduce the following estimation, which is essentially a translation into nearest points projection language of the thinness of geodesic triangles in Gromov hyperbolic spaces:

Corollary 7.2.3. *Let X be a geodesic δ-hyperbolic space. Let $x, y, o \in X$ be three points and $\gamma := [o, y]$ be a geodesic segment. Then for any $p \in \pi_\gamma(x)$,*

$$\langle x, y \rangle_o = d_X(o, p) + O(\delta). \tag{7.2}$$

Proof. Using (8.1) for $d_X(x, o)$ and $d_X(x, y)$ appeared in (2.4) will yield the result. □

Now we can prove the following lemma about the distance between two boundary points under a visual metric:

Lemma 7.2.4. *Let X be a δ-hyperbolic geodesic space. Fix a base point $o \in X$. Let $\rho_{o,b}$ be a visual metric from o with parameter $b \in (1, 1 + \varepsilon)$ for sufficiently small $\varepsilon > 0$. Then for every large enough $E \gg 1$, $\lambda \geq 1$ and $k \geq 0$, there exists a constant $C := C(b, k, E, \delta) > 0$ such that for every two distinct points $\xi, \eta \in \partial\Gamma$ and for any ℓ_1, ℓ_2 continuous (λ, k)-quasi-geodesics connecting o to ξ and η respectively, if*

$$r := \sup\{t \geq 0 \colon d_X(\ell_1(t), \ell_2(t)) \leq E\},$$

then

$$b^{-\lambda r}/C \leq \rho_{o,b}(\xi, \eta) \leq C b^{-r/\lambda}. \tag{7.3}$$

Proof. By continuity of ℓ_1 and ℓ_2, if $x := \ell_1(r)$ and $y := \ell_2(r)$, then $d_X(x, y) = E$. Let $\gamma := [o, y]$ be a geodesic segment and $p \in \pi_\gamma(x)$. Applying the reverse triangle inequality (8.1) twice, we have

$$d_X(p, y) = d_X(x, y) - d_X(x, p) + O(\delta) = E - d_X(x, o) + d_X(o, p) + O(\delta).$$

Using the fact that $d_X(o, y) = d_X(o, p) + d_X(p, y)$, we can further deduce that

$$d_X(o, p) = \frac{1}{2}\left(d_X(x, o) + d_X(y, o) - E\right) + O(\delta).$$

Using the (λ, k)-quasi-geodesicity, we have the inequality

$$\frac{r}{\lambda} - k - E/2 + O(\delta) \leq d_X(o, p) \leq \lambda r + k - E/2 + O(\delta). \tag{7.4}$$

But by (2.8), when E is sufficiently large, we have $\langle x, y \rangle_o = \langle \xi, \eta \rangle_o + O(\delta)$. Combining this fact with (7.2) and (7.4), we can deduce that

$$\frac{r}{\lambda} - k - E/2 + O(\delta) \leq \langle \xi, \eta \rangle_o \leq \lambda r + k - E/2 + O(\delta).$$

By setting $C := 4b^{k + E/2 + O(\delta)}$, we can deduce the desired inequality via applying the esti-

mation (7.1). □

Remark 7.2.5. As mentioned in the proof of Proposition 7.2.1, in a geodesic δ-hyperbolic space, each boundary point $\xi \in \partial X$ can be connected to the base point o by a continuous $(1, N\delta)$-quasi-geodesic. So the λ from Lemma 7.2.4 can be taken as 1.

At this moment, we introduce the following notion:

Definition 7.2.6 (Linear connectedness)**.** A metric space (X, d_X) is *linearly connected* if there is a constant $L > 0$ such that for each pair $x, y \in X$ there is a compact connected set $K \subset X$ containing x, y with diameter at most $Ld_X(x, y)$.

The application of Lemma 7.2.4 is the following proposition inspired by [MS20, Proof of Proposition 5.2] and is an improvement of [Wri23, Proposition 2.2].

For notations, let $S_o(r)$ be the sphere centred at o of radius $r > 0$ and $B_o(r)$ be the ball centred at o of radius $r > 0$.

Proposition 7.2.7. *Let Γ be a Gromov hyperbolic simplicial graph and let $o \in \Gamma$ be a base point. Suppose that the following holds.*

(G1) Every vertex of Γ is adjacent to point of Γ that is 1 farther from o.

If in addition there is some $D > 0$ such that for all $r \geq 0$, the following conditions are satisfied:

(G2) For every $z \in S_o(r)$ and $x, y \in S_o(r+1) \cap B_z(1)$ there exists a path

$$x = x_0,\ x_1,\ \ldots,\ x_\ell = y$$

with

$$x_i \in (\Gamma - B_o(r)) \cap B_x(D)$$

for $0 \leq i \leq \ell$.

(G3) For every adjacent pair $x, y \in S_o(r)$ there exists a path

$$x = x_0,\ x_1,\ \ldots,\ x_\ell = y$$

with

$$x_i \in (\Gamma - B_o(r)) \cap B_x(D)$$

for $0 < i < \ell$.

Then equipped with the visual metric $\rho_{o,b}$ from o with parameter $b \in (1, 1+\varepsilon)$ for some small enough $\varepsilon > 0$, the Gromov boundary $\partial\Gamma$ is linearly connected and path connected.

Proof. The proof is a straightforward adaption of the proof of [Wri23, Proposition 2.2]. Let $\xi_0, \xi_1 \in \partial X$ be any two points on the Gromov boundary. Now connect ξ_0 and ξ_1 to o respectively by continuous $(1,k)$-quasi-geodesic rays γ_0 and γ_1. By the same arguments in [Wri23, Proof of Proposition 2.2], (G1) assures the existence of a dense subset $I \subset [0,1]$ such that for all $t \in I$, there is a geodesic ray $\gamma_t \subset \Gamma$ connecting o to a point $c(t) \in \partial\Gamma$. Since the family $(\gamma_t)_{t\in I\cup\{0,1\}}$ consists of continuous $(1,k)$-quasi-geodesic rays, thus by Lemma 7.2.4, we can deduce from (G2) and (G3) the same estimation as in [Wri23, Proof of Proposition 2.2], *i.e.* there exists $L > 0$ independent of ξ_0 and ξ_1 such that the diameter $\operatorname{diam}\big(c(I)\big) < L\rho_{o,b}(\xi_0, \xi_1)$ and that $c\colon I \to \partial\Gamma$ is uniformly continuous. The desired path is then the closure of $c(I)$. □

Remark 7.2.8. Proposition 7.2.7 cannot be obtained by applying [Wri23, Proposition 2.2].

7.3 Fine curve graph

Let S be an orientable finite-type surface of genus $g \geq 2$. It is a surface carrying a hyperbolic metric. In a recent paper, Bowden, Hensel and Webb design for S its *fine curve graph* [BHW22] as a combinatorial tool for studying $\mathrm{Homeo}(S)$, which is an analogue to the renown curve graph.

An *essential simple closed smooth curve* γ is a proper $\mathcal{C}^\infty$-embedding of the circle $S^1 \hookrightarrow S$ such that γ is not homotopic to a singleton nor bounds a once-punctured disc or an annulus. In this chapter, all curves will be essential on the surface S, simple, closed, and smooth, unless otherwise mentioned.

The *fine curve graph* $\mathcal{C}^\dagger(S)$ is a graph where the vertices are essential simple closed smooth curves on S and two vertices are joined by an edge if they are disjoint. The *non-separating fine curve graph* $\mathcal{NC}^\dagger(S)$ is the subgraph of $\mathcal{C}^\dagger(S)$ consisting of vertices as non-separating curves. If they are equipped with the simplicial metrics, then these two connected graphs are quasi-isometric (with the inclusion map being a $(1,1)$-quasi-isometry) and are simultaneously Gromov hyperbolic [BHW22, Corollary 3.9]. Hence by Proposition 7.2.1, the Gromov boundaries $\partial\mathcal{NC}^\dagger(S)$ is homeomorphic to $\partial\mathcal{C}^\dagger(S)$.

Remark 7.3.1. We remark that it is possible to construct the fine curve graphs by using non-differentiable curves, *i.e.* $\mathcal{C}^0$-embedding of $S^1 \hookrightarrow S$, which turns out to be the *topological version of the fine curve graphs*. However, we remark that any topological curves can be isotopic to a differentiable curve that is disjoint from it: isotopically mollify the curve and then move it away. Hence there is a 1-Lipschitz embedding of the fine curve graphs of differentiable curves into the fine curve graphs of non-differentiable curves and this embedding is 1-dense, *i.e.* the two versions of fine curve graphs are quasi-isometric and have no difference in the view point of large-scale geometry.

In the sequel, we will denote by $d^\dagger$ the simplicial metric on $\mathcal{NC}^\dagger(S)$. Also, we will also denoted by $\mathcal{NC}(S)$ the simplicial graph where the vertices are *isotopic* classes of essential non-separating simple closed curves and two vertices are connected by an edge if they admit disjoint representative. We remark that $\mathcal{NC}(S)$ is also Gromov hyperbolic and is quasi-isometric to the usual curve graph [Ham14, §3].

Definition 7.3.2 (Transversality)**.** Let α, β be two simple closed curves. We say that they are *transverse* if exactly one of the following holds:

- $\alpha \cap \beta = \emptyset$.
- For every $p \in \alpha \cap \beta$, there is an open neighbourhood $N \subset S$ of p and a diffeomorphism onto the unit disk $\varphi \colon N \to D \subset \mathbb{R}^2$ such that $\varphi(\alpha \cap N) = (-1,1) \times \{0\}$ and $\varphi(\beta \cap N) = \{0\} \times (-1,1)$.

Note that if two curves are transverse, then by compactness, they only intersect at finitely many points.

A *bigon* of two transverse curves α, β in a (punctured) surface is a connected component in the complementary $\alpha \cup \beta$ that is homeomorphic to a disk and bounds exactly one subarc of α and one subarc of β. Two transverse curves α, β are in *minimal position* if they have no bigon [FM11, Proposition 1.7].

One advantage of being smooth is that the curves can be taken as geodesic under some hyperbolic metric:

Proposition 7.3.3 (Proposition 10, Exposé 3, [FLP91])**.** *Let S be an orientable finite-type surface of genus at least 2. Let γ be an essential simple closed smooth curve. Then there exists a hyperbolic metric on S such that γ is geodesic. If in addition γ and γ' are two smooth curves in minimal positions, then there exists a hyperbolic metric on S such that γ and γ' are simultaneously geodesic.*

An important metric geometric property of the fine curve graph is that it can be approximated via the curve graph of its punctured subsurfaces:

Proposition 7.3.4 (Lemma 3.4 & Lemma 3.6 [BHW22])**.** *Suppose that $\alpha, \beta \in \mathcal{NC}^\dagger(S)$ are transverse, and that α and β are in minimal position in $S - P$, where $P \subset S$ is finite and disjoint from $\alpha \cup \beta$. Then*

$$d_{\mathcal{NC}(S-P)}([\alpha]_{S-P}, [\beta]_{S-P}) = d^\dagger(\alpha, \beta),$$

where $[\alpha]_{S-P}$ and $[\beta]_{S-P}$ are respectively the isotopic classes of α and β in $S - P$. Moreover, there is a geodesic $\alpha = \nu_0, \nu_1, \dots, \nu_n = \beta$ in $\mathcal{NC}^\dagger(S)$ such that $\nu_i \cap P = \emptyset$ and $[\alpha]_{S-P} = [\nu_0]_{S-P}, [\nu_1]_{S-P}, \dots, [\nu_n]_{S-P} = [\beta]_{S-P}$ is also a geodesic in $\mathcal{NC}(S - P)$.

Remark 7.3.5. The finite set P can be empty. In that case, it yields an isometric embedding of $\mathcal{NC}(S - P) \hookrightarrow \mathcal{NC}^\dagger(S)$. This can be realised by choosing a hyperbolic metric on S and each isotopy class of simple closed curves is sent to its geodesic representative.

Now we will fix an arbitrary base point $o \in \mathcal{NC}^\dagger(S)$. We write $\mathcal{NC}^\dagger_{o,\pitchfork}(S)$ the subgraph of $\mathcal{NC}^\dagger(S)$ where all the vertices are transverse to o. It is *a priori* equipped with the subgraph distance but not the induced distance.

Lemma 7.3.6. *Let $\alpha \in \mathcal{NC}^\dagger(S)$. Let $F \subset \mathcal{NC}^\dagger(S)$ be a finite collection of pairwise transverse curves. Suppose that for every $\gamma \in F$, $\alpha \cap \gamma \neq \emptyset$. Then there exists a curve $\alpha' \in \mathcal{NC}^\dagger(S)$ isotopic to α that is disjoint from α and is transverse to all $\gamma \in F$.*

Proof. Let $A \subset S$ be a small enough annular neighbourhood of the curve $\gamma \in F$. Since α is a proper embedding of S^1, there are only finitely many connected components in $\alpha \cap A$. Hence it is possible to cut A into finitely many pieces $A_1, \ldots, A_n$ so that for each $1 \leq i \leq n$, the piece A_i contains exactly one connected component of $\alpha \cap A$. Note that each A_i is diffeomorphic to a $[-1,1]^2 \subset \mathbb{R}^2$ with the image of $o \cap A_i$ being $[-1,1] \times \{0\}$. It is an elementary exercise to show that under this diffeomorphism, α is isotopic to an α' that is transverse to γ and disjoint from α inside of $[-1,1]^2$. Note that if A is small enough, then α' can be taken so that whenever α is transverse to some other finitely many curves, α' remains transverse to them. Hence by induction, we can prove the desired result. □

We notice that the perturbation in Lemma 7.3.6 can be done within an arbitrarily small annular neighbourhood. All the isotopic perturbation in the sequel will be of the same flavour, although the situation might be slightly different.

Proposition 7.3.7. *The inclusion map $\mathcal{NC}^\dagger_{o,\pitchfork}(S) \hookrightarrow \mathcal{NC}^\dagger(S)$ is a $(1,2)$-quasi-isometry, hence $\partial\mathcal{NC}^\dagger_{o,\pitchfork}(S) \simeq \partial\mathcal{NC}^\dagger(S)$.*

Proof. For any $\alpha \in \mathcal{NC}(S)$, if it is not transverse to o, then by Lemma 7.3.6, there is another $\alpha' \in \mathcal{NC}^\dagger(S)$ that is disjoint from α and is transverse to o. Hence $d^\dagger(\alpha, \alpha') = d^\dagger(\alpha, \mathcal{NC}^\dagger_{o,\pitchfork}(S)) = 1$.

For any two vertices $\alpha, \beta \in \mathcal{NC}^\dagger_{o,\pitchfork}(S)$, if they are not transverse, then by applying Lemma 7.3.6, we can find a β' with $d^\dagger(\beta, \beta') = 1$ that is transverse to both α and o. Now we can find a geodesic $\alpha = \nu_0, \nu_1, \ldots, \nu_n = \beta'$ in $\mathcal{NC}^\dagger(S)$ connecting α to β' with the property that there is a finite subset $P \subset S$ disjoint from o such that $\nu_i \cap P = \emptyset$ for all i and $[\alpha]_{S-P} = [\nu_0]_{S-P}, [\nu_1]_{S-P}, \ldots, [\nu_n]_{S-P} = [\beta]_{S-P}$ is also a geodesic in $\mathcal{NC}(S-P)$. Without loss of generality, we may assume that ν_i's and o are in their minimal positions in $S-P$, by taking a hyperbolic metric making o a geodesic in $S-P$ as per Proposition 7.3.3, which in particular implies that ν_i's are transverse to o and are contained in $\mathcal{NC}^\dagger_{o,\pitchfork}(S)$. Hence the subgraph distance between $\alpha, \beta' \in \mathcal{NC}^\dagger_{o,\pitchfork}(S)$ is exactly their distance induced from $\mathcal{NC}^\dagger(S)$.

Now combine the results of two paragraphs above, we can conclude that the inclusion map $\mathcal{NC}^\dagger_{o,\pitchfork}(S) \hookrightarrow \mathcal{NC}^\dagger(S)$ is a $(1,2)$-quasi-isometry and thus by Proposition 7.2.1, their Gromov boundaries are homeomorphic. □

7.4 Connectivity properties

In order to demonstrate the connectedness of the Gromov boundary of the fine curve graph, we need only to prove that the spheres in the subgraph $\mathcal{NC}^{\dagger}_{o,\pitchfork}(S)$ enjoys connectivity properties as in the assumption of Proposition 7.2.7.

For notations, for any positive integer $r > 0$, let $\mathcal{S}_{o,\pitchfork}(r)$ be the sphere in $\mathcal{NC}^{\dagger}_{o,\pitchfork}(S)$ centred at o of radius r, *i.e.* $\{\alpha \in \mathcal{NC}^{\dagger}_{o,\pitchfork}(S) : d^{\dagger}(\alpha, o) = r\}$.

Lemma 7.4.1. *Let $r > 0$ be any positive integer and let $\alpha, \beta \in \mathcal{S}_{o,\pitchfork}(r)$. If α and β are not transverse, then there exists a $\beta' \in \mathcal{S}_{o,\pitchfork}(r)$ isotopic to β but disjoint from it, which is also transverse to α.*

Proof. In light of Proposition 7.3.4, we can find a finite subset $P \subset S$ so that (o, α) and (o, β) are in minimal positions in $S - P$. Up to a small perturbation as in Lemma 7.3.6, the curve β is isotopic to but disjoint from β' while β' is transverse to α. Since the perturbation can be taken so that β' remains in a minimal position with o in $S - P$. By Proposition 7.3.4, $\beta' \in \mathcal{S}_{o,\pitchfork}(r)$. □

Lemma 7.4.2. *For any positive integer $r > 0$, the sphere $\mathcal{S}_{o,\pitchfork}(r)$ is connected.*

Proof. Let $\alpha, \beta \in \mathcal{S}_{o,\pitchfork}(r)$ be distinct non-separating curves. By Lemma 7.4.1, up to perturbation, we can choose $\beta' \in \mathcal{S}_{o,\pitchfork}(r)$ isotopic to but disjoint from β. Moreover, there exists a finite set $P \subset S$ such that β', α and o are in minimal positions in $S - P$. By taking a hyperbolic metric on $S - P$ as in Proposition 7.3.3, we can even assume that β' (up to perturbation), α and o are geodesics. Moreover, in view of Proposition 7.3.4, there is an isometric embedding via geodesic representations $\iota : \mathcal{NC}(S - P) \hookrightarrow \mathcal{NC}^{\dagger}_{o,\pitchfork}(S)$. Note that $S - P$ is of at least genus 2 and 1 puncture, *i.e.* of high complexity as called in [Wri23]. Then applying [Wri23, Proposition 5.4] and [Wri23, Lemma 2.1] to $\mathcal{NC}(S - P)$, there is a path $[\alpha]_{S-P} = [\nu_0]_{S-P}, [\nu_1]_{S-P}, \ldots, [\nu_n]_{S-P} = [\beta']_{S-P}$ lying inside of the r-sphere centred at $[o]_{S-P}$ in $\mathcal{NC}(S - P)$. Hence the image under the isometric embedding ι via geodesic representations, $\alpha = \nu_0, \nu_1, \ldots, \nu_n = \beta'$, yields a path connecting α to β' in $\mathcal{S}_{o,\pitchfork}(r)$. Since β and β' is disjoint, they are adjacent in $\mathcal{S}_{o,\pitchfork}(r)$. Hence the concatenation $\alpha = \nu_0, \nu_1, \ldots, \nu_n = \beta', \beta$ is a path in $\mathcal{S}_{o,\pitchfork}(r)$ connecting the two vertices. □

Similar arguments will soon yield Theorem 7.1.2:

Proof of Theorem 7.1.2. For any two distinct $\beta, \gamma \in \{\alpha \in \mathcal{NC}^{\dagger}(S) : d^{\dagger}(\alpha, o) = r\}$, as in Lemma 7.4.1, we can similarly perturb them to β' and γ' that are isotopic to but disjoint from β and γ respectively, so that $\{o, \beta', \gamma'\}$ are transverse and $d^{\dagger}(o, \beta') = d^{\dagger}(o, \gamma') = r$. Now by Lemma 7.4.2, we can connect them by path contained in the sphere. □

The following lemma shows that there is no "*dead ends*" in the sense of [BM15] inside of the subgraph $\mathcal{NC}^{\dagger}_{o,\pitchfork}(S)$, *i.e.* this subgraph satisfies the assumption (G1) in Proposition 7.2.7.

Lemma 7.4.3. *For every adjacent pair $\alpha, \beta \in \mathcal{S}_{o,\pitchfork}(r)$ there exists a path*

$$\alpha = \nu_0, \nu_1, \nu_2 = \beta$$

in $\mathcal{NC}^{\dagger}_{o,\pitchfork}(S)$ with $\nu_1 \in \mathcal{S}_{o,\pitchfork}(r+1)$.

Proof. Since α and β are disjoint, take a non-empty finite subset $P \subset S$ so that P is disjoint from $\alpha \cup \beta \cup o$ and that o, α, β are in minimal positions. Take a hyperbolic metric on $S - P$ as in Proposition 7.3.3 so that α and β are geodesics. By [Wri23, Proposition 5.4], we can find a geodesic curve ν with respect to the above hyperbolic metric such that it is disjoint from α and β with $d_{\mathcal{NC}(S-P)}([\nu]_{S-P}, [o]_{S-P}) = r+1$. Hence it lies in a connected component Σ of $S - \alpha \cup \beta$. Now perturb ν isotopically into ν_1 so that it has no bigon with $o \cap \Sigma$ in $\Sigma - P$. This will make sure that ν_1 is still disjoint from α and β, but is in minimal position with o in $S - P$. Note that $[\nu]_{S-P} = [\nu_1]_{S-P}$. Hence $d^{\dagger}(o, \nu_1) = d_{\mathcal{NC}(S-P)}([\nu_1]_{S-P}, [o]_{S-P}) = r+1$ by Proposition 7.3.4. □

Lemma 7.4.4. *Let $r > 0$ and $\alpha, \alpha' \in \mathcal{S}_{o,\pitchfork}(r)$ be isotopic. Then there exists a path on $\mathcal{S}_{o,\pitchfork}(r)$ connecting α to α', in which every vertices is a curve isotopic to α. If $\beta \in \mathcal{NC}^{\dagger}(S)$ is in addition disjoint from both α and α', then so will be the vertices in this path.*

Proof. Let $P \subset S$ be a finite subset so that (o, α) and (o, α') are in minimal positions in $S - P$. Up to a small perturbation as in Lemma 7.3.6, we can take α'' that is isotopic to but disjoint from α and that is transverse to α'. Since the perturbation is small, the curve α'' will remain in minimal position with o on $S - P$ and is disjoint from β. If α'' is disjoint from α', then the proof is done. Otherwise they have bigons. At least one of these bigons bounds a topological disk and contains exactly one subarc of α' and one subarc of α''. By deforming isotopically the subarc of α'' to the other side of α', the curve α'' can be deformed into an isotopic α_1 with strictly less intersection points with α' and also disjoint from α''. Since the support of this deformation on S is disjoint from P and β, so α_1 remains in minimal position with o and is still disjoint from β. Hence we can conclude that $d^{\dagger}(\alpha_1, o) = r$ and $d^{\dagger}(\alpha_1, \alpha'') = 1$. Continuing the construction process, we gain $\alpha_1, \alpha_2, \ldots, \alpha_n$ until α_n is disjoint from α' in a way that $d^{\dagger}(\alpha_i, o) = r$ and $d^{\dagger}(\alpha_i, \alpha_{i+1}) = 1$ for all $1 \leq i \leq n-1$. Hence we get a desired path $\alpha, \alpha'', \alpha_1, \ldots, \alpha_n, \alpha'$. □

If we write $B_r(\gamma) \coloneqq \{\alpha \in \mathcal{NC}^{\dagger}(S) \colon d^{\dagger}(\alpha, \gamma) \leq r\}$ the ball of radius r centred at γ, then we will have the following result:

Lemma 7.4.5. *For every $\gamma \in \mathcal{S}_{o,\pitchfork}(r)$ and $\alpha, \beta \in \mathcal{S}_{o,\pitchfork}(r+1) \cap B_1(\gamma)$, then there exists a path $\alpha = \nu_0, \nu_1, \dots, \nu_\ell = \beta$ in the graph $\mathcal{NC}^\dagger(S)$ with $\nu_i \in \mathcal{S}_{o,\pitchfork}(r+1) \cap B_1(\gamma)$ for $0 \leq i \leq \ell$.*

Proof. Let $P \subset S$ be a non-empty finite subset so that $\{o, \gamma, \alpha\}$ are in minimal positions. If α and β are isotopic in $S - P$, then by Lemma 7.4.4, we are done. Suppose now that they are not. Take a hyperbolic metric on $S - P$ so that o and γ become geodesic. Now we take the geodesic representatives $\alpha' \in [\alpha]_{S-P}$ and $\beta' \in [\beta]_{S-P}$. So $\{o, \gamma, \alpha', \beta'\}$ are pairwise in minimal position on $S - P$. In particular, we conclude that $\alpha', \beta' \in \mathcal{S}_{o,\pitchfork}(r+1) \cap B_1(\gamma)$ by Proposition 7.3.4. By sending the isotopic classes to their geodesic representatives on $S - P$, we can construct an isometric embedding $\iota \colon \mathcal{NC}(S - P) \hookrightarrow \mathcal{NC}^\dagger_{o,\pitchfork}(S)$. We remark that [Wri23, Proposition 5.4] implies that there exists a path in $\mathcal{NC}(S - P)$ connecting $[\alpha']_{S-P}$ to $[\beta']_{S-P}$ in which each vertices has distance $r+1$ to $[o]_{S-P}$ and distance 1 to $[\gamma]_{S-P}$. Their images under ι then yield a path from α' to β' included in $B_1(\gamma)$. Now we connect a path from α to α' as well as from β to β' as in Lemma 7.4.4. Because the vertices from these two paths are disjoint from γ, they also lie in $B_1(\gamma)$. By consequences, the concatenation of the three paths above yield the desired path. □

Now we have finished the proof of all assumptions in Proposition 7.2.7 for the subgraph $\mathcal{NC}^\dagger_{o,\pitchfork}(S) \subset \mathcal{NC}^\dagger(S) \subset \mathcal{C}^\dagger(S)$. Note that they are all quasi-isometric. To prove Theorem 7.1.1, it suffices to apply Proposition 7.2.7 to $\mathcal{NC}^\dagger_{o,\pitchfork}(S)$.

Proof of Theorem 7.1.1. With $D = 1$, for the subgraph $\mathcal{NC}^\dagger_{o,\pitchfork}(S)$, Lemma 7.4.3 verifies the assumption (G1) and (G2) from Proposition 7.2.7 and Lemma 7.4.5 verifies the assumption (G3) from Proposition 7.2.7. □

Remark 7.4.6. If the base point o is chosen as a separating curve on S, then instead of linearly connectivity, we will obtain a *polynomial connectivity*, *i.e.* any two points on the boundary $\xi, \eta \in \partial\mathcal{C}^\dagger(S)$ can be included in a connected compact set with diameter at most $L\rho_{o,b}(\xi, \eta) + (L+1)b$ for some $L > 0$, since the visual metrics verify the inequality $|\rho_{o,b} - \rho_{o',b}| \leq bd^\dagger(o, o')$.

7.5 Group action on the boundary

In this section, let us assume that the curves are only $\mathcal{C}^0$-embedding of S^1 into S, *i.e.* we will be considering the topological version of the fine curve graphs. As per Remark 7.3.1, there is no difference in terms of their boundary at infinity.

Let S be an orientable compact surface as above and let $\mathrm{Homeo}(S)$ be the homeomorphism group of S. In this section, we will make several observations on the action of the topological group $\mathrm{Homeo}(S)$ on the topological space $\partial\mathcal{C}^\dagger(S)$.

We remark that $\mathrm{Homeo}(S)$ acts by isometries on the fine curve graph $\mathcal{C}^\dagger(S)$ and thus on the Gromov bordification of $\mathcal{C}^\dagger(S)$. But the action of $\mathrm{Homeo}(S)$ on $\mathcal{C}^\dagger(S)$ is not continuous: a sequence of homeomophisms $g_n \to \mathrm{Id}$ in compact-open topology (and even isotopic to Id) can send a simple closed curve $\gamma \subset S$ to $g_n\gamma$ such that $g_n\gamma$ is disjoint from γ for all n, *i.e.* $d^\dagger(g_n\gamma, \gamma) = 1$ does not converge to 0 as $n \to \infty$. But since we are only interested in its geometry at large scale, this does not cause any problem:

Proposition 7.5.1. *The action of* $\mathrm{Homeo}(S)$ *on* $\partial\mathcal{C}^\dagger(S)$ *is continuous.*

Proof. It suffices to show that whenever given a sequence $g_n \to \mathrm{Id}$ as $n \to \infty$, for any $M > 0$ and $\xi \in \partial\mathcal{C}^\dagger(S)$, there exists $M' > 0$ and $N > 0$ such that if $\langle \xi, \eta\rangle_o > M'$, then $\langle \xi, g_n\eta\rangle_o > M$ for any $n > N$. Indeed, since the open sets on the boundary is also generated by shadows defined in (2.9), we can find an $\alpha \in \mathcal{C}^\dagger(S)$ such that whenever $\langle \alpha, \eta\rangle_o > R$, we have $\langle \eta, \xi\rangle_o > M$. Moreover, we can assume that $\langle \alpha, \xi\rangle_o$ is much larger than R. Since the topology on $\mathrm{Homeo}(S)$ is the compact-open topology, there exists $N > 0$ such that $g_n\alpha$ is contained in a uniformly small neighbourhood of α for every $n > N$. This implies that for any curve $\gamma \subset S$ that is disjoint from this small neighbourhood and *a fortiori* from α, γ is also disjoint from $g_n\alpha$ for all $n > N$. Hence $d^\dagger(\alpha, g_n\alpha) \leq 2$ for all $n > N$. The same arguments hold for o so that $d^\dagger(o, g_n o) \leq 2$ for $n > N$. Let $\eta \in \partial\mathcal{C}^\dagger(S)$ such that $\langle \eta, \xi\rangle_o > M'$. We compute by (2.5) and (2.8)

$$\begin{aligned}\langle \alpha, g_n\eta\rangle_o &= \langle g_n^{-1}\alpha, \eta\rangle_{g_n^{-1}o} \geq \langle g_n^{-1}\alpha, \eta\rangle_o - 2 - O(\delta)\\ &\geq \min\left(\langle \xi, g_n^{-1}\alpha\rangle_o, \langle \xi, \eta\rangle_o\right) - 2 - O(\delta)\\ &\geq \min\left(\langle \xi, g_n^{-1}\alpha\rangle_o, M'\right) - 2 - O(\delta)\end{aligned}$$

But as $d^\dagger(\alpha, g_n\alpha) \leq 2$, we have $\left|\langle \xi, g_n^{-1}\alpha\rangle_o - \langle \xi, \alpha\rangle_o\right| \leq 2 + O(\delta)$, which means that $\langle \xi, g_n^{-1}\alpha\rangle_o$ is also much larger than R. Hence by letting M' be large enough, we can assure that $\langle \alpha, g_n\eta\rangle_o > R$, which implies that $\langle \xi, g_n\eta\rangle_o > M$ for all $n > N$. □

Recall that for a δ-hyperbolic space X and a subgroup $G < \mathrm{Isom}(X)$. Let $o \in X$ be a (and thus any) point in X. The *limit set* of G, denoted $\Lambda(G)$, is given by

$$\Lambda(G) \coloneqq \left\{\xi \in \partial X : g_n o \to \xi \text{ for some } (g_n) \in G^{\mathbb{N}}\right\}.$$

Now let us examine the limit set of $\mathrm{Homeo}(S)$ on the Gromov boundary of $\mathcal{C}^\dagger(S)$.

Lemma 7.5.2. *Let S be an orientable surface as above. The limit set of* $\mathrm{Homeo}(S)$ *on* $\partial\mathcal{C}^\dagger(S)$ *is* $\partial\mathcal{C}^\dagger(S)$.

Proof. Let us consider $\mathcal{NC}^\dagger(S)$, which is quasi-isometric to $\mathcal{C}^\dagger(S)$. We remark that the homeomorphism $\partial\mathcal{NC}^\dagger(S) \to \partial\mathcal{C}^\dagger(S)$ is $\mathrm{Homeo}(S)$-equivariant. So it suffices to show

that the limit set of $\mathrm{Homeo}(S)$ on $\partial\mathcal{NC}^\dagger(S)$ is $\partial\mathcal{NC}^\dagger(S)$ itself. Indeed, by the theorem of classification of finite-type surfaces, for any two non-separating simple closed curves $\alpha, \beta \subset S$, there exists $\phi \in \mathrm{Homeo}(S)$ such that $\phi(\alpha) = \beta$, *i.e.* the action of $\mathrm{Homeo}(S)$ on $\mathcal{NC}^\dagger(S)$ is transitive. Let $o \in \mathcal{NC}^\dagger(S)$ be any base point and $\xi \in \partial\mathcal{NC}^\dagger(S)$. If α_n is a Cauchy-Gromov sequence converging to ξ, then by transitivity of the action, for each $n > 0$, there exists $\phi_n \in \mathrm{Homeo}(S)$ such that $\phi_n(o) = \alpha_n$. But this means that $\phi_n(o) \to \xi$. Hence we can conclude that $\Lambda\big(\mathrm{Homeo}(S)\big)$ is the entire boundary. □

Recall that an isometric group action on a δ-hyperbolic space X is of *general type* if there exists two independent hyperbolic isometries, or equivalently if the action admits no fixed point on ∂X. For isometric actions of general type on δ-hyperbolic spaces, there is an interesting dynamical property:

Proposition 7.5.3 (Corollary 7.4.3 (ii), [DSU17])**.** *Let X be a δ-hyperbolic space and let G act on X by isometries. Suppose in addition that the action is of general type. Then the limit set $\Lambda(G)$ is the smallest non-empty closed G-invariant subset of ∂X.*

Recall that a topological space X on which G acts is said *G-minimal* if for every point $x \in X$, the closure of the orbit $\overline{Gx}$ is X. Such an action will also be called a *minimal action*.

Proposition 7.5.4. *Let S be an orientable surface as above. Then the action of* $\mathrm{Homeo}(S)$ *on $\partial\mathcal{C}^\dagger(S)$ is minimal.*

Proof. In view of Proposition 7.5.3 and Lemma 7.5.2, it suffices to show that the action of $\mathrm{Homeo}(S)$ on $\mathcal{C}^\dagger(S)$ is of general type. To prove so, it suffices to fix a hyperbolic metric on S and take two independent pseudo-Anosov isometric homeomorphisms on S, which will yield two independent hyperbolic isometries on $\mathcal{C}^\dagger(S)$ as their action of a subgraph of $\mathcal{C}^\dagger(S)$ isometric to $\mathcal{C}(S)$ is of general type. □

To conclude this section, we wish to point out in the interests of curiosity that if S is a compact surface, then $\mathrm{Homeo}(S)$ is a Polish group, *i.e.* a completely metrizable and separable topological group, and its continuous minimal actions on compacta are of special interests. For example, [GTZ21] shows that its *universal minimal flow* (see Definition 3.1.35) is not metrisable.

Part III

Neither Fish, Flesh, nor Fowl

Chapter 8

Ultralimit and Quasi-isometries

夫雞肋，棄之可惜，食之無所得。[1]

楊脩 (175–219), *cf.*《九州春秋》.

In the classical case (X geodesic and proper), we can alternatively define Gromov boundary as equivalence classes of geodesic rays (see Remark 2.2.13). In non proper settings, joining a point in $x \in X$ to a boundary point $\xi \in \partial X$ will be rather problematic, because Arzelà-Ascoli theorem is no longer available. One possibility is to adopt the notions of *roads* and *biroads*, which are substitutes of geodesic rays and geodesic lines in an length hyperbolic space [Väi05]. Here we present another way to overcome this difficulty.

8.1 Ultralimit

Let (X, d) be a metric space. Let $\mathrm{Seq}_b(X)$ be the collection of all bounded sequences in X. We assume that the sequences are indexed by $\mathbb{N}$. Recall that a *non-principal ultrafilter* ω on $\mathbb{N}$ can be regarded as a finitely additive probability measure such that all subsets in $\mathbb{N}$ are measurable (or a *mean* on $\mathbb{N}$) with $\omega(S) \in \{0, 1\}$ and $\omega(S) = 0$ if S is a finite subset of $\mathbb{N}$.

By Zorn's lemma, there is always a non-principal ultrafilter on $\mathbb{N}$. Moreover, given a bounded sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, we can deduce that there exists a unique $l \in \mathbb{R}$ such that for all $\varepsilon > 0$, the measure $\omega(\{n \in \mathbb{N} : |a_n - l| < \varepsilon\}) = 1$. We denote $l = \omega\text{-}\lim_n a_n$.

Given two bounded sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ of points in X, we define

$$d_\omega\big((x_i), (y_i)\big) = \omega\text{-}\lim_i d(x_i, y_i) \in [0, \infty)$$

[1] Translation from classical Chinese: *This is like chicken ribs, discarding them would be a pity, but there is not much to gain by eating them.*

on $\mathrm{Seq}_b(X)$. We remark that as both sequences are bounded, the limit is actually taken inside of a compact set and thus it always exists. We say that $(x_i) \sim_\omega (y_i)$ if and only if $d_\omega\big((x_i),(y_i)\big) = 0$. So $\sim_\omega$ becomes an equivalent relation on $\mathrm{Seq}_b(X)$.

Definition 8.1.1 (*ω-ultralimit*)**.** Let X be a metric space. We denote by X^ω the metric space $\big(\mathrm{Seq}_b(X)/\sim_\omega, d_\omega\big)$ and we call it the *ω-ultralimit* of X.

Remark 8.1.2. There are some non-equivalent ways to define the ultralimit of a metric space. Some authors may not require the sequences in the construction to be bounded, but at the end the ultralimit will not be a metric space but a space with pseudo-metric. See [Kap00, §9.2] for instance.

A *path* in a space X is a topological embedding of the unit real interval $[0,1] \hookrightarrow X$. We write $\gamma\colon x \rightsquigarrow y$ if γ is a path with endpoints x and y. The *length of the path* γ is denoted $\ell(\gamma)$ defined by

$$\ell(\gamma) := \sup\left\{\sum_{i=0}^{n-1} d_X\big(\gamma(t_i),\gamma(t_{i+1})\big) : 0 \le t_0 < t_1 < \cdots < t_n = 1\right\}.$$

Recall that a *length metric space* (X, d_X) is a metric space satisfying the condition:

$$d_X(x,y) = \inf_{\gamma\colon x\rightsquigarrow y} \ell(\gamma).$$

Sometimes, a length metric space is also called an *intrinsic metric space*.

Proposition 8.1.3. *If X is a length space, then for any non-principal ultrafilter ω on $\mathbb{N}$, X^ω is geodesic.*

Proof. Let (x_i) and (y_i) be representatives of two distinct points in X^ω and let $L = d_\omega\big((x_i),(y_i)\big)$. It suffices to find a path $\gamma : [0,L] \to X^\omega$ such that for every $0 \le s \le L$, $d_\omega\big(\gamma(s),(x_i)\big) = s$ and $d_\omega\big(\gamma(s),(y_i)\big) = L - s$. Note that this path will be automatically a geodesic path.

For each $i \in \mathbb{N}$, define $\gamma_i : [0,\ell(\gamma_i)] \to X$ to be a rectifiable path between x_i and y_i, parameterised by length, such that the total length $\ell(\gamma_i)$ is less than $d(x_i,y_i) + 2^{-i}$. Reparameterise γ_i linearly into a path $\gamma_i : [0,L] \to X$. Let $\gamma(s)$ be the equivalent class of the sequence $\big(\gamma_i(s)\big)$ in X^ω. Since $\omega\text{-}\lim_i d(x_i,y_i) = L$ and ω is non-principal, we have

$$d_\omega\big(\gamma(s),(x_i)\big) = \omega\text{-}\lim_i \frac{s}{L}\left(d(x_i,y_i) + \frac{1}{2^i}\right) = s$$

and

$$d_\omega\big(\gamma(s),(y_i)\big) = \omega\text{-}\lim_i \frac{L-s}{L}\left(d(x_i,y_i) + \frac{1}{2^i}\right) = L - s.$$

This proves the desired result. □

Since each geodesic metric space is *a priori* a length metric space, we can conclude the following result:

Corollary 8.1.4. *If X is a geodesic space, then for any non-principal ultrafilter ω on $\mathbb{N}$, X^ω is a geodesic metric space.*

We can define the following isometric embedding $\iota_\omega : X \hookrightarrow X^\omega$ by sending each $x \in X$ to the equivalent class of the constant sequence of x. In particular, we have $\mathrm{Isom}(X) < \mathrm{Isom}(X^\omega)$ in the most natural way by $g(x_n)_{n\in\mathbb{N}} = (gx_n)_{n\in\mathbb{N}}$ for $g \in \mathrm{Isom}(X)$.

If (X, d_X) is a proper metric space, then ι_ω is an isometry between X^ω and X. For any bounded sequence (x_n) in a proper metric space X, it admits several accumulation points in X and in particular, its limit along a fixed non-principal ultrafilter is one of these accumulation points, denoted by $x \in X$. Then the bounded sequence (x_n) is equivalent to the constant sequence of x in X^ω.

But generally, the ultralimit of a non-proper metric space is not isometric to itself and the isometric embedding is not essentially surjective. Here are some examples:

Example 8.1.5 (Countable discrete space)**.** Let the natural numbers $(\mathbb{N}, \delta_\mathbb{N})$ be equipped with the discrete metric. For any non-principal ultrafilter ω on $\mathbb{N}$, we have $\mathbb{N}^\omega = (\mathbb{R}, \delta_\mathbb{R})$, where $\delta_\mathbb{R}$ is the discrete metric on $\mathbb{R}$. Indeed, on one hand we see that $\mathbb{N}^\omega = \mathbb{N}^\mathbb{N}/\sim_\omega$ and as a result $|\mathbb{N}^\omega| \leq |\mathbb{R}|$; on the other hand, the map $\phi : (1, \infty) \to \mathbb{N}^\omega$ by $s \mapsto (\lfloor s^n \rfloor)_{n\in\mathbb{N}}$ is injective, which yields $|\mathbb{R}| \leq |\mathbb{N}^\omega|$ after Cantor-Bernstein theorem. Moreover, since the distance in $(\mathbb{N}, \delta_\mathbb{N})$ is either 0 or 1, so will be its limit taken along the ultrafilter ω, which makes $(\delta_\mathbb{N})_\omega$ a discrete metric.

Example 8.1.6 ($\mathbb{R}$-tree)**.** Recall that the *SNCF metric*, or the *French railway metric*, d_S on $\mathbb{R}^2$ is given by

$$d_S(p, q) = \begin{cases} |p - q|, & \text{if } p, q \text{ are linearly dependent} \\ |p| + |q|, & \text{if else} \end{cases},$$

where $|\cdot|$ is the ambient Euclidean norm in $\mathbb{R}^2$. It is an $\mathbb{R}$-tree. Let X be obtained by gluing together the ends of countably infinite geodesic rays, *i.e.* a tree with one root vertex o and countably many branches of infinite length. Then for any non-principal ultrafilter ω on $\mathbb{N}$, the ultralimit X^ω is $(\mathbb{R}^2, d_S)$, *i.e.* a tree with one root vertex $\iota_\omega(o)$ and $|\mathbb{R}|$ many infinitely long branches. To see this, it suffices to consider the points at distance 1 to o and $\iota_\omega(o)$ and uses the result from Example 8.1.5.

Example 8.1.7 (Real Hilbert space)**.** Recall that the dimension of a real Hilbert space is the cardinal of a maximal orthonormal set in it. Let $\mathcal{H}$ be a real Hilbert space of countable

dimension and ω be a non-principal ultrafilter on $\mathbb{N}$. It is not hard to see that the ultralimit is always complete (see [BH13, §I.5, Lemma 5.53]). The ω-ultralimit of a Hilbert space remains a Hilbert space by defining the inner product $\langle u|v\rangle = \omega\text{-}\lim_n \langle u^n|v^n\rangle$, $u+v = (u^n+v^n)_{n\in\mathbb{N}}$ and $\lambda u = (\lambda u^n)_{n\in\mathbb{N}}$, for any $\lambda \in \mathbb{R}$ and every $u, v \in \mathcal{H}^\omega$ and their respective representatives $(u^n)_{n\in\mathbb{N}}, (v^n)_{n\in\mathbb{N}} \in \mathrm{Seq}_{\mathrm{b}}(\mathcal{H})$.

It is worth noticing that given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$, its ultralimit along ω $\left((e_n)_{n\in\mathbb{N}}\right)^\omega \subset \mathcal{H}^\omega$ is an orthonormal set but it is not maximal. Let $u \in \mathcal{H}^\omega$ be a unitary vector and $(u^n)_{n\in\mathbb{N}}$ be its representative with $u^n = \sum_{k\in\mathbb{N}} x_k^n e_k$. It is not hard to see that u is orthogonal to $(\epsilon_i)_{i\in I}$ if and only if $\omega\text{-}\lim_n \max_k |x_k^n| = 0$.

If we denote by $(\epsilon_i)_{i\in I}$ the ultralimit of $(e_n)_{n\in\mathbb{N}}$ and an orthonormal basis $(\epsilon_j)_{j\in J}$ containing the ultralimit with $I \subset J$, then $|J \setminus I| \geq \aleph_0$. Indeed, for each positive integer $n \geq 1$, we define $u^{(n)} \in \mathcal{H}^\omega$ to be the equivalent class of the sequence $\left(u_m^{(n)}\right)_{m\in\mathbb{N}}$, where

$$u_m^{(n)} = \sum_{k=(n-1)m}^{nm-1} \frac{1}{\sqrt{m}} e_k \in \mathcal{H}.$$

Then $\left\{u^{(n)} \in \mathcal{H}^\omega : n \geq 1\right\}$ is a countable orthonormal family that is also orthogonal to $(\epsilon_i)_{i\in I}$.

Nevertheless, we can still compute explicitly the dimension of $\mathcal{H}^\omega$. First, we should notice that $|\mathcal{H}| \leq \left|\mathbb{R}^{\mathbb{N}}\right| = |\mathbb{R}|$, while $|\mathcal{H}| \geq |\mathbb{R}|$. Thus $|\mathcal{H}| = |\mathbb{R}|$. As a result, we have $|\mathcal{H}^\omega| \leq |\mathcal{H}^{\mathbb{N}}| = |\mathbb{R}| \times |\mathbb{N}| = |\mathbb{R}|$. This yields that $\dim \mathcal{H}^\omega \leq |\mathbb{R}|$. Conversely, $\dim \mathcal{H}^\omega = |J| \geq |I| = |\mathbb{R}|$. Hence $\dim \mathcal{H}^\omega = |\mathbb{R}|$.

Therefore, given a real Hilbert space $\mathcal{H}$ of countably infinite dimension, we can conclude that $\mathcal{H}^\omega$ is a real Hilbert space of dimension $|\mathbb{R}|$.

Example 8.1.8 (Real hyperbolic space)**.** Recall that the algebraic hyperbolic space $\mathbb{H}(\mathcal{H})$ over a field $\mathbf{K}$ is constructed from a Hilbert space $\mathcal{H}$ over $\mathbf{K}$ by projectivising the Minkowski space $\mathcal{L} \subset \mathbf{K} \oplus \mathcal{H}$ equipped with a Lorentzian quadratic form. It is clear from the construction that $\mathbb{H}(\mathcal{H})^\omega = \mathbb{H}(\mathcal{H}^\omega)$ for any non-principal ultrafilter ω on $\mathbb{N}$. In particular, the ultralimit of a real hyperbolic space of countably infinite dimension is a real hyperbolic space of dimension $|\mathbb{R}|$.

We have seen in Example 8.1.8 that the ultralimit of a real hyperbolic space remains a real hyperbolic space but of higher dimension. Both of these two spaces are Gromov hyperbolic. More generally, δ-hyperbolicity is invariant under taking the ultralimit.

Lemma 8.1.9. *If X is a δ-hyperbolic space, then for any non-principal ultrafilter ω on $\mathbb{N}$, X^ω is also a δ-hyperbolic space.*

Proof. Let (x_i), (y_i), (z_i) and (p_i) be representatives of any four points in X^ω. By the δ-hyperbolicity of X, we have $\langle x_i, y_i\rangle_{p_i} \geq \min\left(\langle x_i, z_i\rangle_{p_i}, \langle z_i, y_i\rangle_{p_i}\right) - \delta$, and by passing to the limit along the ultrafilter ω, it soon yields that X^ω is also δ-hyperbolic. □

Finally, we need to point out that ι_ω can be extended to the Gromov boundaries in a natural way, *i.e.* one sets $\iota_\omega(\xi) \in \partial(X^\omega)$ to be the limit of $\iota_\omega(x_n)$ for any Cauchy-Gromov sequence (x_n) converging to ξ. Moreover, it is not hard to see that the embedding $\iota_\omega \colon \partial X \hookrightarrow \partial(X^\omega)$ is continuous.

8.2 Ultracompleteness

Recall that a Gromov hyperbolic space X is *ultracomplete* if any two points in its Gromov bordification can be connected by a geodesic. In this section, we will give a proof of the ultracompleteness of the ultralimit X^ω of a hyperbolic space X. The fact that $\partial X \subset \partial X^\omega$ enjoys visibility property is used in [Cap+15, Proof of Proposition 3.1] and such a geodesic line can be easily constructed by taking ω-limit of geodesic segments, but as $\partial X \neq \partial X^\omega$ in general (see examples in §8.1), this does not imply ultracompleteness of X^ω.

In what follows, we will refer to $O(\delta)$ an additive error at most a multiple of δ, *viz.* if we write $f(x) = g(x) + O(\delta)$, then it means $|f(x) - g(x)| < M\delta$ for some uniform $M > 0$. Errors $O(\delta)$ appear in different places can be different.

For a given geodesic γ in X, we define for a point $x \in X$ its *nearest points projection* $\pi_\gamma(x)$ by the points $y \in \gamma$ satisfying $d_X(x, y) = d_X(x, \gamma)$. Since a geodesic is a locally compact space, such y always exists for any $x \in X$. Nearest point projection appears to be a powerful tool dealing with geodesic Gromov hyperbolic spaces.

The first result is referred as the *reverse triangle inequality*:

Proposition 8.2.1 ([MT18], Proposition 2.2)**.** *Let X be δ-hyperbolic and γ be a geodesic. For any $x \in X$ and any $p \in \pi_\gamma(x)$, we have*

$$d(x, y) = d(x, p) + d(p, y) + O(\delta) \tag{8.1}$$

for all $y \in \gamma$. Moreover, any geodesic connecting y to x must pass within distance $M(1, 2\delta)$ of p.

Let us observe that for any $p, q \in \pi_\gamma(x)$, Proposition 8.2.1 implies $d(p, q) \leq O(\delta)$. This shows the fact that in a δ-hyperbolic space X the nearest points projection $\pi_\gamma(x)$ onto the geodesic γ is coarsely well-defined.

The reverse triangle inequality can be further used to deduce the following estimation.

Proposition 8.2.2 ([Mah10], Proposition 3.4)**.** *Let X be δ-hyperbolic and γ be a geodesic. Let x and y be two points in X. Suppose that* $\operatorname{diam}(\pi_\gamma(x) \cup \pi_\gamma(y))$ *is large enough, then for $p \in \pi_\gamma(x)$ and $q \in \pi_\gamma(y)$, we have $d(x,y) = d(x,p) + d(p,q) + d(q,y) + O(\delta)$.*

Now we can prove the *quasi-ultracompleteness* of geodesic Gromov hyperbolic spaces:

Proposition 8.2.3 (*quasi-ultracompleteness*)**.** *Let (X, d_X) be a δ-hyperbolic geodesic space. Then there exists a $K > 0$, depending only on δ, such that any two points in the Gromov bordification of X can be connected by a continuous $(1,K)$-quasi-geodesic.*

Proof. Let $x \in X$ and $\xi \in \partial X$. Suppose that (x_n) converges to ξ. By passing to a subsequence, we may assume that $d_X(x_n, x) > 100n\delta$ and $d_X(x_n, x_{n+1}) > 100\delta$ for every $n \geq 1$. Let γ_n be the geodesic segment $[x, x_n]$. Using δ-slimness and *thinness of geodesic triangles* [BH13, §III.H, Definition 1.16 & Proposition 1.17], for each $k \geq 1$, we can find a n_k such that the length $\ell(\gamma_{n_k} \cap \mathcal{N}_\delta(\gamma_m)) > k$ for all $m \geq n_k$. Take $y_k \in \gamma_{n_k} \cap \mathcal{N}_\delta(\gamma_m)$ such that $d_X(x, y_k) = k$. Now, for any $i, j \geq 1$, pick $m > \max(n_i, n_j)$, then project y_i, y_j respectively to $p_i, p_j \in \gamma_m$. Note that $d_X(y_i, p_i) < \delta$ and $d_X(y_j, p_j) < \delta$. Applying Proposition 8.2.2, we have $d_X(y_i, y_j) = |i - j| + O(\delta)$. This shows that the continuous path

$$\alpha := [x, y_1] \cup [y_1, y_2] \cup [y_2, y_3] \dots$$

is a $(1,K)$-quasi-geodesic for some $K > 0$ depending only on δ. It is not hard to check that the sequence (y_k) is equivalent to (x_n), hence α connects x to ξ. Similar arguments also imply the existence of a quasi-geodesic connecting two boundary points. □

Now we are ready to show the following theorem, of which the proof seems a little technical:

Theorem 8.2.4 (ultracompleteness of ultralimit)**.** *Let (X, d_X) be a Gromov hyperbolic geodesic space and ω be a non-principal ultrafilter over $\mathbb{N}$. Then the ultralimit X^ω is ultracomplete.*

Proof. We shall show that for any pair of distinct points $\xi_\pm \in \partial(X^\omega)$, there exists a geodesic connecting ξ_- to ξ_+. Let $o \in X$ and $\bar{o} = \iota_\omega(o) \in X^\omega$. By Proposition 8.2.3, we can find a continuous $(1,K)$-quasi-geodesic line γ in X^ω that connects ξ_- to ξ_+. Take $x^n = \gamma(t_n)$ and $y^n = \gamma(-t_n)$ for an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers with $t_n \to \infty$. Due to (2.8), by passing to subsequences, we may assume that $\langle x^n, y^n\rangle_{\bar{o}} \leq \langle \xi_+, \xi_-\rangle_{\bar{o}} + 3\delta \leq M$ for all $n \in \mathbb{N}$ and for some positive $M < \infty$. Let $(x_j^n)_{j\in\mathbb{N}}$ and $(y_j^n)_{j\in\mathbb{N}}$ be respectively representatives of x^n and y^n. Since

$$\omega\left(\left\{j \in \mathbb{N} : \left|\langle x_j^n, y_j^n\rangle_o - \langle x^n, y^n\rangle_{\bar{o}}\right| < 1\right\}\right) = 1,$$

we may choose for each x^n and y^n a representative that $\langle x_j^n, y_j^n \rangle_o \leq M+1$ for all $n, j \in \mathbb{N}$. In the sequel, whenever we write x_j^i or y_j^i for any $i, j \in \mathbb{N}$, it means the j-th component of the representative chosen above for x^i or y^i.

Note that the ω-limit of a sequence of geodesics connecting the components in the representatives of two points in X^ω will yield a geodesic between these two points (cf. Proposition 8.1.3). To avoid any ambiguity, we fix once and for all $[x^n, y^n]$ a geodesic segment in X^ω between x^n and y^n that is obtained by taking the ω-limit of a sequence of geodesic segments $[x_j^n, y_j^n]$ in X. For all $m \leq n$, we denote $\pi_n(x^m)$ and $\pi_n(y^m)$ respectively a nearest point projection of x^m and y^m on $[x^n, y^n]$. Let $\big(p_j^m(n)\big)_{j\in\mathbb{N}}$ and $\big(q_j^m(n)\big)_{j\in\mathbb{N}}$ be representatives of $\pi_n(x^m)$ and $\pi_n(y^m)$.

Now let us define for every $n \geq 1$

$$A_n = \left\{ j \in \mathbb{N} : \left| d\big(x_j^m, p_j^m(n)\big) - d_\omega\big(x^m, \pi_n(x^m)\big) \right| < \frac{1}{n}, \text{ for all } m \leq n \right\},$$

and

$$B_n = \left\{ j \in \mathbb{N} : \left| d\big(y_j^m, q_j^m(n)\big) - d_\omega\big(y^m, \pi_n(y^m)\big) \right| < \frac{1}{n}, \text{ for all } m \leq n \right\}.$$

We further define $J_0 = \mathbb{N}$, $J_1 = A_1 \cap B_1 \cap [1, \infty)$, and inductively

$$J_n = A_n \cap B_n \cap J_{n-1} \cap [\inf(J_{n-1}) + 1, \infty),$$

for all $n > 1$. Since $\omega(A_n) = \omega(B_n) = 1$, we can also deduce that $\omega(J_n) = 1$ for all $n \geq 0$. Let $j_n = \min(J_n \setminus J_{n+1})$. Then the sequence $(j_n)_{n\in\mathbb{N}}$ will be a strictly increasing sequence of integers. For any $k \in \mathbb{N}$, we also define a function $\phi(k) = n$ whenever $j_n \leq k < j_{n+1}$. The function ϕ is non-decreasing and $\phi(k) \leq k$.

Now we wish to construct a sequence of mappings $\ell_k : \mathbb{R} \to X$ so that the ω-limit of this sequence will yield a geodesic line ℓ in X^ω that connects ξ^- to ξ_+. We insist that ℓ_k is not necessarily continuous. Not distinguishing the mapping ℓ_k from its image, we may first assume that the geodesic segment $\left[x_k^{\phi(k)}, y_k^{\phi(k)}\right]$ is contained in ℓ_k and is parameterised by length with $\ell_k(0)$ being a nearest point projection of o to $\left[x_k^{\phi(k)}, y_k^{\phi(k)}\right]$. We extend ℓ_k by the k-th components of a given representative of the continuous quasi-geodesic γ with the segment between $x^{\phi(k)}$ and $y^{\phi(k)}$ removed so that the parameter of ℓ_k becomes a well-defined map $\mathbb{R} \to X$.

The first thing we need to verify is that for any $t \in \mathbb{R}$, the sequence $\big(\ell_k(t)\big)_{k\in\mathbb{N}}$ is bounded. Indeed, for large enough k, the number $\phi(k)$ will also be large enough so that $\min\{d(x_k^{\phi(k)}, o), d(y_k^{\phi(k)}, o)\} \gg |t|$, which implies that $\ell_k(t) \in \left[x_k^{\phi(k)}, y_k^{\phi(k)}\right]$. Recall that $\ell_k(0)$

is a nearest point projection of o on $\left[x_k^{\phi(k)}, y_k^{\phi(k)}\right]$. Then by Proposition 8.2.2, we get

$$\begin{aligned} d(o, \ell_k(t)) &= d\left(o, \left[x_k^{\phi(k)}, y_k^{\phi(k)}\right]\right) + d\big(\ell_k(0), \ell_k(t)\big) + O(\delta) \\ &\leq \langle x_k^{\phi(k)}, y_k^{\phi(k)} \rangle_o + |t| + O(\delta) \\ &\leq M + 1 + |t| + O(\delta) < \infty \end{aligned}$$

for large enough k. Hence the ω-limit of ℓ_k defines a map $\ell : \mathbb{R} \to X^\omega$.

Secondly, we need to show that ℓ is a geodesic line. For any $s, t \in \mathbb{R}$, we may take large enough k so that $\ell_k(t), \ell_k(s) \in \left[x_k^{\phi(k)}, y_k^{\phi(k)}\right]$. As a result,

$$d_\omega\big(\ell(t), \ell(s)\big) = \omega\text{-}\lim_k d\big(\ell_k(t), \ell_k(s)\big) = \omega\text{-}\lim_{k \gg 1} |t - s| = |t - s|.$$

This implies that ℓ is a geodesic line.

Finally, we have to verify whether ℓ connects ξ_- to ξ_+. To prove this, it suffices to show that x^m and y^m are at a uniform bounded distance to ℓ for all m. Let us take any $m \in \mathbb{N}$. Let $\ell_k(t_k) = p_k^m\big(\phi(k)\big)$ be the nearest point projection of x_k^m to $\left[x_k^{\phi(k)}, y_k^{\phi(k)}\right]$ as above. Since the distance between x_k^m and o is bounded, by Proposition 8.2.2, the sequence $(t_k)_{k \in \mathbb{N}}$ is also bounded. Let $T = \omega\text{-}\lim_k t_k$. By picking large enough k, we can assume that $\phi(k) \geq m$. Hence we have

$$\begin{aligned} d_\omega(\ell, x^m) &\leq d_\omega(\ell(T), x^m) \\ &= \omega\text{-}\lim_{k \gg 1} d(\ell_k(T), x_k^m) \\ &\leq \omega\text{-}\lim_{k \gg 1} \Big(d(\ell_k(t_k), x_k^m) + d\big(\ell_k(t_k), \ell_k(T)\big)\Big) \\ &= \omega\text{-}\lim_{k \gg 1} \Big(d(\ell_k(t_k), x_k^m) + |t_k - T|\Big) \\ &= \omega\text{-}\lim_{k \gg 1} d(\ell_k(t_k), x_k^m) \\ &= \omega\text{-}\lim_{k \gg 1} d\left(p_k^m(\phi(k)), x_k^m\right). \end{aligned}$$

Now we need the following set-theoretic remark. Let $S = \{k \in \mathbb{N} : k \in J_{\phi(k)}\}$. We claim that $S = J_1$. On one hand, for any $k \in S$, there exists $n = \phi(k)$ such that $k \in J_k \subset J_1$, which implies that $S \subset J_1$. On the other hand, because $J_1 = \bigcup_{n \geq 1} J_n \setminus J_{n+1}$, for any $k \in J_1$, there exists some $n \geq 1$ such that $k \in J_n \setminus J_{n+1}$, which means $\phi(k) = n$ and $k \in J_{\phi(k)}$, *i.e.* $k \in S$. So we prove the claim.

Since we are taking ω-limit and $\omega(S) = \omega(J_1) = 1$, it is possible to assume that $k \in$

$J_{\phi(k)} \subset A_{\phi(k)}$, so by Morse Lemma,

$$d\left(p_k^m(\phi(k)), x_k^m\right) \leq d_\omega\left(x^m, \pi_{\phi(k)}(x^m)\right) + \frac{1}{\phi(k)} \leq M(1,K) + \frac{1}{\phi(k)},$$

where $M(\cdot,\cdot)$ is the Morse function. By taking the ω-limit, we have

$$d_\omega(\ell, x^m) \leq M(1,K),$$

for all $m \in \mathbb{N}$. This shows that ℓ is at a finite Hausdorff distance from the sequence $(x^m)_{m\in\mathbb{N}}$. The same proof also applies for the sequence $(y^m)_{m\in\mathbb{N}}$. Hence ℓ is a geodesic line in X^ω that connects ξ_- to ξ_+. □

Proposition 8.2.5 (Extended Morse Lemma)**.** *Let (X, d_X) be a Gromov hyperbolic geodesic space and $\overline{X}$ be its Gromov bordification. Then any pair of quasi-geodesics connecting two points $x, y \in \overline{X}$ must be of finite Hausdorff distance.*

Proof. Take a ultralimit X^ω. Using the arguments from the proof of Theorem 8.2.4, for each quasi-geodesic, we can find a sequence of geodesic segments converging to a geodesic that share the same endpoints as the quasi-geodesic. We can conclude the result by applying Morse Lemma to the quasi-geodesics and these geodesic segments. □

Remark 8.2.6. The arguments for converging geodesics is a substitute for Arzelà-Ascoli Theorem

8.3 Ultracomplete supspace

In this section, we will discuss the induced quasi-isometry between ultralimits from quasi-isometry between length Gromov hyperbolic spaces.

Theorem 8.3.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let ω be a non-principal ultrafilter on $\mathbb{N}$. Suppose that $f\colon X \to Y$ is a (λ, K)-quasi-isometry. Then there exists a (λ, K)-quasi-isometry $f^\omega\colon (X^\omega, d_\omega^X) \to (Y^\omega, d_\omega^Y)$ such that $f^\omega|_{\iota_\omega(X)} = \iota_\omega \circ f$, where $\iota_\omega\colon X \hookrightarrow X^\omega$ is the canonical isometric embedding.*

Proof. Let $x \in X^\omega$ and $\psi_X(x) \subset \mathrm{Seq}_b(X)$ be the collection of all sequences in X that converges to x. Similarly, we define $\psi_Y(y)$ for $y \in Y^\omega$. Define for each $x \in X^\omega$

$$\mathcal{Y}_x := \left\{y \in Y^\omega : \exists (x_n) \in \psi_X(x) \text{ such that } \left(f(x_n)\right) \in \psi_Y(y)\right\}.$$

Note that for any $(x_n) \in \psi_X(x)$, the sequence $\left(f(x_n)\right)$ is bounded by the virtue of quasi-isometry, thus $\mathcal{Y}_x$ is not empty. We claim that $\mathcal{Y}_x$ is uniformly bounded for every $x \in X^\omega$.

Indeed, for any two sequences $(x_n), (x'_n) \in \psi_X(x)$, by the definition of quasi-isometry

$$d_Y\big(f(x_n), f(x'_n)\big) \leq \lambda d_X(x_n, x'_n) + K \underset{\omega}{\to} K,$$

as a result $d^Y_\omega(y, y') \leq K$ for any $y, y' \in \mathcal{Y}_x \subset Y^\omega$. Now we define $f^\omega|_{\iota_\omega(X)} := \iota_\omega \circ f$ and for every $x \in X^\omega \setminus \iota_\omega(X)$, we set $f^\omega(x)$ to be any point in $\mathcal{Y}_x$. Since inequality is stable under taking limits, the map $f^\omega \colon X^\omega \to Y^\omega$ is also a (λ, K)-quasi-isometric embedding. We claim that f^ω is essentially surjective. Indeed, let $y \in Y^\omega$ be an arbitrary point. Because $f \colon X \to Y$ is essentially surjective, there exists a constant $C > 0$ such that $\mathcal{N}_C\big(f(X)\big) = Y$. By consequences, for any sequence $(y_n) \in \psi_Y(y)$, we can find a sequence (x_n) in X such that $d_Y\big(f(x_n), y_n\big) < C + 1$ for any n. As a result, by quasi-isometry, for any $n, m > 0$,

$$\begin{aligned}
\frac{1}{\lambda} d_X(x_n, x_m) - K &\leq d_Y\big(f(x_n), f(x_m)\big) \\
&\leq d_Y\big(f(x_n), y_n\big) + d_Y\big(f(x_m), y_m\big) + d_Y(y_n, y_m) \\
&< 2C + 2 + d_Y(y_n, y_m),
\end{aligned}$$

which implies that $(x_n) \in \mathrm{Seq}_b(X)$. Let $x \in X^\omega$ be the associated point to the above sequence $(x_n) \in \mathrm{Seq}_b(X)$. Then $f^\omega(x) \in \mathcal{Y}_x$ by definition. Let $(x'_n) \in \psi_X(x)$ be such a sequence that $\big(f(x_n)\big) \in \psi_Y\big(f^\omega(x)\big)$. Note that (x_n) and (x'_n) induce the same point in X^ω. Now we compute

$$\begin{aligned}
d^Y_\omega\big(f^\omega(x), y\big) &= \omega\text{-}\lim_n d_Y\big(f(x'_n), y_n\big) \\
&\leq \omega\text{-}\lim_n \big[d_Y\big(f(x'_n), f(x_n)\big) + d_Y\big(f(x_n), y_n\big)\big] \\
&\leq \omega\text{-}\lim_n \big[\lambda d_X(x'_n, x_n) + K + d_Y\big(f(x_n), y_n\big)\big] \\
&\leq K + C + 1 < \infty.
\end{aligned}$$

In conclusion, we just show that $\inf_{x \in X^\omega} d^Y_\omega\big(y, f^\omega(x)\big) \leq K + C + 1$ for all $y \in Y^\omega$, *i.e.* the map $f^\omega \colon X^\omega \to Y^\omega$ is essentially surjective, thus a quasi-isometry. □

Now we define the following notion:

Definition 8.3.2 (*ultracomplete supspace*)**.** Let X be a Gromov hyperbolic space. A *ultracomplete supspace* $u(X)$ of X is a ultracomplete space with an isometric embedding $\iota \colon X \hookrightarrow u(X)$.

It turns out that ultracomplete supspace is guaranteed for any Gromov hyperbolic metric space, even not necessarily geodesic. The following fact can be viewed as a partial affirmation to Conjecture 2.2.26:

Proposition 8.3.3. *For any Gromov hyperbolic space X, there exists a ultracomplete supspace $u(X)$.*

Proof. Let $\mathrm{Inj}(X)$ be the injective hull of X. It is a Gromov hyperbolic geodesic space with the same δ as X where X is isometrically embedded [Lan13]. Then by Theorem 8.2.4, the metric space $\mathrm{Inj}(X)^\omega$ is a desired ultracomplete supspace. □

In the follow, we will list some consequences of the existence of ultracomplete supspace.

Corollary 8.3.4. *Let (X, d_X) be a Gromov hyperbolic space and $\overline{X}$ be its Gromov bordification. Then any pair of quasi-geodesics connecting two points $x, y \in \overline{X}$ must be of finite Hausdorff distance. Also, all quasi-geodesic triangles are slim.*

Proof. Take a ultracomplete supspace of X and then apply extended Morse Lemma to $u(X)$. Note that these results pass to subspaces. □

Corollary 8.3.5. *Let (X, d_X) be a Gromov hyperbolic length space and let $o \in X$ be any base point. Then its Gromov boundary ∂X is the equivalent classes of quasi-geodesics ray issued from o up to bounded Hausdorff distance.*

Proof. With Corollary 8.3.4, we can adapt the proof of Proposition 8.2.3 to length space: instead of taking geodesics, one takes closed enough paths which are $(1, \varepsilon)$-quasi-geodesics for sufficiently small $\varepsilon > 0$. As a result, for any point on $\xi \in \partial X$, there exists a quasi-geodesic ray issued from o converging to ξ. By extended Morse Lemma, any of two quasi-geodesic rays issued from o and converges to a same point on the boundary must be of bounded Hausdorff distance. Conversely, by Proposition 2.2.42, any quasi-geodesic ray converges to a point on ∂X, *i.e.* all increasing sequences on a quasi-geodesic ray is Cauchy-Gromov and they are all equivalent. Hence we prove the corollary. □

Remark 8.3.6. The definition of Gromov boundary cannot be defined via geodesic rays when the Gromov hyperbolic space is not ultracomplete. But if the space is indeed ultracomplete, then similar arguments will show that the three models, defined respectively via geodesic rays, quasi-geodesic rays and Cauchy-Gromov sequences, are equivalent.

Example 8.3.7. When the space is far from being length, then Corollary 8.3.5 might fail. Let us consider the standard Cayley graph of free group $\mathbb{F}_2$ on two generators. The subspace $X := \{b^n a^n : n \geq 0\}$ has a unique Gromov boundary point but it contains no quasi-geodesic ray issued from $\mathrm{Id} \in \mathbb{F}_2$.

As reader may notice, some of the above results can also be deduced using injective hulls, in which the original Gromov hyperbolic space is isometrically embedded. But when we try to study the large-scale geometry of these spaces, injective hull will no longer be handy: in general, quasi-isometry between two spaces cannot be extended to a quasi-isometry between their injective hulls.

Example 8.3.8. Let $\Gamma := \{(x,y) \in \mathbb{R}^2 : y = |x|\}$ be a subspace of $\mathbb{R}^2$. It is quasi-isometric to $\mathbb{R}$. However, the injective hull of $\mathbb{R}$ is itself, while $\mathrm{Inj}(\Gamma)$ contains $\{(x,y) \in \mathbb{R}^2 : y \geq |x|\}$ and is not Gromov hyperbolic. As a result, the space $\mathbb{R}$ is not quasi-isometric to $\mathrm{Inj}(\Gamma)$ as geodesic spaces.

Let X and Y be two Gromov hyperbolic spaces. Let $u(X)$ and $u(Y)$ be ultracomplete supspaces respectively. Then $\big(u(X), u(Y)\big)$ is an *admissible pair of ultracomplete supspaces* if any quasi-isometric embedding $f\colon X \to Y$ can be extended to a quasi-isometric embedding $f_u\colon u(X) \to u(Y)$.

Recall that (X, d_X) is a *discretely geodesic metric space* if for any $x, y \in X$, $d_X(x,y) \in \mathbb{N}$ and there exists an isometric embedding $\gamma\colon \{0, 1, \ldots, d_X(x,y)\} \to X$ with $\gamma(0) = x$ and $\gamma\big(d_X(x,y)\big) = y$.

Proposition 8.3.9. *Let X and Y be two Gromov hyperbolic spaces. If X and Y are simultaneously discretely geodesic metric spaces or length spaces, then they have an admissible pair of ultracomplete supspaces $u(X)$ and $u(Y)$. Moreover, if X and Y are quasi-isometric, so will be $u(X)$ and $u(Y)$.*

Proof. Note that a discretely geodesic Gromov hyperbolic is quasi-isometric to its injective hull [Lan13, Proposition 1.3]. If X and Y are simultaneously discretely geodesic, then in light of Theorem 8.2.4, the ultracomplete spaces $\mathrm{Inj}(X)^\omega$ and $\mathrm{Inj}(Y)^\omega$ will be the desired admissible pair of ultracomplete supspaces. If X and Y are both length spaces, then by applying Theorem 8.2.4 twice, we can see that the pair $(X^\omega)^\omega$ and $(Y^\omega)^\omega$ is what we want. □

As a classical result deducing from slimness of (quasi)-geodesic triangles, a quasi-isometric embedding $f\colon X \to Y$ between Gromov hyperbolic geodesic spaces will induce a topological embedding $f_\partial\colon \partial X \to \partial Y$ [CDP90, §3.2, Théorème 2.2]. Using admissible pairs of ultracomplete supspaces will immediately yield the generalisations:

Proposition 8.3.10. *Let X and Y be two Gromov hyperbolic spaces. If X and Y have an admissible pair of ultracomplete supspaces, then any quasi-isometric embedding $f\colon X \to Y$ will yield a topological embedding $f_\partial\colon \partial X \to \partial Y$. If in addition, f is a quasi-isometry, then f_∂ is a homeomorphism.*

In particular, we have:

Corollary 8.3.11. *Let X and Y be two Gromov hyperbolic spaces. If X and Y are simultaneously discretely geodesic metric spaces or length spaces, then any quasi-isometric embedding $f\colon X \to Y$ will yield a topological embedding $f_\partial\colon \partial X \to \partial Y$. If in addition, f is a quasi-isometry, then f_∂ is a homeomorphism.*

Since even when the space is not Gromov hyperbolic, one can still construct its ultralimit and quasi-isometric embedding can be extended to the ultralimits, we can also deduce the following result as generalisation of Theorem 2.2.40:

Proposition 8.3.12. *Let X and Y be two length spaces. Suppose that Y is Gromov hyperbolic. If there exists a quasi-isometric embedding $f\colon X \to Y$, then X will also be Gromov hyperbolic.*

In principle, if a class of Gromov hyperbolic spaces have admissible ultracomplete supspaces, then we can always try to recover the classical geometric results for this class of spaces.

Let us finish this appendix with a remark on the group of isometries of a general Gromov hyperbolic space, of which the proof is just an adaption of proper geodesic settings to ultracomplete supspace. It is essentially the same as [Cap+15, Proposition 3.1] but under the most general settings:

Proposition 8.3.13. *Let (X, d_X) be a Gromov hyperbolic space. Then there is a trichotomy for an isometry $g \in \mathrm{Isom}(X)$. Namely, the element g is:*

- ***elliptic**, if and only if $\langle g \rangle$ has bounded orbits;*
- ***parabolic**, if and only if $d_X(x, g^n x)/n \to 0$ as $n \to \infty$;*
- ***hyperbolic**, if and only $d(x, g^n x)/n \to c > 0$ as $n \to \infty$.*

For any group G acts on X by isometries, there is a full classification for its action. Namely, the action is

- ***elementary** and*
  - ***bounded** if it has bounded orbits;*
  - ***horocyclic** if it is unbounded and has no hyperbolic elements;*
  - ***lineal** if it has hyperbolic elements but any two hyperbolic elements have the same endpoints;*
- ***non-elementary** and*
  - ***focal** if it has hyperbolic elements, is not lineal, and any two of its hyperbolic elements have one common endpoint;*
  - ***general type** if it has hyperbolic elements with no common endpoint.*

Remark 8.3.14. Sometimes in the literature, by *non-elementary action*, people might only talk about the action of general type. This is different from the original nomenclature in [Gro87, §3.1] .

Remark 8.3.15. Let $Y \subset X$ be Gromov hyperbolic space. Suppose that G acts on Y by isometries and each element $g \in G$ can be extended to an isometry on X. Then it is not hard to see that the isometry type of g remains unchanged after extension, *i.e.* if g acts elliptically (*resp.* parabolically or hyperbolically) on Y, then its extension will be an elliptic (*resp.* parabolic or hyperbolic) isometry on X.

Chapter 9

Hyperbolic Rigidity of Amenable Groups

天下國家可均也.....中庸不可能也。[1]

孔子（circa 551–479 BCE），*cf.*《中庸》。

9.1 Introduction

This chapter mainly concerns the following folklore result: *the continuous action by isometries of an amenable group on a Gromov hyperbolic space can never be of general type*. Some special cases of this result are long known to people.

Let G be an amenable group acting continuously by isometries on a Gromov hyperbolic space (X, d). Then this action can be extended continuously to the Gromov boundary ∂X. If X is proper, then the arguments of Lemma 9.3.2 will show that this action cannot be of general type. When the space (X, d) is not proper, this result also holds for locally compact amenable groups, deducing from the existence of a Schottky subgroup, which will witness the non-amenability of locally compact groups [Cap+15].

But in order to study the amenability of big mapping class groups, which are never locally compact (*cf.* Theorem 4.2.5), we will need the generalisation of the above result:

Theorem 9.1.1. *Let X be a separable geodesic δ-hyperbolic space. Let G be an amenable topological group acting continuously on X by isometries. Then the action cannot be of general type.*

[1] A quote of Confucius, extracted from 中庸, *Doctrine of the Mean*. Its translation from classical Chinese to English is: *(Even) the world, the states, and the families may be amenable... the course of the Mean cannot be easily attained to.*

We should note that the *blown-up projection complex* (see §4.2.2 for definition) is a separable geodesic δ-hyperbolic space, on which the big mapping class group acts continuously by isometries.

The proof of Theorem 9.1.1 mainly uses the horicompactification defined as in [Duc23]. It is sometimes also known as *metric functional compactification* [Kar21] or *horofunction boudnary* [BGS85], although these notions have slight nuances in its way of definition as topological spaces. The elements in horicompactification are called *horivectors*. For a separable geodesic Gromov hyperbolic space X, we first show that each horivector is either associated to a bounded subset or a unique point on its Gromov boundary ∂X. For those who are associated to a bounded subset, we call them finite horivectors; otherwise ther are called infinite. We first prove that there is a continuous and surjective mapping from the infinite part of horicompactification to the Gromov boundary. Then we show that an $\mathrm{Isom}(X)$-invariant probability measure on the horicompactification must supported on its infinite part. Hence the push-forward probability measure will yield an $\mathrm{Isom}(X)$-invariant measure on the Gromov boundary and this goes back to the classical case of Lemma 9.3.2.

9.2 Compactifications of metric space

In this section, we will mainly discuss several similar compactifications of metric spaces, especially under the Gromov hyperbolic settings.

9.2.1 Horicompactification and others

First, consider the following mapping from the metric space X to a huge product of compact intervals

$$\begin{aligned} \Phi\colon X &\longrightarrow \prod_{y,z\in X\times X} [-d(y,z), d(y,z)] \\ x &\longmapsto \Big(d(x,y) - d(x,z)\Big)_{(y,z)\in X\times X} \end{aligned}$$

and define a compactification of X, denoted $\overline{X}^v$, by the closure of $\Phi(X)$ in the product space. An immediate application of Tychonoff's theorem implies that $\overline{X}^v$ is indeed compact. Following the usage from [Duc23], this compactification is called the *horicompactification* of X. An element of $\overline{X}^v$ will be called a *horivector* in what follows. One remarks that $v(y,z) = v(y,x) - v(z,x)$ and $v(x,y) = -v(y,x)$ for any horivector v.

Remark 9.2.1. In [Gro87, §7.5.E], such an object is given a name of "differentials of horofunctions" due to the property $v(y,z) = v(y,x) - v(z,x)$.

Another slightly different compactification of metric spaces, denoted $\overline{X}^h$, is the closure of the image of X under the following mapping

$$\begin{aligned}\Psi_o \colon X &\longrightarrow \prod_{y\in X}[-d(y,o),d(y,o)]\\ x &\longmapsto \Big(d(x,y)-d(x,o)\Big)_{y\in X}\end{aligned}$$

with a fixed base point o and it is called the *metric compactification* of X in [Rie02; Gut19] for instances. The elements in $\overline{X}^h$ adopt the name of *metric functional* following [Kar21].

One other related notion is *Busemann function* associated with a geodesic ray γ, namely $\beta_\gamma(x) = \lim_{t\to\infty}[d(x,\gamma(t))-t]$. In fact, the quantity $d(x,\gamma(t))-t$ is monotonically decreasing in $t>0$ and bounded from below by $-d(x,\gamma(0))$, see [BH13, II, Lemma 8.18]. It is by definition a metric functional based at $\gamma(0)$.

The following lemma shows that the horicompactification and metric compactification are essentially the same topological object and that the definition of metric compactification does not depend on the choice of the base point. See also [Duc23, Remark 2.7].

Proposition 9.2.2. *Let (X,d) be a metric space. For any base point $o\in X$, the two spaces $\overline{X}^v$ and $\overline{X}^h$ are homeomorphic.*

Proof. Let $A=\prod_{y\in X}[-d(y,o),d(y,o)]$ and $B=\prod_{y\in X,z\neq o}[-d(y,z),d(y,z)]$. Let π_A be the projection from the product $A\times B$ to A. Since B is compact, the projection π_A is a closed mapping (also referred as Kuratowski's theorem, see [Eng89, Theorem 3.1.16]). Hence, we have the inclusion $\pi_A(\overline{X}^v)=\pi_A\big(\overline{\Phi(X)}\big)\subseteq\overline{\Psi_o(X)}=\overline{X}^h$ per definition of closure. Conversely, any converging net $\big(\Phi(x_a)\big)_a$ in $A\times B$ yields a converging net $\big(\Psi_o(x_a)\big)_a$ in A, which further implies that $\pi_A(\overline{X}^v)\supseteq\overline{X}^h$. Therefore π_A is a continuous sujection from $\overline{X}^v$ to $\overline{X}^h$. As $\overline{X}^v$ is compact and $\overline{X}^h$ is Hausdorff, in order to show that π_A is a homeomorphism, it suffices to show that π_A is injective on $\overline{X}^v$. Indeed, given two horivectors $v,w\in\overline{X}^v$ satisfying $v(x,o)=w(x,o)$ for any $x\in X$, we have

$$v(y,z)-w(y,z)=v(y,o)-v(z,o)-w(y,o)+w(z,o)=0$$

for any $y,z\in X$. □

The following observation is essential for continuous group actions by isometries on metric spaces:

Proposition 9.2.3. *Let (X,d) be a metric space and G be any topological group acting on X continuously by isometries. Then the action of G on $\overline{X}^v$ is continuous, i.e. $\overline{X}^v$ is a G-flow.*

Proof. As horivectors are 1-Lipschitz in both variables, it soon yields that the mapping Φ is continuous and it turns out that $\overline{X}^v$ is an $\mathrm{Isom}(X)$-flow for the pointwise convergence topology of $\mathrm{Isom}(X)$ ([Duc23, Lemma 2.5]). Now let G acts continuously by isometries on (X,d). Then there is a continuous homomorphism $G \to \mathrm{Isom}(X)$, which further makes $\overline{X}^v$ a G-flow. □

horofunctions or Busemann functions have served as replacement for linear functionals when the space is not linear. In a recent paper [Kar21], Karlsson establishes a similar result for metric functionals. Thanks to the homeomorphism that we establish in Proposition 9.2.2, we can reformulate his result in terms of horivectors without difficulty.

Proposition 9.2.4 (Hahn-Banach for horivectors)**.** *Let (Y,d) be a metric space and X be a subspace of Y. Then for every horivector $v \in \overline{X}^v$, there exists a horivector $V \in \overline{Y}^v$ that extends v in the sense that $V(y,z) = v(y,z)$ for all $y,z \in X$.*

9.2.2 Busemann sequences

Now we will adapt some results from [MT18] in terms of horivectors as well as discuss some properties of Busemann sequences in Gromov hyperbolic spaces.

Definition 9.2.5 (Busemann sequence)**.** Let X be a Gromov-hyperbolic space. A sequence $(x_n) \subseteq X$ is said to be a *Busemann sequence* if the formula $d(x_n,x) - d(x_n,y)$ converges for every $x,y \in X$, *i.e.* $\Phi(x_n)$ converges to $v \in \overline{X}^v$.

In what follows, we will refer to $O(\delta)$ an additive error at most a multiple of δ, *viz.* if we write $f(x) = g(x) + O(\delta)$, then it means $|f(x) - g(x)| < M\delta$ for some uniform $M > 0$ independent of x. Errors $O(\delta)$ appear in different places can be different.

Let us introduce the notion of *orientation* of a geodesic γ, which is a strict total order on the points of γ induced by an isometric parameter $\gamma^+ : I \to X$. Suppose that $x = \gamma^+(t)$, $y = \gamma^+(s)$ and we say that $x \geq y$ if and only if $t \geq s$. Then given a fixed orientation, the *signed distance function* on a geodesic is defined by

$$d^+_\gamma(x,y) = \begin{cases} d(x,y), & \text{if } x \leq y \\ -d(x,y), & \text{if } x \geq y \end{cases}.$$

Proposition 9.2.6. *Let X be a δ-hyperbolic space and γ be a geodesic in X. For any horivector $v \in \overline{X}^v$, up to an additive error at most a multiple of δ and independent of γ, either there exists a $p \in \gamma$ such that*

$$v(y,z) = v(p,z) + d(y,p) + O(\delta), \quad (\forall y \in \gamma), \tag{9.1}$$

or there is an orientation for γ so that for every $p \in \gamma$

$$v(y,z) = v(p,z) + d_\gamma^+(y,p) + O(\delta), \quad (\forall y \in \gamma). \tag{9.2}$$

Proof. It follows from [MT18, Proposition 3.6]. □

Remark 9.2.7. From the proof we can see that, whenever the concerned geodesic γ is a segment, the case (9.1) will always hold. The case (9.2) can happen only when a Busemann sequence is fellow-travelling with a geodesic ray, and in that case this sequence will be Cauchy-Gromov.

For a metric space (X,d), we say that a sequence $(x_n)_{n\geq 0}$ is *Busemann at base point o* if $\Psi_o(x_n)$ converges to a metric functional in $\overline{X}^h$; it is *Busemann* if $\Phi(x_n)$ converges to a horivector in $\overline{X}^v$. Note that if a sequence is Busmann, then it is Busmann at any point $o \in X$.

Busemann sequences can be used to give a classification for horivectors, as well as metric functionals, on a δ-hyperbolic space. The first type of Busemann sequence is Cauchy-Gromov.

Lemma 9.2.8. *Let X be a δ-hyperbolic space and $(x_n)_{n\geq 0}$ be a Busemann sequence in it. Assume that $(x_n)_{n\geq 0}$ has a Cauchy-Gromov subsequence, then it is itself also Cauchy-Gromov.*

Proof. Suppose that it has a subsequence $(x_{n_k})_{k\geq 0}$ converging to $\xi \in \partial X$. By assumption, for any $K > 0$, there exists a $x \in X$ with $d(x,o)$ much larger than $2K$ such that $x_{n_k} \in \mathcal{S}_o(x, 2K)$ for large enough k, *i.e.* for any $M > 0$, there exists $m > M$ such that $\langle x_m, x\rangle_o \geq 2K$. Since the sequence is Busemann, there exists $N > 0$ such that for all $n, m > N$

$$|\Phi(x_n)(x,o) - \Phi(x_m)(x,o)| = 2\,|\langle x_n, x\rangle_o - \langle x_m, x\rangle_o| < K.$$

In particular, by taking a particular $x_m \in \mathcal{S}_o(x, 2K)$, we can deduce that $\langle x_n, x\rangle_o \geq K$ and that by consequence $x_n \in \mathcal{S}_o(x, K)$ for all $n \geq N$. This implies that x_n converge to $\xi \in \partial X$ as a Cauchy-Gromov sequence. □

Conversely, we also have the following estimation:

Lemma 9.2.9. *Let X be a δ-hyperbolic space. If $(x_n) \subseteq X$ is a Cauchy-Gromov sequence that admits Busemann subsequences, then for any two horivectors v, v' determined by its Busemann, one has*

$$\left|v(x,y) - v'(x,y)\right| \leq 2\delta$$

for any $x, y \in X$.

Proof. First compute

$$\begin{aligned} &|d(x_n,x)-d(x_n,y)-d(x_m,x)+d(x_m,y)| \\ =&|d(x_n,x)-d(x_n,y)-d(x,y)-d(x_m,x)+d(x_m,y)+d(x,y)| \\ =&2\left|\langle x_m|x\rangle_y-\langle x_n|x\rangle_y\right|. \end{aligned} \tag{9.3}$$

Without loss of generality, one assumes that (9.3) is equal to $2\big(\langle x_m|x\rangle_y-\langle x_n|x\rangle_o\big)$. By δ-hyperbolicity, one gets

$$\begin{aligned} &|d(x_n,x)-d(x_n,y)-d(x_m,x)+d(x_m,y)| \\ =&2\big(\langle x_m|x\rangle_y-\langle x_n|x\rangle_y\big) \\ \leq&2\Big(\langle x_m|x\rangle_y-\min\left\{\langle x_m|x_n\rangle_y,\langle x_m|x\rangle_y\right\}+\delta\Big). \end{aligned}$$

Note that $\langle x_m|x\rangle_y\leq d(x,y)$ while $\langle x_m|x_n\rangle_y\to\infty$ as n,m go to infinity. It soon yields

$$|d(x_n,x)-d(x_n,y)-d(x_m,x)+d(x_m,y)|\leq 2\delta \tag{9.4}$$

for large enough n,m. By taking n tends to infinity alongside one Busemann subsequence and m alongside another Busemann subsequence, one immediately gets the desired inequality. □

The following lemma will finish establishing a dichotomy for Busemann sequences and also horivectors in a δ-hyperbolic space.

Lemma 9.2.10. *Let X be a δ-hyperbolic space and $(x_n)_{n\geq 0}$ be a Busemann sequence in it. Suppose in addition that $(x_n)_{n\geq 0}$ converges to a horivector $v\in\overline{X}^v$ and is not Cauchy-Gromov, then the function $v(\cdot,z)$ is bounded from below for any $z\in X$.*

Proof. Fix any $z\in X$. Suppose for contradiction that $v(\cdot,z)$ is not bounded from below. Then for any $M>0$, there exists an $y\in X$ such that $v(y,z)<-2M$. In terms of limit, it means that there is an $N>0$ such that $d(x_n,y)-d(x_n,z)<-M$ for all $n\geq N$. So for any $n,m\geq N$, we will have

$$\begin{aligned} \langle x_n,x_m\rangle_z=&\frac{1}{2}\left(d(x_n,z)+d(x_m,z)-d(x_n,x_m)\right) \\ >&\frac{1}{2}\left(d(x_n,y)+d(x_m,y)+2M-d(x_n,x_m)\right)>M, \end{aligned}$$

which implies that $(x_n)_{n\geq 0}$ must be Cauchy-Gromov. Contradiction! □

9.2.3 Horiboundary

For horicompactification, we can define the *horiboundary* by $\overline{X}^v \setminus \Phi(X)$, denoted $\partial\overline{X}^v$.

Previously we give a dichotomy for Busemann sequences in a δ-hyperbolic space. The same dichotomy for horivectors can be established by passing to limits.

Let $\overline{X}^v_\infty$ be the subset of $\partial\overline{X}^v$ so that any of its elements is not not bounded from below. Also denote by $\overline{X}^v_f$ the set of horivectors v such that $v(\cdot, z)$ is bounded below for any $z \in X$. We note that $\overline{X}^v = \overline{X}^v_\infty \cup \overline{X}^v_f$ and $\overline{X}^v_\infty \subseteq \partial\overline{X}^v$.

On one hand, each horivector in $\overline{X}^v_f$ is uniquely corresponded to a bounded part in X. Let $v \in \overline{X}^v_f$. Define the *coarse minima based at z of v* by

$$\mathcal{L}(v, z) := \left\{ y \in X : v(y, z) \leq \inf_{x \in X} v(x, z) + 1 \right\}.$$

Similarly, its *coarse minima* $\mathcal{L}(v)$ is then defined as the union of $\mathcal{L}(v, z)$ for all $z \in X$. The same arguments for [MT18, Lemma 3.13] will yield the following result:

Proposition 9.2.11. *Let X be a geodesic δ-hyperbolic space and $v \in \partial\overline{X}^v_f$. Then there exists a constant K only depending on δ such that* $\operatorname{diam}\big(\mathcal{L}(v)\big) \leq K$.

On the other hand, using *minimising sequence* for such horivector $v \in \overline{X}^v_\infty$, *i.e.* a sequence $(y_n)_{n\geq 1}$ such that $v(y_n, z) \to -\infty$ as $n \to \infty$, one can construct a *boundary correspondence* $\Xi\colon \overline{X}^v_\infty \to \partial X$ between the infinite part of horiboundary and Gromov boundary [MT18, Lemma 3.10].

Proposition 9.2.12 ([MT18])**.** *Let X be a separable geodesic Gromov hyperbolic space. Then the boundary correspondence $\Xi\colon \overline{X}^v_\infty \to \partial X$ is continuous and surjective.*

Remark 9.2.13. In terms of *reduced horicompactification* (introduced for example in [BF20]), *i.e.* the quotient space of $\overline{X}^v$ by equivalence relation $v \sim v'$ if there exists $M > 0$ such that $|v(x, y) - v'(x, y)| < M$ for all $x, y \in X$, then the reduced horicompactification will become $\partial X \sqcup \{*\}$, where ∂X is the Gromov boundary of X. But this space is in general not Hausdorff.

9.3 Main result

Let X be a separable geodesic δ-hyperbolic space and D be a dense subset in X. Then the projection

$$\overline{X}^v \to \prod_{x,y \in D} [-d(x, y), d(x, y)]$$

is a homeomorphism. Hence we need only to treat the valuation of horivectors on the dense subset D with base point $o \in D$. As a result, the expression

$$\overline{X}^v_\infty = \bigcap_{N>0} \bigcup_{x \in D} \left\{ v \in \overline{X}^v : v(x,o) < -N \right\}$$

implies that $\overline{X}^v_\infty$ and $\overline{X}^v_f = \overline{X}^v \setminus \overline{X}^v_\infty$ are Borel sets in $\overline{X}^v$.

Suppose that G is an amenable group acting continuously by isometries on X. Proposition 9.2.3 asserts that the action of G on the horicompactification $\overline{X}^v$ is continuous. Hence there exists a G-invariant probability measure μ on $\overline{X}^v$.

Proposition 9.3.1. *Let X be a separable geodesic δ-hyperbolic space. Suppose that G is a group acting continuously by isometries on X and suppose that the action is unbounded. Let μ be a G-invariant probability measure on $\overline{X}^v$. Then $\mu(\overline{X}^v_f) = 0$.*

Proof. Let D be a dense subset in X and $o \in D$ be a base point. For a fixed $q \in \mathbb{Q}$ and two distinct points $x, y \in D$, the open set

$$V(x,y,q) = \overline{X}^v \cap \left([-d(x,o), q) \times (q, d(y,o)] \times \prod_{(z,w) \neq (x,o),(y,o)} [-d(z,w), d(z,w)] \right)$$

is the collection of all horivectors v in $\overline{X}^v$ such that $v(x,o) < q < v(y,o)$. Therefore, the union $V_{x,y} = \bigcup_{q \in \mathbb{Q}} V(x,y,q)$ is the collection of the horivectors $v \in \overline{X}^v$ such that $v(x,o) < v(y,o)$.

Let $R > 0$ be a positive real number that is larger than the uniform diameter K of coarse minima $\mathcal{L}(v)$ from Proposition 9.2.11. For any $z \in D$, we define

$$Y(z,R) = \left\{ v \in \overline{X}^v_f : \mathcal{L}(v) \cap B(z,R) \neq \emptyset \right\}$$

where $B(z,R) = \{p \in X : d(z,p) < R\}$. One remarks that $\mathcal{L}(v)$ contains $\mathcal{L}(v,o)$. So if $v \in Y(z,R)$, then $\mathcal{L}(v,o)$ will be contained in $B(z,2R)$. By consequence, for every $y \notin B(z,2R)$, we will have

$$v(y,o) > \inf_{x \in X} v(x,o) + 1 > \inf_{x \in X} v(x,o) = \inf_{x \in B(z,2R)} v(x,o).$$

By the density of D in X, it turns out that there must be some $x \in B(z,2R) \cap D$ such that $v(x,o) < v(y,o)$. So if one sets $B(z,2R)^c$ to be the complement of $B(z,R)$ in X and

$$Y'(z,R) = \left\{ v \in \overline{X}^v_f : \forall y \in B(z,2R)^c \cap D \; \exists x \in B(z,2R) \cap D \text{ so that } v(x,o) < v(y,o) \right\},$$

then $Y(z,R) \subseteq Y'(z,R)$. In fact, the set $Y'(z,R)$ remains the same even if one removes the constraint of x, y being in D from the definition. But the definition above means that we

have the following expression

$$Y'(z,R) = \bigcap_{y \in B(z,2R)^c \cap D} \left(\bigcup_{x \in B(z,2R) \cap D} V_{x,y} \right).$$

This shows that $Y'(z,R)$ is a Borel set in $\overline{X}^v$.

Let $g \in G$ be any isometry. For any $v \in Y'(z,R)$, we can see that for any $g^{-1}y \notin B(g^{-1}z, 2R)$, there exists a $g^{-1}x \in B(g^{-1}z, 2R)$ such that

$$\begin{aligned}
\big(gv\big)(y,o) &= v(g^{-1}y, g^{-1}o) \\
&= v(g^{-1}y, o) + v(o, g^{-1}o) \\
&> v(g^{-1}x, o) + v(o, g^{-1}o) \\
&= v(g^{-1}x, g^{-1}o) \\
&= \big(gv\big)(x,o),
\end{aligned}$$

which implies that $gv \in Y'(g^{-1}z, R)$. Hence $gY'(z,R) \subseteq Y'(g^{-1}z,R)$. If we apply the same argument for g^{-1} and $Y'(g^{-1}z,R)$, then the inclusion above is actually equality, *i.e.* $gY'(z,R) = Y'(g^{-1}z,R)$.

Now we claim that $\mu\big(Y'(z,R)\big) = 0$ for all $z \in X$. Indeed, suppose *ab absurdo* that there is a point $z \in X$ such that $\mu\big(Y'(z,R)\big) > 0$. Note that if $v \in Y'(z,R)$, then $\mathcal{L}(v,o) \cap B(z,2R) \neq \emptyset$, which further implies that $\mathcal{L}(v) \subset B(z,3R)$. So if $d(z_1,z_2) > 6R$, then $Y'(z_1,R) \cap Y'(z_2,R) = \emptyset$. As the action of G on X is unbounded, by picking a sequence $(g_n)_{n\in\mathbb{N}}$ of elements in G such that $d(g_n z, g_m z) > 6R$ for every $n \neq m$, we will have

$$1 = \mu\left(\overline{X}^v\right) \geq \mu\left(\bigcup_{n\in\mathbb{N}} Y'(g_n z, R)\right) = \sum_{n\in\mathbb{N}} \mu\big(Y'(g_n z, R)\big) = \sum_{n\in\mathbb{N}} \mu\big(Y'(z,R)\big) = \infty,$$

which is not possible!

Finally, by density of D in X, we have $\overline{X}^v_f \subseteq \bigcup_{z\in D} Y(z,R)$. So we get

$$0 \leq \mu\left(\overline{X}^v_f\right) \leq \mu\left(\bigcup_{z\in D} Y(z,R)\right) \leq \sum_{z\in D} \mu\big(Y(z,R)\big) \leq \sum_{z\in D} \mu\big(Y'(z,R)\big) = 0.$$

This completes the proof. □

Now we shall prove the following general result:

Lemma 9.3.2. *Let X be a δ-hyperbolic space and let G act on X by isometries. Suppose that there exists a G-invariant probability measure ν on ∂X. Then the action of G cannot be of general type.*

Proof. Suppose for contradiction that the action of G on X is of general type. Then G has no finite orbit on ∂X. Because ν is G-invariant, so ν is atomless, otherwise the total measure of ∂X under ν will exceed 1. Moreover, there exists at least one hyperbolic isometry g in G. Let $\partial_X\langle g\rangle = \{\xi_\pm\}$. Since $\xi_\pm$ are not atoms, we can find open neighbourhoods $U_\pm$ of $\xi_\pm$ respectively such that $\nu(U_\pm) < 1/3$. This means that $\nu(\partial X \setminus U_-) > 2/3$. Yet by North-South dynamic of g (see [DSU17, §6.1]), there exists $n > 0$ such that $g^n(\partial X \setminus U_-)$ is contained in U_+. Hence we have

$$\frac{2}{3} < \nu(\partial X \setminus U_-) = \nu(g^n(\partial X \setminus U_-)) \leq \nu(U_+) < \frac{1}{3}.$$

This is absurd! □

Now we are able to prove Theorem 9.1.1:

Proof of Theorem 9.1.1. Suppose that G is an amenable group act continuously by isometries on a separable geodesic Gromov hyperbolic space (X, d). Then there is a G-invariant probability measure on $\overline{X}^v$, denoted μ. By Proposition 9.3.1, the probability measure μ must supported on $\overline{X}^v_\infty$. The continuous boundary correspondence $\Xi\colon \overline{X}^v_\infty \to \partial X$ from Proposition 9.2.12 will yield a G-invariant probability measure $\Xi_*(\mu)$ by pushing-forward. Now the result comes immediately from Lemma 9.3.2. □

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List of Symbols

$[x, y]$	geodesic segment between x and y. 23, 28
$B_0(x, x)$	Lorentzian quadratic form. 22
b_μ	barycenter of probability measure μ. 51
∂X	Gromov boundary of Gromov hyperbolic space X. 29
$\mathcal{C}(S)$	isotopic classes of essential simple closed curves on surface S, or curve graph of surface S. 60
$\mathcal{C}^\dagger(S)$	fine curve graph of surface S 120
$\mathrm{Cay}(G, S)$	Cayley graph of G with respect to S. 35
$\overline{\mathrm{co}}(C)$	convex hull of C 99
$d_{\mathbb{H}}$	hyperbolic metric. 22, 23
$D\varphi$	Malliavin derivative of φ 92
$E[x]$	points in X E-closed to x. 40
$\ell(\gamma)$	Length of the path γ. 132
$\mathrm{End}(S)$	end space of surface S. 55
$\mathrm{End}_\infty(S)$	non-planar ends of surface S. 56
$\overline{\mathrm{ess}}(\mathbf{f})$	essential range of $\mathbf{f}$ 100
$\langle x, y\rangle_o$	Gromov product of x and y based at o. 27, 28
$\mathbb{H}^\alpha$	real hyperbolic space of dimension α. 24, 26
$h_K(X)$	support function of (infinite-dimensional) GB convex body K 80
$\mathfrak{H}_n$	n-th Wiener chaos. 92
d_{Haus}	Hausdorff distance. 27
$\mathrm{Homeo}(S)$	homeomorphism group of S. 61
$\mathrm{Homeo}^+(S)$	orientation preserving homeomorphism group of S. 61
$\mathrm{Inj}(X)$	Injective hull of X. 36
$\mathrm{Isom}(X)$	group of isometries. 23
$\mathcal{M}(G)$	universal minimal flow of G. 49
$\mathfrak{M}(V)$	Means over V. 46
$\mathfrak{M}_m(V)$	Multiplicative means over V. 46

$\mathcal{MCG}(S)$	mapping class group of S. 61
$\mathcal{NC}^{\dagger}(S)$	non-separating fine curve graph of surface S 120
$\mathcal{P}_M(\mathbb{Y})$	projection complex of constant M constructed from BBF family $\mathbb{Y}$. 67
$\pi_\gamma(x)$	nearest points projection of x. 135
$\mathcal{PMCG}(S)$	pure mapping class group of S. 63
$\mathcal{QI}(X)$	Quasi-isometry group of X. 35
$\mathcal{N}_r$	r-neighbourhood. 27
$\mathrm{RUCB}(G)$	bounded right uniform continuous functions on G. 47
$\mathcal{S}(X)$	Samuel compactification of X. 45
$\mathrm{Seq}_{\mathrm{b}}(X)$	Collection of bounded sequences in X. 131
$\mathrm{Stein}(K)$	Steiner point of (infinite-dimensional) GB convex body K 80
$u(X)$	ultracomplete supspace of X. 140
$\mathcal{U}(X)$	uniform structure on X. 40
$\mathcal{U}_{\mathrm{l}}(G)$	left uniform structure on G. 41
$\mathcal{U}_{\mathrm{r}}(G)$	right uniform structure on G. 41
$\mathrm{UCB}(X)$	bounded uniform continuous functions on X. 44
$\overline{X}^h$	metric compactification of metric space X 147
X^ω	ω-ultralimit of X. 132
$\overline{X}^v$	horicompactification of metric space X 146
$\xi(S)$	complexity of surface S. 56

List of Terminologies and Their Translations

Alexandrov-Fenchel inequality *inégalité d'Alexandrov-Fenchel* (fr), Alexandrov-Fenchel不等式 (cn). 78

algebraic hyperbolic space *espace hyperbolique algébrique* (fr), 代数双曲空间 (cn). 22, 29, 134

amenability *moyennabilité* (fr), 可均性 (cn). 50

barycenter *barycentre* (fr), 质心 (cn). 51

bounded right uniformly continuous function *fonction bornée uniformément continue à droite* (fr), 有界右一致连续函数 (cn). 47

bounded uniformly continuous function *fonction bornée uniformément continue* (fr), 有界一致连续函数 (cn). 44

Busemann function *fonction de Busemann* (fr), Busemann函数 (cn). 147

Cauchy filter *filtre de Cauchy* (fr), Cauchy滤子 (cn). 41

Cauchy-Gromov sequence *suite de Cauchy-Gromov* (fr), Cauchy-Gromov序列 (cn). 29

Cayley graph *graphe de Cayley* (fr), Cayley图 (cn). 35

cobounded *co-borné* (fr), 余有界 (cn). 34

completeness *complétude* (fr), 完备性 (cn). 42

convex body *corps convexe* (fr), 凸体 (cn). 71

- **Gaußian continuous** *gaussiennement continu* (fr), Gauss连续 (cn). 77
- **Gaußian bounded** *gaussiennement borné* (fr), Gauss有界 (cn). 76

curve graph *graphe de courbes* (fr), 曲线1-复形 (cn). 60

- **fine curve graph** *graphe fin de courbes* (fr), 精细曲线1-复形 (cn). 120
- **non-separating fine curve graph** *graphe fin de courbes non-séparantes* (fr), 精细非分割曲线1-复形 (cn). 120

Dehn twist *twist de Dehn* (fr), Dehn扭转 (cn). 63

$\mathbb{R}$-tree $\mathbb{R}$-*arbre* (fr), $\mathbb{R}$-树 (cn). 28

δ-hyperbolic space *espace δ-hyperbolique* (fr), δ-双曲空间 (cn). 28

δ-slim *δ-fin* (fr), δ-狭窄 (cn). 27

dilation *dilatation* (fr), 等比缩放 (cn). 76

université PARIS-SACLAY

ÉCOLE DOCTORALE
de mathématiques
Hadamard (EDMH)

Titre: Divers aspects de la géométrie hyperbolique et de la dynamique de groupes

Mots clés: géométrie hyperbolique, dynamique de groupes, topologie en basses dimensions, théorie géométrique et ergodique de groupes.

Résumé: Cette thèse explore divers sujets liés à la géométrie hyperbolique et à la dynamique de groupes, dans le but d'étudier l'interaction entre la géométrie et la théorie de groupes. Elle couvre un large éventail de disciplines mathématiques, telles que la géométrie convexe, l'analyse stochastique, la théorie ergodiques et géométriques de groupes, et la topologie en basses dimensions, et cætera. Comme résultats de recherche, la géométrie hyperbolique des corps convexes en dimension infinie est examinée en profondeur, et des tentatives sont faites pour développer la géométrie intégrale en dimension infinie d'un point de vue de l'analyse stochastique. L'étude des gros groupes de difféotopies, un sujet d'actualité en topologie en basses dimensions et en théorie géométrique de groupes, est entreprise avec une détermination complète de leur propriété de point fixe sur les compacts. La thèse étudie la connexité du bord de Gromov des graphes de courbes fins, un outil combinatoire utilisé dans l'étude des groupes d'homéomorphismes des surfaces de type fini. Enfin, la thèse clarifie également certains théorèmes folkloriques concernant les espaces hyperboliques au sens de Gromov et la dynamique des groupes moyennables sur ces espaces.

Title: Diverse aspects of hyperbolic geometry and group dynamics

Keywords: hyperbolic geometry, group dynamics, low-dimensional topology, ergodic and geometric group theory.

Abstract: This thesis explores diverse topics related to hyperbolic geometry and group dynamics, aiming to investigate the interplay between geometry and group theory. It covers a wide range of mathematical disciplines, such as convex geometry, stochastic analysis, ergodic and geometric group theory, and low-dimensional topology, *etc.* As research outcomes, the hyperbolic geometry of infinite-dimensional convex bodies is thoroughly examined, and attempts are made to develop integral geometry in infinite dimensions from a perspective of stochastic analysis. The study of big mapping class groups, a current focus in low-dimensional topology and geometric group theory, is undertaken with a complete determination of their fixed-point on compacta property. The thesis also clarifies certain folklore theorems regarding the Gromov hyperbolic spaces and the dynamics of amenable groups on them. Last but not the least, the thesis studies the connectivity of the Gromov boundary of fine curve graphs, a combinatorial tool employed in the study of the homeomorphism groups of surfaces of finite type.