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THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE BRETAGNE OCCIDENTALE

ÉCOLE DOCTORALE N° 644

Mathématiques et Sciences et Technologies

de l'Information et de la Communication en Bretagne Océane

Spécialité : *Mathématiques et leurs Interactions*

Par

Armand COUDRAY

Asymptotic behaviour of zero rest-mass fields on radiative space-times

Thèse présentée et soutenue à Brest, le 6 Mai 2024

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RÉSUMÉ (FRANÇAIS)

Equations d'Einstein et solution dans le vide

Cette thèse porte sur l'étude asymptotique des champs sans masses dans des espaces-temps purement radiatifs. En 1915, Einstein présente la théorie de la relativité générale dans laquelle l'univers est décrit au moyen d'une variété différentiable de dimension 4, notée \mathcal{M} et munie d'une métrique, g_{ab} lorentzienne de signature $(+, -, -, -)$ et on appelle espace-temps le couple (\mathcal{M}, g) . La relativité générale stipule que la présence de matière et d'énergie dans l'univers (présence encodée dans le tenseur énergie-impulsion et notée T_{ab}) provoque la courbure de l'espace-temps et c'est ce que décrit l'équation d'Einstein :

$$R_{ab} - \frac{1}{2}g_{ab}\text{Scal}_g = 8\pi T_{ab},$$

où R_{ab} est le tenseur de Ricci et Scal_g la courbure scalaire, c'est-à-dire la trace du tenseur de Ricci. On considère dans ce manuscrit que la constante cosmologique $\Lambda = 0$, autrement dit, les solutions de l'équation d'Einstein que l'on étudie ici ne sont pas en expansion. La solution la plus simple à cette équation dans le vide (c'est-à-dire quand $T_{ab} = 0$) correspond à la métrique de l'espace-temps plat, aussi appelée métrique de Minkowski, pour laquelle $R_{ab} = 0$ et qui constitue l'espace-temps de la relativité restreinte. La métrique associée est alors (dans les coordonnées sphériques (t, r, θ, φ) usuelles) :

$$\eta = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Dans cette thèse, la notion de temps retardé est fréquemment employée et nous commençons par la définir dans le cadre de la solution de Minkowski. On appelle u le temps retardé (ou coordonnée d'Eddington-Finkelstein retardée) défini par :

$$u = t - r.$$

Il est également possible de définir v , le temps avancé, comme $v = t + r$. Ces deux coordonnées de temps sont particulièrement adaptées pour décrire les courbes isotropes,

c'est-à-dire les courbes dont le vecteur tangent k^a est un vecteur isotrope (ou de type lumière), dont la norme est nulle :

$$g_{ab}k^ak^b = 0.$$

Un tel vecteur correspond à un observateur qui se déplace à la vitesse de la lumière. En utilisant les coordonnées (u, r, θ, φ) , la métrique de Minkowski, notée η , devient :

$$\eta = du^2 + 2dudr - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

Une autre solution des équations d'Einstein dans le vide est la solution décrite par Karl Schwarzschild en 1916 qui représente le champ gravitationnel à l'extérieur d'une source de gravitation, sphérique, non chargée et statique. La métrique de Schwarzschild est donnée par :

$$g^{\text{Sch}} = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

où M est une constante positive qui correspond à la masse du trou noir de Schwarzschild. La métrique de Schwarzschild exprimée dans les coordonnées sphériques présente deux singularités : une lorsque $r = 0$ et l'autre lorsque $r = 2M$. La première correspond à une réelle singularité, indépendante des coordonnées choisies ; la seconde dépend quant à elle du choix de coordonnées.

En introduisant les coordonnées (u, r, θ, φ) , avec u le temps retardé défini par :

$$u = t - (r + 2M \log(r - 2M)),$$

la métrique devient :

$$g^{\text{Sch}} = \left(1 - \frac{2M}{r}\right) du^2 + 2dudr - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

La métrique est désormais régulière lorsque $r = 2M$ et on appelle *horizon passé des évènements* l'hypersurface isotrope :

$$\mathcal{H}^- = \mathbb{R}_u \times \{2M\}_r \times S^2.$$

De même, en se plaçant dans les coordonnées (v, r, θ, φ) avec :

$$v = t + (r + 2M \log(r - 2M)) ,$$

la métrique devient :

$$g^{\text{Sch}} = \left(1 - \frac{2M}{r}\right) dv^2 - 2dvdr - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) .$$

et l'horizon futur des évènements est défini comme l'hypersurface isotrope :

$$\mathcal{H}^+ = \mathbb{R}_v \times \{2M\}_r \times S^2 .$$

Solutions purement radiatives

Un espace-temps est dit purement radiatif si le tenseur énergie-impulsion T_{ab} est non-nul et qu'il peut s'écrire de la façon suivante :

$$T_{ab} = \rho k_a k_b ,$$

où ρ représente la densité de radiation et $k_a = g_{ab} k^b$ avec k^a qui est un vecteur isotrope. Un tel tenseur énergie-impulsion décrit un champ qui se déplace à la vitesse de la lumière. Il peut s'agir, par exemple, d'un champ électromagnétique ou bien de poussières sans masses. Cette thèse porte sur l'étude de deux types d'espaces-temps radiatifs différents : l'espace-temps de Vaidya et l'espace-temps de Robinson-Trautman.

Espace-temps de Vaidya et horizon des évènements

L'espace-temps de Vaidya est un espace-temps purement radiatif, à symétrie sphérique, qui représente un trou blanc qui perd de la masse via l'émission de sphères de poussières isotropes le long des courbes $\{u = cte\}$. Il est également possible de décrire un trou noir de Vaidya comme un trou noir dont la masse augmente par accrétion de poussière isotropes le long des courbes $\{v = cte\}$. Dans le cas de la métrique de Vaidya représentant un trou blanc en évaporation (ce qui correspond à la situation étudiée dans cette thèse), la masse M devient une fonction dépendant du temps, et en particulier du temps retardé désigné par u . Par conséquent, la métrique de Schwarzschild qui était statique devient pour la solution de Vaidya une métrique dynamique qui dépend désormais du temps retardé u .

La métrique s'écrit alors dans les coordonnées (u, r, θ, φ) :

$$g_V = \left(1 - \frac{2m(u)}{r}\right) du^2 + 2dudr - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

et le tenseur source de l'équation d'Einstein devient :

$$T_{ab} = -\frac{m'(u)}{4\pi r^2} (du)_a (du)_b.$$

Ainsi, afin de garantir que la densité d'énergie dans l'espace-temps

$$\rho = -\frac{m'(u)}{4\pi r^2},$$

est positive, nous devons imposer que la masse soit une fonction décroissante de u sur un intervalle de la forme $[u_-, u_+]$. De plus, dans cette thèse, nous faisons l'hypothèse que la masse admet des limites finies quand u tend vers u_{\pm} .

$$\lim_{u \rightarrow u_{\pm}} m(u) \rightarrow m_{\pm} \text{ avec } 0 \leq m_+ < m_- < +\infty, \quad (1)$$

avec

$$m'(u) < 0 \quad \text{sur }]u_-, u_+[, \quad -\infty \leq u_- < u_+ \leq +\infty, \quad m'(u) = 0 \quad \text{sinon.} \quad (2)$$

Il est alors possible de distinguer deux situations : soit la masse varie sur un intervalle fini, soit l'évaporation se fait pour tout $u \in \mathbb{R}$. Dans ce dernier cas, on a $u_- = -\infty$ et $u_+ = +\infty$.

Une question naturelle qui se pose alors est la suivante : quel est le comportement de l'horizon passé dans l'espace-temps de Vaidya ? Cette question est traitée dans le Chapitre 3, où on rappelle les résultats obtenus par l'auteur et Jean-Philippe Nicolas dans [15], et nous résumons ici les principaux résultats. L'idée générale de cet article est d'étudier le comportement des courbes intégrales des directions isotropes principales entrantes dans la métrique de Vaidya. Du fait de la géométrie de la solution, ces courbes sont des géodésiques. On étudie alors le comportement de la famille de courbes $\gamma(u)$ indexées par $\omega \in S^2$:

$$\gamma(u) = (u, r = r(u), \omega), \quad u \in \mathbb{R}.$$

Étant donné que ces courbes sont isotropes, cela implique que :

$$g(\dot{\gamma}(u), \dot{\gamma}(u)) = 1 - \frac{2m(u)}{r(u)} + 2\dot{r}(u) = 0.$$

Ainsi, la fonction $r(u)$ doit satisfaire l'équation différentielle suivante :

$$\dot{r}(u) = -\frac{1}{2} \left(1 - \frac{2m(u)}{r(u)} \right). \quad (3)$$

On obtient alors le théorème principal de [15] :

Theorem 0.0.1. *Soit m une fonction lisse et décroissante, dépendante du temps retardé u et qui satisfait les hypothèses (1) et (2), alors il n'existe qu'une unique solution maximale r_h de (3) telle que :*

$$\lim_{u \rightarrow -\infty} r_h(u) = 2m_-.$$

- *Dans le cas où $m_+ > 0$ ou bien que $u_+ = +\infty$, r_h existe alors pour tout $u \in \mathbb{R}$, $r_h(u) \rightarrow 2M_+$ quand $u \rightarrow +\infty$ et toute autre solution maximale r de (3) appartient alors à l'une des deux catégories suivantes :*
 1. *r existe pour tout $u \in \mathbb{R}$, $r(u) > r_h(u)$ pour tout $u \in \mathbb{R}$, $\lim_{u \rightarrow -\infty} r(u) = +\infty$ et $\lim_{u \rightarrow +\infty} r(u) = 2m_+$;*
 2. *r existe sur un intervalle de la forme $]u_0, +\infty[$ avec $u_0 \in \mathbb{R}$ et satisfait : $r(u) < r_h(u)$ pour tout $u \in]u_0, +\infty[$, $\lim_{u \rightarrow u_0} r(u) = 0$ et $\lim_{u \rightarrow +\infty} r(u) = 2m_+$.*
- *Si $m_+ = 0$ ou si $u_+ < +\infty$, alors r_h existe sur un intervalle $] - \infty, u_0[$ avec $u_+ \leq u_0 < +\infty$ et $\lim_{u \rightarrow u_0} r_h(u) = 0$. Les autres solutions maximales sont alors de deux types :*
 1. *r existe sur $] - \infty, u_1[$ avec $u_0 \leq u_1 < +\infty$, $r(u) > r_h(u)$ sur l'intervalle $] - \infty, u_0[$, $\lim_{u \rightarrow u_1} r(u) = 0$ et $\lim_{u \rightarrow -\infty} r(u) = +\infty$;*
 2. *r existe sur l'intervalle $]u_1, u_2[$ avec $-\infty < u_1 < u_2 \leq u_0$, $r(u) \rightarrow 0$ quand u tend soit vers u_1 soit vers u_2 et $r(u) < r_h(u)$ sur $]u_1, u_2[$.*

De plus, comme les courbes étudiées ici sont des géodésiques, alors elles génèrent des hypersurfaces et la solution qui satisfait $r = r_h$ génère l'horizon passé des événements dans l'espace-temps de Vaidya. On supposera désormais que $m_+ > 0$, ainsi, $r_h > 0$ pour toute valeur de $u \in \mathbb{R}$, et cela décrit la situation physique suivante : un trou blanc de

masse m_- s'évapore via l'émission de poussières isotropes avant de se stabiliser en un trou blanc (et asymptotiquement en un trou noir) de masse m_+ .

Compactification conforme

L'une des thématiques principales de cette thèse est d'étudier le comportement asymptotique des ondes scalaires dans l'espace-temps de Vaidya.

L'analyse asymptotique est réalisée en utilisant les méthodes de compactification conforme développées par Penrose dans les années 1960 dans une série d'articles ([72], [69], et [70]). Cela consiste à plonger un espace-temps dit "physique" (\mathcal{M}, g) dans un espace-temps *compactifié* plus grand $(\hat{\mathcal{M}}, \hat{g})$, où \mathcal{M} correspond à l'intérieur de $\hat{\mathcal{M}}$, et la métrique se transforme de la façon suivante :

$$\hat{g} = \Omega^2 g;$$

Ω est appelé *facteur conforme* et permet de définir le bord de $\hat{\mathcal{M}}$: $\partial\hat{\mathcal{M}} = \hat{\mathcal{M}} - \mathcal{M}$. Ω vérifie $\Omega > 0$ dans \mathcal{M} , et au bord de l'espace-temps conforme $\partial\hat{\mathcal{M}}$, on a : $\Omega = 0$ et $\nabla_a \Omega \neq 0$.

L'intérêt de se placer dans l'espace-temps compactifié réside dans le fait que l'on considère les points du bord $\partial\hat{\mathcal{M}}$ comme des points à l'infini de l'espace-temps physique. Cela permet donc de transformer les méthodes asymptotiques dans \mathcal{M} en des techniques locales sur une hypersurface qui est le bord du compactifié conforme.

Lorsque l'on compactifie l'espace-temps plat de Minkowski, on obtient un espace-temps conforme dont le bord est un compact composé de deux hypersurfaces isotropes \mathcal{I}^\pm , appelées infini isotrope futur et passé, et de trois points i_0 , l'infini spatial, i_+ l'infini temporel futur et i_- l'infini temporel passé (cf. figure 1).

Ce n'est plus le cas lorsque l'on compactifie les espaces-temps de Schwarzschild ou bien de Vaidya, et les points i_0 , i_+ et i_- deviennent des singularités du bord conforme. On choisit alors le facteur conforme $\Omega = R = 1/r$ et on obtient la métrique de Vaidya compactifiée :

$$\hat{g}^V = R^2 (1 - 2m(u)R) du^2 - 2dudR - (d\theta^2 + \sin^2 \theta d\varphi^2) ,$$

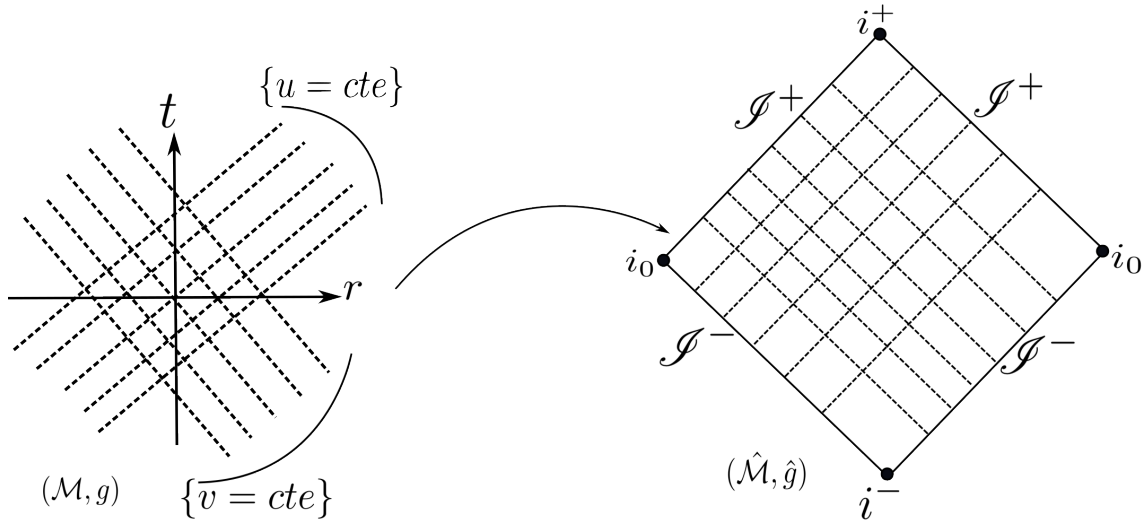


FIGURE 1 : La compactification conforme transforme l'espace-temps physique (à gauche) en un espace-temps compactifié conforme (à droite) avec un bord. Les directions isotropes (par exemple les courbes à $u = cte$ et $v = cte$) sont préservées lors de cette transformation.

On définit alors l'infini isotrope futur \mathcal{I}^+ et l'horizon passé \mathcal{H}^- :

$$\begin{aligned}\mathcal{I}^+ &= \mathbb{R}_u \times \{0\}_R \times S^2, \\ \mathcal{H}^- &= \mathbb{R}_u \times \{1/r_h\}_R \times S^2.\end{aligned}$$

L'infini isotrope passé ainsi que l'horizon futur, du fait de l'hypothèse (1), sont les mêmes que ceux de l'espace-temps de Schwarzschild. Toutefois, les coordonnées (u, r, θ, φ) ne sont pas adaptées pour décrire ces hypersurfaces. C'est pourquoi nous devons effectuer la compactification conforme de la métrique de Schwarzschild dans les coordonnées (v, R, θ, φ) :

$$\hat{g}^{\text{Sch}} = R^2(1 - 2MR)dv^2 + 2dv dR - (d\theta^2 + \sin^2 \theta d\varphi^2).$$

L'infini isotrope passé \mathcal{I}^- s'écrit alors comme

$$\mathcal{I}^- = \mathbb{R}_v \times \{0\}_R \times S_\omega^2.$$

L'horizon futur du trou noir est situé dans l'espace-temps physique à $r = 2m_+$ puisque nous avons assumé que la masse $m(u)$ admettait une limite finie notée m_+ quand $u \rightarrow +\infty$.

Ainsi, l'horizon futur \mathcal{H}^+ est décrit dans l'espace-temps conforme comme l'hypersurface :

$$\mathcal{H}^+ = \mathbb{R}_v \times \{1/2m_+\}_R \times S^2 .$$

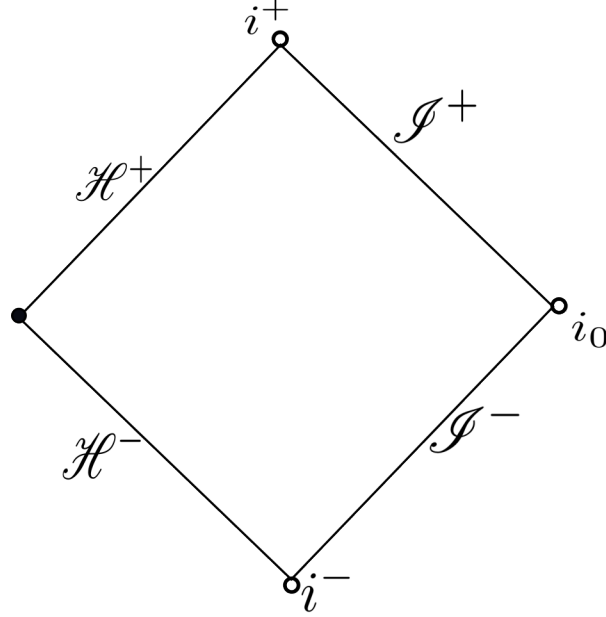


FIGURE 2 : Diagramme de Carter-Penrose de l'espace-temps de Schwarzschild avec i_0, i^+ et i^- qui sont des singularités du bord conforme.

Equation des ondes et flux d'énergie

La mention dans ce manuscrit de l'expression "équations des ondes scalaires" renvoie à l'équation qui régit l'évolution d'un champ scalaire sans masse, noté ϕ , dans un espace-temps donné (\mathcal{M}, g) :

$$\square_g \phi = 0, \tag{4}$$

avec $\square_g = \nabla_a \nabla^a$, où ∇_a est la connexion de Levi-Civita associée à la métrique g . Toutefois, dans un espace-temps de dimension 4, cette équation n'est pas conformément invariante et nous corrigeons cela en prenant en compte la courbure scalaire de la métrique g , notée Scal_g , de la façon suivante :

$$\left(\square_g + \frac{1}{6} \text{Scal}_g \right) \phi = 0. \tag{5}$$

Si nous considérons désormais la transformation conforme telle que $\hat{g} = \Omega^2 g$ et $\hat{\phi} = \Omega^{-1} \phi$, alors l'équation (5) est bien conformément invariante :

$$\left(\square_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\phi} = 0,$$

avec $\text{Scal}_{\hat{g}}$ la courbure scalaire de la métrique conforme \hat{g} et $\square_{\hat{g}} = \hat{\nabla}_a \hat{\nabla}^a$, avec $\hat{\nabla}$ la connexion associée à \hat{g} .

Dans cette thèse, les ondes scalaires sont considérées comme un champ test. Cela signifie que le tenseur énergie-impulsion associé aux ondes scalaires ϕ , donné par :

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \langle \nabla \phi, \nabla \phi \rangle,$$

avec

$$\langle \nabla \phi, \nabla \phi \rangle = \nabla_a \phi \nabla^a \phi,$$

n'est pas la source des équations d'Einstein. Les ondes étudiées ici se propagent donc dans l'espace-temps de Vaidya sans influencer sa géométrie. Il est important de noter que ce tenseur énergie-impulsion n'est pas conformément invariant. Ainsi, cela implique que si le tenseur énergie-impulsion T_{ab} des ondes ϕ dans l'espace-temps physique (\mathcal{M}, g) est sans divergence, cela n'est généralement pas vrai dans l'espace-temps compactifié $(\hat{\mathcal{M}}, \hat{g})$ pour le tenseur énergie impulsion conforme \hat{T}_{ab} associé au champ conforme $\hat{\phi} = \Omega^{-1} \phi$. Nous choisissons dans ce manuscrit de travailler avec le tenseur T_{ab} associé au champ ϕ et qui satisfait :

$$\nabla^a T_{ab} = 0.$$

L'inconvénient de ce choix est que désormais $\hat{\nabla}^a \hat{T}_{ab} \neq 0$, et cela fera apparaître des termes d'erreurs comme nous le verrons plus tard.

Les résultats obtenus dans les chapitres 4, 5 et 6 reposent sur des méthodes de champs de vecteurs, aussi appelées méthodes d'inégalités d'énergie. Ces méthodes nécessitent de calculer le flux d'énergie d'un champ mesuré par un observateur au travers d'une hypersurface. Soit V^a un champ de vecteur dans (\mathcal{M}, g) et soit T_{ab} le tenseur énergie impulsion associé à un champ ϕ , alors on définit le courant d'énergie J_a comme :

$$J_a = V^b T_{ab},$$

Le flux d'énergie mesuré par V^a au travers d'une hypersurface Σ vaut alors :

$$\mathcal{E}_{\Sigma, V}(\phi) = \int_{\Sigma} \star J_a|_{\Sigma},$$

avec $J_a|_{\Sigma}$ la restriction du courant d'énergie à l'hypersurface Σ . $\star J_a$ correspond au dual d'Hodge de la 1-forme J_a et est défini de la façon suivante. Soit x^0, x^1, x^2, x^3 un système de coordonnées dans lequel on exprime la métrique g , alors on définit l'élément de volume de (\mathcal{M}, g) comme la 4-forme notée $d\text{Vol}_g^4$ et donnée par :

$$d\text{Vol}_g^4 = \sqrt{|\det g|} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

Dans ce contexte, le dual d'Hodge de la 1-forme ω_a vaut :

$$\star \omega_a = g^{ab} \omega_a \lrcorner d\text{Vol}_g,$$

où g^{ab} est la métrique inverse.

Peeling et régularité asymptotique

Les chapitres 4 et 5 de cette thèse sont consacrés à l'étude de la propriété de *peeling* pour les ondes scalaires dans l'espace-temps de Vaidya. En d'autres termes, en se plaçant dans l'espace-temps compactifié conforme $(\hat{\mathcal{M}}, \hat{g})$ de l'espace-temps de Vaidya, on cherche à répondre à la question suivante : quelle est la classe de données initiales du champ conforme $\hat{\phi} = \Omega^{-1}\phi$ qui assure la régularité du champ conforme au bord à l'infini isotrope ? La méthode suivie dans cette thèse est celle développée par Mason et Nicolas dans [54] et qui consiste à caractériser la régularité du champ non pas en termes d'espaces \mathcal{C}^k mais plutôt en termes d'espaces d'énergie. Ces espaces sont construits sur une hypersurface Σ comme les espaces de Sobolev sur Σ et dont la norme est donnée par le flux d'énergie d'un champ ϕ mesuré par un observateur au travers de Σ . L'observateur choisi pour obtenir ces normes doit être transverse à l'infini isotrope, cela permet en effet un meilleur contrôle de la norme des dérivées du champ conforme et fournira des inégalités d'énergie plus précises. On choisit pour cela un vecteur purement temporel à l'infini isotrope (c'est-à-dire que sa norme est strictement positive à \mathcal{I}^{\pm}). Le vecteur choisi est un vecteur de "Morawetz"¹

1. Ce vecteur est obtenu à partir de l'expression d'un vecteur défini par Morawetz dans [60] dans l'espace-temps de Minkowski et on garde la même expression dans l'espace-temps de Schwarzschild ou de Vaidya

avec pour expression dans les coordonnées (u, R, θ, φ) :

$$V^a = u^2 \partial_u^a - 2(1 + uR) \partial_R^a,$$

Ce vecteur a pour propriété de n'être un vecteur de Killing² qu'à l'infini isotrope où $\Omega = 0$. Le courant d'énergie considéré pour le champ conforme $\hat{\phi}$ est alors donné³ par :

$$J_a = V^b \hat{T}_{ab},$$

avec

$$\hat{T}_{ab} = \hat{\nabla}_a \hat{\phi} \hat{\nabla}_b \hat{\phi} - \frac{1}{2} \hat{g}_{ab} \hat{\nabla}_c \hat{\phi} \hat{\nabla}^c \hat{\phi}.$$

Enfin, la méthode de Mason et Nicolas se concentre sur la régularité du champ conforme dans un voisinage de l'infini spatial i_0 . L'idée est d'éviter qu'une singularité du champ conforme en i_0 ne se forme en cas de décroissance trop lente de $\hat{\phi}$ et puisse ensuite se propager le long du bord conforme du compactifié.

L'idée générale des chapitres 4 et 5 est d'obtenir des équivalences d'énergies entre l'énergie associée aux données initiales du champ conforme sur une hypersurface des données initiales Σ_0 et l'énergie associée au champ conforme à \mathcal{I}^\pm , le tout en se plaçant dans un voisinage arbitrairement proche de i_0 . La principale difficulté provient du fait que le courant d'énergie J_a n'est pas parfaitement conservé dans (\hat{M}, \hat{g}) :

$$\hat{\nabla}^a J_a = \hat{\nabla}^{(a} V^{b)} \hat{T}_{ab} + V^b \hat{\nabla}^a \hat{T}_{ab} \neq 0.$$

Le premier terme du membre de droite ne s'annule que si le vecteur V est un vecteur de Killing et ne s'annule donc ici qu'au bord du compactifié, le second terme lui ne s'annule pas étant donné que le tenseur énergie-impulsion des ondes conformes n'est pas sans divergence. Ces termes induisent des *termes d'erreurs*, dans la loi de conservation d'énergie, obtenue par application du théorème de Stokes :

$$\mathcal{E}_{S_1, V}(\hat{\phi}) - \mathcal{E}_{S_2, V}(\hat{\phi}) = \int_{\mathcal{B}} \nabla^a J_a d\text{Vol}_{\hat{g}}^4,$$

2. Un vecteur K est dit vecteur de Killing si il génère un groupe qui laisse la métrique invariante. De plus il vérifie $\nabla_{(a} K_{b)} = 0$.

3. Par souci de simplicité il existe des différences de notation entre cette introduction et les notations dans le corps du manuscrit. En effet, des conventions différentes sont choisies et on désigne les quantités conformes sans "•".

où \mathcal{B} représente un domaine délimité par deux hypersurfaces \mathcal{S}_1 et \mathcal{S}_2 . Ces termes d'erreurs, notés $\text{Err}(\hat{\phi})$ nécessitent d'être contrôlés afin d'obtenir les inégalités d'énergie entre $\mathcal{E}_{\Sigma_0, V}(\hat{\phi})$ et $\mathcal{E}_{\mathcal{I}^\pm, V}(\hat{\phi})$. Cela se fait en trois temps :

1. on commence par feuilleter le domaine sur lequel on souhaite établir nos lois de conservation d'énergie par des hypersurfaces \mathcal{H}_s de type spatial de sorte à ce que \mathcal{H}_0 corresponde à l'infini isotrope dans un voisinage de i_0 et \mathcal{H}_1 corresponde à Σ_0 dans un voisinage de i_0 .
2. on montre ensuite qu'on peut majorer l'intégrale des termes d'erreurs sur chaque hypersurface \mathcal{H}_s par l'énergie du champ conforme sur cette même surface \mathcal{H}_s , notée $\mathcal{E}_{\mathcal{H}_s, V}(\hat{\phi})$. En d'autres termes, il existe une constante positive C telle que dans un voisinage de l'infini spatial on a :

$$\int_{\mathcal{H}_s} \text{Err}(\hat{\phi}) \, dud\omega|_{\mathcal{H}_s} \leq C \mathcal{E}_{\mathcal{H}_s, V}(\hat{\phi}),$$

avec $d\omega = \sin^2 \theta d\theta d\varphi$.

3. en appliquant le lemme de Grönwall sur la loi de conservation de l'énergie, on obtient l'équivalence entre la norme de l'espace d'énergie des données initiales associées à $\hat{\phi}$ et la norme du champ conforme à \mathcal{I} dans un voisinage de i_0 .

Une fois ces équivalences obtenues pour $\hat{\phi}$, on cherche à contrôler les dérivées successives de $\hat{\phi}$ de la même manière. L'application des dérivées ∂_u, ∂_R et ∂_ω sur la loi de conservation d'énergie produit alors de nouveaux termes d'erreurs que l'on contrôle de façon similaire à ce qui a été expliqué précédemment. On montre alors le résultat principal des chapitres 4 et 5 : l'énergie des dérivées successives de degré total d du champ conforme $\hat{\phi}$ à l'infini isotrope (passé ou futur) est majorée (à une constante multiplicative près) par l'énergie des dérivées d'ordre total inférieur ou égal à d des données initiales sur Σ_0 . Des inégalités dans l'autre sens sont également obtenue de la même manière et on montre également que la classe de données initiales qui permet d'obtenir de telles équivalences est aussi large que celle qui vérifie les propriétés de peeling dans l'espace-temps de Schwarzschild et de Minkowski. La seule différence qui se manifeste par rapport aux cas des espaces-temps de Schwarzschild et de Minkowski est que le contrôle de la régularité des dérivées d'ordre supérieur du champ conforme nécessite désormais l'utilisation d'une combinaison de dérivées transversales et tangentes (respectivement ∂_R et ∂_u) du champs conforme. Cette différence provient du fait que désormais la métrique, le d'Alembertien $\square_{\hat{g}}$

et la courbure scalaire dépendent de R (comme pour la métrique de Schwarzschild) mais aussi de u . L'application des dérivées successives ∂_u fait donc apparaître des termes d'erreurs qui n'existaient pas lorsque la métrique était statique. Cependant, cela n'entraîne aucune perte de régularité globale.

Peeling dans le passé

Il faut noter qu'il existe une différence entre le contrôle du champ dans le futur et le contrôle du champ dans le passé. Cette différence ne se manifestait pas dans les espaces-temps statiques comme celui de Minkowski et celui de Schwarzschild car il y avait une symétrie entre l'expression des coordonnées u et v . Par exemple, dans l'espace-temps de Schwarzschild, on a :

$$\begin{aligned} u &= t - r_\star, \\ v &= t + r_\star, \end{aligned}$$

avec r_\star la coordonnée de la tortue, définie par $r_\star = r + 2M \log(r - 2M)$ comme évoqué précédemment. Cette coordonnée est obtenue dans la métrique de Schwarzschild en résolvant :

$$dr_\star = \frac{dr}{1 - \frac{2M}{r}}.$$

Dans le cas de la métrique de Vaidya, comme désormais la masse dépend du temps retardé u , la coordonnée de la tortue, notée \tilde{r} , satisfait :

$$d\tilde{r} = \frac{dr}{1 - \frac{2m(u)}{\tilde{r}}},$$

et il n'est pas possible de donner une expression explicite de \tilde{r} . Par ailleurs, on avait la relation suivante entre u, v et r dans la métrique de Schwarzschild :

$$dv = du + \frac{2}{1 - \frac{2M}{r}} dr.$$

Dans l'espace-temps de Vaidya, on montre que cette relation devient :

$$dv = \psi du + \frac{2\psi}{1 - \frac{2m(u)}{r}} dr,$$

avec ψ une fonction positive qui vérifie l'équation aux dérivées partielles suivante :

$$\frac{\partial\psi}{\partial u} - \frac{F(u, r)}{2} \frac{\partial\psi}{\partial r} + \frac{2m'(u)}{Fr} \psi = 0,$$

et on pose $\psi = 1$ à l'infini isotrope passé \mathcal{I}^- (cf. [15] pour plus de détails).

Ainsi, dans l'espace-temps de Schwarzschild, la classe de données initiales qui assurait le peeling des ondes scalaires dans le futur était immédiatement la même que celle qui assurait le peeling dans le passé. Cette équivalence n'est plus évidente pour la métrique de Vaidya. Nous montrons dans le Chapitre 5 comment résoudre cette difficulté en nous plaçant dans les coordonnées (v, R, θ, φ) telles que définies dans [15]. Il apparaît alors que le vecteur de Morawetz V^a exprimé dans les coordonnées (v, R, θ, φ) :

$$V^a = v^2 \partial_v^a + 2(1 - vR) \partial_R^a,$$

ne permet pas d'obtenir une décroissance suffisamment rapide dans les termes d'erreur pour que ceux-ci puissent être contrôlés par l'énergie du champ conforme sur les hypersurfaces \mathcal{H}_s . Le vecteur V^a doit alors subir une légère modification afin de garantir la décroissance des termes d'erreurs à l'infini isotrope passé et ainsi permettre d'établir la propriété de peeling des ondes scalaires à \mathcal{I}^- . Il devient alors :

$$V^a = \psi v^2 \partial_v^a + 2(1 - vR) \partial_R^a.$$

Le reste de la méthode est ensuite complètement identique à ce qui a été réalisé dans le futur, et les résultats obtenus sont identiques.

Scattering Conforme

Le chapitre 6 est dédié à la construction de l'opérateur de scattering conforme pour les ondes scalaires dans l'espace-temps de Vaidya. Une théorie du scattering permet de caractériser l'évolution complète d'un champ, solution d'une équation de propagation, par un opérateur de scattering qui associe le comportement asymptotique du champ dans le passé à son comportement asymptotique dans le futur. L'intérêt principal provient du fait que généralement, le comportement asymptotique d'un champ est plus simple à décrire que le comportement de ce même champ près de la source de rayonnement. L'idée est de s'appuyer sur les méthodes conformes pour construire un tel opérateur. Ainsi, l'opérateur

de scattering conforme associe la trace du champ conforme sur le bord passé de $(\hat{\mathcal{M}}, \hat{g})$ à sa trace sur le bord futur du compactifié conforme. En plus de montrer l'existence d'un tel opérateur, on souhaite s'assurer que la solution est entièrement caractérisée par son comportement asymptotique passé ou futur. La construction d'un tel opérateur nécessite de connaître les lois de décroissance du champ conforme au voisinage des infinis temporels (passé et futur). Si de telles lois sont connues pour les ondes scalaires dans l'espace-temps de Schwarzschild, ce n'est pas actuellement le cas pour l'espace-temps de Vaidya. C'est pourquoi, dans le chapitre 6, nous considérons que la variation de la masse $m(u)$ se fait sur un intervalle de la forme $[u_-, u_+]$ avec $-\infty < u_- < u_+ < +\infty$. Cela implique que la métrique est la métrique de Schwarzschild au voisinage de i_+ et i_- , et nous pouvons alors utiliser les résultats de décroissance existants pour construire l'opérateur de scattering conforme. La méthode de construction est la suivante :

1. On définit le bord passé (resp. futur) de notre espace-temps conforme comme $\mathcal{H}^- \cup \mathcal{I}^-$ (resp. $\mathcal{H}^+ \cup \mathcal{I}^+$) et on feuillette l'espace-temps conforme par des hypersurfaces Σ_t qui ont pour propriété d'être transverses à la fois à l'infini isotrope passé (resp. futur) et à l'horizon passé (resp. futur). Ensuite, on définit des espaces d'énergie sur ces hypersurfaces \mathcal{S} comme l'espace des fonctions lisses à support compact sur \mathcal{S} complété par la norme donnée par le flux d'énergie mesuré par un observateur V^a au travers de \mathcal{S} . Contrairement au peeling où l'on souhaitait avoir un contrôle suffisamment fort de l'énergie de $\hat{\phi}$ au bord, on choisit un vecteur temporel qui n'est pas transverse à l'infini isotrope : $V = \partial_u$.
2. On construit ensuite l'opérateur de trace futur, noté \mathcal{T}^+ , qui associe, aux données initiales lisses et à support compact, les données de scattering futur, c'est-à-dire la restriction de la solution $\hat{\phi}$ sur le bord futur. Nous montrons en utilisant des inégalités d'énergie, que cet opérateur s'étend comme un opérateur borné, linéaire, injectif et d'image fermée entre les espaces d'énergie définis auparavant.
3. Afin de prouver que l'opérateur de trace est un isomorphisme, il reste à prouver que l'image de l'opérateur est dense. Nous démontrons cela en nous fondant sur les résultats de Hörmander [39] et de Nicolas [65], et en résolvant un problème de Goursat pour les données de scattering sur le bord conforme futur $\mathcal{H}^+ \cup \mathcal{I}^+$. Nous montrons alors que l'opérateur \mathcal{T}^+ s'étend comme un isomorphisme entre les espaces d'énergies considérés.

-
4. En appliquant la même méthode pour définir l'opérateur de trace passé \mathcal{T}^- , nous obtenons des résultats similaires. Enfin, nous construisons l'opérateur de scattering \mathcal{S} de la façon suivante :

$$\mathcal{S} := \mathcal{T}^+(\mathcal{T}^-)^{-1},$$

et cet opérateur s'étend comme un isomorphisme entre l'espace d'énergie des données de scattering sur le bord passé et l'espace d'énergie des données de scattering sur le bord futur du compactifié conforme.

Métrie de Robinson-Trautman

Le chapitre 7 est consacré à l'analyse deuxième type d'espace-temps purement radiatif étudié dans cette thèse : la métrie de Robinson-Trautman purement radiative. Cette solution des équations d'Einstein peut être vue comme une généralisation de la métrie de Vaidya. Désormais, la métrie dépend de deux fonctions $m(u)$ et $P(u, \theta, \varphi)$ qui sont reliées à la densité de rayonnement ρ via une équation différentielle d'ordre 4. Du fait de la très grande généralité des solutions de Robinson-Trautman, nous faisons dans cette thèse les hypothèses suivantes : nous supposons que la métrie est de type D dans la classification de Petrov et que la courbure gaussienne sur la sphère ne dépend que du temps retardé u . En introduisant les coordonnées stéréographiques complexes ξ et $\bar{\xi}$:

$$\xi = \cot \theta e^{i\varphi},$$

cela revient à considérer que la fonction P se comporte de la façon suivante :

$$P(u, \xi, \bar{\xi}) = A(u) + B(u)\xi + \bar{B}(u)\bar{\xi} + C(u)\xi\bar{\xi},$$

où A et C sont des fonctions réelles et B est une fonction complexe. Après un rappel de la construction de l'espace-temps de Robinson-Trautman en utilisant le formalisme de coefficients de spins, on donne dans le chapitre 7 les propriétés géométriques de la solution de Robinson-Trautman. Contrairement à l'espace-temps de Vaidya, il n'est pas aisé d'interpréter directement la signification physique de m et de P . En effet, les coordonnées sphériques usuelles (u, r, θ, φ) ne forment pas des coordonnées de Bondi pour la métrie de Robinson-Trautman puisqu'ils ne satisfont pas les conditions asymptotiques nécessaires. On rappelle alors les résultats obtenus par Von der Gönna et Kramer dans [29] et par Cornish et Mickelwright dans [13] et on effectue le changement de variable

qui permet de définir la masse de Bondi de la solution de Robinson-Trautman purement radiative.

On cherche ensuite à étudier l'horizon passé en suivant la même méthode que celle définie pour la métrique de Vaidya. On considère donc les directions isotropes entrantes et on en étudie les courbes intégrales. Toutefois, du fait de la géométrie de la métrique de Robinson-Trautman, les directions isotropes entrantes ne sont pas orthogonales à des hypersurfaces et les courbes intégrales de ces directions isotropes ne sont pas des géodésiques. La méthode qui fonctionnait dans le cadre de la métrique de Vaidya ne permet pas d'étudier l'horizon passé dans la solution de Robinson-Trautman purement radiative.

Le cadre géométrique est le suivant : soit \tilde{P} la fonction :

$$\tilde{P} = \frac{P(u, \xi, \bar{\xi})}{1 + \xi\bar{\xi}},$$

on suppose que $\tilde{P} > 0$ et $\partial_u \tilde{P} > 0$. De plus, on considère que la fonction \tilde{P} se comporte comme :

$$\tilde{P}(u, \xi, \bar{\xi}) = \begin{cases} 1 & \forall u \in]-\infty, u_-], \\ \tilde{P}(u, \xi, \bar{\xi}) & \forall u \in [u_-, u_+], \\ \tilde{P}_+ & \forall u \in [u_+, +\infty[, \end{cases},$$

avec $-\infty \leq u_- < u_+ \leq +\infty$ et $1 < \tilde{P}_+ < +\infty$. On montre en outre, que la masse de Bondi M_B a les limites suivantes :

$$\lim_{u \rightarrow -\infty} M_B(u) = m_-, \quad \lim_{u \rightarrow +\infty} M_B(u) = M_B^+, \quad M_B'(u) \geq 0 \quad \forall u \in \mathbb{R}.$$

En d'autres termes, on étudie un espace-temps qui est l'espace-temps de Vaidya sur $] -\infty, u_-]$ avec une masse $m(u)$ qui converge vers un autre espace-temps de Vaidya sur $[u_+, +\infty[$ dont la masse est donnée par $m(u)/\tilde{P}_+^3$. Sur l'intervalle $[u_-, u_+]$, la métrique est celle de Robinson-Trautman, la fonction \tilde{P} n'est plus constante et la métrique perd sa symétrie sphérique.

Finalement, nous classifions les courbes intégrales⁴ $R(u, \xi, \bar{\xi})$ des directions isotropes entrantes en étudiant l'équation différentielle ordinaire qui les gouverne. Nous observons alors que le comportement des solutions est similaire à celui observé dans l'espace-temps de Vaidya, c'est-à-dire : il existe une unique solution qui admet une limite finie lorsque $u \rightarrow -\infty$. On appelle cette solution R_h et elle a pour limite :

4. avec $R = r/\tilde{P}$.

$$\lim_{u \rightarrow -\infty} R_h(u, \xi, \bar{\xi}) = 2m_- .$$

R_h existe pour tout $u \in \mathbb{R}$ et converge dans le futur vers :

$$\lim_{u \rightarrow +\infty} R_h(u, \xi, \bar{\xi}) = 2M_B^+ .$$

Les autres solutions $R(u, \xi, \bar{\xi})$ peuvent être classées dans les deux catégories suivantes :

1. R existe $\forall u \in \mathbb{R}$, sur l'ensemble de son domaine d'existence $R > R_h$, et R a pour limites :

$$\lim_{u \rightarrow -\infty} R(u, \theta) = +\infty, \quad \lim_{u \rightarrow +\infty} R(u, \theta) = \frac{2m_+}{\tilde{P}_+^3} .$$

2. R existe sur un intervalle de la forme $[u_0, +\infty[$ avec $u_0 > -\infty$. De plus, pour tout $u \in [u_0, +\infty[$, on a $R < R_h$ et R a pour limites :

$$\lim_{u \rightarrow u_0} R(u, \theta) = 0, \quad \lim_{u \rightarrow +\infty} R(u, \theta) = \frac{2m_+}{\tilde{P}_+^3} .$$

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PART I

Introduction and definitions

INTRODUCTION

The results presented in this thesis deal with conformal analysis on two kinds of radiative spacetimes, that are solutions to the Einstein equation with matter : the Vaidya and the Robinson-Trautman spacetimes. The Einstein theory began in 1905 with special relativity as a resolution of an incompatibility between two fundamental theories of classical physics : the classical mechanics of the 17th century, and the electromagnetic laws, unified by Maxwell and presented in their modern version by Heaviside in 1884. This can be exposed in the following way : let \mathcal{R} and \mathcal{R}' , be two frames that are in uniform rectilinear motion relative to each other. Let (t, x, y, z) and (t', x', y', z') be the coordinates respectively associated to \mathcal{R} and \mathcal{R}' . Assuming that \mathcal{R}' translates with a constant velocity v with respect to \mathcal{R} along the x-axis, then, classical mechanic obeys to Galilean transformation between frames :

$$\begin{cases} t' = t, \\ x' = x - vt, \\ y' = y, \\ z' = z. \end{cases}$$

This entails that the velocity of a particle in \mathcal{R}' is obtained from the law of composition of velocities, i.e. as the sum of the velocity in \mathcal{R} and the translation speed between frames. On the other hand, electromagnetism is governed by the Lorentz transformations :

$$\begin{cases} t' = \gamma \left(t - \frac{vx}{c^2} \right), \\ x' = \gamma(x - vt), \\ y' = y, \\ z' = z. \end{cases}$$

With :

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}},$$

and c is the light celerity. Moreover, in the Maxwell equations, light propagates in a vac-

uum with a constant velocity, regardless of the reference frame. This is in stark contrast to classical mechanics, where light is assumed to propagate instantaneously. Thus, this creates an incompatibility between these two formalisms from a theoretical perspective. Physicists' experiments also indicate that light behaves as an electromagnetic wave with a finite constant speed, while classical mechanics accurately predicts the movement of massive objects. Special relativity and Einstein resolved this by adopting the electromagnetic perspective through two fundamental postulates:

1. Physics laws are consistent in every inertial reference frame.
2. There exists a finite speed of information transmission between all these frames, which is constant and the same in every inertial reference frame. This speed is known as the speed of light, denoted as c .

This resulted in the rejection of classical concepts from Newtonian mechanics, including the absolute nature of time and the Galilean transformation law between Galilean frames. The notion of simultaneity also fades away, as a speed limit now exists for the transmission of information. Classical mechanics remains a valid approximation when the speed v is significantly smaller than c .

This theory is generalized in 1915 with general relativity that describes the deformation of space and time in the universe, in terms of the energy and matter distribution expressed by T_{ab} , a tensor of rank 2. Space and time are encoded in a smooth manifold denoted by \mathcal{M} of dimension 4 equipped with a Lorentzian metric (i.e. an object that allows to compute distance in space or time in curved geometries, for more details see section 1.1 where a precise definition of the metric is given), called g_{ab} of signature $(+, -, -, -)$. The curvature of the spacetime (\mathcal{M}, g) is now described using the Riemann tensor¹ R_{abcd} and the Ricci tensor R_{ab} . Then, the Einstein equation reads :

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab},$$

with R being the scalar curvature, obtained from the Ricci tensor :

$$R = \text{Tr}(R_{ab}) = g^{ab}R_{ab}.$$

In the remainder of this thesis, we will use geometrical notations, with units chosen so

1. definitions are given in Section 1.1.

that :

$$G = c = 1 .$$

Several solutions of Einstein's equations exist, beginning by the Minkowski solution, that is the geometrical formulation of special relativity, also called flat spacetime. Here the manifold is \mathbb{R}^4 and the metric, in cartesian coordinates (t, x, y, z) is :

$$\eta = dt^2 - dx^2 - dy^2 - dz^2 .$$

It is also useful to give the expression in terms of spherical coordinates (t, r, θ, φ) :

$$\eta = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) .$$

This model can be understood as the metric of a topologically trivial spacetime, without any curvature, i.e. $R_{abcd} = R_{ab} = 0$ in the vacuum defined by $T_{ab} = 0$. Another vacuum solution to the Einstein equation is the one obtained in 1916 by Karl Schwarzschild [86]. It describes the exterior of a static, spherical star, of constant mass $m > 0$, its expression in spherical coordinates is :

$$g_{\text{Sch}} = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) .$$

Note that this solution has two singularities, i.e., two points where components of the metric become infinite. The first singularity occurs at $r = 0$, and the second is located at $r = 2m$. These two singularities are of different nature; the first one is independent of the coordinate choice and the scalar curvature diverges there. On the other hand, the second singularity comes from an inappropriate choice of coordinate basis. The usual spherical coordinates (t, r, θ, φ) are not well-suited for expressing this metric. This lead Eddington and Finkelstein to introduce new coordinates independently in 1924 and 1958 in [19] and [20]. These coordinates denoted by u and v known as the retarded and the advanced time, resemble those used in electromagnetic theory to describe field propagation between a source and an observer. In the flat case, these expressions are quite similar and read as follows:

$$\begin{aligned} u &= t - r , \\ v &= t + r . \end{aligned}$$

In the Schwarzschild spacetime, these definitions do not apply, and we require more complex expressions involving the mass m of the black hole. We will discuss this in section 2.1.2. Now using coordinates (v, r, θ, φ) or (u, r, θ, φ) as the basis, the Schwarzschild solution is respectively interpreted as a spacetime with a static black hole (white hole) of constant mass, located at $r = 0$ and an event horizon at $r = 2m$ from which physical particles cannot escape (or enter) if their velocity is less than or equal to the speed of light. According to Birkhoff's theorem, stated in 1923 in [6], the static Schwarzschild metric describes the exterior of any spherically symmetric solution of the vacuum field equations ($T_{ab} = 0$). Another vacuum solution is the Kerr solution, published in 1963, [43], which describes the geometry of a spinning, rotating, uncharged axially symmetric black hole in a vacuum.

As alluded to earlier, the aim of this thesis is to study radiative spacetimes; in other words, we focus on solutions to the Einstein equations where $T_{ab} \neq 0$. Before defining radiative spacetimes, let's recall some useful geometric definitions in general relativity.

1.1 Basic definitions

1.1.1 Manifolds, vectors and tensors

Definition 1.1.1. Smooth differential manifold, Wald [94] :

A n -dimensional smooth manifold, or a C^∞ -manifold, is a set \mathcal{M} , and a collection of subsets $\{O_\alpha\}$ of \mathcal{M} satisfying the following properties :

1. *Each point $p \in \mathcal{M}$ lies in at least one O_α , i.e. the collection of $\{O_\alpha\}$ covers \mathcal{M} .*
2. *For each α , there is a map (called a chart or a coordinate system) $\psi_\alpha : O_\alpha \rightarrow U_\alpha$ that is a diffeomorphism and where U_α is an open subset of \mathbb{R}^n .*
3. *If any two sets O_α and O_β overlap, i.e. $O_\alpha \cap O_\beta \neq \emptyset$, we can consider the map :*

$$\psi_\beta \circ \psi_{\alpha^{-1}} : \psi_\alpha(O_\alpha \cap O_\beta) \longrightarrow \psi_\beta(O_\alpha \cap O_\beta),$$

where

$$\psi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha \subset \mathbb{R}^n, \psi_\beta(O_\alpha \cap O_\beta) \subset U_\beta \subset \mathbb{R}^n,$$

with ψ_α and ψ_β two smooth functions (i.e. infinitely continuously differentiable functions).

Definition 1.1.2. Tangent vector and tangent space, Wald [94] *Let \mathcal{M} , be a n -dimensional smooth manifold and let \mathcal{F} be the collection of smooth functions from \mathcal{M} to \mathbb{R} . We define a tangent vector v at point $p \in \mathcal{M}$ to be a map :*

$$v : \mathcal{F} \longrightarrow \mathbb{R},$$

that obeys to :

1. $v(af + bg) = av(f) + bv(g), \forall f, g \in \mathcal{F}; a, b \in \mathbb{R}$,
2. $v(fg) = f(p)v(g) + g(p)v(f)$.

The collection of tangent vector at a point p , called the tangent space of \mathcal{M} at p , denoted by $T_p\mathcal{M}$, has the structure of a vector space under

1. the addition law : $(v_1 + v_2)(f) = v_1(f) + v_2(f)$,
2. the scalar multiplication law : $(av)(f) = av(f)$

and has the following property (see [94] for more details):

$$\dim T_p\mathcal{M} = n.$$

We also define the cotangent space, denoted by $T_p^*\mathcal{M}$, the dual of $T_p\mathcal{M}$, i.e. the space of continuous linear forms acting on $T_p\mathcal{M}$.

Definition 1.1.3. Vector field and 1-form

A vector field on a manifold \mathcal{M} is an assignment of a tangent vector $v(p) \in T_p\mathcal{M}$ at each point $p \in \mathcal{M}$. A 1-form is defined in the same way, respectively with cotangent space $T_p^\mathcal{M}$.*

Definition 1.1.4. Tensor, Wald [94]

Let V be a finite dimensional vector space and let V^ be its dual. Then a tensor T of type (k, l) over V is the multilinear map :*

$$T : \underbrace{V^* \times \cdots \times V^*}_k \times \underbrace{V \times \cdots \times V}_l \longrightarrow \mathbb{R}$$

Such a tensor is said to be k -times contravariant and l -times covariant. The space of tensors of type (k, l) is denoted by $\mathcal{T}(k, l)$.

Definition 1.1.5. Tensor product

Let $T \in \mathcal{T}(k, l)$ and $T' \in \mathcal{T}(k', l')$. Then one can construct another tensor $T \otimes T' \in \mathcal{T}(k+k', l+l')$ with \otimes that is called the tensor product in the following manner: given $(k+k')$ dual vectors $v^1, \dots, v^{k+k'}$ and $(l+l')$ vectors $w_1, \dots, w_{l+l'}$, then we define $T \otimes T'$ acting on these vectors to be the product of $T(v^1, \dots, v^k; w_1, \dots, w_l)$ and $T'(v^{k+1}, \dots, v^{k+k'}; w_{l+1}, \dots, w_{l+l'})$.

Introducing $\{v_\mu\}$ a basis of V and $\{v^\nu\}$ its dual basis, we can decompose T as :

$$T = \sum_{\mu_1, \dots, \nu_l=1}^n T_{\nu_1 \dots \nu_l}^{\mu_1 \dots \mu_k} v_{\mu_1} \otimes \dots \otimes v^{\nu_l},$$

where $T_{\nu_1 \dots \nu_l}^{\mu_1 \dots \mu_k}$ are referred to the *components* of the tensor T with respect to the basis $\{v_\mu\}$. The transformation law of a tensor between two coordinate systems (x^μ) and $(x'^{\mu'})$ of the tensor components reads :

$$T'_{\nu'_1 \dots \nu'_l}^{\mu'_1 \dots \mu'_k} = \sum_{\mu_1, \dots, \nu_l=1}^n T_{\nu_1 \dots \nu_l}^{\mu_1 \dots \mu_k} \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\nu_l}}{\partial x'^{\nu'_l}}.$$

Abstract index notation

The idea of abstract index notation, given by Penrose in [74] is to use the advantage of the concrete indices notation, like the Einstein summation convention, without having to refer to a basis. In this formalism, a tensor of type (k, l) will be denoted by a letter (say T) followed by k contravariant and l covariant lower-case latin indices : $T_{b_1 \dots b_l}^{a_1 \dots a_k}$. It is important to note that this is an intrinsic manner to express tensor fields and this does not refer to a collection of components. Contraction denotes the action of a 1-form on a vector, e.g. $\alpha_a V^a$ denotes $\alpha(V)$. For more details, see [74].

Metric and connection

In classical physics there is no difficulty to measure distance and time in a frame, because these are absolute values, independent of the referential. In general relativity, this is more delicate, because space and time are curved, hence they are not the same at any point of the universe. In Riemann's geometry, distances (in terms of space or time), scalar product, etc. are encoded in a tensor of type $(0, 2)$ called the metric and defined as follows :

Definition 1.1.6. Metric

Let \mathcal{M} be a smooth manifold of dimension n . A metric on \mathcal{M} is a tensor field of type $(0, 2)$ that is :

1. symmetric, i.e. let $v_1, v_2 \in T_p \mathcal{M}$, then $g(v_1, v_2) = g(v_2, v_1)$,
2. non-degenerate, i.e. let $v_1 \in T_p \mathcal{M}$, if $g(v, v_1) = 0 \forall v \in T_p \mathcal{M}$, then $v_1 = 0$.

We shall assume a metric on \mathcal{M} to be at least continuous on \mathcal{M} .

Associated to the metric, we can define an intrinsic derivative operator, called the connection, denoted by ∇ and defined by :

Definition 1.1.7. Connection :

Let $T \in \mathcal{T}$, a connection ∇ is a map :

$$\begin{aligned} \nabla : \mathcal{T}(k, l) &\longrightarrow \mathcal{T}(k, l + 1) \\ T &\longmapsto \nabla_a T, \end{aligned}$$

that satisfies :

1. linearity, for $A, B \in \mathcal{T}(k, l)$ and $\alpha, \beta \in \mathbb{R}$,

$$\nabla(\alpha A + \beta B) = \alpha \nabla A + \beta \nabla B.$$

2. Leibnitz rule : $\forall A \in \mathcal{T}(k, l), B \in \mathcal{T}(k', l')$:

$$\nabla(A \otimes B) = (\nabla A) \otimes B + A \otimes (\nabla B).$$

3. Commutativity with contraction : $\forall A \in \mathcal{T}(k, l)$:

$$\nabla_d (A_{b_1 \dots c \dots b_l}^{a_1 \dots c \dots a_k}) = \nabla_d A_{b_1 \dots c \dots b_l}^{a_1 \dots c \dots a_k}.$$

4. Consistency with the notion of tangent vectors as directional derivatives on scalar fields : $\forall f \in \mathcal{F}$ and $\forall v \in T_p \mathcal{M}$:

$$v(f) = v^a \nabla_a f.$$

Theorem 1.1.1. There exists an unique connection ∇_a such that :

1. it is torsion-free, i.e. for any scalar field f : $\nabla_a \nabla_b f = \nabla_b \nabla_a f$.
2. it commutes with the metric (and the inverse metric) $\nabla_a g_{bc} = 0$ and $\nabla_a g^{bc} = 0$.

This connection is called the Levi-Civita connection.

1.1.2 Maps on manifolds and Lie derivative

Let \mathcal{M} and \mathcal{N} be two manifolds, not necessarily of the same dimension. Consider $\phi : \mathcal{M} \rightarrow \mathcal{N}$ to be a \mathcal{C}^∞ map, and let v be a tangent vector at $p \in \mathcal{M}$. We define the map ϕ^* that *carries along* tangent vectors at p to tangent vectors at $\phi(p) \in \mathcal{N}$ in the following way:

$$\begin{aligned} \phi^* : T_p \mathcal{M} &\longrightarrow T_{\phi(p)} \mathcal{N}, \\ v &\longmapsto \phi^* v, \end{aligned}$$

and for every smooth function $f : \mathcal{N} \rightarrow \mathbb{R}$, we have:

$$(\phi^* v)(f) = v(f \circ \phi).$$

Similarly, we can define ϕ_* to *pull back* dual vectors at $\phi(p)$ to dual vectors at p :

$$\begin{aligned} \phi_* : T_{\phi(p)}^* \mathcal{N} &\longrightarrow T_p^* \mathcal{M}, \\ \mu &\longmapsto \phi_* \mu, \end{aligned}$$

and this acts for all $v^a \in T_p \mathcal{M}$ as :

$$(\phi_* \mu)_a v^a = \mu_a (\phi^* v)^a.$$

Now, let $\phi : \mathcal{M} \rightarrow \mathcal{N}$ be a diffeomorphism, i.e., a \mathcal{C}^∞ map that is one-to-one, onto, and has a \mathcal{C}^∞ inverse. This implies that \mathcal{M} and \mathcal{N} are of the same dimension. Then we can define the action of ϕ^* on a tensor of type (k, l) at p by considering that $\phi_* = (\phi^{-1})^*$ (for more details, see [94]):

$$(\phi^* T)_{a_1 \dots a_l}^{b_1 \dots b_k} (\mu_1)_{b_1} \dots (\mu_k)_{b_k} (t_1)^{a_1} \dots (t_l)^{a_l} = T_{a_1 \dots a_l}^{b_1 \dots b_k} ((\phi^{-1})^* \mu_1)_{b_1} \dots ((\phi^{-1})^* t_l)^{a_l}.$$

Let us define on \mathcal{M} a one-parameter group of diffeomorphisms, denoted by ϕ_t , and generated by a vector field $v \in T_p \mathcal{M}$. As defined above, we can use ϕ_t^* to carry along a smooth tensor field T of type (k, l) . This leads to the definition of the Lie derivative with respect to v , denoted by \mathcal{L}_v , such that:

$$\mathcal{L}_v T = \lim_{t \rightarrow 0} \left(\frac{\phi_t^* T - T}{t} \right),$$

where all the tensor are evaluating at the same point $p \in \mathcal{M}$.

The Lie derivative is independent of the choice of connection and measures the change of a tensor field T of type (k, l) along the flow ϕ_t defined by another vector field v . We do not go further into the details concerning the Lie derivative; for an extensive survey of this notion, we refer to [35]. We state the following useful proposition that established the Lie derivative of the metric along a vector field V :

Proposition 1.1.1. *The Lie derivative of the metric along a vector field V^a is given by*

$$\mathcal{L}_V g_{ab} = g_{cb} \nabla_a V^c + g_{ac} \nabla_b V^c = 2\nabla_{(a} V_{b)},$$

Geodesics and light cones

In classical physics the scalar product of two vectors is still positive, but here, in Riemann's geometry, it could be positive, negative or zero. According to our choice of signature, the squared norm of a vector $v \in T_p \mathcal{M}$, defined by :

$$\langle v, v \rangle = g(v),$$

gives the type of the vector. So v is :

- timelike if its squared norm is positive,
- spacelike if its squared norm is negative,
- null if its squared norm is zero.

We are now able to generalize the notion of physical curves in Newtonian mechanics. This is done by introducing geodesics that are an application of the Fermat principle in curved geometry in the sense that a geodesic can be understood as the lines that curve the least as possible. Naively, a geodesic is to a curved geometry what the straight line is to plane geometry, i.e. the curved version of the free-falling trajectory.

Definition 1.1.8. *Let γ be a curve on a spacetime (\mathcal{M}, g) equipped with the Levi-Civita connection ∇ . Let s be the parameter of the curve. Let $t^a \in T_p \mathcal{M}$ be the tangent vector to γ at each point. Then γ is a geodesic if its tangent vector is parallel propagated along itself, i.e. if :*

$$t^a \nabla_a t^b = \alpha t^b, \quad \alpha \in \mathbb{R}.$$

If $\alpha = 0$, s is said to be an affine parameter of the geodesic. Geodesics are also classified, depending on the squared norm of the tangent vector t^a :

1. if $g(t, t) > 0$, γ is said to be *timelike*.
2. if $g(t, t) < 0$, γ is said to be *spacelike*.
3. if $g(t, t) = 0$, γ is said to be *null or lightlike*.

This classification takes a lot of physical meaning by considering the light cone associated to it :

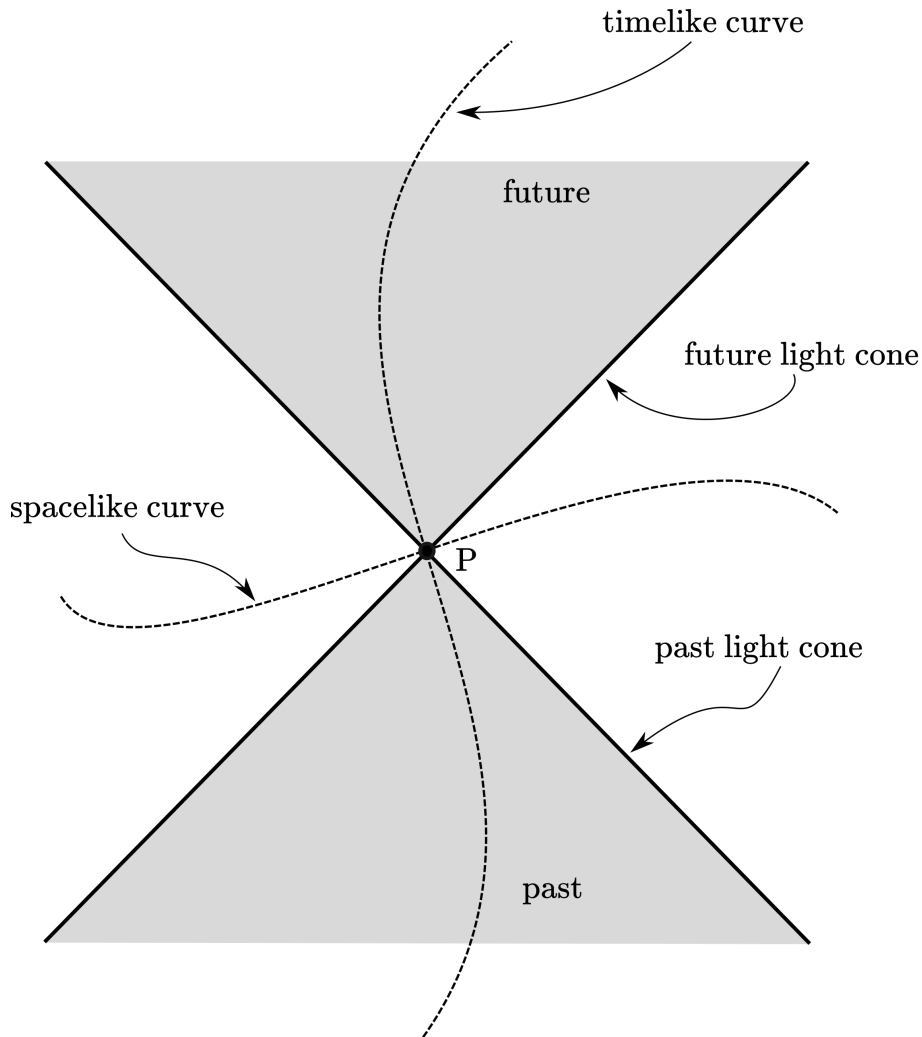


Figure 1.1: Lightcone and causal structure at a point $P \in (\mathcal{M}, g)$.

It is now clear that if a point $p' \in \mathcal{M}$ does not lie in the light cone of $p \in \mathcal{M}$, then there is no physical causality between these two points that can be joined uniquely with an information going faster than the light. The physical trajectories are timelike for massive particles and null concerning the object without mass (light for instance).

Killing vector

A Killing vector field on a manifold \mathcal{M} is a vector field K^a that generates a one-parameter group of isometries $\phi_t : \mathcal{M} \rightarrow \mathcal{M}$, i.e. a group that leaves the metric invariant. In other words, the Lie derivative of the metric along the Killing vector field K^a is zero, $\mathcal{L}_K g_{ab} = 0$. It follows from the Proposition 1.1.1 that a differentiable vector field K^a on (\mathcal{M}, g) is Killing if and only if K satisfies the Killing equation :

$$\nabla_a K_b + \nabla_b K_a = \nabla_{(a} K_{b)} = 0,$$

where ∇_a is the Levi-Civita connection associated to the metric g_{ab} . One of the most useful features of Killing vector fields is the following property.

Proposition 1.1.2. *Let K^a be a Killing vector field on (\mathcal{M}, g) and let γ be a geodesic with a tangent vector u^a . Then the scalar quantity $K_a u^a$ is conserved along γ .*

See [94], Appendix C.3, for the proof.

1.1.3 Radiative spacetimes

As mentioned earlier, solutions to the Einstein equations exist in vacuum; however, these solutions are considered as toy models from a physicist's standpoint due to the absence of matter and energy content in the universe. This is why we dedicate significant attention in this thesis to radiative spacetimes, which involve $T_{ab} \neq 0$. In other words, we will study solutions to the Einstein equations with both matter and energy, not solely vacuum solutions. Deriving a comprehensive general solution to the Einstein field equations is an immensely complex task, leading us to adopt the methodology proposed by Friedmann, Lemaître, Robertson, and Walker ([25], [26], [48], [49], [78], [79], [80] and [95]). We assume that on a large scale (spatial homogeneity), matter and energy in the universe are uniformly distributed, and the universe appears the same in all directions (isotropy). These two assumptions lead the stress-energy tensor T_{ab} to adopt the form of a perfect fluid:

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab},$$

where ρ is the energy density, p is the pressure, and $u^a = g^{ab}u_b$ is a timelike vector representing the 4-velocity of the fluid. When the fluid's velocity approaches that of light, we attain pure radiative spacetimes.

Definition 1.1.9. Pure radiative spacetime : Let T_{ab} be a stress-energy tensor, the source of the Einstein equations for the spacetime (\mathcal{M}, g) . If k^a is a null vector field, i.e. $k^a k_a = 0$, then (\mathcal{M}, g) is classified as a pure radiative spacetime if:

$$T_{ab} = \rho k_a k_b, ,$$

with ρ representing the radiation density.

This describes a field that moves at the speed of light. According to [31], it can be a null electromagnetic field, an incoherent beam of photons, or some forms of idealized (massless) neutrino fields. In this context, we consider it to be null dust, essentially a pressureless perfect fluid with a velocity approaching the speed of light. The two models of pure radiative spacetimes that are studied in this thesis are the Vaidya metric and the Robinson-Trautman metric.

In contrast to the Schwarzschild metric, these two spacetimes are dynamic and this has consequences on the notion of global energy. In general relativity, energy of matter are derived from the stress energy tensor T_{ab} , that can be source of the Einstein equations or not. This defines a local energy measured by a local observer. The way to construct a global conserved energy is to use a timelike Killing observer (see 1.1.2), that defines a conserved energy current. However a timelike Killing observer exists only on stationary spacetimes, i.e. on metrics on which it is possible to choose a function t that is timelike and such that the metric is independent of t , hence the Killing observer reads as $\partial/\partial t$. In dynamical spacetimes, it is not possible to find a such timelike Killing vector and then energy can only be defined locally, and this constitutes a significant topic in this thesis.

Vaidya's spacetime

The Vaidya metric was derived in 1953 by Prahalad Chunnial Vaidya² in [92]. Prior to that, in 1943 and 1951, Vaidya also presented this metric in a more complex form (see [12], [93]). The original intention behind this metric was to describe the outer region of a radiating spherical star whose mass decreases due to the emission of massless dust at the speed of light. Subsequently, the Vaidya metric, with its time orientation reversed, was interpreted as depicting a spherical black hole with increasing mass, arising from the accretion of pure radiation. In simpler terms, null dust particles fall into the black hole

2. Ericourgoulhon informed the author that a very similar expression of the Vaidya metric was discovered by Henri Mineur in 1933 in [56], a full 20 years before Vaidya's formulation.

without breaking its spherical symmetry. Alternatively, the metric can be understood as a white hole emitting pure radiation when the time orientation is reversed again. This latter perspective is adopted in this work.

It is important to clarify that in our study, the pure radiative matter in the Vaidya spacetime is entirely classical and therefore cannot describe phenomena like the evaporation of a black hole, as explained by Hawking radiation. This is due to the fact that if the radiation density is positive (a natural assumption in classical physics), it would imply either a decrease in the black hole's mass or an increase in the white hole's mass. More detailed geometrical analysis of the emission process of the white hole is provided in section 3.1. More precisely, it will be explained that the Vaidya metric can be seen as a radiating Schwarzschild black hole, where the constant mass m becomes a function of advanced or retarded time³.

The attempt to model Hawking radiation using a Vaidya black hole is discussed by Hiscock in 1981 in [37], where he studied a Vaidya spacetime in which the black hole mass $m(v)$, depending on the advanced time, decreases with v , implying physically that negative-energy density falls into the black hole. However, the Vaidya model falls short in explaining Hawking radiation for several reasons. Firstly, the radiation in a Vaidya spacetime consists of pure radiation, with a trace-free stress-energy tensor. However, a semi-classical spacetime is expected to have a non-zero trace for the stress-energy tensor. Secondly, the radiation modeling is incorrect. In a Vaidya black hole with decreasing mass $m(v)$, the negative energy moves along ingoing principal null geodesics, i.e. it originates from the distant past (both in distance and time): past null infinity (see section 2.1 for a definition of this concept in the context of conformal compactification). On the other hand, Hawking evaporation involves energy radiating from the black hole towards future null infinity, i.e. along outgoing null geodesics.

Robinson-Trautman's spacetime

Robinson-Trautman's solution was first developed in the 1960s in vacuum in [83] and [82]. It is geometrically defined by the existence of a geodetic, shear-free, twist-free, expanding null congruence. This metric was later generalized to describe a pure radiation field (as well as other types of spacetimes, which we will not discuss here, see [87] and [31] for an

3. Advanced and retarded time are obtained through coordinate transformations, which naturally resolve coordinate-related issues (such as metric components blowing up) at the horizon of the metric, i.e. at $r = 2m$ in the Schwarzschild spacetime.

extensive survey of this class of solutions). In this thesis, we focus on the pure radiative solution, which is a generalization of the Schwarzschild and Vaidya black holes. In a coordinate basis (u, r, θ, φ) , the level hypersurfaces of u form the geodesic congruence defined above, and they are considered as the hypersurfaces along which pure radiation is emitted, with u being interpreted as a retarded time. r is then an affine parameter along the rays and can be interpreted as the radial distance from the source of radiation.

The formulation of Robinson-Trautman solutions (see [62]) implies that the metric depends on two functions: m and P . Robinson-Trautman solutions offer a broad generalization of other solutions to Einstein's equations, encompassing various scenarios such as vacuum solutions, spacetimes with electromagnetic fields, or pure radiative metrics. For instance, the Robinson-Trautman metric can be specialized to yield the Schwarzschild solution, the Vaidya spacetime, or the Reissner-Nordström metric. The nature of the spacetime, whether vacuum, pure radiative, or containing Einstein-Maxwell fields, is encoded in a fourth-order differential equation, commonly referred to as the Robinson-Trautman equation. Furthermore, these solutions can be extended to spacetimes with a cosmological constant Λ , which may take positive, negative, or zero values. Due to the geometrical construction of Robinson-Trautman solutions, they are all algebraically special, allowing for classification using the Petrov classification. The study of Petrov types and associated principal null directions was conducted in [76]. For an in-depth exploration of the Robinson-Trautman solution family, we direct readers to [87] and [31]. In this thesis, as previously mentioned, our emphasis is on the generalization of the Vaidya solution, specifically focusing on pure radiative Petrov type D spacetimes.

Concerning the pure radiative solutions, the Robinson-Trautman metric depends on two functions, $m = m(u)$ and $P = P(u, \theta, \varphi)$, which are related to the radiation density via a 4th order differential equation. One natural question that arises at this stage is to understand the physical meaning of m and P . The initial formulation of the Robinson-Trautman solution was indeed made in usual (u, r, θ, φ) -coordinates, and there is a term in the metric proportional to r that contradicts the Bondi-Sachs conditions ensuring asymptotic flatness. Thus, it is not possible to interpret m and P directly in terms of Bondi-Sachs quantities. Instead, one needs to use more convenient coordinates. The transformation cannot be expressed in closed-form, as proven by Newman and Unti in [63]. This transformation can be achieved by introducing new coordinates as series of powers of $1/r$. This has been done in [13], [29], and [3] for pure radiative solutions.

A second difficulty arises from the generality of the Robinson-Trautman pure radiative

solutions, necessitating the imposition of assumptions to confine our study to a physically meaningful situation. Initially, we considered the assumption that it is possible to separate the time variable and angular variables for the function P . However, this assumption turned out to be too restrictive, effectively imposing the Vaidya metric on the spacetime. Consequently, we opted to follow the approach of [87] and [31] by assuming that the spacetime is of Petrov type D, and that the Gaussian curvature on the Euclidean sphere, denoted by K , is a function of the retarded time u . In terms of the stereographic coordinates $(\xi, \bar{\xi})$ defined by:

$$\xi = \cot \theta e^{i\varphi},$$

this implies that the function P behaves as:

$$P(u, \xi, \bar{\xi}) = A(u) + B(u)\xi + \bar{B}(u)\bar{\xi} + C(u)\xi\bar{\xi},$$

where A and C are real functions, and B is a complex function.

1.1.4 Conformal analysis and field propagation

The study of field propagation in general relativity is highly relevant from a physical standpoint, as it provides a common approach to gather information about astrophysical objects. The emission of radiation or gravitational fields by a bounded source is of significant interest, as these emissions likely carry information about the source's composition, shape, temperature, and more.

Furthermore, due to the curvature of spacetime, the propagation of fields deviates from that in flat spacetime. This deviation can be exploited to extract geometric data about the studied spacetime, especially when analyzed asymptotically, i.e. at a large distance and at a large time from the bounded source.

In this context, we focus on the asymptotic analysis of field propagation for two reasons:

1. Near the radiation source, the energy distribution is complex, and so is the metric. Analyzing the field becomes challenging due to a loss of information. For example, in the Vaidya spacetime, for a general choice of mass, there is no direct relationship between advanced and retarded time, and the localization of the horizon is not explicit. In contrast, the asymptotic study of a bounded source is comparatively easier. As we move away from the source, the source's influence on spacetime

(i.e., spacetime curvature generated by the source) diminishes with distance. Consequently, over larger scales (in either time or space), the metric can be seen as a perturbation of flat Minkowski spacetime. The goal is to extract enough information from these perturbations to understand the specific characteristics of a given curved spacetime. In this thesis, we consider test fields that do not act as sources of the Einstein equation but are solely propagated through the spacetime without curving it.

2. Physically, observing gravitational sources from Earth occurs in the framework defined above: we are at a significant distance from the signal's emission. Therefore, the mathematical operators developed to associate the asymptotic behaviour of a signal to its behaviour in the past (nearest the source) prove useful for gaining a better understanding of the physics near the source of emission.

An important part of the asymptotic analysis of a field is contained in scattering theory, which describes the entire propagation of a field by associating its asymptotic behaviour in the past with its asymptotic behaviour in the future. These asymptotic behaviours are in principle simpler compared to the dynamics of the field near the source. Furthermore, a complete scattering theory ensures the existence and uniqueness of the solution based on the past or future asymptotic data. Another important question is: how large is the class of initial data that ensures sufficient decay and regularity asymptotically? This question is answered by establishing the peeling-off property of a field and describing the class of initial data that could have physical meaning asymptotically.

The asymptotic analysis of field propagation (particularly regarding scattering theory) has taken two different paths. The first, known as *the analytical way*, was initiated by Lax and Phillips in [47]. The idea is to use a spectral representation of the Hamiltonian associated with the field equation and interpret the evolution as a translation with a parameter t , associated with a Killing vector ∂_t on (\mathcal{M}, g) . Generally, the metric can depend on time and then there is not existence of a such killing vector field, hence the Lax-Phillips approach cannot be easily extended to time-dependent metrics. Other spectral approaches have been used starting in the 1980's with Dimock and Kay [17] and [18] but they are equally ill-suited to generic time dependence. The second path, which we will follow in this thesis, is the *conformal way*, first proposed by Penrose in [70]. It provides an extension of the Lax-Phillips theory to time-dependent metrics.

1.1.5 Conformal compactification

The conformal approach relies on a geometric transformation of the physical spacetime known as conformal compactification, developed by Penrose in the 1960s ([72], [73]). For an overview of conformal methods, we refer to [74] and [71].

The physical spacetime (\mathcal{M}, g) is embedded into a larger spacetime denoted as $(\hat{\mathcal{M}}, \hat{g})$, which is termed the compactified spacetime or unphysical spacetime. \mathcal{M} forms the interior of $\hat{\mathcal{M}}$, and when there is no cosmological constant, the boundary of the compactified spacetime $\partial\hat{\mathcal{M}}$ consists of two null hypersurfaces denoted as \mathcal{I}^\pm , representing future and past null infinities, as well as three "points" denoted as i^\pm, i_0 . These points correspond to future and past timelike infinities and spacelike infinity. The metric \hat{g} is obtained by employing a conformal factor Ω and setting $\hat{g} = \Omega^2 g$. While Ω is positive on \mathcal{M} , at $\partial\hat{\mathcal{M}}$ we have $\Omega = 0$ and $\partial\Omega \neq 0$.

A fundamental structure preserved in a conformal transformation is the null cone structure; two conformally equivalent metrics share the same null directions. Thus, causality remains unchanged by a conformal transformation, and the type of a curve (null, spacelike, or timelike) is preserved.

The advantage of conformal geometry lies in considering the points on the boundary $\partial\hat{\mathcal{M}}$ as points at infinity of the physical metric. This facilitates the transposition of asymptotic methods on \mathcal{M} into local techniques on a hypersurface \mathcal{S} . A more detailed explanation of the conformal compactification of the Minkowski and Schwarzschild metrics is provided in section 2.1.

1.1.6 Ideas of this thesis

The objectives of this thesis are twofold. On the one hand, there is a geometrical analysis of the principal null congruences of two pure radiative spacetimes describes above : the Vaidya spacetime and the pure radiative Robinson-Trautman spacetime. The method employed here, relies on the method developed by Jean-Philippe Nicolas and the author in [15] in the context of the Vaidya spacetime and that is to see incoming principal null geodesics as the solutions of an ordinary differential equation. In this framework , the event horizon is localised using the fact that its null generators are the only geodesics satisfying the differential equation and admitting a finite (non-zero) limit in the past infinity. This method was extended in this thesis to the pure radiative Robinson-Trautman spacetime, where the horizon has an unexpected behaviour compared to the event horizon of a

Schwarzschild or a Vaidya black hole. On the other hand, we perform in this thesis a complete conformal analysis of the wave equation, i.e. the propagation of a zero rest-mass scalar field of spin 0. This study is decomposed in two parts, firstly we establish that the peeling-off property of the wave equation still holds in a dynamical spacetime and that the class of data that peels at infinity in the Vaidya spacetime is as large as in the Schwarzschild spacetime. In other words there is no loss of regularity due to the absence of a timelike Killing vector field. We focus secondly on the conformal scattering of the wave equation in the Vaidya spacetime and we construct a scattering operator on a dynamical spacetime, following the construction done by Nicolas in [64]. As far as the author knows, the only constructions of conformal scattering operator on non-stationary spacetimes was done by Mokdad in the interior of a Reissner–Nordström-like black hole [57], [59] and by Hafner, Mokdad and Nicolas in [32].

- Chapter 3 is devoted to the geometry of the Vaidya spacetime, with a specific focus on the results obtained by the author and Nicolas in [15]. These results relate to the geometry of incoming null geodesics in the Vaidya spacetime, particularly concerning the past event horizon of the dynamical white hole. Due to the time dependence of the metric, usual expressions between advanced and retarded time are no longer valid and we present the method to construct the second optical function v .
- In Chapter 4, we establish the peeling-off property of the wave equation on the Vaidya spacetime in the future. Following the approach of Mason and Nicolas in [54], this peeling property is obtained in a neighbourhood of spacelike infinity i_0 . Spacelike infinity is directly connected to the mass-energy content of the spacetime. When the spacetime contains energy, this point becomes a singularity on the boundary of the conformally compactified spacetime. The decay rate of the propagated field must be sufficiently fast to prevent a singularity at this point from propagating along the entire null infinity. The peeling property of the wave equation in this spacetime is not a straightforward extension of the peeling property observed in the Schwarzschild spacetime due to the different nature of the spacetime—now radiative (i.e., non-empty) and dynamic.
- In Chapter 5, our focus shifts to the peeling property of the wave equation in the past of the initial Cauchy data hypersurface. In static spacetimes like the Schwarzschild spacetime, the relations between u , v , t , and r are well-known, making it easy to

transform results obtained in the (u, r, θ, φ) basis into the (v, r, θ, φ) basis. However, in the Vaidya spacetime, which is time-dependent, this symmetry between the two bases is disrupted. For a general choice of a smooth decreasing mass function $m(u)$, there is no explicit coordinate transformation law. Consequently, extending the peeling-off property is not as immediate as it is in the Schwarzschild spacetime.

- Chapter 6 concludes our conformal analysis of the wave equation in the Vaidya spacetime by examining the conformal scattering of the wave equation. The construction of the conformal scattering operator is carried out on a dynamic spacetime, building upon the method developed by Nicolas in the Schwarzschild spacetime ([64]). However, establishing the scattering operator requires determining the decay rate of the scalar field near timelike infinity. While such a result is known for the wave equation in a Schwarzschild spacetime (see [16]), it is not the case in the Vaidya spacetime. We therefore restrict our study to spacetimes that are dynamic only over a finite interval of the retarded time u , ensuring that a Schwarzschild spacetime exists in the vicinity of both past and future timelike infinity.
- Chapter 7 is dedicated to the study of the pure radiative Robinson-Trautman spacetime. We begin by calculating the metric expression based on geometrical assumptions, following the approach outlined in [62] and utilizing the spin coefficients formalism. Subsequently, we focus on the geometry of incoming principal null geodesics, employing the method developed in [15] with particular attention devoted to the event horizon of the white hole.

In this thesis, we follow the assumptions of [87] and [31], where we assume that the Gaussian curvature on the 2-sphere of the Robinson-Trautman metric depends solely on the retarded time, and we consider the metric to be of type D. This interpretation leads us to view our model as a Vaidya white hole transforming into another white hole by emitting pure radiation during a finite time interval.

A key distinction from the Vaidya solution arises in the pure radiative Robinson-Trautman metric: the incoming null principal direction does not generate geodesics, and thus, it is not hypersurface-forming. Consequently, studying the horizon differs from the approach in [15] and requires analysis via the geodesic equation. Since this equation is a second-order differential equation, a thorough examination will be conducted in future work. In this manuscript, our focus centers on the incoming principal null curves that are tangent to the incoming principal null direction.

In this chapter, we establish that the integral lines of the incoming principal null direction exhibit a behaviour similar to the Vaidya spacetime. Specifically, there is a unique curve that possesses a finite limit in the past, exists on the entire real line, and converges to the Schwarzschild horizon in the future. The other maximal solutions fall into two categories: either they exist on the entire real line and blow up in the past, or they reach zero at a finite retarded time value in the past. In both situations, these solutions converge toward the Schwarzschild horizon in the future.

DEFINITION AND TECHNICAL TOOLS

2.1 Conformal compactification

The conformal compactification was introduced by Penrose in the 1960s in a series of articles ([72], [69], and [70]). Conformal compactification consists of embedding a physical spacetime (\mathcal{M}, g) into a larger "compactified" spacetime $(\hat{\mathcal{M}}, \hat{g})$, where \mathcal{M} is the interior of $\hat{\mathcal{M}}$ and

$$\hat{g} = \Omega^2 g;$$

where Ω is the conformal factor and being a defining function of the boundary of $\hat{\mathcal{M}}$: $\partial\hat{\mathcal{M}} = \hat{\mathcal{M}} - \mathcal{M}$. Ω satisfies $\Omega > 0$ on \mathcal{M} , and at the boundary $\partial\hat{\mathcal{M}}$ we have: $\Omega = 0$ and $\nabla_a \Omega \neq 0$.

2.1.1 Compactification of the Minkowski spacetime

The metric of Minkowski spacetime is given in spherical coordinates (t, r, θ, φ) by:

$$\eta = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \quad (2.1)$$

We introduce new coordinates $(\tau, \zeta, \theta, \varphi)$ such that:

$$\tau = \arctan(v) + \arctan(u), \quad (2.2)$$

$$\zeta = \arctan(v) - \arctan(u), \quad (2.3)$$

with $u = t - r$ and $v = t + r$ being the advanced and retarded coordinates. The metric becomes in these coordinates:

$$\eta = \frac{(1+u^2)(1+v^2)}{4} (d\tau^2 - d\zeta^2) - \frac{(v-u)^2}{4} (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.4)$$

Introducing the conformal factor:

$$\Omega^2 = \frac{4}{(1+u^2)(1+v^2)}, \quad (2.5)$$

and remarking that:

$$\begin{aligned} r^2 \Omega^2 &= \frac{(v-u)^2}{(1+u^2)(1+v^2)}, \\ &= \sin^2 \zeta, \end{aligned}$$

we obtain the compactified metric:

$$\hat{\eta} = \omega^2 \eta = d\tau^2 - d\zeta^2 - \sin^2 \zeta (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.6)$$

which is nothing but:

$$\hat{\eta} = d\tau^2 - \sigma_{S^3}, \quad (2.7)$$

with σ_{S^3} the round sphere metric on the 3-sphere. The conformal spacetime is now described by:

$$\hat{\mathcal{M}} = \{|\tau| + \zeta \leq \pi, \zeta \geq 0, \omega \in S^2\}. \quad (2.8)$$

The boundary $\partial \hat{\mathcal{M}}$ is a compact hypersurface, made of two null hypersurfaces \mathcal{S}^\pm referred as the future null infinity and past null infinity:

$$\begin{aligned} \mathcal{S}^+ &= \{(\tau, \zeta, \omega) | \tau + \zeta = \pi, \zeta \in]0, 2\pi[, \omega \in S^2\}, \\ \mathcal{S}^- &= \{(\tau, \zeta, \omega) | \zeta - \tau = \pi, \zeta \in]0, 2\pi[, \omega \in S^2\}. \end{aligned}$$

The boundary is completed with 3 points i^+, i^-, i_0 that are respectively the future timelike infinity, the past timelike infinity, and the spatial infinity:

$$\begin{aligned} i^\pm &= \{(\tau = \pm\pi, \zeta = 0, \omega); \omega \in S^2\}, \\ i_0 &= \{(\tau = 0, \zeta = \pi, \omega); \omega \in S^2\}. \end{aligned}$$

It is important to notice that these points are smooth points for η and that the boundary is really compact. This will not be the case anymore when we will perform the conformal compactification of curved spacetimes (Schwarzschild, Vaidya). That is why we define a *partial compactification* for Minkowski's spacetime, that brings back the null infinity at a

finite distance; however, it leaves i_0, i^+ and i^- at an infinite distance from the boundary. This is done by using the conformal factor $\Omega = 1/r = R$. We recall the expression of the flat metric in the (u, r, ω) coordinates:

$$\hat{\eta} = du^2 + 2dudr - r^2d\omega^2, \quad (2.9)$$

with,

$$d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Then, the conformal metric is now:

$$\hat{\eta} = R^2du^2 - 2dudR - d\omega^2. \quad (2.10)$$

The null infinity is now defined by:

$$\mathcal{I}^+ = \{(u, R, \omega) | u \in \mathbb{R}, R = 0, \omega \in S^2\}. \quad (2.11)$$

The conformal compactification that gives \mathcal{I}^- can be obtained in the same way by working with (v, R, ω) coordinates.

2.1.2 Compactification of Schwarzschild's spacetime and space-like infinity

The Schwarzschild metric describes a static space-time with a spherical, isolated black hole of constant mass M ; in spherical coordinates (t, r, θ, ϕ) it is given by:

$$g_{\text{Sch}} = F(r)dt^2 - F(r)^{-1}dr^2 - r^2d\omega^2, \quad (2.12)$$

with:

$$F(r) = 1 - \frac{2M}{r}, \quad d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

The metric has a curvature singularity at $r = 0$. The locus $r = 2M$ is a fictitious singularity that can be understood as the union of two null hypersurfaces (the future and the past event horizon) by means of Eddington-Finkelstein coordinates. Outgoing Eddington-Finkelstein coordinates (u, r, θ, ϕ) are defined by $u = t - r_*$, with

$$r_* = r + 2M \log(r - 2M), \quad dr_* = \frac{dr}{F(r)}. \quad (2.13)$$

In these coordinates, the Schwarzschild metric reads:

$$g_{\text{Sch}} = F du^2 + 2 du dr - r^2 d\omega^2. \quad (2.14)$$

This metric is analytic on $\mathbb{R}_u \times]0, +\infty[_r \times S^2$ and remains bounded on this domain. An analogous construction can be done by using the ingoing Eddington-Finkelstein coordinates: (v, r, ω) with $t = v - r_*$. The metric becomes:

$$g_{\text{Sch}} = F dv^2 - 2 dv dr - r^2 d\omega^2, \quad (2.15)$$

and it is analytic on $\mathbb{R}_v \times]0, +\infty[_r \times S^2$. The restriction of these two metrics at $r = 2M$ is degenerate because $F = 0$. However, the determinant of the metric is not zero:

$$\det g_{\text{Sch}} = -r^4 \sin^2 \theta.$$

Hence, the surface at $r = 2M$ is understood as a null hypersurface. The hypersurface $\mathbb{R}_v \times \{2M\}_r \times S^2$ corresponds to the black hole horizon that separates the exterior region and the interior of the black hole where nothing that is timelike or null can escape. Similarly, the hypersurface $\mathbb{R}_u \times \{2M\}_r \times S^2$ corresponds to the separation between the exterior and the interior of a white hole. The conformal compactification of the exterior of the white hole cannot be performed as in the Minkowski spacetime. Due to the mass of the white hole, after compactification, there remains a singularity at spatial infinity, and this point cannot be brought to a finite distance via the conformal compactification. The same difficulties occur at the future and past timelike infinity, and this is why we perform a conformal compactification of the exterior region of a Schwarzschild white hole, using the conformal factor $\Omega = 1/r$. The conformal metric becomes:

$$\hat{g} = \Omega^2 g = R^2 (1 - 2MR) du^2 - 2 du dR - d\omega^2.$$

The inverse metric is:

$$\hat{g}^{-1} = -R^2(1 - 2MR)\partial_R - 2\partial_u\partial_R - \partial_\omega^2,$$

with ∂_ω^2 as the inverse metric on the Euclidean sphere. The (u, R, θ, φ) coordinates are

adapted to define future null infinity \mathcal{I}^+ and the past horizon \mathcal{H}^- :

$$\mathcal{I}^+ = \mathbb{R}_u \times \{0\}_R \times S_\omega^2, \quad \mathcal{H}^- = \mathbb{R}_u \times \{1/2M\}_R \times S_\omega^2.$$

The conformal compactification of the exterior region of the black hole is done in the same way, using (v, R, θ, φ) . The rescaled metric reads:

$$\hat{g} = R^2(1 - 2MR)dv^2 + 2dv dR - d\omega^2.$$

In these coordinates, we define \mathcal{I}^- and \mathcal{H}^+ as:

$$\mathcal{I}^- = \mathbb{R}_v \times \{0\}_R \times S_\omega^2, \quad \mathcal{H}^+ = \mathbb{R}_v \times \{1/2M\}_R \times S_\omega^2.$$

The boundary of the compactified metric is then made up of:

$$\partial\hat{\mathcal{M}} = \mathcal{I}^+ \cup \mathcal{H}^+ \cup \mathcal{H}^- \cup \mathcal{I}^-,$$

and the points i_0, i^+ , and i^- are singularities of the boundary, i.e., they remain at an infinite distance for the rescaled metric \hat{g} .

2.2 Wave equation

2.2.1 Conformally Invariant Wave Equation

In this manuscript, we use the expression "wave equation" to designate the equation that governs the evolution of a massless scalar field denoted by ϕ on a given spacetime (\mathcal{M}, g) . This is equivalent to considering the Klein-Gordon equation for a massless field:

$$\square_g \phi = 0, \tag{2.16}$$

where \square_g is the wave operator (or the d'Alembertian). It is defined using the connection ∇ associated with the metric g by:

$$\square = \nabla^a \nabla_a.$$

Introducing local coordinates $x^{\mathbf{a}} = (x^0, x^1, x^2, x^3)$, its expression becomes:

$$\square_g = \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x^{\mathbf{a}}} \sqrt{|\det g|} g^{\mathbf{ab}} \frac{\partial}{\partial x^{\mathbf{b}}}.$$

The approach chosen in this work is to study asymptotic properties of a field by using conformal methods. If $\dim(\mathcal{M}) \neq 2$, equation (2.16) is not conformally invariant (see Appendix D. in [94]) in the sense defined below:

Definition 2.2.1. Conformal Invariance:

Let Ω be a conformal factor such that the conformal metric \hat{g} is related to the physical metric g by:

$$\hat{g} = \Omega^2 g.$$

An equation for a field ψ is said to be conformally invariant if there exists a real number s such that ψ is a solution for the equation associated with the physical metric g , and $\hat{\psi}$ is the rescaled solution associated with the equation with the metric \hat{g} where:

$$\hat{\psi} = \Omega^s \psi.$$

s is sometimes called the conformal weight of the field.

Let us now slightly change (2.16) so as to have a conformally invariant wave equation. Consider the definition above and study the action of the rescaled d'Alembertian (i.e., the wave operator associated with the conformal metric \hat{g}) on the rescaled field $\hat{\phi}$ with a conformal weight s on a spacetime with dimension n :

$$\begin{aligned} \square_{\hat{g}} \hat{\phi} &= \hat{g}^{ab} \hat{\nabla}_a \hat{\nabla}_b \hat{\phi}, \\ &= \Omega^{-2} g^{ab} \hat{\nabla}_a \nabla_b (\Omega^s \phi), \\ &= \Omega^{-2} g^{ab} (\nabla_a \nabla_b (\Omega^s \phi) - C_{ab}^c \nabla_c (\Omega^s \phi)), \\ &= \Omega^{s-2} g^{ab} \nabla_a \nabla_b \phi + s \Omega^{s-3} \phi g^{ab} \nabla_a \nabla_b \Omega \\ &\quad + (2s + n - 2) \Omega^{s-3} g^{ab} g^{ab} \nabla_a \Omega \nabla_b \phi + s(n + s - 3) \Omega^{s-4} \nabla_a \Omega \nabla_b \Omega. \end{aligned}$$

Note that we use results about the conformal transformation of the connection. They are summed up in Appendix D.1. It is now clear that in dimension 2, setting $s = 0$, the scalar

wave equation is conformally invariant since:

$$\square_{\hat{g}}\hat{\phi} = 0 \Leftrightarrow \square_g\phi = 0.$$

In other dimensions, in particular $n = 4$, we set $s = 1 - n/2 = -1$ in order that:

$$\square_{\hat{g}}\hat{\phi} = \Omega^{-3}g^{ab}\nabla_a\nabla_b\phi - \Omega^{-4}\phi g^{ab}\nabla_a\nabla_b\Omega.$$

We may add the scalar curvature that has the "good" behavior under conformal rescaling, for $n = 4$, (see Appendix D.1 for more details):

$$\begin{aligned} \text{Scal}_{\hat{g}} &= \Omega^{-2} [\text{Scal}_g - 6g^{ac}\nabla_a\nabla_c \ln \Omega - 6g^{ac}(\nabla_a \ln \Omega)(\nabla_c \ln \Omega)] , \\ &= \Omega^{-2} [\text{Scal}_g - 6g^{ab}(\Omega^{-1}\nabla_a\nabla_b\Omega - \Omega^{-2}\nabla_a\Omega\nabla_b\Omega) - 6g^{ac}\Omega^{-2}\nabla_a\Omega\nabla_c\Omega] , \\ &= \Omega^{-2} [\text{Scal}_g - 6g^{ab}\Omega^{-1}\nabla_a\nabla_b\Omega] . \end{aligned}$$

Hence adding the conformal scalar curvature in the wave equation leads to:

$$\square_{\hat{g}}\hat{\phi} + \alpha \text{Scal}_{\hat{g}}\hat{\phi} = \Omega^{-3}g^{ab}\nabla_a\nabla_b\phi - \Omega^{-4}\phi g^{ab}\nabla_a\nabla_b\Omega + \alpha\Omega^{-3}\text{Scal}_g\phi + 6\alpha\Omega^{-4}g^{ab}\nabla_a\nabla_b\Omega.$$

with α any real number. We set $\alpha = 1/6$ in a way that cancels $\nabla_a\nabla_b\Omega$ -terms:

$$\begin{aligned} \square_{\hat{g}}\hat{\phi} + \frac{1}{6}\text{Scal}_{\hat{g}}\hat{\phi} &= \Omega^{-3}g^{ab}\nabla_a\nabla_b\phi + \frac{1}{6}\Omega^{-3}g^{ab}\nabla_a\nabla_b\text{Scal}_g\phi , \\ \left(\square_{\hat{g}} + \frac{1}{6}\text{Scal}_{\hat{g}}\right)\hat{\phi} &= \Omega^{-3}\left(\square_g + \frac{1}{6}\text{Scal}_g\right)\phi . \end{aligned}$$

As an immediate consequence, we state:

Theorem 2.2.1. *Let (\mathcal{M}, g) be a physical spacetime and $(\hat{\mathcal{M}}, \hat{g})$ be the rescaled spacetime associated, with $\hat{g} = \Omega^2g$. There is an equality of operators acting on scalar fields on \mathcal{M} :*

$$\square_g + \frac{1}{6}\text{Scal}_g = \Omega^3\left(\square_{\hat{g}} + \frac{1}{6}\text{Scal}_{\hat{g}}\right)\Omega^{-1}.$$

Then let $\phi \in \mathcal{D}'(\mathcal{M})$, the two following statements are equivalent:

1. ϕ is a solution of:

$$\left(\square_g + \frac{1}{6}\text{Scal}_g\right)\phi = 0 ,$$

in the sense of distributions on \mathcal{M} .

2. $\hat{\phi} = \Omega^{-1}\phi$, referred to as the rescaled solution, satisfies:

$$\left(\square_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\phi} = 0,$$

in the sense of distributions on \mathcal{M} .

2.3 Energy fluxes

Our approach to asymptotic analysis relies entirely on vector fields methods (also called energy estimates methods). The general idea is to work with fields governed by a propagation equation (for instance, the scalar wave equation) and to study their regularity between the hypersurface of initial data on the one hand and the boundary of the conformal spacetime on the other hand. The regularity on the hypersurface is then completely characterized by the energy flux of the field across the hypersurface, defining a Sobolev norm that leads to the energy space where the fields live. We denote by $\mathcal{E}_{S,V}(\phi)$ the energy flux of ϕ measured by an observer V through the hypersurface S . Note that we will drop the observer V when there is no ambiguity about it. In the remainder of this manuscript, we will deal with stress-energy tensors associated with propagation equations. It is important to notice that we consider only test fields, i.e., fields that are not sources in the Einstein equation. Hence, the fields we study do not influence the geometry of the spacetime. Similarly, when we talk about the stress-energy tensor of a field, it corresponds to the stress-energy tensor of the test field and not the source of the Einstein equation. In general, we take the stress-energy tensor to be divergence-free, and this imposes a propagation law for the field. For instance:

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \langle \nabla \phi, \nabla \phi \rangle, \quad (2.17)$$

with :

$$\langle \nabla \phi, \nabla \phi \rangle = \nabla_a \phi \nabla^a \phi = g^{ab} \nabla_a \phi \nabla_b \phi,$$

is the stress-energy tensor for scalar waves and its divergence is :

$$\begin{aligned} \nabla^a T_{ab} &= \nabla^a \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \langle \nabla \phi, \nabla \phi \rangle \right), \\ &= \nabla^a \nabla_a \phi \nabla_b \phi + \nabla_a \phi \nabla^a \nabla_b \phi - \frac{1}{2} \nabla_b (\nabla^a \phi \nabla_a \phi), \end{aligned}$$

$$\begin{aligned}
 &= \square_g \phi \nabla_b \phi + \nabla_a \phi \nabla^a \nabla_b \phi - \nabla_a \phi \nabla_b \nabla^a \phi, \\
 &= \square_g \phi \nabla_b \phi.
 \end{aligned}$$

Where we used the torsion-free nature of the connection:

$$\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi.$$

T_{ab} is divergence-free (in a non-trivial sense) if and only if:

$$\square_g \phi = 0. \quad (2.18)$$

This is the conformally invariant scalar wave equation for a spacetime with $\text{Scal}_g = 0$.

Remark 2.3.1. *In general, the stress-energy tensor is not conformally invariant. This implies that if T_{ab} , the stress-energy tensor in the physical spacetime associated with ϕ , is divergence-free, this still does not remain true in the conformal spacetime $(\hat{\mathcal{M}}, \hat{g})$ for the rescaled stress-energy tensor \hat{T}_{ab} associated with the rescaled field $\hat{\phi}$. Thus, we have chosen to work with a stress-energy tensor T_{ab} that fulfills $\nabla^a T_{ab} = 0$. Unfortunately, the price to pay will be to deal with error terms that arise from $\hat{\nabla}^a \hat{T}_{ab} \neq 0$.*

2.3.1 Volume form and Hodge dual

Definition 2.3.1. (Volume form)

The volume-form on (\mathcal{M}, g) , a 4-dimensional, time oriented manifold, is the 4-form denoted $d\text{Vol}_g^4$, whose expression in a coordinate basis (x^0, x^1, x^2, x^3) is given by :

$$d\text{Vol}_g^4 = \sqrt{|\det g|} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

When there is no ambiguity in the choice of the metric we denoted the volume form $d\text{Vol}^4$.

In the context of conservation laws established in the vector field method, we have to use Stokes' theorem, hence to transform 1-forms into 3-forms. This is done using the Hodge dual:

Definition 2.3.2. (Hodge dual).

Let ω be a p -form, $0 \leq p \leq 4$, the Hodge dual of ω is the $(4 - p)$ -form such that

1. for a 0-form f :

$$(\star f)_{abcd} = f d\text{Vol}^4.$$

2. for a 1-form α :

$$(\star\alpha)_{abc} = \alpha^d (\text{dVol}^4)_{abcd} .$$

3. for a 2-form β :

$$(\star\beta)_{ab} = \frac{1}{2} \beta^{cd} (\text{dVol}^4)_{abcd} .$$

4. for a 3-form γ :

$$(\star\gamma)_a = \frac{1}{6} \gamma^{bcd} (\text{dVol}^4)_{abcd} .$$

5. for a 4-form δ :

$$(\star\delta) = \frac{1}{24} \delta^{abcd} (\text{dVol}^4)_{abcd} .$$

The two following propositions are useful in our study. The first one:

Proposition 2.3.1. *For any two p -forms, $0 \leq p \leq 3$, denoted by α and β ,*

$$\alpha \wedge \star\beta = (-1)^p \frac{(4-p)!}{4!} \langle \alpha, \beta \rangle_g \text{dVol}^4 ,$$

with

$$\langle \alpha, \beta \rangle_g = \alpha_{a_1 \dots a_p} \beta^{a_1 \dots a_p} .$$

The second proposition describes the action of the exterior derivative, denoted by d , on the Hodge dual of a differentiable 1-form:

Proposition 2.3.2. *Let ω be a differentiable 1-form, then:*

$$(d(\star\omega))_{abcd} = -\frac{1}{4} \nabla_e \omega^e (\text{dVol}^4)_{abcd} .$$

Definition 2.3.3. Energy Flux:

Let ϕ be a solution of a propagation equation derived from a stress-energy tensor T_{ab} . Let V^a be a vector field on (\mathcal{M}, g) , and Σ be a hypersurface. Then the energy current 1-form J_a is:

$$J_a = V^b T_{ab} ,$$

The energy flux measured across Σ is:

$$\mathcal{E}_{\Sigma, V}(\phi) = \int_{\Sigma} \star J_a |_{\Sigma} ,$$

where $J_a|_\Sigma$ denotes the restriction of $\star V^b T_{ab}$ to the hypersurface Σ , with \star being the Hodge dual.

2.3.2 Energy conditions

The definition of energy given above is very general and not necessarily reasonable from a physical point of view. This is why there exist energy conditions: *weak energy condition*.

Definition 2.3.4. Weak energy condition:

Let T_{ab} be a stress-energy tensor, let V^a be a timelike observer. The two following assertions are equivalent:

1. T_{ab} satisfies the weak energy conditions.
2. For all timelike observers V^a , the quantity ε , named the energy density, satisfies:

$$\varepsilon = T_{ab} V^a V^b \geq 0.$$

Definition 2.3.5. Dominant energy condition:

Let T_{ab} be a stress-energy tensor, and let V^a be a timelike, future-oriented vector. The two following assertions are equivalent:

1. T_{ab} satisfies the dominant energy condition.
2. For all V^a , the quantity $J^a = T_b^a V^b$ should be a future-directed causal (timelike or null) vector.

In other words, the speed of the energy current (the energy flow of the field) can never be faster than the speed of light.

In the vector fields method, conservation laws are obtained using Stokes's theorem, which is recalled for a 3-form:

Theorem 2.3.1. (Stokes' theorem):

Let Ω be a bounded open subset of \mathcal{M} with a piecewise \mathcal{C}^1 boundary \mathcal{S} . Let ω be a differentiable 3-form on \mathcal{M} , \mathcal{C}^1 on $\bar{\Omega}$. Then,

$$\int_{\mathcal{S}} \omega = \int_{\Omega} d\omega.$$

Now, consider ω as a Hodge dual of a 1-form, denoted by α . Stokes' theorem becomes:

Theorem 2.3.2. (The divergence theorem): *Let Ω be a bounded open subset of \mathcal{M} with a piecewise \mathcal{C}^1 boundary \mathcal{S} . Let l^a be a vector field transverse to \mathcal{S} and outgoing, and n^a be a normal vector field to \mathcal{S} such that it is normalized like $l_a n^a = 1$. Let α be a 1-form \mathcal{C}^1 on $\bar{\Omega}$, then*

$$\int_{\mathcal{S}} \alpha_a n^a (l_{\perp} d\text{Vol}^4) = \int_{\Omega} \nabla_a \alpha^a d\text{Vol}^4.$$

PART II

Conformal analysis in the Vaidya spacetime

GEOMETRY OF THE VAIDYA SPACETIME

3.1 Construction of Vaidya's spacetime

The Vaidya metric is defined from (2.14) by allowing the mass m to depend of the retarded time u .

$$g = F(u, r)du^2 + 2dudr - r^2d\omega^2, \quad F(u, r) = 1 - \frac{2m(u)}{r}. \quad (3.1)$$

Alternatively we can construct the Vaidya metric using ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ) with $v = t + r_*$.

$$g = F(v, r)dv^2 - 2dvdr - r^2d\omega^2, \quad F(v, r) = 1 - \frac{2m(v)}{r}. \quad (3.2)$$

Metrics (3.1) and (3.2) are solutions of Einstein equations with a source. (3.1) describes a white hole that evaporates classically via the emission of null dust whereas (3.2) corresponds to a black hole that mass increases as a result of accretion of null dust. In this work, we will deal with an evaporating white hole, in the coordinates (u, r, θ, ω) with the stress-energy tensor T_{ab} in the Einstein equation $G_{ab} = 8\pi T_{ab}$:

$$T_{ab} = -\frac{m'(u)}{4\pi r^2}(du)_a(du)_b. \quad (3.3)$$

Hence the mass m is a non-increasing function of u to ensure that the energy density $\eta = -m'(u)/4\pi r^2$ is positive. We assume that the mass is a smooth decreasing function of u for $u \in [u_-, u_+]$, with $u_- < u_+$. Another natural assumption is that the mass has finite limits as u tends to u_{\pm} :

$$\lim_{u \rightarrow u_{\pm}} m(u) \rightarrow m_{\pm} \text{ with } 0 \leq m_+ < m_- < +\infty, \quad (3.4)$$

and such that :

$$m'(u) < 0 \quad \text{on }]u_-, u_+[, \quad -\infty \leq u_- < u_+ \leq +\infty , \quad m'(u) = 0 \quad \text{elsewhere} . \quad (3.5)$$

We then distinguish the case in which $m(u)$ is decreasing on a finite interval and the case in which the evaporation of the white hole is $\forall u \in \mathbb{R}$. In the latter case, we have $u_- = -\infty$ and $u_+ = +\infty$.

The Weyl tensor has Petrov type D (see D.2.1 for the Petrov classification of the Weyl tensor in terms of the multiplicities of its principal null directions), i.e. it has two double principal null directions that are given by :

$$V = \frac{\partial}{\partial r} , \quad W = \frac{\partial}{\partial u} - \frac{1}{2}F \frac{\partial}{\partial r} . \quad (3.6)$$

This is well known (see [31]) and can be checked easily by observing that V and W both satisfy the condition ensuring that they are at least double roots of the Weyl tensor (see R. Penrose, W. Rindler [74] Vol. 2, p. 224)

$$C_{abc[d}V_{e]}V^bV^c = C_{abc[d}W_{e]}W^bW^c = 0 .$$

We consider a null tetrad built using the principal null vectors above

$$\begin{aligned} l &= V , \\ n &= W , \\ m &= \frac{1}{r\sqrt{2}} \left(\frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right) , \\ \bar{m} &= \frac{1}{r\sqrt{2}} \left(\frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right) . \end{aligned}$$

It is a normalised Newman-Penrose tetrad, i.e.

$$l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = l_a m^a = n_a m^a = 0 , \quad l_a n^a = -m_a \bar{m}^a = 1 .$$

Let $\{o^A, \iota^A\}$ be the spin-frame (a local basis of the spin-bundle ξ^A that is normalised, i.e. $o_A \iota^A = 1$) defined uniquely up to an overall sign by

$$l^a = o^A \bar{o}^{A'} , \quad n^a = \iota^A \bar{\iota}^{A'} , \quad m^a = o^A \bar{\iota}^{A'} , \quad \bar{m}^a = \iota^A \bar{o}^{A'} .$$

Since the spacetime has Petrov type D, the Weyl spinor Ψ_{ABCD} has only one non-zero component which is

$$\Psi_2 = \Psi_{ABCD} o^A o^B \iota^C \iota^D = -\frac{m(u)}{r^3}.$$

3.1.1 Conformal compactification of the Vaidya spacetime

The conformal compactification is performed with the conformal factor $\Omega = R = 1/r$:

$$\hat{g} = R^2 F(u, R) du^2 - 2du dR - d\omega, \quad F(u, R) = 1 - 2m(u)R. \quad (3.7)$$

The inverse rescaled metric is :

$$\hat{g}^{-1} = -R^2 F(u, R) \partial_R^2 - 2\partial_R \partial_u - \partial_\omega^2. \quad (3.8)$$

The d'Alembertian associated to this compactified metric

$$\square_{\hat{g}} := \nabla_a \nabla^a,$$

where ∇ is the Levi-Civita connection associated to \hat{g} , is given in terms of coordinates (u, R, ω) by :

$$\square_{\hat{g}} = -2\partial_u \partial_R - \partial_R R^2 (1 - 2m(u)R) \partial_R - \Delta_{S^2}. \quad (3.9)$$

The Ricci scalar, also called the scalar curvature of \hat{g} , i.e. the trace of the Ricci tensor with respect to the rescaled metric is :

$$\text{Scal}_{\hat{g}} = \hat{g}^{ab} R_{ab} = 12m(u)R,$$

and the scalar curvature of the physical metric g is :

$$\text{Scal}_g = g^{ab} R_{ab} = 0.$$

The metric (3.1) has the following non-zero Christoffel symbols, with $m'(u) = dm/du$:

$$\begin{aligned} \Gamma_{00}^0 &= -3R^2 m(u) + R, \quad \Gamma_{01}^1 = 3R^2 m(u) - R, \\ \Gamma_{00}^1 &= 6R^5 m^2(u) - 5R^4 m(u) + R^3 m'(u) + R^3, \\ \Gamma_{33}^2 &= -\cos \theta \sin \theta, \quad \Gamma_{32}^3 = \frac{\cos \theta}{\sin \theta}. \end{aligned}$$

3.2 The second optical function on the Vaidya spacetime

We can construct a second optical function analogous to $v = t + r_*$ in the Schwarzschild spacetime. This has been done in details in [15].

On the Schwarzschild metric we have $g = F du dv - r^2 d\omega^2$; for Vaidya's spacetime we have

$$g = F du \left(du + \frac{2}{F} dr \right) - r^2 d\omega^2$$

and the 1-form $du + \frac{2}{F} dr$ is not exact. However, introducing an auxiliary positive function φ we can write :

$$g = \frac{F}{\psi} du \left(\psi du + 2\psi F^{-1} dr \right) - r^2 d\omega^2 \quad (3.10)$$

and arrange for the 1-form, $\psi du + 2\psi F^{-1} dr$ to be exact and null, if we assume :

$$\frac{\partial \psi}{\partial u} - \frac{F(u, r)}{2} \frac{\partial \psi}{\partial r} + \frac{2m'(u)}{Fr} \psi = 0. \quad (3.11)$$

This is an partial differential equation along the integral lines of the second principal null direction (defined in (3.6)). This equation can be solved by setting, say, $\psi = 1$ on \mathcal{I}^- and integrating along incoming principal null geodesics (see [15]). Then we define v by

$$dv = \psi du + 2 \frac{\psi}{F} dr$$

and $v = -\infty$ on \mathcal{I}^- . The gradient of v is given by :

$$\nabla^a v = 2\psi F^{-1} W^a = 2 \frac{\psi}{F} \left(\frac{\partial}{\partial u} \right)^a - \psi \left(\frac{\partial}{\partial r} \right)^a. \quad (3.12)$$

and it is clearly null :

$$g(\nabla v, \nabla v) = 0. \quad (3.13)$$

This ensure that v is optical function. It is useful to define new radial and time variables such that :

$$\begin{cases} t = u + \tilde{r}, \\ t = v - \tilde{r}. \end{cases}$$

The relations between their differentials are

$$d\tilde{r} = \frac{1}{2}(\psi - 1) du + \frac{\psi}{F} dr \quad (3.14)$$

$$dt = \frac{1}{2}(\psi + 1) du + \frac{\psi}{F} dr \quad (3.15)$$

3.2.1 Family of null geodesics

The integral lines of V and W , the principal null directions of Vaidya's spacetime are respectively the outgoing and ingoing principal null congruences, furthermore they are family of outgoing (resp. ingoing) null geodesics. The function u and v are optical functions, which means that their gradient are null vector fields :

$$g(\nabla u, \nabla u) = 0 \quad \text{and} \quad g(\nabla v, \nabla v) = 0. \quad (3.16)$$

An important property of optical functions is that the integral lines of their gradient are null geodesics with affine parametrisation. This is established in [34]. The more complete Propositions (7.1.60) and (7.1.61) in Penrose and Rindler Vol 2 [71] state that for a null congruence, the following three properties are equivalent :

1. it is hypersurface-orthogonal;
2. it is hypersurface-forming;
3. it is geodesic and twist-free.

We recall the proof of the fact that the integral curves of an optical function are null geodesics, as it is a straightforward calculation.

Lemma 3.2.1. *Let ξ be an optical function and denote $\mathcal{L} = \nabla\xi$. The integral curves of \mathcal{L} are geodesics and \mathcal{L} corresponds to a choice of affine parameter, i.e.*

$$\nabla_{\mathcal{L}}\mathcal{L} = 0.$$

Proof. The proof is direct :

$$\begin{aligned} \nabla_{\mathcal{L}}\mathcal{L}^b &= \nabla_{\nabla\xi}\nabla^b\xi, \\ &= \nabla_a\xi\nabla^a\nabla^b\xi, \end{aligned}$$

$$\begin{aligned}
 &= \nabla_a \xi \nabla^b \nabla^a \xi \text{ since the connection is torsion-free,} \\
 &= \nabla^b (\nabla_a \xi \nabla^a \xi) - (\nabla^b \nabla_a \xi) \nabla^a \xi, \\
 &= 0 - \nabla_a \xi \nabla^a \nabla^b \xi \text{ since } \nabla \xi \text{ is null and the connection torsion-free,} \\
 &= -\nabla_{\nabla \xi} \nabla^b \xi. \quad \square
 \end{aligned}$$

Since :

$$\nabla u = V \quad \text{and} \quad \nabla v = 2\psi F^{-1}W,$$

with V and W two distinct principal null directions of the Weyl tensor then a consequence of Lemma 3.2.1 and of (3.16) is the following.

Proposition 3.2.1. *The integral lines of V (resp. W) are affinely parametrised null geodesics; they are the outgoing (resp. ingoing) principal null geodesics of Vaidya's spacetime.*

For more details, see [15].

3.3 The incoming principal null congruence

Here the metric (3.1) describes a white hole with a mass $m(u)$ that decreases from an initial value $m = m_-$ at $u = u_-$ and then converges (in finite or asymptotically) toward $m(u) = m_+$. In the final state, there is a Schwarzschild black hole with a mass m_+ , hence there exists a future horizon located at $r = 2m_+$. The existence and the localisation of the past horizon, i.e. the horizon of the white hole, is more tricky to define. The method developed in [15] was to study the congruence of incoming null radial geodesics and to classify them based on their behaviour when $u \rightarrow -\infty$. The starting point is to obtain the ODE that governs incoming null radial geodesics : let $\gamma(u)$ be the family of curves indexed by $\omega \in S^2$

$$\gamma(u) = (u, r = r(u), \omega), \quad u \in \mathbb{R}, \quad (3.17)$$

and have the property of being null, i.e.

$$g(\dot{\gamma}(u), \dot{\gamma}(u)) = 1 - \frac{2m(u)}{r(u)} + 2\dot{r}(u) = 0. \quad (3.18)$$

Hence, the function $r(u)$ satisfies the following ordinary differential equation

$$\dot{r}(u) = -\frac{1}{2} \left(1 - \frac{2m(u)}{r(u)} \right), \quad (3.19)$$

The main theorem in [15] :

Theorem 3.3.1. *Let m be a smooth decreasing function of the retarded time u satisfying (3.4) and (3.5). Then, there exists a unique maximal solution r_h to (3.19) such that*

$$\lim_{u \rightarrow -\infty} r_h(u) = 2m_- .$$

- *If either $m_+ > 0$ or $u_+ = +\infty$, r_h exists on the whole real line, $r_h(u) \rightarrow 2M_+$ as $u \rightarrow +\infty$ and any other maximal solution r to (3.19) belongs to either of the following two categories:*
 1. *r exists on the whole real line, $r(u) > r_h(u)$ for all $u \in \mathbb{R}$, $\lim_{u \rightarrow -\infty} r(u) = +\infty$ and $\lim_{u \rightarrow +\infty} r(u) = 2m_+$;*
 2. *r exists on $]u_0, +\infty[$ with $u_0 \in \mathbb{R}$ and satisfies: $r(u) < r_h(u)$ for all $u \in]u_0, +\infty[$, $\lim_{u \rightarrow u_0} r(u) = 0$ and $\lim_{u \rightarrow +\infty} r(u) = 2m_+$.*
- *If $m_+ = 0$ and $u_+ < +\infty$, r_h exists on an interval $] - \infty, u_0[$ with $u_+ \leq u_0 < +\infty$ and $\lim_{u \rightarrow u_0} r_h(u) = 0$. The other maximal solutions are of two types:*
 1. *r exists on $] - \infty, u_1[$ with $u_0 \leq u_1 < +\infty$, $r(u) > r_h(u)$ on $] - \infty, u_0[$, $\lim_{u \rightarrow u_1} r(u) = 0$ and $\lim_{u \rightarrow -\infty} r(u) = +\infty$;*
 2. *r exists on $]u_1, u_2[$ with $-\infty < u_1 < u_2 \leq u_0$, $r(u) \rightarrow 0$ as u tends to either u_1 or u_2 and $r(u) < r_h(u)$ on $]u_1, u_2[$.*

The proof is done in [15].

PEELING OF THE WAVE EQUATION IN THE VAIDYA SPACETIME

4.1 Introduction

In 1961, Sachs studied outgoing radiation fields from a source in curved space-times along null geodesics. In [84], he expanded the Riemann tensor in negative powers of r , with r an affine parameter along outgoing null geodesics. The Sachs *peeling property* of a field can be described as the alignment of its principal null directions along the generator of the geodesics when moving away from the source of radiation.

In 1965, Penrose proved in [70], that the peeling property can be reinterpreted in much simpler terms using a conformal compactification. Penrose's conformal methods consist in embedding a physical spacetime (\mathcal{M}, g) into a larger "compactified" spacetime $(\hat{\mathcal{M}}, \hat{g})$, where \mathcal{M} is the interior of $\hat{\mathcal{M}}$ and

$$\hat{g} = \Omega^2 g;$$

the conformal factor Ω being a defining function of the boundary of $\hat{\mathcal{M}}$: $\partial\hat{\mathcal{M}} = \hat{\mathcal{M}} \setminus \mathcal{M}$. In the Ricci-flat case, this boundary is composed of two null hypersurfaces \mathcal{I}^\pm referred to as future and past null infinities. Penrose showed that the Sachs peeling property is equivalent to the continuity of the conformal field at the boundary.

Then comes the question of how large a class of initial data ensures the peeling behaviour. On Minkowski spacetime, there is a simple answer based on a complete compactification in which the rescaled spacetime really is compact. As soon as the spacetime contains energy, it becomes singular at spacelike infinity and the question becomes difficult : if the field does not decay sufficiently fast at spacelike infinity, a singularity could creep up null infinity and make the whole asymptotic behaviour singular.

In 2007, Lionel Mason and Jean-Philippe Nicolas gave a complete description of the

peeling for scalar field on Schwarzschild's spacetime in [54].

They established that the same assumptions of regularity and decay on initial data as in Minkowski's spacetime entailed the same regularity at null infinity of the conformal field. The whole construction is based on a partial compactification with conformal factor $1/r$ including for Minkowski spacetime, which imposes weaker conditions on the fall-off of the data that one would get from a complete compactification. In [41] they extended their results to Dirac and Maxwell fields on Schwarzschild's space-time. Then Pham and Nicolas obtained similar theorems for linear and semi-linear scalar fields on Kerr metrics in [66].

The peeling property defined by Mason and Nicolas is little bit different from the Penrose version because regularity at infinity is characterised by Sobolev norms instead of C^k spaces. More precisely the functions spaces on a Cauchy hypersurface and null infinity are obtained from energy norms associated to well chosen vector fields. The idea is to obtain an equivalence at all orders between the energy of initial data and the energy at the boundary. Then they compared regularity and decay assumptions with the corresponding ones on Minkowski's spacetime and show that they are equivalent.

As alluded to above, the singularity at i_0 is directly linked to the black hole mass. The peeling of massless field via the approach of [54] has so far only been studied in stationary situations. Does the property still hold when the mass of the black hole varies with time? We study this question for Vaidya metrics. Vaidya black holes are spherically symmetric and have a varying mass due to the absorption or the emission of null dust that is transported along null geodesics. In this article we focus on the case of a white hole that evaporates via the emission of null dust.

In [15] Jean-Philippe Nicolas and the author studied the geometry of such spacetimes with a particular emphasis on spherically symmetric optical functions and the corresponding null geodesic congruences. The second optical function (analogous to the advanced time of Eddington-Finkelstein coordinates on Schwarzschild's spacetime) will be useful to us in controlling the geometry of the spacetime near i_0 and \mathcal{I} .

For a completely general mass function, we characterise entirely the peeling property of massless scalar fields and we prove that conditions for peeling at any given order are analogous to those on flat spacetime.

The paper is organised as follows :

1. We recall the definition and properties of Vaidya's spacetime, its conformal compactification and the wave equation on this background. In the remainder of the

paper we work only on the compactified spacetime.

2. We then present then the vector field method for the wave equation. We extend to Vaidya's spacetime the definition of the so-called Morawetz vector field and after a choice of stress-energy tensor, we give the expression of the energy current and its divergence. This divergence is non-zero due to the fact that the Morawetz vector field is only an asymptotic Killing vector and the stress-energy tensor is not exactly conserved. We establish that the divergence terms are controlled by the energy of the field on the leaves of a foliation that we construct.

We obtain a first theorem that gives a "peeling at order 0".

3. We obtain similar results for higher order energies. The main difficulty here is linked to the non-stationarity of Vaidya's spacetime. We establish the peeling at all orders and observe some subtle differences compared to the Schwarzschild case.

4.2 Equivalence between peeling theorems

4.2.1 Sachs' ideas

The first formulation of the peeling-off property of a field in vacuum was introduced by Sachs in 1961 in [84] and then extended by Sachs himself in 1962 in [85] for asymptotically flat spacetimes. In these works, Sachs studied the decay of the outgoing gravitational field and derived this information from a detailed analysis of the Riemann tensor. Sachs' approach was based on earlier ideas in the 1950's by Pirani [75], Trautman [90] and [91], and Bondi, Pirani, and Robinson [8], which posited that at large distances from the source, the Riemann tensor could be approximated as that of a plane wave. Deviations from the asymptotic planar-wave structure become apparent as one moves closer to the source. Sachs accomplished this by expanding the Riemann tensor along null curves in powers of r^{-k} , where r is the radial distance from the localized source of radiation. He stated the following theorem:

Theorem 4.2.1 (Sachs, 1961). *In general, the Riemann tensor of a vacuum metric with geodesic rays i.e., in a spacetime that admits a family of null geodesics, can be expanded along null geodesics as*

$$R_{abcd} = \frac{N_{abcd}}{r} + \frac{III_{abcd}}{r^2} + \frac{II_{abcd}}{r^3} + \frac{I_{abcd}}{r^4} + \mathcal{O}(r^{-5}),$$

where N_{abcd} , III_{abcd} , II_{abcd} , and I_{abcd} correspond to the types in the Petrov classification (see Appendix D.2.1 for more details). Therefore, if k^a defines a null geodesic ray congruence, the following conditions hold:

$$\begin{aligned} N_{abcd}k^d &= 0, \\ \text{II}_{abc[d}k^ck_e] &= 0, \\ \text{III}_{abc[d}k^bk^ck_e] &= 0. \end{aligned}$$

This expansion also holds true in asymptotically flat spacetimes. For more details, we refer to [84] and [85].

To translate this theorem into the Penrose formalism of conformal infinity, we will state a similar property for the Weyl tensor, which is conformally invariant. Let C_{abcd} be the Weyl tensor associated with R_{abcd} , and let γ be an outgoing null geodesic with an affine parameter λ . The Weyl tensor is said to satisfy the peeling property if:

$$C_{abcd} = \frac{C_{abcd}^{(1)}}{\lambda} + \frac{C_{abcd}^{(2)}}{\lambda^2} + \frac{C_{abcd}^{(3)}}{\lambda^3} + \frac{C_{abcd}^{(4)}}{\lambda^4} + O\left(\frac{1}{\lambda^5}\right), \quad (4.1)$$

where the components $C_{abcd}^{(i)}$ can be classified into the Petrov classification using k^a , the tangent vector to the geodesic γ .

- $C_{abcd}^{(1)}$ is type IV.
- $C_{abcd}^{(2)}$ is type III.
- $C_{abcd}^{(3)}$ is type II.
- $C_{abcd}^{(4)}$ is type I.

4.2.2 Penrose's version and the extension to zero rest-mass fields

In 1965, Penrose introduced a new definition for the peeling property of a field in [70], expanding Sachs' definition to zero rest-mass fields, not just the gravitational field, i.e., the radiation of the Riemann tensor. This new definition, developed by Penrose, relies directly on a conformal compactification, as the asymptotic decay studied by Sachs is now seen as the behavior of the field in the neighborhood of conformal infinity. Penrose remarked that any zero rest-mass field of spin s determines at each point p of the spacetime $2s$ principal null directions. The Sachs peeling-off property in this framework is that for $k = 0, 1, \dots, 2s$

and r a linear radial parameter along a radial null direction, the part of the field that decays like r^{-k-1} has $2s - k$ principal null directions aligned along the generator of the null geodesics. By setting $s = 2$ (the spin of the gravitational field), we obtain (4.1).

Proposition 4.2.1. Penrose, 1965 : *Let ϕ be a physical field on (\mathcal{M}, g) and $\hat{\phi}$ be the rescaled field associated with the conformal spacetime $(\hat{\mathcal{M}}, \hat{g})$ such that $\hat{\phi} = \Omega^{-\omega}\phi$, where ω is the conformal weight and Ω is the conformal factor. Then the two following assertions are equivalent:*

1. *The principal null directions of a physical field ϕ peel off.*
2. *The rescaled field $\hat{\phi}$ remains finite and bounded at the conformal infinity and has the regularity C^h at the boundary of the conformal spacetime, $0 \leq h \leq k - 1$ with $k \geq 3$.*

The natural question that arises from this definition is: how large a class of initial data associated with a field ensures the peeling behavior? The answer in Penrose's framework is formulated as follows in [54]:

Definition 4.2.1. *A solution $\tilde{\phi}$ of (4.2) is said to peel at order $k \in \mathbb{N}$ if the rescaled solution $\phi = \Omega^{-1}\tilde{\phi}$ extends as a C^k function on $(\hat{\mathcal{M}}, \hat{g})$. This is satisfied by a solution ϕ of (4.3) arising from initial data $\phi|_{t=0} \in C^k(S^3)$ and $\partial_t\phi|_{t=0} \in C^{k-1}(S^3)^1$. The corresponding class of physical data, associated with the physical field $\tilde{\phi}$ in the physical space (\mathcal{M}, g) , gives us solutions that peel at order k .*

This is equivalent to studying a Cauchy problem; however, this leads to a new difficulty as the regularity of the field is characterized in terms of C^k spaces in Penrose's definition. Indeed, C^k spaces are not suitable for the Cauchy problem for hyperbolic equations. For instance, in Minkowski spacetime, if we study f , a solution of the wave equation, with initial data $f|_{t=0} \in C^k(\mathbb{R}^3)$ and $\partial_t f|_{t=0} \in C^{k-1}(\mathbb{R}^3)$, f is generally not in $C^k(\mathbb{R}^3)$. That is why, in 2009, Mason and Nicolas introduced a new definition of the peeling theory in [54] in which the regularity of the field is given in terms of Sobolev spaces. We can notice that the Definition 4.2.1 is true in the case $k = \infty$.

1. where S^3 denotes the 3-sphere.

4.2.3 Mason and Nicolas' method

In this work, the authors compared the peeling property of scalar waves between Minkowski's spacetime and Schwarzschild's spacetime. By following a new approach to define the regularity of the field, they showed that the class of data that gives a solution that peels on Minkowski spacetime is the same as that on Schwarzschild spacetime. This can be summarized as follows concerning flat spacetime :

1. The conformal compactification of Minkowski spacetime is not the complete one. By using $\Omega = 1/r = R$ instead of the conformal factor (2.5), i_0, i^+ , and i^- are not in the boundary and remain at infinite distance.
2. The regularity of the field is not characterized in terms of \mathcal{C}^k spaces, but in terms of energy spaces. These energy spaces on a hypersurface Σ are constructed with a Sobolev norm that is the energy flux associated with an observer through Σ .
3. In order to obtain these energy norms, they chose a timelike observer that is transverse to null infinity to compute the energy. This leads to a better control in the norm of the derivative of the rescaled field at infinity and provides more precise estimates. On partially compactified Minkowski spacetime, there exists a timelike vector, K^a , that is also a Killing vector, which leads to an exact conservation law for the energy of the rescaled field.

The definition of peeling for a complete conformal compactification, i.e., with a conformal factor:

$$\Omega^2 = \frac{4}{(1+u^2)(1+v^2)},$$

is given by

Definition 4.2.2. *A solution $\tilde{\phi}$ of (4.2) is said to peel at order $k \in \mathbb{N}$ if the trace of $\partial_\tau^k \phi \in H^1(\mathcal{I}^+)$. The set of solutions to (4.3) satisfying this property is exactly the set of solutions whose data at $\tau = 0$ satisfy $\phi|_{\tau=0} \in H^{k+1}(S^3)$ and $\partial_\tau \phi|_{\tau=0} \in H^k(S^3)$. In the physical space (\mathcal{M}, g) with the physical field ϕ , this gives the class of data for (4.2), giving rise to solutions that peel at order k .*

This definition is then modified for a partial conformal compactification $\Omega = R = 1/r$, and we denote by $\mathcal{E}_S(\phi)$ the energy flux measured by an observer K , across a hypersurface S . We explain the method to compute it in Section 4.3. The definition of the peeling becomes then:

Definition 4.2.3. Let $\mathcal{I}_{u_0}^+$ be the part of the future null infinity for $u < u_0$, with $u_0 \ll -1$ to be in a small neighborhood of the spacelike infinity. Then, we say that a solution ϕ to (4.3) peels at order $k \in \mathbb{R}$ if, for all polynomials P in ∂_R and ∇_{S^2} of order lower than or equal to k , we have $\mathcal{E}_{\mathcal{I}_{u_0}^+}(P\phi) < \infty$. This means that for all $p \in \{0, 1, \dots, k\}$, we have for all $q \in \{0, 1, \dots, p\}$, $\mathcal{E}_{\mathcal{I}_{u_0}^+}(\partial_R^q \nabla_{S^2}^{p-q} \phi) < \infty$.

This definition for scalar waves is then extended to Schwarzschild's spacetime and, in the following sections, to Vaidya's spacetime. The extension to Schwarzschild's spacetime is not trivial due to the singularity on the conformal boundary, particularly at spacelike infinity. Furthermore, the purely timelike vector chosen in this context is no longer a Killing vector. In [54], the authors introduce a "Morawetz" observer by expressing the Minkowskian Morawetz vector K^a (see [60] for the original expression) in (u, R, ω) coordinates as:

$$K^a = u^2 \partial_u - 2(1 + uR) \partial_R, ,$$

and keeping this expression on Schwarzschild's spacetime. This observer is transverse to null infinity; however, it is only a Killing vector at infinity when $R = 0$. This implies that some error terms appear and need to be controlled. The same holds for Vaidya's spacetime, which is non-stationary. This will be further explored in the following sections.

Notations :

1. We use the notation \lesssim to signify that the left-hand side of an expression is bounded by the right-hand side, up to a constant independent of the parameters and the functions in the inequality. In other words, $a \lesssim b$ if there exists a positive constant C independent of a and b such that $a \leq Cb$.
2. If $a \lesssim b$ and $b \lesssim a$, we use $a \simeq b$.

4.3 Framework and method

Let Ψ be a solution to the equation

$$(\square_g + \frac{1}{6} \text{Scal}_g) \Psi = 0, \tag{4.2}$$

This is equivalent to $\phi = \Omega^{-1}\Psi$ solution to :

$$(\square_{\hat{g}} + \frac{1}{6}\text{Scal}_{\hat{g}})\phi = 0, \quad (4.3)$$

We will study the peeling for solutions to (4.2) entirely at the level of the rescaled field ϕ by performing geometric energy estimates for solutions to (4.3). This can be obtained in the following manner :

1. choose a stress-energy tensor T_{ab} and a timelike vector field (the observer) T^a ;
2. contract these two quantities to compute the associated energy current;
3. use the divergence Theorem on a closed hypersurface.

The Energy flux of a field ϕ through a hypersurface S is defined by :

$$\mathcal{E}_S(\phi) = \int_S \star T^a T_{ab}$$

where \star denotes the Hodge dual. We choose the energy momentum tensor as :

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} \hat{g}_{ab} \nabla_c \phi \nabla^c \phi, \quad (4.4)$$

where, for the sake of clarity, we denote by ∇ the rescaled connection associated with \hat{g} , the compactified metric. We need an observer that is transverse to null infinity so as to have an energy on \mathcal{I}^+ that controls all tangential derivatives instead of merely $\partial\phi/\partial u$. We choose the Morawetz vector field :

$$T = u^2 \partial_u - 2(1 + uR) \partial_R \quad (4.5)$$

We establish in Appendix A.1 that it is timelike and future-oriented on a future neighbourhood of the spatial infinity denoted by Ω_{u_0} and defined below in Section 4.3.1. Note that we choose ∂_u as the global time orientation for our spacetime.

The corresponding energy current is given by the 3-form :

$$\begin{aligned} \star T^a T_{ab} &= T^a \nabla_a \star \nabla_b \phi - \frac{1}{2} \nabla_c \phi \nabla^c \phi \star T_b \\ &= [u^2 \phi_u^2 + R^2 F(u, R)(u^2 \phi_u \phi_R - (1 + uR) \phi_R^2) + (1 + uR) |\nabla_{S^2} \phi|^2] du \wedge d\omega \\ &\quad + \frac{1}{2} [((2 + uR)^2 - 2m(u)u^2 R^3) \phi_R^2 + u^2 |\nabla_{S^2} \phi|^2] dR \wedge d\omega \end{aligned}$$

Now to obtain the energy flux through an oriented hypersurface, we integrate on it the previous 3-form.

4.3.1 Hypersurfaces and identifying vector

For $u_0 \ll -1$ given, let us consider the neighbourhood of i^0

$$\Omega_{u_0} := \{t \geq 0\} \cap \{u < u_0\}.$$

All our estimates will be established in Ω_{u_0} and u_0 will be chosen large enough in absolute value to ensure a finite number of basic inequalities, such as those given in Proposition 4.3.1. Note that since $t \geq 0$ on Ω_{u_0} and $u_0 \ll -1$, we necessarily have that $R \ll 1$ in this domain.

We define a foliation of Ω_{u_0} by spacelike hypersurfaces \mathcal{H}_s :

$$\mathcal{H}_s = \{u = -s\tilde{r}\} \cap \{u < u_0\}, \quad 0 < s \leq 1, \quad (4.6)$$

The co-normal 1-form on these hypersurfaces is given by:

$$\omega = \left[1 + \frac{s}{2}(\psi - 1)\right] du + s \frac{\psi}{F} dr, .$$

One can easily verify that $g^{-1}(\omega, \omega) > 0$, whence the normal vector $(\hat{g}^{-1}\omega)$ is timelike and the hypersurfaces \mathcal{H}_s are spacelike. The hypersurface \mathcal{H}_1 corresponds to $t = 0$ and this is where we shall set our initial data. We can in addition define the hypersurface \mathcal{H}_0 as the limit of \mathcal{H}_s as $s \rightarrow 0$; for fixed u , as $s \rightarrow 0$, we have $\tilde{r} \rightarrow +\infty$, so \mathcal{H}_0 is the set of points at infinity for which $u < u_0$,

$$\mathcal{H}_0 = \mathcal{I}^+ \cap \{u < u_0\} =: \mathcal{I}_{u_0}^+.$$

We choose an identifying vector ν i.e. a vector that is transverse to all the surfaces of the foliation, such that it crosses each surface only once and it satisfies $\nu(s) = 1$:

$$\nu = \frac{\tilde{r}^2 R^2 F}{\psi|u|} \partial_R \quad (4.7)$$

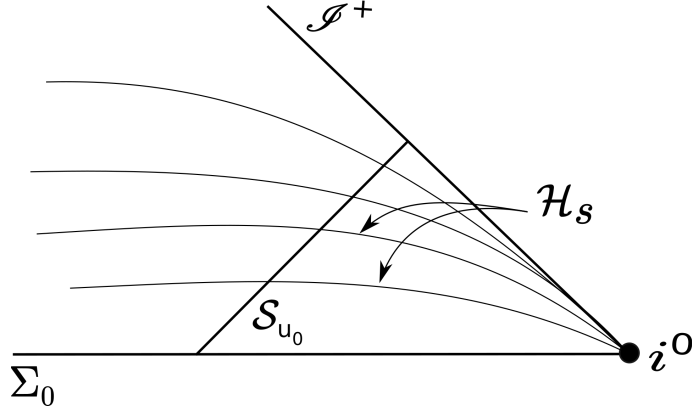


Figure 4.1: Geometrical framework near spacelike infinity

And we have also on \mathcal{H}_s :

$$du|_{\mathcal{H}_s} = \frac{2s\psi}{F[2+s(\psi-1)]} \frac{dR}{R^2} \Big|_{\mathcal{H}_s}, \quad dR|_{\mathcal{H}_s} = \frac{FR^2}{s\psi} \left[1 + \frac{s}{2}(\psi-1) \right] du|_{\mathcal{H}_s} \quad (4.8)$$

The boundary of Ω_{u_0} is made of the hypersurfaces $\mathcal{H}_0 = \mathcal{S}_{u_0}^+$, \mathcal{H}_1 and a third one defined by

$$\mathcal{S}_{u_0} := \{u = u_0\} \cap \{t \geq 0\}. \quad (4.9)$$

.

4.3.2 Energy fluxes and estimates

The expression of energy fluxes across the hypersurfaces \mathcal{H}_s , \mathcal{H}_0 and \mathcal{S}_u is as follows

$$\begin{aligned} \mathcal{E}_{\mathcal{H}_s}(\phi) = \int_{\mathcal{H}_s} & \left\{ u^2 \phi_u^2 + u^2 R^2 F(u, R) \phi_u \phi_R + |\nabla_{S^2} \phi|^2 \left[u^2 \frac{FR^2}{4s\psi} (2 + s(\psi - 1)) + (1 + uR) \right] \right. \\ & \left. + R^2 F \left[\frac{2 + s(\psi - 1)}{4s\psi} ((2 + uR)^2 - 2m(u)u^2 R^3) - (1 + uR) \right] \phi_R^2 \right\} du \wedge d\omega|_{\mathcal{H}_s} \end{aligned}$$

$$\mathcal{E}_{\mathcal{S}_{u_0}^+}(\phi) = \int_{\mathcal{H}_0} [u^2 \phi_u^2 + |\nabla_{S^2} \phi|^2] du \wedge d\omega$$

$$\mathcal{E}_{\mathcal{S}_u}(\phi) = \int_{\mathcal{S}_u} \frac{1}{2} [((2 + uR)^2 - 2m(u)u^2 R^3) \phi_R^2 + u^2 |\nabla_{S^2} \phi|^2] dR \wedge d\omega$$

These expressions are rather complicated but in Ω_{u_0} they can be simplified to obtain more usable approximate forms

Proposition 4.3.1. *Let $\varepsilon > 0$, then for $u_0 \ll -1$ large enough in absolute value, we have in Ω_{u_0} :*

$$\begin{aligned} 1 &\leq \psi < 1 + \varepsilon, \quad 1 - \varepsilon < \tilde{r}R < 1 + \varepsilon, \\ 0 &\leq R|u| < 1 + \varepsilon, \quad 1 - \varepsilon < 1 - 2m(u)R \leq 1. \end{aligned}$$

The proof is given in Appendix A.2 Section A.2.1.

Using this we can compute an equivalent form of the energy flux across \mathcal{H}_s .

Proposition 4.3.2. *In Ω_{u_0} , we have the equivalence:*

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \simeq \int_{\mathcal{H}_s} \left[u^2 |\partial_u \phi|^2 + \frac{R}{|u|} |\partial_R \phi|^2 + |\nabla_{S^2} \phi|^2 \right] du \wedge d\omega|_{\mathcal{H}_s} \quad (4.10)$$

The proof is given in Appendix A.2 Section A.2.2.

4.3.3 Error terms and Stokes' Theorem

Our approach to the Peeling is based on energy estimates. Stokes' Theorem will allow us to obtain inequalities between the energy fluxes through our different hypersurfaces, from a conservation law for the energy current. However, we have to be careful because the conservation law is only approximate. There will be two types of error terms coming from the non-zero divergence of the stress-energy tensor and from the fact that our observer is not a Killing vector field:

$$\nabla^a (T^b T_{ab}) = \nabla^{(a} T^{b)} T_{ab} + T^b \nabla^a T_{ab}.$$

The divergence of (4.4), provided ϕ satisfies equation (4.3), is given by,

$$\nabla^a T_{ab} = \square_{\hat{g}} \phi \nabla_b \phi = -2m(u)R\phi \partial_b \phi. \quad (4.11)$$

The Morawetz vector field is not Killing but its Killing form (or deformation tensor) tends to zero at infinity:

$$\nabla^{(a} T^{b)} = (-u^2 R^3 m'(u) + 2mR^2(3 + uR)) \partial_R^a \partial_R^b \quad (4.12)$$

Given $0 \leq s_1 \leq s_2 \leq 1$, we define the domain

$$\Omega_{u_0, s_1, s_2} = \Omega_{u_0} \cap \{s_1 \leq s \leq s_2\}$$

and the part of S_{u_0} that is in the boundary of Ω_{u_0, s_1, s_2} :

$$S_{u_0, s_1, s_2} = S_{u_0} \cap \{s_1 \leq s \leq s_2\}.$$

The equations (4.11) and (4.12) as well as Stokes' Theorem give us the following identity for any scalar field ϕ on Ω_{u_0}

$$\begin{aligned} \mathcal{E}_{\mathcal{H}_{s_2}}(\phi) + \mathcal{E}_{S_{u_0, s_1, s_2}}(\phi) - \mathcal{E}_{\mathcal{H}_{s_1}}(\phi) \\ = \int_{\Omega_{u_0, s_1, s_2}} (\square_{\tilde{g}}\phi \partial_T \phi + (-u^2 R^3 m'(u) + 2mR^2(3 + uR)) T_{11}(\phi)) \, d\text{Vol}^4. \end{aligned} \quad (4.13)$$

The right-hand side of (4.13) can be decomposed into an integral in s over $[s_1, s_2]$ of integrals over \mathcal{H}_s ; this is done by splitting the 4-volume measure using the identifying vector field ν (see (4.7)) as follows

$$d\text{Vol}^4 = ds \wedge \nu \lrcorner d\text{Vol}^4$$

and

$$ds \wedge \nu \lrcorner d\text{Vol}^4|_{\mathcal{H}_s} = \frac{(\tilde{r}R)^2}{\varphi|u|} du \wedge d\omega.$$

Equation (4.13) then becomes

$$\begin{aligned} \mathcal{E}_{\mathcal{H}_{s_2}}(\phi) + \mathcal{E}_{S_{u_0, s_1, s_2}}(\phi) - \mathcal{E}_{\mathcal{H}_{s_1}}(\phi) \\ = \int_{s_1}^{s_2} \left(\int_{\mathcal{H}_s} \text{Err}(\phi) du \wedge d\omega \right) ds, \end{aligned} \quad (4.14)$$

where

$$\text{Err}(\phi) = \left(\square_{\tilde{g}}\phi \partial_T \phi + (-u^2 R^3 m'(u) + 2mR^2(3 + uR)) T_{11}(\phi) \right) \frac{(\tilde{r}R)^2}{\varphi|u|}. \quad (4.15)$$

In the case where ϕ satisfies equation (4.3), the error term (4.15) becomes

$$\text{Err}(\phi) = [(-u^2 R^3 m'(u) + 2mR^2(3 + uR)) \phi_R^2 - 2mR\phi(u^2 \phi_u - 2(1 + uR)\phi_R)] \frac{(\tilde{r}R)^2}{\varphi|u|} \quad (4.16)$$

where ϕ_u and ϕ_R are respectively $\partial_u \phi$ et $\partial_R \phi$.

4.3.4 Control of the error terms

In this subsection we focus on the control of the error terms. We show that the error term is almost entirely controlled by the energy density on \mathcal{H}_s , except for, a priori, a additional L^2 term.

Proposition 4.3.3. *For $u < u_0$, $u_0 \ll -1$ and $R \rightarrow 0$:*

$$|\text{Err}(\phi)| \lesssim u^2 \phi_u^2 + \frac{R}{|u|} \phi_R^2 + \phi^2.$$

The proof is given in Appendix A.2 Section A.2.3. Then we use a Poincaré-type estimate proved in [54], in order to control the additional term.

Lemma 4.3.1. *For $u_0 < 0$, there exists $C > 0$, in \mathbb{R} such that for all bounded support function $f \in C_0^\infty(\mathbb{R})$ we have :*

$$\int_{-\infty}^{u_0} (f(u))^2 du \leq C \int_{-\infty}^{u_0} u^2 (f'(u))^2 du,$$

with $f'(u) = df/du$. And an immediate consequence is that for $0 \leq s \leq 1$:

$$\int_{\mathcal{H}_{s,u_0}} \phi^2 dud\omega \lesssim \mathcal{E}_{\mathcal{H}_s}(\phi).$$

This lemma together with Propositions 4.3.2 and 4.3.3 entail:

Theorem 4.3.1. *In the domain Ω_{u_0} , $u_0 < -1$ large enough in absolute value, we have :*

$$\int_{\mathcal{H}_s} \text{Err}(\phi) dud\omega|_{\mathcal{H}_s} \lesssim \mathcal{E}_{\mathcal{H}_s}(\phi).$$

4.4 Peeling

To obtain a peeling theory in Vaidya spacetime, we use the approximate conservation law (4.14) for ϕ solution to (4.3) and its successive derivatives, in order to obtain estimates both ways between the energy on \mathcal{I}^+ and \mathcal{H}_1 at all orders of regularity. This is done as follows:

1. we obtain the fundamental estimate for ϕ using the control of the error terms and a Grönwall inequality;
2. then we work out the equations satisfied by ϕ_R , ϕ_u , $\partial_R^k \partial_u^l \phi$ and follow the same procedure to infer higher order estimates.

Remark 4.4.1. *In the following sections, we will perform energy estimates for ϕ , a solution to (4.3) with smooth compactly supported data. The space of smooth compactly supported data on a hypersurface Σ , denoted by $\mathcal{C}_0^\infty(\Sigma)$, is completed by the norm given by the energy flux across Σ to give the function space \mathcal{H}_Σ . Thus, for the remainder of this work, we will focus on data that are compactly supported, and the results will subsequently be extended by density to data with finite energy.*

4.4.1 Fundamental estimates

Theorem 4.4.1. *For $u_0 \ll -1$, $0 \leq s_0 \leq 1$, for any ϕ solution to (4.3) associated with initial data in \mathcal{C}_0^∞ , we have :*

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \lesssim \mathcal{E}_{\mathcal{H}_1}(\phi), \quad (4.17)$$

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \lesssim \mathcal{E}_{\mathcal{I}_{u_0}^+}(\phi) + \mathcal{E}_{S_{u_0,0,s}}(\phi). \quad (4.18)$$

For $s = 0$, (4.17) becomes:

$$\mathcal{E}_{\mathcal{H}_{\mathcal{I}_{u_0}^+}}(\phi) \lesssim \mathcal{E}_{\mathcal{H}_1}(\phi)$$

and for $s = 1$, (4.18) gives

$$\mathcal{E}_{\mathcal{H}_1}(\phi) \lesssim \mathcal{E}_{\mathcal{I}_{u_0}^+}(\phi) + \mathcal{E}_{S_{u_0}}(\phi).$$

Proof. Conservation law (4.14) entails

$$\mathcal{E}_{\mathcal{H}_s}(\phi) + \mathcal{E}_{S_{u_0,s,1}}(\phi) - \mathcal{E}_{\mathcal{H}_1}(\phi) \leq \int_1^s \int_{\mathcal{H}_{\tilde{s}}} |\text{Err}(\phi)| \, dud\omega d\tilde{s}. \quad (4.19)$$

Before continuing the demonstration we have to prove that on surfaces \mathcal{S}_u we have a non-negative energy. As a consequence of the estimates of Proposition 4.3.1, we have in Ω_{u_0} :

$$(2 + uR)^2 - 2m(u)u^2R^3 \simeq (2 + uR)^2 > 0$$

and the density of energy on S_{u_0} is therefore non-negative. Note that this can also be inferred geometrically, since the Morawetz vector field T^a is timelike and future-oriented on the whole of Ω_{u_0} , S_{u_0} is null and we choose the future-oriented normal $-\partial_R$ on it and finally the stress-energy tensor (4.4) satisfies the dominant energy condition.

So (4.14) yields:

$$\mathcal{E}_{\mathcal{H}_s}(\phi) - \mathcal{E}_{\mathcal{H}_1}(\phi) \leq \int_1^s \int_{\mathcal{H}_{\tilde{s}}} |\text{Err}(\phi)| dud\omega d\tilde{s}. \quad (4.20)$$

Theorem 4.3.1 allows to control the error term by the energy density on $\mathcal{H}_{\tilde{s}}$ and so:

$$\mathcal{E}_{\mathcal{H}_s}(\phi) - \mathcal{E}_{\mathcal{H}_1}(\phi) \leq \int_1^s \mathcal{E}_{\mathcal{H}_{\tilde{s}}}(\phi) d\tilde{s}. \quad (4.21)$$

Grönwall's lemma then gives equation (4.17).

The proof of (4.18) is similar. We have

$$\begin{aligned} \mathcal{E}_{\mathcal{H}_s}(\phi) &\leq \mathcal{E}_{\mathcal{I}_{u_0}^+}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0,0,s}}(\phi) + \int_0^s \int_{\mathcal{H}_{\tilde{s}}} |\text{Err}(\phi)| dud\omega d\tilde{s}, \\ \mathcal{E}_{\mathcal{H}_s}(\phi) &\lesssim \mathcal{E}_{\mathcal{I}_{u_0}^+}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0,0,s}}(\phi) + \int_0^s \mathcal{E}_{\mathcal{H}_{\tilde{s}}}(\phi) d\tilde{s}. \end{aligned}$$

and we conclude using Grönwall's lemma. \square

4.4.2 Higher order estimates

Theorem 4.4.1 gives us the basic estimates both ways between \mathcal{I}^+ and the initial data hypersurface. In order to prove estimates for derivatives of the solution, we commute partial derivatives into Equation 4.3. The Vaidya metric is not stationary and this means that ∂_u does not commute with the wave equation. As a consequence, a control of $\partial_u\phi$ will require a joint control of $\partial_R\phi$ because of the relations:

$$[\partial_u, \square_{\hat{g}}] = 2m'(u)R^3\partial_R^2 + 6m'(u)R^2\partial_R, \quad (4.22)$$

$$[\partial_u, \text{Scal}_{\hat{g}}] = 12m'(u)R. \quad (4.23)$$

By contrast, when commuting the R derivative with the d'Alembertian, we only obtain error terms involving ∂_R :

$$[\partial_R, \square_{\hat{g}}] = -2R(1 - 3mR)\partial_R^2 - 2(1 - 6mR)\partial_R, \quad (4.24)$$

$$[\partial_R, \text{Scal}_{\hat{g}}] = 12m(u). \quad (4.25)$$

Hence, just as in the Schwarzschild case, we can control the successive derivatives with respect to R independently of the other variables. However, the energy of $\partial_R^k \partial_u^l \phi$ will need to be controlled by those of $\partial_R^p \partial_u^q \phi$ with $p + q \leq l + k$ and $k \leq p \leq k + 1$. We obtain the following result.

Theorem 4.4.2. *We have the following inequalities:*

1. *The R derivatives, are controlled independently of the others, just like angular derivatives: for all $k \in \mathbb{N}$,*

$$\begin{aligned} \mathcal{E}_{\mathcal{I}_{u_0}^+}(\partial_R^k \phi) &\lesssim \mathcal{E}_{\mathcal{H}_1}(\partial_R^k \phi), \\ \mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^k \phi) &\lesssim \mathcal{E}_{\mathcal{I}_{u_0}^+}(\partial_R^k \phi) + \mathcal{E}_{\mathcal{S}_{u_0}}(\partial_R^k \phi), \\ \mathcal{E}_{\mathcal{I}_{u_0}^+}(\nabla_{S^2}^k \phi) &\lesssim \mathcal{E}_{\mathcal{H}_1}(\nabla_{S^2}^k \phi), \\ \mathcal{E}_{\mathcal{H}_{1,u_0}}(\nabla_{S^2}^k \phi) &\lesssim \mathcal{E}_{\mathcal{I}_{u_0}^+}(\nabla_{S^2}^k \phi) + \mathcal{E}_{\mathcal{S}_{u_0}}(\nabla_{S^2}^k \phi). \end{aligned}$$

2. *The general control on partial derivatives of all orders has the following form: for all $k, l, n \in \mathbb{N}$,*

$$\begin{aligned} \mathcal{E}_{\mathcal{I}_{u_0}^+}(\partial_R^k \partial_u^l \nabla_{S^2}^n \phi) &\lesssim \sum_{p+q \leq k+l} \mathcal{E}_{\mathcal{H}_1}(\partial_R^p \partial_u^q \nabla_{S^2}^n \phi), \\ \mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^k \partial_u^l \nabla_{S^2}^n \phi) &\lesssim \sum_{p+q \leq k+l} \left[\mathcal{E}_{\mathcal{I}_{u_0}^+}(\partial_R^p \partial_u^q \nabla_{S^2}^n \phi) + \mathcal{E}_{\mathcal{S}_{u_0}}(\partial_R^p \partial_u^q \nabla_{S^2}^n \phi) \right]. \end{aligned}$$

The proof is given in Appendix A.2.4.

4.5 Conclusion

The peeling-off property, as described in Section 4.4, gives us the existence of a large class of initial data that satisfies the peeling condition at any order on the Vaidya spacetime. It is important to notice that this class of data is as large as that which peels on the

Schwarzschild spacetime. According to [54], where the authors established that the class of data satisfying peeling on the Schwarzschild spacetime is the same as in Minkowski space, we can state that the peeling-off property on the Vaidya spacetime is also a natural generalization of the definition on Minkowski spacetime. The only difference that arises compared to the cases of Schwarzschild and Minkowski spacetimes is that controlling the higher-order regularity of the field requires the use of a combination of transverse and tangential derivatives instead of simply the order of derivatives of the field. However, this does not result in any loss of global regularity.

A NOTE ABOUT THE PEELING IN THE PAST

5.1 Geometrical framework

In Chapter 4, we establish the peeling-off property of the wave equation in the future of i_0 . This was done by determining the class of data that ensures the field's sufficient regularity on \mathcal{I}^+ . In the Schwarzschild spacetime, this is entirely analogous to the data class that guarantees regularity at the past null infinity : \mathcal{I}^- . This equivalence arises due to the symmetry between past and future null infinity. However, in the Vaidya spacetime, the framework is different, necessitating a careful examination of what occurs in the past, particularly near the past null infinity and i_0 . Due to the mass variation and dynamic behavior of the metric, we do not have an explicit expression for v in the Vaidya spacetime, unlike Schwarzschild's spacetime where $v = u + 2r_* = u + 2r + 4m \log(r - 2m)$. In the Vaidya spacetime, we turn to (v, R, θ, φ) coordinates to establish equivalence between the norm of the field ϕ near \mathcal{I}^- and the norm of the initial data on the hypersurface $\{t = 0\}$. First, we have

$$du = \frac{1}{\psi} dv + \frac{2}{R^2 F} dR, \quad (5.1)$$

hence :

$$\hat{g} = \frac{R^2 F}{\psi^2} dv^2 + \frac{2}{\psi} dv dR - d\omega^2,$$

and the inverse metric becomes :

$$\hat{g}^{-1} = -R^2 F \partial_R^2 + 2\psi \partial_R \partial_v - \partial_\omega^2. \quad (5.2)$$

The mass $m(u)$ in this system of coordinates is now a function $m = m(v, R)$ and because the scalar curvature is coordinate independent, we have :

$$\text{Scal}_{\hat{g}} = 12m(v, R)R. \quad (5.3)$$

The d'Alembertian operator $\square_{\hat{g}}$ expressed in (v, R, θ, φ) coordinates is :

$$\square_{\hat{g}} = 2\psi\partial_v\partial_R - \psi\partial_R\left(\frac{R^2F}{\psi}\right)\partial_R - \Delta_{S^2}. \quad (5.4)$$

The stress energy tensor (6.5) is still the same, however $\langle\nabla\phi, \nabla\phi\rangle$ reads :

$$\langle\nabla\phi, \nabla\phi\rangle = -R^2F\phi_r^2 + 2\psi\phi_v\phi_R - |\nabla_{S^2}\phi|^2. \quad (5.5)$$

A crucial point is to chose the good Morawetz vector. Naively we would like to take the Morawetz vector on the Minkowski spacetime in v, R, ω coordinates :

$$T = v^2\partial_v + 2(1 - vR)\partial_R.$$

However, this will not bring the sufficient decay in the error term to apply the method we develop before. That is why we use the vector :

$$T = \psi v^2\partial_v + 2(1 - vR)\partial_R. \quad (5.6)$$

The sign of the norm near the spacelike infinity will be given later, in Remark 5.2.1.

5.2 Vector field method and energy fluxes.

As in the future we define the neighbourhood of i_0 for $v_0 \gg 1$:

$$\Omega_{v_0} := \{t \leq 0\} \cap \{v > v_0\}.$$

We define a foliation of this domain by spacelike hypersurfaces \mathcal{H}_s :

$$\mathcal{H}_s = \{v = s\tilde{r}\} \cap \{v > v_0\}, 0 < s \leq 1, \quad (5.7)$$

with $v_0 \gg 1$ given. The co-normal 1-form on these hypersurfaces is :

$$\kappa = \left(1 - \frac{s\psi - 1}{2\psi}\right) dv - \frac{s}{F} dr.$$

This form is clearly timelike near i_0 , hence \mathcal{H}_s are spacelike hypersurfaces and $H_{\{s=1\}}$, denoted after \mathcal{H}_1 , corresponds to $\{t = 0\}$ and we set it as the hypersurface of initial data. Furthermore we define the hypersurface $\mathcal{H}_{\{s=0\}}$ as the limit of \mathcal{H}_s as $s \rightarrow 0$, for fixed v , then this corresponds to $\tilde{r} \rightarrow \infty$. Hence \mathcal{H}_0 is the set of points at infinity for which $v > v_0$,

$$\mathcal{H}_0 = \mathcal{I}^- \cap \{v > v_0\} =: \mathcal{I}_{v_0}^-.$$

The identifying vector ν is chosen as in the future to be transverse to all the surfaces \mathcal{H}_s and to cross them only once, by setting $\nu(s) = 1$, this leads to :

$$\nu = \frac{(\tilde{r}R)^2 F}{v} \partial_R, \quad (5.8)$$

and on \mathcal{H}_s we have :

$$dR|_{\mathcal{H}_s} = -\frac{R^2 F}{s} \left(1 - \frac{s}{2} \left(1 - \frac{1}{\psi}\right)\right) dv. \quad (5.9)$$

Finally we define \mathcal{S}_{v_0} the hypersurface that closes the boundary of Ω_{v_0} together with \mathcal{H}_0 and \mathcal{H}_1 :

$$\mathcal{S}_{v_0} := \{v = v_0\} \cap \{t \leq 0\}.$$

All our estimates will be established in Ω_{v_0} and v_0 will be chosen sufficiently large to ensure the following estimates :

Proposition 5.2.1. *Let $\varepsilon > 0$, then for $v_0 \gg 1$ large enough, we have in Ω_{v_0} :*

$$\begin{aligned} 1 &\leq \psi < 1 + \varepsilon, \quad 1 - \varepsilon < \tilde{r}R < 1 + \varepsilon, \\ 0 &\leq Rv < 1 + \varepsilon, \quad 1 - \varepsilon < F = 1 - 2m(v, R)R \leq 1. \end{aligned}$$

Proof : We just have to prove the estimate concerning vR . The three others, introduced in Proposition 4.3.1 are still valid in Ω_{v_0} . The fourth is proven directly :

$$\begin{aligned} 0 &\leq s < 1 \\ 0 &\leq \frac{v}{\tilde{r}} < 1 \end{aligned}$$

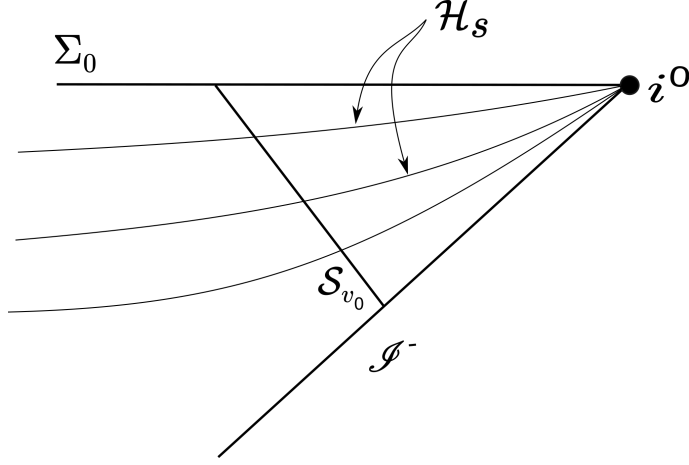


Figure 5.1: Geometrical framework in the past of the spacelike infinity

$$0 \leq \frac{Rv}{\tilde{r}R} < 1$$

$$0 \leq Rv < 1 + \varepsilon. \quad \square$$

Remark 5.2.1. *It is now clear, using Proposition 5.2 that T^a is a timelike vector in a neighbourhood of i_0 . The norm of T is :*

$$\hat{g}(T, T) = v^2 (R^2 v^2 - 4Rv + 4 - 2R^3 v^2 m(v, R)) .$$

The function $f(x) = x^2 - 4x + 4$ is still positive, except for $x = 2$ so if we take $x = vR$, we have $x \in [0, 1]$ with $f(0) = 4$ and $f(1) = 1$, thus :

$$v^2 (1 - 2R^3 v^2 m) \leq \hat{g}(T, T) \leq v^2 (4 - 2R^3 v m(v, R)) .$$

For sufficiently large value of v_0 and $v \geq v_0$ we have $\hat{g}(T, T) > 0$.

The energy current $J_a = T_{ab} T^a$ reads in this context as :

$$J_a = \left[\psi v^2 \phi_v^2 + \frac{R^2 F}{2} \left(\frac{R^2 F v^2}{\psi} + \frac{2(1 - vR)}{\psi} \right) \phi_R^2 - R^2 F v^2 \phi_r \phi_v \right. \\ \left. + \frac{1}{2} \left(\frac{R^2 F v^2}{\psi} + \frac{2(1 - vR)}{\psi} \right) |\nabla_{S^2} \phi|^2 \right] dv \\ + \left[2(1 - vR) \phi_R^2 + \frac{v^2 r^2 F}{2} \phi_R^2 + \frac{v^2}{2} |\nabla_{S^2} \phi|^2 \right] dR ,$$

and the Hodge dual of this current is given by :

$$\begin{aligned} \star J_a &= \left[\psi v^2 \phi_v^2 - R^2 v^2 F \phi_R \phi_v + \frac{R^2 F}{\psi} \left(R^2 F v^2 + \frac{3}{2}(1 - vR) \right) \phi_R^2 \right. \\ &\quad \left. + \frac{1}{\psi} \left(R^2 F v^2 + (1 - vR) \right) |\nabla_{S^2} \phi|^2 \right] dv \wedge d\omega^2 \\ &\quad - \left[\left(\frac{1}{2} v^2 R^2 F + 2(1 - vR) \right) \phi_R^2 + \frac{v^2}{2} |\nabla_{S^2} \phi|^2 \right] dR \wedge d\omega^2, \end{aligned}$$

with in these coordinates : $d\text{Vol}^4 = -\frac{1}{\psi} dv \wedge dR \wedge d\omega$. Then we get the energy fluxes across the hypersurfaces \mathcal{H}_s , \mathcal{S}_{v_0} and \mathcal{I}^- :

$$\mathcal{E}_{\mathcal{I}^-}(\phi) = \int_{\mathcal{I}^-} \left(\psi v^2 \phi_v^2 + \frac{1}{\psi} |\nabla_{S^2} \phi|^2 \right) dv \wedge d\omega^2, \quad (5.10)$$

$$\mathcal{E}_{\mathcal{S}_{v_0}}(\phi) = \int_{\mathcal{S}_{v_0}} \left(\left(\frac{1}{2} v^2 R^2 F + 2(1 - vR) \right) \phi_R^2 + \frac{v^2}{2} |\nabla_{S^2} \phi|^2 \right) dR \wedge d\omega^2. \quad (5.11)$$

and the energy flux through \mathcal{H}_s is :

$$\begin{aligned} \mathcal{E}_{\mathcal{H}_s}(\phi) &= \int_{\mathcal{H}_s} \left\{ \psi v^2 \phi_v^2 - R^2 F v^2 \phi_R \phi_v + R^2 F \left[\frac{R^2 v^2 F}{\psi} + \frac{3}{2}(1 - vR) \right. \right. \\ &\quad \left. \left. + \frac{1}{s} \left(1 - \frac{s}{2} \left(1 - \frac{1}{\psi} \right) \right) \left(2(1 - vR) + \frac{v^2 R^2 F}{2} \right) \right] \phi_R^2 \right. \\ &\quad \left. + \left[\frac{R^2 v^2 F}{\psi} + 2(1 - vR) + \frac{R^2 v^2 F}{2s} \left(1 - \frac{s}{2} \left(1 - \frac{1}{\psi} \right) \right) \right] |\nabla_{S^2} \phi|^2 \right\} dv \wedge d\omega^2. \end{aligned}$$

Remark 5.2.2. Note that energy flux $\mathcal{E}_{\mathcal{S}_{v_0}}(\phi)$ is non-negative for v_0 sufficiently large because, in this context :

$$\frac{1}{2} v^2 R^2 F + 2(1 - vR) \simeq \frac{1}{2} v^2 R^2 + 2(1 - vR).$$

The function $f(x) = 1/2x^2 - 2x + 2$ is always non negative, so let $x = vR \in [0, 1 + \epsilon[$ with $\epsilon > 0$, then :

$$\frac{1}{2} \leq f(vR) \leq 2,$$

and this ensures that $\mathcal{E}_{\mathcal{S}_{v_0}}(\phi) \geq 0$.

Proposition 5.2.2. *The energy flux across \mathcal{H}_s is equivalent to :*

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \simeq \int_{\mathcal{H}_s} \left(v^2 \phi_v^2 + \frac{R}{v} \phi_R^2 + |\nabla_{S^2} \phi|^2 \right) dv \wedge d\omega^2. \quad (5.12)$$

The proof is done in Appendix B.

5.3 Error terms and energy estimates

As we did in subsection 4.3.3 we want to apply Stoke's Theorem on the domain Ω_{v_0} and this requires to control error terms that come from the divergence of the current :

$$\nabla^a J_a = \nabla^a (T^b T_{ab}) = \nabla^a (T^b) T_{ab} + T^b \nabla^a T_{ab}.$$

The divergence of the stress energy tensor is similar to what we obtained in (u, R, ω) coordinates :

$$\nabla^a T_{ab} = \square_{\hat{g}} \phi \nabla_b \phi = -2m(v, R) R \phi \nabla_b \phi. \quad (5.13)$$

The main difference comes from the Killing form of the Morawetz vector that is :

$$\nabla^a (T^b) = K_{vv} \partial_v^2 + K_{vR} \partial_v \partial_R + K_{RR} \partial_R^2, \quad (5.14)$$

with

$$K_{vv} = v^2 \psi \frac{\partial \psi}{\partial R}, \quad (5.15)$$

$$K_{vR} = (2v\psi(\psi - 1) + (2R^3 v^2 m(v, R) - Rv^2 + 2Rv - 2)) \frac{\partial \psi}{\partial R}, \quad (5.16)$$

$$K_{RR} = -R^3 v^2 \psi \frac{\partial m}{\partial v} - 2R\psi + 2(R^3 v - 3R^2) m(v, R) + 2(R^4 v - R^3) \frac{\partial m}{\partial R} + 2R. \quad (5.17)$$

We apply the same method as before, i.e. let Ω_{v_0, s_1, s_2} be the domain defined by :

$$\Omega_{v_0, s_1, s_2} = \Omega_{v_0} \cap \{s_1 \leq s \leq s_2\},$$

with $0 \leq s_1 \leq s_2 \leq 1$. The part of S_{v_0} that is in the boundary of Ω_{v_0, s_1, s_2} is referred as :

$$S_{v_0, s_1, s_2} = S_{v_0} \cap \{s_1 \leq s \leq s_2\}.$$

We obtain then an energy identity for any scalar field ϕ on Ω_{v_0} by applying Stoke's theorem between $\mathcal{H}_{s_2}, \mathcal{H}_{s_1}$ and S_{v_0, s_1, s_2} and we replace $\nabla^a J_a$ by the expressions (5.13) and (5.14):

$$\mathcal{E}_{\mathcal{H}_{s_2}}(\phi) + \mathcal{E}_{S_{v_0, s_1, s_2}}(\phi) - \mathcal{E}_{\mathcal{H}_{s_1}}(\phi) = \int_{\Omega_{v_0, s_1, s_2}} \left(\square_{\hat{g}} \phi \partial_T \phi + \nabla^{(a} T^{b)} T_{ab}(\phi) \right) d\text{Vol}^4. \quad (5.18)$$

The right-hand side of (5.18) is decomposed into an integral in s over $[s_1, s_2]$ of integrals over \mathcal{H}_s ; this is done by splitting the 4-volume measure using the identifying vector field ν (see (5.8)) as follows

$$d\text{Vol}^4 = ds \wedge (\nu \lrcorner d\text{Vol}^4)$$

and

$$\nu \lrcorner d\text{Vol}^4|_{\mathcal{H}_s} = \frac{(\tilde{r}R)^2 F}{\psi v} dv \wedge d\omega.$$

Equation (5.18) then becomes

$$\mathcal{E}_{\mathcal{H}_{s_2}}(\phi) + \mathcal{E}_{S_{v_0, s_1, s_2}}(\phi) - \mathcal{E}_{\mathcal{H}_{s_1}}(\phi) = \int_{s_1}^{s_2} \left(\int_{\mathcal{H}_s} \text{Err}(\phi) dv \wedge d\omega \right) ds, \quad (5.19)$$

where

$$\begin{aligned} \text{Err}(\phi) = & \frac{(R\tilde{r})^2 F}{v\psi} \left[\left(K_{RR} + \frac{1}{2} \left(\frac{R^2 F}{\psi^2} K_{vv} + 2K_{vR} \right) R^2 F \right) \phi_R^2 \right. \\ & + \left(K_{vR} - \psi \left(\frac{R^2 F}{\psi^2} K_{vv} + 2K_{vR} \right) \right) \phi_R \phi_v + K_{vv} \phi_v^2 + \left(\frac{R^2 F}{2\psi^2} K_{vv} + 2K_{vR} \right) |\nabla_{S^2} \phi|^2 \\ & \left. - 2m(v, R)v^2 \psi R \phi \phi_v - 4m(v, R)(1 - vR)R \phi \phi_R \right]. \end{aligned}$$

5.4 Control of the error terms and fundamental estimates

The method is still the same, we want to obtain estimates in the both ways between the energy norms of the initial data and the energy norms of the field on the boundary. This implies to control error terms in the conservation law (5.19) in order to use Grönwall's lemma to obtain fundamental estimates as in section 4.4.1. This is ensured by the following propositions and lemmas (proofs are done in appendix B)

Proposition 5.4.1. *For $v > v_0$, $v_0 \gg 1$ and $R \rightarrow 0$:*

$$|\text{Err}(\phi)| \lesssim \frac{1}{\sqrt{s}} \left(v^2 \phi_v^2 + \frac{R}{v} \phi_R^2 + |\nabla_{S^2} \phi|^2 + \phi^2 \right),$$

Then using a Poincaré-type estimate :

Lemma 5.4.1. *For $v_0 > 0$ there exists a constant $C > 0$ such that for all compactly supported function $f \in C_0^\infty(\mathbb{R})$:*

$$\int_{v_0}^\infty (f(v))^2 dv \leq C \int_{v_0}^\infty v^2 (f'(v))^2 dv,$$

with $f'(v) = df/dv$. This leads for any $s \in [0, 1]$:

$$\int_{\mathcal{H}_{s,v_0}} \phi^2 dv d\omega \lesssim \mathcal{E}_{\mathcal{H}_s}(\phi).$$

We obtain the following property of the error term :

Theorem 5.4.1. *In the domain Ω_{v_0} , for $v_0 \gg 1$ large enough we have :*

$$\int_{\mathcal{H}_s} |\text{Err}(\phi)| \lesssim \frac{1}{\sqrt{s}} \mathcal{E}_{\mathcal{H}_s}(\phi).$$

Remark 5.4.1. *We point out that the choice of the Morawetz vector field is crucial to obtain the theorem 5.4.1. The naive idea is to transform in the Minkowski spacetime the Morawetz vector field :*

$$T^a = u^2 \partial_u^a - 2(1 + uR) \partial_R^a,$$

using the change of coordinates : $v = u + 2r$ and $R = 1/r$ that gives :

$$T^a = v^2 \partial_v^a + 2(1 - vR) \partial_R^a$$

and then to use this expression as the Morawetz vector field on the Vaidya spacetime. This method works in the Schwarzschild case, however here, this leads to :

$$\begin{aligned} \nabla^{(a} T^{b)} &= \left(-v^2 \frac{\partial \psi}{\partial v} + 2(Rv - 1) \frac{\partial \psi}{\partial R} \right) \frac{\partial}{\partial v} \otimes \frac{\partial}{\partial R} \\ &+ \left(-R^3 v^2 \frac{\partial m}{\partial v} + 2R(1 - \psi) + 2(R^3 v - 3R^2)m + 2(R^4 v - R^3) \frac{\partial m}{\partial R} \right) \frac{\partial}{\partial R} \otimes \frac{\partial}{\partial R}. \end{aligned}$$

Thus, because $\nu_0 \simeq \frac{1}{v}$ (see (5.8)), the dominant terms in the error term are like :

$$\text{Err}(\phi) \simeq +\mathcal{O}(v^2) \phi_v^2 + \mathcal{O}(v) \phi_v \phi_R + \mathcal{O}(R) \phi_R^2 + \mathcal{O}(v) |\nabla_{S^2} \phi|^2 + \phi^2$$

and this does not have sufficient decay to be controlled by the energy density due to the cross term $\mathcal{O}(v) \phi_v \phi_R$ that cannot be bounded by $v^2 \phi_v^2 + R/v \phi_R^2$. This is why we make the suitable choice of the vector :

$$T^a = \psi v^2 \partial_v^a + 2(1 - vR) \partial_R^a,$$

that leads to sufficient decay in the error term without any assumptions on the decay of ψ . We only assume that ψ and its derivatives are bounded functions.

5.5 Peeling

5.5.1 Fundamental estimates

Theorem 5.5.1. For $v_0 \gg 1$, $0 \leq s_0 \leq 1$, for ϕ any solution to (6.3) associated with initial data in \mathcal{C}_0^∞ , we have :

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \lesssim \mathcal{E}_{\mathcal{H}_1}(\phi), \quad (5.20)$$

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \lesssim \mathcal{E}_{\mathcal{I}_{v_0}^-}(\phi) + \mathcal{E}_{S_{v_0,0,s}}(\phi). \quad (5.21)$$

For $s = 0$, the first estimate becomes :

$$\mathcal{E}_{\mathcal{I}_{v_0}^-}(\phi) \lesssim \mathcal{E}_{\mathcal{H}_1}(\phi),$$

and for $s = 1$, the second one gives :

$$\mathcal{E}_{\mathcal{H}_1}(\phi) \lesssim \mathcal{E}_{\mathcal{I}_{v_0}^-}(\phi) + \mathcal{E}_{S_{v_0}}(\phi). \quad (5.22)$$

Proof : We control the left hand side of the conservation law (5.19) using the absolute value of the error term :

$$\mathcal{E}_{\mathcal{H}_{s_2}}(\phi) + \mathcal{E}_{S_{v_0,s_1,s_2}}(\phi) - \mathcal{E}_{\mathcal{H}_{s_1}}(\phi) \leq \int_{s_1}^{s_2} \left(\int_{\mathcal{H}_s} |\text{Err}(\phi)| dv \wedge d\omega \right) ds.$$

Due to the non-negativity of the energy across S_v (see Remark 5.2.2), this turns into :

$$|\mathcal{E}_{\mathcal{H}_{s_2}}(\phi) - \mathcal{E}_{\mathcal{H}_{s_1}}(\phi)| \leq \int_{s_1}^{s_2} \left(\int_{\mathcal{H}_s} |\text{Err}(\phi)| dv \wedge d\omega \right) ds.$$

Let $s_1 = s$ and $s_2 = 1$, then Theorem 5.4.1 states that the integral of the absolute value of the error term on \mathcal{H}_s is controlled by the energy flux for the rescaled field ϕ across \mathcal{H}_s :

$$|\mathcal{E}_{\mathcal{H}_1}(\phi) - \mathcal{E}_{\mathcal{H}_s}(\phi)| \leq \int_s^1 \frac{1}{\sqrt{s}} \mathcal{E}_{\mathcal{H}_s}(\phi) ds.$$

Since $1/\sqrt{s}$ is an integrable function on $[0, 1]$, Grönwall's lemma yields :

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \lesssim \mathcal{E}_{\mathcal{H}_1}(\phi).$$

The proof of the second estimate is similar. Let $s_1 = s$ and $s_2 = 0$ we have:

$$\begin{aligned} \mathcal{E}_{\mathcal{H}_s}(\phi) &\leq \mathcal{E}_{S_{v_0, s_1, s_2}}(\phi) + \mathcal{E}_{\mathcal{J}_{v_0}^-}(\phi) + \int_0^s \left(\int_{\mathcal{H}_s} |\text{Err}(\phi)| dv \wedge d\omega \right) ds \\ \mathcal{E}_{\mathcal{H}_s}(\phi) &\lesssim \mathcal{E}_{S_{v_0, s_1, s_2}}(\phi) + \mathcal{E}_{\mathcal{J}_{v_0}^-}(\phi) + \int_0^s \frac{1}{\sqrt{s}} \mathcal{E}_{\mathcal{H}_s}(\phi) ds \end{aligned}$$

Using the Grönwall lemma, this leads to :

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \lesssim \mathcal{E}_{S_{v_0, s_1, s_2}}(\phi) + \mathcal{E}_{\mathcal{J}_{v_0}^-}(\phi). \quad \square$$

5.5.2 Higher order estimates

In the previous subsection we proved an equivalence between the norm of the initial data and the norm of the rescaled field at the boundary, in a neighbourhood of spacelike infinity. We now want to extend this equivalence to the successive derivatives of the solution $\partial_v^l \partial_R^k \phi$. As in section 4.4.2, the difficulty comes from the new terms in the error term $\text{Err}(\partial_v^l \partial_R^k \phi)$ that appear by commuting partial derivatives ∂_v and ∂_R into (6.3). As in the past ∂_v does not commute with the wave equation because of the non-stationary of the Vaidya metric, and the control of $\partial_v \phi$ will require also the control of $\partial_R \phi$ because of :

$$\begin{aligned} [\partial_v, \square_{\hat{g}}] &= 2 \frac{\partial \psi}{\partial v} \partial_v \partial_R + 2R^3 \frac{\partial m}{\partial v} \partial_R^2 \\ &+ \left[R^2 \left(6 \frac{\partial m}{\partial v} - \frac{F}{\psi^2} \frac{\partial \psi}{\partial v} + \frac{F}{\psi} \frac{\partial^2 \psi}{\partial v \partial R} \right) + R^3 \left(2 \frac{\partial^2 m}{\partial v \partial R} - \frac{2}{\psi} \frac{\partial m}{\partial v} \right) \right] \partial_R, \end{aligned}$$

$$[\partial_v, \text{Scal}_{\hat{g}}] = 12 \frac{\partial m}{\partial v} R.$$

Because the function ψ depends on v and R , the expression of the commutator $[\partial_R, \square_{\hat{g}}]$ does not only involve R -derivative and derivatives with respect to v appears unlike the expression in the future :

$$\begin{aligned} [\partial_R, \square_{\hat{g}}] &= 2 \frac{\partial \psi}{\partial R} \partial_R \partial_v - \left(2R - 6mR - 2 \frac{\partial m}{\partial R} R^3 \right) \partial_R^2 \\ &+ \left[-2 + R \left(12m + \frac{2}{\psi} \frac{\partial \psi}{\partial R} \right) + R^2 \left(12 \frac{\partial m}{\partial R} - \frac{6m}{\psi} \frac{\partial \psi}{\partial R} + \frac{1}{\psi^2} \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial R} \right)^2 \right) \right. \\ &+ \left. R^3 \left(2 \frac{\partial^2 m}{\partial R^2} - \frac{2}{\psi} \frac{\partial m}{\partial R} \frac{\partial \psi}{\partial R} - \frac{2m}{\psi} \frac{\partial^2 \psi}{\partial R^2} + \frac{2m}{\psi^2} \left(\frac{\partial \psi}{\partial R} \right)^2 \right) \right] \partial_R, \end{aligned}$$

$$[\partial_R, \text{Scal}_{\hat{g}}] = 12m(v, R) + 12R \frac{\partial m}{\partial R}.$$

However, the peeling at higher order is very similar to this in the future because we can control the successive derivatives with respect to R independently of the other variables. This is not the case concerning the energy of $\partial_R^k \partial_v^l \phi$ that needs to be controlled by those of $\partial_R^p \partial_v^q \phi$, with $p + q \leq l$ and $k \leq p \leq k + 1$. This is detailed in the following theorem.

Theorem 5.5.2. *We have the following estimates :*

1. *Then angular derivatives and the derivatives with respect to R are controlled independently of the others : for all $k \in \mathbb{N}$,*

$$\begin{aligned} \mathcal{E}_{\mathcal{I}_{v_0}^-}(\nabla_{S^2}^k \phi) &\lesssim \mathcal{E}_{\mathcal{H}_{1, v_0}}(\nabla_{S^2}^k \phi), \\ \mathcal{E}_{\mathcal{H}_{1, v_0}}(\nabla_{S^2}^k \phi) &\lesssim \mathcal{E}_{\mathcal{H}_1}(\nabla_{S^2}^k \phi) + \mathcal{E}_{\mathcal{S}_{v_0}}(\nabla_{S^2}^k \phi), \\ \mathcal{E}_{\mathcal{I}_{v_0}^-}(\partial_R^k \phi) &\lesssim \mathcal{E}_{\mathcal{H}_1}(\partial_R^k \phi), \\ \mathcal{E}_{\mathcal{H}_{1, v_0}}(\partial_R^k \phi) &\lesssim \mathcal{E}_{\mathcal{I}_{v_0}^-}(\partial_R^k \phi) + \mathcal{E}_{\mathcal{S}_{v_0}}(\partial_R^k \phi). \end{aligned}$$

2. *The general control on partial derivatives of all orders has the following form : for all $k, l, n \in \mathbb{N}$,*

$$\mathcal{E}_{\mathcal{I}_{v_0}^-}(\partial_R^k \partial_v^l \nabla_{S^2}^n \phi) \lesssim \sum_{p+q \leq k+l} \mathcal{E}_{\mathcal{H}_{1, v_0}}(\partial_R^p \partial_v^q \nabla_{S^2}^n \phi),$$

$$\mathcal{E}_{\mathcal{H}_{1,v_0}}(\partial_R^k \partial_v^l \nabla_{S^2}^n \phi) \lesssim \sum_{p+q \leq k+l} \left[\mathcal{E}_{\mathcal{I}_{v_0}^-}(\partial_R^p \partial_v^q \nabla_{S^2}^n \phi) + \mathcal{E}_{\mathcal{S}_{v_0}}(\partial_R^p \partial_v^q \nabla_{S^2}^n \phi) \right].$$

The proof is very similar to the proof of the theorem 4.4.2 and is done in the Appendix B.

CONFORMAL SCATTERING OF THE WAVE EQUATION IN THE VAIDYA SPACETIME.

6.1 Introduction

The theory of scattering emerges naturally from physics with the objective of characterizing a field by its asymptotic properties, specifically through the observation of distant events (both in terms of distance and time). Constructing a scattering operator is not only a matter of its existence; it is also necessary to prove that the future (or past) behaviour of the field entirely and uniquely characterizes the solution in the rest of the spacetime. Historically, the study of these asymptotic properties has relied on spectral methods that are ill-suited to generic time dependence. An alternative approach, that authorises the time-dependence of the metric, is to use the concept of conformal compactification, as introduced by Penrose in the 1960s in a series of articles ([72], [73]; for a survey of conformal methods, see [74]). The physical spacetime (\mathcal{M}, g) , is embedded into a larger spacetime, denoted by $(\hat{\mathcal{M}}, \hat{g})$, which is called the compactified spacetime or un-physical spacetime. \mathcal{M} is the interior of $\hat{\mathcal{M}}$ and the boundary of the compactified spacetime $\partial\hat{\mathcal{M}}$ is made up of two null hypersurfaces, denoted \mathcal{S}^\pm , referred to as the future and past null infinities, and three "points", denoted by i^\pm, i^0 , which are the future and past timelike infinities and the spacelike infinity. We obtain \hat{g} using a conformal factor Ω and putting $\hat{g} = \Omega^2 g$. On \mathcal{M} , $\Omega > 0$ and at $\partial\hat{\mathcal{M}}$, $\Omega = 0$ and $d\Omega \neq 0$. We also transform the physical field ϕ , to a rescaled field $\hat{\phi}$, given by $\hat{\phi} = \Omega^{-1}\phi$.

The first formulation of conformal time-dependent scattering was developed by Friedlander, who observed a direct link between the concept of radiation fields ([23], [24], [22]) and the scattering theory of Lax and Phillips ([47]). The theory of Lax and Phillips uses a translation representer of the solution that corresponds to Friedlander's radiation field, which is an asymptotic profile of the field along outgoing radial null geodesics. The scattering problem is then understood as solving a Goursat problem on \mathcal{S} using the radiation

field as data. In 1990, L. Hörmander in [39], gave a method to solve the Goursat problem for the wave equation on a general spatially compact spacetime, based on energy estimates.

Following this, L. Mason and JP. Nicolas in [52] formulated conformal scattering as the construction of a scattering operator on the compactified spacetime that associates the trace of the rescaled field on \mathcal{I}^+ to its trace on \mathcal{I}^- . To ensure the 'good properties' of the scattering operator, it needs to be an isomorphism between past and future null infinities so that the trace of the field on the boundary determines the solution in the interior of the spacetime entirely and uniquely. The geometrical framework of this article was a class of non-stationary vacuum space-times admitting a conformal compactification that is smooth at null and timelike infinities. The extension to black holes was done by JP. Nicolas for the wave equation on the Schwarzschild spacetime in [64], the main difficulty being to deal with the singularities at the timelike infinities. The extension possibilities for conformal scattering then took two main directions: either by studying another equation or by modifying the spacetime in which the equation propagates. This was done on the Kerr geometry in [33], on Kerr-de Sitter extremal black holes in [9], in Reissner-Nordstrom metrics in [42], [32]. Note also the extension for non linear equations by Joudioux in [41] and the application for Maxwell potentials with a particular attention to gauge choices by Taujanskas in [88] and by Nicolas and Taujanskas in [67].

In this paper, we construct a scattering theory on a non-empty dynamical spacetime, with a white hole background. The white hole considered is a Schwarzschild white hole with mass m_+ that evolves to a Schwarzschild white hole with a smaller mass m_- . The transition from one white hole to the other is described using the Vaidya metric, which models the emission of null dust along null geodesics by the white hole. The Vaidya metric is defined on a given finite retarded time interval to ensure that the past and future timelike infinities are in a Schwarzschild neighbourhood. We need indeed, in order to construct the scattering operator, to prove that the energy of the field is going to zero near i^\pm . Such decay results are established in the Schwarzschild spacetime by M. Dafermos and I. Rodnianski in [16]. As far as the author knows, similar results do not exist on the Vaidya spacetime. This paper could be seen as a part of the analytic study of the wave equation on Vaidya spacetime, other tools can be found in a separate publication by the same author (see [14]).

The chapter is organised as follows: Section 6.2 deals with the description of the geometrical framework including definitions and properties of the Schwarzschild and Vaidya

spacetimes. In particular we recall the definition of the second optical function presented in [15] by JP. Nicolas and the author. This function, denoted by v is analogous to the advanced time of Eddington-Finkelstein coordinates on Schwarzschild's spacetime. Furthermore, we perform the conformal compactification of the physical spacetime with the conformal factor $\Omega = 1/r$. In Section 6.3, we begin by describing the conformally invariant wave equation and then introduce the vector field method. We choose a causal observer, a stress-energy tensor and compute the associated energy current. We add by hand a zero-order term, involving the scalar curvature to the stress-energy tensor for the wave equation, in order to gain an L^2_{loc} control in our energy current. In the Schwarzschild region, the modified energy current is divergence-free.

The boundary chosen in our framework is composed one the one hand of the future event horizon \mathcal{H}^+ and the future null infinity \mathcal{I}^+ , on the other hand, this is made of the past event horizon \mathcal{H}^- and the past null infinity \mathcal{I}^- . Then, Theorem 6.3.1 establishes an equivalence between the energy of the rescaled field on a Cauchy hypersurface and the energy on the future and past boundary of the compactified spacetime. In Section 6.4, we construct the scattering operator and prove that it is well-defined as an isomorphism between $\mathcal{H}^- \cup \mathcal{I}^-$ and $\mathcal{H}^+ \cup \mathcal{I}^+$. We do this by defining energy spaces for initial data and for scattering data in the future on $\mathcal{H}^+ \cup \mathcal{I}^+$. Then, we define the future trace operator \mathcal{T}^+ that to the initial data defined at $t = 0$ associates the trace of the solution on $\mathcal{H}^+ \cup \mathcal{I}^+$. Finally, we solve the Goursat problem using the ideas of Hörmander in order to prove that \mathcal{T}^+ is an isomorphism. A similar construction holds for the past trace operator \mathcal{T}^- .

6.2 Geometrical framework

6.2.1 Variation of the mass and Penrose diagram

In this work we focus on a Vaidya spacetime that starts from a Schwarzschild spacetime then evolves for a finite retarded time interval towards another Schwarzschild spacetime as we describe in figure 6.1. The global metric on the compactified spacetime is :

$$\hat{g} = R^2(1 - 2m(u)R)du^2 - 2dudR - d\omega^2,$$

with, for $-\infty < u_- < u_+ < +\infty$ given :

$$m(u) = \begin{cases} m_- & \text{for } u \in]-\infty, u_-] \\ m(u) & \text{for } u \in [u_-, u_+] \\ m_+ & \text{for } u \in [u_+, +\infty[\end{cases} . \quad (6.1)$$

The mass $m(u)$ should be decreasing to describe an evaporating white hole, hence $m_+ < m_-$. Boundaries between Vaidya and Schwarzschild areas are two $u = C^{te}$ -hypersurfaces : S_{u_+} and S_{u_-} for respectively $\{u = u_+\}$ and $\{u = u_-\}$. We denote by I and II respectively the past and future Schwarzschild spacetimes and by V the Vaidya domain in between. We have also $\mathcal{H}^- = \mathcal{H}_I^- \cup \mathcal{H}_V^- \cup \mathcal{H}_{II}^-$ (see remark 6.2.1) with :

$$\begin{aligned} \mathcal{H}_I^- &= \mathcal{H}^- \cap \{u \leq u_-\} \\ \mathcal{H}_V^- &= \mathcal{H}^- \cap \{u_- \leq u \leq u_+\} \\ \mathcal{H}_{II}^- &= \mathcal{H}^- \cap \{u \geq u_+\}, \end{aligned}$$

and $\mathcal{I}^+ = \mathcal{I}_I^+ \cup \mathcal{I}_V^+ \cup \mathcal{I}_{II}^+$ with :

$$\begin{aligned} \mathcal{I}_I^+ &= \mathcal{I}^+ \cap \{u \leq u_-\} \\ \mathcal{I}_V^+ &= \mathcal{I}^+ \cap \{u_- \leq u \leq u_+\} \\ \mathcal{I}_{II}^+ &= \mathcal{I}^+ \cap \{u \geq u_+\}. \end{aligned}$$

In the remainder of this article, we shall denote by Σ_t the level hypersurfaces of t , in particular :

$$\begin{aligned} \Sigma_0 &= \{(t, r, \omega) | t = 0, r \in [r_h, +\infty[, \omega \in S^2\}, \\ \Sigma_0^V &= \Sigma_0 \cap \{u_- \leq u \leq u_+\}. \end{aligned}$$

where r_h refers to the past horizon (see Remark 6.2.1).

Remark 6.2.1. *The construction of the past horizon $\{r = r_h(u)\}$ can be found in [15], where the authors study its behaviour in general and devote particular attention to the case where the transition happens in finite retarded time. The function $r_h(u)$ satisfies :*

1. $r_h(u) = 2m_-$, for $u \leq u_-$.
2. $r_h(u)$ is decreasing.

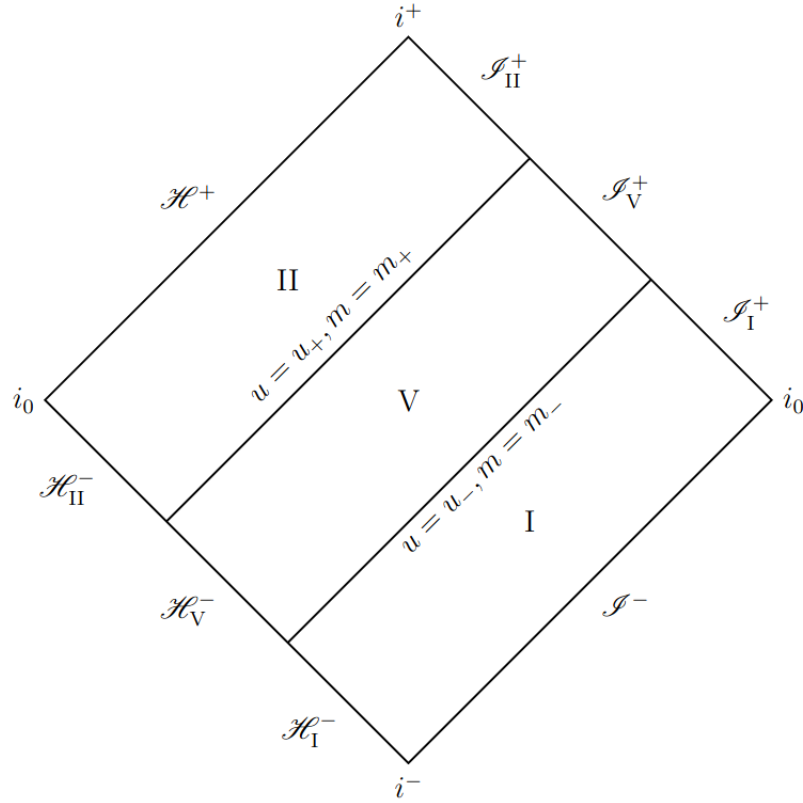


Figure 6.1: Carter-Penrose's diagram of the exterior of the conformal compactified spacetime.

$$3. \lim_{u \rightarrow +\infty} r_h(u) = 2m_+.$$

Generically, $r_h(u) > 2m_+$ on $[u_+, +\infty[$. So, although \mathcal{H}_I^- coincides with the past Schwarzschild horizon in the region I, the same is not in general true of \mathcal{H}_{II}^- and region II.

6.3 Energy estimates

6.3.1 Strategy and Propositions

Let Ψ be a solution of the physical wave equation on the Schwarzschild spacetime :

$$\left(\square_g + \frac{1}{6} \text{Scal}_g \right) \Psi = \square_g \Psi = 0, \quad (6.2)$$

this is equivalent to saying that $\phi = \Omega^{-1}\Psi$ satisfies the conformal wave equation on the rescaled spacetime :

$$\left(\square_{\hat{g}} + \frac{1}{6}\text{Scal}_{\hat{g}}\right)\phi = 0. \quad (6.3)$$

We will study the scattering for (6.2) at the level of the conformal field ϕ by computing energy estimates for the solutions to (6.3). This can be done by choosing a stress-energy tensor T_{ab} and a causal vector field T^a and contracting these two quantities to obtain an energy current :

$$J_a = T^b T_{ab}. \quad (6.4)$$

Finally we define the energy flux $\mathcal{E}_{\mathcal{S}}$ through a hypersurface \mathcal{S} by :

$$\mathcal{E}_{\mathcal{S}} = \int_{\mathcal{S}} \star T_{ab} T^a dx^b = \int_{\mathcal{S}} T_{ab} T^a n^b l_{\perp} d\text{Vol}^4,$$

where \star denotes the Hodge dual, l is a vector field transverse to the hypersurface \mathcal{S} and n is the normal vector to \mathcal{S} such that $\hat{g}(l, n) = 1$. The energy momentum tensor for the rescaled field is taken as :

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} \hat{g}_{ab} \nabla_c \phi \nabla^c \phi + \frac{1}{12} \text{Scal}_{\hat{g}} \phi^2 \hat{g}_{ab}. \quad (6.5)$$

The divergence of the stress-energy tensor in the conformal spacetime $(\hat{\mathcal{M}}, \hat{g})$ is not zero since :

$$\nabla^a T_{ab} = \square_{\hat{g}} \phi \nabla_b \phi + \frac{1}{12} \nabla_b (\text{Scal}_{\hat{g}} \phi^2).$$

Contracting this divergence with the timelike observer $T^b = \partial_u^b$ and using the wave equation (6.3) to replace $\square_{\hat{g}}$, we obtain :

$$T^b \nabla^a T_{ab} = -\frac{1}{6} \text{Scal}_{\hat{g}} \phi \phi_u + \frac{1}{12} \partial_u (\text{Scal}_{\hat{g}} \phi^2).$$

We observe that this is zero if the scalar curvature $\text{Scal}_{\hat{g}}$ does not depend on u . It happens in the Schwarzschild area where consequently :

$$T^b \nabla^a T_{ab}|_{\text{Sch}} = 0, \quad (6.6)$$

However, in the Vaidya region, $\text{Scal}_{\hat{g}} = 12m(u)R$ and then :

$$T^b \nabla^a T_{ab}|_{\text{V}} = m'(u)R\phi^2.$$

Remark 6.3.1. *Let Σ_0 be the hypersurface of initial data at $t = 0$. In this paper, we adopt the following strategy (introduced in [64]) : we take initial data $(\phi|_{t=0}, \partial_t \phi|_{t=0}) \in \mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0)$ and the associated solution ϕ of (6.3). Then we obtain estimates between the energy flux of initial data (denoted $\mathcal{E}_{\Sigma_0}(\phi)$) and the energy flux of the solution at the boundary of the conformal spacetime. Estimates proved for data in $(\phi|_{t=0}, \partial_t \phi|_{t=0}) \in \mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0)$ are then extended by density for any data with finite energy. Finally, the trace operator acts from \mathcal{H}_0 , the energy space of initial data $(\phi|_{t=0}, \partial_t \phi|_{t=0}) \in \mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0)$, completed in the norm $[\mathcal{E}_{\Sigma_0}(\phi)]^{1/2}$ to the energy space of the trace of the solution ϕ at the boundary (see definition 6.4.1 and 6.4.2 for more details).*

Our approach to the scattering is based on energy estimates. By applying Stokes' Theorem, we can derive equalities and inequalities between the energy fluxes through our different hypersurfaces. It is important to notice that on the Schwarzschild spacetime (see section 6.3.2) there exists an exact conservation law coming, firstly from (6.6) where the contraction of the Killing observer ∂_u and the divergence of the stress-energy tensor is zero, and secondly from the Killing equation that constrains $\nabla^{(a} T^{b)}$ to be zero also. This is not true on the Vaidya spacetime (see section 6.3.3) where we have only an approximate conservation law. This leads to the first theorem of this article, which establishes an equivalence between the energy flux at $t = 0$ and the energy fluxes on the boundary.

Theorem 6.3.1. *For $(\phi|_{t=0}, \partial_t \phi|_{t=0}) \in \mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0)$ we have :*

$$\begin{aligned} \mathcal{E}_{\Sigma_0}(\phi) &\simeq \mathcal{E}_{\mathcal{H}^+}(\phi) + \mathcal{E}_{\mathcal{J}^+}(\phi), \\ \mathcal{E}_{\Sigma_0}(\phi) &\simeq \mathcal{E}_{\mathcal{J}^-}(\phi) + \mathcal{E}_{\mathcal{H}^-}(\phi). \end{aligned}$$

The proof of this theorem is decomposed in two parts : firstly we will prove that on the Schwarzschild spacetime we have :

Proposition 6.3.1. *The global energy conservation laws on the Schwarzschild spacetime (see figures 6.2 and 6.3):*

$$\mathcal{E}_{\mathcal{H}_I^-}(\phi) + \mathcal{E}_{\mathcal{J}^-}(\phi) = \mathcal{E}_{S_{u_-}}(\phi) + \mathcal{E}_{\mathcal{J}_I^+}(\phi), \quad (6.7)$$

$$\mathcal{E}_{\mathcal{H}^+}(\phi) + \mathcal{E}_{\mathcal{J}_I^+}(\phi) = \mathcal{E}_{S_{u_+}}(\phi) + \mathcal{E}_{\mathcal{H}_{II}^-}(\phi). \quad (6.8)$$

can be decomposed in region II into two conservation laws between :

1. the hypersurface $\tilde{\Sigma}_0^{II}$ defined by :

$$\tilde{\Sigma}_0^{II} = (\Sigma_0 \cap \{u \geq u_+\}) \cup (\mathcal{S}_{u_+} \cap \{t \geq 0\})$$

2. the future and the past boundary of the II-region referred to respectively as \mathcal{B}_+^{II} and \mathcal{B}_-^{II} :

$$\begin{aligned} \mathcal{B}_+^{II} &= \mathcal{H}^+ \cup \mathcal{I}_{II}^+, \\ \mathcal{B}_-^{II} &= \mathcal{H}_{II}^- \cup (\mathcal{S}_{u_+} \cap \{t \leq 0\}) . \end{aligned}$$

This leads to :

$$\mathcal{E}_{\tilde{\Sigma}_0^{II}}(\phi) = \mathcal{E}_{\mathcal{B}_+^{II}}(\phi) , \quad (6.9)$$

$$\mathcal{E}_{\Sigma_0 \cap \{u \geq u_+\}}(\phi) = \mathcal{E}_{\mathcal{B}_-^{II}}(\phi) . \quad (6.10)$$

The same decomposition holds in the I-region between this time :

1. $\tilde{\Sigma}_0^I$, the hypersurface :

$$\tilde{\Sigma}_0^I = (\Sigma_0 \cap \{u \leq u_-\}) \cup (\mathcal{S}_{u_-} \cap \{t \leq 0\})$$

2. \mathcal{B}_+^I and \mathcal{B}_-^I the future and the past boundary on the I-region :

$$\begin{aligned} \mathcal{B}_+^I &= \mathcal{I}_1^+ \cup (\mathcal{S}_{u_-} \cap \{t \geq 0\}) , \\ \mathcal{B}_-^I &= \mathcal{H}_1^- \cup \mathcal{I}^- . \end{aligned}$$

and the conservation laws are :

$$\mathcal{E}_{\Sigma_0 \cap \{u \leq u_-\}}(\phi) = \mathcal{E}_{\mathcal{B}_+^I}(\phi) , \quad (6.11)$$

$$\mathcal{E}_{\tilde{\Sigma}_0^I}(\phi) = \mathcal{E}_{\mathcal{B}_-^I}(\phi) . \quad (6.12)$$

Then, we will focus our attention on the Vaidya spacetime where we don't have conservation laws but only approximate conservation laws.

Proposition 6.3.2. *The global approximate conservation law in the Vaidya area is given*

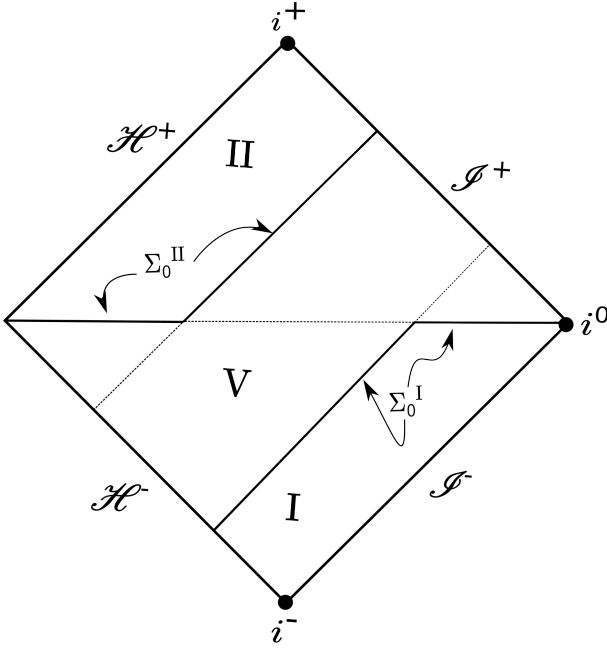


Figure 6.2: Hypersurfaces of "initial data" (Σ_0^I and Σ_0^{II}) on the Schwarzschild area.

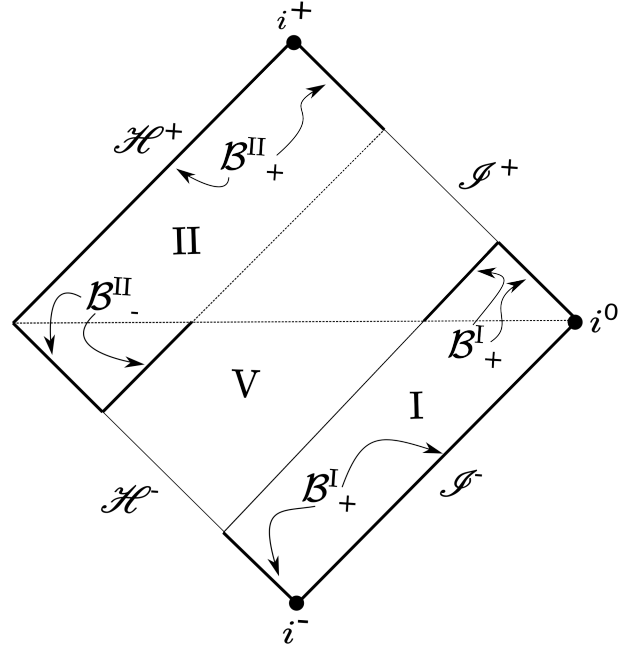


Figure 6.3: Past and future boundary on the Schwarzschild zone.

by :

$$\mathcal{E}_{\mathcal{H}_V^-}(\phi) + \mathcal{E}_{\mathcal{S}_{u_-}}(\phi) \simeq \mathcal{E}_{\mathcal{S}_{u_+}}(\phi) + \mathcal{E}_{\mathcal{S}_V^+}(\phi).$$

This equivalence can be decomposed into two equivalences in the following framework (see figure 6.4 and 6.5) :

1. $\tilde{\Sigma}_{0,+}^V$ and $\tilde{\Sigma}_{0,-}^V$ are on V , the hypersurfaces :

$$\begin{aligned} \tilde{\Sigma}_{0,-}^V &= (\mathcal{S}_{u_+} \cap \{t \leq 0\}) \cup (\Sigma_0 \cap \{u \geq u_-\}) , \\ \tilde{\Sigma}_{0,+}^V &= (\Sigma_0 \cap \{u \leq u_+\}) \cup (\mathcal{S}_{u_-} \cap \{t \geq 0\}) . \end{aligned}$$

2. the past and future boundary (resp. \mathcal{B}_-^V and \mathcal{B}_+^V) are :

$$\begin{aligned} \mathcal{B}_-^V &= \mathcal{H}_V^+ \cup (\mathcal{S}_{u_-} \cap \{t \leq 0\}) , \\ \mathcal{B}_+^V &= \mathcal{S}_V^+ \cup (\mathcal{S}_{u_+} \cap \{t \geq 0\}) . \end{aligned}$$

The conservation law is divided into two equivalences :

$$\mathcal{E}_{\tilde{\Sigma}_{0,-}^V}(\phi) \simeq \mathcal{E}_{\mathcal{B}_-^V}(\phi), \quad (6.13)$$

$$\mathcal{E}_{\tilde{\Sigma}_{0,+}^V}(\phi) \simeq \mathcal{E}_{\mathcal{B}_+^V}(\phi). \quad (6.14)$$

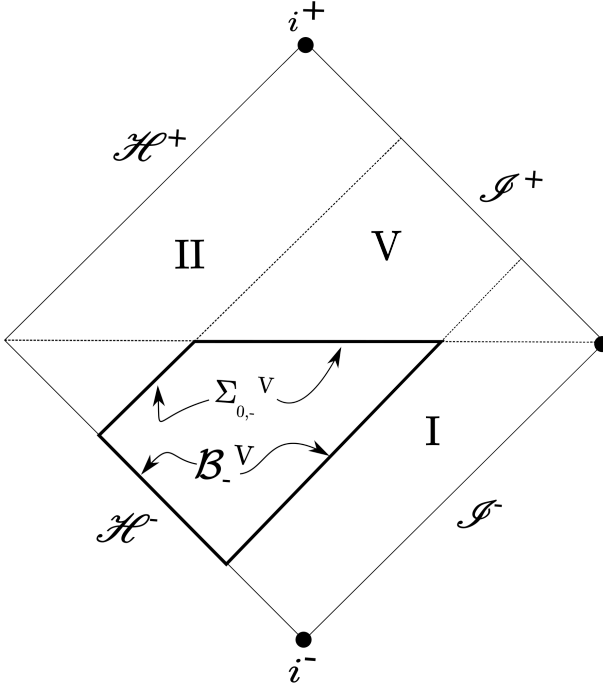


Figure 6.4: Framework in the past on the Vaidya spacetime.

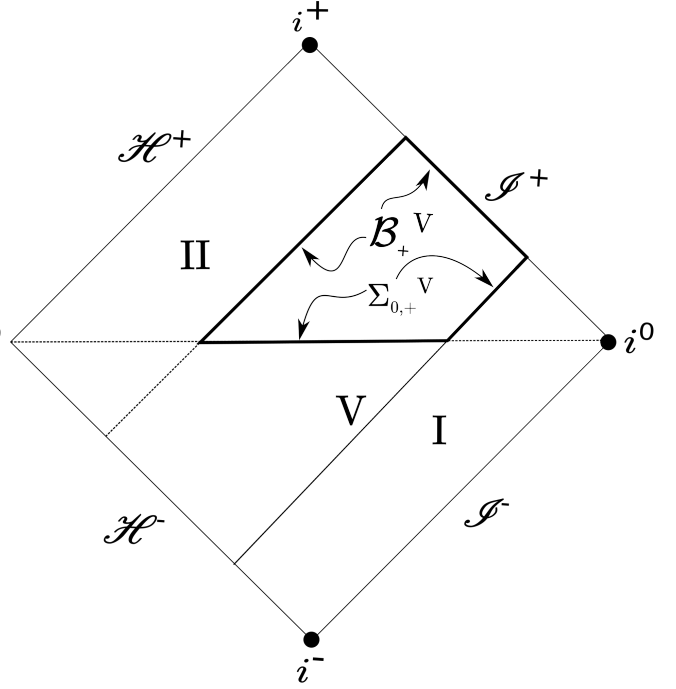


Figure 6.5: Future boundary on the Vaidya spacetime.

Proof : The proof is done in the two following sections, respectively section 6.3.2 for Proposition 6.3.1 and section 6.3.3 for Proposition 6.3.2.

6.3.2 Energy estimates on the Schwarzschild spacetime

Energy estimates on the Schwarzschild spacetime have been obtained in [64], and we adapt these results to our framework. We choose the Killing vector field $T^a = \partial_u^a$ as the observer on the Schwarzschild spacetime. Due to $\hat{g}(T, T) = R^2 F$, it is clearly timelike and it is furthermore future oriented on the rescaled spacetime. The expression of energy fluxes across the hypersurfaces Σ_t , \mathcal{I} , \mathcal{H}^- and \mathcal{S}_u are as follows : first, the energy current is

given by :

$$\star T^a T_{ab} = [\phi_u^2 + R^2 F \phi_u \phi_R] du \wedge d\omega + \left[m(u) R^2 \phi^2 + \frac{R^2 F}{2} \phi_R^2 + \frac{1}{2} |\nabla_{S^2} \phi|^2 \right] dR \wedge d\omega, \quad (6.15)$$

on Σ_t ,

$$du = \frac{dR}{R^2 F},$$

hence,

$$\mathcal{E}_{\Sigma_t}(\phi) = \int_{\Sigma_t} \left[\phi_u^2 + R^2 F \phi_u \phi_R + \frac{R^2 F}{2} (2m(u) R^2 \phi^2 + R^2 F \phi_R^2 + |\nabla_{S^2} \phi|^2) \right] du \wedge d\omega, \quad (6.16)$$

and,

$$\mathcal{E}_{\mathcal{I}^+}(\phi) = \int_{\mathcal{I}^+} \phi_u^2 du \wedge d\omega, \quad (6.17)$$

$$\mathcal{E}_{\mathcal{S}_{u_{\pm}}}(\phi) = \int_{\mathcal{S}_{u_{\pm}}} \left(m_{\pm} R^2 \phi^2 + \frac{R^2 (1 - 2m_{\pm} R)}{2} \phi_R^2 + \frac{1}{2} |\nabla_{S^2} \phi|^2 \right) dR \wedge d\omega. \quad (6.18)$$

$$(6.19)$$

Knowing that on the past horizon we have (see [15]) :

$$dR = \frac{R^2 F}{2} du, \quad (6.20)$$

then :

$$\mathcal{E}_{\mathcal{H}^-}(\phi) = \int_{\mathcal{H}^-} \left[\phi_u^2 + \frac{R_h^2 F}{2} (2\phi_u \phi_R + 2m(u) R_h^2 F \phi^2 + R_h^2 F \phi_R^2 + |\nabla_{S^2} \phi|^2) \right] du \wedge d\omega,$$

with $R_h = \frac{1}{r_h(u)}$. In the region I, on the horizon, $F = 1 - 2m_- R$ is zero, then we decompose the previous expression into :

$$\begin{aligned} \mathcal{E}_{\mathcal{H}^-}(\phi) &= \int_{\mathcal{H}_1^-} \phi_u^2 du \wedge d\omega \\ &+ \int_{\mathcal{H}_V^- \cup \mathcal{H}_{II}^-} \left[\phi_u^2 + \frac{R_h^2 F}{2} (2\phi_u \phi_R + 2m(u) R_h^2 F \phi^2 + R_h^2 F \phi_R^2 + |\nabla_{S^2} \phi|^2) \right] du \wedge d\omega. \end{aligned}$$

The coordinates (u, r, θ, φ) are unsuitable to compute the energy fluxes through \mathcal{H}^+

and \mathcal{I}^- because these hypersurfaces are such that $u = \pm\infty$ and $R = C^{te}$ on them. Since \mathcal{H}^+ and \mathcal{I}^- are in Schwarzschild regions (respectively II and I), there is no difficulty to turn to (v, R, θ, φ) coordinates, with $v = t - r_*$ and $\partial_t^a = \partial_v^a$, then :

$$\star T^a T_{ab} = - [\phi_v^2 - R^2 F \phi_v \phi_R] dv \wedge d\omega - \left[\frac{1}{12} \text{Scal}_{\hat{g}} \phi^2 + \frac{R^2 F}{2} \phi_R^2 + \frac{1}{2} |\nabla_{S^2} \phi|^2 \right] dR \wedge d\omega. \quad (6.21)$$

Hence :

$$\mathcal{E}_{\mathcal{I}^-}(\phi) = \int_{\mathcal{I}^-} \phi_v^2 dv \wedge d\omega, \quad (6.22)$$

$$\mathcal{E}_{\mathcal{H}^+}(\phi) = \int_{\mathcal{H}^+} \phi_v^2 dv \wedge d\omega. \quad (6.23)$$

The energy flux $\mathcal{E}_{\Sigma_t}(\phi)$ is equivalent to :

$$\mathcal{E}_{\Sigma_t}(\phi) \simeq \int_{\Sigma_t} (\phi_u^2 + R^4 \phi_R^2 + R^3 \phi^2 + R^2 |\nabla_{S^2} \phi|^2) du \wedge d\omega$$

On the rescaled Schwarzschild spacetime, T^a is a Killing vector field, i.e. $\nabla^{(a} T^{b)} = 0$ and the stress-energy tensor satisfies (6.6). This ensures that the divergence of the current J_a is zero.

$$\nabla^a J_a = \nabla^{(a} (T^{b)} T_{ab}) = \nabla^{(a} T^{b)} T_{ab} + T^{b} \nabla^a T_{ab} = 0.$$

From this we infer the conservation of the energy between the hypersurfaces of our boundary and we prove Proposition 6.3.1. \square

Remark 6.3.2. *Here we ignored the fact that the boundary is not compact between the past (future) horizon and past (future) null infinity. Due to the singularities at i^- and i^+ , we cannot state directly (6.7) and (6.8) without proof that the energy is going to zero at the singularities. This is done in appendix C.1.*

6.3.3 Energy fluxes on the Vaidya spacetime

In order to perform energy estimates on the compact area of the Vaidya spacetime, we need to use suitable hypersurfaces. The time function t is problematic because $t \in \mathbb{R}$ is unbounded on V . This implies that if we choose the level hypersurfaces of t in our vector field method, we would have to use the Grönwall lemma on an infinite domain. A better idea is to use a family of spacelike hypersurfaces that are transverse to null infinity and the horizon.

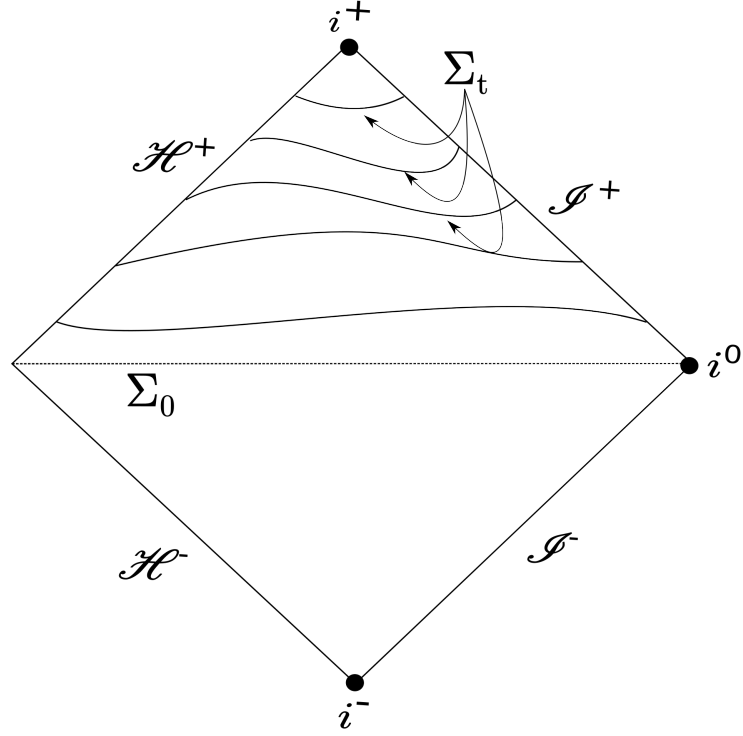


Figure 6.6: Illustration of the foliation of the rescaled spacetime in the future of Σ_0 by spacelike hypersurfaces, transverse to \mathcal{I}^+ and \mathcal{H}^+ .

We denote by Σ_τ these hypersurfaces and we set :

$$\Sigma_{\tau=0} = \Sigma_{t=0}.$$

We introduce the Newman-Penrose tetrad of null vectors on the compactified spacetime $(\hat{l}, \hat{n}, \hat{m}, \hat{\bar{m}})$:

$$\hat{l}^a = \partial_u^a + \frac{R^2 F}{2} \partial_R^a, \quad (6.24)$$

$$\hat{n}^a = -\partial_R^a, \quad (6.25)$$

$$\hat{m}^a = \frac{1}{\sqrt{2}} \left(\partial_\theta^a + \frac{i}{\sin \theta} \partial_\varphi^a \right), \quad \hat{\bar{m}}^a = \frac{1}{\sqrt{2}} \left(\partial_\theta^a - \frac{i}{\sin \theta} \partial_\varphi^a \right). \quad (6.26)$$

It is normalised, i.e.

$$\hat{l}_a \hat{n}^a = -\hat{m}^a \hat{\bar{m}}_a = 1,$$

and the spin coefficients associated to this tetrad are¹ :

$$\begin{aligned}\varepsilon &= \frac{1}{2}R - \frac{3}{2}R^2m(u), \alpha = -\beta = -\frac{\sqrt{2}}{4}\cot\theta, \\ \kappa &= \rho = \gamma = \tau = \sigma = \nu = \lambda = \mu = \pi = 0.\end{aligned}$$

We define ν as :

$$\nu = \frac{1}{\sqrt{2}}(\hat{n} + \hat{l}), \quad (6.27)$$

It is straightforward that ν is timelike and :

$$\hat{g}(\nu, \nu) = 1.$$

We use ν to split the 4-volume measure into :

$$d\text{Vol}^4 = (\nu_a dx^a) \wedge d\Sigma_\tau.$$

If ν is orthogonal to the hypersurfaces Σ_τ , we have $(\nu_a dx^a) = d\tau$. In other situations, we denote by k the orthogonal vector to the hypersurface Σ_τ . The observer ∂_u reads :

$$\partial_u^a = \hat{l}^a + \frac{R^2 F(u, R)}{2} \hat{n}^a. \quad (6.28)$$

and the energy current is given by :

$$J_a = T_{ab} \partial_u^a = \left(\phi_{\hat{l}} + \frac{R^2 F}{2} \phi_{\hat{n}} \right) \nabla_b \phi + \left(\hat{l}_b + \frac{R^2 F}{2} \hat{n}_b \right) \left(\frac{1}{12} \text{Scal}_{\hat{g}} \phi^2 - \frac{1}{2} \langle \nabla \phi, \nabla \phi \rangle \right). \quad (6.29)$$

The energy flux measured by this observer through Σ_τ is now :

$$\mathcal{E}_{\partial_u, \Sigma_\tau}(\phi) = \int_{\Sigma_\tau} J_a \nu^a d\Sigma_\tau = \int_{\Sigma_\tau} T_{ab} \partial_u^a \nu^b k_{\perp} d\text{Vol}^4, \quad (6.30)$$

and,

$$\mathcal{E}_{\partial_u, \Sigma_\tau}(\phi) = \frac{1}{\sqrt{2}} \int_{\Sigma_\tau} \left[|\phi_{\hat{l}}|^2 + \frac{R^2 F}{2} |\phi_{\hat{n}}|^2 + \left(1 + \frac{R^2 F}{2} \right) \left(2 \text{Re}(\nabla_{\hat{m}} \phi \nabla_{\hat{m}} \phi) + \frac{1}{12} \text{Scal}_{\hat{g}} \phi^2 \right) \right] d\Sigma_\tau. \quad (6.31)$$

1. For the sake of simplicity we denote these spin-coefficient without a hat, however they are the spin coefficient associated with the compactified metric \hat{g} and the associated connection $\hat{\nabla}$. We also denote without a hat directional derivatives D, Δ, δ and δ' .

Furthermore we have :

$$\nabla_{\hat{m}}\phi\nabla_{\hat{m}}\phi = \frac{1}{2} \left(|\partial_{\theta}\phi|^2 + \frac{1}{\sin^2\theta} |\partial_{\varphi}\phi|^2 \right) = |\nabla_{S^2}\phi|^2, \quad (6.32)$$

hence $Re(\nabla_{\hat{m}}\phi\nabla_{\hat{m}}\phi)$ is clearly a non negative quantity.

6.3.4 Error terms and energy estimates on the Vaidya spacetime

On the geometrical framework

Our approach to the scattering is based on energy estimates. Stokes' Theorem will allow us to obtain equalities and inequalities between the energy fluxes through our different hypersurfaces, from a conservation law for the energy current $J_a = T^b T_{ab}$. There will be error terms coming from the non-zero divergence of the energy current :

$$\nabla^a J_a = \nabla^a (T^b T_{ab}) = \nabla^a ({}^a T^b) T_{ab} + T^b \nabla^a T_{ab}.$$

Finally we obtain (see Appendix C.2 for more details), for any smooth mass function $m(u)$:

$$\nabla^a ({}^a T^b) = -m'(u) R^3 \hat{n}^{(a} \hat{n}^{b)}.$$

The divergence of the energy momentum tensor T_{ab} given in (6.5) is :

$$\nabla^a T_{ab} = \square_{\hat{g}}\phi \nabla_b \phi + \nabla_b \left(\frac{1}{12} \text{Scal}_{\hat{g}} \phi^2 \right) = \frac{1}{12} \nabla_b (\text{Scal}_{\hat{g}}) \phi^2.$$

Provided that ϕ satisfies (6.3) and using $\text{Scal}_{\hat{g}} = 12m(u)R$, we have :

$$\nabla^a J_a = m'(u) R^3 |\phi_{\hat{n}}|^2 + m'(u) R \phi^2. \quad (6.33)$$

Taking \mathcal{B} a domain closed by two hypersurfaces : \mathcal{S}_{t_2} and \mathcal{S}_{t_1} and foliated by \mathcal{S}_t . The equation (6.33) and Stokes' Theorem give us the following identity for any scalar field ϕ on \mathcal{B} :

$$\mathcal{E}_{\mathcal{S}_{t_2}}(\phi) - \mathcal{E}_{\mathcal{S}_{t_1}}(\phi) = \int_{\mathcal{B}} \nabla^a J_a d\text{Vol}^4. \quad (6.34)$$

The right-hand side of (6.34) can be decomposed into an integral in τ over $[\tau_1, \tau_2]$ of integrals over Σ_{τ} . This is done by splitting the 4-volume measure using the identifying

vector field ν as follows

$$d\text{Vol}^4 = d\tau \wedge \nu \lrcorner d\text{Vol}^4.$$

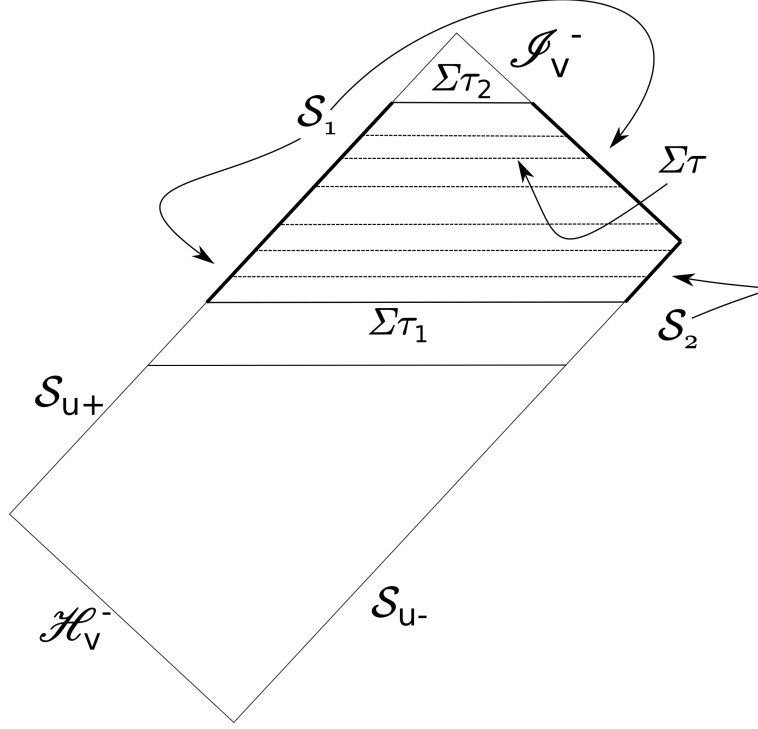


Figure 6.7: Illustration of the foliation by Σ_τ of the Vaidya area.

We decompose the proof in two parts. Firstly we focus on the equivalence in the past (6.13) and secondly in the future (6.14). In the past, we consider $\tau_- \leq \tau_1 \leq \tau \leq \tau_2 \leq 0$ and the boundary is made of : $\Sigma_{\tau_1}, \Sigma_{\tau_2}, \mathcal{S}_1$ and \mathcal{S}_2 :

$$\begin{aligned} \mathcal{S}_1 &= (\mathcal{H}_V^- \cup \mathcal{S}_{u-}) \cap \{\tau_1 \leq \tau \leq \tau_2\}, \\ \mathcal{S}_2 &= (\mathcal{S}_{u+} \cup \mathcal{I}_V^+) \cap \{\tau_1 \leq \tau \leq \tau_2\}. \end{aligned}$$

Stoke's Theorem gives :

$$\mathcal{E}_{\Sigma_{\tau_2}}(\phi) + \mathcal{E}_{\mathcal{S}_2}(\phi) - \mathcal{E}_{\Sigma_{\tau_1}}(\phi) - \mathcal{E}_{\mathcal{S}_1}(\phi) = \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \text{Err}(\phi) d\Sigma d\tau. \quad (6.35)$$

where

$$\text{Err}(\phi) = m'(u)R^3|\phi_{\hat{n}}|^2 + m'(u)R\phi^2, \quad (6.36)$$

From this we infer :

$$|\text{Err}(\phi)| \leq |m'(u)|R^3|\phi_{\hat{n}}|^2 + |m'(u)|R\phi^2. \quad (6.37)$$

Remarking that the term $Re(\nabla_{\hat{m}}\phi\nabla_{\hat{m}}\phi) = |\nabla_{S^2}\phi|^2 \geq 0$ and that $m'(u)$ is a bounded function on V , there exists a positive constant C such that :

$$|\text{Err}(\phi)| \leq C \left[|\phi_{\hat{t}}|^2 + \frac{R^2F}{2}|\phi_{\hat{n}}|^2 + \left(1 + \frac{R^2F}{2}\right) \left(2Re(\nabla_{\hat{m}}\phi\nabla_{\hat{m}}\phi) + \frac{1}{12}\text{Scal}_g\phi^2\right) \right]. \quad (6.38)$$

The right hand side corresponds up to a constant to the integral of the energy flux across Σ_τ given in (6.31). As a direct consequence we have :

$$\int_{\Sigma_\tau} |\text{Err}(\phi)| du \wedge d\omega \leq C^{te} \mathcal{E}_{\Sigma_\tau}(\phi).$$

Then, by bounding the error term in this way, equation (6.35) entails that two estimates :

$$\begin{aligned} \mathcal{E}_{\Sigma_{\tau_2}}(\phi) + \mathcal{E}_{S_2}(\phi) - \mathcal{E}_{\Sigma_{\tau_1}}(\phi) - \mathcal{E}_{S_1}(\phi) &\leq C^{te} \int_{\tau_1}^{\tau_2} \mathcal{E}_{\Sigma_\tau}(\phi) d\tau, \\ \mathcal{E}_{\Sigma_{\tau_1}}(\phi) + \mathcal{E}_{S_1}(\phi) - \mathcal{E}_{\Sigma_{\tau_2}}(\phi) - \mathcal{E}_{S_2}(\phi) &\leq C^{te} \int_{\tau_1}^{\tau_2} \mathcal{E}_{\Sigma_\tau}(\phi) d\tau. \end{aligned}$$

Knowing that S_1 and S_2 are reunion of null hypersurfaces, the dominant energy condition entails that $\mathcal{E}_{S_1}(\phi)$ and $\mathcal{E}_{S_2}(\phi)$ are non negative. Then we obtain :

$$\mathcal{E}_{\Sigma_{\tau_2}}(\phi) + \mathcal{E}_{S_2}(\phi) - \mathcal{E}_{\Sigma_{\tau_1}}(\phi) - \mathcal{E}_{S_1}(\phi) \leq C^{te} \int_{\tau_1}^{\tau_2} (\mathcal{E}_{\Sigma_\tau}(\phi) + \mathcal{E}_{S_2}(\phi)) d\tau. \quad (6.39)$$

$$\mathcal{E}_{\Sigma_{\tau_1}}(\phi) + \mathcal{E}_{S_1}(\phi) - \mathcal{E}_{\Sigma_{\tau_2}}(\phi) - \mathcal{E}_{S_2}(\phi) \leq C^{te} \int_{\tau_1}^{\tau_2} (\mathcal{E}_{\Sigma_\tau}(\phi) + \mathcal{E}_{S_1}(\phi)) d\tau. \quad (6.40)$$

Using Grönwall's lemma on the bounded domain $[\tau_1, \tau_2]$,

$$\mathcal{E}_{\Sigma_{\tau_2}}(\phi) + \mathcal{E}_{S_2}(\phi) \leq C^{te} (\mathcal{E}_{\Sigma_{\tau_1}}(\phi) + \mathcal{E}_{S_1}(\phi)). \quad (6.41)$$

$$\mathcal{E}_{\Sigma_{\tau_1}}(\phi) + \mathcal{E}_{S_1}(\phi) \leq C^{te} (\mathcal{E}_{\Sigma_{\tau_2}}(\phi) + \mathcal{E}_{S_2}(\phi)). \quad (6.42)$$

Taking $\tau_1 = \tau_-$ and $\tau_2 = 0$, hypersurfaces \mathcal{S}_1 and \mathcal{S}_2 become :

$$\Sigma_{\tau_1} + \mathcal{S}_1 = \mathcal{H}_V^- \cup \mathcal{S}_{u_-} \cap \{t \leq 0\} = \mathcal{B}_-^V, \quad (6.43)$$

$$\Sigma_{\tau_2} + \mathcal{S}_2 = \Sigma_0^V + \mathcal{S}_{u_+} \cap \{t \leq 0\} = \Sigma_{0,-}^V. \quad (6.44)$$

hence, (6.41) and (6.42) turn into :

$$\begin{aligned}\mathcal{E}_{\Sigma_{0,-}^V}(\phi) &\leq C^{te} \left(\mathcal{E}_{\mathcal{B}_-^V}(\phi) \right) . \\ \mathcal{E}_{\mathcal{B}_-^V}(\phi) &\leq C^{te} \left(\mathcal{E}_{\Sigma_{0,-}^V}(\phi) \right) .\end{aligned}$$

This concludes the proof for (6.13). □

We apply the same reasoning in the future. Let $0 \leq \tau_1 \leq \tau \leq \tau_2 \leq \tau_+$ and :

$$\begin{aligned}\mathcal{S}_1 &= \mathcal{I}_V^+ \cup \mathcal{S}_{u_+} \cap \{\tau_1 \leq \tau \leq \tau_2\} , \\ \mathcal{S}_2 &= \mathcal{S}_{u_-} \cap \{\tau_1 \leq \tau \leq \tau_2\} .\end{aligned}$$

Then Stoke's theorem and Grönwall's lemma still hold and now taking $\tau_1 = 0, \tau_2 = \tau_+$ we get :

$$\begin{aligned}\Sigma_{\tau_1} + \mathcal{S}_2 &= \Sigma_0^V + \mathcal{S}_{u_-} \cap \{t \geq 0\} = \Sigma_{0,+}^V , \\ \Sigma_{\tau_2} + \mathcal{S}_1 &= \mathcal{I}_V^+ + \mathcal{S}_{u_+} \cap \{t \geq 0\} = \mathcal{B}_+^V .\end{aligned}$$

This leads to :

$$\begin{aligned}\mathcal{E}_{\Sigma_{0,+}^V}(\phi) &\leq C^{te} \left(\mathcal{E}_{\mathcal{B}_+^V}(\phi) \right) . \\ \mathcal{E}_{\mathcal{B}_+^V}(\phi) &\leq C^{te} \left(\mathcal{E}_{\Sigma_{0,+}^V}(\phi) \right) .\end{aligned}$$

and this concludes the proof for (6.14). □

6.4 Conformal scattering

From the energy estimates obtained above, we construct the conformal scattering operator in the following manner :

1. On specified hypersurfaces (typically $\Sigma_t, \mathcal{I}^\pm, \mathcal{H}^\pm$), we define energy spaces that are the completion of the space of smooth compactly supported functions on these hypersurfaces in the norm given by the energy fluxes computed in section 6.3 for a solution of (6.3).
2. We define the future trace operator that to smooth and compactly supported initial data associates the future scattering data, i.e. the restriction of the solution on the

future boundary. Using energy estimates obtained in Theorem 6.3.1 this operator is then extended as a bounded linear operator, one-to-one, with closed range between energy spaces in Proposition 6.4.1.

3. In order to prove that the trace operator is an isomorphism and knowing the previous properties, all we need to prove is that its range is dense. We solve this problem following Hormänder in [39] and Nicolas in [65], [64] by solving a Goursat problem for the scattering data on the null hypersurfaces $\mathcal{H}^+ \cup \mathcal{I}^+$. This leads to Theorem 6.4.1.
4. Finally, we apply the same procedure to define the past trace operator and then obtain the scattering operator in Theorem 6.4.2, which is constructed as an isomorphism between the energy space on the past boundary and the energy space on the future boundary.

6.4.1 Trace operator and energy spaces

Definition 6.4.1. (Energy space of initial data). *Let \mathcal{H}_0 be the completion of $\mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0)$ in the norm :*

$$\|(\phi_0, \phi_1)\|_{\mathcal{H}_0}^2 = \frac{1}{\sqrt{2}} \int_{\Sigma_0} \left[|\phi_i|^2 + \frac{R^2 F}{2} |\phi_{\hat{n}}|^2 + \left(1 + \frac{R^2 F}{2}\right) \left(2 \operatorname{Re}(\nabla_{\hat{m}} \phi \nabla_{\hat{m}} \phi) + \frac{1}{12} \operatorname{Scal}_{\hat{g}} \phi^2\right) \right] d\Sigma_0,$$

or, equivalently in coordinates (u, R, θ, φ) :

$$\|(\phi_0, \phi_1)\|_{\mathcal{H}_0}^2 = \int_{\Sigma_0} \left[\phi_u^2 + \frac{R^2 F}{4\psi} (\psi + 1) \left(\frac{R^2 F}{\psi} \phi_R^2 + |\nabla_{S^2} \phi|^2 + 2m(u) R \phi^2 \right) \right] du \wedge d\omega,$$

with the mass function $m(u)$ as defined in (6.1). The function ψ is the function that appears in the Vaidya metric and satisfies (3.11). In region I, it is clear that $\psi = 1$. In region II, we have $F = 1 - 2m_+/r$ and ψ is a constant (see Remark 6.4.1).

Remark 6.4.1. *From [15], we know that $m'(u) = 0$ ensures that ψ is a constant function along incoming null principal geodesics. Thus on region II, we have :*

$$\frac{d}{du}(\psi \circ \gamma) = 0 \Rightarrow \psi = C^{te},$$

where $\gamma = \gamma(u) = (u, r(u), \omega)$ is an incoming principal geodesic. On the other hand, it is

clear that along outgoing null geodesics, ψ is also a constant function, hence we conclude that on region II, $\psi = C^{te}$.

Definition 6.4.2. We define on $\mathcal{H}^+ \cup \mathcal{I}^+$ the function space for scattering data \mathcal{H}^+ as the completion of $\mathcal{C}_0^\infty(\mathcal{H}^+) \times \mathcal{C}_0^\infty(\mathcal{I}^+)$ in the norm

$$\|(\xi, \zeta)\|_{\mathcal{H}^+}^2 = \frac{1}{2} \left(\int_{\mathcal{H}^+} \xi_v^2 dv \wedge d\omega + \int_{\mathcal{I}^+} \zeta_u^2 du \wedge d\omega \right).$$

We define the future trace operator for the solution ϕ of (6.3) with initial data $(\phi_0 = \phi|_{\Sigma_0}, \phi_1 = \partial_t \phi|_{\Sigma_0})$, as the operator that associates to the initial data, the trace of the solution on the future-part of the boundary $\partial \hat{\mathcal{M}}$, i.e. $(\phi|_{\mathcal{H}^+}, \phi|_{\mathcal{I}^+})$. This can be justified using Leray's theorem (see [50]) that ensures that the solution to (6.3) associated to initial data $(\phi_0, \phi_1) \in \mathcal{H}_0$ exists and is unique.

Definition 6.4.3. (Future trace operator). Let $(\phi_0, \phi_1) \in \mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0)$. Consider $\phi \in \mathcal{C}^\infty(\hat{\mathcal{M}})$ the solution of (6.3) such that :

$$\phi|_{\Sigma_0} = \phi_0, \quad \partial_t \phi|_{\Sigma_0} = \phi_1.$$

We define the trace operator \mathcal{T}^+ from $\mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0)$ to $\mathcal{C}^\infty(\mathcal{H}^+) \times \mathcal{C}^\infty(\mathcal{I}^+)$ as follows :

$$\mathcal{T}^+(\phi_0, \phi_1) = (\phi|_{\mathcal{H}^+}, \phi|_{\mathcal{I}^+}).$$

From Theorem 6.3.1, we infer the following proposition (the same proposition holds for the past trace operator) :

Proposition 6.4.1. The trace operator \mathcal{T}^+ extends uniquely as a bounded linear map from \mathcal{H}_0 to \mathcal{H}^+ that still satisfies :

$$\|\mathcal{T}^+(\phi_0, \phi_1)\|_{\mathcal{H}^+} \simeq \|(\phi_0, \phi_1)\|_{\mathcal{H}_0},$$

is one-to-one and that its range is closed.

Proof : It is clear that \mathcal{T}^+ is a linear operator, since the propagation equation (6.3) is linear; it acts between Hilbert spaces \mathcal{H}_0 and \mathcal{H}^+ . Furthermore,

$$\|\mathcal{T}^+(\phi_0, \phi_1)\|_{\mathcal{H}^+} \simeq \|(\phi_0, \phi_1)\|_{\mathcal{H}_0}, \quad \text{i.e. } C_1 \|(\phi_0, \phi_1)\|_{\mathcal{H}_0} \leq \|\mathcal{T}^+(\phi_0, \phi_1)\|_{\mathcal{H}^+} \leq C_2 \|(\phi_0, \phi_1)\|_{\mathcal{H}_0}$$

with $C_1 < C_2$ two real positive constants. It is now a well-known theorem (see Theorem 2.5 in [1]) that a such operator is one-to-one and has closed range. \square

6.4.2 Goursat problem and Scattering operator

In order to prove that the future trace operator is an isomorphism we need to prove that the range of \mathcal{T}^+ is dense in $H^1(\mathcal{I}^+)$. We follow the method described by Nicolas for the Schwarzschild spacetime in [64] by solving the Goursat problem from $\mathcal{I}^+ \cup \mathcal{H}^+$ for data in $\mathcal{C}_0^\infty(\mathcal{I}^+) \times \mathcal{C}_0^\infty(\mathcal{H}^+)$. This way, the support of data on \mathcal{H}^+ and \mathcal{I}^+ remains away from i^+ . In this context we state the following proposition :

Proposition 6.4.2. *Let $(\phi_{\mathcal{H}^+}^\infty, \phi_{\mathcal{I}^+}^\infty) \in \mathcal{C}_0^\infty(\mathcal{H}^+) \times \mathcal{C}_0^\infty(\mathcal{I}^+)$, there exist :*

$$(\phi_{\Sigma_0}, \partial_t \phi_{\Sigma_0}) \in \mathcal{H}_0,$$

such that :

$$(\phi_{\mathcal{H}^+}^\infty, \phi_{\mathcal{I}^+}^\infty) = \mathcal{T}^+ (\phi_{\Sigma_0}, \partial_t \phi_{\Sigma_0}),$$

With \mathcal{H}_0 the energy space of initial data at $t = 0$, (see definition 6.4.1).

Proof : First and foremost, we remark that the singularity at i^+ is completely avoided here, then we will focus on the singularity i^0 on \mathcal{I}^+ . The compact support of $\phi_{\mathcal{I}^+}^\infty$ on \mathcal{I}^+ ensures that Hormänder's results in [39] hold. Note that Hormänder dealt with a spatially compact spacetime, that is not the case here due to the singularity of the conformal boundary at i^0 . However, according to [64] in Appendix B, it is possible to extend Hormänder's results to the conformal compactified Schwarzschild spacetime. The construction proceeds as follows: data on $\mathcal{H}^+ \cup \mathcal{I}^+$ are compactly supported to ensure that their past support remains away from i^+ . Let \mathcal{S} be a spacelike hypersurface on $\bar{\mathcal{M}}$ that intersects \mathcal{I}^+ and \mathcal{H}^+ in the past of the supported data. Then we consider the future of this hypersurface, denoted by $\mathcal{I}(\mathcal{S})$, and we remove the future of a point lying in the future of the past support of the data and the resulting spacetime is finally extended as a cylindrical globally hyperbolic spacetime. Within this framework, Nicolas proved the uniqueness of the solution of the Goursat problem in the future of \mathcal{S} . With this construction, we can apply Hormänder's results in our study:

Proposition 6.4.3. *(Hormänder, 1990)*

Let \mathcal{S} be a spacelike hypersurface on the rescaled spacetime, that crosses \mathcal{I}^+ in the past of the support of $\phi_{\mathcal{I}^+}^\infty$ and \mathcal{H}^+ in the past of the support of $\phi_{\mathcal{H}^+}^\infty$. We denote by $\mathcal{J}^+(\mathcal{S})$ the causal future of \mathcal{S} . Then there exists a unique solution ϕ of (6.3) such that :

1. $\phi \in H^1(\mathcal{J}^+(\mathcal{S}))$.
2. $\phi|_{\mathcal{I}^+} = \phi_{\mathcal{I}^+}^\infty$ and $\phi|_{\mathcal{H}^+} = \phi_{\mathcal{H}^+}^\infty$.

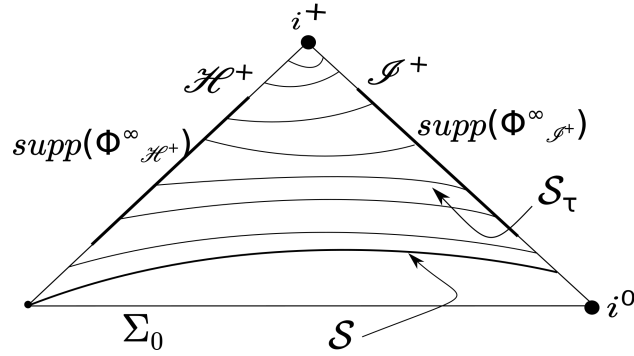


Figure 6.8: Foliation of the future of \mathcal{S} by spacelike hypersurfaces \mathcal{S}_τ transverse to the future null boundary.

Now we can propagate the solution ϕ in the past, down to Σ_0 in a way that avoids the spacelike infinity i^0 .

Due to our construction, with \mathcal{S} in the past of the support of $\phi_{\mathcal{H}^+}^\infty$ and $\phi_{\mathcal{I}^+}^\infty$, the solution $\phi|_{\mathcal{S}}$ vanishes at the boundary, i.e. at $\mathcal{S} \cap \mathcal{I}^+$ and at $\mathcal{S} \cap \mathcal{H}^+$. Then $(\phi|_{\mathcal{S}}, \partial_t \phi|_{\mathcal{S}})$ is naturally in $(H_0^1(\mathcal{S}), L^2(\mathcal{S}))$. We infer from this that we can define two sequences of smooth functions : $(\phi_{0,\mathcal{S}}^n)_n$ and $(\phi_{1,\mathcal{S}}^n)_n$ that converge in the following manner :

$$\begin{aligned} \phi_{0,\mathcal{S}}^n &\xrightarrow{H_0^1(\mathcal{S})} \phi|_{\mathcal{S}}, \\ \phi_{1,\mathcal{S}}^n &\xrightarrow{L^2(\mathcal{S})} \partial_t \phi|_{\mathcal{S}}. \end{aligned}$$

Consider now $(\phi^n)_n$ the sequence of smooth solutions of (6.3) on the rescaled spacetime with initial data $((\phi_{0,\mathcal{S}}^n)_n, (\phi_{1,\mathcal{S}}^n)_n)$ on \mathcal{S} . Because firstly the support of ϕ^n is compact on \mathcal{S} and secondly because the conformal metric remains bounded on $\text{supp}(\phi^n|_{\mathcal{S}})$, then all the weights that appears in the energy flux $\mathcal{E}_{\partial_t, \mathcal{S}}(\phi^n)$ are bounded and :

$$\mathcal{E}_{\partial_t, \mathcal{S}}(\phi^n) \simeq \|\phi^n\|_{H_0^1(\mathcal{S})}^2. \quad (6.45)$$

Furthermore, due to the compact support of the solution on \mathcal{S} and knowing the finite speed propagation of scalar waves, this ensures that ϕ vanishes in a neighbourhood of i^0 . Thus, using Theorem 6.3.1, we obtain the following equivalence between energy fluxes associated to the observer ∂_t on \mathcal{S} and Σ_0 :

$$\mathcal{E}_{\partial_t, \mathcal{S}}(\phi^n) \simeq \mathcal{E}_{\partial_t, \Sigma_0}(\phi^n). \quad (6.46)$$

Denoting :

$$\phi_{\Sigma_0} = \phi|_{\Sigma_0} \text{ and } \partial_t \phi_{\Sigma_0} = \partial_t \phi|_{\Sigma_0},$$

the equivalence (6.46) leads to :

$$(\phi_{\Sigma_0}^n, \partial_t \phi_{\Sigma_0}^n) \xrightarrow{\mathcal{H}_0} (\phi|_{\Sigma_0}, \partial_t \phi|_{\Sigma_0}).$$

Hence :

$$(\phi_{\Sigma_0}, \partial_t \phi_{\Sigma_0}) \in \mathcal{H}_0,$$

and this satisfies :

$$(\phi_{\mathcal{H}^+}^\infty, \phi_{\mathcal{I}^+}^\infty) = \mathcal{T}^+ (\phi_{\Sigma_0}, \partial_t \phi_{\Sigma_0}). \quad \square$$

From Proposition 6.4.2 we infer that the range of \mathcal{T}^+ is dense in \mathcal{H}^+ . Adding to this the Proposition 6.4.1 we obtain the following theorem :

Theorem 6.4.1. *The trace operator \mathcal{T}^+ :*

$$\begin{aligned} \mathcal{T}^+ : \quad \mathcal{H}_0 &\longrightarrow \mathcal{H}^+ \\ (\phi_0, \phi_1) &\longmapsto (\phi|_{\mathcal{H}^+}, \phi|_{\mathcal{I}^+}), \end{aligned}$$

is an isomorphism.

Remark 6.4.2. *The same result holds for \mathcal{T}^- . Let ϕ be a solution of (6.3) with initial data $(\phi_0 = \phi|_{\Sigma_0}, \phi_1 = \partial_t \phi|_{\Sigma_0})$, then the pas trace operator is defined as:*

$$\begin{aligned} \mathcal{T}^- : \quad \mathcal{C}_0^\infty(\Sigma_0) \times \mathcal{C}_0^\infty(\Sigma_0) &\longrightarrow \mathcal{C}^\infty(\mathcal{H}^-) \times \mathcal{C}^\infty(\mathcal{I}^-) \\ (\phi_0, \phi_1) &\longmapsto (\phi|_{\mathcal{H}^-}, \phi|_{\mathcal{I}^-}). \end{aligned}$$

It follows from the second approximate conservation law in Theorem 6.3.1 that \mathcal{T}^- extends

as a bounded linear map :

$$\begin{aligned} \mathcal{T}^- : \quad \mathcal{H}_0 &\longrightarrow \mathcal{H}^- \\ (\phi_0, \phi_1) &\longmapsto (\phi|_{\mathcal{H}^-}, \phi|_{\mathcal{I}^-}), \end{aligned}$$

satisfies :

$$\|\mathcal{T}^-(\phi_0, \phi_1)\|_{\mathcal{H}^-} \simeq \|(\phi_0, \phi_1)\|_{\mathcal{H}_0},$$

hence is one-to-one and has closed range. The resolution of Goursat problem for data compactly supported on $\mathcal{H}^- \cup \mathcal{I}^-$ is similar to what was done for the resolution of the Goursat problem on the future null boundary and entails that \mathcal{T}^- is a isomorphism.

Theorem 6.4.2. (Scattering operator) :

Let ϕ be the solution of (6.3) with initial data $(\phi_0, \phi_1) \in \mathcal{H}_0$. Consider $(\phi_{\mathcal{H}^\pm}^\infty, \phi_{\mathcal{I}^\pm}^\infty) \in \mathcal{H}^\pm$ the trace of ϕ on the respectively future and past boundary of the rescaled spacetime. Then the scattering operator S obtained from future and past trace operators :

$$S = \mathcal{T}^+(\mathcal{T}^-)^{-1},$$

that acts as :

$$\begin{aligned} S : \quad \mathcal{H}^- &\longrightarrow \mathcal{H}^+ \\ (\phi_{\mathcal{H}^-}^\infty, \phi_{\mathcal{I}^-}^\infty) &\longmapsto (\phi_{\mathcal{H}^+}^\infty, \phi_{\mathcal{I}^+}^\infty), \end{aligned}$$

is an isomorphism.

PART III

**Geometry of pure radiative
Robinson-Trautman's spacetime**

GEOMETRY OF THE PURE RADIATIVE ROBINSON-TRAUTMAN'S SOLUTION

7.1 Introduction

Presented for the first time in 1960 in a short paper [83] by Robinson and Trautman, the Robinson-Trautman metric was then analyzed in detail in a second paper published in 1962 [82]. In the same year, Newman and Tamburino, in [62], obtained these solutions to Einstein's equations by solving a formulation of the Einstein equations using the formalism of null tetrad and spin coefficients. This class of solutions has great physical significance since it generalizes many solutions: the Schwarzschild black hole, the Reissner-Nordström spacetime, the Vaidya metric, uniformly accelerated black holes (the C-metric), and expanding spherical gravitational waves propagating on conformally flat spacetimes. Furthermore, for all of these solutions, it is possible to include a cosmological constant, positive or negative.

The beauty of this very general solution to Einstein's equations lies in a very simple geometrical assumption: it is the class of metrics that admit a geodesic, shear-free, twist-free but expanding null congruence. Note that if we modify the condition of expanding to a non-expanding congruence, we obtain the Kundt class of solutions, investigated for the first time in [46] by Kundt in 1961.

All of these different kinds of solutions can be classified algebraically by their Petrov type and are of type II, D, III, N, or conformally flat. This classification is ensured by the Goldberg-Sachs theorem and its generalization due to Kundt and Thompson ([45] in 1962) or Robinson and Schild ([81] in 1963), which state that a shear-free null congruence for vacuum or an aligned (null or non-null) electromagnetic field entails that the spacetime has to be algebraically special. However, it should be noted that this generalization cannot be extended in the most general manner for pure radiative spacetimes, as explained in [87] by the authors in section 7.6. For a complete classification of the Robinson-Trautman

spacetimes, refer to [76].

The vacuum Robinson-Trautman metric is well known and widely analyzed (see [31], [87], for instance). In this context, the metric depends on two functions P and m . The case with $m = 0$ describes type III or N spacetimes with pure gravitational waves. When $m \neq 0$, there is a singularity at $r = 0$, and spacetimes are of type II or D, and m can be related, in particular cases, to the mass of the source (for instance, Schwarzschild's solution belongs to the vacuum type D Robinson-Trautman metric). The type D, in a vacuum Robinson-Trautman spacetime, ensures that the mass is constant, unlike in type II where the mass can depend on time.

Concerning the type II solutions, it was proved by Chruściel and Singleton in [11] that these solutions exist globally for all positive retarded time and converge asymptotically to the Schwarzschild solution. This result was then extended in [5], including a non-vanishing cosmological constant, and it was proved that the Robinson-Trautman solutions of type II, with $\Lambda > 0$ (resp. $\Lambda < 0$), converge to the Schwarzschild-de-Sitter (resp. Schwarzschild-anti-de-Sitter) black hole.

A Robinson-Trautman metric satisfying the geometrical assumptions made above can admit a non-zero Ricci component, corresponding to the presence of pure radiation, i.e., a flow of zero rest-mass matter, propagated along the repeated principal null direction. There is no type N or conformally flat pure radiative solution opposite to the vacuum case. Furthermore, we assume that $\Lambda = 0$. Thus, we focus on the type D pure radiative Robinson-Trautman, which is a generalization of the Vaidya metric. Furthermore, it was proven in 1987 by Bičák and Perjés in [4] that type II pure radiative Robinson-Trautman solutions approach the Vaidya metric for infinite retarded time u (note that these results were then extended by Podolský and Svítek for pure radiative spacetime with a non-vanishing cosmological constant in [77]).

A natural question arising from physics is the localization of the horizon in a such spacetime. From a more general point of view, one may ask about the geometry of the incoming principal null geodesics in the pure radiative Petrov type D Robinson-Trautman spacetime. Concerning the horizon, in Schwarzschild's spacetime, the solution is simple since it depends only on the physical parameter of the black hole: its mass m . When the mass of the black hole varies with the retarded time, this becomes more tricky, and this leads to the study done by Nicolas and the author in the Vaidya spacetime [15]. The method developed in this paper is as follows: considering the past event horizon (i.e., the horizon of a white hole) as a radial ingoing null geodesic $\gamma(u, r(u), \theta, \varphi)$, due to the

spherical symmetry, its localization is entirely defined by the study of $r(u)$. This radius $r(u)$ obeys an ordinary differential equation that is satisfied by all the ingoing radial null geodesics. All of these solutions are then classified depending on their value at $u \rightarrow -\infty$, and it was proven that there exists a unique solution, called $r_h(u)$ that admits a finite limit at $u \rightarrow -\infty$. Because of the assumptions on the mass:

$$\lim_{u \rightarrow \pm\infty} m(u) = m_{\pm} \in \mathbb{R}_+, m_- < m_+,$$

the finite limit of $r(u)$ as $u \rightarrow -\infty$ coincides with the event horizon of a Schwarzschild black hole of mass m_- . Consequently, we interpret this solution as the horizon of the Vaidya white hole. The other solutions fall into two categories: either $r(u)$ exists across the entire real line and has a limit:

$$\lim_{u \rightarrow -\infty} r(u) = +\infty,$$

or $r(u)$ exists only from a finite retarded time in the past \tilde{u} and satisfies:

$$\lim_{u \rightarrow \tilde{u}} r(u) = 0.$$

The idea of this part is to extend the analysis done in the Vaidya spacetime to the pure radiative Robinson-Trautman spacetime. This chapter is organized as follows:

- Section 7.2 is devoted to the derivation of the Robinson-Trautman metric using the spin coefficients as done by Newman and Tamburino in [62].
- In Section 7.3, we present the geometry of the pure radiative Robinson-Trautman solution. In particular, we give the Robinson-Trautman equation, that defines the radiation density, i.e., the source of the Einstein equation. This density depends on P and m , the two functions appearing in the metric. The natural coordinates chosen to compute the Robinson-Trautman metric in the vacuum $(u, r, \xi, \bar{\xi})$ do not ensure the asymptotic flatness of the metric. In particular, the metric blows up when r goes to ∞ . That is why, we begin by following Tod in [89] and we introduce a new coordinate basis $(u, R, \xi, \bar{\xi})$ that removes this divergence at infinity. However, it is not sufficient to define the Bondi-Sachs coordinates with the good behaviour at infinity. This is done by introducing new coordinates (U, ρ, X, \bar{X}) in a power series of $1/r$, as done by Newman and Unti in [63]. We detail this method in Section 7.3.3, and we compute the Bondi mass and the Bondi mass aspect.

Finally, we assume that the metric is Petrov type D, and this leads us to consider that the Gaussian curvature on the 2-sphere, denoted by K , depends only on the retarded time u . Hence, we know from [87] and [31] that the function:

$$P(u, \xi, \bar{\xi}) = A(u) + B(u)\xi + \bar{B}(u)\bar{\xi} + C(u)\xi\bar{\xi},$$

with A and C two real functions and B that is complex, satisfies the previous assumptions.

- In Section 7.4, we begin by investigating the situation $B = 0$, which corresponds to an axisymmetric metric. We focus on the case $A \neq C$, since $A = C$ is proven to be the Vaidya metric (see Proposition 7.4.1). Then we apply a method previously developed by Nicolas and the author in [15], where they investigated the behaviour of the event horizon of the Vaidya white hole by analyzing solutions to an ordinary differential equation governing all the ingoing principal null curves. We derive a similar differential equation for the pure radiative Robinson-Trautman solution and observe that the behaviour of the ingoing principal null curves closely resembles what occurs in the Vaidya spacetime. The main difference lies in the fact that the integral lines of the ingoing principal null directions are no longer geodesics. Then, the ingoing principal null direction is no longer hypersurface-forming; hence, it does not govern the dynamics of the past event horizon as it did in the Vaidya spacetime. In this study, we assume that the Robinson-Trautman radiation begins at $u = u_-$, and that we have the Vaidya metric for $u \leq u_-$. Then, the incoming principal null directions form the horizon until $u = u_-$.

In this section, we analyze the behaviour of the integral lines of the incoming principal null direction, and particularly, we observe that there exists only one solution that converges to the event horizon in the past. This solution, denoted by R_H , is decreasing on the entire real line and converges in the future to the Schwarzschild horizon of the black hole. The other solutions are classified in Theorem 7.4.1.

- Finally, Section 7.5 provides similar results for the $B \neq 0$ situation, i.e., for a solution that does not have any spherical symmetry. We obtain an analogous theorem on the behaviour of the principal null curves, and we classify them in Theorem 7.5.1.

7.2 Derivation of the metric from the spin coefficient equations

7.2.1 Assumptions

As mentioned earlier, we follow the approach laid out in [62]. Consider a vacuum spacetime (\mathcal{M}, g) containing a geodesic ray, corresponding to Petrov classification type I. Then, there exists a principal null direction l^a that is tangent to a congruence of null geodesics, satisfying:

$$l^a l_a = 0, \quad l^b \nabla_b l^a = 0.$$

We denote the coordinate u such that the level hypersurfaces of u are orthogonal to the geodesic ray l^a , i.e.,

$$l_a = \nabla_a u.$$

These assumptions allow us to define Bondi coordinates (u, r, x^A) , where r is the affine parameter along null geodesics, and x^A labels the geodesics on each $\{u = \text{constant}\}$ -hypersurface. Now, let's introduce the Newman-Penrose tetrad associated with this Bondi frame:

$$\begin{aligned} l^a &= \partial_r^a, \\ n^a &= \partial_u^a + U \partial_r^a + X^A \partial_{x^A}^a, \\ m^a &= \omega \partial_r^a + \xi^A \partial_{x^A}^a, \\ \bar{m}^a &= \bar{\omega} \partial_r^a + \bar{\xi}^A \partial_{x^A}^a, \end{aligned}$$

where U , X^A , ξ^A , and ω are chosen such that the only non-zero products between components of the tetrad are:

$$l^a n_a = -m^a \bar{m}_a = 1.$$

By our assumptions, the Robinson-Trautman metric possesses a geodesic null congruence that is shear-free. We can define the complex shear, denoted by σ , such that:

$$\sigma = \nabla_a l_b m^a m^b = 0. \tag{7.1}$$

According to the Goldberg-Sachs theorem, the Robinson-Trautman metric needs to be algebraically special, falling into one of the types II, D, III, N, or conformally flat. Con-

sequently, the components of the Weyl spinor behave as follows:

$$\Psi_0 = \Psi_1 = 0.$$

Additionally, the congruence defined above must be in expansion, meaning it has a non-vanishing divergence, denoted by ρ and defined as follows :

$$\rho = \nabla_a l_b m^a \bar{m}^b .$$

. The existence of a hypersurface orthogonal to a geodesic null congruence ensures that :

$$\rho = \bar{\rho} ,$$

and a non-vanishing divergence leads to :

$$\rho \neq 0.$$

The metric g_{ab} expressed with respect to the tetrad is given by:

$$g_{ab} = l_a n_b + n_a l_b - m_a \bar{m}_b - m_b \bar{m}_a,$$

and it must remain invariant under the following coordinate transformations:

1. Shifting of the radial origin:

$$r' = r + f(u, x^A), \quad u' = u, \quad x'^A = x^A. \quad (7.2)$$

2. Relabelling of hypersurfaces:

$$r' = \frac{r}{\gamma}, \quad u' = \gamma(u), \quad x'^A = x^A. \quad (7.3)$$

3. Relabelling geodesics:

$$r' = r, \quad u' = u, \quad x'^A = x'^A(u, x^A). \quad (7.4)$$

Furthermore, the two following tetrad transformations (rotation) do not affect l^a and, therefore, do not impact the assumptions regarding the Robinson-Trautman geometry:

1. null rotation :

$$\begin{aligned} l^{\mu'} &= l^\mu \\ n^{\mu'} &= n^\mu + \bar{B}m^\mu + B\bar{m}^\mu + B\bar{B}l^\mu, \\ m^{\mu'} &= m^\mu + Bl^\mu. \end{aligned}$$

where B is a complex scalar field that does not depend on the radial distance r .

2. spatial rotation :

$$\begin{aligned} l^{\mu'} &= l^\mu, \\ n^{\mu'} &= n^\mu, \\ m^{\mu'} &= e^{iC} m^\mu. \end{aligned}$$

where C is a real function independent of r .

Finally we define three derivative operators associated to the Newman-Penrose tetrad :

$$D = \frac{\partial}{\partial r}, \tag{7.5}$$

$$\delta = \omega \frac{\partial}{\partial r} + \xi^k \frac{\partial}{\partial x^k}, \tag{7.6}$$

$$\Delta = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X^k \frac{\partial}{\partial x^k}. \tag{7.7}$$

7.2.2 Spin coefficient equations

In this section, we will solve the spin coefficient equations (see Appendix 7.2) under the previous assumptions. We follow Newman and Tamburino in [62]. The method is as follows: we begin with the radial equations, i.e., those that involve $D = \partial/\partial r$. Then we examine the simplifications entailed by the geometrical assumptions and determine the associated spin coefficients. We use Newman-Tamburino's notation f° to denote a function f that does not depend on r . Therefore, considering the hypotheses made above, in the spin coefficient equations (D.35f), (D.35n), and (D.37a), all terms vanish. The radial equation (D.35e) is easily solved:

$$D\rho = \rho^2 \Rightarrow \rho = -\frac{1}{r + \rho^\circ}.$$

Since the metric is invariant under a shift of the origin (7.2), we set:

$$\rho = -\frac{1}{r}. \quad (7.8)$$

In this coordinate system, Newman and Penrose proved in [61] that:

$$\tau = \bar{\alpha} + \beta.$$

Using radial derivative equations (D.35g), (D.35h), and (D.35i), and knowing that $\rho = 1/r$, we get:

$$\tau = \frac{\tau^\circ}{r}, \quad (7.9)$$

$$\alpha = \frac{\alpha^\circ}{r}, \quad (7.10)$$

$$\beta = \frac{\beta^\circ}{r}. \quad (7.11)$$

Then, $\tau^\circ = \bar{\alpha}^\circ + \beta^\circ$. Using null and spatial rotations, it is possible to set $\tau^\circ = 0$. So:

$$\tau = 0, \quad \bar{\alpha} = -\beta.$$

Equation (D.36k) reduces to: $\bar{\lambda}\rho = 0$, and this ensures that $\lambda = 0$, hence (D.35k) is trivial. In the same way, from (D.35o), (D.35j), (D.35l), we obtain:

$$\Psi_2 = \frac{\Psi_2^\circ}{r^3}, \quad (7.12)$$

$$\gamma = \gamma^\circ - \frac{\Psi_2^\circ}{2r^2}, \quad (7.13)$$

$$\mu = \frac{\mu^\circ}{r} - \frac{\Psi_2^\circ}{r^2}. \quad (7.14)$$

Because ρ depends only on r , (7.6) entails that $\delta\rho = \omega D\rho$. Moreover, from (D.36f), we have $\omega D\rho = 0$. This implies that $\omega = 0$, and thus (D.35b) is trivially solved. Furthermore, (D.36c) implies that $Im(\mu) = 0$. Hence, from (7.14):

$$\mu^\circ = \text{real and } \Psi_2^\circ = \text{real}. \quad (7.15)$$

Finally, (D.35c), (D.35a), and (D.35d) give:

$$X^i = X^{\circ i}, \quad (7.16)$$

$$U = U^\circ - (\gamma^\circ + \bar{\gamma}^\circ)r - \frac{1}{2r}(\Psi_2^\circ + \bar{\Psi}_2^\circ), \quad (7.17)$$

$$\xi^i = \frac{\xi^{\circ i}}{r}. \quad (7.18)$$

Using the invariance 7.4 of the metric, we chose $\xi^{\circ i}$ such that:

$$\xi^{\circ 3} = P, \quad \xi^{\circ 4} = iP. \quad (7.19)$$

Knowing that $\omega = 0$ and using the expressions of U and X^k , the derivative operators δ and Δ are now:

$$\begin{aligned} \delta &= \xi^k \frac{\partial}{\partial x^k} = \frac{\xi^{\circ k}}{r} \frac{\partial}{\partial x^k}, \\ \Delta &= \left(U^\circ - (\gamma^\circ + \bar{\gamma}^\circ)r - \frac{1}{r}\Psi_2^\circ \right) \frac{\partial}{\partial r} + \frac{\partial}{\partial u} + X^{\circ k} \frac{\partial}{\partial x^k}. \end{aligned}$$

Equation (D.36a) turns into:

$$\begin{aligned} \delta X^i - \Delta \xi^i &= (\mu + \bar{\gamma} - \gamma) \xi^i, \\ \delta X^i - \Delta \xi^i &= \frac{\xi^{\circ k}}{r} \frac{\partial X^{\circ i}}{\partial x^k} + \left(U^\circ - (\gamma^\circ + \bar{\gamma}^\circ)r - \frac{1}{r}\Psi_2^\circ \right) \frac{\xi^{\circ i}}{r^2} - \frac{1}{r} \frac{\partial \xi^{\circ i}}{\partial u} - \frac{X^{\circ k}}{r} \frac{\partial \xi^{\circ i}}{\partial x^k}, \\ &= \left(\frac{\mu^\circ}{r} - \frac{\Psi_2^\circ}{r^2} + \bar{\gamma}^\circ - \gamma^\circ \right) \frac{\xi^{\circ i}}{r}. \end{aligned}$$

Identifying parts in $1/r$ and $1/r^2$ (note that terms in Ψ_2° and γ° vanish), we have:

$$\mu^\circ = U^\circ, \quad (7.20)$$

$$\xi^{\circ k} \frac{\partial X^{\circ i}}{\partial x^k} - \frac{\partial \xi^{\circ i}}{\partial u} - X^{\circ k} \frac{\partial \xi^{\circ i}}{\partial x^k} = 2\bar{\gamma}^\circ \xi^{\circ i}. \quad (7.21)$$

Now we want to make $X^{\circ i}$ equal to zero. The first thing to remark is that $X^\circ = X^{\circ 3} + iX^{\circ 4}$ is an analytic function that depends only on u, x^3, x^4 . Thus, using the coordinate transformation (7.3):

$$r' = r/\gamma, \quad u' = \gamma(u), \quad \zeta' = x'^3 + ix'^4 = f(\zeta, u).$$

With $\zeta = x^3 + ix^4$, it is possible to transform X° such that:

$$X^{\circ i} = 0.$$

Then, equation (7.21) transforms as:

$$-\frac{\partial \xi^{\circ i}}{\partial u} = 2\bar{\gamma}^\circ \xi^{\circ i},$$

or in other terms:

$$\bar{\gamma}^\circ = -\frac{1}{2} \frac{\partial \ln P}{\partial u}. \quad (7.22)$$

Let us denote by ∇ the operator:

$$\nabla = \frac{\partial}{\partial x^3} + i \frac{\partial}{\partial x^4} \quad (7.23)$$

Using (7.18) and (7.19), the derivative operator δ becomes:

$$\delta = \frac{P}{r} \nabla.$$

Equation (D.37b) reduces to:

$$\begin{aligned} \delta \Psi_2 = 0, & \Rightarrow \xi^k \frac{\partial}{\partial x^k} \left(\frac{\Psi_2^\circ}{r^3} \right) = 0, \\ \nabla \left(\frac{\Psi_2^\circ}{r^3} \right) &= 0. \end{aligned}$$

Furthermore, we know that Ψ_2° is a real function. Then this leads to:

$$\Psi_2^\circ = \Psi_2^\circ(u). \quad (7.24)$$

Equations (D.35p), (D.35q) give Ψ_3 and Ψ_4 :

$$\Psi_3 = \frac{\Psi_3^\circ}{r^2}, \quad (7.25)$$

$$\Psi_4 = \frac{\Psi_4^\circ}{r} \frac{(\bar{P} \bar{\nabla} \Psi_3^\circ + 2\alpha^\circ \Psi_3^\circ)}{r^2}. \quad (7.26)$$

Equation (D.35m) yields:

$$\nu = \nu^\circ - \frac{\Psi_3^\circ}{r}. \quad (7.27)$$

To fully solve the spin coefficient equations, we have to determine: α° , ν° , μ° , Ψ_2° , Ψ_3° , Ψ_4° , P , and γ° . The strategy is to express all of them in terms of Ψ_2° and P . Finally, we just have to obtain a relation between P and Ψ_2° to define the Robinson-Trautman metric. Equations (D.36b), (D.36d), (D.36g), and (D.36h) imply that:

$$\alpha^\circ = \frac{1}{2} \bar{P} \bar{\nabla} \ln P, \quad (7.28)$$

$$\bar{\nu}^\circ = P \nabla (\gamma^\circ + \bar{\gamma}^\circ) = -\frac{1}{2} P \nabla \frac{\partial \ln P \bar{P}}{\partial u}, \quad (7.29)$$

$$\Psi_4^\circ = \bar{P} \bar{\nabla} \nu^\circ + 2\alpha^\circ \nu^\circ, \quad (7.30)$$

$$\mu^\circ = -\frac{1}{2} P \bar{P} \nabla \bar{\nabla} \ln P \bar{P}, \quad (7.31)$$

$$\Psi_3^\circ = \bar{P} \bar{\nabla} \mu^\circ. \quad (7.32)$$

Equation (D.37c) gives the relation between P and Ψ_2° :

$$\left(\frac{\partial}{\partial u} - 3 \frac{\partial P / \partial u}{P} \right) \Psi_2^\circ = P \nabla \Psi_3^\circ - 2\alpha^\circ \Psi_3^\circ. \quad (7.33)$$

Other derivative equations : (D.36i), (D.36j), (D.36l), (D.36m) and (D.37d) are identically solved. Using the spatial rotation, we arrange to make P real:

$$P = \bar{P}.$$

7.3 Geometrical framework

7.3.1 Vacuum solution

Coming back to the inverse metric, the nonzero components of g^{ab} are:

$$\begin{aligned} g^{22} &= 2U = 2U^\circ - 4\gamma^\circ r - \frac{2\Psi_2^\circ}{r}, \\ g^{12} &= 1, \\ g^{ij} &= -2 \frac{P^2}{r^2} \delta^{ij}, \end{aligned}$$

where $i, j = 2, 3$, and δ^{ij} corresponds to the inverse metric on the unitary two-sphere. We change the notation to be consistent with the expression given in [31] (with respect to the

change of the metric):

$$U^\circ = -P^2 \nabla^2 \ln P, \quad \gamma^\circ = -\frac{1}{2} \frac{\partial \ln P}{\partial u}, \quad \Psi_2^\circ = m.$$

where $\nabla^2 = \nabla \bar{\nabla}$. Then, in $(u, r, \xi, \bar{\xi})$ coordinates, the metric is:

$$g = \left(\Delta \ln P - 2r \frac{\partial \ln P}{\partial u} - \frac{2m}{r} \right) du^2 + 2dudr - \frac{2r^2}{P} d\xi d\bar{\xi},$$

where $\Delta = 2P^2 \partial_\xi \partial_{\bar{\xi}}$. It is important to notice that because of equations (7.19) and (7.24), P and m cannot depend on any variables:

$$P = P(u, \xi, \bar{\xi}), \quad m = m(u).$$

In this framework, equation (7.33) turns into the well-known vacuum Robinson-Trautman's equation:

$$\Delta \Delta (\ln P) + 12m \frac{\partial (\ln P)}{\partial u} - 4 \frac{\partial m}{\partial u} = 0.$$

Using the Newman-Penrose tetrad:

$$\begin{aligned} k &= \partial_r, \\ l &= \partial_u - \frac{1}{2} \left(\Delta \ln P - 2r \frac{\partial \ln P}{\partial u} - \frac{2m}{r} \right) \partial_r, \\ m &= \frac{P}{r} \partial_{\bar{\xi}}, \end{aligned}$$

the Weyl components are:

$$\Psi_0 = \Psi_1 = 0, \tag{7.34}$$

$$\Psi_2 = -\frac{m}{r^3}, \tag{7.35}$$

$$\Psi_3 = -\frac{P}{2r^2} \partial_{\bar{\xi}} \Delta \ln P, \tag{7.36}$$

$$\Psi_4 = \frac{1}{2r^2} \partial_{\bar{\xi}} (P^2 \partial_{\bar{\xi}} \Delta \ln P) - \frac{1}{r} \partial_{\bar{\xi}} \left(P^2 \frac{\partial^2 \ln P}{\partial u \partial \bar{\xi}} \right). \tag{7.37}$$

7.3.2 Pure radiative spacetime

Up to now, we have only described the vacuum Robinson-Trautman solution. The original assumptions made in the vacuum still hold; i.e. this metric still admits a shear-free, twist-free non-zero divergence null geodesic congruence. Then the metric reads in the same way:

$$g = \left(\Delta \ln P - 2r \frac{\partial \ln P}{\partial u} - \frac{2m}{r} \right) du^2 + 2dudr - \frac{2r^2}{P} d\xi d\bar{\xi} \quad (7.38)$$

The main difference is that now, the pure radiative solution admits a non-zero Ricci component:

$$\Phi_{22} = \frac{1}{4r^2} \left(\Delta \Delta \ln P + 12m \frac{\partial \ln P}{\partial u} - 4 \frac{\partial m}{\partial u} \right).$$

This describes the presence of pure aligned radiation transported along the repeated principal null direction. The stress-energy tensor T_{ab} , the source of Einstein's equation, becomes:

$$T_{ab} = \rho k_a k_b,$$

where $k^a = \partial_r^a$ is the propagation direction of the radiation, with ρ denoting the radiation density:

$$\rho = \frac{n^2(u, \xi, \bar{\xi})}{r^2}.$$

Here, n^2 is a positive real function that appears in the Robinson-Trautman equation.

$$\Delta \Delta (\ln P) + 12m \frac{\partial \ln P}{\partial u} - 4 \frac{\partial m}{\partial u} = 16\pi n^2. \quad (7.39)$$

7.3.3 Bondi coordinates

It is important to note that the metric (7.38) is not expressed in Bondi coordinates, as g_{uu} diverges when $r \rightarrow +\infty$ due to the term rP'/P . The initial approach to transform the metric is to follow [89] and define a radial distance coordinate, denoted by R :

$$R := \frac{r}{\tilde{P}}. \quad (7.40)$$

where \tilde{P} is :

$$P(u, \xi, \bar{\xi}) = (1 + \xi \bar{\xi}) \tilde{P}(u, \xi, \bar{\xi}).$$

In coordinates $(u, R, \xi, \bar{\xi})$, the metric (7.38) becomes :

$$g = \left(K(u) - \frac{2m(u)}{R\tilde{P}} \right) du^2 + 2\tilde{P}du dR + 2Rdu \left(\frac{\partial\tilde{P}}{\partial\xi} d\xi + \frac{\partial\tilde{P}}{\partial\bar{\xi}} d\bar{\xi} \right) - 2R^2 \frac{d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2}. \quad (7.41)$$

Although this metric no longer has a linear term in R in g_{uu} , it is not expressed in Bondi coordinates because it does not converge to the Minkowski metric as $R \rightarrow \infty$. As demonstrated by Newman and Unti in [63], the coordinate transformations to Bondi-Sachs coordinates cannot be expressed in closed form; instead, it requires introducing new coordinates (U, ρ, X, \bar{X}) obtained as an infinite series in powers of r^{-1} .

This is done by following [63], [29], and [13], where we define the Bondi coordinates (U, ρ, X, \bar{X}) as:

$$\begin{aligned} U &= U_0 + U_1 r^{-1} + U_2 r^{-2} + \mathcal{O}(r^{-3}), \\ \rho &= \rho_0 r + \rho_1 + \rho_2 r^{-1} + \mathcal{O}(r^{-2}), \\ X &= X_0 + X_1 r^{-1} + X_2 r^{-2} + \mathcal{O}(r^{-3}), \end{aligned}$$

All the coefficients in the series are functions of u, ξ , and $\bar{\xi}$. The inverse metric in the Bondi coordinates, denoted by G^{ab} , shall satisfy:

$$\begin{aligned} G^{UU} &= G^{XU} = G^{\bar{X}U} = 0, \\ G^{UR} &= 1 + \mathcal{O}(R^{-1}), G^{XR} = G^{\bar{X}R} = \mathcal{O}(R^{-2}), \\ G^{RR} &= -1 + \mathcal{O}(R^{-1}), \\ (G^{X\bar{X}})^2 - G^{XX} G^{\bar{X}\bar{X}} &= (1 + X\bar{X})^4 R^{-4} + \mathcal{O}(R^{-5}). \end{aligned}$$

In this context, we can define the Bondi mass of the Robinson-Trautman metric :

Definition 7.3.1. Bondi mass aspect and Bondi mass:

The Bondi mass aspect of a spacetime (\mathcal{M}, g) , denoted by \mathcal{M}_B , is obtained from the metric g in the Bondi-Sachs coordinates (U, ρ, X, \bar{X}) (where ρ is the Bondi-Sachs radial coordinate, U is the Bondi-Sachs time, and X and \bar{X} are the variables on the 2-sphere) by expanding the component g_{UU} of the metric as:

$$g_{UU} = 1 - \frac{V}{\rho} + \mathcal{O}(\rho^{-2}).$$

Then, the Bondi mass aspect reads as :

$$\mathcal{M}_B(U, X, \bar{X}) = \frac{1}{2}V(U, X, \bar{X}). \quad (7.42)$$

The Bondi mass is then obtained by integrating the mass aspect over the 2-sphere :

$$M_B(U) = \frac{1}{4\pi} \int_{S_0} \mathcal{M}_B dS_0,$$

where dS_0 is the unit volume element on the 2-sphere and the $1/4\pi$ factor is chosen to ensure that $M_B(U) = m$ for the Schwarzschild black hole.

Knowing that the usual spherical coordinates on the sphere, θ and φ , are defined by:

$$\xi = \cot \theta e^{i\varphi},$$

and using the definitions above and the results of [13], [29], and [3], this gives:

$$\begin{aligned} \mathcal{M}_B(u, \theta, \varphi) = & \frac{m(u)}{\tilde{P}^3} - \frac{\partial U_0}{\partial \theta} \left[\frac{\partial \tilde{c}_1}{\partial \theta} + \frac{3}{2} \cot \theta \tilde{c}_1 + \frac{1}{\sin^2 \theta} \frac{\partial \tilde{c}_2}{\partial \varphi} \right] + \frac{1}{\sin^2 \theta} \frac{\partial U_0}{\partial \varphi} \left[\frac{\partial \tilde{c}_1}{\partial \varphi} - \frac{\partial(\tilde{c}_2 \sin \theta)}{\partial \theta} \right] \\ & + \frac{1}{2\tilde{P}} \frac{\partial \tilde{c}_1}{\partial u} \left[\left(\frac{\partial U_0}{\partial \theta} \right)^2 - \frac{1}{\sin^2 \theta} \left(\frac{\partial U_0}{\partial \varphi} \right)^2 \right] + \frac{1}{\tilde{P} \sin \theta} \left(\frac{\partial U_0}{\partial \varphi} \right) \left(\frac{\partial U_0}{\partial \theta} \right) \left(\frac{\partial \tilde{c}_2}{\partial u} \right) \\ & - \frac{\tilde{c}_1}{2} \left(\frac{\partial^2 U_0}{\partial \theta^2} - \frac{1}{\sin^2 \theta} \frac{\partial^2 U_0}{\partial \varphi^2} \right) - \frac{\tilde{c}_2}{\sin \theta} \frac{\partial^2 U_0}{\partial \varphi \partial \theta}. \end{aligned}$$

with U_0 such that :

$$U_0 = \int \tilde{P} du + f(\theta, \varphi)$$

where $\int P du$ corresponds to the primitive of P with respect to u , $f(\theta, \varphi)$ is an integration function, and $\partial_u \tilde{c}_i$, for $i = 1, 2$, is given by:

$$\frac{\partial \tilde{c}_i}{\partial u} = \frac{1}{\tilde{P}} \frac{\partial c_i}{\partial u}, \quad i = 1, 2.$$

Functions $\partial_u c_1$ and $\partial_u c_2$ are related to the News function N (refer to the definitions in Appendix D.4.3) via:

$$N = \frac{\partial \tilde{c}_1}{\partial u} + i \frac{\partial \tilde{c}_2}{\partial u}.$$

and

$$\frac{\partial c^{(1)}}{\partial u} = \tilde{P} \tilde{c}_1 = \frac{1}{2} \left(\frac{\partial^2 \tilde{P}}{\partial \theta^2} - \cot \theta \frac{\partial \tilde{P}}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \tilde{P}}{\partial \varphi^2} \right), \quad (7.43)$$

$$\frac{\partial c^{(2)}}{\partial u} = \tilde{P} \tilde{c}_2 = \frac{1}{\sin \theta} \left(\frac{\partial \tilde{P}}{\partial \theta \partial \varphi} - \cot \theta \frac{\partial \tilde{P}}{\partial \varphi} \right). \quad (7.44)$$

These terms are then understood as radiative terms that correspond to the emission of gravitational energy from a bounded source. Here, it corresponds to the emission of gravitational waves from the white hole. They lead to a variation of the Bondi mass (see the mass loss formula in Appendix D.4.3) of the white hole, and once specified, they entail the evolution of the system.

The Bondi mass-aspect expression reduces in the axisymmetrical situation to :

$$\mathcal{M}_B(u, \theta) = \frac{m(u)}{\tilde{P}^3} - \frac{\partial U_0}{\partial \theta} \left[\frac{\partial \tilde{c}_1}{\partial \theta} + \frac{3}{2} \cot \theta \tilde{c}_1 \right] + \frac{1}{2\tilde{P}} \frac{\partial \tilde{c}_1}{\partial u} \left(\frac{\partial U_0}{\partial \theta} \right)^2 - \frac{\tilde{c}_1}{2} \left(\frac{\partial^2 U_0}{\partial \theta^2} \right).$$

The details of these computations are given in the Appendix D.4.3 for the sake of clarity.

7.3.4 Assumptions

We denote by K the Gaussian curvature of the 2-surfaces spanned by the complex spatial coordinate ξ :

$$K(u, \xi, \bar{\xi}) = \Delta \ln P,$$

The following proposition, which recalls results obtained in [87] and [31], gives a relation between the Petrov type D and the consequences of the form of the Gaussian curvature K .

Proposition 7.3.1. *Let a spacetime (\mathcal{M}, g) be of Petrov type D, then the components of the Weyl spinor satisfy:*

$$3\Psi_2\Psi_4 = 2\Psi_3^2.$$

Using the expressions (7.35), (7.36), and (7.37) of the Weyl components, this leads to:

$$\frac{\partial}{\partial \bar{\xi}} \left(P^2 \frac{\partial K}{\partial \xi} \right) = 0, \quad (7.45)$$

$$P^2 \left(\frac{\partial K}{\partial \bar{\xi}} \right)^2 = 6m \frac{\partial}{\partial \bar{\xi}} \left(P^2 \frac{\partial^2}{\partial u \partial \bar{\xi}} \ln P \right). \quad (7.46)$$

with $K = \Delta \ln P = 2P^2 \partial_{\xi} \partial_{\bar{\xi}} \ln P$. In other words, (7.45) entails that:

$$P^2 \frac{\partial K}{\partial \xi} = h(u, \xi), \quad (7.47)$$

where h is an arbitrary function. In the pure radiative solution [31], the authors state that $h = 0$ corresponds to the following condition:

$$K = K(u).$$

The proof of the proposition is detailed in [87] and [31]. \square

Assuming that K depends solely on u , it is well known that solutions of this type exist (see [87] or [31]) in the form :

$$P(u, \xi, \bar{\xi}) = A(u) + 2\mathcal{R}e(B(u)\xi) + C(u)\xi\bar{\xi}, \quad (7.48)$$

with $A(u)$ and $C(u)$ being two real functions, and $B(u)$ a complex function. In this framework, the Gaussian curvature is:

$$K(u) = 2(AC - B\bar{B}). \quad (7.49)$$

Because $K = K(u)$, (7.46) ensures that :

$$\frac{\partial}{\partial \bar{\xi}} \left(P^2 \frac{\partial^2}{\partial u \partial \bar{\xi}} \ln P \right) = 0.$$

Hence the Weyl spinor components, defined above in equations (7.34), (7.35), (7.36), and (7.37), become:

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0,$$

and

$$\Psi_2 = -\frac{m}{r^3}. \quad (7.50)$$

Petrov type D solutions have 2 principal null directions that satisfy the following condition:

$$C_{abc[d}V_{e]}V^bV^c = C_{abc[d}W_{e]}W^bW^c = 0,$$

with C_{abcd} being the Weyl tensor. We assume that that the 2-surfaces orthogonal to the principal null directions are spheres. These directions are, for the Robinson-Trautman

metric (see [76]):

$$V = \frac{\partial}{\partial r}, \quad W = \frac{\partial}{\partial u} - \frac{H}{2} \frac{\partial}{\partial r}, \quad (7.51)$$

where :

$$H = K(u) - 2r \frac{\partial}{\partial u} \ln P - \frac{2m}{r}.$$

Because K depends solely on u , the Robinson-Trautman equation (7.39) becomes:

$$-\frac{1}{\tilde{P}^2} \left(\frac{\partial m(u)}{\partial u} - \frac{3m(u)}{\tilde{P}} \frac{\partial \tilde{P}}{\partial u} \right) = n^2,$$

with

$$\tilde{P} = \frac{P}{1 + \xi \bar{\xi}}.$$

The difficulty here is to control the sign of $m(u)$ and of its derivative because it does not represent the physical mass of the system anymore, and we cannot make physical assumptions on its behaviour. This is why we prefer to express (7.39) as:

$$-\tilde{P}(u, \xi, \bar{\xi}) \frac{\partial}{\partial u} \frac{m(u)}{\tilde{P}^3} = 4\pi n^2(u, \xi, \bar{\xi}).$$

Since m/\tilde{P}^3 is the leading term in the Bondi mass aspect \mathcal{M}_B , we assume for the rest of this work that :

$$\tilde{P}(u, \xi, \bar{\xi}) \geq 0.$$

Moreover, we consider that m/\tilde{P}^3 decreases with u , more precisely, we assume that :

$$\frac{\partial m}{\partial u} < 0, \quad \frac{\partial \tilde{P}}{\partial u} > 0, \quad \forall u \in \mathbb{R}. \quad (7.52)$$

Finally, we make the assumptions that m and \tilde{P} are smooth functions on \mathbb{R} , and $m(u)$ satisfies:

$$\lim_{u \rightarrow \pm\infty} m(u) = m_{\pm}, \quad (7.53)$$

where :

$$0 < m_+ < m_- < +\infty.$$

The assumptions on \tilde{P} will be detailed separately in the following sections depending on the value of B .

Geometry and spin coefficient

For the rest of this article, we will work with the metric (7.41) in the $(u, R, \xi, \bar{\xi})$ coordinates with :

$$R = \frac{r}{\tilde{P}}.$$

As we saw in the previous section, these coordinates are not Bondi-Sachs coordinates; however, they are sufficient to remove the coordinate divergence at \mathcal{I} that comes from the term $r\partial_u \log P$:

$$g = Hdu^2 + 2\tilde{P}dudR + 2Rdu \left(\frac{\partial \tilde{P}}{\partial \xi} d\xi + \frac{\partial \tilde{P}}{\partial \bar{\xi}} d\bar{\xi} \right) - 2R^2 \frac{d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2},$$

with $H = K(u) - 2m(u)/R\tilde{P}$. We define the Newman-Penrose tetrad (l, n, m, \bar{m}) made of the two principal null directions and of the two null vectors on the sphere in the coordinates $(u, R, \xi, \bar{\xi})$:

$$\begin{aligned} l &= \frac{1}{\tilde{P}} \partial_R, \\ n &= \partial_u - \frac{H}{2\tilde{P}} \partial_R, \\ m &= -\frac{(1 + \xi\bar{\xi})}{\sqrt{2}\tilde{P}} \frac{\partial \tilde{P}}{\partial \xi} \frac{\partial}{\partial R} + \frac{1 + \xi\bar{\xi}}{R} \frac{\partial}{\partial \xi}, \\ \bar{m} &= -\frac{(1 + \xi\bar{\xi})}{\sqrt{2}\tilde{P}} \frac{\partial \tilde{P}}{\partial \bar{\xi}} \frac{\partial}{\partial R} + \frac{1 + \xi\bar{\xi}}{R} \frac{\partial}{\partial \bar{\xi}}. \end{aligned}$$

This tetrad satisfies :

$$l^a n_a = -m^a \bar{m}_a = 1,$$

and the other inner products are zero. The associated spin coefficients are :

$$\begin{aligned} \varepsilon = \kappa = \sigma = \lambda = \tau = \pi &= 0, \\ \rho &= -\frac{1}{R\tilde{P}}, \alpha = -\bar{\beta} = \frac{\frac{\partial \tilde{P}}{\partial \xi}(1 + \xi\bar{\xi}) + \xi\tilde{P}}{2R\tilde{P}}, \mu = -\frac{RK(u)\tilde{P} - 2m(u)}{2R^2\tilde{P}^2}, \\ \gamma &= -\frac{1}{2} \frac{\frac{\partial \tilde{P}}{\partial u}}{\tilde{P}} + \frac{m(u)}{2R^2\tilde{P}^2}, \nu = -\frac{1 + \xi\bar{\xi}}{\tilde{P}} \left(\tilde{P} \frac{\partial^2 \tilde{P}}{\partial u \partial \bar{\xi}} - \frac{\partial \tilde{P}}{\partial u} \frac{\partial \tilde{P}}{\partial \bar{\xi}} \right). \end{aligned}$$

Remark 7.3.1. Geometry of the principal null geodesics *It is clear that the twist and the shear along the null curves generated by l^a are zero:*

$$\sigma = 0, \omega = \text{Im}(\rho) = 0. \quad (7.54)$$

Furthermore, the twist and the shear along the null curves generated by n^a are also zero. This corresponds in the Newman-Penrose formalism to $\omega' = \text{Im}(\rho')$ for the twist and σ' for the shear, and we know from [74] that $\sigma' = -\lambda$ and $\rho' = \mu$. Then it follows that:

$$\sigma' = 0,$$

and knowing that \tilde{P} , K , and m are real functions:

$$\text{Im}(\rho') = \text{Im}(-\mu) = 0.$$

However, for a null vector field to be hypersurface forming (i.e. hypersurface orthogonal), it must be geodesic and twist-free (see Appendix D.2). This ensures that l^a is hypersurface-orthogonal, while n is not a hypersurface-forming vector field. $\nu \neq 0$ is equivalent to stating that n is not geodesic. Note that:

$$\nu = -\frac{1 + \xi\bar{\xi}}{\tilde{P}} \tilde{P}^2 \frac{\partial}{\partial u} \frac{\partial}{\partial \bar{\xi}} \ln \tilde{P} = -P \frac{\partial}{\partial u} \frac{\partial}{\partial \bar{\xi}} \ln P.$$

Using the Robinson-Trautman equation for $K = K(u)$, we have:

$$3m \frac{\partial}{\partial u} \ln P - m'(u) = 4\pi n^2(u, \xi, \bar{\xi}).$$

Then, remarking that in the vacuum $n^2 = 0$ or in the situation in which $n^2 = n^2(u)$, ν becomes:

$$\nu = 0,$$

hence, the vector field n^a is geodesic, twist-free, and hypersurface-orthogonal.

7.4 Case with $B=0$

Under the assumptions made above, we can state the following proposition that classifies the solutions with $B = 0$ into two categories.

Proposition 7.4.1. *Let $P(u, \xi, \bar{\xi})$ be given by:*

$$P(u, \xi, \bar{\xi}) = A(u) + C(u)\xi\bar{\xi},$$

and $\tilde{P} = P/(1 + \xi\bar{\xi})$. The Gaussian curvature K is $K(u) = A(u)C(u)$, and the Robinson-Trautman pure radiative solution metric (7.41) becomes:

$$g = \left(A(u)C(u) - \frac{2m(u)}{R\tilde{P}} \right) du^2 + 2\tilde{P}dudR + 2Rdu \left(\frac{\partial\tilde{P}}{\partial\xi}d\xi + \frac{\partial\tilde{P}}{\partial\bar{\xi}}d\bar{\xi} \right) - R^2 \frac{d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2}. \quad (7.55)$$

and it is of two kinds:

1. If $A(u) = C(u)$, then (7.55) is the Vaidya solution.
2. If $A(u) \neq C(u)$, then (7.55) is a pure radiative axisymmetrical solution.

Proof : Let us begin by computing the $g_{u\xi}$ and $g_{u\bar{\xi}}$ terms in the metric (7.55):

$$\begin{aligned} \frac{\partial\tilde{P}}{\partial\xi} &= \frac{\bar{\xi}(C - A)}{(1 + \xi\bar{\xi})^2}, \\ \frac{\partial\tilde{P}}{\partial\bar{\xi}} &= \frac{\xi(C - A)}{(1 + \xi\bar{\xi})^2}. \end{aligned}$$

Then, expressing ξ and $\bar{\xi}$ in terms of the usual spherical coordinates θ and φ :

$$\xi = \cot\theta e^{i\varphi},$$

this leads to:

$$\begin{aligned} \frac{\partial\tilde{P}}{\partial\xi}d\xi + \frac{\partial\tilde{P}}{\partial\bar{\xi}}d\bar{\xi} &= \frac{C - A}{(1 + \xi\bar{\xi})^2} (\bar{\xi}d\xi + \xi d\bar{\xi}) \\ &= \frac{C - A}{(1 + \xi\bar{\xi})^2} d(\xi\bar{\xi}) = \frac{C - A}{1 + \cot^2\theta} d(\cot^2\theta) \\ &= -2(C - A) \cot\theta d\theta. \end{aligned}$$

Hence, the metric for $B = 0$ is :

$$g = \left(A(u)C(u) - \frac{2m(u)}{R\tilde{P}} \right) du^2 + 2\tilde{P}dudR - 4R(C - A) \cot\theta dud\theta - R^2 d\omega^2, \quad (7.56)$$

with,

$$\tilde{P} = A(u) \sin^2 \theta + C(u) \cos^2 \theta .$$

Introducing here k, l, m as the generators of the rotations defined by:

$$\begin{aligned} k &= \partial_\varphi , \\ l &= \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi , \\ m &= -\sin \varphi \partial_\theta - \cot \theta \cos \varphi \partial_\varphi , \end{aligned}$$

the Killing equation gives:

$$\begin{aligned} \nabla_{(a} k_{b)} &= 0 , \\ \nabla_{(a} l_{b)} &= \delta \frac{m(u) \cos \theta \sin \theta \cos \varphi}{R \tilde{P}^2} du^2 + \delta \cos \phi \cos \theta \sin \theta dudR + \delta \frac{R \cos \varphi}{\sin^2 \theta} dud\theta \\ &\quad + \delta \cot \theta \sin \varphi dud\varphi . \\ \nabla_{(a} m_{b)} &= \delta \frac{m(u) \cos \theta \sin \theta \sin \varphi}{R \tilde{P}^2} du^2 + \delta \sin \varphi \cos \theta \sin \theta dudR + \delta \frac{R \sin \varphi}{\sin^2 \theta} dud\theta \\ &\quad + \delta \cot \theta \cos \varphi dud\varphi . \end{aligned}$$

where $\delta = A - C$. This implies that the metric is only axisymmetrical for $A \neq C$, and (7.55) is spherically symmetric if and only $A(u) = C(u)$. We assume that A is an increasing function on u . In this case, functions P and \tilde{P} are:

$$P = A(1 + \xi \bar{\xi}), \tilde{P} = A ,$$

and the metric (7.55) becomes :

$$g = A^2 \left(1 - \frac{2m(u)}{A^3 R} \right) du^2 + 2A dudR - R^2 d\omega^2 .$$

Note that the Bondi mass is given by:

$$M_B(u) = \frac{m(u)}{A^3(u)} .$$

Choosing \tilde{u} such that $d\tilde{u} = A(u)du$, turns the metric into :

$$g = \left(1 - \frac{2M_B}{R}\right) d\tilde{u}^2 + 2d\tilde{u}dR - R^2d\omega^2,$$

which is the Vaidya metric.

7.4.1 Axisymmetric solutions

Let us now consider that $B = 0$ but $A \neq C$. Assumptions $\tilde{P} > 0$ and $\partial_u \tilde{P} \geq 0$ on the whole real line ensure that:

$$A(u) \geq 0, C(u) \geq 0, A'(u) \geq 0, C'(u) \geq 0.$$

Finally, we assume that $\tilde{P}(u, \xi, \bar{\xi})$ is a smooth function of $u \in \mathbb{R}$ such that :

$$\tilde{P}(u, \xi, \bar{\xi}) = \begin{cases} 1 & \forall u \in]-\infty, u_-], \\ A(u) \sin^2 \theta + C(u) \cos^2 \theta & \forall u \in [u_-, u_+], \\ \tilde{P}_+ & \forall u \in [u_+, +\infty[, \end{cases}$$

with :

$$1 < \tilde{P}_+ < +\infty.$$

This is equivalent to stating:

$$A(u) = \begin{cases} 1 & \forall u \in]-\infty, u_-], \\ A(u) & \forall u \in [u_-, u_+], \\ \tilde{P}_+ & \forall u \in [u_+, +\infty[, \end{cases}, \quad C(u) = \begin{cases} 1 & \forall u \in]-\infty, u_-], \\ C(u) & \forall u \in [u_-, u_+], \\ \tilde{P}_+ & \forall u \in [u_+, +\infty[, \end{cases}$$

In other words, for $u \in]-\infty, u_-]$ and $u \in [u_+, +\infty[$, the metric (7.56) is the Vaidya metric with respectively a mass $m(u)$ and $m(u)/\tilde{P}_+^3$. The event horizon of the white hole on $] -\infty, u_-]$ is then the Vaidya event horizon as defined in [15] and denoted by R_h^V .

The question is how to study the behavior of the horizon on $[u_-, u_+]$? Because $\nu \neq 0$ for all $u \in [u_-, u_+]$, we know that the incoming principal null direction is not tangent to geodesics; hence, it is not hypersurface-forming. Therefore, the past event horizon is not generated by the principal null directions on $[u_-, +\infty]$. The study of the horizon has to be approached as follows: for $u \leq u_-$, the horizon is generated by the incoming null principal directions; for $u \geq u_+$, we have to use the geodesic equation. Obtaining solutions to the

geodesic equation is more challenging since it is a second-order equation and needs to be addressed in a future project.

Here, we focus on the null curves indexed by $\omega \in S^2$, denoted by $\gamma = \gamma(u, R(u, \theta, \varphi), \omega)$, $u \in \mathbb{R}$ that are tangent to the incoming principal null direction:

$$W = \partial_u - \frac{1}{2\tilde{P}} \left(K(u) - \frac{2m(u)}{R\tilde{P}} \right) \partial_R.$$

This implies that the function $R(u)$ is a solution of the following differential equation:

$$\frac{dR}{du} = -\frac{1}{2\tilde{P}(u, \theta)} \left(K(u) - \frac{2m(u)}{R\tilde{P}(u, \theta)} \right), \quad (7.57)$$

with $R > 0$. The method followed here is the same as in the Vaidya spacetime (see [15]) and is summarized as follows:

1. We observe that there exists a function $R_0(u, \theta)$ that is not a solution to (7.57) such that the right-hand side of (7.57) is zero. We also prove that if $R(u, \theta)$ is a solution to (7.57), it cannot be equal to R_0 on an interval on which $A'(u) \neq 0$ and $C'(u) \neq 0$.
2. Then we prove that if a solution to (7.57) is at a point u_0 such that $R(u_0, \theta) \geq R_0(u_0, \theta)$, then it follows that $R(u, \theta) \geq R_0(u, \theta)$ for $u \in]u_0, u_1[$ with $u_1 \geq u_+$.
3. Finally, we establish a theorem that classifies the solutions to (7.57) into three kinds depending on the value of the limit of R in the past. In particular, we prove that there is only one solution to (7.57) that admits a finite limit when $u \rightarrow -\infty$, that exists on the whole real line, and that is identically the Vaidya horizon on $] -\infty, u_-]$.

Lemma 7.4.1. *Let $]u_0, u_1[$ be an interval on which \tilde{P} is an strictly increasing function of u , and let $R(u)$ be a solution to (7.57). We define R_0 to be the function:*

$$R_0 := \frac{2m(u)}{\tilde{P}(u, \theta)K(u)},$$

hence,

$$\frac{\partial R_0}{\partial u} < 0 \forall u \in]u_0, u_1[.$$

Then, $R(u)$ cannot be identically equal to $R_0(u)$ on $]u_0, u_1[$.

Proof: Let us assume that $R(u)$ is a solution to (7.57) such that $R = R_0$ on $]u_0, u_1[$.

Then, this entails:

$$\frac{dR}{du} = 0.$$

This is in contradiction with the definition of R_0 that is a strictly decreasing function on $]u_0, u_1[$. Thus, it is not possible for a solution to (7.57) to be identically equal to R_0 on $]u_0, u_1[$. \square

The following lemma establishes that R_0 is a boundary that cannot be crossed from above by a solution to (7.57). This can be seen as a direct consequence of the local uniqueness of the solutions to the Cauchy problem.

Lemma 7.4.2. *Let R be a solution to (7.57) on $]u_1, u_2[$ that satisfies at a point $u_0 \in]u_1, u_2[$:*

$$R(u_0, \theta) \geq R_0(u_0, \theta).$$

Then, assuming that R_0 is a decreasing function on $]u_1, u_2[$, we have :

$$R(u) > R_0(u) \forall u \in]u_0, u_2[.$$

Proof: Observing that if $R(u_0) = R_0(u_0)$, we have $\dot{R}(u_0) = 0$. Since R_0 decreases on $]u_1, u_2[$, there exists $\varepsilon > 0$ such that on $]u_0, u_0 + \varepsilon[$, $R > R_0$. This result holds if $R(u_0) > R_0(u_0)$ by continuity.

Let us now consider u_3 the lowest value of $u \in]u_0, u_2[$ such that $R(u) = R_0(u)$. (7.57) implies that $\dot{R}(u_3) = 0$, and we know that $\dot{R}_0(u_3) < 0$. Then, there exists $\delta > 0$ such that on $]u_3 - \delta, u_3[$ we have $R(u) < R_0(u)$. Then, by continuity of R and R_0 , there exists $u_4 \in]u_0, u_3[$ such that $R(u_4) = R_0(u_4)$. This is in contradiction with the assumption on u_3 . Finally, we obtain $R(u) > R_0(u)$ on $]u_0, u_2[$. \square

The final step in our study of the solutions to (7.57) is to classify these solutions depending on the value of their limit in the past. The limit is unstable when $u \rightarrow -\infty$, and only one solution has a finite limit that is $2m_-$. This solution foliates the past event horizon on $] - \infty, u_-]$. This is the subject of the following theorem:

Theorem 7.4.1. *Assuming that $\tilde{P}(u, \theta)$ is a smooth increasing function of u on $[u_-, u_+]$ and that $m(u)$ is a smooth decreasing function on \mathbb{R} with a finite limit when u goes to $\pm\infty$, then there exists a unique solution to (7.57), denoted R_h , such that:*

$$\lim_{u \rightarrow -\infty} R_h(u, \theta) = 2m_-.$$

R_h exists on the whole real line and satisfies:

$$\lim_{u \rightarrow +\infty} R_h(u, \theta) = \lim_{u \rightarrow +\infty} M_B(u) = \frac{2m_+}{\tilde{P}_+^3}.$$

The other solutions to (7.57) are of two kinds:

- R exists $\forall u \in \mathbb{R}$, on its whole domain of existence, it satisfies $R(u) > R_h(u)$ and is a decreasing function that has limits:

$$\lim_{u \rightarrow -\infty} R(u, \theta) = +\infty, \quad \lim_{u \rightarrow +\infty} R(u, \theta) = \frac{2m_+}{\tilde{P}_+^3}.$$

- R exists on $]u_0, +\infty[$ with $u_0 > -\infty$, on its whole domain of existence, it satisfies $R(u) < R_h(u)$ and has limits:

$$\lim_{u \rightarrow u_0} R(u, \theta) = 0, \quad \lim_{u \rightarrow +\infty} R(u, \theta) = \frac{2m_+}{\tilde{P}_+^3}.$$

Proof : The proof is decomposed into three steps: 1. we prove that there exists a unique finite limit for solutions to (7.57), then we prove that there is a unique solution that converges towards this limit in the past; 2. we construct this solution and establish its existence on the whole real line; 3. We classify the other solutions, determine their domain of existence, and compute their limits.

1. *Uniqueness of the maximal solution with a finite limit in the past.* Let us assume that R is a solution to (7.57) that exists on an interval of the form $] - \infty, u_0[$ and that admits a finite limit as u goes to $-\infty$. Then we have:

$$\lim_{u \rightarrow -\infty} R(u) = l, \quad l \in \mathbb{R}.$$

From (7.57), we know that if $R(u)$ has a finite limit as $u \rightarrow -\infty$, then it implies that $\dot{R}(u)$ also has a finite limit, and it should be $\lim_{u \rightarrow -\infty} \dot{R}(u) = 0$. If this is not the case, it follows that $\lim_{u \rightarrow -\infty} |R(u)| = +\infty$. Then,

$$\lim_{u \rightarrow -\infty} \dot{R}(u) = 0.$$

Using (7.57),

$$\lim_{u \rightarrow -\infty} \dot{R}(u) = \lim_{u \rightarrow -\infty} \left(K(u) - \frac{2m(u)}{R(u)\tilde{P}(u, \theta)} \right) = 0,$$

hence,

$$l = 2m_-.$$

Let us prove that there exists a unique solution to (7.57) defined on an interval of the form $] -\infty, u_0[$ such that:

$$\lim_{u \rightarrow -\infty} R(u) = 2m_-.$$

Assuming that there exist two solutions to (7.57), R_1 and R_2 , that have a finite limit $l = 2m_-$ in the past. We introduce ψ :

$$\psi = R_1 - R_2.$$

We have,

$$\dot{\psi} = \frac{m(u)}{\tilde{P}^2(u, \theta)} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = -\frac{m(u)}{\tilde{P}^2(u, \theta)} \frac{\psi}{R_1 R_2},$$

and by assumption,

$$\lim_{u \rightarrow -\infty} \psi = 0.$$

Since,

$$\lim_{u \rightarrow -\infty} \frac{m(u)}{\tilde{P}^2(u, \theta)} \frac{1}{R_1 R_2} = \frac{1}{4m_-} \geq 0,$$

it follows that if $\psi \neq 0$ we have:

$$\lim_{u \rightarrow -\infty} \frac{\dot{\psi}}{\psi} = -\frac{1}{4m_-},$$

Hence, ψ is diverging exponentially fast as u goes to $-\infty$. This is in contradiction with the assumptions that ensure that ψ is going to zero when $u \rightarrow -\infty$, and so $R_1 = R_2$.

2. *Construction of the solution.* We want to construct a solution that converges to $2m_-$ in the past. We focus on the case $u_- \rightarrow -\infty$. For each $n \in \mathbb{N}$, we define R_n to be

the maximal solution to (7.57) that satisfies:

$$R_n(-n) = R_0(-n).$$

R_n exists on an interval of the form $]u_n^1, u_n^2[$ with $u_n^1 < -n < u_n^2$. Let us prove that $u_n^2 = +\infty$. Let $u_n^3 = \min\{u_n^2, u_+\}$, then using Lemma 7.4.2, we have:

$$R > R_0 \forall u \in]-n, u_n^3[.$$

From (7.57), this ensures that $\dot{R}_n < 0$. On $] -n, u_n^3[$, R_n is bounded as:

$$R_0(u) < R_n(u) < R_0(-n).$$

and we know that R_0 is a decreasing function on \mathbb{R} such that:

$$2m_- \geq R_0 \geq \frac{2m_+}{\tilde{P}_+^3}, \forall u \in \mathbb{R}. \quad (7.58)$$

This implies that $u_n^2 \geq u_+$; hence, if $u_+ = \infty$, this leads to $u_n^2 = +\infty$. Let us now consider the case where $u_+ < +\infty$. If $R_n(u_+) = 2m_+/\tilde{P}_+^3$, then from (7.57), we have $\dot{R}_n = 0$ for $u \geq u_+$, hence $u_n^2 = +\infty$, and $R_n(u) = 2m_+/\tilde{P}_+^3$ on $[u_+, +\infty[$. If $R_n(u_+) > 2m_+/\tilde{P}_+^3$, since two solutions cannot cross, it implies that $R_n(u) > 2m_+/\tilde{P}_+^3$ for $u \in [u_+, u_n^2]$, and it follows that $\dot{R}_n < 0$ for $u \in [u_+, u_n^2]$. This ensures that $u_n^2 = +\infty$.

On each compact interval I of \mathbb{R} , there exists $n_0 \in \mathbb{N}$ such that the sequence $R_{n \geq n_0}$ is increasing and bounded, and using Lebesgue's dominated convergence theorem, it converges in $L_{\text{loc}}^1(\mathbb{R})$ toward a decreasing positive function R_h such that

$$R_0(u) < R_h(u) \leq 2m_-, \forall u \in \mathbb{R}. \quad (7.59)$$

Since $R_0 > 0$ on \mathbb{R} , this entails that $1/R_n$ converges also in $L_{\text{loc}}^1(\mathbb{R})$ toward $1/R_h$. Hence, \dot{R}_n converges in $L_{\text{loc}}^1(\mathbb{R})$. By the uniqueness of the limit in the sense of distributions, the limit must be \dot{R}_h , thus R_h is a solution to (7.57) in the sense of distributions. Because $R_0(u)$ has $2m_-$ as limit when $u \rightarrow -\infty$, (7.59) ensures that:

$$\lim_{u \rightarrow -\infty} R_h(u) = 2m_-.$$

Let us study the behaviour of R_h as $u \rightarrow +\infty$. Since $R_h(u) > R_0$ on \mathbb{R} , it follows that $\dot{R}_h(u) < 0$ on \mathbb{R} . This ensures that R_h has a finite limit denoted by l as $u \rightarrow +\infty$, so:

$$\lim_{u \rightarrow +\infty} \dot{R}_h(u) = \lim_{u \rightarrow +\infty} \frac{1}{2\tilde{P}} \left(K(u) - \frac{2m(u)}{R\tilde{P}(u)} \right) = 0.$$

Then,

$$\lim_{u \rightarrow +\infty} R_h(u) = \lim_{u \rightarrow +\infty} 2\mathcal{M}_B(u) = \frac{2m_+}{\tilde{P}_+^3}.$$

In the situation where $u_- > -\infty$, we have the Vaidya metric on $] -\infty, u_-[$, and we know from [15] that there is a unique solution to (7.57) that exists on $] -\infty, u_0[$ and tends to $2m_-$ in the past. This is the past horizon in the Vaidya spacetime, denoted by R_h^V . Then, $R_h(u)$ becomes the solution to (7.57) that satisfies:

$$R_h(u) = R_h^V(u), \forall u \in] -\infty, u_-[.$$

Furthermore, we have :

$$R_0(u) = 2m(u), \forall u \in] -\infty, u_-[.$$

It follows, using results from [15], that $R_h(u) > R_0(u)$ on $] -\infty, u_-[$. Then, the same arguments as for $u_- = -\infty$ hold.

3. *Classification of the other maximal solutions.* We begin with an observation that comes directly from the uniqueness of solutions to the Cauchy problem. Let R be a maximal solution to (7.57) that exists on an interval of the form $]u_1, u_2[$. If $R(u_0) < R_h(u_0)$, with $u_0 \in]u_1, u_2[$, then $R(u) < R_h(u)$, $\forall u \in]u_1, u_2[$. The same result holds respectively for $R(u_0) > R_h(u_0)$.

- If for $u_0 \in]u_1, u_2[$, $R(u_0) > R_h(u_0)$, then it follows that $R(u) > R_0(u)$, $\forall u \in]u_1, u_2[$. Hence, $\dot{R}(u) < 0$ on its whole domain of existence. This ensures $u_2 = +\infty$, and the limit of $R(u)$ has to be finite as u goes to $+\infty$. Then:

$$\lim_{u \rightarrow +\infty} R(u) = \frac{2m_+}{P_+^3}.$$

On $]u_1, +\infty[$, since $R(u) > R_h(u) \geq R_0(u)$, this implies that $\dot{R}(u)$ remains

bounded. Therefore:

$$-\frac{K(u)}{2\tilde{P}} < \dot{R} < 0.$$

Because $1 \leq K(u) \leq \tilde{P}_+^2$ and $1 \leq \tilde{P} \leq \tilde{P}_+$,

$$-\frac{\tilde{P}_+^2}{2} < \dot{R} < 0.$$

Hence, $u_1 = -\infty$, and R exists on the whole real line. It follows that R has a limit when $u \rightarrow -\infty$, and it cannot be a finite limit because the only solution to (7.57) with a finite limit as $u \rightarrow -\infty$ is R_h . This yields:

$$\lim_{u \rightarrow -\infty} R(u) = +\infty.$$

- We consider now the situation in which $R(u_0) < R_h(u_0)$ for $u_0 \in]u_1, u_2[$. Firstly, we assume that $R_0(u_0) < R(u_0) < R_h(u_0)$. Then, using Lemma 7.4.2, $R(u) \geq R_0(u), \forall u \in]u_0, u_2[$, hence $\dot{R}(u) < 0$ for $u \in]u_0, u_2[$. Furthermore, using the uniqueness of solutions to the Cauchy problem, this means that $R(u)$ is bounded above by $R_h(u)$ on $]u_1, u_2[$, and this ensures that $u_2 = +\infty$. Therefore:

$$R_0(u) \leq R(u) < R_h(u), \forall u \in]u_0, +\infty[,$$

and this gives :

$$\lim_{u \rightarrow +\infty} R(u) = \frac{2m_+}{\tilde{P}_+^3}.$$

We want to show that there exists $u_3 \in]u_1, u_0[$ such that $R(u_3) = R_0(u_3)$. Let us assume that $R_0(u) < R(u) < R_h(u)$ on $]u_1, u_2[$. This implies that $u_1 = -\infty$ and that $R(u)$ converges to $2m_-$ as $u \rightarrow -\infty$. However, this is not possible because it contradicts the uniqueness of R_h .

Then, there exists u_3 such that $R(u_3) = R_0(u_3)$ and $\dot{R}(u_3) = 0$. Furthermore, $\dot{R}(u_3) > \dot{R}_0(u_3)$. It follows that $R(u) < R_0(u), \forall u \in]u_1, u_3[$. Therefore, \dot{R} is decreasing on $]u_1, u_3[$, and this entails that $u_1 > -\infty$, or in other terms, that R reaches 0 for a finite value of u_1 :

$$\lim_{u \rightarrow u_1} R(u) = 0.$$

If $R(u_0) \leq R_0(u_0)$, the same arguments hold, but we have to consider that

$$u_3 \in [u_0, u_2[. \quad \square$$

7.5 Case with $B \neq 0$

Let us assume now that $B \neq 0$, then:

$$P(u, \xi, \bar{\xi}) = A(u) + 2\mathcal{R}e(B(u)\xi) + C(u)\xi\bar{\xi},$$

depends on θ and φ in the spherical coordinates. The vector ∂_φ is not Killing anymore, and thus the metric is not axisymmetric anymore:

$$g = \left(K(u) - \frac{2m(u)}{R\tilde{P}(u, \xi, \bar{\xi})} \right) du^2 + 2\tilde{P}du dR + 2Rdu \left(\frac{\partial\tilde{P}}{\partial\xi} d\xi + \frac{\partial\tilde{P}}{\partial\bar{\xi}} d\bar{\xi} \right) - R^2 \frac{d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2}.$$

with :

$$\begin{aligned} \frac{\partial\tilde{P}}{\partial\xi} &= \frac{B(u) + (C - A)\bar{\xi} - \bar{B}(u)\bar{\xi}^2}{(1 + \xi\bar{\xi})^2}, \\ \frac{\partial\tilde{P}}{\partial\bar{\xi}} &= \frac{\bar{B}(u) + (C - A)\xi - B(u)\xi^2}{(1 + \xi\bar{\xi})^2}. \end{aligned}$$

The Gaussian curvature is given by :

$$K(u) = A(u)C(u) - B(u)\bar{B}(u),$$

We make the following assumptions:

$$B(u) = \begin{cases} 0 & \forall u \in]-\infty, u_-], \\ B(u) & \forall u \in [u_-, u_+], \\ 0 & \forall u \in [u_+, +\infty[, \end{cases} \quad (7.60)$$

and

$$A(u) = \begin{cases} 1 & \forall u \in]-\infty, u_-], \\ A(u) & \forall u \in [u_-, u_+], \\ \tilde{P}_+ & \forall u \in [u_+, +\infty[, \end{cases} \quad , \quad C(u) = \begin{cases} 1 & \forall u \in]-\infty, u_-], \\ C(u) & \forall u \in [u_-, u_+], \\ \tilde{P}_+ & \forall u \in [u_+, +\infty[, \end{cases} \quad (7.61)$$

with $\tilde{P}_+ > 1$. Furthermore, we assume that :

$$\frac{\partial}{\partial u} \Delta_{S^2} \ln \tilde{P} \geq 0 \quad \forall u \in \mathbb{R}. \quad (7.62)$$

As we did for $B = 0$, we denote by γ the family of curves indexed by $\omega \in S^2$ generated the incoming principal null vector field:

$$\gamma(u, R, \omega) = \gamma(u, R(u, \xi, \bar{\xi}), \omega), \quad u \in \mathbb{R},$$

with :

$$R(u, \xi, \bar{\xi}) = R_h^V(u); \quad \forall u \in]-\infty, u_-].$$

It follows that $R(u)$ is a solution to:

$$\frac{dR}{du} = -\frac{1}{2\tilde{P}(u, \xi, \bar{\xi})} \left(K(u) - \frac{2m(u)}{R\tilde{P}(u, \xi, \bar{\xi})} \right). \quad (7.63)$$

Observing that there exists a function R_0 :

$$R_0 := \frac{2m(u)}{\tilde{P}(u, \xi, \bar{\xi})K(u)} = \frac{2m(u)}{\tilde{P}^3 (1 + \Delta_{S^2} \ln \tilde{P})},$$

which, under our assumptions, is a decreasing function on \mathbb{R} . This function is analogous to the one obtained for $B = 0$, hence it follows that results obtained for $B = 0$ still hold, and solutions to (7.63) are classified in the following theorem:

Theorem 7.5.1. *Assume that $\tilde{P}(u, \theta, \varphi)$ is a smooth, increasing function of u on \mathbb{R} , satisfying (7.60), (7.61) and (7.62), and that $m(u)$ is a smooth, decreasing function on \mathbb{R} with a finite limit as u goes to $\pm\infty$. Then, there exists a unique solution to (7.63), denoted R_h , such that:*

$$\lim_{u \rightarrow -\infty} R_h(u, \theta, \varphi) = 2m_-.$$

R_h exists on the whole real line and satisfies:

$$\lim_{u \rightarrow +\infty} R_h(u, \theta, \varphi) = \lim_{u \rightarrow +\infty} 2M_B(u, \theta, \varphi) = \frac{2m_+}{\tilde{P}_+^3}.$$

The other solutions to (7.63) fall into two categories:

- R exists for all $u \in \mathbb{R}$, and throughout its entire domain of existence, it satisfies

$R(u) > R_h(u)$. It is a decreasing function with limits:

$$\lim_{u \rightarrow -\infty} R(u, \theta, \varphi) = +\infty, \quad \lim_{u \rightarrow +\infty} R(u, \theta, \varphi) = \frac{2m_+}{\tilde{P}_+^3}.$$

- R exists on $]u_0, +\infty[$ with $u_0 > -\infty$. Throughout its entire domain of existence, it satisfies $R(u) < R_h(u)$ and has limits:

$$\lim_{u \rightarrow u_0} R(u, \theta, \varphi) = 0, \quad \lim_{u \rightarrow +\infty} R(u, \theta, \varphi) = \frac{2m_+}{\tilde{P}_+^3}.$$

Proof: We proceed similarly to Theorem 7.4.1. □

7.6 Conclusions and perspectives

In this section, we have analyzed the behaviour of the integral curves of the principal null ingoing direction in a pure radiative type D Robinson-Trautman spacetime. We observe that these curves behave similarly to the Vaidya spacetimes, following an ordinary differential equation (ODE), and for each value of θ and φ , there exists a unique solution with a finite limit in the past. Furthermore, the other solutions to this ODE can be classified into two categories: either they exist on the entire real line and blow up in the past, or solutions exist from a finite retarded time at which they reach 0. All of these curves converge to the future Schwarzschild horizon, given that the spacetime we considered is asymptotically Schwarzschild with a mass m_+/P_+^3 .

Regarding the solution with a finite limit in the past, denoted by R_h , we remark that it generates the past horizon in the Vaidya region. However, when the Robinson-Trautman radiation is turned on, the integral curves of the ODE are no longer geodesics, and thus, they fail to form hypersurfaces. This implies that R_h cannot be the horizon in the Robinson-Trautman pure radiative solution, and we present ideas for consideration to study the past event horizon:

1. In the method developed in [15], the authors imposed only that the integral lines under study are null, without a priori consideration of whether they are geodesics or not. This was not a problem since these lines are generated by the incoming principal null directions, and due to the spherical symmetry of the Vaidya metric, these curves are geodesics. Therefore, the first idea to explore is to study the behaviour of null

geodesics, denoted by γ , and to impose that they are tangent to the event horizon before "switching on" the Robinson-Trautman radiation at u_- . In other words, from the retarded time at which the radiation becomes active, we study the solutions to the geodesic equation rather than studying the integral lines generated by the incoming principal null direction.

2. As a first step in this project, we may consider the situation $B = 0$, i.e., an axisymmetrical metric. In terms of θ and φ coordinates, this implies that ∂_φ is a killing vector field, ensuring that $\frac{dR}{d\varphi} = 0$ on the horizon. In a second step, we shall generalize our results to the $B \neq 0$ situation.
3. The study of the geodesic equation can be developed in two directions. Firstly, we can impose that the metric for $u \leq u_-$ is the Schwarzschild metric, with a mass M . Then the event horizon is located at $R = 2M$, and we can numerically integrate the geodesic equation, imposing initial conditions at $u = u_-$: $R(u_-) = 2M$ and $\dot{R}(u_-) = 0$, where the dot denotes the derivative with respect to u . We expect that the horizon R decreases with u , as the Bondi mass of the white hole decreases too. Secondly, we aim to perform a complete analytical study, imposing the metric to be the Vaidya metric for $u \leq u_-$.

PART IV

Appendix

PEELING

A.1 On the Morawetz vector field

In this section, we prove that the Morawetz vector field is timelike and future-oriented in the neighbourhood of spacelike infinity and therefore transverse to null infinity. This allows us to define a positive or non-negative energy flux across spacelike or null hypersurfaces whose normals are future-oriented (for instance on \mathcal{H}_s , \mathcal{I}^+ or \mathcal{S}_{u_0}).

The “squared norm” of the vector field T^a is given by:

$$\hat{g}(T, T) = u^2 [F|uR|^2 - 4|uR| + 4] .$$

The expression between square brackets admits two roots denoted by $|uR|_{\pm}$:

$$|ur|_{\pm} = 2 \frac{1 \pm \sqrt{2m(u)R}}{1 - 2m(u)R} . \tag{A.1}$$

As one approaches i^0 , both roots (A.1) tend to 2 and we have seen that in Ω_{u_0} , $0 < |u|R < 1 + \varepsilon$. So this means that $\hat{g}(T, T) > 0$, hence T is timelike, in Ω_{u_0} . Now choosing ∂_u as the global time orientation of our spacetime, we have:

$$g(T, \partial_u) = |uR|^2 F - 2|uR| + 2$$

and this is positive near i^0 . The Morawetz vector field is therefore timelike and future oriented in Ω_{u_0} . □

A.2 Proof of main propositions and theorem

A.2.1 Proof of proposition 4.3.1

Firstly we want to prove that in Ω_{u_0} , provided we take $|u_0|$ large enough, we have

$$1 \leq \psi < 1 + \varepsilon.$$

We need a good understanding of the behaviour of the function ψ near i^0 . To do so, we integrate the differential equation (3.11) along $v = \text{cst.}$ curves parametrised by u . We start from \mathcal{I}^- where we set $\psi \equiv 1$. Equation (3.11) gives

$$(\log(\psi))' = \frac{-2Rm'(u)}{F} \quad (\text{A.2})$$

using the fact that the total derivative of φ along the curve is

$$\frac{d\psi}{du} = \frac{\partial\psi}{\partial u} - \frac{F}{2} \frac{\partial\psi}{\partial r}.$$

From (A.2), we obtain an expression for φF :

$$\psi = \exp \left[- \int_{-\infty}^u \frac{2Rm'(\mu)}{1 - 2mR} d\mu \right] =: \exp \left[- \int_{-\infty}^u f(\mu, R) d\mu \right]. \quad (\text{A.3})$$

And since $m'(\mu) \leq 0$ in the exterior region, we have $f(\mu, R) \leq 0$. Within Ω_{u_0} the function F is bounded below by F_{\min} is arbitrarily close to 1, and bounded above by 1. Hence we have the following estimates for f :

$$0 \leq |f(u, R)| \leq -2 \frac{m'(u)R}{F_{\min}} \leq -m'(u).$$

Moreover, the function f tends to 0 at \mathcal{I}^+ , because $R \rightarrow 0$. Since the mass function has the finite limit m_- as $u \rightarrow -\infty$, it follows that m' is integrable in the neighbourhood of $-\infty$. We can therefore use Lebesgue's dominated convergence Theorem to obtain, uniformly for $u \leq u_0$

$$\begin{aligned} \lim_{\mathcal{I}^+} \int_{-\infty}^u f(\mu, R) d\mu &= 0, \\ \lim_{\mathcal{I}^+}(\psi) &= \lim_{\mathcal{I}^+} \exp \left[\int_{-\infty}^u f(\mu, R) d\mu \right] = 1. \end{aligned}$$

Given $\varepsilon > 0$, Equation (A.3) entails that for $u_0 < 0$, $|u_0|$ large enough and $u \leq u_0$, we have

$$1 \leq \psi < 1 + \varepsilon.$$

Now we want to prove that :

$$1 - \varepsilon < \tilde{r}R < 1 + \varepsilon$$

Let $v_0 \gg 1$, we begin by integrating $r(u, v)$ between v_0 and v , on a $u = cst.$ line, using:

$$\frac{dr}{dv} = \frac{F}{2\psi},$$

and $\tilde{r} = (v - u)/2$, so :

$$\frac{r(u, v)}{\tilde{r}(u, v)} = \frac{2}{v - u} \left(r(u, v_0) + \frac{1}{2} \int_{v_0}^v \frac{F}{\psi} dv \right)$$

Taking $v_0 < v_1 < v$ we have,

$$\frac{r(u, v)}{\tilde{r}(u, v)} - 1 = \frac{2r(u, v_0) + u - v_1}{v - u} + \frac{1}{v - u} \int_{v_0}^{v_1} \frac{F}{\psi} dv + \frac{1}{v - u} \int_{v_1}^v \left(\frac{F}{\psi} - 1 \right) dv \quad (\text{A.4})$$

At $v = v_0$ we have :

$$2r(u, v_0) = 2r(u_0, v_0) - \frac{1}{2} \int_{u_0}^u F du.$$

If we take $u < u_0$, with $u_0 \ll -1$, we know that $1 - \varepsilon \leq F \leq 1$. Let also $v > v_0$ with v_0 large enough,

$$\begin{aligned} \frac{\varepsilon u + 2r(u_0, v_0) + u_0 - v_1}{v - u} &\leq \frac{2r(u, v_0) + u - v_1}{v - u} \leq \frac{2r(u_0, v_0) + u_0 - v_1}{v - u} \\ -\frac{\varepsilon|u|}{v + |u|} &\leq \frac{2r(u, v_0) + u - v_1}{v - u} \leq \varepsilon \\ -\varepsilon &\leq \frac{2r(u, v_0) + u - v_1}{v - u} \leq \varepsilon. \end{aligned}$$

The second term in (A.4) is also bounded by $\varepsilon > 0$ near i_0 because $F/\psi \leq 1$ and we have chosen $v_0 < v_1 < \infty$. So for $v > v_1$ and $u < u_0$, with $u_0 \ll -1$:

$$0 \leq \frac{1}{v - u} \int_{v_0}^{v_1} \frac{F}{\psi} dv \leq \frac{1}{v - u} \int_{v_0}^{v_1} dv$$

$$0 \leq \frac{1}{v-u} \int_{v_0}^{v_1} \frac{F}{\psi} dv \leq \varepsilon$$

For $v > v_1$ and $u < u_0$, with v_1 sufficiently large and $u_0 \ll -1$ we have, using 4.3.1 $|F/\psi - 1| \leq \varepsilon$ so the last term in (A.4) becomes:

$$\begin{aligned} \left| \frac{1}{v-u} \int_{v_1}^v \left(\frac{F}{\psi} - 1 \right) dv \right| &\leq \frac{1}{v-u} \int_{v_1}^v \varepsilon dv \\ &\leq \varepsilon \frac{v-v_1}{v-u} \leq \varepsilon. \end{aligned}$$

This concludes the proof.

Let us now turn to $R|u|$. It is clearly non negative and in Ω_{u_0} , we have $u = -s\tilde{r}$ with $s \in [0, 1]$. Whence

$$0 \leq R|u| = sR\tilde{r} \leq R\tilde{r} < 1 + \varepsilon.$$

Finally, since R is small in the neighbourhood of i^0 and m is positive and bounded, we have trivially that $1 - \varepsilon < 1 - 2m(u)R \leq 1$ in Ω_{u_0} provided $u \ll -1$ is large enough in absolute value. \square

A.2.2 Proof of proposition 4.3.2

We use 4.3.1. Recall that

$$\begin{aligned} \mathcal{E}_{\mathcal{H}_s}(\phi) &= \int_{\mathcal{H}_s} \left\{ u^2 \phi_u^2 + u^2 R^2 F(u, R) \phi_u \phi_R + |\nabla_{S^2} \phi|^2 \left[u^2 \frac{FR^2}{4s\psi} (2 + s(\psi - 1)) + (1 + uR) \right] \right. \\ &\quad \left. + R^2 F \left[\frac{2 + s(\psi - 1)}{4s\psi} ((2 + uR)^2 - 2m(u)u^2 R^3) - (1 + uR) \right] \phi_R^2 \right\} du \wedge d\omega|_{\mathcal{H}_s} \end{aligned}$$

1. We begin with the term in front of $|\nabla_{S^2} \phi|^2$. We use the fact that $s = -u/\tilde{r}$

$$\begin{aligned} \frac{u^2 R^2 F}{2s\psi} (2 + s(\psi - 1)) + (1 + uR) &= 1 + |u|R \left[-1 + \frac{\tilde{r}RF}{2\psi} + \frac{s(\psi - 1)\tilde{r}RF}{4\psi} \right] \\ -\frac{1}{2} - \varepsilon &\leq \left[-1 + \frac{\tilde{r}RF}{2\psi} + \frac{s(\psi - 1)\tilde{r}RF}{4\psi} \right] \leq -\frac{1}{2} + \varepsilon \\ \frac{1}{2} - \varepsilon &\leq 1 + |u|R \left[-1 + \frac{\tilde{r}RF}{2\psi} + \frac{s(\psi - 1)\tilde{r}RF}{4\psi} \right] \leq 1 \end{aligned} \tag{A.5}$$

2. Now we study the term in front of ϕ_R^2 that we denote by f_{RR} .

$$\begin{aligned} f_{RR} &= R^2 F \left[\frac{2 + s(\psi - 1)}{4s\psi} \left((2 + uR)^2 - 2m(u)u^2 R^3 \right) - (1 + uR) \right] \\ &= \frac{R}{|u|} \left[\left(\frac{\tilde{r}RF}{2\psi} + \frac{R^2 F(\psi - 1)}{4\psi} \right) \left((2 + uR)^2 - 2m(u)u^2 R^3 \right) - |u|RF(1 + uR) \right] \\ &\simeq \frac{R}{2|u|} (3(|u|R)^2 - 6|u|R + 4) \end{aligned}$$

Let $x = |u|R$ and $f(x) = 3x^2 - 6x + 4$. If $x \in [0, 1]$ then $1 \leq f(x) \leq 4$, hence,

$$\left(\frac{1}{2} - \varepsilon \right) \frac{R}{|u|} \leq f_{RR} \leq (2 + \varepsilon) \frac{R}{|u|} \quad (\text{A.6})$$

3. Finally we have to estimate the coefficient in front of $\phi_u \phi_R$. This term is exactly the same as for Schwarzschild and so we use similar arguments to [54]

$$\begin{aligned} |R^2 u^2 F(u, R) \phi_u \phi_R| &\leq (R|u|)^{3/2} |u \phi_u| \left| \sqrt{\frac{R}{|u|}} \phi_R \right| \\ &\leq \frac{1}{2} \left(\lambda^2 u^2 \phi_u^2 + \frac{1}{\lambda^2} \frac{R}{|u|} \phi_R^2 \right), \quad \lambda \in \mathbb{R}^* \end{aligned} \quad (\text{A.7})$$

And for an appropriate choice of $\lambda \in \mathbb{R}$ ($\lambda = 3/2$ for instance):

$$\frac{\lambda^2}{2} < 1, \quad \frac{1}{2\lambda^2} < \frac{1}{2} - \varepsilon. \quad (\text{A.8})$$

Using (A.5), (A.6), (A.7) and (A.8), we obtain :

$$\mathcal{E}_{\mathcal{H}_s}(\phi) \simeq \int_{\mathcal{H}_s} \left[u^2 |\partial_u \phi|^2 + \frac{R}{|u|} |\partial_R \phi|^2 + |\nabla_{S^2} \phi|^2 \right] du \wedge d\omega|_{\mathcal{H}_s} \quad \square$$

A.2.3 Proof of proposition 4.3.3

Let us recall the expression of the error term (4.16).

$$\text{Err}(\phi) = \left[(-u^2 R^3 m'(u) + 2mR^2(3 + uR)) \phi_R^2 - 2mR\phi (u^2 \phi_u - 2(1 + uR)\phi_R) \right] \frac{(\tilde{r}R)^2}{\varphi|u|}. \quad (\text{A.9})$$

Using 4.3.1,

$$\begin{aligned}
 |\text{Err}(\phi)| &\leq [|m'(u)|u^2R^3|\phi_R|^2 + 2mR^2|3 + uR|\phi_R|^2 + 2mu^2R|\phi||\phi_u| + 4mR|1 + uR||\phi||\phi_R|] \frac{|\tilde{r}R|^2}{\varphi|u|} \\
 &\lesssim \left[\frac{R}{|u|}|\phi_R|^2 + \frac{R^2}{|u|}|\phi_R|^2 + |\phi||\phi_u| + \frac{R}{|u|}|\phi||\phi_R| \right] \\
 &\lesssim \frac{R}{|u|}|\phi_R|^2 + u^2|\phi_u|^2 + \phi^2
 \end{aligned}$$

where we obtain the last line using the facts that $R < 1$ and $1 < |u|$ in Ω_{u_0} . \square

A.2.4 Proof of Theorem 4.4.2

First observe that the estimates on $\nabla_{S^2}^k \phi$ in the first part of the theorem are direct consequences of the fundamental estimates and the spherical symmetry of the spacetime. To prove the other estimates, we need to understand what new terms appear each time we apply ∂_u and ∂_R to the wave equation and how we can control them. We have :

$$\nabla^a (T^b T_{ab}(\Psi)) = (-u^2 R^3 m'(u) + 2mR^2(3 + uR)) \Psi_R^2 + \square_{\hat{g}} \Psi (u^2 \Psi_u - 2(1 + uR)\Psi_R) \tag{A.10}$$

where Ψ is any smooth function on Ω_{u_0} . In the following estimates, we shall take $\Psi = \partial_u^l \partial_R^k \phi$. When we integrate (A.10) on Ω_{u_0} , we shall split the 4-volume measure as $ds \wedge (\nu \lrcorner d\text{Vol}^4)$. Recall from the proof of the fundamental estimates that :

$$\nu \lrcorner d\text{Vol}^4|_{\mathcal{H}_s} \simeq \frac{1}{|u|} du \wedge d\omega$$

So we shall need to control on each \mathcal{H}_s the error term

$$\text{Err}(\Psi) := \nabla^a (T^b T_{ab}(\Psi)) \frac{1}{|u|}.$$

A first immediate estimate is:

$$|\text{Err}(\Psi)| \lesssim \frac{R}{|u|} |\Psi_R|^2 + |\square_{\hat{g}} \Psi| \left(|u| |\Psi_u| + \frac{1}{|u|} |\Psi_R| \right).$$

The first term is controlled by the energy density on \mathcal{H}_s as well as the square of $|u| |\Psi_u|$.

But this is not the case for the square of $\frac{1}{|u|}|\Psi_R|$. Indeed, we do not have in Ω_{u_0} that $1/|u|^2 \leq R/|u|$ since, R is allowed to tend to zero independently of the behaviour of u . However, using the fact that $s = |u|/\tilde{r}$, we have :

$$\begin{aligned} \frac{1}{|u|}|\Psi_R| &= \frac{1}{\sqrt{s}}\sqrt{\frac{R}{R\tilde{r}|u|}}|\Psi_R| \\ &\simeq \frac{1}{\sqrt{s}}\sqrt{\frac{R}{|u|}}|\Psi_R|. \end{aligned}$$

So we obtain :

$$|\text{Err}(\Psi)| \lesssim \frac{1}{\sqrt{s}} \left(|\square_{\hat{g}}\Psi|^2 + |u|^2|\Psi_u|^2 + \frac{R}{|u|}|\Psi_R|^2 \right)$$

Now, with the expression (4.10) of the energy on \mathcal{H}_s and using the fact that $s^{-1/2}$ is an integrable function near the origin, we can use Grönwall's lemma to prove the theorem, provided we know how to deal with the term $|\square_{\hat{g}}\Psi|^2$.

Control of the wave operator

Now let us consider $\Psi = \partial_R^k \partial_u^l \phi$ where ϕ is a smooth solution of (4.3) with compactly supported data on \mathcal{H}_1 .

The control of $\square_{\hat{g}}\Psi$ in this case is the main difference between Vaidya and Schwarzschild spacetimes. Indeed for Schwarzschild, we have a commutation relation between ∂_u and the d'Alembertian operator, due to the fact that the metric was static :

$$[\partial_u, \square_{\hat{g}}]_{\text{Sch}} = 0$$

Now we have a d'Alembertian that depends both on u and R (see (3.9)). And so :

$$[\partial_R, \square_{\hat{g}}] = -2R(1 - 3mR)\partial_R^2 - 2(1 - 6mR)\partial_R \tag{A.11}$$

$$[\partial_u, \square_{\hat{g}}] = 2m'(u)R^3\partial_R^2 + 6m'(u)R^2\partial_R \tag{A.12}$$

The method to control $\square_{\hat{g}}\partial_R^k \partial_u^l \phi$ is the following :

1. We begin by computing $\square_{\dot{g}}\phi_u$ and $\square_{\dot{g}}\phi_R$:

$$\square_{\dot{g}}\phi_u = -2m'(u)R^3\partial_R\phi_R - 6m'(u)R^2\phi_R - 2m'(u)R\phi - 2m(u)R\phi_u \quad (\text{A.13})$$

$$\square_{\dot{g}}\phi_R = 2(1 - 7mR)\phi_R + 2R(1 - 3mR)\partial_R\phi_R - 2m\phi \quad (\text{A.14})$$

2. For the sake of clarity and simplicity, we decompose our study into 3 parts : the case where $\Psi = \partial_R^k\phi$, the case where $\Psi = \partial_u^l\phi$ and the mixed case where $\Psi = \partial_u^l\partial_R^k\phi$. In each case, $\square_{\dot{g}}\Psi$ will involve some derivatives of the form $\partial_R^m\partial_u^n\phi$ with $m \leq k$ and $n \leq l$ multiplied by bounded smooth coefficients. These terms when squared can be estimated by the L^2 norm of $\partial_R^m\partial_u^n\phi$ which is in turn controlled by the energy of $\partial_R^m\partial_u^n\phi$. So all we need to consider are the terms involving derivatives of ϕ of order $k+l$ or above. In this manner, we shall prove that it is possible to control $|\square_{\dot{g}}\partial_u^l\partial_R^k\phi|^2$ as we have done for the fundamental estimates.

Derivatives with respect to R We start with $\Psi = \partial_R^2\phi$. We obtain $\square\Psi$ by applying ∂_R to (A.14). The only terms of degree higher than 2 are twice the same term

$$2R(1 - 3mR)\partial_R^3\phi$$

coming once from the right-hand side of (A.14) and once from the commutation of ∂_R with $\square_{\dot{g}}$ (see (A.11)) in the left-hand side. By induction, for $\Psi = \partial_R^k\phi$, the only terms of order larger than k will be

$$2kR(1 - 3mR)\partial_R^{k+1}\phi.$$

We can estimate the square of this term as follows :

$$|2kR(1 - 3mR)\partial_R^{k+1}\phi|^2 \lesssim R^2 |\partial_R^{k+1}\phi|^2 \lesssim \frac{R^2|u|}{|u|} |\partial_R^{k+1}\phi|^2 \simeq \frac{R}{|u|} |\partial_R^{k+1}\phi|^2.$$

We can state as an intermediary result that for the derivatives with respect to R we have, as in the Schwarzschild case :

$$\int_{\mathcal{H}_s} |\text{Err}(\partial_R^k\phi)| du \wedge d^2\omega \lesssim \sum_{p=0}^k \mathcal{E}_{\mathcal{H}_s}(\partial_R^p\phi)$$

Derivatives with respect to u Now we set $\Psi = \partial_u^l \phi$. To obtain the wave operator of Ψ we apply ∂_u^{l-1} to (A.13). Similarly to what happened for the ∂_R case, the only term involving derivatives of degree higher than l is

$$-2lm'(u)R^3\partial_R^2\partial_u^{l-1}\phi.$$

We estimate it as follows :

$$|2lm'(u)R^3\partial_R^2\partial_u^{l-1}\phi|^2 \lesssim R^6 |\partial_u^{l-1}\partial_R^2\phi|^2 \lesssim \frac{R}{|u|} |\partial_u^{l-1}\partial_R^2\phi|^2$$

and this is controlled by the energy density on \mathcal{H}_s of $\partial_u^{l-1}\partial_R^2\phi$.

So we can control this term with the energy of a derivative of order l of ϕ , however we notice that we need to introduce a control on derivatives in R in order to control derivatives in u .

Mixed case Now we set $\Psi = \partial_u^l \partial_R^k \phi$. We apply $\partial_u^k \partial_R^l$ to (4.3) and the only terms involving derivatives of order higher than $k+l$ are similar to the terms we found in the previous two cases, namely

$$2kR(1-3mR)\partial_u^l \partial_R^{k+1}\phi - 2lm'(u)R^3\partial_u^{l-1}\partial_R^{k+2}\phi.$$

They are estimated just as before :

$$\begin{aligned} |2kR(1-3mR)\partial_u^l \partial_R^{k+1}\phi - 2lm'(u)R^3\partial_u^{l-1}\partial_R^{k+2}\phi|^2 &\lesssim R^2 |\partial_u^l \partial_R^{k+1}\phi|^2 + R^6 |\partial_u^{l-1}\partial_R^{k+2}\phi|^2 \\ &\lesssim \frac{R}{|u|} \left(|\partial_u^l \partial_R^{k+1}\phi|^2 + |\partial_u^{l-1}\partial_R^{k+2}\phi|^2 \right) \end{aligned}$$

which is in turn controlled by the sum of the energy densities on \mathcal{H}_s of $\partial_u^l \partial_R^k \phi$ and $\partial_u^{l-1} \partial_R^{k+1} \phi$. Hence we have the following estimate :

$$\int_{\mathcal{H}_s} \text{Err}(\partial_u^l \partial_R^k \phi) du d\omega|_{\mathcal{H}_s} \lesssim \frac{1}{\sqrt{s}} \sum_{p+q \leq k+l} \mathcal{E}_{\mathcal{H}_s}(\partial_R^p \partial_u^q \phi).$$

The theorem then follows from a Grönwall estimate. \square

PEELING IN THE PAST

B.1 Proof of Proposition 5.2.2

We obtain the equivalence terms by terms by using the basic estimates in Proposition 5.2:

1. we begin by the term in front of ϕ_v^2 that is ψv^2 . Since ψ is a bounded positive function, we have :

$$\psi v^2 \simeq v^2. \quad (\text{B.1})$$

2. Concerning the term in front of $|\nabla_{S^2}\phi|^2$:

$$\begin{aligned} \frac{R^2 v^2 F}{\psi} + 2(1 - vR) + \frac{R^2 v^2 F}{2s} \left(1 - \frac{s}{2} \left(1 - \frac{1}{\psi}\right)\right) &\simeq R^2 v^2 + 2(1 - vR) + \frac{R^2 v^2}{2s} \\ &\simeq R^2 v^2 + 2(1 - vR) + \frac{R^2 \tilde{r}}{2} \\ &\simeq 1. \end{aligned}$$

3. The term in front of ϕ_R^2 , referred as f_{RR} :

$$f_{RR} = R^2 F \left[\frac{R^2 v^2 F}{\psi} + \frac{3}{2}(1 - vR) + \frac{1}{s} \left(1 - \frac{s}{2} \left(1 - \frac{1}{\psi}\right)\right) \left(2(1 - vR) + \frac{v^2 R^2 F}{2}\right) \right],$$

is equivalent to :

$$\begin{aligned} f_{RR} &\simeq R^2 \left[R^2 v^2 + \frac{3}{2}(1 - vR) + \frac{1}{s} \left(2(1 - vR) + \frac{v^2 R^2}{2}\right) \right] \\ &\simeq R^2 \left(R^2 v^2 + \frac{3}{2}(1 - vR) + \frac{2(1 - vR)\tilde{r}}{v} + \frac{v R^2 \tilde{r}}{2} \right) \\ &\simeq R^2 \left(R^2 v^2 - vR + \frac{3}{2} + \frac{2(1 - vR)\tilde{r}}{v} \right) \end{aligned}$$

Let $f(x) = x^2 - x + \frac{3}{2}$, then f is strictly positive on \mathbb{R} . Now let $x = vR$ then

$x \in [0, 1 + \varepsilon[$ with $\varepsilon > 0$, and $f(0) = 3/2, f(1) = 3/2$. Thus :

$$\frac{3}{2}R^2 \lesssim f_{RR} \lesssim \frac{3}{2}R^2 + 2\frac{R}{v}$$

Since we have $R^2 = R^2v/v \simeq R/v$ this leads to :

$$f_{RR} \simeq \frac{R}{v}.$$

4. We finish our estimates by proving that the term $R^2v^2F\phi_R\phi_v$ does not change the sign of the energy density and that it is controlled by $v^2\phi_v^2 + R/v\phi_R^2$. This is done by bounding the absolute value of it as we do in the past:

$$\begin{aligned} f_{Rv} &= -\frac{R^2v^2F}{\psi} \\ |f_{Rv}\phi_R\phi_v| &\simeq R^2v^2|\phi_R||\phi_v| \\ &\leq (Rv)^{3/2}|v\phi_v| \left| \sqrt{\frac{R}{v}}\phi_R \right| \\ &\leq \frac{1}{2} \left(\lambda^2v^2\phi_v^2 + \frac{1}{\lambda^2}\frac{R}{v}\phi_R^2 \right) \quad \lambda \in \mathbb{R}. \end{aligned}$$

Since the minimal value of f_{RR} is $3R/2v$ we want λ to be :

$$\lambda^2 \leq 2, \quad \frac{1}{2\lambda^2} \leq \frac{3}{2},$$

so let $\lambda \in [1/3, 2]$ and this ensures that :

$$|f_{Rv}\phi_R\phi_v| \lesssim \left(v^2\phi_v^2 + \frac{R}{v}\phi_R^2 \right). \quad \square$$

B.2 Proof of Proposition 5.4.1

In order to bounded the error terms by the integrand of the energy flux through \mathcal{H}_s , we begin by isolating in K_{vv}, K_{vR} and K_{RR} the leading term. This gives :

$$|K_{vv}| = v^2 \left| \psi \frac{\partial \psi}{\partial R} \right| \simeq v^2,$$

$$\begin{aligned}
 |K_{vR}| &= \left| (2v\psi(\psi - 1) + (2R^3v^2m(v, R) - Rv^2 + 2Rv - 2) \frac{\partial\psi}{\partial R}) \right|, \\
 &\lesssim Rv^2 \left| \frac{\partial\psi}{\partial R} \right| \\
 &\lesssim v, \\
 |K_{RR}| &= \left| -R^3v^2\psi \frac{\partial m}{\partial v} - 2R\psi + 2(R^3v - 3R^2)m(v, R) + 2(R^4v - R^3) \frac{\partial m}{\partial R} + 2R \right| \\
 &\lesssim R^3v^2\psi \left| \frac{\partial m}{\partial v} \right| + 2R\psi + 2R \\
 &\lesssim R.
 \end{aligned}$$

Note that we used the fact that $m(v, R)$ and ψ and their derivatives are bounded functions. Then we insert this into the expression of this error term $\text{Err}(\phi)$ that we recalled here :

$$\begin{aligned}
 \text{Err}(\phi) &= \frac{(R\tilde{r})^2 F}{v\psi} \left[\left(K_{RR} + \frac{1}{2} \left(\frac{R^2 F}{\psi^2} K_{vv} + 2K_{vR} \right) R^2 F \right) \phi_R^2 \right. \\
 &\quad + \left(K_{vR} - \psi \left(\frac{R^2 F}{\psi^2} K_{vv} + 2K_{vR} \right) \right) \phi_R \phi_v \\
 &\quad \left. + K_{vv} \phi_v^2 + \left(2K_{vR} + \frac{R^2 F}{2\psi^2} K_{vv} \right) |\nabla_{S^2} \phi|^2 - 2m(v, R)v^2\psi R\phi\phi_v - 4m(v, R)(1 - vR)R\phi\phi_R \right].
 \end{aligned}$$

Hence for $v > v_0$, $v_0 \gg 1$ and $R \rightarrow 0$:

$$|\text{Err}(\phi)| \lesssim \frac{1}{v} [R\phi_R^2 + v|\phi_v\phi_R| + v^2\phi_v^2 + v|\nabla_{S^2}\phi|^2 + \phi^2]$$

Now the only tricky term is the crossed term $|\phi_R\phi_v|$ and needs to be control like :

$$\begin{aligned}
 |\phi_v\phi_R| &= \left| v\phi_v \frac{1}{v} \phi_R \right| \\
 &= \left| v\phi_v \frac{1}{\sqrt{s}} \sqrt{\frac{R}{R\tilde{r}v}} \phi_R \right| \\
 &\simeq \left| v\phi_v \frac{1}{\sqrt{s}} \sqrt{\frac{R}{v}} \phi_R \right| \\
 &\leq v^2\phi_v^2 + \frac{1}{\sqrt{s}} \frac{R}{v} \phi_R^2.
 \end{aligned}$$

Since $1/\sqrt{s}$ is an integrable function on our domain, this will not be problematic when we will use the Grönwall lemma after. Finally we have :

$$|\text{Err}(\phi)| \leq \frac{Cte}{\sqrt{s}} \left[v^2 \phi_v^2 + \frac{R}{v} \phi_R^2 + |\nabla_{S^2} \phi|^2 + \phi^2 \right] \quad \square$$

B.3 Proof of Lemma 5.4.1

Let $v_0 > 0$ and let f be a smooth compactly supported function, $f \in C_0^\infty(\mathbb{R})$, then :

$$\int_{v_0}^{\infty} (f(v))^2 dv = [(v - v_0)(f(v))^2]_{v_0}^{\infty} - \int_{v_0}^{\infty} 2v f'(v) f(v) dv,$$

and the first term on the right hand side is zero because f has a compact support. Thus :

$$\begin{aligned} \int_{v_0}^{\infty} (f(v))^2 dv &\leq \frac{1}{2} \int_{v_0}^{\infty} (f^2 + 4v^2 (f'(v))^2) dv, \\ \int_{v_0}^{\infty} (f(v))^2 dv &\leq 2 \int_{v_0}^{\infty} v^2 (f'(v))^2 dv. \end{aligned}$$

Let $f = \phi$, then we give an equivalent to $d\phi/dv$ in order to prove the second part of the lemma:

$$\frac{d\phi}{dv} = \phi_v + \phi_R \frac{dR}{dv},$$

and on \mathcal{H}_s :

$$\begin{aligned} \frac{dR}{dv} &= -\frac{R^2 F}{s} \left(1 - \frac{s}{2} \left(1 - \frac{1}{\psi} \right) \right) \\ &= \frac{R^2 \tilde{r}}{v} \\ &= \frac{R}{v}. \end{aligned}$$

Thus :

$$\begin{aligned} \int_{v_0}^{\infty} (f(v))^2 dv &\leq 2 \int_{v_0}^{\infty} \left(v^2 \phi_v^2 + v^2 \left(\frac{dR}{dv} \right)^2 \phi_R^2 \right) dv, \\ &\leq C \int_{v_0}^{\infty} \left(v^2 \phi_v^2 + v^2 \frac{R^2}{v^2} \phi_R^2 \right) dv \\ &\leq C \int_{v_0}^{\infty} \left(v^2 \phi_v^2 + \frac{R}{v} \phi_R^2 \right) dv. \end{aligned}$$

Finally, this leads to :

$$\int_{\mathcal{H}_s} \phi^2 dv \wedge d\omega^2 \lesssim \int_{\mathcal{H}_s} \left(v^2 \phi_v^2 + \frac{R}{v} \phi_R^2 \right) dv \wedge d\omega \lesssim \mathcal{E}_{\mathcal{H}_s}(\phi). \quad \square$$

B.4 Proof of Theorem 5.5.2

First observe that the estimates on $\nabla_{S^2}^k \phi$ in the first part of the theorem are direct consequences of the fundamental estimates and the spherical symmetry of the spacetime. To prove second part of the theorem, we will focus on the new terms that appears when we apply ∂_R and ∂_v on the wave equation and in particular the new terms that appear in the error term. Let Ψ be any smooth function on Ω_{v_0} , then :

$$\begin{aligned} \nabla^{(a} (T^b) T_{ab}(\Psi)) &= \left[\left(K_{RR} + \frac{1}{2} \left(\frac{R^2 F}{\psi^2} K_{vv} + 2K_{vR} \right) R^2 F \right) \Psi_R^2 + \square_{\dot{g}} \Psi (v^2 \psi \Psi_v + 2(1 - vR) \Psi_R) \right. \\ &\quad + \left(K_{vR} - \psi \left(\frac{R^2 F}{\psi^2} K_{vv} + 2K_{vR} \right) \right) \Psi_R \Psi_v + K_{vv} \Psi_v^2 \\ &\quad \left. + \left(2K_{vR} + K_{vv} \frac{R^2 F}{2\psi^2} \right) |\nabla_{S^2} \phi|^2 \right]. \end{aligned}$$

where we recalled that :

$$K_{vv} = v^2 \psi \frac{\partial \psi}{\partial R}, \quad (\text{B.2})$$

$$K_{vR} = (2v\psi(\psi - 1) + (2R^3 v^2 m(v, R) - Rv^2 + 2Rv - 2) \frac{\partial \psi}{\partial R}), \quad (\text{B.3})$$

$$K_{RR} = -R^3 v^2 \psi \frac{\partial m}{\partial v} - 2R\psi + 2(R^3 v - 3R^2) m(v, R) + 2(R^4 v - R^3) \frac{\partial m}{\partial R} + 2R. \quad (\text{B.4})$$

When we integrate $\nabla^{(a} (T^b) T_{ab}(\Psi))$ on Ω_{v_0} , we shall split the 4-volume measure as $ds \wedge (\nu \lrcorner d\text{Vol}^4)$. From the fundamental estimates we had :

$$\nu \lrcorner d\text{Vol}^4|_{\mathcal{H}_s} \simeq \frac{1}{v} dv \wedge d\omega,$$

and this leads to the Error term associated to Ψ :

$$\text{Err}(\Psi) := \nabla^{(a} (T^b) T_{ab}(\Psi)) \frac{1}{v}.$$

From now, we take $\Psi = \partial_v^l \partial_R^k \phi$. Knowing that :

$$\begin{aligned} |K_{vv}| &\lesssim v^2, \\ |K_{vR}| &\lesssim v, \\ |K_{RR}| &\lesssim R, \end{aligned}$$

We infer a first estimate :

$$|\text{Err}(\Psi)| \lesssim \frac{R}{v} \Psi_R^2 + v^2 \Psi_v^2 + |\nabla_{S^2} \Psi|^2 + \Psi_R \Psi_v + |\square_{\hat{g}}| \left| v \Psi_v + \frac{1}{v} \Psi_R \right|.$$

The control of the term $\Psi_R \Psi_v$ is similar as what we done for $\phi_v \phi_R$ in the proof of the Proposition 5.4.1 :

$$\begin{aligned} |\Psi_v \Psi_R| &= \left| v \Psi_v \frac{1}{v} \Psi_R \right| \\ &= \left| v \Psi_v \frac{1}{\sqrt{s}} \sqrt{\frac{R}{R\tilde{r}v}} \Psi_R \right| \\ &\leq v^2 \Psi_v^2 + \frac{1}{\sqrt{s}} \frac{R}{v} \Psi_R^2. \end{aligned}$$

The term Ψ_v/v is dealt in the same way, thus :

$$|\text{Err}(\Psi)| \lesssim \frac{1}{\sqrt{s}} \left(v^2 \Psi_v^2 + \frac{R}{v} \Psi_R^2 + |\nabla_{S^2} \Psi|^2 + |\square_{\hat{g}}|^2 \right).$$

Then, knowing the energy of Ψ on \mathcal{H}_s :

$$\mathcal{E}_{\mathcal{H}_s}(\Psi) \simeq \int_{\mathcal{H}_s} \left(v^2 \Psi_v^2 + \frac{R}{v} \Psi_R^2 + |\nabla_{S^2} \Psi|^2 \right) dv \wedge d\omega^2,$$

and the fact that $1/\sqrt{s}$ is an integrable function on Ω_{v_0} , the error term can be controlled with the Grönwall's lemma, provided we have sufficient control on $|\square_{\hat{g}} \Psi|^2$.

B.4.1 Control of the wave operator

The control of the wave operator for $\Psi = \partial_v^l \partial_R^k \phi$, with ϕ a smooth solution of (6.3) with compactly supported data on \mathcal{H}_1 is slightly different in the past than in the future, because

there is always "mixed terms". This comes from :

$$\begin{aligned} [\partial_v, \square_{\hat{g}}] &= 2 \frac{\partial \psi}{\partial v} \partial_v \partial_R + 2R^3 \frac{\partial m}{\partial v} \partial_R^2 \\ &+ \left[R^2 \left(6 \frac{\partial m}{\partial v} - \frac{F}{\psi^2} \frac{\partial \psi}{\partial v} + \frac{F}{\psi} \frac{\partial^2 \psi}{\partial v \partial R} \right) + R^3 \left(2 \frac{\partial^2 m}{\partial v \partial R} - \frac{2}{\psi} \frac{\partial m}{\partial v} \right) \right] \partial_R, \end{aligned}$$

and

$$\begin{aligned} [\partial_R, \square_{\hat{g}}] &= 2 \frac{\partial \psi}{\partial R} \partial_R \partial_v - \left(2R - 6mR - 2 \frac{\partial m}{\partial R} R^3 \right) \partial_R^2 \\ &+ \left[-2 + R \left(12m + \frac{2}{\psi} \frac{\partial \psi}{\partial R} \right) + R^2 \left(12 \frac{\partial m}{\partial R} - \frac{6m}{\psi} \frac{\partial \psi}{\partial R} + \frac{1}{\psi^2} \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial R} \right)^2 \right) \right. \\ &\left. + R^3 \left(2 \frac{\partial^2 m}{\partial R^2} - \frac{2}{\psi} \frac{\partial m}{\partial R} \frac{\partial \psi}{\partial R} - \frac{2m}{\psi} \frac{\partial^2 \psi}{\partial R^2} + \frac{2m}{\psi^2} \left(\frac{\partial \psi}{\partial R} \right)^2 \right) \right] \partial_R. \end{aligned}$$

The method is the following :

1. We begin by compute $\square_{\hat{g}} \phi_v$ and $\square_{\hat{g}} \phi_R$, knowing that $\square_{\hat{g}} \phi = -2m(v, R)R\phi$:

$$\begin{aligned} \square_{\hat{g}} \phi_v &= \left[R^2 \left(6 \frac{\partial m}{\partial v} - \frac{F}{\psi^2} \frac{\partial \psi}{\partial v} + \frac{F}{\psi} \frac{\partial^2 \psi}{\partial v \partial R} \right) + R^3 \left(2 \frac{\partial^2 m}{\partial v \partial R} - \frac{2}{\psi} \frac{\partial m}{\partial v} \right) \right] \partial_R \phi \\ &+ 2 \frac{\partial \psi}{\partial v} \partial_v \partial_R \phi + 2R^3 \frac{\partial m}{\partial v} \partial_R \phi - 2 \frac{\partial m}{\partial v} R \phi - 2m(v, R)R \phi_v, \end{aligned}$$

and

$$\begin{aligned} \square_{\hat{g}} \phi_R &= \left[-2 + R \left(12m + \frac{2}{\psi} \frac{\partial \psi}{\partial R} \right) + R^2 \left(12 \frac{\partial m}{\partial R} - \frac{6m}{\psi} \frac{\partial \psi}{\partial R} + \frac{1}{\psi^2} \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial R} \right)^2 \right) \right. \\ &+ R^3 \left(2 \frac{\partial^2 m}{\partial R^2} - \frac{2}{\psi} \frac{\partial m}{\partial R} \frac{\partial \psi}{\partial R} - \frac{2m}{\psi} \frac{\partial^2 \psi}{\partial R^2} + \frac{2m}{\psi^2} \left(\frac{\partial \psi}{\partial R} \right)^2 \right) \left. \right] \phi_R \\ &+ 2 \frac{\partial \psi}{\partial R} \partial_R \partial_v \phi - \left(2R - 6mR - 2 \frac{\partial m}{\partial R} R^3 \right) \partial_R \phi - 2 \frac{\partial m}{\partial R} R \phi - 2m(v, R)\phi - 2m(v, R)R \phi_R. \end{aligned}$$

2. For $\Psi = \partial_v^l \partial_R^k \phi$, the terms of type $\partial_v^m \partial_R^n \phi$ with $m \leq l, n \leq k$ that appears in $\square_{\hat{g}} \Psi$ are controlled, when squared, using the L^2 norms of $\partial_v^m \partial_R^n \phi$, hence are controlled by the energy of $\partial_v^m \partial_R^n \phi$ on \mathcal{H}_s . This is a direct consequence of the lemma 4.3.1:

$$\int_{\mathcal{H}_s} |\partial_v^l \partial_R^k \phi|^2 dv \wedge d\omega^2 \lesssim \int_{\mathcal{H}_s} \left(v^2 |\partial_v^{l+1} \partial_R^k \phi|^2 + \frac{R}{v} |\partial_v^l \partial_R^{k+1} \phi|^2 \right) dv \wedge d\omega^2 \lesssim \mathcal{E}_{\mathcal{H}_s}(\partial_v^l \partial_R^k \phi).$$

Then :

$$\int_{\mathcal{H}_s} |\partial_v^l \partial_R^k \phi| dv \wedge d\omega^2 \lesssim \mathcal{E}_{\mathcal{H}_s}(\partial_v^l \partial_r^k \phi).$$

3. The control of $\Psi = \partial_R^k \phi$ is done by applying successive derivatives with respect to R in $\square_{\hat{g}} \partial_R^k \phi$. The only terms with degree equal or above to $k + 1$ in $\square_{\hat{g}} \partial_R \Psi$ are :

$$2 \frac{\partial \psi}{\partial R} \partial_v \Psi - \left(2R - 6mR - 2 \frac{\partial m}{\partial R} R^3 \right) \partial_R \Psi,$$

and, once squared, this is controlled in the error term by :

$$v^2 |\partial_v \partial_R \Psi|^2 + \frac{R}{v} |\partial_R \partial_R \Psi|^2.$$

So the error term of $\partial_R^{k+1} \phi$ is bounded as :

$$\int_{\mathcal{H}_s} |\text{Err}(\partial_R^{k+1} \phi)| dv \wedge d\omega^2 \lesssim \sum_{i=0}^k \mathcal{E}_{\mathcal{H}_s}(\partial_R^i \phi).$$

4. Now we consider $\Psi = \partial_v^l \phi$ and we deal with the control of $\partial_v \Psi$ by commuting ∂_v into $\square_{\hat{g}} \Psi$. The only terms of degree higher or equal to $l + 1$ are :

$$\begin{aligned} & \left[R^2 \left(6 \frac{\partial m}{\partial v} - \frac{F}{\psi^2} \frac{\partial \psi}{\partial v} + \frac{F}{\psi} \frac{\partial^2 \psi}{\partial v \partial R} \right) + R^3 \left(2 \frac{\partial^2 m}{\partial v \partial R} - \frac{2}{\psi} \frac{\partial m}{\partial v} \right) \right] \partial_R \Psi + 2 \frac{\partial \psi}{\partial v} \partial_v \partial_R \Psi \\ & + 2R^3 \frac{\partial m}{\partial v} \partial_R \partial_R \Psi - 2m(v, R) R \partial_v \Psi, \end{aligned}$$

This is bounded in the error term by :

$$C^{te} \left(\frac{R}{v} |\partial_R \Psi|^2 + v^2 |\partial_v \partial_R \Psi|^2 + \frac{R}{v} |\partial_R \partial_R \Psi|^2 + v^2 |\partial_v \Psi|^2 \right).$$

Thus :

$$\int_{\mathcal{H}_s} |\text{Err}(\partial_v^{l+1} \phi)| dv \wedge d\omega^2 \lesssim \mathcal{E}_{\mathcal{H}_s}(\partial_v^l \phi) + \mathcal{E}_{\mathcal{H}_s}(\partial_R \partial_v^l \phi)$$

5. Before to conclude the proof, we need to focus on the mixed case, where we both apply derivatives with respect to R and v . This follows from the 2 previous points. Let $\Psi = \partial_v^l \partial_R \phi$. Let us firstly control $\square_{\hat{g}} \partial_R \Psi$. Once again, the only terms of higher

degree than $k + l + 1$ are :

$$2 \frac{\partial \psi}{\partial R} \partial_v \partial_R \Psi - \left(2R - 6mR - \frac{\partial m}{\partial R} R^3 \right) \partial_R^2 \Psi,$$

and they are controlled in the error term in the following way :

$$C^{te} \left(v^2 |\partial_v \partial_R \Psi|^2 + \frac{R}{v} |\partial_R \partial_R \Psi|^2 \right),$$

This ensures that :

$$|\text{Err}(\partial_R \Psi)| \lesssim v^2 |\partial_v \partial_R \Psi|^2 + \frac{R}{v} |\partial_R \partial_R \Psi|^2$$

Thus :

$$\int_{\mathcal{H}_s} |\text{Err}(\partial_R \Psi)| \lesssim \mathcal{E}_{\mathcal{H}_s}(\partial_R \Psi).$$

Let us now consider the action of ∂_v on Ψ . the only terms with degree bigger than $k + l + 1$ in $\square_{\hat{g}} \partial_v \Psi$ are :

$$\begin{aligned} & \left[R^2 \left(6 \frac{\partial m}{\partial v} - \frac{F}{\psi^2} \frac{\partial \psi}{\partial v} + \frac{F}{\psi} \frac{\partial^2 \psi}{\partial v \partial R} \right) + R^3 \left(2 \frac{\partial^2 m}{\partial v \partial R} - \frac{2}{\psi} \frac{\partial m}{\partial v} \right) \right] \partial_R \Psi \\ & + 2 \frac{\partial \psi}{\partial v} \partial_v \partial_R \Psi + 2R^3 \frac{\partial m}{\partial v} \partial_R^{k+2} \partial_v^{l-1} \phi - 2m(v, R) R \partial_v \Psi, \end{aligned}$$

and this is bounded, when squared in the error term by :

$$\frac{R}{v} |\partial_R \Psi|^2 + v^2 |\partial_v \partial_R \Psi|^2 + \frac{R}{v} |\partial_R \partial_R \Psi|^2 + |\partial_v \Psi|^2,$$

and it implies that the error term of $\partial_v \Psi$ is controlled by the energy of $\partial_v \Psi$, $\partial_R \psi$ and all the derivative of the field with degree lower than $k + l$. This is sum up in the following expression :

$$\int_{\mathcal{H}_s} |\text{Err}(\partial_v \Psi)| dv \wedge d\omega^2 \lesssim \mathcal{E}_{\mathcal{H}_s}(\partial_v \Psi) + \mathcal{E}_{\mathcal{H}_s}(\partial_R \Psi).$$

6. Finally we prove the theorem by applying the Grönwall lemma. □

CONFORMAL SCATTERING

C.1 Decay in the Schwarzschild spacetime

The goal of this appendix is to recall results of M. Dafermos and I. Rodnianski in [16] and JP. Nicolas in [64] that ensure that the energy of the rescaled field ϕ is going to zero near i^\pm in the conformally compactified Schwarzschild spacetime. This is essential in order to obtain a conservation law as done in the Proposition 6.3.1. In [16], in Theorem 4.1, the authors give a decay estimate for the physical field. JP. Nicolas in [64], Proposition 3., gives a conformal version of the estimate of Dafermos and Rodnianski.

Theorem C.1.1 (Dafermos-Rodnianski). *On (\mathcal{M}, g) , let :*

1. \mathcal{S}_τ be a family spacelike surfaces that cross both the future horizon \mathcal{H}^+ and future null infinity \mathcal{I}^+ .
2. ∂_τ be the unit timelike observer, normal to \mathcal{S}_τ .
3. Ψ be the physical field solution to (6.2) such that $\Psi \in H_0^4(\mathcal{M}), \Psi' \in H_0^3(\mathcal{M})$, and suppose that

$$\lim_{r \rightarrow i_0} \Psi = 0.$$

4. \tilde{T}_{ab} be the stress-energy tensor associated to the wave equation.

5.

$$\mathcal{E}_{\partial_\tau, \mathcal{S}_\tau}(\Psi) = \int_{\mathcal{S}_\tau} \star T_{ab} \partial_\tau^b,$$

is the energy of the physical field Ψ through \mathcal{S}_τ seen by an observer ∂_τ .

There exist $K > 0$ such that $\mathcal{E}_{\partial_\tau, \mathcal{S}_\tau}(\Psi)$ decays as follow :

$$\mathcal{E}_{\partial_\tau, \mathcal{S}_\tau}(\Psi) \leq \frac{C}{\tau^2}$$

It is not obvious that this theorem gives us the correct information concerning the decay of the rescaled field ϕ near i^\pm on the compactified spacetime. This is due to the fact that the stress-energy tensor for the wave equation is not conformally invariant. In [64], the author sets \mathcal{S}_τ and ∂_τ in order that it fits with his geometrical framework. Hence he formulates the following proposition :

Proposition C.1.1 (Nicolas). *On $(\hat{\mathcal{M}}, \hat{g})$, let :*

1. \mathcal{S}_τ be a spacelike surface such that :

$$\mathcal{S}_\tau = \left\{ (t, r_\star, \omega) \in \mathbb{R} \times \mathbb{R} \times S_\omega^2; t = \tau + \sqrt{1 + r_\star^2} \right\} .$$

2. n is a timelike vector field that approaches ∂_t for r large enough.

3. T_{ab} be the stress-energy tensor for the rescaled field $\phi = \Omega^{-1}\Psi$ solution of (6.3) with initial data $(\phi|_{t=0}, \partial_t \phi|_{t=0}) \in C_0^\infty(\Sigma_0) \times C_0^\infty(\Sigma_0)$, where $\Omega = 1/r$.

4. $\mathcal{E}_{\partial_\tau, \mathcal{S}_\tau}(\phi)$ be the energy flux, associated with the timelike observer ∂_τ , of the rescaled solution ϕ across \mathcal{S}_τ .

There exists $K > 0$ such that $\mathcal{E}_{\partial_\tau, \mathcal{S}_\tau}(\phi)$ decays as follow :

$$\mathcal{E}_{\partial_\tau, \mathcal{S}_\tau}(\phi) \leq \frac{K}{\tau}$$

Using these two results, we define two hypersurfaces $\mathcal{S}_{\tau, \text{I}}, \mathcal{S}_{\tau, \text{II}}$ which corresponds to both the definition of Theorem C.1.1 and Proposition C.1.1. We need indeed to treat separately the two Schwarzschild regions I and II with respectively two different masses m_- and m_+ . so we consider independently the case of the past timelike infinity in region I and the case of the future timelike infinity in region II. We state the following proposition :

Proposition C.1.2. *Let ϕ be a solution of (6.3), let $\mathcal{S}_{\tau, \text{I}}, \mathcal{S}_{\tau, \text{II}}$ such that :*

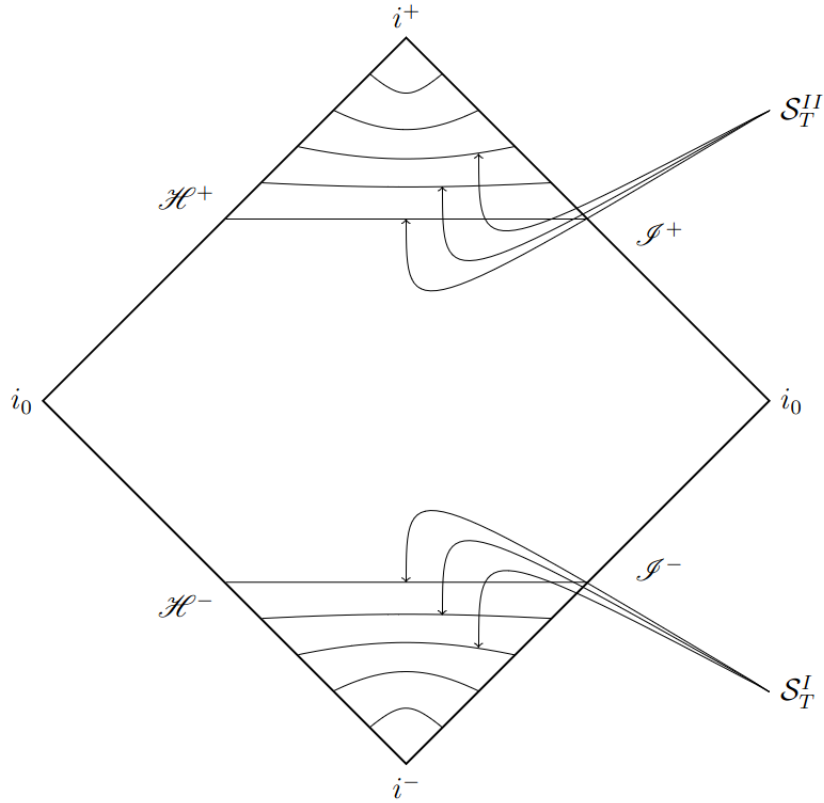
$$\begin{aligned} \mathcal{S}_{\tau, \text{I}} &= \left\{ (t, r_\star, \omega) \in \mathbb{R} \times \mathbb{R} \times S_\omega^2; t = -\tau - \sqrt{1 + r_\star^2} \right\} \cap \{t - r_\star < u_-\}, \\ \mathcal{S}_{\tau, \text{II}} &= \left\{ (t, r_\star, \omega) \in \mathbb{R} \times \mathbb{R} \times S_\omega^2; t = \tau + \sqrt{1 + r_\star^2} \right\} \cap \{t - r_\star > u_+\}, \end{aligned}$$

and consider $\mathcal{E}_{\partial_\tau, \mathcal{S}_\tau^{\text{I}}}(\phi)$ (resp. II) the energy flux associated with the timelike observer ∂_τ , of the rescaled solution ϕ across $\mathcal{S}_\tau^{\text{I}}$ (resp. II), then there exists $K_{\text{I}}, K_{\text{II}} > 0$ such that

these fluxes decay as :

$$\mathcal{E}_{\partial\tau, S^I_\tau}(\phi) \leq \frac{K_I}{\tau}, \quad \mathcal{E}_{\partial\tau, S^II_\tau}(\phi) \leq \frac{K_{II}}{\tau}.$$

Hence, when $\tau \rightarrow \infty$, the energy near the two timelike infinities is going to zero.



The proof is straightforward using the Proposition C.1.1. □

C.2 Error terms in the Vaidya spacetime

C.2.1 Definition of the spin coefficients

In this article we work with a Newman-Penrose tetrad $(\hat{l}, \hat{n}, \hat{m}, \hat{\bar{m}})$ that satisfies :

$$\hat{l}^a \hat{n}_a = -\hat{m}^a \hat{\bar{m}}_a = 1.$$

Because of this normalisation there are only 12 independent spin coefficients to compute :

$$\begin{aligned}
 \kappa &= m^a D l_a, & \varepsilon &= \frac{1}{2} (n^a D l_a + m^a D \bar{m}_a), & \pi &= -\bar{m}^a D n_a. \\
 \rho &= m^a \delta' l_a, & \alpha &= \frac{1}{2} (n^a \delta' l_a + m^a \delta' \bar{m}_a), & \lambda &= -\bar{m}^a \delta' n_a. \\
 \sigma &= m^a \delta l_a, & \beta &= \frac{1}{2} (n^a \delta l_a + m^a \delta \bar{m}_a), & \mu &= -\bar{m}^a \delta n_a. \\
 \tau &= m^a \Delta l_a, & \gamma &= \frac{1}{2} (n^a \Delta l_a + m^a \Delta \bar{m}_a), & \nu &= -\bar{m}^a \Delta n_a.
 \end{aligned}$$

The other spin coefficients are α', β', γ' and ε' and they are related to the previous coefficients with :

$$\alpha' = -\beta, \beta' = -\alpha, \gamma' = -\varepsilon, \varepsilon' = -\gamma.$$

C.2.2 Proof in the Vaidya spacetime

Here are the details to obtain (6.33). We recall that we use the following tetrad :

$$\hat{l}^a = \partial_u^a + \frac{R^2 F}{2} \partial_R^a, \quad (C.1)$$

$$\hat{n}^a = -\partial_R^a, \quad (C.2)$$

$$\hat{m}^a = \frac{1}{\sqrt{2}} \left(\partial_\theta^a + \frac{i}{\sin \theta} \partial_\varphi^a \right), \hat{\bar{m}}^a = \frac{1}{\sqrt{2}} \left(\partial_\theta^a - \frac{i}{\sin \theta} \partial_\varphi^a \right), \quad (C.3)$$

The only non-zero spin coefficients are :

$$\varepsilon = \frac{1}{2} R - \frac{3}{2} R^2 m(u), \alpha = -\beta = -\frac{\sqrt{2}}{4} \cot \theta.$$

The Killing form of $T = \partial_u$ reads :

$$\nabla^{(a} \partial_u^{b)} = \nabla^{(a} \hat{l}^{b)} + \nabla^{(a} \left(\Omega^2 \frac{F}{2} \hat{n}^{b)} \right). \quad (C.4)$$

We decompose the connection along null directions of the tetrad :

$$\nabla^a = \hat{l}^a \Delta + \hat{n}^a D - \hat{m}^a \delta' - \hat{\bar{m}}^a \delta. \quad (C.5)$$

The spin coefficient-equations for \hat{l} and \hat{n} are given by (see section 4.5 in [74]) :

$$\begin{aligned}
 D\hat{l}^a &= (\varepsilon + \bar{\varepsilon})\hat{l}^a - \bar{\kappa}\hat{m}^a - \kappa\hat{\bar{m}}^a, & D\hat{n}^a &= (\gamma' + \bar{\gamma}')\hat{n}^a - \tau'\hat{m}^a - \bar{\tau}'\hat{\bar{m}}^a. \\
 \delta\hat{l}^a &= (\beta + \bar{\alpha})\hat{l}^a - \bar{\rho}\hat{m}^a - \sigma\hat{\bar{m}}^a, & \delta\hat{n}^a &= (\alpha' + \bar{\beta}')\hat{n}^a - \rho'\hat{m}^a - \bar{\sigma}'\hat{\bar{m}}^a. \\
 \delta'\hat{l}^a &= (\alpha + \bar{\beta})\hat{l}^a - \bar{\sigma}\hat{m}^a - \rho\hat{\bar{m}}^a, & \delta'\hat{n}^a &= (\beta' + \bar{\alpha}')\hat{n}^a - \sigma'\hat{m}^a - \bar{\rho}'\hat{\bar{m}}^a. \\
 \Delta\hat{l}^a &= (\gamma + \bar{\gamma})\hat{l}^a - \bar{\tau}\hat{m}^a - \tau\hat{\bar{m}}^a, & \Delta\hat{n}^a &= (\varepsilon' + \bar{\varepsilon}')\hat{n}^a - \kappa'\hat{m}^a - \bar{\kappa}'\hat{\bar{m}}^a.
 \end{aligned}$$

The only non-zero prime coefficients are $\alpha' = -\beta, \beta' = -\alpha$ and $\gamma' = -\varepsilon$. Furthermore, these coefficients are real. Thus the spin-coefficient equations with non-zero right hand side are :

$$D\hat{l}^a = 2\varepsilon\hat{l}^a, \tag{C.6}$$

$$D\hat{n}^a = -2\varepsilon\hat{n}^a. \tag{C.7}$$

Equation (C.4) becomes :

$$\nabla^{(a}\partial_u^{b)} = 2\varepsilon\hat{n}^{(a}\hat{l}^{b)} - \Omega^2 F \varepsilon \hat{n}^{(a}\hat{n}^{b)} + \nabla^{(a} \left(\frac{\Omega^2 F}{2} \right) \hat{n}^{b)}, \tag{C.8}$$

with $\Omega = R$ and:

$$\begin{aligned}
 \nabla^a F &= \hat{n}^a D F + \hat{l}^a \Delta F = - (2m'(u)R - m(u)R^2 F) \hat{n}^a + 2m(u)\hat{l}^a, \\
 \nabla^a \Omega &= \hat{n}^a D \Omega + \hat{l}^a \Delta \Omega = \frac{R^2 F}{2} \hat{n}^a - \hat{l}^a.
 \end{aligned}$$

Finally we get :

$$\nabla^{(a}\partial_u^{b)} = -m'(u)R^3 \hat{n}^a \hat{n}^b. \tag{C.9}$$

□

USEFUL RESULTS

D.1 Conformal transformations

D.1.1 Connection

In this appendix, we present the usual conformal transformations of geometrical and physical quantities (such as the connection, scalar curvature, etc.) to obtain the expressions of rescaled quantities (denoted with a hat) in terms of physical quantities. We summarize the computations made in [94]. Let $(\hat{\mathcal{M}}, \hat{g})$ be an n -dimensional spacetime, and let Ω be a smooth strictly positive function such that $\hat{g} = \Omega^2 g$. Firstly, we provide the relation between the physical connection ∇ associated with the physical metric g and the conformal connection $\hat{\nabla}$ associated with \hat{g} . Considering ω_b as a 1-form, the transformation law for two different connections acting on ω_b is:

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b - C_{ab}^c \omega_c, \quad (\text{D.1})$$

where C_{ab}^c is a tensor of rank $(1, 2)$, as fully characterized by the following theorem:

Theorem D.1.1. *Let g_{ab} be a metric. Then, there exists a unique derivative operator ∇_a that ensures $\nabla_a g_{bc} = 0$.*

Proof : Let $\tilde{\nabla}$ be any derivative operator. Then, for any 1-form ω_b :

$$\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - C_{ab}^c \omega_c. \quad (\text{D.2})$$

And for any vector field t^b :

$$\nabla_a t^b = \tilde{\nabla}_a t^b + C_{ac}^b t^c. \quad (\text{D.3})$$

These two equations are generalized for any tensor of type (k, l) in the following manner:

$$\nabla_a T_{c_1 \dots c_l}^{b_1 \dots b_k} = \tilde{\nabla}_a T_{c_1 \dots c_l}^{b_1 \dots b_k} + \sum_i C_{ad}^{b_i} T_{c_1 \dots c_l}^{b_1 \dots d \dots b_k} - \sum_j C_{ac_j}^d T_{c_1 \dots d \dots c_l}^{b_1 \dots b_k}.$$

Hence, by taking g_{ab} , we obtain:

$$\nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C_{ab}^d g_{dc} - C_{ac}^d g_{bd}, \quad (\text{D.4})$$

And it has to be zero under the assumption of the theorem. By index permutation, we have:

$$C_{cab} + C_{bac} = \tilde{\nabla}_a g_{bc}, \quad (\text{D.5})$$

$$C_{cba} + C_{abc} = \tilde{\nabla}_b g_{ac}, \quad (\text{D.6})$$

$$C_{bca} + C_{acb} = \tilde{\nabla}_c g_{ab}. \quad (\text{D.7})$$

Thus:

$$C_{ab}^c = \frac{1}{2} g^{cd} (\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab}). \quad (\text{D.8})$$

This choice of C_{ab}^c is suitable with $\nabla_a g_{bc} = 0$ and ensures uniqueness. \square

Coming back to the conformal transformation (inverting ∇ and $\tilde{\nabla}$),

$$C_{ab}^c = \frac{1}{2} \hat{g}^{cd} [\nabla_a \hat{g}_{bd} + \nabla_b \hat{g}_{ad} - \nabla_d \hat{g}_{ab}], \quad (\text{D.9})$$

By using $\hat{g} = \Omega^2 g$, one obtains, according to Theorem D.1.1 :

$$\nabla_a \hat{g}_{bc} = \nabla_a (\Omega^2 g_{bc}) = 2\Omega g_{bc} \nabla_a \Omega. \quad (\text{D.10})$$

(D.9) turns into :

$$C_{ab}^c = \Omega^{-1} g^{cd} (g_{bd} \nabla_a \Omega + g_{ad} \nabla_b \Omega - g_{ab} \nabla_d \Omega).$$

Or in other terms :

$$C_{ab}^c = \delta_a^c \nabla_b \ln \Omega + \delta_b^c \nabla_a \ln \Omega - g_{ab} g^{cd} \nabla_d \ln \Omega = 2\delta_a^c (\nabla_b \ln \Omega) - g_{ab} g^{cd} \nabla_d \ln \Omega,$$

With δ_a^b being the Kronecker symbol, the transformation law of the connection is finally:

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b - \omega_c \Omega^{-1} g^{cd} (g_{bd} \nabla_a \Omega + g_{ad} \nabla_b \Omega - g_{ab} \nabla_d \Omega). \quad (\text{D.11})$$

D.1.2 Curvature tensor

The conformal Riemann tensor \hat{R}_{abcd} is associated with the conformal connection $\hat{\nabla}$ and acts on any 1-form ω as follows:

$$\hat{R}_{abc}^d \omega_d = \hat{\nabla}_a \hat{\nabla}_b \omega_c - \hat{\nabla}_b \hat{\nabla}_a \omega_c.$$

Knowing the transformation law of the connection (D.1) :

$$\begin{aligned} \hat{\nabla}_a \hat{\nabla}_b \omega_c &= \nabla_a \nabla_b \omega_c - C_{ab}^d \nabla_d \omega_c - C_{ac}^d \nabla_b \omega_d + \omega_d \nabla_a C_{ac}^d + C_{bc}^d \nabla_a \omega_d \\ &\quad - C_{ab}^e C_{ec}^d \omega_d - C_{ac}^e C_{be}^d \omega_d. \end{aligned}$$

Then, under antisymmetrization and considering that $C_{ab}^c = C_{ba}^c$, we obtain:

$$\begin{aligned} \hat{\nabla}_a \hat{\nabla}_b \omega_c - \hat{\nabla}_b \hat{\nabla}_a \omega_c &= \nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c - 2\nabla_{[a} C_{b]c}^d \omega_d + 2C_{c[a}^e C_{b]e}^d \omega_d, \\ \hat{R}_{abc}^d \omega_d &= R_{abc}^d \omega_d - 2\nabla_{[a} C_{b]c}^d \omega_d + 2C_{c[a}^e C_{b]e}^d \omega_d. \end{aligned}$$

Thus,

$$\begin{aligned} \hat{R}_{abc}^d &= R_{abc}^d + 2\delta_{[a}^d \nabla_{b]} \nabla_c \ln \Omega - 2g^{de} g_{c[a} \nabla_{b]} \nabla_e \ln \Omega + 2(\nabla_{[a} \ln \Omega) \delta_{b]}^d \nabla_c \ln \Omega \\ &\quad - 2(\nabla_{[a} \ln \Omega) g_{b]c} g^{df} \nabla_f \ln \Omega - 2g_{c[a} \delta_{b]}^d g^{ef} (\nabla_e \ln \Omega) \nabla_f \ln \Omega. \end{aligned}$$

Contracting over b and d indices, we obtain the transformation law for the Ricci tensor in an n -dimensional spacetime:

$$\begin{aligned} \hat{R}_{ab} &= R_{ab} - (n-2) \nabla_a \nabla_b \ln \Omega - g_{ab} g^{de} \nabla_d \nabla_e \ln \Omega + (n-2) (\nabla_a \ln \Omega) \nabla_b \ln \Omega \\ &\quad - (n-2) g_{ab} g^{de} (\nabla_d \ln \Omega) \nabla_e \ln \Omega. \end{aligned}$$

Finally, taking the trace of \hat{R}_{ab} , we have the transformation law for the rescaled scalar curvature:

$$\begin{aligned} \text{Scal}_{\hat{g}} &= \hat{g}^{ab} \hat{R}_{ab} = \hat{R}_a^a, \text{ and } \text{Scal}_g = g^{ab} R_{ab} = R_a^a \\ \text{Scal}_{\hat{g}} &= \Omega^{-2} [\text{Scal}_g - 2(n-1) g^{ab} \nabla_a \nabla_b \ln \Omega - (n-2)(n-1) g^{ab} (\nabla_a \ln \Omega) \nabla_b \ln \Omega]. \end{aligned} \tag{D.12}$$

The trace-free part of the Riemann tensor, also known as the Weyl tensor, is conformally invariant and satisfies:

$$\hat{C}_{abcd} = C_{abcd}. \quad (\text{D.13})$$

D.2 Geodesics and geometrical properties

D.2.1 The Petrov classification

Petrov in 1954 proposed to classify the Weyl tensor C_{abcd} into *Petrov type*. The Weyl tensor, *also known as the conformally invariant curvature tensor*, is defined by:

$$C_{abcd} = R_{abcd} - \frac{1}{2}(R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}) + \frac{1}{6}(g_{ac}g_{bd} - g_{ad}g_{bc})R \quad (\text{D.14})$$

In general, there exist four distinct null vectors denoted by k^a satisfying the relation:

$$k^b k^c k_{[e} C_{a]bc[d} k_{f]} = 0 \quad (\text{D.15})$$

These four vectors are called *principal null directions* (PND). Algebraically special spacetimes are those for which there exist fewer than four principal null directions. The Petrov classification of these spacetimes can be done in terms of the multiplicity of the PND:

$$\begin{aligned} I &\longleftrightarrow \text{four distinct PND,} \\ II &\longleftrightarrow \text{one pair of two PND coincides,} \\ D &\longleftrightarrow \text{two pairs of two PND coincide,} \\ III &\longleftrightarrow \text{3 PND coincide,} \\ IV/N &\longleftrightarrow \text{all four PND coincide.} \end{aligned}$$

There also exists a type O, which is conformally flat, i.e., the Weyl tensor is zero. Another way to understand the Petrov classification is to use the Weyl spinor Ψ_{ABCD} and the quantities $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ that are:

$$\Psi_0 = \Psi_{0000}, \Psi_1 = \Psi_{0001}, \Psi_2 = \Psi_{0011}, \Psi_3 = \Psi_{0111}, \Psi_4 = \Psi_{1111}.$$

Let (l, n, m, \bar{m}) be a Newman-Penrose tetrad, we have :

$$\begin{aligned}\Psi_0 &= C_{abcd} l^a m^b l^c m^d, & \Psi_1 &= C_{abcd} l^a m^b l^c n^d, \\ \Psi_2 &= C_{abcd} l^a m^b \bar{m}^c n^d, & \Psi_3 &= C_{abcd} l^a n^b \bar{m}^c n^d, \\ \Psi_4 &= C_{abcd} \bar{m}^a n^b \bar{m}^c n^d.\end{aligned}$$

To summarize, we have the following table inspired by [94]:

| Type | Description | Condition satisfied by (repeated) PND | Conditions on Weyl spinor |
|------|-------------------------------|---|---|
| I | four distinct PND | $k^b k^c k_{[e} C_{a]bc[d} k_{f]} = 0$ | $\Psi_0 = 0$ |
| II | one pair of two PND coincides | $k^b k^c C_{abc[d} k_{e]} = 0$ | $\Psi_0 = \Psi_1 = 0$ |
| D | two pairs of two PND coincide | $k^b k^c C_{abc[d} k_{e]} = 0,$ $l^b l^c C_{abc[d} l_{e]} = 0$ | $\Psi_0 = \Psi_1 = 0,$ $\Psi_3 = \Psi_4 = 0$ |
| III | three PND coincide | $k^c C_{abcd[d} k_{e]} = 0$ | $\Psi_0 = \Psi_1 = \Psi_2 = 0$ |
| N | all four PND coincide | $k^c C_{abcd} = 0$ | $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ |

Where l^a and k^a are two null vectors that are not proportional.

There exist complex scalar polynomial invariants for a vacuum spacetime. Two of them, I and J , are defined (see Chapter 8 in [71]):

$$I := \Psi_{AB}^{CD} \Psi_{CD}^{AB}, \quad J := \Psi_{AB}^{CD} \Psi_{CD}^{EF} \Psi_{EF}^{AB}. \quad (\text{D.16})$$

In terms of the Weyl spinor components, this becomes:

$$\begin{aligned}I &= \Psi_0 \Psi_4 - 4\Psi_1 \Psi_3 + 3\Psi_2^2, \\ J &= 6 \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}.\end{aligned}$$

The condition that the metric is algebraically special, i.e. that at least two principal null directions coincide is (see [76]) :

$$I^3 = 27J^2. \quad (\text{D.17})$$

Moreover, if a metric is algebraically special, it follows that $\Psi_0 = 0$. This turns I and J

into:

$$I = 3\Psi_2^2 - 4\Psi_1\Psi_3, \quad J = \Psi_1(2\Psi_2\Psi_3 - \Psi_1\Psi_4) - 3\Psi_2^3.$$

(D.17) entails that :

$$\Psi_1^2 [27(\Psi_1^2\Psi_4^2 - 4\Psi_1\Psi_2\Psi_3\Psi_4 + 2\Psi_2^3\Psi_4) + 64\Psi_1\Psi_3^3 - 36\Psi_2^2\Psi_3^2] = 0. \quad (\text{D.18})$$

The metric is algebraically special and admits at least two PND that coincide, shall satisfy (D.18). There are two situations:

1. If $\Psi_1 = 0$, then there exists a null vector field $\partial/\partial r$, (where r is the radial distance along a congruence of null geodesics) that is a (repeated) principal null direction.
2. If $\Psi_1 \neq 0$, there exists another double PND, that is not ∂_r (which remains a non-degenerate PND).

Concerning the Petrov type D, the Weyl spinor components have to satisfy the two following conditions:

$$\Psi_1 = 0, \quad (\text{D.19})$$

$$3\Psi_2\Psi_4 = 4\Psi_3^2. \quad (\text{D.20})$$

D.2.2 Geometrical optics

Let us begin by defining a null geodesic congruence (also called a ray) in the following way:

Definition D.2.1. *Let k^a be a null vector field on a spacetime (\mathcal{M}, g) , i.e., $k^a k_a = 0$, and let γ be the integral curves of k^a that define a congruence \mathcal{C} . This congruence is called geodesic (in other words, each of its members is a geodesic curve) if, for u a suitable parameter chosen for each curve of \mathcal{C} , we have:*

$$k^a \nabla_a k^b \propto k^b.$$

If u is such that:

$$k^a \nabla_a k^b = 0,$$

then it is called an affine parameter of the congruence.

There exists a direct link between this definition and the spin coefficients κ and ε :

Proposition D.2.1. *Let \mathcal{C} be a null congruence with k^a as the tangent vector to each curve parametrized by u . We consider (k, l, m, \bar{m}) as a Newman-Penrose tetrad. Let κ and ε be two spin coefficients associated with this tetrad. Then:*

1. \mathcal{C} is geodetic if and only if $\kappa = 0$,
2. \mathcal{C} is geodetic, and u is an affine parameter if and only if $\kappa = 0$ and $\varepsilon + \bar{\varepsilon} = 0$.

Proof : First of all, it is convenient to express the derivatives of the Newman-Penrose tetrad (k, l, m, \bar{m}) as:

$$\begin{aligned}\nabla_b k_a &= -(\gamma + \bar{\gamma})k_a k_b - (\varepsilon + \bar{\varepsilon})k_a l_b + (\alpha + \bar{\beta})k_a m_b + (\bar{\alpha} + \beta)k_a \bar{m}_b + \bar{\tau}m_a k_b + \bar{\kappa}m_a l_b \\ &\quad - \bar{\sigma}m_a m_b - \bar{\rho}m_a \bar{m}_b + \tau\bar{m}_a k_b + \kappa\bar{m}_a l_b - \rho\bar{m}_a m_b - \sigma\bar{m}_a \bar{m}_b, \\ \nabla_b l_a &= (\gamma + \bar{\gamma})l_a k_b + (\varepsilon + \bar{\varepsilon})l_a l_b - (\alpha + \bar{\beta})l_a m_b - (\bar{\alpha} + \beta)l_a \bar{m}_b - \nu m_a k_b - \pi m_a l_b + \lambda m_a m_b \\ &\quad + \mu m_a \bar{m}_b - \bar{\nu}\bar{m}_a k_b - \bar{\pi}\bar{m}_a l_b + \bar{\mu}\bar{m}_a m_b + \bar{\lambda}\bar{m}_a \bar{m}_b, \\ \nabla_b m_a &= -(\gamma - \bar{\gamma})m_a k_b - (\varepsilon - \bar{\varepsilon})m_a l_b + (\alpha - \bar{\beta})m_a m_b + (\beta - \bar{\alpha})m_a \bar{m}_b - \bar{\nu}k_a k_b - \bar{\pi}k_a l_b \\ &\quad + \bar{\mu}k_a m_b + \bar{\lambda}k_a \bar{m}_b + \tau l_a k_b + \kappa l_a l_b - \rho l_a m_b - \sigma l_a \bar{m}_b.\end{aligned}$$

The congruence \mathcal{C} is geodetic if and only if the tangent vector k^a is propagated in parallel:

$$k^b \nabla_b k^a \propto k^a.$$

From the expression of $\nabla_b k_a$, we infer:

$$k^b \nabla_b k^a = (\varepsilon + \bar{\varepsilon})k^a - \bar{\kappa}m^a - \kappa\bar{m}^a, \quad (\text{D.21})$$

Then \mathcal{C} is geodetic if and only if $\kappa = 0$. The parameter u is said to be affine if and only if:

$$k^b \nabla_b k^a = 0$$

For any k^a , using (D.21), this is equivalent to stating:

$$\kappa = 0, \varepsilon + \bar{\varepsilon} = 0. \quad \square \quad (\text{D.22})$$

D.2.3 Shear, Twist and convergence

From the study of κ and ε , we learn about the *geodetic* properties of a congruence. The study of ρ and σ gives us more information about the geometry of \mathcal{C} . For a geodetic congruence, we may define ζ as the displacement between two geodesics in a spacelike plane spanned by m and \bar{m} , the two spacelike vectors of the tetrad that are orthogonal to k^a , the tangent vector to each geodesic. Then, if the tetrad is parallelly propagated along the congruence (in other words, if u is an affine parameter), we have:

$$\frac{D\zeta}{ds} = -\rho\zeta - \sigma\bar{\zeta},$$

where $\frac{D\zeta}{ds} = k^a\nabla_a\zeta$ is the directional derivative of the tetrad along the congruence. Following [71], we put: $\rho = \theta + i\omega$ and $\sigma = |\sigma|e^{2i\psi}$ in order to interpret ρ and σ . Then:

$$\frac{D\zeta}{ds} = -\theta\zeta - i\omega\zeta - |\sigma|e^{2i\psi}\bar{\zeta}. \quad (\text{D.23})$$

1. Let $\omega = |\sigma| = 0$, then (D.23) becomes:

$$\frac{D\zeta}{ds} = -\theta\zeta,$$

where $\theta = -\text{Re}(\rho)$ is interpreted as the *convergence* of the congruence, i.e., the rate of contraction between the rays.

2. Let $\theta = |\sigma| = 0$, and (D.23) turns into:

$$\frac{D\zeta}{ds} = -i\omega\zeta,$$

hence ω measures the rotation or the *twist* of the congruence.

3. Finally, setting $\theta = \omega = 0$, (D.23) reads:

$$\frac{D\zeta}{ds} = -|\sigma|e^{2i\psi}\bar{\zeta}.$$

This indicates a contraction when $\arg(\zeta) = \psi, \psi + \pi$ but an expansion when $\arg(\zeta) = \psi \pm \pi/2$, in other words, $|\sigma|$ measures the *shear* of the congruence \mathcal{C} .

These quantities, particularly concerning the shear, will play an important role in two following results. On the one hand, there exists a useful theorem, by Goldberg and Sachs

in [28], that establishes an equivalence between the Petrov classification of the vacuum metric and the existence of a shear-free null geodetic congruence. On the other hand, the shear is related to the curvature of the spacetime and, therefore, to the Bondi mass of the spacetime, which we will investigate in detail further in section D.4.1.

Concerning the twist, there is an important proposition in [71] that establishes an equivalence between the spin coefficients associated with a null vector and the capacity for this null vector to generate a null hypersurface, i.e., to be null-hypersurface forming.

Proposition D.2.2. *1. A null congruence is hypersurface-orthogonal if and only if it is geodetic and twist-free, i.e., if and only if:*

$$\kappa = 0, \rho = \bar{\rho}.$$

2. A null congruence is hypersurface-orthogonal if and only if it is null hypersurface forming.

D.2.4 Goldberg Sachs' theorem

In 1962, Goldberg and Sachs presented in [28] a theorem that establishes an equivalence between the Petrov classification of the vacuum metric and the geometrical properties of null congruences. In particular, the Goldberg-Sachs theorem is related to the notion of shear-free geodesic null congruences, sometimes called shear-free ray congruence, denoted by SFR and defined as:

Definition D.2.2. Shear-free geodesic null congruence : *Let \mathcal{C} be a null congruence, where curves are integral lines of k^a , a null vector field, and let $(k^a, l^a, m^a, \bar{m}^a)$ be a null Newman-Penrose tetrad associated. The congruence \mathcal{C} is said to be a shear-free null geodesic congruence if:*

$$\kappa = \sigma = 0,$$

where κ and σ are the spin coefficients associated with the tetrad.

In the original paper, the theorem was divided into two parts that we combine to give the following theorem:

Theorem D.2.1. Goldberg-Sachs theorem

A vacuum metric, i.e., $R_{ab} = 0$, is algebraically special if and only if it contains a shear-free null geodesic congruence:

$$\Psi_0 = \Psi_1 = 0 \Leftrightarrow \sigma = \kappa = 0,$$

with Ψ_0 and Ψ_1 being two components of the Weyl tensor, defined in subsection D.2.1. Furthermore, the tangent vector to the congruence, denoted as k^a , satisfies:

$$k_{[a} C_{b]ijc} l^i k^j = 0.$$

Remark D.2.1. Goldberg-Sachs theorem and its generalizations

The original Goldberg-Sachs theorem took place in the vacuum solutions. A generalization of the Goldberg-Sachs theorem was done by Kundt and Thompson in 1962 in [45] and by Robinson and Schild in 1963 in [81]. A vectorial version of this theorem can be found in Chapter 7 of [87]. For a spinorial formulation of the generalized Goldberg-Sachs theorem, see Chapter 7.3 in [71]. These formulations generalize the theorem to the situation in which T_a in the Einstein equations is the energy-momentum tensor of an electromagnetic field. However, there is no generalization that establishes a relation between the geometrical properties of the null congruence and the Petrov type of the spacetime in the pure radiative solutions.

D.3 Asymptotic flatness and Bondi mass

D.3.1 Asymptotic flatness

We introduce the Bondi-Sachs coordinates: (u, r, x^A) . u is a retarded time coordinate (also called the Bondi time) such that the level hypersurfaces of u are null, and r is the luminosity distance that varies along the null rays. It is chosen to be an areal coordinate such that:

$$\det g_{ab} = r^4 \det q_{AB}, \tag{D.24}$$

where $\det q_{AB}$ is the determinant of the unit sphere metric q_{AB} associated with the angular coordinates x^A . For instance, in the usual spherical coordinates (θ, φ) :

$$\det q_{AB} = \sin^2 \theta.$$

In these coordinates, the metric reads:

$$g_{ab} = \frac{V}{r} e^{2\beta} du^2 + 2e^{2\beta} dudr - r^2 h_{AB} (dx^A - U^A du) (dx^B - U^B du) , \quad (\text{D.25})$$

where

$$h_{AB} dx^A dx^B = \frac{e^{2\gamma} + e^{2\delta}}{2} d\theta^2 + 2 \sin \theta \sinh(\gamma - \delta) d\theta d\varphi + \sin^2 \theta \frac{e^{-2\gamma} + e^{-2\delta}}{2} d\varphi^2 .$$

The functions $V, U^A, \beta, \gamma, \delta$ are taken to be any six functions of the coordinates (u, r, θ, φ) .

The asymptotic flatness is defined in terms of the boundary conditions (see section 3.a. in [85]):

1. For some choice of u , one can go to the limit $r \rightarrow \infty$ along each ray.
2. For some choice of θ and ϕ , and the above choice of u , we have:

$$\lim_{r \rightarrow \infty} \beta = \lim_{r \rightarrow \infty} U^A = \lim_{r \rightarrow \infty} \gamma = \lim_{r \rightarrow \infty} \delta = 0, \quad \lim_{r \rightarrow \infty} \frac{V}{r} = 1 ,$$

3. Over the coordinate ranges $u \in [u_0, u_1], r \in [r_0, \infty[, \theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$, all metric components can be expanded in powers of r^{-1} with at most a finite pole at $r = \infty$. Such power series can be freely added, multiplied, differentiated, etc.

In other words, the boundary conditions that we introduce above are sufficient to ensure that the spacetime is asymptotically flat, and its metric (D.25) tends to the Minkowski metric:

$$\eta = du^2 + 2dudr - r^2 d\omega^2 .$$

An alternative definition of asymptotic flatness is given by Penrose in [72] and [70] (see Chapter 9.9 in [71] for a comprehensive survey). This definition, more geometric than the Bondi-Sachs definition, relies on asymptotic simplicity since asymptotically flat spacetimes are a subclass of asymptotically simple spacetimes.

Definition D.3.1. Asymptotic Simplicity:

A spacetime (\mathcal{M}, g) is said to be k -asymptotically simple if there exists a C^{k+1} smooth manifold $\hat{\mathcal{M}}$, with a metric \hat{g} with a boundary $\mathcal{I} = \partial\hat{\mathcal{M}}$, and a conformal factor Ω such that:

1. \mathcal{M} is the interior of $\hat{\mathcal{M}}$,

2. $\hat{g} = \Omega^2 g$ in \mathcal{M} ,
3. Ω and \hat{g} are C^k smooth throughout $\hat{\mathcal{M}}$,
4. $\Omega > 0$ in \mathcal{M} and $\Omega|_{\mathcal{I}} = 0, \hat{\nabla}\Omega|_{\mathcal{I}} \neq 0$,
5. every null geodesic in \mathcal{M} acquires a past and future end-point on \mathcal{I} .

Remark D.3.1. *The last condition entails that the whole null infinity is described by \mathcal{I} . This condition seems to be too strong in order to describe physical solutions, as, for instance, the Schwarzschild solution. As explained in [71], in the Schwarzschild spacetime, there is the photon sphere, i.e., circular closed orbits, at $r = 3m$; hence, there exist null geodesics that do not acquire an endpoint on \mathcal{I} . In order to take account of these situations, Penrose formulated, in 1968, in [68], a weaker definition of asymptotic simplicity. A spacetime is called weakly asymptotically simple if its asymptotic region is diffeomorphic to an asymptotically simple spacetime.*

A natural question arises then: what is the physical meaning of the weakly asymptotic simplicity conditions?

In [36], the asymptotic flatness is depicted by the following property :

Proposition D.3.1. *The Weyl tensor C_{abcd} vanishes at the boundary \mathcal{I} and the rate of approach to zero is given by the peeling theorem.*

D.4 Bondi functions

D.4.1 Bondi mass

The notion of mass in general relativity is not as easy and intuitive as it is in classical mechanics. This comes from that in general, there is no global conservation law for the stress energy tensor T_{ab} , i.e. for the energy of the gravitational field, in the Einstein equation. In other words, it does not exist a meaningful definition for the gravitational energy density in general relativity (see section 11.2 in [94] for a physical interpretation of this statement). It is still possible to define a notion of mass of a spacetime in the framework of asymptotically flat spacetime. This is done by Komar in [44] in the context of stationary asymptotically flat spacetime. With these assumptions, there exists a timelike

Killing field ξ^a and the mass reads as :

$$M = -\frac{1}{8\pi} \int_S \epsilon_{abcd} \nabla^c \xi^d,$$

with ϵ_{abcd} the volume element and S a two-sphere which encloses all the sources. This definition does not hold in general case of non-stationary asymptotically flat spacetime. In this framework we consider two other definition of the mass : the ADM mass and the Bondi mass. These two definitions take place at the boundary of the rescaled spacetime $(\hat{\mathcal{M}}, \hat{g})$. The ADM mass corresponds to the mass seen at the spatial infinity i_0 while the Bondi mass is the mass defined along the null infinity. When the mass is constant over time, ADM and Bondi masses are equals but this is not still the case when the mass varies, e.g. in the Vaidya spacetime. In this situation the ADM mass is the limit of the Bondi mass when $u \rightarrow -\infty$.

Let (u, r, θ, φ) be a coordinate basis that entails the metric to fulfill the Bondi-Sachs asymptotic conditions. Because of the peeling theorem, the Weyl spinor components Ψ_a , in these coordinates are

$$\begin{aligned} \psi_0 &= \psi_0^0 r^{-5} + \mathcal{O}(r^{-6}), \\ \psi_1 &= \psi_1^0 r^{-4} + \mathcal{O}(r^{-5}), \\ \psi_2 &= \psi_2^0 r^{-3} + \mathcal{O}(r^{-4}), \\ \psi_3 &= \psi_3^0 r^{-2} + \mathcal{O}(r^{-3}), \\ \psi_4 &= \psi_4^0 r^{-1} + \mathcal{O}(r^{-2}). \end{aligned}$$

Let $l = \partial_r$, be a null vector field and let σ be the shear of the outgoing null hypersurface generated by l . Then σ behaves as :

$$\sigma = \sigma^0 r^{-2} + \mathcal{O}(r^{-3}).$$

From [7], [36] and [63] we can state the following definition :

Definition D.4.1. Bondi Mass : *The Bondi mass of an asymptotically flat vacuum spacetime, in Bondi coordinates (u, r, θ, φ) is given by :*

$$\mathcal{M} = -\psi_2^0 - \sigma^0 \dot{\bar{\sigma}}^0.$$

An alternative way is to define the Bondi mass aspect as follows :

$$\mathcal{M}_B(u, \theta, \varphi) = -\frac{1}{2} \lim_{r \rightarrow \infty} [V(u, \theta, \varphi) - r] ,$$

with $g_{uu} = V e^{2\beta} / r$ for a metric in Bondi coordinates. The Bondi mass, denoted by M_B is then defined by :

$$M_B(u) = \frac{1}{4\pi} \int_S \mathcal{M}_B(u, \theta, \varphi) dS ,$$

where dS is the volume element on the unit sphere.

D.4.2 Bondi News function

Let q_{AB} be the metric on the euclidean 2-spheres and let h_{AB} be the metric on the sphere in the metric (D.25). Then, since the asymptotic conditions ensure that h_{AB} converges toward q_{AB} when $r \rightarrow \infty$, we introduce c_{AB} such that :

$$h_{AB} = q_{AB} + \frac{c_{AB}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) .$$

Definition D.4.2. *News tensor* The News tensor, denoted by N_{ab} is defined as :

$$N_{ab} = \frac{1}{2} \frac{\partial c_{AB}}{\partial u} .$$

Let q^A be a complex dyad satisfying on the unit sphere such that the inverse metric on the euclidean sphere, q^{AB} is :

$$q^{AB} = (q^A \bar{q}^B + \bar{q}^A q^B) .$$

The News function is then defined, relative to the choice of a polarization dyad :

$$N = q^A q^B N_{AB} .$$

Proposition D.4.1. *News function and shear* Let (u, r, θ, φ) be a coordinate frame and let σ be the shear of the outgoing null hypersurfaces, i.e. the level hypersurfaces of u :

$$\sigma = \frac{\sigma^0}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) .$$

Then, the Bondi News function, denoted by N is :

$$N = \frac{\partial \sigma^0}{\partial u}. \quad (\text{D.26})$$

The News tensor determines the flux of gravitational radiation emitted by a bounded source. If there is no News ($N = 0$), the system is at equilibrium, and the Bondi mass does not vary with time. In other words, if a bounded source emits gravitational waves, then its Bondi mass is decreasing due to this emission, and this gives the mass-loss formula:

$$\frac{d}{du} M_B = \frac{1}{4\pi} \oint_S |N|^2 d\omega^2.$$

For more details on the Bondi-Sachs formalism, see [51].

D.4.3 Bondi mass of Robinson-Trautman solutions

The Robinson-Trautman metric expressed in the usual (u, r, θ, φ) coordinates is given by:

$$g = \left(K - 2r \frac{\partial}{\partial u} \ln u - \frac{2m(u)}{r} \right) du^2 + 2dudr - \frac{2r^2}{\tilde{P}^2} d\omega, \quad (\text{D.27})$$

is not expressed in a Bondi frame since $2r\partial_u \ln P$ diverges at $r \rightarrow \infty$, and the metric does not tend to the Minkowski solution when $r \rightarrow \infty$. Then, the natural question that arises is: does there exist a coordinate transformation that can be expressed in closed form to obtain the Bondi-Sachs coordinate frame? Newman and Unti answered in [63] and proved that no such transformation exists, and that the coordinate transformations are given by an infinite series in powers of r^{-1} . These transformations are given in [29], [13], and [3]. Let (U, R, X, \bar{X}) be the Bondi-Sachs coordinates defined by :

$$U = U_0 + U_1 r^{-1} + U_2 r^{-2} + \mathcal{O}(r^{-3}), \quad (\text{D.28})$$

$$R = R_0 r + R_1 + R_2 r^{-1} + \mathcal{O}(r^{-2}), \quad (\text{D.29})$$

$$\Theta = \Theta_0 + \Theta_1 r^{-1} + \Theta_2 r^{-2} + \mathcal{O}(r^{-3}), \quad (\text{D.30})$$

$$\Phi = \Phi_0 + \Phi_1 r^{-1} + \Phi_2 r^{-2} + \mathcal{O}(r^{-3}), \quad (\text{D.31})$$

where, all the coefficients are functions of u, θ and φ . The inverse metric in (U, R, Θ, Φ) shall behave as:

$$\begin{aligned} g^{UU} &= 0, g^{UR} = 1 + \mathcal{O}(R^{-1}), g^{U\Theta} = g^{U\Phi} = \mathcal{O}(R^{-2}), \\ g^{RR} &= -1 + \mathcal{O}(R^{-1}), g^{R\Theta} = g^{R\Phi} = \mathcal{O}(R^{-2}), \\ g^{\Theta\Phi} &= \mathcal{O}(R^{-2}), g^{\Theta\Theta} g^{\Phi\Phi} - g^{\Theta\Phi} g^{\Phi\Theta} = \frac{1}{R^4 \sin^2 \Theta} + \mathcal{O}(R^{-5}). \end{aligned}$$

The metric components transformation law is given by:

$$g^{AB} = \frac{\partial X^A}{\partial x^i} \frac{\partial X^B}{\partial x^j} g^{ij}, \quad (\text{D.32})$$

where the capital letters refer to the metric in Bondi-Sachs coordinates $((U, R, \Theta, \Phi))$ and the lowercase letters refer to the metric in coordinates (u, r, θ, φ) . It follows that :

$$\begin{aligned} U_0 &= \int \tilde{P} du + h(\theta, \varphi), \\ R_0 &= \left(\frac{\partial U_0}{\partial u} \right)^{-1} = \frac{1}{\tilde{P}}, \\ \Theta_0 &= \theta \\ \Phi_0 &= \varphi. \end{aligned}$$

where $\int \tilde{P} du$ is the primitive of $\tilde{P} = P \sin^2 \theta$ with respect to u .

$$\begin{aligned} U_1 &= -\frac{\tilde{P}}{2} \left[\left(\frac{\partial U_0}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left(\frac{\partial U_0}{\partial \varphi} \right)^2 \right], \\ R_1 &= \frac{1}{2} \Delta_{S^2} U_0, \\ \Theta_1 &= -\tilde{P} \frac{\partial U_0}{\partial \theta}, \\ \Phi_1 &= -\frac{\tilde{P}}{\sin^2 \theta} \frac{\partial U_0}{\partial \varphi}, \end{aligned}$$

with :

$$\Delta_{S^2} = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.$$

Finally :

$$\begin{aligned}
 U_2 &= \frac{\tilde{P}}{2} \left[\left(\frac{\partial U_0}{\partial \theta} \right)^2 \frac{\partial^2 U_0}{\partial \theta^2} - \frac{\cot \theta}{\sin^2 \theta} \frac{\partial U_0}{\partial \theta} \left(\frac{\partial U_0}{\partial \varphi} \right)^2 + \frac{2}{\sin^2 \theta} \frac{\partial U_0}{\partial \theta} \frac{\partial U_0}{\partial \varphi} \frac{\partial^2 U_0}{\partial \theta \partial \varphi} + \frac{1}{\sin^4 \theta} \left(\frac{\partial U_0}{\partial \varphi} \right)^2 \frac{\partial^2 U_0}{\partial \varphi^2} \right], \\
 \Theta_2 &= \tilde{P}^2 \frac{\partial U_0}{\partial \theta} \frac{\partial^2 U_0}{\partial \theta^2} - \frac{\tilde{P}^2}{2 \sin^2 \theta} \left[\cot \theta \left(\frac{\partial U_0}{\partial \varphi} \right)^2 - \frac{\partial U_0}{\partial \varphi} \frac{\partial^2 U_0}{\partial \theta \partial \varphi} \right], \\
 \Phi_2 &= \frac{\tilde{P}^2}{\sin^2 \theta} \left(\frac{\partial U_0}{\partial \theta} \frac{\partial^2 U_0}{\partial \theta \partial \varphi} - \cot \theta \frac{\partial U_0}{\partial \theta} \frac{\partial U_0}{\partial \varphi} + \frac{1}{\sin^2 \theta} \frac{\partial U_0}{\partial \varphi} \frac{\partial^2 U_0}{\partial \varphi^2} \right).
 \end{aligned}$$

and :

$$\begin{aligned}
 R_2 &= -\frac{\tilde{P}}{8} \left[\left(\frac{\partial^2 U_0}{\partial \theta^2} \right)^2 + \frac{4}{\sin^2 \theta} \left(\frac{\partial^2 U_0}{\partial \theta \partial \varphi} \right)^2 + \frac{1}{\sin^4 \theta} \left(\frac{\partial^2 U_0}{\partial \varphi^2} \right)^2 \right] \\
 &+ \frac{1}{4} \left(\frac{\partial U_0}{\partial \theta} \right)^2 \left[\frac{3}{2} \tilde{P} \cot^2 \theta - \frac{1}{\sin^2 \theta} \frac{\partial^2 \tilde{P}}{\partial \varphi^2} + \frac{\partial^2 \tilde{P}}{\partial \theta^2} - \cot \theta \frac{\partial \tilde{P}}{\partial \theta} \right] \\
 &+ \frac{1}{4 \sin^2 \theta} \left[\tilde{P} \frac{\partial^2 U_0}{\partial \theta^2} \frac{\partial^2 U_0}{\partial \varphi^2} - \left(\frac{\partial U_0}{\partial \varphi} \right)^2 \frac{\partial \tilde{P}}{\partial \theta} \cot \theta \right] - \frac{1}{4 \sin^2 \theta} \left(\frac{\partial U_0}{\partial \varphi} \right)^2 \left[2\tilde{P} - \frac{\partial^2 \tilde{P}}{\partial \theta^2} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \tilde{P}}{\partial \varphi^2} \right] \\
 &+ \frac{\cot \theta}{2 \sin^2 \theta} \left(\frac{\partial U_0}{\partial \varphi} \right) \left[\tilde{P} \frac{\partial^2 U_0}{\partial \theta \partial \varphi} - 2 \frac{\partial \tilde{P}}{\partial \varphi} \frac{\partial U_0}{\partial \theta} + \frac{\partial \tilde{P}}{\partial \theta} \frac{\partial U_0}{\partial \varphi} - \frac{2\tilde{P}}{\sin^2 \theta} \frac{\partial^3 U_0}{\partial \varphi^3} \right] \\
 &- \frac{\tilde{P}}{4} \frac{\partial U_0}{\partial \theta} \left[\cot \theta \frac{\partial^2 U_0}{\partial \theta^2} + 2 \frac{\partial^3 U_0}{\partial \theta^3} + \frac{1}{\sin^2 \theta} \left(2 \frac{\partial^3 U_0}{\partial \theta \partial \varphi^2} - 3 \cot \theta \frac{\partial^2 U_0}{\partial \varphi^2} \right) \right] \\
 &- \frac{1}{2 \sin^2 \theta} \frac{\partial U_0}{\partial \varphi} \left[\tilde{P} \frac{\partial^3 U_0}{\partial \varphi \partial \theta^2} - 2 \frac{\partial U_0}{\partial \theta} \frac{\partial^2 \tilde{P}}{\partial \theta \partial \varphi} + \cot^2 \theta \tilde{P} \frac{\partial U_0}{\partial \varphi} \right].
 \end{aligned}$$

For more details, see [13] and [3].

Then, it is possible to define the Bondi mass, by expanding the g^{RR} term in power of $1/R$ as done in [3]. The Bondi mass aspect is given by:

$$\begin{aligned}
 \mathcal{M}_B(u, \theta, \varphi) &= \frac{m(u)}{\tilde{P}^3} - \frac{\partial U_0}{\partial \theta} \left[\frac{\partial \tilde{c}_1}{\partial \theta} + \frac{3}{2} \cot \theta \tilde{c}_1 + \frac{1}{\sin^2 \theta} \frac{\partial \tilde{c}_2}{\partial \varphi} \right] + \frac{1}{\sin^2 \theta} \frac{\partial U_0}{\partial \varphi} \left[\frac{\partial \tilde{c}_1}{\partial \varphi} - \frac{\partial(\tilde{c}_2 \sin \theta)}{\partial \theta} \right] \\
 &+ \frac{1}{2\tilde{P}} \frac{\partial \tilde{c}_1}{\partial u} \left[\left(\frac{\partial U_0}{\partial \theta} \right)^2 - \frac{1}{\sin^2 \theta} \left(\frac{\partial U_0}{\partial \varphi} \right)^2 \right] + \frac{1}{\tilde{P} \sin \theta} \left(\frac{\partial U_0}{\partial \varphi} \right) \left(\frac{\partial U_0}{\partial \theta} \right) \left(\frac{\partial \tilde{c}_2}{\partial u} \right) \\
 &- \frac{\tilde{c}_1}{2} \left(\frac{\partial^2 U_0}{\partial \theta^2} - \frac{1}{\sin^2 \theta} \frac{\partial^2 U_0}{\partial \varphi^2} \right) - \frac{\tilde{c}_2}{\sin \theta} \frac{\partial^2 U_0}{\partial \varphi \partial \theta},
 \end{aligned}$$

where $\partial_u \tilde{c}_1$ and $\partial_u \tilde{c}_2$, are given by:

$$\frac{\partial c^{(1)}}{\partial u} = \tilde{P} \tilde{c}_1 = \frac{1}{2} \left(\frac{\partial^2 \tilde{P}}{\partial \theta^2} - \cot \theta \frac{\partial \tilde{P}}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \tilde{P}}{\partial \varphi^2} \right), \quad (\text{D.33})$$

$$\frac{\partial c^{(2)}}{\partial u} = \tilde{P} \tilde{c}_2 = \frac{1}{\sin \theta} \left(\frac{\partial \tilde{P}}{\partial \theta \partial \varphi} - \cot \theta \frac{\partial \tilde{P}}{\partial \varphi} \right). \quad (\text{D.34})$$

D.5 Spin coefficient equation

Consider the Newman-Penrose tetrad :

$$\begin{aligned} l^a &= \partial_r^a, \\ n^a &= \partial_u^a + U \partial_r^a + X^A \partial_{x^A}^a, \\ m^a &= \omega \partial_r^a + \xi^A \partial_{x^A}^a, \\ \bar{m}^a &= \bar{\omega} \partial_r^a + \bar{\xi}^A \partial_{x^A}^a. \end{aligned}$$

We define directional derivatives associated with this tetrad:

$$D = \frac{\partial}{\partial r}, \Delta = U \frac{\partial}{\partial r} + \frac{\partial}{\partial u} + X^k \frac{\partial}{\partial x^k}, \delta = \omega \frac{\partial}{\partial r} + \xi^k \frac{\partial}{\partial x^k}.$$

Field equations may be divided into three categories: the radial equations (derivatives with respect to D), non-radial equations, and u derivatives (which includes taking into account Bianchi identities).

D.5.1 Radial equations

$$D\xi^i = \rho\xi^i + \sigma\bar{\xi}^i, \quad (\text{D.35a})$$

$$D\omega = \rho\omega + \sigma\bar{\omega} - (\bar{\alpha} + \beta), \quad (\text{D.35b})$$

$$DX^i = (\bar{\alpha} + \beta)\bar{\xi}^i + (\alpha + \bar{\beta})\xi^i, \quad (\text{D.35c})$$

$$DU = (\bar{\alpha} + \beta)\bar{\omega} + (\alpha + \bar{\beta})\omega - (\gamma + \bar{\gamma}), \quad (\text{D.35d})$$

$$D\rho = \rho^2 + \sigma\bar{\sigma}, \quad (\text{D.35e})$$

$$D\sigma = 2\rho\sigma + \Psi_0, \quad (\text{D.35f})$$

$$D\tau = \tau\rho + \bar{\tau}\sigma + \Psi_1, \quad (\text{D.35g})$$

$$D\alpha = \alpha\rho + \beta\bar{\sigma}, \quad (\text{D.35h})$$

$$D\beta = \beta\rho + \alpha\sigma + \Psi_1, \quad (\text{D.35i})$$

$$D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2, \quad (\text{D.35j})$$

$$D\lambda = \lambda\rho + \mu\bar{\sigma}, \quad (\text{D.35k})$$

$$D\mu = \mu\rho + \lambda\sigma + \Psi_2, \quad (\text{D.35l})$$

$$D\nu = \tau\lambda + \bar{\tau}\mu + \Psi_3, \quad (\text{D.35m})$$

$$D\Psi_1 - \bar{\delta}\Psi_0 = 4\rho\Psi_1 - 4\alpha\Psi_0, \quad (\text{D.35n})$$

$$D\Psi_2 - \bar{\delta}\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0, \quad (\text{D.35o})$$

$$D\Psi_3 - \bar{\delta}\Psi_2 = 2\rho\Psi_3 - 2\lambda\Psi_1, \quad (\text{D.35p})$$

$$D\Psi_4 - \bar{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2. \quad (\text{D.35q})$$

D.5.2 Non radial equations

$$\delta X^i - \Delta\xi^i = (\mu + \bar{\gamma} - \gamma)\xi^i + \bar{\lambda}\bar{\xi}^i, \quad (\text{D.36a})$$

$$\delta\xi^i - \bar{\delta}\bar{\xi}^i = (\bar{\beta} - \alpha)\xi^i + (\bar{\alpha} - \beta)\bar{\xi}^i, \quad (\text{D.36b})$$

$$\delta\bar{\omega} - \bar{\delta}\omega = (\bar{\beta} - \alpha)\omega + (\bar{\alpha} - \beta)\bar{\omega} + (\mu - \bar{\mu}), \quad (\text{D.36c})$$

$$\delta U - \Delta\omega = (\mu + \bar{\gamma} - \gamma)\omega + \bar{\lambda}\bar{\omega} - \bar{\nu}, \quad (\text{D.36d})$$

$$\Delta\lambda - \bar{\delta}\nu = 2\alpha\nu + (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda - \Psi_4, \quad (\text{D.36e})$$

$$\delta\rho - \bar{\delta}\rho = (\beta + \bar{\alpha})\rho + (\bar{\beta} - 3\alpha)\sigma - \Psi_1, \quad (\text{D.36f})$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma - 2\alpha\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - \Psi_2, \quad (\text{D.36g})$$

$$\delta\lambda - \bar{\delta}\mu = (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3, \quad (\text{D.36h})$$

$$\delta\nu - \Delta\mu = \gamma\mu - 2\nu\beta + \bar{\gamma}\mu + \mu^2 + \lambda\bar{\lambda}, \quad (\text{D.36i})$$

$$\delta\gamma - \Delta\beta = \tau\mu - \sigma\nu + (\mu - \gamma + \bar{\gamma}\beta + \bar{\lambda}\alpha), \quad (\text{D.36j})$$

$$\delta\tau - \Delta\sigma = 2\tau\beta + (\bar{\gamma} + \mu - 3\gamma)\sigma + \bar{\lambda}\rho, \quad (\text{D.36k})$$

$$\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2, \quad (\text{D.36l})$$

$$\Delta\alpha - \bar{\delta}\gamma = \rho\nu - \tau\lambda - \lambda\beta + (\bar{\gamma} - \gamma - \bar{\mu})\alpha - \Psi_3. \quad (\text{D.36m})$$

D.5.3 u -derivative equations

$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - (4\tau + 2\beta)\Psi_1 + 3\sigma\Psi_2, \quad (\text{D.37a})$$

$$\Delta\Psi_1 - \delta\Psi_2 = (\nu\Psi_0 + (2\gamma - 2\mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3) \quad (\text{D.37b})$$

$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + (-2\tau + 2\beta)\Psi_3 + \sigma\Psi_4, \quad (\text{D.37c})$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - (2\gamma + 4\mu)\Psi_3 + (-\tau + 4\beta)\Psi_4. \quad (\text{D.37d})$$

D.6 Leray's theorem

Leray's theorem gives the existence of a solution to the Cauchy problem for the wave equation on globally hyperbolic spacetime with analytic data. Leray's work can be found in his 1953 lecture notes [50]. Here, we restrict his results to the study of the conformally invariant wave equation, i.e., the wave equation with a potential that comes from the scalar curvature of the spacetime. It is essential to note that Leray's definition of global hyperbolicity is more general than the one provided here:

Definition D.6.1. Cauchy hypersurface:

Let (\mathcal{M}, g) be a time-orientable spacetime (i.e., a spacetime that admits a continuous timelike vector field, vanishing nowhere). A Cauchy hypersurface on (\mathcal{M}, g) is a hypersurface Σ such that:

1. Σ is spacelike everywhere.
2. Every inextendible timelike curve intersects Σ at exactly one point.

This later condition imposes that the domain of influence of Σ is \mathcal{M} .

Definition D.6.2. Global hyperbolicity:

A time-orientable spacetime (\mathcal{M}, g) is said to be globally hyperbolic if it admits a Cauchy hypersurface.

Theorem D.6.1. *Leray, 1953.*

Let (\mathcal{M}, g) be a globally hyperbolic spacetime, and let Σ_0 be a Cauchy hypersurface on \mathcal{M} . Let t be a time function on \mathcal{M} . Then given $\phi_0, \phi_1 \in \mathcal{C}^\infty(\Sigma_0)$, there exists a unique solution $\phi \in \mathcal{C}^\infty(\mathcal{M})$ to the Cauchy problem:

$$\begin{cases} \square_g \phi = 0 \text{ on } \mathcal{M}, \\ \phi|_{\Sigma_0} = \phi_0, \partial_t \phi|_{\Sigma_0} = \phi_1. \end{cases}$$

Corollary D.6.1. *Leray's theorem remains valid if we add F , a first-order differential operator with smooth coefficients. Given $\phi_0, \phi_1 \in \mathcal{C}^\infty(\Sigma_0)$, there exists a unique solution $\phi \in \mathcal{C}^\infty(\mathcal{M})$ to the Cauchy problem:*

$$\begin{cases} (\square_g + F)\phi = 0 \text{ on } \mathcal{M}, \\ \phi|_{\Sigma_0} = \phi_0, \partial_t \phi|_{\Sigma_0} = \phi_1. \end{cases}$$

D.7 Sobolev spaces

The vector fields method used in this manuscript relies on the construction of energy spaces on a given hypersurface Σ , that is spacelike or null to ensure that the energy density is physically meaningful. The norm associated with these spaces is obtained from the energy fluxes for a field across Σ and corresponds to weighted Sobolev norms. In general (see, for instance, Adams' book [2]), the definition of the Sobolev norm is given by:

Definition D.7.1. (The Sobolev norm):

Let m be a positive integer and $1 \leq p < \infty$, then for any function u , the Sobolev norm $\|\cdot\|_{m,p}$ is given by:

$$\|u\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p}^p \right)^{1/p},$$

where $D^\alpha u$ corresponds to the weak partial derivative of u , and $\|\cdot\|_{L^p}$ is the L^p norm.

Sobolev spaces are then defined as follows:

Definition D.7.2. (Sobolev spaces):

Let m be a positive integer and $1 \leq p < \infty$. From the Sobolev norm, we define two types of Sobolev spaces on a domain Ω :

1. $H^{m,p}$, the completion of $\{u \in \mathcal{C}^m(\Omega) : \|u\|_{m,p} < \infty\}$.
2. $W^{m,p}(\Omega)$:

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}.$$

In 1964, Meyers and Serrin in [55] proved that for every domain Ω :

$$H^{m,p}(\Omega) = W^{m,p}(\Omega). \tag{D.38}$$

In this manuscript, we will actually use weighted Sobolev spaces instead of Sobolev spaces. Let $w(x)$ be a locally integrable function on Ω and referred to as a weight; then the weighted Sobolev norm for any function u on a domain Ω is:

$$\|u\|_{H^{m,p}(\Omega,\omega)} = \left(\sum_{0 \leq |\alpha| \leq m} \int_{\omega} |D^{\alpha}u|^p \omega \, dx \right)^{1/p}.$$

On a spacetime (\mathcal{M}, g) , the weak partial derivative D is replaced by the connection ∇ associated with the metric g , weights ω come from the metric components involved in the scalar product. Furthermore, in the context of vector fields methods, one restrains $m = 1, l = 2$, the domain Ω is a hypersurface Σ , and we denote by $H^1(\Sigma)$ the weighted Sobolev space $H^{1,2}(\Sigma, \omega)$. Then on a spacetime (\mathcal{M}, g) , The Sobolev norm now reads as:

$$\|u\|_{H^1(\Sigma)}^2 = \int_{\Sigma} (u^2 + \nabla^a u \nabla_a u) \, d\Sigma, \quad (\text{D.39})$$

with $d\Sigma$ the volume element on Σ and ∇ the Levi-Civita connection associated with the metric g . Hence, $H^1(\Sigma)$ space is:

Definition D.7.3. (H^1 space):

Let Σ be a hypersurface on (\mathcal{M}, g) , then $H^1(\Sigma)$ is the completion of smooth functions on Σ , i.e., functions $u \in \mathcal{C}^{\infty}(\Sigma)$ in the norm defined in (D.39), $\|\cdot\|_{H^1(\Sigma)}$.

We denote by $H_0^1(\Sigma)$ the completion of the space of smooth compactly supported functions u on Σ , i.e., $u \in \mathcal{C}_0^{\infty}(\Sigma)$ in the norm $\|\cdot\|_{H^1(\Sigma)}$.

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Titre : Comportement asymptotique des champs sans masse dans des espaces-temps purement radiatifs.

Mot clés : Analyse asymptotique, trou noir de Vaidya, ondes scalaires, métrique de Robinson-Trautman, propriété de peeling, scattering conforme.

Cette thèse explore deux sujets distincts. La première partie examine le comportement asymptotique des ondes scalaires dans l'espace-temps de Vaidya, décrivant un trou blanc sphérique en évaporation via émission de poussières isotropes. L'analyse se focalise sur la régularité des ondes scalaires conformes au bord (passé et futur) isotrope du compactifié, en fonction des données initiales du champ conforme. Nous construisons également l'opérateur de scattering conforme, montrant qu'il encode toute l'évolution du champ dans l'espace-temps compactifié. Ces résultats reposent sur des méthodes d'inégalités d'énergie et de champs de vecteurs.

La seconde partie se concentre sur l'analyse des courbes isotropes entrantes dans les espaces-temps purement radiatifs de Robinson-Trautman de type D. Contrairement à une étude précédente sur la métrique de Vaidya, ces courbes ne forment pas l'horizon passé en raison de la géométrie de la solution. Le dernier chapitre classe ces courbes, montrant qu'elles présentent un comportement similaire à celui observé dans l'espace-temps de Vaidya.

Title: Asymptotic behaviour of zero rest-mass fields on radiative spacetimes

Keywords: Asymptotic analysis, Vaidya's black hole, scalar waves, Robinson-Trautman's metric, peeling-off property, conformal scattering.

This thesis addresses two distinct subjects. The first part examines the asymptotic behavior of scalar waves in the Vaidya spacetime, describing a spherical white hole evaporating via emission of isotropic dust. The analysis focuses on the regularity of conformal scalar waves at the isotropic boundary (past and future) of the compactified spacetime, depending on the initial data of the conformal field. Additionally, we construct the conformal scattering operator, demonstrating its ability to encode the entire field evolution in the compactified spacetime. These findings rely on energy inequalities and vector field methods.

The second part centers on analyzing incoming isotropic curves in the purely radiative Robinson-Trautman spacetimes of type D. In contrast to a previous study on Vaidya's metric, these curves do not form the past horizon due to the solution's geometry. The final chapter categorizes these curves, revealing a behavior akin to that observed in Vaidya's spacetime.