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Modern Techniques in Gravity and the Structure of Holographic Conformal Field Theories

Gabriele Di Ubaldo

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Modern Techniques in Gravity and the Structure of Holographic Conformal Field Theories

*Techniques modernes en gravité et structures des
théories des champs conformes holographiques*

Thèse de doctorat de l'université Paris-Saclay

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Mots clés : correspondance AdS/CFT, gravité quantique, théorie conforme des champs, matrices aléatoires, chaos quantique

Résumé : Dans cette thèse, nous étudions la relation entre le chaos quantique dans la théorie conforme des champs (CFT) et la présence de trous de ver dans la gravité quantique.

Dans le chapitre 2, nous explorons les connexions entre la théorie des matrices aléatoires, la gravité quantique AdS_3 et les CFTs 2D chaotiques. Nous développons un cadre pour identifier le comportement des matrices aléatoires dans les CFTs 2D et la gravité AdS_3 . Du côté CFT, nous introduisons une formule de trace pour les CFTs 2D, analogue à la formule de trace de Gutzwiller pour les systèmes quantiques. Cette formule nous permet d'identifier et de quantifier l'universalité des matrices aléatoires dans les CFTs chaotiques, menant à une condition nécessaire et suffisante pour qu'un CFT 2D affiche une rampe linéaire dans son Facteur de Forme Spectrale. Du côté de la gravité, nous nous concentrons sur le trou de ver de Cotler-Jensen (CJ) dans la gravité pure AdS_3 . Il est démontré que celui-ci est égal au corrélateur de la théorie des matrices aléatoires, complété minimalement pour respecter la symétrie de Virasoro et l'invariance $SL(2, \mathbb{Z})$. Cela suggère une dualité AdS_3/RMT_2 , généralisant la dualité JT/RMT à 3D. Nous établissons ensuite le dictionnaire AdS/CFT en montrant que les trous de ver AdS_3 peuvent être considérés comme le dual gravitationnel de la projection de Hecke de la fonction de partition au carré dans un CFT 2D. Cela illustre comment les trous de ver peuvent émerger d'un CFT chaotique individuel sans nécessiter de moyenne d'ensemble sur les théories.

Dans le chapitre 3, nous abordons des problèmes non-perturbatifs dans la gravité 3D et le bootstrap modulaire à grande charge centrale. Le calcul de la fonction de partition de la gravité 3D par Maloney et Witten a révélé une non-unitarité à cause de une densité d'états négative près de l'extrémalité. Pour restaurer l'unitarité, nous introduisons des contributions non-perturbatives à la fonction de partition de Maloney-Witten, qui donne lieu à une fonction de partition unitaire et invariante modulaire avec un gap spectral de $\Delta=(c-1)/12$. Ces nouvelles contributions peuvent être interprétées comme des cordes tournantes et des trous noirs "stringy", dont l'entropie, bien que non-perturbativement supprimée, domine près de l'extrémalité et produit une densité d'états positive. Cela suggère une possible réalisation UV en théorie des cordes de la gravité pure AdS_3 . Le gap spectral de notre solution, $\Delta=(c-1)/12$, valide même pour c fini, dépasse la précédente borne numérique du bootstrap de $\Delta=c/9.08$.

Dans le chapitre 4, nous explorons si les trous de ver en gravité 2D peuvent émerger de l'intrication entre les microétats des trous noirs plutôt que de une moyenne d'ensemble. Nous étudions cela en utilisant la Mécanique Quantique des Matrices (MQM), qui peut être interprétée comme une quantification de l'intégrale matricielle duale à la gravité JT. Ainsi, elle sert de modèle pour l'intrication entre les microétats chaotiques des trous noirs. Nous résolvons la MQM pour des courbes spectrales arbitraires en développant une théorie effective hydrodynamique. Nous montrons que l'intrication de l'état fondamental dans la MQM reproduit le noyau sinus non-perturbatif et par conséquent la transition rampe-plateau dans le SFF. Étant donné que ces effets sont capturés par les trous de ver dans la gravité JT, cela suggère qu'ils peuvent être interprétés comme émergeant de l'intrication entre les microétats des trous noirs. Nous étudions également la structure d'intrication dans la MQM, révélant un haut degré d'intrication de l'état fondamental, proportionnel à l'entropie des trous noirs.

Title: Modern Techniques in Gravity and the Structure of Holographic Conformal Field Theories

Keywords : AdS/CFT correspondence, quantum gravity, conformal field theory, random matrix theory, quantum chaos

Abstract : In this thesis we study the relation between quantum chaos in Conformal Field Theory and the presence of wormholes in the gravitational path integral.

In Chapter 2, we explore the connections between random matrix theory, AdS_3 quantum gravity, and chaotic 2D CFTs. We develop a framework for identifying random matrix behavior in 2D CFTs and AdS_3 gravity. On the CFT side, we introduce a trace formula for 2D CFTs, mirroring Gutzwiller's trace formula for quantum chaotic systems. This formula allows us to identify and quantify random matrix universality in chaotic CFTs, leading to a necessary and sufficient condition for a 2D CFT to display a linear ramp in its Spectral Form Factor. On the gravity side, we focus on the Cotler-Jensen (CJ) wormhole in pure AdS_3 gravity. This is shown to be equal to the random matrix theory correlator, minimally completed to respect Virasoro symmetry and $\text{SL}(2, \mathbb{Z})$ invariance. This suggests an $\text{AdS}_3/\text{RMT}_2$ duality, generalizing the JT/RMT duality to 3D. We then establish the AdS/CFT dictionary by showing that AdS_3 wormholes can be viewed as the gravity dual of the Hecke projection of the squared partition function in a 2D CFT. This illustrates how wormholes can emerge from an individual chaotic CFT without requiring an ensemble average over theories.

In Chapter 3, we address non-perturbative issues in 3D gravity and the modular bootstrap at large central charge. Maloney and Witten's computation of the on-shell 3D gravity partition function revealed non-unitarity due to a negative density of states near extremality. To restore unitarity, we introduce non-perturbative contributions to the Maloney-Witten partition function, resulting in a unitary, modular-invariant partition function with a spectral gap of $\Delta=(c-1)/12$. These new contributions can be interpreted as spinning strings and stringy black holes, whose entropy, though non-perturbatively suppressed, dominates near extremality and produces a positive density of states. This is suggestive of a possible UV string theory realization of AdS_3 gravity. The spectral gap of our solution $\Delta=(c-1)/12$ even at finite c , surpasses the previous state-of-the-art numerical bootstrap bound of $\Delta=c/9.08$.

In Chapter 4, we explore whether wormholes in 2D gravity can arise from entanglement between black-hole microstates rather than ensemble averaging. We study this using Matrix Quantum Mechanics (MQM), which can be interpreted as a quantization of the matrix integral dual to JT gravity. Thus it serves as a toy model for the entanglement between chaotic black hole microstates. We solve MQM for arbitrary spectral curves by developing a hydrodynamic effective theory. We show that ground state entanglement in MQM reproduces the non-perturbative sine-kernel and consequently the ramp-plateau transition in the SFF. Since these effects are captured by wormholes in JT gravity, this suggests they may be interpreted as emergent from the entanglement between black hole microstates. We further study the entanglement structure in MQM revealing a high degree of ground state entanglement, proportional to the black hole entropy.

Quantum Chaos in CFT and Black Holes

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Publications

This thesis is based on the following publications [1–3]:

- G. Di Ubaldo and E. Perlmutter, *AdS₃ Pure Gravity and Stringy Unitarity*, *Phys. Rev. Lett.* **132** (2023), no. 4 041602, [[arxiv:2308.017877](#)]
- G. Di Ubaldo and E. Perlmutter, *AdS₃/RMT₂ Duality*, *JHEP* **12** (2023) 179, [[arxiv:2307.03707](#)]
- G. Di Ubaldo and G. Policastro, *Ensemble averaging in JT gravity from entanglement in Matrix Quantum Mechanics*, *JHEP* **07** (2023) 122, [[arxiv:2301.02259](#)]

Chapter 1

Introduction

Black holes lie at the frontier of our understanding of Nature. More than a hundred years after Schwarzschild found the first black hole solution to Einstein's General Relativity, we still lack a complete quantum mechanical understanding of black holes. The most immediate puzzling feature of a black hole is the presence of a classical event horizon, a closed region in spacetime from which nothing can escape, not even light itself, hence the moniker *black hole*. Anything we throw past the horizon would seem to be lost forever, apparently. This feature, when considered together with the fundamental principles of quantum mechanics, results strikingly incompatible with the laws of physics we have discovered so far.

As our first thought experiment, suppose I were to burn this thesis: all of the information contained in it would be present in the form of extremely small correlations between the particles of the resulting smoke and ashes. Information is not destroyed. It might be impossible, for all intents and purposes, to reconstruct the content of my thesis, but it is in there. The underlying physical principle behind this thought experiment is determinism, which states that fully knowing the state of a system at any given instant of time allows us to predict the state at any other time. If we could collect all of the particles of smoke and carefully measure their positions and velocities, we could, in principle, run time backward and restore this manuscript.

Black holes make information unavailable to the rest of the universe, hiding it behind the horizon. Naively this might seem in contradiction with classical determinism, however the equations of General Relativity are time-reversal invariant so the theory is not self-contradictory. However, we will see that when we consider quantum mechanics in the presence of a black hole, the tension between the two will be stark and lead to the so-called black hole information problem.

The desire to reconcile black holes with the known laws of Nature has driven many theoretical physicists for roughly five decades, since the early seventies [4–14]. During this time, significant progress has been achieved and by now we have collected vast amounts of

evidence that black holes do indeed behave like quantum mechanical systems, at least for observers outside the event horizon. Information is believed not to be lost, instead encoded in the black hole microstates that make up the Bekenstein-Hawking entropy. Thanks to black hole evaporation, the information may be, in principle, reconstructed from Hawking radiation quanta [15]. Supporting evidence for this conclusion originates from three complementary approaches to quantum black holes.

String theory. String theory [16–19] is currently the leading candidate for a consistent, UV complete theory of quantum gravity. It aims to provide a full microscopic description of black hole microstates in terms of fundamental degrees of freedom, namely strings and branes, whose statistical mechanical entropy should account for black hole entropy. This reductionist approach is extremely non-trivial to achieve in practice, due to the difficulty of carrying out the necessary computations. Success has been achieved in particular supersymmetric cases, starting from the famous work of Strominger and Vafa [20].¹

AdS/CFT correspondence. The AdS/CFT correspondence [28–31] states that a quantum gravitational theory in an asymptotically Anti-de-Sitter (AdS) spacetime in $d + 1$ dimensions is exactly dual to a non-gravitational conformal field theory (CFT) in d dimensions. In this approach, gravity is emergent from the complex dynamics of a strongly coupled, large N quantum system, the CFT. In this sense, we might think of AdS/CFT as a prime example of the complexity paradigm introduced by Anderson’s *More is different* [32]. Anderson highlighted the importance of complexity in physics as opposed to the extremely successful reductionist paradigm in particle physics: even if we know the fundamental constituents of a complex system, collective behaviour can arise that would have been extremely unexpected and difficult to predict using the fundamental degrees of freedom. Analogously, a CFT a priori has nothing to do with gravity, black holes or extra dimensions, and yet it precisely encodes all of these phenomena. Gravity emerges as if it were a hologram produced from the CFT dynamics. Assuming the AdS/CFT correspondence, black holes are trivially consistent and evolve unitarily as quantum mechanical systems, since the dual CFT is a perfectly well-defined, unitary, quantum mechanical system. However, we do not understand the precise mechanisms necessary to describe a black hole from the CFT point of view. How can we describe unitary black hole evaporation precisely using the CFT? How does a CFT encode the experience of an observer falling behind the horizon? What is the role of the black hole singularity from the CFT point of view? The black hole information problem is solved in principle, by postulation of the AdS/CFT correspondence, but unsolved in practice, which has spurred a large amount of work in this direction, including this thesis.

Gravitational path integral. The third approach does not rely on any external constructions like string theory or CFT and instead strives to understand the Gravitational

¹There has been impressive progress in entropy counts for supersymmetric black holes since Strominger and Vafa, using string theory, AdS/CFT and more recently supergravity localization. We just mention a few reviews and a recent work where one can find a more up-to-date bibliography [21–27].

Path Integral (GPI) from a bottom up point of view [33]. The GPI is not well defined in general, meaning we don't know the precise rules to correctly compute observables. As a consequence, one is often focused on simpler toy models of gravity in two (or three) spacetime dimensions, where there are no UV divergences. In such models one can compute the path integral (almost) exactly and then (hope to) extend the lessons gained to other contexts.

One of the lessons recently learned is that for the GPI to produce answers consistent with quantum mechanics we should include wormholes in the path integral. However there is not a well grounded, a priori reason to include them in the GPI: a posteriori we see that by including them we obtain deep insights into black hole physics and solutions to many of its conundrums. Thus wormholes, while extremely insightful, are still rather exotic objects as we don't fully grasp why we should include them in the path integral. In particular, they connect different asymptotic boundaries of spacetime, consequently their contribution does not factorize. This leads to the so-called factorization problem since, in principle, observables should factorize in a microscopic theory. It is unclear how to understand wormhole configurations from the microscopic point of view, either in string theory or in the dual CFT.

One possibility that has emerged in the past few years is that the GPI does not behave as a full quantum mechanical theory but rather as a statistical ensemble of quantum theories of some kind. This is striking as it is apparently in tension with the microscopic descriptions provided by String Theory and the dual CFT, both of which are individual quantum systems. Understanding whether this is indeed the case, and if so, how we should modify the GPI to recover an individual quantum mechanical description is an active topic of research. Conversely, it is crucial to understand whether it is possible for an individual quantum mechanical system to exhibit an emergent behavior that is statistically equivalent to an average over an ensemble of systems. Studies in many-body quantum systems show that chaos is a possible way to achieve this. Random matrix universality of quantum chaotic systems is the statement that the Hamiltonian behaves statistically like a random matrix. Thus we are led to ask whether puzzling features of the GPI, such as wormholes, can be interpreted as emergent phenomena due to chaos in CFT or string theory. This is one of the questions we will address in this thesis.

The purpose of this thesis is to continue this scientific endeavor. In particular, we will explore connections between the three different approaches (String theory, AdS/CFT, GPI) and propose ways to resolve and understand gravitational puzzles, like wormholes, as emergent new phenomena in quantum mechanical systems.

1.1 Summary of thesis

In **chapter 1**, we introduce various topics relevant to the rest of the thesis. We start by reviewing some basic facts about black holes and introducing the black hole information problem. We proceed to discuss chaos and its relevance to black hole physics. We present two of the main notions of chaos in quantum systems and focus on a specific observable: the Spectral Form Factor (SFF). We review the phenomenology of this observable in chaotic systems, informed by random matrix universality. We review JT gravity as a toy model of quantum gravity and its SFF, which helps set expectations for the case of 3D gravity discussed in this thesis. We end by recalling some relevant facts about the spectrum of 2D CFTs and the symmetries that constrain it, which serve as a basis for chapter 2.

In **chapter 2** we explore and establish connections between random matrix universality, AdS₃ quantum gravity and chaotic 2D CFTs. We develop a framework for identifying and quantifying random matrix behavior of general 2d CFTs and theories of AdS₃ quantum gravity. On the gravity side, we consider as a specific example the Cotler-Jensen (CJ) wormhole of pure AdS₃ gravity. The CJ wormhole was recognized to be related to random matrix theory [34], since in the near-extremal limit it reduces to the JT gravity wormhole, which is equal to a matrix integral [35]. We understand and make precise the relation between the CJ wormhole and random matrices, identifying it as extremal in the space of AdS₃ wormhole amplitudes. We show that the CJ wormhole is equal to the minimal completion of random matrix theory compatible with Virasoro symmetry and $SL(2, \mathbb{Z})$ -invariance. This equality between a wormhole amplitude in AdS₃ and a matrix integral (suitably modified to accommodate the necessary symmetries) is suggestive of an AdS₃/RMT₂ duality, generalizing the JT/RMT duality to 3D [35].

On the CFT side, we explore the mechanism by which an individual chaotic CFT exhibits random matrix universality. In quantum mechanics, the relevant mechanism has been understood using a semiclassical expansion for the path integral: the Gutzwiller trace formula [36]. The trace formula expresses the density of states $\rho(E)$ as a sum over highly oscillatory contributions corresponding to semiclassical periodic orbits. The spectral statistics are then encoded in the interference between these oscillations. Using this framework, Berry reproduced the leading random matrix behaviour via the so-called diagonal approximation [37].

We present a 2d CFT trace formula, precisely analogous to the Gutzwiller trace formula for chaotic quantum systems. The CFT trace formula originates from the $SL(2, \mathbb{Z})$ spectral decomposition of the density of Virasoro primary states. We make the analogy precise by giving specific and, within some assumptions, unique identifications between the $SL(2, \mathbb{Z})$ basis of Eisenstein series and Maass cusp forms and the periodic orbits appearing in Gutzwiller's formula.

We establish the analog of Berry's diagonal approximation in 2d CFT using Hecke

operators for $SL(2, \mathbb{Z})$, resulting in the Hecke projection. This allows to extract spectral statistics of individual 2d CFTs by coarse-graining, and consequently to identify and quantify signatures of chaos and random matrix universality. We find a necessary and sufficient condition for a 2d CFT to display a linear ramp in its spectral form factor. The condition can be stated in terms of the microscopic CFT data as a certain asymptotic behavior in the $SL(2, \mathbb{Z})$ basis. It can be thought of as a sum rule valid in, and only in, chaotic CFTs.

Having explored independently the gravity and CFT manifestations of random matrix universality, we turn to establishing the AdS/CFT dictionary between the two. AdS_3 wormholes are cleanly interpreted as the gravity dual of the Hecke projection of the squared partition function in a microscopic 2d CFT. Schematically:

$$Z_{\text{WH}}(\tau_1, \tau_2) = \mathcal{P}_{\text{Hecke}}[Z_{\text{CFT}}(\tau_1)Z_{\text{CFT}}(\tau_2)] \quad (1.1.1)$$

This demonstrates that wormholes can emerge from an individual chaotic CFT. Non-factorization of the wormhole amplitude can be interpreted as coarse-graining, without the need to invoke an ensemble average over theories. The responsible physical mechanism is the CFT version of Berry's diagonal approximation.

From the holographic point of view, the Cotler-Jensen wormhole is the maximal realization of random matrix universality consistent with the necessary symmetries, a property we name **MaxRMT**. The notion of **MaxRMT** is similar in flavor to the notion of *maximal chaos* introduced by Maldacena, Shenker and Stanford [38]. The latter is expressed as saturation of the bound on the Lyapunov exponent which is an early time, semiclassical diagnostic of chaos. **MaxRMT** instead is expressed as the saturation of the regime of validity of random matrix universality, a late time, fully quantum diagnostic of chaos.

Finally, the completeness of the $SL(2, \mathbb{Z})$ spectral basis as a trace formula indicates a natural factorization of the Cotler-Jensen wormhole. The factorized partition function $Z_{\text{RMT}}(\tau)$ is a microscopic CFT partition function which, upon Hecke projection, produces the CJ wormhole.

This is a candidate for a single boundary contribution to the gravity path integral which captures fine details of the spectrum of BTZ black hole microstates. $Z_{\text{RMT}}(\tau)$ displays very erratic behaviour given by the Riemann zeta function, which suggests it may be interpreted as an AdS_3 half-wormhole. We end the chapter by discussing its implications for the dual CFT and modular bootstrap at large central charge.

In **chapter 3** we explore some well known non-perturbative problems in 3D gravity and the large c modular bootstrap and find an explicit solution.

Maloney and Witten computed the partition function of 3D gravity by summing over all on-shell, smooth geometries (plus loops) and yet found a non-unitary answer due to a negative density of states near extremality. For 3D gravity to be a consistent quantum theory,

it is necessary to include new configurations to the path integral which render the density of states positive, consistently with a unitary theory. A conformal bootstrap avatar of this problem is to either exclude or construct a positive, modular invariant partition function with a spectral gap of $\Delta_{\text{gap}} = \frac{c}{12}$ as in 3D gravity.

We constructed an explicit solution to these problems by adding non-perturbative contributions to the Maloney-Witten (MW) partition function, resulting in a positive, modular invariant partition function with a spectral gap of $\Delta_{\text{gap}} = \frac{c-1}{12}$.

The new contributions can be identified in the bulk as coming from spinning strings and corresponding stringy black holes. Their entropy is non-perturbatively suppressed with respect to the MW entropy except near extremality. In that regime, they dominate and produce a positive density of states. The stringy black hole entropy displays a qualitatively new oscillatory behaviour, as opposed to the exponential Cardy behaviour, indicating its quantum nature. Due to these oscillations, the stringy density is not positive definite by itself, however by combining it with the MW density one obtains an overall positive density.

The gap is exactly $\frac{c-1}{12}$, even down to finite c . At large c we can interpret the large gap as a result of the strings being strongly coupled, which lifts them to the Planck scale. This behaviour serves as a bottom up model to mimic a possible UV string theoretic realization of AdS₃ pure gravity.

From the CFT point of view, this represents the first example of a partition function with such a large gap. The previous state-of-the-art numerical bootstrap bound was $\frac{c}{9.1}$. The twist gap of the solution is also $\frac{c-1}{12}$ except for two states with large spin proportional to c .

The features of the solution highlight the importance of non-perturbative effects in the large N bootstrap, as the $N \rightarrow \infty$ often does not commute with solving bootstrap constraints. We comment on some curious number theoretic aspects of this solution.

In **chapter 4** we explore whether wormholes in 2D gravity can arise from entanglement between microstates, as opposed to ensemble averaging over theories. We show that ensemble averages over random matrices can arise from entanglement in Matrix Quantum Mechanics (MQM).

We consider a 0+1D theory of quantum mechanics with matrix degrees of freedom, interpreted as a quantization of the matrix integral dual to JT gravity. The fundamental difference with the latter is that the density of eigenvalues is promoted to a quantum mechanical operator $\hat{\rho}(E)$ acting on the Hilbert space of MQM. For example, the spectral curve is given by the leading large N ground state expectation value of the density operator $\rho_0(E) = \langle \hat{\rho}(E) \rangle$. The ground state of MQM is highly entangled which provides a toy model for the entanglement of typical black hole microstates.

We solve MQM for arbitrary spectral curves $\rho_0(E)$ by applying recently developed techniques for large N 1+1D quantum systems. This results in a hydrodynamical theory for

the density of eigenvalues. We proceed to quantize the fluctuations around the equilibrium density which results in a 2D free boson BCFT living on a curved background specified by the spectral curve.

The quantum hydrodynamical EFT is very practical and we compute various observables. We show that equal-time correlation functions in the ground state of MQM reproduce the known results of random matrix theory, including non-perturbative effects. In particular, we show that the two point function $\langle \hat{\rho}(E_1, t_1) \hat{\rho}(E_2, t_2) \rangle$ reproduces the non-perturbative sine-kernel of random matrix theory. In JT gravity, the sine kernel is reproduced by resumming wormhole contributions with arbitrary genus [39, 40] and produces the ramp-plateau transition in the spectral form factor. This suggests that wormholes in JT gravity may be interpreted as emergent from the entanglement between black hole microstates.

We proceed to study the entanglement structure of the ground state of MQM in detail. We compute the Rényi entropies associated to a general bipartition of the eigenvalues. We find a large amount of ground state entanglement, proportional to the microcanonical entropy $S = \log(\rho_0(E))$ and localized at the bipartition. This indicates that typical black hole microstates are highly entangled states, with each eigenstate being entangled with all of the other e^S eigenstates. The entropy follows a Page curve, as required by unitarity, only if the eigenvalues have compact support. Instead, by double-scaling a matrix model, we lose information about the high energy part of the spectrum and the Page curve grows forever. The Rényi entropies are naturally finite due to the mean eigenvalue spacing $\frac{1}{\rho_0}$ which acts as a UV cutoff. We also compute the reduced density matrix obtained by tracing out $k < N$ eigenvalues.

As an application, we examine the $c = 1$ MQM known to be dual to a non-critical string theory in two dimensional target space. We reproduce and extend significantly previous results by Hartnoll and Mazenc. From the string theory side, finiteness of the entropy can be understood as g_s effects regulating the divergence. The hydrodynamic effective theory provides a natural picture for the emergence of spacetime in 2D string theory due to condensation of eigenvalues and we derive a previously proposed map between eigenvalue-space and spacetime.

We end by commenting on a possible interpretation of MQM as the boundary dual of recently discussed Universe Field Theory of JT gravity.

1.2 Résumé étendu en Français

Le désir de concilier les trous noirs avec les lois connues de la Nature a conduit de nombreux physiciens théoriciens depuis le début des années soixante-dix [4–14]. Pendant ce temps, des progrès significatifs ont été réalisés et, à ce jour, nous avons recueilli une grande quantité de preuves que les trous noirs se comportent effectivement comme des systèmes mécaniques quantiques, du moins pour les observateurs à l’extérieur de l’horizon des événements. On pense que l’information n’est pas perdue, mais plutôt encodée dans les micro-états du trou noir qui composent l’entropie de Bekenstein-Hawking. Grâce à l’évaporation des trous noirs, l’information peut être, en principe, reconstruite à partir des quanta de radiation de Hawking [15]. Les preuves à l’appui de cette conclusion proviennent de trois approches complémentaires des trous noirs quantiques.

Théorie des cordes. La théorie des cordes [16–19] est actuellement le principal candidat pour une théorie cohérente et complète en UV de la gravité quantique. Elle vise à fournir une description microscopique complète des micro-états des trous noirs en termes de degrés de liberté fondamentaux, à savoir les cordes et les branes, dont l’entropie statistique devrait rendre compte de l’entropie des trous noirs. Cette approche réductionniste est extrêmement difficile à mettre en pratique, en raison de la complexité des calculs nécessaires. Des succès ont été obtenus dans des cas supersymétriques particuliers, à commencer par le célèbre travail de Strominger et Vafa [20].

Correspondance AdS/CFT. La correspondance AdS/CFT [28–31] stipule qu’une théorie gravitationnelle quantique dans un espace-temps asymptotiquement Anti-de-Sitter (AdS) en $d + 1$ dimensions est exactement duale à une théorie des champs conforme (CFT) non gravitationnelle en d dimensions. Dans cette approche, la gravité émerge de la dynamique complexe d’un système quantique fortement couplé et de grand N , la CFT. En ce sens, nous pourrions considérer AdS/CFT comme un exemple primordial du paradigme de la complexité introduit par Anderson dans *More is different* [32]. Anderson a souligné l’importance de la complexité en physique par opposition au paradigme réductionniste extrêmement réussi en physique des particules : même si nous connaissons les constituants fondamentaux d’un système complexe, un comportement collectif peut émerger qui aurait été extrêmement inattendu et difficile à prédire en utilisant les degrés de liberté fondamentaux. De manière analogue, une CFT a priori n’a rien à voir avec la gravité, les trous noirs ou les dimensions supplémentaires, et pourtant il encode précisément tous ces phénomènes. La gravité émerge comme un hologramme produit par la dynamique de la CFT. En supposant la correspondance AdS/CFT, les trous noirs sont trivialement cohérents et évoluent unitairement comme des systèmes mécaniques quantiques, puisque la CFT dual est un système quantique bien défini et unitaire. Cependant, nous ne comprenons pas les mécanismes précis nécessaires pour décrire un trou noir du point de vue de la CFT. Comment pouvons-nous décrire précisément l’évaporation unitaire d’un trou noir en utilisant la CFT ? Comment une CFT encode-t-il

l'expérience d'un observateur qui tombe derrière l'horizon ? Quel est le rôle de la singularité du trou noir du point de vue de la CFT ? Le problème de l'information des trous noirs est résolu en principe, par la postulation de la correspondance AdS/CFT, mais non résolu en pratique, ce qui a suscité de nombreux travaux dans cette direction, y compris cette thèse.

Intégrale de chemin gravitationnelle. La troisième approche ne repose sur aucune construction externe comme la théorie des cordes ou la CFT, mais cherche plutôt à comprendre l'intégrale de chemin gravitationnelle (GPI) d'un point de vue ascendant [33]. La GPI n'est généralement pas bien définie, ce qui signifie que nous ne connaissons pas les règles précises pour calculer correctement les observables. En conséquence, on se concentre souvent sur des modèles jouets plus simples de gravité en deux (ou trois) dimensions spatio-temporelles, où il n'y a pas de divergences UV. Dans de tels modèles, on peut calculer l'intégrale de chemin (presque) exactement et ensuite (espérer) étendre les leçons apprises à d'autres contextes. Une des leçons récemment apprises est que pour que la GPI produise des réponses cohérentes avec la mécanique quantique, nous devons inclure des trous de ver dans l'intégrale de chemin. Cependant, il n'y a pas de raison bien fondée a priori de les inclure dans la GPI : a posteriori, nous voyons qu'en les incluant, nous obtenons des informations profondes sur la physique des trous noirs et des solutions à bon nombre de leurs énigmes. Ainsi, les trous de ver, bien que très instructifs, sont encore des objets plutôt exotiques car nous ne comprenons pas entièrement pourquoi nous devrions les inclure dans l'intégrale de chemin. En particulier, ils connectent différentes frontières asymptotiques de l'espace-temps, par conséquent, leur contribution ne se factorise pas. Cela conduit au problème de la factorisation car, en principe, les observables devraient se factoriser dans une théorie microscopique. Il est difficile de comprendre les configurations de trous de ver du point de vue microscopique, que ce soit dans la théorie des cordes ou dans la CFT dual.

Une possibilité qui a émergé au cours des dernières années est que la GPI ne se comporte pas comme une théorie mécanique quantique complète mais plutôt comme un ensemble statistique de théories quantiques. Cela semble en contradiction avec les descriptions microscopiques fournies par la théorie des cordes et la CFT dual, qui sont toutes deux des systèmes quantiques individuels. Comprendre si c'est effectivement le cas et, dans l'affirmative, comment nous devrions modifier la GPI pour retrouver une description mécanique quantique individuelle est un sujet de recherche actif. Inversement, il est crucial de comprendre s'il est possible qu'un système mécanique quantique individuel présente un comportement émergent statistiquement équivalent à une moyenne sur un ensemble de systèmes. Des études sur des systèmes quantiques à plusieurs corps montrent que le chaos est une façon possible d'y parvenir. L'universalité des systèmes quantiques chaotiques est l'affirmation que l'hamiltonien se comporte statistiquement comme une matrice aléatoire. Nous sommes donc amenés à nous demander si les caractéristiques déroutantes de la GPI, telles que les trous de ver, peuvent être interprétées comme des phénomènes émergents dus au chaos dans la CFT ou la théorie des cordes. C'est l'une des questions que nous aborderons dans cette thèse.

Dans le **chapitre 1**, nous introduisons divers sujets pertinents pour le reste de la thèse. Nous commençons par examiner certains faits de base sur les trous noirs et par introduire le problème de l'information des trous noirs. Nous discutons ensuite du chaos et de sa pertinence pour la physique des trous noirs. Nous présentons deux des principales notions de chaos dans les systèmes quantiques et nous concentrons sur un observable spécifique : le Facteur de Forme Spectrale (FFS). Nous passons en revue la phénoménologie de cet observable dans les systèmes chaotiques, éclairée par l'universalité des matrices aléatoires. Nous examinons la gravité JT comme un modèle simplifié de la gravité quantique et son FFS, ce qui aide à fixer les attentes pour le cas de la gravité en 3D discuté dans cette thèse. Nous terminons en rappelant certains faits pertinents sur le spectre des CFTs 2D et les symétries qui le contraignent, ce qui sert de base pour le chapitre 2.

Dans le **chapitre 2**, nous explorons et établissons des connexions entre l'universalité des matrices aléatoires, la gravité quantique AdS_3 et les CFTs 2D chaotiques. Nous développons un cadre pour identifier et quantifier le comportement des matrices aléatoires des CFTs 2D générales et des théories de la gravité quantique AdS_3 . Du côté de la gravité, nous considérons comme exemple spécifique le trou de ver Cotler-Jensen (CJ) de la gravité pure AdS_3 . Le trou de ver CJ a été reconnu comme étant lié à la théorie des matrices aléatoires [34], puisqu'en limite quasi-extrémal, il se réduit au trou de ver de la gravité JT, qui est égal à une intégrale de matrice [35]. Nous comprenons et précisons la relation entre le trou de ver CJ et les matrices aléatoires, l'identifiant comme extrémal dans l'espace des amplitudes de trous de ver AdS_3 . Nous montrons que le trou de ver CJ est égal à la complétion minimale de la théorie des matrices aléatoires compatible avec la symétrie de Virasoro et l'invariance $SL(2, \mathbb{Z})$. Cette égalité entre une amplitude de trou de ver en AdS_3 et une intégrale de matrice (modifiée de manière appropriée pour accommoder les symétries nécessaires) suggère une dualité $\text{AdS}_3/\text{RMT}_2$, généralisant la dualité JT/RMT à 3D [35].

Du côté des CFTs, nous explorons le mécanisme par lequel une CFT chaotique individuelle exhibe l'universalité des matrices aléatoires. En mécanique quantique, le mécanisme pertinent a été compris en utilisant une expansion semi-classique pour l'intégrale de chemin : la formule de trace de Gutzwiller [36]. La formule de trace exprime la densité d'états $\rho(E)$ comme une somme sur des contributions fortement oscillatoires correspondant à des orbites périodiques semi-classiques. Les statistiques spectrales sont alors encodées dans l'interférence entre ces oscillations. En utilisant ce cadre, Berry a reproduit le comportement principal des matrices aléatoires via l'approximation diagonale [37].

Nous présentons une formule de trace pour les CFTs 2D, précisément analogue à la formule de trace de Gutzwiller pour les systèmes quantiques chaotiques. La formule de trace des CFTs 2D provient de la décomposition spectrale $SL(2, \mathbb{Z})$ de la densité des états primaires de Virasoro. Nous rendons l'analogie précise en fournissant des identifications spécifiques et, sous certaines hypothèses, uniques entre la base $SL(2, \mathbb{Z})$ des séries de Eisenstein et des formes de Maass et les orbites périodiques apparaissant dans la formule de Gutzwiller.

Nous établissons l’analogie de l’approximation diagonale de Berry dans les CFTs 2D en utilisant des opérateurs de Hecke pour $SL(2, \mathbb{Z})$, résultant en la projection de Hecke. Cela permet d’extraire les statistiques spectrales des CFTs 2D individuelles par agrégation grossière, et par conséquent d’identifier et de quantifier les signatures de chaos et d’universalité des matrices aléatoires. Nous trouvons une condition nécessaire et suffisante pour qu’une CFT 2D affiche une pente linéaire dans son facteur de forme spectrale. La condition peut être énoncée en termes de données microscopiques de la CFT comme un certain comportement asymptotique dans la base $SL(2, \mathbb{Z})$. Elle peut être considérée comme une règle de somme valable dans, et seulement dans, les CFTs chaotiques.

Après avoir exploré indépendamment les manifestations de l’universalité des matrices aléatoires dans la gravité et les CFTs, nous nous tournons vers l’établissement du dictionnaire AdS/CFT entre les deux. Les trous de ver AdS₃ sont interprétés proprement comme le dual gravitationnel de la projection de Hecke de la fonction de partition au carré dans une CFT 2D microscopique. Schématiquement :

$$Z_{\text{WH}}(\tau_1, \tau_2) = \mathcal{P}_{\text{Hecke}}[Z_{\text{CFT}}(\tau_1)Z_{\text{CFT}}(\tau_2)] \quad (1.2.1)$$

Cela démontre que des trous de ver peuvent émerger d’une CFT chaotique individuelle. La non-factorisation de l’amplitude de trou de ver peut être interprétée comme une agrégation grossière, sans avoir besoin d’invoquer une moyenne d’ensemble sur les théories. Le mécanisme physique responsable est la version CFT de l’approximation diagonale de Berry.

Du point de vue holographique, le trou de ver Cotler-Jensen est la réalisation maximale de l’universalité des matrices aléatoires compatible avec les symétries nécessaires, propriété que nous appelons **MaxRMT**. La notion de **MaxRMT** est similaire en saveur à la notion de *chaos maximal* introduite par Maldacena, Shenker et Stanford [38]. Cette dernière est exprimée comme la saturation de la borne sur l’exposant de Lyapunov, qui est un diagnostic semi-classique précoce du chaos. **MaxRMT** est plutôt exprimée comme la saturation du régime de validité de l’universalité des matrices aléatoires, un diagnostic de chaos pleinement quantique et tardif.

Enfin, la complétude de la base spectrale $SL(2, \mathbb{Z})$ en tant que formule de trace indique une factorisation naturelle du trou de ver Cotler-Jensen. La fonction de partition factorisée $Z_{\text{RMT}}(\tau)$ est une fonction de partition CFT microscopique qui, après projection de Hecke, produit le trou de ver CJ.

C’est un candidat pour une contribution de frontière unique à l’intégrale de chemin de gravité qui capture les détails fins du spectre des micro-états de trous noirs BTZ. $Z_{\text{RMT}}(\tau)$ affiche un comportement très erratique donné par la fonction zêta de Riemann, ce qui suggère qu’elle peut être interprétée comme un demi-trou de ver AdS₃. Nous terminons le chapitre en discutant de ses implications pour la CFT duale et le bootstrap modulaire à grande charge centrale.

Dans le **chapitre 3**, nous explorons certains problèmes non perturbatifs bien connus en gravité 3D et le bootstrap modulaire de grande c et trouvons une solution explicite.

Maloney et Witten ont calculé la fonction de partition de la gravité 3D en sommant sur toutes les géométries lisses (plus les boucles) en on-shell, et ont trouvé une réponse non unitaire en raison d'une densité d'états négative près de l'extrémalité. Pour que la gravité 3D soit une théorie quantique cohérente, nous devons inclure de nouvelles configurations dans l'intégrale de chemin qui restaurent l'unitarité. Un avatar en bootstrap conforme de ce problème est d'exclure ou de construire une fonction de partition unitaire, invariante modulaire avec un écart spectral de $\Delta_{\text{gap}} = \frac{c}{12}$ comme en gravité 3D.

Nous avons construit une solution explicite à ces problèmes en ajoutant des contributions non perturbatives à la fonction de partition Maloney-Witten (MW), résultant en une fonction de partition unitaire, invariante modulaire avec un écart spectral de $\Delta_{\text{gap}} = \frac{c-1}{12}$.

Les nouvelles contributions peuvent être identifiées dans le bulk comme provenant de cordes en rotation et de trous noirs "stringy" correspondants. Leur entropie est non perturbativement supprimée par rapport à l'entropie MW sauf près de l'extrémalité. Dans ce régime, ces contributions dominent et produisent une densité d'états positive. L'entropie des trous noirs stringy affiche un comportement oscillatoire qualitativement nouveau, contrairement au comportement exponentiel de Cardy, indiquant sa nature quantique. En raison de ces oscillations, la densité stringy n'est pas positive définie par elle-même, cependant en la combinant avec la densité MW, on obtient une densité globale positive.

L'écart est exactement $\frac{c-1}{12}$, même pour des valeurs finies de c . À grand c , nous pouvons interpréter le grand écart comme une conséquence du fort couplage des cordes, ce qui les élève à l'échelle de Planck. Ce comportement sert de modèle bottom-up pour imiter une possible réalisation théorique UV des cordes de la gravité pure AdS₃.

Du point de vue des CFTs, cela représente le premier exemple d'une fonction de partition avec un écart aussi grand. La précédente borne numérique à la pointe était de $\frac{c}{9.1}$. L'écart de twist de la solution est également $\frac{c-1}{12}$ sauf pour deux états avec un grand spin proportionnel à c .

Les caractéristiques de la solution soulignent l'importance des effets non perturbatifs dans le bootstrap à grand N , car le $N \rightarrow \infty$ ne commute souvent pas avec la résolution des contraintes de bootstrap. Nous commentons certains aspects curieux de théorie des nombres de cette solution.

Dans le **chapitre 4**, nous explorons si les trous de ver en gravité 2D peuvent émerger de l'intrication entre les micro-états, plutôt que d'une moyenne d'ensemble sur les théories. Nous montrons que les moyennes d'ensemble sur les matrices aléatoires peuvent découler de l'intrication dans la Mécanique Quantique des Matrices (MQM).

Nous considérons une théorie 0+1D de mécanique quantique avec des degrés de liberté

matriciels, interprétés comme une quantification de l'intégrale matricielle duale à la gravité JT. La différence fondamentale avec cette dernière est que la densité des valeurs propres est promue en un opérateur quantique $\hat{\rho}(E)$ agissant sur l'espace de Hilbert de la MQM. Par exemple, la courbe spectrale est donnée par la valeur moyenne sur l'état fondamental à grande N de l'opérateur de densité $\rho_0(E) = \langle \hat{\rho}(E) \rangle$. L'état fondamental de la MQM est fortement intriqué, ce qui fournit un modèle simplifié pour l'intrication des micro-états typiques des trous noirs.

Nous résolvons la MQM pour des courbes spectrales arbitraires $\rho_0(E)$ en appliquant des techniques récemment développées pour les systèmes quantiques 1+1D à grand N . Cela résulte en une théorie hydrodynamique pour la densité des valeurs propres. Nous procédons à la quantification des fluctuations autour de la densité d'équilibre, ce qui donne une BCFT de boson libre 2D vivant sur un fond courbé spécifié par la courbe spectrale.

La théorie EFT hydrodynamique quantique est très pratique et nous calculons diverses observables. Nous montrons que les fonctions de corrélation à temps égal dans l'état fondamental de la MQM reproduisent les résultats connus de la théorie des matrices aléatoires, y compris les effets non perturbatifs. En particulier, nous montrons que la fonction à deux points $\langle \hat{\rho}(E_1, t_1) \hat{\rho}(E_2, t_2) \rangle$ reproduit le noyau sinus non perturbatif de la théorie des matrices aléatoires. En gravité JT, le noyau sinus est reproduit en résumant les contributions des trous de ver avec un genre arbitraire [39, 40] et produit la transition rampe-plateau dans le facteur de forme spectrale. Cela suggère que les trous de ver en gravité JT peuvent être interprétés comme émergeant de l'intrication entre les micro-états des trous noirs.

Nous procédons à l'étude de la structure d'intrication de l'état fondamental de la MQM en détail. Nous calculons les entropies de Rényi associées à une bipartition générale des valeurs propres. Nous trouvons une grande quantité d'intrication dans l'état fondamental, proportionnelle à l'entropie microcanonique $S = \log(\rho_0(E))$ et localisée à la bipartition. Cela indique que les micro-états typiques des trous noirs sont des états fortement intriqués, chaque état propre étant intriqué avec tous les autres e^S états propres. L'entropie suit une courbe de Page, comme l'exige l'unité, uniquement si les valeurs propres ont un support compact. En revanche, en doublant un modèle matriciel, nous perdons des informations sur la partie haute énergie du spectre et la courbe de Page croît indéfiniment. Les entropies de Rényi sont naturellement finies en raison de l'espacement moyen des valeurs propres $\frac{1}{\rho_0}$ qui agit comme une coupure UV. Nous calculons également la matrice de densité réduite obtenue en traçant $k < N$ valeurs propres.

En application, nous examinons la MQM $c = 1$ connue pour être duale à une théorie des cordes non critique dans un espace cible bidimensionnel. Nous reproduisons et étendons considérablement les résultats précédents de Hartnoll et Mazenc. Du côté de la théorie des cordes, la finitude de l'entropie peut être comprise comme des effets g_s régulant la divergence. La théorie effective hydrodynamique fournit une image naturelle pour l'émergence de l'espace-temps dans la théorie des cordes 2D en raison de la condensation des valeurs propres et

nous dérivons une fonction précédemment proposée entre l'espace des valeurs propres et l'espace-temps.

Nous terminons en commentant sur une possible interprétation de la MQM comme le dual de frontière de la récente Universe Field Theory de la gravité JT.

1.3 Black holes

The first idea of a black hole predates General Relativity by more than a century. In premonitory strokes of imagination, Michell in 1783 and Laplace in 1796, independently presented the concept of an object whose gravity is so strong that not even light can escape it [41]. The mental picture of their 'invisible body' probably wouldn't be that far off from the first picture of a black hole published in 2019 by the Event Horizon Telescope (EHT), more than 230 years later [42].

The recent development of gravitational wave observatories has ushered black hole physics into the realm of experimental science. In 2015 the LIGO-VIRGO scientific collaboration achieved the first detection of a binary black hole merger [43]. This discovery has transformed black hole physics: today we are routinely able to observe black hole mergers and precisely match them with theoretical predictions made possible by the theory of General Relativity. The theory has been extensively tested experimentally over the past century starting from Eddington all the way to LIGO/VIRGO and EHT [44]. Promising future experiments such as LISA [45] will continue to test General Relativity to ever increasing accuracy in a variety of regimes and possibly shed light on new physics.

Black holes are a robust theoretical prediction of General Relativity. Oppenheimer and Snyder first showed that the collapse of matter leads to black hole formation [46]. Penrose and Hawking's singularity theorems showed that forming a singularity was an unavoidable mathematical consequence [4, 5, 14]. We should then take seriously the many puzzles one encounters when trying to make sense of black holes.

The problems that one faces when studying black holes and interpreting them as quantum mechanical systems arise from a fundamental incompatibility between describing a black hole as a smooth geometry and the characteristic discreteness of quantum mechanics. The many avatars of this fundamental tension go under the broad name of *Black hole information problem*. We will now introduce and describe a few selected aspects of this multifaceted problem. There are many wonderful reviews on the topic where further details and references can be found [15, 47–51].

Black hole entropy

Bekenstein and Hawking argued that black holes have an entropy which is proportional to their area [7–13, 52]:

$$S_{BH} = \frac{A}{4L_P^2}, \tag{1.3.1}$$

where $L_P = \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length. The entropy diverges as $\hbar \rightarrow 0$, which suggests that its origin is quantum mechanical. The area of the black hole depends on macroscopic properties such as charge Q , spin J and mass M . For example, for Kerr-Newmann black holes in four spacetime dimensions it is given by:

$$S_{BH} = \frac{4\pi}{4L_P^2} \left(r_+^2 + \left(\frac{J}{Mc} \right)^2 \right), \quad (1.3.2)$$

where $r_+ = GM/c^2 + \sqrt{(GM/c^2)^2 - (G^{1/2}Q/c^2)^2 - J^2/(Mc)^2}$ is the outer horizon. The entropy of any quantum mechanical system is given by Boltzmann's formula:

$$S = k_B \log W \quad (1.3.3)$$

where W is an integer equal to the number of microstates corresponding to a given macrostate. In classical statistical mechanics these could be, for example, all the positions and velocities of N indistinguishable particles such that given macroscopic quantities like energy or pressure are fixed. In quantum mechanics microstates are identified with states in the Hilbert space of the quantum system; entropy is a measure of the dimension of Hilbert space.

We are then immediately led to ask what are the black hole microstates that the Bekenstein-Hawking entropy is counting. The metric $g_{\mu\nu}$ of a black hole and its entropy S_{BH} depend only on the macrostate, namely the total mass M , charge Q and angular momentum J , as exemplified above by the Kerr-Newmann solution. This property has come to be known by the statement "black holes have no hair". Since, in General Relativity, all the information that specifies a black hole is a few numbers, the theory is unable to discern the discreteness of black hole microstates and count them. The first indication of the thermodynamical origin of black hole entropy was obtained by Hawking when he discovered that black holes are not entirely black. Instead, when coupled to quantum fields, they emit thermal radiation with a temperature given by:

$$T_{BH} = \frac{\hbar c}{2\pi} \frac{r_+ - GM/c^2}{r_+^2 + J^2/(Mc)^2} \quad (1.3.4)$$

The way Hawking arrived at this result (for $J = Q = 0$) was by quantizing a quantum field in the presence of a background black hole, see [47] for a modern derivation. The black hole then evaporates by emitting purely thermal blackbody radiation at temperature T_{BH} . In doing so, the black hole loses mass due to evaporation.

Understanding that black holes are thermodynamical objects sheds some light on possibly the simplest (classical) version of the black hole information problem: do black holes violate the laws of thermodynamics? It is possible to show that black holes satisfy the first law of

thermodynamics:

$$dM = \Omega dJ + \Phi dQ + T_{BH} dS_{BH} \quad (1.3.5)$$

where

$$\Omega = \frac{J/(Mc)}{r_+^2 + J^2/(Mc)^2} \quad (1.3.6)$$

is the angular frequency of a test particle dropped into the horizon and

$$\Phi = \frac{r_+ G^{1/2} Q/c^2}{r_+^2 + J^2/(Mc)^2} \quad (1.3.7)$$

is the potential energy of a test charge at the horizon.

The second law of thermodynamics states that the total change in entropy of a system must be positive, namely entropy must always increase. Black holes can satisfy this law by virtue of their entropy, so if we throw an object of mass δM and entropy S into a black hole of mass M we must have:

$$S_{BH}(M + \delta M) \geq S + S_{BH}(M). \quad (1.3.8)$$

The second law of thermodynamics in the presence of black holes is non-trivial and can be used to derive a constraint on the maximum amount of entropy that any system can have, the Bekenstein bound [6]:

$$S \lesssim \frac{2\pi k_B R E}{\hbar c} \quad (1.3.9)$$

where E and R are the rest energy and the radius of the system. The bound follows from recognising that if one could have a system with entropy larger than the bound, it would be possible to lower the total entropy by forming a black hole, see [53] for a review. These simple arguments are not enough to claim that black holes do satisfy the laws of thermodynamics and much work has been since devoted to this topic, see [54, 55]. As an example, there has been recent progress in understanding the third law of thermodynamics for black holes. The third law states that as one goes to zero temperature, the entropy of a system should become a constant independent of the macrostate. The entropy and temperature of a black hole coupled to quantum fields do not satisfy the third law, as the validity of Hawking's calculation breaks down. This requires a more careful study of near-extremal black holes with a very small temperature, where one must correctly account for quantum gravitational effects [56, 57].

When we consider black holes as quantum mechanical systems, beyond classical statistical

mechanics, the black hole information problem takes its true form. In quantum mechanics time evolution is unitary, meaning that if we have a system initially described by a density matrix $\rho(0)$, time evolution is given by:

$$\rho(t) = U\rho(0)U^\dagger, \quad U = e^{-\frac{iHt}{\hbar}} \quad (1.3.10)$$

where U is a unitary matrix generated by the Hamiltonian H and a generic density matrix $\rho(0)$ is defined as a statistical ensemble of states in the Hilbert space:

$$\rho \equiv \sum_{k=1}^{\dim(\mathcal{H})} p_k |k\rangle \langle k|. \quad (1.3.11)$$

Suppose now we form a black hole by collapsing matter in a pure state. Quantum mechanical evolution is unitary and cannot evolve a pure state into a mixed state. Hawking computed the density matrix of the radiation emitted by a black hole and found it to be thermal. Thus Hawking radiation is in a mixed state, regardless of the state of the black hole interior and the matter that formed it by collapse.² The violation of unitarity is at the core of the black hole information problem and it appears time and again in observables other than the density matrix.

Computing a density matrix in gravity is difficult, one should compute a number of entries of order $\dim(\mathcal{H})^2 \approx e^{2S_{BH}} \sim e^{G_N^{-1}}$ which is exponentially large, with enough accuracy to distinguish it from the thermal density matrix. It turns out that the measurements of an observable in a pure state versus a mixed state are exponentially close to each other, as we will now review. As a consequence, measuring a density matrix involves an exponential number of measurements down to exponentially small accuracy, which is clearly beyond the scope of any calculation in QFT on curved spacetimes a la Hawking. It is a genuine, non-perturbative quantum gravity calculation.

Pure and mixed states are very close

We sketch a simple derivation in quantum mechanics which is very relevant to black hole physics and semiclassical attempts to understand Hawking radiation [47]. Suppose we are in a subspace of Hilbert space \mathcal{H}_E of dimension e^S obtained by acting with a projector P . The maximally mixed density matrix in that subspace is

$$\rho_E = \frac{1}{e^S} \sum_{i=1}^{e^S} |i\rangle \langle i|. \quad (1.3.12)$$

²See [47] for a modern review of Hawking's calculation

Consider now a random pure state which is a superposition of eigenstates in the same subspace:

$$|\Psi\rangle = \sum_i^{e^S} a_i |i\rangle \quad (1.3.13)$$

where the complex coefficients a_i are uniformly distributed such that the state has unit norm:

$$d\mu = \frac{1}{V} \delta\left(\sum_i |a_i|^2 - 1\right) \prod_i d^2 a_i \quad (1.3.14)$$

and V is a normalization factor. Suppose now we measure an operator A in both the state $|\Psi\rangle$ and the mixed ensemble ρ_E . We see that the average expectation value of A in the state $|\Psi\rangle$ is equal to the expectation value in the mixed state:

$$\int d\mu \langle \Psi | A | \Psi \rangle = \int d\mu \sum_{i,j}^{e^S} a_i a_j^* A_{i,j} = \frac{1}{e^S} \sum_i A_{ii} = \text{Tr}(\rho_E A) \quad (1.3.15)$$

where we used that

$$\int d\mu a_i a_j^* = \frac{1}{e^S} \delta_{ij}. \quad (1.3.16)$$

Now consider the variance between measuring A in a random pure state and measuring it in the maximally mixed state:

$$\begin{aligned} \int d\mu [\langle \Psi | A | \Psi \rangle - \text{Tr}(\rho_E A)]^2 &= \int \sum_{i \neq j, \ell \neq m} A_{ij} A_{\ell,m} a_i a_j^* a_\ell a_m^* \\ &= \int d\mu \sum_{i \neq j} |A_{i,j}|^2 |a_i|^2 |a_j|^2 = \frac{1}{e^S(e^S + 1)} \sum_{i \neq j} |A_{ij}|^2 \\ &\leq \frac{\sigma_{A,\rho}^2}{e^S + 1} \end{aligned} \quad (1.3.17)$$

where we used that $\int d\mu |a_i|^2 |a_j|^2 = \frac{1}{e^S(e^S + 1)}$ and zero for other index contractions. Moreover the inequality comes from the following fact:

$$\frac{1}{e^S} \sum_{i \neq j} |A_{ij}|^2 = \text{Tr}[\rho(PAP)^2] - \text{Tr}(\rho A)^2 \leq \text{Tr}(\rho A^2) - \text{Tr}(\rho A)^2 \equiv \sigma_{A,\rho}^2. \quad (1.3.18)$$

We conclude then that a measurement of an operator A on a typical pure state $|\Psi\rangle$ in a subspace differs only by an exponentially small amount from measurement in the maximally mixed state ρ_E . Thus the two expectation values may differ significantly only for an exponentially small number of pure states in the subspace.

This elementary result sheds light on Hawking radiation, showing that Hawking’s calculation is not precise enough to establish any kind of paradox. To render Hawking radiation pure instead of thermal we expect we need to add exponentially small corrections $e^{-S} \sim e^{-\frac{1}{G_N}}$, which are non-perturbative in Newton’s constant. At this order of magnitude the notion of a smooth geometry might break down and it’s not clear what gravity even is. There has been much work on precisely formulating a paradox involving black holes and solving it, in particular studying a simpler quantity than the density matrix, the Von Neumann entropy:

$$S_{VN}(\rho) = -\text{Tr}(\rho \log \rho). \tag{1.3.19}$$

We will not review the many developments in this direction since they are not necessary to understand the rest of this thesis and we direct the reader to the reviews [15, 47–51]. We will instead focus on a different version of the information problem first introduced by Maldacena [58].

Before doing so let us comment on a naive puzzle regarding pure state black holes. The entropy of a pure state is zero. However, a black hole has an entropy. How is it possible to have a black hole which is in a pure state? The answer lies in the fact that one should not confuse Von Neumann entropy and thermodynamic entropy. The Von Neumann entropy of a pure state is vanishing. However, it is possible to assign a thermodynamic entropy to a system in a pure state due to ignorance of the precise state. In gravity, it is not possible to identify a specific pure state since doing so would require exponentially complex measurements. Instead, we can only measure low-point correlators of the metric and fields to few orders in G_N . The situation is analogous to the statistical mechanics of a gas with N particles. A gas is perfectly deterministic, its particles at any given time have specific positions and velocities, thus the state of the gas is pure. However, we do assign an entropy to gas, given by how many configurations (microstates) of the gas are available at a given energy, pressure, and volume. When we say that a gas has entropy, we are not saying that the gas is in a statistically mixed state, we are assigning an entropy due to the inability of precisely measuring the state. Thus, the existence of a horizon and its relative entropy for a pure state black hole is a direct consequence of there being many pure states consistent with the macroscopic description of the black hole.

1.4 Chaos

A calculation of a complex quantity like the density matrix of Hawking radiation or the precise microstates of a black hole is almost certainly out of reach, however we can study simpler quantities which still display the fundamental tension between black holes and quantum mechanics. Maldacena first stated a version of the black hole information problem in terms of the two point function of a simple operator as a function of time [58]. In a quantum mechanical system we can write the two point function as follows:

$$G(t) = \frac{1}{Z} \text{tr}(e^{-\beta H} O(t) O(0)) = \frac{1}{Z} \sum_{n,m=1}^{e^S} e^{-it(E_n - E_m)} e^{-\beta E_n} |\langle E_n | O | E_m \rangle|^2 \quad (1.4.1)$$

In AdS/CFT this would be a thermal two-point function of a local operator in the dual CFT. In gravity this object corresponds to the amplitude for a particle in the presence of a black hole to be detected outside the horizon at a later time t . Since things tend to fall into the black hole and they cannot escape back out, one can easily guess that this two point function decays to zero forever. This is confirmed by explicit computations of quantum fields in a black hole background [58]. In a quantum mechanical theory, however, this observable cannot decay forever. It will decay until it is of order e^{-S} with respect to its initial value at $t = 0$. We can see this by taking the long time average of the above:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt G(t) = \frac{1}{Z} \sum_n e^{-\beta E_n} |O_{nn}|^2 \quad (1.4.2)$$

where we implicitly assumed that the energy levels are non-degenerate. This is the case for generic quantum systems unless they enjoy particular symmetries. Since, as we will see, black holes are chaotic systems, it is a well justified assumption. We can also compute the long time average of the square which tells us about fluctuations around the mean value:

$$\frac{1}{T} \int_0^T dt |G(t)|^2 = \frac{1}{Z^2} \sum_{i,j,i',j'} e^{-\beta(E_i + E_{i'})} |O_{ij}|^2 |O_{i'j'}|^2 (\delta_{ij} \delta_{i'j'} + \delta_{ii'} \delta_{jj'}) \quad (1.4.3)$$

We see that there is a nonzero variance and so $G(t)$ cannot decrease monotonically to zero at late times as the black hole calculation would imply. The fact that correlation functions computed in a black hole background decay to zero is Maldacena's version of the black hole information problem. This problem can also be studied in a simpler observable which does not involve the matrix elements of O , the so-called Spectral Form Factor:

$$K_\beta(t) = Z(\beta + it) Z(\beta - it) = \sum_{i,j=1}^{e^S} e^{-\beta(E_i + E_j)} e^{-it(E_i - E_j)} \quad (1.4.4)$$

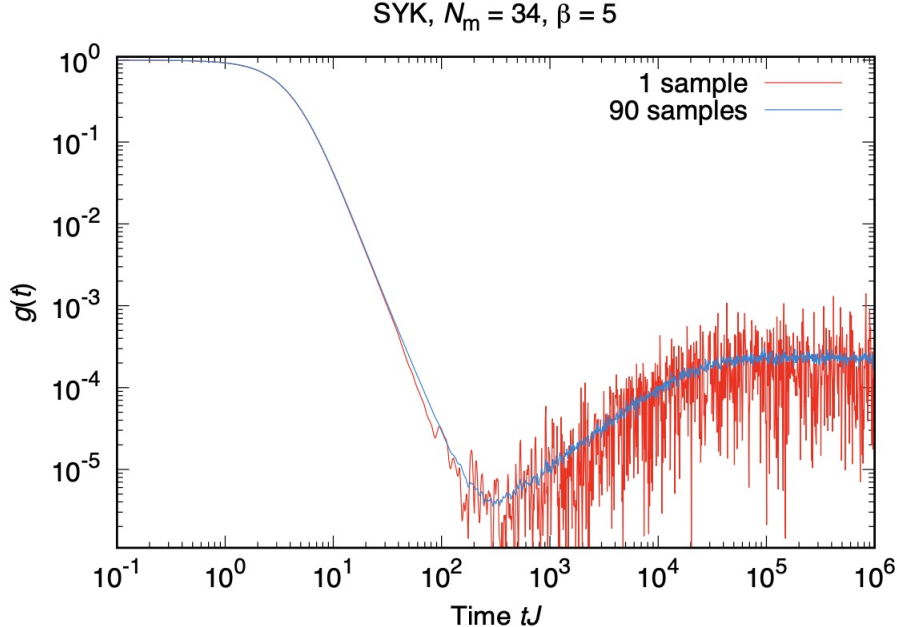


Figure 1.1: The typical behavior of the SFF for a chaotic quantum system, in agreement with random matrix universality. The plot shows the SFF for the SYK model for $N = 34$, $\beta = 5$. The red erratic curve is an individual sample of the model’s random couplings while the blue line is the average over 90 realizations. The figure is taken from [64].

where $Z(\beta) = \text{Tr}(e^{-\beta H})$. The long time average is equal to:

$$\frac{1}{T} \int_0^T dt K_\beta(t) = Z(2\beta) \quad (1.4.5)$$

which is exponentially suppressed with respect to the initial value $K_\beta(0) = Z(\beta)^2$. Without more information about the matrix elements O_{ij} or the energy levels E_i , we cannot say much about these quantities. We can make more precise predictions for the behavior of this observable by observing that black holes are chaotic objects. In chaotic systems, there is a large degree of universality in the statistical behavior of microscopic quantities E_i and O_{ij} . These go by the name of Random Matrix Universality [59, 60] for the energy levels E_i and Eigenstate Thermalization Hypothesis (ETH) [61–63] for the matrix elements O_{ij} . In this thesis, whenever we refer to quantum chaos we will be referring implicitly to random matrix universality and not to ETH, simply because ETH was not an object of study of this thesis even though it is a very rich, interesting and popular subject of study. In chaotic quantum systems, there is a common universal behavior expected for $G(t)$ and $K_\beta(t)$ shown in Fig. 1.1 for $K_\beta(t)$.

Notice that initially $K_\beta(t)$ decays smoothly until an exponentially small value, consistently with the black hole calculation. However, after the so-called dip time, it starts growing again

on average and large erratic fluctuations appear. At exponentially late times, it reaches a plateau where it is constant and exponentially small on average, with a large noise of the same order as the average. The outstanding challenge is then to understand how gravity reproduces this behavior, as expected of a quantum mechanical system. Before focusing on the SFF any further, we should review what chaos is and why it is relevant for black holes.

1.4.1 Semiclassical chaos and scrambling

The notion of chaos in classical mechanics is the fact that a small change in the initial conditions of a system produces a large change at later times. Classical chaos is then the sensitivity of the system to small changes or more poetically, the 'butterfly effect' [65]. This is diagnosed by the following quantity:

$$\frac{\partial q(t)}{\partial q(0)} = e^{\lambda t} \quad (1.4.6)$$

where λ is the Lyapunov exponent and controls the exponential growth of perturbations. The study of classical chaotic systems is an old and rich subject; see, for example, [66]. The above observable can be written as a Poisson bracket:

$$\{q(t), p(0)\} \equiv \frac{\partial q(t)}{\partial q(0)} \quad (1.4.7)$$

A natural generalization to quantum mechanics of this quantity follows by canonical quantization where the Poisson bracket is promoted to a commutator

$$i\hbar\{q(t), p(0)\} \rightarrow [q(t), p]. \quad (1.4.8)$$

It will be convenient to study the commutator squared instead of the commutator itself:

$$C(t) \equiv -\langle [W(t), V(0)]^2 \rangle_\beta \quad (1.4.9)$$

where we evaluate the expectation value in the thermal ensemble $\langle \cdot \rangle = Z^{-1} \text{tr}[e^{-\beta H} \cdot]$. In the classical limit where we recover the Poisson bracket, if we choose $W = q$ and $V = p$ we have schematically that:

$$C(t) \sim \hbar^2 e^{2\lambda_L t}. \quad (1.4.10)$$

We assume that $V(0), W(0)$ are simple operators, meaning that they involve at most a finite product of $\mathcal{O}(1)$ degrees of freedom. For example, in a spin chain of size N they would involve only products of spin operators involving $\mathcal{O}(1)$ spins.

The physical picture of this observable is that it measures how a perturbation due to an operator V affects later measurements of W . In a many-body quantum system, if the initial operators $V(0)$ and $W(0)$ are localized on different degrees of freedom, separated by a distance r , they will commute, so at early times, the commutator will be very small. As the operator $W(t) = e^{-iHt}W(0)e^{iHT}$ spreads throughout the system due to time evolution, it will start interacting with the degrees of freedom on which V is supported and the commutator will grow. In a chaotic system, we expect that time evolution will be very efficient at spreading the operator $W(t)$ throughout the system, and the commutator will grow exponentially fast. This early growth is often of ballistic type in chaotic systems and takes the form:

$$C(r, t) = \frac{1}{N} e^{\lambda(t-r/v_B)} \quad (1.4.11)$$

where N in general counts the number of degrees of freedom in the system, such as the number of spins or the central charge c_T in CFT. The parameter v_B is called the butterfly velocity and it describes the speed of propagation of the operator W in the system. At late times we expect this observable to saturate at a value of order:

$$C(t) \rightarrow 2\langle VV \rangle \langle WW \rangle, \quad t \gtrsim t_* \quad (1.4.12)$$

where $\langle VV \rangle$ and $\langle WW \rangle$ are normalized to $\mathcal{O}(1)$ values. The time t_* is called the scrambling time and it is the time at which $C(t)$ becomes of order $\mathcal{O}(1)$. In the classical limit we have:

$$t_* \sim \frac{1}{\lambda_L} \log\left(\frac{1}{\hbar}\right). \quad (1.4.13)$$

It can be convenient to work with a closely related observable, considered by Maldacena, Shenker and Stanford in [38]:

$$F(t) = \text{Tr}(yV yW(t) yV yW(t)), \quad y^4 = \rho_\beta = \frac{e^{-\beta H}}{Z}, \quad (1.4.14)$$

where we have split the thermal density matrix into four inequivalent insertions, one after each operator insertion. This observable can be related to a regulated version of the commutator squared. The characteristic behavior of this correlator can be similarly understood in terms of operator spreading. At early times $F(t)$ is of order $\mathcal{O}(1)$ and decreases exponentially. In a large N CFT it can be computed holographically by studying high energy scattering near the black hole horizon [67] which produces:

$$F(t) = f_0 - \frac{f_1}{N^2} e^{\frac{2\pi}{\beta} t} + \mathcal{O}(N^{-4}), \quad (1.4.15)$$

where f_0, f_1 are order $\mathcal{O}(1)$ constants which depend on the choice of V, W . We see that the

scrambling time is given by:

$$t_* = \frac{\beta}{2\pi} \log(N^2), \quad (1.4.16)$$

similarly to the one we obtain in the semiclassical limit of single particle systems with $\hbar \rightarrow \frac{1}{N^2}$ and a Lyapunov exponent given by

$$\lambda_{grav} = \frac{2\pi}{\beta}. \quad (1.4.17)$$

We arrive at the conclusion that black holes are chaotic systems, in the sense defined as scrambling operators under time evolution. They actually behave not just like any arbitrary chaotic system, but there is a sense in which they are 'maximally' chaotic. The conjecture made by MSS [38] states that the Lyapunov exponent, as defined by $F(t)$ at large N for any many-body quantum system, is bounded by the black hole Lyapunov exponent:

$$\lambda_L \leq \frac{2\pi}{\beta}. \quad (1.4.18)$$

Black holes saturate this bound and have the largest possible Lyapunov exponent.

1.4.2 Quantum chaos and random matrices

The Lyapunov exponent we just discussed is a semiclassical diagnostic of chaos. It is defined at large N and can be computed perturbatively in $\frac{1}{N}$. It is conceptually related to the classical Lyapunov exponent of a chaotic dynamical system. Quantum systems, however, are much richer and have a much more complex structure than what is captured by the Lyapunov exponent. In particular, we would like to understand the notion of ergodicity in quantum mechanics.

One way to do so is through the idea of random matrix universality of chaotic quantum systems. If one writes down a generic interacting Hamiltonian, it won't be exactly solvable, nor will it have any particular symmetries. Wigner and later Dyson introduced the idea that one can effectively model the Hamiltonian of a complex system as a random matrix. Wigner was trying to understand the energy levels of heavy nuclei observed experimentally. Making a theoretical prediction was extremely difficult since already the simplest case of the Helium atom is a quantum version of the 3-body problem, which is notoriously chaotic. Wigner proposed instead to look at the statistics of the energy levels and compare them to those obtained by an ansatz where the Hamiltonian is a random Hermitian $N \times N$ matrix with Gaussian-distributed entries:

$$\langle \cdot \rangle = \frac{1}{\mathcal{Z}} \int dH \cdot e^{-\frac{N}{2} \text{Tr} H^2}. \quad (1.4.19)$$

Instead of trying to precisely model the specific Hamiltonian, one examines the statistical properties of the spectrum. This involves coarse-graining the microscopic (experimental or numerical) data, for example by binning energy levels into histograms or by smoothing noisy behavior in observables with some weak form of averaging such as averaging over a small time window δt .

In chaotic systems, the Hamiltonian is so complex that studying individual energy levels is not just difficult but also not particularly rewarding since the spectrum is practically featureless, except for some macroscopic coarse-grained properties such as the smooth approximation to the density of states $\bar{\rho}(E)$. A random matrix with independently distributed entries is the most featureless ansatz that one can use as a benchmark to diagnose whether the system is chaotic. In particular, in quantum mechanics, we do not expect degeneracies in interacting systems unless there is a symmetry responsible for the degeneracy due to the phenomenon of avoided crossing of energy levels.

Until this point the statement of random matrix universality has been presented only qualitatively, let us make it more precise. To understand the spectral statistics of a quantum system we look at the density of states:

$$\rho(E) \equiv \sum_i \delta(E - E_i) \quad (1.4.20)$$

and its Laplace transform, the partition function:

$$Z(\beta) = \sum_i e^{-\beta E_i} = \int \rho(E) e^{-\beta E}. \quad (1.4.21)$$

We will be interested in quantum systems with many degrees of freedom, namely a large microcanonical density of states $\bar{\rho}(E) \gg 1$ in an energy interval $E \pm \delta E$. One can obtain the microcanonical density by coarse graining the exact density of states over a large enough window δE :

$$\bar{\rho}(E) = \int_{E-\delta E}^{E+\delta E} \rho(E) dE. \quad (1.4.22)$$

We can define the microcanonical entropy as $S(E) = \log \bar{\rho}(E)$. The Cardy density in 2D CFT is an example of a coarse-grained density of states:

$$\rho_{\text{Cardy}}(h, \bar{h}) = \exp\left(4\pi\sqrt{\xi(h-\xi)}\right) \exp\left(4\pi\sqrt{\xi(\bar{h}-\xi)}\right), \quad \xi \equiv \frac{c-1}{24}. \quad (1.4.23)$$

The spectrum of the 2D CFT is discrete while the Cardy density captures the leading continuous approximation to this dense, yet discrete, spectrum. The density of states in chaotic systems is usually system dependent and there is no random matrix universality

involved. In fact, the density is an input that one should specify to compare the system to a random matrix. The ensemble of random matrices should be chosen with a measure such that the average density of matrix eigenvalues is equal at leading order in N to the microcanonical density of states $\bar{\rho}$. This corresponds to considering random matrices with arbitrary potentials:

$$\langle \cdot \rangle = \frac{1}{\mathcal{Z}} \int dH \cdot e^{-N\text{Tr}V(H)}. \quad (1.4.24)$$

The leading large N average density of eigenvalues is often called the spectral curve of the matrix model:

$$\rho_0(E) = \langle \rho(E) \rangle_{N \rightarrow \infty} \quad (1.4.25)$$

To compare a chaotic quantum system to a random matrix, we choose the spectral curve such that:

$$N\rho_0(E) = \bar{\rho}(E). \quad (1.4.26)$$

We will use $N = e^{S_0}$ from now on, to denote the fact that the dimension N of the matrix is the size of the Hilbert space, which is exponential in the entropy of the system. We are now interested in understanding the spectral statistics, namely the correlations between different energy levels. In random matrix theory, these are described by the connected two-point correlation function of the density of eigenvalues:

$$\langle \rho(E_1)\rho(E_2) \rangle_{\text{conn}} = \langle \rho(E_1)\rho(E_2) \rangle - \langle \rho(E_1) \rangle \langle \rho(E_2) \rangle \quad (1.4.27)$$

This quantity has a universal form in the limit where two eigenvalues are close $|E_1 - E_2| \ll 1$, given by the famous sine-kernel:

$$\langle \rho(E_1)\rho(E_2) \rangle_{\text{conn}} = -\frac{\sin^2(\pi e^{S_0} \rho_0(E)(E_1 - E_2))}{\pi^2 (E_1 - E_2)^2} + e^{S_0} \rho_0(E) \delta(E_1 - E_2). \quad (1.4.28)$$

This is possibly the deepest result of random matrix theory. We see that it depends only on the spectral curve $\rho_0(E)$ of the matrix model.

It captures the phenomenon of eigenvalue repulsion, namely that eigenvalues of a random matrix effectively behave as particles with a repulsive interaction. For eigenvalue separation of order of the level spacing $E_1 - E_2 \ll e^{-S_0}$, the eigenvalues are strongly anti-correlated, while the correlation becomes weaker as the separation between eigenvalues increases.

In a large e^{S_0} expansion, the sine in the numerator is doubly-non-perturbative in the entropy S_0 since it behaves as $e^{ie^{S_0}}$. Thus, it is invisible to all orders in the e^{-S_0} expansion, and all

that one sees is a double pole at $E_1 = E_2$. We can see this by writing the sine kernel as:

$$\langle \rho(E_1)\rho(E_2) \rangle_{\text{conn}} = -\frac{1}{2\pi^2(E_1 - E_2)^2} + \frac{1}{2\pi^2(E_1 - E_2)^2} \cos\left(2\pi e^{S_0} \rho_0(E)(E_1 - E_2)\right) \quad (1.4.29)$$

The first term is perturbative in e^{-S_0} , while the second one is non-perturbative and oscillates very strongly around zero. Thus, perturbatively one sees only the first term. The contact term $\delta(E_1 - E_2)$ is universal since any eigenvalue is correlated with itself.

The statement of random matrix universality is that the correlations between energy levels of a chaotic system follow the sine-kernel at small energy differences of order of the level spacing:

$$\overline{\rho(E_1)\rho(E_2)} \approx -\frac{\sin^2(\pi\bar{\rho}(E)(E_1 - E_2))}{\pi^2(E_1 - E_2)^2}, \quad E_1 - E_2 \sim \mathcal{O}\left(\frac{1}{\bar{\rho}(E)}\right) \quad (1.4.30)$$

A different way to state random matrix universality is in terms of the Spectral Form Factor, introduced in the previous section. We can use the sine-kernel to derive an expression for the spectral form factor. First consider the microcanonical spectral form factor [39, 40]:

$$K_E(T) = \int d\epsilon e^{i\epsilon T} \langle \rho(E + \epsilon/2)\rho(E - \epsilon/2) \rangle_{\text{conn}} \quad (1.4.31)$$

where $\epsilon = E_1 - E_2$. Using the sine-kernel, we obtain an expression valid for time-scales $t \sim e^{S_0}$:

$$K_E(T) = \min\left(\frac{T}{2\pi}, e^{S_0} \rho_0(E)\right) \quad (1.4.32)$$

This is the ramp-plateau structure of the SFF for chaotic systems, see fig. 1.1. The SFF presents a linearly growing in time behavior until the transition to the plateau, which happens at $T = 2\pi e^{S_0} \rho_0(E)$, after which it becomes constant. The linear ramp corresponds to the first perturbative term $1/\epsilon^2$ in the sine-kernel, as can be seen by the Fourier transform. The purpose of the second oscillatory term is to precisely cancel the linear growth after a time $t = e^{S_0} \rho_0(E)$. Finally, the contact term sets the height of the plateau to be $e^{S_0} \rho_0(E)$. We show the relative time behaviors of the Fourier transforms here:

$$-\frac{1}{2\pi^2(E_1 - E_2)^2} \longrightarrow \frac{t}{2\pi}, \quad (1.4.33)$$

$$\frac{\cos\left(2\pi e^{S_0} \rho_0(E)(E_1 - E_2)\right)}{2\pi^2(E_1 - E_2)^2} \longrightarrow \begin{cases} -e^{S_0} \rho_0(E) & \text{if } t < 2\pi e^{S_0} \rho_0(E) \\ -\frac{t}{2\pi}, & t \geq 2\pi e^{S_0} \rho_0(E), \end{cases} \quad (1.4.34)$$

$$e^{S_0} \rho_0(E) \delta(E_1 - E_2) \longrightarrow e^{S_0} \rho_0(E). \quad (1.4.35)$$

Adding all the terms together we obtain the expression for $K_E(T)$. We stress the importance of each term because each of them encodes different physics, and together, they give a universal understanding of quantum chaotic systems. The linear ramp with a precise slope of $\frac{1}{2\pi}$ is a signature of chaos in a quantum mechanical system.

The oscillations on the scale of the level spacing are instead a probe of the discreteness of the spectrum, they would not be present if the spectrum was smooth like $\bar{\rho}(E)$. Thus the plateau in the SFF is an indirect probe of the underlying discreteness of the spectrum of the system, however it is non-perturbative and difficult to access. Even in random matrix theory, the calculation of the plateau is non-trivial. You can see from the above expression for $K_E(T)$ that it is not possible to reach the plateau starting from the ramp and doing perturbation theory, as the expression is not even analytic at the transition.

In two very interesting recent papers [39, 40], it was shown how to obtain the plateau by resumming a convergent expansion around the ramp. Start by considering the SFF in the canonical ensemble $K_\beta(T)$. It was shown that $K_\beta(T)$ in random matrix theory admits a τ -scaling limit where $e^{S_0} \rightarrow \infty$ with $\tau = e^{-S_0} T$ is fixed, given by:

$$\lim_{S_0 \rightarrow \infty} e^{-S_0} K_\beta(\tau e^{S_0}) = \int_0^\infty dE e^{-2\beta E} \min\left(\frac{\tau}{2\pi}, \rho_0(E)\right) \quad (1.4.36)$$

A notable feature of this expression is that it admits a convergent expansion in $\tau \ll 1$, which is given by:

$$\int_0^\infty dE e^{-2\beta E} \min\left(\frac{\tau}{2\pi}, \rho_0(E)\right) = \frac{\tau}{4\pi\beta} + \sum_{g=1} P_g^{(\rho)}(\beta) \tau^{2g+1} \quad (1.4.37)$$

where the coefficients are given by

$$P_g^{(\rho)}(\beta) = -\frac{1}{g(2g+1)(2\pi)^{2g+1}} \oint_0 \frac{dE}{2\pi i} \frac{e^{-2\beta E}}{\rho_0(E)^{2g}}. \quad (1.4.38)$$

It is possible to resum this series inside the radius of convergence and then analytically continue it to arbitrary τ . This will give an expression for the SFF which includes the ramp and the plateau. There are deep mathematical structures hidden behind this expression, explored in [40]. It relies on very specific cancellations at every order in the e^{-S_0} expansion of the matrix integral whenever one takes the τ -scaling limit.

Chaos and symmetry

One important aspect to consider when comparing a chaotic system to random matrix theory is symmetry. A system may be chaotic and yet enjoy a particular symmetry, which is not strong enough to make the system completely integrable. The distinction between chaotic and integrable systems is a complicated subject and it's not clear a priori if a symmetry is enough to make a system integrable. There is no absolute criterion to distinguish and classify physical systems into integrable and chaotic, since there are systems which may exhibit features of both classes.

However, in many interesting cases, a system (or a subsystem of a larger system) lies at one end of the spectrum, meaning it either exhibits most of the defining features of chaos (such as non-zero Lyapunov exponents, random matrix statistics, ETH, etc...) or it is exactly solvable. These systems which are 'maximally' chaotic or 'fully' integrable are an easier playground to understand various phenomena. Often one can think of a mixed system, one with characteristics which are intermediate between chaos and integrability, as having different approximate subsectors which are chaotic or integrable and coexist in one theory. We will discuss $\mathcal{N} = 4$ super Yang-Mills theory as an example of a mixed system, but first, let's understand better the interplay between symmetry and chaos.

We are used to the fact that physical systems can have a wealth of symmetries: spacetime symmetries such as rotational invariance or Lorentz symmetry, gauge symmetries such as electric charge conservation or global symmetries such as \mathbb{Z}_2 symmetry of magnets described by the Ising model, just to give a few elementary examples. None of these symmetries, by itself, is enough to completely solve the system. A system with any of these symmetries may still be chaotic, however the Hamiltonian cannot possibly behave like a random matrix due to the symmetry constraints.

Consider as an example, $SU(2)$ symmetry in quantum mechanics. The Hamiltonian commutes with the three generators $[H, J_i] = 0$ so we organize states $|E, \ell, m\rangle$ by their eigenvalues with respect to H , the Casimir J^2 with eigenvalue $\ell(\ell + 1)$ and one of the generators J_z with eigenvalue $m = -\ell \dots \ell$. Since J^2 and J_z commute with the Hamiltonian we have a selection rule: the Hamiltonian will not connect states with different values of ℓ or m .

This means that the Hamiltonian is of block diagonal form where each block has fixed ℓ, m but possibly different energies. Moreover, we can have degeneracies, namely states with the same energy but different quantum numbers. For each value of E, ℓ we have $2\ell + 1$ states with $m = -\ell, \dots, \ell$. Due to these two features, it is clear that the Hamiltonian cannot be purely a random matrix, since random matrices, as we defined them, are neither block diagonal nor they have degeneracies. However, the $SU(2)$ symmetry does not constrain the actual values of the energies of the highest weight states $m = \ell$ inside a block. This is where we should look for random matrix universality. We should first consider a single fixed spin block of the

Hamiltonian and inside that block look only at the energies of highest-weight states $|E, \ell, \ell\rangle$.

The previous example teaches us the general procedure to establish whether a quantum mechanical system exhibits random matrix universality. We should first understand the symmetries of the system and then study a sector where the matrix elements of the Hamiltonian are unconstrained by the symmetries. Once we have correctly accounted for the symmetries, we can then see whether the system is chaotic or not. Often this is simple to do, such as in the case of $SU(2)$ symmetry where one just needs to work in an individual Hamiltonian block, but more complicated symmetries can be present for which it is not possible to do so.

In this thesis, we will consider 2D Conformal Field Theories. These theories enjoy an infinite dimensional symmetry algebra, the Virasoro algebra, and thus there are infinitely many conserved charges. However, we don't expect all 2D CFTs to be exactly solvable (though many of them are). They also enjoy a special symmetry called modular invariance. This symmetry relates the partition function of the theory at different values of the temperature and angular potential, similarly to a high/low-temperature duality. We will see that this symmetry relates different spin sectors so that the CFT Hamiltonian is no longer block diagonal in spin. This symmetry also relates the low energy and high energy parts of the spectrum so energies across the spectrum are correlated in a nontrivial way. To understand random matrix universality in 2D CFT we must disentangle the energy levels from the constraints of modular invariance.

There are many physically interesting quantum field theories with a rich symmetry structure that we expect to display signatures of chaos. Let's consider the example of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 4D. This is a maximally supersymmetric field theory and it is also conformally invariant. The spectrum organizes itself into representations of the superconformal algebra $PSU(2, 2|4)$ which has an $SO(6)$ global R-symmetry. As a consequence of this symmetry, for example, $\frac{1}{2}$ BPS states have integer conformal dimensions which are clearly not random matrix distributed. The theory also enjoys S-duality in the complexified coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$, meaning that theories with values of the coupling related by $SL(2, \mathbb{Z})$ are dual to each other.

Even though the theory enjoys such a large amount of symmetry, we still expect it to be chaotic and exhibit random matrix universality, at least in certain sectors of the spectrum. One piece of evidence for this chaotic behavior is given by the AdS/CFT duality. $\mathcal{N} = 4$ super Yang Mills theory is conjectured to be exactly dual to type IIB string theory on an asymptotically $AdS_5 \times S^5$ spacetime. The latter theory contains black holes which, as we reviewed, are chaotic. We then expect to see the chaos of black holes in the CFT.

A different, broader perspective, is that an interacting system with a large number of degrees of freedom, which has a sector unconstrained by symmetry, should obey RMT statistics in that sector. If, instead, the system is still integrable, there should be a good explanation for this atypical behaviour, such as a symmetry that we didn't take into account

previously. From this point of view, we expect that the non-BPS, superconformal primary states at fixed spin and R-charge in $\mathcal{N} = 4$ super Yang Mills at large but finite N and finite coupling τ should have RMT statistics. We expect a similar behavior for the t'Hooft limit at strong coupling $\lambda = g_{YM}^2 N$.³ Studying this explicitly by computing anomalous dimensions is of course very difficult, see [68] for recent progress.

On the other hand, we know well other sectors of the theory, and we see that they are integrable, not chaotic. There has been extensive work at leading order in large N where the theory displays a rich integrable structure and it's been possible to compute the spectrum of non-BPS operators with $\Delta \sim \mathcal{O}(N^0)$ exactly in λ . This does not invalidate the previous discussion, since integrability appears strictly at $N = \infty$ and, so far, it has been shown to apply only to light operators that involve a small number of degrees of freedom compared to the central charge $c_T = \frac{N^2-1}{4}$ which is formally infinite in this limit.

1.4.3 The SFF in gravity

There has been a lot of interest in the SFF for the spectrum of black hole microstates. It's a quantity that allows to probe characteristics of the black hole microstates such as chaos, given by the linear ramp, and discreteness, given by the plateau, without having to explicitly compute the microstates E_i which is, in general, beyond our reach.

We will now very briefly review aspects of the SFF from the 2D gravity point of view. In higher dimensions $d \geq 4$, the geometry that has been proposed to capture the ramp is the double-cone, an analytical continuation of the eternal two sided black hole. We will not review it here but the interested reader can consult [69]. As for $d = 3$, the geometry that captures the ramp is the Cotler-Jensen wormhole [34] which will be one of the main topics of this thesis.

We review the theory of 2D quantum gravity named Jackiw-Teitelboim gravity, following [35]. For more details, see the review [70]. The Euclidean action is given in terms of the metric $g_{\mu\nu}$ and a dilaton field ϕ by:

$$I = -\frac{S_0}{2\pi} \left[\frac{1}{2} \int_{\mathcal{M}} \sqrt{g} R + \int_{\partial\mathcal{M}} \sqrt{h} K \right] - \frac{1}{2} \int_{\mathcal{M}} \sqrt{g} \phi (R + 2) + \int_{\partial\mathcal{M}} \sqrt{h} \phi (K - 1) \quad (1.4.39)$$

The first square bracket is a topological term and it's equal to $-S_0 \chi(\mathcal{M})$, the Euler characteristic of the surface \mathcal{M} . The second term is a bulk dilaton term which can be integrated out exactly which imposes $R = -2$ off-shell. The theory describes hyperbolic two-dimensional surfaces. These are Riemann surfaces $\Sigma_{g,n}$ and they are fully classified by their genus g and the number of boundaries n , with a continuous moduli space for fixed g, n . The final term gives a non-trivial dynamics on the boundary of spacetime and can be shown to be equal to

³See section 9 of [64] for a discussion on this topic.

Schwarzian quantum mechanics on the S^1 boundaries of \mathcal{M} . The observables we want to study are the partition functions on $\Sigma_{g,n}$ with asymptotic boundaries. To arrive at a finite result one needs to perform holographic renormalization by introducing regularized lengths β_i/ϵ and regularized boundary dilaton $\phi = 1/2\epsilon$. The partition function on a surface with a single boundary and genus 0, the disk, is equal to the Schwarzian partition function:

$$Z_{\text{Sch}}(\beta) = \frac{e^{\frac{\pi^2}{\beta}}}{4\sqrt{\pi}\beta^{3/2}} \quad (1.4.40)$$

which produces a microcanonical density of states:

$$\rho_{\text{Sch}}(E) = \frac{1}{4\pi^2} \sinh(2\pi\sqrt{E}). \quad (1.4.41)$$

The partition functions admit a topological expansion:

$$Z_n(\beta_1, \dots, \beta_n) = \sum_{g=0} \frac{Z_{g,n}(\beta_1, \dots, \beta_n)}{e^{S_0(2g+n-2)}}. \quad (1.4.42)$$

The way to compute the partition functions $Z_{g,n}$ is to decompose the Riemann surface $\Sigma_{g,n}$ as follows. For each asymptotic boundary, we choose a geodesic homologous to the boundary and cut the surface along the geodesic. We are left with n trumpet geometries which connect the asymptotic boundary of length β_i to the geodesic boundary of length b_i . The interior of the spacetime is now a Riemann surface with n geodesic boundaries. Having performed this decomposition, we weigh the trumpets according to the path integral of the Schwarzian theory on the trumpet, which gives:

$$Z_{\text{Tr}}(\beta, b) = \frac{e^{-\frac{b^2}{4\beta}}}{2\sqrt{\pi}\beta}. \quad (1.4.43)$$

The interior Riemann surface with n geodesic boundaries is weighted by the volume of its moduli space $V_{g,n}(b_1, \dots, b_n)$. The measure on moduli space space, which follows from the JT path integral, is the Weil-Petersson measure. The geometry of moduli spaces of bordered Riemann surfaces is a deep topic of current research in pure mathematics and there are beautiful reviews on the topic [71, 72]. We will now see that crucial mathematical results in this are at the core of the solution of JT gravity. The last step to compute $Z_{g,n}$ is to glue back together the trumpets to the bordered Riemann surface along the geodesics boundaries. The measure for gluing them can be derived to be $d\mu(b) = bdb$. We arrive at the following expression:

$$Z_{g,n}(\beta_1, \dots, \beta_n) = \int_0^\infty b_1 db_1 \cdots \int_0^\infty b_n db_n V_{g,n}(b_1, \dots, b_n) Z_{\text{Tr}}(\beta_1, b_1) \cdots Z_{\text{Tr}}(\beta_n, b_n). \quad (1.4.44)$$

Note that the geometries we are considering, whose partition function $Z_{g,n}$ can be exactly computed, are not solutions to the equation of motion of the theory, they are not saddle-points. They are off-shell configurations which normally we wouldn't know how to include in the gravitational path integral. However, in the case of JT gravity, the theory is so simple that one can compute the path integral exactly on a fixed topology. The computation of $Z_{g,n}$ boils down to the knowledge of $V_{g,n}$. Mirzakhani proved that the Weil-Petersson volumes satisfy a recursion relation which allows to compute them explicitly [73]. This is known as topological recursion, see [72] for a review.

It is possible to compute the SFF by analytically continuing the $n = 2$ boundary partition function:

$$K_\beta^{JT}(T) = Z_2(\beta + iT, \beta - iT). \quad (1.4.45)$$

At leading order in e^{S_0} this is equal to gluing two trumpets together, hence the name double trumpet and gives:

$$Z_{0,2}(\beta_1, \beta_2) = \int b db Z_{\text{Tr}}(b, \beta_1) Z_{\text{Tr}}(b, \beta_2) = \frac{\sqrt{\beta_1 \beta_2}}{2\pi(\beta_1 + \beta_2)} \quad (1.4.46)$$

This result is recognized to be exactly equal to the two-point function of a matrix integral, at leading order at large $N = e^{S_0}$:

$$Z_{0,2}(\beta_1, \beta_2) = \langle \text{Tr} e^{-\beta_1 H} \text{Tr} e^{-\beta_2 H} \rangle_{g=0}. \quad (1.4.47)$$

Under analytic continuation and by taking $T \gg \beta$ this produces the ramp in the SFF a signature of quantum chaos in 2D gravity.

$$K_\beta^{JT}(T) \approx \frac{T}{4\pi\beta} + \mathcal{O}(e^{-2S_0}), \quad T \gg \beta. \quad (1.4.48)$$

Random matrix universality, as we reviewed in the previous section, is the statement that a chaotic quantum system exhibits random matrix behavior on time scales $T \sim e^{S_0}$. We emphasize that JT gravity goes beyond this by being exactly equal to a matrix integral at finite β_1, β_2 . In [35] it was shown that all the $Z_{g,n}$ partition functions of JT gravity are equal to correlation functions of a certain random matrix model and so the full theory is dual to a matrix integral, to all orders in e^{S_0} :

$$Z_n^{JT}(\beta_1, \dots, \beta_n) = \langle \text{Tr} e^{-\beta_1 H} \dots \text{Tr} e^{-\beta_n H} \rangle. \quad (1.4.49)$$

This follows by showing that the topological recursion of Mirzakhani for $V_{g,n}$ can be exactly mapped to the topological recursion for matrix integrals by Eynard and Orantin, [74]. In

particular, the matrix model is specified by

$$N = e^{S_0}, \quad \rho_0(E) = \rho_{\text{Sch}}(E). \quad (1.4.50)$$

One can interpret the matrix model as an ensemble of quantum systems with Hamiltonian H that we average over. From this point of view, JT gravity is then dual, not to an individual quantum system like in AdS/CFT, but to an ensemble average over many quantum mechanical systems.

1.5 Conformal Field Theory

Conformal field theory is a large and well-developed subject, which has been a major focus of theoretical physicists for decades due to its applications in disparate fields, from statistical mechanics and phase transitions to quantum gravity via the AdS/CFT correspondence. Major progress has been achieved since the revival of the conformal bootstrap approach, which aims to solve or constrain either a specific theory or the space of possible theories by exploiting the stringent consistency conditions a CFT must satisfy. I will not attempt to review this progress, there are many excellent reviews for CFT in $d > 2$ such as [75–79]. The work in this thesis focuses mostly on CFTs in $d = 2$. These have been studied in depth, especially ones that can be solved exactly, see [80–84] for excellent reviews. I will now briefly recall some aspects that will be relevant for the rest of the thesis. In particular, we will focus on the partition function of 2D CFTs, which depends only on the spectrum of local operators (h, \bar{h}) and not on OPE coefficients C_{ijk} . Since we will be interested in understanding signatures of chaos in 2D CFT, such as random matrix universality, we need to understand what are the symmetries that act on the spectrum, as emphasized in previous sections.

CFTs in $d = 2$ enjoy a larger symmetry than the global conformal algebra $so(d + 1, 1)$, the infinite-dimensional Virasoro algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}. \quad (1.5.1)$$

A generic 2D CFT is symmetric under two commuting copies of the Virasoro algebra $Vir \times \overline{Vir}$ for the left and right movers with generators:

$$L_n \equiv \frac{1}{2\pi i} \oint dz z^{n+1} T(z), \quad \bar{L}_n \equiv \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}), \quad (1.5.2)$$

where $T(z) = T_{zz}(z)$ and $\bar{T}(\bar{z}) = \bar{T}_{\bar{z}\bar{z}}(\bar{z})$ are the two non-vanishing components of the stress energy tensor. They are respectively holomorphic and antiholomorphic due to the conservation of the stress tensor. The stress tensor is traceless due to scale invariance $T_{z\bar{z}} = 0$. The central charge c appearing as a central extension in the Virasoro algebra is defined in terms of the stress energy tensor two-point function:

$$\langle T(z)T(0) \rangle = \frac{c/2}{z^4}, \quad \langle \bar{T}(\bar{z})T(0) \rangle = \frac{c/2}{z^4}. \quad (1.5.3)$$

We can classify states in the 2D CFT by their eigenvalues with respect to L_0 and \bar{L}_0 which we denote by (h, \bar{h}) :

$$L_0 |h\rangle = h |h\rangle. \quad (1.5.4)$$

The scaling dimension, meaning the eigenvalue with respect to the dilation operator $D = L_0 + \bar{L}_0$ and the spin $J = L_0 - \bar{L}_0$ are:

$$\Delta = h + \bar{h}, \quad j = h - \bar{h} \quad (1.5.5)$$

The rest of the generators of the Virasoro algebra act like raising and lowering operators:

$$[L_0, L_{-n}] = nL_{-n}. \quad (1.5.6)$$

We define a primary state as a state whose dimension cannot be lowered further:

$$L_n |h\rangle_P = 0, \quad \forall n > 0 \quad (1.5.7)$$

Thus the states in the theory are split into primaries and descendants. We will drop the subscript and denote primaries as $|h, \bar{h}\rangle$. Descendants are obtained by:

$$L_{-n_1} \dots L_{-n_k} \bar{L}_{-\bar{n}_1} \dots \bar{L}_{-\bar{n}_k} |h, \bar{h}\rangle. \quad (1.5.8)$$

We can already see that if we wish to find any signatures of random matrix behavior in a 2D CFT, we should not include descendants since their dimensions are $h_D = h_P + \sum_i^k n_i$, exactly correlated with the primary dimensions and integer spaced. Consider now the partition function of a 2D CFT at finite temperature β^{-1} and on a finite interval with periodic boundary conditions. This is equal to putting the CFT on a torus with modular parameter τ :

$$\tau = x + iy = \frac{\theta}{2\pi} + i\frac{\beta}{2\pi}, \quad (1.5.9)$$

where θ is the angular potential. The partition function is then equal to:

$$Z(\tau, \bar{\tau}) = \sum_{h, \bar{h} \in \mathcal{S}} e^{2\pi i \tau (h - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{h} - \frac{c}{24})}, \quad (1.5.10)$$

where the sum runs over all the operators in the spectrum \mathcal{S} , both primaries and descendants. There is a shift in energy at finite volume:

$$E_L = L_0 - \frac{c}{24}. \quad (1.5.11)$$

We have introduced a complex variable $\bar{\tau}$ which allows us to analytically continue beyond the Euclidean signature where $\bar{\tau} = \tau^*$, the complex conjugate of τ . The partition function, in general, is a non-holomorphic function of τ , thus depending on 2 real variables x, y . Upon analytic continuation, we have that $Z(\tau, \bar{\tau})$ is a function of 2 independent complex variables.

In Euclidean signature we have:

$$Z(\tau, \tau^*) = \sum_{h, \bar{h} \in \mathcal{S}} e^{-2\pi y \Delta} e^{2\pi i j x}, \quad (1.5.12)$$

which is the usual grand canonical partition function in quantum mechanics. It's possible to sum all the contributions to the partition function coming from descendant of a given primary. This is achieved by the Virasoro characters:

$$\chi_h(\tau) \equiv \text{Tr}_h(q^{h - \frac{c}{24}}), \quad q = e^{2\pi i \tau}, \quad (1.5.13)$$

where the trace sums only over states in a given representation of highest weight h of the Virasoro algebra. It is possible to compute these characters explicitly since the descendants at level N correspond to partitions of N . For the Virasoro algebra we obtain:

$$\chi_h(\tau) = \frac{q^{h-\xi}}{\eta(\tau)} \quad (1.5.14)$$

where $\xi = \frac{c-1}{24}$ and $\eta(\tau)$ is the Dedekind eta function. We can then write the partition function as:

$$Z(\tau, \bar{\tau}) = \sum_{h, \bar{h} \in \mathcal{P}} \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}), \quad (1.5.15)$$

where the sum now runs only over primaries. We are now ready to discuss the second crucial symmetry which acts on the CFT spectrum: modular invariance. When considering a CFT on the torus, scale invariance implies that observables are independent of the volume of the torus. Thus, observables in the CFT can only depend on the shape of the torus, not its volume.⁴ The torus can be defined by taking the complex plane and identifying points $w \in \mathbb{C}$ by a lattice action with fundamental vectors α_1, α_2 :

$$w \sim w + n\alpha_1 + m\alpha_2, \quad m, n \in \mathbb{Z}, \quad \alpha_1, \alpha_2 \in \mathbb{C} \quad (1.5.16)$$

The shape of the lattice, meaning the angle and the ratio of the sides of a lattice cell, is encoded in the parameter τ :

$$\tau = \frac{\alpha_2}{\alpha_1} \quad (1.5.17)$$

It is clear that there are different choices of vectors α_1, α_2 and β_1, β_2 which define the same

⁴More mathematically, it depends only on the complex structure moduli of T^2 and not the Kahler moduli.

lattice and thus the same torus. These are related by $SL(2, \mathbb{Z})$ transformations:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad (1.5.18)$$

Since tori whose moduli τ are related by $SL(2, \mathbb{Z})$ transformations are identical, the CFT cannot possibly distinguish them, and thus observables must be invariant (or covariant) under $SL(2, \mathbb{Z})$ transformations. In particular, the partition functions must be modular invariant:

$$Z(\tau, \bar{\tau}) = Z(\gamma\tau, \gamma\bar{\tau}). \quad (1.5.19)$$

This is an extremely powerful symmetry that leads to a wealth of results in 2D CFT. It is possible to work in a fundamental domain for the $SL(2, \mathbb{Z})$ action on τ , given by:

$$\mathcal{F} = \left\{ -\frac{1}{2} \leq x \leq \frac{1}{2}, y \geq 0, |\tau| \geq 1 \right\}. \quad (1.5.20)$$

From the physical point of view, it is more of a high-temperature/low-temperature duality than a symmetry, since for the value $\tau = i\frac{\beta}{2\pi}$ it implies:

$$Z(\beta) = Z\left(\frac{4\pi^2}{\beta}\right). \quad (1.5.21)$$

If low and high temperatures are related, it means that the high energy and low energy parts of the spectrum must be related as well. Moreover, different spin sectors are related as well. It is not a symmetry in the conventional sense of having a generator that commutes with the Hamiltonian, and thus, we cannot diagonalize both simultaneously. This means that to understand random matrix statistics in 2D CFT, we will first have to understand how to correctly process modular invariance.

The most famous result about 2D CFTs that can be derived via modular invariance is the Cardy density of states. This is a formula for the asymptotic density of high energy states in a 2D CFT with $c > 1$:

$$\rho_{\text{Cardy}}(h, \bar{h}) = \exp\left(4\pi\sqrt{\xi(h-\xi)}\right) \exp\left(4\pi\sqrt{\xi(\bar{h}-\xi)}\right), \quad \xi \equiv \frac{c-1}{24}. \quad (1.5.22)$$

which is valid asymptotically for $h, \bar{h} \rightarrow \infty$ and fixed central charge c , the so-called Cardy regime.

Chapter 2

AdS₃/RMT₂ duality

This chapter consists of a paper [1] written in collaboration with Eric Perlmutter. The original abstract is as follows:

We introduce a framework for quantifying random matrix behavior of 2d CFTs and AdS₃ quantum gravity. We present a 2d CFT trace formula, precisely analogous to the Gutzwiller trace formula for chaotic quantum systems, which originates from the $SL(2, \mathbb{Z})$ spectral decomposition of the Virasoro primary density of states. An analogy to Berry's diagonal approximation allows us to extract spectral statistics of individual 2d CFTs by coarse-graining, and to identify signatures of chaos and random matrix universality. This leads to a necessary and sufficient condition for a 2d CFT to display a linear ramp in its coarse-grained spectral form factor.

Turning to gravity, AdS₃ torus wormholes are cleanly interpreted as diagonal projections of squared partition functions of microscopic 2d CFTs. The projection makes use of Hecke operators. The Cotler-Jensen wormhole of AdS₃ pure gravity is shown to be extremal among wormhole amplitudes: it is the minimal completion of the random matrix theory correlator compatible with Virasoro symmetry and $SL(2, \mathbb{Z})$ -invariance. We call this **MaxRMT**: the maximal realization of random matrix universality consistent with the necessary symmetries. Completeness of the $SL(2, \mathbb{Z})$ spectral decomposition as a trace formula allows us to factorize the Cotler-Jensen wormhole, extracting the microscopic object $Z_{\text{RMT}}(\tau)$ from the coarse-grained product. This captures details of the spectrum of BTZ black hole microstates. $Z_{\text{RMT}}(\tau)$ may be interpreted as an AdS₃ half-wormhole. We discuss its implications for the dual CFT and modular bootstrap at large central charge.

2.1 Introduction

To establish whether AdS₃ pure gravity exists, one must understand the random matrix behavior of its black hole microstates.

Such is the view suggested by recent work on holographic duality in low dimensions, both for the AdS_3 quantum theory and its semiclassical limit. Perhaps the main justification comes from the celebrated JT/RMT duality in two bulk dimensions [35], in which the boundary theory is an ensemble of (double-scaled) random matrices. This work (and its dilaton gravity generalizations, e.g. [85–89]) was a combined evolution of earlier works drawing direct connections between black hole dynamics and random matrix statistics in AdS/CFT [64, 69] and the emergence of the SYK model as a tractable yet strongly-coupled quantum system [90–94]. In higher-dimensional holography, the boundary theory is a continuum conformal field theory (CFT), endowed with extra structure. For 2d CFTs, this structure includes locality, Virasoro symmetry and modular invariance of the torus partition function, but more generally is comprised of some set of fundamental bootstrap axioms. How is random matrix theory (RMT) “allowed” to manifest itself in the observables of an individual 2d CFT while respecting the necessary constraints?

We focus our attention on $\text{AdS}_3/\text{CFT}_2$ henceforth, and the ongoing quest for AdS_3 pure gravity.¹ As is well-known, the natural idea [97] for computing the semiclassical bulk path integral with a single torus boundary (sum over all smooth bulk saddles \mathcal{M} with $\partial\mathcal{M} = T^2$) fails to produce a unitary result, instead carrying exponentially large negative degeneracies [98]: if a consistent partition function exists, something more must contribute. But what? Reckoning with path integral contours is not a simple endeavor in quantum field theory, much less in gravity. In the proposal of [88] – still fairly implicit, but currently the only one which preserves a spectral gap to the black hole threshold – what ostensibly fixes the problem is a specific infinite family of off-shell geometries (Seifert manifolds), whose circle reductions are identified with JT gravity backgrounds in the presence of defects. This gives an elegant hint of random matrices in AdS_3 pure gravity in the near-extremal regime.

Stronger hints come from wormholes. Two-boundary path integrals have been computed in JT/RMT duality: the double-trumpet wormhole, together with the all-orders genus sum over higher topologies, exhibits the famous RMT level repulsion in the ensemble-averaged density-density correlator [35]. After analytic continuation to complex temperature, the double-trumpet leads to a linear ramp in the spectral form factor (SFF); the wormholes with higher topology, exponentially suppressed in entropy, collectively initiate the transition from ramp to plateau [39, 40, 99, 100]. In seeking an $\text{AdS}_3/\text{CFT}_2$ lesson from (or version of) the JT/RMT ensemble duality, it is an AdS_3 analog of the double-trumpet geometry that one should understand. This was the motivation of Cotler and Jensen (CJ) [34], who computed the contribution of such a geometry – an off-shell, connected, two-boundary torus wormhole – to the AdS_3 gravity path integral. Let us call this $Z_{\text{CJ}}(\tau_1, \tau_2)$. Being off-shell, the computation is non-standard, requiring the technique of constrained instantons instead of familiar-but-unavailable saddle point techniques.

¹We will not spell out the full history of this subject, whose modern incarnation started in [95]; a recent account was given in [96].

The CJ result is at once highly mysterious, remarkably simple, and deceptively rich. It contains unmistakable signs of random matrices or 2d CFT avatars thereof: at leading order in the low-temperature limit, $Z_{\text{CJ}}(\tau_1, \tau_2)$ reproduces the universal result of double-scaled matrix integrals. It also contains infinite series of corrections that are apparently tied to modular invariance, and generalizes the RMT result to include spacetime spin. This indicates the presence of some underlying Virasoro generalization of RMT. Less clear is the sense in which an ensemble interpretation of the result is necessary, and if so, how this squares with reasonable expectations of the space of irrational 2d CFTs as a sporadic, generically discrete set of points. This constellation of ideas was playfully labeled “random CFT” in [34]. Despite the absence of a proper definition, this much seems certain: whatever “random CFT” means, it ought to be relevant for holography.

In a chaotic 2d CFT in general, how do we extract random matrix behavior hiding within?

In a chaotic 2d CFT dual to AdS₃ pure gravity in particular, what does $Z_{\text{CJ}}(\tau_1, \tau_2)$ mean?

Perhaps in spite of appearances, the CJ wormhole does not imply that the boundary dual of semiclassical AdS₃ pure gravity is an ensemble of large central charge CFTs. Given a microscopic large c CFT, by which we mean a $c \rightarrow \infty$ limit of an unbounded sequence of irrational Virasoro CFTs $\{\mathcal{T}_c\}$ with a suitable spectral gap, there may be a coarse-graining procedure or kinematic averaging (e.g. with respect to energy or time), as in quantum systems, which is compatible with the bulk wormhole computation. In general, bulk calculations that imply non-factorizing correlations between disconnected boundaries are agnostic about what kind of boundary averaging gives rise to this correlation [34, 101, 102]; we are not aware of an effective field theory calculation in AdS _{$D \geq 3$} gravity that singles out a boundary ensemble interpretation. The more robust concept, as emphasized in [103], is not ensemble averaging *per se*, but *apparent* averaging, which arises essentially because of the chaotic nature of the high-energy spectrum in the large c limit. A nice discussion of this set of ideas is given in the Introduction of [104]. Also relevant for our work are the comments on the role of wormholes in non-averaged theories in [35, 105].

Condensing the above into a challenge for semiclassical AdS₃ holography, the goal is to show how the bulk theory can be dual to a microscopic large c CFT in a manner consistent with nonvanishing bulk wormhole amplitudes (or, perhaps, to show that it cannot). We emphasize that this is a challenge particularly posed by off-shell wormholes, as on-shell wormholes are instead fixed by suitable gluing of universal asymptotic CFT data (spectrum and OPE coefficients), which are themselves fixed by crossing symmetry in terms of low-energy inputs – insensitive to level statistics of black hole microstates.²

²Recent work establishes an impressive match between partition functions of individual saddle points of AdS₃ gravity (possibly coupled to point particles) of some fixed topology, and certain boundary computations. The latter recast the partition function either as a moment problem of a (near-)Gaussian “large c ensemble” of CFT data [106] or using a novel topological quantum field theory [107]. (See e.g. [108–110] for further developments.) This extends earlier ideas about AdS₃ gravity as an effective field theory, by incorporating a version of ETH for 2d CFTs [101, 106] and allowing multiple disconnected boundaries. Because these works

Independent of applications to AdS₃ wormholes, we would like to develop a quantitative toolkit to derive emergent RMT physics from microscopic 2d CFT data. It may be useful to phrase this yet another way, using the vocabulary of the bootstrap approach to CFTs [111]. The modular and conformal bootstrap have focused so far on constraining single-copy observables. This seems to obscure chaotic microstructure of the spectrum which is revealed in “two-copy observables” like the SFF. One would like to know whether the dip-ramp-plateau structure of the coarse-grained SFF of a chaotic CFT can be *bootstrapped* from a minimal set of CFT data. Lowering our sights by focusing on the linear ramp region, the CJ wormhole suggests a more specific program toward addressing this question vis-à-vis gravity: first, use the constraints of Virasoro symmetry and modular invariance to carve out the space of possible wormhole amplitudes; then, upon imposing the features of a 2d CFT dual to *pure* gravity in particular, determine where the CJ wormhole sits in this space. An emergence of RMT-like physics from this analysis would give one answer to what “random CFT” could mean in two dimensions, compatible with the necessary symmetry constraints.

2.1.1 Summary of results

This work makes some headway on these questions for generic $c > 1$ Virasoro CFTs and their AdS₃ gravity duals, guided in part by the theory of trace formulas for chaotic quantum systems, and by new perspectives on modular invariance.

We begin in **Section 2.2** by recalling the $SL(2, \mathbb{Z})$ spectral decomposition of torus partition functions of parity-invariant Virasoro CFTs (with no extra conserved currents) [112]. A key player in our framework is $Z_{\text{spec}}(\tau)$, a certain subtracted partition function, defined by removing “light” primary operators from the partition function in a modular-invariant way. $Z_{\text{spec}}(\tau)$ admits a decomposition into a complete $SL(2, \mathbb{Z})$ -invariant eigenbasis.³ We review some suggestive hints from AdS₃ gravity and Narain CFT about the physical meaning of $Z_{\text{spec}}(\tau)$; these lead us to view $Z_{\text{spec}}(\tau)$ as the “chaotic part” of the partition function that suitably incorporates the symmetries.

In **Section 2.3** we substantiate this point of view by presenting a 2d CFT trace formula. It mimics the Gutzwiller trace formula for chaotic quantum systems, which we first review. To make the connection, we proceed to transform $Z_{\text{spec}}(\tau)$ to a microcanonical density of states. The total density of spin- j Virasoro primaries splits into two terms:

$$\rho_j(t) = \widehat{\rho}_{L,j}(t) + \rho_{\text{spec},j}(t), \quad t := \min(h, \bar{h}) - \frac{c-1}{24} \quad (2.1.1)$$

establish a match at the level of individual saddles, irrespective of the full sum over topologies and of level statistics, the questions of whether pure gravity exists and what its boundary dual is (e.g. ensemble or not) are of course not addressed.

³Leaving details to the main text, the eigenbasis is comprised of real-analytic Eisenstein series $E_s(\tau)$ with $s = \frac{1}{2} + i\omega$ and $\omega \in \mathbb{R}$, and Maass cusp forms $\phi_n(\tau)$ with $n \in \mathbb{Z}_{\geq 0}$.

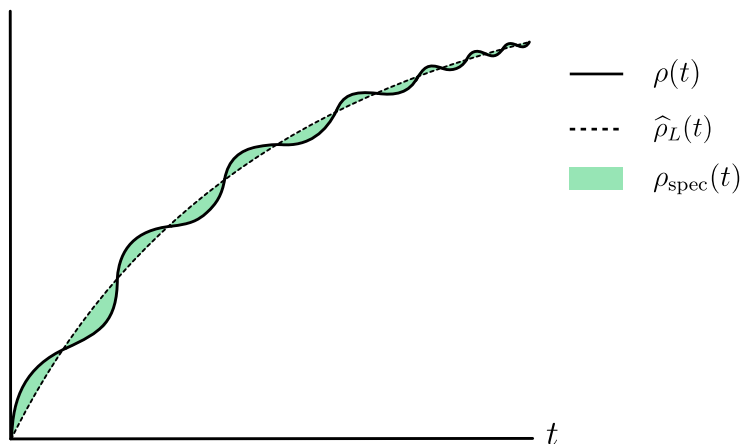


Figure 2.1: The density of heavy states $\rho(t)$ of a compact CFT, here approximated by the smooth black curve (as in a large c limit) for illustrative purposes, is highly oscillatory on wavelengths of order $e^{-S_{\text{Cardy}}(t)}$, the mean level spacing, while the modular completion of light states, $\hat{\rho}_L(t)$, contains the smooth, universal Cardy-like growth (dashed line). Their difference, $\rho_{\text{spec}}(t)$, captures the oscillations, encoding chaotic statistics. We have suppressed the spin index j .

One should think of the spectral density $\rho_{\text{spec},j}(t)$ as having removed all self-averaging contributions $\hat{\rho}_{L,j}(t)$ from the total density $\rho_j(t)$: it is supported only on the chaotic, “heavy” spectrum $t \geq 0$, computing the difference between the exact density and the smooth asymptotic approximation to it. See Figure 2.1.

In seeking a possible analogy to quantum systems, the first term $\hat{\rho}_{L,j}(t)$ would map to a mean density $\bar{\rho}(E)$, while the second term $\rho_{\text{spec},j}(t)$ would map to $\rho_{\text{osc}}(E)$, an oscillatory part that can be expanded over periodic orbits. Indeed, such a relation can be made sharp: the $SL(2, \mathbb{Z})$ spectral decomposition of $\rho_{\text{spec},j}(t)$ is shown to take exactly the form of a Gutzwiller trace formula, for every fixed spin j (see (2.3.20)). Periodic orbits correspond to elements of the $SL(2, \mathbb{Z})$ eigenbasis, labeled by $SL(2, \mathbb{Z})$ spectral frequencies ω , with a clean identification of the orbit actions, periods and one-loop determinants for each element. An important aspect of this trace formula is that the $SL(2, \mathbb{Z})$ eigenbasis, and hence the set of orbits, is complete.

With this in hand, we analyze correlations and define a coarse-graining procedure in analogy to Berry’s diagonal approximation. Mimicking the local energy averaging of quantum systems, coarse-graining a product of spectral densities over mean twist correlates the two copies by pairing the $SL(2, \mathbb{Z})$ eigenvalues – a 2d CFT analog of restricting the double sum over orbits to those of equal action. Inspired by this we define a diagonal partition function in the canonical ensemble. First we define $Z_{\text{diag}}(\tau_1, \tau_2)$, by projecting the factorized product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$ onto the kernel of a difference of Laplacians, thus correlating the eigenvalues

of basis elements. We then introduce an enhanced diagonal projection of $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$ which pairs $SL(2, \mathbb{Z})$ *eigenfunctions*, not just eigenvalues: there are degeneracies between Eisenstein series and Maass cusp forms. This is defined by projecting onto the kernel of a difference of $SL(2, \mathbb{Z})$ Hecke operators, $T_j^{(\tau_1)} - T_j^{(\tau_2)}$, for every spin $j \in \mathbb{Z}_+$: we call this Hecke projection, and the corresponding partition function $Z_{\text{Hecke}}(\tau_1, \tau_2)$. These are given in (2.3.41) and (2.3.43). From the point of view of the trace formula and periodic orbit theory, Hecke projection does the job of properly pairing *identical orbits*. One may view $Z_{\text{Hecke}}(\tau_1, \tau_2)$ as an enhanced form of coarse-graining in 2d CFTs, carrying extra symmetry and arithmeticity, annihilated as it is by an infinite set of commuting Hecke operators.

In **Section 2.4**, we analyze $Z_{\text{Hecke}}(\tau_1, \tau_2)$ for general chaotic 2d CFTs, culminating in a necessary and sufficient condition for the coarse-grained spectral form factor (SFF) to exhibit a linear ramp. In chaotic quantum systems, the SFF, call it $K_\beta(T)$, famously exhibits a linear ramp at times $T \gg \beta$, with coefficient controlled by the particular RMT ensemble governing the late-time dynamics. The diagonal approximation to the SFF is designed to capture this ramp behavior. Having constructed a diagonal partition function for 2d CFTs, with self-averaging terms judiciously subtracted, we are in position to show the same.

Focusing on the scalar Fourier mode of $Z_{\text{Hecke}}(\tau_1, \tau_2)$, it is fully determined by a function we call $\mathcal{R}(z)$, defined as the inverse Mellin transform of $|(Z_{\text{spec}}, E_s)|^2$, the squared spectral overlap of $Z_{\text{spec}}(\tau)$ with the Eisenstein series $E_s(\tau)$. The variable $z := y_1/y_2$ is the ratio of inverse temperatures $y_i := \text{Im}(\tau_i)$. Passing to SFF kinematics via $y_1 = \beta + iT$ and $y_2 = \beta - iT$, we show that the coarse-grained SFF exhibits a linear ramp at times $T \gg \beta$ if and only if $\mathcal{R}(z)$ has a simple pole at $z = -1$, with the correct RMT residue:

$$\mathcal{R}(z \rightarrow -1) \sim \frac{C_{\text{RMT}}}{2\pi} \frac{1}{1+z} \quad (2.1.2)$$

The constant C_{RMT} sets the RMT ensemble (for example, $C_{\text{GOE}} = 2$). This simple pole may be recast as a straightforward falloff condition on the partition function in $SL(2, \mathbb{Z})$ spectral space:

$$|(Z_{\text{spec}}, E_{\frac{1}{2}+i\omega})|^2 \sim e^{-\pi\omega} f(\omega) \quad (\omega \rightarrow \infty) \quad (2.1.3)$$

where $f(\omega)$ approaches C_{RMT} asymptotically. Moreover, Virasoro symmetry and $SL(2, \mathbb{Z})$ -invariance imply a specific set of terms (2.4.25) in $Z_{\text{Hecke}}(\tau_1, \tau_2)$ that necessarily accompany (2.1.2); in SFF kinematics, these are power-law corrections at late times.

In **Section 2.5** we prepare for our descent into the wormhole.

There is a satisfying synergy of the diagonal partition function $Z_{\text{Hecke}}(\tau_1, \tau_2)$ with AdS_3 torus wormholes. In particular, starting from a geometric definition, we demonstrate that torus wormhole amplitudes $Z_{\text{WH}}(\tau_1, \tau_2)$ are Hecke symmetric: that is, they exhibit precisely the

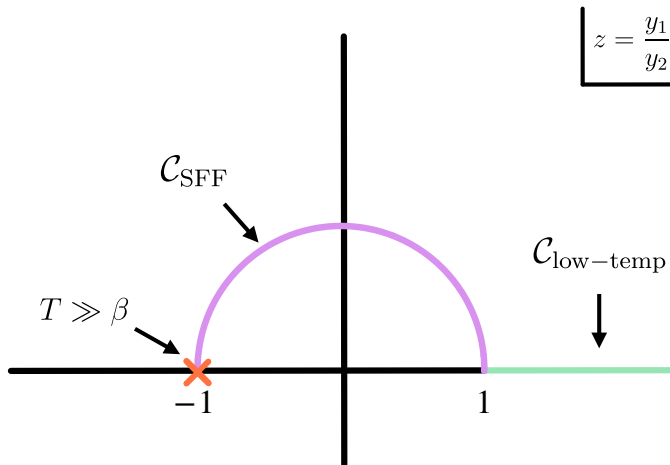


Figure 2.2: Off-shell wormhole amplitudes $Z_{\text{WH}}(\tau_1, \tau_2)$ in semiclassical AdS_3 gravity are fixed by a single function $\mathcal{R}(z)$, where $y_i = \text{Im}(\tau_i)$ are inverse temperatures. The colored contours show Euclidean and Lorentzian kinematics in the low-temperature limit. The spectral form factor (SFF), for which z lies on the unit circle, has a linear ramp at times $T \gg \beta$ if and only if $\mathcal{R}(z)$ has a simple pole at $z = -1$: see (2.1.2).

functional form of $Z_{\text{Hecke}}(\tau_1, \tau_2)$. This signals that wormholes may be viewed microscopically, understood as coarse-grained two-copy partition functions of underlying chaotic CFTs. This forms the basis of our “wormhole Farey tail”: that is, the interpretation of bulk wormhole amplitudes $Z_{\text{WH}}(\tau_1, \tau_2)$, constructed as $SL(2, \mathbb{Z})$ image sums over large diffeomorphisms, as gravitational duals of $Z_{\text{Hecke}}(\tau_1, \tau_2)$ in large c CFTs. This is an AdS_3 realization of the idea of [35, 105] that bulk spacetime wormholes geometrize the diagonal approximation.

Let us expand on this slightly. In a diffeomorphism-invariant theory of semiclassical gravity, a wormhole amplitude $Z_{\text{WH}}(\tau_1, \tau_2)$ that is independently modular-invariant with respect to both τ_1 and τ_2 can be constructed as an $SL(2, \mathbb{Z})$ Poincaré sum of a suitable seed function: the $SL(2, \mathbb{Z})$ transformations at the boundary implement the action of large bulk diffeomorphisms. This is a multi-boundary generalization of the familiar black hole Farey tail for thermal partition functions [97, 113, 114]. Taking this as one definition (given more precisely in Subsection 2.5.1) of an off-shell torus wormhole, we prove that Poincaré sums of this form enjoy a few remarkable properties. Among others, $Z_{\text{WH}}(\tau_1, \tau_2)$ is even more constrained than a generic Hecke projection: its Eisenstein and cusp form spectral overlaps are functionally equal, leading to a highly-constrained functional form in spectral space,

$$Z_{\text{WH}}(\tau_1, \tau_2) = \int_{\mathcal{C}_{\text{crit}}} f_{\text{WH}}(s) E_{1-s}(\tau_1) E_s(\tau_2) + \sum_{n=1}^{\infty} f_{\text{WH}}(s_n) \phi_n(\tau_1) \phi_n(\tau_2) \quad (2.1.4)$$

where $\mathcal{C}_{\text{crit}}$ defined in (2.2.6) denotes the straight contour $s = \frac{1}{2} + i\mathbb{R}$. Given a $Z_{\text{spec}}(\tau)$ of an

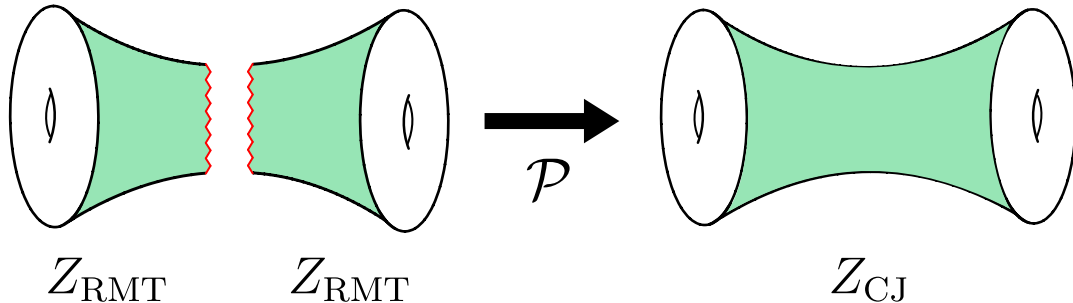


Figure 2.3: The Cotler-Jensen wormhole $Z_{\text{CJ}}(\tau_1, \tau_2)$ is generated by gluing two single boundary partition functions $Z_{\text{RMT}}(\tau_1)$ and $Z_{\text{RMT}}(\tau_2)$. Each factor may be interpreted as a half-wormhole in AdS_3 pure gravity. The gluing is performed by coarse-graining over the spectrum of the dual CFT: the (Hecke) projection \mathcal{P} projects the factorized product onto the diagonal terms with respect to the $SL(2, \mathbb{Z})$ spectral basis, analogously to trace formulas for non-disordered chaotic systems.

underlying CFT, the identification with the wormhole is simply

$$f_{\text{WH}}(s) = |(Z_{\text{spec}}, E_s)|^2. \quad (2.1.5)$$

The form of (2.1.4) makes manifest that wormhole correlations are diagonalized by the $SL(2, \mathbb{Z})$ spectral basis. Since diagonality is basis-dependent, this affirms the $SL(2, \mathbb{Z})$ spectral decomposition as a proper trace formula for 2d CFT.

In **Section 2.6** we turn to AdS_3 pure gravity and the CJ wormhole; see Figure 2.3. This wormhole amplitude was derived in [34] as a Poincaré sum of the above type. Its spectral overlap is very simple:

$$f_{\text{CJ}}(s) = \frac{1}{\pi} \Gamma(s) \Gamma(1-s). \quad (2.1.6)$$

In terms of the function $\mathcal{R}(z)$ defined earlier, $\mathcal{R}_{\text{CJ}}(z)$ not only contains the pole (2.1.2) that generates the linear ramp – it is exactly *equal* to it. Moreover, the corrections prescribed by Virasoro and $SL(2, \mathbb{Z})$ are exactly those found in [34].

This analysis tells us that the CJ wormhole is *extremal* within the space of admissible wormhole amplitudes, in the following quantitative sense. Having incorporated the requisite Virasoro symmetry and modular invariance – that is, upon “quotienting” by the symmetries of 2d CFTs and wormholes – the amplitude is determined solely by the function $\mathcal{R}(z)$. The CJ wormhole of AdS_3 pure gravity then sets $\mathcal{R}(z)$ exactly equal to the double-scaled RMT result. This signature of pure gravity is what we call **MaxRMT**: *the maximal realization of*

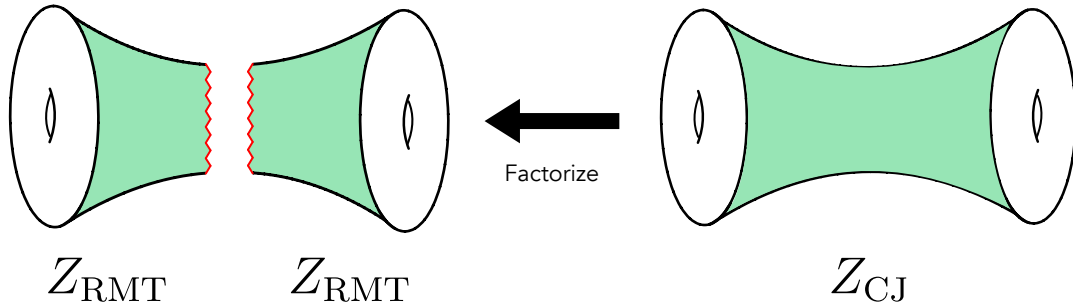


Figure 2.4: Completeness of the $SL(2, \mathbb{Z})$ spectral basis permits factorization of the Cotler-Jensen wormhole. $Z_{\text{RMT}}(\tau)$ encodes quantum substructure of the pure gravity black hole spectrum.

random matrix universality consistent with Virasoro symmetry and modular invariance. The fact that pure gravity exhibits MaxRMT statistics may be viewed as extending the hallmark maximal chaos of pure gravity in the semiclassical, early-time regime of Lyapunov chaos [38] to the quantum, late-time regime as defined by RMT. We expand on this and make related comments in Subsection 2.6.2.

That our formalism fits the CJ wormhole like a glove strongly indicates that the wormhole may be interpreted in terms of a *microscopic* 2d CFT dual, compatible with a traditional holographic interpretation for semiclassical AdS_3 pure gravity. The CJ wormhole is generated dynamically from an underlying CFT upon coarse-graining the chaotic spectral correlations as prescribed above. This gives a concrete actualization of the apparent averaging phenomenon of [103], which was argued on general grounds to emerge from chaos of the semiclassical black hole spectrum.

With this understanding, in Subsection 2.6.3 we take a step further by leveraging the completeness of the $SL(2, \mathbb{Z})$ spectral eigenbasis to factorize the CJ wormhole into its constituent components: see Figure 2.4. We call the resulting microscopic partition function $Z_{\text{RMT}}(\tau)$. The result is unique up to signs, and is given explicitly in (2.6.14) and (2.6.15). $Z_{\text{RMT}}(\tau)$ captures exponentially suppressed fine structure of the black hole spectrum of AdS_3 pure gravity. As is perhaps clear from these results, $Z_{\text{RMT}}(\tau)$ may be meaningfully viewed as a half-wormhole of pure gravity. We substantiate this with comparison to 2D gravity half-wormholes, broken cylinders and branes. One intriguing aspect is that $Z_{\text{RMT}}(\tau)$ carries a conspicuous erratic phase: its spectral overlap is dressed by a Riemann zeta phase $\phi(\omega) = \arg(\zeta(1 + 2i\omega))$. This is a property of $Z_{\text{spec}}(\tau)$ in general. The Riemann zeta function is a famously quantum chaotic object [115–119], and $\phi(\omega)$ varies wildly along the line $\omega \in \mathbb{R}$

(see Figure 2.6); upon gluing to form a wormhole, the phases cancel. This nicely exhibits the erratic nature expected of a higher-dimensional generalization of half-wormholes.

Summarizing the above, the identification of $Z_{\text{RMT}}(\tau)$ gives a new, quantum piece of the torus partition function $Z_{\text{grav}}(\tau)$ of the putative 2d CFT dual to semiclassical AdS_3 pure gravity, which augments the sum over smooth bulk saddles:

$$Z_{\text{grav}}(\tau) \approx Z_{\text{MWK}}(\tau) + Z_{\text{RMT}}(\tau). \quad (2.1.7)$$

$Z_{\text{MWK}}(\tau)$ is the sum over saddles with $\partial\mathcal{M} = T^2$ Maloney:2007ud,Keller:2014xba. There are small, as-yet-undetermined corrections to $Z_{\text{RMT}}(\tau)$, call them $\delta Z_{\text{RMT}}(\tau)$, expected to come from other off-shell configurations in the bulk; on the CFT side, these would give corrections to the spectral statistics encoded in $Z_{\text{RMT}}(\tau)$. We explain why $\delta Z_{\text{RMT}}(\tau)$ must be nonzero due to CFT unitarity, and sketch its origins from the point of view of wormholes and string theory. This further implies that the BTZ black hole threshold lies strictly *below* the naive semiclassical threshold $t = 0$. The structure of the resulting partition function also jibes nicely with the proposal of Maxfield and Turiaci [88]. The inclusion of level statistics of heavy operators poses an interesting challenge for the large c modular bootstrap of 2d CFTs.

We end the paper with a discussion of some future directions, a glossary of partition functions defined in this work, and appendices with details complementing the main text.

2.2 Groundwork

We begin by briefly recalling some notions in the spectral theory of the Laplacian on the fundamental domain $\mathcal{F} = \mathbb{H}/SL(2, \mathbb{Z})$ and their application to 2d CFTs. For more detailed treatments and background, see [112, 120, 121]; for follow-up work, see [122, 123]. We mostly use conventions of [121].

2.2.1 Lightning review of $SL(2, \mathbb{Z})$ spectral theory in 2d CFT

A square-integrable, $SL(2, \mathbb{Z})$ -invariant function admits a spectral decomposition in a complete $SL(2, \mathbb{Z})$ -invariant eigenbasis with three branches:

$$\begin{cases} \Delta_\tau E_s^*(\tau) = s(1-s)E_s^*(\tau) & s = \frac{1}{2} + i\omega, \quad \omega \in \mathbb{R} \\ \Delta_\tau \phi_n(\tau) = s_n(1-s_n)\phi_n(\tau) & s_n = \frac{1}{2} + i\omega_n, \quad \omega_n \in \mathbb{R} \\ \Delta_\tau \phi_0 = 0 \end{cases} \quad (2.2.1)$$

$E_s^*(\tau) = E_{1-s}^*(\tau)$ is the completed Eisenstein series evaluated on the critical line, $\phi_n(\tau)$ is an infinite discrete set of Maass cusp forms labeled by $n \in \mathbb{Z}_+$, and ϕ_0 is the constant function.

We define square integrability with respect to the Petersson inner product with hyperbolic measure $dx dy/y^2$ where $\tau := x + iy$. The cusp forms are unit-normalized. The respective Fourier decompositions are⁴

$$\begin{aligned} E_s^*(\tau) &= \sum_{j=0}^{\infty} (2 - \delta_{j,0}) \mathbf{a}_j^{(s)} \cos(2\pi jx) \sqrt{y} K_{s-\frac{1}{2}}(2\pi jy), \\ \phi_n(\tau) &= \sum_{j=1}^{\infty} 2\mathbf{b}_j^{(n)} \cos(2\pi jx) \sqrt{y} K_{s_n-\frac{1}{2}}(2\pi jy), \end{aligned} \tag{2.2.2}$$

There are also Maass cusp forms odd under $x \rightarrow -x$, but they are not relevant for the applications ahead. The Eisenstein series has Fourier coefficients

$$\mathbf{a}_j^{(s)} = \frac{2\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}} \tag{2.2.3}$$

Its scalar mode can be formally obtained as a smooth $j \rightarrow 0$ limit of the spinning modes (see Appendix 2.10),

$$\begin{aligned} E_{s,0}^*(y) &= \Lambda(s)y^s + \Lambda(1-s)y^{1-s}, \\ &= \lim_{j \rightarrow 0} \mathbf{a}_j^{(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi jy) \end{aligned} \tag{2.2.4}$$

where $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s) = \Lambda\left(\frac{1}{2} - s\right)$ is the completed Riemann zeta function. The cuspidal spectral parameters ω_n and Fourier coefficients $\mathbf{b}_j^{(n)}$ are sporadic real numbers with no known analytic expression. The multiplicity of Maass cusp forms at a given eigenvalue is conjecturally bounded above by one, i.e. the spectrum is “simple” (e.g. [124]).

Given an $f(\tau) \in L^2(\mathcal{F})$, the spectral decomposition is

$$f(\tau) = \langle f \rangle + \int_{\mathcal{C}_{\text{crit}}} \{f, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (f, \phi_n) \phi_n(\tau), \tag{2.2.5}$$

where we integrate over the critical line,

$$\int_{\mathcal{C}_{\text{crit}}} := \frac{1}{4\pi i} \int_{\text{Re}(s)=\frac{1}{2}} ds = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \tag{2.2.6}$$

The (completed) Eisenstein overlaps are

$$\{f, E_s\} := \frac{(f, E_s)}{\Lambda(s)} \tag{2.2.7}$$

⁴With respect to notation in [112, 121], $2\mathbf{b}_j^{(n)}|_{\text{here}} = a_j^{(n)}|_{\text{there}}$.

where (f, E_s) is the Petersson inner product. The constant term, $\langle f \rangle$, is the “modular average” of $f(\tau)$ over \mathcal{F} . In the remainder of this work, we focus on parity-invariant CFT observables, so the sum over cusp forms is taken over even cusp forms only, with $n \in \mathbb{Z}_+$. We will occasionally find it useful to write the spectral decomposition in a unified notation as

$$f(\tau) = \sum_{\omega} \tilde{f}_{\omega} \psi_{\omega}(\tau), \quad \text{where } \psi_{\omega}(\tau) := \{E_{\frac{1}{2}+i\omega}(\tau), \phi_n(\tau), \phi_0\} \quad (2.2.8)$$

are the eigenfunctions and $\tilde{f}_{\omega} := (f(\tau), \psi_{\omega}(\tau))$ is a shorthand for the spectral overlap.

Consider now the torus partition function $Z(\tau)$ of a general (non-holomorphic) parity-invariant 2d CFT, which we assume to have only Virasoro symmetry with $c > 1$. The primary partition function, which strips off Virasoro descendants⁵ while preserving modular invariance, is defined by

$$Z_p(\tau) = \sqrt{y} |\eta(\tau)|^2 Z(\tau). \quad (2.2.9)$$

In [112], it was shown that $Z_p(\tau)$ can be written in a manifestly modular-invariant way:

$$Z_p(\tau) = \widehat{Z}_L(\tau) + Z_{\text{spec}}(\tau), \quad (2.2.10)$$

$\widehat{Z}_L(\tau)$ is the “modular completion” of light states, defined as follows. First, one constructs the partition function of light primaries

$$Z_L(\tau) := \sqrt{y} \sum_{\min(h, \bar{h}) \leq \xi} q^{h-\xi} \bar{q}^{\bar{h}-\xi}, \quad (2.2.11)$$

where in terms of conformal weight Δ and spin j ,

$$\Delta = h + \bar{h}, \quad j = |h - \bar{h}|, \quad \xi = \frac{c-1}{24} \quad (2.2.12)$$

and $q = e^{2\pi i \tau}$. One subsequently “completes” this into a modular-invariant function $\widehat{Z}_L(\tau)$ by suitably adding heavy states. A convenient, and physically distinguished, mode of modular completion is to perform a Poincaré sum over $SL(2, \mathbb{Z})$ images [97, 113, 114, 125]:

$$\widehat{Z}_L(\tau) = \sum_{\gamma \in SL(2, \mathbb{Z})/\Gamma_{\infty}} Z_L(\gamma\tau). \quad (2.2.13)$$

We have modded out by Γ_{∞} , the set of T -transformations, under which the summand is invariant (thanks to spin-quantization). Poincaré summation is physically distinguished because it is the *minimal* modular completion: it adds only the modular images of the light states, and nothing more. For this reason we define $\widehat{Z}_L(\tau)$ using Poincaré summation throughout the paper. Having explicitly constructed $\widehat{Z}_L(\tau)$, we can define $Z_{\text{spec}}(\tau)$ by

⁵The level-one null descendants of the Virasoro vacuum module are included, but all other descendants are removed.

subtraction, which is square-integrable and thus admits an $SL(2, \mathbb{Z})$ spectral decomposition:

$$Z_{\text{spec}}(\tau) = \langle Z_{\text{spec}} \rangle + \int_{\mathcal{C}_{\text{crit}}} \{Z_{\text{spec}}, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (Z_{\text{spec}}, \phi_n) \phi_n(\tau). \quad (2.2.14)$$

Whereas $\widehat{Z}_L(\tau)$ contains the leading-order Cardy asymptotics of $Z_p(\tau)$ – namely (but not only) the identity block and its modular images – $Z_{\text{spec}}(\tau)$ instead probes the chaotic, high-energy spectrum. We will make this sharper below. Note that $Z_{\text{spec}}(\tau)$ has support only on heavy states, but it indirectly “knows about” the light states of the CFT because of the modular completion used to define it.

We point out that the analog of $Z_{\text{spec}}(\tau)$ would be trivial in a holomorphic (or holomorphically factorized) CFT, where the spectrum of states with $h > c/24$ is determined by its complement; this inherent non-holomorphicity is useful for focusing on generic irrational 2d CFTs and their gravity duals (as opposed to toy models such as chiral gravity).

Two technical remarks

In our expressions for $Z_{\text{spec}}(\tau)$, we henceforth set the modular averages $\langle Z_{\text{spec}} \rangle$ to zero. For Virasoro CFT partition functions (2.2.10), this is just a choice of convention to “put” the constant in $\widehat{Z}_L(\tau)$ rather than $Z_{\text{spec}}(\tau)$: a c-number constant may be freely shuffled between the two terms while keeping $Z_p(\tau)$ fixed.⁶ This choice is motivated by the fact that a constant’s contribution to a microcanonical density of states is self-averaging under coarse-graining in energy – a concept that will show up later. Moreover, we note that in many cases of interest, such as the Narain case to be reviewed in Subsection 2.2.2, $\langle Z_{\text{spec}} \rangle = 0$.

Note that in (2.2.11) we have defined “light” operators with respect to the twist threshold, $\min(h, \bar{h}) \leq \xi$, rather than the dimension threshold, $\Delta \leq 2\xi$. The latter is more commonplace, and was used in the original definition of [112]; moreover, from the mathematical point of view, one need only subtract the operators with $\Delta \leq 2\xi$ from $Z_p(\tau)$ in order for $Z_{\text{spec}}(\tau)$ to be square-integrable. On the other hand, operators with $\min(h, \bar{h}) \leq \xi$ – dubbed “censored” operators in [114] – are not part of the black hole spectrum of AdS₃ pure gravity, and more generally, the chaotic spectrum of a 2d CFT (as it is currently understood). Though mathematically unnecessary, it is physically well-motivated to define $\widehat{Z}_L(\tau)$ and hence $Z_{\text{spec}}(\tau)$ with respect to the twist threshold in the manner above, and we will do so in view of the applications of present interest.⁷ See [123] for similar comments.

⁶This may be viewed as a kind of physical implementation of Zagier’s regularization [126], who showed that a constant term in a square-integrable function on \mathcal{F} may be formally renormalized away by introducing a renormalized Rankin-Selberg transform, in which one cuts off the integral near the cusp at $y = L$ and takes the $L \rightarrow \infty$ limit. This amounts to defining a new $Z'_{\text{spec}}(\tau) = Z_{\text{spec}}(\tau) - \langle Z_{\text{spec}} \rangle$, which leaves $Z_p(\tau)$ invariant if we shift $\widehat{Z}_L(\tau)$ oppositely.

⁷Due to cosmic censorship, states with $h < \xi$ but $\bar{h} > \xi$ are dual to neither black holes nor conical defects in gravity. The most familiar such states are multi-twist composites of light primaries with $\min(h, \bar{h}) < \frac{3}{4}\xi$:

2.2.2 Hints

To better understand the physical significance of the two pieces $\widehat{Z}_L(\tau)$ and $Z_{\text{spec}}(\tau)$, the authors of [112] studied the partition function of Narain CFT in this formalism. Let us review a few of the lessons gained. A Narain CFT of c free bosons has a local $U(1)^c \times U(1)^c$ symmetry, and moduli which we collectively denote as m . The $U(1)^c \times U(1)^c$ primary partition function $Z_p^{(c)}(\tau|m)$ can be decomposed as

$$Z_p^{(c)}(\tau|m) = \langle Z_p^{(c)}(\tau|m) \rangle_m + Z_{\text{spec}}^{(c)}(\tau|m) \quad (2.2.15)$$

The first term is the partition function averaged over moduli, and the second term is the spectral partition function. This expression displays two interesting features:

- The modular completion of light states is equal to the average over Narain moduli space:

$$\widehat{Z}_L^{(c)}(\tau) = \langle Z_p^{(c)}(\tau|m) \rangle_m = E_{\frac{c}{2}}(\tau). \quad (2.2.16)$$

The only primary in Narain CFT that is light with respect to the $U(1)^c$ primary threshold, with $\min(h, \bar{h}) \leq (c - c_{\text{currents}})/24 = 0$, is the vacuum itself: the Eisenstein series is simply the Poincaré modular completion of the vacuum state [128, 129].

- The spectral decomposition averages to zero under both modular and ensemble averages:

$$\langle Z_{\text{spec}}^{(c)}(\tau|m) \rangle_\tau = \langle Z_{\text{spec}}^{(c)}(\tau|m) \rangle_m = 0. \quad (2.2.17)$$

Thus $Z_{\text{spec}}^{(c)}(\tau|m)$ captures deviations from the average over moduli space. These deviations encode the higher statistical moments of the Narain ensemble, e.g. the variance

$$\langle Z_p^{(c)}(\tau_1|m) Z_p^{(c)}(\tau_2|m) \rangle_m^{(\text{conn})} = \langle Z_{\text{spec}}^{(c)}(\tau_1|m) Z_{\text{spec}}^{(c)}(\tau_2|m) \rangle_m. \quad (2.2.18)$$

A study of these correlations was undertaken in [130, 131].

Extrapolating these properties to arbitrary 2d CFTs suggests [112] that $\widehat{Z}_L(\tau)$ is an average partition function of some kind, capturing universal contributions of light states to the heavy spectrum implied by crossing. The same picture is suggested by the MWK partition function, the sum over smooth semiclassical saddles of AdS₃ gravity and fluctuations

these composites live on Regge trajectories which asymptote, at large spin, to the Virasoro Mean Field Theory spectrum [127]. This demonstrates that a 2d CFT may have an infinite number of operators with $\min(h, \bar{h}) \leq \xi$, whereas the number of operators with $\Delta \leq 2\xi$ is strictly finite (at finite ξ). The sum (2.2.11) may thus require regularization. The same is in fact true for the Poincaré modular completion of even a single state [97, 114].

around them, which is a (regularized) modular completion of the vacuum and its $SL(2, \mathbb{R})$ null descendants: in particular, this sum generates the smooth, Cardy asymptotics (and non-perturbative corrections thereto) present in any large c CFT density of states. Somewhat similar ideas were subsequently suggested in [103], motivated by chaos. In such a picture, these universal contributions would encode leading-order coarse-grained data of the original CFT. The remainder, $Z_{\text{spec}}(\tau)$, would capture fine-grained fluctuations. For example, one might average the product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$, à la (2.2.18), to extract statistical correlations among microstates.

The challenge in making this precise is to extend it to the generic Virasoro setting, sans moduli. In the next section we will realize a version of this idea for generic 2d CFTs. This will show that $Z_{\text{spec}}(\tau)$ can indeed be viewed as the chaotic part of the partition function in a manner consistent with the symmetries.

2.3 Coarse-graining and Trace Formulas

We start by briefly introducing the Gutzwiller trace formula [36] for chaotic quantum systems which provides useful physical intuition about the density of states of individual non-disordered chaotic systems. We show that the $SL(2, \mathbb{Z})$ spectral decomposition of the density of states is directly analogous to a trace formula. This leads us, via a coarse-graining procedure, to an analog of Berry's diagonal approximation for 2d CFT [37].

2.3.1 Trace formulas for chaotic systems

A well-understood example of quantum chaos is the semiclassical dynamics of few-body quantum systems which are classically chaotic; for a review of trace formulas, see [59, 132–134].⁸ The Gutzwiller trace formula expresses the density of states in the semiclassical limit as

$$\rho(E) = \bar{\rho}(E) + \rho_{\text{osc}}(E) \tag{2.3.1}$$

$\bar{\rho}(E)$ is a mean density. $\rho_{\text{osc}}(E)$, the oscillatory part, is given by a sum over semiclassical periodic orbits γ ,

$$\rho_{\text{osc}}(E) = \frac{1}{\pi} \text{Re} \sum_{\gamma} \mathcal{A}_{\gamma} e^{iS_{\gamma}(E)} \tag{2.3.2}$$

$S_{\gamma}(E)$ is the orbit action and \mathcal{A}_{γ} is called the stability amplitude. The statistical correlations among energy levels are encoded in the highly oscillatory behavior of $\rho_{\text{osc}}(E)$, and can be extracted by microcanonical coarse-graining over an energy window δE .

⁸Since $\hbar = 1$, the semiclassical limit corresponds to $E \gg 1$.

One notable application of the trace formula is the computation of the coarse-grained microcanonical spectral form factor,

$$K_E(T) = \int_{-\infty}^{\infty} d\epsilon e^{i\epsilon T} \overline{\rho_{\text{osc}}\left(E + \frac{\epsilon}{2}\right)} \rho_{\text{osc}}\left(E - \frac{\epsilon}{2}\right) \quad (2.3.3)$$

where T is real Lorentzian time. Inserting the trace formula,

$$K_E(T) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\epsilon e^{i\epsilon T} \sum_{\gamma_1, \gamma_2} \mathcal{A}_{\gamma_1} \mathcal{A}_{\gamma_2}^* e^{i(S_{\gamma_1}(E+\frac{\epsilon}{2}) - S_{\gamma_2}(E-\frac{\epsilon}{2}))} + \text{c.c.} \quad (2.3.4)$$

In the theory of chaotic quantum systems, the difference between periodic orbit actions $\Delta S = S_{\gamma_1} - S_{\gamma_2}$ is the central quantity which allows one to systematically organize the sum over orbits. Periodic orbits with $\Delta S = 0$ give the leading diagonal contribution to level statistics.⁹ Berry [37, 135] showed that the diagonal approximation, which restricts the sum to $\gamma_1 = \gamma_2$, reproduces the linear ramp as $T \rightarrow \infty$:

$$\begin{aligned} K_E(T) \Big|_{\text{diag}} &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\epsilon e^{i\epsilon T} \sum_{\gamma} |\mathcal{A}_{\gamma}|^2 e^{i(S_{\gamma}(E+\frac{\epsilon}{2}) - S_{\gamma}(E-\frac{\epsilon}{2}))} + \text{c.c.} \\ &\approx \frac{T}{2\pi} \mathbf{C}_{\text{RMT}}. \end{aligned} \quad (2.3.5)$$

The factor \mathbf{C}_{RMT} controls the universality class of random matrix behavior (e.g. $\mathbf{C}_{\text{GUE}} = 1$ and $\mathbf{C}_{\text{GOE}} = 2$).

Berry's analysis extracts the random matrix behavior of an individual chaotic quantum system, with no ensemble averaging, from the oscillatory behavior of the density of states. While the original double sum over orbits is manifestly factorized, the diagonal approximation exhibits an emergent non-factorization, by discarding microscopic details of the spectrum.¹⁰

2.3.2 A trace formula for 2d CFT

Is there a decomposition of a 2d CFT density of states of the form (2.3.1), with the structure (2.3.2)? As it happens, the $SL(2, \mathbb{Z})$ spectral decomposition does the trick.

⁹In a chaotic system we expect that $S_{\gamma} = S_{\gamma'}$ only if the orbits are identical $\gamma = \gamma'$ or related by a symmetry, such as time-reversal in the GOE class.

¹⁰It is possible to go beyond the diagonal approximation by systematically including subleading contributions from pairs of orbits with $\Delta S \ll 1$. This is the role of encounter theory [133, 136–138].

Density of states

The density of primary spin- j states $\rho_j(\Delta)$ in a parity-invariant CFT is defined as

$$Z_p(\tau) = \sqrt{y} \sum_{j=0}^{\infty} (2 - \delta_{j,0}) \cos(2\pi jx) \int_j^{\infty} d\Delta e^{-2\pi y(\Delta - 2\xi)} \rho_j(\Delta). \quad (2.3.6)$$

The lower bound is set by unitarity. It will be convenient to introduce the “reduced twist” t , defined as [96]

$$t := \frac{\Delta - j}{2} - \xi = \min(h, \bar{h}) - \xi. \quad (2.3.7)$$

As a first step toward a trace formula, let us transform the spectral decomposition (2.2.5) into the microcanonical ensemble, decomposing the density as

$$\rho_j(t) = \widehat{\rho}_{L,j}(t) + \rho_{\text{spec},j}(t). \quad (2.3.8)$$

By inverse Laplace transform, we obtain a manifestly modular-invariant decomposition

$$\rho_{\text{spec},j}(t) = \int_{\mathcal{C}_{\text{crit}}} \{Z_{\text{spec}}, E_{\frac{1}{2}+i\omega}\} \rho_{\frac{1}{2}+i\omega,j}^*(t) + \sum_{n=1}^{\infty} (Z_{\text{spec}}, \phi_n) \rho_{n,j}(t) \quad (2.3.9)$$

The basis elements are given for $j \neq 0$ by

$$\begin{aligned} \rho_{\frac{1}{2}+i\omega,j}^*(t) &= \mathbf{a}_j^{(s)} \theta(t) \frac{\cos\left(\omega \cosh^{-1}\left(\frac{2t}{j} + 1\right)\right)}{\sqrt{t(t+j)}} \\ \rho_{n,j}(t) &= \mathbf{b}_j^{(n)} \theta(t) \frac{\cos\left(\omega_n \cosh^{-1}\left(\frac{2t}{j} + 1\right)\right)}{\sqrt{t(t+j)}} \end{aligned} \quad (2.3.10)$$

and for $j = 0$ by

$$\rho_{\frac{1}{2}+i\omega,0}^*(t) = \frac{\zeta(2i\omega)}{t} (4t)^{i\omega} + (\omega \rightarrow -\omega) \quad (2.3.11)$$

Note that $\rho_{\frac{1}{2}+i\omega,0}^*(t)$ may be obtained as the $j \rightarrow 0$ limit of (2.3.10) (see Appendix 2.10).¹¹

Let us make a few comments about the physical features of these formulas. While $\rho_j(t)$ is a sum of delta functions in a compact CFT, both $\widehat{\rho}_{L,j}(t)$ and $\rho_{\text{spec},j}(t)$ are continuous functions of t . The latter is highly oscillatory, with the spectral parameters $\{\omega, \omega_n\}$ appearing as frequencies in the basis elements (2.3.10) (hence the notation). We can understand this

¹¹In a more symmetric notation, if we define $\bar{t} = \max(h, \bar{h}) - \xi$ then

$$\rho_{\frac{1}{2}+i\omega,j}^*(t) = \mathbf{a}_j^{(s)} \theta(t) \frac{\cos\left(\omega \cosh^{-1}\left(\frac{t+\bar{t}}{j}\right)\right)}{\sqrt{t\bar{t}}}. \quad (2.3.12)$$

and likewise for $\rho_{n,j}(t)$.

oscillatory behavior as follows. In the asymptotic spectrum (or for $t \gtrsim \xi$ in sparse large c CFTs [139]), the mean level spacing of $\rho_j(t)$ is approximately $e^{-S_{\text{Cardy},j}(t)}$, the inverse of the spin- j Cardy degeneracy, where

$$S_{\text{Cardy},j}(t) = 4\pi\sqrt{\xi t} + 4\pi\sqrt{\xi(t+j)} \quad (2.3.13)$$

Because $\widehat{\rho}_{L,j}(t)$ is continuous and has exponential Cardy growth at $t \rightarrow \infty$, $\rho_{\text{spec},j}(t)$ must oscillate over extremely small wavelengths, of the order of the mean level spacing, to encode the microscopic information about the discrete spectrum.¹² See Figure 2.1 (note that the exact spectrum $\rho_j(t)$ is drawn there as a smooth curve, both for illustrative purposes and to evoke the large c limit).

Trace formula

As explained above, $\widehat{\rho}_{L,j}(t)$ has smooth exponential growth in t , whereas $\rho_{\text{spec},j}(t)$ is a sum/integral over infinitely many oscillatory terms. This dichotomy suggests a direct analogy with trace formulas, with an identification along the following lines:

$$\begin{aligned} \bar{\rho}(E) &\longleftrightarrow \widehat{\rho}_{L,j}(t) \\ \rho_{\text{osc}}(E) &\longleftrightarrow \rho_{\text{spec},j}(t) \end{aligned} \quad (2.3.14)$$

This is to be understood as holding at every fixed spin j .

In fact, this can be made quite precise. Let us return to the Gutzwiller trace formula in (2.3.2). The stability amplitude is canonically decomposed into two pieces:

$$\mathcal{A}_\gamma = A_\gamma T_\gamma(E) \quad (2.3.15)$$

$T_\gamma(E)$ is the period of the orbit,

$$T_\gamma(E) = \frac{\partial S_\gamma(E)}{\partial E}, \quad (2.3.16)$$

while A_γ is a one-loop determinant. We notice that the basis densities (2.3.10) can be written precisely as contributions of individual periodic orbits. Using $\rho_{\frac{1}{2}+i\omega,j}(t)$ to denote either $\rho_{\frac{1}{2}+i\omega,j}^*(t)$ or $\rho_{n,j}(t)$, we have

$$\rho_{\frac{1}{2}+i\omega,j}(t) = \frac{1}{\pi} \text{Re} \left(A_{\omega,j} T_{\omega,j}(t) e^{iS_{\omega,j}(t)} \right), \quad (2.3.17)$$

¹²A nonzero constant term $\langle Z_{\text{spec}} \rangle$ in the spectral decomposition would contribute microcanonically as $\delta_{0,j} \langle Z_{\text{spec}} \rangle \sqrt{2/t}$. This is not oscillatory, so we associate it with $\widehat{\rho}_{L,0}(t)$, cf. Subsection 2.2.1.

with the following identifications:

$$\begin{aligned}
S_{\omega,j}(t) &= \omega \cosh^{-1} \left(\frac{2t}{j} + 1 \right) + \omega \log j \\
T_{\omega,j}(t) &= \frac{\omega}{\sqrt{t(t+j)}} = \frac{\partial S_{\omega,j}}{\partial t} \\
A_{\omega,j} &= \frac{\pi}{\omega} \mathbf{c}_j^{(s)} j^{-i\omega},
\end{aligned} \tag{2.3.18}$$

where $\mathbf{c}_j^{(s)}$ stands for either the Eisenstein ($\mathbf{a}_j^{(s)}$) or cusp form ($\mathbf{b}_j^{(n)}$) Fourier coefficient. The above identifications of the different terms in (2.3.17) follow simply from demanding that $S_{\omega,j}(t) \in \mathbb{R}$ for $\omega \in \mathbb{R}$, and that the $j \rightarrow 0$ limit is smooth for each individual piece. In particular, this instructs us to include $\omega \log j$ in the action: doing so generates $j = 0$ orbit data that correctly reproduces (2.3.11), namely,

$$\begin{aligned}
S_{\omega,0}(t) &= \omega \log(4t) \\
T_{\omega,0}(t) &= \frac{\omega}{t} \\
A_{\omega,0}^* &= \frac{\pi}{\omega} \zeta(2i\omega).
\end{aligned} \tag{2.3.19}$$

Denoting $A_{\omega,j}^*$ and $A_{n,j}$ as the specialization of $A_{\omega,j}$ to Eisensteins and cusp forms, respectively, we land on the rewriting of the $SL(2, \mathbb{Z})$ spectral decomposition as a trace formula for 2d CFTs:

$$\rho_{\text{spec},j}(t) = \frac{1}{\pi} \text{Re} \left(\int_{\mathcal{C}_{\text{crit}}} \{Z_{\text{spec}}, E_{\frac{1}{2}+i\omega}\} A_{\omega,j}^* T_{\omega,j} e^{iS_{\omega,j}(t)} + \sum_{n=1}^{\infty} (Z_{\text{spec}}, \phi_n) A_{n,j} T_{\omega_n,j} e^{iS_{\omega_n,j}(t)} \right) \tag{2.3.20}$$

This holds for any fixed j .

A key fact about this formula, to be used later, is that the set of orbits forms a complete and explicit ($SL(2, \mathbb{Z})$ -invariant) eigenbasis: given a determinant factor $A_{\omega,j}$, one can uniquely reconstruct the corresponding orbit action.

A small but notable point is that $A_{\omega,j}$ is complex by a pure phase. This is reminiscent of something in periodic orbit theory: there exists a phase, the so-called ‘‘Maslov phase’’ $e^{i\frac{\pi}{2}\mu_\gamma}$, that one can choose to absorb into A_γ rather than $S_\gamma(E)$ such that $S_\gamma(E) \in \mathbb{R}$.¹³ This suggests that we identify a CFT analog of the Maslov index, call it $\mu_{\omega,j}$, defined so that

¹³The Maslov index for periodic orbits is a generalized topological invariant which counts windings around a certain submanifold of phase space [59].

$A_{\omega,j} e^{i\frac{\pi}{2}\mu_{\omega,j}} \in \mathbb{R}$:

$$\frac{\pi}{2}\mu_{\omega,j} := \omega \log j. \quad (2.3.21)$$

While (2.3.21) is so far intended to be a purely functional observation, it raises the obvious question of the physical meaning of this CFT Maslov index. In any case, we see that the identification (2.3.18) admits a non-trivial compatibility with the analytic structure of periodic orbits of quantum systems, revealed by imposing smoothness of the $j \rightarrow 0$ limit.

Before making use of this 2d CFT trace formula, we pause to comment on the relation to trace formulas more broadly. The Gutzwiller trace formula can be applied only for systems which admit a semiclassical limit. On the other hand, the $SL(2, \mathbb{Z})$ spectral decomposition is valid for any 2d CFT. Moreover, the Gutzwiller trace formula, when applicable, describes only the high energy ($E \gg 1$) part of the spectrum, while the spectral decomposition is valid for all energies above threshold, thanks to the inherent distinction between light and heavy states due to modular invariance. In these respects, the spectral decomposition is more similar to the Selberg trace formula, an exact relation between the spectrum of the automorphic Laplacian on hyperbolic quotient manifolds and their classical periodic orbits. Nevertheless, we frame the analogy with respect to Gutzwiller, which is more generally applicable, in order to emphasize the universal structure of the $SL(2, \mathbb{Z})$ spectral decomposition.

2.3.3 Berry's diagonal approximation for 2d CFT

Inspired by the treatment of coarse-grained spectral correlations in chaotic systems, we now proceed analogously, in order to motivate a precise and mathematically well-defined version of Berry's diagonal approximation for 2d CFT.

A standard assumption in the theory of periodic orbits is that coarse-graining over an energy window δE smooths out the oscillatory part of the density of states,

$$\overline{\rho_{\text{osc}}(E)} = 0. \quad (2.3.22)$$

The energy window is taken to be a mesoscopic scale, much larger than the mean level spacing $\bar{\rho}(E)^{-1}$ but much smaller than the scale over which the mean density varies noticeably.¹⁴ This assumption of phase incoherence means that any nonzero phase factor appearing in a density correlator is taken to approximately vanish upon coarse-graining.

Naturally, we employ the same approach to the CFT density $\rho_j(t)$. By analogy to (2.3.22), the density of states $\rho_{\text{spec},j}(t)$ is taken to average to zero upon microcanonical coarse-graining over a mesoscopic window δt ,

$$\overline{\rho_{\text{spec},j}(t)} = 0. \quad (2.3.23)$$

We expect that this holds for δt larger than the mean level spacing, yet smaller than the

¹⁴See for example Chapter 10.6 of [59] and Section 2 of [132].

characteristic variation of the mean density, both controlled by $\widehat{\rho}_{L,j}(t)$. One can, for instance, perform microcanonical coarse-graining by convolution against a window function $W(t-t')$ with characteristic width δt :¹⁵

$$\overline{f(t)} := \int_0^\infty W(t-t')f(t')dt', \quad \text{with} \quad \int_0^\infty W(t)dt = 1. \quad (2.3.25)$$

The density of light states $\widehat{\rho}_{L,j}(t)$ is instead approximately self-averaging, as its behavior in its argument is exponential rather than oscillatory. Hence we have that

$$\overline{\rho_j(t)} \approx \widehat{\rho}_{L,j}(t). \quad (2.3.26)$$

We now turn to two-point correlations. Again following a standard approach in quantum chaos, we define the microcanonical coarse-graining for the product of two densities by integrating over the mean twist $t := \frac{t_1+t_2}{2}$ while keeping the difference $\epsilon := \frac{t_1-t_2}{2}$ fixed:

$$\overline{f(t_1)f(t_2)} := \int_0^\infty dt' f(t'+\epsilon)f(t'-\epsilon)W(t-t') \quad (2.3.27)$$

Coarse-graining the product of densities $\overline{\rho_{\text{spec},j_1}(t+\epsilon)\rho_{\text{spec},j_2}(t-\epsilon)}$ produces a sum/integral of terms of the following form:

$$\overline{\rho_{\frac{1}{2}+i\omega_1,j_1}(t+\epsilon)\rho_{\frac{1}{2}+i\omega_2,j_2}(t-\epsilon)} = \frac{A_{\omega_1,j_1}\overline{A_{\omega_2,j_2}}}{4\pi^2} T_{\omega_1,j_1} T_{\omega_2,j_2} \overline{e^{i(S_{\omega_1,j_1}(t+\epsilon)-S_{\omega_2,j_2}(t-\epsilon))}} + \text{c.c.} \quad (2.3.28)$$

By way of the earlier assumption, terms involving the sum of actions are smoothed out by the averaging,

$$\overline{e^{i(S_{\omega_1,j_1}(t+\epsilon)+S_{\omega_2,j_2}(t-\epsilon))}} = 0. \quad (2.3.29)$$

Instead, terms involving the difference in actions $\Delta S = S_{\omega_1,j_1}(t) - S_{\omega_2,j_2}(t)$ can give large contributions if the actions cancel. Concentrating on the leading terms, with $\Delta S = 0$, is the analog of Berry's diagonal approximation.

To see what this approximation means in the case of 2d CFT, we choose to consider the Fourier mode of equal spins, $j_1 = j_2 = j$. The above microcanonical coarse-graining then

¹⁵Coarse-graining the individual basis elements of the $SL(2, \mathbb{Z})$ decomposition gives

$$\overline{\rho_{\frac{1}{2}+i\omega,j}(t)} = \frac{1}{\pi\delta t} \text{Re} \left[A_{\omega,j} \int_{t-\delta t}^{t+\delta t} T_{\omega,j}(t') e^{iS_{\omega,j}(t')} dt' \right] = \frac{1}{\pi\delta t} \text{Re} \left[A_{\omega,j} \int_{S_{\omega,j}(t-\delta t)}^{S_{\omega,j}(t+\delta t)} e^{iS_{\omega,j}(t)} dS_{\omega,j} \right]. \quad (2.3.24)$$

This is a simple oscillatory integral which vanishes for a suitably chosen scale δt , derivable from the action (2.3.18). This scale depends on the frequency ω and spin j . The assumption from periodic orbit theory is that there exists a finite scale δt with respect to which the average of the entire sum vanishes.

selects the terms with equal actions $S_{\omega_1, j}(t) = S_{\omega_2, j}(t)$, which sets $\omega_1 = \omega_2$:

$$\begin{aligned} \overline{\rho_{\frac{1}{2}+i\omega_1, j}(t+\epsilon)\rho_{\frac{1}{2}+i\omega_2, j}(t-\epsilon)} &\propto e^{i(S_{\omega_1, j_1}(t+\epsilon)-S_{\omega_2, j_2}(t-\epsilon))} + \text{c.c.} \\ &\propto \delta_{\omega_1, \omega_2} \end{aligned} \tag{2.3.30}$$

where we used (2.3.29) and took $\epsilon \rightarrow 0$ to focus on nearby energy levels. We call the resulting diagonal density $\rho_{\text{diag}}(t_1, t_2)$, the projection of the product of $SL(2, \mathbb{Z})$ spectral decompositions onto terms with equal eigenvalues. Subsequently, all spin sectors (j_1, j_2) are fixed by $SL(2, \mathbb{Z})$ -invariance of the full density, in terms of the respective basis densities of the Eisensteins and cusp forms:

$$\begin{aligned} \rho_{\text{diag}}^{(j_1, j_2)}(t_1, t_2) &= \int_{\mathcal{C}_{\text{crit}}} \{Z_{\text{spec}}, E_{\frac{1}{2}+i\omega}\}^2 \rho_{\frac{1}{2}+i\omega, j_1}^*(t_1) \rho_{\frac{1}{2}+i\omega, j_2}^*(t_2) + \sum_{n=1}^{\infty} (Z_{\text{spec}}, \phi_n)^2 \rho_{n, j_1}(t_1) \rho_{n, j_2}(t_2) \\ &\quad + \sum_{n=1}^{\infty} \{Z_{\text{spec}}, E_{\frac{1}{2}+i\omega_n}\} (Z_{\text{spec}}, \phi_n) \left(\rho_{\frac{1}{2}+i\omega_n, j_1}^*(t_1) \rho_{n, j_2}(t_2) + \rho_{n, j_1}(t_1) \rho_{\frac{1}{2}+i\omega_n, j_2}^*(t_2) \right) \end{aligned} \tag{2.3.31}$$

We point out that the density is not diagonal in spin: $\rho_{\text{diag}}^{(j_1, j_2)}(t) \neq 0$ for $|j_1| \neq |j_2|$. This is a manifestation of the general fact that modular invariance correlates different spin sectors of 2d CFT data. In Appendix 2.11 we give a quick proof that diagonality in spin is incompatible with modular invariance. In contrast, one expects different spin sectors to be statistically independent in generic chaotic systems.

Note that there are cross-terms on the second line of (2.3.31) because of the spectral degeneracy between cusp forms and Eisenstein series at $\omega = \omega_n$. These cross-terms pair *distinct* orbits, with the same action $S_{\omega, j}(t)$ but different one-loop determinants $A_{\omega, j}$. This suggests that one should really seek a diagonal projection which pairs eigenfunctions rather than eigenvalues.

This concludes our motivation from microcanonical coarse-graining and the diagonal approximation in the periodic orbit approach to quantum systems. We now proceed to rigorously define the 2d CFT diagonal approximation in the canonical ensemble, by constructing diagonal products of partition functions. We first define what we call $Z_{\text{diag}}(\tau_1, \tau_2)$, the canonical counterpart of $\rho_{\text{diag}}(t_1, t_2)$. We then pass to an object that we call $Z_{\text{Hecke}}(\tau_1, \tau_2)$, a diagonal partition function that properly pairs identical orbits.

Diagonal projection I

The density $\rho_{\text{diag}}(t_1, t_2)$ corresponds, in the canonical ensemble, to the following *diagonal partition function*:

$$\begin{aligned} Z_{\text{diag}}(\tau_1, \tau_2) = & \int_{\mathcal{C}_{\text{crit}}} \{Z_{\text{spec}}, E_s\}^2 E_s^*(\tau_1) E_s^*(\tau_2) + \sum_{n=1}^{\infty} (Z_{\text{spec}}, \phi_n)^2 \phi_n(\tau_1) \phi_n(\tau_2) \\ & + \sum_{n=1}^{\infty} \{Z_{\text{spec}}, E_{s_n}\} (Z_{\text{spec}}, \phi_n) \left(\phi_n(\tau_1) E_{s_n}^*(\tau_2) + \phi_n(\tau_2) E_{s_n}^*(\tau_1) \right), \end{aligned} \quad (2.3.32)$$

The Laplace eigenvalues of the $SL(2, \mathbb{Z})$ eigenbasis elements are paired. This is the *diagonal projection*, $Z_{\text{diag}}(\tau_1, \tau_2)$, of the factorized product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$. To formalize this, consider the operator

$$\Delta_{12} := \Delta_{\tau_1} - \Delta_{\tau_2} \quad (2.3.33)$$

acting on functions $f(\tau_1, \tau_2) \in L^2(\mathcal{F} \times \mathcal{F})$, the space of square-integrable functions of two moduli τ_1, τ_2 valued in \mathcal{F} . Such functions admit a double spectral decomposition in a joint basis of eigenfunctions.¹⁶ Since $f(\tau_1, \tau_2)$ has paired eigenvalues if and only if $\Delta_{12}f(\tau_1, \tau_2) = 0$ for all τ_1, τ_2 , we can define $Z_{\text{diag}}(\tau_1, \tau_2)$ as the diagonal projection of $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$ onto $\ker(\Delta_{12})$, the kernel of Δ_{12} :

$$\boxed{Z_{\text{diag}}(\tau_1, \tau_2) := \mathcal{P}_{\text{diag}} [Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)] , \quad \text{where} \quad \mathcal{P}_{\text{diag}} := \mathcal{P}_{\ker(\Delta_{12})}} \quad (2.3.34)$$

$Z_{\text{diag}}(\tau_1, \tau_2)$ is a manifestly modular-invariant analog of Berry's diagonal approximation for 2d CFT. $Z_{\text{diag}}(\tau_1, \tau_2)$ does not factorize, unlike $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$. Factorization can be explicitly restored by including the off-diagonal terms. In Section 2.6.3 we will develop an analogy to the approach to factorization in 2D gravity and random matrix theory.

Diagonal projection II: Hecke projection

In order to eliminate the cross terms in $Z_{\text{diag}}(\tau_1, \tau_2)$ which do not pair identical orbits, we introduce a slightly souped-up diagonal partition function which we call $Z_{\text{Hecke}}(\tau_1, \tau_2)$. Stated succinctly, whereas $Z_{\text{diag}}(\tau_1, \tau_2)$ pairs *eigenvalues*, $Z_{\text{Hecke}}(\tau_1, \tau_2)$ pairs *eigenfunctions*.

The new projection may be defined using $SL(2, \mathbb{Z})$ Hecke operators. Let us recall their definition. Hecke operators T_j exist for every spin $j \in \mathbb{Z}_+$. Their action on $SL(2, \mathbb{Z})$ -invariant

¹⁶A few remarks on the $L^2(\mathcal{F} \times \mathcal{F})$ spectral decomposition, and some technical remarks on scheme-dependence of the projection $\mathcal{P}_{\text{diag}}$, are given in Appendix 2.9.

functions $f(\tau)$ is¹⁷

$$T_j f(\tau) = \frac{1}{\sqrt{j}} \sum_{ad=j, d>0} \sum_{b=0}^{d-1} f\left(\frac{a\tau + b}{d}\right) \quad (2.3.35)$$

In this normalization, the Hecke action on the $SL(2, \mathbb{Z})$ eigenbasis elements is

$$\begin{aligned} T_j E_s^*(\tau) &= \frac{\mathbf{a}_j^{(s)}}{2} E_s^*(\tau) \\ T_j \phi_n(\tau) &= \tilde{\mathbf{b}}_j^{(n)} \phi_n(\tau) \end{aligned} \quad (2.3.36)$$

where $\tilde{\mathbf{b}}_j^{(n)} := \mathbf{b}_j^{(n)} / \mathbf{b}_1^{(n)}$. An equivalent definition, which will be useful momentarily, is

$$T_j f(\tau) = \frac{1}{\sqrt{j}} \sum_{A \in SL(2, \mathbb{Z}) \setminus M_j} f(A\tau) \quad (2.3.37)$$

where M_j denotes the following set of matrices,

$$M_j = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}; ad - bc = j \right\} \quad (2.3.38)$$

and $A\tau$ denotes the fractional linear transformation,

$$A\tau = \frac{a\tau + b}{c\tau + d}, \quad A\bar{\tau} = \frac{a\bar{\tau} + b}{c\bar{\tau} + d}. \quad (2.3.39)$$

The quotient $SL(2, \mathbb{Z}) \setminus M_j$ is taken by identifying elements under the equivalence relation $A \sim B$ if there exists a $\gamma \in SL(2, \mathbb{Z})$ such that $B = \gamma A$. We emphasize that $j \in \mathbb{Z}_+$.

We now introduce $Z_{\text{Hecke}}(\tau_1, \tau_2)$, the projection of $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$ onto the kernel of the *difference* of Hecke operators at every spin. Defining

$$T_j^{(12)} := T_j^{(\tau_1)} - T_j^{(\tau_2)}, \quad (2.3.40)$$

we define the *Hecke projection* onto $\ker(T_j^{(12)})$,

$$Z_{\text{Hecke}}(\tau_1, \tau_2) := \mathcal{P}_{\text{Hecke}} \left[Z_{\text{spec}}(\tau_1) Z_{\text{spec}}(\tau_2) \right], \quad \text{where } \mathcal{P}_{\text{Hecke}} := \mathcal{P}_{\ker(T_j^{(12)})} \quad \forall j \in \mathbb{Z}_+$$

(2.3.41)

The essential feature of $Z_{\text{Hecke}}(\tau_1, \tau_2)$ is seen upon considering the action of $T_j^{(12)}$ on “mixed”

¹⁷Hecke operators for non-prime spins j are fixed in terms of the prime spins via the Hecke multiplication rule. For a recent review of Hecke operators in a physics context see [140].

Eisenstein-cusp form terms in $Z_{\text{diag}}(\tau_1, \tau_2)$:

$$T_j^{(12)} \left(E_{s_n}^*(\tau_1) \phi_n(\tau_2) \right) = \left(\frac{\mathbf{a}_j^{(s)}}{2} - \tilde{\mathbf{b}}_j^{(n)} \right) E_{s_n}^*(\tau_1) \phi_n(\tau_2) \quad (2.3.42)$$

This is nonzero on account of the inequality of Eisenstein and cusp form Fourier coefficients. While such terms are present in $Z_{\text{diag}}(\tau_1, \tau_2)$, they are eliminated by the Hecke projection (2.3.41) by design. Note that the object $Z_{\text{Hecke}}(\tau_1, \tau_2)$ is defined via projection for *all* spins j .¹⁸ The spectral decomposition of $Z_{\text{Hecke}}(\tau_1, \tau_2)$ is thus

$$Z_{\text{Hecke}}(\tau_1, \tau_2) = \int_{\mathcal{C}_{\text{crit}}} |(Z_{\text{spec}}, E_s)|^2 E_s(\tau_1) E_{1-s}(\tau_2) + \sum_{n=1}^{\infty} (Z_{\text{spec}}, \phi_n)^2 \phi_n(\tau_1) \phi_n(\tau_2) \quad (2.3.43)$$

We may rephrase the special property of $Z_{\text{Hecke}}(\tau_1, \tau_2)$ as a statement of enhanced symmetry: employing the terminology of [142], we say that a modular-invariant function $f(\tau_1, \tau_2)$ enjoys *Hecke symmetry* if

$$T_j^{(12)} f(\tau_1, \tau_2) = 0 \quad \forall \quad j \in \mathbb{Z}_+ \quad (\text{Hecke symmetry}) \quad (2.3.44)$$

In this sense, $Z_{\text{Hecke}}(\tau_1, \tau_2)$ carries an enhanced symmetry with respect to both $Z_{\text{diag}}(\tau_1, \tau_2)$ and $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$: it is annihilated by an infinite set of commuting operators, one for each positive integer.

Let us recapitulate the idea behind $Z_{\text{Hecke}}(\tau_1, \tau_2)$. It is a fact of life on the $SL(2, \mathbb{Z})$ fundamental domain that there is spectral degeneracy for the infinite discretuum of Laplace eigenvalues $\omega = \omega_n$ where Maass cusp forms exist. From the point of view of the trace formula and analogy to periodic orbit theory, the Eisensteins and Maass cusp forms with equal eigenvalues appear as different orbits: their actions are equal, but their one-loop determinants differ. The Hecke projection $\mathcal{P}_{\text{Hecke}}$ is designed to account for this innate quirk of 2d CFT. We are proposing Hecke projection as an enhanced form of coarse-graining. It would be nice to understand this more fundamentally, say, from a microcanonical perspective.

For these reasons, we view $Z_{\text{Hecke}}(\tau_1, \tau_2)$ as the “right” two-copy partition function which diagonalizes the chaotic spectral correlations. This view will find further support in Section 2.5, as the Hecke projection really comes to life when we consider its dual gravitational interpretation. In particular, we will show that Hecke symmetry is an emergent feature of Euclidean torus wormholes in semiclassical AdS_3 gravity.

¹⁸We are being rather conservative in defining Hecke projection for all spins j , in order to guard against the possibility that $\mathbf{a}_j^{(s)} = 2\tilde{\mathbf{b}}_j^{(n)}$ for very special choices of n and j – though even this seems extremely unlikely (and perhaps is provably false). On the other hand, it is highly inconceivable that the Fourier coefficients could be equal for *all* j , given the numerical and statistical properties of cusp forms – for example, the available numerical data at finite n [141], and the Sato-Tate conjectures on the statistical distribution of $\tilde{\mathbf{b}}_j^{(n)}$ at $j \rightarrow \infty$ or $n \rightarrow \infty$.

2.4 Random Matrix Universality in Chaotic 2d CFTs

Having identified a 2d CFT trace formula using the $SL(2, \mathbb{Z})$ spectral decomposition of partition functions, we presented a framework for studying coarse-grained CFT correlations with modular invariance and Virasoro symmetry baked in, focusing on the diagonal approximation in particular. We now analyze general properties of the diagonal partition function and the implications of chaos for the CFT. This will lead to a necessary and sufficient condition for the presence of a linear ramp in the spectral form factor (SFF).

The partition functions $Z_{\text{diag}}(\tau_1, \tau_2)$ and $Z_{\text{Hecke}}(\tau_1, \tau_2)$ admit double Fourier decompositions, e.g.

$$Z_{\text{diag}}(\tau_1, \tau_2) = \sum_{j_1=0}^{\infty} (2 - \delta_{j_1,0}) \cos(2\pi j_1 x_1) \sum_{j_2=0}^{\infty} (2 - \delta_{j_2,0}) \cos(2\pi j_2 x_2) Z_{\text{diag}}^{(j_1, j_2)}(y_1, y_2) \quad (2.4.1)$$

and likewise for $Z_{\text{Hecke}}(\tau_1, \tau_2)$. In this section we specialize to the scalar sector, $(j_1, j_2) = (0, 0)$; note that $Z_{\text{Hecke}}^{(0,0)}(y_1, y_2) = Z_{\text{diag}}^{(0,0)}(y_1, y_2)$ because the cusp forms have vanishing scalar component.¹⁹ Besides reasons of clarity and simplicity, we do so because we will later consider a special class of amplitudes – namely, wormhole amplitudes – which are *fully determined* by the scalar sector.

In the spectral basis,

$$Z_{\text{diag}}^{(0,0)}(y_1, y_2) = \frac{1}{4\pi i} \int_{\text{Re } s = \frac{1}{2}} ds \{Z_{\text{spec}}, E_s\} \{Z_{\text{spec}}, E_{1-s}\} E_{1-s,0}^*(y_1) E_{s,0}^*(y_2) \quad (2.4.2)$$

where we have written the integral over the critical line explicitly. Inserting the zero mode

$$E_{s,0}^*(y) = \Lambda(s) \left(y^s + \varphi(s) y^{1-s} \right), \quad \varphi(s) := \frac{\Lambda(1-s)}{\Lambda(s)} \quad (2.4.3)$$

and using symmetry under reflections $s \rightarrow 1-s$ yields

$$Z_{\text{diag}}^{(0,0)}(r, z) = \frac{1}{2\pi i} \int_{\text{Re } s = \frac{1}{2}} ds |(Z_{\text{spec}}, E_s)|^2 \left(r z^{\frac{1}{2}-s} + r^{2s} \varphi(1-s) \right) \quad (2.4.4)$$

where we have defined

$$r := \sqrt{y_1 y_2}, \quad z := \frac{y_1}{y_2} \quad (2.4.5)$$

Recall that $|f(s)|^2 := f(s)f(1-s)$. We can write this as

$$Z_{\text{diag}}^{(0,0)}(r, z) = r\sqrt{z}\mathcal{R}(z) + \mathcal{S}(r) \quad (2.4.6)$$

¹⁹We use the “diag” subscript in this section to emphasize the physical setting.

where

$$\begin{aligned}\mathcal{R}(z) &:= \mathcal{M}^{-1}\left[|(Z_{\text{spec}}, E_s)|^2; z\right] \\ \mathcal{S}(r) &:= \mathcal{M}^{-1}\left[|(Z_{\text{spec}}, E_s)|^2 \varphi(1-s); r^{-2}\right]\end{aligned}\tag{2.4.7}$$

are inverse Mellin transforms of the squared overlap, defined in general as

$$\mathcal{M}^{-1}[f(s); x] = \frac{1}{2\pi i} \int_{\mathcal{C}} ds f(s) x^{-s}\tag{2.4.8}$$

The contour \mathcal{C} is a vertical contour within the critical strip of the inverse Mellin transform [143], which for our purposes may be taken to be $\text{Re } s = \frac{1}{2}$.²⁰

Lest it appear that there are two independent functions in (2.4.6), we emphasize that $\mathcal{S}(r)$ is completely determined by $\mathcal{R}(z)$, since the latter may be used to reconstruct $|(Z_{\text{spec}}, E_s)|^2$. This determinism can be formalized using techniques of Mellin convolution (see Appendix 2.12), giving

$$\mathcal{S}(r) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^2} \int_0^1 \frac{du}{\sqrt{u(1-u)}} \mathcal{R}\left(\frac{u}{n^2 r^2}\right)\tag{2.4.9}$$

where $\phi(n)$ is the Euler totient function. An equivalent representation in terms of residues of $\mathcal{R}(z)$, given in (2.12.11), may be obtained by blowing up the contour. So in total,

$$Z_{\text{diag}}^{(0,0)}(r, z) = r\sqrt{z}\mathcal{R}(z) + \sum_{n=1}^{\infty} \frac{\phi(n)}{n^2} \int_0^1 \frac{du}{\sqrt{u(1-u)}} \mathcal{R}\left(\frac{u}{n^2 r^2}\right)\tag{2.4.10}$$

Note the $z \rightarrow 1/z$ inversion symmetry of $\sqrt{z}\mathcal{R}(z)$, inherited from the inverse Mellin transform.

The above expressions are valid for all temperatures. We now take the low-temperature limit:

$$r \gg 1, \quad z \text{ fixed}.\tag{2.4.11}$$

This limit includes two physically distinct regimes of interest (see Figure 2.2), depending on how we take y_i to infinity in the complex plane:

- In “Euclidean” signature where $y_i \in \mathbb{R}$, this is a low-temperature limit, and $z \in \mathbb{R}_+$.
- In Lorentzian signature, analytically continuing as one does to construct the SFF,

$$y_1 \rightarrow \beta + iT, \quad y_2 \rightarrow \beta - iT.\tag{2.4.12}$$

The limit (2.4.11) is a simultaneous late-time/low-temperature limit of $T \gg 1$ and fixed

²⁰If there are poles on the contour, they must be regulated for the spectral decomposition to formally exist. This can be done, e.g. by shifting the poles to $\text{Re } s = \frac{1}{2} \pm \epsilon$. We neglect this possibility in what follows.

T/β , with $z \in \mathbb{H}$ on the unit circle:

$$z := e^{i\theta}, \quad \theta = 2 \tan^{-1} \left(\frac{T}{\beta} \right) \in [0, \pi] \quad (2.4.13)$$

In either case, applying the limit (2.4.11) to (2.4.4), we must deform the contour of the second term to the left. Consequently the $\mathcal{S}(r)$ term is subleading:²¹

$$Z_{\text{diag}}^{(0,0)}(r, z) \approx r\sqrt{z}\mathcal{R}(z) \quad (r \gg 1, z \text{ fixed}) \quad (2.4.14)$$

This result is interesting. First, the behavior is universally linear in r . More importantly, since $\mathcal{R}(z)$ fixes the entire function $Z_{\text{diag}}^{(0,0)}(r, z)$ via (2.4.10), *the leading result at low temperatures fixes the result at all temperatures*. Indeed, it fixes the entire Eisenstein part of $Z_{\text{diag}}(\tau_1, \tau_2)$. As we will see later, for two-copy partition functions with enhanced symmetries, $\mathcal{R}(z)$ fixes the function completely.

2.4.1 A condition for a linear ramp

The spectral form factor, $K_\beta(T)$, is defined for quantum systems as

$$K_\beta(T) := Z(\beta + iT)Z(\beta - iT) \quad (2.4.15)$$

where $Z(\beta + iT) = \text{Tr}(e^{-(\beta+iT)H})$ is the analytically-continued sum over states. In 2d CFT, it is cleaner to account for Virasoro descendants by defining the SFF with respect to the Virasoro primary partition function,

$$K_\beta(T) := \frac{1}{\sqrt{\beta^2 + T^2}} \langle Z_p(\beta + iT)Z_p(\beta - iT) \rangle \quad (2.4.16)$$

where the brackets denote a coarse-graining of some kind. The prefactor, which amounts to stripping off the $\sqrt{y_1 y_2}$ coming from using the *primary* partition function (before analytic continuation), is a useful convention for the late-time limit and contact with previous literature (e.g. [34, 64, 69]).

We can also define a SFF using $Z_{\text{spec}}(\tau)$ instead,

$$K_{\text{spec}}(\beta; T) := \frac{1}{\sqrt{\beta^2 + T^2}} \langle Z_{\text{spec}}(\beta + iT)Z_{\text{spec}}(\beta - iT) \rangle \quad (2.4.17)$$

Now, since $Z_{\text{spec}}(\tau)$ and $Z_p(\tau)$ differ only by terms which contribute to the slope/dip of the

²¹This is the same mechanism that operates in the spectral decomposition of $\mathcal{N} = 4$ super Yang-Mills observables in the large N 't Hooft limit [121]. There, the spectral integrand contains a sum of terms with powers N^{1-2s} and N^0 . The former is suppressed, so the latter gives the full planar result.

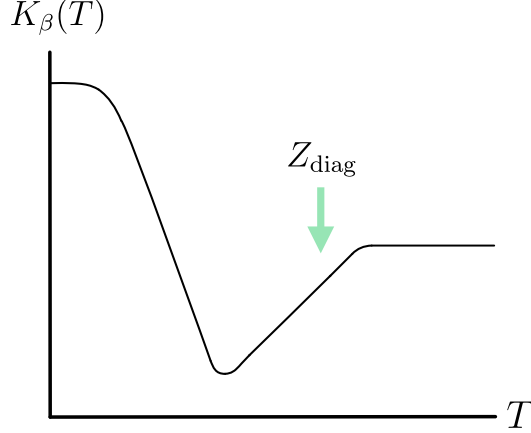


Figure 2.5: Z_{diag} captures the ramp in the spectral form factor $K_\beta(T)$ of a 2d CFT.

SFF [144, 145],²² the difference between $K_{\text{spec}}(\beta; T)$ and $K_\beta(T)$ vanishes at late times:

$$K_\beta(T) \approx K_{\text{spec}}(\beta; T) \quad (T \gg \beta) \quad (2.4.18)$$

Our spectral formalism is tailor-made to study the SFF in the late-time regime: the quantity $Z_{\text{diag}}(\tau_1, \tau_2)$, constructed from $Z_{\text{spec}}(\tau)$, is designed precisely as a coarse-grained product of partition functions in the diagonal approximation, whose point is to produce the ramp. So our claim is that

$$K_\beta(T) \approx \frac{1}{T} Z_{\text{diag}}(\beta; T) \quad (T \gg \beta) \quad (2.4.19)$$

where $Z_{\text{diag}}(\beta; T)$ is the analytic continuation of $Z_{\text{diag}}(\tau_1, \tau_2)$ to SFF kinematics. (Implicitly, here and below, T is bounded above by the plateau time.)

Starting from $Z_{\text{diag}}(\tau_1, \tau_2)$, we construct the SFF by analytically continuing as in (2.4.12). Denote the spin-graded SFF as $K_\beta^{(j_1, j_2)}(T)$. We specialize to the case $(j_1, j_2) = (0, 0)$, whereupon (2.4.14) and (2.4.19) imply that

$$K_\beta^{(0,0)}(T) = \sqrt{\frac{\beta + iT}{\beta - iT}} \mathcal{R} \left(\frac{\beta + iT}{\beta - iT} \right) + (\text{subleading at } T \gg \beta) \quad (2.4.20)$$

We now impose the ramp. From (2.4.13) and (2.4.20), the presence of a *linear* ramp is

²²This is because the difference, $\widehat{Z}_L(\tau)$, is a sum over self-averaging quantities. (If $Z_L(\tau)$ sums over an infinite set of light operators, we are assuming that this sum can be regulated.) Supporting arguments from other perspectives can be found in the literature. In 2D gravity, the ramp is generated by a *new* bulk saddle [69], with similar arguments for one-point wormholes [146] and in higher dimensions [64, 69]. It was argued in [144, 145] in the AdS₃ context that Poincaré completions of individual operators (i.e. sums over smooth saddles) contribute only to the dip. In the Poincaré completion of individual light states, the modular images are self-averaging, with a density that is a sum of smooth functions growing exponentially in energy (see e.g. eq. (2.3) of [96]). Similar arguments can be made for enigmatic AdS₃ black holes, which are also smooth saddles (when they exist).

equivalent to the presence of a simple pole in the analytic continuation of $\mathcal{R}(z)$ to $z = -1$:

$$\boxed{K_\beta^{(0,0)}(T \rightarrow \infty) \sim \frac{T}{4\pi\beta} \mathbf{C}_{\text{RMT}} \quad \Leftrightarrow \quad \mathcal{R}(z \rightarrow -1) \sim \frac{\mathbf{C}_{\text{RMT}}}{2\pi} \frac{1}{1+z}} \quad (2.4.21)$$

\mathbf{C}_{RMT} is a constant that fixes the choice of RMT ensemble governing the ramp.²³ Since $\mathcal{R}(z)$ is simply the inverse Mellin transform (2.4.7) of the spectral overlap, the pole condition (2.4.21) is equivalent to an asymptotic property of the spectral overlap (Z_{spec}, E_s) on the critical line $s = \frac{1}{2} + i\omega$. In the parameterization (2.4.13),

$$2\pi(z+1)\mathcal{R}(z) \Big|_{z=e^{i\theta}} = 2 \cos \frac{\theta}{2} \int_{-\infty}^{\infty} d\omega |(Z_{\text{spec}}, E_{\frac{1}{2}+i\omega})|^2 \cosh \omega\theta \quad (2.4.22)$$

This only satisfies (2.4.21) if the integral diverges linearly. This implies the following condition:

$$\boxed{(\text{Linear ramp}) \quad \Leftrightarrow \quad |(Z_{\text{spec}}, E_{\frac{1}{2}+i\omega})|^2 \sim e^{-\pi\omega} f(\omega) \quad (\omega \rightarrow \infty)} \quad (2.4.23)$$

where $f(\omega)$ is allowed to fluctuate around \mathbf{C}_{RMT} , but is flat at infinity “on average”,

$$\lim_{W \rightarrow \infty} \frac{1}{W} \int_{\omega_0}^{\omega_0+W} d\omega f(\omega) = \mathbf{C}_{\text{RMT}} \quad (2.4.24)$$

for any finite ω_0 .

Equations (2.4.23) and (2.4.24), equivalently (2.4.21), comprise a necessary and sufficient condition for the presence of a linear ramp in the (scalar) SFF of a 2d CFT, with the constant in (2.4.24) set by the chosen RMT ensemble governing the ramp. The linear ramp, a property of the coarse-grained theory, is transmuted into a quantitative property of the microscopic spectrum. The algorithm to detect it from a torus partition function is straightforward: form $Z_{\text{spec}}(\tau)$, then look for the asymptotic (2.4.23) in the spectral basis.

Note for context that the convergence of the spectral decomposition only requires the much weaker falloff $|(Z_{\text{spec}}, E_{\frac{1}{2}+i\omega})| \lesssim \mathcal{O}(1)$ as $\omega \rightarrow \infty$, up to power-law and logarithmic corrections (e.g. [121]). Systems for which this upper bound is saturated would have an exponential rather than linear ramp.²⁴ So we see that in the $SL(2, \mathbb{Z})$ spectral basis, random matrix universality becomes a condition of rapid decay at infinity.

²³The GOE ensemble is the relevant one for parity-invariant 2d CFTs [147].

²⁴This was seen in e.g. the quadratic SYK model [148]. Exponential ramps are also characteristic of arithmetic systems [149, 150]

2.4.2 Universal corrections

The symmetries of 2d CFT prescribe corrections to this result. Having extracted the condition (2.4.23) on the overlap, modular invariance implies an infinite set of other terms in $Z_{\text{diag}}^{(0,0)}(r, z)$, coming from the second term in (2.4.10) (the $\mathcal{S}(r)$ term in (2.4.6)). In SFF kinematics these terms give subleading corrections to the RMT ramp at $T \gg \beta$, but are otherwise present and unsuppressed for generic temperatures.

Specifically, the falloff (2.4.23) implies an infinite set of square-root singularities at $z \in \mathbb{R}_-$. The simplest way to see this is to insert the polar behavior (2.4.21) into the second term in (2.4.10). This yields

$$\mathcal{S}(r) \supset \frac{C_{\text{RMT}}}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \frac{r}{\sqrt{1+n^2 r^2}} \quad (2.4.25)$$

The branch point singularities (2.4.25) are universal in any chaotic 2d CFT. Their existence is required by modular invariance (and Virasoro symmetry) upon imposing the linear ramp in the SFF.

Note that in detecting these singularities, we are probing analytic structure of the partition function in the complex-temperature plane: the $r^2 = -1/n^2$ locus is not a “physical” locus, neither in real time T in SFF kinematics (where $r^2 = \beta^2 + T^2$) nor in Euclidean kinematics (where $r^2 = y_1 y_2$). In the limit of $r \rightarrow \infty$ with fixed z , these singularities indicate an accumulation of branch points at $z \rightarrow -n^2 y_1^2$ and their images under inversion. At leading order in $r \rightarrow \infty$, the sum is constant in r but linearly divergent (see [34] and Appendix 2.12).

We note in passing the generalization to non-linear ramps. At leading order,

$$(T^{1-\alpha} \text{ ramp}) \quad \Leftrightarrow \quad |(Z_{\text{spec}}, E_{\frac{1}{2}+i\omega})|^2 \sim \omega^{-\alpha} e^{-\pi\omega} f(\omega) \quad (\omega \rightarrow \infty) \quad (2.4.26)$$

where $f(\omega)$ can again fluctuate around a constant at infinity. The constraints on $\mathcal{S}(r)$ may be likewise derived from (2.4.9), yielding

$$\mathcal{S}(r) \sim \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \frac{r}{(1+n^2 r^2)^{\frac{1}{2}-\alpha}}, \quad \left(r^2 \rightarrow -\frac{1}{n^2} \quad \forall n \in \mathbb{Z}_+ \right) \quad (2.4.27)$$

We note that if $\alpha > \frac{1}{2}$, the corrections are term-wise finite in r . This singles out \sqrt{T} behavior of the late-time SFF as a somewhat notable threshold from the analyticity point of view.

2.5 Wormholes and Hecke Symmetry

In Section 2.3.3 we introduced a coarse-grained two-copy partition function, $Z_{\text{Hecke}}(\tau_1, \tau_2)$, which projects the factorized product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$ onto its diagonal subspace. The

centrality of $Z_{\text{Hecke}}(\tau_1, \tau_2)$ for analyzing chaotic spectral correlations in 2d CFT really comes to life upon considering its dual gravitational interpretation, as we now address a driving question of this work: where are the wormholes? As we show in this section, Hecke symmetry is an emergent feature of Euclidean torus wormholes in semiclassical AdS₃ gravity.

2.5.1 Interlude: A Wormhole Farey Tail

The next subsection establishes some technical results about $SL(2, \mathbb{Z})$ Poincaré sums over appropriate “seed” functions $f_0(\tau_1, \tau_2)$ of two complex moduli τ_1 and τ_2 . Sums of this form may be taken as one definition of “wormhole amplitudes” in semiclassical theories of gravity.

To motivate this, and to explain the title of this subsection, let us recall the (non-supersymmetric) black hole Farey tail [97, 113, 114]. The partition sum over all smooth saddles \mathcal{M} of semiclassical Einstein gravity with $\partial\mathcal{M} = T^2$ can be written as a (regularized) $SL(2, \mathbb{Z})$ Poincaré sum over the partition function of thermal AdS₃: writing the primary partition function,

$$Z_{\text{MWK}}(\tau) = \sum_{\gamma \in SL(2, \mathbb{Z})/\Gamma_\infty} \sqrt{|\text{Im}(\gamma\tau)|} |q_\gamma^{-\xi} (1 - q_\gamma)|^2 \quad (2.5.1)$$

The sum runs over the full “ $SL(2, \mathbb{Z})$ family” of BTZ black holes, one for each independent $SL(2, \mathbb{Z})$ element. The essential point is that the one-to-one correspondence between smooth saddles and $SL(2, \mathbb{Z})$ images of thermal AdS₃ is not merely a technical observation: these saddles are generated by large bulk diffeomorphisms acting on the vacuum contribution to the path integral, inducing a boundary $SL(2, \mathbb{Z})$ action.

The idea that Poincaré sums are how semiclassical gravity implements boundary modular invariance in AdS₃/CFT₂ is independent of the number of asymptotic boundaries. As long as the bulk theory respects diffeomorphism invariance, large diffeomorphisms will again relate different contributions to the path integral, implementing $SL(2, \mathbb{Z})$ transformations (or the appropriate higher-genus generalization) of boundary moduli. The obvious question is what the analog of the vacuum solution is. Consider now bulk topologies \mathcal{M} with two asymptotic torus boundaries, $\partial\mathcal{M} = T^2 \cup T^2$. In the example of semiclassical Einstein gravity, as we will recall in Section 2.6, there is no smooth on-shell solution with this topology. Nevertheless, diffeomorphism invariance implies that if there exists such an amplitude – perhaps off-shell, justifiable one way or another – it should take the form of a Poincaré sum over a suitable seed function $f_0(\tau_1, \tau_2)$, with the image sum geometrizing the boundary modular invariance.²⁵

²⁵This view receives indirect formal support from [107], where it is argued that the canonical quantization of AdS₃ gravity on hyperbolic manifolds M produces sums of the form $Z(M) = \sum_\gamma Z_0(M_\gamma)$, where Z_0 is the bulk gravity partition function on a fixed manifold and $\gamma \in \text{Map}(\partial M)/\text{Map}(M, \partial M)$, where $\text{Map}(\partial M)$ and $\text{Map}(M, \partial M)$ are the boundary and bulk relative mapping class groups, respectively. Torus wormholes are not hyperbolic. Nevertheless, the sum prescribed above would, if naively applied, lead to the picture in the text. Namely, for $M = T^2 \times I$, the resulting sum would be over the coset $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})/SL(2, \mathbb{Z})$,

This slight generalization of the familiar single-boundary ideology to the case $\partial\mathcal{M} = T^2 \cup T^2$ guides what we will define as a “torus wormhole” in semiclassical gravity, and accordingly begets the *wormhole Farey tail*: that is, *the identification of torus wormholes, i.e. $SL(2, \mathbb{Z})$ Poincaré sums over suitable seed functions, as gravitational duals of $Z_{\text{Hecke}}(\tau_1, \tau_2)$ in large c CFTs.*

2.5.2 Properties of wormholes

We will study $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ -invariant functions $f(\tau_1, \tau_2) \in L^2(\mathcal{F} \times \mathcal{F})$ which admit representations as Poincaré sums over relative modular transformations²⁶

$$f(\tau_1, \tau_2) = \sum_{\gamma \in SL(2, \mathbb{Z})} f_0(\tau_1, \gamma\tau_2). \quad (2.5.2)$$

As a consequence, the seed function $f_0(\tau_1, \tau_2)$ is invariant under simultaneous modular transformations,

$$f_0(\tau_1, \tau_2) = f_0(\gamma\tau_1, \gamma\tau_2), \quad \gamma \in SL(2, \mathbb{Z}). \quad (2.5.3)$$

The results to follow may be tweaked to accommodate seeds that instead obey

$$f_0(\tau_1, \tau_2) = f_0(\gamma\tau_1, M\gamma M\tau_2), \quad \gamma \in SL(2, \mathbb{Z}), \quad M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.5.4)$$

where $M\tau = -\tau$ implements an orientation reversal.

In Subsection 2.5.2 we will assume slightly more of the seed, namely, an invariance under simultaneous $SL(2, \mathbb{R})$ transformations.²⁷ One may justify why this is sensible from different points of view. As we will see, it will play nicely with the result of Subsection 2.5.2, where we will prove that Hecke symmetry of $f(\tau_1, \tau_2)$ implies that the seed $f_0(\tau_1, \tau_2)$ is not only fully $SL(2, \mathbb{R})$ -invariant, but is solely a function of the \mathbb{H} -invariant distance between τ_1 and τ_2 . This being a very natural criterion from a physical standpoint, one may reasonably demand that the seed amplitudes depend only on this distance in the first place, justifying the assumption of simultaneous $SL(2, \mathbb{R})$ -invariance. At any rate, we expect that the logical connections among these ideas will be clear in the proofs to follow.

We take $f(\tau_1, \tau_2)$ as an operational definition of torus wormhole amplitudes. What properties do these possess?

which can be parametrized as a sum over relative modular transformations of a seed which is invariant under simultaneous modular transformations.

²⁶Depending on context, the sum may instead be taken over $PSL(2, \mathbb{Z})$. This distinction is not relevant for what follows, and indeed, in Section 2.6 we will study an $SL(2, \mathbb{Z})$ sum.

²⁷Actually, we will not use the full $SL(2, \mathbb{R})$, but only the subset of matrices whose entries are given by \sqrt{j} times integers, where $j \in \mathbb{Z}_+$.

Hecke symmetry

Result I. *If $f(\tau_1, \tau_2)$ is a Poincaré sum (2.5.2) with a seed invariant under simultaneous $SL(2, \mathbb{R})$ transformations,*

$$f_0(\tau_1, \tau_2) = f_0(\gamma\tau_1, \gamma\tau_2), \quad \gamma \in SL(2, \mathbb{R}) \quad (2.5.5)$$

then $f(\tau_1, \tau_2)$ is Hecke symmetric.

A modular-invariant function $f(\tau_1, \tau_2) \in L^2(\mathcal{F} \times \mathcal{F})$ is entirely specified by its spectral overlaps with the basis elements: if two such functions have the same overlaps, they are equal. Using this logic, we prove Hecke symmetry by showing that

$$(T_j^{(\tau_1)} f(\tau_1, \tau_2), \psi_\omega(\tau_2)) = (T_j^{(\tau_2)} f(\tau_1, \tau_2), \psi_\omega(\tau_2)) \quad (2.5.6)$$

where $\psi_\omega(\tau_2) := \{E_{\frac{1}{2}+i\omega}(\tau_2), \phi_n(\tau_2)\}$ is an $SL(2, \mathbb{Z})$ eigenfunction. We start from the expression

$$(T_j^{(\tau_1)} f(\tau_1, \tau_2), \psi_\omega(\tau_2)) = \frac{1}{\sqrt{j}} \sum_{A \in SL(2, \mathbb{Z}) \setminus M_j} \int_{\mathbb{H}} \frac{dx_2 dy_2}{y_2^2} f_0(A\tau_1, \tau_2) \psi_\omega(\tau_2) \quad (2.5.7)$$

where we have used the unfolding trick for $f(\tau_1, \tau_2)$, since it is given by a Poincaré sum. We now use the invariance of the seed,²⁸

$$f_0(A\tau_1, \tau_2) = f_0(\tau_1, A^{-1}\tau_2), \quad A \in SL(2, \mathbb{Z}) \setminus M_j \quad (2.5.8)$$

Next, we perform a change variables $\tau'_2 = A^{-1}\tau_2$ in the integral, which leaves the Poincaré measure on \mathbb{H} invariant.²⁹ This results in

$$\begin{aligned} (T_j^{(\tau_1)} f(\tau_1, \tau_2), \psi_\omega(\tau_2)) &= \frac{1}{\sqrt{j}} \sum_{A \in SL(2, \mathbb{Z}) \setminus M_j} \int_{\mathbb{H}} \frac{dx'_2 dy'_2}{y'^2_2} f_0(\tau_1, \tau'_2) \psi_\omega(A\tau'_2) \\ &= \int_{\mathbb{H}} \frac{dx'_2 dy'_2}{y'^2_2} f_0(\tau_1, \tau'_2) T_j^{(\tau'_2)} \psi_\omega(\tau'_2) \\ &= (f(\tau_1, \tau'_2), T_j^{(\tau'_2)} \psi_\omega(\tau'_2)) \end{aligned} \quad (2.5.9)$$

In the second line we brought the sum inside the integral. Using self-adjointness of the Hecke operators,

$$(T_j^{(\tau_2)} f(\tau_1, \tau_2), \psi_\omega(\tau_2)) = (f(\tau_1, \tau_2), T_j^{(\tau_2)} \psi_\omega(\tau_2)) \quad (2.5.10)$$

²⁸Elements of M_j may be mapped to elements of $SL(2, \mathbb{R})$ by rescaling: given a matrix $B \in M_j$ with $\det B = j$, there exists a matrix $B' := B/\sqrt{j}$ with $\det B' = 1$. Since $j \in \mathbb{Z}_+$, and B is integer-valued, $B' \in SL(2, \mathbb{R})$. Note that both B and B' act on τ via the same fractional linear transformation.

²⁹Similar manipulations can be found in Theorem 3.6.4, Point 4 of [120].

Noting equality of (2.5.9) and (2.5.10), we have

$$T_j^{(12)} f(\tau_1, \tau_2) = 0 \quad (2.5.11)$$

which is the desired result. The proof for a seed invariant under (2.5.4) is essentially identical. Following the same steps, the final result follows from the invariance of the right-hand side of (2.3.35) upon taking $b \rightarrow -b$ in the summand.

Strong Spectral Determinacy

Result II. *If $f(\tau_1, \tau_2)$ is a Hecke-symmetric Poincaré sum (2.5.2) with seed obeying (2.5.3), then the Eisenstein and cusp form overlaps are equal.*

A Hecke-symmetric function has spectral overlaps proportional to the respective basis elements, as seen in (2.3.43). We parameterize these as

$$\begin{aligned} (f(\tau_1, \tau_2), E_s(\tau_2)) &= f_E(s) E_{1-s}(\tau_1) \\ (f(\tau_1, \tau_2), \phi_n(\tau_2)) &= f_\phi(s_n) \phi_n(\tau_1) \end{aligned} \quad (2.5.12)$$

In general, $f_E(s)$ and $f_\phi(s_n)$ are different functions of their arguments. Using the unfolding trick for $f(\tau_1, \tau_2)$, we write these overlaps explicitly as

$$\begin{aligned} f_E(s) E_{1-s}(\tau_1) &= \int_{\mathbb{H}} \frac{dx_2 dy_2}{y_2^2} f_0(\tau_1, \tau_2) E_{1-s}(\tau_2) \\ f_\phi(s_n) \phi_n(\tau_1) &= \int_{\mathbb{H}} \frac{dx_2 dy_2}{y_2^2} f_0(\tau_1, \tau_2) \phi_n(\tau_2) \end{aligned} \quad (2.5.13)$$

We now expand the functions on the right-hand side in a Fourier series, and then project onto a spin $k \neq 0$ with respect to τ_1 . In doing so we employ the Fourier decomposition of a seed $f_0(\tau_1, \tau_2)$ invariant under simultaneous $SL(2, \mathbb{Z})$ transformations (2.5.3),

$$f_0(\tau_1, \tau_2) = \sum_{\ell \in \mathbb{Z}} e^{4\pi i \ell x_+} f_0^{(\ell)}(x_-, y_1, y_2) \quad \text{where} \quad x_{\pm} := \frac{x_1 \pm x_2}{2} \quad (2.5.14)$$

Changing variables to x_{\pm} , we can perform the integral over x_+ which sets $j = |k - 2\ell|$ and yields (recall that $\mathbf{a}_j^{(s)} = \mathbf{a}_{-j}^{(s)}$ and likewise for $\mathbf{b}_j^{(n)}$)

$$\begin{aligned} \mathbf{a}_k^{(1-s)} f_E(s) \left(\sqrt{y_1} K_{s-\frac{1}{2}}(2\pi k y_1) \right) &= \sum_{\ell \in \mathbb{Z}} (2 - \delta_{k-2\ell, 0}) \mathbf{a}_{k-2\ell}^{(1-s)} \mathcal{I}_{k, \ell}^{(s)}(y_1) \\ \mathbf{b}_k^{(n)} f_\phi(s_n) \left(\sqrt{y_1} K_{s_n-\frac{1}{2}}(2\pi k y_1) \right) &= 2 \sum_{\ell \in \mathbb{Z}} \mathbf{b}_{k-2\ell}^{(n)} \mathcal{I}_{k, \ell}^{(s_n)}(y_1) \end{aligned} \quad (2.5.15)$$

where

$$\mathcal{I}_{k,\ell}^{(s)}(y_1) := \int \frac{dy_2}{y_2^2} \int dx_- e^{2\pi i x_- (-k+|k-2\ell|)} f_0^{(\ell)}(x_-, y_1, y_2) \sqrt{y_2} K_{s-\frac{1}{2}}(2\pi|k-2\ell|y_2) \quad (2.5.16)$$

The factor in parentheses in (2.5.15) is the spin- k Fourier mode with the coefficient stripped off. Evaluating the first line of (2.5.15) at $s = s_n$ and taking the ratio of the two equations, the left-hand side is independent of y_1 . This is incompatible with the right-hand side unless

$$f_0^{(\ell)}(x_-, y_1, y_2) \propto \delta_{\ell,\ell^*} \quad (2.5.17)$$

for some fixed ℓ^* . In fact, the only possible choice is $\ell^* = 0$, such that the Fourier coefficients cancel completely, because the equation must be satisfied for *any* $k \in \mathbb{Z}_+$. Therefore,

$$f_\phi(s_n) = f_E(s_n) \quad (2.5.18)$$

That is, the cusp form overlap is equal to the Eisenstein overlap evaluated at $s = s_n$.

As a statement about Fourier decomposition, this means that the $(j_1, j_2) = (0, 0)$ mode of $f(\tau_1, \tau_2)$, which determines the Eisenstein part of the spectral decomposition, actually determines the entire function. Building on the language of [112, 151], we call this *strong spectral determinacy*. This is compatible with the general result of [123] because we are imposing extra structure on $f(\tau_1, \tau_2)$.

Dependence on hyperbolic distance

Result III. *A function $f_0(\tau_1, \tau_2)$ that obeys (2.5.3) and has spin $\ell = 0$ under the Fourier decomposition in $x_1 + x_2$ is solely a function of the invariant hyperbolic distance,*

$$f_0(\tau_1, \tau_2) = f_0(\sigma(\tau_1, \tau_2)), \quad \text{where } \sigma(\tau_1, \tau_2) := \frac{|\tau_1 - \tau_2|^2}{y_1 y_2}. \quad (2.5.19)$$

This result applies to wormhole amplitudes because the seed for (2.5.2) is such a function (cf. below (2.5.17)).

The spin $\ell = 0$ property implies that $f_0(\tau_1, \tau_2) = f_0(x_-, y_1, y_2)$. Without loss of generality we can write this as $f_0(\tau_-, y_1, y_2)$, where $\tau_- := (\tau_1 - \tau_2)/2$. This is invariant under simultaneous T -transformations. Under simultaneous S -transformation,

$$f_0(\tau_-, y_1, y_2) \mapsto f_0\left(\frac{\tau_-}{\tau_1 \tau_2}, \frac{y_1}{|\tau_1|^2}, \frac{y_2}{|\tau_2|^2}\right) \quad (2.5.20)$$

Demanding simultaneous S -invariance then fixes the functional form to be as in (2.5.19),³⁰

³⁰This is a two-variable generalization of the fact that a modular-invariant function $f(y)$ must be a constant.

where $\sigma(\tau_1, \tau_2)$ is the \mathbb{H} -invariant distance between τ_1 and τ_2 , related to the geodesic distance $d(\tau_1, \tau_2)$ as

$$\sigma(\tau_1, \tau_2) = 4 \sinh^2 \left(\frac{d(\tau_1, \tau_2)}{2} \right). \quad (2.5.21)$$

This concludes the proof. Essentially the same result holds for a seed obeying the condition (2.5.4). In this case, $f_0(\tau_1, \tau_2)$ has a Fourier decomposition (2.5.14) with the swap $x_+ \leftrightarrow x_-$. The same argument then leads to

$$f_0(\sigma_-(\tau_1, \tau_2)), \quad \text{where} \quad \sigma_-(\tau_1, \tau_2) := \sigma(\tau_1, -\tau_2) = \frac{|\tau_1 + \tau_2|^2}{y_1 y_2} \quad (2.5.22)$$

That is, the seed depends on geodesic distance with the orientation-reversal $\tau_2 \rightarrow -\tau_2$.

That a Hecke-symmetric Poincaré sum of the form (2.5.2) has a seed which depends only on the \mathbb{H} -invariant distance ties nicely into the earlier proof of Hecke symmetry: $\sigma(\tau_1, \tau_2)$ is fully $SL(2, \mathbb{R})$ -invariant.

2.5.3 Summary and comments

The results of the previous subsection suggest the following general nomenclature: we define “wormhole amplitudes” $Z_{\text{WH}}(\tau_1, \tau_2)$ as those functions which have the following spectral decomposition:

$$Z_{\text{WH}}(\tau_1, \tau_2) = \int_{\mathcal{C}_{\text{crit}}} f_{\text{WH}}(s) E_s(\tau_1) E_{1-s}(\tau_2) + \sum_{n=1}^{\infty} f_{\text{WH}}(s_n) \phi_n(\tau_1) \phi_n(\tau_2) \quad (2.5.23)$$

In our spectral framework, $Z_{\text{WH}}(\tau_1, \tau_2)$ are to be regarded as instances of $Z_{\text{Hecke}}(\tau_1, \tau_2)$ with enhanced structure. Given a $Z_{\text{spec}}(\tau)$ of a CFT,

$$f_{\text{WH}}(s) = |(Z_{\text{spec}}, E_s)|^2. \quad (2.5.24)$$

The property $Z_{\text{Hecke}}(\tau_1, \tau_2) = Z_{\text{WH}}(\tau_1, \tau_2)$ thus holds if and only if

$$(Z_{\text{spec}}, \phi_n)^2 = |(Z_{\text{spec}}, E_{s_n})|^2 \quad (2.5.25)$$

Poincaré sums of the form (2.5.2) are but one example of such amplitudes. We may also view this as a prediction for large c CFTs: if we take the Farey tail sum (2.5.2) as a wormhole ansatz in semiclassical gravity, (2.5.25) should hold in the dual large c CFT.

The functions $Z_{\text{WH}}(\tau_1, \tau_2)$ furnish a highly-symmetric class of square-integrable $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ -invariant functions: not only are they Hecke-symmetric, but they are fixed by a

The proof in that case is similar: an S -transformation introduces x -dependence.

single overlap function, $f_{\text{WH}}(s)$. Let us put this last point in sharper focus. In the language of Section 2.4, recall that the leading low-temperature term of the $(j_1, j_2) = (0, 0)$ Fourier mode was controlled by the function $\mathcal{R}(z)$. This term fixes other Fourier modes of our various amplitudes, *away* from the low-temperature limit, according to their degree of symmetry:

Amplitude	Modes (j_1, j_2) fixed by $\mathcal{R}(z)$
$Z_{\text{diag}}(\tau_1, \tau_2)$	$(0, 0)$
$Z_{\text{Hecke}}(\tau_1, \tau_2)$	$(0, j)$
$Z_{\text{WH}}(\tau_1, \tau_2)$	All

For $Z_{\text{WH}}(\tau_1, \tau_2)$ in particular, $\mathcal{R}_{\text{WH}}(z) = \mathcal{M}^{-1}[f_{\text{WH}}(s); z]$ fixes *all* Fourier modes, and hence the entire function (“strong spectral determinacy”).

While the notation $Z_{\text{WH}}(\tau_1, \tau_2)$ invokes bulk language to reflect the properties of Poincaré sums derived in the previous subsection, other functions could in principle assume the functional form (2.5.23). However, there is clearly a gravitational nature to $Z_{\text{WH}}(\tau_1, \tau_2)$, and to $Z_{\text{Hecke}}(\tau_1, \tau_2)$ more generally. The projection $\mathcal{P}_{\text{Hecke}}$ is in some sense capturing the “wormhole part” of the factorized product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$: it isolates the correlations between the two copies which, when there is an AdS₃ bulk dual, are geometrized by a smooth, two-boundary, connected Euclidean spacetime. This interpretation is on firmest footing at large c , where it becomes a semiclassical geometric statement that is supported by our proofs above for wormholes-qua- $SL(2, \mathbb{Z})$ Poincaré sums.

These results suggest the following conceptual interpretation. In gravitational language, the Hecke operators $T_j^{(\tau)}$, which act on a single boundary component, are AdS₃ “half-wormhole detectors.” That $T_j^{(\tau)}Z_{\text{spec}}(\tau) \neq 0$ reflects the fact that $Z_{\text{spec}}(\tau)$ by definition receives no contribution from smooth geometric saddles in gravity: it only counts off-shell, or non-geometric (e.g. matter), degrees of freedom. Then forming the difference operator $T_j^{(12)} = T_j^{(\tau_1)} - T_j^{(\tau_2)}$, the projection onto its kernel via $\mathcal{P}_{\text{Hecke}}$ implements the pairing of these degrees of freedom to form a smooth two-boundary spacetime. To say this slightly differently, $T_j^{(12)}$ has the flavor of a non-local bulk operator, which can be freely moved to the “left” or “right” boundary tori: Hecke symmetry is then the statement that these operators are, in a sense, topological. Conversely, a nonzero action of $T_j^{(12)}$ detects non-geometric contributions to the wormhole amplitude. It would of course be nice to find an explicit *bulk* construction of Hecke operators, or of $\mathcal{P}_{\text{Hecke}}$ as a topology-changing operator.

We will next make these ideas explicit in semiclassical AdS₃ pure gravity.

2.6 Pure Gravity as MaxRMT

We now apply our framework to semiclassical AdS₃ pure gravity. We begin our treatment with the two-boundary torus wormhole of Cotler and Jensen [34].

2.6.1 Torus wormhole

The Cotler-Jensen (CJ) wormhole was originally presented in [34] as a Poincaré sum of the form considered in Section 2.5,³¹

$$Z_{\text{CJ}}(\tau_1, \tau_2) = \sum_{\gamma \in SL(2, \mathbb{Z})} f_{0, \text{CJ}}(\tau_1, \gamma \tau_2) \quad (2.6.1)$$

with seed given by the inverse of the (orientation-reversed) hyperbolic distance:

$$f_{0, \text{CJ}}(\tau_1, \tau_2) = \frac{1}{2\pi^2} \sigma_-^{-1}(\tau_1, \tau_2) = \frac{1}{2\pi^2} \frac{y_1 y_2}{|\tau_1 + \tau_2|^2} \quad (2.6.2)$$

The CJ wormhole is an off-shell contribution to the path integral of AdS₃ pure gravity with the topology of a torus times an interval, $T^2 \times I$. It is a constrained instanton, namely it becomes on-shell after adding a constraint [152]. The CJ computation uses a dynamical theory of only boundary gravitons [153]. This differs from the corresponding quantity in the Chern-Simons formulation of AdS₃ gravity [34, 107, 154].

The point is now to understand $Z_{\text{CJ}}(\tau_1, \tau_2)$ as an instance of $Z_{\text{Hecke}}(\tau_1, \tau_2)$: that is, as a coarse-grained two-copy partition function of an underlying chaotic CFT.

Since $Z_{\text{CJ}}(\tau_1, \tau_2)$ is a Hecke-symmetric wormhole amplitude, the entire amplitude is fixed by the (0, 0) Fourier mode. From [34] eq. (4.12) (and doubling the result), we read off the functions in (2.4.6) as

$$\begin{aligned} \mathcal{R}_{\text{CJ}}(z) &= \frac{1}{\pi} \frac{1}{1+z} \\ \mathcal{S}_{\text{CJ}}(r) &= \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \frac{r}{\sqrt{1+n^2 r^2}} \end{aligned} \quad (2.6.3)$$

with $r = \sqrt{y_1 y_2}$ and $z = y_1 / y_2$. Recall that $\mathcal{R}_{\text{CJ}}(z)$ is the leading low-temperature term, which fixes the entire zero mode by (2.4.10).

By the general proofs of Section 2.5.2 for Poincaré sums of the form (2.6.1), $Z_{\text{CJ}}(\tau_1, \tau_2)$

³¹ [34] wrote this as a $PSL(2, \mathbb{Z})$ image sum, but considerations of discrete symmetries suggest that the result should be doubled [147]. For this reason, and to streamline some of the expressions to follow, we work with the GOE version of the CJ wormhole, which amounts to overall multiplication of the expressions in [34] by $C_{\text{GOE}} = 2$.

admits a spectral decomposition³² of wormhole form (2.5.23): we need only determine the spectral overlap, call it $f_{\text{CJ}}(s)$, by Mellin transform of $\mathcal{R}_{\text{CJ}}(z)$. A simple integral yields

$$\begin{aligned} f_{\text{CJ}}(s) &= \mathcal{M}[\mathcal{R}_{\text{CJ}}(z); s] \\ &= \frac{1}{\pi} \Gamma(s) \Gamma(1-s) \end{aligned} \tag{2.6.4}$$

This determines the full $Z_{\text{CJ}}(\tau_1, \tau_2)$. Let us write it explicitly for the sake of clarity:

$$\pi Z_{\text{CJ}}(\tau_1, \tau_2) = \int_{\mathcal{C}_{\text{crit}}} \Gamma(s) \Gamma(1-s) E_s(\tau_1) E_{1-s}(\tau_2) + \sum_{n=1}^{\infty} \Gamma(s_n) \Gamma(1-s_n) \phi_n(\tau_1) \phi_n(\tau_2) \tag{2.6.5}$$

It will prove useful to rewrite (2.6.5) in terms of the completed Eisenstein series:

$$\boxed{\pi Z_{\text{CJ}}(\tau_1, \tau_2) = \int_{\mathcal{C}_{\text{crit}}} \frac{\pi}{\zeta(2s)\zeta(2-2s)} E_s^*(\tau_1) E_s^*(\tau_2) + \sum_{n=1}^{\infty} \Gamma(s_n) \Gamma(1-s_n) \phi_n(\tau_1) \phi_n(\tau_2)} \tag{2.6.6}$$

The Eisenstein overlap takes an intriguing form in terms of the Riemann zeta function.

The spectral decomposition (2.6.5) is a very clean way to package the Fourier modes for arbitrary spins, determined as they are by the low-temperature scalar term $\mathcal{R}_{\text{CJ}}(z)$. (For comparison, the modes $Z_{\text{CJ}}^{(j_1, j_2)}(y_1, y_2)$ are given in eq. (4.20) of [34].) Note that there is no constant term in the decomposition. This follows our general discussion earlier in the paper, but is additionally bound up with the fact that the (0,0) mode is actually divergent [34] (cf. the $r \rightarrow \infty$ limit of $\mathcal{S}_{\text{CJ}}(r)$).³³ Given this, the constant term is subject to the choice of regularization scheme, and the finite constant may be set to zero; this is the meaning of (2.6.5).

2.6.2 MaxRMT

The CJ wormhole has some very special features. Notice that (2.6.4) realizes the RMT ramp falloff condition (2.4.23), with $f(\omega) = C_{\text{GOE}} = 2$ exactly – in particular, with no fluctuations. Likewise, not only does (2.6.3) manifestly realize the universal singularities (2.4.21) and (2.4.25) required by the presence of a linear ramp in the scalar SFF, but the CJ wormhole is given *exactly* as the sum over those singularities!

This last property is remarkable. Having shown that general wormhole amplitudes are fully determined by the single function $\mathcal{R}(z)$ after accounting for Virasoro and modular

³²We thank Scott Collier for sharing a note on the spectral decomposition of the CJ wormhole, which helped inform our general perspective.

³³The divergence, and subsequent regularization, of [34] may be recovered in the spectral formalism for functions $f(\tau_1, \tau_2) \in L^2(\mathcal{F} \times \mathcal{F})$ by taking the appropriate residues of the Eisenstein overlap; see Appendix 2.9. Note also that this divergence is not special to the CJ wormhole, as shown in (2.12.10).

symmetries, $\mathcal{R}_{\text{CJ}}(z)$ is exactly equal to the double-scaled RMT result. Stating this from the Lorentzian point of view, the scalar SFF of the CJ wormhole is, for *all* fixed T/β , given by “just the ramp,” plus corrections fully fixed by the symmetries; in turn, the entire amplitude for *all* moduli τ_1 and τ_2 , including Euclidean and Lorentzian temperatures, is its minimal completion.

These properties of the torus wormhole reveal that the spectrum of semiclassical AdS_3 pure gravity and its dual 2d CFT exhibit random matrix statistics to the maximal extent possible. As stated in the Introduction, this is a signature of what we call

MaxRMT: *the maximal realization of random matrix universality consistent with Virasoro symmetry and modular invariance.*

This extends the sense in which AdS_3 pure gravity may be understood as an extremal theory: within the space of consistent theories of AdS_3 gravity, the statistical correlations among black hole microstates of pure gravity are as random as possible (in the sense of RMT).

At early times, well before random matrix behavior sets in, theories of semiclassical Einstein gravity (in any spacetime dimension) saturate the chaos bound on the Lyapunov exponent of out-of-time-order correlators [38]. What our analysis shows is that pure gravity is also maximally chaotic at late times: viewing $Z_{\text{CJ}}(\tau_1, \tau_2)$ as computing coarse-grained spectral correlations of a dual microscopic CFT at large c , its level statistics are “maximally random” for a 2d CFT.³⁴ In other words, random matrix universality enjoys a maximally extended regime of validity. This is a refinement, incorporating chaos, of the statement [139] that the Cardy density of states enjoys an extended regime of validity beyond the asymptotic regime $\Delta \rightarrow \infty$, all the way down to $\Delta \sim c/6$ (or possibly to $c/12$, for certain exceptional theories), in a sparse CFT at leading order in large c .

Rephrasing this slightly, we have recovered the CJ wormhole amplitude as an extremal two-copy partition function. In particular, the CJ wormhole amplitude is the unique solution to the following large c CFT bootstrap problem: find a $Z_{\text{Hecke}}(\tau_1, \tau_2)$ that *i*) admits a Poincaré sum representation of wormhole form (2.5.2), and *ii*) reproduces the double-scaled RMT SFF at late times T for any fixed T/β . The first criterion specializes to large c , though not to pure gravity specifically, by imposing the geometric structure of semiclassical wormholes (see Subsection 2.5.1). Upon imposing the SFF of RMT, Virasoro symmetry and modular invariance take care of the rest, leading to $Z_{\text{CJ}}(\tau_1, \tau_2)$.³⁵ This is similar in spirit to the bootstrap approach to four-point functions in $\text{AdS}_5 \times S^5$ supergravity in [155].

³⁴To give some indicative timescales, the early-time Lyapunov chaos happens well before the scrambling time, logarithmic in entropy, whereas the RMT ramp occurs after the dip time, exponential in entropy.

³⁵The CJ wormhole was also “bootstrapped” in [131] from a different set of conditions, the main one being a specific prescription for the zero mode volume in the two-boundary gravity path integral. The origin of that important factor is rather subtle [107, 147, 154].

Comments

Ensembles, RMT wormholes and ETH wormholes

Is semiclassical pure gravity dual to an ensemble of CFTs? A more grounded statement, supported by explicit computations in the literature, is that saddle point partition functions of semiclassical pure gravity are dual to ensembles of CFT data. This appears to hold for saddles of arbitrary boundary topology. On the other hand, we are able to interpret $Z_{\text{CJ}}(\tau_1, \tau_2)$, a bulk *off-shell* wormhole amplitude encoding spectral data alone, in terms of a *microscopic* CFT. This provides a realization of the large c notions articulated in [103]: indeed, our formalism amounts to a dynamical mechanism of the “apparent averaging” of [103] in $\text{AdS}_3/\text{CFT}_2$, by showing how wormholes can emerge from the high-energy spectral statistics of a large c limit of a family of chaotic CFTs. It is satisfying that the torus wormhole admits a clean microscopic interpretation in bona fide CFT_2 , without requiring an ensemble interpretation (though it does not prohibit one, if desired).

Note that, in contrast, saddle point wormholes in AdS_3 gravity capture different physics. Unlike what one might call “RMT wormholes” that encode spectral statistics, such as the CJ wormhole, on-shell wormholes [106, 107, 109, 156] are “ETH wormholes”: they are instead computing averaged matrix elements that are in turn fixed by leading-order spectral (Cardy) and OPE (DOZZ) asymptotics of Virasoro primaries [101, 157, 158], independent of level statistics. All on-shell wormholes in AdS_3 gravity are of this type Maldacena:2004rf.³⁶ The partition function of any such wormhole saddle can be reduced to a moment problem for a probability distribution of CFT data that encodes universal asymptotics [106],³⁷ insensitive to chaos and the structure of black hole microstates. For this reason, on-shell AdS_3 wormholes and their possible ensemble interpretations are of a rather different nature than wormholes in JT gravity.

Quantum chaos from arithmetic chaos

The CJ amplitude exhibits a linear ramp in all spin sectors [34]. Given the amplitude (2.6.5), we can compute the Eisenstein contribution to any Fourier mode $Z_{\text{CJ}}^{(j_1, j_2)}(y_1, y_2)$. The necessary

³⁶As an illustrative example, for a two-boundary torus wormhole to be on-shell in gravity, one must instead consider a one-point wormhole with boundary operators \mathcal{O} inserted, where $\Delta_{\mathcal{O}} < 2\xi$ is dual to a bulk matter field Maldacena:2004rf. The trace over Hilbert space is an OPE sum, and the result is fixed by Virasoro ETH asymptotics. Consistent with this, if one studies the late-time behavior of the “one-point SFF”, the on-shell one-point wormhole does not produce a ramp [147].

³⁷This is not fully crossing symmetric on average, but is (almost) S -invariant on average. Small violations of average S -invariance may be fixed using a matrix-tensor model with a potential controlled by the variance under S [110]. For other proposals on AdS_3 gravity with a similar treatment of partial crossing symmetry, see [153, 159].

integral was performed in [123] for $j_1 > 0$, yielding

$$Z_{\text{CJ}}^{(j_1, j_2)}(y_1, y_2) \Big|_{\text{Eis}} \approx \lambda_{j_1} \delta_{j_1, j_2} e^{-2\pi(j_1 y_1 + j_2 y_2)} \sqrt{\frac{y_1 y_2}{y_1 + y_2}} + \dots, \quad (y_1, y_2 \rightarrow \infty) \quad (2.6.7)$$

where λ_{j_1} are $\mathcal{O}(1)$ numerical factors. After analytic continuation and stripping off a factor $\sqrt{y_1 y_2}$ (see (2.4.19)), this gives an $\mathcal{O}(1)$ contribution to the SFF. Therefore, the late-time linear ramp at nonzero spin comes from the cusp form contribution alone. This observation establishes a relation between the *arithmetic* chaos of Maass cusp forms and *quantum* chaos of AdS₃ pure gravity as defined by RMT.³⁸

On c -independence of wormhole amplitudes

The CJ wormhole amplitude is independent of Newton’s constant $G_N = 3L_{\text{AdS}}/2c$. This is consistent with the fact that it is not a semiclassical saddle in the bulk, but otherwise calls for a fundamental explanation (in contrast to the JT double-trumpet, whose $\mathcal{O}(1)$ scaling follows from the topological expansion of 2D gravity).

Our formalism gives an indirect explanation of this fact: it is a consequence of the linear ramp in the SFF and the “wormhole” form of the amplitude. In short, if there were a c -dependent factor in the wormhole amplitude, it would violate random matrix universality. Demanding a linear ramp in the scalar SFF with RMT coefficient implies (2.4.21), a c -independent condition. In general, given a family of chaotic CFTs $\{\mathcal{T}_c\}$ which admits a $c \rightarrow \infty$ limit, any dependence on c , or other parameters (e.g. exactly marginal couplings) which we collectively denote as λ , must sit in an additive correction term,

$$\mathcal{R}^{(c)}(z|\lambda) = \frac{\mathbf{C}_{\text{RMT}}}{2\pi} \frac{1}{1+z} + \delta\mathcal{R}^{(c)}(z|\lambda) \quad (2.6.8)$$

At large c , this is finite. Since any wormhole amplitude $Z_{\text{WH}}(\tau_1, \tau_2)$ of the form (2.5.23) is fixed completely by this quantity, as shown in Section 2.5, the entire amplitude is independent of c . What sets the CJ wormhole apart is that the correction exactly vanishes – a property of MaxRMT.

Wormholes in AdS₃ \times \mathcal{M} gravity

In a theory of AdS₃ pure gravity, corrections to MaxRMT behavior should be non-perturbative in G_N . We will discuss these in Subsection 2.6.3. On the other hand, in theories of gravity coupled to matter, such as AdS₃ \times \mathcal{M} string or M-theory compactifications with large extra dimensions \mathcal{M} , the corrections will generically be $\mathcal{O}(1)$. These theories should not have MaxRMT statistics. The RMT wormholes with $T^2 \times I$ topology will receive large matter

³⁸This connection has been developed further in [160, 161].

contributions to the quantity $\delta\mathcal{R}^{(c)}(z|\lambda)$ written above, thus modifying the geometry on macroscopic scales. By virtue of diffeomorphism invariance their partition functions should take the form (2.5.23) and, being chaotic, preserve the linear ramp of the SFF, but we expect their overlaps $f_{\text{WH}}(s)$ to differ substantially from $f_{\text{CJ}}(s)$.

We may parameterize these overlaps as

$$f_{\text{WH}}(s) = f_{\text{CJ}}(s)g(s) \quad (2.6.9)$$

with $\mathcal{R}_{\text{WH}}(z)$ and $G(z) = \delta(z-1) + \delta G(z)$ the corresponding inverse Mellin transforms, respectively. Mellin convolution implies

$$\mathcal{R}_{\text{WH}}(z) = \frac{1}{\pi} \frac{1}{1+z} + \delta\mathcal{R}_{\text{WH}}(z) \quad (2.6.10)$$

where

$$\delta\mathcal{R}_{\text{WH}}(z) := \int_0^\infty dy \frac{\delta G(y)}{1+zy} \quad (2.6.11)$$

This correction must be finite at $z = -1$. Singularities of $\delta\mathcal{R}_{\text{WH}}(z)$ for $z \in \mathbb{C} \setminus \{-1\}$ can arise either from delta functions (and possibly their derivatives) in $\delta G(y)$ on the contour, or poles off the contour. These singularities must not violate the physical requirement that $\mathcal{R}_{\text{WH}}(z)$ be regular at $|z| = 1$ (no spurious poles in the SFF) and $z \in \mathbb{R}_{\geq 0}$ (finite Euclidean low-temperature limit).³⁹

In Appendix 2.13, we provide a toy family of RMT wormholes in $\text{AdS}_3 \times \mathcal{M}$ gravity, for which both $f_{\text{WH}}(s)$ and $\mathcal{R}_{\text{WH}}(z)$ can be computed explicitly. The spectral overlaps are of the form (2.6.9) with

$$g(s) = \frac{\mathbf{C}_{\text{RMT}}}{4} \left(\zeta(s+p) + \zeta(p+1-s) \right), \quad \text{where } p \in \mathbb{Z}_+, \quad (2.6.12)$$

while their dual functions $\mathcal{R}_{\text{WH}}(z)$ in (2.13.6) are harmonic sums over an infinite set of simple poles, with the correctly normalized RMT ramp (2.4.21) at $z \rightarrow -1$. We note here that $g(s)$ non-trivially satisfies the falloff condition (2.4.23), which follows from the asymptotic $|\zeta(\sigma + i\omega)| \rightarrow 1$ as $\omega \rightarrow \infty$ for any $\sigma > 1$.

2.6.3 Black hole microstates of AdS_3 pure gravity

The above framework actually lets us extract *non-coarse-grained* microstructure of the black hole spectrum of AdS_3 pure gravity by factorizing the CJ wormhole; see Figure 2.4. In particular, we can deduce what $Z_{\text{spec}}(\tau)$ gives rise to the correlations encoded in $Z_{\text{CJ}}(\tau_1, \tau_2)$

³⁹In SFF kinematics, a pole $1/(z+z_*) \supset \delta\mathcal{R}_{\text{WH}}(z)$ gives a decaying contribution $\sim 1/T$ as $T \rightarrow \infty$. Note that these are not the same as the non-ergodic modes/massive modes of [162, 163], which decay as $Te^{-\#T}$ in the SFF. Those are exponential corrections to the dip, whereas ours are power-law corrections to the ramp.

upon coarse-graining by using the powerful fact, highlighted earlier, that the 2d CFT trace formula expands $Z_{\text{spec}}(\tau)$ over a complete eigenbasis. Combined with the high degree of symmetry of wormhole amplitudes, the result is nearly unique.

We wish to solve

$$Z_{\text{CJ}}(\tau_1, \tau_2) = \mathcal{P}_{\text{Hecke}} [Z_{\text{RMT}}(\tau_1)Z_{\text{RMT}}(\tau_2)] \quad (2.6.13)$$

That is, $Z_{\text{RMT}}(\tau)$ is the $Z_{\text{spec}}(\tau)$ that generates the CJ wormhole upon coarse-graining. We deduce the spectral overlaps (Z_{RMT}, E_s) and (Z_{RMT}, ϕ_n) , and hence $Z_{\text{RMT}}(\tau)$, from (2.6.5). The cusp form overlaps are fixed by (2.5.25) in terms of the Eisenstein overlap, because of the wormhole form (2.5.23) of the CJ amplitude. The Eisenstein overlap, in turn, is easily determined using (2.6.6) in which the reflection symmetry is manifest:

$$\boxed{\{Z_{\text{RMT}}, E_s\}^2 = \frac{1}{\zeta(2s)\zeta(2-2s)}} \quad (2.6.14)$$

On the critical line, $\{Z_{\text{RMT}}, E_{\frac{1}{2}+i\omega}\}^2 = |\zeta(1+2i\omega)|^{-2}$. So a putative CFT dual to AdS₃ pure gravity has a spectral partition function $Z_{\text{spec}}(\tau) \approx Z_{\text{RMT}}(\tau)$ where

$$Z_{\text{RMT}}(\tau) = \int_{\mathcal{C}_{\text{crit}}} (Z_{\text{RMT}}, E_s)E_s(\tau) + \sum_{n=1}^{\infty} |(Z_{\text{RMT}}, E_{s_n})|\phi_n(\tau), \quad (2.6.15)$$

with overlaps given above. In other words, by factorizing the wormhole we have extracted a contribution to the torus partition function of a CFT dual to pure gravity,

$$Z_{\text{grav}}(\tau) \approx Z_{\text{MWK}}(\tau) + Z_{\text{RMT}}(\tau). \quad (2.6.16)$$

This holds up to exponentially small corrections in c , discussed further in Subsection 2.6.3.

$Z_{\text{RMT}}(\tau)$ is almost, but not quite, uniquely determined: its overlaps are fixed up to overall signs. This is inherent in our derivation-by-factorization of the two-boundary wormhole, which is quadratic in $Z_{\text{RMT}}(\tau)$. One would thus like to identify a computation, or a physical criterion, which selects one branch of the square root. While we so far lack a proof, the trace formula (2.3.20) suggests that the positive branch may be the correct one, both for the Eisenstein and cusp form overlaps, because the spectral overlaps play the role of orbit densities: just as semiclassical periodic orbit densities must be positive for physical systems, one may reasonably expect the same to hold for their 2d CFT analog. A full determination of these signs is important. Either way, it is quite intriguing that $\{Z_{\text{RMT}}, E_s\}$ is *sign-definite* on the critical line, a feature which is surprising from the point of view of the individual functions themselves: neither $Z_{\text{RMT}}(\tau)$ nor $E_s(\tau)$ obeys any manifest sign constraint for $\tau \in \mathcal{F}$ (for example, the integral of $E_s(\tau)$ over \mathcal{F} vanishes). This appears to be a special

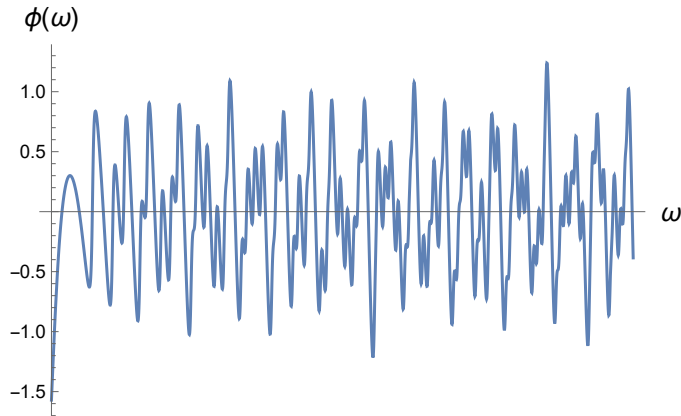


Figure 2.6: A plot of the Riemann zeta phase $\phi(\omega) = \arg(\zeta(1 + 2i\omega))$ for $\omega \in [0, 100]$.

feature of wormhole amplitudes under the CFT identification (2.5.24).⁴⁰ It suggests that the chaotic spectrum $Z_{\text{spec}}(\tau)$ of any large c CFT must obey such a sign constraint.

We view $Z_{\text{RMT}}(\tau)$ as a half-wormhole of AdS₃ pure gravity: it is a single-boundary, non-self-averaging quantity which, when squared and glued appropriately, generates the smooth spacetime wormhole. See Figure 2.3. As is by now hopefully clear, the right procedure is prescribed by the 2d CFT trace formula, gluing the half-wormholes in $SL(2, \mathbb{Z})$ spectral space. The detailed form of $Z_{\text{RMT}}(\tau)$ is “jiggly,” with spectral overlaps characterized by erratic fluctuations. We can make this manifest by writing the Eisenstein overlap on the critical line as (choosing the positive branch for definiteness)

$$(Z_{\text{RMT}}, E_{\frac{1}{2}+i\omega}) = \frac{1}{\sqrt{\cosh(\pi\omega)}} e^{i(\chi(\omega)+\phi(\omega))} \quad (2.6.17)$$

where

$$\chi(\omega) = \arg\left(\pi^{-i\omega}\Gamma\left(\frac{1}{2} + i\omega\right)\right), \quad \phi(\omega) = \arg(\zeta(1 + 2i\omega)) \quad (2.6.18)$$

In addition to the regular “background” phase $\chi(\omega)$, the overlap is dressed by the “Riemann zeta phase” $\phi(\omega)$, a quantum-chaotic object: $\phi(\omega)$ oscillates erratically along the critical line (see Figure 2.6), producing the non-self-averaging behavior characteristic of half-wormholes.⁴¹ Upon squaring and projecting $Z_{\text{RMT}}(\tau)$ to form the CJ wormhole, the phase cancels, leaving the smooth integrand of (2.6.5).

This dovetails nicely with expectations of how the wormholes and half-wormholes of 2D

⁴⁰One easily verifies that the positivity holds for Narain wormholes \mathcal{C} as well. Conversely, one may ask whether imposing the positivity of $f_{\text{WH}}(\frac{1}{2} + i\omega)$, as in the identification (2.5.24), may be used to constrain the allowed seeds of Poincaré sums (2.5.2) representing wormhole amplitudes in gravity (or, possibly, whether not all reasonable wormholes actually possess such positivity). We thank David Berenstein for this question, and Scott Collier for a subsequent discussion.

⁴¹The Riemann zeta phase also appears in the Gutzwiller model of chaotic scattering on the “leaky torus” as a phase in the S-matrix, producing poles at the locations of non-trivial Riemann zeros [164].

gravity should generalize to higher dimensions. The factorized product $Z_{\text{RMT}}(\tau_1)Z_{\text{RMT}}(\tau_2)$ explicitly contains the wormhole $Z_{\text{CJ}}(\tau_1, \tau_2)$, isolated by the Hecke projection $\mathcal{P}_{\text{Hecke}}$; the terms that are projected out, presumably to be thought of as “linked half-wormholes” of some kind, combine with the wormhole to restore factorization. We can say this in the closely related but some what more general “broken cylinders” picture of [165] (see Sections 3.6 and 5.2). A broken cylinder in the “ b basis” is geometrically a trumpet with a geodesic boundary carrying a random boundary condition, $\Psi(b)$. Two broken cylinders are glued into a smooth wormhole by averaging over these boundary conditions, integrating over b with measure

$$\langle \Psi(b)\Psi(b') \rangle_{\Psi} = \frac{1}{b} \delta(b - b') \quad (2.6.19)$$

The random function $\Psi(b)$ is a bulk effective description of coarse-graining the dual theory. There is an intuitive analogy between AdS_2 broken cylinders and AdS_3 half-wormholes:

$$\begin{array}{ccc} b & \longleftrightarrow & \omega \\ \Psi(b) & \longleftrightarrow & e^{i\phi(\omega)} \\ (2.6.19) & \longleftrightarrow & \mathcal{P}_{\text{Hecke}} \end{array}$$

We recall that $\mathcal{P}_{\text{Hecke}}$ projects onto the diagonal the factorized double sum over eigenfunctions,

$$\mathcal{P}_{\text{Hecke}} \left[\sum_{\omega} \tilde{f}_{\omega} \psi_{\omega}(\tau_1) \sum_{\omega'} \tilde{f}_{\omega'}^* \psi_{\omega'}^*(\tau_2) \right] = \sum_{\omega} |\tilde{f}_{\omega}|^2 \psi_{\omega}(\tau_1) \psi_{\omega}^*(\tau_2) \quad (2.6.20)$$

where we used the unified notation (2.2.8). An interesting aspect here is that ω is not manifestly a geometric quantity in AdS_3 , unlike the geodesic length b in AdS_2 .

This picture emphasizes that, while the CJ wormhole is topologically an annulus $\times S^1$, it should be viewed as a gluing of two constituent topologies, namely, of two “deformed solid tori” (i.e. deformed annuli times a circle). See Figures 2.3 and 2.7. This is somewhat schematic because we do not yet have a bulk geometric derivation of $Z_{\text{RMT}}(\tau)$. What we do know is that $Z_{\text{RMT}}(\tau)$ probes the part of the black hole Hilbert space that is precisely *not* captured by semiclassical saddles: whatever off-shell configurations should be included in the path integral with a single torus boundary, they must generate this term.

We note that the Riemann zeta phase $\phi(\omega)$ is the phase for any $(Z_{\text{spec}}, E_{\frac{1}{2}+i\omega})$, as one can show from its functional equation in ω . This supports our earlier half-wormhole characterization of $Z_{\text{spec}}(\tau)$ in the discussion of Hecke projection (Subsection 2.5.3), even absent a large c limit. For this reason, $Z_{\text{spec}}(\tau)$ is closely analogous to the “brane one-point function” $Z_{\text{brane}}(\beta)$ in the 2D gravity model of [166], an additive correction to the JT disk partition function that is non-perturbatively exact in the entropy S_0 . What we are finding is that this same mechanism operates in the bona fide $\text{AdS}_3/\text{CFT}_2$ setting in broad generality,

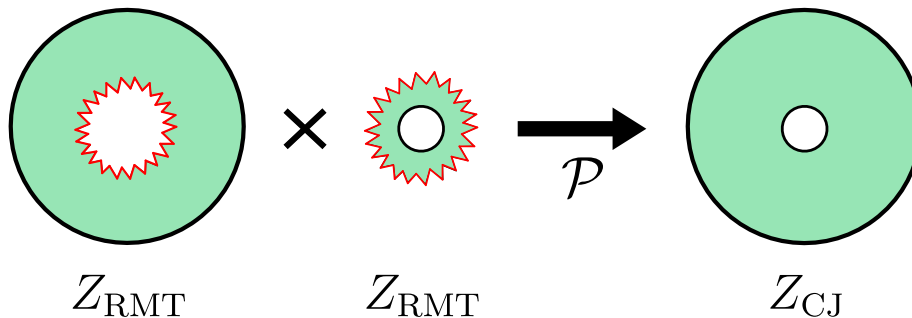


Figure 2.7: An alternate depiction of Figure 2.3 emphasizing the analogy to 2D gravity, with the extra S^1 direction suppressed. Each half-wormhole is viewed as a deformed annulus times a circle, in which one of the annular boundaries is replaced by an erratic/random boundary. In the spectral representation this boundary is precisely the Riemann zeta phase $e^{i\phi(\omega)}$. The CJ wormhole is formed by gluing along the deformed boundaries. We have drawn the second deformed annulus “inside out” for visual aid.

with $Z_{\text{spec}}(\tau)$ playing the role of the brane.

Pure gravity path integral

The identification of $Z_{\text{RMT}}(\tau)$ augments our knowledge of $Z_{\text{grav}}(\tau)$, the primary partition function of semiclassical AdS_3 pure gravity with a single torus boundary, as shown in (2.6.16). Any primary partition function $Z_p(\tau)$ for which the vacuum is the only Virasoro primary with⁴² $t < 0$ can, in general, be written

$$Z_p(\tau) = Z_{\text{MWK}}(\tau) + Z_{\text{spec}}(\tau) \quad (2.6.21)$$

By leveraging $Z_{\text{CJ}}(\tau_1, \tau_2)$, we have shown that $Z_{\text{spec}}(\tau)$ is not a free function in semiclassical pure gravity, leading to (2.6.16). Including further corrections,

$$Z_{\text{grav}}(\tau) = Z_{\text{MWK}}(\tau) + Z_{\text{RMT}}(\tau) + \delta Z_{\text{RMT}}(\tau) \quad (2.6.22)$$

There are three contributions:

- $Z_{\text{MWK}}(\tau)$ sums over smooth bulk saddles \mathcal{M} with $\partial\mathcal{M} = T^2$.
- $Z_{\text{RMT}}(\tau)$ encodes the leading RMT fluctuations of the black hole spectrum.
- $\delta Z_{\text{RMT}}(\tau)$ gives possible corrections to $Z_{\text{RMT}}(\tau)$.

⁴²Recall our definitions $t = \min(h, \bar{h}) - \xi$ and $\xi = (c - 1)/24$.

This refines the usual point of view taken in the modular bootstrap approach to $Z_{\text{grav}}(\tau)$, by adding the (off-shell) fine-grained spectral fluctuations of the black hole/heavy spectrum to the sum over (on-shell) BTZ black hole geometries. The “brane” terms $Z_{\text{RMT}}(\tau) + \delta Z_{\text{RMT}}(\tau)$ are exponentially suppressed relative to the sum over saddles.

The corrections $\delta Z_{\text{RMT}}(\tau)$ are important, even though they are expected to be exponentially small in $c \sim 1/G_N$. Were they perturbative instead, they would lead, upon squaring and projecting, to $1/c$ corrections to the CJ wormhole: this would violate reasonable expectations inferred from the genus expansion of 2D gravity for the structure of AdS_3 wormholes, as well as the method of computation of [34].⁴³ Instead, $\delta Z_{\text{RMT}}(\tau)$ should represent “higher genus” terms that encode UV data of a (say) strongly-coupled string theory, or M-theory, compactification. Indeed, it could well be the case that the string landscape hosts many compactifications to AdS_3 pure gravity which share a parametrically identical spectral gap, differing only in the detailed correlations among black hole microstates: the quantity $\delta Z_{\text{RMT}}(\tau)$ encodes exactly these contributions. When glued to form a wormhole, these would give $\mathcal{O}(e^{-\alpha c^\beta})$ corrections to $Z_{\text{CJ}}(\tau_1, \tau_2)$ for some $\alpha, \beta > 0$, perhaps corresponding to wormholes with torus boundary but higher topology in the interior.

Actually, not only should such corrections be present on general grounds, they *must* be present in a consistent CFT: if $\delta Z_{\text{RMT}}(\tau) = 0$, then the partition function (2.6.22) is non-unitary. To see this, recall that the MWK partition function $Z_{\text{MWK}}(\tau)$ contains an exponentially large negative density of states in a near-extremal limit of $t \rightarrow 0_+$ and $j \rightarrow \infty$ [98] for odd j : suppressing power-law prefactors,

$$\int_0^{e^{-\frac{1}{2}S_0(j)}} dt \rho_{\text{MWK},j}(t) \approx (-1)^j e^{\frac{1}{4}S_0(j)}, \quad \text{where } S_0(j) := 4\pi\sqrt{\xi|j|} \quad (2.6.23)$$

is the extremal spin- j Cardy entropy. But this cannot be cured by $Z_{\text{RMT}}(\tau)$, which is independent of the central charge.

The above results altogether lead us to a rather interesting conclusion. If a unitary $Z_{\text{grav}}(\tau)$ exists, the non-perturbative corrections $\delta Z_{\text{RMT}}(\tau)$ must not only be present, but must resum in some manner to give a contribution to the density of states below the semiclassical black hole threshold. Were this not the case, then for all values of the modulus τ we could safely drop $\delta Z_{\text{RMT}}(\tau)$ in the large c limit; but as reviewed above, the resulting partition function would be non-unitary. To avoid this order-of-limits issue between large c and near-extremality, the corrections $\delta Z_{\text{RMT}}(\tau)$ should resum to shift the black hole threshold slightly below $t = 0$, giving *non-square-integrable* contributions to $Z_{\text{grav}}(\tau)$.⁴⁴ In other words, the BTZ black hole

⁴³It should again be emphasized, given the unusual nature of the computation [34], that it is not known how (or whether) higher topologies beyond $T^2 \times I$ organize themselves in a small G_N expansion, and what their moduli dependence is. However, they should be present, and we are giving some expectations for their scaling.

⁴⁴We thank Daniel Jafferis for a discussion.

threshold is strictly below $t = 0$.

On the CFT side, the large c modular bootstrap quest is thus transformed to the following: *find $\delta Z_{\text{RMT}}(\tau)$ such that (2.6.22) is unitary.* In the bulk, $\delta Z_{\text{RMT}}(\tau)$ is an off-shell contribution to the AdS_3 path integral that presumably captures non-perturbative degrees of freedom.

On the Maxfield-Turiaci proposal

There is one proposal in the literature for at least some such off-shell contributions, due to Maxfield and Turiaci [88], which realizes precisely the resummation mechanism described above. They proposed that $Z_{\text{grav}}(\tau)$ should include a sum over a class of Seifert manifolds. These smooth 3-manifolds are S^1 fibrations over a disc D_2 with orbifold points.⁴⁵ Their inclusion in $Z_{\text{grav}}(\tau)$ was motivated by studying the near-extremal limit, where the AdS_3 theory admits an effective dimensionally-reduced description as JT gravity coupled to conical defects. Maxfield and Turiaci pointed out that, from the dimensionally-reduced point of view, the MWK negativity (2.6.23) is cured upon summing over multiple-defect configurations, not just the saddle points. This shifts the density near extremality to⁴⁶

$$\rho_{\text{MT},j}(t) \approx e^{S_0(j)} \sqrt{2(t - t_0(j))}, \quad \text{where } t_0(j) \approx -\frac{1}{2(2\pi)^2} (-1)^j e^{-\frac{1}{2}S_0(j)} \quad (2.6.24)$$

plus exponentially small corrections to the shift. The corrections, which are required for modular invariance of the AdS_3 result, would come from including the full set of defects. More explicitly, each species of JT defect of [88] is labeled by an integer q (corresponding to a \mathbb{Z}_q orbifold point on the base D_2), and an associated coupling

$$\lambda_{1/q} \sim e^{-S_0(j)(1-\frac{1}{q})} \quad (2.6.25)$$

In [88], the approximation $t_0(j) \approx \lambda_{1/2}$ is made: that is, the sum is only over the gas of $q = 2$ defects. The full sum over arbitrary numbers of arbitrary species of defects was proposed to have an S^1 uplift to a sum over Seifert manifolds,

$$Z_{\text{MT}}(\tau) = Z_{\text{MWK}}(\tau) + Z_{\text{Seifert}}(\tau) + \dots \quad (2.6.26)$$

The \dots represents other, non-Seifert contributions which near-extremal physics does not constrain. $Z_{\text{Seifert}}(\tau)$ has not been computed.

There has been ample reason to believe that the Maxfield-Turiaci proposal is correct. Our argument above supports this further. If it is correct, the Seifert manifolds of [88] should contribute to $Z_{\text{RMT}}(\tau) + \delta Z_{\text{RMT}}(\tau)$. Understanding how this works is a very worthwhile

⁴⁵An excellent review of Seifert manifolds may be found in [167].

⁴⁶The result of [88] formally holds for $j \rightarrow \infty$, but the functions appearing in the calculation depend on the combination ξj , which partially justifies an extrapolation to all nonzero spins at large ξ .

direction for future work. Let us take some initial steps toward understanding this connection. First, recall that $Z_{\text{RMT}}(\tau)$ is independent of c , and hence of the entropy. A generic defect configuration term has coefficient

$$e^{S_0(j)} \prod_{q=2}^{\infty} (\lambda_{1/q})^{n_q} \sim e^{S_0(j) \left(1 - \sum_q n_q \left(1 - \frac{1}{q}\right)\right)}, \quad n_q \in \mathbb{Z}_{\geq 0} \quad (2.6.27)$$

It is rather tidy that the only term which is independent of entropy is the $q = n_2 = 2$ term. So we see that $Z_{\text{RMT}}(\tau)$ should include only the Seifert manifold with $n_2 = 2$ exceptional fibers, each over a \mathbb{Z}_2 orbifold point on the base. On the other hand, $Z_{\text{RMT}}(\tau)$ is $SL(2, \mathbb{Z})$ -invariant, whereas the $q = n_2 = 2$ Seifert partition function is not. This confirms that $Z_{\text{grav}}(\tau)$ should include other, non-Seifert, off-shell topologies which give an $\mathcal{O}(1)$ contribution, as anticipated in [88]. (More explicitly, in the Maxfield-Turiaci realization of (2.6.22), $\delta Z_{\text{RMT}}(\tau)$ would include the other Seifert manifolds that are further entropically suppressed, while $Z_{\text{RMT}}(\tau)$ would include non-Seifert manifolds.)

2.7 Future Directions

We have presented a framework for quantifying random matrix statistics of microscopic 2d CFT spectra, manifestly compatible with the requisite symmetries. The identification of a 2d CFT trace formula suggests deep and robust connections with random matrix theory and chaotic quantum systems. While we focused mostly on the diagonal approximation to the product of partition functions, a clear path forward is to understand how the full non-perturbative structure of RMT embeds itself in the dynamics of high-energy states in 2d CFTs and their dual black holes of AdS_3 gravity, and how to excavate it within our spectral framework.

Let us highlight a few specific directions of interest.

One target therein is to understand how encounter theory [59, 133, 136, 138], which initiates the transition from ramp to plateau in the SFF of quantum chaotic systems, should be phrased in the language of the $SL(2, \mathbb{Z})$ spectral decomposition. This subject was nicely reviewed and ported over to the JT gravity setting in [39]. The goal vis-à-vis 2d CFT is to go beyond the diagonal approximation to $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$, organizing the off-diagonal terms in a way that mimics encounter theory. Our framework gives a clear meaning to “off-diagonal terms”: they are correlations among $SL(2, \mathbb{Z})$ eigenfunctions with unequal spectral parameters, $\omega_1 \neq \omega_2$. The question is how to develop a systematic approach to these correlations. and whether they give rise to different types of corrections to (say) the SFF.

A related, more specific starting point is the following. The SFF of double-scaled random matrix ensembles admits a “ τ -scaling limit” in which $T \rightarrow \infty$ with $\tau := T e^{-S_0}$ held fixed. The

expansion in $\tau \ll 1$ has been reproduced to all orders by encounter theory [133, 138]. Is there a τ -scaling limit in 2d CFT? If so, how do we compute it in our formalism? Understanding this could help to extend the meaning of MaxRMT to the non-perturbative level [168–175].

It is interesting to notice that there are two distinct classes of corrections to the SFF in the $SL(2, \mathbb{Z})$ spectral framework: corrections to the spectral overlaps themselves, and the inclusion of terms beyond the diagonal approximation. The former give corrections to the SFF beyond the strict ramp term, even *without* incorporating off-diagonal terms in $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$. The latter are obviously crucial to reproduce the constant late-time plateau; but in the interpolation from ramp to plateau, the two types of corrections should in principle combine to give the full SFF. It is important to disentangle and stratify these various effects, and to connect them to non-perturbative corrections to the CJ wormhole of AdS_3 pure gravity.

The condition (2.4.23) for the SFF to have an RMT linear ramp at $T \gg \beta$ is an asymptotic condition on $Z_{\text{spec}}(\tau)$ in $SL(2, \mathbb{Z})$ spectral space. Given the torus partition function of a CFT, this condition can be checked unambiguously. If we accept that generic irrational CFTs are chaotic, then to the extent that one believes (a CFT version of) the Bohigas-Giannoni-Schmit conjecture [176] that ties chaos to random matrix statistics, the condition (2.4.23) should be obeyed by generic irrational CFTs [177]. It is worth asking whether this condition can be tied to other, ideally low-energy, properties of irrational CFT data. For example, it is not known whether a nonzero primary twist gap above the vacuum state is sufficient to imply that the CFT is chaotic in an appropriate sense.⁴⁷ Perhaps one can determine whether the twist gap is related to (2.4.23).

On the bootstrap front, it is of clear interest to further pursue the large c modular bootstrap program [179–183] for the partition function (2.6.22). This raises the larger question of how to incorporate RMT spectral statistics into modular and conformal bootstrap algorithms.

Our formalism is not restricted to studying partition functions. One immediate generalization is to study coarse-grained products of torus one-point functions of local primary operators, $\langle \mathcal{O} \rangle_\tau$. One-point functions are modular-covariant and, like partition functions, can be dressed and massaged to be modular-invariant and square-integrable on \mathcal{F} , hence admitting a $SL(2, \mathbb{Z})$ spectral decomposition and thus a trace formula. The spectral overlaps encode the OPE statistics. In the AdS_3 bulk, the projections $\mathcal{P}_{\text{Hecke}}[\langle \mathcal{O} \rangle_{\tau_1} \langle \mathcal{O} \rangle_{\tau_2}]$ would be holographically dual to “one-point RMT wormholes”: that is, off-shell connected configurations with a profile for the field dual to \mathcal{O} threading the wormhole, with local sources at each torus boundary. Essentially this quantity was studied in the 2D gravity context

⁴⁷It is not clear to us that the twist gap is a relevant criterion. A twist gap can vanish due either to extra conserved currents, or an accumulation to zero twist at asymptotically large spin. One should always use the maximally extended chiral algebra to formulate the notion of twist gap [178], so we ignore the first possibility. But we are not aware of any solid argument that a zero-twist accumulation point implies an absence of chaos/RMT statistics or vice-versa (even though it does invalidate standard Cardy-type arguments about spectral asymptotics).

in [146, 184, 185]. These generalizations of the CJ wormhole are not the Maldacena-Maoz one-point ETH wormholes [186], instead incorporating the level statistics of the states in the OPE. One can formulate a straightforward one-point generalization of the CJ wormhole (2.6.5) that combines the RMT level statistics with Virasoro ETH asymptotics; this will be presented elsewhere.

We have introduced a notion of MaxRMT in 2d CFTs. How does it generalize to higher dimensions? A natural conjecture that we would like to make is that *the black hole spectrum of semiclassical Einstein gravity in $AdS_{D \geq 3}$ exhibits MaxRMT statistics*. The meaning of this in $D > 3$ is somewhat vague. In the AdS_3/CFT_2 context, the MaxRMT nature of pure gravity is a *quantitative* statement, built on the infrastructure of the $SL(2, \mathbb{Z})$ spectral decomposition as a trace formula. That allowed us to fully process the symmetries of the spectrum, leaving only those spectral correlations that are unrelated by symmetry; MaxRMT then states that these correlations are exactly those of RMT. This latter characterization in principle generalizes MaxRMT to higher dimensions. What, then, is the CFT_d trace formula for the density of states? Given such a formula, one could construct a diagonal approximation to the two-point correlator and proceed as we did here. It would be very interesting to make this idea concrete, and to ask how MaxRMT relates to the familiar holographic CFT conditions of large N and large higher-spin gap.

Our work may also have something to say about lower dimensions. In JT gravity an analogous description of wormholes as emergent from non-disordered systems via a trace formula has been anticipated since [35]. Given the existence of an emergent Schwarzian sector in the near-extremal limit of 2d CFTs [88, 187, 188], it seems reasonable to seek a dimensional reduction of the CJ wormhole that yields the JT double-trumpet. One could then hope to translate the diagonality of the CJ wormhole (2.6.5) to a diagonal approximation interpretation of the JT double-trumpet. In that case, identifying the reduction of $Z_{\text{RMT}}(\tau)$ would yield new microscopic information about the microstates of JT gravity in the form of a trace formula for $\rho(E)$. It would be interesting to compare it to other proposals for restoring factorization in JT gravity [99, 165, 166, 189–192].

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2.8 Appendix A: Glossary

The following is the short exact sequence of partition functions appearing in this work. All objects below are modular-invariant with respect to each independent argument.

- $Z_p(\tau)$: Virasoro primary-counting torus partition function.
- $Z_{\text{spec}}(\tau)$: Spectral partition function, obtained by subtracting light states and their modular completions from $Z_p(\tau)$. Admits an $SL(2, \mathbb{Z})$ spectral decomposition.
- $Z_{\text{diag}}(\tau_1, \tau_2)$: Diagonal projection of the factorized product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$, defined by projection onto the kernel of $\Delta_{12} := \Delta_{\tau_1} - \Delta_{\tau_2}$, where Δ_τ is the hyperbolic Laplacian. Pairs eigenvalues of the Laplacian.
- $Z_{\text{Hecke}}(\tau_1, \tau_2)$: Hecke projection of the factorized product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$, defined by projection onto the kernel of $T_j^{(12)} := T_j^{(\tau_1)} - T_j^{(\tau_2)}$, where $T_j^{(\tau)}$ is a spin- j Hecke operator for $SL(2, \mathbb{Z})$, for all spins $j \in \mathbb{Z}_+$. Pairs eigenfunctions of the Laplacian.
- $Z_{\text{WH}}(\tau_1, \tau_2)$: A name for $Z_{\text{Hecke}}(\tau_1, \tau_2)$ when Eisenstein and cusp form overlaps are equal. Includes off-shell torus wormhole amplitudes in semiclassical AdS₃ gravity.
- $Z_{\text{CJ}}(\tau_1, \tau_2)$: Cotler-Jensen torus wormhole amplitude of AdS₃ pure gravity. An instance of $Z_{\text{WH}}(\tau_1, \tau_2)$.
- $Z_{\text{RMT}}(\tau)$: The $Z_{\text{spec}}(\tau)$ whose coarse-grained product generates $Z_{\text{CJ}}(\tau_1, \tau_2)$. That is, a solution of $Z_{\text{CJ}}(\tau_1, \tau_2) = \mathcal{P}_{\text{Hecke}}[Z_{\text{RMT}}(\tau_1)Z_{\text{RMT}}(\tau_2)]$.

2.9 Appendix B: Spectral decomposition on $L^2(\mathcal{F} \times \mathcal{F})$

We present a few details regarding the functional spaces appearing in the discussion and their different spectral decompositions. Consider the manifold $\mathcal{F} = \mathbb{H}/SL(2, \mathbb{Z})$ and the associated space of square-integrable functions $f(\tau) \in L^2(\mathcal{F})$ with respect to the Petersson inner product defined by the measure $dx dy/y^2$. A complete basis of functions for $L^2(\mathcal{F})$ is given by the eigenfunctions of the Laplacian:

$$\Delta_\tau \psi_\omega(\tau) = \lambda_\omega \psi_\omega(\tau), \quad f(\tau) = \sum_\omega \tilde{f}_\omega \psi_\omega(\tau) \quad (2.9.1)$$

where

$$\psi_\omega(\tau) = \{\phi_0, E_{\frac{1}{2}+i\omega}(\tau), \phi_n(\tau)\} \quad (2.9.2)$$

with $\phi_0 = \text{vol}(\mathcal{F})^{-\frac{1}{2}} = \sqrt{3/\pi}$ and $n \in \mathbb{Z}_+$. The overlaps are denoted here by $\tilde{f}_\omega = (f(\tau), \psi_\omega(\tau))$, with implicit degeneracy for every $\omega = \omega_n$ for which a Maass cusp form exists. The tensor product space $L^2(\mathcal{F}) \otimes L^2(\mathcal{F})$ of factorized functions $f(\tau_1)g(\tau_2)$ admits a simple decomposition which is just given by the product of the individual decompositions of $f(\tau_1)$ and $g(\tau_2)$.

Consider now the space of square integrable functions of two moduli τ_1, τ_2 , i.e. $L^2(\mathcal{F}_1 \times \mathcal{F}_2)$. A function $f(\tau_1, \tau_2) \in L^2(\mathcal{F}_1 \times \mathcal{F}_2)$ is in general not factorizable. The functions appearing in the body of the text ($Z_{\text{diag}}, Z_{\text{Hecke}}, Z_{\text{WH}}, Z_{\text{CJ}}$) are elements of this space. The Stone-Weierstrass theorem [193] states that the subspace of factorized functions $f(\tau_1)g(\tau_2) \in L^2(\mathcal{F}) \otimes L^2(\mathcal{F})$ is dense in $L^2(\mathcal{F}_1 \times \mathcal{F}_2)$. We then have a complete basis of functions for $L^2(\mathcal{F}_1 \times \mathcal{F}_2)$ given by:

$$f(\tau_1, \tau_2) = \sum_{\omega_1, \omega_2} \tilde{f}_{\omega_1, \omega_2} \psi_{\omega_1}(\tau_1) \psi_{\omega_2}(\tau_2) \quad (2.9.3)$$

The overlaps $\tilde{f}_{\omega_1, \omega_2}$ are computed by taking the inner product with the basis elements, with the “double inner product” defined by the product metric

$$(f, g) := \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \int_{\mathcal{F}_2} \frac{dx_2 dy_2}{y_2^2} f(\tau_1, \tau_2) \bar{g}(\tau_1, \tau_2), \quad (2.9.4)$$

As an example, the constant term in the spectral decomposition can be computed by taking residues of a double Rankin-Selberg transform:

$$\langle f(\tau_1, \tau_2) \rangle = \frac{1}{\text{vol}(\mathcal{F})^2} \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \int_{\mathcal{F}_2} \frac{dx_2 dy_2}{y_2^2} f(\tau_1, \tau_2) = \text{Res}_{s_1=1} \text{Res}_{s_2=1} R_{s_1, s_2}[f], \quad (2.9.5)$$

where the double Rankin-Selberg transform is defined as:

$$R_{s_1, s_2}[f] := \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \int_{\mathcal{F}_2} \frac{dx_2 dy_2}{y_2^2} f(\tau_1, \tau_2) E_{s_1}(\tau_1) E_{s_2}(\tau_2). \quad (2.9.6)$$

The overlaps with a single Eisenstein can also be computed in terms of the Rankin-Selberg transform on the critical line $\text{Res} = \frac{1}{2}$:

$$\begin{aligned} (f, \phi_0 E_s(\tau_1)) &= \frac{1}{\sqrt{\text{vol}(\mathcal{F})}} \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \int_{\mathcal{F}_2} \frac{dx_2 dy_2}{y_2^2} f(\tau_1, \tau_2) E_s(\tau_1) \\ &= \sqrt{\text{vol}(\mathcal{F})} \text{Res}_{r=1} R_{1-s, r}[f], \end{aligned} \quad (2.9.7)$$

and similarly for $(f, \phi_0 E_s(\tau_2))$.

2.9.1 Regularization of the Cotler-Jensen wormhole

As an application, we show how the divergence of the Cotler-Jensen wormhole can be obtained and regularized from the constant term in the spectral decomposition. The constant term is given by taking residues of the double Rankin-Selberg transform:

$$\begin{aligned} \langle Z_{\text{CJ}} \rangle &= \text{Res}_{s_1=1} \text{Res}_{s_2=1} \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \int_{\mathcal{F}_2} \frac{dx_2 dy_2}{y_2^2} Z_{\text{CJ}}(\tau_1, \tau_2) E_{s_1}(\tau_1) E_{s_2}(\tau_2) \\ &= \frac{1}{\text{vol}(\mathcal{F})} \text{Res}_{s_2=1} \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \frac{\Gamma(s_2) \Gamma(1-s_2)}{\pi} E_{s_2}(\tau_1). \end{aligned} \quad (2.9.8)$$

The Rankin-Selberg transform has a double pole at $s_2 = 1$, indicating the need for regularization. Simply taking the residue one obtains

$$\langle Z_{\text{CJ}} \rangle = -\frac{1}{\pi \text{vol}(\mathcal{F})} \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \hat{E}_1(\tau_1). \quad (2.9.9)$$

where

$$\hat{E}_1(\tau) := \lim_{s \rightarrow 1} \left(E_s(\tau) - \frac{3}{\pi} \frac{1}{s-1} \right) \quad (2.9.10)$$

is the “finite part” of $E_s(\tau)$ as $s \rightarrow 1$. The integral (2.9.9) is divergent, as $\hat{E}_1(\tau_1)$ diverges linearly as $y_1 \rightarrow \infty$. We can regularize the overlaps by splitting the pole with a new parameter r in the following way, preserving the $s \rightarrow 1-s$ reflection symmetry:

$$\begin{aligned} \langle Z_{\text{CJ}}^{(\text{reg})} \rangle &= \frac{1}{\pi \text{vol}(\mathcal{F})} \text{Res}_{s=1} \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \Gamma(s+r-1) \Gamma(r-s) E_s(\tau_1) \\ &= \frac{1}{\pi \text{vol}(\mathcal{F})^2} \int_{\mathcal{F}_1} \frac{dx_1 dy_1}{y_1^2} \Gamma(r) \Gamma(r-1) \\ &= \frac{1}{\pi \text{vol}(\mathcal{F})} \Gamma(r) \Gamma(r-1) \end{aligned} \quad (2.9.11)$$

As $r \rightarrow 1$, this has a divergence that is constant in τ_1, τ_2 , so we can regularize by subtracting this term:

$$Z_{\text{CJ}}^{(\text{reg})} = \lim_{r \rightarrow 1} \left(Z_{\text{CJ}}(\tau_1, \tau_2) - \frac{3}{\pi^2} \frac{1}{r-1} \right) \quad (2.9.12)$$

This is the same regularization originally proposed by [34], now obtained from regularizing the constant term in $L^2(\mathcal{F} \times \mathcal{F})$. The remaining finite constant is a free parameter of the regularization scheme.

2.9.2 Diagonal projection

Let us make a comment on the diagonal projection of Section 2.4, defined as projection of $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$ onto $\ker(\Delta_{12})$. Note that functions for which $\Delta_\tau f(\tau)$ is constant are annihilated by Δ_{12} . When $\Delta_\tau f(\tau) = 0$, $f(\tau)$ is a constant. When $\Delta_\tau f(\tau) \neq 0$, the unique solution up to rescaling is given by $\widehat{E}_1(\tau)$, which obeys

$$\Delta_\tau \widehat{E}_1(\tau) = -\frac{3}{\pi} \quad (2.9.13)$$

Thus, given a $Z_{\text{diag}}(\tau_1, \tau_2)$, we may define

$$Z'_{\text{diag}}(\tau_1, \tau_2) = Z_{\text{diag}}(\tau_1, \tau_2) + c_1(\widehat{E}_1(\tau_1) + \widehat{E}_1(\tau_2)) + c_2 \quad (2.9.14)$$

which also obeys $Z'_{\text{diag}}(\tau_1, \tau_2) \in \ker(\Delta_{12})$, where c_1 and c_2 are constants. The choice of these constants is thus a choice of scheme for the diagonal projection. In the main text, we choose $c_1 = c_2 = 0$. One way to motivate this choice is to note that $Z'_{\text{diag}}(\tau_1, \tau_2) \notin L^2(\mathcal{F} \times \mathcal{F})$ when $c_1 \neq 0$: in particular, $\widehat{E}_1(\tau)$ is not part of the basis (2.9.2). So the choice $c_1 = c_2 = 0$ may be rephrased as the condition that the diagonal projection preserve the square integrability, present in the original product $Z_{\text{spec}}(\tau_1)Z_{\text{spec}}(\tau_2)$, and moreover sets the constant term to zero.

2.10 Appendix C: Continuity of the $j \rightarrow 0$ limit

In this appendix we show that the scalar Fourier mode of the Eisenstein series $E_{s,0}(y)$ can be obtained from a smooth $j \rightarrow 0$ limit of the spin j mode $E_{s,j}(y)$, and similarly for the Eisenstein density $\rho_{s,j}(t)$.

We want to evaluate the following limit

$$\lim_{j \rightarrow 0} E_{s,j}^*(y) = \lim_{j \rightarrow 0} \frac{2\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi jy) \quad (2.10.1)$$

and show its equality with

$$E_{s,0}^*(y) = \Lambda(s)y^s + \Lambda(1-s)y^{1-s} \quad (2.10.2)$$

The limit $j \rightarrow 0$ for fixed y corresponds to the small argument expansion for the Bessel function which is, at leading order

$$K_{s-\frac{1}{2}}(2\pi jy) \approx \frac{(2\pi jy)^{s-\frac{1}{2}}}{2^{s+\frac{1}{2}}} \Gamma\left(\frac{1}{2} - s\right) + (s \rightarrow 1-s) \quad (2.10.3)$$

Using $\Lambda(s) = \Lambda\left(\frac{1}{2} - s\right) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2} - s\right)\zeta(1 - 2s)$ we can simplify various factors to obtain

$$\lim_{j \rightarrow 0} \frac{2\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi jy) = \lim_{j \rightarrow 0} \left(\Lambda(s) y^s \frac{\sigma_{2s-1}(j)}{\zeta(1-2s)} + (s \rightarrow 1-s) \right) \quad (2.10.4)$$

Remembering that the divisors of zero are all the natural numbers we have

$$\lim_{j \rightarrow 0} \sigma_{2s-1}(j) = \zeta(1-2s) \quad (2.10.5)$$

which correctly gives the scalar result.

The same manipulations may be applied to the corresponding density. The spin- j Eisenstein density is

$$\rho_{\frac{1}{2}+i\omega, j}^*(t) = 2 \frac{\sigma_{2i\omega}(j)}{j^{i\omega}} \theta(t) \frac{\cos\left(\omega \cosh^{-1}\left(\frac{2t}{j} + 1\right)\right)}{\sqrt{t(t+j)}} \quad (2.10.6)$$

To arrive at the scalar density we need to compute the following limit:

$$\lim_{j \rightarrow 0} \frac{\sigma_{2i\omega}(j)}{j^{i\omega}} e^{i\omega \cosh^{-1}\left(\frac{2t}{j} + 1\right)} = (4t)^{i\omega} \lim_{j \rightarrow 0} \frac{\sigma_{2i\omega}(j)}{j^{2i\omega}}. \quad (2.10.7)$$

We use the identity

$$\frac{\sigma_{2i\omega}(j)}{j^{2i\omega}} = \sigma_{-2i\omega}(j) \quad (2.10.8)$$

which, together with the previous limit (2.10.5), recovers the $j = 0$ density from the $j \rightarrow 0$ limit,

$$\begin{aligned} \lim_{j \rightarrow 0} \rho_{\frac{1}{2}+i\omega, j}^*(t) &= \rho_{\frac{1}{2}+i\omega, 0}^*(t) \\ &= \frac{\zeta(2i\omega)}{t} (4t)^{i\omega} + (\omega \rightarrow -\omega). \end{aligned} \quad (2.10.9)$$

2.11 Correlations among spins

In this appendix we prove that modular invariance correlates unequal spins in the diagonal approximation:

$$Z_{\text{diag}}^{(j_1, j_2)}(y_1, y_2) \propto \delta_{j_1, j_2} \iff Z_{\text{diag}}(\tau_1, \tau_2) = 0 \quad (2.11.1)$$

We proceed by contradiction, assuming that $Z_{\text{diag}}^{(j_1, j_2)}(y_1, y_2) \propto \delta_{j_1, j_2}$.

First suppose the scalar sector is non-vanishing, $Z_{\text{diag}}^{(0,0)}(y_1, y_2) \neq 0$. Then the Eisenstein

overlaps are non-vanishing:

$$Z_{\text{diag}}^{(0,0)}(y_1, y_2) \neq 0 \quad \Rightarrow \quad (Z_{\text{spec}}, E_s) \neq 0. \quad (2.11.2)$$

The spin-(0, j) mode, which by assumption is equal to zero, is given by

$$\begin{aligned} 0 = Z_{\text{diag}}^{(0,j)}(y_1, y_2) &= \int_{\mathcal{C}_{\text{crit}}} \{Z_{\text{spec}}, E_s\}^2 E_{s,0}^*(y_1) \mathbf{a}_j^{(s)} \sqrt{y_2} K_{s-\frac{1}{2}}(2\pi j y_2) \\ &+ \sum_{n=1}^{\infty} \{Z_{\text{spec}}, E_{s_n}\} (Z_{\text{spec}}, \phi_n) \left(E_{s_n,0}^*(y_1) \mathbf{b}_j^{(n)} \sqrt{y_2} K_{s_n-\frac{1}{2}}(2\pi j y_2) + (y_1 \leftrightarrow y_2) \right) \end{aligned} \quad (2.11.3)$$

We can single out the first line by using the Bessel orthogonality relation

$$\int_0^{\infty} dx \frac{K_{it}(x) K_{i\omega}(x)}{x} = \frac{\pi^2}{2t \sinh(\pi t)} \delta(t - \omega), \quad (2.11.4)$$

which allows us to project onto a chosen, non-degenerate eigenvalue $\omega \neq \omega_n$ on the critical line. This implies $\{Z_{\text{spec}}, E_s\} = 0$, in contradiction with the starting assumption that $Z_{\text{diag}}^{(0,0)}(y_1, y_2) \neq 0$.

If instead $Z_{\text{diag}}^{(0,0)}(y_1, y_2) = 0$, then the Eisenstein overlaps vanish. But nonzero cusp form overlaps imply $Z_{\text{diag}}^{(j_1, j_2)}(y_1, y_2) \neq 0$ for $j_1 \neq j_2$, which is in contradiction with the δ_{j_1, j_2} behavior.

Thus the only modular-invariant $Z_{\text{diag}}(\tau_1, \tau_2)$ proportional to δ_{j_1, j_2} is zero.

2.12 Appendix D: Derivation of (2.4.9)

Here we derive (2.4.9), the ‘‘corrections’’ to the RMT ramp that are mandated by $SL(2, \mathbb{Z})$ invariance, starting from the definition of $\mathcal{S}(r)$ in (2.4.7).

To prove (2.4.9) we first write

$$\mathcal{S}(r) = r^2 \mathcal{M}^{-1}[\mathcal{M}[\mathcal{R}(z); 1-s] \varphi(s); r^2] \quad (2.12.1)$$

We now use the Mellin convolution theorem: given two functions $f(s)$ and $g(s)$ with inverse Mellin transforms⁴⁸ $F(z)$ and $G(z)$, respectively,

$$\mathcal{M}^{-1}[f(s)g(s); x] = \int_0^{\infty} \frac{dy}{y} F\left(\frac{x}{y}\right) G(y) \quad (2.12.2)$$

⁴⁸Defined within the relevant strips.

If $g(s) = g(1 - s)$, then $G(x) = x^{-1}G(x^{-1})$, and we can instead write

$$\mathcal{M}^{-1}[f(s)g(s); x] = \int_0^\infty dy F(xy)G(y) \quad (2.12.3)$$

Since $\mathcal{M}[\mathcal{R}(z); s] = \mathcal{M}[\mathcal{R}(z); 1 - s]$ we can apply (2.12.3) to write

$$\mathcal{S}(r) = r^2 \int_0^\infty dy \mathcal{M}^{-1}[\varphi(s); r^2 y] \mathcal{R}(y) \quad (2.12.4)$$

The function $\varphi(s)$, defined in (2.4.3), may be written as

$$\varphi(s) = \frac{\sqrt{\pi}\Gamma\left(s - \frac{1}{2}\right) \zeta(2s - 1)}{\Gamma(s) \zeta(2s)} \quad (2.12.5)$$

We next use

$$\frac{\zeta(2s - 1)}{\zeta(2s)} = \sum_{n=1}^\infty \frac{\phi(n)}{n^{2s}} \quad (2.12.6)$$

where $\phi(n)$ is the Euler totient function. Using (2.12.5) and (2.12.6),

$$\begin{aligned} \mathcal{M}^{-1}[\varphi(s); r^2 y] &= \sum_{n=1}^\infty \phi(n) \mathcal{M}^{-1}\left[n^{-2s} \frac{\sqrt{\pi}\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}; r^2 y\right] \\ &= \sum_{n=1}^\infty \frac{\phi(n)}{n} \frac{\Theta(1 - n^2 r^2 y)}{\sqrt{r^2 y (1 - n^2 r^2 y)}} \end{aligned} \quad (2.12.7)$$

Plugging into (2.12.4) and moving the sum over n outside the integral, this can be massaged into the form

$$\mathcal{S}(r) = \sum_{n=1}^\infty \frac{\phi(n)}{n^2} \int_0^1 \frac{du}{\sqrt{u(1-u)}} \mathcal{R}\left(\frac{u}{n^2 r^2}\right) \quad (2.12.8)$$

This is (2.4.9).

As a check, we plug in the CJ/MaxRMT result, $\mathcal{R}_{\text{CJ}}(z) = \pi^{-1}/(1+z)$, which gives

$$\mathcal{S}_{\text{CJ}}(r) = \sum_{n=1}^\infty \frac{\phi(n)}{n} \frac{r}{\sqrt{1+n^2 r^2}} \quad (2.12.9)$$

This reproduces the correct result (2.6.3).

Note that if $\mathcal{R}(0)$ is finite and nonzero, then $\mathcal{S}(r)$ has a linear, r -independent, divergence as $r \rightarrow \infty$:

$$\mathcal{S}(r) \approx \pi \mathcal{R}(0) \sum_{n=1}^\infty \frac{\phi(n)}{n^2} \quad (r \rightarrow \infty) \quad (2.12.10)$$

This is the case for the CJ/MaxRMT wormhole, where $\mathcal{R}_{\text{CJ}}(0) = \pi^{-1}$.

We can also rewrite (2.12.8) as a sum over residues by contour deformation. Since the integral is along the branch cut from zero to one,⁴⁹ we can replace it with a contour surrounding the branch cut, which we then blow up to pick up residues elsewhere in \mathbb{C} . For example, simple poles of $\mathcal{R}(z)$ contribute as

$$\mathcal{S}(r) \supset \pi \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{z_*} \sqrt{\frac{r^2}{z_*(n^2 r^2 z_* - 1)}} \operatorname{Res}_{z=z_*} \mathcal{R}(z) \quad (2.12.11)$$

This may of course be generalized to include higher-order poles or non-meromorphic behavior.

2.13 Appendix F: Toy wormholes in $\text{AdS}_3 \times \mathcal{M}$ gravity

In this appendix we construct an explicit family of torus wormhole amplitudes which are toy models (or perhaps candidates) for off-shell wormholes of AdS_3 gravity with light matter, such as $\text{AdS}_3 \times \mathcal{M}$ compactifications with large \mathcal{M} . These theories are not MaxRMT theories, and their wormholes should reflect this. Specifically, the function $\mathcal{R}_{\text{WH}}(z)$, which determines the entire amplitude (see (2.4.7) and (2.5.23)), should have poles in the complex z plane away from the ramp pole at $z = -1$ with coefficients of $\mathcal{O}(c^0)$, thus augmenting the CJ pure gravity wormhole with large corrections. The examples given below have the added property that both the spectral overlap $f_{\text{WH}}(s)$ and their Mellin transforms $\mathcal{R}_{\text{WH}}(z)$ are computable, with a linear ramp.

We first register a useful Mellin transform,

$$\mathcal{M} \left[\frac{1}{1+nz}; s \right] = n^{-s} \Gamma(s) \Gamma(1-s) \quad (2.13.1)$$

Note that reflection symmetry $f_{\text{WH}}(s) = f_{\text{WH}}(1-s)$ requires the poles to appear in pairs, at $z = z_*$ and $z = z_*^{-1}$. The reflection-symmetric combination is

$$\mathcal{M} \left[\frac{1}{1+nz} + \frac{1}{n+z}; s \right] = (n^{-s} + n^{s-1}) \Gamma(s) \Gamma(1-s) \quad (2.13.2)$$

A well-known [143] set of Mellin dual pairs involves harmonic sums in z , i.e. taking $F_{\text{WH}}(z)$ to be an infinite sum over simple poles with no regular term. So long as we include the pole at $z = -1$ with the correct normalization, the wormhole reproduces the linear ramp. In accordance with the comments in Section 2.6.2, the remaining poles should avoid $z \in \mathbb{R}_+$ and $|z| = 1$. For this reason, harmonic sums over poles along $z \in \mathbb{R}_-$ is not only mathematically, but also physically, well-motivated here.

⁴⁹ $\mathcal{R}(z)$ is analytic for all $z > 0$ because it is the low-temperature limit $y_1, y_2 \rightarrow \infty$ for fixed $z = y_1/y_2$, which is non-singular on physical grounds.

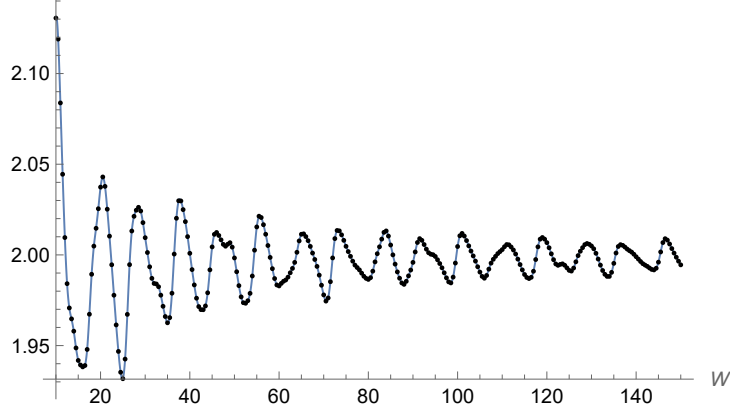


Figure 2.8: A plot of the integral in (2.13.8) for $p = 1, \omega_0 = 100$ and $W \in [10, 150]$, exhibiting oscillations around the asymptotic value of two. Black points are numerical data points, and the blue line is an interpolation.

Summing over n in (2.13.2) with an insertion of n^{-p} , we of course have

$$\sum_{n=1}^{\infty} n^{-p-s} = \zeta(s+p) \quad (2.13.3)$$

and its reflection. For integer $p \in \mathbb{Z}_+$, the inverse Mellin transform of (2.13.3) defines a convergent harmonic sum, plus a simple p -dependent shift:

$$\sum_{n=1}^{\infty} \frac{n^{-p}}{1+nz} = (-z)^{p-1} H(z^{-1}) + \sum_{q=2}^p \zeta(q) (-z)^{p-q} \quad (2.13.4)$$

So the previous two equations are Mellin partners. Upon adding the reflected term, we have an explicit pair for every $p \in \mathbb{Z}_+$. That is, written in terms of $f_{\text{CJ}}(s) = \pi^{-1} \Gamma(s) \Gamma(1-s)$,

$$f_{\text{WH}}(s) = f_{\text{CJ}}(s) \times \frac{\mathbf{C}_{\text{RMT}}}{4} (\zeta(s+p) + \zeta(p+1-s)) \quad (2.13.5)$$

and its inverse Mellin transform

$$\mathcal{R}_{\text{WH}}(z) = \frac{\mathbf{C}_{\text{RMT}}}{4\pi} \left((-z)^{p-1} H(z^{-1}) + \sum_{q=2}^p \zeta(q) (-z)^{p-q} + (-z)^{-p} H(z) - \sum_{q=2}^p \zeta(q) (-z^{-1})^{p+1-q} \right) \quad (2.13.6)$$

Inserting this overlap into the wormhole ansatz (2.5.23) defines a family of toy model RMT wormholes for a theory of gravity with light matter, one for every $p \in \mathbb{Z}_+$.⁵⁰

⁵⁰It would be nice to ask whether one can construct an amplitude of this form from a Poincaré sum over relative modular transformations. The structure of such Poincaré sums might also constrain the allowed singularity structures in the complex z plane.

By construction, this has the RMT ramp:

$$\mathcal{R}_{\text{WH}}(z \rightarrow -1) \sim \frac{\mathbf{C}_{\text{RMT}}}{2\pi} \frac{1}{1+z} \quad (2.13.7)$$

On the Mellin side, one readily verifies that

$$\frac{1}{W} \int_{\omega_0}^{\omega_0+W} d\omega \left(\zeta\left(p + \frac{1}{2} + i\omega\right) + \zeta\left(p + \frac{1}{2} - i\omega\right) \right) \approx 2 \quad (W \rightarrow \infty) \quad (2.13.8)$$

for all $p \geq 1$ and any finite ω_0 , because $|\zeta(\sigma + i\omega)| \rightarrow 1$ asymptotically as $\omega \rightarrow \infty$ for any $\sigma > 1$. This can be seen numerically in Figure 2.8. Together with the exactly constant falloff of $f_{\text{CJ}}(s)$, the total $f_{\text{WH}}(s)$ satisfies the linear ramp condition (2.4.23).

Chapter 3

AdS₃ Pure Gravity and Stringy Unitarity

This chapter consists of a paper [2] written in collaboration with Eric Perlmutter. The original abstract is as follows:

We construct a unitary, modular-invariant torus partition function of a two-dimensional conformal field theory with a Virasoro primary spectral gap of $\Delta_* = \frac{c-1}{12}$ above the vacuum. The twist gap is identical, apart from two states \mathcal{O}_* with spin scaling linearly in the central charge c . These states admit an AdS₃ interpretation as strongly coupled strings. All other states are black hole microstates.

3.1 The quest for AdS₃ pure gravity

The quest for AdS₃ pure gravity still beckons.

It is not fully known whether, or in what precise sense, a consistent such theory exists, either quantum mechanically or in the semiclassical limit. The latter is of particular physical interest, due to the existence of black holes and the emergence of spacetime.

Holographically speaking, the outstanding spectral problem is to find a torus partition function of a two-dimensional conformal field theory (CFT) that is mutually compatible with unitarity (a non-negative Virasoro primary spectral density) and modularity (exact $SL(2, \mathbb{Z})$ -invariance of the partition function), while preserving the spectral gaps of a dual bulk theory with only black holes above a normalizable AdS₃ ground state. No known partition function satisfies these basic requirements.

There exists a diverse set of approaches to this problem which, famous as it is, we describe in condensed fashion. Summing over all smooth on-shell 3-manifolds \mathcal{M} with $\partial\mathcal{M} = T^2$ [97], namely the $SL(2, \mathbb{Z})$ family of BTZ black holes, generates a negative density of states in two

regimes [97, 98, 114]: at large spin $j \rightarrow \infty$ near extremality,

$$\int_0^{t_0} dt \rho_{\text{MWK},j}(t) \sim (-1)^j e^{\pi\sqrt{\xi j}}, \quad t_0 \sim e^{-2\pi\sqrt{\xi j}} \quad (3.1.1)$$

where

$$t := \min(h, \bar{h}) - \xi, \quad j = h - \bar{h}, \quad \xi := \frac{c-1}{24}, \quad (3.1.2)$$

and at the scalar black hole threshold,

$$\rho_{\text{MWK},0}(t) = -6\delta(t) + (t > 0 \text{ continuum}). \quad (3.1.3)$$

The property (3.1.1) is especially severe: an exponentially large negative density despite an exponentially small window. From the bulk perspective, seeking a consistent pure gravity path integral requires reckoning with the sum over topologies; for related work, see [34, 113, 154, 194–198]. (We note here some recent work in AdS₃/CFT₂ that studies fixed bulk topologies [101, 106–108, 147, 199].)

Some valuable progress has been made. Explicit restoration of unitarity may be achieved by retreating from pure gravity [96, 200], adding heavy point-particle matter which admits a geometric bulk interpretation. The construction of [88], which preserves the pure gravity spectrum, uses dimensional reduction to JT gravity to fix (3.1.1) with an infinite sum over off-shell Seifert manifolds, though it remains a mostly [1] implicit construction away from extremality and leaves (3.1.3) intact. Other approaches that forego a subset of the above conditions include [153, 159, 201].

3.2 Partition function

The Virasoro primary partition function is defined as

$$Z_p(\tau) = \sqrt{y} |\eta(\tau)|^2 Z(\tau) \quad (3.2.1)$$

where $Z(\tau) = \text{Tr}_{\mathcal{H}}(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}})$ is the torus partition function (non-holomorphic) and $\tau := x + iy$. The following modular-invariant $Z_p(\tau)$ is unitary at sufficiently large ξ :

$$\mathcal{Z}(\tau) = Z_{\text{MWK}}(\tau) + Z_{\text{string}}(\tau) \quad (3.2.2)$$

where

$$\begin{aligned} Z_{\text{MWK}}(\tau) &:= \sum_{\gamma \in SL(2, \mathbb{Z})/\Gamma_\infty} \sqrt{\text{Im}(\gamma\tau)} |q_\gamma^{-\xi} (1 - q_\gamma)|^2 \\ Z_{\text{string}}(\tau) &:= \sum_{\gamma \in SL(2, \mathbb{Z})/\Gamma_\infty} \sqrt{\text{Im}(\gamma\tau)} \left(2q_\gamma^{\xi/4} \bar{q}_\gamma^{-\xi/4} + \text{c.c.} \right) \end{aligned} \quad (3.2.3)$$

with $q_\gamma := e^{2\pi i \gamma \tau}$. These are Poincaré sums over $SL(2, \mathbb{Z})$ modulo Γ_∞ , the set of modular T -transformations¹. As we substantiate below, the unitary range of ξ includes $\xi \gg 1$, and provisionally appears to hold for all $\xi \in 2\mathbb{Z}_+$. The reason for the “string” moniker will be explained momentarily.

From a CFT point of view, $Z_{\text{string}}(\tau)$ is a Poincaré sum over two copies of a Virasoro primary seed state \mathcal{O}_* with quantum numbers

$$(\Delta_*, j_*) = \left(2\xi, \frac{\xi}{2}\right) \Leftrightarrow (t_*, \bar{t}_*) = \left(-\frac{\xi}{4}, \frac{\xi}{4}\right) \quad (3.2.4)$$

and its parity image with $h_* \leftrightarrow \bar{h}_*$. We have employed the “reduced twist” variable t along with its partner $\bar{t} := \max(h, \bar{h}) - \xi$. We have chosen the state in (3.2.3) to be doubly-degenerate, a natural choice that preserves integrality, but $\mathcal{Z}(\tau)$ is unitary for a finite range of degeneracies $d_* > 1$ (see Appendix B).

Let us state the spectral properties of the partition function $\mathcal{Z}(\tau)$, deferring its unitarity to the next subsection. The spectrum is shown in Fig. 3.1. The gap in conformal dimension above the vacuum is exactly

$$\Delta_* = \frac{c-1}{12} \quad (3.2.5)$$

with no corrections. This is the value anticipated by the Virasoro modular bootstrap program (e.g. [179–183]) as the optimal gap at *large* c , on the basis of black hole universality: the conformal dimension (3.2.5) corresponds to the massless limit of the semiclassical BTZ black hole. The state-of-the-art bootstrap upper bound on the spectral gap at large c is the numerical result [182]

$$\Delta_* \lesssim \frac{c}{9.08} \quad (c \gg 1) \quad (3.2.6)$$

with a slightly weaker analytical bound [183]. (See [127, 139, 145, 151, 202–218] for further bootstrap work on Virasoro spectra at large c .) The explicit realization by $\mathcal{Z}(\tau)$ of the gap (3.2.5) while preserving unitarity at $\xi \gg 1$ (the first such example, to our knowledge) is also noteworthy because of the paucity of pure CFT arguments that a gap this large is possible. Conversely, $\mathcal{Z}(\tau)$ shows constructively that without incorporating discreteness into the modular bootstrap [151], the optimal bound on the gap cannot be lower than $\Delta_* = 2\xi$. This statement applies for all values of ξ for which $\mathcal{Z}(\tau)$ is unitary.

As for the twist spectrum, all Virasoro primaries besides the identity and \mathcal{O}_* have $t \geq 0$. There is a positive integer number of scalar states at $t = 0$ (see (3.2.10)). The spectrum of $t > 0$ states is continuous. This can be understood rather generally in terms of coarse-graining. At large ξ , this can be thought of as a consequence of ignorance of exponentially small effects

¹We thus formally demand that $\xi \in 2\mathbb{Z}_+$, though we note that $\mathcal{Z}(\tau)$ remains unitary if we shift the seed quantum numbers while preserving T -invariance. In particular, $\mathcal{Z}(\tau)$ remains unitary if we slightly shift $\Delta_* > 2\xi$ while keeping $t_* = -\frac{\xi}{4}$ fixed; for simplicity, we focus on the case $\Delta_* = 2\xi$ in the main text.

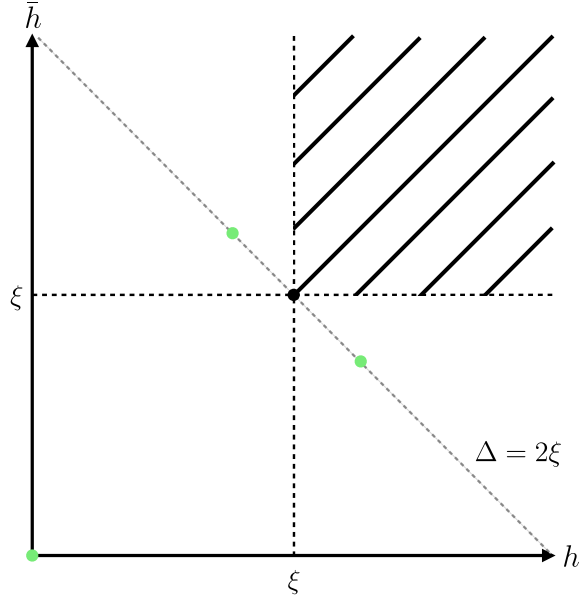


Figure 3.1: The Virasoro primary spectrum of $\mathcal{Z}(\tau)$. Green dots denote the vacuum state and two (parity-invariant) states with $(h_*, \bar{h}_*) = (\frac{5}{4}\xi, \frac{3}{4}\xi)$, interpretable in AdS_3 as strongly coupled strings. All other states exceed the semiclassical black hole threshold: $\min(h, \bar{h}) \geq \xi$. The density of states is positive.

in c – for example, smearing over the mean level spacing $\sim e^{-S_{\text{Cardy},j}(t)}$. We explain these points of interpretation in Sec. 3.5.

3.2.1 Density of states

The corresponding Virasoro primary density of states, related to our partition function as

$$\frac{\mathcal{Z}(\tau)}{\sqrt{y}} = \sum_{j=0}^{\infty} (2 - \delta_{j,0}) \cos(2\pi jx) \int_{\mathbb{R}} d\Delta e^{-2\pi y(\Delta - 2\xi)} \rho_j(\Delta) \quad (3.2.7)$$

can be derived straightforwardly using existing methods for Poincaré sums. We have, in terms of reduced twist t ,

$$\rho_j(t) = \rho_{\text{MWK},j}(t) + \rho_{\text{string},j}(t) \quad (3.2.8)$$

for every spin j . The MWK density $\rho_{\text{MWK},j}(t)$ is recalled in Appendix A. The new term is, for $j \neq 0$,

$$\begin{aligned} \rho_{\text{string},j}(t) &= \frac{4}{\sqrt{t\bar{t}}} \sum_{s=1}^{\infty} f_{j,j_*,s} \cos\left(\frac{2\pi}{s}\sqrt{\xi\bar{t}}\right) \cosh\left(\frac{2\pi}{s}\sqrt{\xi t}\right) \\ &+ (j_* \rightarrow -j_*, t \leftrightarrow \bar{t}) \end{aligned} \quad (3.2.9)$$

where $f_{j,j_*;s} := S(j, j_*; s)/s$ with $S(j, j_*; s)$ a Kloosterman sum. For $j = 0$, such sums must be regularized; using standard methods nicely summarized in [96], the result is the $j = 0$ specialization of the $j \neq 0$ densities, augmented by a constant subtraction; see Appendix A.

There are two hurdles to establishing positivity: one must cancel the negativity of the MWK partition function in the $j \rightarrow \infty$ regime, and at the scalar threshold $t = 0$, both without introducing new negativity.

At $j \rightarrow \infty$, the negativity (3.1.1) is resolved by construction: we have added states with reduced twist $t_* = -\xi/4$, designed precisely to avoid the large-spin negativity in accordance with the arguments of [98] and the subsequent approach of [96, 200]. (We added two such states, but any number $d_* > 1$ would do; we review this in Appendix B.) The states \mathcal{O}_* have asymptotically large spin as $\xi \rightarrow \infty$. It is exactly this property which admits the novelty of a spectral gap $\Delta_* = 2\xi$ without introducing further negativity elsewhere in the spectrum – and indeed, as we now show, curing the scalar negativity (3.1.3) in the process.

The scalar density of states is

$$\rho_0(t) = \delta(t + \xi) + (-6 + 8\sigma_0(j_*))\delta(t) + \tilde{\rho}_0(t). \quad (3.2.10)$$

The first term is the vacuum state. The second, formerly problematic, term has been rendered strictly positive, for any $j_* = \xi/2$. Happily, it is also an integer, a welcome surprise. Unlike previous approaches to this negativity, its resolution does not require the addition of an “extra” ad hoc $+6\delta(t)$ [114], instead coming for free in $\rho_{\text{string},0}(t)$. We note a number-theoretic feature of this degeneracy: if j_* is prime, then $\sigma_0(j_*) = 2$.

The last term, $\tilde{\rho}_0(t)$, is the continuum with support on $t > 0$, given explicitly in Appendix A. Its positivity requires a more careful analysis because various large- ξ suppression factors are absent when $j = 0$, i.e. $t = \bar{t}$, as can be seen in (3.2.9); however, $\tilde{\rho}_0(t)$ is indeed positive for all $t \geq 0$. We provide details in Appendix B, but can sketch the essential point here. In the regime $\xi t \gg 1$, the scalar MWK density is positive and exponentially larger in magnitude than the string density. As $\xi t \sim \mathcal{O}(1)$, positivity is non-trivial as both densities are of the same order and the string density is term-wise oscillatory in t . With an eye toward semiclassical gravity, we focus on $\xi \gg 1$, taking the regime of $x := 2\pi\sqrt{\xi t}$ fixed. The proof of positivity proceeds in two steps: first, we show that the sum of the $s = 1$ and $s = 2$ terms in the regularized density is positive; and second, we show that the $s \geq 2$ terms are individually positive (see the Appendix B)

Numerical evaluation of the scalar density at large but finite ξ confirms these analytic results, as shown in Fig. 3.2. Indeed, we see that positivity appears to hold all the way down to $j_* = 1$, i.e. $\xi = 2$, formally the smallest central charge in our construction ².

²One can generalize our construction to the case of $j_* \in \mathbb{Z}_+ - \frac{1}{2}$, whereupon the stabilizer group Γ_∞ in (3.2.3) would instead be the set of T^2 transformations. Analogous extensions to supersymmetric partition functions would also be straightforward. It would be interesting to ask whether the requisite positivity is

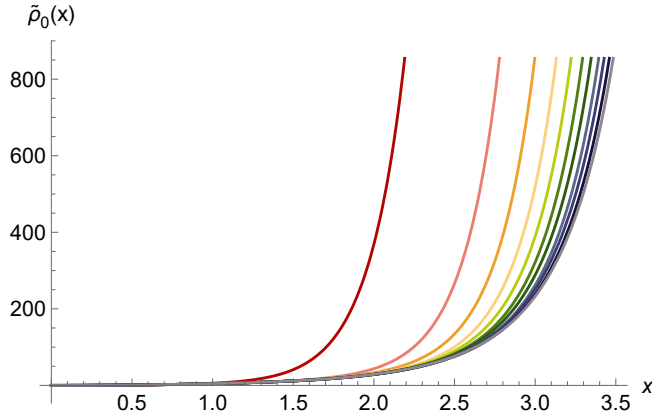


Figure 3.2: Plot of the regularized scalar primary density of states $\tilde{\rho}_0(x)$ of the partition function $\mathcal{Z}(\tau)$, as a function of $x = 2\pi\sqrt{\xi\bar{t}}$, with ξ ranging from $\xi = 2$ (red) to $\xi = 102$ (blue) in steps of 10. The curves are positive for all $x \geq 0$. (Obtained by summing over $s \leq 200$ in the regularized density, see Appendix A.)

3.3 A bulk string interpretation

The above construction is purely on the CFT side. Is there an AdS_3 gravity interpretation of the highly-spinning operator \mathcal{O}_* and its modular images?

One appealing answer is that \mathcal{O}_* is a strongly coupled string, and its modular images, stringy contributions to the black hole spectrum. While an operator like \mathcal{O}_* with $t < 0$ and $\bar{t} > 0$ cannot be dual to a smooth BTZ black hole nor to a conical defect (such solutions with real mass and angular momenta do not exist in semiclassical AdS_3 gravity coupled to point particles), spinning strings in AdS_3 can, and indeed do, satisfy this condition.

The spectrum of folded, spinning Nambu-Goto strings coupled to gravity in AdS_3 was studied in [220] in the classical limit. The Virasoro primary string spectrum is parameterized by a string tension λ and an angular velocity ω . The string tension is given in terms of AdS, string and Planck scales as

$$\lambda = \frac{1}{2\pi} \frac{L_{\text{AdS}} \ell_p}{\ell_s \ell_s} \quad (3.3.1)$$

where $\ell_p = 8\pi G_N$.

For a given λ , the string spectrum starts at the origin $t = \bar{t} = -\xi$ and ends at the extremality bound $t = 0$ or $\bar{t} = 0$. Matching the string spectrum to the quantum numbers (3.2.4) of the operator \mathcal{O}_* yields the unique result

$$\lambda_* = 1, \quad \omega_* = 2. \quad (3.3.2)$$

satisfied in each of these cases. See [97, 203, 208, 215, 219] for related material.

This string is strongly coupled: from (3.3.1), an AdS-sized string with $\lambda_* = 1$ requires $\ell_p/\ell_s \sim \mathcal{O}(1)$, which is the ratio that defines an effective string coupling $g_s = (\ell_p/\ell_s)^{>0}$ (where the exponent depends on the details of the putative string background ³).

The specific value $\lambda_* = 1$ happens to enjoy a certain synergy with the equations of [220]. For generic λ and ω , the solutions of [220] are given in terms of elliptic integrals. However, at $\lambda = 1$ – and only at $\lambda = 1$ – the solution simplifies dramatically, as the string embedding equation becomes algebraic. It is simple enough to recall explicitly in a few lines. The Lorentzian spacetime metric outside the string is locally AdS₃ with the corresponding mass and angular momentum,

$$ds^2 = \frac{1}{16}(-du^2 + dv^2) - \left(z - \frac{1}{256z}\right) dudv + \frac{dz^2}{4z^2} \quad (3.3.3)$$

where, in the conventions of [220], the conformal boundary is at $z \rightarrow \infty$. The string embedding is determined by functions $u(\sigma, \tau)$, $v(\sigma, \tau)$ and $z(\sigma)$, where (σ, τ) are worldsheet coordinates with induced metric

$$h = \Omega^2(\sigma)(-d\tau^2 + d\sigma^2) \quad (3.3.4)$$

$$\Omega^2(\sigma) = 3 \frac{(z_L - z(\sigma))(z(\sigma) - z_R)}{z(\sigma)}$$

where (3.3.2) implies $z_L = \frac{3}{16}$, $z_R = -\frac{1}{48}$, and

$$z(\sigma) = \frac{(32 - 25 \cos^2 \sigma + 5\sqrt{25 \cos^4 \sigma - 64 \cos^2 \sigma + 64})}{384} \quad (3.3.5)$$

Opposite points on the string are identified, “sewing up” the spacetime ⁴. The outermost radius of the string (where it folds) is at $z = z_L$, while the center of the string is at $z = z(0) = 1/12$. The spacetime ends at the string, avoiding a naked singularity.

So we see that the state \mathcal{O}_* admits an interpretation as a highly-spinning string coupled to gravity in AdS₃. That it is strongly coupled dovetails nicely with how AdS₃ pure gravity could possibly arise in string theory: strong coupling is necessary to gap the light string modes to the Planck scale.

³For example, in the D1-D5 system $\lambda \sim g_6/\sqrt{c}$ where g_6 is the six-dimensional effective string coupling, so $\lambda = 1$ is highly quantum. In general, $\lambda \sim \mathcal{O}(1)$ captures a “very strongly coupled” limit of fixed coupling g_s , rather than a ’t Hooft-type double-scaling limit. One can see this by rewriting $\lambda = 6(L/\ell_s)^2/c$ using the Brown-Henneaux relation [31].

⁴This identification does not happen at fixed t , so the spacetime does not “pinch off” on a constant time slice.

3.3.1 Black hole microstates

Our construction adds not only the states \mathcal{O}_* , but their $SL(2, \mathbb{Z})$ images too. These states are heavy, but are *not* BTZ black holes (fully captured by the MWK sum over smooth Euclidean saddles) nor their orbifolds (which are modular images of conical defect geometries).

Instead, these are new black hole microstates made of strongly coupled strings. The Euclideanized, modular-transformed solutions of [220] are small black strings, in the following specific sense: whereas a BTZ/conical defect solution with the same quantum numbers would be nakedly singular, the strings shroud this region by terminating the spacetime. These geometries may be thought of as quantum AdS_3 versions of the stringy cloak of [221] and other small black strings (e.g. [222, 223]). That they are “small” – the modular transforms of a single string, rather than a parametrically large number of them – is also visible thermodynamically in the different functional forms of the stringy and BTZ microcanonical entropies: $\rho_{\text{string},j}(t)$ is oscillatory as a function of t , unlike the BTZ density $\rho_{\text{MWK},j}(t)$, and is exponentially subleading to $\rho_{\text{MWK},j}(t)$, term-by-term in the modular sum, away from the near-extremal regime $\xi t \lesssim \mathcal{O}(1)$ where the BTZ black hole becomes highly quantum⁵. This fluctuating behavior signals that the stringy degrees of freedom give genuinely new contributions to the black hole Hilbert space, distinct from the semiclassical BTZ geometries or quotients thereof.

3.4 $SL(2, \mathbb{Z})$ spectral representation

As a slight detour, it is enlightening to give an alternative representation of $Z_{\text{string}}(\tau)$. The spectral gap condition $\Delta_* = 2\xi$ implies that $Z_{\text{string}}(\tau) \in L^2(\mathcal{F})$, and hence admits a harmonic decomposition in the $SL(2, \mathbb{Z})$ spectral eigenbasis, comprised of the completed Eisenstein series $E_{\frac{1}{2}+i\omega}^*(\tau)$ with $\omega \in \mathbb{R}$ and Maass cusp forms $\phi_n(\tau)$ (e.g. [112, 120, 121]). Denoting their spin- j Fourier coefficients as $\mathbf{a}_j^{(s)}$ and $\mathbf{b}_j^{(n)}$, respectively, and using the conventions of [1], we have

$$\begin{aligned} Z_{\text{string}}(\tau) &= \int_{\mathcal{C}_{\text{crit}}} \mathbf{a}_{j_*}^{(s)} \frac{\Gamma\left(\frac{\frac{1}{2}-s}{2}\right) \Gamma\left(\frac{s-\frac{1}{2}}{2}\right)}{\Lambda(s)\Lambda(1-s)} E_s^*(\tau) \\ &\quad + \sum_{n=1}^{\infty} \mathbf{b}_{j_*}^{(n)} \Gamma\left(-\frac{i\omega_n}{2}\right) \Gamma\left(\frac{i\omega_n}{2}\right) \phi_n(\tau) \end{aligned} \tag{3.4.1}$$

where $\mathcal{C}_{\text{crit}}$ denotes $((4\pi i)^{-1}$ times) contour integration along $s = \frac{1}{2} + i\omega$, and $\Lambda(s) := \pi^{-s}\Gamma(s)\zeta(2s)$ is the completed Riemann zeta function. For details see Appendix C.

Presenting $Z_{\text{string}}(\tau)$ in spectral form reveals some interesting features and curiosities.

⁵This is within the “enigmatic” regime [139] where such corrections may be large, consistent with modularity and sparseness.

First, the modular average of $Z_{\text{string}}(\tau)$ vanishes:

$$\langle Z_{\text{string}} \rangle := \int_{\mathcal{F}} \frac{dx dy}{y^2} Z_{\text{string}}(\tau) = 0. \quad (3.4.2)$$

This follows from the vanishing of the Eisenstein spectral overlap in (3.4.1) at $s = 0$, which defines the modular average in general. We note that this property is shared by Narain CFTs [112].

Next, $Z_{\text{string}}(\tau)$ may be written as the action of an $SL(2, \mathbb{Z})$ Hecke operator $T_{\xi/2}$ ⁶ on a “primitive” partition function, $\mathcal{Z}_{\text{string}}(\tau)$, defined as $Z_{\text{string}}(\tau)$ but with the Fourier coefficients evaluated at $j_* = 1$:

$$\mathcal{Z}(\tau) = Z_{\text{MWK}}(\tau) + T_{\xi/2} \mathcal{Z}_{\text{string}}(\tau). \quad (3.4.3)$$

In this way, the entire family of unitary partition functions indexed by ξ may be generated by a Hecke action, implementing shifts in central charge. This shares a superficial likeness with Witten’s construction of holomorphic extremal CFT partition functions [95], with obvious differences.

Finally, there is a profound conjecture in number theory, the “horizontal” Sato-Tate conjecture for Maass cusp forms of $SL(2, \mathbb{Z})$, which has interesting consequences for the spectral decomposition [224–227]. The conjecture states that for prime $j \rightarrow \infty$ and any fixed n , the normalized Fourier coefficients $\mathbf{b}_j^{(n)}/\mathbf{b}_1^{(n)}$ are equidistributed with respect to Wigner’s semicircle distribution. This (and $\mathbf{b}_1^{(n)} \neq 0$, which follows from Hecke relations applied to Hecke-Maass cusp forms) implies that

$$\lim_{j \rightarrow \infty} \langle \langle \mathbf{b}_j^{(n)} \rangle \rangle = 0 \quad (\text{fixed } n) \quad (3.4.4)$$

where $\langle \langle \cdot \rangle \rangle$ indicates a statistical average. Therefore, even though $(Z_{\text{string}}, \phi_n) \propto \mathbf{b}_{j_*}^{(n)} \neq 0$, they vanish on average in the large central charge limit $j_* \rightarrow \infty$ (for prime j_*)⁷. In this sense, the Eisenstein term seems to more directly underlie the unitarity of $\mathcal{Z}(\tau)$. It would be nice to understand this from a physical, quantum chaos point of view.

3.5 Summary and Random (matrix) comments

Our main result is the construction of the unitary partition function $\mathcal{Z}(\tau)$ given in (3.2.2), with the spectral gaps depicted in Fig. 3.1.

⁶Acting on non-holomorphic $SL(2, \mathbb{Z})$ -invariant functions, $T_j f(\tau) = \frac{1}{\sqrt{j}} \sum_{a,b,d} f\left(\frac{a\tau+b}{d}\right)$ where $ad = j$, $d > 0$ and $0 \leq b \leq d - 1$.

⁷The conjecture is widely held to be true on the basis of stringent numerical checks, limited proofs, and relation to Sato-Tate conjectures in other contexts. Note that at finite j_* , the “vertical” Sato-Tate conjecture [228], would only imply vanishing coefficients as $n \rightarrow \infty$.

From the AdS₃ gravity point of view, despite the dimension gap above the vacuum state to the black hole threshold $\Delta_* = \frac{c-1}{12}$, this is not a semiclassical pure gravity path integral in the strict sense, due to the spinning states \mathcal{O}_* with sub-threshold twist. At any finite spin, these states are not visible, and the theory contains only black hole states. The degeneracies of all discrete states are integers.

We have advanced a bulk interpretation of \mathcal{O}_* as a strongly coupled spinning string, though other interpretations may well be possible (or preferred). We view this as an indicative toy model for a genuine string theory compactification to AdS₃ pure gravity. A complete approach would include higher Regge trajectories; corrections to the spectrum from excitations around the spin- j ground states of [220]; and the other ingredients, such as fluxes and their brane sources, required to solve the strongly coupled string field equations (whatever they may be). Nevertheless, the problem of finding a direct, physically sensible quantization of AdS₃ gravity which would lead to our partition function (or another one with the desired properties) remains outstanding.

3.5.1 Randomness

Our construction cures the negativity from the sum over smooth bulk saddles semiclassically, rather than quantum mechanically. Quantum effects are not just present in a consistent theory, but are expected to be crucial in the engineering of a bona fide theory of AdS₃ pure gravity: there are strong indications that if such a theory exists, off-shell geometries encoding random matrix behavior of the chaotic spectrum play a central role in unitarizing the spectrum [34, 35, 88]. An explicit determination of the leading-order random matrix contribution to the semiclassical path integral of pure gravity with torus boundary, denoted as $Z_{\text{RMT}}(\tau)$, was made in [1].

In any theory of semiclassical AdS₃ gravity (pure or otherwise), the black hole spectrum is chaotic, and its path integral should encode random fluctuations for quantum consistency. Such random matrix contributions are absent in $\mathcal{Z}(\tau)$. We may explain this fact, as well as the continuous spectrum in the chaotic regime $t > 0$, by interpreting $\mathcal{Z}(\tau)$ as the partition function of a microscopic compact CFT that has been subject to coarse-graining.

As shown in [1] using a formalism built on a CFT trace formula, the random matrix contribution to the density of states, properly understood, vanishes upon coarse-graining the spectrum over a suitable microcanonical window in twist, δt ⁸. Because this window is necessarily larger than the exponentially small mean level spacing of the chaotic spectrum, the coarse-graining simultaneously explains both the absence of random matrix contributions to (3.2.2) and its continuous spectrum while remaining compatible with a microscopic CFT

⁸This coarse-graining may be performed using convolution as $\overline{f(t)} := \int_0^\infty dt' W(t-t')f(t')$ where $W(t)$ has characteristic width δt and obeys $\int_0^\infty dt W(t) = 1$.

interpretation. Given our explicit construction, we can determine δt : it is the characteristic wavelength of the oscillations of $\rho_{\text{string},j}(t)$ in (3.2.9), namely, $\delta t \sim 1/\xi$. We emphasize that this coarse-graining interpretation does not rely on a $\xi \gg 1$ limit, and is compatible with compactness of a putative underlying CFT; there could, of course, be as-yet-unknown bootstrap constraints that rule this possibility out.

Note that in a $\xi \gg 1$ limit, $\mathcal{Z}(\tau)$ is also compatible with other interpretations, in particular with a hypothetical ensemble average over (possibly near-)CFTs, or with other, perhaps independent, constructions of “approximate CFT” [103]. While we have presented a microscopic CFT interpretation in part to emphasize that a departure from standard AdS/CFT physics is not required at this level, semiclassical gravity seems unable to distinguish among these [102–104], at least perturbatively in G_N .

A complementary view on this coarse-grained interpretation comes from the formalism of [1]. Since $Z_{\text{string}}(\tau)$ is the modular completion of a non-black hole state, we do not expect it to encode random matrix behavior per se [1, 64, 103, 123]. Applying the results of [1] to $Z_{\text{string}}(\tau)$ helps to ratify this perspective. In (3.4.1) we provided the $SL(2, \mathbb{Z})$ spectral decomposition of $Z_{\text{string}}(\tau)$. A canonical diagnostic of random matrix universality is the presence of a linear ramp in the coarse-grained spectral form factor, with a specific coefficient prescribed by the random matrix ensemble. We can ask whether $Z_{\text{string}}(\tau)$ generates this ramp after squaring and taking the diagonal approximation. A necessary and sufficient condition for the ramp was derived in [1], as an exponential decay condition on the spectral overlaps at $\omega \rightarrow \infty$. One readily checks that $Z_{\text{string}}(\tau)$ does not satisfy this criterion, instead decaying as a power law ⁹.

3.5.2 Stringiness

On the other hand, $Z_{\text{string}}(\tau)$ exhibits some behavior that lies somewhere “in between” chaotic and non-chaotic. Define a microcanonical coarse-graining over mean twist,

$$\overline{f(t_1)f(t_2)} := \int_0^\infty dt' f(t' + \epsilon)f(t' - \epsilon)W(t - t') \quad (3.5.1)$$

where $t = \frac{t_1+t_2}{2}$ and $\epsilon = \frac{t_1-t_2}{2}$. Applying this to $f(t) = \rho_{\text{string},j}(t)$ at fixed j using (3.2.9) produces a non-zero variance upon coarse-graining over windows $\delta t \gtrsim \frac{1}{\xi}$. However, its oscillatory behavior leads to suppression relative to the disconnected average. In particular, at $\xi \gg 1$,

$$\frac{\text{Var}(\rho_j(t))}{\bar{\rho}_j(t)^2} \approx e^{-4\pi\sqrt{\xi(t+j)}} \quad (\xi \gg 1) \quad (3.5.2)$$

where $\bar{\rho}_j(t) = \rho_{\text{MWK},j}(t)$. In the extremal limit $t \rightarrow 0$, the suppression factor is $e^{-S_{0,j}}$, where $S_{0,j} = 4\pi\sqrt{\xi j}$ is the extremal spin- j BTZ black hole entropy. In contrast, wormholes encoding

⁹Similarly, the Hecke projection [1] of $Z_{\text{string}}(\tau_1)Z_{\text{string}}(\tau_2)$ is not a wormhole amplitude.

chaotic behavior are suppressed as $e^{-2S_{0,j}}$ in the extremal limit [1, 34, 35, 69, 187]. It would be worthwhile to understand this intermediate behavior as a non-perturbative effect, possibly associated to strongly coupled strings, in a UV complete AdS₃ gravity path integral.

3.6 Appendix A: Density of states

We write the total spin- j density of states as

$$\rho_j(t) = \rho_{\text{MWK},j}(t) + \rho_{\text{string},j}(t) \quad (3.6.1)$$

The MWK density of states may be written as [96, 98]

$$\begin{aligned} \rho_{\text{MWK},j}(t) = & \\ & \frac{2}{\sqrt{t\bar{t}}} \sum_{s=1}^{\infty} \left[f_{j,0;s} \cosh\left(\frac{4\pi}{s}\sqrt{\xi\bar{t}}\right) \cosh\left(\frac{4\pi}{s}\sqrt{\xi t}\right) \right. \\ & - f_{j,-1;s} \cosh\left(\frac{4\pi}{s}\sqrt{\xi\bar{t}}\right) \cosh\left(\frac{4\pi}{s}\sqrt{(\xi-1)t}\right) \\ & - f_{j,1;s} \cosh\left(\frac{4\pi}{s}\sqrt{(\xi-1)\bar{t}}\right) \cosh\left(\frac{4\pi}{s}\sqrt{\xi t}\right) \\ & \left. + f_{j,0;s} \cosh\left(\frac{4\pi}{s}\sqrt{(\xi-1)\bar{t}}\right) \cosh\left(\frac{4\pi}{s}\sqrt{(\xi-1)t}\right) \right] \end{aligned} \quad (3.6.2)$$

where $f_{j,k;s} := S(j, k; s)/s$ and $S(j, k; s)$ is a Kloosterman sum,

$$S(j, k; s) = \sum_{0 \leq d \leq s-1, (d,s)=1} e^{2\pi i \left(\frac{d}{s}j + \frac{d^{-1}}{s}k \right)} \quad (3.6.3)$$

where $d^{-1} \in \mathbb{Z}$ is the multiplicative inverse of $d \bmod s$. The scalar density requires regularization [97, 114]. It is comprised of a delta function piece given by (3.1.3), and a continuous piece which we denote by $\tilde{\rho}_{\text{MWK},0}(t)$:

$$\begin{aligned} \tilde{\rho}_{\text{MWK},0}(t) = & \\ & \frac{2}{t} \sum_{s=1}^{\infty} \left\{ \frac{\phi(s)}{s} \left[\sinh^2\left(\frac{4\pi}{s}\sqrt{\xi t}\right) + \sinh^2\left(\frac{4\pi}{s}\sqrt{(\xi-1)t}\right) \right] \right. \\ & \left. - 2 \frac{\mu(s)}{s} \left[\cosh\left(\frac{4\pi}{s}\sqrt{\xi t}\right) \cosh\left(\frac{4\pi}{s}\sqrt{(\xi-1)t}\right) - 1 \right] \right\} \end{aligned} \quad (3.6.4)$$

where $\phi(s) = S(0, 0; s)$ is the Euler totient function and $\mu(s) = S(0, 1; s)$ is the Mobius function.

The string density for $j \neq 0$ was given in (3.2.9), which we repeat here for convenience:

$$\begin{aligned} \rho_{\text{string},j}(t) &= \frac{4}{\sqrt{t\bar{t}}} \sum_{s=1}^{\infty} f_{j,j_*,s} \cos\left(\frac{2\pi}{s}\sqrt{\xi\bar{t}}\right) \cosh\left(\frac{2\pi}{s}\sqrt{\xi t}\right) \\ &+ (j_* \rightarrow -j_*, t \leftrightarrow \bar{t}) \end{aligned} \quad (3.6.5)$$

Similarly to the MWK case, the scalar density requires regularization. The regularized density contains a delta function piece given in (3.2.10), and a continuous piece which we denote by $\tilde{\rho}_{\text{string},0}(t)$:

$$\begin{aligned} \tilde{\rho}_{\text{string},0}(t) &= \frac{8}{t} \sum_{s=1}^{\infty} \frac{c_s(j_*)}{s} \times \\ &\left[\cos\left(\frac{2\pi}{s}\sqrt{\xi t}\right) \cosh\left(\frac{2\pi}{s}\sqrt{\xi t}\right) - 1 \right] \end{aligned} \quad (3.6.6)$$

where $c_s(j_*)$ is a Ramanujan sum,

$$c_s(j_*) = \sum_{1 \leq d \leq s, (d,s)=1} e^{2\pi i \frac{d}{s} j_*} \quad (3.6.7)$$

The total regularized scalar density is given by the sum of these two contributions, together with the delta functions in (3.2.10):

$$\rho_0(t) = \delta(t + \xi) + (-6 + 8\sigma_0(j_*))\delta(t) + \tilde{\rho}_0(t) \quad (3.6.8)$$

where

$$\tilde{\rho}_0(t) = \tilde{\rho}_{\text{MWK},0}(t) + \tilde{\rho}_{\text{string},0}(t). \quad (3.6.9)$$

3.7 Appendix B: Positivity

We treat in turn the positivity of the large spin $j \rightarrow \infty$, finite spin $j \geq 1$, and scalar $j = 0$ sectors of the density (3.6.1), focusing mostly on the regime $\xi \gg 1$. Actually, we consider a more general case in which we have d_* string states: namely, $Z_p(\tau) = Z_{\text{MWK}}(\tau) + \frac{d_*}{2} Z_{\text{string}}(\tau)$, and correspondingly for the densities. For the partition function $\mathcal{Z}(\tau)$ defined in the main text, $d_* = 2$.

3.7.1 Positivity at large spin

The MWK density (3.6.2) is known to be negative in the extremal limit $t \rightarrow 0$ of large spin $|j| \rightarrow \infty$ (more precisely, for $t \lesssim t_0 \sim e^{-2\pi\sqrt{\xi}j}$, dropping a numerical prefactor). We cure this negativity with the states \mathcal{O}_* by design, having chosen their twist to be $t_* = -\frac{\xi}{4}$. The mechanism is the same as described in Sec. 2.1 of [96], building on [98, 200]. In the regime $t < t_0$ with $j \rightarrow \infty$, the MWK density is approximately equal to [96]

$$\rho_{\text{MWK},j}(t) \approx \frac{(-1)^j}{\sqrt{j}t} \left(e^{2\pi\sqrt{\xi}j} + e^{2\pi\sqrt{(\xi-1)j}} \right) \quad (3.7.1)$$

In the same regime, the string density (3.6.5) has the same exponential behavior but with a positive coefficient, coming from the state of spin $-j_*$:

$$\rho_{\text{string},j}(t) \approx \frac{d_*}{\sqrt{j}t} e^{2\pi\sqrt{\xi}j}. \quad (3.7.2)$$

The string density cancels the odd-spin negativity for $d_* > 1$. Whereas at $d_* = 1$ there are subleading negativities to take care of, requiring the addition of higher-twist states [96], choosing $d_* = 1 + \delta$ for any finite δ gives a positive extremal density. In the construction of $\mathcal{Z}(\tau)$ in (3.2.2), we chose $d_* = 2$, the smallest integer degeneracy which guarantees positivity, as a matter of naturalness. The above discussion applies equally to the regime of large negative spin $j \rightarrow -\infty$, whereupon the negativity is cured by the state \mathcal{O}_* with spin $+j_*$.

3.7.2 Positivity at $j \geq 1$

We now consider finite spin $j \geq 1$. We also take $\xi \gg 1$. There are two regimes of twist t to consider: the extremal limit $t \rightarrow 0$, and fixed t .

In the extremal limit, the arguments given just above are again sufficient to guarantee positivity. In particular, we have $\xi j \gg 1$ in the present regime of interest; one may confirm upon inspection that the ξ - and j -dependence of $\rho_{\text{MWK},j}(t)$ and $\rho_{\text{string},j}(t)$ are such that at $\xi j \gg 1$, even for finite j , the result of the previous subsection carries through.

Now we consider the regime of fixed t . Since $\xi j \gg 1$, terms of the form $\cosh\left(\frac{4\pi}{s}\sqrt{\xi(t+j)}\right)$ for $s \geq 2$ are exponentially suppressed with respect to the $s = 1$ term. At fixed t , the MWK density is therefore well-approximated by the $s = 1$ term:

$$\begin{aligned} \frac{\sqrt{t\bar{t}}}{2} \rho_{\text{MWK},j}(t) &\approx \left[\cosh\left(4\pi\sqrt{\xi t}\right) - \cosh\left(4\pi\sqrt{(\xi-1)t}\right) \right] \\ &\times \left[\cosh\left(4\pi\sqrt{\xi(t+j)}\right) - \cosh\left(4\pi\sqrt{(\xi-1)(t+j)}\right) \right] \end{aligned} \quad (3.7.3)$$

This is manifestly positive, and scales as $\sim e^{4\pi\sqrt{\xi j}}$ times an $\mathcal{O}(1)$ coefficient. The string density at leading order in $\xi j \gg 1$ is

$$\begin{aligned} \frac{\sqrt{t\bar{t}}}{2d_*} \rho_{\text{string},j}(t) &= \cos\left(2\pi\sqrt{\xi t}\right) \cosh\left(2\pi\sqrt{\xi(t+j)}\right) \\ &+ \sum_{s=1}^{\infty} f_{j,j_*;s} \cos\left(\frac{2\pi}{s}\sqrt{\xi(t+j)}\right) \cosh\left(\frac{2\pi}{s}\sqrt{\xi t}\right) \end{aligned} \quad (3.7.4)$$

In the first line we dropped the exponentially-suppressed $s > 1$ terms (this is allowed because the sum over s cannot lead to exponential enhancement), whereas no such suppression is present in the second line. Noting that (3.7.4) scales as $\sim e^{2\pi\sqrt{\xi j}}$, we see that the sum of (3.7.3) and (3.7.4) is positive, as the latter is exponentially suppressed in $\xi j \gg 1$. As an aside, note that this hierarchy can be overcome if d_* is exponentially large in ξ , a possibility that we discard (in the next subsection we bound d_* by an $\mathcal{O}(1)$ number).

Summarizing so far, we have shown that $\rho_j(t) > 0$ for $j \geq 1$ and all t at $\xi \gg 1$.

3.7.3 Positivity at $j = 0$

The scalar sector requires slightly more attention since there is no longer a parametrically large scale that suppresses $s > 1$ terms in the density. Since the MWK density is exponentially large and positive as $\xi t \gg 1$, any possible negativity will arise only for $\xi t \lesssim \mathcal{O}(1)$. We can then study the density at fixed $x := 2\pi\sqrt{\xi t}$, where we also take $\xi \gg 1$.

We divide the proof into two parts: showing that the sum of ($s = 1$) and ($s = 2$) terms is positive for d_* below a critical value; and showing that $s > 2$ terms are individually positive.

$$(s = 1) + (s = 2)$$

The sum of the $s = 1, 2$ terms of (3.6.4) and (3.6.6) (times $\frac{d_*}{2}$) is, at leading order in large ξ ,

$$\begin{aligned} \frac{t}{2} \tilde{\rho}_0(t) \Big|_{s \leq 2} &= 2 \sinh^2 x + 2d_* (\cos x \cosh x - 1) \\ &+ d_* (-1)^{j_*} \left(\cos\left(\frac{x}{2}\right) \cosh\left(\frac{x}{2}\right) - 1 \right). \end{aligned} \quad (3.7.5)$$

One can easily see numerically that upon increasing d_* , this function develops a minimum x_{\min} which eventually becomes negative, approximately given by

$$\begin{aligned} d_* &\lesssim 4.910, & x_{\min} &\approx 1.851 & (j_* \text{ even}) \\ d_* &\lesssim 5.236, & x_{\min} &\approx 1.847 & (j_* \text{ odd}) \end{aligned} \quad (3.7.6)$$

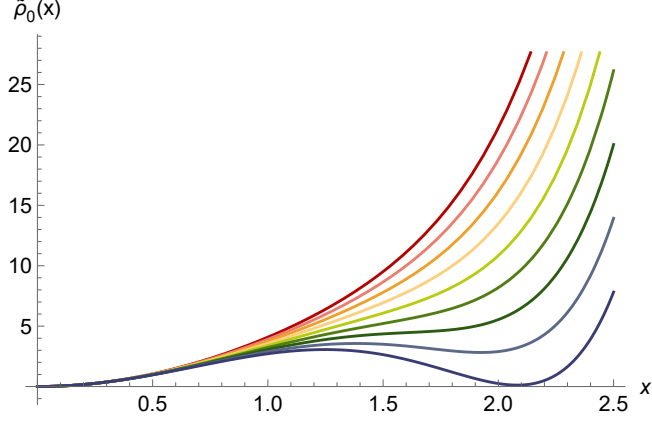


Figure 3.3: Plot of the regularized scalar density of states $\tilde{\rho}_0(x)$ with $\xi = 1000$, as a function of $x = 2\pi\sqrt{\xi t}$, with degeneracy ranging from $d_* = 3.3$ (red) to $d_* = 7.3$ (blue) in half-integer steps. For $d_* \gtrsim 7.3$, the density develops a negative region. (Obtained by summing over $s \leq 200$ in (3.6.9).)

If d_* obeys these bounds, then (3.7.5) is positive. We can check how these bounds compare to the full sum over s at finite but large central charge: see Fig. 3.3. Summing up to $s = 200$ for $\xi = 1000$, which easily ensures convergence, we observe numerically that the density becomes negative for $d_* \gtrsim 7.3$, not far from the limited analytic bound obtained above. The growth of the upper bound as we include more terms in the sum is due to the positivity of the $s > 2$ terms, as we will show next. Note that at smaller ξ , the upper bound actually grows, as seen in Fig. 3.4: for the smallest value $\xi = 2$ allowed within our construction, we observe positivity for $d_* \lesssim 19.2$.

$s \geq 3$ terms

At leading order in $\xi \gg 1$, the density of states for $s \geq 3$ is:

$$\begin{aligned} \frac{ts}{2} \tilde{\rho}_0(t) \Big|_{s \geq 3} &= 2(\phi(s) - \mu(s)) \sinh^2(2x_s) \\ &+ 2d_* c_s(j_*) (\cos x_s \cosh x_s - 1), \end{aligned} \quad (3.7.7)$$

where

$$x_s := \frac{x}{s}. \quad (3.7.8)$$

Denoting the right-hand side of (3.7.7) as $f(x_s)$, we observe that there is a minimum at $x_s = 0$ for which $f(0) = f'(0) = 0$. As a consequence, one way to ensure positivity is to demand convexity, $f''(x_s) > 0$, for all $x_s > 0$; this is of course not a necessary condition, but it is sufficient to achieve our goal of demonstrating existence of a range of d_* in which these

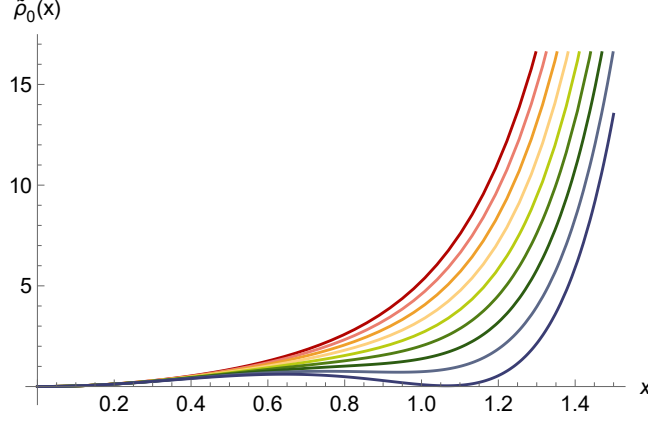


Figure 3.4: Plot of the regularized scalar density of states $\tilde{\rho}_0(x)$ with $\xi = 2$, as a function of $x = 2\pi\sqrt{\xi t}$, with degeneracy ranging from $d_* = 3.2$ (red) to $d_* = 19.2$ (blue) in steps of two. For $d_* \gtrsim 19.2$, the density develops a negative region – a larger critical value than for $\xi = 1000$. (Obtained by summing over $s \leq 200$ in (3.6.9).)

terms are positive. Imposing convexity gives the inequality

$$16(\phi(s) - \mu(s)) \cosh(4x_s) - 4d_*c_s(j_*) \sin x_s \sinh x_s > 0. \quad (3.7.9)$$

Using the bounds

$$|c_s(j_*)| < \phi(s), \quad |\mu(s)| \leq 1, \quad \phi(s \geq 3) \geq 2 \quad (3.7.10)$$

gives rise to the strongest inequality,

$$d_* < 2 \frac{\cosh(4x_s)}{\sinh(x_s)}. \quad (3.7.11)$$

If this is satisfied for all x_s , then so is (3.7.9). Minimizing the right-hand side gives

$$d_* \lesssim 11.888. \quad (3.7.12)$$

This ensures positivity of each individual $s \geq 3$ term in the density. This is compatible with the previously derived bounds for the $s = 1, 2$ terms.

Altogether, we conclude that for a finite range of $d_* > 1$, the density of states is positive, $\rho_j(t) > 0$, for all spins j and twists t at $\xi \gg 1$.

3.8 Appendix C: Spectral decomposition of $Z_{\text{string}}(\tau)$

In this appendix we derive (3.4.1). We directly present the relevant calculations, directing the reader to [112, 120, 121] for details on the $SL(2, \mathbb{Z})$ spectral formalism, and [1, 122, 123]

for its further use in the 2d CFT context.

We wish to compute the Petersson inner product

$$(Z_{\text{string}}, \psi_\omega) := \int_{\mathcal{F}} \frac{dx dy}{y^2} Z_{\text{string}}(\tau) \overline{\psi_\omega}(\tau) \quad (3.8.1)$$

where $\psi_\omega(\tau) = \{E_{\frac{1}{2}+i\omega}(\tau), \phi_n(\tau)\}$ are the $SL(2, \mathbb{Z})$ eigenbasis elements. Since $Z_{\text{string}}(\tau)$ is a Poincaré sum, the overlaps with the Eisenstein series and Maass cusp forms can be easily computed using the “unfolding trick.” This results in the following integral for the Eisenstein series:

$$(Z_{\text{string}}, E_{\frac{1}{2}+i\omega}) = \frac{4\mathbf{a}_{j^*}^{(\frac{1}{2}+i\omega)}}{\Lambda(\frac{1}{2}-i\omega)} \int_0^\infty \frac{dy}{y} K_{i\omega}(2\pi j^* y) \quad (3.8.2)$$

where $\mathbf{a}_j^{(\frac{1}{2}+i\omega)}$ are the Eisenstein Fourier coefficients

$$\mathbf{a}_j^{(\frac{1}{2}+i\omega)} = \frac{2\sigma_{2i\omega}(j)}{j^{i\omega}}, \quad (3.8.3)$$

which obey reflection symmetry, $\mathbf{a}_j^{(\frac{1}{2}+i\omega)} = \mathbf{a}_j^{(\frac{1}{2}-i\omega)}$. The cusp form overlap is obtained similarly:

$$(Z_{\text{string}}, \phi_n) = 4\mathbf{b}_j^{(n)} \int_0^\infty \frac{dy}{y} K_{i\omega}(2\pi j^* y), \quad (3.8.4)$$

where $\mathbf{b}_j^{(n)}$ are the cusp form Fourier coefficients, known only approximately via numerics [141]. The integral is divergent at the origin. Regularizing the divergence is straightforward: introducing

$$\begin{aligned} (Z_{\text{string}}, E_{\frac{1}{2}+i\omega})_\epsilon &:= \frac{4\mathbf{a}_{j^*}^{(\frac{1}{2}+i\omega)}}{\Lambda(\frac{1}{2}-i\omega)} \int_0^\infty \frac{dy}{y^{1-\epsilon}} K_{i\omega}(2\pi j^* y) \\ &= \frac{(\pi j^*)^{-\epsilon} \mathbf{a}_{j^*}^{(\frac{1}{2}+i\omega)}}{\Lambda(\frac{1}{2}-i\omega)} \Gamma\left(\frac{\epsilon-i\omega}{2}\right) \Gamma\left(\frac{\epsilon+i\omega}{2}\right) \end{aligned} \quad (3.8.5)$$

the overlaps may be defined by removing the regulator,

$$\begin{aligned} (Z_{\text{string}}, E_{\frac{1}{2}+i\omega}) &= \lim_{\epsilon \rightarrow 0} (Z_{\text{string}}, E_{\frac{1}{2}+i\omega})_\epsilon \\ &= \frac{\mathbf{a}_{j^*}^{(\frac{1}{2}+i\omega)}}{\Lambda(\frac{1}{2}-i\omega)} \Gamma\left(-\frac{i\omega}{2}\right) \Gamma\left(\frac{i\omega}{2}\right). \end{aligned} \quad (3.8.6)$$

and likewise for the cusp form overlap (3.8.4). This yields the spectral decomposition (3.4.1). We note that the regularization used here is equivalent to the following standard regularization of Poincaré sums over seed primaries of fixed dimensions,

$$Z_{h,\bar{h}}^\epsilon(\tau) := \sum_{\gamma \in SL(2,\mathbb{Z})/\Gamma_\infty} \text{Im}(\gamma\tau)^{\frac{1}{2}+\epsilon} q_\gamma^{h-\xi} \bar{q}_\gamma^{\bar{h}-\xi}, \quad (3.8.7)$$

where $(h, \bar{h}) = (\frac{5}{4}\xi, \frac{3}{4}\xi)$ for our state \mathcal{O}_* .

3.8.1 Re-deriving the scalar density

As a consistency check, we can re-derive the scalar density $\rho_{\text{string},0}(t)$ from the spectral decomposition. The scalar piece of the regularized partition function is

$$\frac{Z_{\text{string},0}^\epsilon(y)}{2\sqrt{y}(\pi j_*)^{-\epsilon}} = \int_{\mathcal{C}_{\text{crit}}} \frac{y^{i\omega} \mathfrak{a}_{j_*}^{(\frac{1}{2}+i\omega)}}{\Lambda(\frac{1}{2}-i\omega)} \Gamma\left(\frac{\epsilon+i\omega}{2}\right) \Gamma\left(\frac{\epsilon-i\omega}{2}\right) \quad (3.8.8)$$

where $\int_{\mathcal{C}_{\text{crit}}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega$ is the integration along the critical line. In writing (3.8.8) we have used the scalar Fourier modes $E_{s,0}^*(y) = \Lambda(s)y^s + \Lambda(1-s)y^{1-s}$ and $\phi_{n,0}(y) = 0$. We now perform contour integration for complex $z := i\omega$ by deforming to a new contour, \mathcal{C} , a semicircle in the left half plane $\text{Re}(z) < 0$ such that y^z decays at infinity. The integrand vanishes factorially on the arc at infinity due to the $\Lambda(\frac{1}{2}-i\omega)$ in the denominator. The poles inside the contour come from $\Gamma(\frac{\epsilon+z}{2})$ at $z = -2k - \epsilon$ with $k = 0, 1, \dots$. The integral (3.8.8) is then given as a sum over residues,

$$\begin{aligned} \frac{Z_{\text{string},0}^\epsilon(y)}{\sqrt{\pi}(\pi j_*)^{-\epsilon}} &= (k=0 \text{ term}) + \\ &\sqrt{y} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{4\sigma_{4k+2\epsilon}(j_*)\Gamma(k+\epsilon)}{\Gamma(\frac{1}{2}+\epsilon+2k)\zeta(1+2\epsilon+4k)} \left(\frac{\pi}{j_*y}\right)^{2k+\epsilon} \end{aligned} \quad (3.8.9)$$

where we used $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$ and the explicit Fourier coefficients (3.8.3). We have separated the $k=0$ term because as we remove the regulator $\epsilon \rightarrow 0$, two simple poles at $z = \pm\epsilon$ coalesce into a double pole at $z=0$, on the contour. We will thus treat separately the $k>0$ terms, for which the regulator can be trivially removed, and the $z = \pm\epsilon$ poles.

Let us now Laplace transform to the density of states,

$$Z_{\text{string},0}(y) = \sqrt{y} \int_0^\infty dt e^{-4\pi y t} \rho_{\text{string},0}(t). \quad (3.8.10)$$

We have written the density in terms of the reduced twist, $t = \Delta/2 - \xi$ for scalars, which is related to the density as a function of dimension Δ through $\rho(\Delta)d\Delta = \rho(t)dt$. The regularized

partition function (3.8.9) gives a regularized density

$$\rho_{\text{string},0}(t) = (k = 0 \text{ term}) + \frac{8\sqrt{\pi}}{t} \sum_{k=1}^{\infty} \frac{(-1)^k \sigma_{4k}(j_*)}{\Gamma(1+2k)\Gamma(\frac{1}{2}+2k)\zeta(1+4k)} \left(\frac{4\pi^2 t}{j_*}\right)^{2k} \quad (3.8.11)$$

where we have removed the regulator in the second line. Using the identity

$$\sigma_z(j) = \zeta(z+1)j^z \sum_{s=1}^{\infty} \frac{c_s(j)}{s^{z+1}}, \quad (3.8.12)$$

swapping the order of the sums and performing some simplifications,

$$\rho_{\text{string},0}(t) = (k = 0 \text{ term}) + \frac{8}{t} \sum_{s=1}^{\infty} \frac{c_s(j_*)}{s} \sum_{k=1}^{\infty} \frac{(-1)^k}{(4k)!} \left(\frac{2\pi\sqrt{2\xi t}}{s}\right)^{4k}. \quad (3.8.13)$$

The second line can be resummed to reproduce (3.6.6), the continuous part of the scalar density.

Finally, we return to the $k = 0$ term, still with the regulator, which is explicitly given by

$$\rho_{\text{string},0}(t)\Big|_{k=0} = \frac{4\sqrt{\pi}}{t(\pi j_*)^\epsilon} \frac{\sigma_{2\epsilon}(j_*)}{\Gamma(\frac{1}{2}+\epsilon)\zeta(1+2\epsilon)} \left(\frac{4\pi^2 t}{j_*}\right)^\epsilon \quad (3.8.14)$$

To regulate the divergence as $\epsilon \rightarrow 0$ and $t \rightarrow 0$, similarly to [96] we integrate from $t = 0$ up to some $t_* > 0$ and use $\epsilon\zeta(1+2\epsilon) \rightarrow \frac{1}{2}$ as $\epsilon \rightarrow 0$ to arrive at

$$\lim_{\epsilon \rightarrow 0} \left[\int_0^{t_*} dt \rho_{\text{string},0}(t)\Big|_{k=0} \right] = 8\sigma_0(j_*). \quad (3.8.15)$$

Together with the continuous part of the density derived above, this reproduces the full result (3.6.8).

Chapter 4

Ensemble Averaging in JT gravity from Entanglement in Matrix Quantum Mechanics

This chapter consists of a paper [3] written in collaboration with Giuseppe Policastro. The original abstract is as follows:

We consider the generalization of a matrix integral with arbitrary spectral curve $\rho_0(E)$ to a 0+1D theory of matrix quantum mechanics (MQM). Using recent techniques for 1D quantum systems at large- N , we formulate a hydrodynamical effective theory for the eigenvalues. The result is a simple 2D free boson BCFT on a curved background, describing the quantum fluctuations of the eigenvalues around $\rho_0(E)$, which is now the large- N limit of the quantum expectation value of the eigenvalue density operator $\hat{\rho}(E)$.

The average over the ensemble of random matrices becomes a quantum expectation value. Equal-time density correlations reproduce the results (including non-perturbative corrections) of random matrix theory. This suggests an interpretation of JT gravity as dual to a *one-time-point* reduction of MQM.

As an application, we compute the Rényi entropy associated to a bipartition of the eigenvalues. We match a previous result by Hartnoll and Mazenc for the $c = 1$ matrix model dual to two-dimensional string theory and extend it to arbitrary $\rho_0(E)$. The hydrodynamical theory provides a clear picture of the emergence of spacetime in two dimensional string theory. The entropy is naturally finite and displays a large amount of short range entanglement, proportional to the microcanonical entropy. We also compute the reduced density matrix for a subset of $n < N$ eigenvalues.

4.1 Introduction and Motivation

Random Matrix Models have been studied for a long time, as they provide a powerful computational tool and a great source of insight with many applications in different fields, from nuclear physics to condensed matter to high-energy physics.¹

One of the most intriguing applications arises from the connection to quantum gravity. It was first observed by 't Hooft [244] that a theory with matrix degrees of freedom can be interpreted as a theory of random surfaces in the large- N limit and thus, in many cases, it can be connected to some 2d quantum gravity/string theory. The perturbative expansion in Feynman diagrams can be reorganized as a topological expansion in the genus of the surface, with $1/N$ playing the role of the expansion parameter (string coupling). This idea has found its most concrete realization so far in the AdS/CFT correspondence [28–30]. In its most basic and well-understood instance, this correspondence relates a gravity theory on 5D Anti-de Sitter space to a $SU(N)$ gauge theory on the 4D boundary. Despite the fact that the correspondence has a very precise formulation and has been tested to great accuracy, its perhaps most striking conceptual aspect, namely the emergence of spacetime from the matrix degrees of freedom, is still poorly understood. The correspondence gives in principle a complete definition of quantum gravity in AdS, since the boundary theory is well defined non-perturbatively (e.g. by the CFT axioms); however, because it is a weak/strong coupling duality, it is still difficult to use it in order to find detailed answers to fundamental questions, such as the information loss paradox, and the statistical interpretation of the Bekenstein-Hawking entropy in terms of black hole microstates.

JT gravity

Driven by the desire to understand these questions in a simplified setting, there have been many recent developments in low dimensional holography. The SYK model [91, 245] is composed of a large number of fermions interacting with disordered couplings. In the large- N limit the low-energy sector of the model is described holographically by 2-dimensional Jackiw-Teitelboim (JT) gravity, or equivalently by a 1D Schwarzian theory [92, 246, 247]. The SYK model and JT gravity were shown to exhibit quantum chaos as universally described by random matrix theory [64, 69]. The paper [35] showed a much stronger connection to random matrices: the partition function of JT gravity on a surface of arbitrary genus and number of boundaries agrees with the perturbative expansion of a certain matrix integral, thus solving the theory to all orders in the genus expansion. The matrix integral is interpreted as an average over an ensemble of Hamiltonians and the matrix eigenvalues as the energy

¹There are many reviews on the subject. For general aspects see, in rough order of complexity [229–234]. For applications to low dimensional gravity and string theory see [235–242]. For applications to chaotic systems see [59, 243].

levels dual to gravitational microstates. It was also noted that JT gravity can be seen as the $p \rightarrow \infty$ limit of $(2, p)$ minimal strings which were long known to be dual to matrix models [238, 241, 248–250]. The study of non-perturbative effects in these models can then help us understand the detailed structure of gravitational microstates. As a consequence much effort has been devoted to this pursuit (see [70] for a review). However many interconnected questions still remain. The matrix integral does not provide a unique non-perturbative completion of JT gravity [251]. The bulk theory does not seem to have, at first sight, a well defined dual quantum mechanical system, but rather an ensemble of them. The presence of connected geometries and the consequent lack of factorization pose a deep puzzle about the nature of the gravitational path integral [189]. A more explicit understanding of the emergence of spacetime from the dual degrees of freedom remains to be attained [252].

In this work we discuss some of these issues by considering a generalization of the kind of matrix integral dual to JT-gravity, given by a $0 + 1D$ theory of matrix quantum mechanics (MQM) [253]:

$$S = \int dt \operatorname{tr} \left(\frac{1}{2} \dot{H}^2 + V(H) \right). \quad (4.1.1)$$

with an arbitrary potential $V(H)$. The classical average over the matrix ensemble becomes a quantum path integral:

$$\langle \cdot \rangle_H = \int dH e^{-N \operatorname{tr} V(H)} \rightarrow \int DH(t) e^{-S}. \quad (4.1.2)$$

We can think of the original matrix integral as a matrix quantum mechanics with *one-time-point* (as discussed in [190] for SYK) meaning that we look at a single instant of time where the dynamics are frozen. We will make this statement precise and show that we can reproduce matrix integrals from equal time correlations in the ground state of matrix quantum mechanics. In particular, the ground state expectation value of the eigenvalue density operator $\hat{\rho}(E)$ in MQM is equal to the ensemble-averaged density of eigenvalues:

$$\langle \rho(E) \rangle_H = \langle \Omega | \hat{\rho}(E) | \Omega \rangle \quad (4.1.3)$$

and similarly for higher correlations at equal times. The average over the ensemble of random matrices is equal to a quantum expectation value in Hilbert space. The statistical fluctuations due to ensemble averaging might be interpreted as quantum fluctuations of a single quantum mechanical system. We show that this holds for arbitrary spectral curves $\rho_0(E)$, defined as the large N limit of the eigenvalue density, by treating MQM with arbitrary potentials $V(H)$. If we specialize to the case of JT gravity's spectral curve $\rho_0^{JT}(E)$, this equality suggests a different perspective on JT gravity as dual to a *one-time-point* matrix quantum mechanics.

Matrix quantum mechanics is, however, richer than a matrix integral, simply since it is a quantum mechanical theory, endowed with a Hilbert space. The eigenvalues are described by a wavefunction $\psi_N(E_1, \dots, E_N)$ instead of a classical probability distribution

$\rho_N(E_1, \dots, E_N)$ as in random matrix theory. In the duality between JT gravity and a matrix integral, the eigenvalues are identified as microstates of a quantum mechanical theory $\{E_i\}$. MQM describes the wavefunction $\psi_N(E_1, \dots, E_N)$ in the basis of energy eigenstates $|E_i\rangle$. It is then natural to consider the entanglement between eigenvalues. Such entanglement is responsible for the correlations between eigenvalues, as they would factorize in its absence. It is in this sense that ensemble averaging arises from entanglement.

$c = 1$ String theory

In two spacetime dimensions there is another instance of holographic duality: the duality between two-dimensional string theory and the $c = 1$ matrix model [254], a theory of matrix quantum mechanics with a specific potential tuned to a critical point. This duality precedes AdS/CFT and has been extensively checked both perturbatively and non-perturbatively.² Two-dimensional string theory and JT gravity form part of the same family of theories. A minimal string consists of a Liouville CFT with $c_L > 25$ and a minimal model with $c_M < 1$ coupled by anomaly cancellation. In the limit $p \rightarrow \infty$ of the $(2, p)$ minimal string, which corresponds to JT gravity, we have that $c_M \rightarrow -\infty$. Instead, two dimensional string theory consists of a $c_L = 25$ Liouville theory and a $c_M = 1$ free boson. The two theories then lie at opposite ends of the same spectrum of worldsheet CFTs given by Liouville theory coupled respectively to $c_M = -\infty$ and $c_M = 1$.³ This limit indicates that we should think of spacetime in JT as analogous to the worldsheet in minimal string theory. As a consequence, wormholes in JT correspond to connected worldsheets with multiple boundaries. Achieving a better understanding of the relation between the two dualities could produce new insights.⁴

Two-dimensional string theory is a very rich arena to understand string theory and dualities. The worldsheet description allows for a detailed study of non-perturbative effects [257, 258, 260, 269–276] and their matching to the matrix model. The duality between the $c = 1$ matrix model and two-dimensional string theory is a perfect playground to study the emergence of spacetime from matrix degrees of freedom in gauge theories since, at large N , the eigenvalues form a continuum that is directly related to the dual spacetime. Locality and the relation between spacetime and eigenvalue-space can be tested in various ways, *e.g.* using local observables, scattering of excitations, or using entanglement entropy, as in [277]. The motivation of this last paper was to apply to the $c = 1$ matrix model the insight, gained in AdS/CFT with the Ryu-Takayanagi formula, of the essential role that entanglement plays in the emergence of spacetime [278, 279]. In this chapter we will give a different and more comprehensive perspective on the eigenvalue/spacetime relation by explicitly constructing the geometry of eigenvalue-space that corresponds to the spacetime geometry in a natural

²See [236–239] for reviews. See [255–260] for extensive recent work on matching scattering amplitudes. See [261–265] for recent related work on black holes

³For a review of the Liouville approach, see [242]

⁴See [89, 266–268] for related work

way and characterizing its entanglement properties.

The entanglement between eigenvalues is an example of entanglement in target space for which recently there has been a growing interest, see [280–287].

4.1.1 Overview and results

We start sec. 4.2 by recalling some basic facts about Matrix Quantum Mechanics with arbitrary potential. Eigenvalue repulsion enforces fermionic statistics for the eigenvalues which can be mapped to a system of fermions in an external potential. We introduce a second-quantized fermionic field $\Psi(E)$ which gives the eigenvalue density operator $\hat{\rho}(E) = \Psi^\dagger(E)\Psi(E)$. The density of eigenvalues is the expectation value $\rho(E) = \langle \hat{\rho}(E) \rangle$ which is, at leading order in the large N limit, equal to the spectral curve $\rho(E) \approx \rho_0(E)$. We then proceed in sec. 4.2.2 to illustrate the construction of an effective hydrodynamical theory for the eigenvalues valid for arbitrary $\rho_0(E)$. The construction follows from recent developments in the study of 1D many body quantum systems in external potentials [288–290]. It can be seen as a generalization of the collective field theory approach [240, 291] to arbitrary potentials. In sec. 4.2.3 we discuss quantum fluctuations of the eigenvalues in the effective theory. One can show that the quantum hydrodynamical fluctuations of the eigenvalues are described by a 2D free boson CFT on a curved background determined by $\rho_0(E)$ with boundaries at the edge of the spectrum where $\rho_0(E^*) = 0$.

In section 4.3 we proceed to use the 2D CFT to study the different properties of the eigenvalues. We start by computing spectral correlations in sec. 4.3.1 which are now given by correlation functions of the density operator: $\langle \hat{\rho}(E) \rangle$ and $\langle \hat{\rho}(E_1)\hat{\rho}(E_2) \rangle$. These are given by correlation functions of vertex operators in the CFT. We reproduce the leading non-perturbative corrections to the density of states $\rho(E)$ and to the level-correlation $\rho(E_1, E_2)$ as described in sec. 5 of [35] by considering equal-time correlations. In other words, we reproduce the oscillations of $\rho(E)$ around the semiclassical density $\rho_0(E)$ and the terms in $\rho(E_1, E_2)$ corresponding to the ramp and plateau in the spectral form factor (i.e. the sine kernel). In matrix quantum mechanics these spectral correlations arise due to quantum fluctuations of a single quantum mechanical system, as opposed to statistical fluctuations due to ensemble averaging. This matching provides evidence to support the idea that a matrix integral and consequently JT gravity might be interpreted as a *one-time-point* matrix quantum mechanics with the same spectral curve $\rho_0(E)$.

In sec. 4.3.2 we consider the entanglement between the eigenvalues. We compute the Rényi entropies for a bipartition $(0, E) \cup (E, E_R)$, where E_R is the right edge of the eigenvalue density, finding some interesting features. For non double-scaled matrix models, where the density has a right edge E_R , we see that the entanglement entropy follows a “Page curve” (as a function of the length of the interval) as required by unitarity and comes down instead of

growing indefinitely. This feature is lost in double-scaled models where $E_R \rightarrow \infty$ indicating that indeed we are missing states from the spectrum. The entanglement entropy is naturally finite due to the mean spacing between the eigenvalues $\frac{1}{\rho_0(E)} \sim e^{-S_0}$ acting as a UV cutoff.⁵ In two-dimensional string theory we can interpret the finiteness of the entropy as due to g_s stringy effects regulating the divergence as first noted in [277]. We notice that the entanglement entropy $S_{ent}(E)$ present a leading contribution proportional to the microcanonical entropy in the window $E \pm dE$ such that $S_{ent}(E) \propto S_0(E) = \log(\rho_0(E))$, indicating a large amount of short range entanglement between eigenvalues close to the boundary. We also compute the entanglement entropy for an interval bipartition (E_1, E_2) , extending the results of [277] for the $c = 1$ matrix model to arbitrary spectral curves $\rho_0(E)$. We provide constructive evidence for the proposed map between the eigenvalue-space and the emergent spacetime in two-dimensional string theory [277, 293] and the identification of the spacetime geometry with the geometry of the Fermi surface.

In sec. 4.3.3 we compute the one eigenvalue reduced density matrix obtained by tracing out $N - 1$ eigenvalues, corresponding to the fermion one-body density matrix $g(E, E') = \langle \Psi^\dagger(E) \Psi(E') \rangle$. We also write the general expression for the n eigenvalue density matrix.

We conclude in sec. 4.4 with a discussion of open questions and possible future work.

4.2 Quantum hydrodynamics of random matrix eigenvalues

We study the quantum mechanics of a random $N \times N$ hermitian matrix $H(t)$ in a generic potential $V(H)$ with the following action:

$$S = N \int dt \operatorname{tr} \left[\frac{1}{2} \dot{H}^2 + V(H) \right]. \quad (4.2.1)$$

The eigenvalues $(E_1 \dots E_N)$ of H no longer obey a classical probability distribution as in Random Matrix Theory. Instead they are now described by a quantum mechanical wavefunction $\psi_N(E)$. We will now briefly summarize some well known facts about matrix quantum mechanics (MQM) and derive the Schrodinger equation for the N -eigenvalue wavefunction $\psi_N(E)$. More details can be found in the above mentioned reviews [230, 235–239].

⁵While finishing the paper that this chapter is based on, the work [292] appeared which discusses the finiteness of the entanglement entropy in matrix quantum mechanics. Their methods are different and the discussion is complementary. In particular, they discuss a vanishing potential $V = 0$ while we treat arbitrary potentials.

4.2.1 Eigenvalues as fermions

To study the eigenvalues we diagonalize the matrix H :

$$H = \Omega^\dagger E \Omega \quad (4.2.2)$$

where $\Omega \in SU(N)$ and $E = \text{diag}(E_1, \dots, E_N)$. This change of variables has a non-trivial jacobian which modifies the path integral measure $\mathcal{D}H(t)$:

$$\int \mathcal{D}H = \int \mathcal{D}\Omega \prod_i \mathcal{D}E_i \Delta^2(E), \quad (4.2.3)$$

where $\Delta(E) = \prod_{i < j} (E_i - E_j)$ is the well known Vandermonde determinant which causes eigenvalue repulsion in random matrix theory. We will now see that in MQM, eigenvalue repulsion becomes the Pauli exclusion principle resulting in fermionic eigenvalues [253]. Due to the non-trivial Jacobian, the kinetic term for the eigenvalues becomes:

$$-\frac{1}{2} \sum_{i=1}^N \frac{1}{\Delta^2(E)} \frac{d}{dE_i} \Delta^2(E) \frac{d}{dE_i}. \quad (4.2.4)$$

Thanks to the fact that $\sum_i \frac{d^2 \Delta}{dE_i^2} = 0$, this is equal to:

$$-\frac{1}{2\Delta} \sum_i \frac{d^2}{dE_i^2} \Delta. \quad (4.2.5)$$

The Hamiltonian \mathcal{H} of matrix quantum mechanics, after diagonalization of H , is then [236,253]:

$$\mathcal{H} = -\frac{1}{2\Delta} \sum_i \frac{d^2}{dE_i^2} \Delta + \sum_i V(E_i) + \sum_{i < j} \frac{L_{ij}^2 + \tilde{L}_{ij}^2}{(E_i - E_j)^2}. \quad (4.2.6)$$

The first term is the kinetic term for the eigenvalues we just discussed. The matrix potential $V(H)$ becomes a single particle potential for the eigenvalues $V(E_i)$ due to invariance of the trace. The last term is the kinetic term for the angular degrees of freedom Ω , where L_{ij}, \tilde{L}_{ij} are the angular momenta and $(E_i - E_j)^2$ plays the role of a radius in the direction ij . Let us denote a generic wavefunction for the Hamiltonian \mathcal{H} as $\chi_N(E, \Omega)$, which will depend on both the eigenvalues E and the angular variables Ω . We use the subscript N as a reminder that the wavefunctions depend on all the eigenvalues E_1, \dots, E_N . We will restrict ourselves to scalar configurations which are invariant under $SU(N)$ rotations, namely the singlet sector. Thus we consider wavefunctions $\chi_N(E)$ which are independent of the angular variables. The singlet wavefunctions $\chi_N(E)$ are the relevant ones and correctly describe MQM in the following regimes:

- **Ground state.** Since the angular term is positive definite, the ground state of the system is given by the singlet sector ground state. Thus the singlet wavefunction describes the collective ground state of the N eigenvalues.

- **Low temperature phase.** Considering MQM at finite temperature, we have a Berezinski-Kosterlitz-Thouless transition at β_{BKT} . The singlet sector describes the low temperature phase $\beta > \beta_{BKT}$ [294–297].

Consider now the Schrodinger equation for the singlet sector $\mathcal{H}\chi_N(E) = \epsilon\chi_N(E)$. The wavefunctions $\chi_N(E)$ are clearly symmetric functions of the eigenvalues. By defining a completely anti-symmetric wavefunction $\psi_N(E) = \Delta(E)\chi_N(E)$, the Schrodinger equation now reads:

$$\sum_{i=1}^N \mathcal{H}_i \psi_N(E) = \sum_i^N \epsilon_i \psi_N(E), \quad \mathcal{H}_i = -\frac{1}{2} \frac{d^2}{dE_i^2} + V(E_i). \quad (4.2.7)$$

The Hamiltonian acting on $\psi_N(E)$ is now a sum of single-particle Hamiltonians. The wavefunction $\psi_N(E)$ is completely antisymmetric by construction due to the antisymmetry of the Vandermonde and vanishes whenever $E_i = E_j$. We see that the eigenvalue repulsion of random matrices enforces the Pauli exclusion principle. The eigenvalues E_i are then equivalent to a system of N fermions each in an external potential $V(E)$, interacting only through the exclusion principle/eigenvalue repulsion.

The many-body ground state wavefunction $\psi_N(E)$ can be obtained by first solving for the single particle wavefunctions $\psi_\epsilon(E)$ and building the Slater determinant $\psi_N(E) = \frac{1}{\sqrt{N!}} \det_{ij}(\psi_{\epsilon_i}(E_j))$ which involves a single fermion in each energy level up to the Fermi energy ϵ_F , the energy of the last (N -th) fermion. We will not do this as it involves solving the Schrodinger equation for a specific choice of potential with the resulting Slater wavefunctions ψ_N being complicated expressions for large N . We will instead describe the system in second quantization by introducing a fermionic field $\Psi(E)$ [277, 298] with the following Hamiltonian \mathbb{H} :

$$\mathbb{H} = N \int dE \Psi^\dagger(E) \left(-\frac{1}{2N^2} \frac{d^2}{dE^2} + V(E) \right) \Psi(E). \quad (4.2.8)$$

The fermionic field $\Psi(E)$ can be expressed as a mode expansion with creation/annihilation operators $a_\epsilon, a_\epsilon^\dagger$ weighted by the single particle wavefunctions $\psi_\epsilon(E)$:

$$\Psi(t, E) = \int d\epsilon e^{-i\epsilon t} a_\epsilon \psi_\epsilon(E). \quad (4.2.9)$$

The fermions fill the potential $V(E)$ up to the Fermi energy ϵ_F . We can control how the potential is filled by introducing a chemical potential $\mu = N\epsilon_F$ in the Hamiltonian $\mathbb{H} - \mu\Psi^\dagger\Psi$.

The system forms a Fermi surface $|\mu\rangle$ on which the operators $a_\epsilon, a_\epsilon^\dagger$ act as follows:

$$\begin{aligned} a_\epsilon |\mu\rangle &= 0 & \epsilon > \mu, \\ a_\epsilon^\dagger |\mu\rangle &= 0 & \epsilon < \mu. \end{aligned} \tag{4.2.10}$$

The presence of a Fermi sea corresponds to having a finite density of eigenvalues $\rho_0(E)$ in RMT. In what follows we will employ recent techniques from condensed matter [299] describing the quantum fluctuations of the Fermi surface by a 2D effective hydrodynamical theory .

Correspondingly, one can develop a quantum hydrodynamical theory for the eigenvalue density $\rho(E)$, describing the fluctuations around a semiclassical background $\rho_0(E)$ given by the RMT spectral curve. Quantum fluctuations on top of the Fermi surface involving the creation/annihilation of a single eigenvalue will produce non-perturbative effects in $\frac{1}{\rho_0(E)} \sim e^{-S_0}$ (similarly as described in sec. 5 of [35]). The effective theory will be a simple free boson CFT on a curved background with a boundary. This simple description allows us to study many interesting features of Matrix Quantum Mechanics. We can access non-perturbative physics like the oscillations in the density of states $\rho(E)$ and the plateau in the two level correlation $\rho(E_1, E_2)$ which are a consequence of the underlying discreteness of the spectrum of H . We can also compute observables that do not have a clear classical counterpart such as the reduced density matrix obtained by tracing out k -out-of- N eigenvalues and the spectrum of Renyi entropies for arbitrary bipartition (E_1, E_2) .

Incorporating the chemical potential, we arrive at the following Hamiltonian, which is the starting point for the rest of discussion:

$$\mathbb{H} = \int dE \Psi^\dagger(t, E) \left(-\frac{1}{2} \frac{d^2}{dE^2} + (V(E) - \mu) \right) \Psi(t, E). \tag{4.2.11}$$

4.2.2 Effective hydrodynamics of the eigenvalue density $\rho(E)$

We now give a self-contained review of some recent developments in the study of 1D many-body quantum systems in external potentials via hydrodynamics [288, 300–302]. The hydrodynamics approach to 1D quantum systems was introduced a few years ago in [289, 290] and has been rapidly developing ever since, see the recent lectures [303] for a review. We will only introduce the necessary tools for our purposes.

Conformal Field Theory in 2D is a well-proven technique in addressing 1D critical quantum systems [304]. It is commonly used to describe low energy excitations around a fixed energy scale (such as the Fermi energy ϵ_F) and so it is not a priori possible to apply it to inhomogeneous systems, where there is a varying energy scale due to an external potential or out-of-equilibrium dynamics. On the other hand, hydrodynamics is useful to describe

inhomogeneous systems at mesoscopic scales, large enough to contain a macroscopic number of degrees of freedom but smaller than the characteristic scale of the inhomogeneities. In [288], they obtained a 2D CFT describing inhomogeneous 1D quantum systems using hydrodynamics. The CFT lives on a non-trivial background metric encoding the inhomogeneities of the original system.

Let us start by considering a many-body quantum system composed of N particles with a finite density $\rho(x)$ in the large N limit in an interval $x \in (x_L, x_R)$. This means that the quantum density operator $\hat{\rho}(x) = \Psi^\dagger(x)\Psi(x)$ acquires a VEV $\rho(x) \equiv \langle \Psi^\dagger(x)\Psi(x) \rangle$. The VEV introduces a length scale in the system corresponding to the local average spacing between particles $d(x) = \frac{1}{\rho(x)}$. In Random Matrix Theory there is a finite density of eigenvalues $\rho(E)$ due to eigenvalue repulsion, which is analogous to the non-zero VEV of the quantum density operator $\hat{\rho}$. We will make this correspondence precise in MQM: since the eigenvalues are fermions we have $x = E$ and we have that $\rho(E) = \langle \hat{\rho}(E) \rangle$. The mean level spacing is then equal to $d = \frac{1}{\rho(E)} \sim e^{-S_0}$.⁶ The key assumption to develop hydrodynamics for inhomogeneous systems is the separation of scales, meaning there exists an intermediate mesoscopic scale ℓ such that:

$$d \ll \ell \ll \frac{\rho(x)}{\partial_x \rho(x)}, \quad (4.2.12)$$

where $\frac{\rho}{\partial \rho}$ is the characteristic length scale of the inhomogeneities. The scale ℓ is then small enough such that the system is quasi-homogeneous and large enough to contain a thermodynamically large number of particles. These scales provide both UV and IR cutoffs in the effective theory defined at energy scales Λ such that:

$$\frac{\partial_x \rho(x)}{\rho(x)} = \Lambda_{IR} \ll \Lambda \ll \Lambda_{UV} = \frac{1}{d} = \rho(x), \quad (4.2.13)$$

Having understood the characteristic scales and the regime of validity of the effective theory, we will now focus on a specific system: the Lieb-Liniger gas of interacting bosons in an external potential. It is defined by the following Hamiltonian:

$$H = \int dx \left(\Phi^\dagger \left(\frac{\hbar^2 \partial_x^2}{2m} + V(x) \right) \Phi + \frac{g}{2} \Phi^{\dagger 2} \Phi^2 \right), \quad (4.2.14)$$

where $\Phi(x)$ is a bosonic field $[\Phi(x), \Phi(x')] = \delta(x - x')$. This system can be solved exactly via Bethe-Ansatz in the homogeneous $V = 0$ case [305]. In the limit of hard-core bosons $g \rightarrow \infty$ it is equivalent to a system of free fermions in the potential $V(x)$ and thus describes the eigenvalues of MQM. This limit is often referred to as the Tonks-Girardeau gas in the literature. The mapping between hard-core bosons and free fermions is made via a

⁶This is not the first instance where the spectral density $\rho(E)$ is identified with a VEV, in [163] $\rho(E)$ is identified as the order parameter responsible for Causal Symmetry Breaking in the universal late time behaviour of chaotic systems.

Jordan-Wigner string:

$$\Psi^\dagger(x) = e^{i\pi \int_{y<x} \Phi^\dagger(y)\Phi(y)dy} \Phi^\dagger(x). \quad (4.2.15)$$

$\Psi(x)$ is now a fermionic field $\{\Psi^\dagger(x), \Psi(x')\} = \delta(x - x')$. We then obtain the free fermion hamiltonian:

$$\mathbb{H} = \int dx \Psi^\dagger \left(-\frac{\hbar^2 \partial_x^2}{2m} + V(x) \right) \Psi. \quad (4.2.16)$$

The hydrodynamic description of the homogeneous case ($V = 0$) was first presented in [306] where the authors developed the hydrodynamics of out-of-equilibrium systems with an infinite number of conserved charges. Let us now proceed with the case of a general external potential $V(x)$. We start by writing down the Euler equations for a Galilean invariant fluid in the presence of an external potential:

$$\begin{aligned} \partial_t \rho + \partial_x j &= 0, \\ \partial_t u + u \partial_x u + \frac{1}{\rho} \partial_x P &= -\partial_x V, \end{aligned} \quad (4.2.17)$$

where ρ is the particle density, u is the mean velocity given by $u = \frac{j}{\rho}$ with j the momentum density and P is the pressure. To close the system of equations we need the equation of state at zero temperature which expresses the pressure as a function of the density $P(\rho)$. This can be obtained from the energy density ρ_E by the thermodynamic relation $P(\rho) = -\rho_E + \rho \left(\frac{\partial \rho_E}{\partial \rho} \right)_{T=0}$. In the Lieb-Liniger model, these equations follow from the conservation of the following charges:

$$\begin{aligned} \hat{\rho}(x) &= \Phi^\dagger(x)\Phi(x) \\ \hat{\rho}_P(x) &= -i\hbar\Phi^\dagger(x)\partial_x\Phi(x) \\ \hat{\rho}_E(x) &= \frac{\hbar^2}{2} \left(\partial_x\Phi^\dagger(x)\partial_x\Phi(x) \right) + \frac{g}{2}\Phi^{\dagger 2}(x)\Phi^2(x) \end{aligned} \quad (4.2.18)$$

with associated currents $\hat{j}, \hat{j}_P, \hat{j}_E$. The Euler equations describe the expectation values of the charges and currents $\langle \rho \rangle = \rho, \langle j \rangle = j$.⁷

In particular, consider the quantum expectation value of the density operator for the fermionic field Ψ :

$$\rho(x) \equiv \langle \Psi^\dagger(x)\Psi(x) \rangle. \quad (4.2.19)$$

In the hydrodynamic description where the system is quasi-homogeneous at the scales we are probing, we will have a semiclassical background density $\rho_0(x)$ at leading order in the large N limit such that $\rho(x) \approx \rho_0(x)$. We can then describe fluctuations of this semiclassical

⁷Since we will be considering zero temperature and thus zero entropy hydrodynamics, the continuity equation for the energy density is trivially satisfied and we have not displayed it in the text.

density which will give both subleading corrections to $\rho(x)$ and correlations $\langle \hat{\rho}(x)\hat{\rho}(x') \rangle$. The semiclassical density sets the scales for which the effective theory is valid and self-consistent:

$$\frac{\partial_x \rho_0(x)}{\rho_0(x)} \ll \Lambda \ll \rho_0(x). \quad (4.2.20)$$

In Random Matrix Theory the density $\rho_0(x)$ is given by the leading density of eigenvalues $\rho_0(E)$ in the large N limit, meaning the spectral curve/the disk density of states.

We can obtain an approximate expression for $\rho_0(x)$ as a function of the potential $V(x)$ in the hydrodynamical effective theory. Since $\rho_0(x)$ is the density at equilibrium, meaning $\partial_t \rho = 0, u = 0$, the Euler equation reduces to $\frac{1}{\rho} \partial_x P = -\frac{1}{m} \partial_x V$. Using the thermodynamic relation $dP = \rho S dT + \frac{\rho}{m} d\mu$ at $T = 0$ we have that $\partial_x(\mu(x) + V(x)) = 0$. The local chemical potential is then $\mu(x) = \mu - V(x)$ where μ is the fixed chemical potential appearing in the Hamiltonian. For a homogeneous system, the equilibrium density is just a function of the chemical potential $\rho_{hom} = \rho_{hom}(\mu)$ so for scales where the system is quasi-homogeneous we can simply substitute the local chemical potential $\mu(x)$ in the homogeneous density. We have then that the semiclassical density $\rho_0(x)$ in the hydrodynamic description is given by:

$$\rho_0(x) \stackrel{hydro}{\approx} \rho_{hom}(\mu(x)), \quad \mu(x) = \mu - V(x). \quad (4.2.21)$$

The theory will be entirely defined in terms of the density $\rho_0(x)$, without making reference to the potential so it is not necessary to use the relation above although it can be useful if we wish to define our MQM by the potential $V(x)$ instead of the spectral curve.

In particular, the density for free fermions with $V(x) = 0$ is:

$$\rho_{hom}(\mu) = \frac{\sqrt{2\mu}}{\pi \hbar}, \quad (4.2.22)$$

we then have that the semiclassical density is:

$$\rho_0(x) \approx \frac{1}{\pi \hbar} \sqrt{2(\mu - V(x))}. \quad (4.2.23)$$

This expression is usually called the Local Density Approximation (LDA) and it is well-known that it correctly describes the bulk density (sufficiently away from edges where $\rho \approx 0$) of a Fermi gas in the large N limit [307, 308]. We can see immediately that for a Gaussian potential $V(x) = \frac{x^2}{2}$ it correctly reproduces Wigner's semicircle law. We can also see that this is exactly the expression for the momentum $p(x)$ of a particle with energy μ appearing in the WKB approximation:

$$\psi_{WKB} \approx \frac{A}{\sqrt{p(x)}} \exp\left(\pm \frac{i}{\hbar} \int^x p(x')\right), \quad p(x) = \pi \hbar \rho_0(x). \quad (4.2.24)$$

As a final consistency check note that the density of free fermions scales as $\rho \sim \mathcal{O}(\hbar^{-1})$ so that the mean spacing $d = \frac{1}{\rho} \sim \mathcal{O}(\hbar)$ while the scale of inhomogeneities is $\frac{\rho}{\partial_x \rho} \mathcal{O}(1)$. Thus for $\hbar \rightarrow 0$ there is indeed separation of scales and we can always find a regime $\ell \sim \mathcal{O}(\hbar^\nu)$ with $0 < \nu < 1$ where the hydrodynamic description is valid. If we send $\hbar \rightarrow 0$, the total number of particles $N = \int \rho(x) dx$ diverges so we should take the large N limit with $N\hbar = \mathcal{O}(1)$. This is the well known property that large N limits are semiclassical.

4.2.3 2D CFT for the quantum fluctuations of $\rho(E)$

We are now ready to build the field theory for the hydrodynamical fluctuations of the density $\rho(x)$. Let us now consider a microscopic correlation function of local operators $O(x)$ in the ground state:

$$\langle O(x_1) \dots O(x_n) \rangle \equiv \lim_{\beta \rightarrow \infty} \frac{\text{tr}[O(x_1) \dots O(x_n) e^{-\beta \mathbb{H}}]}{\text{tr} e^{-\beta \mathbb{H}}}, \quad (4.2.25)$$

where \mathbb{H} is the fermion Hamiltonian in eq. 4.2.11.

In the hydrodynamic limit where $\frac{1}{N} \sim \hbar \rightarrow 0$ we can compute the correlation function by doing a path integral over the hydrodynamic fields $\rho(x, \tau)$ and $j(x, \tau)$ with an Euclidean action $S_E[\rho, j]$:

$$\langle O(x_1) \dots O(x_n) \rangle \stackrel{\hbar, \frac{1}{N} \rightarrow 0}{=} \frac{1}{Z} \int D\rho D j \delta(\partial_\tau \rho + \partial_x j) O(x_1) \dots O(x_n) e^{-S_E[\rho, j]}, \quad (4.2.26)$$

where the continuity equation is a constraint in the space of configurations (ρ, j) of the path integral⁸. The task now is to determine the action $S_E[\rho, j]$ that computes these correlation functions in the hydrodynamic limit. To do so we will proceed by first finding an action which gives the Euler equations as its equations of motion. We consider the following action:

$$S[\rho, j] = \int dx dt \left(\frac{j^2}{2\rho} + \rho_E(\rho) + (V(x) - \mu)\rho \right). \quad (4.2.27)$$

We now perform a variation of the action $(\bar{\rho} + \delta\rho, \bar{j} + \delta j)$ starting from a configuration $(\bar{\rho}, \bar{j})$ which satisfies the Euler and continuity equations. To perform a variation consistent with the constraint $\partial_t \rho + \partial_x j = 0$ we write the fluctuations as:

$$\delta\rho(x, t) = \frac{1}{2\pi} \partial_x h(x, t), \quad \delta j(x, t) = -\frac{1}{2\pi} \partial_t h(x, t). \quad (4.2.28)$$

We have introduced a new field $h(x, t)$ such that the constraint is now trivially satisfied due

⁸The same type of path integral has also appeared recently in [309] as the action for ballistic MFT, although in that case it is supposed to apply at finite temperature and describe statistical fluctuations (thanks to T. Yoshimura for pointing it out).

to the fact that partial derivatives commute. The second order variation $\delta^2 S_E[\bar{\rho} + \delta\rho, \bar{j} + \delta j]$ gives a quadratic action for the quantum fluctuations described by the field $h(x, t)$. The action is the following:

$$S[h] = \frac{1}{8\pi} \int \frac{\sqrt{g} d^2x}{K(x)} g^{ab} \partial_a h \partial_b h, \quad (4.2.29)$$

where K , known as the Luttinger parameter, is a function of the density $\bar{\rho}(x)$

$$K(x) = \frac{\pi \hbar \bar{\rho}(x)}{v(x)} \quad \text{with} \quad v(x) = \sqrt{\bar{\rho}(x) \partial_\rho^2 \rho_E}, \quad (4.2.30)$$

and the metric is given by

$$ds^2 = (dx - (u + v)dt)(dx - (u - v)dt). \quad (4.2.31)$$

where $u(x) = \frac{\bar{j}}{\bar{\rho}}$ is the background local fluid velocity. We see from the metric that $v(x)$ is the local speed of sound in the fluid since excitations which propagate along light-rays correspond to sound waves propagating at velocity $u \pm v$. The system exhibits curved lightcones due to the dependence on the local value of the density $\bar{\rho}(x)$ [310]. This metric is the effective geometry of the Fermi surface seen by the excitations.

The above action thus describes quantum fluctuations around non-trivial hydrodynamical backgrounds $\bar{\rho}(x, t), \bar{j}(x, t)$ for 1D inhomogenous quantum systems specified by their microscopic equation of state $\rho_E(\rho)$, from which we obtain the Luttinger parameter K and the sound velocity v .

There can be corrections to the effective action for the fluctuations 4.2.29 by expanding to higher order the hydrodynamic action 4.2.27. There can also be hydrodynamic gradient corrections, recently discussed in [311].

We will now restrict ourselves to the case of free fermions since we wish to describe fluctuations of the eigenvalues of random matrices. We also restrict ourselves to equilibrium configurations given by the saddle point $(\bar{\rho}, \bar{j}) = (\rho_0(x), 0)$ where $\rho_0(x)$ is the semiclassical density. We leave the study of out-of-equilibrium dynamics of the eigenvalues for future work. The equation of state for free fermions at zero temperature is:

$$\rho_E = \frac{\pi^2 \hbar^2 \rho^3}{6}. \quad (4.2.32)$$

We have that the sound velocity is simply proportional to the density

$$v(x) = \pi \hbar \rho_0(x), \quad (4.2.33)$$

and the Luttinger parameter is simply $K = 1$. Notice that $v(x)$ is equal to the classical momentum $p(x)$ appearing in the WKB approximation. We arrive then at the following Euclidean action, describing the quantum hydrodynamical fluctuations of free fermions in an

external potential:

$$S[h] = \frac{1}{8\pi} \int \sqrt{g} d^2x g^{ab} \partial_a h \partial_b h, \quad (4.2.34)$$

with metric given by (now in units where $\hbar = 1$):

$$ds^2 = \pi^2 \rho_0^2(x) d\tau^2 + dx^2, \quad (4.2.35)$$

The action $S[h]$ provides a description of the eigenvalue fluctuations around the semiclassical spectral density $\rho_0(E)$ in terms of quantum hydrodynamics of the Fermi surface. Analogously to the $\frac{1}{N}$ genus expansion of matrix models, which is completely fixed by Topological Recursion in terms of the spectral curve $\rho_0(E)$ (see [231] for an extensive explanation), the theory of hydrodynamic fluctuations of the eigenvalues is completely determined by the matrix model spectral curve $\rho_0(E)$. The theory is defined on the domain $(x, \tau) \in (x_L, x_R) \times \mathbb{R}$, where (x_L, x_R) are the points where the semiclassical density vanishes $\rho_0(x_{L,R}) = 0$ ⁹. As long as we work at scales Λ inside the range of validity of the hydrodynamic effective theory given in eq. 4.2.20, we can apply the theory to a matrix model specified by its semiclassical density of eigenvalues $\rho_0(E)$. In particular, to apply this description to JT gravity, $(2, p)$ minimal strings and related models it is enough to use the spectral density of the desired model, i.e. $\rho(E) = \rho_{JT}(E)$. On the other hand, if one wishes to specify the matrix model potential $V(E)$, we can obtain the spectral density in the hydrodynamic approximation as:

$$\rho_0(E) = \sqrt{2(\mu - V(E))}. \quad (4.2.36)$$

We will use this expression for the density to study the $c = 1$ matrix model which corresponds to an inverted oscillator potential $V(E) = -\frac{E^2}{2}$.

The hydrodynamic theory is valid in the domain of non-vanishing particle density so the theory has a boundary at points x_L, x_R such that $\rho(x_{L,R}) = 0$. To summarise, we have an effective theory for the quantum hydrodynamical fluctuations of the Fermi surface of free fermions corresponding to the fluctuations of the eigenvalues of a random matrix H . They are described by a 2D free boson BCFT on a curved geometry determined by the spectral density $\rho_0(E)$. In this formalism it is straightforward to consider double scaled matrix models, it is enough to use the double scaled spectral density which has only one zero at $x_L = 0$ and $x_R = \infty$ so we have a BCFT on the half-line.

To complete the correspondence we are in need of a prescription to relate local operators \mathcal{O}_F in the microscopic fermionic theory (4.2.11), such as Ψ, Ψ^\dagger , to local operators \mathcal{O}_{Eff} in the effective theory (4.2.34). A single operator in the microscopic theory $\mathcal{O}_F(x)$ can be expanded

⁹We are treating the case of a single interval with non-zero density, corresponding to single cut matrix models

as a sum of local operators $O_{\text{Eff}}(x)$:

$$\mathcal{O}_F(x) = \sum_i \tilde{A}_{\mathcal{O}, O_i} O_i(x), \quad (4.2.37)$$

where $\tilde{A}_{\mathcal{O}, O_i}$ are dimensionful coefficients $[\tilde{A}_{\mathcal{O}, O_i}] = \Delta_{O_i} - \Delta_{\mathcal{O}}$ and we dropped the subscript from the operators $O_{\text{Eff}}(x)$ in the right hand side. We define dimensionless coefficients $A_{\mathcal{O}, O_i}$ using the characteristic length scale of the microscopic system $d = (\rho_0(x))^{-1}$, we have then

$$\tilde{A}_{\mathcal{O}, O_i} = \frac{A_{\mathcal{O}, O_i}}{\rho_0(x)^{\Delta_{O_i} - \Delta_{\mathcal{O}}}}. \quad (4.2.38)$$

Using this prescription we can in principle write any correlation function in the microscopic model as a sum of CFT correlators which we know explicitly given the simplicity of the CFT:

$$\langle \mathcal{O}_F(x_1) \dots \mathcal{O}_F(x_n) \rangle = \sum_{i_1, \dots, i_n} \frac{A_{\mathcal{O}, O_{i_1}}}{\rho_0(x_1)^{\Delta_{O_{i_1}} - \Delta_{\mathcal{O}}}} \dots \frac{A_{\mathcal{O}, O_{i_n}}}{\rho_0(x_n)^{\Delta_{O_{i_n}} - \Delta_{\mathcal{O}}}} \langle O_{i_1}(x_1) \dots O_{i_n}(x_n) \rangle_{\text{CFT}}. \quad (4.2.39)$$

The sum can be organized according to the relevance of the operators in the effective theory as each term in the sum is suppressed by the UV scale $d(x)$ as $d^{\sum_k \Delta_{O_k} - n\Delta_{\mathcal{O}}}$. The dimensionless coefficients $A_{\mathcal{O}, O_i}$ are determined by matching to the microscopic theory, as usual in effective theories.

Consider the fermionic fields Ψ^\dagger, Ψ in the microscopic theory, they are charged under a global $U(1)$ symmetry with charge $q = 1$ so the the corresponding CFT operators should also be. The bosonic $U(1)$ charge is the winding or magnetic number q of vertex operators $V_{p,q}$. Thus the corresponding operators are the CFT vertex operators $V_{p,q=1}$ and their descendants, where we define a (p, q) vertex operator by

$$V_{p,q}(z, \bar{z}) =: e^{i(p-\frac{q}{2})\phi(z) + i(p+\frac{q}{2})\bar{\phi}(\bar{z})} :. \quad (4.2.40)$$

We have used chiral factorization of the CFT to write the boson field $h(x, t)$ as a sum of holomorphic and antiholomorphic fields $h(x, \tau) = \phi(z) + \bar{\phi}(\bar{z})$.

Considering only the most relevant most relevant operator we have then:

$$\Psi^\dagger(x) \approx \tilde{A}_{\Psi^\dagger, \mathcal{V}_{0,1}}(x) \mathcal{V}_{0,1}(x). \quad (4.2.41)$$

Since the fermion fields have dimension $\frac{1}{2}$ and the vertex operator has dimension $\frac{1}{4}$ the coefficient $\tilde{A}_{\Psi^\dagger, \mathcal{V}_{0,1}}$ has dimensions $-\frac{1}{4}$ such that:

$$\tilde{A}_{\Psi^\dagger, \mathcal{V}_{0,1}}(x) = A_{\Psi^\dagger, \mathcal{V}_{0,1}} \rho_0(x)^{1/4}. \quad (4.2.42)$$

The dimensionless coefficient $A_{\Psi^\dagger, \mathcal{V}_{0,1}}$ is the same as in the homogeneous $V(x) = 0$ case, and

so it can be calculated analytically by Bethe-Ansatz to obtain:

$$|A_{\Psi^\dagger, \nu_{0,1}}|^2 = \frac{G^4(3/2)}{\sqrt{2\pi}}, \quad (4.2.43)$$

where G indicates Barnes' G -function. This completes the prescription for how to compute observables in the fermion theory using the hydrodynamical effective theory. We can now proceed to apply this framework to study the quantum mechanics of random matrix eigenvalues.

4.3 Spectral correlations and entanglement

We now apply the effective theory describing the quantum hydrodynamical fluctuations of the eigenvalue density (eq. 4.2.34). The theory is a free boson 2D CFT on a non trivial background with boundaries. Thanks to its simplicity, we can easily compute many quantities of interest straightforwardly and reproduce previous results obtained via less trivial methods. Let us first summarize the results we derive.

We compute corrections to the semiclassical spectral density $\rho_0(E)$ reproducing the leading non-perturbative correction to $\rho(E)$ as described, for example, in sec. 5 and appendix A of [35].

We compute the two-level correlation between eigenvalues $\rho(E_1, E_2)$ and reproduce the ramp and plateau contributions to the Spectral Form Factor in the limit $|E_1 - E_2| \ll 1$.

We compute the spectrum of Renyi entropies for an arbitrary interval bipartition of the eigenvalues $S_n(E_1, E_2)$ which generalises the results of [277] to an arbitrary spectral curve $\rho_0(E)$ and reproduces their results in the appropriate limit.

Finally we compute the $n < N$ eigenvalue reduced density matrix obtained by integrating out $(N - n)$ eigenvalues.

Let us start by defining a new coordinate X :

$$X(x) = \int_{x_L}^x \frac{dx'}{\pi \rho_0(x')}, \quad dX = \frac{dx}{\pi \rho_0(x)} \quad (4.3.1)$$

such that the metric $g_{ab}(x)$ becomes conformally flat:

$$ds^2 = \pi^2 \rho_0(x)^2 (dX^2 + dt^2). \quad (4.3.2)$$

The domain of the coordinate X is $X \in (0, L)$, where $L \equiv X(x_R)$ is given by the mapping the right boundary point. Since $\pi \rho_0(x)$ is the Fermi velocity, we can think of the coordinate $X(x)$ as the time it takes for an eigenvalue to go from the boundary x_L to the point x .

4.3.1 Spectral correlations

We start by considering correlations of spectral densities $\langle \rho(E) \rangle$ and $\langle \rho(E_1)\rho(E_2) \rangle$. The spectral correlations in RMT are computed by averaging the discrete density $\rho(E)$ over the ensemble of random matrices H :

$$\rho(E) \equiv \sum_{i=1}^N \delta(E - E_i) \rightarrow \langle \rho(E) \rangle_H = \int dH e^{\text{tr}V(H)} \rho(E). \quad (4.3.3)$$

In Matrix Quantum Mechanics the average over random matrices becomes a quantum expectation value of the density operator $\hat{\rho}(E)$:

$$\hat{\rho}(E) \equiv \Psi^\dagger(E)\Psi(E) \rightarrow \langle \hat{\rho}(E) \rangle. \quad (4.3.4)$$

The n -eigenvalue correlation is then given by the n -point correlation function of the density operator. The quantum hydrodynamics effective theory allows us to easily compute these density correlations [302] in terms of free CFT correlation functions. We are able to reproduce the leading non-perturbative corrections to $\langle \rho(E) \rangle$ and $\langle \rho(E_1)\rho(E_2) \rangle$ discussed in sec. 5 of [35]. These matrix integrals are equal to ground state correlations in matrix quantum mechanics at equal times.

¹⁰ We start by expanding the density operator $\hat{\rho}(x)$ into CFT operators:

$$\hat{\rho}(x, t) \approx \rho_0(x)\text{Id} + \frac{\partial_x h(x, t)}{2\pi} + \sum_{p=1}^{\infty} \left(A_{\rho, \mathcal{V}_{p,0}} \mathcal{V}_{p,0}(x, t) + A_{\rho, \mathcal{V}_{-p,0}} \mathcal{V}_{-p,0}(x, t) \right). \quad (4.3.5)$$

The first two operators follow directly from the construction of the effective theory since $\rho_0(x)$ is the saddle point value of the density and the linear variation around the saddle is $\delta\rho \equiv \frac{\partial_x h(x, t)}{2\pi}$. The expansion of $\hat{\rho}$ only includes vertex operators $\mathcal{V}_{p,q}$ with $q = 0$ since $\hat{\rho}$ does not change the total number of eigenvalues. Keeping only the most relevant operators in the expansion we have:

$$\hat{\rho}(x, t) \approx \rho_0(x)\text{Id} + \frac{\partial_x h(x, t)}{2\pi} + A_{\rho, \mathcal{V}_{1,0}} \mathcal{V}_{1,0}(x, t) + A_{\rho, \mathcal{V}_{-1,0}} \mathcal{V}_{-1,0}(x, t). \quad (4.3.6)$$

The coefficients $A_{\rho, \mathcal{V}_{\pm 1,0}}$ are naturally dimensionless since both $\hat{\rho}$ and $\mathcal{V}_{\pm 1,0}$ have scaling dimension $\Delta = 1$.

They are given by the following expression

$$A_{\rho, \mathcal{V}_{\pm 1,0}} = \frac{1}{2\pi} e^{\pm 2\pi i \theta(x)}, \quad \theta(x) = \int_0^x \rho_0(x') dx' - \frac{1}{2}. \quad (4.3.7)$$

¹⁰For the reader interested in the $c = 1$ matrix model, the exact density correlations can be written in terms of hypergeometric functions and evaluated numerically [267, 298].

The absolute value $|A_{\rho, \mathcal{V}_{\pm 1, 0}}| = \frac{1}{2\pi}$ can be obtained exactly from Bethe-Ansatz form factors (see appendix B of [302]). The phase $\theta(x)$ is a WKB phase.

Non perturbative corrections to density of eigenvalues $\langle \rho(E) \rangle$

The quantum hydrodynamical fluctuations will give the leading non-perturbative corrections to the semiclassical spectral density $\rho_0(x)$. These corrections produce oscillations on top of the semiclassical density which, from the fermionic point of view, can be identified as Friedel oscillations. Let us now compute $\langle \hat{\rho}(x) \rangle$ by taking the expectation value of the previous expression for $\hat{\rho}$. We have that $\langle \partial_x h \rangle = 0$ due to \mathbb{Z}_2 symmetry. The vertex operator VEV is obtained again by first changing coordinates $X(x)$ so the metric is conformally flat, performing a Weyl transformation to go to flat space and using a conformal transformation $w(z) = e^{i\frac{\pi}{L}z}$ to map the strip to the upper half plane \mathbb{H} :

$$\langle \mathcal{V}_{\pm 1, 0}(z) \rangle_g = (\pi \rho_0(x))^{-1} \langle \mathcal{V}_{\pm 1, 0}(z) \rangle_{strip} = (\pi \rho_0(x))^{-1} \left| \frac{dw}{dz} \right| \langle \mathcal{V}_{\pm 1, 0}(w(z)) \rangle_{\mathbb{H}}. \quad (4.3.8)$$

The expectation value on the upper half plane can be computed by the method of images and is given by

$$\langle \mathcal{V}_{\pm 1, 0}(w(z)) \rangle_{\mathbb{H}} = e^{\frac{1}{2}G_D(w)} \quad G_D(w) = -\log |w - \bar{w}|^2, \quad (4.3.9)$$

where $G_D(w)$ is the regularised Green's function at coincident points with Dirichlet boundary conditions. Setting $t = 0$ in $z = X + it$ and using the mapping $w(z) = e^{i\frac{\pi}{L}z}$ we have:

$$G_D(X) = -\log \left| 2 \sin \left(\frac{\pi}{L} X \right) \right|^2. \quad (4.3.10)$$

We arrive at the following expression for $\langle \hat{\rho} \rangle$:

$$\langle \hat{\rho}(x) \rangle = \rho_0(x) - \frac{\cos(2\pi \int^x \rho_0(x') dx')}{2\pi L \rho_0(x) \sin\left(\frac{\pi X}{L}\right)}. \quad (4.3.11)$$

As a consistency check, this expression precisely matches with the large N limit of the exact solution of the Gaussian matrix model, given by Hermite polynomials.¹¹ As a function of the eigenvalues E the density is:

$$\langle \hat{\rho}(E) \rangle = \rho_0(E) - \frac{\cos\left(2\pi \int^E \rho_0(E') dE'\right)}{2\pi L \rho_0(E) \sin\left(\frac{1}{L} \int^E \frac{dE'}{\rho_0(E')}\right)}, \quad (4.3.12)$$

¹¹For the GUE the map to free fermions in a harmonic potential is actually exact [312], since the probability density $|\psi_N(x_1, \dots, x_N)|^2$ is equal to the GUE joint eigenvalue probability density $|\psi_N(x_1, \dots, x_N)|^2 = \rho_{GUE}(x_1, \dots, x_N)$.

This gives the first quantum correction to the spectral density $\rho(E)$. In a double scaled matrix model, where $L = \infty$, we have:

$$\langle \hat{\rho}(E) \rangle = \rho_0(E) - \frac{\cos\left(2\pi \int^E \rho_0(E') dE'\right)}{2\pi \rho_0(E) \left(\int^E \frac{dE'}{\rho_0(E')}\right)}, \quad (4.3.13)$$

This is a non-perturbative correction to the density of states since it is of the form $\cos(e^{S_0}) = e^{ie^{S_0}}$. It reproduces the leading non-perturbative correction to the density of states.

As an example, take the Airy spectral curve $\rho_0(E) = \sqrt{E}$ for which we obtain:

$$\langle \hat{\rho}(E) \rangle = \rho_0(E) - \frac{\cos\left(2\pi \int^E \rho_0(E') dE'\right)}{4\pi E}, \quad (4.3.14)$$

this is exactly the expression in eq. (155) of [35].

The ramp and plateau in $\langle \rho(E_1)\rho(E_2) \rangle$

We can now compute the two-point function of the density of eigenvalues $\langle \hat{\rho}(E_1)\hat{\rho}(E_2) \rangle$. One has to multiply the expansions for the density operators and take the expectation value. Two point functions of vertex operators and the height field are entirely determined in terms of the Green function with Dirichlet boundary conditions $G_D(w_1, w_2)$ on \mathbb{H} which is given by:

$$G_D(w_1, w_2) = \log \left| \frac{w_1 - w_2}{w_1 - \bar{w}_2} \right|^2. \quad (4.3.15)$$

Evaluating it at equal times $t_1 = t_2 = 0$ and using the mapping $w(z) = e^{i\frac{\pi}{L}z}$ we have:

$$G_D(\bar{X}_1, \bar{X}_2) = \log \left| \frac{\sin\left(\frac{\bar{X}_1 - \bar{X}_2}{2}\right)}{\sin\left(\frac{\bar{X}_1 + \bar{X}_2}{2}\right)} \right|^2, \quad \bar{X} = \frac{\pi}{L}X. \quad (4.3.16)$$

The two-point correlation of the spectral density is given by:

$$\begin{aligned} \langle \hat{\rho}(x_1)\hat{\rho}(x_2) \rangle_c = \frac{1}{\pi^2 \rho_0(x_1)\rho_0(x_2)} & \left\{ - \frac{\partial_{\bar{X}_1} \partial_{\bar{X}_2} G_D(\bar{X}_1, \bar{X}_2)}{4\pi^2} + \right. \\ & \left[\partial_{\bar{X}_1} G_D(\bar{X}_1, \bar{X}_2) \sin(2\pi\theta(x_2)) e^{\frac{1}{2}G_D(X_2)} + (\bar{X}_1 \leftrightarrow \bar{X}_2) \right] + \\ & e^{\frac{1}{2}(G_D(X_1)+G_D(X_2))} \left(e^{G_D(\bar{X}_1, \bar{X}_2)} - 1 \right) \cos [2\pi(\theta(x_1) + \theta(x_2))] + \\ & \left. e^{\frac{1}{2}(G_D(X_1)+G_D(X_2))} \left(e^{-G_D(\bar{X}_1, \bar{X}_2)} - 1 \right) \cos [2\pi(\theta(x_1) - \theta(x_2))] \right\}. \end{aligned} \quad (4.3.17)$$

The derivatives appearing in the expression are the following:

$$\partial_{\bar{X}_1} G_D(\bar{X}_1, \bar{X}_2) = \left[\cot\left(\frac{\pi(X_1 - X_2)}{2L}\right) - \cot\left(\frac{\pi(X_1 + X_2)}{2L}\right) \right] \quad (4.3.18)$$

$$\partial_{\bar{X}_2} G_D(\bar{X}_1, \bar{X}_2) = - \left[\cot\left(\frac{\pi(X_1 - X_2)}{2L}\right) + \cot\left(\frac{\pi(X_1 + X_2)}{2L}\right) \right] \quad (4.3.19)$$

$$\partial_{\bar{X}_1} \partial_{\bar{X}_2} G^D(\bar{X}_1, \bar{X}_2) = \frac{1}{2} \left[\sin^{-2}\left(\frac{\pi(X_1 - X_2)}{2L}\right) + \sin^{-2}\left(\frac{\pi(X_1 + X_2)}{2L}\right) \right]. \quad (4.3.20)$$

For finite L it is enough to substitute the expressions for the Green functions and their derivatives, which we won't do explicitly. For a double scaled matrix model where $L \rightarrow \infty$ we have:

$$\begin{aligned} \langle \hat{\rho}(x_1) \hat{\rho}(x_2) \rangle = & \frac{1}{2\pi^2} \left(\frac{1}{\pi^2 \rho_0(x_1) \rho_0(x_2)} \right) \left\{ - \left(\frac{1}{(X_1 - X_2)^2} + \frac{1}{(X_1 + X_2)^2} \right) + \right. \\ & \left[\frac{\sin(2\pi\theta(x_2))}{X_2} \left(\frac{1}{(X_1 - X_2)} - \frac{1}{(X_1 + X_2)} \right) + (X_1 \leftrightarrow X_2) \right] + \\ & \left. \left(- \frac{\cos(2\pi(\theta(x_1) + \theta(x_2)))}{(X_1 + X_2)^2} + \frac{\cos(2\pi(\theta(x_1) - \theta(x_2)))}{(X_1 - X_2)^2} \right) \right\}. \end{aligned} \quad (4.3.21)$$

Approximating the integral $\int_0^E \frac{1}{\rho_0(E')} \approx \frac{E}{\rho_0(E)}$ we have that $X \approx \frac{E}{\pi \rho_0(E)}$. Moreover, in the limit where $|E_2 - E_1| \ll 1$ we can write $\rho_0(E_1) = \rho_0(E_2) = \rho_0(E)$ where $E = \frac{E_1 + E_2}{2}$. We have then:

$$\begin{aligned} \langle \hat{\rho}(E_1) \hat{\rho}(E_2) \rangle = & \frac{1}{2\pi^2} \left\{ - \left(\frac{1}{(E_1 - E_2)^2} + \frac{1}{(E_1 + E_2)^2} \right) + \right. \\ & \frac{1}{(E_1 - E_2)} \left(\frac{\cos\left(2\pi \int^{E_2} \rho_0(E') dE'\right)}{E_2} - \frac{\cos\left(2\pi \int^{E_1} \rho_0(E') dE'\right)}{E_1} \right) + \\ & \left. \left(- \frac{\cos\left(2\pi \left(\int^{E_2} \rho_0(E') dE' + \int^{E_1} \rho_0(E') dE' \right)\right)}{(E_1 + E_2)^2} + \frac{\cos\left(2\pi \int_{E_1}^{E_2} \rho_0(E') dE'\right)}{(E_1 - E_2)^2} \right) \right\}. \end{aligned} \quad (4.3.22)$$

For $|E_2 - E_1| \ll 1$ we recover the well known Sine kernel expression for the two-point

correlation of eigenvalues in Random Matrix Theory:

$$\begin{aligned} \langle \hat{\rho}(E_1)\hat{\rho}(E_2) \rangle &= -\frac{1}{2\pi^2} \frac{1}{(E_1 - E_2)^2} + \frac{1}{2\pi^2} \frac{\cos\left(2\pi \int_{E_1}^{E_2} \rho_0(E')dE'\right)}{(E_1 - E_2)^2} + \text{reg} \\ &= -\frac{1}{\pi^2} \frac{\sin^2\left(\pi \int_{E_1}^{E_2} \rho_0(E')dE'\right)}{(E_1 - E_2)^2}. \end{aligned} \quad (4.3.23)$$

We have reproduced the known matrix integral expressions for $\langle \rho(E) \rangle$ and $\langle \rho(E_1)\rho(E_2) \rangle$ by considering equal time quantum expectation values of the eigenvalue density operator $\hat{\rho}(E)$ in Matrix Quantum Mechanics. This shows that, in this sense, we can think of a matrix integral as a fixed time instance of a corresponding quantum mechanical theory of matrices. The statistical fluctuations given by integrating over an ensemble of matrices can now be understood as quantum fluctuations of a single matrix in the large N limit.

4.3.2 Entanglement entropy

An observable present in MQM that has no analogue in RMT is the entanglement between eigenvalues. Since the eigenvalues are quantum mechanical with a collective wavefunction $\Psi_N(E_1, \dots, E_N)$ we can consider the entanglement entropy for a bipartition of eigenvalue space. Thanks to the CFT description of the hydrodynamical fluctuations we can compute the entanglement entropy using the universal Cardy-Calabrese formula [313]. Thus we don't need to map any microscopic operators to the effective theory in this case. The Renyi entropies for a subsystem A are defined as:

$$S_n \equiv \frac{1}{1-n} \log(\text{Tr}(\rho_A^n)), \quad (4.3.24)$$

where ρ_A is the reduced density matrix.

Half-space bipartition $(0, E) \cup (E, \infty)$

We consider the Renyi entropies for a bipartition $(x_L, x) \cup (x, x_R)$ which in X coordinates is $(0, X(x)) \cup (X(x), L)$. In a 2D CFT the Renyi entropies for such a bipartition can be computed by the expectation value of a single twist field [313]:

$$S_n(x) = \frac{1}{1-n} \log\left(\epsilon^{\Delta_n} \langle \mathcal{T}_n(x, t=0) \rangle\right), \quad (4.3.25)$$

where Δ_n is the dimension of the twist operator:

$$\Delta_n = \frac{c}{12} \left(n - \frac{1}{n}\right), \quad (4.3.26)$$

and ϵ is a UV cut-off for the formally divergent entanglement entropy. The cut-off is known to encode the divergent amount of short-range entanglement in continuum Quantum Field Theories. This divergence is an issue when attempting to give a rigorous definition of entanglement entropy in QFT (see [314, 315]). However, as illustrated in section 2, in the effective hydrodynamical description we have a microscopic length scale, the mean particle spacing $d(x) = \rho_0^{-1}(x)$ which gives a natural UV cut-off $\Lambda_{UV} = \rho_0(x)$. Since the system is inhomogeneous, the UV cutoff scale is position dependent and we have:

$$\epsilon(x) = \frac{\epsilon_0}{\rho_0(x)}, \quad (4.3.27)$$

where ϵ_0 is a dimensionless constant and the cutoff is evaluated at the boundary point x of the bipartition. We have then a UV-finite expression for the entanglement entropy. This can be interpreted as a consequence of having a finite density of eigenvalues $\rho_0(E)$ in RMT due to eigenvalue repulsion. Equivalently, it is a consequence of having a non-zero VEV for the density $\rho(x) = \langle \Psi^\dagger \Psi \rangle$. In the next section we will see that in the duality between the $c = 1$ matrix model and two-dimensional string theory the finiteness of the entanglement entropy can be interpreted as due to stringy effects [277]. String theory, as expected, regulates the UV-divergence to give a finite answer for the entropy.

We work in complex coordinates $z = X + it$ defined on the infinite strip $(0, L) \times \mathbb{R}$. The metric in complex coordinates is $ds^2 = \pi^2 \rho_0(x)^2 dz d\bar{z}$. We can perform a Weyl transformation $g_{ab} \rightarrow (\pi \rho_0(x))^{-2} g_{ab}$ to go to flat space. Under the Weyl transformation the twist operator scales as $\mathcal{T}_n \rightarrow (\pi \rho_0(x))^{-\Delta_n} \mathcal{T}_n$. Next we map the z -strip to the upper half-plane \mathbb{H} via a conformal transformation $g(z) = e^{i\pi \frac{z}{L}}$. Under this map the twist field transforms as $\mathcal{T}_n(z)_{\text{Strip}} \rightarrow \left| \frac{dg(z)}{dz} \right|^{\Delta_n} \mathcal{T}_n(g(z))_{\mathbb{H}}$. The last ingredient is the expectation value of the twist field on the upper-half plane which is $\langle \mathcal{T}_n(g(z)) \rangle_{\mathbb{H}} = (\text{Im}g(z))^{-\Delta_n}$. We combine everything together to arrive at:

$$S_n(x) = \frac{n+1}{12n} \log \left[\frac{\Omega(x)}{\epsilon(x)} \left| \frac{dg(z)}{dz} \right|^{-1} \text{Im}g(z) \right], \quad (4.3.28)$$

where

$$\left| \frac{dg(z)}{dz} \right| = \frac{\pi}{L}, \quad (4.3.29)$$

$$\text{Im}g(z) = \sin\left(\frac{\pi X}{L}\right). \quad (4.3.30)$$

We then obtain the entanglement entropy for a bipartition $(x_L, x) \cup (x, x_R)$ of eigenvalues in a model with spectral density $\rho_0(x)$:

$$S_n = \frac{n+1}{12n} \log \left(\frac{L}{\pi} \rho_0^2(x) \sin\left(\frac{\pi X(x)}{L}\right) \right) + \text{const.} \quad (4.3.31)$$

Writing this explicitly in the eigenvalue coordinate $x = E$ for a bipartition $(E_L, E) \cup (E, E_R)$ we have:

$$S_n(E) = \frac{n+1}{12n} \log \left(\frac{L}{\pi} \rho_0^2(E) \sin \left(\frac{1}{L} \int_{E_L}^E \frac{dE'}{\rho_0(E')} \right) \right) + \text{const.} \quad (4.3.32)$$

We see that, due to the term $\sin\left(\frac{\pi}{L}X\right)$ with $X \in (0, L)$, the entropy has the expected Page-curve behaviour as a function of the bipartition point E , increasing until a turning point and decreasing afterwards. This is a consequence of unitarity in a system with a finite number of states $N = \int \rho_0(E)dE$ and does not survive the double scaling limit. As $N \rightarrow \infty$ in the double scaling limit we lose unitarity of the entanglement entropy $S_n(E)$ and we have information loss. There are many examples of the tension between unitarity and the large N limit [58, 103, 314, 316–318].

For a double scaled matrix model we have that $x_R = \infty$ and thus $L = \infty$. Expanding the sine we have that the L dependence drops out and we obtain:

$$S_n(E) = \frac{n+1}{12n} \log \left(\frac{1}{\pi} \rho_0^2(E) \int_{E_L}^E \frac{dE'}{\rho_0(E')} \right) + \text{const.} \quad (4.3.33)$$

Approximating the integral by assuming an almost constant density $\int_{E_L}^E \frac{dE'}{\rho_0(E')} \approx \frac{E-E_L}{\rho_0(E)}$ we obtain:

$$S_n(E) \approx \frac{n+1}{12n} \log \left(\frac{1}{\pi} \rho_0(E)(E - E_L) \right) + \text{const.} \quad (4.3.34)$$

We see that the entanglement entropy presents a leading term proportional to the micro-canonical entropy:

$$S_n(E) \propto \log(\rho_0(E)) = S_0(E). \quad (4.3.35)$$

This shows that the entanglement entropy contains a large amount of short-range entanglement coming from neighbouring eigenvalues separated by the bipartition, since the entropy $S_0(E)$ counts the number of states in a window $(E - dE, E + dE)$.

The second term $S_n(E) \propto \log(E - E_L)$ is the usual Cardy-Calabrese behaviour in two dimensions. Note that we should not extrapolate this result to the edge of the spectral density $E \sim E_L$. In RMT, there are distinct 'bulk' and 'edge' limits for the spectral density with different universal kernels describing them [234] and we should not extrapolate a bulk result to the edge region. Hydrodynamics requires the derivative of the density to be small, thus it describes the 'bulk' region.

Interval bipartition (E_1, E_2)

We now calculate the Renyi entropies for a subregion $A = (X_1, X_2) \in (0, L)$ equal to $A = (x_1, x_2) \in (x_L, x_R)$, where $X_{1,2} \equiv X(x_{1,2})$. The Renyi entropies are given by the two

point function of the twist field at the extrema of the interval:

$$S_n \propto \log\left(\epsilon^{\Delta_n} \langle \mathcal{T}_n(z_1) \tilde{\mathcal{T}}_n(z_2) \rangle\right). \quad (4.3.36)$$

$\tilde{\mathcal{T}}_n$ is the conjugate twist field [313]. The cutoff now is $\epsilon = \frac{\epsilon_0}{\rho(x_1)\rho(x_2)}$. Repeating the same manipulations as before we can reduce the computation to a correlation on the flat upper half plane \mathbb{H} :

$$\begin{aligned} \langle \mathcal{T}_n(z_1) \tilde{\mathcal{T}}_n(z_2) \rangle_{\text{strip}} &= \left(\Omega(x_1) \left| \frac{dg(z_1)}{dz_1} \right| \right)^{\Delta_n} \left(\Omega(x_2) \left| \frac{dg(z_2)}{dz_2} \right| \right)^{\Delta_n} \\ &\times \langle \mathcal{T}_n(g(z_1)) \tilde{\mathcal{T}}_n(g(z_2)) \rangle_{\mathbb{H}}. \end{aligned} \quad (4.3.37)$$

The two point function on the upper half plane \mathbb{H} is a priori a non trivial calculation corresponding to a four-point function on the plane. In the case of a free boson theory it is known [313, 319, 320] and we have:

$$\langle \mathcal{T}_n(z_1) \tilde{\mathcal{T}}_n(z_2) \rangle = \left[\Omega(x_1) \left| \frac{dg(z_1)}{dz_1} \right| \Omega(x_2) \left| \frac{dg(z_2)}{dz_2} \right| \frac{|g^*(z_1) - g(z_2)|^2}{\text{Im}g(z_1)\text{Im}g(z_2)|g(z_1) - g(z_2)|^2} \right]^{\Delta_n}. \quad (4.3.38)$$

The Renyi entropies are:

$$S_n \propto \log \left[\left(\frac{L}{\pi} \right)^2 \rho_0^2(x_1) \rho_0^2(x_2) \sin\left(\pi \frac{X_1}{L}\right) \sin\left(\pi \frac{X_2}{L}\right) \frac{|g(z_1) - g(z_2)|^2}{|g^*(z_1) - g(z_2)|^2} \right], \quad (4.3.39)$$

where the last term is equal to:

$$\frac{|g(z_1) - g(z_2)|^2}{|g^*(z_1) - g(z_2)|^2} = \frac{1 - \cos\left(\frac{\pi}{L}(X_2 - X_1)\right)}{1 - \cos\left(\frac{\pi}{L}(X_1 + X_2)\right)} = \frac{\sin^2\left(\frac{\pi}{2L}(X_2 - X_1)\right)}{\sin^2\left(\frac{\pi}{2L}(X_1 + X_2)\right)}. \quad (4.3.40)$$

The final result is then:

$$S_n(x_1, x_2) = \frac{n+1}{12n} \log \left(\left(\frac{L}{\pi} \right)^2 \rho_0^2(x_1) \rho_0^2(x_2) \sin\left(\pi \frac{X_1}{L}\right) \sin\left(\pi \frac{X_2}{L}\right) \frac{\sin^2\left(\frac{\pi}{2L}(X_2 - X_1)\right)}{\sin^2\left(\frac{\pi}{2L}(X_1 + X_2)\right)} \right) + \text{const.} \quad (4.3.41)$$

Writing this in terms of the eigenvalues (E_1, E_2) we have:

$$\begin{aligned} S_n(x_1, x_2) &= \frac{n+1}{12n} \log \left[\left(\frac{L}{\pi} \right)^2 \rho_0^2(E_1) \rho_0^2(E_2) \sin\left(\frac{1}{L} \int_{E_L}^{E_1} \frac{dE'}{\rho_0(E')}\right) \sin\left(\frac{1}{L} \int_{E_L}^{E_2} \frac{dE'}{\rho_0(E')}\right) \right. \\ &\quad \left. \times \frac{\sin^2\left(\frac{1}{2L} \int_{E_1}^{E_2} \frac{dE'}{\rho_0(E')}\right)}{\sin^2\left(\frac{1}{2L} \left(2 \int_{E_L}^{E_1} \frac{dE'}{\rho_0(E')} + \int_{E_1}^{E_2} \frac{dE'}{\rho_0(E')} \right)\right)} \right] + \text{const.} \end{aligned} \quad (4.3.42)$$

For a double scaled matrix model $L = \infty$ we have:

$$S_n(x_1, x_2) = \frac{n+1}{12n} \log \left(\rho_0^2(x_1) \rho_0^2(x_2) X_1 X_2 \frac{(X_2 - X_1)^2}{(X_1 + X_2)^2} \right) + \text{const}, \quad (4.3.43)$$

which in terms of the eigenvalues is:

$$S_n(E_1, E_2) = \frac{n+1}{12n} \log \left[\left(\frac{1}{\pi} \right)^2 \rho_0^2(E_1) \rho_0^2(E_2) \left(\int_{E_L}^{E_1} \frac{dE'}{\rho_0(E')} \right) \left(\int_{E_L}^{E_2} \frac{dE'}{\rho_0(E')} \right) \right. \\ \left. \times \frac{\left(\int_{E_1}^{E_2} \frac{dE'}{\rho_0(E')} \right)^2}{\left(2 \int_{E_L}^{E_1} \frac{dE'}{\rho_0(E')} + \int_{E_1}^{E_2} \frac{dE'}{\rho_0(E')} \right)^2} \right] + \text{const}. \quad (4.3.44)$$

Approximating the integrals by assuming an almost constant density $\int_{E_L}^E \frac{dE'}{\rho_0(E')} \approx \frac{E-E_L}{\rho_0(E)}$ we obtain:

$$S_n(E_1, E_2) \approx \frac{n+1}{12n} \log \left[\left(\frac{1}{\pi} \right)^2 \rho_0(E_1) \rho_0(E_2) (E_1 - E_L) (E_2 - E_L) \frac{(E_2 - E_1)^2}{((E_2 + E_1) - 2E_L)^2} \right] \quad (4.3.45)$$

In the limit of RMT univiersality where $(E_1 - E_2) \ll 1$ we obtain the simple expression:

$$S_n(E_1, E_2) \approx \frac{n+1}{12n} \log \left(\frac{\rho_0^2(E) (E_1 - E_2)^2}{4\pi^2} \right) \quad (4.3.46)$$

where $E = \frac{E_1 + E_2}{2}$ is the average energy. We see again that there is a leading contribution proportional to the average microcanonical entropy $S_n \propto S_0(E) = \log(\rho_0(E))$ of the interval (E_1, E_2) . The entanglement entropy derived via the hydrodynamic CFT has been checked against numerical simulations in [288] for a double well potential finding excellent agreement.

Emergence of spacetime in 2D string theory

Two-dimensional string theory is a non-critical bosonic string theory in $D = 2$ flat spacetime with a linear dilaton background and a massless tachyon. The worldsheet is a Liouville CFT with $c_L = 25$ and a free boson $c_M = 1$ which cancel the ghost central charge $c_g = -26$ [236, 237]. The low energy effective action is [321, 322]:

$$S = \frac{1}{2} \int dt dx \sqrt{-g} e^{-2\Phi} \left(\frac{R}{2} + 2(\partial\Phi)^2 + 8 - (\partial T)^2 + 4T^2 - 2V(T) \right), \quad (4.3.47)$$

where $V(T)$ is a potential for the tachyon. The background has a tachyon condensate with a free parameter $\bar{\mu}$ which determines the effective string coupling $g_{eff} \sim \bar{\mu}^{-1}$, thus we have a perturbative string theory for $\mu \gg 1$. Equivalently, $\bar{\mu}$ is the cosmological constant of the

worldsheet Liouville theory. Two-dimensional string theory is dual to a theory of matrix quantum mechanics with an inverted oscillator potential $V(x) = -\frac{x^2}{2}$ and chemical potential $\mu = -\bar{\mu} < 0$. The potential arises from a double scaling limit of a potential $U(x) = -\frac{x^2}{2} + gx^3$ by taking $N \rightarrow \infty$ and $g \rightarrow 0$ while keeping $\mu = -N\epsilon_F$ fixed, where ϵ_F is the Fermi energy of the fermions filling up the well in the potential. This zooms into the vicinity of the maximum of the potential which gives the universal critical behaviour. The supersymmetric string is described instead by a double well potential [323]. In the hydrodynamic approach it is perfectly possible to treat both the general case with potential $U(X)$ but we will focus on the inverted oscillator. The leading density of eigenvalues is given by 4.2.36:

$$\rho_0(x) = \frac{1}{\pi} \sqrt{2(-V(x) - \bar{\mu})} = \frac{1}{\pi} \sqrt{x^2 - 2\bar{\mu}}. \quad (4.3.48)$$

It has a left edge $x_L = \sqrt{2\bar{\mu}}$ and extends to infinity as it is a double scaled model. The density of states gives the geometry of the Fermi surface on which the 2D CFT (4.2.34) describing the quantum hydrodynamical fluctuations lives:

$$ds^2 = (x^2 - 2\bar{\mu})d\tau^2 + dx^2. \quad (4.3.49)$$

The coordinate transformation which renders the metric conformally flat is explicitly:

$$X(x) = \int_{\sqrt{2\bar{\mu}}}^x \frac{d\lambda}{\sqrt{-2\bar{\mu} + \lambda^2}} = \cosh^{-1} \left(\frac{x}{\sqrt{2\bar{\mu}}} \right) = \log \left(\frac{x + \sqrt{x^2 - 2\bar{\mu}}}{\sqrt{2\bar{\mu}}} \right), \quad (4.3.50)$$

It was argued first in [293] that the transformation $X(x)$ gives, at the semiclassical level, the map from the eigenvalue-space x to the string theory spacetime X since physically it is the "time-of-flight", meaning the WKB time it takes for an eigenvalue to go from x_L to a point x . Then in [277] a consistency argument identifying $X(x)$ with the map between the eigenvalues and the emergent spacetime was given based on entanglement. They computed the entanglement entropy of the eigenvalues using the techniques of [324] and the WKB wavefunctions. Then they argued that the entanglement entropy in spacetime should be given by the Cardy-Calabrese formula and found that the relation $X(x)$ produced the right matching. The hydrodynamic approach instead provides a natural and more constructive point of view: the geometry of the Fermi surface is dual to the spacetime geometry and the mapping $X(x)$ is simply the map between the two metrics. The Cardy-Calabrese formula follows immediately from the fact that the quantum hydrodynamical theory of the eigenvalues is a 2D CFT. We will now show explicitly the match with the entanglement entropy derived in [277]. We can invert eq. 4.3.50 to write the density $\rho_0(X)$ in spacetime coordinates:

$$\pi^2 \rho_0^2(x) = -2\mu + x^2 = 2\mu(-1 + \cosh^2(X)) = 2\mu \sinh^2(X). \quad (4.3.51)$$

One can define a string coupling given by:

$$\frac{1}{\tilde{g}_s(X)} \equiv 2\bar{\mu} \sinh^2(X) \equiv \pi^2 \rho_0^2(X), \quad (4.3.52)$$

which at weak coupling $\bar{\mu} \gg 1$ is equal to the string coupling in the linear dilaton background $\tilde{g}_s(X) = \frac{g_s(X)}{2\mu}$. The entanglement entropy for a spacetime bipartition $(0, X)U(X, L)$ in two-dimensional string theory is then:

$$S_n(X) \Big|_{\text{2D String}} = \frac{n+1}{12n} \log \left[\frac{L}{\pi} \frac{1}{\tilde{g}_s(X)} \sin\left(\frac{\pi X}{L}\right) \right] + \text{const}, \quad (4.3.53)$$

while for a spacetime interval (X_1, X_2) it is given by:

$$S_n = \frac{n+1}{12n} \log \left(\left(\frac{L}{\pi}\right)^2 \frac{\sin\left(\pi \frac{X_1}{L}\right) \sin\left(\pi \frac{X_2}{L}\right) \sin^2\left(\frac{\pi}{2L}(X_2 - X_1)\right)}{\tilde{g}_s(X_1)\tilde{g}_s(X_2) \sin^2\left(\frac{\pi}{2L}(X_1 + X_2)\right)} \right) + \text{const}. \quad (4.3.54)$$

This reproduces the results of [277] when considering the Von Neumann entropy $n = 1$ and $L \rightarrow \infty$:

$$S_1 = \frac{1}{6} \log \left[\frac{X_1 X_2}{\tilde{g}_s(X_1)\tilde{g}_s(X_2)} \frac{(X_2 - X_1)^2}{(X_1 + X_2)^2} \right] + \text{const} = \frac{1}{3} \log \left(\frac{X_2 - X_1}{\sqrt{\tilde{g}_s(X_1)\tilde{g}_s(X_2)}} \right) + \frac{1}{6} \log \left(\frac{X_1 X_2}{(X_1 + X_2)^2} \right) + \text{const}. \quad (4.3.55)$$

Since they were working in the microscopic fermion field theory (4.2.11), they were able to determine the additive constant for $L \rightarrow \infty$. However the constant can depend on the system size L so we cannot use their results to fix the constant in the general L case. Often in in two-dimensional string theory a cut-off is introduced for the inverted oscillator potential which comes from the potential $U(x)$ before the double-scaling limit. The cut-off is at a distance $x_R \sim \frac{1}{g} \sim N$ so at large N it is effectively not there. Since our results are valid for finite L we can also probe the region close to the cut-off or directly do the computation for the potential $U(x)$.

4.3.3 Reduced density matrix for $n < N$ eigenvalues

The one-particle density matrix in the fermionic field theory 4.2.11 is computed by the two-point function of the fermion field [301, 325, 326]:

$$g_1(x, x') \equiv \langle \Psi^\dagger(x) \Psi(x') \rangle. \quad (4.3.56)$$

We have seen that the fermionic operators Ψ, Ψ^\dagger can be expanded as an infinite sum of CFT primary operators and their descendants consistent with the symmetries. In particular,

Ψ, Ψ^\dagger corresponds to vertex operators $\mathcal{V}_{p,\pm 1}$ and their derivatives. Considering only the most relevant most relevant operator we have:

$$\Psi^\dagger(x) \approx A_{\Psi^\dagger, \mathcal{V}_{0,1}} \rho_0(x)^{1/4} \mathcal{V}_{0,1}(x), \quad (4.3.57)$$

where the dimensionless coefficient is given by

$$|A_{\Psi^\dagger, \mathcal{V}_{0,1}}|^2 = \frac{G^4(3/2)}{\sqrt{2\pi}}. \quad (4.3.58)$$

The one-eigenvalue reduced density matrix is then given, at leading order in the hydrodynamic effective theory simply by a two-point function of vertex operators:

$$g_1(x, x') = |A_{\psi, \mathcal{V}_{0,-1}}|^2 \rho_0(x)^{1/4} \rho_0(x')^{1/4} \langle \mathcal{V}_{0,1}(x), \mathcal{V}_{0,-1}(x') \rangle_{\text{CFT}}. \quad (4.3.59)$$

We make use of coordinates $X(x)$ such that the geometry is conformally flat. We map the correlator to the flat space infinite strip:

$$\langle \mathcal{V}_{0,1}(x), \mathcal{V}_{0,1}(x') \rangle_g = \left(\frac{dX}{dx} \right)^{1/4} \left(\frac{dX}{dx} \right)^{1/4} \langle \mathcal{V}_{0,1}(X), \mathcal{V}_{0,1}(X') \rangle_{\text{flat}}. \quad (4.3.60)$$

The two-point function of vertex operators on an infinite strip $(0, L) \times \mathbb{R}$ is known to be:

$$\langle \mathcal{V}_{0,1}(X), \mathcal{V}_{0,1}(X') \rangle_{\text{CFT, flat}} = \frac{\left| \sin\left(\frac{\pi X}{L}\right) \sin\left(\frac{\pi X'}{L}\right) \right|^{1/4}}{\left| \frac{2L}{\pi} \sin\left(\frac{\pi(X-X')}{2L}\right) \sin\left(\frac{\pi(X+X')}{2L}\right) \right|^{1/2}}. \quad (4.3.61)$$

We arrive at the following result for the one-eigenvalue density matrix:

$$g_1(x, x') = \frac{|A_{\psi, \mathcal{V}_{0,-1}}|^2}{\sqrt{\pi}} \frac{\left(\sin\left(\frac{\pi X}{L}\right) \sin\left(\frac{\pi X'}{L}\right) \right)^{1/4}}{\left| \frac{2L}{\pi} \sin\left(\frac{\pi(X-X')}{2L}\right) \sin\left(\frac{\pi(X+X')}{2L}\right) \right|^{1/2}}. \quad (4.3.62)$$

We can easily generalize this result to obtain the n -eigenvalue density matrix:

$$g_n(\{x\}, \{x'\}) = \frac{|A_{\psi, \mathcal{V}_{0,-1}}|^{2n}}{\pi^{n/2}} \prod_{i=1}^n \left| \sin\left(\pi \frac{X_i}{L}\right) \sin\left(\pi \frac{X'_i}{L}\right) \right|^{\frac{1}{4}} \times \frac{\prod_{k < l} \left| \left(\frac{2L}{\pi}\right)^2 \sin\left(\pi \frac{(X_k - X_l)}{2L}\right) \sin\left(\pi \frac{(X_k + X_l)}{2L}\right) \sin\left(\pi \frac{(X'_k - X'_l)}{2L}\right) \sin\left(\pi \frac{(X'_k + X'_l)}{2L}\right) \right|^{1/2}}{\prod_{i,j} \left| \frac{2L}{\pi} \sin\left(\pi \frac{(X_i - X'_j)}{2L}\right) \sin\left(\pi \frac{(X_i + X'_j)}{2L}\right) \right|^{1/2}}. \quad (4.3.63)$$

For a double scaled matrix model $L = \infty$, the one-eigenvalue density matrix is given by:

$$g_1(x, x') = \frac{\sqrt{2}|A_{\psi, \nu_{0, -1}}|^2}{\sqrt{\pi}} \frac{|XX'|^{1/4}}{|(X - X')(X + X')|^{1/2}}, \quad (4.3.64)$$

writing it explicitly in terms of the eigenvalues we have:

$$g_1(E, E') = \frac{\sqrt{2}|A_{\psi, \nu_{0, -1}}|^2}{\sqrt{\pi}} \frac{\left| \left(\int_{E_L}^E \frac{dE''}{\rho_0(E'')} \right) \left(\int_{E_L}^{E'} \frac{dE''}{\rho_0(E'')} \right) \right|^{1/4}}{\left| \left(\int_E^{E'} \frac{dE''}{\rho_0(E'')} \right) \left(\int_{E_L}^E \frac{dE''}{\rho_0(E'')} + \int_{E_L}^{E'} \frac{dE''}{\rho_0(E'')} \right) \right|^{1/2}}, \quad (4.3.65)$$

Approximating the integrals by assuming an almost constant density $\int_{E_L}^E \frac{dE''}{\rho_0(E'')} \approx \frac{E - E_L}{\rho_0(E)}$ we obtain:

$$g_1(E, E') \approx \sqrt{2}|A_{\psi, \nu_{0, -1}}|^2 \frac{|\rho_0(E)\rho_0(E')(E - E_L)(E' - E_L)|^{1/4}}{|E' - E|^{1/2}|E + E' - 2E_L|^{1/2}}. \quad (4.3.66)$$

In the limit of RMT universality $|E - E'| \ll 1$ we have the simple expression:

$$g_1(E, E') \approx |A_{\psi, \nu_{0, -1}}|^2 \frac{|\rho_0(E)\rho_0(E')|^{1/4}}{|E - E'|^{1/2}}. \quad (4.3.67)$$

In the double scaling limit $L = \infty$ the n -eigenvalue density matrix is:

$$g_n(\{x\}, \{x'\}) = |A_{\psi, \nu_{0, -1}}|^{2n} \left(\frac{2}{\pi}\right)^{n/2} \prod_{i=1}^n |X_i X'_i|^{1/4} \times \frac{\prod_{k < l} |(X_k^2 - X_l^2)(X'_k{}^2 - X'_l{}^2)|^{1/2}}{\prod_{i,j} |(X_i^2 - X_j^2)|^{1/2}}. \quad (4.3.68)$$

The factors of L cancel exactly.

We recognize the Vandermonde determinant $\Delta(X^2) = \prod_{i < j} (X_i^2 - X_j^2)$ of the matrix $X_i^{2(j-1)}$:

$$g_n(\{x\}, \{x'\}) = |A_{\psi, \nu_{0, -1}}|^{2n} \left(\frac{2}{\pi}\right)^{n/2} \prod_{i=1}^n |X_i X'_i|^{1/4} \times \frac{|\Delta(X^2)\Delta(X'^2)|^{1/2}}{\prod_{i,j} |(X_i - X'_j)(X_i + X'_j)|^{1/2}}. \quad (4.3.69)$$

To obtain the expression in terms of the eigenvalues it is again enough to substitute $x_i = E_i$ and $X_i = \int_{E_L}^{E_i} \frac{dE}{\pi\rho_0(E)}$. Approximating the integrals by assuming an almost constant density

$\int_{E_L}^E \frac{dE'}{\rho_0(E')} \approx \frac{E-E_L}{\rho_0(E)}$ we obtain:

$$g_n(\{E\}, \{E'\}) \approx |A_{\psi, \nu_0, -1}|^{2n} 2^{n/2} \prod_{i=1}^n \left| \frac{(E_i - E_L)(E'_i - E_L)}{\rho_0(E_i)\rho_0(E'_i)} \right|^{1/4} \prod_{i,j} \left| \frac{(E_i - E_L)^2}{\rho_0(E_i)^2} - \frac{(E'_j - E_L)^2}{\rho_0(E'_j)^2} \right|^{-1/2} \\ \prod_{k<l} \left| \frac{(E_k - E_L)^2}{\rho_0(E_k)^2} - \frac{(E_l - E_L)^2}{\rho_0(E_l)^2} \right|^{1/2} \left| \frac{(E'_k - E_L)^2}{\rho_0(E'_k)^2} - \frac{(E'_l - E_L)^2}{\rho_0(E'_l)^2} \right|^{1/2}. \quad (4.3.70)$$

These expressions have been checked against numerical simulations performed via Density Matrix Renormalization Group (DMRG) methods for harmonic and double-well potentials in [301]. The hydrodynamic CFT accurately matches the numerical results already for $N = 15$ and improves as $N \gg 1$.

4.4 Open questions and future work

We conclude with several questions and possibilities for future work.

Universe field theory

It seems possible to think of Matrix Quantum Mechanics as a universe field theory for JT gravity. We can define an operator $\hat{Z}(\beta)$ given by:

$$\hat{Z}(\beta) = \int dE e^{-\beta E} \hat{\rho}(E). \quad (4.4.1)$$

This operator creates a spacetime with a boundary of length β and gives a realization of the operators acting on the Hilbert space of baby universes discussed in [99, 191, 327].

In particular, understanding the connection between the universe field theory defined by MQM and the one proposed in [99, 327] is an interesting prospect which we plan to investigate further. From this point of view, the time t in MQM corresponds to the time evolution $e^{i\mathcal{H}_{BU}t}$ generated by the third quantised Hamiltonian \mathcal{H}_{BU} on the baby universe Hilbert space. This Hamiltonian should then be identified with the Hamiltonian of the MQM in eq. 4.2.6.

Similarly to $\hat{Z}(\beta)$, the eigenvalue wavefunction $\psi_N(E_1 \dots E_N)$ and the reduced density matrix $g_n(E, E')$ describe the Wheeler–DeWitt wavefunction of universes with specified boundaries. It would be interesting to understand better the implications of the reduced density matrix in this context. In two-dimensional string theory the operators $\hat{Z}(\beta)$ are known as loop operators and their third quantised interpretation in the $c = 1$ matrix model has been discussed in [267].

Out-of-equilibrium spectral density $\rho_0(E, t)$

We have only considered fluctuations of the eigenvalues around an equilibrium spectral density $\rho_0(E)$ which is time independent. In MQM, the matrix $H(t)$ will evolve in time, thus it is natural to consider a time-dependent density $\rho_0(E, t)$. In the hydrodynamic description this corresponds to out-of-equilibrium physics. In two-dimensional string theory the time t corresponds to the time direction in target space [236]. It would be interesting to understand if there is a physical interpretation of time t in connection with JT gravity. This time variable is different from the time τ coming from analytical continuation of the Euclidean boundary circle $\beta \rightarrow \beta + i\tau$.

It would be interesting to understand out-of-equilibrium evolution of the density of eigenvalues in terms of bulk physics, as a toy model of black hole dynamics and evaporation. One could consider coupling the system to a bath or performing a quench and computing the entanglement entropy as a function of time to see if one obtains the desired Page curve behaviour.

The hydrodynamic approach could also be useful for studying time-dependent backgrounds in two-dimensional string theory as in [328]. More recently quantum quenches in the $c = 1$ matrix model and their string theory interpretation were considered in [329, 330].

Topological recursion in MQM

The duality between JT gravity and a matrix integral was established at all orders in $\frac{1}{N}$ thanks to topological recursion [35, 231, 331]. It would then be good to understand topological recursion from the point of view of MQM. In particular, the $\frac{1}{N} \sim \hbar$ corrections in MQM are given by higher orders in the WKB expansion of the eigenvalue wavefunction $\psi(E)$. It has been shown, for certain classes of spectral curves, that the WKB expansion of an associated quantum mechanical system satisfies topological recursion [332].¹² This connection between WKB and topological recursion might shed light on MQM and its *one-time-point* reduction to the matrix integral dual to JT gravity.

Finite temperature and non-singlet sector

We can consider matrix quantum mechanics at finite temperature by compactifying the time direction t with period $2\pi R$ (see secs. 8, 9 and 10 of [236]). It is well known that a Berezinskii–Kosterlitz–Thouless (BKT) phase transition takes place: for $R < R_{BKT}$ vortices

¹²The class of spectral curves for which this has been shown does not include JT gravity’s spectral curve but it does include the Airy case $\rho(E) = \sqrt{E}$.

condense and the non-singlet degrees of freedom dominate the free energy [294–297]. The physics of the non-singlet sector is very rich, involving 2D black holes and long strings [261, 333, 334]. Thus it would be interesting to understand the transition by incorporating vortices into the hydrodynamic effective theory. Moreover, at high temperatures $R \rightarrow 0$, fluctuations along the thermal circle are suppressed and we recover a 0-dimensional matrix integral with potential $V(H)$. If we consider the case of JT gravity, this proposes a different interpretation of the matrix integral dual to JT gravity as a high temperature limit of Matrix Quantum Mechanics.

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