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# Couplage et turbulence à l'interface océan-atmosphère

François Legeais

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# THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES

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*Mathématiques, Télécommunications, Informatique, Signal, Systèmes,  
Électronique*

Spécialité : *Mathématiques et leurs interactions*

Par

**François LEGAIS**

**Couplage et turbulence à l'interface océan-atmosphère**

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## Rapporteurs avant soutenance :

Pauline LAFITTE Professeur des Universités, Centrale-Supélec  
Carole ROSIER Professeur des Universités, Université du littoral côte d'opale

## Composition du Jury :

Président : Fabrice BETHUEL Professeur des Universités, Sorbonne Université

Examineurs : Pauline LAFITTE Professeur des Universités, Centrale-Supélec  
Loïc LE MARREC Maître de Conférences, Université de Rennes  
Julien MATHIAUD Professeur des Universités, Université de Rennes  
Carole ROSIER Professeur des Universités, Université du littoral côte d'opale

Dir. de thèse : Roger LEWANDOWSKI Professeur des Universités, Université de Rennes



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# PRÉAMBULE

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Ce manuscrit est divisé en deux parties. La première contenant les chapitres 2 et 3 traite de systèmes couplés de deux fluides, tant d'un point de vue théorique que numérique. La deuxième contenant les chapitres 4 et 5, propose l'étude de deux problèmes de couches limites scalaires issus de modèles de turbulence.

Pour arriver à modéliser des phénomènes aussi complexes que les interactions entre l'océan et l'atmosphère, il est nécessaire de s'intéresser à des "modèles jouet" qui réduisent drastiquement le nombre de difficultés. Plutôt que de considérer des dizaines de variables et d'équations différentes, on réduit l'étude à ce qui nous semble être un système simple relativement à la situation de départ, mais pertinent à étudier. L'intérêt étant que comprendre un tel modèle simplifié permettra de mieux appréhender le modèle plus complexes et plus en correspondance avec les observations physiques et chimiques des phénomènes observés. Ainsi, les interactions océans-atmosphère dans notre étude se limiteront à considérer deux fluides caractérisés par leur vitesse et leur pression respectivement appelés  $(\mathbf{u}_1, p_1)$  et  $(\mathbf{u}_2, p_2)$ .

Les équations vérifiées par ces fluides seront les équations de Stokes incompressibles dans le chapitre 2, et de Navier-Stokes incompressible dans le chapitre 3. A cela, des équations de condition au bord et d'interface entre les deux fluides seront rajoutés, conditions linéaires de type Robin en 2 et non-linéaires en 3.

Quant au domaine, dans l'optique de pouvoir effectuer des simulations numérique au moyen d'éléments finis, sur Freefem++, nous avons opté pour un domaine borné verticalement et "infini" aux conditions periodiques horizontalement; ce qui revient à considérer le produit cartésien d'un tore et d'un segment. Ce choix est cohérent vis-à-vis de la modélisation des interactions entre l'océan et les sous-couches inférieures de l'atmosphère. On aurait également pu considérer un domaine borné, dont l'altitude est négligeable par rapport aux directions horizontales. C'est d'ailleurs en pratique comme cela que sera codé notre boîte de calculs sur Freefem++, avec des conditions periodiques au bord. Enfin, nous avons décidé de concentrer nos efforts sur le cas stationnaire correspondant à une évolution en temps long du modèle. L'étude du cas d'évolution constitue évidemment une suite logique aux résultats décrits ici.

Les deux chapitres sont pensés de manière similaire. Dans un premier temps une étude théorique des problèmes en question avec l'étude de l'existence de solutions faibles, d'éventuelle unicité, suivis d'algorithmes pour obtenir des simulations sur Freefem++, dont les convergences sont étudiées théoriquement et numériquement. L'une des difficultés est d'ailleurs venue de la partie numérique: comment réussir à simuler deux fluides évoluant dans deux sous-domaines différents, avec une condition de bord impliquant les deux fluides ? Les méthodes de décomposition de domaine et notamment un algorithme de Schwarz a permis cette résolution numérique avec le principe de va-et-vient du haut vers le bas et du bas vers le haut pour résoudre récursivement nos problèmes.

La seconde partie est consacrée à des problèmes de couche limite et de viscosité turbulente. On n'aborde plus cette fois l'idée de couplages de fluides, mais du comportement de l'air dans la sous-couche la plus basse de l'atmosphère en contact avec l'océan appelée sous-couche visqueuse. Dans cette couche, la viscosité turbulente dépend de l'altitude, ce qui rajoute un terme "perturbateur" au système considéré. Cela a pour conséquence des difficultés à définir la trace des fonctions en jeu au bord. Pour obtenir les résultats d'existence de solutions faibles, il a donc fallu définir avec minutie les espaces fonctionnels<sup>1</sup> dans lesquels se placer pour résoudre les formulations variationnelles. On a notamment exhibé des exemples de solutions qui "explosaient" pour des données mal choisies.

Ce travail a donné lieu a quatre articles:

- *Continuous boundary condition at the interface for two coupled fluids* [55], écrit avec R. Lewandowski, publié dans Appl. Math. Lett., 135, 2023.
- *Surface boundary layers through a scalar equation with an eddy viscosity vanishing at the ground* [15], écrit avec L. Berselli et R. Lewandowski, à paraître dans ESAIM: Mathematical Modelling and Numerical Analysis. Cet article constitue le chapitre 4.
- *Singular boundary condition for a degenerated turbulent toy model* [3], écrit avec C. Amrouche, L. Berselli, G. Leloup et R. Lewandowski, soumis et accepté en 2024 dans PFA: *Pure and Applied Functional Analysis*, qui constitue le chapitre 5.
- Un quatrième article, *A coupling of stationary Navier-Stokes equations for two fluids with non-linear interface condition*, correspondant au chapitre 3, a été soumis [56].

---

1. espaces obtenus à partir d'espaces de Sobolev à poids pour la plupart

# INTRODUCTION GÉNÉRALE (FRANÇAIS)

---

## 1.1 Motivation

Motivons ce qui va suivre par quelques domaines d'application de l'amélioration des modèles existant de couches limites ou d'interaction fluide-fluide.

### **Météorologie**

La sous-couche visqueuse intervient dans la modélisation des phénomènes atmosphériques à petite échelle. Les processus de diffusion et de transfert de chaleur à la surface terrestre sont étroitement liés à cette couche, affectant ainsi la formation des nuages, la dispersion des polluants et la compréhension des microclimats urbains. Une meilleure compréhension des phénomènes évoluant dans cette zone permettrait d'améliorer la précision des prévisions climatiques locales et donc des modèles météorologiques.

### **Energie**

Prenons l'exemple des hydroliennes ou encore des éoliennes installées sur le littoral. Les pales immergées dans l'eau ou tournant à quelques mètres au dessus de l'eau sont précisément situées dans cette zone critique d'échanges thermodynamiques. Une compréhension approfondie de ces interactions est essentielle pour concevoir des dispositifs plus efficaces et durables, contribuant ainsi à l'exploitation optimale de l'énergie marine et éolienne.

### **Ingénierie navale**

La sous-couche visqueuse influe sur les performances hydrodynamiques des navires, notamment par l'intermédiaire des forces de traînée. L'optimisation de la conception des coques et des propulseurs nécessite une analyse approfondie de ces phénomènes visqueux pour réduire la consommation de carburant, améliorer la manœuvrabilité et garantir la sécurité des navires.

### **Modélisation des interactions fluide-fluide**

Outre l'étude de la sous-couche visqueuse, il est impératif de mieux modéliser les inter-

actions fluide-fluide, en particulier dans des scénarios où des fluides de différentes natures coexistent. Ces interactions revêtent une importance cruciale dans des domaines variés tels que le mélange de gaz et de liquides, la dispersion de particules et la simulation de phénomènes multiphasiques. Cela concerne beaucoup d'industries aux domaines variés: pétro-chimie, agro-alimentaire, métallurgie, etc... Une modélisation précise de ces interactions permet d'affiner les prédictions et d'améliorer la simulation de scénarii complexes.

## 1.2 Description des modèles

En simplifiant beaucoup les modèles existant, les grandeurs physiques qui vont nous intéresser pour un fluide  $i$  sont

- le champ de vitesse  $\mathbf{u}_i$  en  $m.s^{-1}$ ,
- la pression  $p_i$  en  $Pa$  (ou  $kg.m^{-1}.s^{-2}$ ),
- la masse volumique  $\rho_i$  en  $kg.m^{-3}$ ,
- la viscosité dynamique  $\eta_i$  en  $kg.m^{-1}.s^{-1}$ ,

en laissant de côté des grandeurs telles que la température, la salinité ou encore l'humidité de l'air, qui trouvent leur place notamment dans les systèmes d'équations primitives (voir [63],[58],[76]).

Les équations que l'on va obtenir sont gouvernées par les deux principes physiques que sont la conservation de la masse du fluide, qui impose la relation

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot \rho_i \mathbf{u}_i = 0,$$

ainsi que la conservation de la quantité de mouvement, donnée par

$$\rho_i \left( \frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right) = \rho_i \mathbf{f}_i + \nabla \cdot \boldsymbol{\sigma}_i,$$

où  $\boldsymbol{\sigma}_i$  est le tenseur des contraintes exercées sur le fluide  $i$ . Le terme de gauche représente l'accélération d'une particule. Il regroupe variation temporelle du champ de vitesse, qui correspond à un écoulement instationnaire, et variation spatiale (accélération convective). Pour obtenir les équations de Navier-Stokes, il reste à exprimer le tenseur de contraintes  $\boldsymbol{\sigma}_i$ , dont l'expression dépend des conditions rencontrées. Lorsque l'on choisit le cadre des

fluides newtoniens incompressibles, la divergence de ce tenseur s'écrit

$$\nabla \cdot \boldsymbol{\sigma}_i = -\nabla p_i + \eta_i \Delta \mathbf{u}_i,$$

et la conservation de la quantité de mouvement se réécrit sous la forme

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{1}{\rho_i} \nabla p_i + \nu_i \Delta \mathbf{u}_i + \mathbf{f}_i, \quad (1.1)$$

où  $\nu_i = \frac{\eta_i}{\rho_i}$  est appelée viscosité cinématique, exprimée en  $m^2.s^{-1}$ . Cette quantité dépend du fluide considéré mais également de la température. On note par exemple que pour une température de  $25^\circ\text{C}$ , la viscosité cinématique de l'air vaut  $\nu_{air} = 1.5 \times 10^{-5} m^2.s^{-1}$ , celle de l'eau douce  $\nu_{eau} = 8.8 \times 10^{-7} m^2.s^{-1}$ , et que pour une température de  $50^\circ\text{C}$ ,  $\nu_{eau} = 5.5 \times 10^{-7} m^2.s^{-1}$ . Le terme  $\nu_i \Delta \mathbf{u}_i$  est d'ailleurs appelé régulièrement *terme de viscosité*.

La condition d'incompressibilité permet également de réécrire la conservation de la masse sous la forme

$$\nabla \mathbf{u}_i = 0. \quad (1.2)$$

Les deux équations (1.1) et (1.2) constituent alors les bien connues équations de Navier-Stokes<sup>1</sup>

$$\begin{cases} \frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{1}{\rho_i} \nabla p_i + \nu_i \Delta \mathbf{u}_i + \mathbf{f}_i, \\ \nabla \mathbf{u}_i = 0, \end{cases} \quad (1.3)$$

dont l'une des sources de difficultés est la non-linéarité du terme de transport  $\mathbf{u}_i \cdot \nabla \mathbf{u}_i$ . On peut noter que lorsque la viscosité  $\nu_i$  est prise "très faible" relativement aux autres quantités, on retrouve les équations d'Euler, et inversement, lorsqu'elle est prise "très grande", on obtient les équations de Stokes. Cela conduit à introduire le nombre de Reynolds  $Re$ , quantité sans unité, défini par le rapport des forces d'inerties sur les forces visqueuses:

$$Re := \frac{\rho \|\mathbf{u} \cdot \nabla \mathbf{u}\|}{\eta \|\Delta \mathbf{u}\|}.$$

Pour un écoulement unidimensionnel, si l'on approche  $\mathbf{u}$  par son ordre de grandeur  $U$  sur

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1. de l'ingénieur français Claude Navier 1785-1836 et du physicien britannique George Stokes 1819-1903.



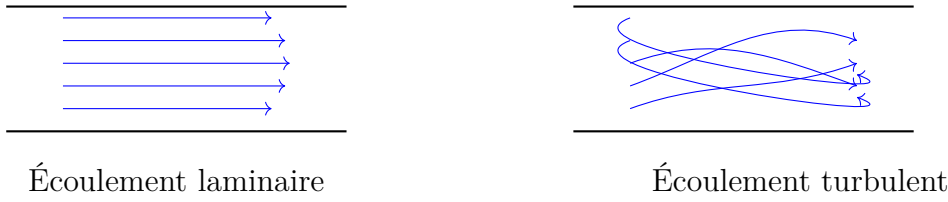


Figure 1.1 – Ecoulements laminaires et turbulents

la distance caractéristique de l'écoulement  $L$  dans la direction correspondante, on obtient

$$Re = \frac{LU}{\nu},$$

où  $L$  et  $V$  sont respectivement les longueurs et les volumes caractéristiques de l'écoulement, et  $\nu$  la viscosité cinématique du fluide.

Ainsi, le nombre de Reynolds  $Re$  se comporte de façon inverse à la viscosité: de grands nombres de Reynolds traduisent la prédominance du terme de transport sur le terme de viscosité (viscosité faible); et de petits nombres de Reynolds la prédominance de la viscosité sur le transport. Cela permet d'ailleurs de définir les différents régimes d'écoulement. Prenons l'exemple d'écoulements à conduite circulaire. Lorsque  $Re < 2000$ , l'écoulement est dit *laminaire*, et lorsque  $Re > 3000$ , il est dit *turbulent*. Entre ces deux valeurs de Reynolds, l'écoulement est instable.

### 1.2.1 Conditions aux bords

Dans la suite, on considérera deux fluides (l'air et l'eau par exemple) évoluant dans leurs domaines respectifs que l'on notera  $\Omega_1$  et  $\Omega_2$ , séparés par une interface  $\Gamma_{Int}$ . Les vitesses et pressions en tout point de ces fluides vérifient les équations de Navier-Stokes (1.3), et il reste à choisir les conditions aux bords, et notamment à l'interface, pour fermer le système de couplages d'équations. Les différentes conditions aux bords que l'on verra par la suite sont au nombre de 4:

- conditions de Dirichlet,
- conditions périodiques,
- conditions de Navier,
- conditions de friction.

## Conditions de non-glissement

On considère ici l'exemple de l'océan représenté par le domaine  $\Omega$  et du plancher océanique noté  $\Gamma$  (voir 1.2).

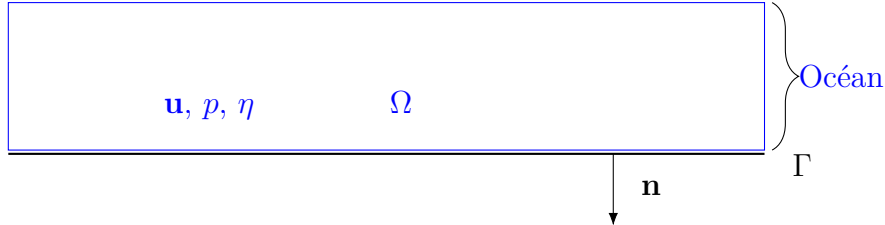


Figure 1.2 – Océan et plancher océanique

D'un point de vue macroscopique, le plancher océanique est relativement régulier, voire même lisse. Si l'on privilégie plutôt l'approche microscopique, le relief est en fait très irrégulier. De plus, ces irrégularités sont d'une taille bien plus importantes que les particules en mouvement: celles-ci sont freinées et même arrêtées par les obstacles rencontrés. La condition de non-glissement s'écrit alors

$$\mathbf{u} = 0, \text{ sur } \Gamma.$$

Dans le contexte de résolution des EDP ou de simulations numériques on parlera plutôt de *conditions de Dirichlet*<sup>2</sup>; homogènes dans le cas précis, puisque le second membre est nul.

## Conditions périodiques

Il s'agit seulement de réduire un domaine "infini", ou considéré comme très grand; l'océan par exemple, à un domaine borné, en ayant supposé que certains phénomènes se répétaient spatialement. Ces conditions sont peut-être les moins physiquement acceptables, il n'empêche qu'elles ont l'avantage de permettre d'appliquer l'analyse de Fourier, ce qui n'est pas négligeable des points de vue théorique comme numérique. Dans l'optique de faire nos études sur des couches de surface atmosphériques et océaniques, dont la hauteur de l'ordre du mètre sont négligeables par rapport à la longueur et largeur, on choisira des conditions périodiques uniquement sur les horizontales. Ainsi, le domaine considéré pour la sous-couche atmosphérique sera  $\mathbb{T}_2 \times [0, z_0]$ , où  $z_0$  est la hauteur de la sous-couche et

2. du nom du mathématicien allemand Johann Peter Gustav Lejeune Dirichlet, 1805-1859.

où  $\mathbb{T}_2$  est le tore de dimension 2 défini par

$$\mathbb{T}_2 = \frac{[0, L_1] \times [0, L_2]}{\mathbb{Z}^2},$$

avec  $L_1$  et  $L_2$  les longueur et largeur de la couche.

### Conditions de Navier

La condition de Navier représente un compromis entre les conditions de glissement et les conditions de frottement du fluide. Considérons alors une frontière  $\Gamma$ , qui peut aussi bien être dans notre modèle le fond de l'océan, que l'interface de séparation entre l'océan et l'atmosphère. Regardons d'abord le cas d'écoulement de l'eau contre une surface, correspondant à nouveau au schéma 1.2. Nous regarderons ensuite cette même condition à l'interface entre océan et atmosphère.

Soit  $\mathbf{u}$  le champ des vitesses de l'eau défini sur la frontière  $\Gamma$ . On note  $\mathbf{u}_\tau$  sa composante tangentielle définie par

$$\mathbf{u}_\tau = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$$

avec  $\mathbf{n}$  le vecteur normal à la frontière, dirigé vers l'extérieur du domaine considéré. Si

l'on note  $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ , alors  $\mathbf{u}_\tau = \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix}$ .

L'une des premières hypothèses de la condition de Navier est que le fluide ne traverse pas son domaine. Cela se traduit par la condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ sur } \Gamma,$$

qui se réécrit  $\mathbf{u} = \mathbf{u}_\tau$ .

La deuxième hypothèse porte sur les frictions du fluide. Les forces qui s'exercent sur lui au niveau de la frontière  $\Gamma$  sont égales à  $\boldsymbol{\sigma} \cdot \mathbf{n}$ , où l'on rappelle que lorsque le fluide est considéré comme newtonien, le tenseur des contraintes  $\boldsymbol{\sigma}$  qui s'exerce sur lui s'écrit

$$\boldsymbol{\sigma} = 2\eta D\mathbf{u} - p\mathbf{I},$$

où  $D\mathbf{u} := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$  (voir [7] ou [80]).

Les forces de friction correspondent alors à la partie tangentielle de  $\boldsymbol{\sigma} \cdot \mathbf{n}$ , que l'on notera

$(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau$ . La condition de glissement de Navier repose sur l'observation suivante: le fluide est ralenti par les forces de friction exercées en  $\Gamma$ , ce qui donne

$$\mathbf{u}_\tau = \mathbf{u} = -\beta(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau \text{ sur } \Gamma,$$

pour  $\beta > 0$ . En résumé, les conditions au bord de Navier s'expriment sous la forme

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = 0 \text{ sur } \Gamma, \\ \mathbf{u} + \beta(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau = 0 \text{ sur } \Gamma. \end{cases} \quad (1.4)$$

Toujours en choisissant  $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ , la première équation de (1.4) donne la composante

verticale à la frontière  $u_z = 0$ . Le terme  $\boldsymbol{\sigma} \cdot \mathbf{n}$  égal à  $\eta \begin{pmatrix} \partial_z u_x + \partial_x u_z \\ \partial_z u_y + \partial_y u_z \\ 2\partial_z u_z \end{pmatrix} - p \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ , devient

alors égal à  $\begin{pmatrix} \eta \partial_z u_x \\ \eta \partial_z u_y \\ -p \end{pmatrix}$ , puis si l'on considère ses composantes tangentielles, on a alors

$$(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau = \begin{pmatrix} \eta \partial_z u_x \\ \eta \partial_z u_y \\ 0 \end{pmatrix}, \quad (1.5)$$

et la deuxième équation de (1.4) donne le système

$$\begin{cases} \partial_z \mathbf{u}_x + \frac{1}{\beta\eta} \mathbf{u}_x = 0 \text{ sur } \Gamma, \\ \partial_z \mathbf{u}_y + \frac{1}{\beta\eta} \mathbf{u}_y = 0 \text{ sur } \Gamma. \end{cases}$$

**Remark 1.2.1.** Lorsque  $\beta$  tend vers 0, la condition de Navier "converge" vers une condition de non-glissement. Inversement, lorsque  $\beta$  tend vers  $+\infty$ , la partie tangentielle des contraintes  $(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau$  devient égale à 0, ce qui correspond à une condition de glissement sans frottement et à l'hypothèse des fluides parfaits.

Adaptons cette condition de Navier à l'exemple qui nous intéresse: l'écoulement de

l'air et de l'eau à la surface de l'océan. On note  $\Omega_a$  et  $\Omega_o$  les domaines correspondant à l'atmosphère et l'océan, ainsi que  $\mathbf{u}^a$ ,  $\mathbf{u}^o$  les vitesses de l'air et de l'eau dans ces domaines, et  $p^a$ ,  $p^o$  les pressions correspondantes. Les frontières (verticales) sont au nombre de trois:  $\Gamma_a$  qui correspond au sommet de la sous-couche d'atmosphère qui nous intéresse,  $\Gamma_o$  à la sous-couche d'océan étudiée et  $\Gamma_{Int}$ , la surface entre les deux. On note également  $\mathbf{n}^a = -\mathbf{n}^o$  le vecteur unitaire normal à la surface, dirigé extérieurement au domaine  $\Omega_a$ . La situation est schématisée sur la figure 1.3.

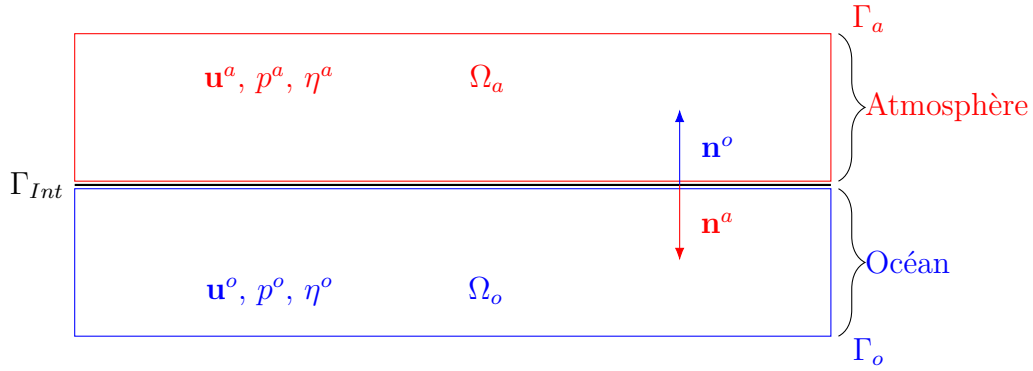


Figure 1.3 – Schématisation des domaines océan et atmosphère

La première hypothèse de la loi de Navier précédente est la même: l'eau et l'air sont contraints à rester dans leur domaine, sans modification de géométrie de celui-ci. Cela s'écrit

$$\begin{cases} \mathbf{u}^a \cdot \mathbf{n}^a = 0 \text{ sur } \Gamma_{Int}, \\ \mathbf{u}^o \cdot \mathbf{n}^o = 0 \text{ sur } \Gamma_{Int}. \end{cases}$$

La deuxième hypothèse concernant les forces de frictions inter-fluides doit maintenant prendre en compte la différence de vitesses entre les deux fluides à l'interface (principe d'action-réaction) et s'écrit:

$$\begin{cases} \mathbf{u}_\tau^a - \mathbf{u}_\tau^o = -\beta(\boldsymbol{\sigma}_a \cdot \mathbf{n}^a)_\tau \text{ sur } \Gamma_{Int}, \\ \mathbf{u}_\tau^o - \mathbf{u}_\tau^a = -\beta(\boldsymbol{\sigma}_o \cdot \mathbf{n}^o)_\tau \text{ sur } \Gamma_{Int}. \end{cases}$$

En posant  $\alpha = \frac{1}{\beta}$ , on obtient alors le système d'équations

$$\begin{cases} \eta^a \frac{\partial \mathbf{u}_h^a}{\partial z} + \alpha \mathbf{u}_h^a = \alpha \mathbf{u}_h^o \text{ sur } \Gamma_{Int}, \\ \eta^o \frac{\partial \mathbf{u}_h^o}{\partial z} + \alpha \mathbf{u}_h^o = \alpha \mathbf{u}_h^a \text{ sur } \Gamma_{Int}, \end{cases}$$

où l'on définit les champs de vitesses horizontales  $\mathbf{u}_h^a$  et  $\mathbf{u}_h^o$  par

$$\mathbf{u}_h^a := \begin{pmatrix} u_x^a \\ u_y^a \end{pmatrix}, \quad \text{et} \quad \mathbf{u}_h^o := \begin{pmatrix} u_x^o \\ u_y^o \end{pmatrix}.$$

**Remark 1.2.2.** Dans le contexte des EDP, et des formulations variationnelles, les conditions de Navier appartiennent à la catégorie des conditions dites mixtes ou conditions aux limites de Robin<sup>3</sup>. C'est cette dernière expression que l'on utilisera principalement dans le chapitre 2.

### Conditions de friction

La encore, commençons par regarder un exemple typique d'application de cette loi, avant de considérer l'adaptation au couplage océan-atmosphère. Considérons donc un solide de centre de gravité  $G$ , se déplaçant à une vitesse uniforme  $\mathbf{U}$  dans l'eau (un sous-marin, une baleine, etc...). Sa surface de contact avec les forces d'écoulement est notée  $S$ . Supposons que le centre de gravité  $G$  du solide est à l'abscisse  $x$  au temps  $t$ . Alors, un court instant après, au temps  $t + \delta t$ , sa position sera  $x + \mathbf{U}\delta t$ , comme illustré par le schéma 1.4.

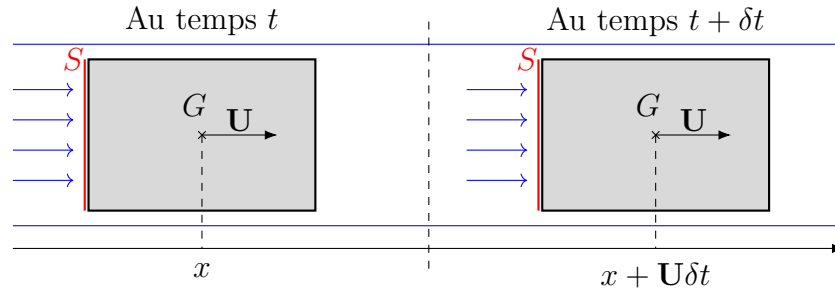


Figure 1.4 – Déplacement d'un solide dans un écoulement

Le volume d'eau déplacé est alors égal à  $S|\mathbf{U}|\delta t$  et la masse correspondante égale à

3. du nom du mathématicien français Victor Gustave Robin 1855-1897

$\delta m = \rho S |\mathbf{U}| \delta t$ . La quantité de mouvement  $\delta \mathbf{p}$  étant par définition égale à  $\mathbf{U} \delta m$ , elle est alors égale à  $\rho S |\mathbf{U}| \mathbf{U} \delta t$ . Cependant, si l'on prend en compte les frottements entre le solide et l'eau, une partie de la quantité de mouvement se "perd" en frottement. La quantité de mouvement de fluide déplacée est donc égale à

$$\delta \mathbf{p} = C \rho S |\mathbf{U}| \mathbf{U} \delta t,$$

où  $C \in ]0, 1[$  est une constante mesurant l'importance des frottements et déterminée expérimentalement. Lorsque l'on fait tendre  $\delta t$  vers 0, on retrouve l'expression des forces exercées par le fluide sur le solide:

$$\lim_{\delta t \rightarrow 0} \frac{\delta \mathbf{p}}{\delta t} = \boldsymbol{\sigma} \cdot \mathbf{n} = C \rho S |\mathbf{U}| \mathbf{U}. \quad (1.6)$$

Dans le contexte de la modélisation océan-atmosphère, si l'on s'intéresse à la vitesse de l'eau  $\mathbf{u}^o$  à la surface, et que l'on prescrit la vitesse de l'air à la surface constante égale à  $\mathbf{V}$ , les conditions limites à considérer sont alors:

$$\left\{ \begin{array}{l} \mathbf{u}^o \cdot \mathbf{n}^o = 0 \text{ sur } \Gamma_{Int}, \\ (\boldsymbol{\sigma}^o \cdot \mathbf{n}^o)_\tau = C |\mathbf{V} - \mathbf{u}_\tau^o| (\mathbf{V} - \mathbf{u}_\tau^o) \text{ sur } \Gamma_{Int}, \end{array} \right.$$

et en reprenant la valeur de  $(\boldsymbol{\sigma}^o \cdot \mathbf{n}^o)_\tau$  qui a la même expression qu'en (1.5), on obtient l'équation

$$\eta^o \frac{\partial \mathbf{u}_h^o}{\partial z} = C |\mathbf{V}_h - \mathbf{u}_h^o| (\mathbf{V}_h - \mathbf{u}_h^o).$$

Lorsque la vitesse de l'air notée  $\mathbf{u}^a$  est cette fois une inconnue du problème et que l'on modélise les interactions couplées de l'écoulement de l'air et de l'eau, on obtient en adaptant le même raisonnement les conditions limites

$$\left\{ \begin{array}{l} \mathbf{u}^o \cdot \mathbf{n}^o = 0 \text{ sur } \Gamma_{Int}, \\ \mathbf{u}^a \cdot \mathbf{n}^a = 0 \text{ sur } \Gamma_{Int}, \\ \eta^o \frac{\partial \mathbf{u}_h^o}{\partial z} = C |\mathbf{u}_h^a - \mathbf{u}_h^o| (\mathbf{u}_h^a - \mathbf{u}_h^o) \text{ sur } \Gamma_{Int}, \\ \eta^a \frac{\partial \mathbf{u}_h^a}{\partial z} = C |\mathbf{u}_h^o - \mathbf{u}_h^a| (\mathbf{u}_h^o - \mathbf{u}_h^a) \text{ sur } \Gamma_{Int}. \end{array} \right.$$

**Remark 1.2.3.** Cette loi de friction est aussi appelée loi Gauckler-Manning, du nom des

ingénieurs qui l'ont établie ([42], [69]).

## 1.2.2 Modèles étudiés

On a choisi d'étudier deux modèles, faisant intervenir deux fluides (l'air et l'eau), chacun définis uniquement sur leur domaine de définition (les sous-couches atmosphériques et océaniques). La situation est représentée par le schéma suivant :

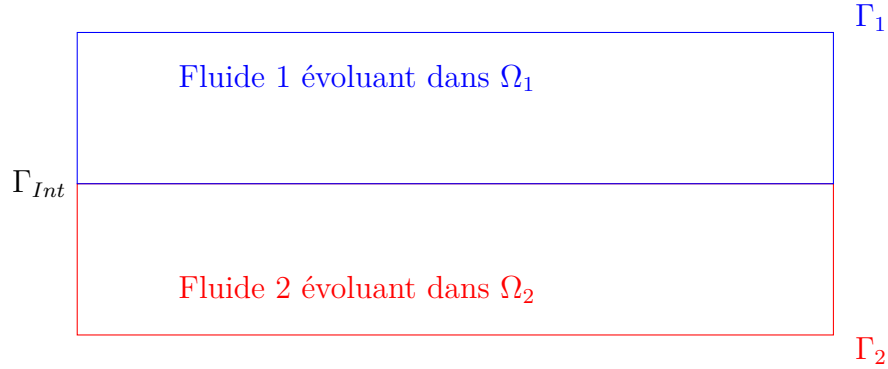


Figure 1.5 – Domaine d'étude

Le premier modèle étudié est le couplage de fluides vérifiant les équations de Stokes stationnaires incompressibles avec condition de Navier à l'interface:

$$\begin{cases} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, & \text{dans } \Omega_i, \\ \nabla \cdot \mathbf{u}_i = 0, & \text{dans } \Omega_i, \\ \mathbf{u}_{1,h}|_{\Gamma_I} = \mathbf{u}_{2,h}|_{\Gamma_I}, \\ \mathbf{u}_i \cdot \mathbf{n}_i|_{\Gamma_I} = 0, \\ \mathbf{u}_{1|\Gamma_1} = \mathbf{u}_{2|\Gamma_2} = 0, \end{cases} \quad (1.7)$$

où  $i \in \{1, 2\}$ , et

- $\mathbf{u}_i$  désigne le champ de vitesse du fluide  $i$ ,
- $p_i$  désigne la pression du fluide  $i$ ,
- $\nu_i$  sa viscosité cinématique,
- $\mathbf{f}_i$  une force exercée sur lui.

Les conditions  $\nabla \cdot \mathbf{u}_i = 0$  et  $\mathbf{u}_i \cdot \mathbf{n}_i|_{\Gamma_I} = 0$  traduisent l'incompressibilité des deux fluides et l'hypothèse de "toit rigide" (rigid lid) qui astreint les fluides à ne pas déborder de leurs domaines respectifs, contrairement à une situation de surface libre. Dans le contexte d'une modélisation de l'océan et de l'atmosphère, le caractère incompressible n'est pas



pertinent si l'on considère l'atmosphère dans son ensemble, mais l'est beaucoup plus si l'on s'intéresse uniquement à une couche de surface d'une dizaine de mètres seulement. L'hypothèse de toit rigide l'est aussi, principalement pour un écoulement laminaire, ce qui correspond à des nombres de Reynolds petits et donc des coefficients de viscosité grand. Enfin la relation  $\mathbf{u}_1|_{\Gamma_1} = \mathbf{u}_2|_{\Gamma_2} = 0$  correspond à une condition de continuité, plutôt propre aux problèmes à surface libre.

Ce modèle doit être considéré comme un "modèle jouet": beaucoup d'approximations ont été faites, bien que raisonnables. Il est cependant important de l'analyser et de le comprendre pour espérer traiter des systèmes plus difficiles.

Le deuxième couplage de fluides étudiés fait intervenir les équations de Navier-Stokes stationnaires incompressibles, avec conditions de frictions à l'interface et sur les bords supérieurs et inférieurs des domaines; correspondant aux limites des couches de surfaces atmosphériques et océaniques considérées.

$$\left\{ \begin{array}{ll} (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{dans } \Omega_i \\ \nabla \cdot \mathbf{u}_i = 0, & \text{dans } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h})|\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{sur } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{sur } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{sur } \Gamma_{Int} \cup \Gamma_i, \end{array} \right. \quad (1.8)$$

pour  $i = 1, 2$ , où  $C_D$  et  $c_{d,i}$  sont des constantes positives, et  $\mathbf{V}_i$  des champs de vitesses fixés en haut et en bas des domaines (des vitesses considérées comme constante aux limites des couches de surface atmosphériques et océaniques).

Ce système est plus complexe que le précédent, notamment en raison de la double non-linéarité induite par le terme de transport de Navier-Stokes, et de la condition de friction à l'interface. Cette condition de friction correspond à des écoulements qui ne peuvent plus être considérés comme laminaires, et donc à des vitesses en jeu plus importantes.

Considérer les états stationnaires pour ces deux modèles n'est pas inutile: les vitesses considérées peuvent être vues comme des vitesses moyennes obtenues en "temps long". Dans le contexte de la météorologie, cela peut être de l'ordre de la dizaine d'heures, et correspond par exemple à des vents établis le long d'un littoral.

Enfin, l'idée derrière cette étude de modèles "simples" est d'arriver à obtenir des approximations suffisamment satisfaisantes, tant d'un point de vue théorique que numérique et au prix d'un coût de calcul très réduit de ce qui se passe à l'interface océan-atmosphère, sans utiliser les équations primitives ni les simulations numériques au coût de calcul important qui leur sont associées.

Ces deux modèles seront étudiés dans les chapitres 2 et 3.

### 1.2.3 Couches limites et modèles logarithmiques

Du point de vue de la mécanique des fluides, on appelle couche limite la zone de l'écoulement influencée par la proximité de la paroi. Cette influence est multiple mais notamment caractérisée par des propriétés dynamiques et thermiques avec des échanges de quantités de mouvement et des flux de chaleur.

On s'intéressera principalement dans ce qui va suivre aux propriétés dynamiques induites par la couche limite. On peut la caractériser de manière plus précise comme étant la région de l'espace dans laquelle négliger les forces de frottements surfaciques sur l'air serait une approximation physique trop grossière.

La couche limite qui va nous intéresser dans ce manuscrit est la couche limite atmosphérique, dans laquelle le frottement de l'air avec la surface de la Terre doit être pris en compte. Cette couche limite dont la hauteur peut varier de quelques dizaines de mètres à quelques kilomètres, dépend de nombreux paramètres. Nous pouvons par exemple citer:

- l'influence du relief,
- l'influence de la température et des phénomènes de convection: la couche limite atmosphérique peut être 10 fois plus importante le jour que la nuit (voir [93]).

Prenons l'exemple d'une couche atmosphérique de hauteur 1km pour fixer les idées. Elle peut être séparée en deux sous-couches:

- une sous couche inférieure, appelée *couche de surface*, comprise entre la surface et une altitude de 100m environ. C'est dans cette couche que se fait le plus sentir le frottement de l'air avec la surface. On peut y négliger les effets de la rotation de la Terre et donc la force de Coriolis. Ainsi, les contraintes de cisaillement sont quasiment constantes sur la verticale, et l'écoulement est dirigé par la friction turbulente au sol (et par le gradient vertical de température).
- une sous-couche supérieure, appelée *couche d'Ekman*, occupant 90% de la hauteur

de la couche, c'est-à-dire comprise entre des altitudes de 100m et 1km. Dans cette couche, l'effet des forces de frottement à la surface est encore présent bien que beaucoup moins important que dans la couche précédente. La force de Coriolis possède une influence de plus en plus forte à mesure que l'on se situe en altitude dans cette couche.

Pour résumer, la couche de surface est la couche ne prenant en compte que les effets des forces de frottement et la couche d'Ekman est la zone de compétition entre forces surfaciques et force de Coriolis. Les profils de vitesse ne suivront donc pas les mêmes lois dans ces deux couches. Une étude en détail multicouche avec modélisation et simulations numériques a notamment été réalisée par C.Pelletier, F.Lemarié, S.Thery et E.Blayo [76], [99], [77].

Nous limiterons notre étude à la couche limite surfacique. On peut encore distinguer à la base de celle-ci une sous-couche rugueuse que l'on appellera *sous-couche visqueuse*, où les phénomènes de turbulence sont les plus importants, tant d'un point de vue dynamique que thermique.

La hauteur de cette couche est directement liée à la hauteur du relief et à une grandeur physique appelée longueur de rugosité, que l'on note  $z_0$ , dont les valeurs caractéristiques sont de l'ordre de  $0.0002m$  pour l'océan,  $0.01m$  pour une prairie, de  $0.5m$  à  $1m$  pour une forêt, et supérieure à  $1m$  pour des terrains très accidentés.

Empiriquement, la hauteur d'une sous-couche visqueuse est comprise de manière large entre 0.1 et 0.03 fois la hauteur moyenne des "obstacles" au sol [94].

Dans ce paragraphe nous allons rappeler certains éléments de la théorie des couches limites turbulentes et définir le cadre général du modèle que l'on étudiera plus en détail dans le chapitre 4. Le domaine considéré est un espace de dimension 3: deux dimensions horizontales caractérisées par les coordonnées  $x$  et  $y$  et une dimension verticale représentée par l'altitude  $z$ . La vitesse moyenne constante du fluide dans une telle couche limite, notée *Boundary Layer* ( $BL$ ), est représentée dans un cadre cartésien bidirectionnel par

$$\mathbf{u} = \mathbf{u}(\mathbf{x}_h, z) = (\mathbf{u}_h, w)$$

au lieu de la vitesse moyenne  $\bar{\mathbf{u}}$ , où

$$\mathbf{u}_h = (u, v), \quad \mathbf{x}_h = (x, y) \in \mathbb{R}^2, \quad z \in ]0, z_{top}[,$$

où  $z_{top} > 0$  est le sommet de ( $BL$ ). Par exemple, si ( $BL$ ) modélise la couche limite de

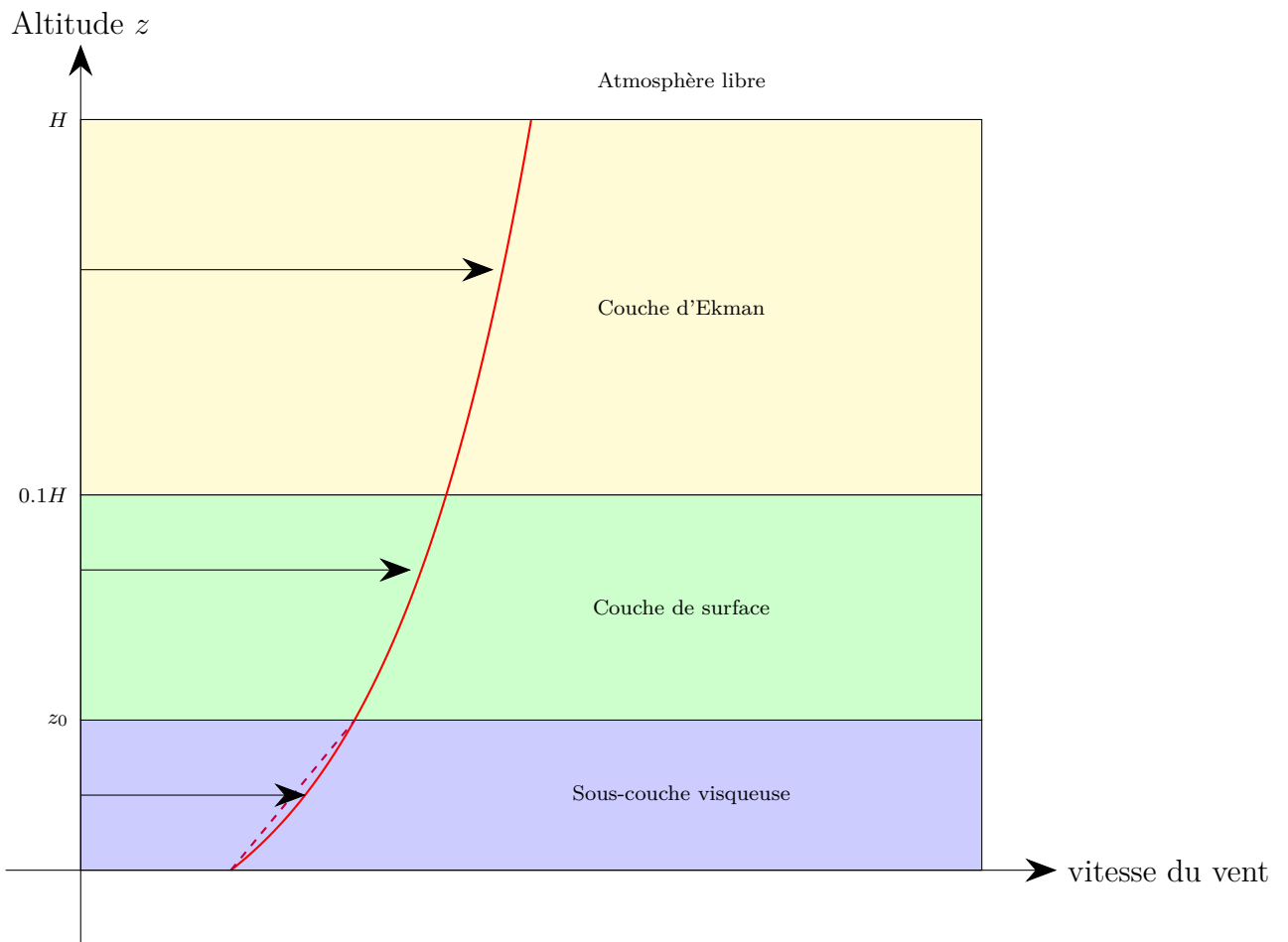


Figure 1.6 – Profil de vitesse du vent dans l’atmosphère, avec modèle logarithmique et modèle linéaire-logarithmique

surface,  $z_{top} \approx 100$  m. Nous devons également prendre en compte la longueur de rugosité  $z_0$ , qui dépend de la nature du sol.

La nature du domaine autorise la séparation des variables et des inconnues en coordonnées horizontales et verticales. Cela va être utile pour transformer les équations physique induites en un système plus simple.

Soit  $\nu > 0$  la viscosité cinématique du fluide. On suppose généralement que dans une couche limite standard ( $BL$ ), les conditions suivantes sont respectées :

- la pression est constante,
- la composante verticale de la vitesse moyenne s'annule, c'est-à-dire  $w = 0$  et, même si cela signifie un changement de coordonnées, on peut supposer  $v = 0$  ;
- la vitesse moyenne  $\mathbf{u} = (u, 0, 0)$  dépend uniquement de l'altitude, c'est-à-dire  $u = u(z)$  ;
- la viscosité turbulente  $\nu_{\text{turb}}$  dépend de  $z$  et de  $u^* = \sqrt{\nu |\partial_z u(0)|}$ , la vitesse de friction intrinséquement lié au régime de turbulence (voir [23, 79]), ce qui donne

$$\nu_{\text{turb}} = \nu_{\text{turb}}(z) = C_\nu C_* u^* z, \quad (1.9)$$

où  $C_\nu \approx 15$ ,  $C_* \approx 10$  sont des constantes adimensionnelles que nous avons calibrées par des simulations numériques. Les valeurs typiques de  $u_*$  varient de 2 à 10 m/s.

**Remark 1.2.4.** *Normalement, dans les modèles industriels ou météorologiques habituels,  $C_\nu$  représente la constante de von Kármán, dont la valeur est égale à 0.4, et  $C_* = 1$ . Cependant, en raison des échelles de nos simulations, nous devons prendre d'autres valeurs de ces constantes pour obtenir des résultats numériques liés aux données physiques de la couche limite atmosphérique.*

- Tous les termes dans l'équation du fluide sont négligeables par rapport au terme de diffusion turbulente.

Ces hypothèses conduisent à l'équation suivante pour la vitesse moyenne  $\mathbf{u} = (u, 0, 0)$ ,

$$-\frac{d}{dz} \left( \nu_{\text{turb}}(z) \frac{du}{dz} \right) = 0,$$

cette équation, une fois intégrée entre la longueur de rugosité  $z_0$  et  $z_{top}$ , avec des conditions aux limites appropriées, donne la loi logarithmique bien connue, uniforme en

$\mathbf{x}_h$ , en utilisant les constantes d'étalonnage  $C_\nu$  et  $C_\star$  :

$$u(z) = \frac{C_\star u_\star}{C_\nu} \left( \log \left( \frac{z}{z_0} \right) + 1 \right) \quad z \in [z_0, z_{top}].$$

Généralement, pour  $z \in [0, z_0]$ , appelée sous-couche visqueuse, un profil linéaire est considéré de telle sorte que  $u = u(z)$  soit continu sur  $]0, z_{top}[$ , et  $u(0) = 0$ , ce qui signifie

$$u(z) = \frac{C_\star u_\star}{C_\nu} z, \quad z \in [0, z_0].$$

Les profils logarithmiques et linéaire-logarithmique sont représentés sur la figure 1.6.

Soit  $u_{\text{Log}}$  la fonction définie sur  $]0, z_{top}[$  par (1.2.3)-(1.2.3). Lorsque la stabilité de l'atmosphère n'est pas à l'équilibre et que les effets de convection sont pris en compte, un terme source non nul est présent dans (1.2.3), et des fonctions de stabilisation introduites dans [74] et développées dans [76] doivent être ajoutées à  $u_{\text{Log}}$  pour obtenir le profil de vitesse correct, conformément à la théorie de Monin-Obukhov [73], ce qui signifie que

$$u(z) = u_{\text{Log}}(z) + \Psi(\mathbf{x}_h, z),$$

où la fonction  $\Psi(\mathbf{x}_h, z)$  est déduite d'arguments de similarité ou de données expérimentales. Des exemples de telles fonctions stabilisantes peuvent être trouvés dans [76].

Notre objectif est de trouver un modèle d'équations aux dérivées partielles complet

1. où  $u$  est défini sur  $]0, z_{top}[$ ,
2. qui inclut une viscosité turbulente de la même forme que celle donnée par (1.9), où le profil  $u = u(\mathbf{x}_h, 0)$  dépend également de la variable horizontale,
3. où l'on est capable de calculer des fonctions stabilisantes telles que dans (1.2.3) dans divers régimes atmosphériques.

Avant de se lancer dans les équations 3D non linéaires et plus complexes, nous considérons en premier lieu le "toy model" elliptique suivant dans  $BL = \mathbb{R}^2 \times (0, z_{top})$  :

$$\lambda u - \nu_h \Delta_h u - \partial_z (\nu_{\text{turb}}(z) \partial_z u) = f,$$

pour certains  $\lambda > 0, \nu_h > 0$ ,  $\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ . Le terme  $\lambda u$  dans (1.2.3) représente un substitut d'une dérivée temporelle  $\partial_t u$  et permet de régulariser le système, en particulier dans le cas  $\alpha = 1$ . Il peut être pris égal à zéro dans la simulation par éléments finis grâce à la dissipation numérique due à la discrétisation.

Dans les applications physiques, le terme source  $f$  est la force de Boussinesq, à savoir une différence de température entre le sol et le fluide exprimée par

$$f = g\alpha(T_0 - T),$$

où  $T$  est la température du fluide,  $T_0$  sa valeur au sol,  $g \approx 10 \text{ ms}^{-1}$  est le coefficient de gravité, et  $\alpha$  le coefficient de dilatation thermique, dont la valeur typique pour l'air sec est d'environ  $0,02 \text{ K}^{-1}$ .

## 1.3 Organisation du manuscrit

### 1.3.1 Chapitre 2

Dans le chapitre 2, on s'intéresse au système couplé

$$\begin{cases} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, & \text{dans } \Omega_i, \\ \nabla \cdot \mathbf{u}_i = 0, & \text{dans } \Omega_i \\ \mathbf{u}_{1,h}|_{\Gamma_I} = \mathbf{u}_{2,h}|_{\Gamma_I}, \\ \mathbf{u}_i \cdot \mathbf{n}_i|_{\Gamma_I} = 0, \\ \mathbf{u}_1|_{\Gamma_1} = \mathbf{u}_2|_{\Gamma_2} = 0, \end{cases} \quad (1.10)$$

où  $i \in \{1, 2\}$ , et

- $\mathbf{u}_i$  désigne le champ de vitesse du fluide  $i$ ,
- $p_i$  désigne la pression du fluide  $i$ ,
- $\nu_i$  sa viscosité cinématique,
- $\mathbf{f}_i$  une force exercée sur lui.

L'objectif de ce chapitre est d'obtenir l'existence de solutions faibles de ce problème comme limites de solutions du second problème:

$$\begin{cases} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, & \text{dans } \Omega_i, \\ \nabla \cdot \mathbf{u}_i = 0, & \text{dans } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i}|_{\Gamma_I} = -\alpha(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) \\ \mathbf{u}_i \cdot \mathbf{n}_i|_{\Gamma_I} = 0, \\ \mathbf{u}_1|_{\Gamma_1} = \mathbf{u}_2|_{\Gamma_2} = 0, \end{cases} \quad (1.11)$$

où la condition de continuité à l'interface a été remplacée par une condition du type Robin

(ou loi de Navier).

De tels problèmes ont été étudiés dans [12, 27, 28, 61] et l'existence et l'unicité de solutions faibles découlent directement du théorème de Lax-Milgram. Le but est de voir comment le champ de vitesse  $\mathbf{u}^\alpha = (\mathbf{u}_1^\alpha, \mathbf{u}_2^\alpha)$  solution faible du problème de friction (1.11) évolue lorsque  $\alpha \rightarrow \infty$  et que la condition de Navier "tend" vers la condition de continuité.

Le résultat principal de ce chapitre est le suivant :

**Theorem 1.3.1.** *La famille  $(\mathbf{u}^\alpha)_\alpha \in W$  solution faible du problème de friction (1.11), converge fortement lorsque  $\alpha \rightarrow +\infty$  vers un champ de vitesse  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in W_0$  solution faible et unique de (1.10),*

où  $W$  et  $W_0$  sont deux espaces de Sobolev dont la définition est donnée dans la partie 3.2.1.

La simulation numérique du problème va être guidée par la forme naturelle du couplage d'équations (1.11), qui invite à appliquer des algorithmes de décomposition de domaine de type Schwarz. En effet, le domaine  $\Omega = \Omega_1 \times \Omega_2$  est par définition dès le départ décomposé en deux sous-domaines qui ne s'intersectent pas. Les algorithmes correspondant à ce type de décomposition font partie de la catégorie "non-overlapping Schwarz algorithms" et de nombreuses études ont été réalisées pour en étudier la convergence [67], [39], [40], [32], [95], etc...

Le but de ce chapitre est de démontrer la convergence des algorithmes introduits et de l'illustrer par des simulation numériques obtenues à l'aide de Freefem++.

Dans un premier temps, une introduction générale aux algorithmes de décomposition sera proposée. On s'intéressera au problème du Laplacien

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.12)$$

sur un domaine  $\Omega$  décomposable en deux sous domaines  $\Omega_1$  et  $\Omega_2$ . La méthode de décomposition de domaine consiste à résoudre ce problème de manière alternée en résolvant plutôt les deux sous-problèmes

$$\begin{cases} -\Delta u_1^n = f & \text{dans } \Omega_1, \\ (\frac{\partial}{\partial \mathbf{n}_1} + \mathcal{L}_1)u_1^n = (\frac{\partial}{\partial \mathbf{n}_2} + \mathcal{L}_2)u_2^{n-1} & \text{sur } \partial\Omega_1, \end{cases} \quad (1.13)$$



et

$$\begin{cases} -\Delta u_2^n = f & \text{dans } \Omega_2, \\ (\frac{\partial}{\partial \mathbf{n}_2} + \mathcal{L}_2)u_2^n = (\frac{\partial}{\partial \mathbf{n}_2} + \mathcal{L}_2)u_1^n & \text{sur } \partial\Omega_2, \end{cases} \quad (1.14)$$

où  $\mathcal{L}_1$  and  $\mathcal{L}_2$  sont deux opérateurs linéaires bien choisis.

La Transformée de Fourier permet d'exhiber la vitesse de convergence de tels problèmes et de calculer l'erreur  $\|u_1 - u_1^n\| + \|u_2 - u_2^n\|$  entre la solution du problème de départ et celle du problème approché, où  $\|\cdot\|$  peut désigner la norme  $L^2$  ou  $H^1$ . Une telle étude sera faite avec ou sans intersection des domaines  $\Omega_1$  et  $\Omega_2$ , en considérant les opérateurs  $\mathcal{L}_i$  choisis comme étant proportionnels à la trace à l'interface. L'influence du coefficient de proportionnalité  $\alpha$  sera en particulier utile pour l'étude de nos problèmes couplés. Des simulations numériques viendront étayer les résultats de convergence pour finir.

Il s'agit ensuite de faire le lien entre cette première partie introductive qui présente la méthode sur un cas relativement simple, et le problème de friction (1.11) que l'on souhaite résoudre numériquement. Cela permettrait alors lorsque  $\alpha$  est pris très grand, d'obtenir une solution approchée raisonnable du problème initial (1.10).

Une résolution alternée du problème de friction (1.11) entre les domaines  $\Omega_1$  et  $\Omega_2$  s'écrit

$$\begin{cases} -\nu_1 \Delta \mathbf{u}_1^{\alpha, n+1} + \nabla P_1^{n+1} = \mathbf{f}_1, \\ \nabla \cdot \mathbf{u}_1^{\alpha, n+1} = 0, \\ \nu_1 \frac{\partial \mathbf{u}_{1,h}^{\alpha, n+1}}{\partial \mathbf{n}_1} |_{\Gamma_I} = -\alpha (\mathbf{u}_{1,h}^{\alpha, n+1} - \mathbf{u}_{2,h}^{\alpha, n}), \\ \mathbf{u}_{1,h}^{\alpha, n+1} |_{\Gamma_1} = 0, \quad \mathbf{u}_{1,h}^{\alpha, n+1} \cdot \mathbf{n}_1 = 0, \end{cases} \quad (1.15)$$

et

$$\begin{cases} -\nu_2 \Delta \mathbf{u}_2^n + \nabla P_2^n = \mathbf{f}_2, \\ \nabla \cdot \mathbf{u}_2^n = 0, \\ \nu_2 \frac{\partial \mathbf{u}_{2,h}^n}{\partial \mathbf{n}_2} |_{\Gamma_I} = -\alpha (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n), \\ \mathbf{u}_{2,h}^n |_{\Gamma_2} = 0, \quad \mathbf{u}_{2,h}^n \cdot \mathbf{n}_2 = 0, \end{cases} \quad (1.16)$$

où les puissances " $n$ " désignent les itérés à l'ordre  $n$ .

Les différences principales avec le problème test du Laplacien sont:

- les inconnus ne sont plus des scalaires mais des vecteurs;
- la vitesse n'est plus la seule inconnue puisque la pression est également présente dans l'équation;
- la condition d'interface que l'on pouvait choisir de façon optimale, même artifi-

ciellement, est ici imposée par la loi de Navier à l'interface entre les deux fluides. En notant  $\mathbf{e}_1^k$  et  $\mathbf{e}_2^k$  les erreurs de vitesse à l'ordre  $k$ , on montrera le résultat de convergence:

**Theorem 1.3.2.** *la suite  $(\mathbf{e}_1^k, \mathbf{e}_2^k)_{k \in \mathbb{N}}$  converge vers  $(0, 0)$  dans  $W_1 \times W_2$ ,*

où  $W_1$  et  $W_2$  sont deux espaces de Sobolev.

Cette démonstration très générale sera complétée par des résultats plus précis de convergence, propres aux domaines considérés (des bandes aux conditions périodiques), s'appuyant à nouveau sur l'analyse de Fourier et en s'inspirant de [41] et [17].

L'influence des viscosités  $\nu_1$  et  $\nu_2$  sera en particulier illustrée par différentes simulations sur Freefem++, tout comme l'influence de  $\alpha$ , notamment lorsqu'il devient très grand par rapport aux autres paramètres introduits.

Une étude similaire sera également proposée pour un système d'équations avec une condition différente à l'interface, faisant intervenir des petites perturbations de dérivées directionnelles.

### 1.3.2 Chapitre 3

Dans le chapitre 3, nous détaillerons un problème de couplage plus complexe que le précédent. Il s'agit d'un couplage d'équations de Navier-Stokes stationnaires, dans le même cadre géométrique, avec des conditions de bord et d'interface différentes. Le système d'équations considérées est le suivant:

$$\left\{ \begin{array}{ll} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{dans } \Omega_i \\ \nabla \cdot \mathbf{u}_i = 0, & \text{dans } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{sur } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i} (\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{sur } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{sur } \Gamma_{Int} \cup \Gamma_i, \end{array} \right. \quad (1.17)$$

pour  $i = 1, 2$ , où  $C_D$  et  $c_{d,i}$  sont des constantes positives, et  $\mathbf{V}_i$  des champs de vitesse fixés en haut et en bas du domaine.

La condition de continuité à l'interface proposée dans le système (1.10) est remplacée par la condition de Gauckler-Manning

$$\nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, \text{ sur } \Gamma_{Int},$$

et des conditions d'entraînement de type Robin en haut et en bas du domaine viennent s'y ajouter.

Des problèmes similaires à un fluide ont été étudiés notamment dans [23], avec des résultats d'existence de solutions faibles. On se donne comme objectif dans ce chapitre, d'obtenir des résultats d'existence de solutions faibles, pour ce problème à deux fluides.

Pour cela, on introduira les bons espaces fonctionnels pour donner un sens au problème considéré et aboutir à une formulation variationnelle mettant en jeu différentes formes linéaires, bilinéaires et trinéaires.

La difficulté principale de l'étude est multiple:

- le terme de transport  $(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i$  ainsi que le terme découlant de la condition de Gauckler-Manning sont non linéaires et interdisent des applications immédiates du théorème de Lax-Milgram,
- sans condition de Dirichlet aux bords, aucune inégalité de Poincaré n'est possible, ce qui est un frein pour prouver la coercivité de certaines formes bilinéaires et complique à nouveau la tâche pour utiliser Lax-Milgram,
- Les théorèmes classiques d'existence de solutions pour les équations de Navier-Stokes reposent sur un argument de De Rham pour obtenir la pression à partir du champ de vitesse. Ici, le cadre ne le permet pas, il va donc falloir réussir à "découpler" le problème vitesse/pression en deux problèmes séparés.

Pour contourner ces différentes difficultés, on va opérer deux transformations successives au problème initial. La première va consister à régulariser la condition d'incompressibilité  $\nabla \cdot \mathbf{u}_i = 0$  et à la remplacer par

$$-\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0, \quad \text{dans } \Omega_i,$$

où  $\varepsilon > 0$  très petit a vocation à tendre vers 0.

Il s'agit d'une manipulation très courante en analyse numérique des équations de Navier-Stokes, qui est d'ailleurs recommandée lorsqu'on réalise ce genre de code sur Freefem++ [47]. L'intérêt de cette modification est double:

- ce terme va permettre d'obtenir la pression comme solution d'un problème de Neumann indépendant de (1.17) et s'écrira  $p_i = p_i(\mathbf{u}_i, \varepsilon)$ . Ainsi, on pourra exprimer la pression dans (1.17) par un terme ne dépendant que des vitesses et de  $\varepsilon$ ;
- régulariser les futurs codes numériques.

La deuxième modification va être de linéariser le système en introduisant un champ de

vitesse fixé  $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$  et de remplacer les termes non-linéaires par des termes dépendant de  $\mathbf{w}$ , ce qui aura comme conséquence d'aboutir à une formulation variationnelle autorisant le théorème de Lax-Milgram.

Il s'agira ensuite de remonter jusqu'au problème initial, par convergence lorsque  $\varepsilon \rightarrow 0$  et grâce à un théorème de point fixe de Schauder. On aura alors montré le théorème

**Theorem 1.3.3.** *Le système (1.17) admet une solution faible  $(\mathbf{u}, \mathbf{p}) \in W \times X$ ,*

où  $W$  et  $X$  sont des espaces de Sobolev définis en section 3.2.1 et où  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$  et  $\mathbf{p} = (p_1, p_2)$ .

Une deuxième partie de ce chapitre sera consacrée à la mise en oeuvre d'un algorithme permettant de simuler numériquement ce problème.

On s'appuiera notamment sur [45].

### 1.3.3 Chapitre 4

Ce chapitre est le premier de la seconde partie. Il contient l'article [15]. On s'intéresse ici au problème

$$\begin{cases} \lambda u - \nu_h \Delta_h u - \mu \partial_z (z^\alpha \partial_z u) = f & \text{in } BL \\ \mu z^\alpha \frac{\partial u}{\partial z} = C_D (V(\mathbf{x}_h) - u(\mathbf{x}_h, z_{top})) & \text{on } \Gamma_{top}, \\ u = 0 & \text{on } \Gamma_b, \end{cases} \quad (1.18)$$

où

$$BL = \Pi_2 \times [0, z_{top}], \quad \Gamma_b = \Pi_2 \times \{z = 0\}, \quad \Gamma_{top} = \Pi_2 \times \{z = z_{top}\},$$

où le tore de dimension 2 est noté  $\Pi_2$ . Les autres données sont:

- $\lambda > 0$  une constante de régularisation,
- $\nu_h > 0$  la viscosité cinématique horizontale,
- $\mu > 0$  et  $C_D > 0$  deux paramètres donnés,
- la source  $f$  et la vitesse  $V$  sont également fixées,
- enfin  $\alpha$  est compris entre 0 et 1 et constitue le coeur de l'étude qui va suivre.

Les conditions du problème sont donc de type Robin sur la partie supérieure du domaine, de type Dirichlet sur le bas du domaine et périodiques horizontalement.

Il s'agit d'une modélisation des phénomènes de turbulence dans la couche limite surfacique de l'atmosphère, où la viscosité turbulente  $\nu_{\text{turb}}$  est donnée par  $\nu_{\text{turb}} = \mu z^\alpha$ , le cas  $\alpha = 1$  correspondant à la réalité physique. (voir [23, 58, 76])

Le but de ce chapitre est de vérifier si le modèle est conforme à la théorie de Monin-Obukhov [73], i.e, si l'on observe des profils de vitesse verticale lineaires-logarithmiques, à d'éventuelles fonctions stabilisantes près.

Tout d'abord nous allons montrer l'existence de solutions faibles au problème (1.18) lorsque  $0 < \alpha < 1$ , ce qui constitue le théorème 4.3.1:

**Theorem 1.3.4.** *Le problème (1.18) admet une solution faible  $u \in V_\alpha$ . De plus, cette solution satisfait l'inégalité d'énergie*

$$\lambda \int_{BL} u^2 + \nu_h \int_{BL} |\nabla_h u|^2 + \mu \int_{BL} z^\alpha |\partial_z u|^2 + C_D \int_{\Gamma_{top}} |u|^2 - \int_{\Gamma_{top}} u G \leq \int_{BL} f u,$$

où  $V_\alpha$  est défini dans la partie 4.3.1.

L'une des principales difficultés de cette étude a justement été de définir correctement l'espace  $V_\alpha$  pour avoir des solutions faibles bien définies sur le bord du domaine.

Ce théorème se démontre par régularisation, en ajoutant le terme de diffusion verticale  $-\varepsilon \partial^2 z z u$ , où  $\varepsilon > 0$  est fixé. On est donc amené à étudier le système:

$$\begin{cases} \lambda u - \nu_h \Delta_h u - \mu \partial_z (z^\alpha \partial_z u) - \varepsilon \partial_{zz}^2 u = f, \\ C_D u + \mu z^\alpha \frac{\partial u}{\partial z} = G \quad \text{sur } \Gamma_{top}, \\ u = 0 \quad \text{sur } \Gamma_b. \end{cases} \quad (1.19)$$

Celui-ci admet une unique solution faible  $u_\varepsilon$  par le théorème de Lax-Milgram. En choisissant comme fonction test  $u_\varepsilon$  et en passant à la limite dans le problème régularisé (1.19), on montre alors l'existence et l'unicité d'une solution faible de (1.18) qui vérifie bien l'inégalité d'énergie énoncée.

Le cas  $\alpha = 1$  fait l'objet d'un traitement à part dans ce chapitre. On considère à présent le système

$$\begin{cases} \lambda u - \nu_h \Delta_h u - \mu \partial_z (z \partial_z u) = f \quad \text{dans } BL, \\ C_D u + \mu z \frac{\partial u}{\partial z} = G \quad \text{sur } \Gamma_{top}. \end{cases} \quad (1.20)$$

Contrairement au système précédent (1.18), nous n'imposons pas ici de condition au bord sur la frontière inférieure  $\Gamma_b$  du domaine. En effet, nous ne pouvons pas donner un sens à la trace des fonctions en jeu, puisqu'il n'y a pas d'estimation dans un meilleur espace que  $H^{1/2}$ . Le résultat principal de cette section est d'ailleurs le résultat d'estimation suivant:

**Theorem 1.3.5.** *Soit  $f \in L^2$  and  $G \in L^2$ . Alors il existe une solution  $u \in V_1 \subset H^{1/2}(BL)$  du problème (1.20), qui vérifie*

$$\|v\|_{V_1} \leq C(\|f\|_{2,BL} + \|G\|_{2,\Gamma_{top}}).$$

Enfin, la dernière partie de ce chapitre est consacrée au traitement numérique des systèmes étudiés. Nous avons simulé sur Freefem++ le système

$$\begin{cases} \kappa u^* z_{top} \frac{du}{dz}(x, z_{top}) = C_D (V(x) - u(x, z_{top})), & \text{en } z = z_{top}, \\ u(x, 0) = 0, & \text{en } z = 0, \end{cases} \quad (1.21)$$

où

$$V(x) = u_{\text{Log}}(z_{top})(1 + \epsilon(x)), \quad (1.22)$$

$x \mapsto \epsilon(x)$  étant une petit perturbation, et

$$u_{\text{Log}}(z) = \begin{cases} \frac{C_* u^*}{C_\nu} z, & z \in [0, z_0] \\ \frac{C_* u^*}{C_\nu} \left( \log\left(\frac{z}{z_0}\right) + 1 \right), & z \in [z_0, z_{top}]. \end{cases} \quad (1.23)$$

Nous avons enfin évaluer numériquement pour plusieurs valeurs de  $\alpha$  comprises entre 0 et 1, la différence entre les solutions de (1.21) et les profils logarithmiques de la forme  $u_{\text{Log}}$ . Pour réduire l'écart entre  $u$  et  $u_{\text{Log}}$ , nous avons obtenu par interpolation des fonctions de stabilisation  $\Psi$  (voir [74], [76]) qui donne pour  $u + \Psi - u_{\text{Log}}$  une erreur bien plus minime, quelque soit le poids  $\alpha$  choisi.

### 1.3.4 Chapitre 5

Dans ce chapitre on va considérer un autre modèle de turbulence. Les deux principales caractéristiques d'un modèle de turbulence pour la simulation d'un fluide d'une paroi solide sont la viscosité turbulente et la condition limite de frottement à la paroi. Cette dernière est souvent appelée *loi de paroi*. Dans de nombreux cas, la viscosité turbulente  $\nu_{\text{turb}} \geq 0$  est proportionnelle à une puissance donnée de la longueur de mélange  $\varrho$ , qui, dans plusieurs modèles physiques [81], [83], s'apparente à la distance au bord.

Si nous supposons que la diffusion moléculaire est négligeable par rapport à la diffusion

turbulente, nous sommes amenés à écrire la loi de paroi comme suit:

$$\nu_{\text{turb}} \frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}} = f(\mathbf{u}_\tau), \quad \text{sur } \Gamma, \quad (1.24)$$

où  $\Gamma$  représente la frontière du domaine d'écoulement  $\Omega$  et  $\mathbf{u}_\tau$  la vitesse tangentielle du fluide à  $\Gamma$ .

Rédigée de cette manière, cette condition n'a pas beaucoup de sens puisque  $\nu_{\text{turb}}$  est nul sur  $\Gamma$ , ce qui signifierait que  $\frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}}$  est infini sur  $\Gamma$ , à moins que  $f(\mathbf{u}_\tau)$  ne s'annule, et que  $f$  satisfasse des conditions appropriées (voir dans [23]). Cela signifie qu'aucun frottement ne se produit en quelque sorte lorsque  $f$  s'annule seulement en 0 tel que  $f(\mathbf{w}) = C_D \mathbf{w}|\mathbf{w}|$ , et nous sommes alors confrontés à une condition de non-glisement à la paroi. Le paradoxe de cette situation est que dans une formulation variationnelle classique, nous avonss formellement un terme de bord de la forme

$$\int_{\Gamma} f(\mathbf{u}_\tau) \cdot \mathbf{v}_\tau$$

qui est bien défini (pour des  $f, \mathbf{u}, \mathbf{v}$  suffisamment réguliers), même si  $\mathbf{u}_\tau \neq 0$  à  $\Gamma$ .

Suivant une approche classique introduite par J.-L. Lions [65], il est naturel d'introduire un problème de *perturbations singulières* pour comprendre ce paradoxe. En particulier, dans ce cas présent, nous sommes amenés à considérer une viscosité de la forme  $\nu_{\text{turb}} + \varepsilon$ , avec  $\varepsilon > 0$ , ce qui revient à reconsidérer la viscosité moléculaire, et à poser la question du problème limite et de sa solution, lorsque  $\varepsilon \rightarrow 0$ .

Dans ce chapitre, nous étudions un modèle scalaire simplifié, en ne considérant que la vitesse tangentielle notée  $u$ , le frottement étant linéaire  $f(u) = u$ ,  $\nu_{\text{turb}}(x) = \varrho(x)$ , et seulement un terme de diffusion avec un terme source.

Le problème est alors une sorte de problème de perturbation singulière avec condition de Robin et il s'écrit :

$$\begin{cases} -\nabla \cdot ((\varrho + \varepsilon)\nabla u) = f & \text{sur } \Omega, \\ -\varepsilon \frac{\partial u}{\partial n} = u & \text{sur } \Gamma, \end{cases} \quad (1.25)$$

où  $\Omega \subset \mathbb{R}^N$  ( $N \leq 3$ ) est un domaine borné  $\mathcal{C}^2$  et  $\varrho : \Omega \rightarrow \mathbb{R}^+$  est une fonction  $\mathcal{C}^2(\overline{\Omega}) \cap$

$W^{2,\infty}(\Omega)$  qui satisfait

$$\lim_{\substack{d(x,\Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x,\Gamma)} = 1, \quad \text{et} \quad \inf_{\substack{d(x,\Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0, \quad (1.26)$$

de façon à avoir le même comportement que la fonction distance au bord.

Nous montrons dans cet article que lorsque le terme source satisfait des conditions de compatibilité appropriées, alors il est possible de passer à la limite dans ce problème, d'une certaine manière. Un problème majeur est la vérification de la condition limite, qui sera *formellement*  $u = 0$ , lorsque  $\varepsilon = 0$ . Pour ce faire, nous définirons soigneusement l'espace fonctionnel dans lequel nous pourrons passer à la limite. Le résultat principal de cette section est le théorème

**Theorem 1.3.6.** *Soit  $u = u_\varepsilon \in C^1(\overline{\Omega}) \cap H^2(\Omega)$  la solution du problème (1.25) pour un certain  $\varepsilon > 0$ . Alors, il existe une suite  $(\varepsilon_n)_{n \in \mathbb{N}}$  qui converge vers 0, et une fonction  $u \in W$ , telle que  $(u_{\varepsilon_n})_{n \in \mathbb{N}}$  converge faiblement dans  $W$  vers  $u$ , qui satisfait  $\forall v \in W$ ,*

$$(u, v)_W =_{W'} \langle f, u \rangle_W, \quad (1.27)$$

où l'espace fonctionnel  $W$  est défini dans la partie 5.2.2, pour donner un sens aux traces des fonctions étudiées.

Ensuite, nous fournissons plusieurs exemples analytiques de cas d'explosion pour des données mal préparées, ainsi que des exemples pour des données bien préparées, pour lesquelles la convergence vers le problème de Dirichlet homogène est vérifiée. On peut noter que dans tous les exemples qui "fonctionnent", en passant à la limite  $\varepsilon \rightarrow 0$ , nous trouvons  $u = 0$  à  $\Gamma$ , un résultat que nous ne sommes pas en mesure de prouver théoriquement à l'heure actuelle et qui semble échapper à la théorie classique des problèmes variationnels dans des espaces de Sobolev à poids.

## 1.4 Perspectives

Nous listons ici les différentes perspectives et possibilités offertes pour continuer ce travail.

1. Etudier les cas d'évolution dans tous les modèles considérés. En effet, sans remettre en question la pertinence des modèles stationnaires étudiés, les cas d'évolutions



correspondent à des modèles physiques actualisés en permanence donc plus précis mais aussi plus complexes.

2. Généraliser le couplage de deux systèmes fluide-fluide à un système multi-couches de plusieurs fluides, qui serait plus représentatif de la très grande diversité de phénomènes physiques et de conditions différentes à l'interface entre l'océan et l'atmosphère. Il s'agirait alors de trouver le juste milieu entre un modèle simplifié comme ceux décrits dans les chapitres 2 et 3, et un modèle comportant "trop" de couplages et donc trop lourd à simuler.
3. Etablir les résolutions numériques en trois dimensions au lieu de deux, ce qui a priori ne devrait pas changer grand chose.
4. Trouver de meilleures fonctions de stabilisation que celles présentées dans le chapitre 4, obtenues par ailleurs de façon très empirique.

PART I

# Linear and non linear coupling problems

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# CONTINUOUS BOUNDARY CONDITION AT THE INTERFACE FOR TWO COUPLED FLUIDS

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This chapter is dedicated to the study of the coupling of two Stokes equations for two different fluids and is following the article [55]. We start with theoretical results about existence of weak solutions and we continue with a numerical analysis of the Schwarz algorithms we used to compute our problems. We will give theoretical convergence rates and perform simulations in Freefem++ to illustrate our results.

## 2.1 Existence of weak solutions

### 2.1.1 Introduction

We consider the two coupled fluids problem with a rigid lid assumption, given by two 3D Stokes equations,

$$-\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, \quad \nabla \cdot \mathbf{u}_i = 0, \quad (2.1)$$

$$\nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} \Big|_{\Gamma_I} = -\alpha (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}), \quad w_i|_{\Gamma_I} = \mathbf{u}_i \cdot \mathbf{n}_i|_{\Gamma_I} = 0, \quad (2.2)$$

$$\mathbf{u}_i|_{\Gamma_i} = 0, \quad (2.3)$$

for  $i = 1, 2$ , where the velocities  $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{x}_h, z_1), \mathbf{u}_2(\mathbf{x}_h, z_2))$  are decomposed as  $\mathbf{u}_i = (\mathbf{u}_{i,h}, w)$ ,  $\mathbf{u}_{i,h} = (u_{i,x}, u_{i,y})$ . Moreover,  $\mathbf{x}_h \in \mathbb{T}_2$ , where  $\mathbb{T}_2$  is the two dimensional torus, which means that we consider horizontal periodic boundary conditions. The interface  $\Gamma_I$  is given by  $\Gamma_I = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}$ , the boundaries  $\Gamma_i$  are given by  $\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\}$ ,  $\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\}$ ,  $z_1 \in J_1 = [0, z_1^+]$ ,  $z_2 \in J_2 = [z_2^-, 0]$ , where  $z_1^+ > 0$  and  $z_2^- < 0$ . The coefficient  $\nu_i > 0$  is the viscosity of the fluid  $i$ ,  $p_i$  its pressure and  $\alpha$  the friction

coefficient of the Navier law. The domain is described in the figure 2.1.

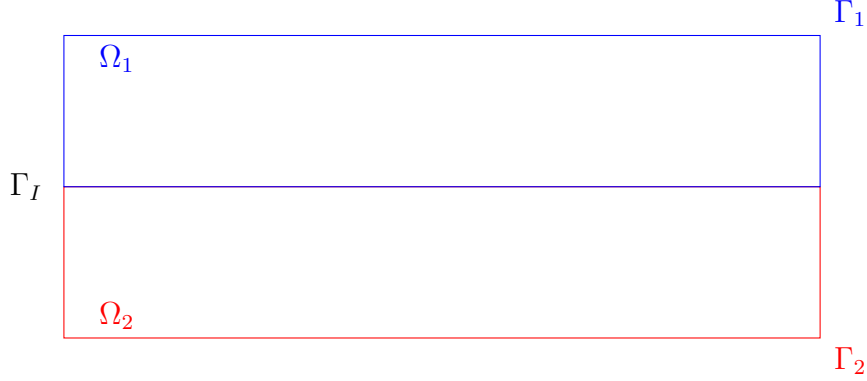


Figure 2.1 – 2D-view of the domains

Similar problems have been already studied before, see [12, 27, 28, 61], and the existence and uniqueness of a weak solution is guaranteed. Notice that the rigid lid assumption we consider is reasonable for laminar coupled flows, as well as for large scales. We aim to investigate how Problem (2.1)-(2.2)-(2.3) approaches problem

$$-\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, \quad \nabla \cdot \mathbf{u}_i = 0, \quad (2.4)$$

$$\mathbf{u}_{1,h}|_{\Gamma_I} = \mathbf{u}_{2,h}|_{\Gamma_I}, \quad w_i|_{\Gamma_I} = \mathbf{u}_i \cdot \mathbf{n}_i|_{\Gamma_I} = 0, \quad (2.5)$$

$$\mathbf{u}_1|_{\Gamma_1} = \mathbf{u}_2|_{\Gamma_2} = 0, \quad (2.6)$$

when the friction coefficient  $\alpha$  goes to infinity. Such question has already been addressed in [1] for a single fluid, where it is proved that the corresponding solution strongly converges to a solution to the corresponding Stokes (Navier-Stokes) equations with a no slip boundary condition when  $\alpha \rightarrow \infty$ . We show in this paper the convergence in  $H^1$  space type of the solution of (2.1)-(2.2)-(2.3) to a solution of (3.1)-(2.5)-(2.6) (see Theorem 2.1.1).

The continuity boundary condition (2.5) of the limit problem is natural and physical [7], and usually considered for free interfaces [52]. In this paper we tackle the question of the existence and uniqueness of a weak solution to Problem (3.1)-(2.5)-(2.6), given as the limit of "frictional solutions", for which we can write a numerical Schwarz-like algorithm (see 2.2).

As we will see, numerical simulations are easily carried out by (2.1)-(2.2)-(2.3) thanks to domain decomposition coupling algorithms, which do not work for (3.1)-(2.5)-(2.6).

This method has already been successfully implemented for coupled problems, see for example [96].

Let us mention that this result raises an interesting open question from the practical point of view. Indeed, in computer simulations that rely on air-sea interaction, the friction parameter  $\alpha$  is chosen based on modeling considerations for boundary-layer interactions, using empirical data, which may be in contradiction with the result of this convergence analysis. However, simulations of air-sea interaction use turbulence models that parametrize statistical means or large scales of the computed fields, for which nonlinear friction laws are modeled through Reynolds stresses analysis [23]. This opens the question to know if the convergence result still holds in a non linear case with models such as  $k - \varepsilon$  models. Nevertheless, we conjecture that it still holds when we use the Navier-Stokes equations instead of the Stokes equation, which is a work in progress, and that the method initiated in this work by the convergence result, can be used in a context of direct numerical simulations (DNS) with a continuity condition at the interface, which opens a way for coupled fluids analysis with free boundaries.

The work is organized as follows. In the first part we set the functional framework and then we prove the convergence result, namely Theorem 2.1.1. In the second part, we describe our algorithm and show some numerical results in the 2D case. In particular we check the numerical convergence of the algorithm.

## 2.1.2 Convergence analysis

### Energy balance

This section is devoted to the derivation of the main a priori estimate, which is standard. Let  $(\mathbf{u}_1, \mathbf{u}_2)$  be any enough smooth solution to Problem (2.1)-(2.2)-(2.3). Taking the scalar product of equation (2.1)<sub>*i*</sub> by  $\mathbf{u}_i$  in integrating over  $\mathbb{T}_2 \times J_i$  over yields by (2.3)<sub>*i*</sub>, because of the periodic boundary conditions in the  $x - y$  axes, the incompressibility condition and  $\mathbf{u}_i \cdot \mathbf{n}_i = 0$  at  $\Gamma_I$ ,

$$\nu_i \int_{\mathbb{T}_2 \times J_i} |\nabla \mathbf{u}_i|^2 - \nu_i \int_{\Gamma_I} \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = \int_{\mathbb{T}_2 \times J_i} \mathbf{f}_i \cdot \mathbf{u}_i,$$

giving by (2.2),  $\nu_i \int_{\mathbb{T}_2 \times J_i} |\nabla \mathbf{u}_i|^2 + \alpha \int_{\Gamma_I} \mathbf{u}_{i,h} \cdot (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) = \int_{\mathbb{T}_2 \times J_i} \mathbf{f}_i \cdot \mathbf{u}_i$ . Summing up the two equalities yields the following energy balance,

$$\nu_1 \int_{\mathbb{T}_2 \times J_1} |\nabla \mathbf{u}_1|^2 + \nu_2 \int_{\mathbb{T}_2 \times J_2} |\nabla \mathbf{u}_2|^2 + \alpha \int_{\Gamma_I} |\mathbf{u}_{1,h} - \mathbf{u}_{2,h}|^2 = \int_{\mathbb{T}_2 \times J_1} \mathbf{f}_1 \cdot \mathbf{u}_1 + \int_{\mathbb{T}_2 \times J_2} \mathbf{f}_2 \cdot \mathbf{u}_2. \quad (2.7)$$

### Functions spaces, variational formulation

Let  $\mathcal{W}_i = \{\mathbf{u} \in C^\infty(\mathbb{T}_2 \times J_i), \mathbf{u}|_{\Gamma_i} = 0, \mathbf{u} \cdot \mathbf{n}_i|_{\Gamma_i} = 0, \nabla \cdot \mathbf{u}_i = 0\}$ , equipped with  $\|\mathbf{u}\|_{i,1} = \|\nabla \mathbf{u}\|_{L^2(\mathbb{T}_2 \times J_i)}$  which is indeed a norm due to the condition  $\mathbf{u}|_{\Gamma_i} = 0$ . Let  $W_i$  denotes the completion of  $\mathcal{W}_i$  with respect to this norm,

$$W = W_1 \times W_2, \quad W_0 = \{(\mathbf{u}_1, \mathbf{u}_2) \in W, \mathbf{u}_{1,h}|_{\Gamma_I} = \mathbf{u}_{2,h}|_{\Gamma_I} \text{ a.e. in } \Gamma_I\}. \quad (2.8)$$

We equip  $W$  with the scalar product, for any  $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W$ ,

$$\Lambda(\mathbf{U}, \mathbf{V}) = \nu_1 \int_{\mathbb{T}_2 \times J_1} \nabla \mathbf{u}_1 \cdot \nabla \mathbf{v}_1 + \nu_2 \int_{\mathbb{T}_2 \times J_2} \nabla \mathbf{u}_2 \cdot \nabla \mathbf{v}_2. \quad (2.9)$$

The space  $W_0$  is the kernel of the form  $L : (\mathbf{u}_1, \mathbf{u}_2) \rightarrow \mathbf{u}_{1,h}|_{\Gamma_I} - \mathbf{u}_{2,h}|_{\Gamma_I}$ , which is continuous by the trace theorem. Therefore  $W_0$  is a closed hyperplane of  $W$ . Let  $P$  denotes the orthogonal projection over  $W_0$ , and  $\Phi = (\phi_1, \phi_2)$  a unit orthogonal vector to  $W_0$ , so that  $W_0^\perp = \text{vect} \Phi$ .

**Definition 2.1.1.** (*weak solution*) A couple  $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2) \in W$  is a weak solution to Problem (2.1)-(2.2)-(2.3) when  $\forall \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W$ ,

$$\Lambda(\mathbf{U}, \mathbf{V}) + \alpha \int_{\Gamma_I} (\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) \cdot (\mathbf{v}_{1,h} - \mathbf{v}_{2,h}) = \int_{\mathbb{T}_2 \times J_1} \mathbf{f}_1 \cdot \mathbf{v}_1 + \int_{\mathbb{T}_2 \times J_2} \mathbf{f}_2 \cdot \mathbf{v}_2 = (\mathbf{F}, \mathbf{v}). \quad (2.10)$$

Throughout the rest of the paper, we assume that  $\mathbf{f}_i \in L^2(\mathbb{T}_2 \times J_i)$ ,  $i = 1, 2$ . The existence and the uniqueness of a weak solution to Problem (2.1)-(2.2)-(2.3) that satisfies the energy balance (2.7) is straightforward by the Lax-Milgram Theorem for any given  $\alpha > 0$ . Notice that work remains to be done about the pressures, by a suitable adaptation of a De Rham like theorem in this framework, which is an open problem.

## Convergence

Let  $\mathbf{U}^\alpha = (\mathbf{u}_1^\alpha, \mathbf{u}_2^\alpha) \in W$  be the solution of (2.1)-(2.2)-(2.3). We study in this section the convergence of the family  $(\mathbf{U}_\alpha)_{\alpha>0}$  when  $\alpha \rightarrow \infty$ , proving the following result.

**Theorem 2.1.1.** *The family  $(\mathbf{U}^\alpha)_{\alpha>0}$  strongly converges in  $W$  to a weak solution  $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2) \in W_0$  of Problem (3.1)-(2.5)-(2.6) when  $\alpha \rightarrow \infty$ , in the sense:*

$$\forall \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W_0, \quad \Lambda(\mathbf{u}, \mathbf{v}) = (\mathbf{F}, \mathbf{v}). \quad (2.11)$$

Moreover, the solution of (2.11) is unique.

*Proof.* Let  $\mathbf{u}^\alpha = (\mathbf{u}_1^\alpha, \mathbf{u}_2^\alpha) \in W = W_1 \times W_2$  be the solution of (2.1)-(2.2)-(2.3). We first show that the family  $(\mathbf{u}^\alpha)_{\alpha>0}$  is bounded in  $W$ . We have, by (2.7),

$$\|\mathbf{u}^\alpha\|_W^2 + \alpha \int_{\Gamma_I} |\mathbf{u}_{1,h}^\alpha - \mathbf{u}_{2,h}^\alpha|^2 = \int_{\mathbb{T}_2 \times J_1} \mathbf{f}_1 \cdot \mathbf{u}_1^\alpha + \int_{\mathbb{T}_2 \times J_2} \mathbf{f}_2 \cdot \mathbf{u}_2^\alpha, \quad (2.12)$$

which yields

$$\|\mathbf{u}^\alpha\|_W^2 \leq \int_{\mathbb{T}_2 \times J_1} \mathbf{f}_1 \cdot \mathbf{u}_1^\alpha + \int_{\mathbb{T}_2 \times J_2} \mathbf{f}_2 \cdot \mathbf{u}_2^\alpha. \quad (2.13)$$

We deduce from Poincaré and Cauchy-Schwarz inequalities that  $(\mathbf{u}^\alpha)_{\alpha>0}$  is indeed bounded in  $W$ . Therefore, we can extract a subsequence  $(\mathbf{u}^{\alpha_n})_{n \in \mathbb{N}}$  ( $\alpha_n \rightarrow \infty$  as  $n \rightarrow \infty$ ) which converges weakly in  $W$  to some  $U \in W$ . Moreover, by the trace theorem and usual Sobolev compactness results, the corresponding traces are strongly convergent in  $L^2(\Gamma_I)$ . As by (2.12)  $\lim_{n \rightarrow \infty} \text{tr}(\mathbf{u}_{1,h}^{\alpha_n} - \mathbf{u}_{2,h}^{\alpha_n}) = 0$  in  $L^2(\Gamma_I)$ , then  $U \in W_0$ . Finally, take  $\mathbf{v} \in W_0$  in (2.10) as test, so that the boundary term vanishes. By passing to the limit in this case when  $\alpha \rightarrow \infty$ , we obtain that  $U$  is a weak solution to (3.1)-(2.5)-(2.6). Uniqueness is straightforward, which in addition guarantees that the entire family does converge to  $\mathbf{u}$ .

It remains to show the strong convergence. Let  $\lambda_\alpha \in \mathbb{R}$ , be such that  $\mathbf{u}^\alpha = P\mathbf{u}^\alpha + \lambda_\alpha \Phi$  ( $\Phi$  being given in section 3.2.1). We first show the strong convergence of  $(P\mathbf{u}^\alpha)_{\alpha>0}$  to  $\mathbf{u}$  by taking  $P\mathbf{u}^\alpha$  as a test in (2.10) which gives, by using the orthogonal decomposition of  $\mathbf{u}^\alpha$ ,

$$\Lambda(\mathbf{u}^\alpha, P\mathbf{u}^\alpha) = \|P\mathbf{u}^\alpha\|_W^2 = (\mathbf{F}, P\mathbf{u}^\alpha), \quad (2.14)$$

since the boundary term on  $\Gamma_I$  equals to zero by orthogonality. Therefore  $(P\mathbf{u}^\alpha)_{\alpha>0}$  is bounded in  $W_0$ , then converges weakly -up to a subsequence (keeping the same notation)- to a limit  $\mathbf{W}$ , strongly in  $L^2((\mathbb{T}_2 \times J_1) \times (\mathbb{T}_2 \times J_2))$ . Taking  $\mathbf{v} \in W_0$  in (2.10) as test, noting that in this case  $\Lambda(\mathbf{u}^\alpha, \mathbf{v}) = \Lambda(P\mathbf{u}^\alpha, \mathbf{v})$ , and passing to the limit when  $\alpha \rightarrow \infty$ , we



see that  $\mathbf{W}$  is solution of the problem (3.1)-(2.5)-(2.6), hence  $\mathbf{W} = \mathbf{u}$  by uniqueness, and the entire sequence converges. Therefore, passing to the limit in (2.14) yields

$$\lim_{\alpha \rightarrow \infty} \|P\mathbf{u}^\alpha\|_W^2 = (\mathbf{F}, \mathbf{u}) = \Lambda(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|_W^2,$$

which, together with the weak convergence, ensures the strong convergence as claimed. To conclude, it remains to prove that  $\lim_{\alpha \rightarrow \infty} \lambda_\alpha = 0$ . By the energy balance (2.12), we have

$$\limsup_{\alpha \rightarrow \infty} \|\mathbf{u}^\alpha\|_W^2 + \limsup_{\alpha \rightarrow \infty} \alpha \int_{\Gamma_I} |\mathbf{u}_{1,h}^\alpha - \mathbf{u}_{2,h}^\alpha|^2 = (\mathbf{F}, \mathbf{u}). \quad (2.15)$$

However, we have  $\int_{\Gamma_I} |\mathbf{u}_{1,h}^\alpha - \mathbf{u}_{2,h}^\alpha|^2 = \lambda_\alpha^2 \int_{\Gamma_I} |\phi_{1,h} - \phi_{1,h}|^2$ . Therefore, since  $(\|\mathbf{u}_\alpha\|_W^2)_{\alpha > 0}$  is bounded, by (2.15) we have  $(\alpha \lambda_\alpha^2)_{\alpha > 0}$  is bounded, which can happen only if  $\lambda_\alpha \rightarrow 0$  as  $\alpha \rightarrow \infty$ , concluding the proof.  $\square$

We will solve the problems (2.1)-(2.2)-(2.3) for large values of  $\alpha$ , with a coupling Schwarz like algorithm, using the software `Freefem++`, by the finite element method.

In the next sections, we will deal with domain decomposition methods and use them on the coupling problems introduced in section 2.1. We will first give some general explanations on the Schwarz algorithms in the section 2.2, with a simple Laplacian problem example.

We will prove that these algorithms are converging for our Stokes coupling problem, for a given  $\alpha > 0$ .

## 2.2 Introduction of the Schwarz algorithms

### 2.2.1 Historical example

To introduce the domain decomposition methods, and more particularly the Schwarz algorithms, we will first consider the Dirichlet problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.16)$$

where the domain  $\Omega$  can be seen as a disk  $\Omega_1$  overlapping a rectangle  $\Omega_2$ , with the interfaces  $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$  and  $\Gamma_2 = \partial\Omega_2 \cap \Omega_1$ . We show on figure 2.2 this original domain decomposition, introduce as an example by Schwarz himself.

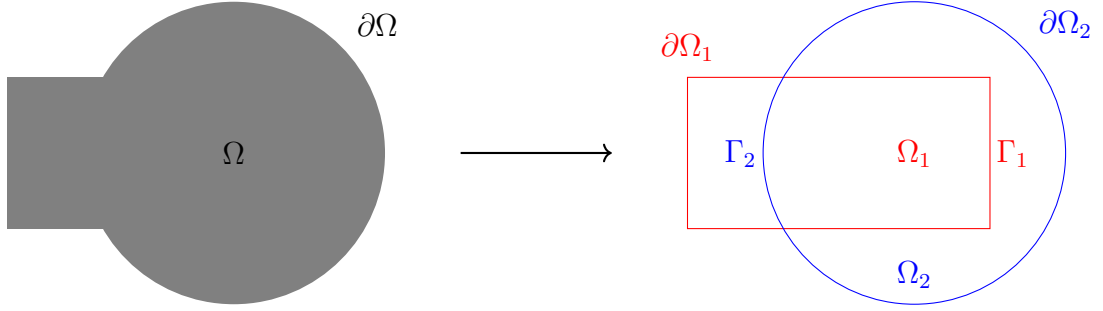


Figure 2.2 – Domain introduced by Schwarz

Instead of solving (2.16) directly in the complex domain  $\Omega$ , Schwarz decided to solve it by iterations, from the disk to the rectangle and inversely, since the solutions were known both for the disk domain and for the rectangle domain. The new problems to solve on both domains are

$$\begin{cases} -\Delta u_1^n = f & \text{in } \Omega_1, \\ u_1^n = u_2^{n-1} & \text{on } \partial\Omega_1, \end{cases} \quad (2.17)$$

in the first subdomain and

$$\begin{cases} -\Delta u_2^n = f & \text{in } \Omega_2, \\ u_2^n = u_1^n & \text{on } \partial\Omega_2, \end{cases} \quad (2.18)$$

in the second domain, where the initial value  $u_2^0$  is fixed. To sum up:

---

**Algorithm 1:** Domain Decomposition Algorithm

---

**Data:** Initial condition:  $u_2^0 \in H^1(\Omega_2)$

**Result:**  $u_1^n$  and  $u_2^n$  solutions of (2.17) (2.18)

- 1 **while** *not converged* **do**
  - 2     **for**  $n \leftarrow 1$  **to**  $N$  **do**
  - 3         └ Solve problems (2.17) and (2.18) in subdomains  $\Omega_i$ ;
  - 4         └ Update the values  $\mathbf{u}_i^k$  Check convergence criteria;
- 

Such a decomposition domain is called nowadays a "Schwarz alternating method". The convergence of the method in a bounded domain can be found in [82] or [32]. To have an idea of the convergence rate of the method, even in a partially-bounded domain<sup>1</sup>, we can use Fourier analysis on the whole space.

In the following, we are using the method developed by M.Gander in [39] and we consider

---

1. Indeed the problem studied in the section 2.1 is defined in the cartesian product of a torus and of a bounded interval

that the domain is  $\mathbb{R}^2$ , where the two subdomains are the upper strip  $\Omega_1 = \mathbb{R} \times (0, +\infty)$  and the lower strip  $\Omega_2 = \mathbb{R} \times (-\infty, \delta)$ , as in 2.3.

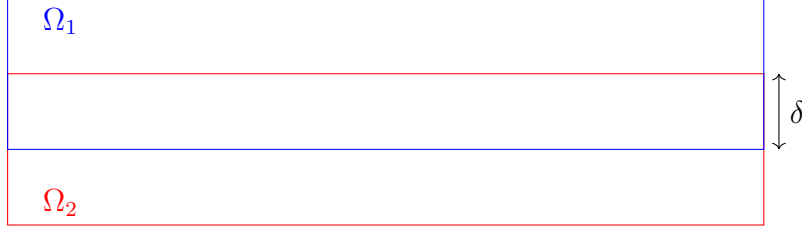


Figure 2.3 – Domain splitting with overlap  $\delta$

We also impose that the solutions decay at infinity. By linearity, we can suppose that  $f = 0$ . Let introduce now the error factors, which will enable us to check if the algorithm is converging.

**Definition 2.2.1.** (*Error factors*)

We define the error factors after the  $n$ -th step on the domains  $\Omega_1$  and  $\Omega_2$  by

$$e_1^n = u_1^n - u|_{\Omega_1} \quad \text{and} \quad e_2^n = u_2^n - u|_{\Omega_2},$$

where  $u_1^n$  and  $u_2^n$  are the solutions of the problems (2.17),(2.18) and  $u|_{\Omega_1}$ ,  $u|_{\Omega_2}$  are the solutions of the initial problem (2.16).

We can now give a convergence result.

**Proposition 2.2.1.** *The Schwarz algorithm 4 is converging, which means that the error factors sequences  $(e_1^n)_{n \in \mathbb{N}}$  and  $(e_2^n)_{n \in \mathbb{N}}$  are converging to 0 when  $n$  goes to infinity.*

*Moreover, the error terms after the  $n$ -th step are verifying*

$$\|e_1^n(\cdot, k)\|_{0,2,x} = \rho^n \|e_0^2(\cdot, k)\|_{0,2,x} \quad \text{and} \quad \|e_2^n(\cdot, k)\|_{0,2,x} = \rho^n \|e_0^2(\cdot, k)\|_{0,2,x},$$

where  $\rho$  equals

$$\rho = \rho(\delta, k) := e^{-2\delta|k|}, \tag{2.19}$$

and is defined as the convergence factor of the Schwarz algorithm.

**Remark 2.2.1.** *The convergence rate  $\rho$  is strictly smaller than 1 when the overlap  $\delta$  is positive, independently of  $k$ .*

*Proof.* Rewriting the Schwarz algorithm gives:

$$\begin{cases} -\Delta e_1^n = 0 & \text{in } \Omega_1, \\ e_1^n(x, 0) = e_2^{n-1}(x, 0) & , x \in \mathbb{R} \end{cases} \quad (2.20)$$

$$\begin{cases} -\Delta e_2^n = 0 & \text{in } \Omega_2, \\ e_2^n(x, \delta) = e_1^n(x, \delta) & , x \in \mathbb{R} \end{cases} \quad (2.21)$$

Define the fourier transform in the  $x$ -direction:

$$\hat{g}(k) = \mathcal{F}(g)(k) = \int_{\mathbb{R}} g(y) e^{-iky} dy, \quad (2.22)$$

where  $k \in \mathbb{R}$  is the frequency. Taking the Fourier transform of the Schwarz algorithm (2.20) (2.21) in the  $x$ -direction, and using the Fourier property that transforms the derivative in  $x$  on multiplication by the factor  $ik$ , we obtain the differential equations in the variable  $y$ :

$$\begin{cases} -(-k^2 + \partial_{yy}^2) \hat{e}_1^n = 0, & y > 0, k \in \mathbb{R} \\ \hat{e}_1^n(k, 0) = \hat{e}_2^{n-1}(k, 0) & , k \in \mathbb{R} \end{cases} \quad (2.23)$$

$$\begin{cases} -(-k^2 + \partial_{yy}^2) \hat{e}_2^n = 0, & y < \delta, k \in \mathbb{R} \\ \hat{e}_2^n(k, \delta) = \hat{e}_1^n(k, \delta) & , k \in \mathbb{R} \end{cases} \quad (2.24)$$

For a given  $k$ , the solutions of the previous linear differential systems are of the form:

$$\hat{e}_j^n(k, y) = A_j^n(k) e^{|k|y} + B_j^n(k) e^{-|k|y}, \quad j = 1, 2, \quad (2.25)$$

where  $A_j^n(k) \in \mathcal{C}$  and  $B_j^n(k) \in \mathcal{C}$ , for  $j = 1, 2$ ,  $n \in \mathbb{N}$ ,  $k \in \mathbb{R}$ . By the condition of decay at infinity of the solutions on the subdomains, we get  $A_1^n = 0$  and  $B_2^n = 0$ , hence

$$\begin{cases} \hat{e}_1^n = B^n(k) e^{-|k|y}, & y > 0, \\ \hat{e}_2^n = A^n(k) e^{|k|y}, & y < \delta. \end{cases} \quad (2.26)$$

Finally, using the interface condition on  $y = 0$  and  $y = \delta$ , we obtain the recurrence relation

$$\begin{cases} B^n = A^{n-1} \\ A^n e^{-\delta|k|} = B^n e^{\delta|k|}, & k \in \mathbb{R}, \end{cases} \quad (2.27)$$

which gives  $B^n = \rho B^{n-1}$ , and the same iteration relation for  $A_n$ .

The Parseval isometry is finishing the proof.

□

## 2.2.2 Alternative Schwarz algorithms

Another possibility has been developed by P.L Lions in [67], consisting in fixing both the initial starting values  $u_1^0$  and  $u_2^0$ , and to solve the "Schwarz paralled method":

$$\begin{cases} -\Delta u_1^n = f & \text{in } \Omega_1, \\ u_1^n = u_2^{n-1} & \text{on } \partial\Omega_1, \end{cases} \quad (2.28)$$

in the first subdomain and

$$\begin{cases} -\Delta u_2^n = f & \text{in } \Omega_2, \\ u_2^n = u_1^{n-1} & \text{on } \partial\Omega_2, \end{cases} \quad (2.29)$$

in the second domain. The only modification is the interface condition on  $\partial\Omega_2$ . It does not change the convergence rate but it is easier to generalise when more than two domains are used. The main drawback of the original method developed by Schwarz is the fact that the method is not converging when the domains are not overlapping, i.e, where  $\Gamma_1 = \Gamma_2$ . To tackle this issue, PL Lions improved the algorithm in replacing the natural Dirichlet conditions by Robin conditions. The starting alternative schwarz algorithm is becoming:

$$\begin{cases} -\Delta u_1^n = f & \text{in } \Omega_1, \\ (\frac{\partial}{\partial \mathbf{n}_1} + \mathcal{L}_1)u_1^n = (\frac{\partial}{\partial \mathbf{n}_2} + \mathcal{L}_1)u_2^{n-1} & \text{on } \partial\Omega_1, \end{cases} \quad (2.30)$$

and

$$\begin{cases} -\Delta u_2^n = f & \text{in } \Omega_2, \\ (\frac{\partial}{\partial \mathbf{n}_2} + \mathcal{L}_2)u_2^n = (\frac{\partial}{\partial \mathbf{n}_2} + \mathcal{L}_2)u_1^n & \text{on } \partial\Omega_2, \end{cases} \quad (2.31)$$

where  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are linear operators. The choice of these operators is crucial to the convergence rate of the method (see [39],[40]). The physics underlying often imposes constraints on these operators.

We will take in our study  $\mathcal{L}_j = \alpha Tr()$ , where  $\alpha \geq 0$ , i.e, we will consider the problems:

$$\begin{cases} -\Delta u_1^n = f & \text{in } \Omega_1, \\ (\frac{\partial}{\partial \mathbf{n}_1} + \alpha)u_1^n = (\frac{\partial}{\partial \mathbf{n}_2} + \alpha)u_2^{n-1} & \text{on } \partial\Omega_1, \end{cases} \quad (2.32)$$

and

$$\begin{cases} -\Delta u_2^n = f & \text{in } \Omega_2, \\ (\frac{\partial}{\partial \mathbf{n}_2} + \alpha)u_2^n = (\frac{\partial}{\partial \mathbf{n}_1} + \alpha)u_1^n & \text{on } \partial\Omega_2. \end{cases} \quad (2.33)$$

The algorithm to solve (2.32) and (2.33) becomes:

---

**Algorithm 2:** Domain Decomposition with Robin condition

---

**Data:** Initial condition:  $u_2^0 \in H^1(\Omega_2)$

**Result:**  $u_1^n$  and  $u_2^n$  solutions of (2.32) and (2.33)

```

1 while not converged do
2   for  $n \leftarrow 1$  to  $N$  do
3     Solve problems (2.32) and (2.33) in subdomains  $\Omega_i$ ;
4     Update the values  $\mathbf{u}_i^k$  Check convergence criteria;
```

---

We can prove the convergence of the algorithm and we can give the speed of convergence as before.

**Proposition 2.2.2.** *The error sequences  $(\|e_1^n\|_{0,2,\Omega_1})_{n \in \mathbb{N}}$  and  $(\|e_2^n\|_{0,2,\Omega_2})_{n \in \mathbb{N}}$  are converging to 0 and the convergence factor  $\rho$  is defined by*

$$\rho = \left( \frac{|k| - \alpha}{|k| + \alpha} \right)^2 e^{-2\delta|k|}, \quad (2.34)$$

for  $k \in \mathbb{R}$ .

*Proof.* We use the same Fourier analysis as before, which leads to

$$\begin{cases} -(-k^2 + \partial_{yy}^2)\hat{e}_1^n = 0, & y > 0, & k \in \mathbb{R}, \\ (|k| + \alpha)\hat{e}_1^n(k, 0) = (-|k| + \alpha)\hat{e}_2^{n-1}(k, 0), & k \in \mathbb{R}, \end{cases} \quad (2.35)$$

and

$$\begin{cases} -(-k^2 + \partial_{yy}^2)\hat{e}_2^n = 0, & y < \delta, & k \in \mathbb{R}, \\ (|k| + \alpha)\hat{e}_2^n(k, \delta) = (-|k| + \alpha)\hat{e}_1^n(k, \delta), & k \in \mathbb{R}, \end{cases} \quad (2.36)$$

where  $\hat{e}_1^n$  and  $\hat{e}_2^n$  are still defined by.

$$\begin{cases} \hat{e}_1^n = B^n(k)e^{-|k|y}, & y > 0, \\ \hat{e}_2^n = A^n(k)e^{|k|y}, & y < \delta. \end{cases} \quad (2.37)$$

From the interface conditions of (2.35) and of (2.36), we get the relation

$$\begin{cases} (|k| + \alpha)B^n = (|k| - \alpha)A^{n-1}, \\ (|k| + \alpha)A^n = (|k| - \alpha)B^n, \end{cases} \quad (2.38)$$

which gives the relation  $B^n = \rho B^{n-1}$ , where

$$\rho = \left( \frac{|k| - \alpha}{|k| + \alpha} \right)^2 e^{-2\delta|k|}.$$

□

**Remark 2.2.2.** *The benefit of this "symmetrical" Robin condition on both sides is the fractional term  $\frac{|k| - \alpha}{|k| + \alpha}$  which is smaller than 1 even when the overlap  $\delta$  is taken 0, which will be very important for us. Indeed, when we will simulate the behaviour of our two fluids, the rigid lid condition states that there is no mixing allowed, i.e no overlapping between the two domains.*

The figure 2.4 shows the comparison between the convergence rate of the method with the Dirichlet condition in blue and with the Robin condition in red, for  $\alpha \in \{1, 10\}$  and for the overlap  $\delta \in \{0.01, 0.1\}$ .

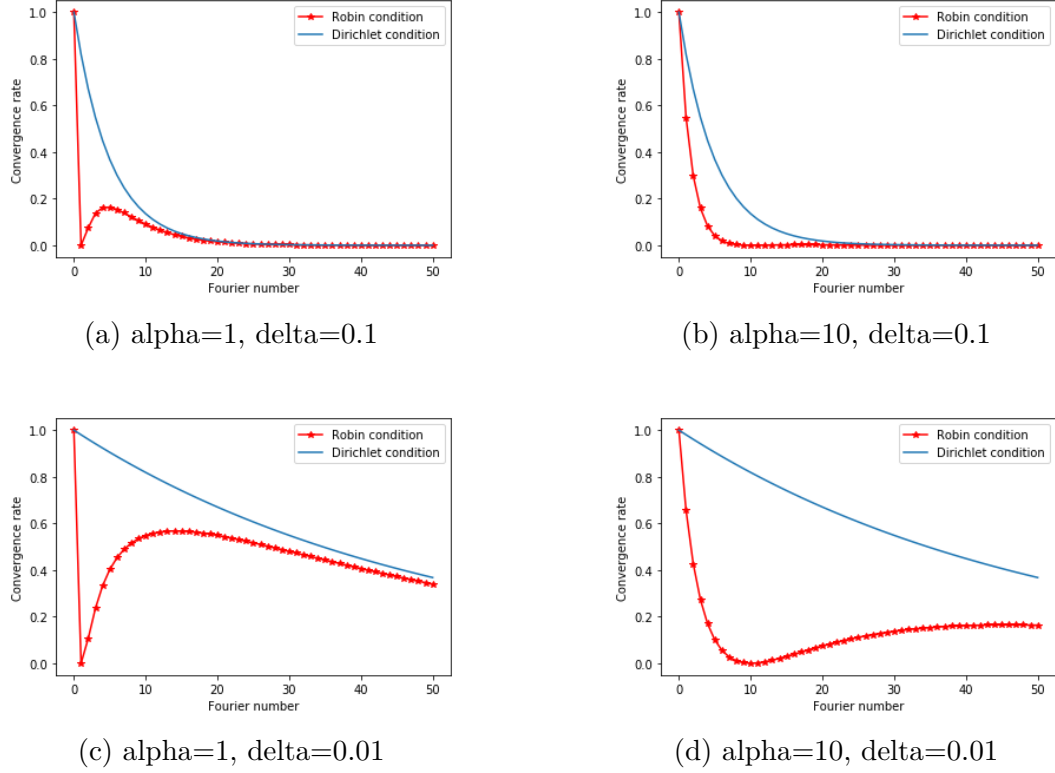


Figure 2.4 – Rate convergences for different values of delta and alpha

**Remark 2.2.3.** Pay attention to the fact that the Fourier convergence rates we exhibit are "theoretical ones", regarding that all our simulations are carried out on a bounded domain.

We show different steps of the algorithm 4 with overlap.

- The four images in 2.5 correspond to the Dirichlet boundary conditions case on  $\Gamma_1$  and  $\Gamma_2$ .
- The four images in 2.6 correspond to periodic boundary conditions.

The simulations have been obtained via Freefem++, with  $P2$ -finite elements. The two domains  $\Omega_1$  and  $\Omega_2$  have been chosen equal here, with the same triangular mesh composed of  $100 \times 100$  cells.

The size of the overlap  $\delta$  is equal to 0.01, while the size of the box is equal to 1.

We skip the details and the difficulties encountered about the implementation in Freefem++. They are explained in the section 2.3.4. The values scale is "orange" < "yellow" < "green" < "blue" < "purple" < "pink", and the reading order is



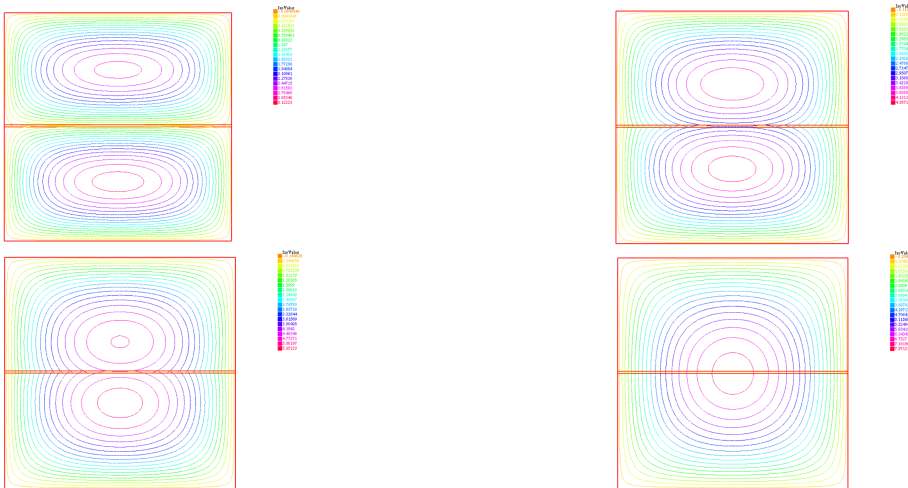
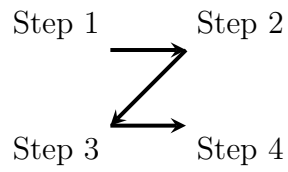


Figure 2.5 – Dirichlet boundary conditions

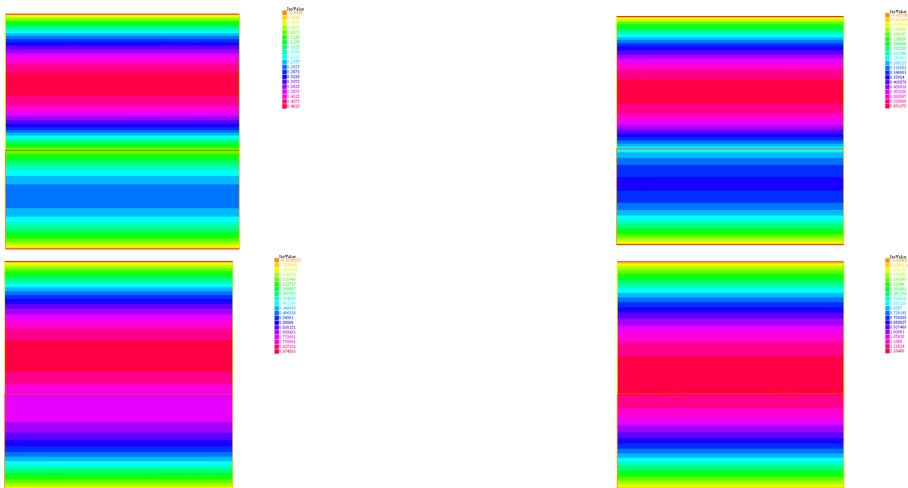


Figure 2.6 – Periodic boundary conditions

We can see in both simulations 2.5 and 2.6 that the last picture corresponds to the situation we would have obtained in the case of a unique domain with a unique PDE.

### 2.2.3 Benchmark test for overlapping and non-overlapping domains, with Robin condition

In general, we won't necessarily have exact solutions to compare with our approximate solutions. We present a particular case with a known solution and we display the errors between the exact and the numerical solutions to check that the algorithm 4 is doing the job.

**Remark 2.2.4.** *Since we study the non-overlapping case, we have to use a Robin condition and the converging algorithm associated with. We recall that the classical Dirichlet condition algorithm 4 is not converging in the non-overlapping case.*

We consider the source function

$$f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y).$$

An exact solution of the Laplacian problem on the square  $[0, 1] \times [0, 1]$  with Dirichlet conditions is given by

$$u(x, y) = \sin(\pi x) \sin(\pi y).$$

The functions  $u_1$  and  $u_2$  we consider are also equal to  $u$ , and we denote by  $u_1^n$  and  $u_2^n$  their approximation at the  $n$ -th step obtained by the algorithm 4.

We choose to stop the computation whenever  $\|u_1^{n+1} - u_1^n\|_{L^2} + \|u_2^{n+1} - u_2^n\|_{L^2} \leq 10^{-6}$ . The first case we study is the overlapping one with  $\delta = 0.01$ , which corresponds to the table 2.1 and the second one is the non-overlapping algorithm, which corresponds to the results displayed in 2.2. Each table is arranged as follows:

- In the first column is given the first value of the  $n$ -th iteration which fulfills the stopping criteria,
- In the column 2 and 3 are given the relative  $L^2$  and  $H^1$  errors

$$relerr_{L^2} := \left( \frac{\|u_1^n - u_1\|_{L^2}^2 + \|u_2^n - u_2\|_{L^2}^2}{\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2} \right)^{1/2},$$

and

$$relerr_{H^1} := \left( \frac{\|u_1^n - u_1\|_{H^1}^2 + \|u_2^n - u_2\|_{H^1}^2}{\|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2} \right)^{1/2}.$$

| $\alpha$ | First value of $n$ | $relerr_{L^2}$         | $relerr_{H^1}$         |
|----------|--------------------|------------------------|------------------------|
| 100      | 139                | $6.165 \times 10^{-3}$ | $6.853 \times 10^{-3}$ |
| 500      | 217                | $1.773 \times 10^{-3}$ | $2.098 \times 10^{-3}$ |
| 1000     | 235                | $9.259 \times 10^{-4}$ | $1.224 \times 10^{-3}$ |
| 5000     | 251                | $1.476 \times 10^{-4}$ | $6.823 \times 10^{-4}$ |

 Table 2.1 – Relative errors in the overlapping case,  $\delta = 0.01$ .

| $\alpha$ | First value of $n$ | $relerr_{L^2}$         | $relerr_{H^1}$         |
|----------|--------------------|------------------------|------------------------|
| 100      | 257                | $3.373 \times 10^{-3}$ | $4.032 \times 10^{-3}$ |
| 500      | 1033               | $7.655 \times 10^{-4}$ | $1.175 \times 10^{-3}$ |
| 1000     | 1865               | $9.539 \times 10^{-4}$ | $1.360 \times 10^{-3}$ |
| 5000     | 7005               | $4.904 \times 10^{-3}$ | $5.864 \times 10^{-3}$ |

Table 2.2 – Relative errors in the non-overlapping case.

Obviously, the convergence is way faster in the overlapping case as in the non-overlapping one, since the convergence factor in the overlapping case is improved by a factor  $e^{-2\delta}$ . The relative errors at the end of the computation are smaller than  $10^{-2}$  in every scenario. We can conclude that the stopping criteria we have used is quite relevant and we will keep in the following this criteria to end our algorithms. Note that it doesn't depend on the exact solution that we won't have in general.

#### 2.2.4 Comparison between the Dirichlet and the Robin conditions in the overlapping case

We detail in the following table the influence of the overlapping length  $\delta$  on the convergence rate for the Dirichlet and Robin conditions.

Since we won't have access to the exact solution in the future simulations, we will consider the stopping criteria  $\|u_1^{n+1} - u_1^n\|_{L^2} + \|u_2^{n+1} - u_2^n\|_{L^2} \leq \varepsilon_{tol}$ , where  $\varepsilon_{tol} = 10^{-6}$ .

We consider the domains  $\Omega_1 = (0, 1) \times (0, z_1)$ , and  $\Omega_2 = (0, 1) \times (\delta, -z_2)$ , where  $z_1 = 0.6$  and  $z_2 = 0.4$ , and the problems

$$\begin{cases} -\nu_1 \Delta u_1^n = f_1 & \text{in } \Omega_1, \\ u_1^n = u_2^{n-1} & \text{on } \partial\Omega_1, \end{cases} \quad (2.39)$$

$$\begin{cases} -\nu_2 \Delta u_2^n = f_2 & \text{in } \Omega_2, \\ u_2^n = u_1^{n-1} & \text{on } \partial\Omega_2. \end{cases} \quad (2.40)$$

The sources  $f_1$  and  $f_2$  are chosen constant equal to 1, the viscosity coefficients  $\nu_1$  and  $\nu_2$  will be fixed equal to 0.1 and the length  $\delta$  will be chosen between 0.1 and 0.001. For each  $\delta$  we display in 2.3 the first iteration step  $n$  such that the tolerance error is reached, and the  $L^2$  and  $H^1$  norms of  $u_1^n$  and  $u_2^n$ .

| $\delta$ | First value of $n$ | $\ u_1^n\ _{L^2}$ | $\ u_2^n\ _{L^2}$ | $\ u_1^n\ _{H^1}$ | $\ u_2^n\ _{H^1}$ |
|----------|--------------------|-------------------|-------------------|-------------------|-------------------|
| 0.1      | 33                 | 0.754187          | 2.18376           | 0.645497          | 2.14087           |
| 0.05     | 61                 | 0.754186          | 2.18375           | 0.582288          | 2.12168           |
| 0.02     | 138                | 0.754182          | 2.18374           | 0.542013          | 2.10791           |
| 0.01     | 257                | 0.754176          | 2.18373           | 0.528241          | 2.10268           |
| 0.005    | 478                | 0.754163          | 2.1837            | 0.521288          | 2.09988           |
| 0.001    | 1996               | 0.75406           | 2.18344           | 0.51562           | 2.09724           |

Table 2.3 – Influence of  $\delta$  in the convergence speed for the Dirichlet model

We have to compare the convergence of (2.39) (2.40) with Dirichlet boundary condition with the Robin model

$$\begin{cases} -\nu_1 \Delta u_1^n = f & \text{in } \Omega_1, \\ (\nu_1 \frac{\partial}{\partial \mathbf{n}_1} + \alpha) u_1^n = (\nu_2 \frac{\partial}{\partial \mathbf{n}_2} + \alpha) u_2^{n-1} & \text{on } \partial\Omega_1, \end{cases} \quad (2.41)$$

and

$$\begin{cases} -\nu_2 \Delta u_2^n = f & \text{in } \Omega_2, \\ (\nu_2 \frac{\partial}{\partial \mathbf{n}_2} + \alpha) u_2^n = (\nu_1 \frac{\partial}{\partial \mathbf{n}_1} + \alpha) u_1^n & \text{on } \partial\Omega_2. \end{cases} \quad (2.42)$$

We assume that  $\nu_1 = \nu_2 = 0.1$  and  $\alpha = 10$ . The influence of the overlapping length  $\delta$  is given in the table 2.4. We can remark that the model (2.41) (2.42) with Robin condition at the interface is clearly better than (2.39) (2.40) with Dirichlet condition for very small values of  $\delta$ . which is no surprise according to the theoretical convergence rates (2.19) and (2.34). The log-log plot 2.7 is summing up the data given in the two previous tables.

| $\delta$ | First value of $n$ | $\ u_1^n\ _{L^2}$ | $\ u_2^n\ _{L^2}$ | $\ u_1^n\ _{H^1}$ | $\ u_2^n\ _{H^1}$ |
|----------|--------------------|-------------------|-------------------|-------------------|-------------------|
| 0.1      | 45                 | 0.765038          | 2.21031           | 0.653686          | 2.16709           |
| 0.05     | 81                 | 0.76441           | 2.20877           | 0.588674          | 2.14501           |
| 0.02     | 154                | 0.763085          | 2.20552           | 0.546569          | 2.12597           |
| 0.01     | 226                | 0.761487          | 2.2016            | 0.531254          | 2.11495           |
| 0.005    | 299                | 0.759766          | 2.19738           | 0.522819          | 2.1062            |
| 0.001    | 405                | 0.757425          | 2.19166           | 0.515214          | 2.09555           |

 Table 2.4 – Influence of  $\delta$  in the convergence speed for the Robin model

We can do the same analysis when the overlapping length  $\delta$  is taken equal to 0, studying the influence of  $\alpha$  on the convergence. The numerical results are shown in 2.5.

| $\alpha$ | First value of $n$ | $\ u_1^n\ _{L^2}$ | $\ u_2^n\ _{L^2}$ | $\ u_1^n\ _{H^1}$ | $\ u_2^n\ _{H^1}$ |
|----------|--------------------|-------------------|-------------------|-------------------|-------------------|
| 10       | 441                | 0.756507          | 2.18942           | 0.513014          | 2.09161           |
| 20       | 809                | 0.755317          | 2.18651           | 0.513638          | 2.09422           |
| 30       | 1169               | 0.754902          | 2.1855            | 0.513835          | 2.09504           |
| 40       | 1507               | 0.754683          | 2.18496           | 0.513923          | 2.09542           |
| 50       | 1833               | 0.754541          | 2.18462           | 0.513968          | 2.0956            |
| 100      | 3359               | 0.754185          | 2.18375           | 0.513999          | 2.09573           |
| 200      | 6111               | 0.753827          | 2.18288           | 0.513866          | 2.09517           |
| 300      | 8633               | 0.753548          | 2.1822            | 0.513688          | 2.09443           |
| 500      | 13271              | 0.753036          | 2.18095           | 0.513307          | 2.09284           |
| 1000     | 23517              | 0.751814          | 2.17797           | 0.512326          | 2.08873           |
| $10^5$   |                    |                   |                   |                   |                   |

 Table 2.5 – convergence rate of the robin model, influence of  $\alpha$ 

**Remark 2.2.5.** *We can observe that the more  $\alpha$  is big, the more the convergence is slow. This can be a problem since we are supposed to consider  $\alpha \rightarrow +\infty$ . In reality, if we scale all our constants,  $\alpha = 10^3$  is already bigger by far than the other data.*

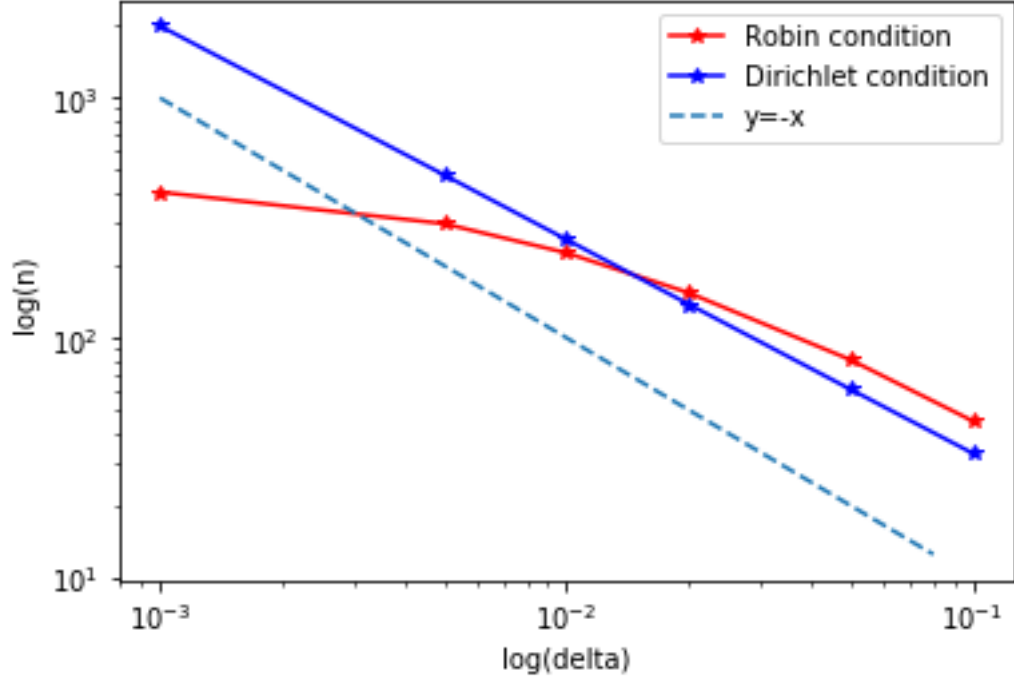


Figure 2.7 – Log-log convergence rate

## 2.3 Schwarz Algorithms for the Stokes coupling problem

The aim of this part is to prove the convergence of the Schwarz algorithm of the problem we have introduced in section 2.1, i.e:

$$\left\{ \begin{array}{l} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, \quad \text{in } \Omega_i \\ \nabla \cdot \mathbf{u}_i = 0, \quad \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -\alpha (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}), \quad \text{on } \Gamma_{Int} \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0, \quad \text{on } \Gamma_{Int} \\ \mathbf{u}_i = 0, \quad \text{on } \Gamma_i. \end{array} \right. \quad (2.43)$$

Contrary to the problems we have studied in this section, we have three major differences:

- we will work with vector fields instead of scalar values,
- there is an additional pressure term to deal with.
- the Robin condition is not "symmetrical" as in (2.41) (2.42). The directional derivative operators are only on the left hand term.

### 2.3.1 Algorithm convergence

To approach (2.43) thanks to Schwarz-like algorithm, we consider the two problems respectively in  $\Omega_1$  and  $\Omega_2$ :

$$\left\{ \begin{array}{l} -\nu_1 \Delta \mathbf{u}_1^{n+1} + \nabla P_1^{n+1} = \mathbf{f}_1, \quad \text{in } \Omega_1, \\ \nabla \cdot \mathbf{u}_1^{n+1} = 0, \quad \text{in } \Omega_1, \\ \nu_1 \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} = -\alpha(\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_{2,h}^n), \quad \text{on } \Gamma_{Int}, \\ \mathbf{u}_{1,h}^{n+1} \cdot \mathbf{n}_1 = 0, \quad \text{on } \Gamma_{Int}, \\ \mathbf{u}_{1,h}^{n+1} = 0, \quad \text{on } \Gamma_1, \end{array} \right. \quad (2.44)$$

and

$$\left\{ \begin{array}{l} -\nu_2 \Delta \mathbf{u}_2^{n+1} + \nabla P_2^{n+1} = \mathbf{f}_2, \quad \text{in } \Omega_2 \\ \nabla \cdot \mathbf{u}_2^{n+1} = 0, \quad \text{in } \Omega_2 \\ \nu_2 \frac{\partial \mathbf{u}_{2,h}^{n+1}}{\partial \mathbf{n}_2} = -\alpha(\mathbf{u}_{2,h}^{n+1} - \mathbf{u}_{1,h}^{n+1}), \quad \text{on } \Gamma_{Int} \\ \mathbf{u}_{2,h}^{n+1} = 0, \quad \text{on } \Gamma_{Int} \\ \mathbf{u}_{2,h}^{n+1} \cdot \mathbf{n}_2 = 0, \quad \text{on } \Gamma_2. \end{array} \right. \quad (2.45)$$

Writing this as an algorithm, we have to follow these lines:

---

#### Algorithm 3: Domain Decomposition Algorithm

---

**Data:** Initial condition:  $\mathbf{u}_2^0 \in W_2$

**Result:** approached velocities and pressures  $\mathbf{u}_i^N$  and  $P_i^N$

```

1 while not converged do
2   for k ← 1 to N do
3     | Solve problem (2.44) and (2.45) in subdomains  $\Omega_i$ ;
4     | Update the velocity and pressure values  $\mathbf{u}_i^k$  and  $P_i^k$ ;
5     | Check convergence criteria;
```

---

We have to be sure that this algorithm is converging. It is the purpose of the next theorem.

**Theorem 2.3.1.** *Let define the velocity errors between the exact solution and the iterative solution at iteration k:*

$$\mathbf{e}_1^k = \mathbf{u}_1^k - \mathbf{u}_1, \quad (2.46)$$

and

$$\mathbf{e}_2^k = \mathbf{u}_2^k - \mathbf{u}_2. \quad (2.47)$$

The sequence  $(\mathbf{e}_1^k, \mathbf{e}_2^k)_{k \in \mathbb{N}}$  converges to  $(0, 0)$  in the space  $W_1 \times W_2$ .

**Remark 2.3.1.** *The result of the theorem and the convergence of the algorithm 3 do not depend on the initial value  $\mathbf{u}_2^0$  we choose.*

*Proof.* Let also define the pressure error terms

$$\eta_i^k = P_i^k - P_i.$$

We will show that the serie  $\sum \left( \|\mathbf{e}_1^{k+1}\|_{W_1}^2 + \|\mathbf{e}_2^{k+1}\|_{W_2}^2 \right)$  converges.

By linearity,  $\mathbf{e}_1^{k+1}$  and  $\mathbf{e}_2^{k+1}$  are solutions of the equations:

$$-\nu_1 \Delta \mathbf{e}_1^{k+1} + \eta_1^{k+1} = 0 \tag{2.48}$$

$$\nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} \Big|_{\Gamma_{Int}} = -\alpha (\mathbf{e}_{1,h}^{k+1} - \mathbf{e}_{2,h}^k), \tag{2.49}$$

$$\mathbf{e}_{1|\Gamma_1}^{k+1} = 0, \tag{2.50}$$

$$\nabla \cdot \mathbf{e}_1^{k+1} = 0, \tag{2.51}$$

$$\mathbf{e}_1^{k+1} \cdot \mathbf{n}_{1|\Gamma_{Int}} = 0, \tag{2.52}$$

and

$$-\nu_2 \Delta \mathbf{e}_2^{k+1} + \eta_2^{k+1} = 0 \tag{2.53}$$

$$\nu_2 \frac{\partial \mathbf{e}_{2,h}^{k+1}}{\partial \mathbf{n}_2} \Big|_{\Gamma_{Int}} = -\alpha (\mathbf{e}_{2,h}^{k+1} - \mathbf{e}_{1,h}^{k+1}), \tag{2.54}$$

$$\mathbf{e}_{2|\Gamma_2}^{k+1} = 0, \tag{2.55}$$

$$\nabla \cdot \mathbf{e}_2^{k+1} = 0, \tag{2.56}$$

$$\mathbf{e}_2^{k+1} \cdot \mathbf{n}_{2|\Gamma_{Int}} = 0. \tag{2.57}$$

Taking the scalar product in (2.48) by  $\mathbf{e}_1^{k+1}$  and using the Green Riemann formula gives

$$\|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2 = - \int_{\Gamma_{Int}} \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} \mathbf{e}_{1,h}^{k+1},$$

since the pressure term is vanishing thanks to the incompressibility condition.



Using the identity  $AB = \frac{1}{4\alpha} ((A + \alpha B)^2 - (A - \alpha B)^2)$ , we have

$$\begin{aligned} \nu_1 \int_{\Gamma_{Int}} \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} \mathbf{e}_{1,h}^{k+1} &= \frac{1}{4\alpha} \int_{\Gamma_{Int}} \left( \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} + \alpha \mathbf{e}_{1,h}^{k+1} \right)^2 - \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} - \alpha \mathbf{e}_{1,h}^{k+1} \right)^2 \right) \\ &= \frac{1}{4\alpha} \int_{\Gamma_{Int}} \left( (\alpha \mathbf{e}_{2,h}^k)^2 - \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} - \alpha \mathbf{e}_{1,h}^{k+1} \right)^2 \right), \end{aligned}$$

using the Robin boundary condition (2.49).

We have obtained the equality

$$-\nu_1 \|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2 = \frac{1}{4\alpha} \int_{\Gamma_{Int}} \left( (\alpha \mathbf{e}_{2,h}^k)^2 - \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} - \alpha \mathbf{e}_{1,h}^{k+1} \right)^2 \right). \quad (2.58)$$

Taking the scalar product in (2.54) by  $\mathbf{e}_2^{k+1}$  and repeating the same operations, we get for the second equation:

$$-\|\nabla \mathbf{e}_2^{k+1}\|_{L^2(\Omega_2)}^2 = \frac{1}{4\alpha} \int_{\Gamma_{Int}} \left( (\alpha \mathbf{e}_{1,h}^{k+1})^2 - \left( \nu_2 \frac{\partial \mathbf{e}_{2,h}^{k+1}}{\partial \mathbf{n}_2} - \alpha \mathbf{e}_{2,h}^{k+1} \right)^2 \right). \quad (2.59)$$

Adding (2.58) and (2.59) and developing the square terms, we have:

$$\begin{aligned} \nu_1 \|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2 + \nu_2 \|\nabla \mathbf{e}_2^{k+1}\|_{L^2(\Omega_2)}^2 + \frac{\alpha}{4} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^k)^2 &= \\ \frac{\alpha}{4} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^{k+1})^2 - \frac{1}{2} \int_{\Gamma_{Int}} \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} \mathbf{e}_{1,h}^{k+1} + \nu_2 \frac{\partial \mathbf{e}_{2,h}^{k+1}}{\partial \mathbf{n}_2} \mathbf{e}_{2,h}^{k+1} \right) & \\ + \frac{1}{4\alpha} \int_{\Gamma_{Int}} \left( \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} \right)^2 + \left( \nu_2 \frac{\partial \mathbf{e}_{2,h}^{k+1}}{\partial \mathbf{n}_2} \right)^2 \right). & \quad (2.60) \end{aligned}$$

We use again the Green Riemann equation to get

$$\begin{aligned} \frac{1}{2} \nu_1 \|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \nu_2 \|\nabla \mathbf{e}_2^{k+1}\|_{L^2(\Omega_2)}^2 + \frac{\alpha}{4} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^k)^2 & \\ = \frac{\alpha}{4} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^{k+1})^2 + \frac{1}{4\alpha} \int_{\Gamma_{Int}} \left( \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} \right)^2 + \left( \nu_2 \frac{\partial \mathbf{e}_{2,h}^{k+1}}{\partial \mathbf{n}_2} \right)^2 \right), & \end{aligned}$$

and the continuity of the trace operator on  $\Gamma_{Int}$ , which states that there exists  $C_1 > 0$

and  $C_2 > 0$  such that

$$\int_{\Gamma_{Int}} \left( \nu_1 \frac{\partial \mathbf{e}_{1,h}^{k+1}}{\partial \mathbf{n}_1} \right)^2 \leq \nu_1^2 C_1 \|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2,$$

and

$$\int_{\Gamma_{Int}} \left( \nu_2 \frac{\partial \mathbf{e}_{2,h}^{k+1}}{\partial \mathbf{n}_2} \right)^2 \leq \nu_2^2 C_2 \|\nabla \mathbf{e}_2^{k+1}\|_{L^2(\Omega_2)}^2,$$

to get

$$\begin{aligned} & \nu_1 \|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2 + \nu_2 \|\nabla \mathbf{e}_2^{k+1}\|_{L^2(\Omega_2)}^2 + \frac{\alpha}{2} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^k)^2 \\ & \leq \frac{1}{2\alpha} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^{k+1})^2 + \frac{2}{\alpha} \left( \nu_1^2 C_1 \|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2 + \nu_2^2 C_2 \|\nabla \mathbf{e}_2^{k+1}\|_{L^2(\Omega_2)}^2 \right), \end{aligned}$$

and we obtain

$$\begin{aligned} & \left( \nu_1 - \frac{\nu_1^2 C_1}{2\alpha} \right) \|\nabla \mathbf{e}_1^{k+1}\|_{L^2(\Omega_1)}^2 + \left( \nu_2 - \frac{\nu_2^2 C_2}{2\alpha} \right) \|\nabla \mathbf{e}_2^{k+1}\|_{L^2(\Omega_2)}^2 + \frac{\alpha}{2} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^k)^2 \\ & \leq \frac{\alpha}{2} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^{k+1})^2. \end{aligned}$$

Remark that  $\alpha$  is supposed to be very large compared to the other constants. We may always assume that  $\frac{\nu_i^2 C_i}{\alpha} < \nu_i$ , for  $i \in \{1, 2\}$ . Summing over  $k$  from  $k = 0$  to  $N > 0$  gives

$$K \sum_{k=1}^{N-1} \left( \|\nabla \mathbf{e}_1^k\|_{L^2(\Omega_1)}^2 + \|\nabla \mathbf{e}_2^k\|_{L^2(\Omega_2)}^2 \right) + \frac{\alpha}{2} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^0)^2 \leq \frac{\alpha}{2} \int_{\Gamma_{Int}} (\mathbf{e}_{2,h}^N)^2, \quad (2.61)$$

where  $K = \min(\nu_1 - \frac{\nu_1^2 C_1}{2\alpha}, \nu_2 - \frac{\nu_2^2 C_2}{2\alpha})$ . Consequently, the serie  $\sum_{k=0}^{\infty} \|\mathbf{e}^k\|_W$  is convergent, and  $(\mathbf{e}^k)_{k \in \mathbb{N}} = (\mathbf{e}_1^k, \mathbf{e}_2^k)_{k \in \mathbb{N}}$  goes to  $(0, 0)$  in  $W = W_1 \times W_2$ .

□

**Remark 2.3.2.** *Unfortunately, we don't have yet any convergence result on the pressure term. Numerically, it seems to have very few impact on the simulations.*

The theorem 2.3.1 only gives the convergence of the velocities. We would like to have more information, especially on the speed of convergence. This is why we will follow the same steps as before, considering Fourier transformations.

### 2.3.2 Fourier convergence rate

We will consider in this subsection the case of 2-dimension domains 2.8.

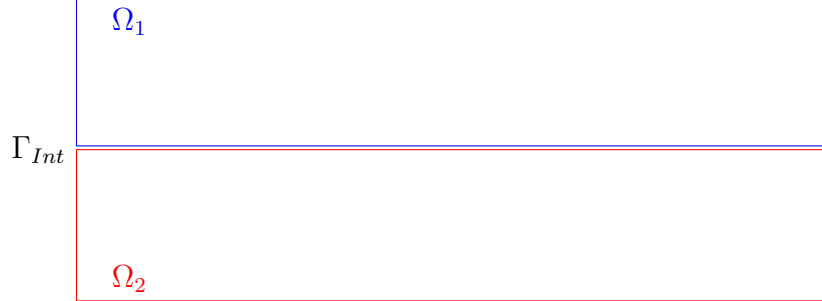


Figure 2.8 – No overlap between  $\Omega_1$  and  $\Omega_2$

Remind that we calculate  $\mathbf{u}_1^{n+1}$  and  $\mathbf{u}_2^{n+1}$  by solving

$$\left\{ \begin{array}{l} -\nu_1 \Delta \mathbf{u}_1^{n+1} + \nabla P_1^{n+1} = \mathbf{f}_1, \\ \nabla \cdot \mathbf{u}_1^{n+1} = 0, \\ \nu_1 \partial_y u_{1,x}^{n+1} + \alpha u_{1,x}^{n+1} = \alpha u_{2,x}^n, \\ \mathbf{u}_1^{n+1}|_{\Gamma_1} = 0, \end{array} \right. \quad (2.62)$$

and

$$\left\{ \begin{array}{l} -\nu_2 \Delta \mathbf{u}_2^{n+1} + \nabla P_2^{n+1} = \mathbf{f}_2, \\ \nabla \cdot \mathbf{u}_2^{n+1} = 0, \\ -\nu_2 \partial_y u_{2,x}^{n+1} + \alpha u_{2,x}^{n+1} = \alpha u_{1,x}^{n+1}, \\ \mathbf{u}_2^{n+1}|_{\Gamma_1} = 0. \end{array} \right. \quad (2.63)$$

We define the velocity errors and the pressure errors as

$$\mathbf{e}_j^n = \mathbf{u}_j^n - \mathbf{u}_j,$$

and

$$\eta_j^n = P_j^n - P_j.$$

The method we will follow is developed in [41] for the resolution of a Stokes-Darcy coupling problem and in [17] for the resolution of a Navier-Stokes system.

**Proposition 2.3.1.** *The velocity errors  $(\|\mathbf{e}_1^n\|_{0,2,\Omega_1})_{n \in \mathbb{N}}$  and  $(\|\mathbf{e}_2^n\|_{0,2,\Omega_2})_{n \in \mathbb{N}}$  are converging*

to 0 and the convergence factor  $\rho$  is defined by

$$\rho(k, \nu_1, \nu_2, \alpha) = \frac{\alpha^2}{(\nu_1|k| + \alpha)(\nu_2|k| + \alpha)}, \quad (2.64)$$

for  $k \in \mathbb{R}$ .

*Proof.* We use the Fourier transform in the  $x$  variable for the error equations, which gives:

$$\begin{cases} -\left(-\nu_j k^2 \hat{e}_{j,x}^n + \nu_j \frac{d^2 \hat{e}_{j,x}^n}{dy^2}\right) + ik \hat{\eta}_j^n = 0, & \text{in } \Omega_j, \\ -\left(-\nu_j k^2 \hat{e}_{j,y}^n + \nu_j \frac{d^2 \hat{e}_{j,y}^n}{dy^2}\right) + \frac{d\hat{\eta}_j^n}{dy} = 0, & \text{in } \Omega_j, \\ ik \hat{e}_{j,x}^n + \frac{d\hat{e}_{j,y}^n}{dy} = 0 & \text{in } \Omega_j, \end{cases} \quad (2.65)$$

We are looking for solutions using the vector field

$$E_j^n := \begin{pmatrix} \hat{e}_{j,x}^n \\ \hat{e}_{j,y}^n \\ \hat{\eta}_j^n \end{pmatrix} (y) = \Phi_j^n e^{\xi y}.$$

Using the previous system, we obtain:

$$\begin{pmatrix} \nu_j k^2 - \nu_j \xi^2 & 0 & ik \\ 0 & \nu_j k^2 - \nu_j \xi^2 & \xi \\ ik & \xi & 0 \end{pmatrix} \Phi_j^n = 0. \quad (2.66)$$

The sufficient and necessary condition to retrieve non-trivial solutions is that the previous  $3 \times 3$  matrix is singular, i.e, its determinant must be equal to zero. The determinant is equal to  $\nu_i(k^2 - \xi^2)^2$ , which leads to two different possibilities for  $\xi$ :  $\xi_1 = |k|$  or  $\xi_2 = -|k|$ . Denote  $\Phi_m$ , where  $1 \leq m \leq 2$ , the eigenvector associated with the eigenvalue 0 and corresponding with each  $\xi_m$ .

The solutions of (2.88) are linear combinations of the two eigenvectors  $\Phi_m$ , and can be expressed by

$$E_j^n = \gamma_{j,1}^n \Phi_1 e^{\xi_1 y} + \gamma_{j,2}^n \Phi_2 e^{\xi_2 y},$$

where

$$\Phi_1 = \begin{pmatrix} ik \\ |k| \\ 0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} ik \\ -|k| \\ 0 \end{pmatrix}.$$

Due to the bounded condition at the top, since  $\xi_1 \geq 0$  and  $\xi_2 \leq 0$ , only one term is possible for each subdomain, leading to

$$E_1^n(y) = \gamma_{1,1}^n \Phi_2 e^{-|k|y}, \quad E_2^n(y) = \gamma_{2,1}^n \Phi_1 e^{|k|y}. \quad (2.67)$$

We can rewrite this in

$$\begin{cases} \hat{e}_{1,x}^n = ik\gamma_{1,1}^n e^{-|k|y}, & \text{in } \Omega_1, \\ \hat{e}_{1,y}^n = -|k|\gamma_{1,1}^n e^{-|k|y}, & \text{in } \Omega_1, \end{cases} \quad (2.68)$$

and

$$\begin{cases} \hat{e}_{2,x}^n = ik\gamma_{2,1}^n e^{|k|y}, & \text{in } \Omega_2, \\ \hat{e}_{2,y}^n = |k|\gamma_{2,1}^n e^{|k|y}, & \text{in } \Omega_2. \end{cases} \quad (2.69)$$

Recall that the interface conditions are

$$\begin{cases} -\nu_1 \partial_y \hat{e}_{1,x}^{n+1} + \alpha \hat{e}_{1,x}^{n+1} = \alpha \hat{e}_{2,x}^{n+1}, \\ \nu_2 \partial_y \hat{e}_{2,x}^{n+1} + \alpha \hat{e}_{2,x}^{n+1} = \alpha \hat{e}_{1,x}^n, \end{cases} \quad (2.70)$$

we get

$$\begin{cases} (\nu_1 |k| + \alpha) \gamma_1^{n+1} = \alpha \gamma_2^{n+1}, \\ (\nu_2 |k| + \alpha) \gamma_2^{n+1} = \alpha \gamma_1^n, \end{cases} \quad (2.71)$$

which gives

$$\gamma_1^{n+1} = \rho(k, \nu_1, \nu_2, \alpha) \gamma_1^n, \quad (2.72)$$

where the geometrical convergence factor  $\rho$  is defined by

$$\rho(k, \nu_1, \nu_2, \alpha) = \frac{\alpha^2}{(\nu_1 |k| + \alpha)(\nu_2 |k| + \alpha)}. \quad (2.73)$$

□

**Remark 2.3.3.** *First note that  $\rho < 1$  for  $k \neq 0$ .*

*We couldn't have considered the non-overlapping Schwarz algorithm without the Robin condition. Indeed, the boundary condition*

$$\begin{cases} \hat{e}_{1,x}^{n+1} = \hat{e}_{2,x}^{n+1}, \\ \hat{e}_{2,x}^{n+1} = \hat{e}_{1,x}^n, \end{cases} \quad (2.74)$$

*immediately gives a convergence factor  $\rho = 1$ , which prevents the algorithm from converg-*

ing.

**Remark 2.3.4.** *The error of the pressure terms is equal to zero at the first iteration, which seems to be unrealistic.*

### 2.3.3 Overlapping algorithms

An alternative is to consider a thin overlapping zone between the two domains  $\Omega_1$  and  $\Omega_2$  as in 2.3. We assume that the two regions are sharing a strip of height  $\delta > 0$ . When we consider the small overlapping length  $\delta$ , we add another approximation to the simulation of the problem (2.43). Nevertheless, keeping even a very small  $\delta$  will significantly increase the convergence factor as we will see.

**Proposition 2.3.2.** *The algorithm is converging with a convergence factor  $\rho$  defined by*

$$\rho = \frac{\alpha^2}{(\nu_1|k| + \alpha)(\nu_2|k| + \alpha)} e^{-2|k|\delta}.$$

*Proof.* Following the same Fourier analysis and the same notations as in the previous subsection, we get as boundary conditions:

$$\begin{cases} -\nu_1 \partial_y \hat{e}_{1,x}^{n+1}(k, 0) + \alpha \hat{e}_{1,x}^{n+1} = \alpha \hat{e}_{2,x}^{n+1}(k, 0), \\ \nu_2 \partial_y \hat{e}_{2,x}^{n+1}(k, \delta) + \alpha \hat{e}_{2,x}^{n+1}(k, \delta) = \alpha \hat{e}_{1,x}^{n+1}(k, \delta), \end{cases} \quad (2.75)$$

which gives

$$\begin{cases} (\nu_1|k| + \alpha) \gamma_1^{n+1} = \alpha \gamma_2^{n+1}, \\ (\nu_2|k| + \alpha) \gamma_2^{n+1} e^{k\delta} = \alpha \gamma_1^n e^{-k\delta}, \end{cases} \quad (2.76)$$

and the convergence factor  $\rho$  is defined by

$$\rho = \frac{\alpha^2}{(\nu_1|k| + \alpha)(\nu_2|k| + \alpha)} e^{-2|k|\delta}. \quad (2.77)$$

□

**Remark 2.3.5.** *The convergence factor of the overlapping algorithm is the product of the convergence factor without overlap by the term  $e^{-2|k|\delta}$ , which significantly increase the speed of the algorithm, as we can see in figure 2.11.*

### 2.3.4 Implementation in Freefem++

#### Implementation issues

We have to mention a huge difficulty to implement the algorithm 3 and numerically simulate the problems (2.44) and (2.45). The issue is coming from the Robin conditions at the interface

$$\begin{cases} \nu_1 \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} + \alpha \mathbf{u}_{1,h}^{n+1} = \alpha \mathbf{u}_{2,h}^n \\ \nu_2 \frac{\partial \mathbf{u}_{2,h}^{n+1}}{\partial \mathbf{n}_2} + \alpha \mathbf{u}_{2,h}^{n+1} = \alpha \mathbf{u}_{1,h}^{n+1} \end{cases}$$

The bilinear and linear forms associated to the variational formulations of (2.44) and (2.45) are

$$\begin{aligned} a_{\text{Robin}}^i : H^1(\Omega_i) \times H^1(\Omega_i) &\rightarrow \mathbb{R} \\ (\mathbf{u}_i, \mathbf{v}_i) &\mapsto \int_{\Omega_i} (\nu_i \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i) + \int_{\partial\Omega_i \cap \Omega_j} \alpha \mathbf{u}_i \mathbf{v}_i, \end{aligned}$$

$$\begin{aligned} b^i : H^1(\Omega_i) \times Q_i &\rightarrow \mathbb{R} \\ (\mathbf{v}_i, p_i) &\mapsto \int_{\Omega_i} p_i \nabla \cdot \mathbf{v}_i, \end{aligned}$$

where  $i, j \in \{1, 2\}$  and  $j \neq i$ , and finally the linear form

$$\begin{aligned} l_{\text{Robin}}^i : H^1(\Omega_i) &\rightarrow \mathbb{R} \\ \mathbf{v}_i &\mapsto \int_{\Omega_i} \mathbf{f}_i \mathbf{v}_i + \underbrace{\int_{\partial\Omega_i \cap \Omega_j} \alpha \mathbf{u}_j^n \mathbf{v}_i}_{\text{problematic term}}. \end{aligned}$$

The implementation issue comes from this linear term. In short, we will have to compute this integral in the domain  $\Omega_i$  while the velocity  $\mathbf{u}_j^n$  doesn't belong to it.

The variational formulation of the Robin problems (2.44) and (2.45) reads:

$$\begin{aligned} &\text{Find } (\mathbf{u}_i, p_i) \in H^1(\Omega_i) \times Q_i \text{ such that} \\ \forall (\mathbf{v}_i, q_i) \in H^1(\Omega_i) \times Q_i, &\quad \begin{cases} a_{\text{Robin}}^i(\mathbf{u}_i, \mathbf{v}_i) + b(\mathbf{v}_i, p_i) = l_{\text{Robin}}^i(\mathbf{v}_i), \\ b(\mathbf{u}_i, q_i) = 0. \end{cases} \end{aligned}$$

The finite elements discretization is obtained by replacing the infinite Sobolev spaces

$H^1(\Omega_i)$  and  $Q_i$  by finite dimensional ones. Since we are working with mixed variational formulations for Stokes problems, we can consider the classical  $P2$  and  $P1$  finite elements spaces denoted by  $V_h(\Omega_i)$  and  $Q_h(\Omega_i)$ . It consists of approximating the velocity by  $P2$  Lagrange finite elements, and the pressure by  $P1$  Lagrange finite elements (see [25] for instance).

The variational formulation with finite elements becomes

$$\text{Find } (\mathbf{u}_i^h, p_i^h) \in V_h(\Omega_i) \times Q_h(\Omega_i) \text{ such that} \quad (2.78)$$

$$\forall (\mathbf{v}_i^h, q_i^h) \in V_h(\Omega_i) \times Q_h(\Omega_i), \quad \begin{cases} a_{\text{Robin}}^i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b(\mathbf{v}_i^h, p_i^h) = l_{\text{Robin}}^i(\mathbf{v}_i^h), \\ b(\mathbf{u}_i^h, q_i^h) = 0. \end{cases} \quad (2.79)$$

We can consider the finite elements bases

$$\{\Phi_i^k, k \in N_2\} = V_h(\Omega_i) \quad \text{and} \quad \{q_i^k, k \in N_1\} = Q_h(\Omega_i),$$

where  $N_2$  is the set of degrees of freedom for the  $P2$  elements and  $N_1$  for the  $P1$  elements. The matrix  $A_{\text{Robin}}^i$  of the bilinear form  $a_{\text{Robin}}^i$  in this finite elements decomposition is given by

$$A_{\text{Robin},k,l}^i = a_{\text{Robin}}^i(\Phi_i^k, \Phi_i^l) = \int_{\Omega_i} (\nu_i \nabla \Phi_i^k \cdot \nabla \Phi_i^l) + \int_{\partial\Omega_i \cap \Omega_j} \alpha \Phi_i^k \Phi_i^l$$

The problem comes from the discretization of the right hand term  $\alpha \mathbf{u}_j^n$  of the linear form  $l_{\text{Robin}}$ . Indeed, when we solve the variational formulation (2.78), we can only use the mesh of  $\Omega_i$  and more particularly the nodes of this triangulation to do our calculation. For instance, if we consider  $P1$ -elements to simplify, the calculation of the trace term

$$\int_{\Delta} \alpha \mathbf{u}_2^n \mathbf{v}_1,$$

where  $\Delta \in \partial\Omega_1 \cap \overline{\Omega_2}$  is the red segment shown in 2.9, will be obtained by the basis decomposition over the  $\Phi_i^k$ , given by the three vertices (black dots). The velocity  $\mathbf{u}_2^n$  belongs to the discretized space  $V_h(\Omega_2)$  and can be decomposed on the  $\{\Phi_2^k, k \in N_2\}$  bases, but cannot be decomposed in the basis corresponding to the mesh of  $\Omega_1$ . This is the major issue.



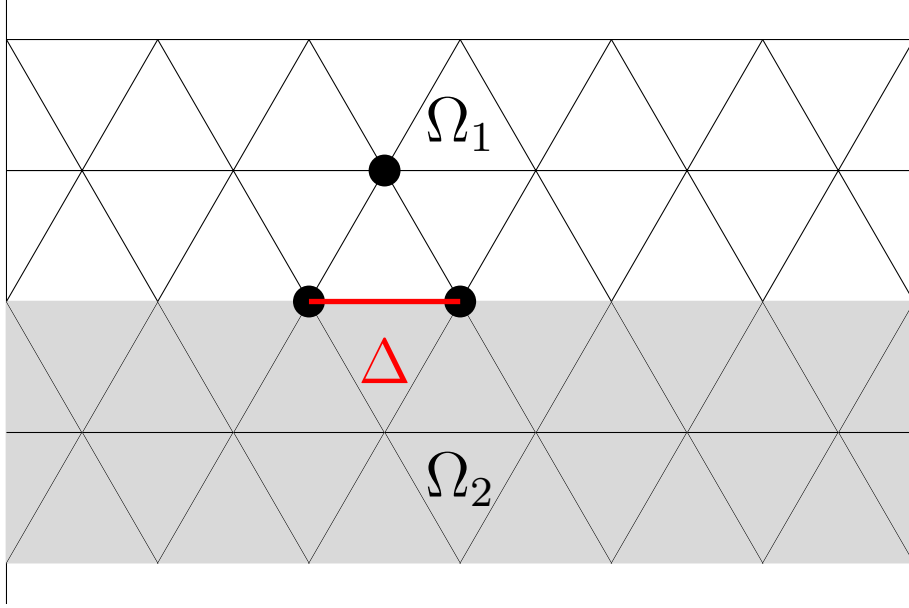


Figure 2.9 – Stencil at the interface for two non-overlapping domains  $\Omega_1$  and  $\Omega_2$ .

### A first solution

The two domains  $\Omega_1$  and  $\Omega_2$  are sharing the same length but not necessarily the same height. We denote by  $z_1$  the height of  $\Omega_1$  and  $z_2$  the height of  $\Omega_2$ .

Basically, we would be happy to make the term  $\mathbf{u}_2^n$  cross the border and belong to  $V_h(\Omega_1)$  and respectively for  $\mathbf{u}_1^n$ . To solve this issue about the calculation on the boundary, we had the idea of considering the "symmetrical velocities" of the solutions  $\mathbf{u}_1^n \in V_h(\Omega_1)$  and  $\mathbf{u}_2^n \in V_h(\Omega_2)$ .

**Definition 2.3.1.** We denote by  $\mathbf{u}_{1,sym}^n$  the velocity field belonging to  $V_h(\Omega_2)$  and given by

$$\mathbf{u}_{1,sym}^n := \mathbf{u}_1^n(x, -\lambda z), \quad (2.80)$$

for almost every  $x, z \in \Omega_1$ , where  $\lambda$  is the ratio

$$\lambda = \frac{z_1}{z_2}.$$

The symmetrical term  $\mathbf{u}_{2,sym}^n \in V_h(\Omega_1)$  is given as well by

$$\mathbf{u}_{2,sym}^n := \mathbf{u}_2^n\left(x, -\frac{1}{\lambda}z\right) \quad (2.81)$$

**Remark 2.3.6.** It may be confusing but  $\mathbf{u}_{1,sym}^n$  belongs to  $V_h(\Omega_2)$  and  $\mathbf{u}_{2,sym}^n$  belongs to  $V_h(\Omega_1)$ , as shown in figure 2.10.

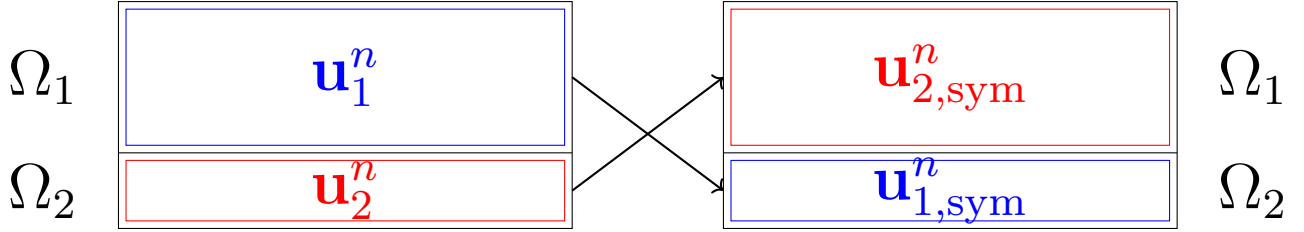


Figure 2.10 – Symmetrical velocities

The point is we cannot consider the velocity  $\mathbf{u}_2^n$  in the calculation of  $\int_{\partial\Omega_1 \cap \overline{\Omega_2}} \alpha \mathbf{u}_2^n \cdot \mathbf{v}_1$ , but we can replace  $\mathbf{u}_2^n$  by  $\mathbf{u}_{2,sym}^n$  and consider the term  $\int_{\partial\Omega_1 \cap \overline{\Omega_2}} \alpha \mathbf{u}_{2,sym}^n \cdot \mathbf{v}_1$  instead, since  $\mathbf{u}_{2,sym}^n$  belongs to  $V_h(\Omega_1)$  and can be decomposed in the same basis as the other terms.

**Remark 2.3.7.** Only the values around the boundary are useful for the calculation of the trace term, which corresponds to a very narrow layer around the interface.

The ratio of the altitudes  $\lambda = \frac{z_1}{z_2}$  may have an impact on the values of the symmetrical velocities over the whole domain, but as we are only interested in the layer around the boundary, the terms  $\lambda z$  and  $\frac{1}{\lambda} z$  are almost equal to 0.

This guarantees the fact that

$$\int_{\partial\Omega_1 \cap \overline{\Omega_2}} \alpha \mathbf{u}_{2,sym}^n \cdot \mathbf{v}_1 \approx \int_{\partial\Omega_1 \cap \overline{\Omega_2}} \alpha \mathbf{u}_2^n \cdot \mathbf{v}_1,$$

and

$$\int_{\partial\Omega_2 \cap \overline{\Omega_1}} \alpha \mathbf{u}_{1,sym}^n \cdot \mathbf{v}_2 \approx \int_{\partial\Omega_2 \cap \overline{\Omega_1}} \alpha \mathbf{u}_1^n \cdot \mathbf{v}_2.$$

The algorithm to solve our problems becomes

**Remark 2.3.8.** One of the advantages of the algorithm 4 is the fact that we can use it with or without overlap. We will see in the section 2.4 another alternative to solve this boundary issue, which will only work in the non-overlapping case, but will be faster and easier to compute.

---

**Algorithm 4:** Algorithm with symmetrical velocities
 

---

**Data:** Initial condition:  $\mathbf{u}_2^0 \in V_h(\Omega_2)$   
 1 Define the symmetrical velocity  $\mathbf{u}_{2,\text{sym}}^0 \in V_h(\Omega_1)$   
 2 **while** *not converged* **do**  
 3     **for**  $k \leftarrow 1$  **to**  $N$  **do**  
 4         Solve problem (2.44) in the subdomain  $\Omega_1$  which gives  $(\mathbf{u}_1^k, P_1^k)$ ,  
 5         Define the symmetrical velocity  $\mathbf{u}_{1,\text{sym}}^k$ ,  
 6         Solve problem (2.45) in the subdomain  $\Omega_2$  which gives  $(\mathbf{u}_2^k, P_2^k)$ ,  
 7         Define the symmetrical velocity  $\mathbf{u}_{2,\text{sym}}^k$ ;  
 8     Update the velocity and pressure values  $\mathbf{u}_i^k$  and  $P_i^k$ ;  
 9     Check convergence criteria;

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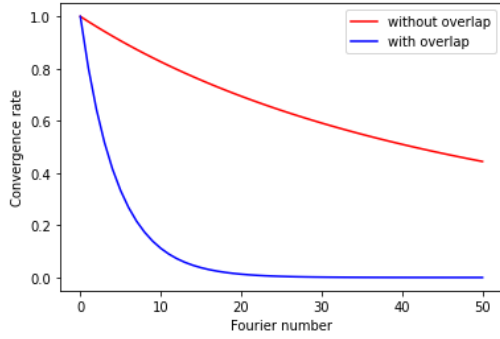
### 2.3.5 Comparison of the convergence rate with and without overlap

We have chosen  $\nu_1 = \nu_2 = 0.1$  and we compare for different values of  $\alpha$  and  $\delta$  the convergence speed of the "symmetrical velocities" algorithm (4) with overlap in blue and without overlap in red.

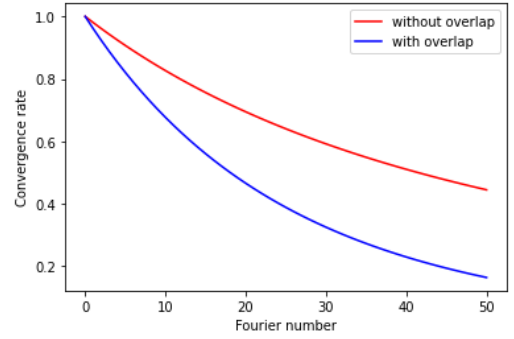
The non-overlapping algorithm is more precise despite the fact it is converging slower. To illustrate this slow convergence, we consider the case where  $\alpha$  is taking the values 10, 100 and 1000, and the couple of viscosities  $(\nu_1, \nu_2)$  is taking the values  $(0.1, 0.1)$ , which corresponds to the laminar regime, and  $(\nu_1, \nu_2) = (0.001, 0.001)$ , the turbulent regime. Note that the scales are not the same in the two figures in 2.12.

Finally, we are numerically simulating with Freefem++ the problem (2.44) and (2.45) thanks to the algorithm (4). We consider  $P2$  finite elements for the velocities and  $P1$  finite elements for the pressure. We also decide to consider different viscosities:  $\nu_1 = 0.04$  and  $\nu_2 = 0.01$ . Three situations corresponding to different boundary conditions have been considered:

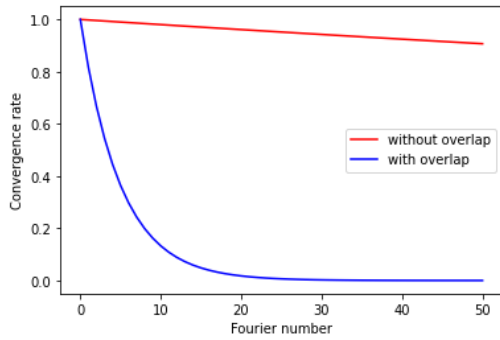
- with horizontal periodic boundary conditions in 2.13, where  $\Omega_1 = 1 \times 0.6$  and  $\Omega_2 = 1 \times 0.4$ ,
- with inlet velocity on the entrance and horizontal periodic boundary conditions 2.14 where  $\Omega_1 = 1 \times 0.6$  and  $\Omega_2 = 1 \times 0.4$ ,
- with driven lid at the top, with Dirichlet conditions on the left, right, at the bottom (lid-driven cavity flow benchmark), where  $\Omega_1 = 1 \times 0.5$  and  $\Omega_2 = 1 \times 0.5$  2.15.



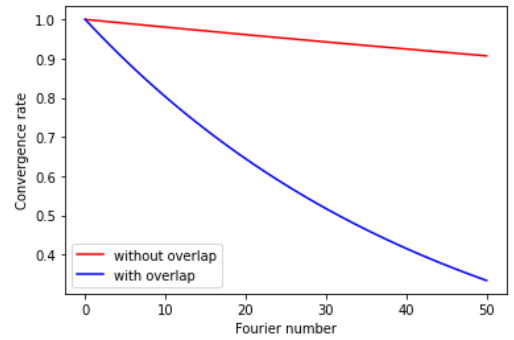
(a)  $\alpha = 10, \delta = 0.1$



(b)  $\alpha = 10, \delta = 0.01$



(c)  $\alpha = 100, \delta = 0.1$



(d)  $\alpha = 100, \delta = 0.01$

Figure 2.11 – Convergence rates with and without overlap for different values of  $\alpha$  and  $\delta$ .

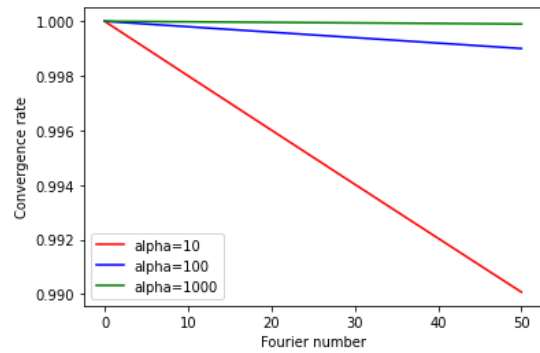
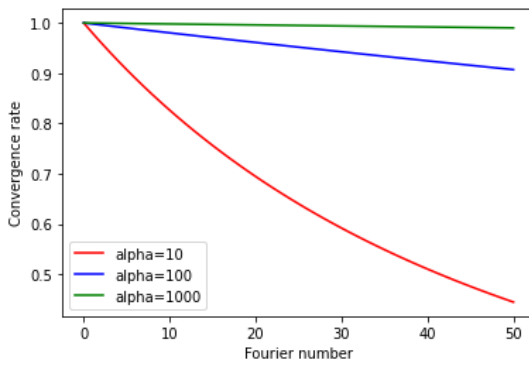


Figure 2.12 – Without overlap,  $\nu_1 = \nu_2 = 0.1$  on the left,  $\nu_1 = \nu_2 = 0.001$  on the right.

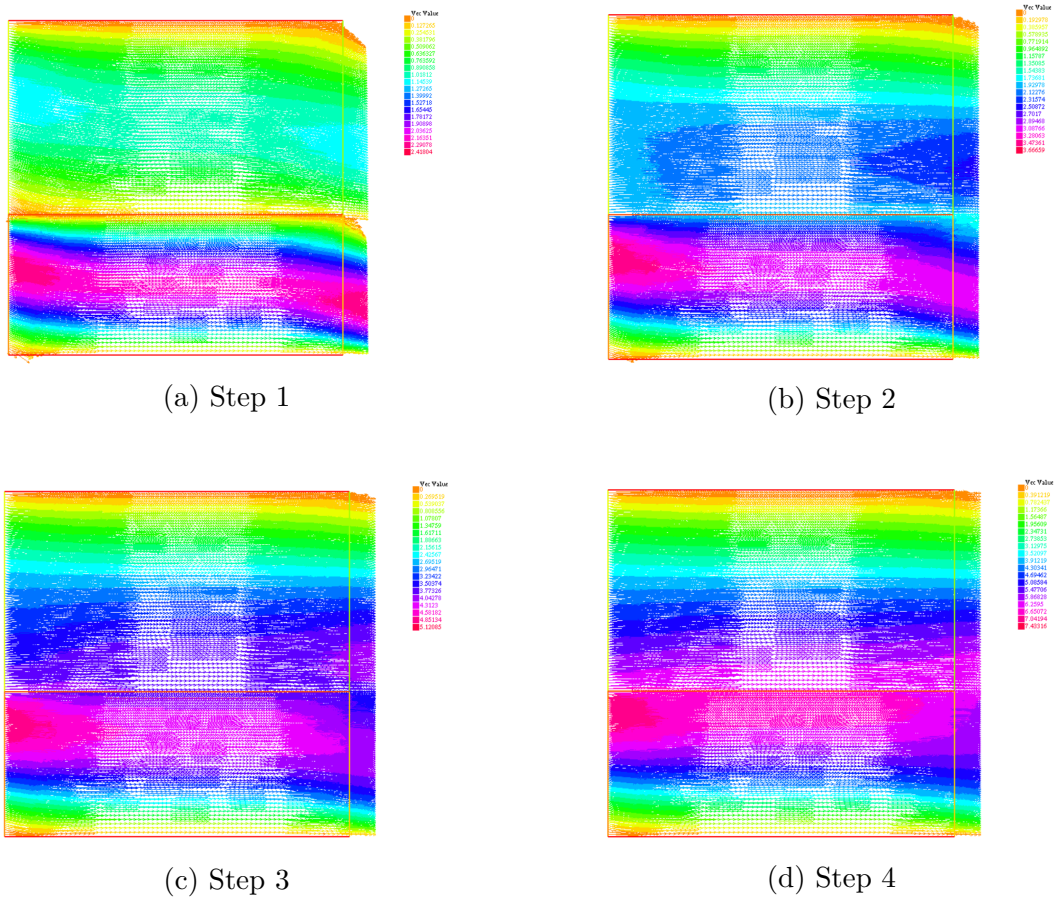


Figure 2.13 – Stokes coupling with periodic boundary conditions

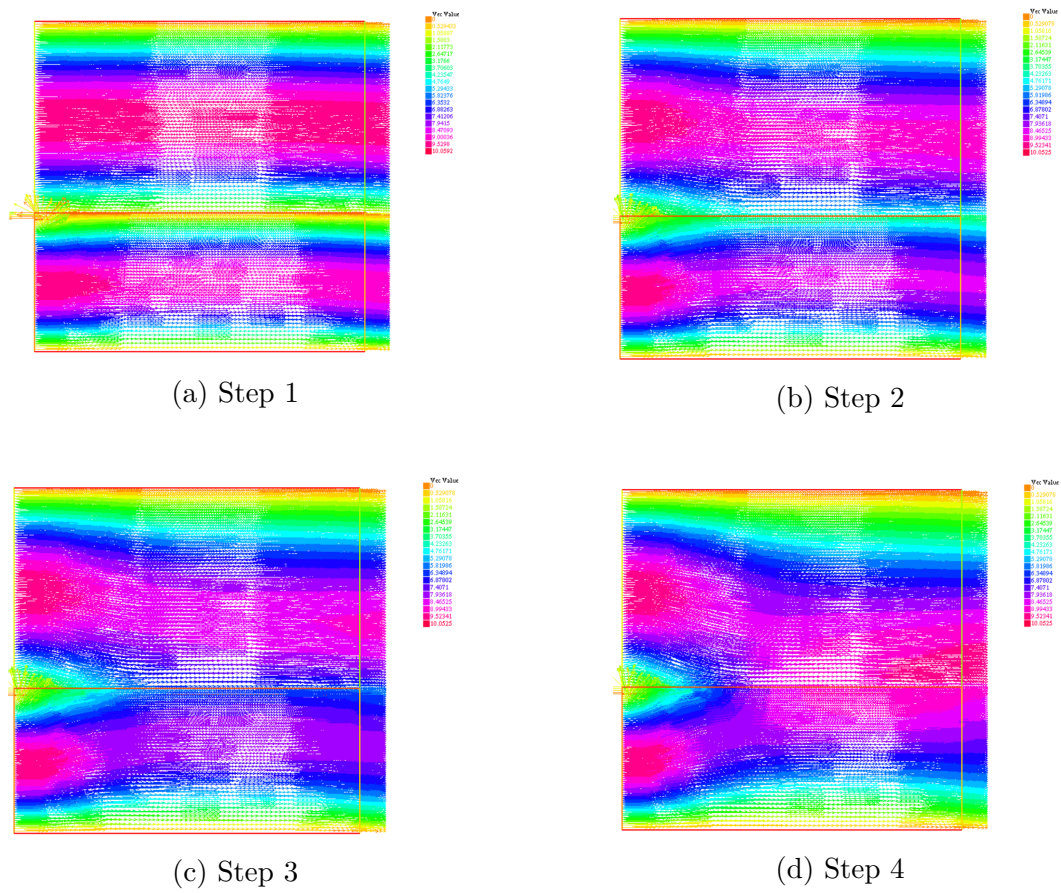


Figure 2.14 – Inlet and periodic boundary conditions

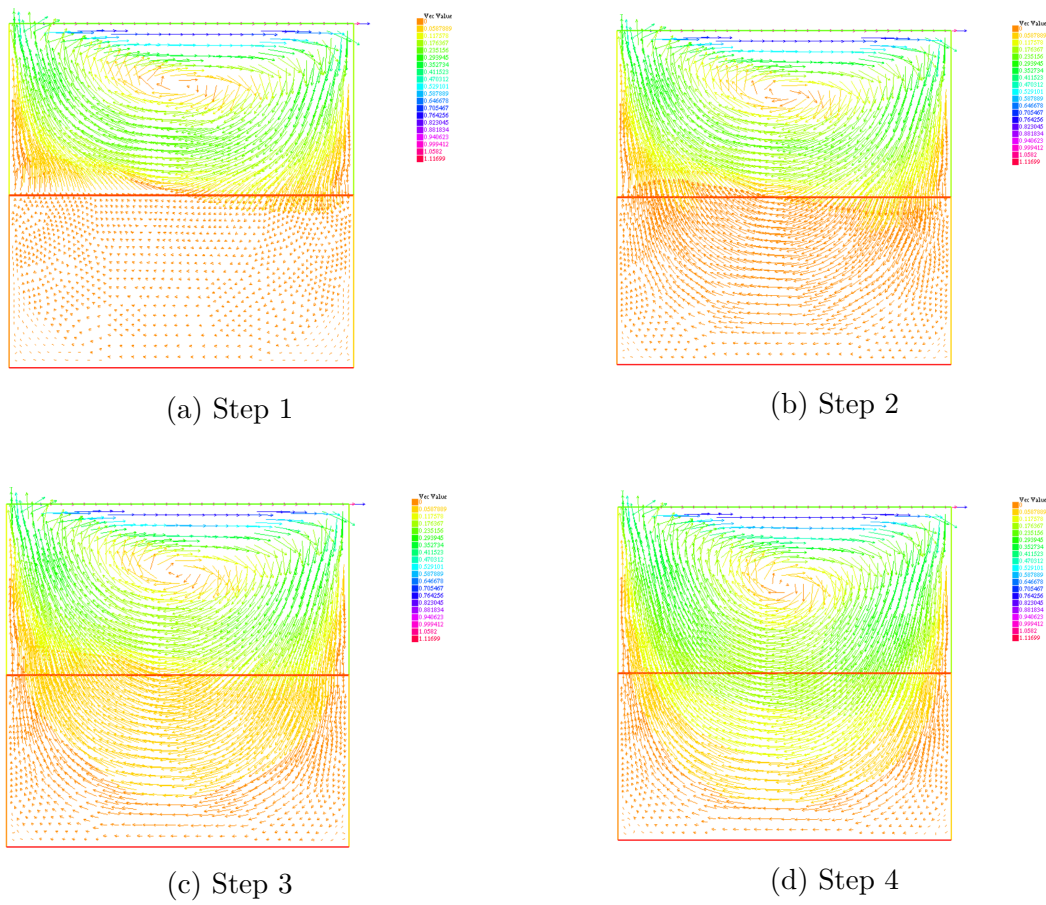


Figure 2.15 – Driven lid cavity simulation



In the following table, we display the first value  $n$  such that the difference

$$\frac{\|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{H^1}}{\|\mathbf{u}^n\|_{H^1}} < \varepsilon_{tol},$$

where the tolerance error  $\varepsilon_{tol} = 10^{-6}$ . Note that the stopping criteria is based on the relative error.

| $\alpha$ | First value of $n$ | $\ \mathbf{u}_1^n\ _{L^2}$ | $\ \mathbf{u}_2^n\ _{L^2}$ | $\ \mathbf{u}_1^n\ _{H^1}$ | $\ \mathbf{u}_2^n\ _{H^1}$ |
|----------|--------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| 10       | 1159               | 2.95798                    | 8.37296                    | 2.77434                    | 11.7425                    |
| 100      | 6529               | 2.85098                    | 8.07494                    | 2.68607                    | 11.436                     |
| 1000     | 15039              | 2.76636                    | 7.66296                    | 2.61665                    | 10.841                     |

Table 2.6 – Without overlap

| $\alpha$ | First value of $n$ | $\ \mathbf{u}_1^n\ _{L^2}$ | $\ \mathbf{u}_2^n\ _{L^2}$ | $\ \mathbf{u}_1^n\ _{H^1}$ | $\ \mathbf{u}_2^n\ _{H^1}$ |
|----------|--------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| 10       | 518                | 3.0553                     | 8.62882                    | 2.85615                    | 12.0086                    |
| 100      | 1004               | 3.16941                    | 8.96841                    | 2.951267                   | 12.3454                    |
| 1000     | 1124               | 3.20478                    | 9.07237                    | 2.97996                    | 12.4503                    |

Table 2.7 – With overlap  $\delta = 0.01$

## 2.4 An alternative method

We will see in this section another way to solve our implementation issues, based on a fixed-point argument. It will only work in the non-overlapping case, but it will also be easier and faster to compute than the "symmetrical velocities" algorithm. The method we develop is adapted from [32] for the fixed-point idea, and from [41], [17] for the Stokes coupling aspects.

### 2.4.1 Perturbed boundary condition

We consider in this part that  $\Omega_1$  and  $\Omega_2$  are not overlapping, as in figure 2.8.



The key is to add in the Robin boundary conditions the small derivatives terms  $\varepsilon \frac{\partial \mathbf{u}_{1,h}^n}{\partial \mathbf{n}_1} |_{\Gamma_I}$  and  $\varepsilon \frac{\partial \mathbf{u}_{2,h}^n}{\partial \mathbf{n}_2} |_{\Gamma_I}$ , where  $\varepsilon > 0$  is very small compared to  $\nu_1$  or  $\nu_2$ . The problems (2.44) and (2.45) are replaced with the new perturbed problems:

$$\left\{ \begin{array}{l} -\nu_1 \Delta \mathbf{u}_1^{n+1} + \nabla P_1^{(n+1)} = \mathbf{f}_1, \\ \nabla \cdot \mathbf{u}_1^{n+1} = 0, \\ \nu_1 \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} |_{\Gamma_I} + \alpha \mathbf{u}_{1,h}^{n+1} = -\varepsilon \frac{\partial \mathbf{u}_{2,h}^n}{\partial \mathbf{n}_2} |_{\Gamma_I} + \alpha \mathbf{u}_{2,h}^n, \\ \mathbf{u}_{1,h}^{n+1} |_{\Gamma_1} = 0, \quad \mathbf{u}_{1,h}^{n+1} \cdot \mathbf{n}_1 = 0. \end{array} \right. \quad (2.82)$$

and

$$\left\{ \begin{array}{l} -\nu_2 \Delta \mathbf{u}_2^{n+1} + \nabla P_2^{(n+1)} = \mathbf{f}_2, \\ \nabla \cdot \mathbf{u}_2^{n+1} = 0, \\ \nu_2 \frac{\partial \mathbf{u}_{2,h}^{n+1}}{\partial \mathbf{n}_2} |_{\Gamma_I} + \alpha \mathbf{u}_{2,h}^{n+1} = -\varepsilon \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} |_{\Gamma_I} + \alpha \mathbf{u}_{1,h}^{n+1}, \\ \mathbf{u}_{2,h}^{n+1} |_{\Gamma_2} = 0, \quad \mathbf{u}_{2,h}^{n+1} \cdot \mathbf{n}_2 = 0. \end{array} \right. \quad (2.83)$$

**Remark 2.4.1.** *The reason of a very small  $\varepsilon$  is to guarantee the fact that the solutions of these problems will be very close to the ones of the problems (2.44) (2.45).*

The drawback is the discretization of the term  $\frac{\partial \mathbf{u}_{1,h}^n}{\partial \mathbf{n}_1} |_{\Gamma_I}$  when we solve the problem on  $\Omega_2 \cap \Gamma_I$  and the discretization of the term  $\frac{\partial \mathbf{u}_{2,h}^n}{\partial \mathbf{n}_2} |_{\Gamma_I}$  on  $\Omega_2 \cap \Gamma_I$ . Indeed, in Freefem++ we cannot solve by finite elements PDEs with conditions like

$$\underbrace{\nu_1 \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} |_{\Gamma_I} + \alpha \mathbf{u}_{1,h}^{n+1}}_{\in V_h(\Omega_1)} = \underbrace{-\varepsilon \frac{\partial \mathbf{u}_{2,h}^n}{\partial \mathbf{n}_2} |_{\Gamma_I} + \alpha \mathbf{u}_{2,h}^n}_{\notin V_h(\Omega_1)} \quad (2.84)$$

or

$$\underbrace{\nu_2 \frac{\partial \mathbf{u}_{2,h}^{n+1}}{\partial \mathbf{n}_2} |_{\Gamma_I} + \alpha \mathbf{u}_{2,h}^{n+1}}_{\in V_h(\Omega_2)} = \underbrace{-\varepsilon \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} |_{\Gamma_I} + \alpha \mathbf{u}_{1,h}^{n+1}}_{\notin V_h(\Omega_2)}, \quad (2.85)$$

for the reasons mentionned in 2.3.4.

In order to avoid the issue of computing the interface terms in the wrong domain, we use the same method as in chapter 6 in [32]. The trick is to replace the right hand term sides involving the "wrong" derivatives and to solve a fixed point problem. After dividing (2.84) and (2.85) by the respective viscosities  $\nu_1$  and  $\nu_2$  we define by  $\lambda_1^n$  and  $\lambda_2^{n+1}$  the

right hand sides of (2.84) and (2.85), i.e

$$\lambda_1^n = -\frac{\varepsilon}{\nu_1} \frac{\partial \mathbf{u}_{2,h}^n}{\partial \mathbf{n}_2} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_1} \mathbf{u}_{2,h}^n,$$

and

$$\lambda_2^n = -\frac{\varepsilon}{\nu_2} \frac{\partial \mathbf{u}_{1,h}^n}{\partial \mathbf{n}_1} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_2} \mathbf{u}_{1,h}^n.$$

**Lemma 2.4.1.** *We have the recursive relations for every  $n \geq 0$ :*

$$\lambda_1^{n+1} = -\frac{\varepsilon}{\nu_1} \lambda_2^{n+1} + \frac{\alpha}{\nu_1} \left(1 + \frac{\varepsilon}{\nu_2}\right) \mathbf{u}_{2,h}^{n+1},$$

and

$$\lambda_2^{n+1} = -\frac{\varepsilon}{\nu_2} \lambda_1^n + \frac{\alpha}{\nu_2} \left(1 + \frac{\varepsilon}{\nu_1}\right) \mathbf{u}_{1,h}^{n+1}.$$

*Proof.* We use the equations (2.84) and (2.85), which gives

$$\begin{aligned} \lambda_1^{n+1} &:= -\frac{\varepsilon}{\nu_1} \frac{\partial \mathbf{u}_{2,h}^{n+1}}{\partial \mathbf{n}_2} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_1} \mathbf{u}_{2,h}^{n+1} \\ &= -\frac{\varepsilon}{\nu_1} \left( \frac{\partial \mathbf{u}_{2,h}^{n+1}}{\partial \mathbf{n}_2} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_2} \mathbf{u}_{2,h}^{n+1} \right) + \frac{\alpha}{\nu_1} \mathbf{u}_{2,h}^{n+1} + \frac{\alpha\varepsilon}{\nu_1\nu_2} \mathbf{u}_{2,h}^{n+1} \\ &= -\frac{\varepsilon}{\nu_1} \left( -\frac{\varepsilon}{\nu_2} \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_2} \mathbf{u}_{1,h}^{n+1} \right) + \frac{\alpha}{\nu_1} \left(1 + \frac{\varepsilon}{\nu_2}\right) \mathbf{u}_{2,h}^{n+1} \\ &= -\frac{\varepsilon}{\nu_1} \lambda_2^{n+1} + \frac{\alpha}{\nu_1} \left(1 + \frac{\varepsilon}{\nu_2}\right) \mathbf{u}_{2,h}^{n+1} \end{aligned}$$

and

$$\begin{aligned} \lambda_2^{n+1} &:= -\frac{\varepsilon}{\nu_2} \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_2} \mathbf{u}_{1,h}^{n+1} \\ &= -\frac{\varepsilon}{\nu_2} \left( \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_1} \mathbf{u}_{1,h}^{n+1} \right) + \frac{\alpha}{\nu_2} \mathbf{u}_{1,h}^{n+1} + \frac{\alpha\varepsilon}{\nu_2\nu_1} \mathbf{u}_{1,h}^{n+1} \\ &= -\frac{\varepsilon}{\nu_2} \left( -\frac{\varepsilon}{\nu_1} \frac{\partial \mathbf{u}_{2,h}^n}{\partial \mathbf{n}_2} \Big|_{\Gamma_I} + \frac{\alpha}{\nu_1} \mathbf{u}_{2,h}^n \right) + \frac{\alpha}{\nu_2} \left(1 + \frac{\varepsilon}{\nu_1}\right) \mathbf{u}_{1,h}^{n+1} \\ &= -\frac{\varepsilon}{\nu_2} \lambda_1^n + \frac{\alpha}{\nu_2} \left(1 + \frac{\varepsilon}{\nu_1}\right) \mathbf{u}_{1,h}^{n+1}. \end{aligned}$$

□

As a consequence, we have the proposition:

**Proposition 2.4.1.** *Instead of solving the problems (2.82) (2.83), we can solve the problems*

$$\left\{ \begin{array}{l} -\nu_1 \Delta \mathbf{u}_1^{n+1} + \nabla P_1^{(n+1)} = \mathbf{f}_1, \\ \nabla \cdot \mathbf{u}_1^{n+1} = 0, \\ \nu_1 \frac{\partial \mathbf{u}_{1,h}^{n+1}}{\partial \mathbf{n}_1} |_{\Gamma_I} + \alpha \mathbf{u}_{1,h}^{n+1} = \lambda_1^n, \\ \lambda_2^{n+1} = -\frac{\varepsilon}{\nu_2} \lambda_1^n + \frac{\alpha}{\nu_2} \left(1 + \frac{\varepsilon}{\nu_1}\right) \mathbf{u}_{1,h}^{n+1}, \\ \mathbf{u}_{1,h}^{n+1} |_{\Gamma_1} = 0, \quad \mathbf{u}_{1,h}^{n+1} \cdot \mathbf{n}_1 = 0, \end{array} \right. \quad (2.86)$$

$$\left\{ \begin{array}{l} -\nu_2 \Delta \mathbf{u}_2^{n+1} + \nabla P_2^{(n+1)} = \mathbf{f}_2, \\ \nabla \cdot \mathbf{u}_2^{n+1} = 0, \\ \nu_2 \frac{\partial \mathbf{u}_{2,h}^{n+1}}{\partial \mathbf{n}_2} |_{\Gamma_I} + \alpha \mathbf{u}_{2,h}^{n+1} = \lambda_2^{n+1}, \\ \lambda_1^{n+1} = -\frac{\varepsilon}{\nu_1} \lambda_2^{n+1} + \frac{\alpha}{\nu_1} \left(1 + \frac{\varepsilon}{\nu_2}\right) \mathbf{u}_{2,h}^{n+1}, \\ \mathbf{u}_{2,h}^{n+1} |_{\Gamma_2} = 0, \quad \mathbf{u}_{2,h}^{n+1} \cdot \mathbf{n}_2 = 0, \end{array} \right. \quad (2.87)$$

and the corresponding algorithm:

---

**Algorithm 5:** Algorithm with perturbed derivatives

---

- Data:** Initial condition:  $\mathbf{u}_2^0 \in V_h(\Omega_2)$
- 1 Compute the right hand term  $\lambda_1^0$
  - 2 **while** not converged **do**
  - 3     **for**  $k \leftarrow 1$  **to**  $N$  **do**
  - 4         Solve problem (2.86) in the subdomain  $\Omega_1$  which gives  $(\mathbf{u}_1^k, P_1^k)$ ,
  - 5         Compute the right hand side term  $\lambda_2^k$ ,
  - 6         Solve problem (2.87) in the subdomain  $\Omega_2$  which gives  $(\mathbf{u}_2^k, P_2^k)$ ,
  - 7         Compute the right hand side term  $\lambda_1^k$ ;
  - 8     Update the velocity and pressure values  $\mathbf{u}_i^k$  and  $P_i^k$  Check convergence criteria.
- 

## 2.4.2 Convergence rate

Using the same Fourier analysis as in section 2.3.2, we obtain:

$$\begin{pmatrix} \nu_j k^2 - \nu_j \xi^2 + \kappa_j^{-1} & 0 & ik \\ 0 & \nu_j k^2 - \nu_j \xi^2 + \kappa_j^{-1} & \xi \\ ik & \xi & 0 \end{pmatrix} \Phi_j^n = 0. \quad (2.88)$$

The sufficient and necessary condition to retrieve non-trivial solutions is that the

previous 3x3 matrix is singular, i.e, its determinant must be equal to zero. The determinant is equal to  $(k^2 - \xi^2) (\nu_j(k^2 - \xi^2) + \kappa_j^{-1})$ , which leads to four different possibilities for  $\xi$ :  $\xi_1 = |k|$ ,  $\xi_2 = -|k|$ ,  $\xi_{3,j} = \lambda_j$  and  $\xi_{4,j} = -\lambda_j$ , with  $\lambda_j = \sqrt{k^2 + \kappa_j^{-1}\nu_j^{-1}}$ . Denote  $\Phi_m$ , where  $1 \leq m \leq 4$ , the eigenvector associated with the eigenvalue 0 and corresponding with each  $\xi_m$ . The solutions of (2.88) are linear combinations of the four eigenvectors  $\Phi_m$ , and can be expressed by

$$E_j^n = \gamma_{j,1}^n \Phi_1 e^{\xi_1 y} + \gamma_{j,2}^n \Phi_2 e^{\xi_2 y} + \gamma_{j,3}^n \Phi_3 e^{\xi_{3,j} y} + \gamma_{j,4}^n \Phi_4 e^{\xi_{4,j} y},$$

where

$$\Phi_1 = \begin{pmatrix} -|k| \\ -ik \\ \kappa_j^{-1} \end{pmatrix}, \Phi_2 = \begin{pmatrix} |k| \\ -ik \\ \kappa_j^{-1} \end{pmatrix}, \Phi_{3,j} = \begin{pmatrix} ik \\ -\lambda_j \\ 0 \end{pmatrix} \text{ and } \Phi_{4,j} = \begin{pmatrix} ik \\ \lambda_j \\ 0 \end{pmatrix}.$$

Due to the bounded condition top, since  $\xi_1, \xi_{3,j} \geq 0$  and  $\xi_2, \xi_{4,j} \leq 0$ , only two terms are possible for each subdomain, leading to

$$E_1^n(y) = \gamma_{1,1}^n \Phi_2 e^{-|k|y} + \gamma_{1,2}^n \Phi_{4,1} e^{-\lambda_1 y}, \quad E_2^n(y) = \gamma_{2,1}^n \Phi_1 e^{|k|y} + \gamma_{2,2}^n \Phi_{3,2} e^{\lambda_2 y}. \quad (2.89)$$

We can rewrite this in

$$\begin{cases} \hat{e}_{1,x}^n = |k| \gamma_{1,1}^n e^{-|k|y} + ik \gamma_{1,2}^n e^{-\lambda_1 y} \text{ in } \Omega_1, \\ \hat{e}_{1,y}^n = -ik \gamma_{1,1}^n e^{-|k|y} + \lambda_1 \gamma_{1,2}^n e^{-\lambda_1 y} \text{ in } \Omega_1, \\ \hat{\eta}_1^n = \kappa_1^{-1} \gamma_{1,1}^n e^{-|k|y} \end{cases} \quad (2.90)$$

and

$$\begin{cases} \hat{e}_{2,x}^n = -|k| \gamma_{2,1}^n e^{|k|y} + ik \gamma_{2,2}^n e^{\lambda_2 y} \text{ in } \Omega_2, \\ \hat{e}_{2,y}^n = -ik \gamma_{2,1}^n e^{|k|y} - \lambda_2 \gamma_{2,2}^n e^{\lambda_2 y} \text{ in } \Omega_2, \\ \hat{\eta}_2^n = \kappa_2^{-1} \gamma_{2,1}^n e^{|k|y} \end{cases} \quad (2.91)$$

The boundary condition at the interface (constraint tensor equals zero):

$$\begin{cases} -\frac{\partial \hat{e}_{1,y}^n}{\partial y} + \hat{\eta}_1^n = 0, \\ \frac{\partial \hat{e}_{2,y}^n}{\partial y} + \hat{\eta}_2^n = 0, \end{cases} \quad (2.92)$$

which yields the relation

$$\begin{cases} \gamma_{1,2}^n = \left( \frac{ik^2 - \kappa_1^{-1}}{\lambda_1^2} \right) \gamma_{1,1}^n, \\ \gamma_{2,2}^n = \left( \frac{\kappa_2^{-1} - ik^2}{\lambda_2^2} \right) \gamma_{2,1}^n \end{cases} \quad (2.93)$$

and

$$\begin{cases} \hat{e}_{1,x}^n = \gamma_{1,1}^n \left( |k| e^{-|k|y} + \frac{ik(ik^2 - \kappa^{-1})}{\lambda_1^2} e^{-\lambda_1 y} \right), \\ \hat{e}_{2,x}^n = \gamma_{2,1}^n \left( -|k| e^{|k|y} - \frac{ik(ik^2 - \kappa^{-1})}{\lambda_2^2} e^{\lambda_2 y} \right) \end{cases} \quad (2.94)$$

Since the boundary condition at the interface are:

$$\begin{cases} (-\nu_1 \partial_y + \alpha) \hat{e}_{1,x}^{n+1} = (-\varepsilon \partial_y + \alpha) \hat{e}_{2,x}^n, \\ (\nu_2 \partial_y + \alpha) \hat{e}_{2,x}^{n+1} = (\varepsilon \partial_y + \alpha) \hat{e}_{1,x}^{n+1} \end{cases} \quad (2.95)$$

we obtain the relation

$$\begin{cases} \left( -\nu_1 \left( -k^2 + ik\lambda_1^{-1}(\kappa^{-1} - ik^2) \right) + \alpha \left( |k| + ik\lambda_1^{-2}(ik^2 - \kappa^{-1}) \right) \right) \gamma_{1,1}^{n+1} \\ = \left( -\varepsilon \left( -k^2 + ik\lambda_2^{-1}(\kappa^{-1} - ik^2) \right) - \alpha \left( |k| + ik\lambda_2^{-2}(ik^2 - \kappa^{-1}) \right) \right) \gamma_{2,1}^n, \\ \left( -\nu_2 \left( -k^2 + ik\lambda_2^{-1}(\kappa^{-1} - ik^2) \right) - \alpha \left( |k| + ik\lambda_2^{-2}(ik^2 - \kappa^{-1}) \right) \right) \gamma_{2,1}^{n+1} \\ = \left( \varepsilon \left( -k^2 + ik\lambda_1^{-1}(\kappa^{-1} - ik^2) \right) + \alpha \left( |k| + ik\lambda_1^{-2}(ik^2 - \kappa^{-1}) \right) \right) \gamma_{1,1}^{n+1} \end{cases} \quad (2.96)$$

Hence,

$$\begin{cases} \gamma_{1,1}^{n+1} = \rho_{1,2} \gamma_{2,1}^n, \\ \gamma_{2,1}^{n+1} = \rho_{2,1} \gamma_{1,1}^{n+1}, \end{cases} \quad (2.97)$$

where

$$\begin{cases} |\rho_{1,2}|^2 = \frac{((\alpha\lambda_2^{-2} - \varepsilon\lambda_2^{-1})k^2 + \varepsilon k - \alpha)^2 + \kappa^{-2}(\varepsilon\lambda_2^{-1} - \alpha\lambda_2^{-2})^2}{(-(\nu_1\lambda_1^{-1} + \alpha\lambda_1^{-2})k^2 + \nu_1 k + \alpha)^2 + \kappa^{-2}(\nu_1\lambda_1^{-1} + \alpha\lambda_1^{-2})^2}, \\ |\rho_{2,1}|^2 = \frac{((\alpha\lambda_1^{-2} - \varepsilon\lambda_1^{-1})k^2 + \varepsilon k - \alpha)^2 + \kappa^{-2}(\varepsilon\lambda_1^{-1} - \alpha\lambda_1^{-2})^2}{(-(\nu_2\lambda_2^{-1} + \alpha\lambda_2^{-2})k^2 + \nu_2 k + \alpha)^2 + \kappa^{-2}(\nu_2\lambda_2^{-1} + \alpha\lambda_2^{-2})^2}, \end{cases} \quad (2.98)$$

which finally gives

$$\gamma_{1,1}^{n+1} = \rho \gamma_{1,1}^n, \quad (2.99)$$

where  $\rho = \rho_{1,2}\rho_{2,1}$ .

It would remain to check that  $\rho < 1$  uniformly in  $k$ , which is not that clear for every case possible.

It will depend on the values of  $\nu_1, \nu_2, \kappa, \alpha$  and  $\varepsilon$ .

Different cases can occur, corresponding to the different regimes of the fluids:

- the flow is turbulent in  $\Omega_1$  and laminar in  $\Omega_2$ :  $\nu_1 \in (0.001, 0.01)$  and  $\nu_2 \in (0.1, 10)$  (and the contrary),
- both regimes are laminar :  $(\nu_1, \nu_2) \in (0.1, 10) \times (0.1, 10)$ ,
- both regimes are turbulent:  $(\nu_1, \nu_2) \in (0.001, 0.01) \times (0.001, 0.01)$ ,
- the regimes are at the limit.

We have numerically studied the cases where the two fluids share the same kinematic viscosity, for a wide range of values.

We can see in the figures 2.16 2.17 2.18 the convergence rates for different Fourier numbers  $k \in \{2, 10, 50\}$ . The theoretical convergence is way faster for laminar flows  $(\nu_1, \nu_2) \in [0.1, 1]^2$  than in the turbulent regime  $(\nu_1, \nu_2) \in [0.001, 0.01]^2$ .

To conclude this work, it remains to be seen whether or not the convergence is uniform in  $k$  for every scenario possible, or at least, every reasonable one.

The advantages of this method is that we could generalize it for other fluid problems where the perturbed boundary condition at the interface with  $\varepsilon$  is in fact a physical boundary condition with a kinematic viscosity  $\nu$  replacing  $\varepsilon$ , and which is not supposed to be very small compared to the other constants.

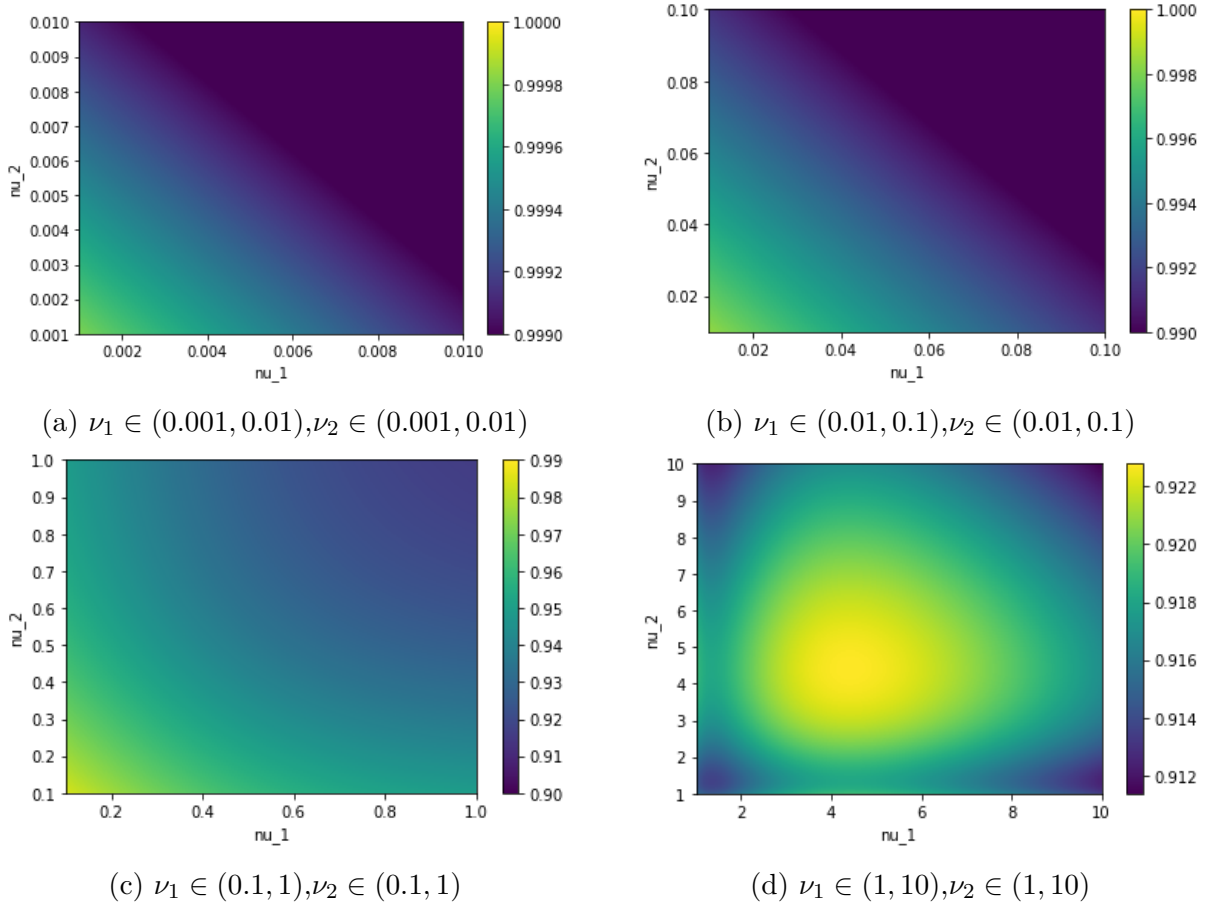


Figure 2.16 – convergence rate for  $k = 2$

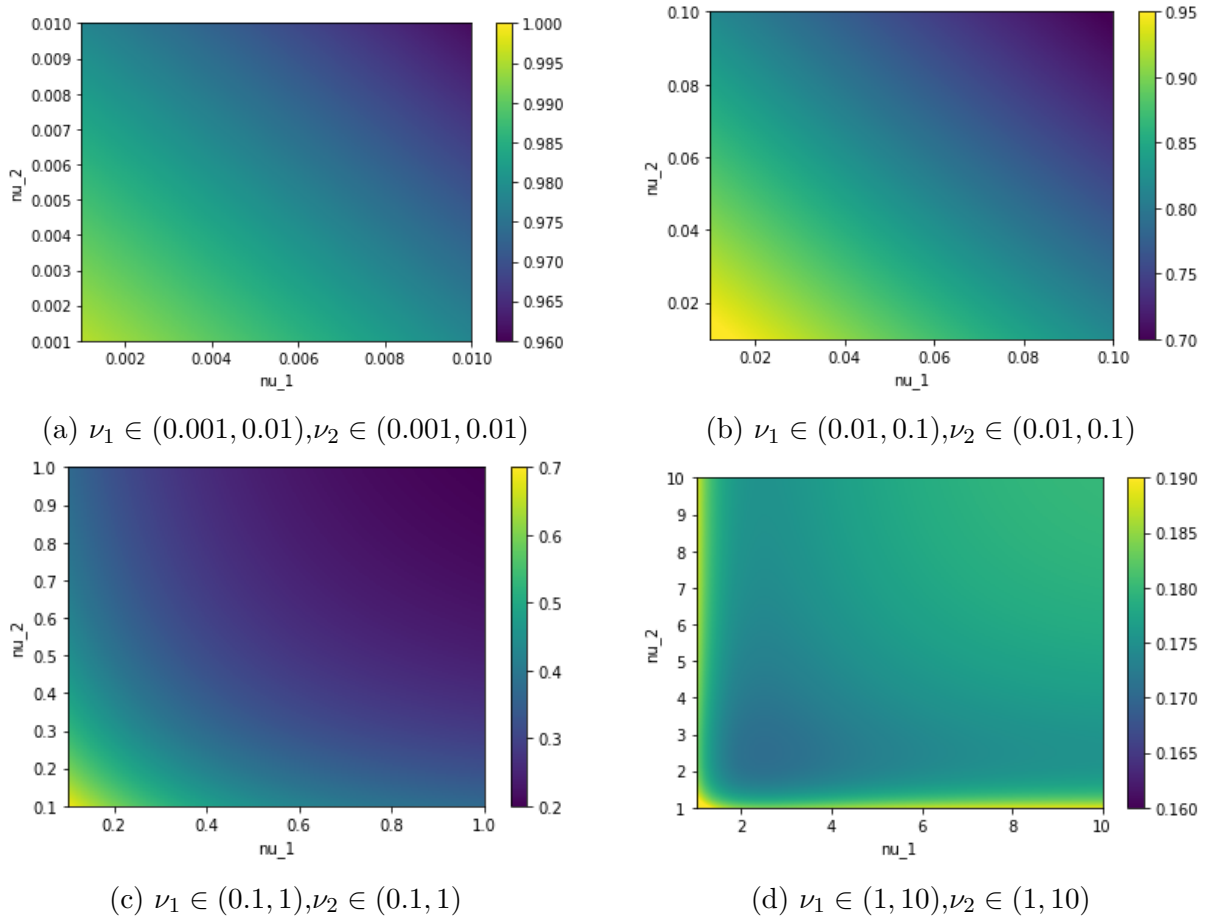
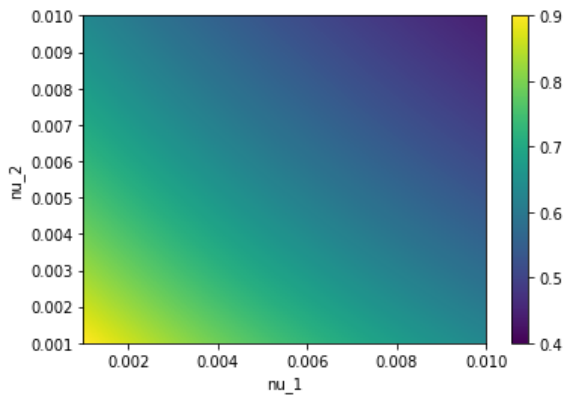
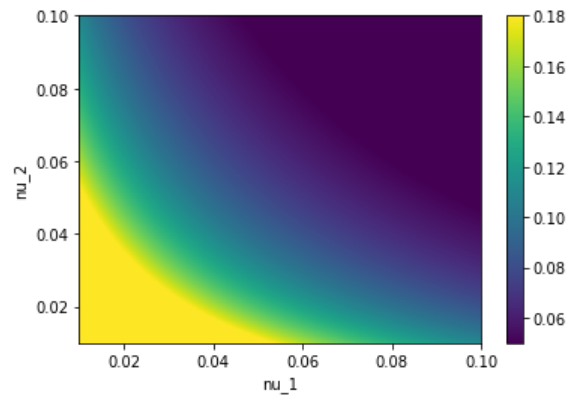


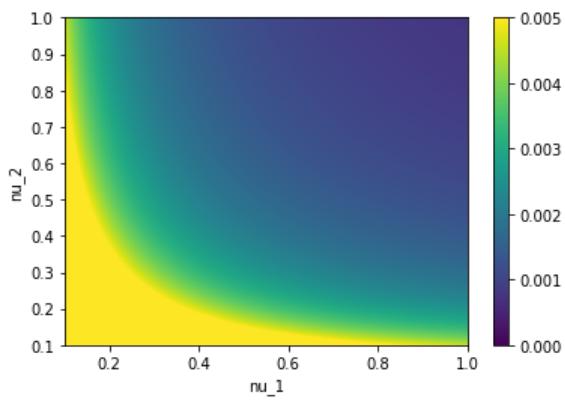
Figure 2.17 – convergence rate for  $k = 10$



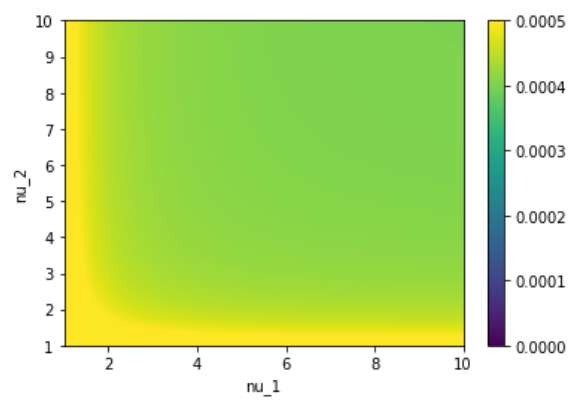
(a)  $\nu_1 \in (0.001, 0.01), \nu_2 \in (0.001, 0.01)$



(b)  $\nu_1 \in (0.01, 0.1), \nu_2 \in (0.01, 0.1)$



(c)  $\nu_1 \in (0.1, 1), \nu_2 \in (0.1, 1)$



(d)  $\nu_1 \in (1, 10), \nu_2 \in (1, 10)$

Figure 2.18 – convergence rate for  $k = 50$





# STATIONARY NAVIER-STOKES COUPLING FOR TWO FLUIDS WITH NONLINEAR INTERFACE CONDITION

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## 3.1 Introduction

We consider a two coupled fluids problem with a rigid lid assumption, driven by two 3D steady Navier-Stokes equations with a Gauckler-Manning condition, also called friction law, at the interface [42] [69], and with a Navier condition at the top and bottom boundaries of the domains  $\Omega_1$  and  $\Omega_2$ .

The equations are the following.

$$\left\{ \begin{array}{ll} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i \\ \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i} (\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \end{array} \right. \quad (3.1)$$

for  $i = 1, 2$ , where the velocities  $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{x}_h, z_1), \mathbf{u}_2(\mathbf{x}_h, z_2))$  are decomposed as

$$\mathbf{u}_i = (\mathbf{u}_{i,h}, w), \quad \mathbf{u}_{i,h} = (u_{i,x}, u_{i,y}).$$

Moreover,  $\mathbf{x}_h = (x, y) \in \mathbb{T}_2$ , where

$$\mathbb{T}_2 = \frac{[0, L_1] \times [0, L_2]}{\mathbb{Z}^2},$$

is a two dimensional torus, which means that for the sake of the simplicity, we consider

horizontal periodic boundary conditions:

$$\forall (n, k, q) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{Z}, \quad D^n \mathbf{u}(x + kL_1, y + qL_2, z) = D^n \mathbf{u}(x, y, z),$$

in the sense of the distributions. The interface  $\Gamma_{Int}$  is given by

$$\Gamma_{Int} = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}.$$

The boundaries  $\Gamma_i$  are given by

$$\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\},$$

which is the top of fluid 1 and

$$\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\},$$

the bottom of the second fluid. For the simplicity we set

$$J_1 = [0, z_1^+], \quad z_2 \in J_2 = [z_2^-, 0],$$

where  $z_1^+ > 0$  and  $z_2^- < 0$ . In other word, the domains  $\Omega_i$  can be defined as

$$\Omega_i = \mathbb{T}_2 \times J_i.$$

Finally, the remaining terms are

- $p_i$  the pressure of the fluid  $i$ ,
- $\mathbf{f}_i$  a source term,
- $\mathbf{V}_i$  a fixed velocity, at the bottom and at the top,
- $\nu_i > 0$  is the cinematic viscosity of the fluid  $i$ ,
- the coefficients  $C_D > 0$  and  $c_{d,i} > 0$  from the Gauckler-Manning's law at the interface and from Navier condition at the top and bottom.

This model suits particularly well to the ocean-atmosphere coupling, especially with the Gauckler-Manning assumptions and Navier assumptions at the top, and bottom, where the fixed Velocities  $\mathbf{V}_i$  can be considered as the speeds of the wind, and of the current flow in higher or lower layers, as depicted in figure 3.1. The heights  $z_1$  and  $z_2$  of the two layers can be chosen from centimeters to couple of meters.

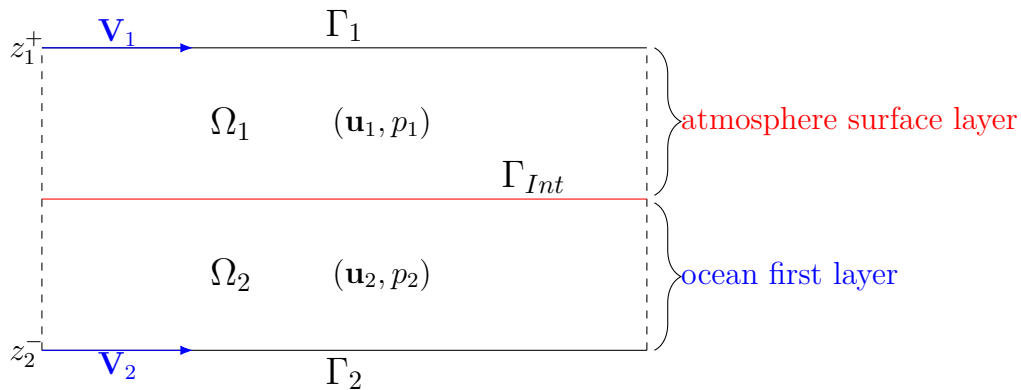


Figure 3.1 – Geometry of the domain

The stationary Navier-Stokes problem (3.1) is hard to solve because of the non-linearity coming from transport terms and the Gauckler-Manning condition. The steady Navier-Stokes equations have been studied a lot, but not with such interface conditions. We can find in [23] the study of such an equation for one fluid and without the Navier condition at the top/bottom boundaries.

The aim is to prove the existence of weak solutions of system (3.1) and to give some numerical tools through algorithms and error estimates, to get a better understanding of this model. We recall that we have deliberately chosen this model for ocean-atmosphere interactions. It is less precise than primitive equations, since we don't consider many physical aspects, (see for instance [62], [76], [58]), but also clearly easier to implement. It is a more realistic and complex toy model than the stationary Stokes equations with continuous Navier conditions at the boundary studied in [55].

The paper is organised in two main sections. The first one 3.2 is dedicated to establish a variational formulation of (3.1) and to prove the existence of weak solutions. The second part 3.3 is devoted to the numerical analysis of (3.1).

Different iterative finite element methods have been developed to obtain numerical simulations of stationary Stokes and Navier-Stokes equations: see for instance [10], [98], [43], [44], [45] or [102]. We follow this steps with a Galerkin finite elements scheme and a linearization algorithm to bypass the non-linearity coming from the convection term and the interface friction condition. We will get theoretical error estimates and Freefem++ numerical simulations for different choices of data.

## 3.2 Problem formulation

Before we start introducing the functional spaces we need, we give some details of the different steps. Let denote with  $\mathcal{VP}$  the variational formulation of (3.1) given in 3.2.2. The different operators involved in  $\mathcal{VP}$ , corresponding to transport, diffusion, pressure, friction terms, are analysed in 3.2.3, especially the compactness properties which will be useful during the whole study. Besides, the subsection 3.2.4 is dedicated to recall all the useful properties satisfied by bounded sequences in Sobolev spaces (existence of weak subsequential limit, compactness in  $L^q$ , results on traces, ...).

We will then give in section 3.2.5 and 3.2.6 *a priori estimates* for the velocity by standard *energy equality method*, and for the pressure by *potential vector methods* (as developed in [20]). The point is to express the pressure as a function of the velocity. To do this, we will kill two birds with one stone and tackle another concerning issue coming from the non-local free divergence term  $\nabla \cdot \mathbf{u}_i = 0$ . We will approximate this condition by the new system

$$\begin{cases} -\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0 & \text{in } \Omega_i, \\ \frac{\partial p_i}{\partial \mathbf{n}_i} = 0 & \text{on } \Gamma_i \cup \Gamma_{Int}, \\ \int_{\Omega_i} p_i(\mathbf{x}) d\mathbf{x} = 0, \end{cases}$$

where  $\varepsilon > 0$  is small. This system admits a unique solution  $p_i = p_i^\varepsilon$  for a given velocity field  $\mathbf{u}_i$  belonging to the suitable functional space. Therefore, we can introduce the system where the free divergence condition is replaced with the approximation introduced above and we denote with  $\mathcal{VP}_\varepsilon$  the corresponding variational formulation, where only  $\mathbf{u}_i = \mathbf{u}_i^\varepsilon$  is involved. We will prove that the weak solutions of  $\mathcal{VP}_\varepsilon$  are converging to the weak solutions of  $\mathcal{VP}$ . Thus it will remain to achieve our goal to show that  $\mathcal{VP}_\varepsilon$  admits weak solutions. This will be done in 3.2.8 by linearization of the non-linear transport and friction terms, and by use of Schauder's fixed point theorem, as summarized in the scheme below:

### 3.2.1 Functional spaces

We define the functional space  $\mathcal{W}_i$  by

$$\mathcal{W}_i = \{\mathbf{u} \in C^\infty(\mathbb{T}_2 \times J_i), \mathbf{u} \cdot \mathbf{n}_i|_{\Gamma_{Int} \cap \Gamma_i} = 0\}, \quad (3.2)$$

equipped with

$$\|\mathbf{u}\|_{\mathcal{W}_i} = \|\nabla \mathbf{u}\|_{L^2(\mathbb{T}_2 \times J_i)} + \|\text{tru}\|_{L^2(\Gamma_i)}, \quad (3.3)$$

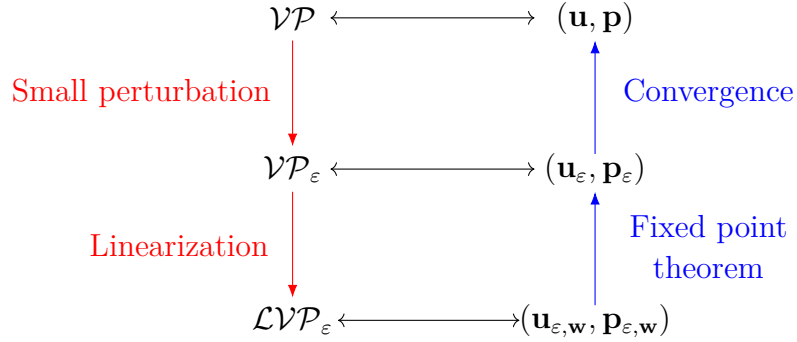


Figure 3.2 – Scheme of the proof

where  $\mathbf{u} \rightarrow \text{tr}\mathbf{u}$  denotes the trace operator. When there is no ambiguity, we will write  $\mathbf{u}$  instead of  $\text{tr}\mathbf{u}$ ; this will be the case every time we use the  $L^2$  norms on the boundary.

Let  $W_i$  denotes the completion of  $\mathcal{W}_i$  with respect to this norm. We define the space  $W = W(\Omega)$  as

$$W = W_1 \times W_2,$$

endowed with the norm

$$\|\mathbf{u}\|_W^2 = \|\mathbf{u}_1\|_{W_1}^2 + \|\mathbf{u}_2\|_{W_2}^2 = \|\nabla\mathbf{u}\|_{0,2,\Omega}^2 + \|\mathbf{u}\|_{0,2,\Gamma}^2, \quad (3.4)$$

where we use the notations

$$\|\nabla\mathbf{u}\|_{0,2,\Omega}^2 := \|\nabla\mathbf{u}_1\|_{0,2,\Omega_1}^2 + \|\nabla\mathbf{u}_2\|_{0,2,\Omega_2}^2$$

and

$$\|\mathbf{u}\|_{0,2,\Gamma}^2 := \|\mathbf{u}_1\|_{0,2,\Gamma_1}^2 + \|\mathbf{u}_2\|_{0,2,\Gamma_2}^2.$$

$W$  is a closed subspace of  $H^1(\Omega_1) \times H^1(\Omega_2)$  so the  $W$ -norm  $\|\cdot\|_W$  is equivalent over  $W$  to the classical  $H^1$ -norm  $\|\cdot\|_{1,2,\Omega}$ .

**Proposition 3.2.1.** *The  $W$ -norm  $\|\cdot\|_W$  and the  $H^1$  norm  $\|\cdot\|_{1,2}$  are equivalent. Moreover, there exists a constant  $C_{eq} > 0$  such that for every  $\mathbf{v} \in W$ :*

$$\frac{1}{C_{eq}} \|\mathbf{v}\|_{1,2} \leq \|\mathbf{v}\|_W \leq C_{eq} \|\mathbf{v}\|_{1,2}. \quad (3.5)$$

**Remark 3.2.1.** *We cannot use the semi-norm  $\|\nabla\mathbf{u}\|_{0,2,\Omega}$  because we don't have a Poincaré inequality in our domain.*

We will seek the pressure in the space  $X := X(\Omega) = L^2(\Omega_1) \times L^2(\Omega_2)$  which is standard for the Navier Stokes equations. Since the pressure is up to a constant, we can even consider the average space  $X_0(\Omega) := L_0^2(\Omega_1) \times L_0^2(\Omega_2)$ , where

$$L_0^2(\Omega_i) = \{p_i \in L^2(\Omega_i), \int_{\Omega_i} p_i = 0\}. \quad (3.6)$$

### 3.2.2 Variational formulation

We will introduce some notations of the involved operators in order to give the variational formulation of the problem (3.1).

First we recall that the velocity  $\mathbf{u}$  we consider in the following is the couple  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ , where  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are the velocities of the fluids 1 and 2 on the domain  $\Omega_1$  and  $\Omega_2$ . We have in particular

$$\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times \mathbb{R}^3.$$

We will also use the notation  $\mathbf{p} = (p_1, p_2)$ , where  $p_1$  and  $p_2$  are the pressure of the fluids on their respective domains. Thus

$$\mathbf{p} = (p_1, p_2) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} \times \mathbb{R}.$$

To give a variational formulation of (3.1), we will use different operators:

- a bilinear diffusion operator  $a$ ,
- a trilinear transport operator  $b$ ,
- a bilinear pressure term  $d$ ,
- a trilinear interface term  $g$ .

#### 1. Diffusion term

We introduce the bilinear diffusion operator  $a$  defined by

$$a(\mathbf{u}, \mathbf{v}) = a_1(\mathbf{u}_1, \mathbf{v}_1) + a_2(\mathbf{u}_2, \mathbf{v}_2), \quad (3.7)$$

where  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ , and where the diffusion operators  $a_1$  and  $a_2$  are defined by

$$a_i(\mathbf{u}_i, \mathbf{v}_i) := \nu_i \int_{\Omega_i} \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i, \quad (3.8)$$

for  $i \in \{1, 2\}$ .

We denote by  $A$  the continuous operator  $W \rightarrow W'$  associated with  $a$ , given by:

$$\langle A(\mathbf{u}), \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}). \quad (3.9)$$

## 2. Transport term

Let  $b$  be the trilinear form defined by:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = b_1(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1) + b_2(\mathbf{u}_2, \mathbf{v}_2, \mathbf{w}_2), \quad (3.10)$$

where

$$b_i(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) = \frac{1}{2} \left( \int_{\Omega_i} (\mathbf{u}_i \cdot \nabla) \mathbf{v}_i \cdot \mathbf{w}_i - \int_{\Omega_i} (\mathbf{u}_i \cdot \nabla) \mathbf{w}_i \cdot \mathbf{v}_i \right). \quad (3.11)$$

We will see in the next subsection different properties of this operator.

We also will consider  $B : W \times W \rightarrow W'$  which satisfies

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = b(\mathbf{u}, \mathbf{u}, \mathbf{u}). \quad (3.12)$$

## 3. Pressure term

We define the pressure term  $d(\mathbf{u}, \mathbf{p})$  by

$$d(\mathbf{u}, \mathbf{p}) = d_1(\mathbf{u}_1, p_1) + d_2(\mathbf{u}_2, p_2), \quad (3.13)$$

where

$$d_i(\mathbf{u}_i, p_i) = \int_{\Omega_i} p_i \nabla \cdot \mathbf{u}_i, \quad (3.14)$$

for  $i \in \{1, 2\}$ .

We can also consider  $D : X \rightarrow X'$  which verifies

$$\langle D(\mathbf{p}), \mathbf{v} \rangle := d(\mathbf{v}, \mathbf{p}), \quad (3.15)$$

for every  $\mathbf{v} \in W$ .

## 4. Interface term

We define the non-linear interface term  $g(\mathbf{u}, \mathbf{v}, \mathbf{w})$  by

$$g(\mathbf{u}, \mathbf{v}, \mathbf{w}) = C_D \int_{\Gamma_{Int}} |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}| (\mathbf{v}_{i,h} - \mathbf{v}_{j,h}) \cdot (\mathbf{w}_{i,h} - \mathbf{w}_{j,h}), \quad (3.16)$$



and  $G : W \rightarrow W'$  the interface operator by

$$\langle G(\mathbf{u}), \mathbf{v} \rangle = g(\mathbf{u}, \mathbf{u}, \mathbf{v}). \quad (3.17)$$

## 5. Bottom and top terms

The term we have to consider is

$$c_{d,1} \int_{\Gamma_1} (\mathbf{u}_1 - \mathbf{V}_1) \cdot \mathbf{v}_1 + c_{d,2} \int_{\Gamma_2} (\mathbf{u}_2 - \mathbf{V}_2) \cdot \mathbf{v}_2 = \sum_{i=1}^2 c_{d,i} \langle \mathbf{u}_i, \mathbf{v}_i \rangle_{\Gamma_i} - \sum_{i=1}^2 c_{d,i} \langle \mathbf{V}_i, \mathbf{v}_i \rangle_{\Gamma_i}, \quad (3.18)$$

where  $\mathbf{V}_i$  is a given velocity at the bottom and at the top of the domain. We will split (3.18) into two different terms: the first one

$$c_{d,1} \langle \mathbf{u}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,2} \langle \mathbf{u}_2, \mathbf{v}_2 \rangle_{\Gamma_2}$$

will be associated with a bilinear operator and the remaining term

$$c_{d,1} \langle \mathbf{V}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,2} \langle \mathbf{V}_2, \mathbf{v}_2 \rangle_{\Gamma_2}$$

to a linear one. Let  $h$  be defined by

$$h(\mathbf{u}, \mathbf{v}) = h_1(\mathbf{u}_1, \mathbf{v}_1) + h_2(\mathbf{u}_2, \mathbf{v}_2), \quad (3.19)$$

where

$$h_i(\mathbf{u}_i, \mathbf{v}_i) = c_{d,i} \int_{\Gamma_i} \mathbf{u}_i \cdot \mathbf{v}_i, \quad (3.20)$$

for  $i \in \{1, 2\}$ .

It is a bilinear operator and we can define  $H : W \rightarrow W'$  as

$$\langle H(\mathbf{u}), \mathbf{v} \rangle = h(\mathbf{u}, \mathbf{v}). \quad (3.21)$$

The remaining term

$$c_{d,1} \langle \mathbf{V}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,2} \langle \mathbf{V}_2, \mathbf{v}_2 \rangle_{\Gamma_2}$$

will be added to the source term in the next paragraph.

**Remark 3.2.2.** *This term can be seen as a drive speed term at the top and at the bottom. It is classical in lid driven cavity test frameworks for instance.*

## 6. Source term

As explained, we will need to take into consideration the top and bottom given velocities  $\mathbf{V}_i$ , which can be seen as source terms as well. Thus, we define  $(\mathbf{f}, \mathbf{v})_\Omega$  as

$$(\mathbf{f}, \mathbf{v})_\Omega = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} + (\mathbf{f}_2, \mathbf{v}_2)_{\Omega_2} + c_{d,1} \langle \mathbf{V}_1, \mathbf{v}_1 \rangle_{\Gamma_1} + c_{d,2} \langle \mathbf{V}_2, \mathbf{v}_2 \rangle_{\Gamma_2}, \quad (3.22)$$

where

$$(\mathbf{f}_i, \mathbf{v}_i)_{\Omega_i} := \int_{\Omega_i} \mathbf{f}_i \mathbf{v}_i, \quad (3.23)$$

for  $i \in \{1, 2\}$ .

Instead of writing  $(\mathbf{f}, \mathbf{v})_\Omega$  we will use  $(\mathbf{f}, \mathbf{v})$  whenever there is no ambiguity.

We are now able to give the variational formulation of the problem (3.1) and the definition of what we mean by weak solutions.

### Definition 3.2.1. (weak solution)

We say that  $(\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{p} = (p_1, p_2)) \in W \times X$  is a weak solution to problem (3.1) if for every  $(\mathbf{v}, \mathbf{q}) \in W \times X$ ,

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - d(\mathbf{v}, \mathbf{p}) + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega, \\ d(\mathbf{u}, \mathbf{q}) = 0. \end{cases} \quad (\mathcal{VP})$$

In other words, Problem (3.1) can be written as: find  $(\mathbf{u}, \mathbf{p}) \in W \times X$ , such that

$$\begin{cases} A(\mathbf{u}) + B(\mathbf{u}, \mathbf{u}) + G(\mathbf{u}, \mathbf{u}) + H(\mathbf{u}) - D(\mathbf{p}) = \mathbf{f} \in W', \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } X. \end{cases} \quad (3.24)$$

## 3.2.3 Properties of the operators

### Diffusion term

In the following we will consider  $\nu = \inf(\nu_1, \nu_2)$ .

We have the straightforward inequality we will use a lot in the future estimates:

### Proposition 3.2.2.

$$\nu \|\nabla \mathbf{u}\|_{0,2,\Omega} \leq a(\mathbf{u}, \mathbf{u}), \quad (3.25)$$

for every  $\mathbf{u} \in W$ .

### Transport term

This trilinear form verifies different properties and many different inequalities. In particular the non-linear transport term  $b$  verifies the lemma

**Lemma 3.2.1.** *Let  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in W^3$ , the trilinear form  $b$  verifies the following properties:*

1.  $b$  is continuous on  $W \times W \times W$  and verifies the inequality

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C_b \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W, \quad (3.26)$$

where  $C_b = C_b(\Omega) > 0$  is the best constant satisfying the inequality.

2.  $b$  is antisymmetric, i.e

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad (3.27)$$

in particular  $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$ , and  $b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$ .

3. When the fluids are incompressible and verify the condition  $\nabla \cdot \mathbf{u}_i = 0$ , then

$$b_i(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) = \int_{\Omega_i} (\mathbf{u}_i \cdot \nabla) \mathbf{v}_i \cdot \mathbf{w}_i, \quad (3.28)$$

which is exactly the transport term in (3.1).

*Proof.* 1. We will mainly use the Cauchy-Schwarz inequality with the Sobolev embedding  $H^1(\Omega_i) \hookrightarrow L^p(\Omega_i)$ ,  $p \leq 6$ . It gives

$$\begin{aligned} |((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i}| &\leq C \|\mathbf{u}\|_{0,4,\Omega_i} \|\nabla \mathbf{v}\|_{0,2,\Omega_i} \|\mathbf{w}\|_{0,4,\Omega_i} \\ &\leq C \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W. \end{aligned}$$

The same estimate can be obtained for the term  $((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{u}_i)_{\Omega_i}$ , hence (3.26).

2. It follows from the definition of  $b$  and  $b_i$ .
3. We have

$$((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i} = \sum_{k,j} \int_{\Omega_i} u_i^k(\mathbf{x}) \partial_k v_i^j(\mathbf{x}) w_i^j(\mathbf{x}) d\mathbf{x},$$

where  $u_i^k$  is the  $k$ -th component of  $\mathbf{u}_i$ . We have thanks to the Stokes formula:

$$\begin{aligned} ((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i} &= \sum_{k,j} \left( \int_{\Gamma_i} v_i^j(\mathbf{x}) w_i^j(\mathbf{x}) u_i^k(\mathbf{x}) n^k(\mathbf{x}) d\Gamma(\mathbf{x}) - \int_{\Omega_i} v_i^j(\mathbf{x}) \partial_k (u_i^k w_i^j)(\mathbf{x}) d\mathbf{x} \right) \\ &= (\mathbf{u}_i \cdot \mathbf{n}, \mathbf{v}_i \cdot \mathbf{w}_i)_{\Gamma_i} - (\nabla \cdot \mathbf{u}_i, \mathbf{w}_i \cdot \mathbf{v}_i)_{\Omega_i} - ((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{v}_i)_{\Omega_i}, \end{aligned}$$

which holds for any  $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \in C^1(\overline{\Omega}_i)^3$ .

Taking  $\mathbf{u}_i \in \mathcal{W}_i$ , the term  $(\mathbf{u}_i \cdot \mathbf{n}, \mathbf{v}_i \cdot \mathbf{w}_i)_{\Gamma_i}$  vanishes and we get

$$((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i, \mathbf{w}_i)_{\Omega_i} = -(\nabla \cdot \mathbf{u}_i, \mathbf{w}_i \cdot \mathbf{v}_i)_{\Omega_i} - ((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{v}_i)_{\Omega_i}. \quad (3.29)$$

It remains to show that (3.29) is still valid when  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  belong to  $W$ , without additional regularity. Thanks to the density of  $\mathcal{W}_i$  in  $W_i$  and of  $C^1(\overline{\Omega}_i)^3$  in  $H^1(\Omega_i)^3$  and thanks to the continuity estimates

$$\begin{aligned} |(\nabla \cdot \mathbf{u}_i, \mathbf{w}_i \cdot \mathbf{v}_i)_{\Omega_i}| &\leq C_1 \|\mathbf{u}_i\|_{1,2,\Omega_i} \|\mathbf{v}_i\|_{1,2,\Omega_i} \|\mathbf{w}_i\|_{1,2,\Omega_i}, \\ |((\mathbf{u}_i \cdot \nabla) \mathbf{w}_i, \mathbf{v}_i)_{\Omega_i}| &\leq C_2 \|\mathbf{u}_i\|_{1,2,\Omega_i} \|\mathbf{v}_i\|_{1,2,\Omega_i} \|\mathbf{w}_i\|_{1,2,\Omega_i}, \end{aligned}$$

where  $C_i$  depends only on  $\Omega_i$ , the relation (3.29) holds for any  $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \in W$ . Finally, the incompressibility condition  $\nabla \cdot \mathbf{u}_i = 0$  gives (3.28).  $\square$

**Remark 3.2.3.** *The choice of the operator  $b_i$  instead of the more classical  $((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i \cdot \mathbf{w}_i)_{\Omega_i}$  comes from the incompressibility condition. Even without the condition  $\nabla \cdot \mathbf{u}_i = 0$ , the operator  $b_i$  satisfies (3.28), which would have not been the case with  $((\mathbf{u}_i \cdot \nabla) \mathbf{v}_i \cdot \mathbf{w}_i)_{\Omega_i}$ .*

### Interface term

We recall that the interface terms  $g$  and  $G$  are defined in (3.16) and (3.17) by

$$g(\mathbf{u}, \mathbf{v}, \mathbf{w}) = C_D \int_{\Gamma_{Int}} |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}| (\mathbf{v}_{i,h} - \mathbf{v}_{j,h}) \cdot (\mathbf{w}_{i,h} - \mathbf{w}_{j,h}),$$

and

$$\langle G(\mathbf{u}), \mathbf{v} \rangle = g(\mathbf{u}, \mathbf{u}, \mathbf{v}).$$

We will show in this part that the interface terms are well defined in the variational formulation.

**Lemma 3.2.2.** *For every  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ , the term  $g(\mathbf{u}, \mathbf{v}, \mathbf{w})$  does exist and checks the inequality*

$$|g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_g \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W, \quad (3.30)$$

where the constant  $C_g > 0$  depends only on  $C_D$  and  $\Omega$ .

*Proof.* We have thanks to the Cauchy-Schwarz inequality

$$|g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_D \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,4,\Gamma_I} \|\mathbf{v}_1 - \mathbf{v}_2\|_{0,4,\Gamma_I} \|\mathbf{w}_1 - \mathbf{w}_2\|_{0,2,\Gamma_I},$$

and because of the triangular inequality:

$$\begin{aligned} & |g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \\ & \leq C_D (\|\mathbf{u}_1\|_{0,4,\Gamma_I} + \|\mathbf{u}_2\|_{0,4,\Gamma_I}) (\|\mathbf{v}_1\|_{0,4,\Gamma_I} + \|\mathbf{v}_2\|_{0,4,\Gamma_I}) (\|\mathbf{w}_1\|_{0,2,\Gamma_I} + \|\mathbf{w}_2\|_{0,2,\Gamma_I}). \end{aligned}$$

Using the Sobolev trace embedding results  $H^1(\Omega_i) \hookrightarrow L^2(\Gamma_I)$  and  $H^1(\Omega_i) \hookrightarrow L^4(\Gamma_I)$  and the corresponding injection constants  $C_{i,2}$  and  $C_{i,4}$ , we get

$$\begin{aligned} |g(\mathbf{u}, \mathbf{v}, \mathbf{w})| & \leq C_D (C_{1,4} \|\nabla \mathbf{u}_1\|_{0,\Omega_1} + C_{2,4} \|\nabla \mathbf{u}_2\|_{0,\Omega_2}) (C_{1,4} \|\nabla \mathbf{v}_1\|_{0,\Omega_1} + C_{2,4} \|\nabla \mathbf{v}_2\|_{0,\Omega_2}) \\ & \quad (C_{1,2} \|\nabla \mathbf{w}_1\|_{0,\Omega_1} + C_{2,2} \|\nabla \mathbf{w}_2\|_{0,\Omega_2}), \end{aligned}$$

and we obtain

$$|g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_t \|\nabla \mathbf{u}\|_{0,2,\Omega} \|\nabla \mathbf{v}\|_{0,2,\Omega} \|\nabla \mathbf{w}\|_{0,2,\Omega},$$

where  $C_t = C_D (C_{1,4} + C_{2,4})^2 (C_{1,2} + C_{2,2})$ , which finally leads to

$$|g(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_g \|\mathbf{u}\|_W \|\mathbf{v}\|_W \|\mathbf{w}\|_W,$$

using the norms equivalence (3.5), where  $C_g = C_t C_{eq}$ .

□

It is convenient to introduce the function  $\tilde{g} \in W$  defined by

$$\tilde{g}(\mathbf{v}) = \mathbf{v}|\mathbf{v}|, \tag{3.31}$$

which gives in particular

$$g(\mathbf{u}, \mathbf{u}, \mathbf{v}) = C_D \int_{\Gamma_I} \tilde{g}(\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2),$$

for every  $\mathbf{u}, \mathbf{v} \in W$ .

The application  $\tilde{g}$  satisfies the following properties:

**Lemma 3.2.3.** *There exists a constant  $c$  depending only on  $\Omega$  such that*

1.  $\tilde{g}$  is positive and satisfies

$$0 \leq \tilde{g}(\mathbf{u}) \cdot \mathbf{u} = |\mathbf{u}|^2, \quad (3.32)$$

for every  $\mathbf{u} \in \mathbb{R}^3$ .

2.  $\tilde{g}$  is monotone:

$$0 \leq (\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})) \cdot (\mathbf{u} - \mathbf{v}), \quad (3.33)$$

and satisfies

$$|\nabla \tilde{g}(\mathbf{u})| \leq c|\mathbf{u}|, \quad (3.34)$$

for every  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ .

3. Let  $\mathbf{u} \in W$ : then  $\tilde{g}(\mathbf{u}) \in L^2(\Gamma_I)$  and

$$\|\tilde{g}(\mathbf{u})\|_{0,2,\Gamma_I} \leq c\|\mathbf{u}\|_W^2. \quad (3.35)$$

4. For any  $\mathbf{u}, \mathbf{v} \in W$ ,

$$\|\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})\| \leq c(\|\mathbf{u}\|_W + \|\mathbf{v}\|_W)\|\mathbf{u} - \mathbf{v}\|_W. \quad (3.36)$$

*Proof.* 1. The result is straightforward.

2. It is straightforward as well.

3. The proof is similar to the proof of the lemma (3.2.2) with the use of Sobolev embedding trace theorems.

4. Doing a Taylor expansion, we obtain for every  $\mathbf{x} \in \Gamma_I$

$$\tilde{g}(\mathbf{u}(\mathbf{x})) - \tilde{g}(\mathbf{v}(\mathbf{x})) = \int_0^1 \nabla \tilde{g}(t\mathbf{u}(\mathbf{x}) + (1-t)\mathbf{v}(\mathbf{x})) \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})) dt.$$

Using the control of the gradient (3.34), we have

$$|\tilde{g}(\mathbf{u}(\mathbf{x})) - \tilde{g}(\mathbf{v}(\mathbf{x}))| \leq \frac{c}{2}(|\mathbf{u}(\mathbf{x})| + |\mathbf{v}(\mathbf{x})|)|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})|,$$

which gives by integration and thanks to the Cauchy-Schwarz inequality

$$\|\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})\| \leq \frac{c}{2} \|\mathbf{u} + \mathbf{v}\|_{0,4,\Gamma_I} \|\mathbf{u} - \mathbf{v}\|_{0,4,\Gamma_I}, \quad (3.37)$$

and hence to (3.36) using again the injection  $H^1(\Omega_i) \hookrightarrow L^4(\Gamma_I)$ .

□

**Remark 3.2.4.** When we will look for estimates in the variational formulation, we will take as a test  $\mathbf{v} = \mathbf{u}$  and consider the term  $g(\mathbf{u}, \mathbf{u}, \mathbf{u})$  which verifies

$$g(\mathbf{u}, \mathbf{u}, \mathbf{u}) \geq 0, \quad (3.38)$$

because of the property (3.32) satisfied by  $\tilde{g}$ .

We can now give and prove some properties satisfied by  $G$ :

**Lemma 3.2.4.** 1. The application  $G : \mathbf{u} \rightarrow G(\mathbf{u})$  maps  $W$  to  $W'$  and verifies

$$\|G(\mathbf{u})\|_{W'} \leq C \|\mathbf{u}\|_W^2. \quad (3.39)$$

2.  $G$  is positive, i.e.,  $\langle G(\mathbf{u}), \mathbf{u} \rangle \geq 0$  for every  $\mathbf{u} \in W$ .

3. The friction operator  $G$  satisfies

$$\|G(\mathbf{v}) - G(\mathbf{w})\|_{W'} \leq C (1 + \|\mathbf{v}\|_W + \|\mathbf{w}\|_W) \|\mathbf{v} - \mathbf{w}\|_W, \quad (3.40)$$

4.  $G$  is monotone.

5.  $G$  is continuous and compact.

*Proof.* 1. By definition of  $G$  and  $\tilde{g}$ , we have the duality relationship

$$\langle G(\mathbf{u}), \mathbf{v} \rangle = (\tilde{g}(\mathbf{u}), \mathbf{v})_{\Gamma_I},$$

which gives

$$|\langle G(\mathbf{u}), \mathbf{v} \rangle| \leq \|\tilde{g}(\mathbf{u})\|_{0,2,\Gamma_I} \|\mathbf{v}\|_{0,2,\Gamma_I}.$$

Combining it with (3.35) and the Sobolev trace embedding theorem, this gives

$$|\langle G(\mathbf{u}), \mathbf{v} \rangle| \leq c \|\nabla \tilde{g}(\mathbf{u})\|_{0,2,\Omega}^2 \|\nabla \mathbf{v}\|_{0,2,\Omega}.$$

2. The results comes from (3.38).

3. We have for every  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W$

$$|\langle G(\mathbf{u}) - G(\mathbf{v}), \mathbf{w} \rangle| \leq \|\tilde{g}(\mathbf{u}) - \tilde{g}(\mathbf{v})\|_{0,2,\Gamma_I} \|\mathbf{w}\|_{0,2,\Gamma},$$

and thanks to the estimate on  $\tilde{g}$  (3.36),

$$|\langle G(\mathbf{u}) - G(\mathbf{v}), \mathbf{w} \rangle| \leq c(\|\nabla \mathbf{u}\|_{0,2,\Omega} + \|\nabla \mathbf{v}\|_{0,2,\Omega}) \|\nabla(\mathbf{u} - \mathbf{v})\|_{0,2,\Omega} \|\nabla \mathbf{w}\|_{0,2,\Omega}.$$

4. The monotonicity of  $G$  is derived directly from the monotonicity of  $\tilde{g}$ .
5. The continuity of  $G$  comes from the property (3.40), which states that  $G$  is Lipschitz over any bounded set of  $W$ .
6. Let  $(\mathbf{u}_n)_n \in W^{\mathbf{N}}$  which weakly converges to  $\mathbf{u} \in W$ . We will show that  $(G(\mathbf{u}_n))_n$  is strongly converging in  $W'$ , up to a subsequence. We will need compact Sobolev embedding to "level up" the weak convergence of  $(\mathbf{u}_n)_n$  into strong convergence of  $(G(\mathbf{u}_n))_n$ . Since  $(\mathbf{u}_n)_n$  is weakly converges to  $\mathbf{u}$  in  $W$ , by definition of the  $W$ -norm, we have the weak convergence of  $(\mathbf{u}_n)_n$  to  $\mathbf{u}$  in the space  $H^{1/2}(\Gamma_I)$ . Moreover, we have the compact embedding  $H^{1/2}(\Gamma_I) \hookrightarrow L^3(\Gamma_I)$ , which means that up to a subsequence, the sequence  $(\|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I})_n$  is converging to zero. We can make appear such  $L^3$ -norms using the Hölder and Cauchy-Schwarz inequality:

$$\begin{aligned} |\langle G(\mathbf{u}_n) - G(\mathbf{u}), \mathbf{v} \rangle| &\leq c(|\mathbf{u}_n| + |\mathbf{u}|) \|\mathbf{u}_n - \mathbf{u}\|_{0,3/2,\Gamma_I} \|\mathbf{v}\|_{0,3,\Gamma_I} \\ &\leq c\|\mathbf{u}_n| + |\mathbf{u}|\|_{0,3,\Gamma_I} \|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I} \|\mathbf{v}\|_{0,3,\Gamma_I}. \end{aligned}$$

The term  $\|\mathbf{u}_n| + |\mathbf{u}|\|_{0,3,\Gamma_I}$  can be bounded by triangular inequality since the sequence  $(\|\mathbf{u}_n\|_W)_n$  is bounded. Still denoting the constant  $c$  which depends on  $\Omega$ ,  $\|\mathbf{u}\|_{0,3,\Gamma_I}$  and on  $\sup_{n \in \mathbf{N}} \|\mathbf{u}_n\|_W$ , we get

$$|\langle G(\mathbf{u}_n) - G(\mathbf{u}), \mathbf{v} \rangle| \leq c \|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I} \|\nabla \mathbf{v}\|_{0,2,\Omega},$$

using one more time the Trace Sobolev embedding  $H^1(\Omega_i) \hookrightarrow L^3(\Gamma_I)$ . Finally

$$\|G(\mathbf{u}_n) - G(\mathbf{u})\|_{W'} \leq c \|\mathbf{u}_n - \mathbf{u}\|_{0,3,\Gamma_I} \xrightarrow{n \rightarrow \infty} 0,$$

which concludes the proof according to the previous remarks. □

**Remark 3.2.5.** *In the proof of the compactness result, we haven't used the estimate of  $\tilde{g}$  with  $L^4(\Gamma_I)$ -norms (3.37) because there is no compact embedding of  $H^{1/2}(\Gamma_I)$  to  $L^4(\Gamma_I)$  for  $n = 3$ . Nevertheless it would have worked for  $n = 2$ .*



### 3.2.4 Compactness tools: VESP and PESP

In what follows, we will often have to use the same kind of arguments. We will have to consider velocity sequences  $(\mathbf{v}_n)_{n \in \mathbb{N}}$  belonging to  $W$ , which are solutions of an approximate problem, and we would like to pass to the limit in the equations. It will be also the case for pressure sequences  $(\mathbf{p}_n)_{n \in \mathbb{N}}$  belonging to  $X$ .

Lets start with the velocities and introduce the Velocity Extracting Subsequences Principle (VESP).

#### 1. VESP.

Assume that we have an priori estimate obtained from energy inequalities of the form

$$\|\mathbf{v}_n\| \leq C, \quad \forall n \in \mathbb{N},$$

where  $C > 0$  does not depend on  $n$ . Different properties can be retrieved from this inequality. Denote by  $B_C \subset W$  the ball or radius  $C$  centered in zero, we get

- (a) From the Banach Alaoglu theorem: the ball  $B_C$  is relatively compact in  $W$ , wich means there exists a velocity  $\mathbf{v} \in B_C$  and subsequence  $(\mathbf{v}_{n_k})_{k \in \mathbb{N}}$  of  $(\mathbf{v}_n)_{n \in \mathbb{N}}$  such that  $\mathbf{v}_{n_k})_{k \in \mathbb{N}}$  weakly converges to  $\mathbf{v}$  when  $k \rightarrow \infty$ . Moreover thanks to the Sobolev embedding theorem, the subsequence  $(\mathbf{v}_{n_k})_{k \in \mathbb{N}}$  strongly converges to  $\mathbf{v}$  in  $L^p(\Omega)$ , when  $1 \leq p < 6$  in dimension 3, and when  $1 \leq p$  in dimension 2.
- (b) From the Riesz-Fischer theorem, it also converges a.e in  $\Omega$ , and there exists  $A_p \in L^p(\Omega)$  such that  $\forall k \in \mathbb{N}, |\mathbf{v}_{n_k}| \leq A_p$ , a.e in  $\Omega$ .
- (c) The sequence of the traces  $(\gamma(\mathbf{v}_{n_k}))_{k \in \mathbb{N}}$  weakly converges to the trace  $\gamma(\mathbf{v})$  in  $H^{1/2}(\Gamma)$  and from the trace continuity theorem and the Sobolev embedding results, it also strongly converges in  $L^q(\Gamma)$ , for  $1 \leq q < 4$  in dimension 3 (the dimension of  $\Gamma$  is 2 in that case).
- (d) Again, from the Riesz-Fischer theorem, the sequence  $(\gamma(\mathbf{v}_{n_k}))_{k \in \mathbb{N}}$  converges a.e in  $\Gamma$ , and there exists  $B_q \in L^q(\Gamma)$  such that  $\forall k \in \mathbb{N}, |\gamma(\mathbf{v}_{n_k})| \leq B_q$  a.e in  $\Gamma$ .

**Remark 3.2.6.** *To simplify, we will always write  $\mathbf{v}_n$  instead of  $\mathbf{v}_{n_k}$  considering that all the convergences are up to a subsequence. To sum up, we have:*

$$\begin{aligned} \mathbf{v}_n &\xrightarrow[n \rightarrow \infty]{W} \mathbf{v}, \\ \mathbf{v}_n &\xrightarrow[n \rightarrow \infty]{L^p(\Omega)} \mathbf{v}, \quad 1 \leq p < 6, \\ \gamma(\mathbf{v}_n) &\xrightarrow[n \rightarrow \infty]{H^{1/2}(\Gamma)} \gamma(\mathbf{v}), \\ \gamma(\mathbf{v}_n) &\xrightarrow[n \rightarrow \infty]{L^q(\Gamma)} \gamma(\mathbf{v}), \quad 1 \leq q < 4. \end{aligned}$$

**Definition 3.2.2.** We say that the limit  $\mathbf{v}$  of the subsequence  $(\mathbf{v}_n)_n$  is a VESP-limit of  $(\mathbf{v}_n)_n$  when it satisfies all the previous properties.

2. **PESP.** The Pressure Extracting Subsequence Principle (PESP) is defined as well.
3. **Convergence lemma.** We will consider non-linear terms in our equations. It is more difficult to pass to the limit in these terms than in the linear ones. We give a small lemma to easily pass to the limit in the non linear terms  $b$  and  $g$  whenever we apply the VESP to velocity sequences.

**Lemma 3.2.5.** Let  $(\mathbf{v}_n)_{n \in \mathbb{N}}$  and  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be two bounded sequences in  $W$ . Let  $\mathbf{v}$  and  $\mathbf{w}$  be VESP-limits of these sequences. We also assume that  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  strongly converges to  $\mathbf{w}$  in  $W$ . Then

$$\lim_{n \rightarrow \infty} b(\mathbf{v}_n, \mathbf{v}_n, \mathbf{w}_n) = b(\mathbf{v}, \mathbf{v}, \mathbf{w}), \quad (3.41)$$

$$\lim_{n \rightarrow \infty} g(\mathbf{v}_n, \mathbf{v}_n, \mathbf{w}_n) = g(\mathbf{v}, \mathbf{v}, \mathbf{w}). \quad (3.42)$$

*Proof.* Let give the proof of the first claim on  $b$ . we have by definition

$$b(\mathbf{v}_n, \mathbf{v}_n, \mathbf{w}_n) = \frac{1}{2} \left( \int_{\Omega} \mathbf{v}_n \otimes \mathbf{w}_n : \nabla \mathbf{v}_n - \int_{\Omega} \mathbf{v}_n \otimes \mathbf{v}_n : \nabla \mathbf{w}_n \right),$$

and

$$b(\mathbf{v}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \left( \int_{\Omega} \mathbf{v} \otimes \mathbf{w} : \nabla \mathbf{v} - \int_{\Omega} \mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{w} \right).$$

According to the a-property of the VESP, we have in particular

$$\begin{aligned} \mathbf{v}_n &\xrightarrow[n \rightarrow \infty]{(L^4(\Omega))^3} \mathbf{v}, \\ \mathbf{w}_n &\xrightarrow[n \rightarrow \infty]{(L^4(\Omega))^3} \mathbf{w}. \end{aligned}$$

Thus, we have the strong convergences

$$\begin{cases} \mathbf{v}_n \otimes \mathbf{w}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \mathbf{v} \otimes \mathbf{w}, \\ \mathbf{v}_n \otimes \mathbf{v}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \mathbf{v} \otimes \mathbf{v}. \end{cases} \quad (3.43)$$

Moreover, still from the a-property of the VESP, we have the weak convergences

$$\begin{cases} \nabla \mathbf{v}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \nabla \mathbf{v}, \\ \nabla \mathbf{w}_n & \xrightarrow[n \rightarrow \infty]{(L^2(\Omega))^9} \nabla \mathbf{w} \end{cases} \quad (3.44)$$

which gives combined with (3.43) the convergence (3.41). The second claim about the convergence of the  $g$ -term is a consequence of the compactness property given in 3.2.4.  $\square$

### 3.2.5 A priori estimate for the velocity

**Proposition 3.2.3.** *Let  $(\mathbf{u}, \mathbf{p})$  be any weak solution of the problem (3.1). Then  $\mathbf{u}$  satisfies the energy equality*

$$a(\mathbf{u}, \mathbf{u}) + g(\mathbf{u}, \mathbf{u}, \mathbf{u}) + h(\mathbf{u}, \mathbf{u}) = \langle \mathbf{f}, \mathbf{u} \rangle, \quad (3.45)$$

leading to the estimate

$$\mu \|\mathbf{u}\|_W^2 \leq \frac{C_{eq}^2}{\mu} \|\mathbf{f}\|_{0,2}^2 + \frac{(\max(c_{d,1}, c_{d,2}))^2}{\min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2 \quad (3.46)$$

where the constant  $\mu$  is defined by

$$\mu = \min\left(\nu, \frac{1}{2}c_{d,1}, \frac{1}{2}c_{d,2}\right),$$

and where the constant  $C_{eq}$  is given in the norm equivalence result 3.2.1. In the following, it will be more convenient to use the simplified a priori estimate

$$\|\mathbf{u}\|_W^2 \leq C_{vel}, \quad (3.47)$$

where  $C_{vel}$  is given by

$$C_{vel} = \frac{C_{eq}^2}{\mu^2} \|\mathbf{f}\|_{0,2}^2 + \frac{(\max(c_{d,1}, c_{d,2}))^2}{\mu \min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2.$$

*Proof.* We take  $\mathbf{v} = \mathbf{u} \in W$  as a test in  $(\mathcal{VP})$  and we consider each term one after another. Since  $\nabla \cdot \mathbf{u} = 0$ , the pressure term  $\langle \mathbf{p}, \nabla \cdot \mathbf{u} \rangle = 0$ . We also have thanks to antisymmetry property 3.2.1 of the trilinear form  $b$  that  $b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$ . This gives us

$$a(\mathbf{u}, \mathbf{u}) + g(\mathbf{u}, \mathbf{u}, \mathbf{u}) + h(\mathbf{u}, \mathbf{u}) = \langle \mathbf{f}, \mathbf{u} \rangle,$$

which is the energy equality (3.45). Moreover, the non-linear term  $g$  satisfies  $g(\mathbf{u}, \mathbf{u}, \mathbf{u}) \geq 0$ . Thus:

$$a(\mathbf{u}, \mathbf{u}) + h(\mathbf{u}, \mathbf{u}) \leq \langle \mathbf{f}, \mathbf{u} \rangle.$$

Recall that the left hand term equals

$$\nu_1 \|\nabla \mathbf{u}_1\|_{0,2,\Omega_1}^2 + \nu_2 \|\nabla \mathbf{u}_2\|_{0,2,\Omega_2}^2 + c_{d,1} \|\mathbf{u}_{1,h}\|_{0,2,\Gamma_1}^2 + c_{d,2} \|\mathbf{u}_{2,h}\|_{0,2,\Gamma_2}^2$$

and the right hand term is

$$\int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{u}_1 + \int_{\Omega_2} \mathbf{f}_2 \cdot \mathbf{u}_2 + c_{d,1} \int_{\Gamma_1} \mathbf{V}_1 \cdot \mathbf{u}_{1,h} + c_{d,2} \int_{\Gamma_2} \mathbf{V}_2 \cdot \mathbf{u}_{2,h},$$

according to (3.22).

Consequently, by definition of the product norms on  $\Omega$ , we get thanks to the Cauchy-Schwarz inequality

$$\nu \|\nabla \mathbf{u}\|_{0,2,\Omega}^2 + \min(c_{d,1}, c_{d,2}) \|\mathbf{u}\|_{0,2,\Gamma}^2 \leq \|\mathbf{f}\|_{0,2} \|\mathbf{u}\|_{1,2,\Omega} + \max(c_{d,1}, c_{d,2}) \|\mathbf{V}\|_{0,2,\Gamma} \|\mathbf{u}\|_{0,2,\Gamma}.$$

Using the Young's inequality, we obtain

$$\nu \|\nabla \mathbf{u}\|_{0,2,\Omega}^2 + \frac{1}{2} \min(c_{d,1}, c_{d,2}) \|\mathbf{u}\|_{0,2,\Gamma}^2 \leq \|\mathbf{f}\|_{0,2} \|\mathbf{u}\|_{1,2,\Omega} + \frac{(\max(c_{d,1}, c_{d,2}))^2}{2 \min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2,$$

We use the equivalence result (3.5) between the two norms  $\|\cdot\|_{1,2,\Omega}$  and  $\|\cdot\|_W$  and the Young's inequality to get

$$\mu \|\mathbf{u}\|_W^2 \leq C_{eq} \|\mathbf{f}\|_{0,2} \|\mathbf{u}\|_W + \frac{(\max(c_{d,1}, c_{d,2}))^2}{2 \min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2,$$

where  $\mu = \min(\nu, \frac{1}{2}c_{d,1}, \frac{1}{2}c_{d,2})$ . Finally we get using the Young's inequality again:

$$\mu \|\mathbf{u}\|_W^2 \leq \frac{C_{eq}^2}{\mu} \|\mathbf{f}\|_{0,2}^2 + \frac{(\max(c_{d,1}, c_{d,2}))^2}{\min(c_{d,1}, c_{d,2})} \|\mathbf{V}\|_{0,2,\Gamma}^2.$$

□

### 3.2.6 A priori estimate for the pressure

In this section, we will use the potential vector method to express the pressure as a function of the velocities.

To do this, we first need a lemma on the elliptic regularity of a Neumann problem on the Torus.

**Lemma 3.2.6.** *Let  $u \in H^2(\Omega)$  be the unique solution of the Neumann problem*

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \Gamma \\ \int_{\Omega} u = 0 \end{cases} \quad (3.48)$$

where  $\Omega = \Pi_2 \times [0, 1]$ . Then we have the estimate:

$$\|u\|_{2,2} \leq C \|f\|_{0,2}. \quad (3.49)$$

*Proof.* See the section 3.2.9 at the end of this chapter. □

We will use this lemma to obtain an a priori estimate on the pressure. We recall that

$$X = L^2(\Omega_1) \times L^2(\Omega_2) \quad (3.50)$$

is the pressure space equipped with the norm:

$$\|\mathbf{p}\|_X = \left( \|p_1\|_{L^2(\Omega_1)}^2 + \|p_2\|_{L^2(\Omega_2)}^2 \right)^{1/2},$$

where  $\mathbf{p} = (p_1, p_2) \in X$ .

**Lemma 3.2.7.** *We can find a test  $\mathbf{w} \in W$  and a constant  $C_{\Omega}$  depending only on  $\Omega$  such*

that

$$\langle \mathbf{p}, \nabla \cdot \mathbf{w} \rangle = -\|\mathbf{p}\|_X^2, \quad (3.51)$$

and

$$\|\mathbf{w}\|_W \leq C_\Omega \|\mathbf{p}\|_X. \quad (3.52)$$

*Proof.* Let  $v_i \in H^2(\Omega_i)$  be the unique solution of the Neumann problem

$$\begin{cases} -\Delta v_i = P_i & \text{in } \Omega_i, \\ \frac{\partial v_i}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_i \cap \Gamma_{int}, \\ \int_{\Omega_i} v_i = 0. \end{cases} \quad (3.53)$$

for  $i = 1, 2$ . We have

$$\|v_i\|_{H^2(\Omega_i)} \leq C_i \|P_i\|_{L^2(\Omega_i)},$$

where  $C_i$  only depends on  $\Omega_i$ . Let  $\mathbf{w}_i = \nabla v_i$ , which satisfies

$$\mathbf{w}_i \in H^1(\Omega_i)^3, \quad \mathbf{w}_i \cdot \mathbf{n}|_{\Gamma_i \cap \Gamma_{int}} = \frac{\partial v_i}{\partial \mathbf{n}}|_{\Gamma_i \cap \Gamma_{int}},$$

and

$$\|\mathbf{w}_i\|_{H^1(\Omega_i)} \leq C_i \|P_i\|_{L^2(\Omega_i)}.$$

Let  $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ . Then,  $\mathbf{w} \in W$  and there exists a constant  $C_\Omega$  such that

$$\|\mathbf{w}\|_W \leq C_\Omega \|\mathbf{p}\|_X.$$

Taking  $\mathbf{w}$  as test in (3.1) gives us that

$$\langle \mathbf{p}, \nabla \cdot \mathbf{w} \rangle = -\|\mathbf{p}\|_X^2.$$

□

This will allow us to show the following a priori estimate for the pressure.

**Proposition 3.2.4.** *Let  $(\mathbf{u}, \mathbf{p})$  be any weak solution of the problem (3.1). Then there exists a positive constant  $C_p$  depending on the domain and on the data such that*

$$\|\mathbf{p}\|_X^2 \leq C_p. \quad (3.54)$$

*Proof.* Let take  $\mathbf{w}$  obtained in the lemma 3.2.6 as test in (3.1). We have the following equality

$$\|\mathbf{p}\|_X^2 = \langle \mathbf{f}, \mathbf{w} \rangle - a(\mathbf{u}, \mathbf{w}) - b(\mathbf{u}, \mathbf{u}, \mathbf{w}) - g(\mathbf{u}, \mathbf{u}, \mathbf{w}) - h(\mathbf{u}, \mathbf{w}). \quad (3.55)$$

We will give an upper bound of each term of the right hand side. Let  $\rho > 0$  to be fixed later. By using the operator inequalities (3.26) 3.2.2, Cauchy-Schwarz inequality and Young's inequality, we get

$$|a(\mathbf{u}, \mathbf{w})| \leq \frac{\max(\nu_1, \nu_2)^2}{2\rho} \|\mathbf{u}\|_W^2 + \frac{\rho}{2} \|\mathbf{w}\|_W^2, \quad (3.56)$$

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{w})| \leq C_b \|\mathbf{u}\|_W^2 \|\mathbf{w}\|_W \leq \frac{C_b^2}{2\rho} \|\mathbf{u}\|_W^4 + \frac{\rho}{2} \|\mathbf{w}\|_W^2, \quad (3.57)$$

$$|g(\mathbf{u}, \mathbf{u}, \mathbf{w})| \leq \frac{C_g^2}{2\rho} \|\mathbf{u}\|_W^4 + \frac{\rho}{2} \|\mathbf{w}\|_W^2, \quad (3.58)$$

$$|h(\mathbf{u}, \mathbf{w})| \leq \frac{\max(c_{d,1}, c_{d,2})^2}{2\rho} \|\mathbf{u}\|_W^2 + \frac{\rho}{2} \|\mathbf{w}\|_W^2, \quad (3.59)$$

and

$$|\langle \mathbf{f}, \mathbf{w} \rangle| \leq \frac{C_{eq}^2}{2\rho} \|\mathbf{f}\|_{0,2}^2 + \frac{\max(c_{d,1}, c_{d,2})^2}{2\rho} \left( \|\mathbf{V}_1\|_{0,2,\Gamma_1}^2 + \|\mathbf{V}_2\|_{0,2,\Gamma_2}^2 \right) + \rho \|\mathbf{w}\|_W^2. \quad (3.60)$$

According to the lemma 3.2.6,  $\mathbf{w}$  verifies the inequality

$$\|\mathbf{w}\|_W \leq C_\Omega \|\mathbf{p}\|_X.$$

Putting (3.55), (3.56), (3.57), (3.58), (3.59) and (3.60) together, we obtain

$$\|\mathbf{p}\|_X^2 \leq +3\rho C_\Omega \|\mathbf{p}\|_X^2 + \frac{1}{2\rho} \left( 2 \max(\nu_1^2, \nu_2^2, c_{d,1}^2, c_{d,2}^2) \|\mathbf{u}\|_W^2 + (C_b + C_g) \|\mathbf{u}\|_W^4 + C_{data} \right), \quad (3.61)$$

where  $C_{data}$  is given by

$$C_{data} := C_{eq}^2 \|\mathbf{f}\|_{0,2}^2 + \max(c_{d,1}, c_{d,2})^2 \left( \|\mathbf{V}_1\|_{0,2,\Gamma_1}^2 + \|\mathbf{V}_2\|_{0,2,\Gamma_2}^2 \right).$$

We can choose

$$\rho = \frac{1}{4C_\Omega},$$

and replacing it in (3.61) gives

$$\|\mathbf{p}\|_X^2 \leq 8C_\Omega \left( 2 \max(\nu_1^2, \nu_2^2, c_{d,1}^2, c_{d,2}^2) \|\mathbf{u}\|_W^2 + (C_b + C_g) \|\mathbf{u}\|_W^4 + C_{data} \right). \quad (3.62)$$

Using the a priori estimate on the velocity (3.47), we finally obtain

$$\|\mathbf{p}\|_X^2 \leq C_p,$$

where  $C_p$  is given by

$$C_p = 8C_\Omega \left( 2 \max(\nu_1^2, \nu_2^2, c_{d,1}^2, c_{d,2}^2) C_{vel} + (C_b + C_g) C_{vel}^2 + C_{data} \right).$$

□

### 3.2.7 Uniqueness

**Theorem 3.2.1.** *Recall that  $\nu = \min(\nu_1, \nu_2)$ , we also define the constants  $C_{1,\nu}$  and  $C_{2,\nu}$  by*

$$C_{1,\nu} := \frac{C_b}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2},$$

and

$$C_{2,\nu} := \frac{2C}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}.$$

If  $\nu$  verifies the condition

$$\nu > C_{1,\nu} + C_{2,\nu}, \quad (3.63)$$

a weak solution  $(\mathbf{u}, \mathbf{p}) \in W \times X_0$  of the problem  $(\mathcal{VP})$  is unique.

*Proof.* Let  $(\mathbf{u}^*, \mathbf{p}^*)$  and  $(\mathbf{u}^{**}, \mathbf{p}^{**})$  be two different solutions of  $(\mathcal{VP})$  and let define  $\mathbf{u} = \mathbf{u}^* - \mathbf{u}^{**}$  and  $\mathbf{p} = \mathbf{p}^* - \mathbf{p}^{**}$ .

We subtract the equations  $(\mathcal{VP})$  corresponding to  $(\mathbf{u}^*, \mathbf{p}^*)$  and  $(\mathbf{u}^{**}, \mathbf{p}^{**})$  and we obtain:

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}^*, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}^*, \mathbf{v}) - d(\mathbf{p}, \nabla \cdot \mathbf{v}) + \langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{v} \rangle \\ + \langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{v} \rangle = 0, \\ d(\mathbf{q}, \nabla \cdot \mathbf{v}) = 0, \end{cases} \quad (3.64)$$

for every  $(\mathbf{v}, \mathbf{q}) \in W \times Q$ .

We take  $\mathbf{v} = \mathbf{u}$  in (3.64), and we study the different terms.



— The transport term verifies  $b(\mathbf{u}^*, \mathbf{u}, \mathbf{u}) = 0$  thanks to 3.2.1, and we also get

$$|b(\mathbf{u}, \mathbf{u}^*, \mathbf{u})| \leq C_b \|\nabla \mathbf{u}\|_{0,2}^2 \|\nabla \mathbf{u}^*\|_{0,2}. \quad (3.65)$$

Since  $\mathbf{u}^*$  is a solution of  $(\mathcal{VP})$ , we can use the estimate (3.46) in (3.65) which gives:

$$|b(\mathbf{u}, \mathbf{u}^*, \mathbf{u})| \leq \frac{C_b}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}. \quad (3.66)$$

— Since  $\nabla \cdot \mathbf{u} = 0$ , The pressure term  $d(\mathbf{p}, \nabla \cdot \mathbf{u}) = 0$ .

— The friction term  $\langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{u} \rangle$  satisfies the inequality

$$\begin{aligned} |\langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{u} \rangle| &\leq C (\|\nabla \mathbf{u}^*\|_{0,2} + \|\nabla \mathbf{u}^{**}\|_{0,2}) \|\nabla \mathbf{u}\|_{0,2}^2 \\ &\leq \frac{2C}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2} \end{aligned} \quad (3.67)$$

thanks to the continuity of  $G$  given in (3.2.3).

— Finally the top and bottom boundary term  $\langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{u} \rangle$  verifies

$$\begin{aligned} \langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{u} \rangle &= \int_{\Gamma_1} (\mathbf{u}_1)_\tau^2 + \int_{\Gamma_2} (\mathbf{u}_2)_\tau^2 \\ &= c_{d,1} \|\mathbf{u}_1\|_{0,2,\Gamma_1}^2 + c_{d,2} \|\mathbf{u}_2\|_{0,2,\Gamma_2}^2. \end{aligned}$$

Putting this together, we get by triangular inequality

$$a(\mathbf{u}, \mathbf{u}) + \langle H(\mathbf{u}^*) - H(\mathbf{u}^{**}), \mathbf{u} \rangle \leq |b(\mathbf{u}, \mathbf{u}^*, \mathbf{u})| + |\langle G(\mathbf{u}^*) - G(\mathbf{u}^{**}), \mathbf{u} \rangle|,$$

hence

$$\begin{aligned} \nu \|\nabla \mathbf{u}\|_{0,2}^2 + \gamma_d \|\mathbf{u}\|_{0,2,\Gamma}^2 &\leq \frac{C_b}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2} \\ &\quad + \frac{2C}{\nu^{1/2}} \|\nabla \mathbf{u}\|_{0,2}^2 \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}. \end{aligned}$$

where  $\gamma_d = \min(c_{d,1}, c_{d,2})$ .

We recognize the constants  $C_{1,\nu}$  and  $C_{2,\nu}$ , which gives

$$(\nu - C_{1,\nu} - C_{2,\nu})\|\nabla \mathbf{u}\|_{0,2}^2 + \gamma_d \|\mathbf{u}\|_{0,2,\Gamma}^2 \leq 0,$$

and finally

$$\min(\nu - C_{1,\nu} - C_{2,\nu}, \gamma_d) \|\mathbf{u}\|_W^2 \leq 0.$$

Using the condition on the data (3.63), we obtain  $\|\mathbf{u}\|_W = 0$ , hence  $\mathbf{u}^* = \mathbf{u}^{**}$ .

Taking as a pressure test  $\mathbf{q} = \mathbf{p}$  gives  $d(\mathbf{p}, \mathbf{w}) = 0$  for every  $\mathbf{w} \in W$ . By using a density result, this remains true for every  $\mathbf{w} \in \mathcal{D}(\Omega)^3$ . Then,  $\mathbf{p} = 0$  a.e in  $\Omega$ , and equal to zero since it belongs to  $X_0$ . □

### 3.2.8 Approximated Linearized Problem

The aim of this section is to prove the existence theorem:

**Theorem 3.2.2.** *If  $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega_1)^3 \times L^2(\Omega_2)^3$  and  $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2) \in L^2(\Gamma_1)^3 \times L^2(\Gamma_2)^3$ , there exists a weak solution  $(\mathbf{u}, \mathbf{p}) \in W \times X_0$  of the initial problem (3.1).*

Since we cannot use "De Rham" arguments to retrieve the pressure from the velocity, we will add a small perturbation on the divergence term. This will allow us to "decouple" the pressure and the velocity, and to obtain  $\varepsilon$ -solutions (approximate solutions).

For a given  $\varepsilon > 0$ , We introduce the following approximate problem

$$\left\{ \begin{array}{ll} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i \\ -\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{v}_{i,h} - \mathbf{v}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i} (\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \frac{\partial p_i}{\partial \mathbf{n}_i} = 0, & \text{on } \Gamma_{Int} \cup \Gamma_i \\ \int_{\Omega_i} p_i = 0 & . \end{array} \right. \quad (3.68)$$

The point is that this new system formally converges to the initial system (3.1) when  $\varepsilon \rightarrow 0$ . We will now consider a pressure problem which will allow us to eliminate the pressure term  $p$  in the system (3.68). Let  $\varepsilon > 0$  and  $\mathbf{u} \in W$  be fixed. We extract some

equations involving the pressure in (3.68) and put them together which gives the Neumann problem:

$$\begin{cases} -\varepsilon \Delta p_i + \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i, \\ \frac{\partial p_i}{\partial \mathbf{n}_i} = 0, & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \int_{\Omega_i} p_i = 0 & . \end{cases} \quad (3.69)$$

This problem is well-posed when we consider the space  $Q_0 = Q_0(\Omega)$  defined by

$$Q_0 := \{q = (q_1, q_2) \in Q = H^1(\Omega_1) \times H^1(\Omega_2), \int_{\Omega_i} q_i = 0, \forall i \in \{1, 2\}\},$$

equipped with the semi-norm  $\|\nabla \mathbf{q}\|_{0,2,\Omega} := \left( \|\nabla q_1\|_{0,2,\Omega_1}^2 + \|\nabla q_2\|_{0,2,\Omega_2}^2 \right)^{1/2}$  (equivalent to the classical Sobolev norm  $\|\cdot\|_{1,2,\Omega}$  thanks to the Poincaré-Wirtinger inequality). According to the classical Lax-Milgram theorem, the problem (3.69) admits a unique weak solution  $\mathbf{p} \in Q_0 \cup H^2(\Omega)$  which verifies the variational formulation: for every  $\mathbf{q} \in H^1(\Omega)^3$ ,

$$\varepsilon \langle \nabla \mathbf{p}, \nabla \mathbf{q} \rangle_\Omega + \langle \nabla \cdot \mathbf{u}, \mathbf{q} \rangle_\Omega = 0. \quad (3.70)$$

Since we have existence and uniqueness of the pressure  $\mathbf{p}$  solution of (3.69), we can define the map

$$P_\varepsilon : \begin{cases} W \rightarrow Q_0, \\ \mathbf{u} \mapsto \mathbf{p}, \text{ the unique weak solution of (3.69).} \end{cases} \quad (3.71)$$

**Lemma 3.2.8.** *The map  $P_\varepsilon$  is linear and continuous. Moreover, we have the estimate*

$$\|P_\varepsilon\| = O\left(\frac{1}{\varepsilon}\right).$$

*Proof.* The linearity is easy to check. Let us prove that  $P_\varepsilon$  is continuous. Let  $\mathbf{u} \in W$  and  $\mathbf{p} = P_\varepsilon(\mathbf{u})$ . We chose  $\mathbf{q} = \mathbf{p}$  as a test in the variational formulation (3.81). Thanks to the Stokes formula, we get

$$\varepsilon \|\nabla \mathbf{p}\|_{0,2,\Omega}^2 + \langle \nabla \cdot \mathbf{u}, \mathbf{p} \rangle_\Omega = 0, \quad (3.72)$$

and using the Neumann condition

$$\frac{\partial p_i}{\partial \mathbf{n}_i} = 0 \text{ on } \Gamma_{Int} \cup \Gamma_i$$

, we get

$$\varepsilon \|\nabla \mathbf{p}\|_{0,2,\Omega}^2 - \langle \mathbf{u}, \nabla \mathbf{p} \rangle_\Omega = 0.$$

Thus by Cauchy-Schwarz inequality:

$$\|\nabla \mathbf{p}\|_{0,2,\Omega} \leq \frac{C}{\varepsilon} \|\mathbf{u}\|_W,$$

where  $C > 0$ . □

A direct consequence of the energy equality (3.72) is the following result:

**Lemma 3.2.9.** *Let  $\mathbf{u} \in W$  and  $\mathbf{p} = P_\varepsilon(\mathbf{u})$ . We have the inequality*

$$\langle \nabla \cdot \mathbf{u}, \mathbf{p} \rangle_\Omega \leq 0.$$

We can now "transform" the problem (3.68) using the linear map  $P_\varepsilon$ , to "get rid of" the pressure  $\mathbf{p}$ . This problem becomes

$$\begin{cases} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i + \frac{1}{2} \mathbf{u}_i (\nabla \cdot \mathbf{u}_i) - \nu_1 \Delta \mathbf{u}_i + \nabla P_\varepsilon(\mathbf{u}_i) = \mathbf{f}_i & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{v}_{i,h} - \mathbf{v}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \end{cases} \quad (3.73)$$

where the only unknown is the velocity field  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ .

The next step is to study the variational formulation associated with (3.73).

### Perturbed variational formulation

Following our plan, the variational problem  $\mathcal{VP}_\varepsilon$  associated with (3.73) is:

find  $\mathbf{u} \in W$  such that,  $\forall \mathbf{v} \in W$ ,

$$a(\mathbf{u}, \mathbf{v}) + c_d \langle \mathbf{u}, \mathbf{v} \rangle_\Gamma + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \langle P_\varepsilon(\mathbf{u}), \nabla \cdot \mathbf{v} \rangle_\Omega + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + c_d \langle \mathbf{V}, \mathbf{v} \rangle_\Gamma. \quad (\mathcal{VP}_\varepsilon)$$

In the future, we will have to pass to the limit. Therefore, we need some a priori estimates to keep going on.

**Lemma 3.2.10.** *Let  $\varepsilon > 0$ ,  $\mathbf{u}_\varepsilon$  be any solution of  $\mathcal{VP}_\varepsilon$  and  $\mathbf{p}_\varepsilon = P_\varepsilon(\mathbf{u}_\varepsilon)$ . We have the same kind of estimates as in the previous section, i.e.:*

$$\|\mathbf{u}_\varepsilon\|_W \leq C_v, \quad (3.74)$$

and

$$\|\mathbf{p}_\varepsilon\|_{0,2,\Omega} \leq C_p, \quad (3.75)$$

where  $C_v$  and  $C_p$  are the constants obtained in 3.46 and 3.2.4.

*Proof.* The proof is following the same steps as in 3.2.3 and 3.2.4.

We are taking  $\mathbf{u}_\varepsilon$  as a test in  $(\mathcal{VP}_\varepsilon)$  which leads to

$$\begin{aligned} a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + c_d \|\mathbf{u}_\varepsilon\|_{0,2,\Gamma}^2 + b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) - \langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{u}_\varepsilon \rangle_\Omega + g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \\ = \langle \mathbf{f}, \mathbf{u}_\varepsilon \rangle_\Omega + c_d \langle \mathbf{V}, \mathbf{u}_\varepsilon \rangle_\Gamma. \end{aligned} \quad (3.76)$$

The difference between (3.76) and the previous a priori estimates is the pressure term  $-\langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{u}_\varepsilon \rangle_\Omega$  which was vanishing according to free divergence condition in (3.1). Hopefully, we can get rid of this term using the lemma 3.2.9 stating that  $-\langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{u}_\varepsilon \rangle_\Omega \leq 0$ , which gives

$$a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + c_d \|\mathbf{u}_\varepsilon\|_{0,2,\Gamma}^2 + b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \leq \langle \mathbf{f}, \mathbf{u}_\varepsilon \rangle_\Omega + c_d \langle \mathbf{V}, \mathbf{u}_\varepsilon \rangle_\Gamma. \quad (3.77)$$

The remaining steps of the proof are exactly the same as in subsections 3.2.5 and 3.2.6, which yields to the given estimates. □

The next step is to show that if we are able to have a solution  $\mathbf{u}_\varepsilon$  of the approximated problem  $\mathcal{VP}_\varepsilon$ , we can consider the couple  $(\mathbf{u}_\varepsilon, P_\varepsilon(\mathbf{u}_\varepsilon))$ , pass to the limit and get a solution of the initial problem  $\mathcal{VP}$ . This leads to the proposition:

**Proposition 3.2.5.** *Let  $\varepsilon > 0$ ,  $\mathbf{u}_\varepsilon$  be a solution of  $\mathcal{VP}_\varepsilon$  and  $\mathbf{p}_\varepsilon = P_\varepsilon(\mathbf{u}_\varepsilon)$ . We consider  $\mathbf{u}$  and  $\mathbf{p}$  any VESP-limit and PESP-limit of the sequences  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  and  $(\mathbf{p}_\varepsilon)_{\varepsilon>0}$ . Then,  $(\mathbf{u}, \mathbf{p})$  is a solution to  $\mathcal{VP}$ .*

*Proof.* By definition of the approximated problem  $\mathcal{VP}_\varepsilon$ , we have for all  $\mathbf{v} \in W$ :

$$a(\mathbf{u}_\varepsilon, \mathbf{v}) + c_d \langle \mathbf{u}_\varepsilon, \mathbf{v} \rangle_\Gamma + b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) - \langle \mathbf{p}_\varepsilon, \nabla \cdot \mathbf{v} \rangle_\Omega + g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + \langle \mathbf{V}, \mathbf{v} \rangle_\Gamma. \quad (3.78)$$

According to the lemma 3.2.10,  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$  is bounded in  $W \times Q_0$ . We can then use the VESP and PESP to obtain VESP and PESP limits  $\mathbf{u}$  and  $\mathbf{p}$ . It remains to prove that  $(\mathbf{u}, \mathbf{p})$  is solution of the initial variational problem  $\mathcal{VP}$ . We will pass to the limit in two different

problems. First, in the variational formulation (3.78) to retrieve the problem  $\mathcal{VP}$ . Second, in the perturbation pressure problem 3.69.

**First passage to the limit.** Let  $\mathbf{v} \in W$ . The weak convergence of  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  gives

$$\begin{cases} a(\mathbf{u}_\varepsilon, \mathbf{v}) & \xrightarrow{\varepsilon \rightarrow 0} a(\mathbf{u}, \mathbf{v}), \\ \langle \mathbf{u}_\varepsilon, \mathbf{v} \rangle_\Gamma & \xrightarrow{\varepsilon \rightarrow 0} \langle \mathbf{u}, \mathbf{v} \rangle_\Gamma, \\ \langle p_\varepsilon, \nabla \cdot \mathbf{v} \rangle_\Omega & \xrightarrow{\varepsilon \rightarrow 0} \langle p, \nabla \cdot \mathbf{v} \rangle_\Omega \end{cases} \quad (3.79)$$

Moreover, using the convergence properties of lemma 3.2.5, we get

$$\begin{cases} b(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) & \xrightarrow{\varepsilon \rightarrow 0} b(\mathbf{u}, \mathbf{u}, \mathbf{v}), \\ g(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) & \xrightarrow{\varepsilon \rightarrow 0} g(\mathbf{u}, \mathbf{u}, \mathbf{v}). \end{cases} \quad (3.80)$$

Since  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  is solution of (3.78) for all  $\mathbf{v} \in W$ , we deduce from (3.79) and (3.80) that  $(\mathbf{u}, p)$  satisfies the variational formulation

$$a(\mathbf{u}, \mathbf{v}) + c_d \langle \mathbf{u}, \mathbf{v} \rangle_\Gamma + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \langle p, \nabla \cdot \mathbf{v} \rangle_\Omega + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + \langle \mathbf{V}, \mathbf{v} \rangle_\Gamma,$$

for all  $\mathbf{v} \in W$ , which is exactly the first equation in  $\mathcal{VP}$ .

**Second passage to the limit.**

We now consider the singular pressure problem 3.69. Let  $\mathbf{q} \in Q$ . The variational formulation (3.81) gives

$$-\varepsilon \langle \mathbf{p}_\varepsilon, \Delta \mathbf{q} \rangle_\Omega + \langle \nabla \cdot \mathbf{u}_\varepsilon, \mathbf{q} \rangle_\Omega = 0. \quad (3.81)$$

Again, using the weak convergences of  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  to  $\mathbf{u}$  in  $W$  and of  $(\mathbf{p}_\varepsilon)_{\varepsilon>0}$  to  $\mathbf{p}$  in  $Q$ , we have

$$\begin{cases} \langle \nabla \cdot \mathbf{u}_\varepsilon, \mathbf{q} \rangle_\Omega & \xrightarrow{\varepsilon \rightarrow 0} \langle \nabla \cdot \mathbf{u}, \mathbf{q} \rangle_\Omega, \\ \langle \mathbf{p}_\varepsilon, \Delta \mathbf{q} \rangle_\Omega & \xrightarrow{\varepsilon \rightarrow 0} \langle \mathbf{p}, \Delta \mathbf{q} \rangle_\Omega. \end{cases} \quad (3.82)$$

This yields  $-\varepsilon \langle \mathbf{p}_\varepsilon, \Delta \mathbf{q} \rangle_\Omega \xrightarrow{\varepsilon \rightarrow 0} 0$  and from (3.81) we obtain

$$\langle \nabla \cdot \mathbf{u}, \mathbf{q} \rangle_\Omega = 0, \quad (3.83)$$

for every  $\mathbf{q} \in Q$ . We recognize the second equation of the variational problem  $\mathcal{VP}$ .

**Conclusion**

We have proved that for a given solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$  of (4.33), we can retrieve a couple  $(\mathbf{u}, \mathbf{p}) \in W \times Q$ , which satisfies the two equations of the variational formulation  $\mathcal{VP}$ .

It remains to prove that we can find solutions of the perturbed problem. □

### Linearized problem

A way to obtain solutions of the problem (4.33) is to linearize it. The Lax-Milgram theorem will enable us to obtain weak solutions of the linearized problem, and the Schauder fixed-point theorem will allow us to retrieve solutions of the perturbed problem.

The non-linear terms  $b$  and  $g$  will be linearized as follows: let  $\mathbf{w} \in W$  be a fixed velocity, we replace the terms  $b(\mathbf{u}, \mathbf{u}, \mathbf{v})$  and  $g(\mathbf{u}, \mathbf{u}, \mathbf{v})$  respectively by  $b(\mathbf{w}, \mathbf{u}, \mathbf{v})$  and  $g(\mathbf{w}, \mathbf{u}, \mathbf{v})$ . We then solve the problem

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}, \mathbf{v}) - d(P_\varepsilon(\mathbf{u}), \mathbf{v}) + g(\mathbf{w}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega, \\ \varepsilon a(\mathbf{p}, \mathbf{q}) + d(\mathbf{q}, \mathbf{u}) = 0, \end{cases} \quad (\mathcal{LP}_{\varepsilon, \mathbf{w}})$$

for every  $(\mathbf{v}, \mathbf{q}) \in W \times Q_0$ . According to the previous construction of  $P_\varepsilon(\mathbf{u})$ , it remains to prove the proposition

**Proposition 3.2.6.** *Let  $\mathbf{w} \in W$ . The variational formulation: find  $\mathbf{u} \in W$  such that*

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}, \mathbf{v}) - d(P_\varepsilon(\mathbf{u}), \mathbf{v}) + g(\mathbf{w}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega, \quad (3.84)$$

for every  $\mathbf{v} \in W$ , admits a unique solution  $\mathbf{u}_{\varepsilon, \mathbf{w}} \in W$ .

*Proof.* We will use the Lax-Milgram to prove it. We can rewrite the previous formulation in: find  $\mathbf{u}$  such that

$$\Lambda_\varepsilon(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}),$$

for every  $\mathbf{v} \in W$ , where the bilinear form  $\Lambda_\varepsilon$  is defined by

$$\Lambda_\varepsilon(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{u}, \mathbf{v}) - d(P_\varepsilon(\mathbf{u}), \mathbf{v}) + g(\mathbf{w}, \mathbf{u}, \mathbf{v}) + \tilde{h}(\mathbf{u}, \mathbf{v}),$$

and the linear form  $L$  by

$$L(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega + h_{\mathbf{V}}(\mathbf{v}).$$

According to the properties of the different operator involved shown in 3.2.3,  $\Lambda_\varepsilon$  is bilinear continuous. Moreover, since

1.  $b(\mathbf{w}, \mathbf{u}, \mathbf{u}) = 0$ , according to lemma 3.2.1
2.  $-d(P_\varepsilon(\mathbf{u}), \mathbf{u}) \geq 0$  according to 3.2.9,
3.  $g(\mathbf{w}, \mathbf{u}, \mathbf{u}) \geq 0$  according to lemma 3.2.3,

we have

$$\begin{aligned} \Lambda_\varepsilon(\mathbf{u}, \mathbf{u}) &\geq a(\mathbf{u}, \mathbf{u}) + \tilde{h}(\mathbf{u}, \mathbf{u}) \\ &\geq \nu \|\nabla \mathbf{u}\|_{0,2,\Omega}^2 + \min(c_{d,1}, c_{d,2}) \|\mathbf{u}\|_{0,2,\Gamma}^2 \\ &\geq \min(\nu, c_{d,1}, c_{d,2}) \|\mathbf{u}\|_W^2 \end{aligned}$$

hence  $\Lambda_\varepsilon$  is also verifying the coercivity property. We can use the Lax-Milgram theorem which concludes the proof. □

We can also give an priori estimate for the velocity:

**Proposition 3.2.7.** *Let  $\mathbf{w} \in W$  and  $\mathbf{u}_{\varepsilon,\mathbf{w}}$  the unique solution of  $(\mathcal{LP}_{\varepsilon,\mathbf{w}})$ . Then we have the same estimate as in 3.2.10, which does not depend nor on  $\varepsilon$ , neither on  $\mathbf{w}$ .*

$$\|\mathbf{u}_{\varepsilon,\mathbf{w}}\|_W \leq C_v. \tag{3.85}$$

We have proved the existence of a weak solution  $\mathbf{u}_{\varepsilon,\mathbf{w}}$  depending on the small perturbation  $\varepsilon$  and on the the given velocity field  $\mathbf{w}$ . The next step is to use a fixed point theorem to get a solution of the problem  $\mathcal{LP}_\varepsilon$ .

We introduce the application  $T_\varepsilon$  which maps the fixed velocity  $\mathbf{w}$  to the unique solution  $\mathbf{u}_{\varepsilon,\mathbf{w}}$  of previous problem  $(\mathcal{LP}_{\varepsilon,\mathbf{w}})$ .

$$T_\varepsilon : \begin{cases} W \rightarrow W, \\ \mathbf{w} \rightarrow \mathbf{u}_{\varepsilon,\mathbf{w}}. \end{cases} \tag{3.86}$$

The unique solution of  $(\mathcal{LP}_{\varepsilon,\mathbf{w}})$  will be denoted  $(\mathbf{u}_{\varepsilon,\mathbf{w}}, \mathbf{p}_{\varepsilon,\mathbf{w}})$ .

We aim to show that  $T_\varepsilon$  verifies the assumptions of the Schauder theorem to get a fixed point and a solution of the problem  $\mathcal{LP}_\varepsilon$ .

We have to check the following properties:

1.  $T_\varepsilon$  is continuous over  $W$ ,
2. There exists a positive constant  $C$  such that  $T_\varepsilon(B_C) \subset B_C$  for  $B_C$  the closed ball of  $W$  of radius  $C$ ,



3.  $T_\varepsilon$  is compact.

To prove the first point, we will show that  $T_\varepsilon$  is weakly continuous, then use it to show that it is continuous.

We need a first weak convergence lemma:

**Lemma 3.2.11.** *Let  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be weakly convergent to  $\mathbf{w}$  in  $W$ . Then the sequence of the solutions  $(\mathbf{u}_{\varepsilon, \mathbf{w}_n}^n)_n$  of the linear problem  $\mathcal{LP}_{\varepsilon, \mathbf{w}_n}$  associated with  $\mathbf{w}_n$  converges weakly to the solution  $\mathbf{u}_{\varepsilon, \mathbf{w}}$  of the linear problem associated ( $\mathcal{LP}_{\varepsilon, \mathbf{w}}$ ) with  $\mathbf{w}$ .*

To simplify the notation, we will denote by  $\mathbf{u}$  the term  $\mathbf{u}_{\varepsilon, \mathbf{w}}$  and by  $\mathbf{u}^n$  the term  $\mathbf{u}_{\varepsilon, \mathbf{w}_n}$  whenever there is no ambiguity.

*Proof.* We can apply the VESP to  $(\mathbf{w}_n)_{n \in \mathbb{N}}$ : there exists a unique VESP-limit  $\mathbf{w}$ . Let  $(\mathbf{u}^n, \mathbf{p}_n)$  denote the solution of the linear problem  $\mathcal{LP}_{\varepsilon, \mathbf{w}_n}$  associated with  $\mathbf{w}_n$ . We deduce from the estimate (3.85) that  $(\mathbf{u}^n, \mathbf{p}_n)_{n \in \mathbb{N}}$  is bounded in  $W \times X$  uniformly on  $n$ . We can apply the VESP to  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  and the PESP to  $(\mathbf{p}_n)_{n \in \mathbb{N}}$ . Let  $\mathbf{u}$  be a VESP-limit and  $\mathbf{p}$  a PESP-limit.

We want to prove that  $(\mathbf{u}, \mathbf{p})$  is the solution of  $\mathcal{LP}_{\varepsilon, \mathbf{w}}$ . Let  $(\mathbf{v}, \mathbf{q}) \in W \times Q$ . Using the properties (3.41), (3.42) we have the convergences:

$$\begin{aligned} a(\mathbf{u}^n, \mathbf{v}) &\xrightarrow{n \rightarrow \infty} a(\mathbf{u}, \mathbf{v}), \\ b(\mathbf{w}_n, \mathbf{u}^n, \mathbf{v}) &\xrightarrow{n \rightarrow \infty} b(\mathbf{w}, \mathbf{u}, \mathbf{v}), \\ g(\mathbf{w}_n, \mathbf{u}^n, \mathbf{v}) &\xrightarrow{n \rightarrow \infty} g(\mathbf{w}, \mathbf{u}, \mathbf{v}), \\ d(\mathbf{p}_n, \mathbf{v}) &\xrightarrow{n \rightarrow \infty} d(\mathbf{p}, \mathbf{v}), \\ d(\mathbf{q}, \mathbf{u}^n) &\xrightarrow{n \rightarrow \infty} d(\mathbf{q}, \mathbf{u}^n), \end{aligned}$$

which implies that  $(\mathbf{u}, \mathbf{p})$  is the solution of  $LP_{\varepsilon, \mathbf{w}}$ . □

We can even get a stronger result based on the energy method:

**Lemma 3.2.12.** *Let  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be weakly convergent to  $\mathbf{w}$  in  $W$ . Then  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  strongly converges in  $W$  to  $\mathbf{u}$ , where  $\mathbf{u}^n = T_\varepsilon(\mathbf{w}_n)$ , and  $\mathbf{u} = T_\varepsilon(\mathbf{w})$ .*

*Proof.* We know from the previous lemma 3.2.11 that  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  is weakly converging in  $W$  to  $\mathbf{u}$ . It remains to show that  $(\|\mathbf{u}^n\|_W)_{n \in \mathbb{N}}$  is converging to  $\|\mathbf{u}\|$  and we will have the

strong convergence since  $W$  is an Hilbert space. According to the energy equalities, we have

$$\begin{aligned} a(\mathbf{u}^n, \mathbf{u}^n) + g(\mathbf{w}_n, \mathbf{u}^n, \mathbf{u}^n) &= \langle \mathbf{f}, \mathbf{u}^n \rangle, \\ a(\mathbf{u}, \mathbf{u}) + g(\mathbf{w}, \mathbf{u}, \mathbf{u}) &= \langle \mathbf{f}, \mathbf{u} \rangle. \end{aligned}$$

According to the convergence property (3.42) and to the weak convergence, we get

$$\begin{aligned} g(\mathbf{w}_n, \mathbf{u}^n, \mathbf{u}^n) &\xrightarrow{n \rightarrow \infty} g(\mathbf{w}, \mathbf{u}, \mathbf{u}), \\ \langle \mathbf{f}, \mathbf{u}_n \rangle &\xrightarrow{n \rightarrow \infty} \langle \mathbf{f}, \mathbf{u} \rangle. \end{aligned}$$

We deduce

$$a(\mathbf{u}^n, \mathbf{u}^n) \xrightarrow{n \rightarrow \infty} a(\mathbf{u}, \mathbf{u}).$$

The norm induced by the bilinear form  $a$  is equivalent to  $\|\cdot\|_W$  which finally gives

$$\|\mathbf{u}^n\|_W \xrightarrow{n \rightarrow \infty} \|\mathbf{u}\|_W,$$

and which combined with the weak convergence of  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  in  $W$  to  $\mathbf{u}$  gives the strong convergence.  $\square$

**Lemma 3.2.13.** *The application  $T_\varepsilon$  is weakly continuous.*

*Proof.* Let  $(\mathbf{w}_n)_{n \in \mathbb{N}}$  be weakly convergent to  $\mathbf{w}$  in  $W$ , and  $\mathbf{u}^n = T(\mathbf{w}_n)$ . We have to prove that  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  weakly converges to  $\mathbf{u} = T(\mathbf{w})$ . We already know from the previous lemma that a subsequence can be extracted from  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  that converges to  $\mathbf{u}$ . But as the solution of the linear problem associated with  $\mathbf{w}$  is unique,  $\mathbf{u}$  is the unique weak limit of  $(\mathbf{u}^n)_{n \in \mathbb{N}}$ .  $\square$

**Lemma 3.2.14.** *The application  $T_\varepsilon$  is continuous over  $W$ .*

*Proof.* This is a direct consequence of the weak continuity of  $T_\varepsilon$  and of the lemma 3.2.12.  $\square$

**Proposition 3.2.8.** *The application  $T_\varepsilon$  has a fixed point. Therefore, the non-linear problem  $(\mathcal{VP}_\varepsilon)$  admits a weak solution  $\mathbf{u}_\varepsilon$ .*

*Proof.* We have proved that  $T_\varepsilon$  is continuous. Let  $C > 0$  the constant given by the a priori estimate  $\|\mathbf{u}\|_W \leq C$  obtained in (3.85). Let  $B_C \subset W$  be the closed ball of radius  $C$

centered in 0. We have by construction  $T(B_C) \subset B_C$ . It remains to prove that  $T(B_C)$  is compact. As  $T(B_C)$  is a closed subspace of the metric space  $W$ , we will show that from any sequence  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  in  $T(B_C)$ , we can extract a subsequence which converges in  $W$ . Let  $\mathbf{w}_n \in B_C$  be such that  $\mathbf{u}^n = T(\mathbf{w}_n)$ . Since these sequences are bounded, we can extract subsequences that weakly converge respectively to  $\mathbf{w}$  and  $\mathbf{u}$ . We deduce from lemma 3.2.11 that  $\mathbf{u} = T(\mathbf{w}) \in B_C$  and from lemma 3.2.14 that  $(\mathbf{u}^n)_{n \in \mathbb{N}}$  strongly converges to  $\mathbf{u}$ , which proves that  $T(B_C)$  is compact.

We can then use the Schauder's fixed point theorem which concludes the proof. □

### Conclusion

To summarize what we have done and proved, we recall the scheme of the existence proof in figure 3.3.

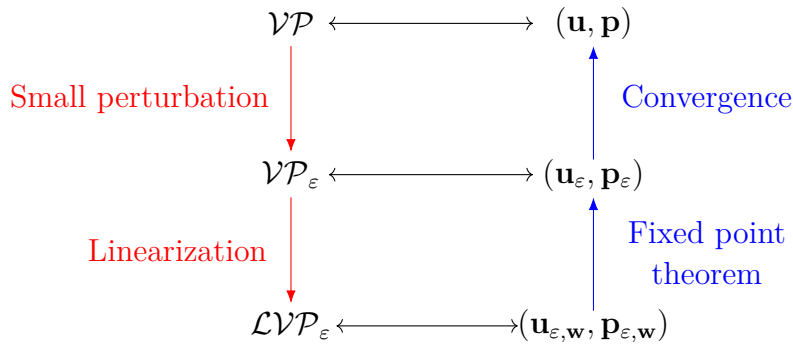


Figure 3.3 – Scheme of the proof

In the previous subsections, we have proved the proposition 3.2.6, which grants the existence and uniqueness of a weak solution  $(\mathbf{u}_{\varepsilon, \mathbf{w}}, \mathbf{p}_{\varepsilon, \mathbf{w}})$  of the linearized approximate problem  $(\mathcal{L}\mathcal{P}_{\varepsilon, \mathbf{w}})$ , for a given  $\mathbf{w} \in W$ . Then, we deduce from the Schauder's fixed point the existence of a weak solution  $\mathbf{u}_\varepsilon$  of  $(\mathcal{V}\mathcal{P}_\varepsilon)$  (proposition 3.2.8) and consequently the existence of a weak solution  $(\mathbf{u}_E, \mathbf{p}_E)$  to approximate problem (3.68).

Finally, the proposition 3.2.5 establishes the existence of weak solutions of the initial problem (3.1) by passing to the limit of  $\varepsilon$  towards 0.

Putting this together with the uniqueness theorem 3.2.1, we have finally proved the existence and uniqueness theorem:

**Theorem 3.2.3.** *Suppose that  $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in L^2(\Omega_1)^3 \times L^2(\Omega_2)^3$  and  $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2) \in L^2(\Gamma_1)^3 \times L^2(\Gamma_2)^3$ .*

There exists a unique weak solution  $(\mathbf{u}, \mathbf{p}) \in W \times X_0$  of the system (3.1), under the condition (3.63). The different given parameters have to ensure the relation:

$$\nu > C_{1,\nu} + C_{2,\nu},$$

where  $\nu = \min(\nu_1, \nu_2)$  and

$$C_{1,\nu} := \frac{C_b}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2},$$

and

$$C_{2,\nu} := \frac{2C}{\nu^{1/2}} \left( c_{d,1} \|\mathbf{V}_1\|_{0,\Gamma_1}^2 + c_{d,2} \|\mathbf{V}_2\|_{0,\Gamma_2}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{0,2}^2 \right)^{1/2}.$$

**Remark 3.2.7.** This condition comes from the uniqueness theorem 3.2.1. It is verified whenever the viscosities coefficients  $\nu_1$  and  $\nu_2$  are "big" compared to the other terms involved.

### 3.2.9 Technical lemma

**Lemma 3.2.15.** Let  $u \in H^2(\Omega)$  be the unique solution of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma \\ \int_{\Omega} u = 0 \end{cases} \quad (3.87)$$

where  $\Omega = \Pi_2 \times [0, 1]$ . Then we have the estimate:

$$\|u\|_{2,2} \leq C \|f\|_{0,2}. \quad (3.88)$$

*Proof.* We consider the Fourier decomposition of  $u$  and  $f$ :

$$u(x, z) = \sum_{k \in \mathbb{Z}} u_k(z) e^{ikx},$$

and

$$f(x, z) = \sum_{k \in \mathbb{Z}} f_k(z) e^{ikx}.$$

Define now the  $H^2$  norm associated with the space  $\Omega = \Pi_2 \times [0, 1]$ :

$$\|u\|_{2,2}^2 = \sum_{k \in \mathbb{Z}} \int_0^1 \left( |u_k''(z)|^2 + k^2 |u_k'(z)|^2 + k^4 |u_k|^2 \right) dz. \quad (3.89)$$

The Fourier decomposition of (3.87) gives for every  $k \in \mathbb{Z}$ :

$$\int_0^1 -u_k''(z) + k^2 u_k(z) dz = \int_0^1 f_k. \quad (3.90)$$

Taking  $v \in H_0^1$  as a test function and by the Green's formula, we get

$$\int_0^1 u_k' v' + k^2 \int_0^1 u_k v = \int_0^1 f_k v.$$

Taking  $v = u_k$  and thanks to the Young inequality, we obtain the estimate

$$\int_0^1 |u_k'|^2 + k^2 \int_0^1 |u_k|^2 \leq \frac{k^2}{2} \int_0^1 |u_k|^2 + \frac{1}{2k^2} \int_0^1 |f_k|^2,$$

hence

$$\int_0^1 |u_k'|^2 + \frac{k^2}{2} \int_0^1 |u_k|^2 \leq \frac{1}{2k^2} \int_0^1 |f_k|^2,$$

and

$$\int_0^1 k^4 |u_k|^2 \leq \int_0^1 |f_k|^2 - 2 \int_0^1 k^2 |u_k'|^2. \quad (3.91)$$

According to (3.90), we have

$$\begin{aligned} \int_0^1 |u_k''|^2 + k^2 |u_k'|^2 + k^4 |u_k|^2 &= \int_0^1 |k^2 u_k - f_k|^2 + k^2 |u_k'|^2 + k^4 |u_k|^2 \\ &= \int_0^1 2k^4 |u_k|^2 + |f_k|^2 + k^2 |u_k'|^2 \\ &\leq 3 \int_0^1 |f_k|^2 - 3 \int_0^1 k^2 |u_k'|^2 \\ &\leq 3 \int_0^1 |f_k|^2 \end{aligned}$$

using (3.91). Summing this inequality for  $z \in \mathbb{Z}$  gives the estimate we want. □

### 3.3 Iterative element method for steady Navier-Stokes equations with friction conditions

#### 3.3.1 Introduction

We will consider in the following a problem slightly different from the study we have done previously. Instead of considering the spaces  $\Omega_1$  and  $\Omega_2$  as  $\Pi_2 \times [0, z_i]$ , we will assume that they are both bounded domains in  $\mathbb{R}^d$ , where the boundary is at least Lipschitz. Therefore we will also replace the periodic boundary conditions "on the sides" by the Dirichlet condition  $\mathbf{u}_i = 0$  on  $\Gamma_{i,s}$ ,  $1 \leq i \leq 2$ , where  $\Gamma_{i,s}$  denotes the "left" and "right" sides of the boundary of  $\Omega_i$  (see figure 3.4).

$$\left\{ \begin{array}{ll} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i \\ \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) |\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h} - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \mathbf{u}_i = 0 & \text{on } \Gamma_{i,s}, \end{array} \right. , \quad (3.92)$$

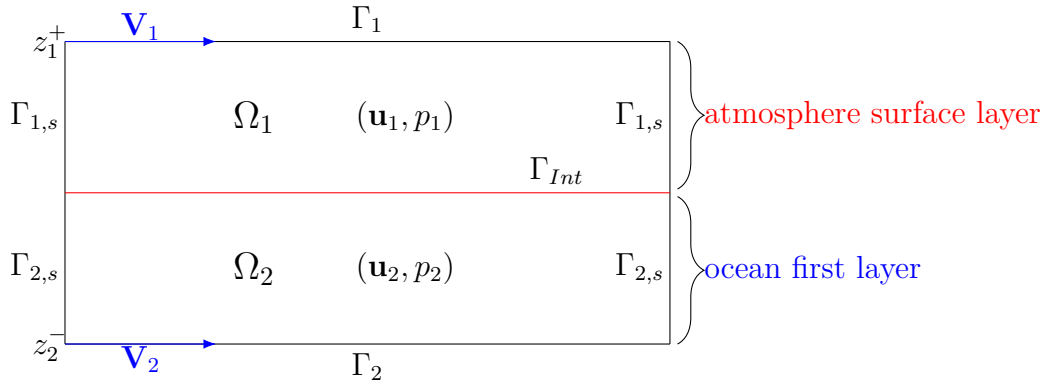


Figure 3.4 – Geometry of the domain

We will give the proper functional setting of this problem in the section 3.3.2 and introduce a finite element Galerkin approximation in section 3.3.3.

Thanks to the Oseen iterative method which consists in a linearization of the trilinear transport term, we will be able to do some numerical simulations in Freefem++. We

establish in section 3.3.4 the stability and error estimates relatively to the iterative method we have chosen. The plan of what we will do can be summarized in the scheme 3.5.

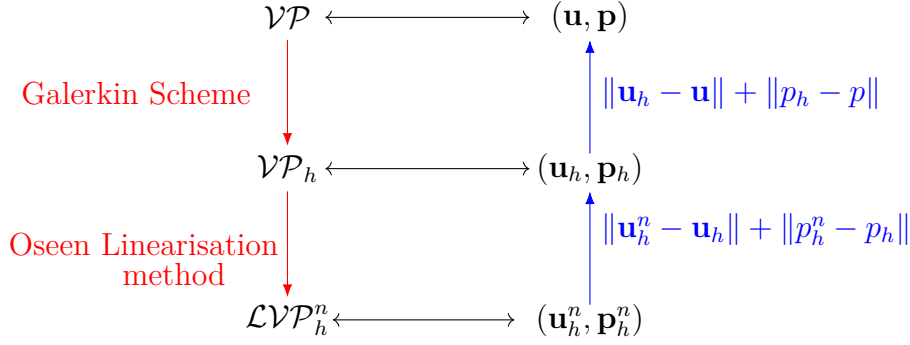


Figure 3.5 – Scheme of the error estimates

### 3.3.2 Functional setting

The benefit of considering this new problem relies on the Poincaré inequality, which will enable us to change the functional spaces we have introduced in the section 3.2, and to consider "classical" Sobolev spaces associated with the norm coming from the scalar product.

**Definition 3.3.1.** We introduce the Sobolev space  $H_{0,s}^1(\Omega_i)^d$  by

$$H_{0,s}^1(\Omega_i)^d := \left\{ \mathbf{u}_i \in H^1(\Omega_i)^d, \text{tr}(\mathbf{u}_i) = 0 \text{ on } \Gamma_{i,s} \right\}.$$

To simplify the notations, we will denote by  $W_i$  the space  $H_{0,s}^1(\Omega_i)^d$  and we finally introduce the Hilbert product space

$$W = W_1 \times W_2.$$

Thanks to the Poincaré inequality, the spaces  $W_i$  are equipped with the usual scalar product  $\langle \nabla \mathbf{u}_i, \nabla \mathbf{v}_i \rangle$  and with the equivalent norm  $\|\mathbf{u}\|_{W_i} = \|\nabla \mathbf{u}\|_0$ , for  $\mathbf{u}_i, \mathbf{v}_i \in W_i$ .

We denote by  $Y$  the space

$$Y := L^2(\Omega_1)^2 \times L^2(\Omega_2)^2,$$

and by  $X$  the space

$$X := L_0^2(\Omega_1)^2 \times L_0^2(\Omega_2)^2.$$

It remains to take into consideration the free divergence condition. We can introduce two other spaces: we denote by  $V$  the closed subset of  $W$  given by

$$V = \{\mathbf{u} \in W, \quad \nabla \cdot \mathbf{u}_i = 0, \quad \forall 1 \leq i \leq 2\},$$

and we denote by  $H$  the closed subset of  $Y$  defined by

$$H = \{\mathbf{u} \in Y, \quad \nabla \cdot \mathbf{u}_i = 0, \mathbf{u}_i \cdot \mathbf{n}_i = 0 \text{ on } \Gamma_{Int} \cup \Gamma_i, \quad \forall 1 \leq i \leq 2\}.$$

More details on the spaces can be found in [98] [43]. We denote the Stokes operator by  $A = -P\Delta$ , where  $P$  is the  $L^2$ -orthogonal projection of  $Y$  onto  $H$ . Following the notations of the section 3.2, the variational formulation of problem ... reads: find  $(\mathbf{u}, p) \in W \times X$  such that for all  $(\mathbf{v}, q) \in W \times X$ ,

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + d(\mathbf{u}, q) - d(\mathbf{v}, p) + g(\mathbf{u}, \mathbf{u}, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle. \quad (3.93)$$

The following existence and uniqueness of solution of (3.93) can be easily adapted from the results of section 3.2 and from [43], [98].

**Theorem 3.3.1.** *Assume that  $\Omega_1$  and  $\Omega_2$  are smooth enough (if they are convex polygons/polyedra in dimension 2 and 3 for instance). Given  $\mathbf{f}$  in  $W'$ , there exists at least a solution  $(\mathbf{u}, p) \in W \times X$  wich satisfies (3.93) and*

$$\|\nabla \mathbf{u}\|_0 \leq \frac{c}{\nu} \|\mathbf{f}\|_{-1}, \quad \|\mathbf{f}\|_{-1} = \sup_{\mathbf{v} \in W} \frac{\langle \mathbf{f}, \mathbf{v} \rangle}{\|\nabla \mathbf{v}\|_0}, \quad (3.94)$$

where  $\nu = \min(\nu_1, \nu_2)$ . Moreover, if  $\nu$  and  $\mathbf{f}$  satisfy the condition of theorem 3.2.1, the solution pair is unique.

We have another regularity result we can adapt from He and Wang [46].

**Theorem 3.3.2.** *Assume that  $\mathbf{f} \in Y$ . The solution pair  $(\mathbf{u}, p)$  of (3.93) satisfies*

$$\nu \|A\mathbf{u}\|_0 + \|\nabla p\|_0 \leq c \|\mathbf{f}\|_0.$$

### 3.3.3 Finite element Galerkin approximation

Let  $h > 0$  be a real positive parameter, which will be the length of the mesh triangulation edge. We introduce  $W_h \times X_h$ , the finite element subspace of  $W \times X$  characterized



by  $J_h$ , a partitioning of  $\Omega_1 \times \Omega_2$  into triangles. Examples of such finite elements subspaces and partitioning can be found in Girault and Raviart [43] or in Ciarlet [25]. We can also define the subspace  $V_h$  of  $W_h$ , given by

$$V_h = \{\mathbf{v}_h \in W_h, d(\mathbf{v}_h, \mathbf{Q}_h) = 0, \forall \mathbf{Q}_h \in X_h\}, \quad (3.95)$$

and the  $L^2$  orthogonal Leray projector  $P_h$  on  $V_h$ .

We assume that the couple  $W_h \times X_h$  satisfies the approximation properties:

1. For each  $\mathbf{v} \in D(A)$  and  $\mathbf{Q} \in H^1(\Omega_1) \times H^1(\Omega_2)$ , there exists a velocity approximation  $\pi_h \mathbf{v} \in W_h$  and a pressure approximation  $\rho_h \mathbf{Q} \in X_h$  such that

$$\|\nabla(\mathbf{v} - \pi_h \mathbf{v})\|_0 \leq ch \|A\mathbf{v}\|_0, \quad \|\mathbf{Q} - \rho_h \mathbf{Q}\|_0 \leq ch \|\mathbf{Q}\|_1, \quad (3.96)$$

2. we have the inverse inequality

$$\|\nabla \mathbf{v}_h\|_0 \leq ch^{-1} \|\mathbf{v}_h\|_0, \quad \forall \mathbf{v}_h \in W_h \quad (3.97)$$

3. and we have the inf-sup inequality: for each  $\mathbf{Q}_h \in X_h$ , there exists  $\mathbf{v}_h \in W_h$  (different from zero) such that

$$d(\mathbf{v}_h, \mathbf{Q}_h) \geq \beta \|\mathbf{Q}_h\|_0 \|\nabla \mathbf{v}_h\|_0, \quad (3.98)$$

where  $\beta$  is a positive constant depending on  $\Omega$ .

This kind of condition is classical; see [19] for instance.

Examples of spaces  $W_h$  and  $X_h$  such that the approximation hypothesis is verified can be found in Girault Raviart [43] or in Bercovier-Pironneau [10].

Let  $(\mathbf{v}, \mathbf{Q}) \in W \times X$  be given, we will denote by  $(R_h(\mathbf{v}, \mathbf{Q}), \mathbf{Q}_h(\mathbf{v}, \mathbf{Q})) \in W_h \times X_h$  the Stokes projection defined by

$$(\nabla R_h, \nabla \mathbf{v}_h) - d(\mathbf{v}_h, \mathbf{Q}_h) + d(R_h, q_h) = (\nabla \mathbf{v}, \nabla \mathbf{v}_h) - d(\mathbf{v}_h, q) + d(\mathbf{v}, q_h), \quad (3.99)$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ . We have the following inequalities (proved in [98] or [43]):

$$\|R_h - \mathbf{v}\|_0 + h \|\nabla(R_h - \mathbf{v})\|_0 \leq ch (\|\nabla \mathbf{v}\|_0 + \|q\|_0), \quad (3.100)$$

for all  $(\mathbf{v}, q) \in W \times X$  and

$$\|R_h - \mathbf{v}\|_0 + h(\|\nabla(R_h - \mathbf{v})\|_0 + \|Q_h - q\|_0) \leq ch^2(\|A\mathbf{v}\|_0 + \|q\|_1), \quad (3.101)$$

for all  $(\mathbf{v}, q) \in D(A) \times (H^1(\Omega_1) \times H^1(\Omega_2)) \cup X$ . Finally we can define a discrete version of the Stokes operator  $A$  by  $A_h = -P_h\Delta_h$ , given by  $(-\Delta_h\mathbf{u}_h, \mathbf{v}_h) = (\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)$ , for all  $(\mathbf{u}_h, \mathbf{v}_h) \in W_h^2$ . We make the additional assumption that the projected laplacian operator  $A_h = -P_h\Delta_h$  verifies that

$$\langle G(\mathbf{u}_h), A_h\mathbf{u}_h \rangle = 0, \quad \langle H(\mathbf{u}_h), A_h\mathbf{u}_h \rangle = 0 \quad (3.102)$$

We get the gradient control inequalities

$$\gamma_0\|\mathbf{v}_h\|_0^2 \leq \|\nabla\mathbf{v}_h\|_0^2, \quad \gamma_0\|\nabla\mathbf{v}_h\|_0^2 \leq \|A_h\mathbf{v}_h\|_0^2, \quad (3.103)$$

for all  $\mathbf{v}_h \in W_h$ .

We will use some inequalities verified by the trilinear form  $b$  ([44]):

**Lemma 3.3.1.** *The trilinear form  $b$  satisfies the estimates*

$$\begin{aligned} & |b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w})| + |b(\mathbf{v}_h, \mathbf{u}_h, \mathbf{w})| + |b(\mathbf{w}, \mathbf{u}_h, \mathbf{v}_h)| \\ & \leq \frac{1}{2}c_0\|A_h\mathbf{v}_h\|_0^{1/2}\|\nabla\mathbf{v}_h\|_0^{1/2}\|\mathbf{u}_h\|_0^{1/2}\|\nabla\mathbf{u}_h\|_0^{1/2}\|\mathbf{w}\|_0 \\ & + \frac{1}{2}c_0\|A_h\mathbf{v}_h\|_0^{1/2}\|\mathbf{v}_h\|_0^{1/2}\|\nabla\mathbf{u}_h\|_0^{1/2}\|\mathbf{w}\|_0, \end{aligned} \quad (3.104)$$

and

$$\begin{aligned} & |b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w})| + |b(\mathbf{v}_h, \mathbf{u}_h, \mathbf{w})| + |b(\mathbf{w}, \mathbf{u}_h, \mathbf{v}_h)| \\ & \leq c_0\|\nabla\mathbf{v}_h\|_0^{1/2}\|\nabla\mathbf{u}_h\|_0^{1/2}\|\mathbf{w}\|_0, \end{aligned} \quad (3.105)$$

for all  $\mathbf{u}_h, \mathbf{v}_h \in V_h, \mathbf{w} \in W$ , where  $c_0 > 0$  depends only on  $\Omega$ .

The Galerkin approximation corresponding to the problem (3.92) reads as follows: find  $(\mathbf{u}_h, p_h) \in W_h \times X_h$  such that for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ ,

$$\begin{aligned} & a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + d(\mathbf{u}_h, q_h) - d(\mathbf{v}_h, p_h) + h(\mathbf{u}_h, \mathbf{v}_h) + g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\ & = (\mathbf{f}, \mathbf{v}_h). \end{aligned} \quad (3.106)$$

We can now prove the following stability and convergence results of the approximation.

**Theorem 3.3.3.** *Under the assumptions (3.96),(3.97) and (3.98),  $(\mathbf{u}_h, p_h)$  satisfies the stability and error estimates:*

$$\|\nabla \mathbf{u}_h\|_0 \leq c_1, \quad (3.107)$$

$$\|A_h \mathbf{u}_h\|_0 \leq \frac{2c}{\nu} \|\mathbf{f}\|_0, \quad (3.108)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h(c\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0) \leq ch^2, \quad (3.109)$$

where  $c_1$  depends on the constants  $c_{d,1}, c_{d,2}$ , on the viscosities  $\nu_1$  and  $\nu_2$ , and on the functions  $\|\mathbf{f}\|_{-1}, \|\mathbf{V}_1\|_{0,\Gamma_1}, \|\mathbf{V}_2\|_{0,\Gamma_2}$ .

*Proof.* By taking  $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h) \in W_h \times X_h$  in (3.106), we have the same inequality as in (3.46), which gives  $\|\nabla \mathbf{u}_h\|_0 \leq c_1$ .

Then, we take  $\mathbf{v}_h = A_h \mathbf{u}_h \in W_h$  and  $q_h = 0$  in (3.106) and we use (3.103), (3.104), (3.107), the hypothesis on  $A_h$  (3.102) and the uniqueness condition (3.64) to get:

$$\begin{aligned} \nu \|A_h \mathbf{u}_h\|_0^2 &\leq \frac{1}{4} c_0 \left( \|A_h \mathbf{u}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0^{1/2} \|\mathbf{u}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0^{1/2} \|A_h \mathbf{u}_h\|_0 \right. \\ &\quad \left. + \|A_h \mathbf{u}_h\|_0^{1/2} \|\mathbf{u}_h\|_0^{1/2} \|\nabla \mathbf{u}_h\|_0 \|A_h \mathbf{u}_h\|_0 \right) + \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h\|_0 \\ &\leq \frac{1}{2} c_0 \gamma_0^{-1/4} \|A_h \mathbf{u}_h\|_0^{3/2} \|\nabla \mathbf{u}_h\|_0^{3/2} + \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h\|_0, \end{aligned}$$

which gives thanks to the Young inequality and(3.107):

$$\begin{aligned} \nu \|A_h \mathbf{u}_h\|_0 &\leq \frac{\nu}{2} \|A_h \mathbf{u}_h\|_0 + \frac{1}{4} c_0^2 \gamma_0^{-1/2} \|\nabla \mathbf{u}_h\|_0^3 + \|\mathbf{f}\|_0 \\ &\leq \frac{\nu}{2} \|A_h \mathbf{u}_h\|_0 + \frac{1}{4} c_0^2 \gamma_0^{-1/2} c_1^3 + \|\mathbf{f}\|_0 \\ &\leq \frac{\nu}{2} \|A_h \mathbf{u}_h\|_0 + c \|\mathbf{f}\|_0. \end{aligned}$$

It remains to prove the last inequality. We use the Stokes projection in( $\mathcal{VP}$ ), which gives

$$\begin{aligned} &\nu_1 (\nabla R_h^1, \nabla \mathbf{v}_h)_{\Omega_1} + \nu_2 (\nabla R_h^2, \nabla \mathbf{v}_h)_{\Omega_2} - d(\mathbf{v}_h, Q_h^1)_{\Omega_1} - d(\mathbf{v}_h, Q_h^2)_{\Omega_2} \\ &+ d(\nu_1 R_h^1, q_h)_{\Omega_1} + d(\nu_2 R_h^2, q_h)_{\Omega_2} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) + \langle G(\mathbf{u}), \mathbf{v}_h \rangle + \langle H(\mathbf{u}), \mathbf{v}_h \rangle \\ &= (\mathbf{f}, \mathbf{v}_h), \end{aligned} \quad (3.110)$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ , where  $(R_h^i, Q_h^i) = \left( R_h^i(\nu_i \mathbf{u}, p), Q_h^i(\nu_i \mathbf{u}, p) \right)$ ,  $i \in \{1, 2\}$ , and  $(R_h, Q_h) = \left( (R_h^1, R_h^2), (Q_h^1, Q_h^2) \right)$ . Subtracting (3.106) from (3.110) and setting  $(\mathbf{e}_h, \eta_h) = (R_h - \mathbf{u}_h, Q_h - p_h)$ , we get

$$\begin{aligned} & a(\mathbf{e}_h, \mathbf{v}_h) - d(\mathbf{v}_h, \eta_h) + d(\nu_1 \mathbf{e}_h, q_h)_{\Omega_1} + d(\nu_2 \mathbf{e}_h, q_h)_{\Omega_2} + b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}, \mathbf{v}_h) \\ & + b(\mathbf{u}_h, \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{v}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{v}_h \rangle = 0 \end{aligned} \quad (3.111)$$

We take  $\mathbf{v}_h = \mathbf{e}_h$  and  $q_h = (\nu_1^{-1} \eta_h^1, \nu_2^{-1} \eta_h^2)$  in (3.111) which gives, by calculations and property of the trilinear form  $b$ :

$$\begin{aligned} & a(\mathbf{e}_h, \mathbf{e}_h) + b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h) \\ & + \langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle = 0. \end{aligned} \quad (3.112)$$

We will use the following estimates (more details can be found in [44]):

$$\begin{aligned} \|\mathbf{e}_h\|_{L^6} &\leq c \|\nabla \mathbf{e}_h\|_0, \\ \|\nabla \mathbf{u}_h\|_{L^3} + \|\mathbf{u}_h\|_{L^\infty} &\leq c \|A_h \mathbf{u}_h\|_0, \\ \|\nabla \mathbf{u}\|_{L^3} + \|\mathbf{u}\|_{L^\infty} &\leq c \|A \mathbf{u}\|_0. \end{aligned} \quad (3.113)$$

— The three different transport terms satisfy the inequalities:

$$\begin{aligned} |b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h)| &\leq C_b \|\nabla \mathbf{u}\|_0 \|\nabla \mathbf{e}_h\|_0^2 \\ &\leq C_b c_1 \|\nabla \mathbf{e}_h\|_0^2, \end{aligned}$$

thanks to (3.107).

Besides,

$$\begin{aligned} |b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h)| &\leq C_b \left( \|\nabla \mathbf{u}\|_{L^3} \|\mathbf{e}_h\|_{L^6} + \|\mathbf{u}\|_{L^\infty} \|\nabla \mathbf{e}_h\|_0 \right) \|\mathbf{u} - R_h\|_0 \\ &\leq C_b c \|A \mathbf{u}\|_0 \|\nabla \mathbf{e}_h\|_0 \|\mathbf{u} - R_h\|_0 \\ &\leq ch^2 \|\nabla \mathbf{e}_h\|_0, \end{aligned}$$

according to the Stokes projection result (3.101). Last but not least:

$$\begin{aligned} |b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h)| &\leq C_b \left( \|\nabla \mathbf{u}_h\|_{L^3} \|\mathbf{e}_h\|_{L^6} + \|\mathbf{u}_h\|_{L^\infty} \|\nabla \mathbf{e}_h\|_0 \right) \|\mathbf{u} - R_h\|_0 \\ &\leq C_b c \|A_h \mathbf{u}_h\|_0 \|\nabla \mathbf{e}_h\|_0 \|\mathbf{u} - R_h\|_0 \\ &\leq ch^2 \|\nabla \mathbf{e}_h\|_0, \end{aligned}$$

thanks again to (3.101) and to (3.108).

— The  $G$ -term verifies

$$\begin{aligned} |\langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle| &\leq C \left( \|\nabla \mathbf{u}\|_0^2 + \|\nabla \mathbf{u}_h\|_0^2 \right) \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 \|\nabla \mathbf{e}_h\|_0 \\ &\leq 2Cc_1^2 \|\nabla \mathbf{e}_h\|^2 + 2Cc_1^2 \|\nabla(\mathbf{u} - R_h)\|_0 \|\nabla \mathbf{e}_h\|_0 \\ &\leq 2Cc_1^2 \|\nabla \mathbf{e}_h\|^2 + ch \|\nabla \mathbf{e}_h\|_0, \end{aligned}$$

still thanks to (3.101).

— Finally, the  $H$ -term satisfies

$$\begin{aligned} |\langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle| &\leq c \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 \|\nabla \mathbf{e}_h\|_0 \\ &\leq c \|\nabla \mathbf{e}_h\|_0^2 \|\nabla(\mathbf{u} - R_h)\|_0 \|\nabla \mathbf{e}_h\|_0 + c \|\nabla \mathbf{e}_h\|_0^2 \\ &\leq ch \|\nabla \mathbf{e}_h\|_0 + c \|\nabla \mathbf{e}_h\|_0^2. \end{aligned}$$

Putting these five previous inequalities in (3.112), we obtain

$$(\nu - C_b c_1 - 2Cc_1^2 - C) \|\nabla \mathbf{e}_h\|_0 \leq ch + 2ch^2.$$

i.e,

$$\|\nabla \mathbf{e}_h\|_0 \leq ch + O(h^2). \quad (3.114)$$

Getting back to the variational formulation (3.111) and taking  $\mathbf{v}_h = \mathbf{e}_h, q_h = 0$ , we obtain

$$\begin{aligned} a(\mathbf{e}_h, \mathbf{e}_h) - d(\mathbf{e}_h, \eta_h) + b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h) \\ + \langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle = 0. \end{aligned} \quad (3.115)$$

Using the inf-sup condition (3.98) in (3.115), we get

$$\begin{aligned} \beta \|\eta_h\|_0 \|\nabla \mathbf{e}_h\|_0 &\leq \nu \|\nabla \mathbf{e}_h\|_0^2 + |b(\mathbf{e}_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u} - R_h, \mathbf{u}, \mathbf{e}_h) + b(\mathbf{u}_h, \mathbf{u} - R_h, \mathbf{e}_h)| \\ &\quad + |\langle G(\mathbf{u}) - G(\mathbf{u}_h), \mathbf{e}_h \rangle + \langle H(\mathbf{u}) - H(\mathbf{u}_h), \mathbf{e}_h \rangle|, \end{aligned}$$

Hence from the previous inequalities:

$$\beta \|\eta_h\|_0 \|\nabla \mathbf{e}_h\|_0 \leq c \|\nabla \mathbf{e}_h\|_0^2 + ch \|\nabla \mathbf{e}_h\|_0 + ch^2 \|\nabla \mathbf{e}_h\|_0^2. \quad (3.116)$$

We can finally prove the last inequality. Thanks to the triangular inequality:

$$\begin{aligned} \nu \|\mathbf{u} - \mathbf{u}_h\|_0 + h(\nu \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0) &\leq \nu \|\mathbf{e}_h\|_0 + h(\nu \|\nabla(\mathbf{e}_h)\|_0 + \|\eta_h\|_0) \\ + \nu \|\mathbf{u} - R_h\|_0 + h(\nu \|\nabla(\mathbf{u} - R_h)\|_0 + \|p - Q_h\|_0), \end{aligned} \quad (3.117)$$

where

$$\nu \|\mathbf{e}_h\|_0 + h(\nu \|\nabla(\mathbf{e}_h)\|_0 + \|\eta_h\|_0) \leq ch + O(h^2)$$

comes from the inequalities (3.114) and (3.116).

The other term verifies (3.101), thus

$$\nu \|\mathbf{u} - R_h\|_0 + h(\nu \|\nabla(\mathbf{u} - R_h)\|_0 + \|p - Q_h\|_0) \leq ch^2.$$

Putting these two estimates in (3.117), we finally obtain

$$\nu \|\mathbf{u} - \mathbf{u}_h\|_0 + h(\nu \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0) \leq ch + O(h^2).$$

□

**Remark 3.3.1.** *We only have a first order estimate in  $h$ , compared to second order in "more classic" Navier Stokes problems (see for instance [50],[45]).*

We can add another control inequality

**Theorem 3.3.4.**

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0 \leq ch + O(h^2) \quad (3.118)$$

which comes straightforwardly from the previous inequations.

### 3.3.4 Stability of the method and error analysis

We define the linearized numerical scheme corresponding to the problem (3.92) with:

$$\left\{ \begin{array}{ll} (\mathbf{u}_i^n \cdot \nabla) \mathbf{u}_i^{n-1} - \nu_i \Delta \mathbf{u}_i^n + \nabla p_i^n = \mathbf{f}_i & \text{in } \Omega_i \\ \nabla \cdot \mathbf{u}_i^n = 0, & \text{in } \Omega_i \\ \nu_i \frac{\partial \mathbf{u}_{i,h}^n}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n) |\mathbf{u}_{i,h}^{n-1} - \mathbf{u}_{j,h}^{n-1}|, & \text{on } \Gamma_{Int} \\ \nu_i \frac{\partial \mathbf{u}_{i,h}^n}{\partial \mathbf{n}_i} = -c_{d,i}(\mathbf{u}_{i,h}^n - \mathbf{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i^n \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \\ \mathbf{u}_i^n = 0 & \text{on } \Gamma_{i,s}, \end{array} \right. , \quad (3.119)$$

for  $n \geq 1$ .

The weak formulation associated with (3.119) can be seen as an Oseen iterative method (see [45]):

We start from the couple  $(\mathbf{u}_h^0, p_h^0) \in W_h \times X_h$  defined by the discrete Stokes problem on  $\Omega = \Omega_1 \times \Omega_2$

$$a(\mathbf{u}_h^0, \mathbf{v}_h) - d(\mathbf{v}_h, p_h^0) + d(\mathbf{u}_h^0, q_h) = (\mathbf{f}, \mathbf{v}_h), \quad (3.120)$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ .

**Remark 3.3.2.** *The two Stokes problems for the two different fluids can be solved separately on  $\Omega_1$  and  $\Omega_2$  since there is no boundary condition involving the two fluids at this point.*

The iterative method at order  $n$  reads:

$$\begin{aligned} a(\mathbf{u}_h^n, \mathbf{v}_h) - d(\mathbf{v}_h, p_h^n) + d(\mathbf{u}_h^n, q_h) + b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) + \langle \tilde{G}(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}), \mathbf{v}_h \rangle \\ + \langle H(\mathbf{u}_h^n), \mathbf{v}_h \rangle = (\mathbf{f}, \mathbf{v}_h), \end{aligned} \quad (3.121)$$

for all  $(\mathbf{v}_h, q_h) \in W_h \times X_h$ , where the map  $\tilde{G}$  is the discrete version of  $G$  and verifies

$$\langle \tilde{G}(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}), \mathbf{v}_h \rangle = \int_{\Gamma_I} (\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^n) |\mathbf{u}_{1,h}^{n-1} - \mathbf{u}_{2,h}^{n-1}| (\mathbf{v}_{1,h} - \mathbf{v}_{2,h}). \quad (3.122)$$

This leads to the the stability result:

**Theorem 3.3.5.** *The iterative method is unconditionally stable, meaning that the couple*

$(\mathbf{u}_h^n, p_h^n)$  defined by the scheme (3.121) satisfies

$$\nu \|\nabla \mathbf{u}_h^n\|_0 \leq \|\mathbf{f}\|_{-1}, \quad (3.123)$$

and

$$\|A_h \mathbf{u}_h^n\|_0 \leq c_2, \quad (3.124)$$

for all  $n \geq 0$ , where  $c_2$  depends on  $\nu_1, \nu_2, \|\mathbf{f}\|_0$  (and on  $\Omega$  through different Sobolev embedding theorems).

*Proof.* The result is verified for  $n = 0$ , by definition of  $\mathbf{u}_h^0$  as a solution of a standard Stokes problem. We assume that the first inequality is true at the step  $n$  and we will prove it for the step  $n + 1$ . Taking  $\mathbf{v}_h = \mathbf{u}_h^{n+1}, q_h = p_h^{n+1} \in W_h \times X_h$  in (3.121) at the step  $n + 1$ , we get

$$a(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) + \langle \tilde{G}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^n), \mathbf{u}_h^{n+1} \rangle + \langle H(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1} \rangle = (\mathbf{f}, \mathbf{u}_h^{n+1}).$$

Besides, both  $\tilde{G}$  and  $H$ -terms satisfy

$$\langle \tilde{G}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^n), \mathbf{u}_h^{n+1} \rangle \geq 0,$$

and

$$\langle H(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1} \rangle \geq 0.$$

Hence, we get the estimate

$$\nu \|\nabla \mathbf{u}_h^{n+1}\|_0^2 \leq \|\mathbf{f}\|_{-1} \|\nabla \mathbf{u}_h^{n+1}\|_0,$$

and the first claim is proved.

The second claim will be proved without iteration argument. Taking  $\mathbf{v}_h = A_h \mathbf{u}_h^{n+1}$  and  $q_h = 0$  in (3.121) at the step  $n + 1$ , recalling that  $A_h$  satisfy the inequality (3.108) and cancel the  $G$  and  $H$ -terms, we get by Cauchy-Schwarz and transport trilinear term (3.104) inequalities:

$$\nu \|A_h \mathbf{u}_h^{n+1}\|_0^2 \leq \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h^{n+1}\|_0 + \frac{1}{2} c_0 \gamma_0^{-1/4} \|\nabla \mathbf{u}_h^n\|_0 \|\nabla \mathbf{u}_h^{n+1}\|_0^{1/2} \|A_h \mathbf{u}_h^{n+1}\|_0^{3/2},$$

and from the Young inequality:

$$\nu \|A_h \mathbf{u}_h^{n+1}\|_0^2 \leq \|\mathbf{f}\|_0 \|A_h \mathbf{u}_h^{n+1}\|_0 + \frac{1}{4\nu} c_0^2 \gamma_0^{-1/2} \|\nabla \mathbf{u}_h^n\|_0^2 \|\nabla \mathbf{u}_h^{n+1}\|_0 \|A_h \mathbf{u}_h^{n+1}\|_0 + \frac{\nu}{2} \|A_h \mathbf{u}_h^{n+1}\|_0^2.$$



Thanks to (3.123), we obtain

$$\frac{\nu}{2} \|A_h \mathbf{u}_h^{n+1}\|_0 \leq \|\mathbf{f}\|_0 + \frac{1}{4\nu^4} c_0^2 \gamma_0^{-1/2} \|\mathbf{f}\|_{-1}^3,$$

for all  $n \geq 0$ , which gives the stability estimate (3.124) and concludes the proof.  $\square$

A corollary from the stability theorem 3.3.5 and the *a priori estimates* from the theoretical section is the boundedness result:

**Lemma 3.3.2.** *There exists  $C > 0$  depending only on the data such that*

$$\|\mathbf{u}_h\|_{W_h} \leq C \quad \text{and} \quad \|\mathbf{u}_h^n\|_{W_h} \leq C, \quad (3.125)$$

for all  $n \geq 0$ .

We define the errors between the Galerkin theoretical velocities and the Galerkin approximated velocities obtained from the Oseen scheme  $\mathbf{e}_h^n = \mathbf{u}_h^n - \mathbf{u}_h$  and the pressure errors  $\eta_h^n = p_h^n - p_h$ .

This leads to the error convergence result:

**Theorem 3.3.6.** *Provided the condition on the data*

$$\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D \geq 0 \quad \text{and} \quad \frac{C(C_b + C_D)}{\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D} < 1, \quad (3.126)$$

the method is converging and we have the error estimate:

$$\|\mathbf{e}_h^n\|_{W_h} \leq \frac{2}{\nu} \left( \frac{C(C_b + C_D)}{\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D} \right)^{n+1} \|\mathbf{f}\|_{-1}.$$

*Proof.* We consider the difference at the  $n$ -th step, between the equations satisfied by  $(\mathbf{u}_h, p_h)$  and by  $(\mathbf{u}_h^n, p_h^n)$ , which gives the relation

$$a(\mathbf{e}_h^n, \mathbf{v}_h) + b(\mathbf{e}_h^{n-1}, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h^{n-1}, \mathbf{e}_h^n, \mathbf{v}_h) + g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \quad (3.127)$$

$$+ h(\mathbf{e}_h^n, \mathbf{v}_h) - d(\mathbf{v}_h, \eta_h^n) + d(\mathbf{e}_h^n, q_h) = 0, \quad (3.128)$$

for  $n \geq 1$  and  $(\mathbf{v}_h, q_h) \in W_h \times Q_h$ .

Consider the difference  $g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{v}_h)$ , we have

$$\begin{aligned} & g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{v}_h) \\ &= C_D \int_{\Gamma_I} \left( (\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^n) |\mathbf{u}_{1,h}^{n-1} - \mathbf{u}_{2,h}^{n-1}| - (\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) |\mathbf{u}_{1,h} - \mathbf{u}_{2,h}| \right) (\mathbf{v}_{1,h} - \mathbf{v}_{2,h}), \end{aligned}$$

and denoting  $\mathbf{U}_h^n := \mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^n$  and  $\mathbf{U}_h := \mathbf{u}_{1,h} - \mathbf{u}_{2,h}$ ,  $\mathbf{V}_h = \mathbf{v}_{1,h} - \mathbf{v}_{2,h}$ , we get

$$\begin{aligned} g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{v}_h) &= C_D \int_{\Gamma_I} (\mathbf{U}_h^n |\mathbf{U}_h^{n-1}| - \mathbf{U}_h |\mathbf{U}_h|) \mathbf{V}_h \\ &= C_D \int_{\Gamma_I} (\mathbf{U}_h^n |\mathbf{U}_h^{n-1}| - \mathbf{U}_h^n |\mathbf{U}_h| + \mathbf{U}_h^n |\mathbf{U}_h| - \mathbf{U}_h |\mathbf{U}_h|) \mathbf{V}_h \\ &= C_D \int_{\Gamma_I} \left( \mathbf{U}_h^n (|\mathbf{U}_h^{n-1}| - |\mathbf{U}_h|) + |\mathbf{U}_h| (\mathbf{U}_h^n - \mathbf{U}_h) \right) \mathbf{V}_h. \end{aligned}$$

We get by the triangular inequality

$$\begin{aligned} \|g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - g(\mathbf{u}_h, \mathbf{v}_h)\|_{W_h} &\leq C_D \left( \|\mathbf{U}_h^n\|_{W_h} \|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{U}_h\|_{W_h} \|\mathbf{e}_h^n\| \right) \|\mathbf{V}_h\|_{W_h} \\ &= C_D \left( \|\mathbf{u}_h^n\|_{W_h} \|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{u}_h\|_{W_h} \|\mathbf{e}_h^n\|_{W_h} \right) \|\mathbf{v}_h\|_{W_h} \\ &\leq CC_D \left( \|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{e}_h^n\|_{W_h} \right) \|\mathbf{v}_h\|_{W_h}, \end{aligned}$$

using the stability result 3.3.2..

Choosing  $\mathbf{v}_h = \mathbf{e}_h^n$  and  $q_h = \eta_h^n$  in (3.127), we get

$$a(\mathbf{e}_h^n, \mathbf{e}_h^n) + b(\mathbf{e}_h^{n-1}, \mathbf{u}_h, \mathbf{e}_h^n) + b(\mathbf{u}_h^{n-1}, \mathbf{e}_h^n, \mathbf{e}_h^n) + g(\mathbf{u}_h^n, \mathbf{u}_h^{n-1}, \mathbf{e}_h^n) - g(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}_h^n) + h(\mathbf{e}_h^n, \mathbf{e}_h^n) = 0.$$

Using the properties of the different operations, this yields to

$$\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) \|\mathbf{e}_h^n\|_{W_h}^2 \leq C_b \|\mathbf{u}_h\|_{W_h} \|\mathbf{e}_h^n\|_{W_h} \|\mathbf{e}_h^{n-1}\|_{W_h} + CC_D \left( \|\mathbf{e}_h^{n-1}\|_{W_h} + \|\mathbf{e}_h^n\|_{W_h} \right) \|\mathbf{e}_h^n\|_{W_h},$$

hence

$$(\min(\nu_1, \nu_2, c_{d,1}, c_{d,2}) - CC_D) \|\mathbf{e}_h^n\|_{W_h} \leq (CC_b + CC_D) \|\mathbf{e}_h^{n-1}\|_{W_h}.$$

Using again the stability result 3.3.2, we conclude thanks to the recursivity principle.  $\square$

**Remark 3.3.3.** *The major issue of this convergence result is the fact that it is difficult to have access to the constants  $C$  and  $C_b$  involved. Nevertheless, choosing small values of  $C_D$  relatively to the range of values of  $\nu_1, \nu_2, c_{d,1}$  and  $c_{d,2}$  guarantees both conditions. We will see in the numerical simulations an example of this criteria being not fulfilled.*

### 3.3.5 Numerical simulations

Finally, we are numerically simulating with Freefem++ the problem (3.119) thanks to the Oseen linearisation method.

We consider  $P2$  finite elements for the velocities and  $P1$  finite elements for the pressure. The figure 3.6 shows the mesh we have used in Freefem++.

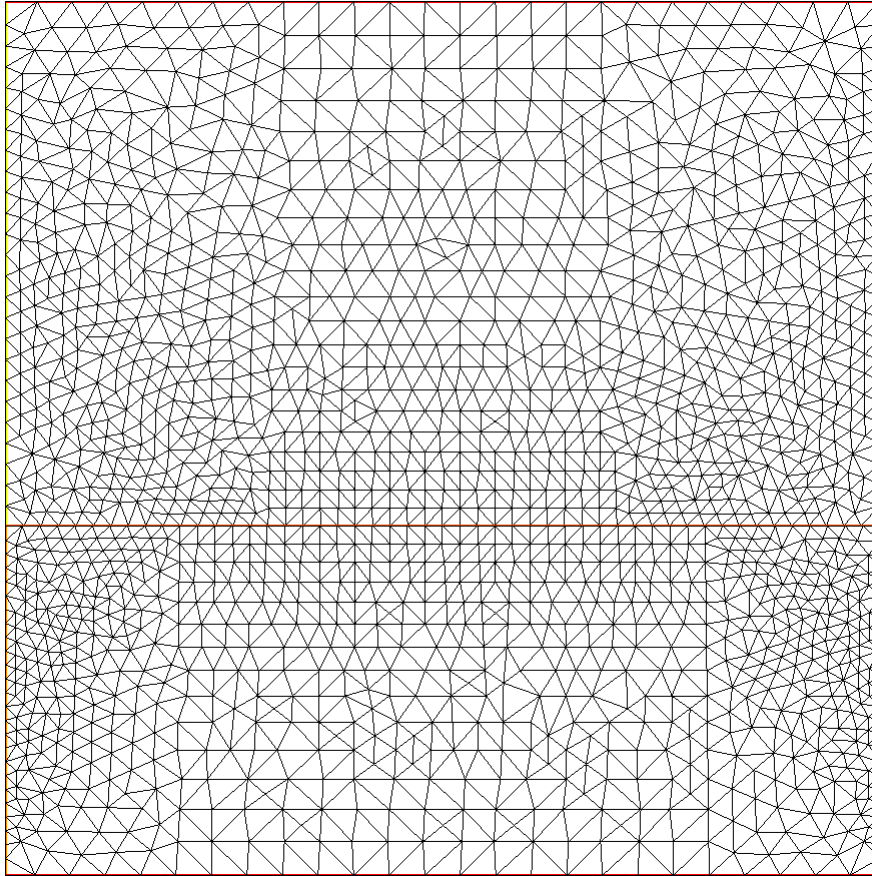


Figure 3.6 – Mesh for the domain  $\Omega = \Omega_1 \times \Omega_2$

We also decide to consider different viscosities:  $\nu_1 = 0.4$  and  $\nu_2 = 0.08$  and horizontal periodic boundary conditions, where  $\Omega_1 = 1 \times 0.6$  and  $\Omega_2 = 1 \times 0.4$ , The convergence criteria we consider is

$$\frac{\|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{H^1}}{\|\mathbf{u}^n\|_{H^1}} < \varepsilon_{tol}, \quad (3.129)$$

where the tolerance error  $\varepsilon_{tol} = 10^{-3}$ . Note that the stopping criteria is based on the

relative error. The source  $\mathbf{f}$  has been chosen constant equal to 1 everywhere in the domains and the constants  $c_{d,1}$  and  $c_{d,2}$  equal to 1. The velocity  $\mathbf{V}$  has been chosen such that the air velocity at the top  $\mathbf{V}_1 = 1$  and the water velocity at the bottom  $\mathbf{V} = 0.1$ .

The table 3.1 shows the number of iterations until the stopping criteria (3.129) is fulfilled, for different values of  $C_D$ .

We have proved that there is convergence if the condition (3.126) is fulfilled; we don't know what is happening when it is not fulfilled.

In this configuration, we have seen that there was quite a fast convergence when  $C_D < 2$ , but when  $C_D \rightarrow 2.1$ , the simulation is "exploding" and the numerical results are not what we expect. The figure 3.7 shows the simulation of the flows after 144 iterations for  $C_D = 1$

| Value of $C_D$ | First value of $n$ | $\ \mathbf{u}_1^n\ _{L^2}$ | $\ \mathbf{u}_2^n\ _{L^2}$ | $\ \mathbf{u}_1^n\ _{H^1}$ | $\ \mathbf{u}_2^n\ _{H^1}$ |
|----------------|--------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| 0.1            | 160                | 5.48849                    | 16.1656                    | 4.42758                    | 15.9186                    |
| 0.2            | 152                | 5.48479                    | 16.1702                    | 4.38152                    | 16.0235                    |
| 0.5            | 146                | 5.47764                    | 16.1937                    | 4.32994                    | 16.3304                    |
| 1              | 144                | 5.46818                    | 16.2515                    | 4.29199                    | 16.7047                    |
| 2              | 461                | 5.45242                    | 16.4263                    | 4.25692                    | 17.2604                    |
| 2.01           | 558                | 5.45227                    | 16.4292                    | 4.2564                     | 17.2688                    |
| 2.02           | 722                | 5.45214                    | 16.4313                    | 4.25593                    | 17.2738                    |
| 2.03           | 1325               | 5.45208                    | 16.4322                    | 4.25607                    | 17.2753                    |
| 2.05           | 8250               | 5.45192                    | 16.4341                    | 4.25618                    | 17.2771                    |
| 2.1            | $+\infty$          |                            |                            |                            |                            |

Table 3.1 – Convergence rate for different values of  $C_D$

and the figure 3.8, the case  $C_D = 2.5$  after only 3 iterations.

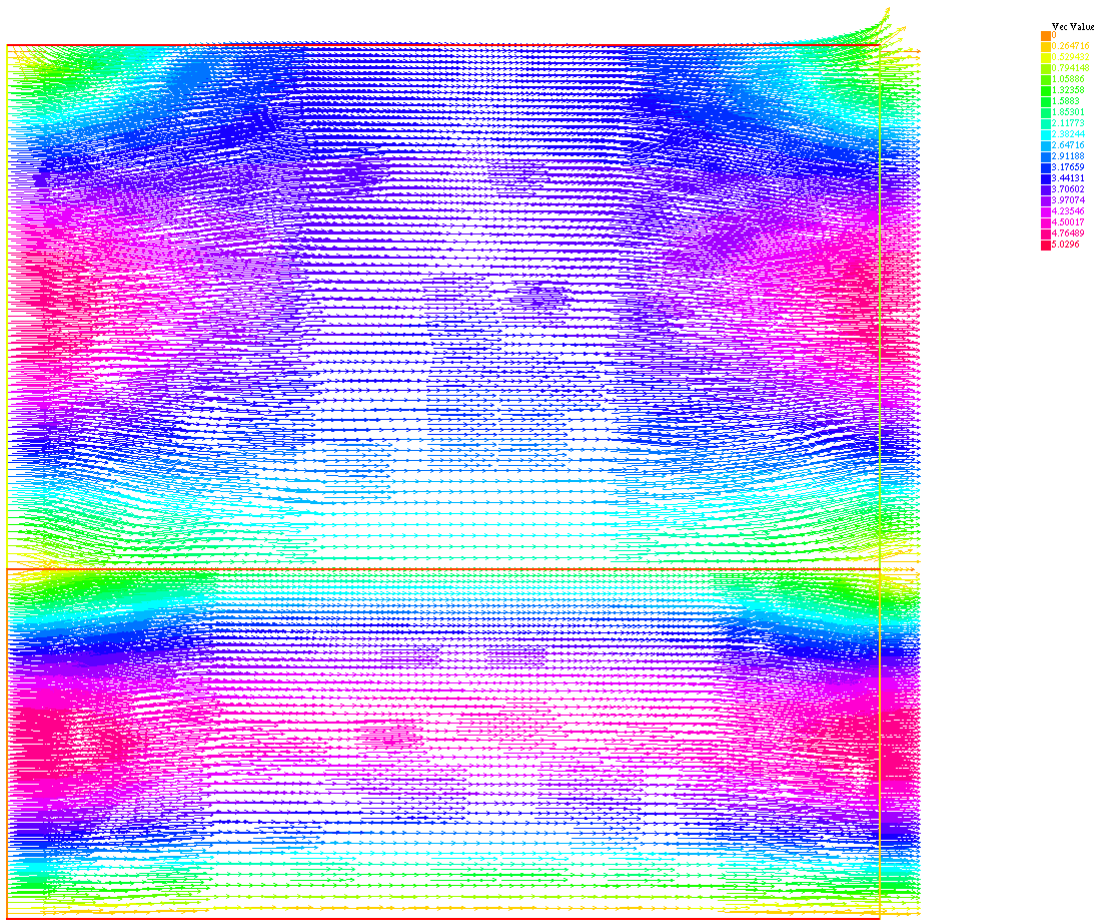


Figure 3.7 – Two fluids with  $\nu_1 = 0.4$ ,  $\nu_2 = 0.08$ ,  $C_D = 1$ ,  $c_{d,1} = c_{d,2} = 1$

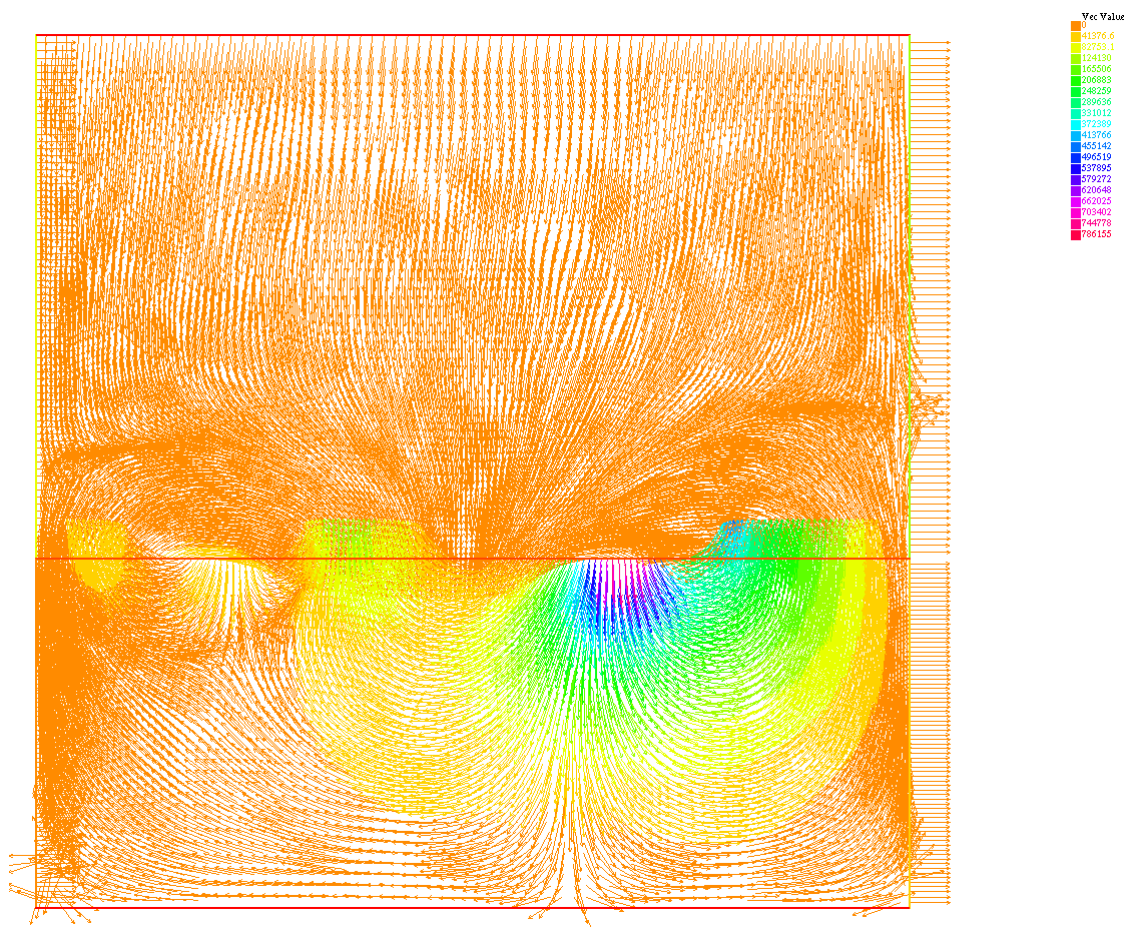


Figure 3.8 – Two fluids with  $\nu_1 = 0.4$ ,  $\nu_2 = 0.08$ ,  $C_D = 2.5, c_{d,1} = c_{d,2} = 1$



PART II

# Surface boundary layers problems

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# SURFACE BOUNDARY LAYERS THROUGH A SCALAR EQUATION WITH AN EDDY VISCOSITY VANISHING AT THE GROUND

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## 4.1 Introduction

This paper is devoted to study a scalar elliptic equation which parameterize the mean velocity of the air in the atmospheric Surface Boundary Layer (SBL), where the turbulent stresses are balanced with the friction forces on the ground. This is part of the more general framework of the turbulent boundary layers, initially developed by L. Prandtl [81], then by T. von Kármán [100], who highlighted the role of logarithmic profiles relative to the height in such layers (see also in [23, 72, 88]), called the log-law, which was validated by several numerical simulations, for instance by using turbulence models such as the  $k - \varepsilon$  model (see [79] and further references inside) and-or by stochastic models [78].

The Monin-Obukhov theory [73] states that under non-neutral conditions, the mean velocity profile differs slightly from the log-law, the difference being determined by stabilization functions. This theory is used in much more general (SBL) regimes [76], and is the basis of most atmospheric flow simulations near the ground, which raises the question of the determination of the stabilization functions.

The starting point is the 1D differential equation that yields the log-law from a theoretical point of view [23], namely

$$u_* \kappa \frac{d}{dz} \left( z \frac{du}{dz} \right) = 0,$$

together with appropriate boundary conditions, where  $u = u(z)$  denotes the mean horizontal velocity component that is supposed to only depend on the height  $z > 0$ ; in this

framework,  $u_*$  is the friction velocity and  $\kappa$  the von Kármán constant. We wonder if a similar simple multi dimensional PDE satisfied by  $u = u(\mathbf{x}_h, z)$ ,  $\mathbf{x}_h \in \mathbb{R}^d$  ( $d = 1, 2$ ), is able to yield Monin-Obukhov profiles type, which suggests to introduce the equation,

$$-\nu_h \Delta_h u - \partial_z(\nu_{\text{turb}}(z) \partial_z u) = f, \quad (4.1)$$

where  $f$  is the Boussinesq force specified by a temperature supposed to be constant in this paper,  $\nu_{\text{turb}} = \nu_{\text{turb}}(z)$  is an eddy viscosity and  $\nu_h > 0$  an horizontal viscosity coefficient. According to standard assumptions and dimensional analysis [29, 58, 74], we should have

$$\nu_{\text{turb}}(z) = \kappa u_* z, \quad (4.2)$$

in the domain  $0 < z < z_{\text{top}}$ , where  $z_{\text{top}}$  denotes the height of the (SBL). However, we know that eddy viscosities that vanish at the boundary are source of serious mathematical issues [2, 16, 14] and are often studied by the mean of weighted Sobolev spaces [51]. Moreover, the case given by (4.2) is critical as we will see later in this article, giving very weak solutions with only  $H^{1/2}$  regularity, which does not allow to set a boundary condition at  $z = 0$ . This is why in many models,  $\nu_{\text{turb}}$  is taken to be constant in a viscous sublayer  $[0, z_0]$ . The same issue occurs in the case of the Smagorinsky model [22], where the eddy viscosity denoted by  $\nu_{\text{smag}}$  is given by  $\nu_{\text{smag}} = \kappa z^2 |\partial_z u|$  near the ground  $z = 0$ . This is why in the Smagorinsky case, several authors have suggested to replace the physical  $\nu_{\text{smag}}$  by  $\nu_{\text{smag}} = \kappa z_0^{2(1-\alpha)} z^{2\alpha} |\partial_z u|$  for some  $0 < \alpha < 1$  [16, 83, 84, 85], to obtain more regular solutions and to be able to take into account appropriate boundary conditions. This suggests to consider in our case general eddy viscosities of the form

$$\nu_{\text{turb}}(z) = \kappa z_0^{1-\alpha} u_* z^\alpha. \quad (4.3)$$

There is the question of the boundary conditions. It is natural to set  $u = 0$  at the ground  $z = 0$ . Following [55], we take a Navier friction condition at the top of the boundary layer  $z = z_{\text{top}}$  (also named Robin law), which is a fairly transparent condition, easy to deal with in a variational formulation. In order to be consistent with the numerical simulations, we take periodic boundary conditions in the horizontal directions. Therefore,

the PDE problem we consider in this paper is the following<sup>1</sup>,

$$\begin{cases} \lambda u - \nu_h \Delta_h u - \mu \partial_z (z^\alpha \partial_z u) = f & \text{in } BL, \\ \mu z_{top}^\alpha \frac{\partial u(\mathbf{x}_h, z)}{\partial z} = C_D (V(\mathbf{x}_h) - u(\mathbf{x}_h, z_{top})) & \text{on } \Gamma_{top}, \\ u = 0 & \text{on } \Gamma_b, \end{cases} \quad (4.4)$$

where  $\mathbf{x}_h = (x, y)$  and

$$BL = \Pi_2 \times [0, z_{top}], \quad \Gamma_b = \Pi_2 \times \{z = 0\}, \quad \Gamma_{top} = \Pi_2 \times \{z = z_{top}\}, \quad (4.5)$$

$0 < \alpha \leq 1$ , for a given 2D torus denoted by  $\Pi_2$ , where the term  $\lambda u$ ,  $\lambda \geq 0$ , is a stabilizing term, useful only in the case  $\alpha = 1$ ,  $\mu > 0$  and  $C_D > 0$  are given coefficients that will be calibrated by numerical experiments.

In this paper we prove the existence of a weak solution to Problem (4.4) in an appropriate weighted space for  $0 \leq \alpha < 1$ , see Theorem 4.3.1 below. Then for  $\alpha = 1$ , we prove the existence of a very weak solution  $u \in H^{1/2}(BL)$  for  $\lambda > 0$ , in Theorem 4.4.1, which is the main result of this paper. In this result, we do not impose  $u = 0$  at  $z = 0$ . It is based on a Nečas lemma type (also known as Lions Lemma), Lemma 4.4.1. In this case, the difficulty is due to the mixed boundary condition and we cannot directly apply the results of [4, 35]. We had to make many efforts to prove this essential result in this problem, based on the interpolation theorem [64]. Finally we carry out several numerical simulations based on a `Freefem++` code [47], which allows to evaluate the difference between the solution and the log-law. In particular we observe that, despite the lack of theoretical regularity, the physical case  $\alpha = 1$  remains the most accurate to parameterize the SBL, and we are able to calculate numerically in several different regimes the stabilizing functions, given by formula (4.86) below, which validates model (4.4) in terms of the Monin-Obukhov theory.

The paper is organized as follows. In a first part we develop the modeling sketched in the introduction, and we set the physical constants involved in the simulation. Then we study the case  $0 \leq \alpha < 1$ , by viscous regularization and proving that the natural weighted space related to the problem is embedded in a standard Sobolev space  $W^{1,\gamma}(BL)$ . Then we focus on the case  $\alpha = 1$ . A large part of the paper is devoted to the study of the function spaces and Nečas Lemma by the mean of Fourier series, which allows to prove that the natural weighted space related to the problem is embedded in  $H^{1/2}(BL)$ . The

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1. the acronym SBL has been replaced by BL for simplicity.

last section of the paper is devoted to the numerical results.

## 4.2 Modeling

This section is devoted to recall some basic elements of the theory of turbulent boundary layers, and to fix the general framework of the model which one studies. The steady mean fluid velocity in such boundary layer, denoted ( $BL$ ), is denoted by

$$\mathbf{u} = \mathbf{u}(\mathbf{x}_h, z) = (\mathbf{u}_h(\mathbf{x}_h, z), w(\mathbf{x}_h, z))$$

instead of  $\bar{\mathbf{u}}$  for the simplicity, where

$$\mathbf{u}_h = (u, v), \quad z \in ]0, z_{top}[,$$

$z_{top} > 0$  being the bottom of ( $BL$ ). For instance, if ( $BL$ ) models the surface boundary layer,  $z_{top} \approx 100$  m. We also will need to consider the roughness length  $z_0$ , which depends on the nature of the ground, and varies from 0.0002 m in open sea, to 1 m for city centre with high- and low-rise buildings.

Note that we are in a flat domain and the splitting of both variables and unknowns into horizontal and vertical will be of particular use to identify the problem and give a clear formulation.

### 4.2.1 assumptions, general equation and issues

Let  $\nu > 0$  denotes the kinematic viscosity of the fluid. It is commonly supposed that in standard (statistical) steady BL it hold the following:

- the pressure is constant and the vertical part of the mean velocity vanishes, that is  $w = 0$  and, even if it means making a change of coordinates, we can assume  $v = 0$ ;
- the mean velocity  $\mathbf{u} = (u, 0, 0)$  depends only on the altitude, that is  $u = u(z)$ ;
- the eddy viscosity  $\nu_{\text{turb}}$  depends on  $z$  and  $u^* = \sqrt{\nu |\partial_z u(0)|}$ , the so-called frictional velocity, which is the tuning parameter of the system (see [23, 79]), which yields

$$\nu_{\text{turb}} = \nu_{\text{turb}}(z) = C_\nu C_* u^* z, \quad (4.6)$$

where  $C_\nu \approx 15$ ,  $C_* \approx 10$  are non dimensional constants, that we have calibrated by numerical simulations. Typical values of  $u_*$  range from 2 to 10  $\text{ms}^{-1}$ .

— all terms in the fluid equation are negligible compared to the turbulent diffusion term.

These assumptions lead to the following equation for the mean velocity  $\mathbf{u} = (u, 0, 0)$ ,

$$-\frac{d}{dz} \left( \nu_{\text{turb}}(z) \frac{du}{dz} \right) = 0, \quad (4.7)$$

which formula, once integrated between a given  $z_0$  and  $z_{\text{top}}$  with appropriate boundary conditions, yields the well known log-law, uniform in  $\mathbf{x}_h$ , using the calibration constants  $C_\nu$  and  $C_\star$ :

$$u(z) = \frac{C_\star u_\star}{C_\nu} \left( \log \left( \frac{z}{z_0} \right) + 1 \right), \quad z \in [z_0, z_{\text{top}}]. \quad (4.8)$$

Generally, for  $z \in [0, z_0]$ , called the viscous sub-layer, a linear profile is considered such that  $u = u(z)$  is continuous over  $]0, z_{\text{top}}[$ , and  $u(0) = 0$ , which means

$$u(z) = \frac{C_\star u_\star}{C_\nu} z, \quad z \in [0, z_0]. \quad (4.9)$$

Let  $u_{\text{Log}}$  denotes the function defined over  $]0, z_{\text{top}}[$  by (4.8)-(4.9). When the stability of the atmosphere is non-neutral and due to the effect of convection, which means that there is a non zero source term in (4.7), stabilizing functions must added to  $u_{\text{Log}}$  to get the right velocity profile, according to the Monin-Obukhov theory [73], which means that

$$u(z) = u_{\text{Log}}(z) + \Psi(\mathbf{x}_h, z), \quad (4.10)$$

where the function  $\Psi(\mathbf{x}_h, z)$  is deduced from similarity arguments or from experimental data. Examples of such stabilizing functions can be found in [76].

**Remark 4.2.1.** *Normally in usual industrial models,  $C_\nu$  stands for the von Kármán constant, the value of which being equal to 0.4, and  $C_\star = 1$ . However, due to the scales of our simulations, we have to take other values of these constants to get numerical results related to the physical data in the atmospheric SBL.*

Our aim is to find a comprehensive PDE model, such that:

1. is defined over  $]0, z_{\text{top}}[$ ;
2. includes an eddy viscosity of the same form as that given by (4.6), where the profile  $u = u(\mathbf{x}_h, 0)$  also depends on the horizontal variable;

3. is able to calculate stabilizing functions such as in (4.10) in various atmospheric regimes.

Before embarking on nonlinear complicated 3D equations of fluid mechanics, we consider as a first step the following elliptic toy-model in  $BL = \mathbb{R}^2 \times (0, z_{top})$ :

$$\lambda u - \nu_h \Delta_h u - \partial_z(\nu_{\text{turb}}(z) \partial_z u) = f, \quad (4.11)$$

for some  $\lambda > 0, \nu_h > 0$ ,  $\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ . The term  $\lambda u$  in (4.11) stands for a numerical artefact of an evolutionary term  $\partial_t u$ , and serves as a system stabilizer, especially in the case  $\alpha = 1$ . It can be taken equal to zero in the finite element simulation thanks to the numerical dissipation, due to the discretization.

In physical applications, the source term  $f$  is the Boussinesq force, namely

$$f = g\beta(T_0 - T), \quad (4.12)$$

where  $T$  is the temperature of the fluid,  $T_0$  its value at the ground,  $g \approx 10 \text{ ms}^{-1}$  is the gravity coefficient,  $\beta$  is the coefficient of thermal expansion, a typical value of which for dry air is varies between  $0.002 \text{ K}^{-1}$  and  $0.003 \text{ K}^{-1}$ .

## 4.2.2 Boundary conditions

The choice of the boundary conditions may be an issue, and there are many options. We consider the case of a BL that flows over a rigid wall, which means that we take an homogeneous boundary condition on the bottom  $\Gamma_b$ ,  $u(\mathbf{x}_h, 0) = 0$ . Moreover, we consider that this BL is coupled on top with a layer of fluid which exerts a frictional force on it. Therefore, as in [55] one can take a linear Navier-Boundary condition like

$$C_\nu u^* z_{top} \frac{du}{dz}(\mathbf{x}_h, z_{top}) = C_D (V(\mathbf{x}_h) - u(\mathbf{x}_h, z_{top})) \quad (4.13)$$

where  $C_D > 0$  is a frictional coefficient and  $V = V(\mathbf{x}_h)$  is the velocity of the top layer. In the numerical simulations we have taken  $V(\mathbf{x}_h) = u_{\text{Log}}(z_{top})(1 + \varepsilon(\mathbf{x}_h))$ , where  $u_{\text{Log}}$  is that given by (4.8)-(4.9). The coefficient  $C_D$  will be numerically optimized in function of  $u_*$ ,  $\varepsilon(\mathbf{x}_h)$  is a small perturbation term.

**Remark 4.2.2.** According to the results of [55], we expect that for large values of  $C_D$ , the boundary condition (4.13) converges (in some sense) to the continuity condition  $u(\mathbf{x}_h) =$

$V(\mathbf{x}_h)$  at  $\Gamma_{top}$ , which is well confirmed in this framework by the numerical simulations.

### 4.2.3 Alternatives and general framework

As we will see in the following, the eddy viscosity given by (4.6) yields variational (or weak) solutions to Problem (4.11) that are in  $H^{1/2}$ , and not much more. In particular the homogeneous Dirichlet boundary condition at the bottom cannot be checked, which is consistent with the log-law. This is why we ask the question whether or not it is possible to identify alternate eddy viscosities, close to (4.6) but giving more regularity to the system, being not critical for the notion of trace, and so on. We wonder if that gives good approximations of the usual BL profiles. In this way, it is natural to consider  $\nu_{\text{turb}}$  of the form

$$\nu_{\text{turb}} = \nu_{\text{turb}}(z) = z_0^{1-\alpha} C_\nu u^* z^\alpha, \quad \alpha \in ]0, 1], \quad (4.14)$$

the main feature of which is that it degenerates at the ground, but with a different velocity. The parameter  $z_0$  has the dimensions of a length and it is needed to have a consistent expression for the viscosity. We take as boundary conditions on the bottom and on the top, we write the friction law (4.13) like a standard Robin condition in the form:

$$\begin{cases} C_D u(\mathbf{x}_h, z_{top}) + \mu z_{top}^\alpha \frac{\partial u}{\partial z}(\mathbf{x}_h, z_{top}) = G(\mathbf{x}_h) \\ u(x_h, 0) = 0, \end{cases} \quad (4.15)$$

for  $\mu = z_0^{1-\alpha} C_\nu u^*$  and where  $G = C_D V$ .

It remains to clarify the boundary conditions in the  $\mathbf{x}_h$ -axis. For practical numerical simulations, we have to limit ourselves to a finite computational box  $[0, L_x] \times [0, L_y] \times ]0, z_{top}[$  for given scales  $L_x$  and  $L_y$ , which raises the question of the boundary conditions at the entrance, exit and sides of the computational box, namely

$$\begin{aligned} \Gamma_{in} &= \{0\} \times [0, L_y] \times [0, z_{top}], & \Gamma_{out} &= \{L_x\} \times [0, L_y] \times [0, z_{top}], \\ \Gamma_{s,1} &= [0, L_x] \times \{0\} \times ]0, z_{top}[, & \Gamma_{s,2} &= [0, L_x] \times \{L_y\} \times [0, z_{top}]. \end{aligned}$$

In [59] we have considered at  $\Gamma_{in}$  a fixed given field in  $H_{00}^{1/2}(\Gamma_{in})$  and nonlinear Neumann transparent boundary conditions at  $\Gamma_0$  which yields serious technical issues, both theoretically and numerically. In this paper, we opt for horizontal periodic boundary conditions,

which means that  $u$  must satisfy for all formal derivative operator  $D^n$ ,  $n \geq 0$ ,

$$D^n u(x + L_x, y + L_y, z) = D^n u(x, y, z), \quad \forall \mathbf{x}_h = (x, y) \in \mathbb{R}^2, \quad \forall z \in [0, z_{top}], \quad (4.16)$$

and imposing the invariance for translations implies working with the torus  $\Pi_2$  given by

$$\Pi_2 = ([0, L_x] \times [0, L_y]) / \mathcal{T}_2, \quad (4.17)$$

where  $\mathcal{T}_2$  denotes the set of wave vectors given by:

$$\mathcal{T}_2 = \frac{2\pi}{L_x} \mathbb{Z} \times \frac{2\pi}{L_y} \mathbb{Z} \quad (4.18)$$

This setting is usual in practical numerical simulations for technical convenience but not only since also it eliminates the problem of an infinite domain in the analytical studies. For instance, it was used in [79] for simulating a boundary layer by a RANS turbulent model.

### 4.3 The $0 < \alpha < 1$ case

Throughout this section, we assume that  $0 < \alpha < 1$  and  $\lambda > 0$  is fixed and we study the problem in a simpler setting. In fact, in this case one can reduce the problem to another one in a setting of unweighted Sobolev spaces for which both existence and interpretation of the solution are standard. Later on we will see the possible treatment of the limiting case.

We will use standard Lebesgue and Sobolev spaces and from now on

$$BL = \Pi_2 \times [0, z_{top}], \quad \Gamma_b = \Pi_2 \times \{z = 0\}, \quad \Gamma_{top} = \Pi_2 \times \{z = z_{top}\}. \quad (4.19)$$

We specify the function spaces we are working with, and then we prove the existence of a solution by a viscous regularization. To start with, we observe that any strong solution to Problem (4.4) satisfies the energy balance:

$$\lambda \int_{BL} u^2 + \nu_h \int_{BL} |\nabla_h u|^2 + \mu \int_{BL} z^\alpha (\partial_z u)^2 + C_D \int_{\Gamma_{top}} |u|^2 - \int_{\Gamma_{top}} uG = \int_{BL} fu, \quad (4.20)$$



### 4.3.1 Function spaces

In this section, we define the function spaces to be utilized throughout the paper. We employ the conventional notations within the framework of distributions, adapted to accommodate the mixed boundary conditions under consideration.

- Let  $\mathcal{D}_b(BL)$  denote the space of functions  $u \in C^\infty(BL)$  such that there exists  $\delta = \delta(u) \in ]0, 1[$ , ensuring that the support of  $u$  is contained within  $\Pi_2 \times [\delta(u), 1]$ . This space is endowed with the standard Schwartz topology, characterized by the semi-norms  $\|D^n u\|_\infty$ . In simpler terms,  $\mathcal{D}_b(BL)$  consists of  $C^\infty$  functions that vanish in a neighborhood of the bottom  $\Gamma_b$  of the boundary layer.
- Let  $\mathcal{D}'_b(BL)$  denote the topological dual of  $\mathcal{D}_b(BL)$ , a sort of distributional space that inspires our chosen notation.
- $W_{0,b}^{1,\gamma}(BL)$  ( $\gamma > 1$ ) denotes the closure of  $\mathcal{D}_b(BL)$  for the norm  $\|\nabla_h u\|_{0,\gamma} + \|\partial_z u\|_{0,\gamma}$ , where  $\|\cdot\|_{s,p}$  stands for the usual  $W^{s,p}$  norm.

In particular,

$$W_{0,b}^{1,\gamma}(BL) = \{u \in W^{1,\gamma}(BL), u = 0 \text{ on } \Gamma_b\}. \quad (4.21)$$

According to the energy balance (4.20), we are led to consider the natural weighted space  $V_\alpha$  defined as the closure of  $\mathcal{D}_b(BL)$  equipped with the norm

$$\forall u \in \mathcal{D}_b(BL), \quad \|u\|_{V_\alpha} = (\|\nabla_h u\|_{0,2}^2 + \|z^{\alpha/2} \partial_z u\|_{0,2}^2)^{1/2}. \quad (4.22)$$

**Proposition 4.3.1.** *For all  $\alpha \in [0, 1[$  we have the embedding,*

$$\forall \gamma \in \left]1, \frac{2}{\alpha+1}\right[ , \quad V_\alpha \subset W_{0,b}^{1,\gamma}(BL),$$

and the following inequality holds  $\forall u \in V_\alpha$ ,

$$\|u\|_{1,\gamma} \leq C \left( \frac{2-\gamma}{2-(1+\alpha)\gamma} \right)^{\frac{2}{\gamma}-1} \|u\|_{V_\alpha}, \quad (4.23)$$

for some constant  $C > 0$ .

*Proof.* Let  $\gamma > 0$  and  $\rho > 0$  that will be fixed later. Let  $u \in \mathcal{D}_{0,b}(BL)$ . The Hölder

inequality yields:

$$\begin{aligned} \int_0^1 |\partial_z u|^\gamma dz &= \int_0^1 z^\rho |\partial_z u|^\gamma \frac{1}{z^\rho} dz \\ &\leq \left( \int_0^1 z^{2\rho/\gamma} |\partial_z u|^2 dz \right)^{\gamma/2} \left( \int_0^1 \frac{dz}{z^{2\rho/2-\gamma}} \right)^{\frac{2-\gamma}{2}}, \end{aligned}$$

with  $\frac{\gamma}{2} + \frac{2-\gamma}{2} = 1$ . The second integral is well-defined if and only if  $\frac{2\rho}{2-\gamma} < 1$ . Then, choosing  $\rho$  such that  $\frac{2\rho}{\gamma} = \alpha$  yields the condition  $\gamma < \frac{2}{\alpha+1}$ . Inequality (4.23) follows after an elementary calculation and integration with respect to the  $d\mathbf{x}_h$  variables.  $\square$

It follows from Proposition 4.3.1 and standard reasoning on Sobolev spaces, that functions in  $V_\alpha$  have a trace at  $z = 0$  equal to zero, and also we have the following characterization

$$V_\alpha = \{u \in \mathcal{D}'_b(BL) \text{ s.t. } \nabla_h u \in L^2(BL), z^{\alpha/2} \partial_z u \in L^2(BL), u = 0 \text{ on } \Gamma_b\}. \quad (4.24)$$

### 4.3.2 Weak formulation

Proposition 4.3.1 can be rephrased, to work in standard (unweighted) Sobolev spaces, as follows

$$V_\alpha \subset W_{0,b}^{1,(\frac{2}{1+\alpha})^-} := \bigcap_{1 < \gamma < \frac{2}{1+\alpha}} W_{0,b}^{1,\gamma}(BL), \quad (4.25)$$

which is put in duality with the set

$$W_{0,b}^{1,(\frac{2}{1-\alpha})^+} = \bigcup_{\eta > \frac{2}{1-\alpha}} W_{0,b}^{1,\eta}(BL). \quad (4.26)$$

Throughout the section, we assume that

$$f \in L^2(BL), \quad (4.27)$$

$$G \in L^2(\Gamma_{top}). \quad (4.28)$$

The following definition of a weak solution to Problem (4.4) is motivated by standard rules about integration by parts, combined with the boundary conditions under consideration.

**Definition 4.3.1.** For  $\alpha \in [0, 1[$  we say that  $u \in V_\alpha$  is a weak solution to Problem (4.4),

if  $\forall v \in W_0^{1,(\frac{2}{1-\alpha})^+}$ ,

$$\lambda \int_{\Omega} u v + \nu_h \int_{\Omega} \nabla_h u \cdot \nabla_h v + \mu \int_{\Omega} z^\alpha \partial_z u \partial_z v + \int_{\Gamma_{top}} (C_D u - G)v = \int_{\Omega} f v. \quad (4.29)$$

**Remark 4.3.1.** Note that all the terms in the integrals written in (4.29) are well-defined. However, the solution  $u$  cannot be a priori taken as test function, which is an issue. As a consequence, we are not able to prove the uniqueness of this solution, even if the problem is linear.

Before all, we notice that combining the energy balance (4.20) and (4.23) with standard calculus inequalities, yields for any  $1 < \gamma < \frac{2}{1+\alpha}$  the following estimate in  $W_{0,b}^{1,\gamma}(BL)$ , satisfied by any given regular solution  $u$  to the variational problem (4.29):

$$\|u\|_{0;1,\gamma} \leq \frac{C_\gamma}{\inf\{\nu_h, \mu\}} (\|f\|_{0,2} + \|G\|_{1-\frac{1}{\gamma},\gamma;\Gamma_{top}}), \quad (4.30)$$

where  $C_\gamma \rightarrow \infty$  as  $\gamma \rightarrow \frac{2}{1+\alpha}$ .

The aim of the rest of this section is proving the following existence result.

**Theorem 4.3.1.** Problem (4.4) admits a weak solution  $u \in V_\alpha$ . Moreover, the solution satisfies the energy inequality

$$\lambda \int_{BL} u^2 + \nu_h \int_{BL} |\nabla_h u|^2 + \mu \int_{BL} z^\alpha |\partial_z u|^2 + C_D \int_{\Gamma_{top}} |u|^2 - \int_{\Gamma_{top}} G u \leq \int_{BL} f u \quad (4.31)$$

**Remark 4.3.2.** The existence of a solution still holds when  $\lambda = 0$ .

**Remark 4.3.3.** Assumption (4.28) about  $G$  is not optimal and could be weakened by taking for instance  $G \in W^{-s,p}(\Gamma_{top})$  for some  $s > 0$ ,  $p > 1$  depending on  $\alpha$ , so that it is put in duality with traces on  $\Gamma_{top}$  of test functions in  $W_{0,b}^{1,(\frac{2}{1-\alpha})^+}$ .

If we still get in this case the existence of a weak solution, we do not know whether the energy inequality (4.31) still holds, or even if it makes sense because of the boundary term  $\int_{\Gamma_{top}} uG$ . It seems that there is an interesting theoretical issue at this point.

**Remark 4.3.4.** The result still holds if one takes the Navier law (4.13) for a given  $V \in L^2(\Gamma_{top})$ ,  $\alpha > 0$ .

### 4.3.3 Viscous regularization and proof of the existence result

From now we set

$$\gamma^* = \frac{2}{1 + \alpha}, \quad (4.32)$$

since this value is the critical one for the embedding of weighted Sobolev spaces.

As a technical tool to prove existence of weak solutions, we regularize Problem (4.4) by adding a viscous term in the  $z$ -direction, which means that we consider the following problem, for a given  $\varepsilon > 0$ :

$$\begin{cases} \lambda u - \nu_h \Delta_h u - \mu \partial_z (z^\alpha \partial_z u) - \varepsilon \partial_{zz}^2 u = f, \\ C_D u + \mu z_{top}^\alpha \frac{\partial u}{\partial z} = G \quad \text{on } \Gamma_{top}, \\ u = 0 \quad \text{on } \Gamma_b, \end{cases} \quad (4.33)$$

**Definition 4.3.2.** *Let  $\varepsilon > 0$ . We say that  $u_\varepsilon \in W_{0,b}^{1,2}(BL)$  is a weak solution to Problem (4.33), if  $\forall v \in W_{0,b}^{1,2}(BL)$ ,*

$$\begin{aligned} \lambda \int_{\Omega} u_\varepsilon v + \nu_h \int_{\Omega} \nabla_h u_\varepsilon \cdot \nabla_h v + \mu \int_{\Omega} z^\alpha \partial_z u_\varepsilon \partial_z v + \\ + \varepsilon \int_{\Omega} \partial_z u_\varepsilon \partial_z v + C_D \int_{\Gamma_{top}} u_\varepsilon v - \int_{\Gamma_{top}} G v = \int_{\Omega} f v. \end{aligned} \quad (4.34)$$

The existence and uniqueness of a weak solution to Problem (4.34) is straightforward by the Lax-Milgram theorem. Moreover, as  $u_\varepsilon$  can be taken as test function, it satisfies the following energy balance(equality):

$$\underbrace{\lambda \int_{BL} u_\varepsilon^2}_{I_{1,\varepsilon}} + \underbrace{\nu_h \int_{BL} |\nabla_h u_\varepsilon|^2 + \int_{BL} (\varepsilon + \mu z^\alpha) |\partial_z u_\varepsilon|^2}_{I_{2,\varepsilon}} + \underbrace{C_D \int_{\Gamma_{top}} |u_\varepsilon|^2}_{I_{3,\varepsilon}} - \underbrace{\int_{\Gamma_{top}} G u_\varepsilon}_{I_{4,\varepsilon}} = \underbrace{\int_{BL} f u_\varepsilon}_{I_{5,\varepsilon}}. \quad (4.35)$$

From this, we are able to finish the proof of Theorem 4.3.1 by taking the limit in (4.34) when  $\varepsilon \rightarrow 0$ .

We deduce from (4.35) and standard calculus inequalities that the family  $(u_\varepsilon)_{\varepsilon > 0}$  is uniformly bounded in  $V_\alpha$ , as well as in  $W_{0,b}^{1,\gamma}(BL)$  for any  $1 < \gamma < \gamma^*$  say

$$\|u_\varepsilon\|_{0,1,\gamma} \leq C(\gamma, G, f, \nu_h, \mu). \quad (4.36)$$

Arguing as in [23, Chapter 7], we can extract a (sub)sequence  $(u_{\epsilon_n})_{n \in \mathbb{N}}$  that weakly converges to some  $u \in V_\alpha$ , which is also weakly converging in  $W_{0,b}^{1,\gamma}(BL)$  for all  $1 < \gamma < \gamma^*$ , and which is strongly converging in  $L^2(BL)$ .

Moreover, by the trace theorem and the Sobolev theorem, the sequence  $(\epsilon_n)$  can be chosen such that in addition  $(tr[u_{\epsilon_n}])_{n \in \mathbb{N}}$  strongly converges to  $tr[u]$  in  $L^2(\Gamma_{top})$ . Let  $1 < \gamma < \gamma^*$  and take as test function

$$v \in W_{0,b}^{1,\gamma'}(BL) \subset W_{0,b}^{1,(\frac{2}{1-\alpha})^+} \subset W_{0,b}^{1,2}(BL),$$

in formulation (4.34). We have to take the limit in the various terms of (4.34), which we do step by step, starting with the diffusion term.

Let  $w \in V_\alpha$ , and let the linear form  $\Psi_v$  given by

$$\langle \Psi_v, w \rangle = \nu_h \int_{\Omega} \nabla_h w \cdot \nabla_h v \, dx + \mu \int_{\Omega} z^\alpha \partial_z w \partial_z v \, dx.$$

By the Hölder inequality we obtain

$$|\langle \Psi_v, w \rangle| \leq \sup\{\nu_h, \mu z_{top}^\alpha\} \|w\|_{0;1,\gamma} \|v\|_{0;1,\gamma'},$$

therefore  $\Psi_v \in W_{0,b}^{1,\gamma}(BL)'$ , which leads to

$$\lim_{n \rightarrow \infty} \langle \Psi_v, u_{\epsilon_n} \rangle = \langle \Psi_v, u \rangle = \nu_h \int_{\Omega} \nabla_h u \cdot \nabla_h v + \mu \int_{\Omega} z^\alpha \partial_z u \partial_z v.$$

Moreover,

$$\left| \epsilon_n \int_{\Omega} \partial_z u_{\epsilon_n} \partial_z v \right| \leq \epsilon_n \|u_{\epsilon_n}\|_{0;1,\gamma} \|v\|_{0;1,\gamma'} \leq C(\gamma, G, f, \nu_h, \mu) \epsilon_n \|v\|_{0;1,\gamma'},$$

giving

$$\lim_{n \rightarrow \infty} \epsilon_n \int_{\Omega} \partial_z u_{\epsilon_n} \partial_z v = 0.$$

In addition, considering the properties of the sequence  $(u_{\epsilon_n})_{n \in \mathbb{N}}$ , we have

$$\lim_{n \rightarrow \infty} \lambda \int_{\Omega} u_{\epsilon_n} v = \lambda \int_{\Omega} u v, \quad \lim_{n \rightarrow \infty} C_D \int_{\Gamma_{top}} u_{\epsilon_n} v = C_D \int_{\Gamma_{top}} u v.$$

Therefore  $u$  satisfies (4.29).

It remains to show that the energy inequality (4.31) holds. Starting from (4.35), we

have on one hand

$$\begin{aligned} \lim_{n \rightarrow \infty} I_{1,\varepsilon_n} &= \lambda \int_{\Omega} |u|^2, & \lim_{n \rightarrow \infty} I_{3,\varepsilon_n} &= C_D \int_{\Gamma_{top}} |u|^2, \\ \lim_{n \rightarrow \infty} I_{4,\varepsilon_n} &= \int_{\Gamma_{top}} u G, & \lim_{n \rightarrow \infty} I_{5,\varepsilon_n} &= \int_{BL} f u, \end{aligned}$$

and on the other hand by lower semi-continuity

$$\nu_h \int_{BL} |\nabla_h u|^2 + \mu \int_{BL} z^\alpha |\partial_z u|^2 \leq \liminf_{n \rightarrow \infty} I_{2,\varepsilon_n},$$

hence the energy balance (4.31), which concludes the proof.

**Remark 4.3.5.** *The solution we exhibit is obtained by viscous regularization. We also can think to directly get a solution by applying the Lax-Milgram theorem in the space  $V_\alpha$ , which requires a different approach that we have voluntarily skipped here. As already stressed in Remark 4.3.1, we do not know if they are completely equivalent; we conjecture that this is the case.*

## 4.4 The case $\alpha = 1$

We now consider the system:

$$\begin{cases} \lambda u - \nu_h \Delta_h u - \mu \partial_z (z \partial_z u) = f & \text{in } BL, \\ C_D u + \mu z_{top} \frac{\partial u}{\partial z} = G & \text{on } \Gamma_{top}. \end{cases} \quad (4.37)$$

As we shall see, the best we can get is an estimate in an  $H^{1/2}$ -like space near the bottom  $\Gamma_b$ . Therefore, we are not able to define the trace of such functions at  $\Gamma_b$ . This is why we do not impose any boundary condition there, contrary to what we did for the case  $0 < \alpha < 1$ . Nevertheless, we still impose  $u|_{\Gamma_b} = 0$  in the numerical code, since the finite element formulation yields additional numerical dissipation that sufficiently regularizes the system.

In order to derive this  $H^{1/2}$  estimate, in the same spirit as in [2], we first need the following version of Nečas Lemma, the proof of which will be complete by the end of this section. To define the function spaces we will utilize, let  $\mathcal{D}(BL)$  denote the space of functions  $u \in C^\infty(BL)$  such that there exists  $\delta = \delta(u) \in ]0, 1[$ , ensuring that the support of  $u$  is contained within  $\Pi_2 \times [\delta(u), 1 - \delta(u)]$ . This space is endowed with the standard

Schwartz topology, characterized by the semi-norms  $\|D^n u\|_\infty$ . In simpler terms,  $\mathcal{D}(BL)$  consists of  $C^\infty$  functions that vanish in the vicinity of both the bottom and the top of the boundary layer. We use  $\mathcal{D}'(BL)$  to denote the corresponding space of distributions. Additionally, so far no risk of confusion arises, we still set

$$H_0^1(BL) = \{u \in H^1(BL), u = 0 \text{ on } \Gamma_b \cup \Gamma_{top}\}, \quad (4.38)$$

and  $H^{-1}(BL) \subset \mathcal{D}'(BL)$  is the topological dual of  $H_0^1(BL)$ .

**Lemma 4.4.1.** *Let  $v \in \mathcal{D}'(BL)$  such that  $v \in H^{-1}(BL)$  and also  $\nabla v \in H^{-1}(BL)$ . Then  $v \in L^2(BL)$  and one has:*

$$\|v\|_{L^2(BL)} \leq \|\nabla v\|_{H^{-1}(BL)} + \|v\|_{H^{-1}(BL)}. \quad (4.39)$$

The main consequence of Lemma 4.4.1 is the following.

**Corollary 1.** *Let  $V_1$  denotes the space:*

$$V_1 = \{u \in \mathcal{D}'_b(BL) \text{ s.t. } u \in L^2(BL), \nabla_h u \in L^2(BL), z^{1/2} \partial_z u \in L^2(BL)\}, \quad (4.40)$$

*Then*

$$V_1 \subset H^{1/2}(BL). \quad (4.41)$$

The space  $V_1$  is the space that naturally suits Problem (4.37) according to the energy balance (4.20) that still holds when  $\alpha = 1$ . It must be stressed that:

- showing that any solution of problem (4.37) is well-defined in  $V_1$  by using (4.20) requires  $\lambda > 0$ ;
- we have

$$V_1 \subset H^1(\Pi_2 \times [\delta, z_{top}]) \quad (4.42)$$

for any  $\delta > 0$ , which makes consistent the boundary condition at  $\Gamma_{top}$ .

**Definition 4.4.1.** *We say that  $u \in V_1$  is a very weak solution to (4.37) if for all  $v \in V_1$ ,*

$$\lambda \int_{\Omega} u v + \nu_h \int_{\Omega} \nabla_h u \cdot \nabla_h v + \mu \int_{\Omega} z \partial_z u \partial_z v + \int_{\Gamma_{top}} (C_D u - G) v = \int_{\Omega} f v. \quad (4.43)$$

The main result of this section is the following.

**Theorem 4.4.1.** *Let  $f \in L^2$  and  $G \in L^2$ . Then, there exists a very weak solution  $u \in V_1 \subset H^{1/2}(BL)$  of problem (4.37), which satisfies*

$$\|v\|_{V_1} \leq C(\|f\|_{2,BL} + \|G\|_{2,\Gamma_{top}})$$

We argue by approximation by using Fourier series expansions which allows for some explicit and direct computations. This is the reason why, before proving 4.4.1 and Theorem 4.4.1, we need to prove a bunch of convergence results about Fourier series in this context of mixed boundary conditions, periodic in the  $\mathbf{x}_h$ -axis but not in the  $z$ -axis. This will be the purpose of the next sections.

#### 4.4.1 Framework

Let  $\mathcal{T}_2 = \frac{2\pi}{L_x}\mathbb{Z} \times \frac{2\pi}{L_y}\mathbb{Z}$  and let  $\mathbf{k} = (k_x, k_y) \in \mathcal{T}_2$  any wave vector. In the following, we set

$$\mathcal{T}_{2,n} := \left\{ \mathbf{k} = (k_x, k_y) \in \mathcal{T}_2, |\mathbf{k}| \leq 2\pi n \sqrt{1/L_x^2 + 1/L_y^2} \right\}.$$

Let  $u \in L^1(BL)$  and let  $c_{\mathbf{k}} = c_{\mathbf{k}}(u; z)$  denotes the ‘‘horizontal’’ Fourier’s coefficient at the wave vector  $\mathbf{k}$ , namely

$$c_{\mathbf{k}}(u; z) := \frac{1}{\rho} \int_{\Pi_2} u(\mathbf{x}_h, z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h,$$

where  $\rho = \sqrt{L_x^2 + L_y^2}$ . Let  $u_n$  be its partial sum of the Fourier series defined by

$$u_n(\mathbf{x}_h, z) := \sum_{\mathbf{k} \in \mathcal{T}_{2,n}} c_{\mathbf{k}}(z) e^{i\mathbf{k} \cdot \mathbf{x}_h}. \quad (4.44)$$

A natural question is what assumptions are needed about  $u$  to prove the convergence of the sequence  $(u_n)_{n \in \mathbb{N}}$  toward  $u$  (in some given topology). Before tackling this question, we consider the following spaces  $\mathcal{D}(]0, z_{top}[) := C_0^\infty(]0, z_{top}[)$  and

$$\mathcal{D}_n(BL) = \left\{ u = u_n(\mathbf{x}_h, z) = \sum_{\mathbf{k} \in \mathcal{T}_{2,n}} c_{\mathbf{k}}(z) e^{i\mathbf{k} \cdot \mathbf{x}_h}, c_{\mathbf{k}} \in \mathcal{D}(]0, z_{top}[) \right\}, \quad (4.45)$$

$$V_n = \left\{ u = u_n(\mathbf{x}_h, z) = \sum_{\mathbf{k} \in \mathcal{T}_{2,n}} c_{\mathbf{k}}(z) e^{i\mathbf{k} \cdot \mathbf{x}_h}, c_{\mathbf{k}} \in H_0^1(]0, z_{top}[) \right\}, \quad (4.46)$$

$$V_n^{-1} = \left\{ v = \sum_{\mathbf{k} \in \mathcal{T}_{2,n}} \psi_{\mathbf{k}}(z) e^{i\mathbf{k} \cdot \mathbf{x}_h}, \psi_{\mathbf{k}} \in H^{-1}(]0, z_{top}[) \right\}. \quad (4.47)$$



We will prove the following result

**Lemma 4.4.2.** *Let  $u \in H_0^1(BL)$  and  $u_n$  be given by (4.44). Then, for all  $n$ ,  $u_n \in V_n$  and  $(u_n)_{n \in \mathbb{N}}$  converges to  $u$  in  $H_0^1(BL)$ .*

We first notice that the following isometries hold:

$$\mathcal{D}_n(BL) \simeq \mathcal{D}([0, z_{top}]^n), \quad V_n \simeq H_0^1([0, z_{top}]^n), \quad V_n^{-1} \simeq H^{-1}([0, z_{top}]^n). \quad (4.48)$$

Therefore, from the standard Nečas inequality (see [35]), we easily get the following result. Notice that the explicit computations allow us to prove that the right hand side bounds the left hand side with multiplicative coefficient equal to one, independently of  $n \in \mathbb{N}$ .

**Lemma 4.4.3.** *Let  $u_n \in \mathcal{D}'_n(BL)$  such that  $u_n, \nabla u_n \in V_n^{-1}$ . Then  $u_n \in L^2(BL)$  and one has*

$$\|u_n\|_{0,2} \leq \|u_n\|_{1,-1} + \|\nabla u_n\|_{1,-1} \quad (4.49)$$

It remains to pass to the limit in (4.49) when  $n \rightarrow \infty$  to prove Lemma 4.4.1. We first must prove Lemma 4.4.2, which will be done step by step in the following subsections.

## 4.4.2 $L^2$ convergence

We prove in this section the following convergence result, which is well-known but we give a self-contained treatment of all the results in this section.

**Lemma 4.4.4.** *Let  $u \in H^2(BL)$ ,  $u_n$  its Fourier's series expansion as given by (4.44). Then  $u_n \rightarrow u$  in  $L^2(BL)$  as  $n \rightarrow \infty$ .*

*Proof. Step 1. Estimate of the  $c_{\mathbf{k}}$ .* Let  $\mathbf{k} \in \mathcal{T}_{2,n} \setminus \{\mathbf{0}\}$ . By Fubini's Theorem and two integration by parts, we get

$$\begin{aligned} c_{\mathbf{k}}(u; z) &= \frac{1}{\rho k_x^2} \int_{\Pi_2} \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_h, z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h, \quad \text{if } k_x \neq 0, \\ c_{\mathbf{k}}(u; z) &= -\frac{1}{\rho k_y^2} \int_{\Pi_2} \frac{\partial^2 u}{\partial y^2}(\mathbf{x}_h, z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h, \quad \text{if } k_y \neq 0. \end{aligned} \quad (4.50)$$

Consequently, we get

$$|c_{\mathbf{k}}(u; z)| \leq \frac{1}{\rho} \inf \left\{ \frac{1}{k_x^2}, \frac{1}{k_y^2} \right\} \int_{\Pi_2} |\nabla_h u(\mathbf{x}_h, z)|^2 d\mathbf{x}_h \leq \frac{C}{\rho |\mathbf{k}|^2} \|\nabla_h u(\cdot, z)\|_{L^2(\Pi_2)}^2. \quad (4.51)$$

Therefore, as  $u \in H^1(BL)$ ,  $z \rightarrow c_{\mathbf{k}}(u; z) \in L^2([0, z_{top}])$  and is finite almost everywhere.

*Step 2. Convergence.* We deduce from classical results that for almost all  $z \in ]0, z_{top}[$ ,

$$u_n(\cdot, z) \rightarrow u(\cdot, z) \quad \text{in } L^2(\Pi_2). \quad (4.52)$$

Put another way:

$$\varepsilon_n(z) = \int_{\Pi_2} |u_n(\mathbf{x}_h, z) - u(\mathbf{x}_h, z)|^2 d\mathbf{x}_h \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.53)$$

Moreover, by the previous step,

$$0 \leq \varepsilon_n \leq \sum_{|\mathbf{k}| \geq \lambda n} |c_{\mathbf{k}}(u; z)|^2 \leq \frac{C}{\rho} \|\nabla_h u\|_{L^2(\Pi_2)}^2 \left( \sum_{|\mathbf{k}| \geq \lambda n} \frac{1}{|\mathbf{k}|^4} \right) = R_n \|\nabla_h u\|_{L^2(\Pi_2)}^2, \quad (4.54)$$

giving

$$\|u - u_n\|_{0,2}^2 = \int_0^{z_{top}} \varepsilon_n(z) dz \leq R_n \|\nabla_h u\|_{0,2}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4.55)$$

which concludes the proof.  $\square$

### 4.4.3 Differentiability of the coefficient $c_{\mathbf{k}}(u; z)$

**Lemma 4.4.5.** *Let  $u \in H^1(BL)$  such that  $\frac{\partial u}{\partial z} \in H^1(BL)$ . Then  $c_{\mathbf{k}}(u; \cdot) \in H^1(]0, z_{top}[)$  and one has for almost all  $z \in ]0, z_{top}[$ ,*

$$\frac{d}{dz} c_{\mathbf{k}}(u; z) = \int_{\Pi_2} \frac{\partial u}{\partial z}(\mathbf{x}_h, z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h. \quad (4.56)$$

*Proof.* Let us write

$$\frac{\partial u}{\partial z}(\mathbf{x}_h, z) = \frac{\partial u}{\partial z}(\mathbf{x}_h, 0) + \int_0^{z_{top}} \frac{\partial^2 u}{\partial z^2}(\mathbf{x}_h, z') dz'. \quad (4.57)$$

By the trace theorem,

$$\frac{\partial u}{\partial z}(\mathbf{x}_h, 0) \in H^{\frac{1}{2}}(\Gamma_b) \hookrightarrow L^2(\Gamma_b) \simeq L^2(\Pi_2)$$

hence,  $\forall z \in ]0, z_{top}[$ ,

$$\left| \frac{\partial u}{\partial z}(\mathbf{x}_h, z) \right| \leq \left| \frac{\partial u}{\partial z}(\mathbf{x}_h, 0) \right| + \int_0^{z_{top}} \left| \frac{\partial^2 u}{\partial z^2}(\mathbf{x}_h, z') \right| dz' \in L^1(\Pi_2).$$

Therefore, formula (4.56) is a classical consequence of the Lebesgue Theorem.  $\square$

**Lemma 4.4.6.** *Let  $u \in H_0^1(BL)$ . Then  $c_{\mathbf{k}}(u; \cdot) \in H_0^1(]0, z_{top}[)$  and (4.56) holds a.e in  $]0, z_{top}[$ .*

*Proof. Step 1. Approximations.* Let  $\varepsilon > 0$ , and  $u_\varepsilon \in H_0^1(BL) \cap H^2(BL)$  such that

$$-\varepsilon \Delta u_\varepsilon + u_\varepsilon = u \tag{4.58}$$

According to standard results about the Helmholtz equation, we know that that  $u_\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0$ , strongly in  $L^2(BL)$ , weakly in  $H_0^1(BL)$ . Moreover, by Lemma 4.4.5, we also know that

$$\frac{d}{dz} c_{\mathbf{k}}(u_\varepsilon; z) = \int_{\Pi_2} \frac{\partial u_\varepsilon}{\partial z}(\mathbf{x}_h, z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h. \tag{4.59}$$

as well as  $c_{\mathbf{k}}(u_\varepsilon; z) \rightarrow c_{\mathbf{k}}(u; z)$  in  $L^2(]0, z_{top}[)$ .

*Step 2. Derivative in the sense of the distributions.* Let  $\varphi \in \mathcal{D}(]0, z_{top}[)$ . The starting point is the identity

$$-\int_0^{z_{top}} \frac{d\varphi}{dz}(z) c_{\mathbf{k}}(u_\varepsilon; z) dz = \int_0^{z_{top}} \frac{d}{dz} c_{\mathbf{k}}(u_\varepsilon; z) \varphi(z) dz \tag{4.60}$$

From the results of step 1, we already know that

$$\int_0^{z_{top}} \frac{d\varphi}{dz}(z) c_{\mathbf{k}}(u_\varepsilon; z) dz \xrightarrow{\varepsilon \rightarrow 0} \int_0^{z_{top}} \frac{d\varphi}{dz}(z) c_{\mathbf{k}}(u; z) dz. \tag{4.61}$$

We must pass to the limit in the r.h.s of (4.60). By (4.56), we have

$$\int_0^{z_{top}} \frac{d}{dz} c_{\mathbf{k}}(u_\varepsilon; z) \varphi(z) dz = \int_{BL} \frac{\partial u_\varepsilon}{\partial z}(\mathbf{x}_h, z) \varphi(z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h dz.$$

Therefore, as  $u_\varepsilon \rightarrow u$  weakly in  $H_0^1(BL)$ ,

$$\int_{BL} \frac{\partial u_\varepsilon}{\partial z}(\mathbf{x}_h, z) \varphi(z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h dz \xrightarrow{\varepsilon \rightarrow 0} \int_{BL} \frac{\partial u}{\partial z}(\mathbf{x}_h, z) \varphi(z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h dz = \int_0^{z_{top}} \varphi(z) \int_{\Pi_2} \frac{\partial u}{\partial z}(\mathbf{x}_h, z) e^{-i\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h dz. \tag{4.62}$$

By combining (4.60), (4.61) and (4.62), we see that (4.56) holds in the sense of the distribution, which concludes the proof because

$$z \rightarrow \int_{\Pi_2} \frac{\partial u}{\partial z}(\mathbf{x}_h, z) e^{i-\mathbf{k} \cdot \mathbf{x}_h} d\mathbf{x}_h \in L^2(]0, z_{top}[.$$

□

By using the same argument, we also have the following, where we note  $P_n u = u_n$ .

**Lemma 4.4.7.** *Let  $u \in L^2(BL)$ . Then  $P_n u$  converges to  $u$  in  $L^2(BL)$  as  $n \rightarrow \infty$ .*

*Proof.* Let  $u \in L^2(BL)$  and  $u_\varepsilon \in H^2(BL)$  given by (4.58). Then, we know from Lemma 4.4.4 that for any  $\varepsilon > 0$  be fixed,  $P_n u_\varepsilon$  converges to  $u_\varepsilon$  in  $L^2(BL)$ . We write

$$P_n u - u = P_n(u - u_\varepsilon) + (P_n u_\varepsilon - u_\varepsilon) + (u - u_\varepsilon).$$

Then, since  $\|P_n\| \leq 1$ , we get

$$\|P_n u - u\|_{0,2} \leq 2\|u - u_\varepsilon\|_{0,2} + \|P_n u_\varepsilon - u_\varepsilon\|_{0,2}.$$

The rest of the proof is straightforward. □

#### 4.4.4 $H^1$ Convergence

We are now able to prove Lemma 4.4.2, stating the convergence of the sequence  $(u_n)_{n \in \mathbb{N}}$  to  $u$  in  $H^1(BL)$ . We already know from Lemma 4.4.7 that  $u_n = P_n u \xrightarrow[n \rightarrow \infty]{} u$  in  $L^2(BL)$ . Let  $\mathcal{F}_n = \mathcal{F}_n(\mathbf{x}_h)$  denotes the Fejer kernel over  $\Pi_2$ . Then, for a.e  $z \in ]0, z_{top}[$ ,  $u_n$  is given by

$$u_n(\mathbf{x}_h, z) = \int_{\Pi_2} \mathcal{F}_n(\mathbf{y}_h) u(\mathbf{x}_h + \mathbf{y}_h, z) d\mathbf{y}_h = \mathcal{F}_n(\cdot) \star u(\cdot, z)(\mathbf{x}_h). \quad (4.63)$$

As the Lebesgue measure over  $BL$  is  $\sigma$ -finite, we deduce from the Lebesgue-Fubini theorem that for a.e  $z \in ]0, z_{top}[$ ,  $u(\cdot, z) \in H^1(\Pi_2)$ . Therefore, always a.e  $z \in ]0, z_{top}[$ ,

$$\nabla_h u_n(\mathbf{x}_h, z) = \mathcal{F}_n(\cdot) \star \nabla_h u(\cdot, z)(\mathbf{x}_h), \quad (4.64)$$

and by standard results,

$$\varepsilon_n(z) = \int_{\Pi_2} |\nabla_h u_n(\mathbf{x}_h, z) - \nabla_h u(\mathbf{x}_h, z)|^2 d\mathbf{x}_h \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.65)$$

Moreover, by the Young inequality,

$$\begin{aligned} 0 \leq \varepsilon_n(z) &\leq (\|\mathcal{F}_n\|_{\Pi_2;0,1} \|\nabla_h u(\cdot, z)\|_{\Pi_2;0,2} + \|\nabla_h u(\cdot, z)\|_{\Pi_2;0,2})^2 \\ &\leq 4\|\nabla_h u(\cdot, z)\|_{\Pi_2;0,2}^2 \in L^1(]0, z_{top}[), \end{aligned} \quad (4.66)$$

where  $\|\cdot\|_{\Pi_2;s,p}$  denotes the usual norm of the Sobolev space  $W^{s,p}(\Pi_2)$ . Therefore, by the Lebesgue Theorem, we get

$$\int_0^{z_{top}} \varepsilon_n(z) = \int_{BL} |u_n(\mathbf{x}_h, z) - u(\mathbf{x}_h, z)|^2 d\mathbf{x}_h dz \xrightarrow{n \rightarrow \infty} 0, \quad (4.67)$$

which, put in another way, yields  $\nabla_h u_n \xrightarrow{n \rightarrow \infty} \nabla_h u$  in  $L^2(BL)$ . Similarly, we deduce from Lemma 4.4.6 that

$$\frac{\partial u_n}{\partial z}(\mathbf{x}_h, z) = \mathcal{F}_n(\cdot) \star \frac{\partial u}{\partial z}(\cdot, z)(\mathbf{x}_h), \quad (4.68)$$

and by an analysis similar to the previous one, we can conclude that

$$\frac{\partial u_n}{\partial z} \xrightarrow{n \rightarrow \infty} \frac{\partial u}{\partial z},$$

in  $L^2(BL)$ , which finishes the proof of Lemma 4.4.2.

#### 4.4.5 Interpolation spaces

As a consequence of Lemma 4.4.2 and Lemma 4.4.7, combined with the interpolation Theorem proved by J.-L. Lions and E. Magenes [64], we have the following.

**Lemma 4.4.8.** *Let  $s \in [0, 1]$ , and let*

$$H^s(BL) = [L^2(BL), H^1(BL)]_s = \mathbb{D}(\nabla^s).$$

*Then,  $u = \sum_{\mathbf{k} \in \mathcal{T}_2} c_{\mathbf{k}}(z) e^{i\mathbf{k} \cdot \mathbf{x}_h} \in H^s(BL)$  if and only if  $\forall \mathbf{k} \in \mathcal{T}_2$ ,  $c_{\mathbf{k}} \in H^s(]0, z_{top}[)$  and*

$$\|u\|_{s,2}^2 = \sum_{\mathbf{k} \in \mathcal{T}_2} \left( |\mathbf{k}|^{2s} \int_0^{z_{top}} |c_{\mathbf{k}}(z)|^2 dz + \|c_{\mathbf{k}}\|_{s,2}^2 \right) < \infty.$$

We now need the definition of the space  $H_{00}^{1/2}(BL)$  suited to our geometry, and its dual space. Let  $\rho = \rho(z)$  be a  $C^\infty$  non negative function on  $]0, z_{top}[$ , and such that

$$\lim_{z \rightarrow 0} \frac{\rho(z)}{z} = \lim_{z \rightarrow z_{top}} \frac{\rho(z)}{z_{top} - z} = 1.$$

Then in our case,

$$H_{00}^{1/2}(BL) = \{u \in H^{1/2}(BL) \text{ s.t. } \rho^{-1/2}u \in L^2(BL)\},$$

equipped with the norm

$$\|u\|_{H_{00}^{1/2}} = (\|u\|_{1/2,2}^2 + \|\rho^{-1/2}u\|_{0,2}^2)^{\frac{1}{2}}.$$

According to Lemma 4.4.8, we have the following

**Lemma 4.4.9.** *Let  $u = \sum_{\mathbf{k} \in \mathcal{T}_2} c_{\mathbf{k}}(z)e^{i\mathbf{k} \cdot \mathbf{x}_h}$ . Then,  $u \in H_{00}^{1/2}(BL)$  if and only if  $\forall \mathbf{k} \in \mathcal{T}_2$ ,  $c_{\mathbf{k}} \in H_{00}^{1/2}(]0, z_{top}[)$  and*

$$\|u\|_{H_{00}^{1/2}}^2 = \|u\|_{1/2,2}^2 + \sum_{\mathbf{k} \in \mathcal{T}_2} \int_0^{z_{top}} \frac{|c_{\mathbf{k}}(z)|^2}{\rho(z)} dz < \infty.$$

Now following [64], let us consider the linear operator  $\Lambda = \nabla^{-1} : L^2 \rightarrow H^{-1}$ , defined such that

$$\forall u, v \in L^2, \quad (u, v)_2 = (\Lambda u, \Lambda v)_{-1}, \quad (4.69)$$

with natural notations to denote the various scalar products. Then, the space  $[H^{-1}, L^2]_{1/2}$  is the domain of  $\Lambda^{1/2}$ , and one has, according to the above,

$$[H^{-1}, L^2]_{1/2} = (H_{00}^{1/2}(BL))'. \quad (4.70)$$

In order to complete the interpolation tool box, let us consider  $W^{-1}$  as the closure of  $C^\infty(BL)$  subjected to the norm

$$\|v\|_{W^{-1}} = \|\nabla v\|_{H^{-1}(BL)} + \|v\|_{H^{-1}(BL)},$$

$W^0$  is the closure with respect to the norm

$$\|\nabla v\|_{L^2(BL)} + \|v\|_{L^2(BL)}.$$

Note that  $W^0 \approx H^1(BL)$ . The question is the characterization of the interpolation space  $[W^{-1}, W^0]_{1/2}$ . From (4.69), we have

$$\forall u, v \in W^0, \quad (u, v)_2 + (\nabla u, \nabla v)_2 = (\Lambda u, \Lambda v)_{-1} + (\Lambda \nabla u, \Lambda \nabla v)_{-1},$$

which allows to characterize the space  $[W^{-1}, W^0]_{1/2}$  thanks to (4.70) by its norm given by:

$$\|v\|_{[W^{-1}, W^0]_{1/2}} = \|v\|_{(H_0^{1/2}(BL))'} + \|\nabla v\|_{(H_0^{1/2}(BL))'}. \quad (4.71)$$

#### 4.4.6 Nečas Lemma and consequences

We are now able to prove in this section Lemma 4.4.1, which we recall that it states the following inequality,

$$\|v\|_{L^2(BL)} \leq \|\nabla v\|_{H^{-1}(BL)} + \|v\|_{H^{-1}(BL)}. \quad (4.72)$$

satisfied by any distribution  $v \in \mathcal{D}'(BL)$  such that  $v, \nabla v \in H^{-1}(BL)$ , the dual of the space  $H_0^1(BL)$  as defined by (4.38). Then, we will prove Corollary 1, namely

$$V_1 \subset H^{\frac{1}{2}}(BL), \quad (4.73)$$

with continuous injection, where  $V_1$  is defined by (4.40), equipped with the norm

$$\|u\|_{V_1} = \|u\|_{L^2(BL)} + \|\nabla_h u\|_{L^2(BL)} + \|z^{1/2} \partial_z u\|_{L^2(BL)}.$$

In particular, being  $V_1$  a closed subspace of  $H^{1/2}(BL)$  turns out to be an Hilbert spaces, allowing to use all the machinery of complete vector spaces, which makes possible the proof of Theorem 4.4.1.

*Proof of Lemma 4.4.1.* Let  $v \in \mathcal{D}'(BL)$  such that  $v, \nabla v \in H^{-1}$ , and  $v_n, \nabla v_n \in V_n^{-1}$  be given by

$$\forall u \in V_n, U \in V_n^3, \quad \langle v_n, u \rangle = \langle v, u \rangle, \quad \langle \nabla v_n, U \rangle = \langle \nabla v, U \rangle,$$

where  $V_n^{-1}$  is defined by (4.47). We have, for all  $n \in \mathbb{N}$ ,

$$\|\nabla v_n\|_{H^{-1}(BL)} \leq \|\nabla v\|_{H^{-1}(BL)}, \quad \|v_n\|_{H^{-1}(BL)} \leq \|v\|_{H^{-1}(BL)},$$

and note that (by explicit computation) it easily follows that the constants on the right-hand side are equal to 1, independently of  $n \in \mathbb{N}$ . Therefore, according to inequality (4.49),

$$\|v_n\|_{L^2(BL)} \leq \|\nabla v\|_{H^{-1}(BL)} + \|v\|_{H^{-1}(BL)}. \quad (4.74)$$

Then, the sequence  $(v_n)_{n \in \mathbb{N}}$  is bounded in  $L^2(BL)$ , and we can extract a subsequence (still denoted by  $(v_n)_{n \in \mathbb{N}}$ ) that weakly converges to some  $\tilde{v}$ , which is equal to  $v$  in  $H^{-1}(BL)$ , therefore in  $L^2(BL)$ , which yields  $v \in L^2(BL)$  and

$$\|v\|_{L^2(BL)} \leq \liminf_{n \rightarrow \infty} \|v_n\|_{L^2(BL)} \leq \|\nabla v\|_{H^{-1}(BL)} + \|v\|_{H^{-1}(BL)},$$

hence (4.72). □

*Proof of (4.73).* Note that a similar result was already obtained in [2], using a former result of [4] combined with an interpolation argument. Going back to basics, we give here a self-contained and combined proof, by using the Nečas lemma and the interpolation theory. Indeed, From Nečas inequality (4.72), we deduce that the injection

$$Id : W^{-1} \rightarrow L^2$$

is continuous. Moreover,  $Id : W^0 \rightarrow H^1$  is also continuous. Therefore, by the interpolation theorem [64], also the restriction of the identity (denoted still by  $Id$ )

$$Id : [W^{-1}, W^0]_{1/2} \rightarrow [L^2, H^1]_{1/2} = H^{1/2},$$

is continuous. In particular, by (4.71), there exists  $C > 0$ , such that

$$\forall v \in [W^{-1}, W^0]_{1/2}, \quad \|v\|_{H^{1/2}} \leq C(\|v\|_{(H_{00}^{1/2}(BL))'} + \|\nabla v\|_{(H_{00}^{1/2}(BL))'}). \quad (4.75)$$

*Conclusion of the proof.* Thanks to (4.75), in order to prove (4.73), all we have to do from now is proving that the following inclusion holds true

$$V^1 \subset [W^{-1}, W^0]_{1/2}, \quad (4.76)$$

with continuous injection. In the case relevant for our problem, we have

$$\|z^{1/2}\varphi\|_{L^2} \leq C\|\varphi\|_{H_{00}^{1/2}}.$$



Let  $u \in V_1$ , and  $\varphi \in H_{00}^{1/2}(BL)$ . Then,

$$\left| \int_{BL} \partial_z u \varphi \, dx \right| = \left| \int_{BL} \sqrt{z} \partial_z u \frac{\varphi}{\sqrt{z}} \, dx \right| \leq \|\sqrt{z} \partial_z u\|_{L^2} \|z^{-1/2} \varphi\|_{L^2} \leq C \|\sqrt{z} \partial_z u\|_{L^2} \|\varphi\|_{H_{00}^{1/2}},$$

hence  $\partial_z u \in (H_{00}^{1/2}(BL))'$ , and

$$\|\partial_z u\|_{(H_{00}^{1/2}(BL))'} \leq C \|\sqrt{z} \partial_z u\|_{L^2}.$$

In the same way, we have also

$$\|u\|_{(H_{00}^{1/2}(BL))'} \leq C \sqrt{z_{top}} \|u\|_{L^2}, \quad \|\nabla_h u\|_{(H_{00}^{1/2}(BL))'} \leq C \sqrt{z_{top}} \|\nabla_h u\|_{L^2}.$$

Combining all the previous inequalities yields

$$\|u\|_{[W^{-1}, W^0]_{1/2}} \leq C \|u\|_{V_1},$$

hence (4.76), which concludes the proof.  $\square$

The functional setting for the proof of Theorem 4.4.1 is more sophisticated than that used in the non-limiting cases  $0 \leq \alpha < 1$ . Despite the proof being a rather standard application of the Lax-Milgram lemma, the choice of the underlying function spaces is obliged by the nature of the problem and the fact that  $\alpha = 1$  implies that the function spaces do not embed in any standard Sobolev space with trace at the boundary. For this point cf. Proposition 4.3.1, which we recall is false for  $\alpha = 1$  and a counter-examples is easily built by means of a double logarithmic function. This is why we resort to the space  $V_1$  and, despite the abstract simplicity of the result, the interpretation of the notion of the solution is of particular difficulty, since the solution satisfies a weak formulation which is not the same as the strong one. As we will discuss later, the obtained solution has problem in the interpretation of the value at  $z = 0$ , for which the functional setting is not proper. Nevertheless, we cannot change it, since it is determined by the equations themselves, so we have to extract the maximum of information possible from the solution.

*Proof of Theorem 4.4.1.* We are now ready for the proof of the main result of the paper, that is the proof of existence of weak solutions in the limiting case  $\alpha = 1$ . In this case once we have the adapted functional setting we can apply Lax-Milgram in the space  $V_1$

with the weak formulation defined by:  $\forall v \in V_1$ ,

$$\lambda \int_{\Omega} uv + \nu_h \int_{\Omega} \nabla_h u \cdot \nabla_h v + \mu \int_{\Omega} z \partial_z u \partial_z v + C_D \int_{\Gamma_{top}} uv - \int_{\Gamma_{top}} v G = \int_{\Omega} f v, \quad (4.77)$$

and eventually observe that the proof can be made fully rigorous again by approximation obtained adding  $-\epsilon \partial_z^2 u$  to the equations.

□

**Remark 4.4.1.** *Note that the trace of  $v \in V_1$  is not defined at  $z = 0$ , where the weight vanishes. On the other hand  $v|_{z=z_{top}}$  is well defined in  $H^{1/2}(\Gamma_{top})$  since the function  $v$  belongs to  $W^{1,2}$  in a neighborhood of the top part of the boundary. Hence, the integrals  $\int_{\Gamma_{top}} v G$  and  $\int_{\Gamma_{top}} uv$  are properly defined.*

Moreover, as is standard for these problems the function  $u \in V_1$  can be taken as test function, proving the energy equality

$$\lambda \|u\|^2 + \nu_h \|\nabla_h u\|^2 + \mu \|\sqrt{z} \partial_z u\|^2 + C_D \int_{\Gamma_{top}} |u|^2 - \int_{\Gamma_{top}} G u = \int_{\Omega} f u.$$

Moreover, this can be used also to show uniqueness of the weak solution. The drawback is the impossibility of controlling the trace.

The only missing point is to observe that we proved negative norm lemma for functions vanishing at  $\{z = 0\} \cup \{z = z_{top}\}$ , while now we have a friction law at the upper boundary. This can be easily overcome by using the fact that the conditions are of Neumann (Navier) type at the top. Hence, considering the space

$$H_{0,\tau}^1(BL) = \{u \in H^1(BL); u = 0 \text{ on } \Gamma_b \text{ and } \partial_z u = 0 \text{ on } \Gamma_{top}\}, \quad (4.78)$$

instead of  $H_0^1(BL)$  as in (4.38), we can easily convert to the case of a Dirichlet problem in a doubled domain  $\Pi \times (0, 2z_{top})$ , with a reflection along the line  $z = z_{top}$ . Then, the proof remains the same as in the previous case. Note that since we have  $V_1$  functions for which the trace is not well-defined, we cannot expect that our weak solution satisfies  $u = 0$  at the bottom boundary. So in this case the weak and strong formulation are not giving the same result and the same have been noted, in a slightly different setting, by Rappaz and Rochat [84], for the von Kármán problem. They also noted as the trace evaluated numerically is strongly depending on the mesh-size, as is expected, and that the value at the boundary is not under control.

## 4.5 Numerical experiments

In this part, we aim to check if the model gives a good approximation of the Monin-Obukhov law (4.8), depending on the values of  $\alpha$ . We solve the problem (4.4) in two dimensions, using the software [Freefem++](#). We consider a rectangular box  $[0, L] \times [0, z_{top}]$  with  $z_{top} = 100$ ,  $L = 1000$  and we add periodic conditions on the left and right sides  $\{0\} \times [0, z_{top}]$  and  $\{L\} \times [0, z_{top}]$ .

The other boundary conditions are Dirichlet at the bottom, and a linear Navier condition at the top involving the roughness coefficient  $C_D$  :

$$\begin{cases} \kappa u^* z_{top} \frac{du}{dz}(x, z_{top}) = C_D (V(x) - u(x, z_{top})), & \text{at } z = z_{top}, \\ u(x, 0) = 0, & \text{at } z = 0, \end{cases} \quad (4.79)$$

where

$$V(x) = u_{\text{Log}}(z_{top})(1 + \epsilon(x)), \quad (4.80)$$

and

$$u_{\text{Log}}(z) = \begin{cases} \frac{C_* u^*}{C_\nu} z, & z \in [0, z_0] \\ \frac{C_* u^*}{C_\nu} \left( \log\left(\frac{z}{z_0}\right) + 1 \right), & z \in [z_0, z_{top}] \end{cases} \quad (4.81)$$

The aim of this numerical study is to check if the numerical solution  $u$  of the problem (4.4) approaches the known log-law, in a sense we will develop below.

First we will explain in subsection 4.5.3 how to get vertical velocities from the [Freefem++](#) resolution of the problem (4.4) and the tools to compare it with the log-law  $u_{\text{Log}}$ .

Then, we will discuss the influence of the different parameters ( $\alpha$  in Subsection 4.5.2),  $C_D$  and  $u^*$  in the Subsection 4.5.3 and calculate the difference between the numerical solution and  $u_{\text{Log}}$ , which allow us in Subsection 4.5.4 to derive an analytical formula of the stabilization function  $\Psi$  deduced from the numerical results by interpolation.

Finally, we will consider the influence on the  $x$ -axis of a small perturbations  $\epsilon$ , as involved in the wind at the top given by (4.80).

**Remark 4.5.1.** *We use in the code the command  $u = 0$  at  $z = 0$ , and we take  $\lambda = 0$ . Contrary to what the analysis predicts, the case  $\alpha = 1$  works very well, even at high resolutions. As [Freefem++](#) is a finite element software, we think that the numerical simulation involves a numerical dissipation which sufficiently regularizes the equation, even in the*

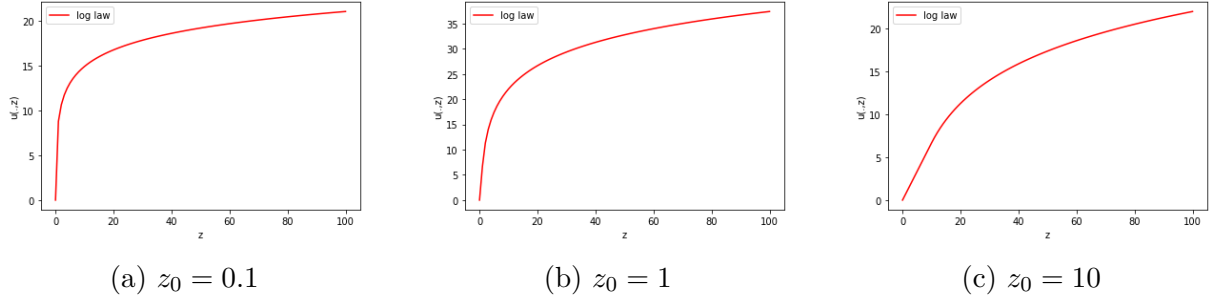


Figure 4.1 – log-law

case  $\alpha = 1$ . However, we did not have studied yet this numerical aspect of the problem.

### 4.5.1 Settings of the analysis

#### Parameters

Different parameters will have influence on the simulation: some will be fixed, and some will be specifically studied. The size of the box will always be in the following  $[0, L] \times [0, z_{top}]$ , where

$$L = 1000 \text{ m} \quad \text{and} \quad z_{top} = 100 \text{ m}.$$

- After several simulations to get velocities that can be measured in situ, it looks like that the best values for the calibration constants  $C^*$  and  $C_\nu$  were  $C^* = 10$  and  $C_\nu = 15$ , as already mentioned in section 4.2.1. Examples of log profiles have been plotted in figure 4.1 with these values,  $u^* = 10$ ,  $z_{top} = 100$ , and respectively  $z_0 = 1$  and  $z_0 = 10$ .

- The height of the viscous sub-layer  $z_0$  in the following simulations will be very small compared to  $z_{top}$ , with a ratio smaller than 0.01. Besides, the viscous sub-layer height  $z_0$  will taken equal to 0.1, which respects a ratio

$$\frac{z_0}{z_{top}} = 10^{-3} < 0.01.$$

- The parameter  $\lambda$  is chosen equal to 0.
- The source  $f$  will be considered as constant:  $f = 5$ . According to formula (4.12), this means that  $\delta T \approx -23^\circ\text{C}$  for  $\beta = 2.10^{-3}$ , for instance in the case where the ground is at  $0^\circ\text{C}$  and the air is dry and cold at a constant temperature equal to  $-23^\circ\text{C}$ , a situation that can happen in the mountains.

- The perturbation  $\epsilon$  which appears in the expression (4.80) of  $V$  will be taken equal to 0 in the next subsections, except in subsection 4.5.5 (see (4.87) below).

The velocity constant  $u_*$  can be seen as a "wind regime switch" belonging to the speed range  $[2m.s^{-1}, 10m.s^{-1}]$ , which corresponds to what is generally measured for flows over rough grounds. This is the main parameter of our simulations, the influence of which will be studied in the subsection 4.5.3.

- Finally, the frictional coefficient  $C_D$  will be chosen according to the  $u^*$ . In the subsection 4.5.3 we will show that  $C_D \simeq 10^6$  is giving convincing results.

## Errors

Let  $u_k$  denotes the vertical velocities at  $x_k = kL$ , given by, for every  $z \in [0, z_{top}]$ :

$$u_k(z) = u(x_k, z), \quad (4.82)$$

where  $k \in \{0, \frac{1}{10}, \frac{2}{10}, \dots, 1\}$ . These  $u_k$  will be compared with the log profile  $u_{\text{Log}}$  defined by (4.81). To achieve these comparisons, we introduce the errors  $err$  at a point  $x$

$$err(x) = \frac{1}{N} \sum_{i=1}^N |u(x, z_i) - u_{\text{Log}}(z_i)|, \quad (4.83)$$

and the relative error  $relerr$  at  $x$

$$relerr(x) = \sum_{i=1}^N \frac{|u(x, z_i) - u_{\text{Log}}(z_i)|}{|u(x, z_i)|}, \quad (4.84)$$

which will be more relevant than the error because of the importance of the velocities, when  $u^* > 7m.s^{-1}$  for instance.

Without the perturbation  $\epsilon$ , the vertical velocities  $u_k$  are very close each other, as we can see in figure 4.2 and in the table 4.1. Even if the difference between the vertical profiles is small, we consider the mean vertical velocity

$$\bar{u}(z) := \frac{1}{p} \sum_{k=1}^p u(x_k, z), \quad (4.85)$$

and we will use it instead of the  $u_k$ . To perform the simulations,  $p = 11$  and  $x_k$  will belongs to  $\{0, 100, 200, \dots, 1000\}$ .

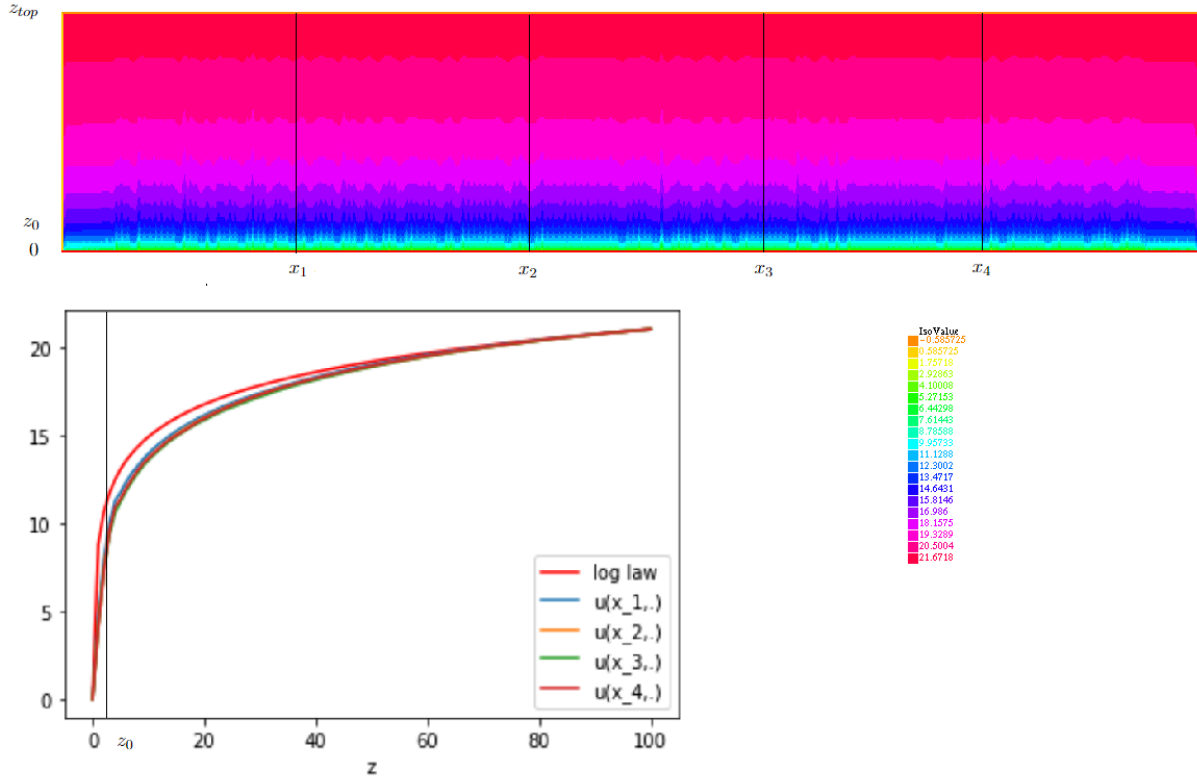


Figure 4.2 – Velocity profiles for different horizontal values

| $u_1$      |               | $u_2$      |               | $u_3$      |               | $u_4$      |               |
|------------|---------------|------------|---------------|------------|---------------|------------|---------------|
| <i>err</i> | <i>relerr</i> | <i>err</i> | <i>relerr</i> | <i>err</i> | <i>relerr</i> | <i>err</i> | <i>relerr</i> |
| 0.37       | 0.042         | 0.49       | 0.053         | 0.53       | 0.056         | 0.47       | 0.047         |

Table 4.1 – Errors and relative errors between the vertical velocities  $u_k$  and  $u_{\text{Log}}$ . The parameters taken are  $C_D = 10^6$ ,  $\alpha = 1$ ,  $u^* = 4$ ,  $x_1 = 200$ ,  $x_2 = 400$ ,  $x_3 = 600$ ,  $x_4 = 800$ .

### 4.5.2 Influence of alpha

We observe that the more  $\alpha$  is getting close to 1, the better the model is, as we can see in figure 4.3: this is correct in the sense that the difference between the calculated profile and the log profile is smaller. The blue curves correspond here to the values  $\bar{u}(z)$ , where  $z \in \{1, 2, \dots, 100\}$ .

As a result, the model is relevant only for  $\alpha = 1$ . We will take  $\alpha = 1$  in the next simulations.

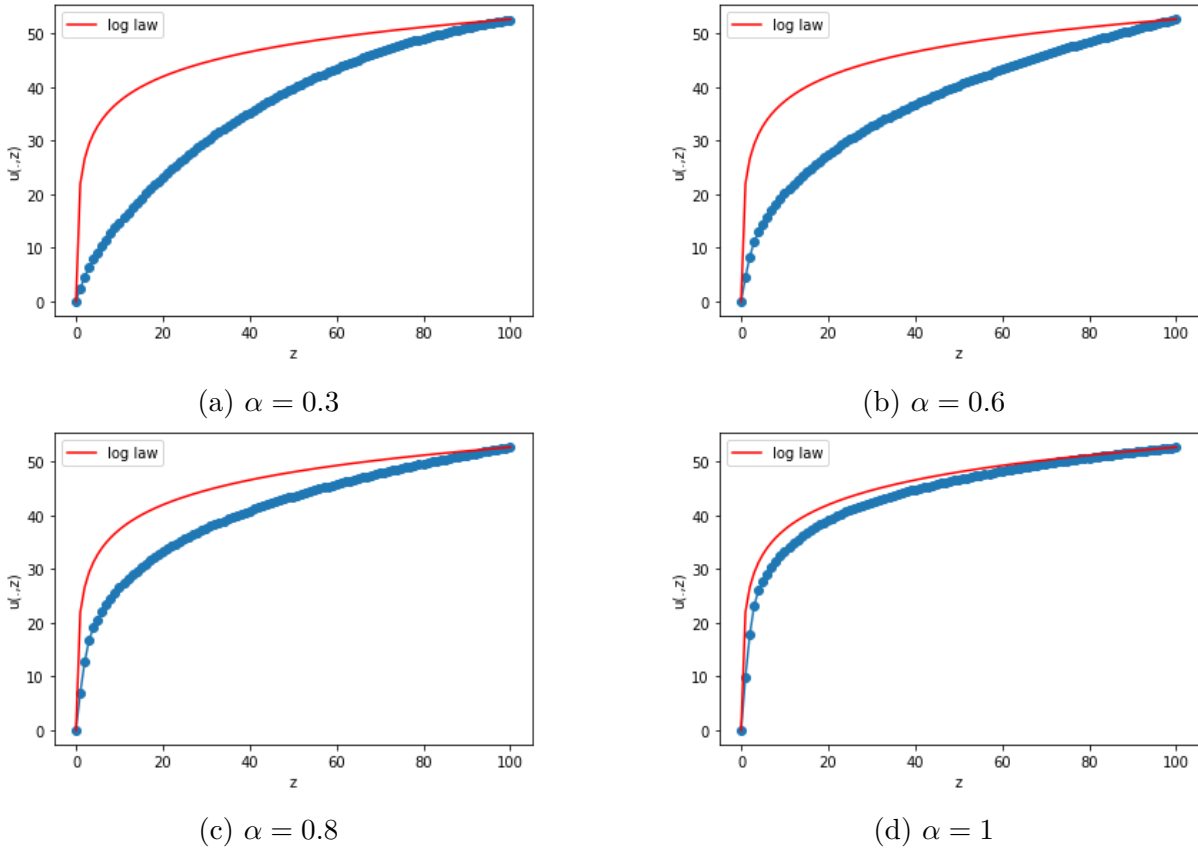


Figure 4.3 – Vertical velocities

### 4.5.3 The three different regimes: influence of $C_D$

We have calibrated the constant values to get wind velocities which are physically relevant (in  $m.s^{-1}$ ). We have observed three different regimes for  $u_*$  respectively equal to  $4m.s^{-1}$ ,  $7m.s^{-1}$  and  $10m.s^{-1}$  corresponding to small wind, medium wind, and storm wind. We consider for each case the errors and relative errors we get in function of the  $C_D$  coefficients.

We can see in figure 4.4 that the vertical velocity  $\bar{u}$  in blue is close to the log-law when  $C_D$  is big and far when  $C_D$  is small. To quantify this, the errors and relative errors corresponding in the table 4.2 show that the bigger  $C_D$  is, the smallest the errors are.

Nevertheless, it is getting steady at some point as we can see in figure 4.5, where the errors and the relative errors have been plotted for  $u^* = 4$ ,  $u^* = 7$  and  $u^* = 10$ . The value  $10^6$  seems to be the threshold value for the coefficient  $C_D$ .

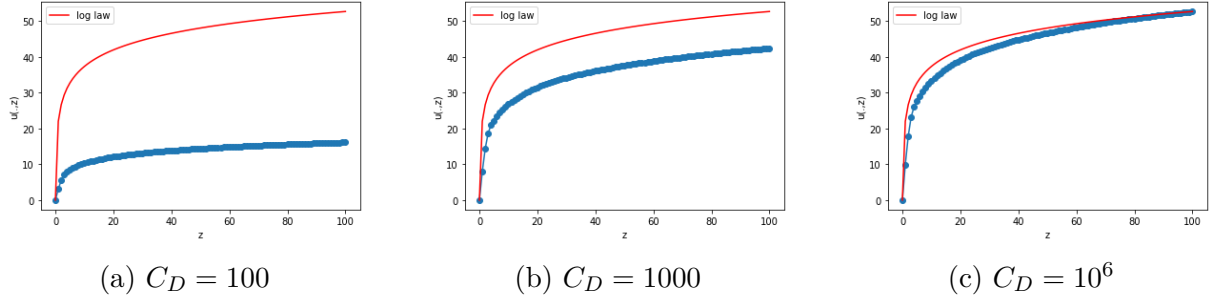


Figure 4.4 – Influence of  $C_D$  for  $u^* = 10$

|        | $u^* = 4$  |               | $u^* = 7$  |               | $u^* = 10$ |               |
|--------|------------|---------------|------------|---------------|------------|---------------|
| $C_D$  | <i>err</i> | <i>relerr</i> | <i>err</i> | <i>relerr</i> | <i>err</i> | <i>relerr</i> |
| $10^2$ | 7.58       | 0.73          | 19.59      | 1.59          | 32.21      | 2.42          |
| $10^3$ | 1.79       | 0.13          | 5.50       | 0.24          | 10.34      | 0.32          |
| $10^4$ | 0.63       | 0.061         | 1.68       | 0.083         | 2.85       | 0.094         |
| $10^5$ | 0.50       | 0.054         | 1.23       | 0.067         | 1.90       | 0.071         |
| $10^6$ | 0.48       | 0.053         | 1.18       | 0.066         | 1.81       | 0.069         |
| $10^9$ | 0.48       | 0.053         | 1.18       | 0.065         | 1.79       | 0.069         |

Table 4.2 – CD calibration for the different regimes

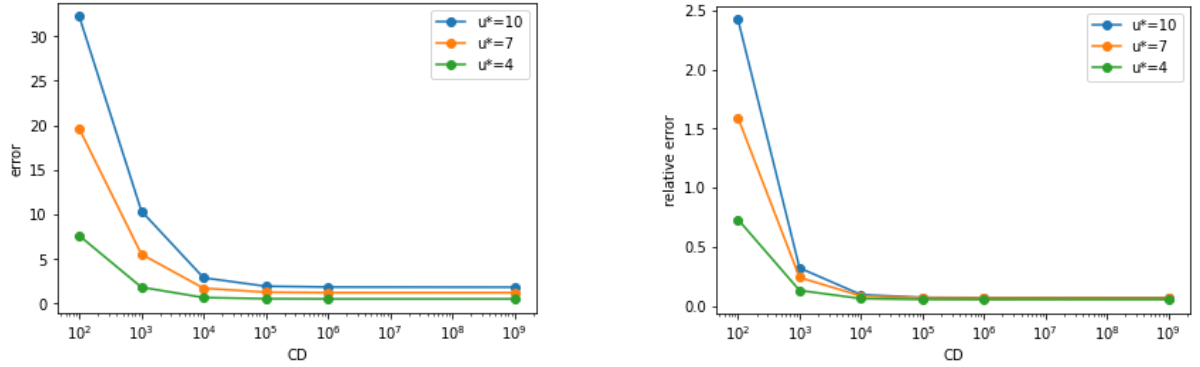


Figure 4.5 – Errors and relative errors dependence on  $C_D$

### 4.5.4 Stabilization functions

From our numerical results, by an empirical method of extrapolation by successive approximations, we have found the following stabilization function

$$\Psi(z) = 2u^*(e^{(z_0-z)} - e^{-2z}) - \frac{u^*}{200}z + 0.4u^*. \tag{4.86}$$



so that, if  $u$  denotes the numerical result,

$$u + \psi \approx u_{\text{Log}}$$

It gives for the different values of  $u^*$  the curves shown in figure 4.6. The peak we can see corresponds to the the height of the viscous sub-layer  $z_0$ .

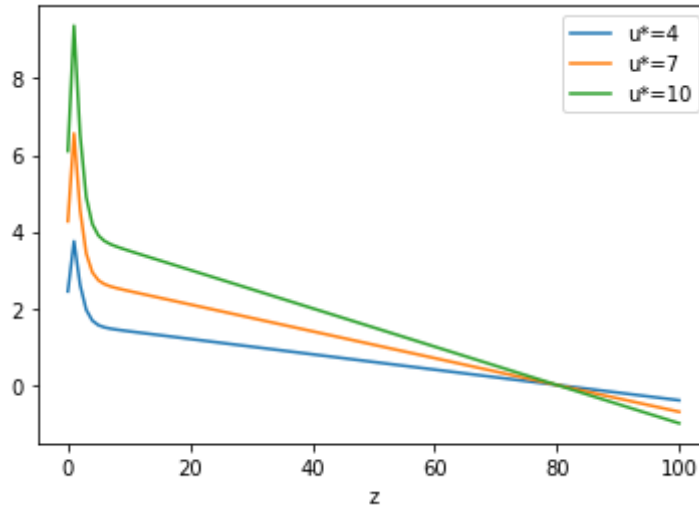


Figure 4.6 – Stabilization functions for different values of  $u^*$

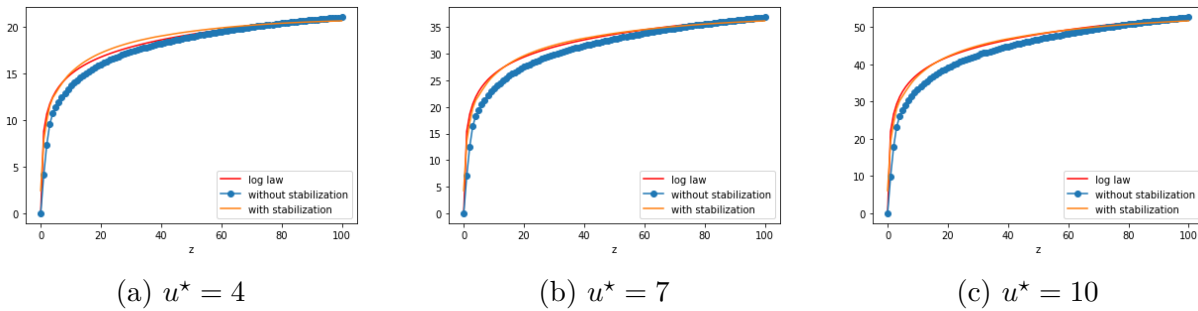


Figure 4.7 – Velocity profile with and without stabilization compared with the log-law

We can compare the errors and relative errors between the raw velocity  $u$  and the log-law  $u_{\text{Log}}$  on the one hand, and between the stabilized velocity  $u + \Psi$  and  $u_{\text{Log}}$  on the other hand, for the different regimes given by  $u^*$ . We can see in the table 4.3 and in the figure 4.7 that the stabilization is a better approximation of the log-law, especially around the viscous sub-layer  $z_0$ .

| $u^*$ | $C_D$  | <i>err</i> w/o stab | <i>err</i> with stab | <i>relerr</i> w/o stab | <i>relerr</i> with stab |
|-------|--------|---------------------|----------------------|------------------------|-------------------------|
| 4     | $10^6$ | 0.49                | 0.30                 | 0.053                  | 0.026                   |
| 7     | $10^6$ | 1.18                | 0.35                 | 0.066                  | 0.021                   |
| 10    | $10^6$ | 1.81                | 0.47                 | 0.069                  | 0.020                   |

Table 4.3 – Errors and relative errors between the stabilized/unstabilized velocities and the log-law

### 4.5.5 With horizontal perturbation

In this part we study the effect of a small oscillation in the horizontal direction, given by

$$\epsilon(x) = 0.01 \sin(11 \frac{2\pi x}{L}), \quad (4.87)$$

so that  $V(x) = u_{\text{Log}}(z_{\text{top}})(1 + \epsilon(x))$ .

We keep the value  $C_D = 10^6$ ,  $\alpha = 1$ , and we will compare the vertical velocities  $\bar{u}$  we get with the ones we had without this perturbation for the three different  $u^*$ . We obtain the table 4.4. This shows that even for a small perturbation, the error variation is quite large compared to the values we have. When we add the stabilization function  $\Psi$ , we can manage to keep the errors still small, even with the perturbation  $\epsilon$ . This opens an interesting stability problem à la "Lyapunov".

| $u^*$ | <i>err</i>     |                 |                            | <i>relerr</i>  |                 |                            |
|-------|----------------|-----------------|----------------------------|----------------|-----------------|----------------------------|
|       | w/o $\epsilon$ | with $\epsilon$ | with $\epsilon$ and $\Psi$ | w/o $\epsilon$ | with $\epsilon$ | with $\epsilon$ and $\Psi$ |
| 4     | 0.49           | 0.62            | 0.27                       | 0.053          | 0.061           | 0.024                      |
| 7     | 1.18           | 1.42            | 0.41                       | 0.066          | 0.073           | 0.023                      |
| 10    | 1.81           | 2.15            | 0.65                       | 0.069          | 0.078           | 0.024                      |

Table 4.4 – Errors and relative errors between the velocities and the log-law

### 4.5.6 Conclusion and perspectives

We have shown numerically that the solution  $u$  of the problem (4.4) is a very good approximation for the known Monin-Obukhov log-law when  $\alpha = 1$  for adiabatic flows, with a large calibration constant  $C_D$ , imposing a Dirichlet condition at  $\Gamma_{\text{top}}$ . This is valid for a large range of wind regimes. The numerical code seems to be a good tool

for calculating the stabilization function, which could be improved by using a formal mathematical calculation tool, which we have not done yet.

The next step is to couple system (4.4) with the equation for the temperature  $T$ , where the source term  $f$  is given by (4.12), which is a work under progress.

# SINGULAR BOUNDARY CONDITION FOR A DEGENERATED TURBULENT TOY MODEL

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## 5.1 Introduction

The two main characteristics of a turbulence model for the simulation of a fluid in the layer near a solid wall are the turbulent viscosity and the boundary condition of friction at the wall; the latter is often called the wall law. In many cases, the turbulent viscosity  $\nu_{\text{turb}} \geq 0$  is proportional to a given power of the mixing length  $\varrho$ , which is, in several physical models, of the order of the distance from the wall. If we assume that molecular diffusion is negligible compared to turbulent diffusion, we are led to write as wall law:

$$\nu_{\text{turb}} \frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}} = f(\mathbf{u}_\tau), \quad \text{on } \Gamma, \quad (5.1)$$

where  $\Gamma$  denotes the boundary of the flow domain  $\Omega$  and  $\mathbf{u}_\tau$  the tangential fluid velocity at  $\Gamma$ .

Obviously written in this way, this condition doesn't make much sense because  $\nu_{\text{turb}}$  is zero over  $\Gamma$  (according to the above discussion), which would mean that  $\frac{\partial \mathbf{u}_\tau}{\partial \mathbf{n}}$  is infinite over  $\Gamma$ , unless  $f(\mathbf{u}_\tau)$  vanishes, and  $f$  satisfies appropriate conditions (see in [23]). This means that no friction occurs in some sense when  $f$  vanishes only at 0 such as  $f(\mathbf{w}) = C_D \mathbf{w}|\mathbf{w}|$ , and we are left with a no-slip boundary condition. This is a classical and old debate. The paradox here is that in a classical variational formulation, we will formally have a boundary term of the form

$$\int_{\Gamma} f(\mathbf{u}_\tau) \cdot \mathbf{v}_\tau$$

which is totally well-defined (for smooth enough  $f, \mathbf{u}, \mathbf{v}$ ), even if  $\mathbf{u}_\tau \neq 0$  at  $\Gamma$ .

Following a classical approach introduced by J.-L. Lions [65], it is natural to introduce a problem of *singular perturbations* to understand this paradox. In particular, in this

present case, we are led to consider a viscosity of the form  $\nu_{\text{turb}} + \varepsilon$ , with  $\varepsilon > 0$ , which amounts to reconsidering the molecular viscosity, and to asking the question of the limit problem and solution, when  $\varepsilon \rightarrow 0$ .

In this paper, we study a scalar toy model, considering only the tangential velocity denoted by  $u$ , the friction is linear  $f(u) = u$ ,  $\nu_{\text{turb}}(x) = \varrho(x)$ , and only a diffusion term with a source term; the problem is then a sort of singular perturbed Robin problem. Therefore, the singular perturbation problem we consider is the following:

$$\begin{cases} -\nabla \cdot ((\varrho + \varepsilon)\nabla u) = f & \text{over } \Omega, \\ -\varepsilon \frac{\partial u}{\partial n} = u & \text{at } \Gamma, \end{cases} \quad (5.2)$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \leq 3$ ) is a  $\mathcal{C}^2$ -bounded domain and the properties of  $\varrho$  are described by (5.3) below. We show in this paper that when the source term satisfies appropriate compatibility conditions, then it is possible to pass to the limit in this problem, in a certain sense. One main problem is the verification of the boundary condition, which will be *formally*  $u = 0$ , when  $\varepsilon = 0$ . To do so, we carefully define the space function in which we pass to the limit.

Then, we provide several analytical examples of blowing up cases for not well prepared data, as well as examples as for well prepared data, for which convergence to the homogeneous Dirichlet holds true. Note that in all these positive examples, when passing to the limit  $\varepsilon \rightarrow 0$ , we find  $u = 0$  at  $\Gamma$ , a result which we are not able to prove theoretically at present and which seems out of the classical theory of variational problems in weighted spaces.

## 5.2 The Limit problem

In this section we describe the limit problem and we give the main theoretical results.

### 5.2.1 Well prepared source term and energy balance

We first need to clarify the hypotheses on the degenerate viscosity, to define what is a well-prepared source term and how it influences the energy balance.

### Mixing length

We assume that  $\varrho : \Omega \rightarrow \mathbb{R}^+$  is a  $C^2(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$  function that satisfies

$$\lim_{\substack{d(x,\Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x,\Gamma)} = 1, \quad \text{and} \quad \inf_{\substack{d(x,\Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0, \quad (5.3)$$

so that it behaves in the same way as the distance function from the boundary. In what follows, we set

$$\varrho_\varepsilon := \varrho + \varepsilon, \quad (5.4)$$

which will be used for the perturbed problem.

### Well prepared source term

We take as a source term  $f \in C^0(\overline{\Omega}) \cap L^\infty(\Omega)$  such that

$$\int_{\Omega} f \, dx = 0. \quad (5.5)$$

Let  $\mathbf{g} \in C^1(\overline{\Omega})$  be the solution to the lifting problem

$$\begin{cases} \nabla \cdot \mathbf{g} = f & \text{in } \Omega, \\ \mathbf{g} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \quad (5.6)$$

Notice that  $\mathbf{g}$  does exist and it is unique up to a free divergence fields quotient, by the compatibility condition (5.5) (see in the proof of Proposition 6.2 in [23]).

We also assume that there exists a constant  $C > 0$  such that one has

$$\int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho} \, dx \leq C. \quad (5.7)$$

The reason for this choice will be clear in what follows and note that it is a sort of request that  $g$  has to vanish fast enough when  $x$  approaches the boundary  $\Gamma$ .

### Energy balance

We write now the standard variational formulation for the perturbed problem. Assume that (5.5), (5.6), and (5.7) hold. We say that  $u \in H^1(\Omega)$  is a weak solution to (5.2) if

$\forall v \in H^1(\Omega)$ , we have

$$\int_{\Omega} \varrho_{\varepsilon} \nabla u \cdot \nabla v + \int_{\Gamma} uv = \int_{\Omega} fv. \quad (5.8)$$

We now give the main existence theorem.

**Lemma 5.2.1.** *Problem (5.2) has a unique weak solution  $u \in H^1(\Omega)$ . In addition,  $u \in H^2(\Omega) \cap C^1(\overline{\Omega})$  and we have the following estimate*

$$\int_{\Omega} \varrho_{\varepsilon} |\nabla u|^2 + \int_{\Gamma} |u|^2 \leq 4 \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_{\varepsilon}} \leq 4C. \quad (5.9)$$

*Proof.* Given  $\varepsilon > 0$ , the existence and the uniqueness of a weak solution to Problem (5.2) is straightforward by Lax-Milgram Theorem. To check the regularity, we note that if  $f$  is in  $L^2(\Omega)$ , then  $u \in H^2(\Omega)$  (see in [5]) and we can write that almost everywhere  $x \in \Omega$

$$-\Delta u = \frac{1}{\varrho_{\varepsilon}} (\nabla \varrho_{\varepsilon} \cdot \nabla u + f). \quad (5.10)$$

In a first analysis, we see that when  $N = 2$ , hence  $u \in H^2(\Omega) \subseteq C^{0,\alpha}(\overline{\Omega}) \cap W^{1,p}(\Omega)$ , for all  $p < \infty$ . Hence,  $\Delta u \in L^p(\Omega)$ , which implies  $u \in W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$ . When  $N = 3$ , we get from the Sobolev embedding that  $\nabla u \in W^{1,6}(\Omega)$ , then  $\nabla u \in C^0(\overline{\Omega})$ , and the proof then follows as for  $N = 2$ , since this implies again that  $\Delta u \in L^p(\Omega)$  for all  $p < \infty$ , due to the fact that  $\nabla \varrho_{\varepsilon} \in C^0$ .

We now check the energy balance (5.9), which explains the choice for the assumption (5.7). Taking  $u$  as test function, and by the particular choice of the source term, we have

$$\left| \int_{\Omega} uf \right| = \left| - \int_{\Omega} \mathbf{g} \cdot \nabla u \right| \leq \left( \int_{\Omega} \frac{|\mathbf{g}|^2}{\varrho_{\varepsilon}} \right)^{\frac{1}{2}} \left( \int_{\Omega} \varrho_{\varepsilon} |\nabla u|^2 \right)^{\frac{1}{2}}, \quad (5.11)$$

and estimate (5.9) follows from (5.7),  $\rho \leq \rho_{\varepsilon}$  and Young inequality.  $\square$

## 5.2.2 Functional space

### Definitions

Beside the classical Sobolev spaces, we define in this section the functional setting, adapted to the problem we will consider, taking care to define the notion of trace that we will use in the following.

Let  $\mathcal{W}$  be the space  $\mathcal{C}^1(\overline{\Omega})$  equipped with the norm

$$\|u\|_{1,\varrho,\Gamma} = \left( \int_{\Omega} \varrho |\nabla u|^2 + \int_{\Gamma} |u|^2 \right)^{\frac{1}{2}}, \quad (5.12)$$

which derives from the scalar product

$$(u, v)_{1,\varrho,\Gamma} = \int_{\Omega} \varrho \nabla u \cdot \nabla v + \int_{\Gamma} uv. \quad (5.13)$$

This space is not complete. Indeed, let  $(u_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in this space. Then the sequence of traces on  $\Gamma$ , say  $(\text{tr}u_n)_{n \in \mathbb{N}}$ , is a Cauchy sequence in  $L^2(\Gamma)$ , which is a complete space. Therefore, it converges to some  $g$  in  $L^2(\Gamma)$ . Furthermore, by (5.3) we can claim that given any  $\omega \subset\subset \Omega$ , the sequence  $(\nabla u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\omega)^N$ , which is also complete, let  $\mathbf{z}_\omega \in L^2(\omega)^N$  be its limit. From there we can construct  $\mathbf{z} \in L^2_{loc}(\Omega)^N$  such that  $\forall \omega \subset\subset \Omega$ ,  $\mathbf{z}|_\omega = \mathbf{z}_\omega$ . However nothing allows us to deduce that  $\mathbf{z} = \nabla u$  for a given  $u$  such that  $\sqrt{\rho} \nabla u \in L^2(\Omega)$ .

From there, let

$$\mathcal{W}_{\mathcal{F}} = \{u = (u_n)_{n \in \mathbb{N}} \in \mathcal{W}^{\mathbb{N}}, u \text{ is a Cauchy sequence}\}$$

with the equivalence relation

$$u \mathcal{R} v \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \|u_n - v_n\|_{1,\varrho,\Gamma} = 0.$$

**Remark 5.2.1.** Let  $u = (u_n)_{n \in \mathbb{N}}$  and  $v = (v_n)_{n \in \mathbb{N}}$  be such that  $u \mathcal{R} v$ . Then we have in  $L^2(\Gamma)$ ,  $\lim_{n \rightarrow \infty} \text{tr}u_n = \lim_{n \rightarrow \infty} \text{tr}v_n$ .

Let  $W = \mathcal{W}_{\mathcal{F}}/\mathcal{R}$  be the quotient space, equipped with the scalar product defined by, for any given  $\bar{u} \in W$  and  $\bar{v} \in W$

$$(\bar{u}, \bar{v})_W = \lim_{n \rightarrow \infty} (u_n, v_n)_{1,\varrho,\Gamma}. \quad (5.14)$$

It is easily checked that the sequence  $((u_n, v_n)_{1,\varrho,\Gamma})_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$  and that the limit in (5.14) does not depend on the choice of the representatives in the class of  $\bar{u}$  and  $\bar{v}$ . Furthermore,

$$\|\bar{u}\|_W = \lim_{n \rightarrow \infty} \|u_n\|_{1,\varrho,\Gamma}, \quad (5.15)$$

where also the limit does not depend of the representatives in the class  $\bar{u}$ . We know that  $W$



is an Hilbert space for this scalar product (see [53]). Moreover, the space  $\mathcal{W}$  is embedded in  $W$  and is everywhere dense in  $W$ .

**Definition 5.2.1.** Let  $\bar{u} = \overline{(u_n)_{n \in \mathbb{N}}} \in W$ . Let  $g \in L^2(\Gamma)$  be the limit in  $L^2(\Gamma)$  of the sequence  $(\text{tr}u_n)_{n \in \mathbb{N}}$ . Then  $g$  does not depend on the representatives in the class of  $\bar{u}$  (see Remark 5.2.1). We say that  $g$  is the generalized trace of  $\bar{u}$  over  $\Gamma$ , and we still denote  $g = \text{tr}\bar{u}$ , so far no risk of confusion occurs. We also set  $\|\text{tr}\bar{u}\|_{0,2,\Gamma} = \|g\|_{0,2,\Gamma}$ .

From now we note  $u \in W$  instead of  $\bar{u}$ . The space  $W$  is the Cauchy completion of  $\mathcal{W}$  (see also in [31]).

**Remark 5.2.2.** As  $\rho$  is a bounded function on  $\Omega$ , we have  $\|u\|_{1,\rho,\Gamma} \leq C\|u\|_{1,2,\Omega}$  for all  $u \in \mathcal{C}^1(\bar{\Omega})$ . Therefore, as  $\mathcal{C}^1(\bar{\Omega})$  is everywhere dense in  $H^1(\Omega)$ , we have a natural embedding  $H^1(\Omega) \hookrightarrow W$ .

**Remark 5.2.3.** Let  $f$  such that (5.6), (5.5) and (5.7) are satisfied. Then for  $u \in \mathcal{C}^1(\bar{\Omega})$  and arguing as in (5.11), we obtain

$$\left| \int_{\Omega} fu \right| \leq C \left( \int_{\Omega} \rho |\nabla u|^2 \right)^{1/2} \leq C \|u\|_{1,\rho,\Gamma}.$$

Then as  $\mathcal{C}^1(\bar{\Omega})$  is dense in  $W$ ,  $[u \mapsto \int_{\Omega} fu]$  can be extended to  $W$ , so that in this sense we have  $f \in W'$  and we write the extension

$$u \in W \mapsto_{W'} \langle f, u \rangle_W.$$

### More about the structure of the space $W$

We aim at a better understanding of the structure of this space. To do so we introduce the quotient space  $\mathcal{C}^1(\bar{\Omega})/\mathbb{R}$  equipped the quotient norm

$$\|u\|_V = \inf \left\{ \left( \int_{\Omega} \rho |\nabla v|^2 \right)^{\frac{1}{2}}, v - u = Cte \right\}$$

knowing that if  $v = u + Cte$ , then  $\int_{\Omega} \rho |\nabla v|^2 = \int_{\Omega} \rho |\nabla u|^2$ . Therefore,  $\|\bar{u}\|_V = \|\sqrt{\rho} \nabla u\|_{0,2,\Omega}$ , and this does not depend on the representative in the class  $\bar{u}$ . Then let  $V$  be the Cauchy completion of  $\mathcal{C}^1(\bar{\Omega})/\mathbb{R}$  for this norm, following the same construction as in the previous section. In particular, writing  $u = (u_n)_{n \in \mathbb{N}}$  instead of  $\bar{u} = (\bar{u}_n)_{n \in \mathbb{N}}$ , where  $(u_n)_{n \in \mathbb{N}}$  is a

Cauchy sequence for the  $V$ -norm, so far no risk of confusion occurs,

$$\|u\|_V = \lim_{n \rightarrow \infty} \left( \int_{\Omega} \rho |\nabla u_n|^2 \right)^{\frac{1}{2}}, \quad (5.16)$$

and for  $v = (v_n)_{n \in \mathbb{N}}$ ,

$$(u, v)_V = \lim_{n \rightarrow \infty} \int_{\Omega} \rho \nabla u_n \cdot \nabla v_n, \quad (5.17)$$

which is easily checked that both sequence involed in the r.h.s of (5.16) and (5.17) are Cauchy sequences in  $\mathbb{R}$ . Therefore,  $V$  is a Hilbert space.

As the  $V$ -norm does not depends on the representative in the quotient by the constants, and because a Cauchy sequence of  $\mathcal{C}^1(\overline{\Omega})$  for the  $W$ -norm is also a Cauchy sequence for the  $V$ -norm, we obviouly have  $W \hookrightarrow V$ , and we see that

$$\forall u \in W, \quad \|u\|_V + \|\text{tru}\|_{0,2,\Gamma} = \|u\|_W, \quad (5.18)$$

where  $\text{tru} \in L^2(\Gamma)$  denotes the generalized trace given in Definition (5.2.1). Moreover, for  $u, v \in W$ ,

$$\forall u, v \in W, \quad (u, v)_V + (\text{tru}, \text{trv})_{0,2,\Gamma} = (u, v)_W, \quad (5.19)$$

which allows to define the projection of  $u \in W$  on  $V$ , which amounts to "forgetting" the trace of u, and see  $W$  as the product of a closed subspace of  $W$  by  $L^2(\Gamma)$ . Unfortunately it is a little more complicated than that. The following construction aims to try to explain this complexity.

The issue is due to the orthogonal of  $\overline{\mathcal{D}(\Omega)}$ , where  $\overline{\mathcal{D}(\Omega)}$  is the adherence of  $\mathcal{D}(\Omega)$  in  $V$ . Let  $T$  denotes this orthogonal, which may be not reduced to 0. Consider indeed the two dimensional torus  $\Omega = \{x \in \mathbb{R}^2, 0 < R_m < |x| < R_M\}$ ,

$$\rho(r) = \frac{(r - R_m)(R_M - r)}{R_M - R_m}.$$

Then it is easily checked that there is a non trivial radial solution  $u(r) \in \mathcal{C}^1(\overline{\Omega})$  to the equation  $\text{div}(\rho \nabla u) = 0$  (see section 5.4 for such calculations). Therefore, for all  $v \in \mathcal{D}(\Omega)$ ,

$$(u, v)_V = \int_{\Omega} \rho \nabla u \cdot \nabla v = 0,$$

hence  $u \in T$ .

We are led to introduce the quotient space  $V/T$  equipped with the quotient norm

$$\|\bar{u}\|_{V/T} = \inf\{\|v\|_V, u - v \in T\}.$$

We are able to states the following structure result.

**Lemma 5.2.2.** *The space  $W$  is continuously embedded in the space*

$$Z = V/T \times L^2(\Gamma). \quad (5.20)$$

*Proof.* We equip the space  $V/T$  with the standard quotient norm, the space  $Z$  with the product norm. Let

$$\psi : \begin{cases} W & \rightarrow & Z \\ u & \mapsto & (\bar{u}, \text{tr}u), \end{cases} \quad (5.21)$$

From the definition of the quotient norm,  $\|\bar{u}\|_{V/T} \leq \|u\|_V$ . Therefore by (5.18)

$$\|\psi(u)\|_Z \leq \|u\|_W.$$

□

Unfortunately, we are not able to prove that  $\text{Im}(\psi)$  is closed in  $Z$ . At this stage, we are able to prove the following result.

**Proposition 5.2.1.** *Let  $(\bar{u}_n, h_n)_{n \in \mathbb{N}}$  be a sequence in  $\text{Im}(\psi)$  that converges to some  $(\bar{u}, h) \in Z$ . Then there exists  $v \in W$  such that  $\bar{u} = \bar{v}$ , and a function  $\tilde{h} \in L^2(\Gamma)$  such that  $\text{tr}v = h + \tilde{h}$ .*

*Proof.* Since  $(\bar{u}_n, h_n) \in \text{Im}(\psi)$ , there exists  $v_n \in W$  such that  $(\bar{u}_n, h_n) = (\bar{v}_n, \text{tr}v_n)$ . Therefore,  $u_n - v_n \in T$ , which means

$$\forall \zeta \in \mathcal{D}(\Omega), \quad (v_n, \zeta)_V = (u_n, \zeta)_V. \quad (5.22)$$

Moreover, by the convergence of  $(\bar{u}_n)_{n \in \mathbb{N}}$ , the sequence  $(v_n)_{n \in \mathbb{N}}$  is bounded in  $W$ . Therefore we can extract a subsequence, still denoted  $(v_n)_{n \in \mathbb{N}}$  that weakly converges to some  $v$  in  $W$ , which means that for all  $\varphi \in \mathcal{C}^1(\bar{\Omega})$ , we have by (5.19)

$$(v_n, \varphi)_V + \int_{\Gamma} v_n \varphi \xrightarrow{n \rightarrow +\infty} (v, \varphi)_V + \int_{\Gamma} v \varphi. \quad (5.23)$$

Taking  $\zeta \in \mathcal{D}(\Omega)$  in (5.23) combined with (5.22) and the convergence of  $(\bar{u}_n)_{n \in \mathbb{N}}$  to  $\bar{u}$  in  $V$ , yields

$$(v_n, \zeta)_V = (u_n, \zeta)_V \xrightarrow{n \rightarrow +\infty} (u, \zeta)_V = (v, \zeta)_V.$$

Therefore,  $\bar{u} = \bar{v}$ , which is the first part of the claim.

Moreover, for all  $g \in L^2(\Gamma)$ ,

$$\int_{\Gamma} \text{tr} v_n g = \int_{\Gamma} h_n g \xrightarrow{n \rightarrow +\infty} \int_{\Gamma} h g \quad (5.24)$$

It is natural to ask whether  $h = \text{tr} v$ , or if not what is missing. The idea is to combine (5.23) and (5.24), starting from a given  $g \in L^2(\Omega)$ . The problem is that such a  $g$  is not necessarily the trace of something defined on  $\Omega$ . Our strategy is then to start from a trace space  $H^{s-1/2}(\Gamma)$  which is everywhere dense in  $L^2(\Gamma)$  for  $s$  large enough, and to consider  $\varphi \in H^s(\Omega)$  in (5.23) the trace of which is  $g$  and which is of class  $\mathcal{C}^1$  at least.

Let  $s > 5/2$  such that  $s - 3/2 \notin \mathbb{N}$ . As  $\Omega$  is of class  $\mathcal{C}^2$ , then  $H^{s-1/2}(\Gamma)$  is everywhere dense in  $L^2(\Gamma)$ , and  $H^s(\Omega)$  is everywhere dense in  $\mathcal{C}^1(\Omega)$  (see [70, chapter 4]). Let  $g \in H^{s-1/2}(\Gamma)$ , and let  $\varphi \in H^s(\Omega)$  be such that  $\text{tr} \varphi = g$ . Then let  $\zeta_\varepsilon \in \mathcal{C}^\infty(\bar{\Omega})$  be a cutoff function, be such that  $\text{supp } \zeta_\varepsilon \subset \{x \in \Omega, \rho(x) \leq \varepsilon\} = C_\varepsilon$ ,  $\zeta_\varepsilon = 1$  over  $\Gamma$ , with  $|\nabla \zeta_\varepsilon| = O(1/\varepsilon)$ . From (5.23) and (5.24), we get

$$\int_{\Gamma} v g = \int_{\Gamma} h_n g + (v_n - v, \zeta_\varepsilon \varphi)_V \quad (5.25)$$

Let us focus on the term

$$F_{n,\varepsilon} = (v_n - v, \zeta_\varepsilon \varphi)_V.$$

Of course, if we would be able to prove the convergence of  $(v_n)_{n \in \mathbb{N}}$  to  $v$  in  $V$ , then the term  $F_{n,\varepsilon}$  would go to zero when  $n \rightarrow \infty$ , and from (5.25) we deduce that for all  $g \in H^{s-1/2}(\Gamma)$

$$\int_{\Gamma} h g = \int_{\Gamma} h \text{tr} v,$$

then  $h = \text{tr} v$  by the density of  $H^{s-1/2}(\Gamma)$  in  $L^2(\Gamma)$ , hence  $(\bar{u}, h) = \psi(v)$ . But things does not work that easily. Indeed from  $u_n - v_n \in T$ , we only can decompose  $v_n$  as  $v_n = u_n + r_n$ . Here so far we have no more control about  $r_n$ , we cannot take advantage from the convergence of  $(u_n)_{n \in \mathbb{N}}$  to  $u$  in  $V$ . On the other hand, we observe that we have  $\zeta_\varepsilon \varphi \in H^s(\Omega) \subset H^1(\Omega)$ , the support of which is in  $C_\varepsilon$ . Therefore, as  $\mathcal{C}^1(\bar{\Omega})$  is everywhere dense in  $H^1(\Omega)$  and  $\rho$  is

bounded,  $H^1$  is naturally embedded in  $V$  and we have

$$\|\zeta_\varepsilon \varphi\|_V \leq \left( \int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 \right)^{1/2}.$$

Hence, we deduce from the Cauchy-Schwarz inequality, combined with the previous inequality,

$$|F_{n,\varepsilon}| \leq \|v_n - v\|_V \left( \int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 \right)^{1/2} \leq C \left( \int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 \right)^{1/2}. \quad (5.26)$$

We have

$$\int_{C_\varepsilon} \rho |\nabla(\zeta_\varepsilon \varphi)|^2 = O \left( \int_{C_\varepsilon} \rho \zeta_\varepsilon^2 |\nabla \varphi|^2 + \frac{2}{\varepsilon} \int_{C_\varepsilon} \zeta_\varepsilon \rho \varphi \nabla \varphi + \frac{1}{\varepsilon^2} \int_{C_\varepsilon} \rho \varphi^2 \right) \\ I_1 + I_2 + I_3$$

Because  $\rho \leq \varepsilon$  in  $C_\varepsilon$ , we get  $I_1 = O(\varepsilon^2)$ ,  $I_2 = O(\varepsilon)$ . The third integral is more problematic. Indeed we have

$$I_3 \leq \frac{1}{\varepsilon} \int_{C_\varepsilon} \varphi^2 \xrightarrow{\varepsilon \rightarrow 0} \int_{\Gamma} g^2,$$

by the mean value theorem, and also because  $\text{tr} \varphi = g$  on  $\Gamma$ . Therefore,  $F_{n,\varepsilon}$  is uniformly bounded in  $n$  and  $\varepsilon$ , and at fixed  $n$ , eventually up to a subsequence,  $F_{n,\varepsilon} \rightarrow \mu_n(g)$  as  $\varepsilon$  goes to 0, and  $\mu_n$  satisfies  $|\mu_n(g)| \leq C \|g\|_{L^2(\Gamma)}$ . Then, up to a subsequence, when  $n \rightarrow \infty$ , we deduce from (5.24) and (5.25) that there exists a continuous linear operator on  $L^2(\Gamma)$  such that for all  $g \in L^2(\Gamma)$ ,

$$\int_{\Gamma} \text{tr} v g = \int_{\Gamma} h g + \mu(g).$$

But as  $\mu$  is linear continuous on  $L^2(\Gamma)$ , there exists  $\tilde{h} \in L^2(\Gamma)$  such that  $\mu(g) = (\tilde{h}, g)$ , hence  $\text{tr} v = h + \tilde{h}$ , which concludes the proof.  $\square$

### 5.2.3 Passing to the limit: main result

We still assume that (5.5), (5.6) and (5.7) hold. We study in this section how to pass to the limit when  $\varepsilon \rightarrow 0$ . We still denote by  $\psi$  the isomorphism defined by (5.21). We assume in addition that  $\Omega$  is bounded. Our main result is the following.

**Theorem 5.2.1.** *Let  $u = u_\varepsilon \in \mathcal{C}^1(\overline{\Omega}) \cap H^2(\Omega)$  be the solution of Problem (5.2) for a given  $\varepsilon > 0$ . Then, there exists a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0, and a function  $u \in W$ ,*

such that  $(u_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges in  $W$  to  $u$ , which satisfies  $\forall v \in W$ ,

$$(u, v)_W =_{W'} \langle f, u \rangle_W \quad (5.27)$$

*Proof.* The function  $u_\varepsilon$  verifies the estimate (5.9), i.e.<sup>1</sup>

$$\int_{\Omega} \varrho_\varepsilon |\nabla u_\varepsilon|^2 + \int_{\Gamma} |u_\varepsilon|^2 \leq C, \quad (5.28)$$

which yields, since  $\varrho \geq 0$

$$\varepsilon \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C. \quad (5.29)$$

In particular as  $u_\varepsilon \in \mathcal{C}^1(\overline{\Omega})$ ,  $u_\varepsilon \in W$ , then we can rewrite (5.28) as

$$\|u_\varepsilon\|_W \leq C. \quad (5.30)$$

Therefore, there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0,  $u \in W$ , such that  $(u_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges to  $u$  in  $W$ . It remains to check what variational problem is satisfied by  $u$ . From the weak convergence in  $W$  we deduce that  $\forall v \in \mathcal{C}^1(\overline{\Omega})$ ,

$$\int_{\Omega} \varrho \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Gamma} u_{\varepsilon_n} v \xrightarrow{n \rightarrow \infty} (u, v)_W. \quad (5.31)$$

Since  $u_{\varepsilon_n} \in \mathcal{C}^1(\overline{\Omega}) \cap H^2(\Omega)$  is a weak solution to (5.2), we have  $\forall v \in \mathcal{C}^1(\overline{\Omega})$ ,

$$\int_{\Omega} \varrho_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Gamma} u_{\varepsilon_n} v = \int_{\Omega} f v. \quad (5.32)$$

In particular,

$$\int_{\Omega} \varrho_{\varepsilon_n} \nabla u_{\varepsilon_n} \cdot \nabla v = \varepsilon_n \int_{\Omega} \nabla u_{\varepsilon_n} \cdot \nabla v + \int_{\Omega} \varrho \nabla u_{\varepsilon_n} \cdot \nabla v. \quad (5.33)$$

We deduce from (5.29)

$$\varepsilon_n \left| \int_{\Omega} \nabla u_{\varepsilon_n} \cdot \nabla v \right| \leq \varepsilon_n \|\nabla u_{\varepsilon_n}\|_{L^2} \|\nabla v\|_{L^2} \leq \varepsilon_n^{1/2} C^{1/2} \|\nabla v\|_{L^2} \xrightarrow{n \rightarrow +\infty} 0, \quad (5.34)$$

hence (5.27) follows by (5.31), (5.34) and the density of  $\mathcal{C}^1(\overline{\Omega})$  in  $W$ .  $\square$

**Remark 5.2.4.** We could have first remarked that Problem (5.27) has a unique solution by the Lax-Milgram Theorem. Theorem 5.2.1 states that this solution is a limit of solutions

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1.  $C$  denotes any generic constant

of partial differential equations. But nothing indicates that it is a solution to a PDE. At this stage it is only a solution to a variational problem.

## 5.3 Some explicit 1D Examples

In this section we provide some explicit examples which illustrate the different behavior of the solutions, even of a further simplification of the toy problem, restricting to a 1D case in order to be able to perform the explicit and exact computation of the solution, obtaining results which are not provable by the classical variational methods for weak solutions and weak limits, as those constructed in the previous section.

### 5.3.1 General solutions on an interval

On the interval  $\Omega = ]0, 1[$ , the problem (5.2) becomes

$$\begin{aligned} \forall z \in ]0, 1[, \quad & -\frac{d}{dz} ((\varrho(z) + \varepsilon)u'_\varepsilon(z)) = f(z), \\ & \varepsilon u'_\varepsilon(0) = u_\varepsilon(0), \\ & -\varepsilon u'_\varepsilon(1) = u_\varepsilon(1), \end{aligned} \tag{5.35}$$

where  $\varrho \in C^1([0, 1])$  satisfies

$$\varrho(z) \underset{z \rightarrow 0^+}{\sim} z \quad \text{and} \quad \varrho(z) \underset{z \rightarrow 1^-}{\sim} 1 - z. \tag{5.36}$$

In this case, the function  $g$  of (5.6) is given by

$$\forall z \in [0, 1], \quad g(z) = \int_0^z f(s) ds. \tag{5.37}$$

And (5.5) yields

$$g(1) = \int_0^1 f(s) ds = 0. \tag{5.38}$$

**Lemma 5.3.1.** *The problem (5.35) has a unique solution on  $[0, 1]$ , which is given by*

$$u_\varepsilon : z \mapsto R_\varepsilon(z) \int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt - \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt, \tag{5.39}$$

Where  $R_\varepsilon$  is given by

$$\forall z \in [0, 1], \quad R_\varepsilon(z) = \frac{1 + \int_0^z \frac{dt}{\varrho(t) + \varepsilon}}{2 + \int_0^1 \frac{dt}{\varrho(t) + \varepsilon}}. \quad (5.40)$$

*Proof.* Let  $u_\varepsilon$  satisfy (5.35). The solution of first equation is

$$\forall z \in [0, 1], \quad u_\varepsilon(z) = d_0 + c_0 \int_0^z \frac{dt}{\varrho(t) + \varepsilon} - \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt. \quad (5.41)$$

The first boundary condition of (5.35) directly yields  $d_0 = c_0$ , which means

$$u_\varepsilon(z) = c_0 \left( 1 + \int_0^z \frac{dt}{\varrho(t) + \varepsilon} \right) - \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt. \quad (5.42)$$

In (5.42), the constant  $c_0$  cannot be chosen arbitrarily. Indeed, it comes from the second boundary condition of (5.35).

$$\begin{cases} u_\varepsilon(1) = c_0 \left( 1 + \int_0^1 \frac{dt}{\varrho(t) + \varepsilon} \right) - \int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt, \\ -\varepsilon u'_\varepsilon(1) = -c_0. \end{cases} \quad (5.43)$$

Therefore,  $c_0$  satisfies

$$c_0 \left( 2 + \int_0^1 \frac{dt}{\varrho(t) + \varepsilon} \right) = \int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt. \quad (5.44)$$

This expression of  $c_0$  injected in (5.42) yields (5.39).  $\square$

Before considering explicit examples, there is an interesting convergence result.

**Lemma 5.3.2.** *Let  $\varrho$  satisfy (5.36) and  $g \in \mathcal{C}^1([0, 1])$  such that  $g(0) = g(1) = 0$ . Then the function  $\left[ t \mapsto \frac{g(t)}{\varrho(t)} \right]$  belongs to  $L^1(0, z)$  for  $0 \leq z \leq 1$ , and we have*

$$\int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^z \frac{g(t)}{\varrho(t)} dt. \quad (5.45)$$

*Proof.* Let  $z \in [0, 1[$ . It is clear that  $\left[ t \mapsto \frac{|g(t)|}{\varrho(t)} \right]$  is continuous on  $]0, z]$  since  $\varrho$  is strictly



positive in the interior. Moreover, we have

$$\left| \frac{g(t)}{\varrho(t)} \right| = \frac{|g(t)|}{\varrho(t)} \quad \text{and also} \quad \lim_{t \rightarrow 0^+} \frac{|g(t)|}{\varrho(t)} = \frac{|g'(0)|}{|\varrho'(0)|} = |g'(0)|, \quad (5.46)$$

and the limit vanishes if  $g'(0) = 0$ , .

Consequently, the function  $g(t)/\varrho(t)$  extended by 0 at  $z = 0$  is continuous on  $[0, z]$ , then  $\left[ t \mapsto \frac{g(t)}{\varrho(t)} \right]$  is in  $L^1(0, z)$  for every  $z$  such that  $0 \leq z < 1$ .

The Lebesgue-Levi theorem on monotone convergence then yields

$$\int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^z \frac{g(t)}{\varrho(t)} dt. \quad (5.47)$$

If  $z = 1$ , we use a similar argument to prove

$$\left| \frac{g(t)}{\varrho(t)} \right| = \frac{|g(t)|}{\varrho(t)} \quad \text{and} \quad \lim_{t \rightarrow 1^-} \frac{|g(t)|}{\varrho(t)} = \frac{|g'(1)|}{|\varrho'(1)|} = |g'(1)|. \quad (5.48)$$

If  $g'(1) = 0$ , the limit vanishes and again

$$\int_0^1 \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^1 \frac{g(t)}{\varrho(t)} dt. \quad (5.49)$$

□

**Corollary 2.** *With the same assumptions as before, we have*

$$\int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt \xrightarrow[\varepsilon \rightarrow 0]{L^p} \int_0^z \frac{g(t)}{\varrho(t)} dt, \quad (5.50)$$

for  $1 \leq p < +\infty$ .

*Proof.* The previous Lemma has shown the pointwise convergence, then only the domination has to be proved. Let  $\varepsilon > 0$ , we have

$$\left| \int_0^z \frac{g(t)}{\varrho(t) + \varepsilon} dt - \int_0^z \frac{g(t)}{\varrho(t)} dt \right| = \left| \int_0^z \frac{\varepsilon g(t)}{\varrho(t)(\varrho(t) + \varepsilon)} dt \right| \leq \int_0^z \frac{|g(t)|}{\varrho(t)} dt \leq \int_0^1 \frac{|g(t)|}{\varrho(t)} dt.$$

By the previous lemma, this constant is finite, then the  $L^p$  Lebesgue Theorem yields the result.

□

### 5.3.2 Examples with $\varrho(z) = z(1 - z)$

We now give some explicit examples when the function  $\varrho$  vanishes linearly at both  $z = 0$  and  $z = 1$  and we use the explicit form to study the regularity of the solution to (5.35).

**Lemma 5.3.3.** *Let us suppose that  $\varrho(z) = z(1 - z)$ . Then, the function  $R_\varepsilon$  is continuous on  $[0, 1]$  and  $\lim_{\varepsilon \rightarrow 0^+} R_\varepsilon = \frac{1}{2}$  in  $L^p(0, 1)$ , for  $1 \leq p < +\infty$ .*

*Proof.* We calculate for  $0 \leq z \leq 1$  :

$$\int_0^z \frac{dt}{\varrho(t) + \varepsilon} = \int_0^z \frac{dt}{\varepsilon + t - t^2}. \quad (5.51)$$

The two zeros of the polynomial are

$$r_1(\varepsilon) = \frac{1 + \sqrt{1 + 4\varepsilon}}{2} > 1 \quad \text{and} \quad r_2(\varepsilon) = \frac{1 - \sqrt{1 + 4\varepsilon}}{2} < 0. \quad (5.52)$$

that yields

$$\begin{aligned} \int_0^z \frac{dt}{\varrho(t) + \varepsilon} &= \int_0^z \frac{dt}{(r_1(\varepsilon) - t)(t - r_2(\varepsilon))} \\ &= \frac{1}{r_1(\varepsilon) - r_2(\varepsilon)} \left( \int_0^z \frac{dt}{r_1(\varepsilon) - t} + \int_0^z \frac{dt}{t - r_2(\varepsilon)} \right) \\ &= \frac{1}{\sqrt{1 + 4\varepsilon}} \left( \ln \left( \frac{r_1(\varepsilon)}{r_1(\varepsilon) - z} \right) + \ln \left( \frac{z - r_2(\varepsilon)}{-r_2(\varepsilon)} \right) \right). \end{aligned} \quad (5.53)$$

This function is continuous on  $[0, 1]$ , which implies  $R_\varepsilon \in \mathcal{C}^0([0, 1])$ . By observing that  $1 - r_2(\varepsilon) = r_1(\varepsilon)$ , we choose to write  $r(\varepsilon) := r_1(\varepsilon)$ , and after some manipulations the expression of  $R_\varepsilon(z)$  for  $0 < z < 1$  becomes the following

$$R_\varepsilon(z) = \frac{1 + \frac{1}{\sqrt{1+4\varepsilon}} \left( \ln \left( \frac{r(\varepsilon)}{r(\varepsilon)-z} \right) + \ln \left( \frac{r(\varepsilon)-(1-z)}{r(\varepsilon)-1} \right) \right)}{2 + \frac{2}{\sqrt{1+4\varepsilon}} \ln \left( \frac{r(\varepsilon)}{r(\varepsilon)-1} \right)}. \quad (5.54)$$

Then, we can calculate the following limit

$$R_\varepsilon(z) \underset{\varepsilon \rightarrow 0}{\sim} \frac{-\ln(r(\varepsilon) - 1)}{-2 \ln(r(\varepsilon) - 1)} = \frac{1}{2}. \quad (5.55)$$

We deduce that  $R_\varepsilon$  converges to  $1/2$  almost everywhere on  $[0, 1]$ .

Then (5.54) yields the following estimate

$$\left| R_\varepsilon(z) - \frac{1}{2} \right| = \frac{1}{2\sqrt{1+4\varepsilon}} \times \frac{\left| \ln \left( \frac{r(\varepsilon)-(1-z)}{r(\varepsilon)-z} \right) \right|}{1 + \frac{1}{\sqrt{1+4\varepsilon}} \ln \left( \frac{r(\varepsilon)}{r(\varepsilon)-1} \right)}. \quad (5.56)$$

Which allows to obtain the simpler upper bound

$$\left| R_\varepsilon(z) - \frac{1}{2} \right| \leq \frac{1}{2}, \quad (5.57)$$

which shows, by applying the Lebesgue dominated convergence theorem the result.  $\square$

The above result has the following consequence on the characterization of the limit of  $u_\varepsilon$ .

**Corollary 3** (Condition for the boundary conditions of the limit). *If  $\varrho(z) = z(1-z)$ , the sequence of solutions  $(u_\varepsilon)_{\varepsilon>0}$  converges to  $u \in L^p(0,1)$ , for all  $1 \leq p < +\infty$ .*

*Moreover, the limit  $u$  satisfies  $u(0) = u(1) = 0$  if and only if  $\int_0^1 \frac{g(t)}{t(1-t)} dt = 0$ .*

*Proof.* Lemmas 5.3.2 and 5.3.3 yield the expression for  $u$ :

$$\forall z \in ]0, 1[, \quad u(z) = \frac{1}{2} \int_0^1 \frac{g(t)}{t(1-t)} dt - \int_0^z \frac{g(t)}{t(1-t)} dt. \quad (5.58)$$

In particular

$$u(0) = \frac{1}{2} \int_0^1 \frac{g(t)}{t(1-t)} dt \quad \text{and} \quad u(1) = -\frac{1}{2} \int_0^1 \frac{g(t)}{t(1-t)} dt, \quad (5.59)$$

from which the result follows.  $\square$

### 5.3.3 An example with less regularity

We give now an example when the function  $g$  is less regular at  $z = 0$ . In particular, we assume that it vanishes logarithmically, instead than linearly. To be more precise let us assume that  $g : [0, 1] \rightarrow \mathbb{R}$  satisfies the following assumptions

- $g(0) = 0$
- $g(t) = \frac{1}{\ln(t)}$  if  $0 < t < e^{-1}$
- $g(t) = 1 - t$  if  $\frac{1}{2} < t \leq 1$
- $g \in \mathcal{C}^1(]0, 1])$ .

It is clear that the class above is non empty, since it is enough to extend  $g$  in the interval  $[e^{-1}, 1/2]$  in a differentiable way matching values of function and derivative.

**Theorem 5.3.1.** *Let  $g$  satisfy the previous assumptions, and  $s \in [0, 1]$ . Then, it follows that  $g \in H^s(0, 1)$  if and only if  $0 \leq s \leq \frac{1}{2}$ .*

*Proof.* First observe that in the interval  $[e^{-1}, 1]$  the function is smooth so it belongs to  $H^1(e^{-1}, 1)$  and the only problem is the behavior at zero. The derivative of  $g$  on  $]0, e^{-1}[$  is

$$\forall t \in ]0; e^{-1}[, \quad g'(t) = -\frac{1}{t \ln(t)^2}. \quad (5.60)$$

This function is not in  $L^2(0, e^{-1})$ , which means  $g \notin H^1(0; e^{-1})$ , then  $g \notin H^1(0, 1)$ . Let  $s \in ]0, 1[$ , since  $g$  is of class  $\mathcal{C}^1$  on  $]0, 1]$ , it is sufficient to prove the convergence of the double integral

$$\int_{x=0}^{e^{-1}} \int_{y=0}^{e^{-1}} \frac{|f(x) - f(y)|^2}{|x - y|^{2s+1}} dx dy, \quad (5.61)$$

which will give, if finite, the Sobolev semi-norm in  $H^s(0, e^{-1})$ .

This integral can be explicitly rewritten as follows

$$\int_{x=0}^{e^{-1}} \int_{y=0}^{e^{-1}} \frac{\ln(y/x)^2}{\ln(x)^2 \ln(y)^2 |x - y|^{2s+1}} dx dy. \quad (5.62)$$

Applying the change of variables

$$\begin{cases} u = \ln(y/x) \\ v = \ln(x) \end{cases} \quad \text{hence} \quad \begin{cases} x = e^v \\ y = e^{u+v}, \end{cases} \quad (5.63)$$

which sends the domain  $(0, e^{-1})^2 \in \mathbb{R}_x \times \mathbb{R}_y$  into  $\mathbb{R} \times (-\infty, 1) \in \mathbb{R}_u \times \mathbb{R}_v$ , the integral (5.62) becomes

$$\begin{aligned} & \int_{u=-\infty}^{+\infty} \int_{v=-\infty}^{-1} \frac{u^2 e^{u+2v} \mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2 e^{(2s+1)v} |e^u - 1|^{2s+1}} dudv \\ &= \int_{-\infty}^{+\infty} \frac{u^2 e^u}{|e^u - 1|^{2s+1}} \left( \int_{-\infty}^{-1} \frac{e^{(1-2s)v} \mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2} dv \right) du. \end{aligned} \quad (5.64)$$

If  $1 - 2s < 0$ , the middle term of (5.64) does not converge. Therefore, the integral is not defined if  $1 - 2s < 0$ , which means  $s > \frac{1}{2}$ . On the other hand, if  $s < \frac{1}{2}$  the integral converges as it easily follows.

Let us now focus on the limiting case  $s = \frac{1}{2}$ . In this case the integral (5.64) then becomes

$$\int_{-\infty}^{+\infty} \frac{u^2 e^u}{|e^u - 1|^2} \left( \int_{-\infty}^{-1} \frac{\mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2} dv \right) du. \quad (5.65)$$

We calculate the middle integral, by using the following decomposition by simple rational functions. Let  $u, v \in \mathbb{R}$  satisfy  $v < -1$  and  $u + v < -1$ , then

$$\frac{1}{v^2 (u+v)^2} = \frac{1}{u^2} \left( \frac{1}{v^2} + \frac{1}{(u+v)^2} \right) + \frac{2}{u^3} \left( \frac{1}{u+v} - \frac{1}{v} \right). \quad (5.66)$$

This implies that we obtain for  $u < 0$

$$\int_{-\infty}^{-1} \frac{\mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2} dv = \int_{-\infty}^{-1} \frac{dv}{v^2 (u+v)^2} = \frac{1}{u^2} \left( 1 + \frac{1}{1-u} \right) + \frac{2}{u^3} \ln(1-u), \quad (5.67)$$

and for  $u > 0$

$$\int_{-\infty}^{-1} \frac{\mathbb{1}_{\{u+v < -1\}}}{v^2 (u+v)^2} dv = \int_{-\infty}^{-(1+u)} \frac{dv}{v^2 (u+v)^2} = \frac{1}{u^2} \left( 1 + \frac{1}{1+u} \right) - \frac{2}{u^3} \ln(1+u). \quad (5.68)$$

By a Taylor series expansion, we can evaluate the two limits as follows

$$\frac{1}{u^2} \left( 1 + \frac{1}{1-u} \right) + \frac{2}{u^3} \ln(1-u) \xrightarrow{u \rightarrow 0^-} \frac{1}{3}$$

and

$$\frac{1}{u^2} \left( 1 + \frac{1}{1+u} \right) - \frac{2}{u^3} \ln(1+u) \xrightarrow{u \rightarrow 0^+} \frac{1}{3}.$$

Then it is easy to show the convergence of the integral (5.65). □

Once proved the preliminary regularity of  $g$ , let  $f$  be defined as the derivative of  $g$  on  $]0, 1[$ . This nonsmooth function satisfies (in particular)

$$\begin{aligned} & - f \in \mathcal{C}^0(]0, 1]) & - f(t) = -1 & \text{if } \frac{1}{2} < t \leq 1. \\ & - f(t) = \frac{-1}{t \ln(t)^2} & \text{if } 0 < t \leq e^{-1} \end{aligned}$$

We now try to solve the problem (5.35) with  $f$  defined this way, and  $\varrho(z) = z(1 - z)$ . Since  $g$  is absolutely continuous in  $[0, 1]$  (note that it is differentiable in  $]0, 1]$  and

$\int_0^{e^{-1}} |t \log^2(t)|^{-1} dt$  is finite and  $g(0) = 0$ ), we can write

$$\forall t \in ]0, 1[, \quad g(t) = \int_0^t f(s) ds.$$

Therefore, a calculation as in Lemma 5.3.1 can be applied and yields for  $0 < z < 1$

$$u_\varepsilon(z) = R_\varepsilon(z) \int_0^1 \frac{g(t)}{t(1-t) + \varepsilon} dt - \int_0^z \frac{g(t)}{t(1-t) + \varepsilon} dt,$$

which can be rewritten as follows

$$u_\varepsilon(z) = (R_\varepsilon(z) - 1) \int_0^{e^{-1}} \frac{g(t)}{t(1-t) + \varepsilon} dt + R_\varepsilon(z) \int_{e^{-1}}^1 \frac{g(t)}{t(1-t) + \varepsilon} dt - \int_{e^{-1}}^z \frac{g(t)}{t(1-t) + \varepsilon} dt.$$

On one hand, the Lebesgue theorem yields for  $0 < z < 1$ , the following convergence

$$R_\varepsilon(z) \int_{e^{-1}}^1 \frac{g(t)}{t(1-t) + \varepsilon} dt - \int_{e^{-1}}^z \frac{g(t)}{t(1-t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} \int_{e^{-1}}^1 \frac{g(t)}{t(1-t)} dt - \int_{e^{-1}}^z \frac{g(t)}{t(1-t)} dt.$$

On the other hand, the Beppo Levi theorem yields for  $0 < z < 1$

$$- \int_0^{e^{-1}} \frac{g(t)}{t(1-t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^{e^{-1}} \frac{dt}{t |\ln(t)|(1-t)} = +\infty.$$

Hence, since  $R_\varepsilon(z) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2}$ :

$$(R_\varepsilon(z) - 1) \int_0^{e^{-1}} \frac{g(t)}{t(1-t) + \varepsilon} dt \xrightarrow{\varepsilon \rightarrow 0} +\infty. \quad (5.69)$$

This shows that the sequence of the approximated solutions does not converge almost everywhere.

## 5.4 A 2D problem: The Case of the Disk

In this section we consider a 2D case with circular symmetry (which in turn reduces again to a 1D problem) and we illustrate other interesting phenomena.

In this section we assume that the domain  $\Omega$  is the unit ball  $\mathbb{B}(0, 1) \subset \mathbb{R}^2$ . What is convenient in this geometry is the fact that one can look for radial solutions as long as the source  $f$  is radial. In particular, choosing  $\varrho(x) = d(x, \partial\mathbb{B}(0, 1))$  the function  $\varrho$  will be

given by

$$\text{if } |x| = r \quad \text{then} \quad \varrho(r) = 1 - r. \quad (5.70)$$

The equation (5.2) becomes in polar variables  $(r, \theta)$ , when the data are independent of  $\theta$  by circular symmetry:

$$-\frac{\partial u_\varepsilon}{\partial r} + (1 - r + \varepsilon) \left( \frac{\partial^2 u_\varepsilon}{\partial r^2} + \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} \right) = f(r), \quad (5.71)$$

and the boundary condition is  $-\varepsilon \frac{\partial u}{\partial r} = u$  on  $r = 1$ .

### 5.4.1 The associated Homogeneous equation

We start by studying the the corresponding homogeneous equation

$$\frac{\partial^2 u_\varepsilon}{\partial r^2} + \left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} = 0, \quad (5.72)$$

for every  $r > 0$ . As

$$\left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} = \frac{1}{r} - \frac{1}{1 + \varepsilon - r},$$

we find that the solution of (5.72) is the function  $\frac{\partial u_\varepsilon}{\partial r}$

$$r \mapsto \frac{1}{r(1 + \varepsilon - r)},$$

### 5.4.2 Blow up example

Assume now that the source  $f$  is constant and it is equal to 1. Note that with these assumptions  $f$  does not longer satisfies (5.5), (5.6), and (5.7). The equation (5.71) becomes

$$\frac{\partial^2 u_\varepsilon}{\partial r^2} + \left( \frac{1 + \varepsilon - 2r}{1 + \varepsilon - r} \right) \frac{1}{r} \frac{\partial u_\varepsilon}{\partial r} = \frac{1}{1 - r + \varepsilon}. \quad (5.73)$$

Thanks to the variation of constants method, the solutions of (5.73) are

$$r \mapsto \frac{K_1}{r(1 + \varepsilon - r)} + \frac{1}{2} \frac{r}{(1 + \varepsilon - r)}. \quad (5.74)$$

Since

$$\frac{K_1}{r(1 + \varepsilon - r)} = \frac{K_1}{1 + \varepsilon} \left( \frac{1}{r} + \frac{1}{1 + \varepsilon - r} \right),$$

and

$$\frac{r}{1 + \varepsilon - r} = \frac{1 + \varepsilon}{1 + \varepsilon - r} - 1,$$

the radial solution  $u_\varepsilon$  is given by

$$u_\varepsilon(r) = \frac{K_1}{1 + \varepsilon} \log\left(\frac{r}{1 + \varepsilon - r}\right) - \frac{r}{2} - \frac{(1 + \varepsilon)}{2} \log(1 + \varepsilon - r) + K_2, \quad (5.75)$$

where the constants  $K_1$  and  $K_2$  can be retrieved from the boundary conditions. From the condition

$$-\varepsilon \frac{\partial u_\varepsilon}{\partial r}(1) = u_\varepsilon(1),$$

we get

$$-\left(K_1 + \frac{1}{2}\right) = \frac{K_1}{1 + \varepsilon} \log\left(\frac{1}{\varepsilon}\right) - \frac{1}{2} - \frac{(1 + \varepsilon)}{2} \log(\varepsilon) + K_2,$$

and the relation

$$K_1 \left(1 + \frac{\log(\varepsilon)}{1 + \varepsilon}\right) + K_2 = \frac{(1 + \varepsilon)}{2} \log(\varepsilon).$$

We want  $u_\varepsilon \in H^1(\Omega)$ . In particular, as

$$u'_\varepsilon(r) \sim \frac{K_1}{(1 + \varepsilon)r}, \quad \text{for } r \rightarrow 0^+,$$

we have to fix  $K_1 = 0$  to have  $u'_\varepsilon \in L^2(\Omega)$ .

It gives :

$$K_2 = \frac{1 + \varepsilon}{2} \log(\varepsilon),$$

and consequently

$$u_\varepsilon(r) = -\frac{r}{2} + \frac{(1 + \varepsilon)}{2} \log\left(\frac{\varepsilon}{1 + \varepsilon - r}\right) = -\frac{r}{2} + \frac{(1 + \varepsilon)}{2} \log\left(1 + \frac{r - 1}{1 + \varepsilon - r}\right). \quad (5.76)$$

We observe that in this case, the solution blows up everywhere when  $\varepsilon \rightarrow 0$ . Here the problem is not that  $u$  is bounded near the boundary, but that  $u$  itself, as limit  $\varepsilon \rightarrow 0$ .

### 5.4.3 Example of a case with a limit

We now consider a source term which is compatible with the weak formulation and in particular, we show how to choose  $f$  by directly finding an appropriate  $\mathbf{g}$  to satisfy (5.5),



(5.6) and (5.7) in order to pass to the limit. The polar basis  $(\mathbf{e}_r, \mathbf{e}_\theta)$  is defined by

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{pmatrix}. \quad (5.77)$$

We are looking for  $\mathbf{g}$  such that

$$\mathbf{g} \cdot \mathbf{n}|_\Gamma = 0,$$

To avoid any singularities at 0, we take  $\mathbf{g}$  of the following form

$$\mathbf{g} = r\varrho(r)\mathbf{e}_r.$$

Therefore,  $f = \nabla \cdot \mathbf{g}$  is given by:

$$\begin{aligned} f(r) &= \frac{1}{r} \left( \frac{\partial}{\partial r} (r^2 \varrho(r)) \right) \\ &= 2\varrho(r) - r. \end{aligned}$$

We take this function as a source term. Using the same notation as before, this gives

$$\frac{K'_1(r)}{r\varrho_\varepsilon(r)} = \frac{f(r)}{\varrho_\varepsilon(r)},$$

and

$$K'_1(r) = 2r - 3r^2.$$

Thus,  $K_1(r) = r^2 - r^3 = r^2\varrho(r)$ , which gives as a solution:

$$u'_\varepsilon(r) = r \frac{\varrho(r)}{\varrho_\varepsilon(r)}. \quad (5.78)$$

In particular,  $u'_\varepsilon(1) = 0$ . The boundary condition implies that  $u_\varepsilon(1) = 0$ . We finally get

$$u_\varepsilon(r) = - \int_1^r r \frac{\varrho(r)}{\varrho_\varepsilon(r)} dr, \quad (5.79)$$

and by the Lebesgue dominated convergence theorem,

$$u_\varepsilon(r) \xrightarrow{\varepsilon \rightarrow 0} \frac{1 - r^2}{2}, \quad (5.80)$$

which is a  $C^\infty$  function that satisfies  $u = 0$  at  $\Gamma = \{r = 1\}$ .

#### 5.4.4 A general class of solutions

We consider now a family of source terms  $f_\varepsilon$  depending on  $\varepsilon$ . Let  $\alpha$  and  $\beta$  be two real numbers, with  $\alpha \geq 0$  and  $\beta \geq 1$ . The idea is to take the source  $f_\varepsilon$  such that the product  $rf_\varepsilon(r)$  gives a derivative with “good” properties. We define the source  $f_\varepsilon = f_{\varepsilon,\alpha,\beta}$  as

$$\forall r \in (0, 1) \quad f_\varepsilon(r) := (\alpha + 1)r^{\alpha-1}(1 + \varepsilon - r)^\beta - \beta(1 + \varepsilon - r)^{\beta-1}r^\alpha.$$

In particular, when applying the variation of constants method, we obtain the following relation

$$K_1'(r) = f_\varepsilon(r)r = (\alpha + 1)r^\alpha(1 + \varepsilon - r)^\beta - \beta(1 + \varepsilon - r)^{\beta-1}r^{\alpha+1},$$

which gives

$$K_1(r) = r^{\alpha+1}(1 + \varepsilon - r)^\beta,$$

and the solutions of (5.71) are the functions

$$r \mapsto \frac{K_1}{r(1 + \varepsilon - r)} + r^\alpha(1 + \varepsilon - r)^{\beta-1}.$$

The integrability of  $|u'_\varepsilon|^2$  implies that  $K_1 = 0$ , so  $u'_\varepsilon$  is given by

$$u'_\varepsilon(r) = r^\alpha(1 + \varepsilon - r)^{\beta-1}. \tag{5.81}$$

If  $\beta$  is an integer, we can give an expression of  $u$  as a sum with a *finite* number of terms). In order to find relevant properties and avoid infinite summation of singular terms, from now on we will make this assumption; we rename  $\beta = k \in \mathbb{N}$  to make it more clear.

We can write

$$u'_\varepsilon(r) = \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i r^{\alpha+i} (1 + \varepsilon)^{k-1-i},$$

which gives by integration

$$u_\varepsilon(r) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{\alpha + i + 1} r^{\alpha+i+1} (1 + \varepsilon)^{k-1-i} + K_2, \tag{5.82}$$

where  $K_2 \in \mathbb{R}$  is a constant we will determine thanks to the boundary condition  $-\varepsilon u'_\varepsilon(1) =$

$u_\varepsilon(1)$ . On the one hand,

$$\varepsilon u'_\varepsilon(1) = \varepsilon^k,$$

on the other hand,

$$u_\varepsilon(1) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{\alpha+i+1} (1+\varepsilon)^{k-i-1} + K_2.$$

Thus,  $K_2$  is a polynomial function in  $\varepsilon$  of degree  $k$ :

$$K_2 = \varepsilon^k + \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^{i+1}}{\alpha+i+1} (1+\varepsilon)^{k-i-1} = \frac{(-1)^k}{\alpha+k} + O(\varepsilon).$$

For instance, if we take  $k = 1$ , we have:

$$u_\varepsilon(r) = \frac{r^{\alpha+1}}{\alpha+1} + \varepsilon - \frac{1}{\alpha+1}$$

and for  $k = 2$ :

$$u_\varepsilon(r) = -\frac{r^{\alpha+2}}{\alpha+2} + \left(\frac{1+\varepsilon}{1+\alpha}\right) r^{\alpha+1} + \varepsilon^2 - \frac{1+\varepsilon}{1+\alpha} + \frac{1}{\alpha+2}.$$

By direct computations we can show that when  $\varepsilon \rightarrow 0^+$

$$u_\varepsilon(r) \rightarrow \frac{(-1)^k}{\alpha+k} (1 - r^{\alpha+k})$$

and in all cases, it vanishes at the boundary.

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**Titre :** Couplage et turbulence à l'interface océan-atmosphère

**Mot clés :** Mécanique des fluides, équations aux dérivées partielles, couplage, turbulence, condition de bord, Freefem++

**Résumé :** Dans ce manuscrit, nous étudions différents problèmes de turbulence de couches limites.

La première partie décrit un modèle simple de couplage de deux fluides caractérisés par le même problème de Stokes incompressible stationnaire. Le système d'équation est notamment fermé par une condition de continuité à l'interface, qui peut être vu comme une condition limite d'un système caractérisé par une condition de Robin à l'interface. Des algorithmes de décomposition de domaine de

type Schwarz permettent d'établir des simulations numériques de ce modèle. Enfin, un modèle plus complexe d'équations de Navier-Stokes stationnaires avec conditions au bord non-linéaires est également étudié et simulé numériquement.

Dans une deuxième partie, on analyse des systèmes d'équations elliptiques scalaires à poids, typiques de sous-couches turbulentes. On établit notamment l'existence de solutions faibles et la correspondance avec la théorie de Monin-Obukhov.

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**Title:** Coupling and turbulence at the ocean-atmosphere interface

**Keywords:** Fluid mechanics, partial differential equations, coupling, turbulence, boundary conditions, Freefem++

**Abstract:** In this manuscript, we investigate various problems related to boundary layer turbulence.

The first part describes a simple model for coupling two fluids characterized by the same stationary incompressible Stokes problem. The equation system is closed with a continuity condition at the interface, which can be viewed as a boundary condition of a system characterized by a Robin condition at the interface. Domain decomposition algorithms of

Schwarz type are used to perform numerical simulations of this model. Finally, a more complex model of stationary Navier-Stokes equations with nonlinear boundary conditions is also studied and numerically simulated.

In the second part, we analyze systems of scalar elliptic equations with weights, typical of turbulent sublayers. We establish the existence of weak solutions and their correspondence with the Monin-Obukhov theory.