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Rémy Larue

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THÈSE

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Spécialité : Physique Théorique

Unité de recherche : Laboratoire de Physique Subatomique et Cosmologie

Quelques aspects de théorie effective des champs et anomalies quantiques en gravité

Aspects of Effective Field Theory and Quantum Anomalies in Gravity

Présentée par :

Rémy LARUE

Direction de thèse :

Christopher SMITH

DIRECTEUR DE RECHERCHE, CNRS DELEGATION ALPES

Directeur de thèse

Jérémie QUEVILLON

CHARGE DE RECHERCHE HDR, CNRS DELEGATION ALPES

Co-directeur de thèse

Rapporteurs :

EMILIAN DUDAS

DIRECTEUR DE RECHERCHE, CNRS DELEGATION ILE-DE-FRANCE SUD

PIERRE VANHOVE

DIRECTEUR DE RECHERCHE, CEA CENTRE DE PARIS-SACLAY

Thèse soutenue publiquement le **14 juin 2024**, devant le jury composé de :

EMILIAN DUDAS,

DIRECTEUR DE RECHERCHE, CNRS DELEGATION ILE-DE-FRANCE SUD

Rapporteur

PIERRE VANHOVE,

DIRECTEUR DE RECHERCHE, CEA CENTRE DE PARIS-SACLAY

Rapporteur

DAVID CARPENTIER,

DIRECTEUR DE RECHERCHE, CNRS DELEGATION RHONE AUVERGNE

Examineur

PIOTR TOURKINE,

CHARGE DE RECHERCHE, CNRS DELEGATION ALPES

Examineur

AURELIEN BARRAU,

PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES

Président

PHILIPPE BRAX,

INGENIEUR HDR, CEA CENTRE DE PARIS-SACLAY

Examineur

SABINE KRAML,

DIRECTRICE DE RECHERCHE, CNRS DELEGATION ALPES

Examinatrice

GENEVIEVE BELANGER,

DIRECTRICE DE RECHERCHE, CNRS DELEGATION ALPES

Examinatrice



Résumé

La Théorie Quantique des Champs (TQC) est un formalisme riche et complexe qui s'est avéré remarquablement fructueux au cours des dernières décennies. Multitudes de travaux ont permis d'améliorer considérablement notre compréhension de la TQC, même s'il reste encore beaucoup à découvrir. L'objectif de cette thèse de doctorat est de contribuer à une meilleure compréhension de certains aspects de la TQC, notamment les Théories Effectives des Champs (TEC) en gravité et les anomalies quantiques (gravitationnelles).

Tout au long de cette thèse, notre principal outil sera l'Intégrale de Chemin, qui est particulièrement adaptée pour traiter les TEC et les anomalies en gravité. Le premier chapitre est donc consacré à l'introduction du concept d'Intégrale de Chemin, sa construction, son interprétation et son utilisation en TQC.

Le paradigme TEC est en plein essor depuis une dizaine d'années en raison de l'absence de détection directe de nouvelle Physique dans les collisionneurs. Cependant, le Modèle Standard (MS) de la Physique des Particules présente des énigmes non résolues, qui pointent vers de la Physique Au-delà du MS. Cela indique par ailleurs que le MS est une théorie incomplète qui n'est plus valide au-dessus d'une certaine échelle d'énergie, et est donc par définition une TEC. Les effets de la gravité dans les TEC concernent de nombreux scénarios (TQC autour des trous noirs, inflation, systèmes de matière condensée, etc...), cependant, les calculs en gravité peuvent rapidement devenir inextricables. Cela motive le développement d'outils de calcul puissants pour contourner cette difficulté. Le deuxième chapitre de cette thèse est donc consacré à l'introduction à la TEC et à la présentation de nos résultats concernant les calculs de TEC en gravité. De plus, nous comblons un manque dans la littérature concernant les fermions chiraux en gravité dans l'Intégrale de Chemin, et obtenons de nouveaux opérateurs effectifs qui étaient omis auparavant.

Le sujet du dernier chapitre de cette thèse est l'étude des anomalies quantiques, qui sont la brisure de symétries classiques par la quantification d'une théorie. Les anomalies apparaissent dans les TEC à basse énergie, et occupent une place prépondérante en raison de leur lien aux invariants topologiques. En conséquence, les anomalies topologiques sont indépendantes de l'échelle d'énergie de la TEC, et fournissent un aperçu direct des effets à haute énergie. Outre leur nature topologique, elles ont d'importantes implications phénoménologiques, l'exemple historique étant la désintégration des pions en deux photons. Les anomalies sont centrales dans la compréhension des TQC, et ont fait l'objet de nombreux débats jusqu'à très récemment. Comme nous le verrons, les difficultés sont principalement dues à leur lien crucial à des divergences qui nécessitent une régularisation et une renormalisation. Ces difficultés sont exacerbées en gravité lorsque plusieurs symétries sont imbriquées. En plus de ces discussions, plusieurs de nos résultats sont présentés dans le chapitre 3. Nous proposons d'abord une méthode efficace pour extraire les anomalies des TEC tout en maintenant sous contrôle les symétries non-anomales. Nous contribuons ensuite à résoudre une controverse concernant la présence

de violation de parité dans l'anomalie de trace d'un fermion de Weyl. Enfin, nous étendons notre résultat précédent pour conclure sur l'absence de violation de parité dans l'anomalie de trace, indépendamment d'un modèle.

Mots clés: Théorie quantique des champs, théories ds champs effectives, intégrale de chemin, anomalies quantiques, gravité.

Abstract

Quantum Field Theory (QFT) is a rich and complex formalism that has proved to be tremendously fruitful over the past decades. Collective endeavor has allowed to greatly enhance our understanding of QFT, yet there remains much to unravel. The aim of this Ph.D. thesis is to help achieve a better understanding of some aspects of QFT, namely Effective Field Theories (EFTs) in curved spacetime and quantum (gravitational) anomalies.

Throughout this thesis, our main tool will be the Path Integral, which is particularly suited when dealing with EFTs and anomalies in gravity. The first Chapter is thus dedicated to introducing the concept of Path Integral, its construction, its interpretation, and its use in QFT.

The EFT paradigm has been in full swing for about a decade due to the lack of direct detection of new Physics in experiments. However, the Standard Model (SM) of Particle Physics exhibits unsolved puzzles, which call for Beyond the SM (BSM) models to resolve them. This indicates that the SM is an incomplete theory that breaks down above some energy scale, which is by definition an EFT. The effects of gravity in EFTs pertain to many scenarii (QFT around black holes, inflation, condensed matter systems, etc...), however computations in curved spacetime can quickly become untractable. This motivates the development of powerful computational tools to circumvent that difficulty. The second Chapter of this thesis is thus dedicated to introducing the EFT paradigm, and to presenting our results concerning EFT computations in gravity. As an interesting novelty, we fill a dearth in the literature concerning chiral fermions in gravity in the Path Integral, and obtain new effective operators that were omitted before.

The subject of the last Chapter of this thesis is the study of quantum anomalies, which are the breaking of classical symmetries upon quantisation of the theory. Anomalies occur in low energy EFTs, and hold a prominent place due to their relation to topological invariants. As a result, topological anomalies are independent from the energy scale of the EFT, and provide direct insight into high energy effects. Besides their topological nature, they have important phenomenological implications, the historical example being the decay of pions into two photons. Anomalies are central in the understanding of QFTs, and have been the subject of many debates up until very recently. As we will see, difficulties are mainly due to their crucial link to divergences that need regularising and renormalising. These difficulties are exacerbated in curved spacetime when several symmetries are intertwined. Along with these discussions, several of our results are presented in Chapter 3. We first propose an efficient method to extract anomalies from EFTs while keeping non-anomalous symmetries under control. We then help solving a controversy regarding the presence of parity violating effects in the trace anomaly of a Weyl fermion. Finally, we extend our previous result to conclude on the absence of parity violating contributions to the trace anomaly in a model-independent manner.

Key words: Quantum Field Theory, Effective Field Theories, Path Integral, quan-

tum anomalies, gravity.

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I am only getting started on the road to become an accomplished physicist, of which this Ph.D. thesis is the first milestone. I wouldn't have achieved this first step without the help of my family, friends, colleagues, supervisor, and some acknowledgements are in order.

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¹Who didn't make it to my defense but okay it's fine.

²That wasn't Corentin.

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A true master, is an eternal student.

- Master Yi

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List of Publications

- [1] **Gravity-gauge anomaly constraints on the energy-momentum tensor**, R. Larue, J. Quevillon, R. Zwicky, *Accepted for publication* [[arXiv:2312.13222 \[hep-th\]](#)]
- [2] **Trace anomaly of Weyl fermions via the path integral**, R. Larue, J. Quevillon, R. Zwicky, *JHEP* **12** (2023) 64, [[arXiv:2309.08670 \[hep-th\]](#)]
- [3] **The one-loop universal effective action with gravity**, R. Larue, J. Quevillon, *JHEP* **11** (2023) 45, [[arXiv:2303.10203 \[hep-th\]](#)]
- [4] **Anomalies from an effective field theory perspective**, B. Filoche, R. Larue, J. Quevillon, P. N.H. Vuong, *Phys. Rev. D* **107** (2023) 025017, [[arXiv:2205.02248 \[hep-th\]](#)].

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Introduction

Quantum Field Theory is a vast and complex area of Physics, which required decades of work to build and understand properly. It probably has culminated with the experimental validation of the Standard Model (SM) of Particle Physics, which closed with the discovery of the Higgs boson in 2012. But from the height of its success, the depth of its shortcomings and intricacies become even more ostensible. That is, despite the indisputable fact that the SM breaks down at higher energies, experiments struggle to find disagreements with the SM.

One of the most the difficult puzzles we encountered surely is the theory of quantum gravity, which has been eluding us for decades. Nevertheless, this does not prevent us to make the most of the effects of classical gravity on quantum particles, and conversely, to obtain new observables that complement collider data. After all, it is only thanks to gravitational effects that Dark Matter (DM) was discovered in the 1930s. Similar observables can arise from cosmology, early universe or black hole Physics.

Despite that new Physics remain out of reach of the experiments, theoretical physicist have found ways to probe higher energies, with and without gravity. This thesis is dedicated to the study of Effective Field Theories (EFTs) in gravity, which aim at describing the effects of high energy Physics at lower energy on a curved background; and quantum gravitational anomalies, which have the remarkable property of being independent of the energy scale, and thus provide direct insight into high energies. Hopefully, our work will help nudge out of the way, even slightly, the shadows that loom of over some areas of QFT.

There are different manners to study QFTs. The path integral is one of them, and a very powerful and elegant one. A reformulation of the Schrödinger equation shows that the motion of a quantum particle can be described by taking into account all the possible paths that a classical particle could take. Quantum observables can thus be obtained by summing over all possible paths that a classical particle could follow, which is formally achieved in the path integral. The first Part of this thesis is dedicated to constructing the path integral, and how it can be interpreted (Chapt. 1). Beyond describing quantum particles, the path integral is particularly well suited to describe quantum fields, and extract relevant quantities such as Vacuum Expectation Values (VEVs) which pertain to Physical observables. Although a complete evaluation of the path integral is extremely difficult to achieve, perturbative computations can be efficiently carried out, such as the Heat Kernel and the Covariant Derivative Expansion (CDE) (Chapt. 2). An example of computation at two-loop is provided in App. A.

The second Part of this thesis is dedicated to introducing the EFT paradigm. Knowing the laws of Nature down to the smallest scales is not necessary to be able to do predictions of larger scales. The boundaries of our understanding of the elementary constituents of Nature have been pushed further and further, down to the knowledge of the SM, which remains itself an EFT with an unknown UV completion. The EFT paradigm as two main applications. The first one, the top-down

approach, assumes a specific UV completion, from which low energy observables are derived. The second one, the bottom-up approach, considers the most generic low energy theories, and aims at constraining a wide range of UV completions from low energy measurements (Chapt. 3). The path integral is particularly well-suited to study the top-down approach, as one can formally integrate out heavy degrees of freedom to obtain the low energy EFT. This is conveniently achieved using the CDE, which can lead to the so-called Universal One-Loop Effective Action (UOLEA) (Chapt. 4).

The effects of gravity are described in the context of General Relativity by the curvature of spacetime. Once we have some basic knowledge of differential geometry and QFT in curved spacetime (Chapt. 5), we may take into account the effects of gravity in EFTs. The aim of our work [3] was to introduce a novel approach to constructing EFTs in curved spacetime, based on the CDE. Its main advantages compared to other existing methods, are its relative computational simplicity, its coordinate-independent momentum representation, which is important when studying gravitational anomalies, and its ability to handle chiral fermions in gravity in the path integral, which was missing in the literature (Chapt. 6). The main results of our work are the bosonic and fermionic UOLEA in gravity (Chapt. 7 and 8).

A summary of Part II can be found in Chapt. 9. Details of computations for the CDE in gravity are deferred to App. B and C.

As mentioned above, quantum anomalies are singular in QFT in that they do not run with the energy, and thus provide direct measurements of high energy effects. Anomalies are the quantum breaking of classical symmetries, and play a major role in modern QFT. The most famous one is the so-called ABJ anomaly, which explains the experimentally observed decay of the neutral pion into two photons. The formulation of anomalies in the path integral permits to understand their remarkable topological nature. It also gives a good grasp on the ambiguities that are inherent to them, whether we consider a global or a gauge anomaly (Chapt. 10). Since global and gauge anomalies arise elegantly in the path integral, we may as well take advantage of existing powerful EFT methods to compute them. This was the goal of our work [4], in which an anomaly is interpreted as the difference between two EFTs. The computation of different types of anomalies is then efficiently achieved using our novel method (Chapt. 11).

Anomalies also have their saying in curved spacetime, and occur in what we call gravitational anomalies, that is, the breaking of the symmetries of General Relativity in the presence of quantum matter. The symmetries at stakes are the diffeomorphism, the Lorentz, and the Weyl groups. The first two are local symmetries of gravity and should not be violated, whereas the latter is a global symmetry, whose anomaly is harmless. Once the path integral is properly formulated in curved spacetime, they can be efficiently computed and understood in the same manner as flat spacetime anomalies (Chapt. 12). Parity violating effects in the Weyl anomaly, which are important to many phenomenological models, have been under controversy for about a decade when it comes to Weyl fermions. In our work [2], we help closing the controversy by showing the absence of parity violation in the Weyl anomaly of a Weyl fermion, using the path integral, while taking great care to manipulate only with well-defined quantities, and keeping diffeomorphism and Lorentz symmetries under control (Chapt. 13). This result was extended to any model in our following paper [1]. In this work, we constrain the generic form of the Weyl anomaly under requiring the *finiteness* of the diffeomorphism and Lorentz anomalies. We are thus able to conclude, for any model which is compatible with dimensional regularisation, that no parity violation can arise in the Weyl anomaly (Chapt. 14).

The results of Part III are summarised in Chapt. 15. Details about Weyl fermions in curved spacetime and computations can be found in App. D, E, F and G.

Part I

Path Integral

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Chapter 1

Path Integral in Quantum Mechanics

The path integral was introduced by Dirac [6] and Feynman [7] in the context of Quantum Mechanics. But the basic idea can be traced back to Wiener, who used it to describe Brownian motions and diffusion [8]. It then proved to be a formidable tool in Quantum Field Theory (QFT), quantum gravity [9], string theory [10], condensed matter [11–15], statistical [16, 17] and polymer physics [17, 18], financial market [18], etc. . . In this Chapter, we review the foundation of the path integral in Quantum Mechanics and Quantum Field Theory (QFT).

In Quantum Mechanics, we have systems that are described by vectors $|\psi\rangle$ in a Hilbert space. The physical quantities are contained in operators \hat{O} . The Hamiltonian \mathcal{H} is such an operator and measures the energy of the system. It also describes the dynamics of the system via the Schrödinger equation. The physical quantities we are interested in are quantum average of operators, and the probability of measuring the quantum system in a specific state. These quantities can be extracted from the path integral that we will build in this Section.

1.1 Schödinger's equation

Let us consider a quantum system described by the Hamiltonian \mathcal{H} , which can be written as

$$\mathcal{H}(\hat{x}, \hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x}) . \quad (1.1)$$

The first term is the kinetic energy operator for a particle of mass m , while the second term is the potential energy operator. \hat{p} and \hat{x} are respectively the momentum and the position operators. Since \hat{x} is an hermitian operator, its eigenvectors $|x\rangle$ define an orthonormal basis:

$$\hat{x} |x\rangle = x |x\rangle , \quad \langle x|x'\rangle = \delta(x - x') , \quad \int dx |x\rangle \langle x| = \mathbf{1} . \quad (1.2)$$

The matrix elements of \hat{p} then follow from the Heisenberg uncertainty principle

$$\begin{aligned} [\hat{x}, \hat{p}] = i\hbar &\Rightarrow \langle x|[\hat{x}, \hat{p}]|x'\rangle = (x - x') \langle x|\hat{p}|x'\rangle = i\hbar \delta(x - x') \\ &\Rightarrow \langle x|\hat{p}|x'\rangle = i\hbar \frac{\delta(x - x')}{x - x'} = -i\hbar \partial_x \delta(x - x') . \end{aligned} \quad (1.3)$$

From (1.3) we can verify that \hat{p} is hermitian, hence the eigenvectors $|p\rangle$ such that $\hat{p}|p\rangle = p|p\rangle$ define an orthonormal basis as well. It will be useful later on to use their expression in position space

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}} , \quad (1.4)$$

which can be derived from (1.3) and the orthonormality condition $\langle p|p'\rangle = \delta(p - p')$.

A quantum state $|\psi(t)\rangle$ (in the Schrödinger picture) can be decomposed on each of these basis. For example we have

$$|\psi(t)\rangle = \int dx \psi(t, x) |x\rangle , \quad (1.5)$$

where $\psi(t, x) = \langle x|\psi(t)\rangle$ is the associated wave function.

Physically, if $|\psi(t)\rangle$ represents the quantum state of a particle, say an electron, then $|\psi(t, x)|^2$ is the probability of measuring the electron at the position x at the time t . On the other hand, $|\langle p|\psi(t)\rangle|^2$ is the probability of measuring its momentum to be p at the time t . These two measurements are not independent due to the Heisenberg uncertainty principle, which implies that

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} , \quad (1.6)$$

where $\sigma_{\hat{O}}$ is the variance of the operator \hat{O}

$$\sigma_x = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2} , \quad (1.7)$$

and $\langle \hat{O} \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle$ is the average value of \hat{O} at the time t . In practice, (1.6) means that if we know precisely the position of our electron (i.e σ_x is small) then we have an important uncertainty on its momentum (i.e $\sigma_p \geq \hbar/\sigma_x$ is large), and conversely.

1.2 Evolution operator

We will now build the path integral describing the motion of the quantum state $|\psi(t)\rangle$. Its time evolution follows from the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \mathcal{H} |\psi(t)\rangle . \quad (1.8)$$

For simplicity let us suppose that the Hamiltonian bears no explicit dependence in t . This equation can be solved to obtain

$$|\psi(t_f)\rangle = \hat{U}(t_f, t_0) |\psi(t_0)\rangle , \quad (1.9)$$

where we defined the time evolution operator of the theory

$$\hat{U}(t_f, t_0) = e^{-\frac{i}{\hbar}(t_f - t_0)\mathcal{H}} , \quad (1.10)$$

between the times t_0 and t_f . In general, a time evolution operator satisfies the following properties

$$\hat{U}(t, t) = \mathbf{1} , \quad \hat{U}(t_0, t) = \hat{U}(t, t_0)^{-1} , \quad \hat{U}(t_0, t) = \hat{U}(t, t_1)\hat{U}(t_1, t_0), \quad \forall t_1$$

Returning to the evolution equation Eq. (1.9), let us introduce an intermediary time $t_0 \leq t_1 \leq t_f$ and use the completeness property $\int dx \langle x|x\rangle = 1$ to write

$$\psi(x_f, t_f) = \langle x_f | \hat{U}(t_f, t_1) \hat{U}(t_1, t_0) | \psi(t_0) \rangle = \int dx_1 dx_0 U(t_f, x_f; t_1, x_1) U(t_1, x_1; t_0, x_0) \psi(x_0, t_0) ,$$

where $U(t_f, x_f; t_0, x_0) = \langle x_f | \hat{U}(t_f, t_0) | x_0 \rangle$.

This equation relates the wavefunction at (t_f, x_f) to the wavefunction at (t_0, x_0) , while considering that at t_1 the particle may have passed by any position x_1 . This equation can be schematically represented by Fig. 1.1a. We can iterate the process and add as many intermediary checkpoints as we like. For example we can insert N checkpoints at the times $t_k = k \Delta t + t_0$ with $\Delta t = t_k - t_{k-1} = (t_f - t_0)/N$, and at each checkpoint insert $\int dx_k \langle x_k | x_k \rangle = 1$ to obtain

$$\psi(x_f, t_f) = \int dx_{N-1} \dots dx_0 U(t_f, x_f; t_{N-1}, x_{N-1}) \dots U(t_1, x_1; t_0, x_0) \psi(x_0, t_0), \quad (1.11)$$

which can be schematically represented by Fig. 1.1b. We have achieved a representation of the evolution of $|\psi(t)\rangle$ by considering all the paths that are broken lines passing by each of the possible x_k at t_k . The goal will be to take the limit $\Delta t \rightarrow 0$ in order to obtain a sum over all paths $x(t)$ (Fig. 1.1c).

Let us first refine (1.11) by using the expressions of \hat{U} and \mathcal{H} . Using the orthonormality of $|x\rangle$ and $|p\rangle$ we may write

$$\begin{aligned} U(t_k, x_k; t_{k-1}, x_{k-1}) &= \langle x_k | e^{-\frac{i\Delta t}{\hbar} \frac{\hat{p}^2}{2m}} e^{-i\frac{\Delta t}{\hbar} V(\hat{x})} | x_{k-1} \rangle \\ &= \langle x_k | \int dp_k | p_k \rangle \langle p_{k-1} | e^{-\frac{i\Delta t}{\hbar} \frac{\hat{p}^2}{2m}} \int dx | x \rangle \langle x | e^{-i\frac{\Delta t}{\hbar} V(\hat{x})} | x_{k-1} \rangle \\ &= \sqrt{\frac{2\pi m \hbar}{i\Delta t}} e^{i\frac{\Delta t}{\hbar} \left(\frac{1}{2} m \left(\frac{x_k - x_{k-1}}{\Delta t} \right)^2 - V(x_{k-1}) \right)}, \end{aligned} \quad (1.12)$$

where we have made use of (1.4). We can then insert (1.12) into (1.11) to obtain

$$\psi(x_f, t_f) = \int \prod_{k=0}^{N-1} \left(dx_k \sqrt{\frac{2\pi m \hbar}{i\Delta t}} \right) e^{i \sum_{k=0}^{N-1} \frac{\Delta t}{\hbar} \left(\frac{1}{2} m \left(\frac{x_k - x_{k-1}}{\Delta t} \right)^2 - V(x_{k-1}) \right)} \psi(x_0, t_0), \quad (1.13)$$

The next bold step is to take the continuous limit $\Delta t \rightarrow 0$, while postponing discussions about convergence and definiteness, to obtain the path integral

$$U(t_f, x_f; t_0, x_0) = \int \mathcal{D}x(t) e^{\frac{i}{\hbar} S[x(t)]}, \quad (1.14)$$

where we identified the action functional

$$S[x(t)] = \int_{t_0}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2 - V(x) \right), \quad (1.15)$$

and formally

$$\mathcal{D}x(t) \equiv \lim_{\Delta t \rightarrow 0} \prod_{k=0}^{N-1} \left(dx_k \sqrt{\frac{2\pi m \hbar}{i\Delta t}} \right). \quad (1.16)$$

Throughout this construction we assumed the Hamiltonian to be time independent, but the same reasoning can be applied without this assumption by introducing the time ordering operator T

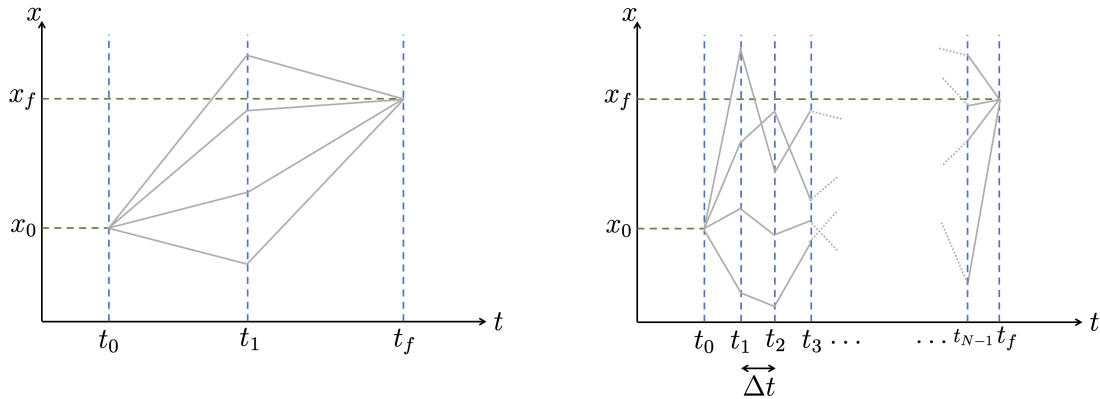
$$T\{f(t)g(t')\} = \begin{cases} f(t)g(t') & \text{if } t \leq t' \\ g(t')f(t) & \text{if } t' < t \end{cases}, \quad (1.17)$$

and the evolution operator thus reads

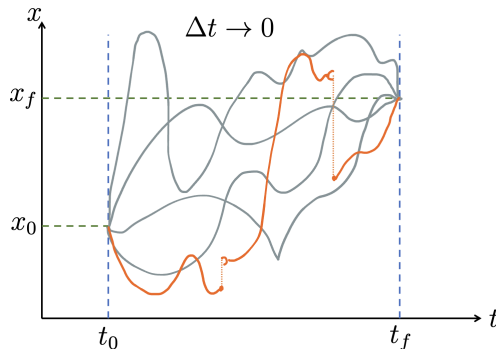
$$\hat{U}(t_f, t_0) = T e^{\int_{t_0}^{t_f} \mathcal{H}(t') dt'}, \quad (1.18)$$

from which the path integral can be constructed similarly.

Before moving on, let us say a word about the meaning of the path integral. The evolution operator $U(t, x; t_0, x_0)$ describes how a quantum particle propagates. This quantum particle is not localised as it is represented by a wavefunction $\psi(t, x)$. The path integral representation of $U(t, x; t_0, x_0)$ unravels that the propagation of a quantum particle that is measured at (t_0, x_0) and then at (t, x) , is equivalent to taking into account all the paths that a classical (off-shell, i.e. not solution to its equations of motion) particle could take between these points. The weight of each of these classical paths $x(t)$ is the phase $\exp(\frac{i}{\hbar}S[x(t)])$. We will see in Sec. 2.1 another interpretation of this weight.



(a) All broken lines paths with checkpoints at t_1 . (b) All broken lines paths with checkpoints at each t_k , with $\Delta t = (t_k - t_{k-1})/N$.



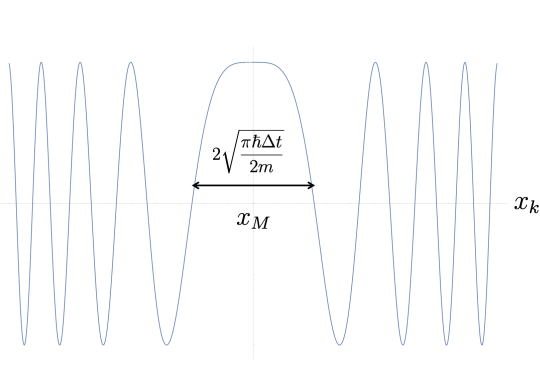
(c) All paths obtained in the $\Delta t \rightarrow 0$ limit. The path integral also includes discontinuous paths such as the one in orange.

1.3 Which paths contribute?

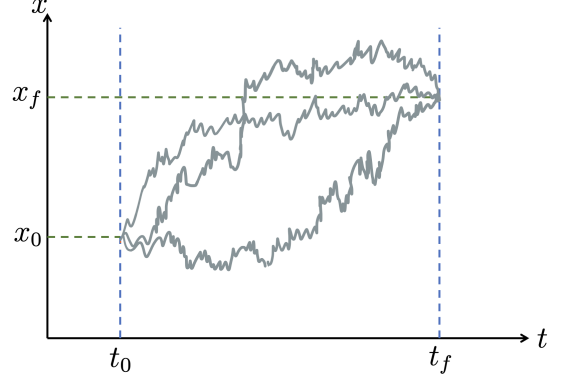
We have achieved the construction of the path integral representation of the time evolution operator

$$U(t_f, x_f; t_0, x_0) = \int \mathcal{D}x(t) e^{\frac{i}{\hbar}S[x(t)]}, \quad (1.19)$$

which consists in a weighted sum over all possible paths $x(t)$ from $x(t_0) = x_0$ to $x(t_f) = x_f$. In particular, this sum includes discontinuous paths, such as the one drawn in orange in Fig. 1.1c. In fact, in the space of all paths, most paths are not continuous at any point. This comes as a



(a) Plot of $f(x_k) = \text{Re} \left[\exp \left(\frac{i}{\hbar} \Delta t \frac{m}{2} \left(\frac{x_k - x_M}{\Delta t} \right)^2 \right) \right]$.



(b) The paths that contribute to the path integral are Brownians.

surprise since we interpreted the path integral as being the sum of the paths of classical particles, and classical particles follow continuous paths. This interpretation is correct and we will show that discontinuous paths are suppressed due to the presence of the kinetic term in the weight $\exp(iS/\hbar)$. This Section mostly follows Matthieux Tissier's lecture "Path Integral and Representation" given at the Ecole Normale Supérieure of Cachan.

Let us first return to the discretised version of the path integral (1.13) and focus on one point x_k

$$\int dx_k e^{\frac{i}{\hbar} \Delta t \frac{m}{2} \left(\frac{x_{k+1} - x_k}{\Delta t} \right)^2} e^{\frac{i}{\hbar} \Delta t \frac{m}{2} \left(\frac{x_k - x_{k-1}}{\Delta t} \right)^2} \propto \int dx_k e^{\frac{i}{\hbar} \Delta t \frac{m}{2} \left(\frac{x_k - x_M}{\Delta t} \right)^2} = \int dx_k f(x_k), \quad (1.20)$$

with obvious definition for f , and $x_M = (x_{k+1} - x_{k-1})/2$. We omitted the potential $V(x_k)$ for simplicity, but including it does change the conclusion. We schematically plotted $\text{Re}[f(x_k)]$ in Fig. 1.2a. Since the oscillating part is suppressed in the integral, the only points that contribute to (1.20) are such that

$$|x_k - x_M| \sim \sqrt{\frac{\hbar \Delta t}{m}}. \quad (1.21)$$

Firstly, since

$$|x_k - x_M| \sim \sqrt{\Delta t} \xrightarrow{\Delta t \rightarrow 0} 0, \quad (1.22)$$

we can conclude that only the continuous paths contribute to the path integral. However, the velocity of the path is divergent

$$v = \frac{|x_k - x_M|}{\Delta t} \sim \sqrt{\frac{\hbar}{m \Delta t}} \xrightarrow{\Delta t \rightarrow 0} \infty. \quad (1.23)$$

This is a striking feature of the path integral: the paths that contribute are continuous but not derivable at any point, that is to say they are Brownian motion (Fig. 1.2b)! Physically, the reason behind this feature is the Heisenberg uncertainty principle. Indeed, it is possible to show that

$$\begin{aligned} \langle x_f | [\hat{x}, \hat{p}] | \psi(t_f) \rangle &= \langle x_f | [\hat{x}, \hat{p}] \hat{U}(t_f, t_f - \Delta t) | \psi(t_f - \Delta t) \rangle \\ &\propto \int dx e^{\frac{im}{2\hbar\Delta t} (x_f - x)^2} \frac{(x_f - x)^2}{\Delta t^{3/2}} \psi(x, t_f - \Delta t), \end{aligned} \quad (1.24)$$

where we see that if the paths that contribute would be such that $|x_f - x| \sim \Delta t$, the integral would vanish in the limit $\Delta t \rightarrow 0$, implying that $[\hat{x}, \hat{p}] = 0$. However, if the motion is Brownian: $|x_f - x| \sim \sqrt{\Delta t}$ then $[\hat{x}, \hat{p}] \neq 0$.

Chapter 2

Path Integral in Quantum Field Theory

Quantum Mechanics is a powerful framework to describe the quantum behavior of a single non-relativistic particle interacting with a potential. However, it does not explain why every electron in the universe have exactly the same properties (mass, charges, spin), neither can it explain how particles can annihilate each other to form new particles, or decay. These are the main motivations behind Quantum Field Theory (QFT), where instead of following a single electron we now have an electron field over the whole space. Local excitations of the electron field produce electrons, which all have the same properties since they arise from the same field. On top of that, QFTs naturally fit in the framework of special relativity. In this Section, we outline the construction of the path integral for Quantum Fields, and how it can be used to compute the relevant physical observables.

The Hamiltonian of a QFT is written in terms of field operators $\hat{\phi}(x) = \hat{\phi}(t, \vec{x})$ (in the Heisenberg picture). We also call a field the eigenvalue of this operator: $\hat{\phi}(x) |\phi\rangle = \phi(x) |\phi\rangle$, where $\phi(x)$ is a function from the base spacetime manifold \mathcal{M} to some vector space (depending on the gauge and spin structures).

The physical processes we are interested in are scattering, that is to say a set of free particles that arrive from $x \rightarrow -\infty$, interact, leaving another set of free particles going to $x \rightarrow +\infty$. By free particle, we mean that the particles do not interact together since they are too far away from one another. These processes are encoded in the S -matrix (S for scattering). Schematically, the S -matrix operator is given by the evolution operator between times $-\infty$ and $+\infty$

$$S \sim \hat{U}(+\infty, -\infty) . \quad (2.1)$$

Using the Lehmann–Symanzik–Zimmermann (LSZ) formula, its matrix elements can be conveniently computed via the time-ordered correlation functions

$$\langle T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \rangle , \quad (2.2)$$

where $\langle \cdot \rangle$ denotes the Vacuum Expectation Value (VEV), defined further below. To illustrate this formula, we consider a theory of a self-interacting scalar field ϕ of mass M . The amplitude of the transition between the free state $|i\rangle$ corresponding to n scalars coming from $-\infty$ with momenta $\{p_1, \dots, p_n\}$, and the free state $|f\rangle$ corresponding to m scalars at $+\infty$ with momenta $\{q_1, \dots, q_m\}$, is given by the LSZ formula as

$$S_{fi} = \langle f|i\rangle \propto \prod_{i=1}^n (p_i^2 - M^2) \prod_{j=1}^m (q_j^2 - M^2) \int \prod_{i=1}^n (d^4x_i e^{ip_i x_i}) \langle T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \rangle , \quad (2.3)$$

up to some known normalisation factor. The probability of this interaction is simply $|S_{fi}|^2$. We thus see that knowing the probability of transition between free states amounts to knowing the correlation functions of the theory.

In the following, we will show how the correlation functions can be extracted from the path integral for fields.

2.1 Path Integral for fields

Our aim to generalise the path integral (1.14) for fields, that is to say, write the evolution operator between two configurations $|\phi_0(\vec{x})\rangle$ at t_0 and $|\phi_f(\vec{x})\rangle$ at t_f . In general it can be expressed as

$$U(t_f, \phi_f(\vec{x}); t_0, \phi_0(\vec{x})) = \int_{b.c} \mathcal{D}\phi(x) e^{i \times \text{phase}[\phi]}, \quad (2.4)$$

where $b.c$ denotes the boundary conditions: $\phi(t_0, \vec{x}) = \phi_0(\vec{x})$ and $\phi(t_f, \vec{x}) = \phi_f(\vec{x})$, where $\hat{\phi}(t_0, \vec{x}) |\phi_0(\vec{x})\rangle = \phi_0(t_0, x) |\phi_0(\vec{x})\rangle$ and likewise for ϕ_f . Note that inside the path integral, there are no operators but only classical (off-shell) fields, and a path represents a configuration $\phi(x)$.

From our previous analysis, we can guess the phase to be the action functional. However, we can adopt a different argument following [19]. Let us assume the phase has a single minimum at ϕ_{cl} . We can thus evaluate the path integral using the stationary phase approximation to obtain the path that contributes the most to the evolution of $\hat{\phi}$

$$U(t_f, \phi_f(\vec{x}); t_0, \phi_0(\vec{x})) \simeq e^{i \times \text{phase}[\phi_{cl}]}. \quad (2.5)$$

Naturally, the configuration that contributes the most must be the classical (on-shell, i.e solution to its equations of motion) one, which must then be the minimum of $\text{phase}[\phi]$. The action functional $S[\phi]$ fulfills this requirement by principle of least action, hence

$$\text{phase}[\phi] \propto S[t_0, t_f, \phi] = \int_{t_0}^{t_f} d^4x \mathcal{L}(\phi(x)), \quad (2.6)$$

is the most natural choice. The classical limit (2.5) must be valid insofar as $S \gg \hbar$, we thus use S/\hbar for the phase, and we recover the result from the previous section for fields

$$U(t_f, \phi_f(\vec{x}); t_0, \phi_0(\vec{x})) = \int_{b.c} \mathcal{D}\phi(x) e^{\frac{i}{\hbar} S_{t_0, t_f}[\phi]}, \quad (2.7)$$

where

$$S_{t_0, t_f}[\phi] = \int_{t_0}^{t_f} d^4x \mathcal{L}(\phi(x)). \quad (2.8)$$

2.2 Vacuum Expectation Value

Since we are interested in the transition between states at $t_0 \rightarrow -\infty$ and $t_f \rightarrow +\infty$, we define the generating functional

$$Z[J] = \int \mathcal{D}\phi(x) e^{\frac{i}{\hbar} S[\phi] + iJ \cdot \phi}, \quad (2.9)$$

where the action is integrated from $t \rightarrow -\infty$ to $t \rightarrow \infty$, and the scalar product is $J \cdot \phi = \int d^4x J(x)\phi(x)$ (for real fields). We omit writing the boundary conditions on the path integral,

but from now on we keep in mind that they are at infinity.¹ From Z we can derive VEVs and in particular correlation functions by taking the derivative with respect to J and then setting $J = 0$

$$\frac{1}{Z[0]} \frac{\delta^n Z}{\delta J(x_1) \dots \delta J(x_n)} \Big|_{J=0} = i^n \langle T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \rangle, \quad (2.10)$$

where the VEV of an operator $\mathcal{O}[\hat{\phi}]$ is defined by

$$\langle T \mathcal{O}[\hat{\phi}] \rangle = \frac{1}{Z[0]} \int \mathcal{D}\phi \mathcal{O}[\phi] e^{\frac{i}{\hbar} S[\phi]} = \frac{\int \mathcal{D}\phi \mathcal{O}[\phi] e^{\frac{i}{\hbar} S[\phi]}}{\int \mathcal{D}\phi e^{\frac{i}{\hbar} S[\phi]}}. \quad (2.11)$$

Remarkably, the path integral automatically provides a time-ordered VEV of the operators, due to the time-discretisation [20].

The reason why the VEV is normalised by $Z[0]$ is because the path integral measure is in fact defined up to an infinite constant. However, it should not play a role in VEVs which are involved in physical observables such as probabilities of interactions. The normalisation cancels such irrelevant constants. $Z[0]$ in fact generates all the vacuum diagrams that are not connected to external legs and only clutter the physical processes without participating (see Tab. 2.1).

As the name indicates, the VEV of an operator corresponds to the average value of this operator over the quantum fluctuations, in the vacuum of the theory. Indeed, before taking the $t_0 \rightarrow -\infty$ and $t_f \rightarrow \infty$ limits, we may write the quantum transition between the states $|\phi_0\rangle$ and $|\phi_f\rangle$ as

$$\langle T \mathcal{O}[\hat{\phi}] \rangle_{t_0, t_f} = \langle \phi_f | T e^{-i\mathcal{H}\Delta t} \mathcal{O}[\hat{\phi}] e^{-i\mathcal{H}\Delta t} | \phi_0 \rangle, \quad (2.12)$$

where $\Delta t = t_f - t_0$ (\mathcal{H} is taken to be time-independent here). By decomposing the quantum states on the eigenbasis of the Hamiltonian $\mathcal{H} |n\rangle = E_n |n\rangle$ we may write²

$$e^{-i\mathcal{H}\Delta t} | \phi_0 \rangle = \sum_n e^{-iE_n \Delta t} |n\rangle \langle n | \phi_0 \rangle, \quad (2.13)$$

and likewise for $\langle \phi_f |$. Only the ground state $|0\rangle$ ³ with $E_0 = 0$ survives in the $\Delta t \rightarrow \infty$ limit, by virtue of the Riemann-Lebesgue lemma which states that for any well-behaved (integrable and measurable) function f

$$\lim_{|\Delta t| \rightarrow \infty} \int_b^a dx f(x) e^{i\Delta t x} = 0. \quad (2.14)$$

This leaves us with

$$\langle T \mathcal{O}[\hat{\phi}] \rangle = \langle 0 | T \mathcal{O}[\hat{\phi}] | 0 \rangle = \frac{\int \mathcal{D}\phi \mathcal{O}[\phi] e^{\frac{i}{\hbar} S[\phi]}}{\int \mathcal{D}\phi e^{\frac{i}{\hbar} S[\phi]}}, \quad (2.15)$$

which is evaluated in the vacuum of the theory $|0\rangle$, i.e the lowest energy state. In a free theory, the vacuum corresponds to having no particles at all. This is not true anymore in an interacting theory where the vacuum has a non-zero energy due to the spontaneous creation and annihilation of particles, i.e vacuum bubbles.

In terms of Feynman diagrams, the correlation function (2.10) corresponds to all the processes that involve n external legs (with zero momenta), including disconnected diagrams (see Tab. 2.1).

¹In general the boundary conditions should be on the boundary of the base manifold \mathcal{M} , which we assume to be at infinity here.

² \sum_n is in fact rather a continuous sum $\int dn$.

³Here $|0\rangle$ denotes the vacuum of the full interacting theory. In textbooks it is often denoted by $|\Omega\rangle$, while $|0\rangle$ is the vacuum of the free theory.

These disconnected diagrams represent scenarii where there is actually no scattering between the incoming and outgoing particles. For physical observables these diagrams should be subtracted. This is achieved by the quantum Action $W[J] = -i \log Z[J]$

$$\langle T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \rangle_c = (-i)^n \frac{\delta^n W}{\delta J(x_1) \dots \delta J(x_n)} \Big|_{J=0}, \quad (2.16)$$

where c stands for “connected”.

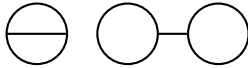
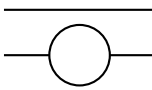

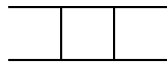
Functional	Examples of generated diagrams for a scalar interacting theory $\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi + g\phi^3$
$Z[J=0]$: vacuum bubbles.	
$\frac{1}{Z[0]} \frac{\delta^4 Z}{\delta J^4} \Big _{J=0}$: four-legged diagrams, no vacuum bubbles.	Includes disconnected diagrams: 
$\frac{\delta^2 W}{\delta J^2} \Big _{J=0}$: two-legged connected diagrams, no vacuum bubbles.	
$\frac{\delta^4 W}{\delta J^4} \Big _{J=0}$: four-legged connected diagrams, no vacuum bubbles.	

Table 2.1: Examples of diagrams generated by each functional.

2.3 Effective action

The quantum Effective Action is defined as the Legendre transform of $W[J]$

$$\Gamma[\varphi] = \text{Min}_J \{W[J] - J \cdot \varphi\} = W[J_\varphi] - J_\varphi \cdot \varphi, \quad (2.17)$$

where $\text{Min}_{x \in I} \{f(x)\}$ is the minimum of f over I , and is naturally generalised to functionals. J_φ minimises $W[J] - J \cdot \varphi$, hence is solution to $\frac{\delta W}{\delta J}[J_\varphi] - \varphi = 0$. $W[J]$ and $\Gamma[\varphi]$ can be understood as follows: when using $W[J]$, we fix an external current J and can deduce the associated VEV of $\hat{\phi}$

$$\langle \hat{\phi} \rangle_J \equiv \frac{\delta W}{\delta J}[J], \quad (2.18)$$

whereas when using $\Gamma[\varphi]$ we fix the VEV as $\langle \hat{\phi} \rangle_{J_\varphi} = \varphi$ and deduce the external current J_φ that yields such a VEV, which is solution to

$$\frac{\delta W}{\delta J}[J_\varphi] = \varphi. \quad (2.19)$$

From $\Gamma[\varphi]$ we define the effective potential

$$V_{eff}[\varphi] = -\frac{\Gamma[\varphi]}{\mathcal{V}}, \quad (2.20)$$

where \mathcal{V} is some fiducial spacetime volume. The minima of this potential represent the stable VEV of $\hat{\phi}$. In general, these solutions are constant, i.e independent of x .⁴

In the context of Effective Field Theory (EFT), we will be interested in the case where the quantum field $\hat{\phi}$ decouples from the other background fields. It means that it reaches its VEV without external source

$$\varphi(x) = \langle \hat{\phi} \rangle = \frac{\delta W}{\delta J} [J = 0]. \quad (2.21)$$

Henceforth, in that context, the quantity we are interested in is

$$\Gamma[\varphi = \langle \hat{\phi} \rangle] = W[J = 0]. \quad (2.22)$$

In terms of Feynman diagrams, it includes all the diagrams that only have background fields on the external legs.

2.4 Quantum equations of motion

The path integral allows to obtain the correlation functions while bypassing the traditional quantisation: upgrade the fields to operators on a Hilbert space with their commutation relations, solve their equations of motion (EoM), compute the correlation functions. Since the quantum fields respect their EoM, so do the correlation functions. For example, if we take the simple case of a real quantum scalar field

$$S = -\frac{1}{2} \int d^4x \hat{\phi}(\partial^2 + m^2)\hat{\phi}, \quad (2.23)$$

then the quantum EoM impose

$$(\partial^2 + m^2)\hat{\phi} = 0. \quad (2.24)$$

Consequently, the two-point function obeys⁵

$$(\partial_x^2 + m^2)\langle T \hat{\phi}(x)\hat{\phi}(y) \rangle = -i\delta(x - y). \quad (2.25)$$

Similar equations can be obtained for n -point correlation functions.

It may thus come as a surprise that no notion of quantum EoM were necessary to obtain the correlation functions via the path integral. The reason is that the quantum EoM are automatically included. They follow from the fact that for any functional $F[\phi]$, its generic variation vanishes under the integral

$$\int \mathcal{D}\phi \delta F[\phi] = 0. \quad (2.26)$$

In particular, taking $F[\phi] = e^{iS+iJ\cdot\phi}$ and the variation $\delta/\delta\phi$ we obtain the Dyson-Schwinger equation [21–23]

$$\left(\frac{\delta S}{\delta \phi} \left[-i \frac{\delta}{\delta J} \right] + J \right) Z[J] = 0. \quad (2.27)$$

For the case of the real scalar field, by taking the derivative of (2.27) with respect to $J(y)$ and then setting it to zero we obtain

$$(\partial_x^2 + m^2)\langle T \hat{\phi}(x)\hat{\phi}(y) \rangle = -i\delta(x - y), \quad (2.28)$$

which is identical to (2.25). We thus see that the quantum EoM are indeed automatically enforced in the path integral formulation.

⁴Counter-examples are instanton solutions.

⁵The contact term $\delta(x - y)$ arise because ∂^2 also acts on the Heaviside functions θ in $\langle T \hat{\phi}(x)\hat{\phi}(y) \rangle = \langle \hat{\phi}(x)\hat{\phi}(y) \rangle \theta(x^0 - y^0) + \langle \hat{\phi}(y)\hat{\phi}(x) \rangle \theta(y^0 - x^0)$.

2.5 Path integral methods

The path integral is a formidable tool to extract all the physical quantities we may be interested into. However, it remains at this stage very formal, and it may seem unclear how to compute explicitly the integration over paths. In this Section we will show the main methods employed to compute path integrals. For simplicity, from now on we use the system of units such that $\hbar = c = 1$. We also drop the $\hat{\cdot}$ on the quantum fields.

Computing the path integral exactly is challenging, hence most methods rely on the stationary phase approximation. It consists in changing the path integral variable to $\eta = \phi - \phi_0$ and then performing a Taylor expansion of the action

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi e^{iS[\phi] + iJ \cdot \phi} \\ &= e^{iS[\phi_0] + iJ \cdot \phi_0} \int \mathcal{D}\eta e^{i\frac{1}{2}\eta^2 \cdot \frac{\delta^2 S}{\delta \eta^2}[\phi_0] + i \sum_{n \geq 3} \frac{1}{n!} \eta^n \cdot \delta^n S[\phi_0]} \\ &= e^{iS[\phi_0] + iJ \cdot \phi_0} \int \mathcal{D}\eta e^{i\frac{1}{2}\eta^2 \cdot \frac{\delta^2 S}{\delta \eta^2}[\phi_0]} \sum_{m \geq 0} \frac{i^m}{m!} \left(\sum_{n \geq 3} \frac{1}{n!} \eta^n \cdot \delta^n S[\phi_0] \right)^m \end{aligned} \quad (2.29)$$

where we used the EoM in the presence of an external source

$$\frac{\delta S}{\delta \phi}[\phi_0] + J = 0, \quad (2.30)$$

This equation can be solved to obtain the propagator Π (or Green function)

$$\phi_0[J](x) = - \int d^4y \Pi(x, y) J(y). \quad (2.31)$$

The Taylor expansion we performed is an expansion in powers of \hbar , i.e. in loops, therefore a good approximation of Z can be obtained by cropping the expansion at the desired loop order. The leading order consisting of one-loop diagrams is often sufficient, hence most efforts in the literature were dedicated to computing one-loop effective actions.

2.5.1 Functional methods: one-loop

In the one-loop approximation, only second order term is kept in (2.29) leaving us with

$$Z[J] \simeq e^{iS[\phi_0] + iJ \cdot \phi_0} \int \mathcal{D}\eta e^{i\frac{1}{2}\eta^2 \cdot \frac{\delta^2 S}{\delta \eta^2}[\phi_0]}. \quad (2.32)$$

Let us perform what is called in the literature a Wick rotation to Euclidean signature. It consists in rotating the time variable to the imaginary axis: $t \rightarrow \pm it$, and the resulting path integral reads

$$\int \mathcal{D}\eta e^{-\frac{1}{2}\eta^2 \cdot \frac{\delta^2 S}{\delta \eta^2}[\phi_0]}. \quad (2.33)$$

In analogy with a gaussian integral (with A positive definite)

$$\int d\vec{x} e^{-\frac{1}{2}\vec{x}^T \cdot A \cdot \vec{x}} = \sqrt{\det(2\pi A^{-1})}, \quad (2.34)$$

we obtain

$$W[J] = -i \log Z[J] \simeq S[\phi_0] + J \cdot \phi_0 + i c_s \text{Tr} \log \left(-\frac{\delta^2 S}{\delta \phi^2}[\phi_0] \right), \quad (2.35)$$

after Wick rotating back to Minkowski, and using $\log \det = \text{Tr} \log$, Tr being the functional trace. The factor c_s depends on the spin of ϕ , for example for real scalars, complex scalars, Dirac fermions, gauge bosons, and Fadeev-Popov ghosts $c_s = 1/2, 1, -1/2, 1/2$, and -1 , respectively. The gaussian integration is permitted insofar as $\delta^2 S$ is invertible. For now we assume it is the case, but the discussion will be particularly relevant when dealing with quantum anomalies in Part III. It amounts to assuming that the configuration of the external fields is trivial, i.e there are no instanton configurations, i.e we are a perturbative set-up.

The heart of the computation now consists in calculating a functional determinant (or trace). We will assume a generic ansatz for the second derivative of the action which is the following

$$W_{1\text{-loop}}[J] = ic_s \text{Tr} \log [D^2 + m^2 + U[\phi_0]] . \quad (2.36)$$

where D is a covariant derivative, m a mass and U some matrix that may depend on other fields and on ϕ_0 .

From now on we drop the argument ϕ_0 of U and keep in mind that the expansion is around the classical value of the field. To obtain the correlation functions at one-loop, one has to insert the solution $\phi_0[J]$ into (2.36), then take the functional derivatives with respect to J before setting $J = 0$.

There exists several methods in the literature to compute such a functional determinant, we briefly outline the two most commonly used in the following.

2.5.1.1 Heat kernel

Historically, the first method to be used was the heat kernel method, introduced by Schwinger and De Witt [24, 25]. The idea relies on the following identity⁶

$$\log \lambda = - \int_0^\infty \frac{d\tau}{\tau} e^{-\tau\lambda} + cte , \quad (2.37)$$

where cte is an infinite constant independent of λ thus irrelevant here. We extend this formula to the functional trace

$$W_{1\text{-loop}}[J] = -ic_s \int_0^\infty \frac{d\tau}{\tau} \text{Tr} e^{-\tau(D^2+m^2+U)} . \quad (2.38)$$

Let us introduce the kernel of a functional operator $\mathcal{O}(x, y)$

$$K(\tau, x, y, \mathcal{O}) = e^{-\tau\mathcal{O}(x,y)} \quad (2.39)$$

It is solution to what is called a heat equation

$$\begin{cases} (\partial_\tau + \mathcal{O})K(\tau, x, y, \mathcal{O}) = 0 \\ K(0, x, y, \mathcal{O}) = \delta(x - y) \end{cases} . \quad (2.40)$$

For $\mathcal{O}_0 = \partial^2 + m^2$ it can be solved exactly on a general background manifold. On a flat d -dimensional manifold its expression is rather simple and reads

$$K(\tau, x, y, \mathcal{O}_0) = \frac{1}{(4\pi\tau)^{d/2}} e^{-\frac{(x-y)^2}{4\tau} - \tau m^2} . \quad (2.41)$$

⁶To prove this relation, one has to differentiate both sides with respect to λ . Setting $\lambda = 1$ we obtain the infinite constant $cte = \int_0^\infty \frac{d\tau}{\tau} e^{-\tau}$.

For a more general operator \mathcal{O} , it is possible to express the kernel in a power series in τ

$$K(\tau, x, y, \mathcal{O}) = K(\tau, x, y, \mathcal{O}_0) \sum_{n \geq 0} \tau^n b_n(x, y, \mathcal{O}) . \quad (2.42)$$

where the $b_n(x, y, \mathcal{O})$ are called heat kernel coefficients. The effective action $W_{1\text{-loop}}$ can be recovered from the trace of the kernel

$$W_{1\text{-loop}}[J] = -ic_s \int_0^\infty \frac{d\tau}{\tau} \text{tr} K(\tau, x, x, D^2 + m^2 + U) , \quad (2.43)$$

where tr denotes the trace over internal degrees of freedom (spin, gauge indices). The computation of the functional trace (2.36) now amounts to computing the heat kernel coefficients in their coincidence limit $b_n(x, x)$. It can be showed that they are finite in such limit, and are expressed as polynomials in the background fields and their derivatives. They can be computed iteratively using the heat equation (2.40). Since they cannot be solved for $n \rightarrow \infty$, the result can be approximated by truncating the series (2.42) at some order. In practice, the heat kernel coefficients are of the form

$$b_n \propto \frac{1}{m^{2n}} , \quad (2.44)$$

such that the truncation is valid at energies smaller than the mass, which provides the upper validity bound of the EFT.

Although the heat kernel coefficients are finite, $W_{1\text{-loop}}$ bears divergences in $d = 4$ due to the τ integration. The divergences at $\tau \rightarrow 0$ are called ultraviolet (UV) divergences, they need to be regularised for example by introducing a cut-off on the lower bound of the integral, or more commonly by using dimensional regularisation. There may also be divergences on the upper bound of the integral, for $\tau \rightarrow \infty$. These divergences occur if $D^2 + m^2 + U$ has negative or vanishing eigenvalues. These infrared (IR) divergences are of particular relevance when dealing with anomalies (see Part III).

The heat kernel coefficients have been computed in curved spacetime, in the presence of a gauge sector, and in the case where U is a simple matrix. This permits to obtain the EFT for scalar fields, vector-like fermions⁷ and spin-1 bosons. The case of chiral fermions has not been treated in a generic manner.

2.5.1.2 Covariant Derivative Expansion

The Covariant Derivative Expansion (CDE) is a more recent approach [26, 27]. This is the method that we will employ for most of our computations. The main idea is to perform a Taylor expansion of the functional determinant and then do a truncation to compute it at some order. It has the advantage of being more transparent and thus more intuitive than the heat kernel method, for which the computation of the heat kernel coefficients is quite involved. Curved spacetime generalisation of the CDE is treated in details in Part II. We briefly outline here the CDE in flat spacetime.

We choose the same ansatz as above for the second derivative of the action, which is a diagonal functional and reads⁸

$$\frac{\delta^2 \mathcal{S}}{\delta\phi(x)\delta\phi(y)} = (D_x^2 + m^2 + U) \delta(x - y) , \quad (2.45)$$

⁷By vector-like fermions we mean fermions for which their left and right-handed chiralities couple identically to other fields.

⁸For local UV theories, the UV Lagrangian is always diagonal in position space.

where $D_x \supset \frac{\partial}{\partial x}$. The functional trace of some functional $F(x, y)$ is defined by

$$\text{Tr } F = \int d^4x F(x, x) . \quad (2.46)$$

Applied to (2.45), we obtain⁹

$$W_{1\text{-loop}}[J] = ic_s \int d^4x \lim_{x \rightarrow y} \log (D_x^2 + m^2 + U) \delta(x - y) . \quad (2.47)$$

Next we express the Dirac distribution as the Fourier transform of the identity

$$\delta(x - y) = \int \frac{d^4q}{(2\pi)^4} e^{iq \cdot (x-y)} , \quad (2.48)$$

which does not commute with the covariant derivative

$$[D_x, e^{iq \cdot (x-y)}] = e^{iq \cdot (x-y)} (D_x + iq) . \quad (2.49)$$

We thus obtain an expression of $W_{1\text{-loop}}$

$$W_{1\text{-loop}}[J] = ic_s \int \frac{d^4x d^4q}{(2\pi)^4} \log \{ (D + iq)^2 + m^2 + U \} \quad (2.50)$$

$$= ic_s \int \frac{d^4x d^4q}{(2\pi)^4} \log \{ -\Delta^{-1} [1 - \Delta (D^2 + 2iq \cdot D + U)] \} , \quad (2.51)$$

where $\Delta^{-1} = q^2 - m^2$, and we assumed $[U, q] = 0$. This assumption does not hold for the case of chiral fermions, which is treated in Part II.

An extra step can be undertaken here, which consists in sandwiching the log with $e^{\pm iD \cdot \partial / \partial q}$. This trick, proposed by Gaillard and Cheyette [26, 27], has the advantage of stowing the covariant derivatives D within commutators, hence providing a manifestly covariant expansion. We skip this step as it makes the computation significantly heavier. Besides, a convenient choice of gauge also allows to only manipulate covariant quantities all throughout the computation.

The final step is to discard the infinite irrelevant constant $\log(-\Delta^{-1})$, and perform a Taylor expansion of the logarithm to obtain an expansion in terms of the covariant derivative

$$W_{1\text{-loop}}[J] = -ic_s \int \frac{d^4x d^4q}{(2\pi)^4} \sum_{n \geq 1} \frac{1}{n} [\Delta (D^2 + 2iq \cdot D + U)]^n . \quad (2.52)$$

This last step is slightly modified if $[D, \Delta] \neq 0$, this concerns the cases where m is a matrix (i.e several fields are integrated out), and curved spacetime.

This quantity has to be regularised, which is conveniently achieved in dimensional regularisation. All that remains to be done is to compute the expansion up to a given order and form covariant quantities, which can be greatly simplified by making a convenient choice of gauge. The momentum integration in dimensional regularisation is given by tabulated master integrals.

⁹We used the fact that for any diagonal functional we have $\log(A(x)\delta(x-y)) = \log(A(x))\delta(x-y)$, in other words $\log \mathbf{1} = 0$ where $\mathbf{1}(x, y) = \delta(x-y)$.

2.5.2 Functional methods: beyond one-loop

Functional methods beyond one-loop have seen a regrowth in interest recently [28, 29], although it had been treated before (see e.g [30]). In the following, we outline the computation of the path integral beyond-one loop, and in App. A we show how to obtain the two-loop β -function in a scalar ϕ^4 theory.

We return to (2.29), and for convenience perform the change of variable $\chi = \Pi^{-1/2} \cdot \eta$ where $\Pi^{-1}(x, y) = -\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}[\phi_0]$ is the propagator (Green function) of the theory [28]. The resulting Jacobian is none other than the one-loop determinant $\det^{-c_s}[-\delta^2 S[\phi_0]]$ and we obtain

$$Z[J] = e^{\frac{i}{\hbar}S[\phi_0] + iJ \cdot \phi_0 - c_s \text{Tr} \log(-\delta^2 S[\phi_0])} \int \mathcal{D}\chi e^{\frac{i}{2}\chi^2} \sum_{m \geq 0} \frac{i^m}{m!} \left(\sum_{n \geq 3} \frac{\hbar^{n/2-1}}{n!} (\Pi^{1/2} \cdot \chi)^n \cdot \delta^n S[\phi_0] \right)^m, \quad (2.53)$$

where we reintroduced \hbar to help with the counting.

So far, no approximation was made and this expression is exact, i.e non-perturbative. Although it looks intricate, the sum in factor of the Gaussian term is simply a series of monomials, and can be recast as

$$Z[J] = e^{\frac{i}{\hbar}S[\phi_0] + iJ \cdot \phi_0 + iW_{1\text{-loop}}[J]} \int \mathcal{D}\chi e^{\frac{i}{2}\chi^2} \sum_{k \geq 0} a_k[\phi_0] \cdot \chi^k, \quad (2.54)$$

where a_k depends on ϕ_0 and the backgrounds fields of the theory. To proceed we commute the integral and the series as such

$$Z[J] = e^{\frac{i}{\hbar}S[\phi_0] + iJ \cdot \phi_0 + iW_{1\text{-loop}}[J]} \sum_{k \geq 0} a_k[\phi_0] \cdot \int \mathcal{D}\chi \chi^k e^{\frac{i}{2}\chi^2}. \quad (2.55)$$

This step is however non-trivial, since this series is in fact divergent (even when regularised). Hence (2.54) and (2.55) are not equivalent: whereas (2.54) is exact, (2.55) is only correct in the perturbative regime and the series has a radius of convergence of zero. This a feature of most QFTs which are solved perturbatively. Nonetheless, it does not mean that there is no physics to extract from it. The lowest order terms remain a good approximation of an exact solution, but one has to keep in mind that summing the terms order by order will start to blow up and drift away from the non-perturbative result.¹⁰

We can thus compute perturbatively the path integral to some finite order using

$$\int \mathcal{D}\chi e^{\frac{i}{2}\chi^2} \prod_i^n \chi^{a_i}(x_i) = \begin{cases} \delta^{a_1 a_2} \delta(x_1 - x_2) \dots \delta^{a_{2p-1} a_{2p}} \delta(x_{2p-1} - x_{2p}) + \text{permutations, if } n = 2p \\ 0, \text{ if } n = 2p + 1, \end{cases} \quad (2.56)$$

where we made the internal indices a_i explicit. In the first line, there are $(2p)!!$ terms, it corresponds to the Wick theorem encountered in the diagrammatic approach.

To help fix the ideas, let us consider a simple model and write down all the terms that contribute at two-loops. We consider a real scalar field with a ϕ^4 self interaction

$$S[\phi] = \int d^4x \frac{1}{2} \phi \left[-\partial^2 - m^2 - \frac{\lambda}{12} \phi^2 \right] \phi. \quad (2.57)$$

¹⁰In fact, for generic QFTs solved perturbatively as $S_n = \sum_{k \geq 0}^n b_k$ where k is the loop order, it is known that $\lim_{n \rightarrow \infty} S_n = \infty$ not only because the b_k don't decrease sufficiently fast (e.g $b_k \underset{k \rightarrow \infty}{\sim} 1/k$), but because $\{b_k\}$ itself is divergent: $\lim_{k \rightarrow \infty} b_k = \infty$. In some cases, the non-perturbative result can be obtained from this divergent series via a Borel resummation.

The propagator of the theory is

$$\Pi(x, y) := (-\delta^2 S[\phi_0])^{-1}(x, y) = \frac{1}{\partial^2 + m^2 + \lambda\phi_0^2/2} \delta(x - y). \quad (2.58)$$

The expansion will come from

$$\begin{aligned} \frac{\delta^3 S}{\delta\phi(x_1) \dots \delta\phi(x_3)}[\phi_0] &= -\lambda\delta(x_1 - x_2)\delta(x_2 - x_3)\phi_0(x_1) \\ \frac{\delta^4 S}{\delta\phi(x_1) \dots \delta\phi(x_4)}[\phi_0] &= -\lambda\delta(x_1 - x_2)\delta(x_2 - x_3)\delta(x_3 - x_4), \end{aligned} \quad (2.59)$$

and any higher derivatives vanish. At two-loop, i.e at $\mathcal{O}(\hbar)$, we obtain the terms

$$e^{iW_{2\text{-loop}}[\mathcal{J}]} = \int \mathcal{D}\chi e^{\frac{i}{2}\chi^2} \left(\frac{i\hbar}{1!4!} (\Pi^{1/2} \cdot \chi)^4 \cdot \delta^4 S[\phi_0] + \frac{i^2\hbar}{2!(3!)^2} \left((\Pi^{1/2} \cdot \chi)^3 \cdot \delta^3 S[\phi_0] \right)^2 \right). \quad (2.60)$$

We will not go any further here, but we proceed in App. A and show how to obtain the two-loop β -function of λ . To the best of our knowledge, such a computation from the path integral is absent in the literature.

Part II

Effective Field Theory in Gravity

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Chapter 3

Motivation behind Effective Field Theories

Does one need to know the details of a theory down to the smallest scale to be able to do predictions at larger scale? Fortunately the answer to this question is negative. If that were the case doing Physics would not be possible. Let us take for example basic hydrodynamics, described by the Navier-Stokes equation. This theory successfully describes the motion of water and similar fluids at our scale, although we do not need to have any knowledge of the microscopic behavior of the molecules of water, in fact we do not even need to know that water is not continuous but is a collection of molecules. Obviously, we know that the Navier-Stokes equation is meaningless if we look at water at the scale of its molecules, which means that our theory breaks down below some length scale. This is the basic concept of effective theories: it is a theory that is able to do predictions below some cut-off energy scale Λ , but breaks down at energies larger than Λ due to our lack of knowledge of Physics beyond Λ .

This Chapter is dedicated to the concept of Effective Field Theory (EFT). We will start by introducing a historical example of an EFT.

3.1 Fermi's theory of weak interactions

In 1933, Fermi proposed a model to describe the radioactive β^- decay of a neutron into a proton, an electron and an electron neutrino [31, 32]. It is described by the Lagrangian

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu, \quad (3.1)$$

where the current J^μ is

$$J^\mu = \bar{\nu}_e \gamma^\mu P_L e + \bar{u} \gamma^\mu P_L d. \quad (3.2)$$

u , d , e and ν_e are the 4-components spinors of respectively the up quark, the down quark, the electron and the electron neutrino. The Dirac matrices respect the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad (3.3)$$

where η is the Minkowski metric, and we define

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad (3.4)$$

which anticommutes with the Dirac matrices: $\{\gamma_5, \gamma^\mu\} = 0$. Since $\gamma_5^2 = 1$, we obtain the chiral projectors

$$\begin{aligned} P_L &= \frac{1 - \gamma_5}{2}, & P_R &= \frac{1 + \gamma_5}{2}, \\ P_{L/R}^2 &= P_{L/R}, & P_L P_R &= 0. \end{aligned} \quad (3.5)$$

The Fermi constant can be measured experimentally via the muon lifetime

$$G_F = 1,17.10^{-5} \text{ GeV}^{-2}. \quad (3.6)$$

Fermi's theory has a four-fermions interaction Fig. 3.1, which describes the β^- decay.

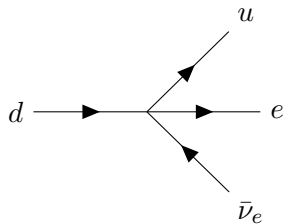


Figure 3.1: Four-fermions interaction that is involved in the β^- decay.

Despite successfully predicting decay rates at low energy, this theory suffers shortcomings:

- Firstly, it is a non-renormalisable theory. The propagators of the theory become divergent at some loop order, and as a result new divergences arise, which require an infinite number of counterterms to be cancelled. This implies that the theory loses its predictivity above some energy cut-off Λ .
- The interaction cross-sections rise with the energy $\sigma(E) \sim G_F^2 E^2$. However, unitarity of the theory, i.e conservation of probabilities, requires that $\sigma(e) \sim E^{-2}$ at most. Therefore the theory breaks unitarity when $\sigma(E) > E^{-2}$, i.e when $E > G_F^{-1/2}$. This provides the cut-off of the theory $\Lambda = G_F^{-1/2}$.

Fermi's theory is thus an EFT with cut-off $\Lambda = G_F^{-1/2}$. Non-renormalisability is often encountered in EFTs. It can be understood by the presence of a dimensionful coupling, G_F here which has mass dimension -2 . For the same reason, quantum gravity is non-renormalisable.

The Standard Model (SM) includes a refined description of the interaction that is at stakes here as we will see, which makes Fermi's theory an EFT of the SM.

3.2 The Standard Model . . .

In this Section, we review briefly the construction of the SM of Particle Physics. Starting from Fermi's theory, attempts to describe the fundamental interactions of nature on elementary particles led to theories that were a refined version of the previous ones, effectively pushing higher the EFT cut-off. Eventually, a renormalisable theory was obtained: the SM.

Throughout the 20th century, the elementary constituents of matter were observed experimentally, unraveling that it is constituted of three generations of spin- $\frac{1}{2}$ fermion fields (see Fig. 3.1). These fermions interact with each other via the fundamental interactions, which are carried by spin-1 bosons.

	1st generation	2nd generation	3rd generation	Q
Quarks	u	c	t	$2/3$
	d	s	b	$-1/3$
Leptons	e^-	μ^-	τ^-	-1
	ν_e	ν_μ	ν_τ	0

 Table 3.1: Fermion fields in the SM. Q is the electromagnetic charge.

Electromagnetic interaction Let us consider a simple fermionic theory

$$S = \int d^4x \bar{\psi} i \not{\partial} \psi , \quad (3.7)$$

with the notation $\not{\psi} = \gamma^\mu v_\mu$ and $\bar{\psi} = \psi^\dagger \gamma^0$. If we consider the S -matrix of this theory, it is clear that it is non-interacting, that is to say a set of fermions and anti-fermions in the *in* state arriving with some known momentum, will only lead to the same fermions having the same momentum in the *out* state.

It is worth noticing that this action is invariant when transforming the fermions as

$$\psi \rightarrow e^{ie\theta} \psi , \quad \bar{\psi} \rightarrow \bar{\psi} e^{-ie\theta} , \quad (3.8)$$

which is a $U(1)$ transformation, and e is some number. If we try to improve that symmetry to a local one, i.e take $\theta(x)$, then the action is not invariant since we have

$$\delta_\theta S = -e \int d^4x \bar{\psi} (\not{\partial} \theta) \psi . \quad (3.9)$$

However, if we modify the action by including a spin-1 field A_μ that transforms as

$$A_\mu \rightarrow A_\mu - \partial_\mu \theta , \quad (3.10)$$

then the symmetry is restored in the action

$$S' = \int d^4x \bar{\psi} i (\not{\partial} + ie\not{A}) \psi , \quad \delta_\theta S' = 0 \quad (3.11)$$

Considering the S -matrix of this new theory S' , we notice that we now have an interacting theory, that is to say if we consider for example an electron and a positron in the *in* state with respective momenta p_1 and p_2 , the *out* state may consist of an electron and a positron with different momenta (obviously the sum of the momenta is conserved). These fermions have interacted together via a creation and annihilation of a boson A_μ , which is diagrammatically represented in Fig. 3.2. This theory represents the fundamental interaction of fermions charged under a $U(1)$ local symmetry, with charge e . This in fact describes the electromagnetic interaction of fermions with electric charge e , with the photon field A_μ . The quantum theory of photons and electrons is called Quantum Electrodynamics (QED).

We can further note that our theory S' exhibits an additional invariance. Let us consider the action

$$S'' = \int d^4x \bar{\psi} i (\not{\partial} + i\not{A} + i(\not{\partial}\alpha)) \psi . \quad (3.12)$$

By transforming the boson as $A_\mu \rightarrow A_\mu - \partial_\mu \alpha$, we recover the same action as previously S' . This implies that by adding A_μ to our action S we have introduced an additional degree of freedom,

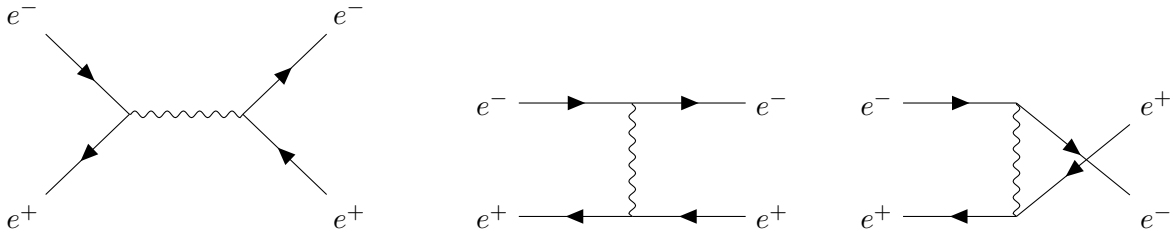


Figure 3.2: Feynman diagrams representing all the classical interactions between the free states: one electron and one positron. The solid lines represent the electron/positron, while the wavy line represent the spin-1 boson A_μ .

since we can arbitrarily choose to couple our fermion to $A_\mu + \partial_\mu \alpha$, $\forall \alpha$. This degree of freedom is called a gauge freedom, and physical observables should not depend on it. In fact, all the fundamental interactions exhibit this gauge freedom. For this reason we call the interaction boson “gauge bosons”, and they are studied in the framework of gauge theory.

Strong interaction The strong interaction is described by a gauge theory as well, except that the symmetry group is not $U(1)$, and that only the quarks are charged. Historically, quark were introduced by Gell-Mann to explain the hadrons spectrum [33]. The baryons are described as bound states of quarks and anti-quarks. In 1951, a new baryon was discovered [34] (denoted by $\Delta^{++}(J = 3/2)$), which seemed to be composed of three quarks in the same quantum state, hence violating Pauli’s exclusion principle. An additional quantum number was then postulated, called the color. The work of Yang and Mills [35] allowed to understand the color number as a gauge theory of a non-abelian group: $SU(3)$. The gauge boson associated to this symmetry is called the gluon field. It describes the strong interaction between quarks and explains their confinement in nuclei. This quantum theory is called Quantum Chromodynamics (QCD).

Weak interaction As described earlier, the theory of weak interaction was introduced to explain radiative decays. The first attempt being Fermi’s theory, which turned out to break down above some energy scale. A refined theory was later introduced by Feynman and Gell-Mann [36], introducing couplings that depend on the chirality of the fermions, hence called $V - A$ (vector – axial) theory. It was motivated by [37] pointing towards the non-conservation of parity, and then the experimental breakthrough in the discovery that neutrinos are only left-handed polarised [38]. Similarly as Fermi’s theory, the $V - A$ theory revealed to break down above $\lambda \sim G_F^{-1}$. Later improvements arose with the work of Glashow [39] called intermediate vector boson theory, who postulated the presence of new massive electrically charged W^\pm bosons, with a mass of about 100GeV, so that the theory would coincide with the $V - A$ theory at low energy. Another massive neutral boson Z has to be introduced to cure some divergences. Despite its success, the theory still lacked an explanation for the origin of the massive bosons, and some further divergences needed the introduction of a scalar field (the Higgs field) to be cured.

Electroweak interaction It was with the ideas of Spontaneous Symmetry Breaking (SSB) that the electromagnetic and weak interactions were understood to be unified as the gauge theory of a $SU(2)_L \otimes U(1)_Y$ group (L for left, and Y for hypercharge).

- **Spontaneous Symmetry Breaking** Inspired from condensed matter work, the concept of SSB was formalised in QFT [40–42]. It states that if the ground state of a theory breaks a

symmetry group to a subgroup, then for each generator of the broken symmetry a massless degree of freedom appears in the theory, called a Nambu-Goldstone boson. Weinberg and Salam then applied this concept to unify the electromagnetic and weak interactions [43, 44]. They introduced a gauge group $SU(2)_L \otimes U(1)_Y$, that is spontaneously broken below the electroweak energy scale to a $U(1)_Q$ gauge group. Before SSB, the gauge theory includes three massless gauge bosons, whereas after SSB only one remains, which is the photon associated to the $U(1)_Q$ electromagnetic interaction. The three generators of $SU(2)_L$ thus give rise to three Nambu-Goldstone bosons which are absorbed to give their mass to the three bosons of the weak interaction: W^\pm and Z .

- Higgs boson** This mechanism presents however a caveat: before SSB, the group $SU(2)_L \otimes U(1)_Y$ can only be a symmetry of the theory if all fermions are massless. However, after SSB all the fermions are measured to have a mass. Higgs, Brout and Englert thus postulated the existence of a scalar field, that would both trigger the the SSB when relaxing to its equilibrium state, and provide the fermions their mass [45, 46].

Finally, these works led to the construction of the SM, which exhibits a gauge symmetry group

$$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y, \quad (3.13)$$

before SSB. The particle content of the SM and their quantum numbers are summarised in Tab. 3.2.

	Fields	Representation $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$	Isospin I_3	Electric charges $Q = Y + I_3$
Spin $\frac{1}{2}$	$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3}, \mathbf{2}, 1/6)$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$
	u_R	$(\mathbf{3}, \mathbf{1}, 2/3)$	0	2/3
	d_R	$(\mathbf{3}, \mathbf{1}, -1/3)$	0	-1/3
	$l_L = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, -1/2)$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
	e_R	$(\mathbf{1}, \mathbf{1}, -1)$	0	-1
	Spin 0	$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, 1/2)$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$
Spin 1	G_μ^A	$(\mathbf{8}, \mathbf{1}, 0)$	0	0
	W_μ^I	$(\mathbf{1}, \mathbf{3}, 0)$	$(1, -1, 0)$	$(1, -1, 0)$
	B_μ	$(\mathbf{1}, \mathbf{1}, 0)$	0	0

Table 3.2: Fields and their quantum numbers under the SM gauge group. For simplicity, we omitted the color and generation indices. The notations $(\mathbf{8}, \mathbf{3}, \mathbf{2}, \mathbf{1})$ denote the octet, triplet, doublet and singlet representations of the $SU(N)$ group. I_3 stands for the third generator of $SU(2)_L$. The breaking of electroweak symmetry, $SU(2)_L \otimes U(1)_Y \rightarrow U(1)_Q$ implies the relation $Q = I_3 + Y$.

The Standard Model Lagrangian Before SSB, the SM Lagrangian reads

$$\mathcal{L}_{\text{SM}} = -\frac{1}{4}G_{\mu\nu}^A G^{A,\mu\nu} - \frac{1}{4}W_{\mu\nu}^I W^{I,\mu\nu} - \frac{1}{4}B_{\mu\nu} B^{\mu\nu} \quad (3.14)$$

$$+ (\bar{q}_L i \not{D} q_L + \bar{u}_R i \not{D} u_R + \bar{d}_R i \not{D} d_R + \bar{l}_L i \not{D} l_L + \bar{e}_R i \not{D} e_R) \quad (3.15)$$

$$+ (D_\mu H)^\dagger (D^\mu H) + \mu^2 H^\dagger H - \frac{\lambda}{2} (H^\dagger H)^2 \quad (3.16)$$

$$- (\bar{l}_L \Gamma_e e_R H + \bar{q}_L \Gamma_u u_R \tilde{H} + \bar{q}_L \Gamma_d d_R H + \text{h.c.}), \quad (3.17)$$

where $\tilde{H}^j = \epsilon_{jk} (H^*)^k$, (i, j, k) . We provide some precisions that will be useful in the following discussions.

- The second line (3.15) includes the kinetic terms of the fermions. D is the covariant derivative and includes the gauge coupling as in (3.11). For example, if a fermion ψ is charged under the gauge group with representation $(\mathbf{3}, \mathbf{2}, Y)$, its covariant derivative reads

$$D_\mu \psi = (\partial_\mu + ig_G G_\mu + ig_W W_\mu + ig_Y Y B_\mu) \psi, \quad (3.18)$$

where g_G , g_W and g_Y are dimensionless coupling constants.

- The gauge bosons that appear in the covariant derivative are: the gluon field $G_\mu = G_\mu^A T^A$, where $\{T^A\}_{1 \leq A \leq 8}$ are the generators of the $SU(3)_c$ Lie algebra; $W_\mu = W_\mu^I T^I$ is the $SU(2)_L$ gauge boson with $\{T^I\}_{1 \leq I \leq 3}$ the generators of its Lie algebra; and B_μ the Abelian gauge boson of $U(1)_Y$. Their kinetic term is given by their field strength in the first line (3.14) such that for $X = G, W, B$

$$X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu - [X_\mu, X_\nu]. \quad (3.19)$$

The commutator vanishes for the Abelian gauge field B_μ . It can be obtained in terms of components using their associated Lie algebra

$$[T^A, T^B] = i f^{ABC} T^C, \quad [T^I, T^J] = i \epsilon^{IJK} T^K, \quad (3.20)$$

f and ϵ being the associated structure constants. The $SU(3)_c/SU(2)_L$ Lie algebra can be expressed in terms of the Gell-Mann/Pauli matrices $T^A = \frac{\lambda^A}{2} / T^I = \frac{\sigma^I}{2}$.

- The third line (3.16) represents the Higgs kinetic term and self coupling. Its potential is called the Mexican hat potential

$$V(x) = \mu^2 x^2 - \frac{\lambda}{2} x^4. \quad (3.21)$$

It triggers the SSB when the VEV of the Higgs relaxes at the minimum of the potential.

- The fourth line (3.17) is the Yukawa interactions between the fermions and the Higgs field. They provide the fermions their mass when the Higgs reaches its equilibrium VEV.

In the quantum theory, an additional line should be added to account for the quantisation of the gauge fields, to include the gauge fixing and ghosts terms.

As we saw, the description of elementary particles and their interactions went through different stages of description, each being an EFT of the next one. The completeness of the SM is a legitimate question, and we will see in the next Section its shortcomings.

3.3 ... as an Effective Field Theory

The SM is considerably successful in describing Physics at the level of the strong and electroweak interactions. It allowed to predict the existence of the W^\pm and Z bosons, which were discovered thanks to the UA1 experiment at CERN in the 80's. The Higgs boson was predicted and then observed in 2012 at the LHC [47], marking a milestone in the history of Particle Physics. Besides these, the SM has found agreement with numerous experimental measurements, up to very high precisions.

Nonetheless, the SM has its shortcomings, indicating that the theory is incomplete. Firstly, gravity is absent of the SM. Its effects are insignificant at energies probed by current experiments in Particle Physics, but not at energies closer to the Planck scale $M_{Pl} \sim 10^{19}$ GeV. We know from our previous discussions that a theory with some energy cut-off is an EFT. The SM is an EFT, and an upper bound on its cut-off is given by the Planck scale: $\Lambda_{SM} = M_{Pl}$.

Despite the absence of gravity, the SM has some other unsolved puzzles. Some of them are:

- **Hierarchy problem:** The mass of the Higgs boson receives radiative corrections, as any massive interacting particle. However in this case, it can be shown that the loop corrections to Higgs mass should scale as $\Delta M_H^2 \sim \Lambda_{SM}^2 y_t^2$, where only the dominant contribution from the top quark which couples to the Higgs with coupling constant $y_t \sim 1$ is taken into account. This implies that the Higgs mass should lie around the Planck mass $\Lambda_{SM} \sim M_{Pl} \sim 10^{19}$ GeV, whereas it was measured to be about $M_H = 125$ GeV. There is thus a tremendous hierarchy between the observed Higgs mass and the Planck mass that is unexplained by the SM.
- **Neutrino mass:** In the SM, the neutrinos are massless. This is in disagreement with experimental measurement of neutrino oscillations, which predicts a small but non-vanishing mass [48].
- **Strong CP problem:** When constructing the Lagrangian of a theory, one should in principle include all the renormalisable operators that are consistent with the fundamental symmetries of the theory. For QCD this implies that the topological term

$$\mathcal{L}_{QCD} \supset \frac{\theta}{32\pi^2} G_{\mu\nu}^A \tilde{G}^{A,\mu\nu}, \quad (3.22)$$

should be present, where G is the gluon field strength, and A denotes the color indices. $\tilde{G}^{A,\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}^A$ is the dual field strength. Despite being a boundary term, the topology of the gauge group $SU(3)_c$ makes it non-vanishing when integrated against the whole manifold. Since this operator changes sign under CP transformation, it would give rise to CP -violating processes in the QCD sector. Such effects are not observed, since in fact the θ angle was experimentally determined, via measurements of neutron electric dipole moment (nEDM) [49–51], to be extremely small: $\theta < 10^{-10}$. The SM lacks a mechanism to explain the smallness of the θ parameter.

Many Beyond the SM (BSM) solutions to these problems were proposed in the literature, often involving new symmetries and particles at energy scales that were not probed by experiments ($> \text{TeV}$). Unfortunately, except for some small deviations from SM predictions, no new Physics was discovered in current experiments since 2012. The possible extensions of the SM are almost limitless, and current experimental data does not select very specific directions. The EFT paradigm is a well suited framework for this situation. There are two main approaches: consider a vast range of BSM operators, and use experimental data to constrain their coefficients (bottom-up approach), or connect UV models to low energy precision measurements (top-down approach). In the following,

we will concentrate on the top-down approach, but let us first say a word about the bottom-up approach.

3.4 Bottom-up approach

An EFT that breaks down at a scale Λ can be expressed in terms of renormalisable operators independent of Λ , supplemented by non-renormalisable operators suppressed by powers of Λ

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{renorm}} + \sum_{n \geq 1} \frac{1}{\Lambda^n} \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}, \quad (3.23)$$

where the $c_i^{(n)}$ are called Wilson coefficients, and the $\mathcal{O}_i^{(n)}$ are effective operators. The Wilson coefficients depend on the renormalisation energy scale. Whereas the top-down approach consists in obtaining these effective operators and Wilson coefficients from a specific UV theory, the bottom-up approach consists in writing down all the possible independent effective operators to cover a wide range of possible UV completions. The Wilson coefficients are then constrained using experimental data.

As one would expect, the range of possible effective operators increases tremendously with the order n of the operator. In practice, the basis can be reduced by considering operators that respect a chosen gauge symmetry group. There are two main choices of EFTs that have been extensively studied in the literature.

- **The Standard Model Effective Field Theory (SMEFT)** is the EFT obtained by introducing all the possible effective operators built out of the SM fields, while respecting the SM gauge symmetry group before SSB $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. It is a direct generalisation of the SM. The Higgs field is identical to the SM one, and the only source for the electroweak SSB. At a given order $1/\Lambda^n$, it is possible to identify a minimal basis of independent operators (using EoMs, integrations by parts, Bianchi identities, Fierz transformation and field redefinitions). Significant effort was put in identifying a basis for dimension 6 operators (i.e. suppressed by $1/\Lambda^2$) [52–54].
- **The Higgs Effective Field Theory (HEFT)** includes effective operators built out of the SM fields with minimal assumptions, after SSB. It is more generic than the SMEFT, but more complicated to study. It includes a Higgs field that may not necessarily be identical as in the SM, such that the Goldstone bosons and the Higgs-like scalar singlet after SSB are independent from each other. The gauge symmetry group is only $SU(3)_c \otimes U(1)_Q$, which increases the variety of effective operators. Finally, the power counting is not as straightforward as in the SMEFT and has been the subject of debates in the literature [55–57].

3.5 Effective Field Theory and gravity

Any renormalisable QFT may exhibit some cut-off scales such as the ones mentioned above for the SM. These problems may be solved by introducing beyond the SM fields and symmetries. These BSM models, however elaborate they may be, cannot push the cut-off beyond the Planck scale. The reason is that beyond the Planck scale, effects of quantum gravity become important, and so far a complete theory of quantum gravity remains elusive.

Although the largeness of the Planck scale puts quantum gravity way beyond the reach of experiments, it also means that any EFT including effects of gravity is a very good approximation

of quantum gravity for a wide range of energies. This includes EFTs of quantum gravity (see e.g [58, 59] for comprehensive reviews), or EFTs where gravity is treated as a background field.

One example of applications of EFTs with a gravitational background field is the computations of back-reactions effects, that is to say the effects of quantum fields on a classical gravitational background. This finds application in very early universe Physics, inflation, quantum black holes, quark-gluon plasma, dynamical symmetry breaking, etc. . . [60–63]. Other examples of application arise in condensed matter where the effects of a non-uniform temperature can be described as a curvature of spacetime [64–66] (see e.g [67] for a recent review). Chapter 6 will be mainly dedicated to constructing EFTs in a background gravitational field.

Chapter 4

Effective Field Theory Formalism

In this Section, we present the top-down EFT approach, and delve into the recent functional methods that led to the Universal One-Loop Effective Action.

The top-down approach consists in comparing the predictions of a UV theory to low energy measurements. The procedure can be broken down in three steps:

- **Matching:** Starting from a UV theory, the goal is to build the EFT that matches the UV theory at the cut-off scale Λ , while removing the heavy degrees of freedom from the UV theory. In practice, the UV theory can be a BSM model at a new physics scale Λ , and the relevant EFT will be valid below Λ , where the new BSM states were integrated out. Typically, Λ is the mass of the BSM fields.

This matching can be performed using Feynman diagrams, but recent developments allow to efficiently use the path integral directly. In general, it is very difficult to compute the effective action as a whole, since it is non-local. However, it can be expanded as a series of local operators and computed up to some power in $1/\Lambda$ and some loop order.

The EFT thus obtained includes effective non-renormalisable operators, with their associated Wilson coefficient at the renormalisation scale $\mu = \Lambda$: $c_i(\mu = \Lambda)$, such as in (3.23). We will be mostly interested in the effective operators and Wilson coefficients at tree and one-loop levels, up to $1/\Lambda^4$ usually.

- **Running:** The UV theory and the EFT are matched at the energy scale Λ . The EFT must then be run down to the energy scale of the experiment (e.g the electroweak scale) in order to be confronted to experimental measurements. This is done accordingly to the Renormalisation Group Equations.
- **Mapping:** Finally, once the EFT is known at the energy scale of the experiments, physical observables have to be extracted and compared with the experimental data, or the SM predictions. For example, the decay width of the Higgs singlet into two photons $h \rightarrow \gamma\gamma$ in the SMEFT can be compared to the SM prediction and expressed in terms of the SMEFT Wilson coefficients [68]

$$\frac{\Gamma_{h \rightarrow \gamma\gamma}^{\text{SMEFT}}}{\Gamma_{h \rightarrow \gamma\gamma}^{\text{SM}}} - 1 \propto (c_{WW} + c_{BB} - c_{WB}), \quad (4.1)$$

where the Wilson coefficients are associated with the effective operators are $|H|^2 W_{\mu\nu} W^{\mu\nu}$, $|H|^2 B_{\mu\nu} B^{\mu\nu}$, and $(H^\dagger \sigma^I H) W_{\mu\nu}^I B^{\mu\nu}$ respectively.

4.1 Matching via Feynman diagrams

Let us present an example of matching using Feynman diagrams. Consider an interacting theory of a massive real scalar field Φ , and a light fermion ψ

$$\mathcal{L}_{\text{UV}} = \bar{\psi}i\not{\partial}\psi - \frac{1}{2}\Phi(\partial^2 + m^2)\Phi - \lambda\Phi\bar{\psi}\psi. \quad (4.2)$$

Our goal is to find the EFT \mathcal{L}_{EFT} built out of the fermion only, and that reproduces the same S -matrix elements as those from \mathcal{L}_{UV} in the limit where external particles have momentum much smaller than m . For example, let us consider the interaction $\psi\psi \rightarrow \psi\psi$ at tree level. The corresponding tree level amplitude in the UV theory corresponds to the diagrams in Fig. 4.1, and can be computed to obtain

$$\mathcal{A}_{\text{UV}} = \frac{-i\lambda^2}{(p_1 - p_3)^2 - m^2} \bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2) - (p_3 \leftrightarrow p_4). \quad (4.3)$$

In the limit $p_i^2 \ll m^2$, the amplitude can be expanded as

$$\mathcal{A}_{\text{UV}} \sim \frac{-i\lambda^2}{m^2} \bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2) - (p_3 \leftrightarrow p_4) + \mathcal{O}\left(\frac{p_i p_j}{m^2}\right). \quad (4.4)$$

At the level of the EFT, the scalar field is absent, which means that there should be an effective 4-fermion operator to reproduce this interaction

$$\mathcal{L}_{\text{EFT}}^{\text{tree}} = \bar{\psi}i\not{\partial}\psi + \frac{c_1}{2}(\bar{\psi}\psi)^2, \quad (4.5)$$

where c_1 is a Wilson coefficient to be determined by matching. This EFT produces a 4-fermion interaction amplitude

$$\mathcal{A}_{\text{EFT}} = [ic_1] \bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2) - \{p_3 \leftrightarrow p_4\}. \quad (4.6)$$

The amplitudes (4.3) and (4.6) should match in the limit $p_i^2 \ll m^2$, which allows us to identify

$$c_1 = \frac{\lambda^2}{m^2}. \quad (4.7)$$

The same procedure can be applied at one-loop, by comparing the one-loop amplitude of the $\psi\psi \rightarrow \psi\psi$ scattering in the low momentum limit, to obtain the one-loop corrections to c_1 .

Now if one would like to obtain the EFT from another UV theory, the corresponding Feynman diagrams would have to be computed from scratch. This has to be done not only any time a new UV theory is considered, but for every effective operator that may arise in the EFT. Needless to say that this becomes cumbersome quite fast. As we will see now, the path integral approach allows one to perform the matching once and for all, for very generic classes of UV theories.

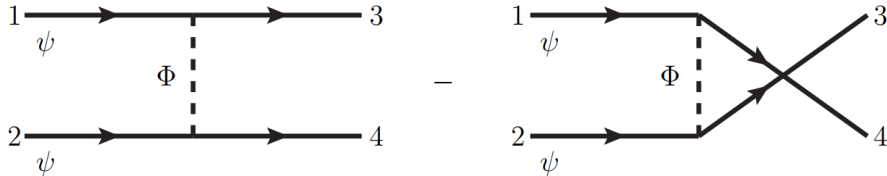


Figure 4.1: Tree-level diagrams proportional to λ^2 that contribute to $\psi\psi \rightarrow \psi\psi$ scattering process, Ref. [5].

4.2 Matching via the path integral

The matching of an EFT onto a UV theory is an exercise where the path integral approach shines in comparison with a diagrammatic approach. As we will soon see, it allows to unravel the universal structure of the one-loop effective action, in the so-called Universal One-Loop Effective Action (UOLEA) [69–73].

Let us start from a UV action

$$S_{\text{UV}}[\phi, B] = \int d^4x \mathcal{L}[\phi, B], \quad (4.8)$$

which depends on a quantum field ϕ , and some background fields B . The generating functional reads

$$Z_{\text{UV}}[J, B] = \int \mathcal{D}\phi e^{iS_{\text{UV}} + iJ \cdot B}, \quad (4.9)$$

and $W_{\text{UV}}[J, B] = -i \log Z_{\text{UV}}[J, B]$. Its Legendre transform is

$$\Gamma_{\text{UV}}[\varphi, B] = W_{\text{UV}}[J_\varphi, B] - J_\varphi \cdot \varphi, \quad (4.10)$$

where J_φ is solution to

$$\frac{\delta W_{\text{UV}}}{\delta J}[J_\varphi, B] = \varphi. \quad (4.11)$$

In the EFT framework, we are interested in the case where ϕ decouples at low energy. We consider the case where it is heavy, such that its mass m is much larger than the typical energy scale reached by experiments. This implies that the relevant scattering amplitudes involve no heavy field ϕ in the *in* and *out* states, which boils down to taking $J = 0$ in the generating functional, or equivalently $\varphi = \langle \phi \rangle$ after the Legendre transform. This leaves us with the effective action

$$\Gamma_{\text{UV}}[\langle \phi \rangle, B] = W_{\text{UV}}[0, B]. \quad (4.12)$$

Our goal is then to find the low energy EFT $S_{\text{EFT}}[B]$ such that its effective action

$$\Gamma_{\text{EFT}}[B] = -i \log \int \mathcal{D}\phi e^{iS_{\text{EFT}}}, \quad (4.13)$$

verifies

$$\Gamma_{\text{EFT}}[B] = \Gamma_{\text{UV}}[\langle \phi \rangle, B]. \quad (4.14)$$

Note that Γ_{UV} is called the effective action, which is not the same as the EFT obtained after integrating out the heavy fields. Nonetheless, we will see that in the absence of external sources and mixed heavy-light terms Γ_{UV} is in fact also the EFT.

Let us start by evaluating Γ_{UV} at one-loop using the stationary phase approximation introduced in Part I

$$\Gamma_{\text{UV}}[\langle \phi \rangle, B] \simeq S_{\text{UV}}[\phi_c, B] + ic_s \log \det \left(-\frac{\delta^2 S_{\text{UV}}}{\delta \phi^2}[\phi_c, B] \right), \quad (4.15)$$

where ϕ_c is solution to the classical EoM

$$\frac{\delta S_{\text{UV}}}{\delta \phi}[\phi_c, B] = 0. \quad (4.16)$$

In that case, the matching is rather trivial, and we directly find the EFT at tree and one-loop levels

$$S_{\text{EFT}}[B] = \Gamma_{\text{UV}}[\langle \phi \rangle, B] \simeq S_{\text{UV}}[\phi_c, B] + ic_s \log \det \left(-\frac{\delta^2 S_{\text{UV}}}{\delta \phi^2}[\phi_c, B] \right). \quad (4.17)$$

The matching is less straightforward in the presence of a source for a light quantum field

$$Z_{\text{UV}}[\Phi, \phi, B] = \int \mathcal{D}\Phi \mathcal{D}\phi e^{iS_{\text{UV}} + iJ \cdot \Phi + ij \cdot \phi}, \quad (4.18)$$

where Φ is a heavy field and ϕ a light field. In that case we find the matching condition

$$S_{\text{EFT}}[\phi] = S_{\text{UV}}[\Phi_c, \phi] + ic_s \log \det Q_{\text{UV}}[\Phi_c[\phi], \phi] - ic_s \log \det \left(-\frac{\delta^2 S_{\text{UV}}[\Phi_c[\phi], \phi]}{\delta \phi^2} \right), \quad (4.19)$$

where Q_{UV} is the Hessian (matrix of the second derivatives) of $-S_{\text{UV}}[\Phi, \phi]$. In terms of diagrams, this formula accounts for loops that includes both light and heavy fields, called mixed heavy-light. We will now show how the path integral formalism allows to obtain universal results in the simpler case where there are no mixed heavy-light loops, although it can be done too [71, 72].

4.2.1 Bosonic UOLEA

To obtain the Universal One-Loop Effective Action (UOLEA), we need to assume a generic ansatz for S_{UV} ¹

$$S_{\text{UV}}[\phi, B] = \int d^4x \phi (D^2 + m^2 + U[B]) \phi, \quad (4.20)$$

where D is the covariant derivative that bears the gauge connections, and U is some local operator that bears no open derivatives acting on the rightmost ϕ . At one-loop, the effective action thus reads

$$S_{\text{1loop}}^{\text{eff}} = ic_s \text{Tr} \log (D^2 + m^2 + U), \quad (4.21)$$

Despite its bosonic form, it is remarkably quite generic. For bosons such as real and complex scalars, spin-1 and even spin-2 fields, it is straightforward to put the effective action under the form (4.21). For a vector-like fermion² with action

$$S_{\text{UV}}[\psi, \bar{\psi}, B] = \int d^4x \bar{\psi} (i\not{D} - m - Q[B]) \psi, \quad (4.22)$$

the one-loop effective action can be squared to obtain a bosonic form such as (4.21)

$$\begin{aligned} S_{\text{1loop}}^{\text{eff}} &= -i \log \det (i\not{D} - m - Q) \\ &= -\frac{i}{2} \log \det (i\not{D} - m - Q)^2 \\ &= -\frac{i}{2} \log \det (-i\not{D} - m - Q) \det (i\not{D} - m - Q) \\ &= -\frac{i}{2} \log \det (D^2 + m^2 + U_{\text{ferm}}), \end{aligned} \quad (4.23)$$

where $U_{\text{ferm}} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} + 2mQ + Q^2 + [i\not{D}, Q]$, $F_{\mu\nu} = [D_\mu, D_\nu]$. In the third line we used the fact that the trace of an odd number of Dirac matrices vanishes, so that the trace is invariant under changing the sign of all the Dirac matrices. Note also that this derivation requires that Q has an even number of Dirac matrices. As we will see below, this does not hold for chiral fermions.

¹Note that U can also depend on ϕ , but then the second derivative of the action slightly differs. For example, for a ϕ^4 theory, we have $U \propto \phi^2$ and thus $\frac{\delta^2 S}{\delta \phi^4} = D^2 + m^2 + 12U[\phi_c]$. To be more generic, the starting ansatz should be (4.21) rather than (4.20).

²By vector-like fermion, we mean that all other fields couple to it regardless of its chirality, as opposed to a chiral fermion.

Eq. (4.21) hence holds for bosonic fields as well as vector-like fermions. U depends on the nature of the field that is integrated out, and so does the spin factor c_s as can be seen in (4.23). For a real scalar, complex scalar, vector-like fermion, gauge boson or Fadeev-Popov ghost $c_s = 1/2, 1, -1/2, 1/2$ or -1 respectively.

Applying the CDE as described in Part I we obtain

$$S_{\text{1loop}}^{\text{eff}} = -ic_s \int d^d x \frac{d^d q}{(2\pi)^d} \sum_{n \geq 1} \frac{1}{n} \left[\Delta (D^2 + 2iq \cdot D + U) \right]^n. \quad (4.24)$$

Conveniently, a factorisation between the momentum integrals and the operator part occurs which is the origin of the computation of the UOLEA [69, 70], and allows to derive the Wilson coefficients in terms of master integrals (App. B). The bosonic UOLEA up to dimension 6 operators reads [69]

$$\begin{aligned} S_{\text{eff}}^{\text{bos}} = & \frac{c_s}{16\pi^2} \int d^4 x \text{tr} \left\{ -m^4 \frac{1}{2} \left(\log \left(\frac{m^2}{\mu^2} \right) - \frac{3}{2} \right) + m^2 \left(1 - \log \left(\frac{m^2}{\mu^2} \right) \right) U \right. \\ & + \log \left(\frac{m^2}{\mu^2} \right) \left[-\frac{1}{6} (\square U) - \frac{1}{2} U^2 - \frac{1}{12} F^2 \right] \\ & + \frac{1}{m^2} \left[\frac{1}{60} (D_\mu F^{\mu\nu})^2 - \frac{1}{90} F_\mu^\nu F_\nu^\rho F_\rho^\mu - \frac{1}{12} U (\square U) - \frac{1}{6} U^3 - \frac{1}{12} U F^2 \right] \\ & \left. + \mathcal{O}(1/m^4) \right\}. \end{aligned} \quad (4.25)$$

The matching step for models involving heavy bosons and vector-like fermions is thus done once and for all in the UOLEA.

The UOLEA can also be obtained using the CDE when the mass matrix is non-degenerate [70] (i.e several massive fields of different mass are integrated out), and to encompass mixed heavy-light loops [71, 72]. A diagrammatic approach also exists to help with the expansion [74].

As mentioned in Part I, there is an optional step that can be undertaken in (4.24) that is called the Gaillard and Cheyette trick [27]. It consists in sandwiching the sum in (4.24) by $e^{\pm iD \cdot \frac{\partial}{\partial q}}$. This insertion is allowed since the derivatives in $e^{\pm iD \cdot \frac{\partial}{\partial q}}$ act on 1 to the right, and yield a boundary term on the left, therefore it amounts to sandwich (4.24) with 1. Despite not changing the result, it changes the organisation of the computation such that all the derivatives are stowed within commutators, and thus the expansion is manifestly covariant, i.e covariant at every step of the computation. Sandwiching an operator with $e^{\pm iv \cdot D}$ for some vector v amounts to doing a covariant Taylor expansion, for example

$$e^{iv \cdot D} U e^{-iv \cdot D} = \sum_{n \geq 0} \frac{1}{n!} (D^{\mu_1} \dots D^{\mu_n} U) v_{\mu_1} \dots v_{\mu_n}. \quad (4.26)$$

In our computation we skip this extra step, although it makes the expansion manifestly covariant, it also makes it much more tedious. Besides, a convenient use of gauge fixing can provide a manifestly covariant expansion while simplifying the computation.

4.2.2 Fermionic UOLEA

As mentioned earlier, this bosonic UOLEA does not apply to chiral fermions. In this Section we show how to deal with them to obtain the fermionic UOLEA [73, 75].

If we integrate out a massive chiral fermion³ which UV action is

$$S = \int d^4x \bar{\psi} (i\mathcal{D} - m - Q) \psi, \quad (4.27)$$

we obtain a one-loop effective action of the form

$$S_{\text{1loop}}^{\text{eff}} = -i \log \det (i\mathcal{D} - m - Q). \quad (4.28)$$

As previously, it can be bosonised by squaring it.⁴ We split Q into Q_e which has an even number of Dirac matrices and Q_o with an odd number,

$$\begin{aligned} \log \det (i\mathcal{D} - m - Q) &= \frac{1}{2} \log \det (i\mathcal{D} - m - Q_e - Q_o) \det (-i\mathcal{D} - m - Q_e + Q_o) \\ &= \frac{1}{2} \log \det \left(D^2 + m^2 + \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} + 2mQ_e \right. \\ &\quad \left. + Q(Q_e - Q_o) - [i\mathcal{D}, Q_e] + \{i\mathcal{D}, Q_o\} \right), \end{aligned} \quad (4.29)$$

where in the first line we used the vanishing of the trace of an odd number of Dirac matrices, therefore the invariance under flipping their sign.

Eq. (4.29) reduces to a determinant of the form (4.21) only if $Q_o = 0$, with $U = \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} + 2mQ_e + Q_e^2 - [i\mathcal{D}, Q_e]$. However, the usual ansatz used in the heat kernel method, previous CDE approaches [77, 78] and the worldline formalism [79], assumes that $e^{-iq \cdot x} U e^{iq \cdot x} = U$ (i.e no open derivative), which is not always true depending on Q_e (for example if $Q_e \propto \gamma_5$).

In other words, the usual ansatz (4.21) (used in heat kernel, CDE, worldline methods) does not encompass chiral fermions since it is not equivalent to (4.29). One could then proceed to work with (4.29), however it is simpler to directly work with (4.28) using the fermionic CDE. We proceed as for the CDE presented in Part I to express the functional trace in momentum space. The difference arises in the form of the propagator Δ

$$\begin{aligned} S_{\text{1loop}}^{\text{eff}} &= -i \int d^d x \frac{d^d q}{(2\pi)^d} \text{tr} \log \left(\Delta^{-1} (1 - \Delta (-i\mathcal{D} + Q)) \right) \\ &= -i \int d^d x \frac{d^d q}{(2\pi)^d} \text{tr} \sum_{n \geq 1} \frac{1}{n} \left[\Delta (-i\mathcal{D} + Q) \right]^n, \end{aligned} \quad (4.30)$$

where $\Delta = m/(q^2 - m^2) - \not{q}/(q^2 - m^2)$.

If we assume a general form for Q , namely a scalar W_0 , pseudo-scalar $W_0 \gamma_5$, vector $\gamma^\mu V_\mu$ and pseudo-vector part $\gamma^\mu A_\mu \gamma_5$, we can derive the so-called fermionic UOLEA [73]. The case of UV theories involving derivative couplings (such as axion models) requires extra care and is treated in details in [75].

In the following, our goal will be to extend the bosonic and fermionic UOLEA to include the effects of gravity.

³Let us make a comment on massive chiral fermions. The mass term is a hard breaking source of axial symmetries (local or global). These symmetries can be made manifest at tree-level by implementing their spontaneous breaking and introducing their associated Goldstone bosons. Here, we choose for convenience to work within the unitary basis and loose manifest tree-level axial invariance.

⁴Note that it can also be bosonised by multiplying by the hermitian conjugate, which amounts to computing the modulus of the determinant. Compared to the square of the determinant, only a phase is omitted. This phase is relevant for computing consistent anomalies [4, 76]. We discuss that point further in Part III.

Chapter 5

Quantum Field Theory in Curved Spacetime

Before building EFTs in gravity, let us provide an introduction to QFT in curved spacetime.

5.1 A glimpse of differential geometry

Let us start with a brief introduction to differential geometry and General Relativity.

Manifolds The equivalence principle, that is to say the equality of gravitational and inertial mass, implies that gravity can be entirely described by the curvature of spacetime. The suitable mathematical object to study curved spacetimes are manifolds.

A differentiable manifold \mathcal{M} of dimension d is a topological space provided with an atlas $\{(U_i, \varphi_i)\}_i$, where the U_i are open sets such that

$$\bigcup_i U_i = \mathcal{M} , \quad (5.1)$$

and the φ_i are smooth bijective maps (homeomorphisms)

$$\varphi_i : \mathcal{M} \rightarrow \mathbb{R}^d . \quad (5.2)$$

The final requirement to have a differentiable manifold is that $\forall i, j$ such that $U_i \cap U_j \neq \emptyset$, $\phi_i \circ \phi_j^{-1}$ is infinitely differentiable.

Using a chart (U_i, φ_i) , $\forall p \in U_i$ we can associate a vector $\varphi_i(p) \in \mathbb{R}^d$, which we call coordinates. It is convenient to write them as $\varphi_i(p)^\mu = x^\mu$, where $1 \leq \mu \leq d$. This simply means that \mathcal{M} resembles \mathbb{R}^d *locally*, which is a mathematical representation of the equivalence principle.

At each point of $p \in \mathcal{M}$, we can consider the tangent space at p : $T_p\mathcal{M}$, which is a vector space. Given coordinates x^μ , any vector $X \in T_p\mathcal{M}$ can be expressed as

$$X = X^\mu \partial_\mu . \quad (5.3)$$

This is called a coordinate basis. We will see an example of a non-coordinate basis when introducing the tangent frame later on. The dual to $T_p\mathcal{M}$ is called the cotangent space $T_p^*\mathcal{M}$. We can also write its elements on the coordinate basis

$$\theta = \theta_\mu dx^\mu . \quad (5.4)$$

A tensor is an element of a tensor product of the tangent and cotangent spaces, and can be expressed as

$$T = T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_n} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_m} . \quad (5.5)$$

We provide \mathcal{M} with the metric tensor g , which can be expressed on the coordinate basis as

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu . \quad (5.6)$$

It defines the inner product of two vectors $X, Y \in T_p\mathcal{M}$ and the norm

$$X \cdot Y = g_{\mu\nu} X^\mu Y^\nu , \quad X^2 = g_{\mu\nu} X^\mu X^\nu . \quad (5.7)$$

Likewise, its inverse

$$g^{-1} = g^{\mu\nu} \partial_\mu \otimes \partial_\nu , \quad (5.8)$$

defines the inner product and the norm on the cotangent space. The metric also provides the manifold with a local volume element

$$d\mathcal{V}(x) = \sqrt{\det g(x)} d^d x , \quad (5.9)$$

which we use for integration. Note that integration of forms over manifold is performed by decomposing the integral over the patches U_i which are mapped onto \mathbb{R}^d [80].

Diffeomorphisms If two charts overlap, i.e $p \in U_i \cap U_j$ $i \neq j$, then there are two choices of coordinates possible: $\varphi_i(p)$ and $\varphi_j(p)$. It goes without saying that physical results should be independent from this arbitrary choice. This is compatible with the fact that norms and inner products of vectors and forms are independent from a choice of coordinate. Changing from a set of coordinates to another is achieved via a diffeomorphism (also called general coordinate transformation). The diffeomorphism is a group and locally admits two representations [20]

- **Passive coordinate transformations:** both the tensors and the coordinates are transformed: $T(x) \rightarrow T'(x')$, and the volume element transforms as $d\mathcal{V}(x) \rightarrow d\mathcal{V}(x')$.
- **Active coordinate transformations:** the coordinates are transformed back, so that only the tensors are transformed: $T(x) \rightarrow T'(x)$, and the volume element is invariant $d\mathcal{V}(x) \rightarrow d\mathcal{V}(x)$.

For example, under an infinitesimal diffeo

$$x \rightarrow x' = x - \xi(x) , \quad (5.10)$$

the passive diffeomorphism transformation of tensors is

$$\begin{aligned} V'_\mu(x') &= V_\nu(x) \frac{\partial x^\nu}{\partial x'^\mu} = V_\mu(x) + (\partial_\mu \xi^\nu)(x) V_\nu(x) + \mathcal{O}(\xi^2) \\ V'^\mu(x') &= \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x) = V^\mu(x) - (\partial_\nu \xi^\mu)(x) V^\nu(x) + \mathcal{O}(\xi^2) , \end{aligned} \quad (5.11)$$

and by Taylor expanding the left-hand side at $\mathcal{O}(\xi^2)$ we obtain the active diffeomorphism transformations

$$\begin{aligned} \delta_\xi^d V_\mu &= V'_\mu - V_\mu = \xi^\nu \partial_\nu V_\mu + (\partial_\mu \xi^\nu) V_\nu + \mathcal{O}(\xi^2) \\ \delta_\xi^d V^\mu &= V'^\mu - V^\mu = \xi^\nu \partial_\nu V^\mu - (\partial_\nu \xi^\mu) V^\nu + \mathcal{O}(\xi^2) , \end{aligned} \quad (5.12)$$

where all quantities are evaluated at x . The transformations are easily generalised to higher rank tensors. For example for the particular case of the metric we find

$$\delta_\xi^d g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu . \quad (5.13)$$

Note that the active diffeomorphism δ_ξ^d of a tensor is none other than its Lie derivative along ξ . Diffeo-invariant quantities can be built out of fully contracted tensors. For example, if we take a vector $V = V^\mu \partial_\mu \in T_p \mathcal{M}$ and a vector in the dual space $U = U_\mu dx^\mu \in T_p^* \mathcal{M}$, then by definition $U \cdot V \in \mathbb{R}$ is a diffeo-scalar, i.e it is invariant under diffeomorphisms.

Covariant derivatives Note that partial derivatives of tensors are not tensors, that is to say they do not transform covariantly under diffeomorphism transformations. Similarly as for gauge groups, we thus introduce the covariant derivative ∇ such that ∇T transforms covariantly if T does. On the manifolds we consider, there exists a unique connection that is both torsion-less¹ and metric compatible

$$\nabla g = 0 , \quad (5.14)$$

which is called the Levi-Civita connection and is expressed in terms of the Christoffel symbols

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) . \quad (5.15)$$

The covariant derivative of vectors thus reads

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}^\rho V_\rho , \quad \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}^\nu V^\rho . \quad (5.16)$$

It is easily generalised to higher rank tensors.

Note that there is an abuse of notation above. V_μ is just the component of the vector $V = V_\mu dx^\mu$, hence it is a function. The covariant derivative acting on functions is simply the partial derivative: $\nabla V_\mu = \partial V_\mu$. It is when acting on a vector that the connection arises $\nabla V \neq \partial V$. In components it reads

$$(\nabla V)_{\mu\nu} = \partial_\mu V_\nu - \Gamma_{\mu\nu}^\rho V_\rho , \quad (5.17)$$

which we write as $(\nabla V)_{\mu\nu} \equiv \nabla_\mu V_\nu$ by abuse of notations.

Curvatures On a flat manifold, the covariant derivative is simply the partial derivative, and we thus have

$$[\nabla_\mu, \nabla_\nu] = [\partial_\mu, \partial_\nu] = 0 . \quad (5.18)$$

This no longer holds on a curved manifold, and we define the Riemann tensor as

$$[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma , \quad (5.19)$$

where

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma_{\nu\sigma}^\mu - \partial_\sigma \Gamma_{\nu\rho}^\mu + \Gamma_{\rho\lambda}^\mu \Gamma_{\nu\sigma}^\lambda - \Gamma_{\sigma\lambda}^\mu \Gamma_{\nu\rho}^\lambda . \quad (5.20)$$

Henceforth, the Riemann tensor is a measure of the local curvature of the manifold.

$R_{\mu\nu\rho\sigma}$ is anti-symmetric in exchanging $\mu \leftrightarrow \nu$, and $\rho \leftrightarrow \sigma$, and symmetric in exchanging $(\mu\nu) \leftrightarrow (\rho\sigma)$. It respects the Bianchi identities, which are purely geometrical identities

$$R_{\mu(\nu\rho\sigma)} = 0 , \quad R_{\mu\nu(\rho\sigma;\lambda)} = 0 , \quad (5.21)$$

¹For any vectors $X, Y \in T_p \mathcal{M}$, the connection is torsion-less if, and only if, $\nabla_X Y - \nabla_Y X = [X, Y]$ where $\nabla_X = X^\mu \nabla_\mu$ is the covariant derivative along X .

where $R_{\mu\nu\rho\sigma;\lambda} \equiv \nabla_\lambda R_{\mu\nu\rho\sigma}$, and the parenthesis denote the symmetrisation of the indices, e.g $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$.

From the Riemann tensor we can construct the Ricci tensor and scalar

$$R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\nu\sigma} = R_{\nu\mu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \quad (5.22)$$

Tangent frame We will see in the next Section that fermions can only be introduced on a curved manifold by referring to the tangent frame, which we introduce here.

Since manifolds are locally flat, it is always possible to find at a given point $p \in \mathcal{M}$ a set of coordinates $\{\xi^a\}_{1 \leq a \leq d}$ ($d = \dim \mathcal{M}$) such that the metric at p looks locally flat

$$g_p = \eta_{ab} d\xi^a \otimes d\xi^b, \quad (5.23)$$

η being the Minkowski metric. These inertial coordinates are also referred to as Riemann Normal Coordinates (RNC). The metric in non-inertial coordinates $\{x^\mu\}_{1 \leq \mu \leq d}$ is related to the one in inertial coordinates by a diffeomorphism

$$g_p = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} e^a{}_\mu e^b{}_\nu dx^\mu \otimes dx^\nu, \quad (5.24)$$

where

$$e^a{}_\mu = \frac{\partial \xi^a}{\partial x^\mu}, \quad (5.25)$$

are called the vielbein (zweibein in $d = 2$, vierbein in $d = 4$). It is conventional to use the greek indices to refer to the non-inertial coordinates and the latin indices to refer to the inertial ones. We can introduce the inverse vielbein $e_a{}^\mu$ such that

$$e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu, \quad e_a{}^\mu e^b{}_\mu = \delta_a^b. \quad (5.26)$$

The vielbein and their inverse define coordinate basis for the cotangent and tangent spaces

$$e^a = d\xi^a = e^a{}_\mu dx^\mu \in T_p^* \mathcal{M}, \quad e_a = \frac{\partial}{\partial \xi^a} = e_a{}^\mu \partial_\mu \in T_p \mathcal{M}, \quad (5.27)$$

in which the metric is flat

$$g_p = \eta_{ab} e^a \otimes e^b. \quad (5.28)$$

This choice of basis for tensors is called the tangent frame² and we have for example

$$V = V^\mu \partial_\mu = V^a e_a, \quad V^a = e^a{}_\mu V^\mu. \quad (5.29)$$

In practice, greek indices are lowered and raised with the metric g , and latin indices with η .

Let us now perform the following transformation on the vielbein

$$(e')^a{}_\mu = (L^{-1})^a{}_b e^b{}_\mu, \quad (5.30)$$

such that $L \in SO(1, d-1)$, hence satisfies

$$\eta_{ab} L^a{}_c L^b{}_d = \eta_{cd}. \quad (5.31)$$

²Note that any coordinate system $\{x^\mu\}$ defines a basis for the tangent space $T_p \mathcal{M}$: $\{\partial_\mu\}$. However this basis is not orthonormal $\partial_\mu \cdot \partial_\nu = g_{\mu\nu}$ (see the inner product defined in (5.7)). What we call a tangent frame is an orthonormal basis for the tangent space: $e_a \cdot e_b = \eta_{ab}$.

We recognise this transformation as being a local Lorentz transformation. It is easy to verify that the metric is flat as well

$$g_p = \eta_{ab} (e')^a \otimes (e')^b, \quad (5.32)$$

in this (non-coordinate) basis

$$(e')^a = (e')^a{}_\mu dx^\mu = (L^{-1})^a{}_b d\xi^b, \quad (e')_a = L^b{}_a \frac{\partial}{\partial \xi^b}, \quad (5.33)$$

This implies that the vielbein are defined up to local Lorentz transformations. This local Lorentz transformation is physical and theories expressed on a curved manifold should in general respect it.

Note that although the vielbein transform under local Lorentz transformations, they also transform under diffeomorphism as vectors

$$e_a{}^\mu(x) \rightarrow (e')_a{}^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} e_a{}^\nu. \quad (5.34)$$

Since $V^\mu = e^\mu{}_a V^a$, it implies that V^a is a diffeomorphism scalar. Likewise, it is possible to show that V^μ is a Lorentz scalar, but V^a transforms as a Lorentz vector.

The covariant derivative of tensors can be expressed in the tangent frame as

$$(\nabla v) = (\partial_\mu v^\nu + \Gamma_{\mu\rho}^\nu v^\rho) dx^\mu \otimes \partial_\nu \quad (5.35)$$

$$= (\partial_\mu v^\nu + \Gamma_{\mu\rho}^\nu v^\rho) e^a{}_\nu dx^\mu \otimes e_a \quad (5.36)$$

$$= (\partial_\mu v^a + \omega_\mu{}^a{}_b v^b) dx^\mu \otimes e_a, \quad (5.37)$$

where we introduced the spin-connection defined as

$$\omega_\mu{}^a{}_b = -e_b{}^\nu \partial_\mu e^a{}_\nu + e^a{}_\nu e^\rho{}_b \Gamma_{\mu\rho}^\nu. \quad (5.38)$$

which is antisymmetric $\omega_{\mu ab} = -\omega_{\mu ba}$. As a result, the vielbein are metric compatible

$$\begin{aligned} \nabla_\mu e^a{}_\nu &= \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma_{\mu\nu}^\rho e^a{}_\rho = 0 \\ \nabla_\mu e_a{}^\nu &= \partial_\mu e_a{}^\nu + \omega_{\mu a}{}^b e_b{}^\nu + \Gamma_{\mu\rho}^\nu e_a{}^\rho = 0. \end{aligned} \quad (5.39)$$

The spin-connection will play a crucial role when introducing fermions in curved spacetime. Its Lorentz transformation can be worked out from the vielbein transformation and reads

$$\omega_\mu{}^a{}_b \rightarrow (L^{-1})^a{}_c \omega_\mu{}^c{}_d L^d{}_b, \quad (5.40)$$

which will be useful later on. Finally, the diffeomorphism invariant volume element can also be expressed in the tangent frame

$$d\mathcal{V} = e^1 \wedge \dots \wedge e^d = e d^d x, \quad (5.41)$$

with $e \equiv \det(e^a{}_\mu) = \sqrt{|g|}$ and \wedge is the Wedge product.

General Relativity Let us quickly introduce the theory that describes gravity, namely General Relativity (GR). As emphasised earlier, the effects of a gravitational field are equivalent to the curvature of spacetime. It is then natural that the action depends on a measure of the curvature. Secondly, it should be independent from a choice of coordinates, it must then be built out of diffeomorphism invariant quantities. Finally, in the non-relativistic limit, it should reduce to the classical theory of gravity, which involves Newton's universal constant $G \simeq 6,7 \cdot 10^{-11} \text{ N.m}^2.\text{kg}^{-2}$. The action should

thus depend on G , and involve the relativistic scale c which is the speed of light in vacuum. On dimensional grounds, we thus obtain the Einstein-Hilbert action

$$S_g[g] = \frac{1}{2\kappa} \int d^4x \sqrt{|g|} (R - 2\Lambda) , \quad (5.42)$$

where $\kappa = 8\pi Gc^{-4}$, and Λ is the so-called cosmological constant.

If we include the action for matter S_{mat} , the equations of motion for the metric are obtained by minimising the action and read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} , \quad (5.43)$$

where $\sqrt{|g|}T_{\mu\nu} = \delta S_{\text{mat}}/\delta g_{\mu\nu}$ is the matter energy-momentum tensor.

Let us say a word about coordinate independence and diffeo-invariance, which may be source of confusion. Coordinate independence is not a special feature of GR, all laws of physics should be independent from the choice of coordinates in which we expressed them. Einstein introduced the theory of General Relativity, referring to it as being background independent, that is to say there exists “no prior geometry”. This amounts to saying that the metric is *dynamical*. S_g is invariant under diffeomorphisms (and the EoM (5.43) is covariant), since the metric is transformed as well. This is the special feature of GR.

For example, let us consider the action for a real scalar field coupled to a background *non-dynamical* metric (which we will detail in the next Section)

$$S[\phi] = \int d^4x \mathcal{L}(\phi(x), x) = -\frac{1}{2} \int d^4x \sqrt{|g|} \phi (g^{\mu\nu} \partial_\mu \partial_\nu + m^2) \phi . \quad (5.44)$$

This is obviously independent from the coordinates in which we expressed the Lagrangian

$$S = \int d^4x \mathcal{L}(\phi(x), x) = \int d^4x' \mathcal{L}(\phi(x'), x') . \quad (5.45)$$

However, diffeo-invariance is spoiled

$$\delta_\xi^d S[\phi] = \int d^4x \delta_\xi^d \phi \frac{\delta S}{\delta \phi} \neq 0 , \quad (5.46)$$

because the metric is a background field that does not transform, i.e the geometry is fixed.

5.2 Fields in curved spacetime

In this Section, we couple matter fields to gravity. Let us introduce some notations that will be helpful later on. The total covariant derivative is denoted by \mathcal{D} , it includes all the connections. Its commutator is denoted by $\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$. The diffeo-covariant derivative ∇ only includes the connection that contract Lorentz indices (Christoffel and spin-connection for the tangent frame). D only includes the gauge connections, and fermionic spin-connection ω_μ that we introduce below. Its commutator is denoted by $F_{\mu\nu} = [D_\mu, D_\nu]$.

5.2.1 Spin ≤ 1 bosons

Scalar fields Scalar fields are by definition, scalars with respect to passive diffeomorphism transformations³

$$\phi(x) \rightarrow \phi'(x') = \phi(x) . \quad (5.47)$$

³But they do transform under active diffeomorphism transformation: $\delta_\xi^d \phi = \xi^\mu \partial_\mu \phi$.

The coordinate-covariant derivative is thus trivial and we have

$$\mathcal{D}\phi = D\phi = (\partial + iV)\phi, \quad \mathcal{F}_{\mu\nu}\phi = F_{\mu\nu}\phi = (i(\partial_\mu V_\nu) - i(\partial_\nu V_\mu) - [V_\mu, V_\nu])\phi, \quad (5.48)$$

for a scalar field charged under a gauge group with gauge connection V . The action for a complex field hence takes the form

$$S_\phi = \int d^4x \sqrt{|g|} \phi^\dagger (g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu + m^2 + \xi R) \phi, \quad (5.49)$$

where we may introduce a coupling to the Ricci scalar, since it is allowed on dimensional grounds.

Gauge fields Spin-1 gauge fields are Lie algebra valued vector fields, hence transform as vectors under the diffeomorphism group. The covariant derivative reads

$$\mathcal{D}_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}^\rho V_\rho + [V_\mu, V_\nu]. \quad (5.50)$$

Their action in gravity is written as

$$S_V = \int d^4x \sqrt{|g|} \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (5.51)$$

where the field strength is⁴

$$F_{\mu\nu} = i \partial_\mu V_\nu - i \partial_\nu V_\mu - [V_\mu, V_\nu]. \quad (5.52)$$

5.2.2 Spin- $\frac{1}{2}$ fermions

Fermions are trickier to couple to gravity. The reason is that they are represented by spinor fields, and the Lie algebra of the diffeomorphism group $GL(d, \mathbb{R})$ ⁵ admits no (finite dimensional) spinor representation [20]. However, spinors are subject to the Lorentz group, which arises as the freedom to choose the tangent frame up to local Lorentz transformations. Whereas the actions (5.49) and (5.51) are directly invariant under Lorentz transformations, the fermionic action in gravity requires an additional connection to preserve Lorentz invariance.

Under a local Lorentz transformation $L(x)$, the spinors transform with a representation $\rho(L(x))$

$$\begin{aligned} \psi(x) &\rightarrow \rho^{-1}(L(x))\psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x)\rho(L(x)). \end{aligned} \quad (5.53)$$

For an infinitesimal Lorentz transformation $L^a_b(x) = \delta^a_b + \alpha^a_b(x)$ with $\alpha_{ab} = -\alpha_{ba}$, we have

$$\rho(L(x)) = 1 + \frac{1}{2} \alpha_{ab} \sigma^{ab}, \quad \rho^{-1}(L(x)) = 1 - \frac{1}{2} \alpha_{ab} \sigma^{ab}, \quad \sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]. \quad (5.54)$$

σ^{ab} are the generators of the spinor representation of the Lorentz group, they satisfy the Lie algebra

$$[\sigma^{ab}, \sigma^{cd}] = \eta^{ad} \sigma^{bc} - \eta^{ac} \sigma^{bd} + \eta^{bc} \sigma^{ad} - \eta^{bd} \sigma^{ac}. \quad (5.55)$$

The partial derivative does not transform covariantly under this transformation

$$\partial_\mu \psi \rightarrow \rho^{-1} \partial_\mu \psi + (\partial_\mu \rho^{-1}) \psi. \quad (5.56)$$

⁴Note that $F_{\mu\nu} = i\nabla_\mu V_\nu - i\nabla_\nu V_\mu - [V_\mu, V_\nu]$, however $F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \neq i\partial^\mu V^\nu - i\partial^\nu V^\mu - [V^\mu, V^\nu]$.

⁵For an infinitesimal diffeomorphism $x' = x - \xi(x)$ we have $V'^\mu(x') = V^\mu(x) - \frac{\partial \xi^\mu}{\partial x^\nu} V^\nu(x) + \mathcal{O}(\xi^2)$, so we see that the generators of the Lie algebra are d dimensional matrices $\frac{\partial \xi^\mu}{\partial x^\nu} \in GL(d, \mathbb{R})$.

However, if we introduce a connection ω and require that the covariant derivative transforms covariantly

$$\mathcal{D}_\mu \psi \equiv (\partial_\mu + \omega_\mu) \psi \rightarrow \rho^{-1} (\partial_\mu + \omega_\mu) \psi , \quad (5.57)$$

we must have

$$\omega_\mu \rightarrow \rho^{-1} \partial_\mu \rho + \rho^{-1} \omega_\mu \rho , \quad (5.58)$$

which reminds us the transformation of the spin-connection $\omega_\mu^a{}_b$ (5.40). This is sufficient to identify

$$\omega_\mu = \frac{1}{2} \sigma^{ab} \omega_{\mu ab} , \quad (5.59)$$

which we call (fermionic) spin-connection.

In order to build a diffeo-invariant action, we need to introduce a vector G^μ to contract with $\mathcal{D}_\mu \psi$ to form a diffeomorphism scalar

$$G \cdot \mathcal{D}\psi = G^\mu \mathcal{D}_\mu \psi . \quad (5.60)$$

Obviously, we should recover the flat spacetime limit, so G^μ should depend on the Dirac flat spacetime matrices γ^μ which are constant (independent of the coordinate x). Let us naively take $G^\mu = \gamma^\mu$, such that G^μ is independent of x . Then in another set of coordinates x'^μ , we would have $G'^\mu(x') = \frac{\partial x^\mu}{\partial x'^\nu} \gamma^\nu$ which depends on x' . But if we had started from the coordinates x'^μ and defined $G'^\mu = \gamma^\mu$ to be independent of x' , this time we would have $G^\mu(x) = \frac{\partial x'^\mu}{\partial x^\nu} \gamma^\nu$ depending on x , which is different from our first choice. Since the choice of coordinate is totally arbitrary, this procedure is inconsistent.

Let us alternatively take $G^\mu(x) = e_a{}^\mu(x) \gamma^a$, where again γ^a are the x -independent flat spacetime Dirac matrices. After changing coordinates we simply obtain $G'^\mu(x') = (e')_a{}^\mu(x') \gamma^a$, which is the definition we would have obtained had we started directly from the coordinates x'^μ . This choice is consistent and recovers the flat spacetime limit. G^μ can be seen as the curved spacetime Dirac matrices, which satisfy the Clifford algebra

$$\{G^\mu(x), G^\nu(x)\} = 2g^{\mu\nu}(x) . \quad (5.61)$$

In the following, for simplicity γ^a refers to the flat spacetime (x -independent) Dirac matrices, and

$$\gamma^\mu(x) \equiv G^\mu(x) = e_a{}^\mu(x) \gamma^a , \quad (5.62)$$

refers to the curved spacetime Dirac matrices. Importantly, they are metric compatible

$$\mathcal{D}_\mu \gamma^\nu = \partial_\mu \gamma^\nu + [\omega_\mu, \gamma^\nu] + \Gamma_{\mu\rho}^\nu \gamma^\rho = 0 . \quad (5.63)$$

Before getting to the fermionic action, let us say a word about $\bar{\psi}$, the conjugate of ψ . The inner product of spinors is defined such that $\bar{\psi}\psi$ transforms as a scalar. In flat spacetime, this is achieved by taking $\bar{\psi} = \psi^\dagger \gamma^0$. In curved spacetime, $\bar{\psi}\psi$ transforms as a scalar provided $\bar{\psi} = \psi^\dagger \epsilon$ with ϵ satisfying [81]

$$\epsilon - \epsilon^\dagger = 0 , \quad \epsilon \gamma^\mu - (\gamma^\mu)^\dagger \epsilon = 0 , \quad \nabla_\mu \epsilon = 0 . \quad (5.64)$$

It can be shown that keeping $\epsilon = \gamma^0$ the flat spacetime Dirac matrix is a suitable choice [82].

Finally, we can write down the gauge, diffeomorphism and Lorentz invariant action for fermions coupled to gravity and a gauge group

$$S_\psi = \int d^4x \sqrt{|g|} \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu - m) \psi , \quad (5.65)$$

where

$$\begin{aligned}\gamma^\mu &= e^\mu{}_a \gamma^a, \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab} \\ \bar{\psi} &= \psi^\dagger \gamma^0 \\ \mathcal{D}_\mu \psi &= D_\mu \psi = (\partial_\mu + \omega_\mu + iV_\mu)\psi.\end{aligned}\tag{5.66}$$

Due to the presence of the fermionic spin-connection, a curvature term arises in the fermion field strength

$$\mathcal{F}_{\mu\nu}\psi = F_{\mu\nu}\psi = \frac{1}{4}\gamma^\rho\gamma^\sigma R_{\mu\nu\rho\sigma}\psi + (i(\partial_\mu V_\nu) - i(\partial_\nu V_\mu) - [V_\mu, V_\nu])\psi,\tag{5.67}$$

and we note the useful identity

$$\sigma^{\mu\nu} F_{\mu\nu} = \frac{1}{4}R.\tag{5.68}$$

Fortunately, chiral fermions pose no further difficulty to introduce since the flat spacetime γ_5 matrix is metric compatible

$$\mathcal{D}_\mu \gamma_5 = [\omega_\mu, \gamma_5] = 0.\tag{5.69}$$

5.3 Quantum Fields in curved spacetime

As emphasised in Part I, the traditional quantisation of fields can be bypassed by writing down directly the path integral. For any field Φ with action $S[\Phi, B]$, where B are some background fields (i.e not quantised) including the metric, the quantum theory of Φ is given by

$$Z[B] = \int \mathcal{D}\Phi e^{iS[\Phi, B] + iJ \cdot \Phi}.\tag{5.70}$$

From which the correlation functions

$$\langle T \hat{\Phi}(x_1) \dots \hat{\Phi}(x_n) \rangle,\tag{5.71}$$

can be obtained. Recall that in the path integral $\phi(x)$ is a (off-shell) classical field, whereas it is a quantum operator in the correlation functions. We postpone for Part III discussions about the definition of the path integral measure, which is particularly interesting in curved spacetime.

Although the quantisation is straightforward, and the curvature of spacetime seems to not play a role, several very intriguing effects arise when quantisation fields on a curved background.

As shown in Part I, $\hat{\phi}$ obeys the quantum EoM

$$\frac{\delta S}{\delta \hat{\phi}} = 0.\tag{5.72}$$

Let us denote the space of solutions to this equation by \mathcal{S} .

Notion of particle In flat spacetime, the modes

$$u_{\vec{p}} \propto e^{-ip \cdot x},\tag{5.73}$$

define an orthonormal basis of \mathcal{S} ⁶, with positive frequency with respect to ∂_t

$$\partial_t u_{\vec{p}} = -ip^0 u_{\vec{p}}, \quad p^0 > 0.\tag{5.74}$$

⁶It is possible to define an inner product on \mathcal{S} [83].

The solutions to the EoM can then be expressed as

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} \left(a_{\vec{p}} u_{\vec{p}} + b_{\vec{p}}^\dagger u_{\vec{p}}^\dagger \right), \quad (5.75)$$

where $a_{\vec{p}}$ and $b_{\vec{p}}$ are annihilation operators and p^0 is fixed by the EoM. For simplicity, we take in the following the case of a scalar field for which $a_{\vec{p}} = b_{\vec{p}}$. The annihilation operator defines a vacuum

$$a_{\vec{p}} |0\rangle = 0, \quad (5.76)$$

that is Poincaré invariant, and is interpreted as the absence of particles. The positive frequency modes are interpreted as particles with definite momentum (hence delocalised in space).

Importantly, in flat spacetime there is not a unique vacuum, but there is a *preferred* vacuum, and *preferred* positive frequency modes, on which inertial observers can agree.

In curved spacetime, the main difference is that in general there exists no timelike killing vector such as ∂_t , therefore there is no preferred definition of positive frequency modes, nor of vacuum.

For example, we could choose a set of orthonormal modes $u_{\vec{p}}$ such that the solution to (5.72) involves the annihilation operator $a_{\vec{p}}$, and this defines a vacuum $|0_a\rangle$.

However, another equally valid choice would be another set $v_{\vec{p}}$, with annihilation operator $b_{\vec{p}}$, and vacuum $|0_b\rangle$. It is possible to relate these two sets of solution by the so-called Bogolyubov transformations [84].

The problem is that the vacuum in the u modes contains particles in the v modes since

$$b_{\vec{p}} |0_a\rangle \neq 0, \quad (5.77)$$

and conversely. This implies that different observers cannot agree on the definition of particles, and when one observer may see particles, another one may not. Note that this does not in any case invalidate QFT in curved spacetime, but only makes the concept of particle observer dependent.

Nonetheless, if there exists a timelike killing vector ξ , then there exists a preferred definition of positive frequency modes, and hence of vacuum, given by the eigenmodes of the Lie derivative along ξ

$$\mathcal{L}_\xi u_{\vec{p}} = -ip^0 u_{\vec{p}}, \quad p^0 > 0. \quad (5.78)$$

This is the case for stationary spacetimes.

Particle creation in non-stationary spacetime An interesting consequence of the arbitrariness of the definition of particles is the creation of particles in non-stationary spacetimes. Let us consider a globally hyperbolic manifold that is stationary before some time t_- , then non-stationary between t_- and t_+ , then stationary again after t_+ . The manifold can thus be broken into three pieces $\mathcal{M} = \mathcal{M}_- \cup \mathcal{M}_0 \cup \mathcal{M}_+$, where \mathcal{M}_0 is the non-stationary manifold between times t_- and t_+ .

Since \mathcal{M}_- and \mathcal{M}_+ are stationary, both admit a preferred choice of positive frequency modes and vacuum that are different, and that we respectively denote by $\{u_i^\pm\}_i$ and $|0_\pm\rangle$, and the annihilation operators are a_i^\pm . Note that we take discrete indices i for simplicity, but in practice they are continuous. The modes are related by a Bogolyubov transformation

$$u_i^+ = \sum_j (A_{ij} u_j^- + B_{ij} u_j^{\dagger,-}), \quad (5.79)$$

where A and B are some transformation matrices.

If at earlier time there are no particles, the initial state is unambiguous and is $|0^-\rangle$. Surprisingly, at later times $t > t_+$, some particles are measured. The particle number operator $N_i^+ = a_i^{\dagger,+} a_i^+$ in the mode i has a non-vanishing expectation value in the initial vacuum

$$\langle 0^- | N_i^+ | 0^- \rangle = \text{tr } BB^\dagger. \quad (5.80)$$

This implies that particles were created in the non-stationary transient spacetime. This interpretation in terms of particle creation is possible because the spacetime is stationary at early and late times. In particular, this process is at the origin of the Hawking radiation and the Unruh effect.

We refer the reader interested in more in depth discussions about QFT in curved spacetime to [81, 83, 85].

Chapter 6

Covariant Derivative Expansion in curved spacetime

Now that we have introduced quantum fields on a curved background manifold, we are ready to compute EFTs in gravity. In this Chapter, we present our new approach to the CDE in gravity [3], but let us first say a word about previous existing functional methods in curved spacetime.

Functional methods are well fit to comprise the effects of the curvature of spacetime. It was undertaken in both the heat kernel [24, 25, 86–92] and the CDE [77, 78]. As opposed to a Feynman diagram approach (see for example [93, 94]), gravity needs not be linearised to obtain the gravitational loop corrections. Even though they are more attainable, the higher order corrections remain a computational challenge to obtain.

The previous CDE procedures in curved spacetime relied on the use of the so-called Gaillard-Cheyette sandwich [27] to form covariant operators. Although it provides a manifestly covariant expansion, it also makes the computation much more intricate. In the CDE presented in [3], this step is avoided. Together with the use of convenient choice of gauge and coordinate system, it makes the computation of higher order corrections in curved spacetime more systematic and thus easier to compute. For the first time, the non-renormalisable corrections are obtained within the framework of the CDE, and on a generic spacetime background.

Although a specific choice of coordinates can simplify the computation, our expansion is coordinate independent. Particularly, the question of the Fourier transform in curved spacetime is treated so as to obtain a diffeomorphism invariant expansion, whereas former approaches were mostly relying on a specific choice of coordinate, the Riemann Normal Coordinates (RNC), to define it [77, 81, 95]. As a result, the method can also be used to obtain non-covariant results such as consistent gravitational anomalies.

Another novelty of [3] is the derivation of a fermionic CDE in curved spacetime. Previous methods (heat kernel, CDE and more recently using the worldline formalism [79]) always relied on a generic bosonic form of the functional determinant. It can describe the effective action after integrating out real and complex scalar fields, massless and massive gauge bosons, the spin-2 metric field, and even vector-like fermions. Despite its generic form, it cannot describe a chiral fermion, as was pointed out in [73]. For the first time, a chiral fermion in curved spacetime is integrated out within the functional approach in a universal form, and leads to new renormalisable and non-renormalisable operators that were not computed before. It also provides a new alternative to the use of Feynman diagrams. Besides the computational simplicity that is proper to the path integral approach with respect to the use of Feynman diagrams, it has the advantage of not needing to perturb the metric around a flat background, which significantly simplifies the calculations.

Our result is the one-loop action in curved spacetime up to six dimensional operators in the

bosonic CDE, and up to five in the fermionic CDE, given in a close form universal formula. Our systematic procedure can be used in practice to obtain much higher dimensional operators. This Gravitational version of the UOLEA should be significantly useful to study low energy consequences of the UV completion of gravity, or generical models including heavy degrees of freedom in gravity (see for example [96]).

6.1 Fourier transform in curved spacetime

As seen in Sections 4.2.1 and 4.2.2 the CDE in flat spacetime is performed using a momentum representation, that is to say a Fourier transform of the functional determinant. The Fourier transform in curved spacetime needs to be addressed thoroughly since it presents some difficulties. The first one is that contrary to flat spacetime, the covariant derivative ∇_μ and the momentum q_μ do not commute. Secondly, the Fourier transform is not unique, in fact there are as many choices of momentum representation as there are choices of coordinate representations. Indeed, consider a manifold \mathcal{M} , provided with an atlas $\{U_i, \varphi_i\}$. The points in a subspace U_i are mapped into $\mathbb{R}^{\dim \mathcal{M}}$, so for the coordinates given by φ_i there exists a momentum conjugate which is simply the momentum representation in flat space. Now consider another subspace U_j , $j \neq i$, such that $U_i \cap U_j \neq \emptyset$, then φ_j provides another set of coordinates for the points in $U_i \cap U_j$ and hence another set of momentum conjugates.

Previous literature [77, 81, 95] involving momentum representation in curved spacetime relied on a specific choice of coordinate, the RNC where spacetime is locally flat around a point, to define the Fourier transform. However, we would like to define the Fourier transform without relying on a specific choice of coordinate (which is of relevance when dealing with quantum anomalies for example). It does not seem trivial to us that the usual Fourier transform in curved spacetime leads to a coordinate independent result, since the choice of momentum representation depends on the choice of coordinate. We explain our procedure for defining the Fourier transform in curved spacetime, and show that it is indeed independent of the choice of coordinate.

6.1.1 Momentum representation on a manifold

We consider a manifold \mathcal{M} of dimension d , provided with an atlas $\{(U_i, \phi_i)\}$. Let p be a point in \mathcal{M} . There exists a chart (U, ϕ) in the atlas such that $p \in U$. $\phi(p) \in \mathbb{R}^d$ is the coordinate representation of p , we note $\phi^\mu(p) = x^\mu(p)$. We can choose a set of d functions from $\mathbb{R}^d \rightarrow \mathbb{R}$: $q_\mu(x)$, $\mu \in \{1, \dots, d\}$, such that

$$\frac{\partial q_\mu}{\partial x^\nu} = (\partial_\nu q_\mu) = 0. \quad (6.1)$$

This is simply the momentum conjugate to x in the flat space \mathbb{R}^d . We can thus provide the points in U with a momentum representation with

$$e^{-iq \cdot x} \frac{\partial}{\partial x^\mu} e^{iq \cdot x} = \partial_\mu + iq_\mu. \quad (6.2)$$

One would be tempted to define the 1-form $Q = q_\mu dx^\mu$ and the vector $X = x^\mu \frac{\partial}{\partial x^\mu}$ so that $q \cdot x$ is coordinate invariant. But X does not define a vector: suppose we have a second chart such that $p \in V$ with a coordinate $\varphi^\mu(p) = y^\mu(p)$, then x and y are related by a diffeomorphism $y^\mu = f^\mu(x)$ and we do not have in general $y^\mu = x^\nu \frac{\partial y^\mu}{\partial x^\nu}$.

The second chart (V, φ) provides another momentum representation r^μ such that $\frac{\partial r^\mu}{\partial y^\nu} = 0$.

Consider a vector $T = T^\mu \frac{\partial}{\partial x^\mu} = \tilde{T}^\mu \frac{\partial}{\partial y^\mu}$. We can write

$$e^{-ir \cdot y} T e^{ir \cdot y} = \tilde{T}^\mu e^{-ir_\nu y^\nu} \frac{\partial}{\partial y^\mu} e^{ir_\nu y^\nu} = \tilde{T}^\mu \left(\frac{\partial}{\partial y^\mu} + ir_\mu \right). \quad (6.3)$$

On the other hand

$$\begin{aligned} e^{-ir \cdot y} T e^{ir \cdot y} &= T^\mu e^{-ir_\nu f^\nu(x)} \frac{\partial}{\partial x^\mu} e^{ir_\nu f^\nu(x)} \\ &= T^\mu \left(\frac{\partial}{\partial x^\mu} + i \frac{\partial r_\nu}{\partial x^\mu} f^\nu(x) + ir_\nu \frac{\partial f^\nu(x)}{\partial x^\mu} \right) \\ &= T^\mu \left(\frac{\partial}{\partial x^\mu} + ir_\nu \frac{\partial f^\nu(x)}{\partial x^\mu} \right), \end{aligned} \quad (6.4)$$

where we used

$$\frac{\partial r_\nu}{\partial x^\mu} = \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial r_\nu}{\partial y^\rho} = 0. \quad (6.5)$$

Since we defined the vector $Q = r_\mu dy^\mu = q_\mu dx^\mu$, q_μ transforms covariantly: $q_\mu = r_\nu \frac{\partial y^\nu}{\partial x^\mu} = r_\nu \frac{\partial f^\nu(x)}{\partial x^\mu}$. Hence

$$e^{-ir \cdot y} T e^{ir \cdot y} = T^\mu \left(\frac{\partial}{\partial x^\mu} + iq_\mu \right) = e^{-iq \cdot x} T e^{iq \cdot x}. \quad (6.6)$$

In summary, if we have two coordinates $\{x^\mu\}$ and $\{y^\mu\}$ for a given point $p \in \mathcal{M}$, we can define the vector $Q = q_\mu dx^\mu = r_\mu dy^\mu$ such that $\frac{\partial q_\mu}{\partial x^\nu} = \frac{\partial r_\mu}{\partial y^\nu} = 0$, and we have for a vector $T = T^\mu \frac{\partial}{\partial x^\mu} = \tilde{T}^\mu \frac{\partial}{\partial y^\mu}$

$$e^{-iq \cdot x} T e^{iq \cdot x} = T^\mu \left(\frac{\partial}{\partial x^\mu} + iq_\mu \right) = e^{-ir \cdot y} T e^{ir \cdot y} = \tilde{T}^\mu \left(\frac{\partial}{\partial y^\mu} + ir_\mu \right). \quad (6.7)$$

Note however that $q \cdot x \neq r \cdot y$

$$q \cdot x = q_\mu x^\mu = r_\nu \frac{\partial y^\nu}{\partial x^\mu} x^\mu \neq r_\nu y^\nu. \quad (6.8)$$

Besides since $y = f(x)$, we have

$$d^d y = dy^1 \wedge \cdots \wedge dy^d = d(f^1(x)) \wedge \cdots \wedge d(f^d(x)) = \det \left(\frac{\partial f(x)}{\partial x} \right) d^d x, \quad (6.9)$$

and

$$d^d q = d \left(\frac{\partial f^{\mu_1}(x)}{\partial x^1} r_{\mu_1} \right) \wedge \cdots \wedge d \left(\frac{\partial f^{\mu_d}(x)}{\partial x^d} r_{\mu_d} \right) = \det \left(\frac{\partial f(x)}{\partial x} \right) d^d r. \quad (6.10)$$

We thus have the invariance of the measure

$$d^d x d^d q = d^d y d^d r. \quad (6.11)$$

Finally, consider a vector T and the two charts (U, ϕ) and (V, φ) with respective coordinate and momentum (x, q) and (y, r) . If they have an intersection $V \cap U \neq \emptyset$, we have,

$$\int_{U \cap V} d^d x \frac{d^d q}{(2\pi)^d} e^{-iq \cdot x} T e^{iq \cdot x} = \int_{U \cap V} d^d y \frac{d^d r}{(2\pi)^d} e^{-ir \cdot y} T e^{ir \cdot y}. \quad (6.12)$$

This result can be generalised without difficulty to any tensor $H = H^{\mu_1 \cdots \mu_n} \partial_{\mu_1} \cdots \partial_{\mu_n}$ by inserting $1 = e^{-iq \cdot x} e^{iq \cdot x}$ between each derivative and using the transformation of a rank n tensor under a diffeomorphism.

6.1.2 Functional trace on a manifold

We now seek to write the functional trace of the logarithm of an operator \mathcal{O} that is covariant. \mathcal{O} is quadratic in covariant derivatives, it can be written under the form $\mathcal{O} = A^{\mu\nu} \partial_\mu \partial_\nu + B^\mu \partial_\mu + C$ where A , B and C depend on the connections in the covariant derivative. The logarithm can be expanded in a series

$$\log \mathcal{O} = \sum_{n \geq 0} T_n^{\mu_1 \dots \mu_n} \partial_{\mu_1} \dots \partial_{\mu_n}. \quad (6.13)$$

Since \mathcal{O} is covariant, T_n must be a rank n tensor. According to the previous section

$$e^{-iq \cdot x} \log \mathcal{O}(x, i \partial_x) e^{iq \cdot x} = \sum_{n \geq 0} T_n^{\mu_1 \dots \mu_n} (\partial_{\mu_1} + iq_{\mu_1}) \dots (\partial_{\mu_n} + iq_{\mu_n}) = \log \mathcal{O}(x, i \partial_x - q), \quad (6.14)$$

is coordinate independent. By choosing a chart at each point of the manifold, we can write the functional trace

$$\text{Tr} \mathcal{O} = \int_{p \in \mathcal{M}} d^d x_i(p) \frac{d^d q_i}{(2\pi)^d} \text{tr} \mathcal{O}(x_i(p), i \partial_{x_i} - q_i), \quad (6.15)$$

where i refers to a chart (U_i, ϕ_i) such that $p \in U_i$, $x_i(p) = \phi_i(p)$ and q_i is the associated momentum. As shown above, the integration is independent of the choice of chart (i.e coordinate) for each point p , which allows us to define the functional trace on a generic manifold \mathcal{M}

$$\text{Tr} \mathcal{O} = \int_{p \in \mathcal{M}} d^d x(p) \frac{d^d q}{(2\pi)^d} \text{tr} \mathcal{O}(x(p), i \partial_x - q). \quad (6.16)$$

Throughout the expansion we will extensively use

$$\frac{\partial q_\mu}{\partial x^\nu} = 0, \quad (\partial_\mu q^2) = (\partial_\mu g^{\alpha\beta}) q_\alpha q_\beta. \quad (6.17)$$

Note that we have chosen to define the vector Q such that $\frac{\partial q_\mu}{\partial x^\nu} = 0$. But we could have chosen $\frac{\partial q^\mu}{\partial x^\nu} = 0$, which is not equivalent. Suppose we make the second choice: $\frac{\partial q^\mu}{\partial x^\nu} = 0$. We thus have,

$$\begin{aligned} e^{-iq \cdot x} T e^{iq \cdot x} &= T^\mu \left(\frac{\partial}{\partial x^\mu} + iq^\nu \frac{\partial x_\nu}{\partial x^\mu} \right) \\ &= T^\mu \left(\frac{\partial}{\partial x^\mu} + iq^\nu \frac{\partial g_{\nu\rho} x^\rho}{\partial x^\mu} \right) \\ &= T^\mu \left(\frac{\partial}{\partial x^\mu} + iq^\nu \frac{\partial g_{\nu\rho}}{\partial x^\mu} x^\rho + iq_\mu \right), \end{aligned} \quad (6.18)$$

which does not yield the desired outcome.

6.2 Bosonic CDE in curved spacetime

We seek to compute a functional trace of the form

$$S_{\text{eff}}^{\text{boson}} = ic_s \text{Tr} \log \left(\sqrt{|g|} (g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu + m^2 + U) \right), \quad (6.19)$$

where \mathcal{D}_μ is again the general covariant derivative (notations introduced in Sec. 5.2). U is some local operator without open derivatives, by open derivative we mean derivatives that act on everything to their right. The trace above is both over internal spaces and spacetime. c_s depends on the

nature of the field that is integrated out (see Sec. 4.2.1). As explained in Sec. 6.1, the functional trace in curved spacetime is written in a diffeomorphism invariant manner as

$$S_{\text{eff}}^{\text{boson}} = ic_s \int d^d x \frac{d^d q}{(2\pi)^d} e^{-iq \cdot x} \text{tr} \log \sqrt{|g|} (\mathcal{D}^2 + m^2 + U) e^{iq \cdot x}. \quad (6.20)$$

Similarly as in the flat spacetime case, we introduce the propagator $\Delta = 1/(q^2 - m^2)$

$$S_{\text{eff}}^{\text{boson}} = ic_s \int d^d x \frac{d^d q}{(2\pi)^d} \text{tr} \log \sqrt{|g|} (-\Delta^{-1}(1 - \Delta(\mathcal{D}^2 + iq \cdot \mathcal{D} + \mathcal{D} \cdot iq + U))), \quad (6.21)$$

In curved spacetime one significant novelty and difficulty comes from the fact that Δ and \mathcal{D} do not commute anymore since

$$[\mathcal{D}_\mu, \Delta] = -(\partial_\mu q^2) \Delta^2. \quad (6.22)$$

Therefore one cannot expand the *log* directly as in flat spacetime. We thus rely on the following trick to perform the expansion¹: we rewrite the *log* as the primitive of the inverse function (see e.g [69])

$$\log \sqrt{|g|} (\mathcal{D}^2 + m^2 + U(m)) = \int^{m^2} dm'^2 \frac{1}{(\mathcal{D}^2 + m'^2 + U(m))}. \quad (6.23)$$

Note that the spacetime measure $\sqrt{|g|}$ disappears in the expansion, it is discarded as an infinite irrelevant constant. As we will see in Part III, it is related to the choice of path integral measure and plays a role when dealing with gravitational anomalies. It is then expanded using

$$\frac{1}{A^{-1}(1 - AB)} = \sum_{n \geq 0} A(AB)^n, \quad (6.24)$$

which does not require A and B to commute. We thus obtain

$$\begin{aligned} S_{\text{eff}}^{\text{boson}} &= ic_s \int d^d x \frac{d^d q}{(2\pi)^d} \int^{m^2} dm'^2 \text{tr} \frac{1}{-\Delta^{-1}(1 - \Delta(\mathcal{D}^2 + iq \cdot \mathcal{D} + \mathcal{D} \cdot iq + U))} \\ &= -ic_s \int d^d x \frac{d^d q}{(2\pi)^d} \int^{m^2} dm'^2 \text{tr} \sum_{n \geq 0} [\Delta(\mathcal{D}^2 + 2iq \cdot \mathcal{D} - g^{\mu\nu} \Gamma_{\mu\nu}^\rho q_\rho + U)]^n \Delta, \end{aligned} \quad (6.25)$$

where now $\Delta = 1/(q^2 - m'^2)$. If the masses are non-degenerate, we can multiply the mass matrix by a parameter and integrate over this parameter instead (see e.g [70]).

We did not use the Gaillard and Cheyette trick mentioned in Sec. 4.2.1 since it makes the computation more tedious already in flat spacetime. Besides, in curved spacetime, it is not so simple to apply as shown in [78].

Note that when all the Lorentz indices to the right of a covariant derivative \mathcal{D} are contracted among themselves, the Christoffel connection cancels (i.e $(\mathcal{D}_\mu v^\nu u_\nu) = (\partial_\mu v^\nu u_\nu)$). In Eq. (6.25) the only covariant derivative that has uncontracted indices to its right is \mathcal{D}_μ in $\mathcal{D}^2 = g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu$. Acting on a field ϕ we have,

$$\mathcal{D}^2 \phi = g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu \phi = g^{\mu\nu} D_\mu D_\nu \phi - g^{\mu\nu} \Gamma_{\mu\nu}^\rho D_\rho \phi. \quad (6.26)$$

¹It is also possible to expand directly the *log* of non-commuting operators using the Baker-Campbell-Hausdorff formula.

Once \mathcal{D}^2 is written as such, all the \mathcal{D} have only contracted indices to their right, hence they can be replaced by D ,

$$S_{\text{eff}}^{\text{boson}} = -ic_s \int d^d x \frac{d^d q}{(2\pi)^d} \int dm'^2 \text{tr} \sum_{n \geq 0} \left[\Delta (g^{\mu\nu} D_\mu D_\nu - g^{\mu\nu} \Gamma_{\mu\nu}^\rho D_\rho + 2iq \cdot D - ig^{\mu\nu} \Gamma_{\mu\nu}^\rho q_\rho + U) \right]^n \Delta, \quad (6.27)$$

Both (6.25) and (6.27) can be used for the expansion. In the former, \mathcal{D} contracts the Lorentz indices yielding Christoffel connections, but commutes with the metric. In the latter, D does not contract the Lorentz indices but does not commute with the metric (which in the end yields Christoffel connections in virtue of $(\partial_\mu g^{\alpha\beta}) = -\Gamma_{\mu\lambda}^\alpha g^{\lambda\beta} - \Gamma_{\mu\lambda}^\beta g^{\alpha\lambda}$).

Note that both in (6.25) and (6.27), if $U \supset [\gamma^\mu, \gamma^\nu] \mathcal{F}_{\mu\nu}$ from the bosonisation of a vector-like fermion, it is possible to rewrite it as $[\gamma^\mu, \gamma^\nu] F_{\mu\nu}$ since all the indices to the right of $\mathcal{F}_{\mu\nu}$ are contracted.

The master integrals produced by the bosonic expansion are of the form,

$$\int \frac{d^d q}{(2\pi)^d} q_{\mu_1} \cdots q_{\mu_{2l}} \int^{m^2} dm'^2 \frac{1}{(q^2 - m'^2)^n} = \sqrt{|g|} \mathcal{J}[q^{2l}]^n g_{\mu_1 \dots \mu_{2l}}, \quad (6.28)$$

where $g_{\mu_1 \dots \mu_{2l}}$ is the fully symmetrised metric. They are related to the usual master integrals in flat spacetime (see App. B).

6.3 Fermionic CDE in curved spacetime

As explained in Sec. 4.2.2, chiral fermions cannot be dealt with using the bosonic CDE and need the use of the so-called fermionic CDE. We thus seek to perform the CDE on a determinant of the form

$$S_{\text{eff}}^{\text{fermion}} = -i \log \text{Tr} \left(\sqrt{|g|} (i\mathcal{D} - m - Q) \right). \quad (6.29)$$

If the fermion has a coupling to an axial (gauge) field, it is convenient to put the axial field in $Q \supset \not{A} \gamma_5$ and keep $(\mathcal{D}_\mu \psi) = (\partial_\mu + iV_\mu + \omega_\mu) \psi$.

The functional trace is expressed as in the bosonic CDE and leads to

$$S_{\text{eff}}^{\text{fermion}} = -i \int d^d x \frac{d^d q}{(2\pi)^d} \text{tr} \log \sqrt{|g|} (i\mathcal{D} - \not{q} - m - Q). \quad (6.30)$$

We make the inverse function appear by integrating over the mass instead of integrating over the mass squared as previously²

$$\begin{aligned} S_{\text{eff}}^{\text{fermion}} &= i \int d^d x \frac{d^d q}{(2\pi)^d} \int^m dm' \text{tr} \frac{1}{i\mathcal{D} - \not{q} - m' - Q(m)} \\ &= i \int d^d x \frac{d^d q}{(2\pi)^d} \int^m dm' \text{tr} \sum_{n \geq 0} [\Delta (-i\mathcal{D} + Q)]^n \Delta, \end{aligned} \quad (6.31)$$

²Since all Lorentz indices are contracted, we could extract the Christoffel connection as previously and perform the expansion with D instead of \mathcal{D} after writing either $\mathcal{D} = \gamma^\mu D_\mu$ or $\mathcal{D} = D_\mu \gamma^\mu + \Gamma_{\mu\nu}^\mu \gamma^\nu$. However it does not simplify the expansion since $[D_\mu, \gamma^\nu] \neq 0$.

where now $\Delta = -1/(\not{q} + m')$, which can be split as

$$\Delta = \frac{m'}{q^2 - m'^2} + \frac{-\not{q}}{q^2 - m'^2}. \quad (6.32)$$

Again if the mass matrix is non-degenerate one just has to multiply the mass matrix by a parameter and integrate over this parameter instead.

This expansion will produce master integrals of the form

$$\int \frac{d^d q}{(2\pi)^d} q_{\mu_1} \cdots q_{\mu_{2l}} \int^m dm' \frac{m'^k}{(q^2 - m'^2)^n} = \sqrt{|g|} \mathcal{K}[q^{2l}]_n^k g_{\mu_1 \dots \mu_{2l}}, \quad (6.33)$$

with $n \geq k$. $g_{\mu_1 \dots \mu_{2l}}$ is the fully symmetrised metric. They are related to the usual master integrals in flat spacetime \mathcal{I} (see App. B).

6.4 A systematic procedure

In flat spacetime, a factorisation of the momentum integrals from the operator part occurs, which is key in deriving a universal formula. In curved spacetime, the momentum dependence does not commute anymore with the covariant derivatives. Nevertheless, we can recover the factorisation of the momentum integration after commuting carefully the momentum part through the covariant derivatives. Using $(\partial_\mu q_\nu) = 0$, we derive a set of useful commutation relations presented in Tab. 6.1.

General covariant derivative: \mathcal{D}	Gauge (and spin-connection) covariant derivative: D
$[\mathcal{D}_\mu, q_\nu] = -\Gamma_{\mu\nu}^\rho q_\rho$ $[\mathcal{D}_\mu, q^\nu] = -g^{\nu\sigma} \Gamma_{\mu\sigma}^\rho q_\rho$ $[\mathcal{D}_\mu, \gamma^\nu] = 0$ $[\mathcal{D}_\mu, g^{\nu\rho}] = 0$ $[\mathcal{D}_\mu, \Gamma_{\rho\sigma}^\nu] = (\nabla_\mu \Gamma_{\rho\sigma}^\nu)$	$[D_\mu, q_\nu] = 0$ $[D_\mu, q^\nu] = (\partial_\mu g^{\rho\nu}) q_\rho$ $[D_\mu, \gamma^\nu] = (\partial_\mu \gamma^\nu) + [\omega_\mu, \gamma^\nu] = -\Gamma_{\mu\rho}^\nu \gamma^\rho$ $[D_\mu, g^{\nu\rho}] = (\partial_\mu g^{\nu\rho})$ $[D_\mu, \Gamma_{\rho\sigma}^\nu] = (\partial_\mu \Gamma_{\rho\sigma}^\nu)$
$[D_\mu, \delta] = [\mathcal{D}_\mu, \delta] = (\partial_\mu \delta) = -(\partial_\mu q^2) \delta^2 = -(\partial_\mu g^{\alpha\beta}) q_\alpha q_\beta \delta^2$ $[\mathcal{D}_\mu, -\not{q}\delta] = -\gamma^\alpha ((\nabla_\mu q_\alpha) \delta + q_\alpha (\partial_\mu \delta)) = -\gamma^\alpha (-\Gamma_{\mu\alpha}^\lambda q_\lambda - q_\alpha (\partial_\mu g^{\rho\sigma}) q_\rho q_\sigma \delta^2)$	

Table 6.1: Set of commutation rules. With the notation $\delta = 1/(q^2 - m^2)$.

Once the momentum dependence is commuted to the left of the covariant derivatives, the integration over momentum and mass can be performed.³ Then the different terms have to be combined together to form covariant quantities. This last point may seem a tedious task since our expansion is not manifestly covariant, but we will see in the examples that the use of Riemann Normal Coordinates (RNC) and Fock-Schwinger (FS) gauge effortlessly provide the result in terms

³The mass integration variable commutes with every operators, so no difficulty arises in that regard.

of covariant quantities. Besides, the use of RNC from the beginning of the computation greatly reduces the number of terms to compute, and simplifies the commutations through the covariant derivatives.

In the following, we give examples to illustrate the systematic procedure to perform the CDE in curved spacetime: commute the momentum dependence to the left using Table 6.1, perform the mass and momentum integration, form covariant quantities. We will show that the use of RNC is not only useful to form covariant curvature quantities, but also from the first step of the procedure it reduces the number of terms that contribute at a given order, and simplifies the commutation of the momentum.

6.4.1 Example of computation - Bosonic CDE

We will compute the first order term ($\propto m^2$) of the bosonic UOLEA in curved spacetime to illustrate the procedure. We will use the expansion from (6.27). For simplicity, we denote $D^2 = g^{\mu\nu} D_\mu D_\nu$, $\Gamma D = g^{\mu\nu} \Gamma_{\mu\nu}^\rho D_\rho$, and $\Gamma q = g^{\mu\nu} \Gamma_{\mu\nu}^\rho q_\rho$, and we take $U = 0$. The contributions at order m^2 are

$$\mathcal{L}_{\text{eff}}^{\text{bos}} \Big|_{\mathcal{O}(m^2)} = -ic_s \int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm'^2 \text{tr} \left(\Delta D^2 \Delta + \Delta(2iq \cdot D - \Gamma D - i\Gamma q) \Delta(2iq \cdot D - \Gamma D - i\Gamma q) \Delta \right). \quad (6.34)$$

The first step is to commute the momentum dependence to the left, that is to say, commute the covariant derivatives to the right. For example, consider the first term of Eq. (6.34)

$$\text{tr} \Delta D^2 \Delta = \text{tr} \left(\Delta(\Delta D^2 + (D^2 \Delta) + 2g^{\mu\nu} (D_\mu \Delta) D_\nu) \right). \quad (6.35)$$

We then use the commutation relations from Table 6.1

$$\begin{aligned} \text{tr} (D^2 \Delta) &= \text{tr} g^{\mu\nu} (\partial_{\mu\nu} \Delta) = \text{tr} g^{\mu\nu} \left(-(\partial_{\mu\nu} q^2) \Delta^2 + 2(\partial_\mu q^2)(\partial_\nu q^2) \Delta^3 \right) \\ \text{tr} 2g^{\mu\nu} (D_\mu \Delta) D_\nu &= -2g^{\mu\nu} \text{tr} (\partial_\mu q^2) \Delta^2 D_\nu. \end{aligned} \quad (6.36)$$

Now we can perform the integration over mass and momentum and express it in terms of master integrals (App. B). The same procedure has to be applied for the 9 other terms from Eq. (6.34). Once the master integrals are explicitly written, all the terms can be combined together to form covariant quantities.

As emphasised earlier, since our expansion is diffeomorphism invariant we can choose a specific coordinate system to simplify the computation. We will use the Riemann Normal Coordinates (RNC) around a point x_0 . At $x = x_0 + y$, the metric and the Christoffel symbols can be expanded around as

$$\begin{aligned} g_{\mu\nu}(y) &= \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}(x_0) y^\alpha y^\beta + \mathcal{O}(y^3) \\ \Gamma_{\nu\rho}^\mu(y) &= -\frac{1}{3} (R^\mu{}_{\nu\rho a} + R^\mu{}_{\rho\nu a})(x_0) y^a + \mathcal{O}(y^2). \end{aligned} \quad (6.37)$$

We can use the RNC to help form covariant quantities after the expansion performed above, but simplifications occur starting from (6.34). Since the Christoffel symbols vanish at x_0 , only their derivatives survive. We can thus already rule out from (6.34) the terms that have Christoffel symbols without derivative to their left

$$\mathcal{L}_{\text{eff}}^{\text{bos}} \Big|_{\mathcal{O}(m^2)} = -ic_s \int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm'^2 \text{tr} \left(\Delta D^2 \Delta + \Delta 2iq \cdot D \Delta(2iq \cdot D - \Gamma D - i\Gamma q) \Delta \right). \quad (6.38)$$

We then commute the momentum dependence to the left in RNC. For the term $\Delta D^2 \Delta$, we obtained Eq. (6.35). But the term including $(D_\mu \Delta)$ is proportional to a first derivative of the metric hence it vanishes. We are left with

$$\begin{aligned} \int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm' \operatorname{tr} \Delta D^2 \Delta &= \int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm' \operatorname{tr} (\Delta^2 D^2 + (D^2 \Delta)) \\ &= \int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm' \operatorname{tr} \left(\Delta^2 D^2 - \Delta^2 q_\alpha q_\beta g^{\mu\nu} (\partial_{\mu\nu} g^{\alpha\beta}) \right) \\ &= \sqrt{|g|} \operatorname{tr} \left(\mathcal{J}[q^0]^2 D^2 - \mathcal{J}[q^2]^3 \frac{2}{3} R \right). \end{aligned} \quad (6.39)$$

The second term from Eq. (6.38) is also rather simple. Using Table 6.1 in RNC it reduces to

$$\begin{aligned} &\int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm' \operatorname{tr} \Delta 2iq \cdot D \Delta (2iq \cdot D - \Gamma D - i\Gamma q) \\ &= \int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm' \operatorname{tr} \left(\Delta^2 (2i)^2 q^\mu q^\nu D_\mu D_\nu - 2iq^\mu g^{\alpha\beta} (\partial_\mu \Gamma_{\alpha\beta}^\rho) D_\rho + 2q^\mu g^{\alpha\beta} (\partial_\mu \Gamma_{\alpha\beta}^\rho) q_\rho \right) \\ &= \sqrt{|g|} \mathcal{J}[q^2]^3 \operatorname{tr} \left(\frac{4}{3} R - 4D^2 \right). \end{aligned} \quad (6.40)$$

Note that the integration with an odd power in q in the numerator vanishes.

The next step which is to combine the different terms to form covariant quantities is avoided as far as the Christoffel part is concerned since the RNC provide directly the covariant quantities. There remains to form covariant quantities with the covariant derivatives, which can also be simplified using the Fock-Schwinger (FS) gauge (see details in App. C).

Combining Eqs. (6.39) and (6.40), we obtain the one-loop effective action at order m^2

$$\mathcal{L}_{\text{eff}}^{\text{bos}} \Big|_{\mathcal{O}(m^2)} = \sqrt{|g|} \frac{c_s}{16\pi^2} m^2 \left(1 - \log \left(\frac{m^2}{\mu^2} \right) \right) \operatorname{tr} \frac{R}{6}. \quad (6.41)$$

The remaining trace is over gauge and spin degrees of freedom. μ is the renormalisation scale from dimensional regularisation. We used the \overline{MS} scheme, and will do so throughout.

6.4.2 Example of computation - Fermionic CDE

For completion, we briefly outline the computation of the m^2 term in the fermionic expansion, although it is similar to the procedure of the bosonic CDE. We take $Q = 0$ and omit the gauge sector for simplicity, the contribution at this order is

$$\mathcal{L}_{\text{eff}}^{\text{ferm}} \Big|_{\mathcal{O}(m^2)} = -i \int \frac{d^d q}{(2\pi)^d} \int^m dm' \operatorname{tr} \Delta \not{D} \Delta \not{D} \Delta. \quad (6.42)$$

We first commute the momentum dependence to the left, we can do it in RNC to keep it simple. Let us keep in mind that the propagators here bear a Dirac matrix

$$\mathcal{L}_{\text{eff}}^{\text{ferm}} \Big|_{\mathcal{O}(m^2)} = -i \int \frac{d^d q}{(2\pi)^d} \int^m dm' \operatorname{tr} (\Delta \gamma^\mu \Delta \gamma^\nu \Delta \mathcal{D}_\mu \mathcal{D}_\nu + \Delta \gamma^\mu \Delta \gamma^\nu (\mathcal{D}_\mu \mathcal{D}_\nu \Delta)). \quad (6.43)$$

We split the propagators according to (6.32), we denote $\Delta_f = -\not{q}/(q^2 - m'^2)$ and $\Delta_b = m'/(q^2 - m'^2)$, and then commute the momentum dependence to the left. Since the integration with an odd power

in q in the numerator vanishes, we can only have an even power in Δ_f . We then perform the integration in terms of the fermionic master integrals.

An extra step that arises in the fermionic CDE is to perform the Dirac trace, or at least simplify the contractions in-between Dirac matrices, in order to be able to form covariant operators. Extra care must be taken for the terms that bear open covariant derivatives to the right since they carry the spin-connection.

Let us focus on the first term of (6.43). After splitting the propagators we obtain

$$\begin{aligned}
 & -i \int \frac{d^d q}{(2\pi)^d} \int^m dm' \operatorname{tr} \left((\Delta_b \gamma^\mu \Delta_b \gamma^\nu \Delta_b + \Delta_f \gamma^\mu \Delta_f \gamma^\nu \Delta_b \right. \\
 & \quad \left. + \Delta_f \gamma^\mu \Delta_b \gamma^\nu \Delta_f + \Delta_b \gamma^\mu \Delta_f \gamma^\nu \Delta_f) \mathcal{D}_\mu \mathcal{D}_\nu \right) \\
 & = i \sqrt{|g|} \operatorname{tr} \left(\left(\mathcal{K}[q^0]_3^3 \gamma^\mu \gamma^\nu + \mathcal{K}[q^2]_3^1 g_{\alpha\beta} \left(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu + \gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\beta + \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \right) \right) \mathcal{D}_\mu \mathcal{D}_\nu \right).
 \end{aligned} \tag{6.44}$$

The first possibility is to simplify the contractions among Dirac matrices using the Clifford algebra, and then form covariant quantities

$$\begin{aligned}
 & i \sqrt{|g|} \operatorname{tr} \left(\mathcal{K}[q^0]_3^3 \mathcal{D}^2 + d \mathcal{K}[q^2]_3^1 \mathcal{D}^2 - 4 \mathcal{K}[q^2]_3^1 \mathcal{D}^2 \right) \\
 & = \sqrt{|g|} \frac{-1}{16\pi^2} \frac{m^2}{2} \left(1 - \log \left(\frac{m^2}{\mu^2} \right) \right) \operatorname{tr} \left(\mathcal{D}^2 - \mathcal{D}^2 \right) \\
 & = \sqrt{|g|} \frac{1}{16\pi^2} \frac{m^2}{2} \left(1 - \log \left(\frac{m^2}{\mu^2} \right) \right) R \operatorname{tr} \mathbf{1}_{\text{gauge}},
 \end{aligned} \tag{6.45}$$

including a factor 4 coming from the trace over spin indices. From the first to the second line we discarded the pole $2/\bar{\epsilon}$, although we took care not to forget the finite contribution obtained when multiplied by $d = 4 - \epsilon$ in the first line. From the second to the last line we used (5.68).

Another possibility is to make explicit the spin-connection in (6.44), directly compute the Dirac trace, and then form covariant quantities with the explicit spin-connections. When the open derivatives are on the far right we can write: $\mathcal{D}_\mu \mathcal{D}_\nu = (\partial_\mu \omega_\nu) + \omega_\mu \omega_\nu$. In RNC it takes a simple form as explained in Appendix C

$$\mathcal{D}_\mu \mathcal{D}_\nu = -\frac{1}{8} \gamma^\alpha \gamma^\beta R_{\nu\mu\alpha\beta}. \tag{6.46}$$

Using the RNC for the spin-connection is the most efficient method to get covariant quantities when the computation involves more terms.

The systematic procedure is the same as for the bosonic CDE: commute momentum dependence to the left, perform the mass and momentum integrations, then form covariant quantities. The last step is slightly more involved for the fermionic CDE, but with the use of RNC it is straightforward.

Chapter 7

Bosonic UOLEA in curved spacetime

As opposed to the fermionic CDE in curved spacetime, the results from the bosonic CDE presented here are well-known. Indeed our results can be matched for example with those from the heat kernel approach [92]¹, or more recently using the worldline formalism [79]. Nevertheless, the CDE has the advantage of being systematic and in fact algorithmic, thus the expansion is easy to automatise. In addition, the CDE being based on an inverse mass expansion, its physical interpretation is always enlightening compared to a quite formal heat kernel approach.

The CDE in curved spacetime has already been approached in [77, 78]. However, as opposed to our current method, these works use the Gaillard and Cheyette sandwich mentioned in Section 4.2.1. As explained earlier, it has the advantage of making the computation manifestly covariant, but at the cost of making the expansion significantly more involved. When the curvature of spacetime is introduced, such computation can quickly become untractable. By avoiding this step, we make the computation simpler which allows us to compute higher order corrections in a straightforward way. The computation of non-renormalisable operators (order $1/m^2$) using the CDE in curved spacetime are presented for the first time. More than that, our systematic method could easily be implemented in a code which would allow to generate even higher dimensional operators associated to generic UV theories involving gravity.² The drawback of our method would then be to form the covariant operators at the end, but thanks to the RNC and the FS gauge (see Appendix C for additional details) it turns out to be straightforward and algorithmic as well. Another advantage of our method is that it is independent of a choice of coordinate system, as opposed to [77], therefore it can be used to compute non-covariant quantities such as consistent gravitational anomalies (see for example [4]).

Note that the result from this section can apply to the integration of a (real or complex) scalar, a vector-like fermion, massive and massless vector bosons, as well as ghosts [69]. The factor c_s and the content of U depend on the nature of the heavy field. One significant asset of our approach is that one can straightforwardly incorporate all the improvements on the CDE in flat spacetime EFTs, such as non-degenerate mass matrix [70], mixed heavy light loops [71] and UV theories involving derivative couplings [75].³

¹Note that due to the difference in conventions in the one-loop effective action of [92], we have $m_{\text{HK}}^2 = -m_{\text{CDE}}^2$ and $S_{\text{eff}}^{\text{CDE}} = 2S_{\text{eff}}^{\text{HK}}$.

²Several codes exist that perform the CDE without gravity, e.g [97–99].

³The covariant diagrams [74] can also be used to enumerate the terms of the expansion, but it does not account for the commutation of the momentum dependence to the left of the derivatives so most of their properties must be dropped.

7.1 Result

We skip computational details and directly write down the bosonic universal one-loop effective action in curved spacetime, up to dimension 6 operators

$$\begin{aligned}
 S_{\text{eff}}^{\text{bos}} = & \frac{c_s}{16\pi^2} \int \sqrt{|g|} d^4x \text{tr} \left\{ m^2 \left(1 - \log \left(\frac{m^2}{\mu^2} \right) \right) \left(\frac{1}{6} R + U \right) \right. \\
 & + \log \left(\frac{m^2}{\mu^2} \right) \left[-\frac{1}{72} R^2 + \frac{1}{180} R_{\mu\nu} R^{\mu\nu} - \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{30} (\square R) \right. \\
 & \quad \left. \left. - \frac{1}{6} R U - \frac{1}{6} (\square U) - \frac{1}{2} U^2 - \frac{1}{12} F^2 \right] \right. \\
 & + \frac{1}{m^2} \left[-\frac{1}{72} R F^2 - \frac{1}{90} R_{\mu\nu} F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{180} R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right. \\
 & \quad + \left(\frac{1}{90} - \frac{a}{2} \right) (\mathcal{D}_\mu F^{\mu\nu})^2 + a F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu + \left(\frac{1}{360} + \frac{a}{4} \right) (\mathcal{D}_\mu F_{\nu\rho}) (\mathcal{D}^\mu F^{\nu\rho}) \\
 & \quad \left. - \frac{1}{12} U (\square U) - \frac{1}{36} R (\square U) - \frac{1}{12} R U^2 - \frac{1}{6} U^3 - \frac{1}{12} U F^2 + \mathcal{O}(R^2) \right] \\
 & \left. + \mathcal{O}(1/m^4) \right\}. \tag{7.1}
 \end{aligned}$$

The result is independent of a which is a freedom in the choice of basis, and only the terms linear in curvature were computed at order $1/m^2$. The remaining trace is over gauge and spin indices. This result is in agreement with [92].

We should comment on the fact that the terms of order m^2 and m^0 are divergent and these divergences can be absorbed by the renormalisation as it is well known. We used dimensional regularisation to compute the divergent momentum integrals with \overline{MS} scheme, and μ is the renormalisation scale. In practice, these contributions can conveniently be used to compute the RGE running of the EFT operators at one-loop (see [68] for example).

Dimension 6 operators are expressed using the following basis

$$B_{\mathcal{D},F} = \{ (\mathcal{D}_\mu F_{\nu\rho}) (\mathcal{D}^\mu F^{\nu\rho}), F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu, (\mathcal{D}_\mu F^{\mu\nu})^2 \}, \tag{7.2}$$

which is both diffeomorphism and gauge invariant. We recall that D and F are the gauge and spin-connection covariant derivative and field strength, whereas \mathcal{D} and \mathcal{F} bear the gauge connection and spin-connection as well as the Christoffel connection. Note that the basis

$$B_{D,F} = \{ (D_\mu F_{\nu\rho}) (D^\mu F^{\nu\rho}), F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu, (D_\mu F^{\mu\nu})^2 \}, \tag{7.3}$$

is not diffeomorphism invariant, and the basis

$$B_{\mathcal{D},\mathcal{F}} = \{ (\mathcal{D}_\mu \mathcal{F}_{\nu\rho}) (\mathcal{D}^\mu \mathcal{F}^{\nu\rho}), \mathcal{F}_\mu{}^\nu \mathcal{F}_\nu{}^\rho \mathcal{F}_\rho{}^\mu, (\mathcal{D}_\mu \mathcal{F}^{\mu\nu})^2 \}, \tag{7.4}$$

is not gauge invariant.

7.2 Integrating out the graviton

So far the computation were performed on a fixed spacetime background. It is of interest to treat the metric as a dynamical quantum field, and integrate over its configurations. As we would do for a spin-1 gauge field, we use the background field method: the metric is split as $g + \delta g$, where g is a fixed background, and δg is a fluctuation integrated over in the path integral. Indices are lowered and raised with the background metric. The quantum field $\delta g_{\mu\nu}$ inherits a gauge invariance from the diffeomorphism invariance of the background, which is dealt with using the Fadeev-Popov procedure. The action is then expanded up to quadratic order in δg so that the path integral can be performed.⁴

There is however a discrepancy when dealing with a spin-2 field with respect to fields of smaller spin: the field-space has a non-trivial metric G . A consequence is that the second derivative of the action is not a scalar under a redefinition of δg . This issue was solved in [103] by introducing a field-space covariant derivative. For a generic field ϕ with kinetic term $\dot{\phi}_a \cdot A^{ab}(\phi) \cdot \dot{\phi}_b$, the field-space metric is $G^{ab}(x, y) = A^{ab}(\phi)\delta(x - y)$ and the connection is $\hat{\Gamma}_{bc}^a = \frac{1}{2}(G^{-1})^{ad}(\frac{\delta G_{bd}}{\delta \phi^c} + \frac{\delta G_{cd}}{\delta \phi^b} - \frac{\delta G_{bc}}{\delta \phi^d})$. We can directly see that if the action is at most quadratic in ϕ , then A^{ab} is independent of ϕ and $\hat{\Gamma}$ vanishes. Therefore, for spin 0, 1/2, 1 and 3/2 matter fields, the second derivative of the action is a scalar and no field-space covariant derivative is needed.

The one-loop effective action is covariantly defined as

$$iS_{\text{eff}[g]} = \log \int \sqrt{G} [\mathcal{D}\delta g] e^{iS[g] + \frac{i}{2}\delta g(\hat{\mathcal{D}}^2 S)[g]\delta g + \mathcal{O}(\delta g^3)} \simeq iS[g] - \frac{1}{2}\text{Tr} \log \left(-(\hat{\mathcal{D}}^2 S)[g] \cdot G^{-1} \right), \quad (7.5)$$

where the background g is solution to the equations of motion, that is $(\hat{\mathcal{D}}S)[g] = \frac{\delta S}{\delta g_{\mu\nu}}[g] = 0$. $\hat{\mathcal{D}}$ is the field-space covariant derivative such that⁵

$$(\hat{\mathcal{D}}^2 S) = \frac{\delta^2 S}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} + \hat{\Gamma}_{\alpha\beta}^{\mu\nu, \rho\sigma} \frac{\delta S}{\delta g_{\alpha\beta}}. \quad (7.6)$$

The field-space metric and its inverse are

$$G^{\mu\nu, \rho\sigma} = \frac{1}{4}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}) \quad \text{and} \quad (G_{\mu\nu, \rho\sigma}^{-1}) = g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}. \quad (7.7)$$

Eq. (7.6) evaluated on g solution to the equations of motion reduces to the usual second derivative of the action. But if one wanted to vary the background g , then the variation of $\hat{\mathcal{D}}^2 S$ would cancel exactly against the variation of \sqrt{G} from the measure, making the theory invariant under a redefinition of δg .

We consider the UV theory

$$S = \int d^4x \sqrt{|g|} \left(\frac{1}{4\kappa} (2\Lambda - R) + \mathcal{L}_{\text{mat}} \right), \quad (7.8)$$

where $\kappa = 8\pi/M_P^2$ with M_P the Planck mass, and \mathcal{L}_{mat} is the matter Lagrangian. With the background g on-shell, the second derivative of the action with respect to the metric, including the

⁴The use of the saddle point approximation is only possible around a background g that is a saddle point of the space of metric configurations. We assume g fulfills that requirement. The complete gravitational path integral would then be the sum of the saddle point contributions, supplemented by other non-perturbative configurations, such as instantons, which cannot be treated following the presented procedure. For recent literature see for example [100–102]

⁵We leave implicit the summation over spacetime indices in the second term. $(\hat{\mathcal{D}}^2 S)(x, y)$ remains proportional to $\delta^d(x - y)$.

ghosts c^μ and \bar{c}^μ , and the gauge-fixing in harmonic gauge⁶, reads

$$\begin{aligned} \int d^4x \frac{1}{2} \delta g \frac{\delta^2 S}{\delta g^2} \delta g &= - \int d^4x \sqrt{|g|} \bar{c}^\mu (g_{\mu\nu} \mathcal{D}^2 + R_{\mu\nu}) c^\nu \\ &\quad - \int d^4x \sqrt{|g|} \delta g_{\alpha\beta} \left(\frac{1}{2} g_{(\rho}^\alpha g_{\sigma)}^\beta \mathcal{D}^2 + R^\alpha{}_{(\rho}{}^\beta{}_{\sigma)} - g^{\alpha\beta} R_{\rho\sigma} + \Lambda g^{\alpha\beta} g_{\rho\sigma} \right) G^{\gamma\delta, \rho\sigma} \delta g_{\gamma\delta} \\ &\quad - \int d^4x \sqrt{|g|} \delta g_{\alpha\beta} \mathcal{O}^{\alpha\beta}{}_{\rho\sigma} G^{\gamma\delta, \rho\sigma} \delta g_{\gamma\delta}, \end{aligned} \quad (7.9)$$

where \mathcal{O} corresponds to the matter part. The parenthesis around the indices denotes the symmetrisation: $T_{(\mu\nu)} = T_{\mu\nu} + T_{\nu\mu}$. Including loops of graviton only, \mathcal{O} is a local operator and reads

$$\mathcal{O} \cdot G = \frac{\kappa}{\sqrt{|g|}} (\hat{\mathcal{D}} \sqrt{|g|} T), \quad (7.10)$$

with T the matter energy-momentum tensor such that $\sqrt{|g|} T = -2 \frac{\delta S}{\delta g}$. More details about the derivation can be found in [58, 78]. In practice, \mathcal{O} can be computed by taking the second derivative of the matter action with respect to the metric.

According to Eq. (7.5), the graviton piece must be contracted with G^{-1} , which yields the identity when contracted against G from Eq. (7.9). Finally, we obtain the one-loop effective action after integrating out the graviton (first line) and ghost (second line)

$$\begin{aligned} S_{\text{eff}}^{\text{graviton}} &= \frac{i}{2} \text{Tr} \log \left(\frac{1}{2} g_{(\rho}^\mu g_{\sigma)}^\nu \mathcal{D}^2 + R^\mu{}_{(\rho}{}^\nu{}_{\sigma)} - g^{\mu\nu} R_{\rho\sigma} + \Lambda g^{\mu\nu} g_{\rho\sigma} + \mathcal{O}^{\mu\nu}{}_{\rho\sigma} \right) \\ &\quad - i \text{Tr} \log (g_{\mu\nu} \mathcal{D}^2 + R_{\mu\nu}), \end{aligned} \quad (7.11)$$

Both are of the form $\text{Tr} \log (\mathbb{1} \mathcal{D}^2 + U)$, therefore they can be obtained from the bosonic UOLEA derived previously. Note that the trace is also performed over Lorentz indices. The graviton piece is traced over by contracting the indices with the identity for order 4 Lorentz tensors

$$\mathbb{1}_{\rho\sigma}^{\mu\nu} = G^{\mu\nu, \alpha\beta} (G^{-1})_{\alpha\beta, \rho\sigma} = \frac{1}{2} g_{(\rho}^\mu g_{\sigma)}^\nu. \quad (7.12)$$

Note that since the ghost and the graviton are massless, the effective action contains infrared divergences. They can be regulated by inserting a mass term $\mathcal{D}^2 \rightarrow \mathcal{D}^2 + m^2$ [69]. For loops of graviton only, \mathcal{O} is a local operator and we can apply the results from the bosonic UOLEA (7.1), with

$$c_s^{\text{graviton}} = 1/2, \quad U_{\text{graviton}} = R^\mu{}_{(\rho}{}^\nu{}_{\sigma)} - g^{\mu\nu} R_{\rho\sigma} + \Lambda g^{\mu\nu} g_{\rho\sigma} + \mathcal{O}^{\mu\nu}{}_{\rho\sigma} \quad (7.13)$$

$$c_s^{\text{ghost}} = -1, \quad U_{\text{ghost}} = R_{\mu\nu}. \quad (7.14)$$

Note that if mixed species loops are allowed, then the bosonic UOLEA does not apply to the graviton since \mathcal{O} includes open derivatives, the CDE has to be performed from scratch following the mixed heavy-light methods [71, 72, 78].

⁶Note that it corresponds to the gauge invariance associated to δg . We still have the freedom to choose the coordinate system, such as the RNC.

Chapter 8

Fermionic UOLEA in curved spacetime

We now turn to the fermionic CDE in curved spacetime. The one-loop effective action that we obtain after integrating out a fermion is of the form

$$S_{\text{eff}}^{\text{ferm}} = -i \text{Tr} \log \left(\sqrt{|g|} (i\mathcal{D} - m - Q) \right), \quad (8.1)$$

In a similar manner as the bosonic determinant, it can be expanded as

$$S_{\text{eff}}^{\text{ferm}} = i \int d^d x \frac{d^d q}{(2\pi)^d} \int^m dm' \text{tr} \sum_{n \geq 0} [\Delta (-i\mathcal{D} + Q)]^n \Delta, \quad (8.2)$$

where $\Delta = m'/(q^2 - m'^2) - \not{q}/(q^2 - m'^2)$. The mass dimension of each term is straightforward to see, contrary to the bosonic expansion, since the n -th term of the sum is proportional to m^{4-n} .

We can split $Q = Q_e + Q_o$ where Q_e (resp. Q_o) has an even (resp. odd) number of Dirac matrices. Following [73], we can assume the general form

$$Q_e = W_o + iW_1\gamma_5, \quad Q_o = X_\mu\gamma^\mu + iA_\mu\gamma^\mu\gamma_5. \quad (8.3)$$

We choose to limit ourselves to the case of a scalar, pseudo-scalar, vector and pseudo-vector term, but the computation can be performed for any type of operator.

For a chiral fermion coupling to an axial (gauge) field, it is simpler to separate it from the covariant derivative $(\mathcal{D}_\mu\psi) = (\partial_\mu + iV_\mu + \omega_\mu)\psi$ where ω is the spin-connection and V a vector gauge field, and include it in $Q_o \supset -\not{A}\gamma_5$.

To compute the traces, we use the Breitenlohner-Maison-'t Hooft-Veltman (BMHV) scheme for γ_5 [104, 105]. The choice of scheme for γ_5 can have consequences which should not be disregarded, especially in the context of quantum anomalies [4].

We emphasise again that others approaches to compute one-loop effective actions in gravity, such as heat kernel, CDE and worldline formalism [77–79, 92, 106], always applied to a bosonic determinant, therefore were restrained to $Q = W_0 + \gamma^\mu X_\mu$, as explained in Section 4.2.2. To our knowledge, the fermionic Universal One-Loop Effective Action on a general spacetime manifold was never computed before.

Improvements on the CDE can also be applied (namely, non-degenerate mass matrix[70], mixed heavy-light [71]). The covariant diagrams can be used to enumerate the terms of the expansion [74], but it does not account for the commutation of the momentum dependence to the left of the derivatives so most of their properties must be dropped.

8.1 Result

Let us skip computational details that can be found in [3] and give directly the fermionic universal one-loop effective action in curved spacetime up to dimension 5 operators. Displaying only the curvature dependent operators it reads

$$\begin{aligned}
 S_{\text{eff}}^{\text{ferm}} \supset & \frac{-1}{16\pi^2} \int \sqrt{|g|} d^4x \text{tr} \left\{ -m^2 \frac{1}{6} R \left(1 - \log \left(\frac{m^2}{\mu^2} \right) \right) + m \frac{1}{3} \log \left(\frac{m^2}{\mu^2} \right) R W_0 \right. \\
 & + \log \left(\frac{m^2}{\mu^2} \right) \left[-\frac{1}{144} R^2 + \frac{1}{90} R_{\mu\nu} R^{\mu\nu} + \frac{7}{720} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{60} (\square R) \right] \\
 & + \frac{1}{3} R W_0^2 \left(1 + \frac{1}{2} \log \left(\frac{m^2}{\mu^2} \right) \right) + \frac{1}{3} R W_1^2 \left(-1 + \frac{1}{2} \log \left(\frac{m^2}{\mu^2} \right) \right) \\
 & - \frac{2}{3} A^\mu A^\nu R_{\mu\nu} \log \left(\frac{m^2}{\mu^2} \right) \\
 & + \frac{1}{m} \left[W_1 \left(-\frac{1}{48} \varepsilon_{\mu\nu\rho\sigma} R_{\alpha\beta}{}^{\mu\nu} R^{\alpha\beta\rho\sigma} \right) \right. \\
 & + W_0 \left(\frac{1}{45} R^{\mu\nu} R_{\mu\nu} - \frac{1}{72} R^2 + \frac{7}{360} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \frac{1}{3} (\square R) \right) \\
 & + R^{\mu\nu} \left(-\frac{4}{3} W_0 A_\mu A_\nu + i \frac{4}{3} A_\mu (\mathcal{D}_\nu W_1) + 2i W_1 (\mathcal{D}_\mu A_\nu) + \frac{2}{3} A_\mu [X_\nu, W_1] \right) \\
 & \left. + R \left(-\frac{1}{3} A^\mu [X_\mu, W_1] + \frac{1}{9} W_0^3 + \frac{1}{3} W_0 W_1^2 - i \frac{1}{3} W_1 (\mathcal{D}_\mu A^\mu) \right) \right] \\
 & + \mathcal{O}(1/m^2) \left. \right\}. \tag{8.4}
 \end{aligned}$$

where $\varepsilon_{\mu\nu\rho\sigma} = \sqrt{|g|} \bar{\varepsilon}_{\mu\nu\rho\sigma}$ with $\bar{\varepsilon}_{\mu\nu\rho\sigma}$ the Levi-Civita tensor in flat spacetime. The remaining trace is over gauge degrees of freedom. Note that the computation was performed without specific choice of coordinates (i.e without use of RNC) to ensure to covariance of the result.

The terms that involve only W_0 and curvatures invariants can be recovered from the bosonic UOLEA (7.1) by bosonising the functional determinant involving vector-like fermions, they are in agreement with [92]. The rest consist in new operators and involve the curvature invariants and the fields A and W_1 , that chirally couple to the heavy fermion. To the best of our knowledge, these operators are new and were never computed before.

Let us point out that the covariant derivative bears the gauge vector field V , and X is a generic vector field. It means that X can also be a gauge field $X = V'$, and in that case it seems that in the last two lines of (8.4) the vector gauge symmetry associated to V' is broken. In fact, this is not the case. If from the start we have a Lagrangian with two vector gauge fields such that $\bar{\psi}(i\cancel{D} - \cancel{V} - V')\psi$, we can choose to proceed with the expansion by keeping $i\cancel{D} \supset -\cancel{V}$ and deal with the other gauge field with $X = V'$ (recall that in (8.1) Q comes in with a minus sign already). Using $i(\mathcal{D}\mathcal{O}) = i(\nabla\mathcal{O}) - [V, \mathcal{O}] + i[\omega, \mathcal{O}]$ for any operator \mathcal{O} , and the cyclicity of the gauge trace, one can show that the last two lines of Eq. (8.4) can be rewritten without the explicit $X = V'$ terms and replacing V by $V + V'$ in $i(\mathcal{D}W_1)$ and $i(\mathcal{D}A)$. Therefore, the vector gauge symmetry is preserved.

One may notice that there are some terms in (8.4) that look like the axial-gravitational and trace/Weyl anomalies. In Part III, we will see how to extract them from (8.4).

Chapter 9

Conclusion

In this Part II, we have introduced and motivated the concept of Effective Field Theory (EFT), with a particular emphasis on EFTs in gravity. After a brief introduction to differential geometry Quantum Field Theory (QFT) in curved spacetime, we have presented our work [3] in which a novel method is described to build EFTs in curved spacetime. It relies on the Covariant Derivative Expansion (CDE) technique, and presents several advantages compared to existing methods.

The main upshot of our method is the relative computational simplicity, systematicity and transparency, especially when using a convenient choice of gauge for the gravitational and gauge fields. The method would very fit in a code that performs the CDE such as [97–99].

On top of that, we filled a gap in the literature concerning the integration of chiral fermions in curved spacetime, via the path integral. This is of great importance since most of the SM fermions are both massive and chiral. We may thus easily imagine BSM models involving massive chiral fermions that need to be integrated out. This is the case for example of axion models, which may introduce new massive chiral fermions (KSVZ model [107, 108]).

Finally, the Fourier transform that we introduced does not rely on a specific choice of coordinate, which allows the expansion to be performed without picking a gauge. This is particularly useful in the context on anomalies, which may break diffeomorphism or Lorentz invariance, thus making illegal to choose specific coordinates such as using the Riemann Normal Coordinates. We will see in the next Part computations of gravitational anomalies that make use of this.

Part III

Quantum Anomalies

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Chapter 10

Introduction to Quantum Anomalies

10.1 Global anomalies

The pions $\{\pi_0, \pi^\pm\}$ are the Nambu-Goldstone bosons resulting from the spontaneous chiral symmetry breaking of QCD, and can be understood in the context of Chiral Perturbation Theory [109, 110]. Although, the neutral pion π^0 does not couple directly to the photon, it can interact with electrically charged fermions, and indeed, it is found experimentally that the π^0 mainly decays into two photons. Let us have a look at the corresponding amplitude.

We start by writing the coupling of π^0 to an SM fermion ψ of mass m coupling to the photon gauge field via the covariant derivative¹

$$\mathcal{L}_1 = -\frac{1}{2}\pi^0(\partial^2 + m_\pi^2)\pi^0 + \bar{\psi}(i\not{D} - m)\psi + 2i\frac{m}{f_\pi}\pi^0\bar{\psi}\gamma_5\psi, \quad (10.1)$$

where f_π is the pion decay constant and m_π its mass. We thus find that the neutral pion couples to fermions via the pseudo-scalar current

$$P = \bar{\psi}\gamma_5\psi. \quad (10.2)$$

The matrix element that describes the pion decay into two photon is defined by [20]

$$\langle A_\mu(k_1), A_\nu(k_2) | \pi^0(q) \rangle \propto \delta(q - k_1 - k_2) 2m T_{\mu\nu}(q, k_1, k_2), \quad (10.3)$$

and the LSZ formula helps us obtain the amplitude in terms of the VEV

$$T_{\mu\nu} = i \int d^4x d^4y d^4z e^{ik_1y + ik_2z - iqx} \langle TP(x) j_\mu(y) j_\nu(z) \rangle, \quad (10.4)$$

where the vector current is $j_\mu = \bar{\psi}\gamma_\mu\psi$. The decay rate of the pion is thus found to be [111]

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = 7.77 \text{ eV}, \quad (10.5)$$

in agreement with the experimental value $7.773 \pm 0.16 \text{ eV}$. We may content ourselves with this result, but let us dig a bit deeper to unravel something interesting.

If we reparametrise the fermion field as

$$\psi \rightarrow \psi e^{i\frac{\pi^0}{f_\pi}\gamma_5} \quad \bar{\psi} \rightarrow e^{i\frac{\pi^0}{f_\pi}\gamma_5} \bar{\psi}, \quad (10.6)$$

¹In Part II, we used the notation \mathcal{D} for the total covariant derivative, whereas D was the covariant derivative on internal indices only. In Part III here, these different notations are not useful, hence we simply take D to be the total covariant derivative.

we obtain after integrating by parts

$$\mathcal{L}_2 = -\frac{1}{2}\pi^0(\partial^2 + m_\pi^2)\pi^0 + \bar{\psi}(i\not{D} - m)\psi + \frac{1}{f_\pi}\pi^0\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi), \quad (10.7)$$

and we find that the neutral pion couples to fermions via the divergence of the axial current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi. \quad (10.8)$$

Similarly, we can obtain the amplitude of the pion decay into two photons

$$\langle A_\mu(k_1), A_\nu(k_2)|\pi^0(q)\rangle \propto \delta(q - k_1 - k_2)q^\rho T_{\mu\nu\rho}(q, k_1, k_2), \quad (10.9)$$

where

$$T_{\mu\nu\rho} = i \int d^4x d^4y d^4z e^{ik_1y + ik_2z - iqx} \langle T j_\mu^5(x) j_\nu(y) j_\rho(z) \rangle. \quad (10.10)$$

Eqs. (10.3) and (10.9) imply the Ward identity

$$q^\rho T_{\mu\nu\rho} = 2mT_{\mu\nu}. \quad (10.11)$$

As pointed by Sutherland and Veltman [112, 113], the left-hand side vanishes in the $q \rightarrow 0$ limit since $T_{\mu\nu\rho}$ has no poles (there exists no physical states between the pion and the vacuum). This implies that if we neglect the pion mass we find that it cannot decay into two photons

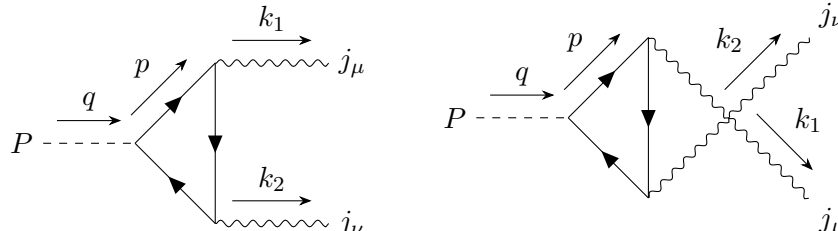
$$\Gamma(\pi^0 \rightarrow \gamma\gamma, q^2 = 0) = 0, \quad (10.12)$$

which is in disagreement with (10.5) and the experimental result. If the pion mass m_π is taken into account, the decay obtained at $q^2 = m_\pi^2$ is still a 1000 times smaller than the experimental result. This puzzle is related to the equation of conservation of j_5^μ , and is known as the Partially Conserved Axial Current (PCAC) puzzle. It was solved a few years later with the discovery of the Adler-Bell-Jackiw (ABJ) anomaly [114, 115] that we present in the following Section.

10.1.1 ABJ anomaly – diagrammatic approach

A great source of debates and confusion in QFT takes its source in the fact that it is not a well-defined mathematical framework since it is riddled with infinities. These infinities are physically well-understood and can be dealt with by regularising and renormalising the theory. Let us see how this pertains to the PCAC puzzle.

Clearly, there is something wrong with (10.11) so let us explicitly compute $T_{\mu\nu}$ and $T_{\mu\nu\rho}$. We omit most details of calculations which can be found in [20]. $T_{\mu\nu}$ is given by the sum of the diagrams



$$(10.13)$$

and reads

$$T_{\mu\nu} = -i \int \frac{d^4p}{(2\pi)^4} \text{tr} \frac{i}{\not{p} - m} \gamma_5 \frac{i}{\not{p} - \not{q} - m} \gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m} \gamma_\mu + \left(\begin{matrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{matrix} \right), \quad (10.14)$$

with $q = k_1 + k_2$ by conservation of momentum of the external legs. Using

$$\forall q_\mu, \quad \frac{1}{\not{q} - m} = \frac{\not{q} + m}{q^2 + m^2}, \quad (10.15)$$

and the traces

$$\text{tr } \gamma_5 = \text{tr } \gamma_5 \gamma^\mu \gamma^\nu = 0, \quad \text{tr } \gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = -4i \epsilon^{\mu\nu\rho\sigma}, \quad (10.16)$$

we readily verify that $T_{\mu\nu}$ is finite and of the form $T_{\mu\nu} \propto \frac{1}{m} \epsilon_{\mu\nu\rho\sigma} q^\rho k_1^\sigma$.

Let us now turn to $T_{\mu\nu\lambda}$ which is the tricky one. It is represented by the sum of the diagrams

$$(10.17)$$

and we have

$$q^\lambda T_{\mu\nu\lambda} = i \int \frac{d^4 p}{(2\pi)^4} \text{tr} \frac{i}{\not{p} - m} \not{q} \gamma_5 \frac{i}{\not{p} - \not{q} - m} \gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m} \gamma_\mu + \left(\begin{matrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{matrix} \right). \quad (10.18)$$

Using

$$\not{q} \gamma_5 = \gamma_5 (\not{p} - \not{q} - m) + (\not{p} - m) \gamma_5 + 2m \gamma_5, \quad (10.19)$$

we readily obtain

$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + R_{\mu\nu}^1 + R_{\mu\nu}^2, \quad (10.20)$$

with

$$R_{\mu\nu}^1 = \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left[\frac{1}{\not{p} - \not{k}_2 - m} \gamma_5 \gamma_\nu \frac{1}{\not{p} - \not{q} - m} \gamma_\mu - \frac{1}{\not{p} - m} \gamma_5 \gamma_\nu \frac{1}{\not{p} - \not{k}_1 - m} \gamma_\mu \right], \quad (10.21)$$

and R^2 is R^1 with $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. Crucially, $R^{1/2}$ are of the form

$$R^{1/2} \sim \int \frac{d^4 p}{(2\pi)^4} (f(p - k_{2/1}) - f(p)), \quad (10.22)$$

where

$$\int_{-\infty}^{+\infty} \frac{d^4 p}{(2\pi)^4} f(p), \quad (10.23)$$

is linearly divergent, that is to say f converges at infinity with $f(\pm\infty) \neq 0$, and $\forall n \geq 1, f^{(n)}(\pm\infty) = 0$. Let us take a one-dimensional example to show how to deal with the difference of linearly divergent integrals

$$\Delta(a) = \int_{-\infty}^{+\infty} dx (f(x+a) - f(x)). \quad (10.24)$$

By Taylor expanding $f(x+a)$ around x we find

$$\Delta(a) = \int_{-\infty}^{+\infty} dx \sum_{n \geq 1} \frac{f^{(n)}(x)}{n!} a^n = \sum_{n \geq 1} \int_{-\infty}^{+\infty} dx \frac{f^{(n)}(x)}{n!} a^n = a (f(+\infty) - f(-\infty)), \quad (10.25)$$

and we verify that $\Delta(a)$ is indeed convergent. Note that commuting the integral and the sum is allowed since either way the result is finite. However, note that

$$\Delta(a) \neq \int_{-\infty}^{+\infty} dx f(x+a) - \int_{-\infty}^{+\infty} dx f(x) , \quad (10.26)$$

since each term is separately divergent. The same method can be applied in $d = 4$ to obtain

$$R_{\mu\nu}^1 + R_{\mu\nu}^2 = -\frac{1}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma . \quad (10.27)$$

There is however a problem here, in that the result depends on the arbitrary choice of relative momentum routing inside the loop between the two diagrams in (10.17). Indeed, if we change the momentum routing only in the first diagram by

$$p \rightarrow p + \delta p , \quad \delta p = \alpha k_1 + (\alpha - \beta) k_2 , \quad (10.28)$$

for some arbitrary α and β , then we find

$$R^{1/2} \sim \int \frac{d^4 p}{(2\pi)^4} (f(p + \delta p - k_{2/1}) - f(p)) , \quad (10.29)$$

and as the 1-dimensional example shows, the difference of linearly divergent integrals, although finite, depends on this shift:

$$\begin{aligned} R_{\mu\nu}^1 + R_{\mu\nu}^2 &= -\frac{1-\beta}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma \\ \Rightarrow q^\lambda T_{\mu\nu\lambda} &= 2m T_{\mu\nu} - \frac{1-\beta}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma . \end{aligned} \quad (10.30)$$

This is concerning, we have an arbitrariness in the computation, despite being linked to a physical observable (the pion decay rate). To understand what is happening, let us take a step back and analyse the meaning of (10.30).

Consider the following theory

$$S = \int d^4 x \bar{\psi} (i\not{D} - m) \psi . \quad (10.31)$$

Let us consider two transformations of the fermions. The first one is the $U(1)_V$ (V for vector) transformation

$$\psi + \delta_V \psi = e^{i\theta(x)} \psi , \quad \bar{\psi} + \delta_V \bar{\psi} = \bar{\psi} e^{-i\theta(x)} , \quad \delta_V S = \int d^4 x \theta(x) \partial_\mu j^\mu , \quad (10.32)$$

with $\theta(\pm\infty) = 0$ so that we can integrate by parts. The other one is the $U(1)_A$ (A for axial) transformation

$$\psi + \delta_A \psi = e^{i\theta(x)\gamma_5} \psi , \quad \bar{\psi} + \delta_A \bar{\psi} = \bar{\psi} e^{i\theta(x)\gamma_5} , \quad \delta_A S = \int d^4 x \theta(x) (\partial_\mu j_5^\mu - 2imP) . \quad (10.33)$$

We recall

$$P = \bar{\psi} \gamma_5 \psi , \quad j^\mu = \bar{\psi} \gamma^\mu \psi , \quad j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi . \quad (10.34)$$

Upon using the EoM of the classical fields $\bar{\psi}$ and ψ , we find the following relations

$$\partial_\mu j^\mu = 0 , \quad \partial_\mu j_5^\mu = 2imP . \quad (10.35)$$

At the quantum level, we would naively expect to find

$$\langle \partial_\mu j^\mu \rangle = 0, \quad \langle \partial_\mu j_5^\mu \rangle = 2im\langle P \rangle. \quad (10.36)$$

However, when taking two derivatives with respect to the gauge field this implies

$$\begin{aligned} \langle T(\partial^\mu j_\mu)(x)j_\nu(y)j_\lambda(z) \rangle &= 0 && \Leftrightarrow k_1^\mu T_{\mu\nu\lambda} = 0 \\ \langle T(\partial^\lambda j_{5,\lambda})(x)j_\mu(y)j_\nu(z) \rangle &= 2im\langle TP(x)j_\mu(y)j_\nu(z) \rangle && \Leftrightarrow q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}, \end{aligned} \quad (10.37)$$

which we identified to be the source of the error, and is in disagreement with the previous computation

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma, \quad (10.38)$$

Similarly, a direct computation of $k_1^\mu T_{\mu\nu\lambda}$ with the same choice of momentum routing (10.28) reveals a disagreement [20]

$$k_1^\mu T_{\mu\nu\lambda} = \frac{1+\beta}{8\pi^2} \epsilon_{\nu\lambda\alpha\beta} k_1^\alpha k_2^\beta. \quad (10.39)$$

If we trust the direct computation, the quantum conservation laws for the $U(1)_V$ and $U(1)_A$ are in fact

$$\langle \partial_\mu j^\mu \rangle = \frac{1+\beta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}, \quad \langle \partial_\mu j_5^\mu \rangle = 2im\langle P \rangle - \frac{1-\beta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (10.40)$$

where $F_{\mu\nu} = i\partial_\mu A_\nu - i\partial_\nu A_\mu$.

We now have all keys in hand to understand what is happening here. The conservation laws for the $U(1)_V$ and $U(1)_A$ transformations show an anomalous behavior since they are violated at the quantum level, and there is no choice of momentum routing that allows to recover both of them. There is however a preferred choice that shows itself. Indeed, the SM and experiments show that the $U(1)_V$ transformation is gauged to become the electromagnetic interaction. If the symmetry associated with the electromagnetic interaction would be violated at the quantum level, this would imply that QED is inconsistent, and it would amount to a non-conservation of the electric charge, which is not observed in nature. We are thus very inclined to enforce the conservation of $U(1)_V$ at the quantum level, hence choosing $\beta = -1$ in the momentum routing procedure (10.28). This results in the quantum anomalous Ward identities, in the presence of the so-called Adler-Bell-Jackiw (ABJ) anomaly [114, 115]

$$\langle \partial_\mu j^\mu \rangle = 0 \quad (10.41)$$

$$\langle \partial_\mu j_5^\mu \rangle = 2im\langle P \rangle + \frac{1}{8\pi^2} F\tilde{F}, \quad (10.42)$$

where we introduce the dual field strength $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

Let us now return to the decay of the pion into two photons with the correct Ward identities. Using \mathcal{L}_1 (10.1), we obtain the correct decay rate as confirmed by experiments. To obtain \mathcal{L}_2 (10.7), we performed a $U(1)_A$ reparametrisation of the fermions, which we now know to be anomalous. In fact, we performed an integration by parts before regularising the theory and we missed a crucial boundary term that is the anomaly [20] (we will see later on that $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ is indeed a boundary term). Nonetheless, the decay rate given by the anomalous Ward identity (10.42) is in agreement the experimental results, hence solving the PCAC puzzle.²

In the next Section, we will see how to properly perform the axial reparametrisation (10.6) that misguided us, within the Path Integral and we will show how the anomaly shows itself.

²It was then pointed out that Sutherland and Veltman [112, 113] used the naive Ward identity $\langle \partial_\mu j_5^\mu \rangle = 2im\langle P \rangle$ in their paper.

10.1.2 ABJ anomaly - path integral approach

According to the previous Section, the quantum theory given by (10.1)

$$Z_1 = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^4x \mathcal{L}_1} , \quad (10.43)$$

is not equivalent to the quantum theory given by (10.7)

$$Z_2 = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^4x \mathcal{L}_2} , \quad (10.44)$$

because of the anomaly. A reparametrisation of the fermions in the Path Integral turns out to be a change of variable as in a regular integral, so let us perform the change of variable

$$\psi' = e^{i\theta\gamma_5}\psi , \quad \bar{\psi}' = \bar{\psi} e^{i\theta\gamma_5} , \quad (10.45)$$

in Z_1 , and for $\theta = \frac{\pi^0}{f\pi}$ we obtain

$$Z_1 = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}' e^{i\int d^4x \mathcal{L}_2} . \quad (10.46)$$

Comparing Z_1 and Z_2 , we see that only the measure differs by a Jacobian

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = J[\theta]\mathcal{D}\psi\mathcal{D}\bar{\psi} . \quad (10.47)$$

Since the fields in the Path Integral are classical (i.e not operator valued), the quantum part of the Path Integral truly comes from the measure, so it would make sense that the anomaly, a purely quantum feature, arises due to the measure. This guess is correct as was shown by Fujikawa in his seminal work [116, 117], and we show in the following how the anomaly can be obtained from the Jacobian of the transformation.

Let us start with the Path Integral

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\int d^4x \bar{\psi}(i\mathcal{D}-m)\psi} . \quad (10.48)$$

We first perform a Wick rotation to Euclidean time which is done as follows: we take a vector $u = u^\mu \partial_\mu$ in a spacetime with metric $\eta = \text{diag}(-1, 1, 1, 1)$ such that the norm is $u \cdot u = \eta_{\mu\nu} u^\mu u^\nu$; we change the spacetime coordinate x^μ such that $x_E^4 = ix^0$ (the spatial components are unchanged $x_E^i = x^i$) and $\partial_0 = \frac{\partial}{\partial x^0} = i \frac{\partial}{\partial x^4} = i \partial_4$, and by defining $u_E^4 = iu^0$ we have $u = \sum_{0 \leq \mu \leq 3} u^\mu \partial_\mu = \sum_{1 \leq \mu \leq 4} u_E^\mu \partial_\mu$, and consequently the norm is $u \cdot u = \delta_{\mu\nu} u_E^\mu u_E^\nu$ where $\delta = \text{diag}(1, 1, 1, 1)$ is the Euclidean metric. The spacetime measure is $d^4x = dx^0 \wedge \dots \wedge dx^3 = idx^1 \wedge \dots \wedge dx^4 = id^4x_E$.

The same procedure has to be applied for all vectors, including the Dirac matrices γ^μ . There is a freedom in the choice of representation for the Dirac matrices, which obviously does not alter physical results, and we can find a representation such that they are hermitian [20, 118]

$$(\gamma^\mu)^\dagger = \gamma^\mu . \quad (10.49)$$

As a result, the Dirac operator $i\mathcal{D}$ is hermitian too:

$$\langle i\mathcal{D}\psi_1 | \psi_2 \rangle \equiv \int d^4x (i\mathcal{D}\psi_1)^\dagger(x) \psi_2(x) \quad (10.50)$$

$$= \int d^4x (i\mathcal{D}\psi_1(x) - V\psi_1(x))^\dagger \psi_2(x) \quad (10.51)$$

$$= \int d^4x \psi_1(x) (i\mathcal{D}\psi_2)(x) = \langle \psi_1 | i\mathcal{D}\psi_2 \rangle , \quad (10.52)$$

where V is some real abelian gauge field under which the fermions are charged (e.g QED).

Omitting the E subscript, the Path Integral is then given by

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int d^4x \bar{\psi}(i\mathcal{D}-m)\psi} . \quad (10.53)$$

Let us motivate the use of the Wick rotation to Euclidean time. There are several advantages to working in Euclidean time, the most important one is that formal quantities such as the Path Integral are more well-defined than in Minkowski signature. In Chapter 1 we mentioned that path Integrals with quadratic action can be computed in analogy with a gaussian integral, we show below that this analogy holds insofar as spacetime is Euclidean. Secondly, the Dirac operator becomes hermitian in Euclidean time, which will prove useful.

To compute a gaussian integral we start by diagonalising the matrix in the exponential, which we do here by introducing the eigenmodes of the Dirac operator

$$i\mathcal{D}\phi_n = \lambda_n \phi_n , \quad \lambda_n \in \mathbb{R} , \quad \int d^4x \phi_m^\dagger(x) \phi_n(x) = \delta_{mn} , \quad (10.54)$$

where the last two points follow from the hermiticity of $i\mathcal{D}$. Since the eigenmodes of the Dirac operator define a basis, we can expand the Path Integral variables as

$$\psi(x) = \sum_n a_n \phi_n , \quad \bar{\psi}(x) = \sum_n b_n \phi_n^\dagger . \quad (10.55)$$

Importantly, the a_n and b_n are just (Grassman) numbers and not functions. Note also that the eigenfunctions are fixed, hence integrating over all the possible functions $\psi(x)$ and $\bar{\psi}(x)$ amounts to integrating over all possible values for the a_n and b_n . This allows us to write the measure with the functional piece factored out. Let us show it by discretising spacetime

$$\mathcal{D}\psi \rightarrow \prod_x d\psi_x = \prod_x \sum_n \phi_{n,x} da_n = \det(\phi_{k,x})^{-1} \prod_n da_n . \quad (10.56)$$

$\phi_{k,x}$ can be understood as a matrix changing from the continuous basis $\psi(x)$ to the discrete basis a_n , this is how we can successfully transform the functional integration into a simple integration of Grassmann numbers. Likewise, we have

$$\mathcal{D}\bar{\psi} \rightarrow \det(\phi_{k,x}^\dagger)^{-1} \prod_n db_n , \quad (10.57)$$

and the Path Integral measure reads

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} = \left(\det \phi^\dagger \cdot \phi \right)^{-1} \prod_n da_n db_n = \prod_n da_n db_n , \quad (10.58)$$

where we used

$$\det(\phi^\dagger \cdot \phi)_{kl} = \det \int d^4x \phi_k^\dagger(x) \phi_l(x) = \det \delta_{kl} = 1 , \quad (10.59)$$

by virtue of the orthonormality of the eigenbasis. The Path Integral can then be computed as a gaussian integral over Grassman variables (Berezin integral)³

$$Z = \int \prod_n da_n db_n e^{-\sum_n (\lambda_n - m) a_n b_n} = \prod_n (\lambda_n - m) \equiv \det(i\mathcal{D} - m) . \quad (10.60)$$

³Note that the minus sign in the exponential, which follows from the Wick rotation, is what allows us to make the analogy with a gaussian integral $\int d\vec{x} e^{-\frac{1}{2}\vec{x}^T \cdot A \cdot \vec{x}} = \sqrt{\det(2\pi A^{-1})}$.

This functional determinant still needs to be regularised to obtain a perfectly well-defined quantity.

Without delaying any more, now that we have a definition of the measure, let us compute the Jacobian of the $U(1)_A$ transformation, which was our goal for this Section. We can expand the transformed variables (10.45) on the eigenbasis as in (10.55) with coefficients that we label a'_n and b'_n , which verify

$$\psi'(x) = \sum_n a'_n \phi_n = \sum_n a_n (1 + i\theta\gamma_5) \phi_n + \mathcal{O}(\theta^2) \quad (10.61)$$

$$\bar{\psi}'(x) = \sum_n b'_n \phi_n^\dagger = \sum_n b_n \phi_n^\dagger (1 + i\theta\gamma_5) \phi_n + \mathcal{O}(\theta^2), \quad (10.62)$$

for θ infinitesimal. By multiplying on the left/right by ϕ_m^\dagger/ϕ_m the first/second line, integrating over spacetime and using the orthonormality condition (10.54), we obtain the transformation matrices

$$a'_m = \sum_n A_{mn} a_n + \mathcal{O}(\theta^2), \quad A_{mn} = \delta_{mn} + \int d^4x \phi_m^\dagger i\theta\gamma_5 \phi_n \quad (10.63)$$

$$b'_m = \sum_n B_{mn} b_n + \mathcal{O}(\theta^2), \quad B_{mn} = \delta_{mn} + \int d^4x \phi_n^\dagger i\theta\gamma_5 \phi_m, \quad (10.64)$$

and at last the transformation of the measure

$$\mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \prod_n da'_n db'_n = \det A^{-1} \det B^{-1} \mathcal{D}\psi \mathcal{D}\bar{\psi}. \quad (10.65)$$

The determinant can be simplified using $\det = \exp \text{Tr} \log$ and Taylor expanding in θ

$$\det A^{-1} = \exp \left(- \sum_n \int d^4x i\theta \phi_n^\dagger \gamma_5 \phi_n \right) + \mathcal{O}(\theta^2) = \det B^{-1}, \quad (10.66)$$

and we finally obtain an expression for the Jacobian

$$\log J[\theta] = -2i \lim_{N \rightarrow \infty} \sum_{n=1}^N \int d^4x \theta(x) \phi_n^\dagger(x) \gamma_5 \phi_n(x). \quad (10.67)$$

This quantity is *a priori* divergent because the sum runs over arbitrary large eigenstates. It can be regularised by introducing a smooth function f that decreases sufficiently fast at infinity

$$f(0) = 1, \quad f(x) \xrightarrow{x \rightarrow \infty} 0, \quad x f'(x)|_{x=0} = x f'(x)|_{x \rightarrow \infty} = 0. \quad (10.68)$$

and a cut-off Λ such that modes below Λ are unaffected whereas those above are cut. The Jacobian is then obtained by taking the limit $\Lambda \rightarrow \infty$

$$\log J[\theta] = -2i \lim_{\Lambda \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{n=1}^N f\left(\frac{\lambda_n}{\Lambda}\right) \int d^4x \theta(x) \phi_n^\dagger(x) \gamma_5 \phi_n(x) \quad (10.69)$$

$$= -2i \lim_{\Lambda \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{n=1}^N \int d^4x \theta(x) \phi_n^\dagger(x) \gamma_5 f\left(\frac{i\not{D}}{\Lambda}\right) \phi_n(x), \quad (10.70)$$

where importantly, the limits do not commute. If that were the case, then the regulator would disappear since $\lim_{\Lambda \rightarrow \infty} f\left(\frac{\lambda_n}{\Lambda}\right) = 1$. It may come as a surprise that the Jacobian is regularised

by taking the limit $\Lambda \rightarrow \infty$, since in the end we do not cut any eigenmode. The reason is that, although it does not converge, the Jacobian is bounded, hence no infinities arise. This is similar to the difference of linearly divergent integrals that yield a finite result as seen in Sec. 10.1.1. We will delve further into this specific point in the next Sections.

Since the eigenmodes define a complete basis, the regularised Jacobian can simply be recast as a functional trace

$$J[\theta] = -2i \lim_{\Lambda \rightarrow \infty} \text{Tr} \theta \gamma_5 f \left(\frac{i\mathcal{D}}{\Lambda} \right). \quad (10.71)$$

To continue with the calculation, we need to pick a choice for f , for example we may take $f(x) = e^{-x^2}$ or $f(x) = 1/(1+x^2)$ which both verify (10.68). Note that the result is independent of the choice of function f [118]. Since a computation is detailed later on in this Chapter, we do not dwell on it. Without surprise, the result is in agreement with the diagrammatic approach and we find

$$\log J[\theta] = \frac{-i}{8\pi^2} \int d^4x \theta(x) F \tilde{F}. \quad (10.72)$$

After Wick rotating back to Minkowski, the factor $-i$ drops [20]. This computation can be generalised to non-abelian gauge fields (with θ remaining abelian for now), to obtain the non-abelian ABJ/singlet anomaly [20]. In (10.72) it amounts to replace $F \tilde{F}$ by $\text{tr} F \tilde{F}$ where $F_{\mu\nu} = i \partial_\mu A_\nu - i \partial_\nu A_\mu - [A_\mu, A_\nu]$.

Let us return to the pion-fermion Lagrangians \mathcal{L}_1 (10.1) and \mathcal{L}_2 (10.7). Although these theories are classically equivalent, at the quantum level the $U(1)_A$ transformation that relates them produces an anomaly that arises in the transformation of the Path Integral measure

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \mathcal{L}_1} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \left(\mathcal{L}_2 - \frac{i}{8\pi^2} \frac{\pi^0}{f_\pi} F \tilde{F} \right)}, \quad (10.73)$$

using $\theta = \pi^0/f_\pi$.

10.1.3 Infinities and anomalies

As mentioned above, it is rather striking that the anomaly is finite despite arising from divergent quantities. In the diagrammatic approach Sec. 10.1.1 it comes from the finite difference between two linearly divergent integrals, in Sec. 10.1.2 it comes as introducing a cut-off that is pushed to infinity in the end, such that nothing is really cut. As another example, consider (10.67) which can be formally written as a functional trace

$$\log J[\theta] = -2i \text{Tr} \theta \gamma_5. \quad (10.74)$$

The functional trace of a functional $F(x, y)$ is given by

$$\text{Tr} F = \int d^4x F(x, x). \quad (10.75)$$

In our case, $\theta \gamma_5$ is proportional to the identity $\delta(x - y)$ such that

$$\log J[\theta] = -2i \text{tr} \gamma_5 \int d^4x \theta(x) \delta(0). \quad (10.76)$$

This expression, though formal, gives an insight on why the anomaly may be UV finite. Clearly, the integral is an infinite quantity, however it is multiplied by $\text{tr} \gamma_5 = 0$, where tr denotes the trace

over internal indices. It is precisely this “ $0 \times \infty$ ” that gives a finite result once regularised. As pointed out by Fujikawa [118], the anomaly is an ill-defined series of the form

$$\log J[\theta] \sim +1 - 1 + 1 - 1 + 1 - 1 \dots , \quad (10.77)$$

due to the infinite number of degrees of freedom. Depending on how the calculation is ordered it may lead to different results. For example one may compute it as $(+1 - 1) + (+1 - 1) + \dots \stackrel{?}{=} 0$, and one may compute it as $1 + (-1 + 1) + (-1 + 1) \dots \stackrel{?}{=} 1$ and get a different result. This ambiguity is the reason why several results may be obtained, as seen in (10.38) and (10.39) with the dependence on the arbitrary choice of momentum routing, but each result corresponds to different physical situations.

10.1.4 Topological interpretation

Remarkably, the ABJ anomaly is independent from the mass of the quantum fermion that originates it, but only depends on the gauge sector. Another striking feature that we did not prove here is that it is one-loop exact, as was shown by Adler and Bardeen in [119]. Anomalies provide QFT amplitudes that we can straightforwardly fully compute, whereas usually physical observables are computed up to some order in perturbation theory. In that context we say the ABJ anomaly is non-perturbative. It was in the 70s that these aspects were better understood along with the topological nature of the ABJ anomaly [120–124], which we investigate in this Section.

Firstly, to be more rigorous, we consider the case where the spacetime manifold is compact. This ensures that the Dirac operator has a discrete spectrum with finite degeneracy. In the case where θ is constant in (10.67)⁴ we have

$$\log J[\theta] = -2i\theta \sum_n \int d^4x \phi_n^\dagger \gamma_5 \phi_n , \quad (10.78)$$

which is finite on a compact manifold. Using the anticommutation relation $\{i\cancel{D}, \gamma_5\} = 0$, we can easily show that if ϕ_n has eigenvalue λ_n , then $\gamma_5 \phi_n$ has eigenvalue $-\lambda_n$. Since ϕ_n and $\gamma_5 \phi_n$ have different eigenvalues for $\lambda_n \neq 0$, they are orthogonal (10.54) and we thus have

$$\langle \phi_n | \gamma_5 \phi_n \rangle = \int d^4x \phi_n^\dagger \gamma_5 \phi_n = 0 \text{ for } \lambda_n \neq 0 . \quad (10.79)$$

This implies that in the Jacobian (10.78) only the zero modes remain. Then, since $i\cancel{D}\phi_n = 0 \Rightarrow i\cancel{D}\gamma_5\phi_n = 0$, we can pick the zero modes to be eigenfunctions of γ_5 as well, which has eigenvalues $\{+1, -1\}$ depending on the chirality of the field. As a result, we have

$$\int d^4x \phi_n^\dagger \gamma_5 \phi_n = \begin{cases} 0 & \text{if } \lambda_n \neq 0 \\ \pm 1 & \text{if } \lambda_n = 0 \text{ and } \gamma_5 \phi_n = \pm \phi_n \end{cases} , \quad (10.80)$$

and this implies that

$$\log J[\theta] = -2i\theta (n_+ - n_-) , \quad (10.81)$$

where n_\pm is the number of (independent) zero modes of chirality \pm .

⁴Taking θ constant may seem trivial, but it is not. Perturbative (diagrammatic and Path Integral) computations require to have $\theta(|x| \rightarrow \infty) = 0$ for integrations by parts. The θ dependence on x is crucial in order to bring in the bulk the anomaly which is a boundary term (and a trivial one when zero modes are excluded) as we will see below. Therefore, by taking $\theta \neq 0$ and constant we assume that we can generalise our perturbative computation to a non-perturbative one (e.g include instanton configurations).

To express (10.81) in terms of a topological quantity, let us introduce the chiral projectors $P_{\pm} = (1 \pm \gamma_5)/2$ (also called $P_{R/L}$ throughout this manuscript), and the projected Dirac operators $i\mathcal{D}P_+$ and $i\mathcal{D}P_- = (i\mathcal{D}P_+)^\dagger$. It is easily proved that $i\mathcal{D}P_+\phi = 0 \Leftrightarrow P_+\phi = 0$ or $i\mathcal{D}\phi = 0$.⁵ In other words we have

$$\ker i\mathcal{D}P_+ = \ker i\mathcal{D} \cup \ker P_+, \quad (10.82)$$

and therefore

$$\dim \ker i\mathcal{D}P_+ = \dim \ker i\mathcal{D} + \dim \ker P_+ - n_-, \quad (10.83)$$

where

$$n_- = \dim(\ker i\mathcal{D} \cap \ker P_+), \quad (10.84)$$

is the number of zero modes of negative chirality. We follow the same procedure for $i\mathcal{D}P_-$ to find

$$\dim \ker i\mathcal{D}P_- = \dim \ker i\mathcal{D} + \dim \ker P_- - n_+. \quad (10.85)$$

Finally, using the fact that $\dim \ker P_+ = \dim \ker P_-$,⁶ we find

$$\frac{1}{-2i\theta} \log J[\theta] = n_+ - n_- = \dim \ker i\mathcal{D}P_+ - \dim \ker i\mathcal{D}P_- \equiv \text{Ind } i\mathcal{D}P_+, \quad (10.86)$$

where the index of a differential operator d is defined by

$$\text{Ind } d = \dim \ker d - \dim \ker d^\dagger. \quad (10.87)$$

When the index is finite (e.g Fredholm operator) and in compact Euclidean spacetimes, the Atiyah-Singer index theorem gives the index in terms of the Pontryagin topological invariant [20, 80]

$$\text{Ind } i\mathcal{D}P_+ = n_+ - n_- = \frac{1}{16\pi^2} \int d^4x \text{tr } F\tilde{F}, \quad (10.88)$$

such that we recover the ABJ anomaly (10.72) for a non-abelian gauge group G [20, 118]. This relation is remarkable since it relates $F\tilde{F}$ that is locally defined on a manifold, to a topological invariant $n_+ - n_-$ which is related to the global structure of the gauge group. A first step in understanding this is to realise that the Pontryagin density is in fact a boundary term

$$F\tilde{F} = 4 \text{tr } \partial_\mu C^\mu, \quad C^\mu = \epsilon^{\mu\nu\rho\sigma} \left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right), \quad (10.89)$$

where C^μ is the so-called Chern-Simons current [20, 80]. As a result, the Pontryagin density integrated over the spacetime manifold \mathcal{M} is

$$\int_{\mathcal{M}} d^4x \text{tr } F\tilde{F} = \int_{\partial\mathcal{M}} d^3x n \cdot C, \quad (10.90)$$

where we used Stokes theorem and n is the vector normal to the boundary $\partial\mathcal{M}$. In this formulation, we can understand that what is relevant to the anomaly is the configuration of the gauge group on the boundary, which in a physical set-up is at $|x| \rightarrow \infty$. Let us study more closely these

⁵ $P_+\phi = 0$ or $i\mathcal{D}\phi = 0 \Rightarrow i\mathcal{D}P_+\phi = 0$ is trivial since $i\mathcal{D}P_+ = P_-i\mathcal{D}$. For the converse, we suppose $i\mathcal{D}P_+\phi = P_-i\mathcal{D}\phi = 0$. It implies that $i\mathcal{D}\phi = 0$ or $i\mathcal{D}\phi$ is of positive chirality, since $i\mathcal{D}$ maps positive onto negative chiralities and conversely (it is a consequence of $\{i\mathcal{D}, \gamma_5\} = 0$) this second point means that ϕ is of negative chirality. Therefore $i\mathcal{D}P_+\phi = 0 \Rightarrow i\mathcal{D}\phi = 0$ or $P_+\phi = 0$.

⁶For $P_+ = \text{diag}(\mathbb{1}_2, 0)$ and thus $P_- = \text{diag}(0, \mathbb{1}_2)$, we can take $\phi = (0, 0, a, b)^T \in \ker P_+$, and the map $(0, 0, a, b)^T \rightarrow (a, b, 0, 0)^T$ from $\ker P_+$ to $\ker P_-$ is a bijection.

configurations of the gauge field at infinity. Firstly, for the integral (10.90) to be finite, we must clearly have

$$\lim_{|x| \rightarrow \infty} F \tilde{F} = 0 . \quad (10.91)$$

This, however, is not equivalent to having $\lim_{|x| \rightarrow \infty} A_\mu = 0$, but it is equivalent to having a so-called pure gauge configuration

$$A_\mu \xrightarrow{|x| \rightarrow \infty} g^{-1}(x) \partial_\mu g(x) , \quad (10.92)$$

where g is an element of the gauge group G . A simple configuration of the gauge field that respects this limit at infinity is the so-called BPST instanton [125] and has the form

$$A_\mu(x) = \frac{|x|^2}{|x|^2 + \lambda^2} g^{-1}(x) \partial_\mu g(x) , \quad (10.93)$$

where λ is the size of the instanton. The gauge element g effectively represents a mapping between the boundary of the spacetime manifold and the gauge group

$$g : \partial \mathcal{M} \longrightarrow G , \quad (10.94)$$

and classifies all the gauge configurations that are inequivalent on the boundary. The relevant equivalence class is the homotopy group of the map. Configurations that provide a non-trivial Pontryagin number are called instanton solutions and play a major role in non-abelian gauge theories. To summarise, instanton configurations of the gauge field are the configurations such that $\text{Ind} i \not{D} P_+ = n_+ - n_- \neq 0$. For more literature on the subject, our references are [20, 80, 118].

Finally, let us reflect on our comments from the first paragraph of this Section. From the fact that the anomaly depends only on the structure of the gauge group, we can understand that it cannot depend on the mass of the fermion. Secondly, the one-loop exactness can be understood from its topological origin. The anomaly is related to a topological *integer* (the Pontryagin number). Higher loop corrections to the anomaly would have to be small corrections, but at the same time integer quantities, these two points are incompatible.

10.2 Gauge Anomalies

In the previous Section, we introduced quantum anomalies by the ABJ anomaly. In a theory of vector-like fermions coupled to a gauge group, such as QED, the chiral symmetry $U(1)_A$ is broken at the quantum level. Since the axial symmetry is not gauged, we talk about a global symmetry, and hence a global anomaly. In this Section, we consider case where we have an axial gauge symmetry.

Consider the following theory

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^4x \bar{\psi} i \not{D} \psi} , \quad (10.95)$$

where the fermion is charged under both an $SU(N)_V$ and an $SU(N)_A$ gauge symmetry

$$D_\mu \psi = \partial_\mu \psi + i V_\mu \psi + i A_\mu \gamma_5 \psi , \quad (10.96)$$

with $V_\mu = V_\mu^a T^a$ and likewise for A , where T^a are the generators of the $SU(N)$ Lie algebra. It is invariant under the vector

$$\begin{cases} \psi + \delta_\theta^V \psi = e^{i\theta} \psi , & \bar{\psi} + \delta_\theta^V \bar{\psi} = \bar{\psi} e^{-i\theta} \\ \delta_\theta^V V_\mu = -(\partial_\mu \theta) - i[V_\mu, \theta] \\ \delta_\theta^V A_\mu = -i[A_\mu, \theta] \end{cases} . \quad (10.97)$$

and axial

$$\begin{cases} \psi + \delta_\theta^A \psi = e^{i\theta\gamma_5} \psi, & \bar{\psi} + \delta_\theta^A \bar{\psi} = \bar{\psi} e^{i\theta\gamma_5} \\ \delta_\theta^A V_\mu = -i[A_\mu, \theta] \\ \delta_\theta^A A_\mu = -(\partial_\mu \theta) - i[V_\mu, \theta] \end{cases}. \quad (10.98)$$

gauge transformations, of infinitesimal parameter $\theta(x) \in SU(N)$. This theory is not exotic at all since the fermions of the SM are charged under non-abelian chiral gauge symmetries. Both symmetries are conserved at the classical level and the associated conserved currents are

$$D_\mu j_V^\mu = 0, \quad D_\mu j_A^\mu = 0, \quad (10.99)$$

with $j_V^\mu = -i(\bar{\psi}\gamma^\mu T^a \psi)T^a$ and $j_A^\mu = -i(\bar{\psi}\gamma^\mu \gamma_5 T^a \psi)T^a$. Nonetheless, our knowledge of the ABJ anomaly makes us suspicious concerning their quantum counterparts

$$\langle D_\mu j_V^\mu \rangle \stackrel{?}{=} 0, \quad \langle D_\mu j_A^\mu \rangle \stackrel{?}{=} 0. \quad (10.100)$$

Our suspicions are founded since indeed, as for the ABJ anomaly, it is impossible to maintain both symmetries at the quantum level and we have a gauge anomaly. However, contrary to the ABJ anomaly where it made sense to enforce the conservation of the vector gauge symmetry and make the global axial symmetry anomalous, here both symmetries are gauged. This implies that the anomaly can equally be found in the vector or the axial symmetry.

The vector and axial gauge anomalies have properties in common with the ABJ anomaly. In particular, they are topological [126] and thus one-loop exact and UV finite. Different results may occur depending on the regularisation procedure, which correspond to different physical situations. In particular, the vector and axial symmetries are entangled at the quantum level. Further ambiguities occur which do not appear for the ABJ anomaly, that is to say even when one of the two symmetries is preserved at the quantum level, there remains some ambiguity in the computation, which gives rise to different forms of the anomalies.

10.2.1 Consistent vs covariant anomalies

Consider a theory with quantum effective action

$$\mathcal{W}[B] = -i \log \int \mathcal{D}\phi e^{iS[\phi, B]}, \quad (10.101)$$

where B is some background gauge field. The theory is classically invariant under the gauge transformation δ_θ^g

$$\delta_\theta^g S = \int d^4x \left(\delta_\theta^g \phi \frac{\delta S}{\delta \phi} + \delta_\theta^g B \frac{\delta S}{\delta B} \right) = 0, \quad (10.102)$$

where $\theta(x)$ is an infinitesimal parameter and $\theta(|x| \rightarrow \infty) = 0$. Since ϕ is solution to its EoM, the first term vanishes and we obtain the conserved gauge current

$$\delta_\theta^g B \frac{\delta S}{\delta B} = 0. \quad (10.103)$$

For example, with the theory (10.95) and the gauge transformations δ_θ^V and δ_θ^A we obtain Eq. (10.99) after integrating by parts.

If the gauge symmetry is not a symmetry of the quantum effective action, we have a gauge anomaly given by

$$\delta_\theta^g \mathcal{W}[B] = \int d^4x \delta_\theta^g B \frac{\delta \mathcal{W}}{\delta B} = \int d^4x \left\langle \delta_\theta^g B \frac{\delta S}{\delta B} \right\rangle \equiv \int d^4x \theta(x) \mathcal{A}_{\text{cons}}(x). \quad (10.104)$$

The anomaly as defined respects some conditions associated with the algebra of the gauge group as we shall prove. For any functional F , the algebra of the gauge group imposes the following condition

$$[\delta_{\theta_1}^g, \delta_{\theta_2}^g] F[B] = \delta_{[\theta_1, \theta_2]}^g F[B]. \quad (10.105)$$

If we apply this formula to the effective action, we obtain the so-called Wess-Zumino consistency conditions (WZcc) [127]

$$\delta_{\theta_1}^g G(\theta_2) - \delta_{\theta_2}^g G(\theta_1) = G([\theta_1, \theta_2]), \quad (10.106)$$

where used

$$G(\theta) = \int d^4x \theta(x) \mathcal{A}_{\text{cons}}(x), \quad (10.107)$$

as a shorthand. This anomaly is called the *consistent* anomaly. The interpretation of the WZcc (10.106) is very simple, it is a necessary and sufficient conditions for being integrable, i.e

$$f \text{ solution to the WZcc} \Leftrightarrow \exists F \text{ such that } \delta_\theta^g F = f. \quad (10.108)$$

The Wess-Zumino consistency conditions (10.106) take on a very simple form if we upgrade the gauge parameter θ to an anticommuting Fadeev-Popov “ghosts” v .⁷ The gauge algebra condition (10.105) simply becomes

$$\delta_v^2 = 0, \quad (10.109)$$

and the WZcc reduce to

$$\delta_v G(v) = 0. \quad (10.110)$$

Basically, the commutators that occur in (10.106) are automatically taken into account by the anticommutation of the ghosts.

For the vector-axial theory (10.95), the computation of the consistent anomaly of the axial current while the vector current remains conserved was first performed by Bardeen in [130] using Feynman diagrams. The computation is rather involved and we just provide the result here

$$\begin{aligned} \mathcal{A}_{\text{cons}}^a = & \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} T^a \left[F_{\mu\nu}^V F_{\rho\sigma}^V + \frac{1}{3} F_{\mu\nu}^A F_{\rho\sigma}^A \right. \\ & \left. + \frac{8}{3} (A_\mu A_\nu F_{\rho\sigma}^V + A_\mu F_{\nu\rho}^V A_\sigma + F_{\mu\nu}^V A_\rho A_\sigma) + \frac{32}{3} A_\mu A_\nu A_\rho A_\sigma \right], \end{aligned} \quad (10.111)$$

where the Bardeen curvatures are

$$\begin{aligned} F_{\mu\nu}^V &= i \partial_\mu V_\nu - i \partial_\nu V_\mu - [V_\mu, V_\nu] - [A_\mu, A_\nu] \\ F_{\mu\nu}^A &= i \partial_\mu A_\nu - i \partial_\nu A_\mu - [V_\mu, A_\nu] - [A_\mu, V_\nu]. \end{aligned} \quad (10.112)$$

⁷Ghosts are introduced in the context of Fadeev-Popov gauge field quantisation [87, 128] and BRS symmetry [129]. When a gauge field is quantised, it is necessary to introduce the ghost and gauge fixing terms. The latter breaks gauge invariance, however its upgraded version, the so-called BRS transformation, is a symmetry of the gauge fixed action.

For the particular case of a purely left-coupled gauge field

$$S = \int d^4x \bar{\psi}(i\cancel{D} - \cancel{L}P_L)\psi, \quad (10.113)$$

i.e taking $V = L/2$ and $A = -L/2$ in (10.96), the anomaly has almost the same expression as for the ABJ anomaly given in terms of the Chern-Simons current in (10.89)

$$\langle D_\mu j_{5,\mu}^a \rangle = \mathcal{A}_{\text{cons}}^a = \frac{1}{24\pi^2} \text{tr} T^a \epsilon^{\mu\nu\rho\sigma} \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma \right). \quad (10.114)$$

One may remark that the consistent anomaly is not covariant, and we may wonder what happens if we use a regularisation procedure that enforces covariance all throughout. In that case, the gauge anomaly that is obtained is indeed covariant and differs from (10.111)

$$\mathcal{A}_{\text{cov}}^a = \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} T^a (F_{\mu\nu}^V F_{\rho\sigma}^V + F_{\mu\nu}^A F_{\rho\sigma}^A). \quad (10.115)$$

This result was the one found by several authors, e.g Fujikawa in [131] (and [76] for gravitational anomalies).

Note that both the consistent and the covariant anomalies can be computed using the Fujikawa method. There is however an obstruction in that the Dirac operator

$$i\cancel{D} = i(\cancel{D} + i\cancel{V} + i\cancel{A}\gamma_5), \quad (10.116)$$

is not hermitian due to the presence of the axial field. The main difficulty consists in obtaining an hermitian operator that shares the same spectrum as the Dirac operator. This can be achieved in two manners (at least). One possibility is to perform an analytic continuation $A \rightarrow iA$ to restore the hermiticity of $i\cancel{D}$ [132–137]. This clearly does not modify the spectrum and thus provides an adequate hermitian operator to work with. The path integral variables can then be expanded using its eigenfunctions. This procedure leads to the consistent form of the anomaly. The other main approach is to use the hermitian conjugate of $i\cancel{D}$ to obtain a set of two hermitian operators $(i\cancel{D})^\dagger i\cancel{D}$ and $i\cancel{D}(i\cancel{D})^\dagger$ [138–141]. It is straightforward to show that these hermitian operators have the same spectrum as $i\cancel{D}$. They provide two sets of eigenfunctions, one is used to expand ψ while the other is used for $\bar{\psi}$. This procedure leads to the covariant form of the anomaly. Other similar methods have been applied in [131, 142].

The gauge anomalies also have a topological interpretation as was shown in [20, 142, 143], but we will not expand further on it.

In the next Section, we explain this ambiguity in the form of the gauge anomaly, and how the covariant and consistent anomalies are related to each other.

10.2.2 Ambiguities

There are several level of ambiguities in the computation of gauge anomalies.

Effective action counterterms The first one is related to the renormalisation of the theory. As emphasised earlier, the anomaly needs to be regularised to be properly computed, however it is finite after regularisation, i.e there are no infinities to remove by renormalisation. Nonetheless, other physical quantities that arise in the same theory may need the theory to be renormalised. The renormalisation of a theory is performed by introducing counterterms \mathcal{W}_{ct} at the level of the effective action \mathcal{W}

$$\mathcal{W}_{\text{ren}} = \mathcal{W} + \mathcal{W}_{\text{ct}}, \quad (10.117)$$

such that \mathcal{W}_{ren} is fully finite. These counterterms have to be local polynomials of the background fields. A gauge anomaly defined by the variation of the renormalised effective action may obtain contributions from the counterterms

$$\int d^4x \theta \mathcal{A} = \delta_\theta \mathcal{W}_{\text{ren}} = \delta_\theta \mathcal{W} + \delta_\theta \mathcal{W}_{\text{ct}} . \quad (10.118)$$

In general, we consider that an anomaly arising in $\delta_\theta \mathcal{W}$ that can be cancelled by a local polynomial in \mathcal{W}_{ct} is a spurious anomaly, that is to say it is not a true anomaly since a specific choice of renormalisation may cancel it. For example, suppose

$$\delta_\theta \mathcal{W} \supset \int d^4x \theta \mathcal{A}_{\text{spurious}} , \quad (10.119)$$

such that $\mathcal{A}_{\text{spurious}}$ integrates into a local polynomial P

$$\delta_\theta P = \int d^4x \theta \mathcal{A}_{\text{spurious}} , \quad (10.120)$$

then taking $\mathcal{W}_{\text{ct}} \supset -P$ will cancel $\mathcal{A}_{\text{spurious}}$ in $\delta_\theta \mathcal{W}_{\text{ren}}$. Spurious anomalies are said to be trivial solutions to the Wess-Zumino consistency conditions. A true anomaly is a non-trivial solution, i.e. it integrates into a non-local quantity such as the effective action.

It may not always be an easy task to find which piece of an anomaly is spurious and which is not. Suppose two computations of the same anomaly, with different renormalisation lead to two different results \mathcal{A}_1 and \mathcal{A}_2 . How can we determine whether they differ by a spurious anomaly or not, i.e.

$$\mathcal{A}_1 = \mathcal{A}_2 + \mathcal{A}_{\text{spurious}} ? \quad (10.121)$$

In a mathematical language, we are interested in the equivalence class $[\mathcal{A}_1]$, such that $\forall \mathcal{A}_2 \in [\mathcal{A}_1]$

$$\exists \mathcal{A}_{\text{spurious}} : \mathcal{A}_1 - \mathcal{A}_2 = \mathcal{A}_{\text{spurious}} . \quad (10.122)$$

In the ghost formalism (10.110), this simply translates to

$$\exists P \text{ local polynomial: } G_1(v) - G_2(v) = \delta_v P . \quad (10.123)$$

By luck, δ_v defines a boundary operator since $\delta_v^2 = 0$ (10.110), and therefore the set of all equivalence classes defined as above is a cohomology group, and they have been extensively studied in the context of anomalies [144–149].

When several symmetries are at play, such as for theory (10.95), the renormalisation counterterms will also shift the anomaly between different currents. For example, the Bardeen anomaly (10.111) is obtained while the vector symmetry is conserved, but we may as well obtain an anomalous vector symmetry and a conversed axial symmetry.

Current counterterms The choice of counterterms to the effective action may explain some discrepancies in different computations of anomalies, but not all of them. There are ambiguities that arise in the regularisation by itself, and that are decoupled from a choice of renormalising counterterms. This is the case of the consistent and covariant anomalies. In fact, Bardeen and Zumino showed in [150] that this ambiguity shows itself as the addition of a local polynomial at the level of the variation of \mathcal{W} , which is called the Bardeen-Zumino (BZ) polynomial \mathcal{P}

$$\int d^4x \theta \mathcal{A}_{\text{cov}} = \delta_\theta \mathcal{W} + \int d^4x \theta D_\mu \mathcal{P}^\mu , \quad (10.124)$$

where

$$\delta_\theta \mathcal{W} = \int d^4x \theta \mathcal{A}_{\text{cons}} . \quad (10.125)$$

Importantly, there exists no local polynomial P such that

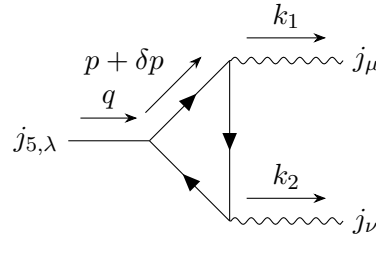
$$\delta_\theta P = \int d^4x \theta D_\mu \mathcal{P}^\mu . \quad (10.126)$$

This is the reason why the BZ polynomial is not equivalent to a choice of renormalising counterterm. In Sec. 11, we will compute the covariant and consistent anomalies from the Path Integral, using dimensional regularisation. We will in particular show that the BZ polynomial arises via the ambiguity in defining γ_5 in dimensional regularisation.

In the next Section, we give physical insight into each type of anomalies, and their implications in a quantum theory.

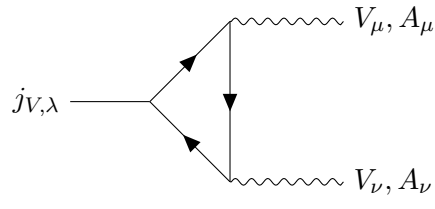
10.3 Physical interpretation of the gauge anomalies

Ambiguities in the calculation of anomalies should be fixed upon physical considerations, in particular by enforcing symmetries. Usually, we will enforce the conservation of the vector current. But as we saw above, this does not suffice to fix the ambiguities, and we may as well obtain the consistent or the covariant anomaly. It is clear that the covariant anomaly is obtained by enforcing the gauge covariance of the gauge anomaly. So which symmetry should be enforced in order to obtain the consistent anomaly? To answer this question, let us briefly return to the case of the ABJ anomaly. The anomaly is computed from $T_{\mu\nu\lambda}$ which is represented by the diagram



$$(10.127)$$

and the one with the legs j_μ and j_ν swapped, and δp is the arbitrary choice of momentum routing defined in (10.28). As we showed, the anomaly is distributed between $q^\lambda T_{\mu\nu\lambda}$ and $k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda}$. This means that the anomaly can be distributed among the different vertices of the diagram. The situation is similar for the gauge anomaly. The Bardeen anomaly is given by several diagrams (triangle, square and pentagons), but we can reason on the triangle diagrams only. Firstly, the vector gauge anomaly involves the triangle diagrams



$$(10.128)$$

This includes the diagrams with (V_μ, V_ν) , (V_μ, A_ν) , (A_μ, V_ν) or (A_μ, A_ν) as external legs. We can fix some of the ambiguity in the computation by choosing a regularisation and renormalisation schemes

that preserve the vector gauge symmetry $\langle D_\mu j_V^\mu \rangle = 0$, such that the above diagram vanishes, as well as the associated square and pentagon diagrams.

As explained earlier, there remains some ambiguity in the computation of $\langle D_\mu j_A^\mu \rangle$, which involves the triangle diagram

$$(10.129)$$

Since we want to preserve the vector gauge symmetry, the vertices that involve V should not be anomalous. This means that the anomaly should be distributed among the A vertices and the j_A vertex.

Physically, we may interpret the axial transformation δ_θ^A as an external global symmetry, i.e. not the gauge symmetry. In that case, we can preserve the axial gauge symmetry by putting all the anomaly in the external current j_A . This procedure leads to the covariant form of the anomaly, and is quite similar to the ABJ anomaly, since all the anomaly is directed to an external global symmetry.

On the other hand, if we consider that the axial transformation that we performed is the axial gauge transformation, then j_A should be treated on equal footing with the A legs. In that case, the anomaly should be evenly distributed between all the axial (A and j_A) vertices, which amounts to enforcing the Bose symmetry. This procedure leads to the consistent form of the anomaly.

In practice, this means that if we have a vector-axial theory (10.95) supplemented by a global $SU(N)$ symmetry (e.g. a non-abelian axion), then the theory should have a covariant anomaly. Such a global anomaly is not harmful for the theory. If there is no global symmetry, then the theory should have a consistent anomaly. Such a gauge anomaly is harmful for the theory, it implies a loss of unitarity of the theory. But let us be more precise, what actually happens is that the theory becomes non-renormalisable. One may wonder whether it is so harmful, after all we have encountered plenty of non-renormalisable theories which we called EFTs. So to be precise, a theory with a gauge consistent anomaly is only inconsistent above some cut-off [141, 151]. Schematically, the argument is as follows. Consider a UV theory with fermions chirally coupled to massless gauge bosons, and a Higgs field. The fermion species are such that all the gauge anomalies cancel. When the Higgs takes its VEV, some of the gauge fields become massive, as well as some of the fermions. In the IR, i.e. below the fermions mass $\sim m$, the theory is described by the EFT obtained after integrating out the massive fermions. The EFT is composed of the remaining light fermions, the massless and the massive bosons (their mass is $\sim gm < m$ where $g < 1$ is the coupling of the fermions to the gauge bosons), and effective operators inherited from the massive fermions. Among these effective operators there is indeed the gauge anomaly that cancels against the gauge anomaly of the light fermions. However, if we start from the IR, and have no knowledge of massive fermions, we will only see light fermions that have uncancelled gauge anomalies. According to the previous reasoning, if we also have massive gauge bosons in the IR, we should not directly conclude that the theory is inconsistent, but that the theory requires the presence of heavy chiral fermions that will cancel the gauge anomalies.

That being said, what conclusions can we draw on the SM? As we know, the SM fermions are chiral and thus produce gauge (consistent anomalies). Fortunately, the fermions species conspire beautifully to cancel any gauge anomaly that may occur. The SM is thus free from any harmful anomaly and is, as announced before, a renormalisable theory. This is truly remarkable that the three families of fermions are charged such that all the gauge anomalies cancel. This also implies

that any BSM model should preserve the gauge symmetries at the quantum level, lest it is itself an EFT of some higher energy theory. This provides very useful constraints when building BSM theories.

Chapter 11

Anomalies from an Effective Field Theory Perspective

In this Section, we present our work [4], in which we propose a novel method to compute anomalies within the path integral. Instead of computing directly the Jacobian of the transformation, as we did previously using the Fujikawa procedure, we identify the Jacobian as the ratio of two EFTs. We may then use the EFT methods presented in Part II, and in particular the CDE. This procedure allows to short-circuit considerations of Dirac operator spectrum and hermiticity. The ambiguities intrinsic to anomalies are then dealt with by enforcing the desired Ward identities.

In the following, we detail the method applied to the calculation of the ABJ anomaly. We then outline how the procedure should be applied to obtain covariant or consistent gauge anomalies.

11.1 Outline of the new method

In this section we explain our method for the simple case of the ABJ anomaly.

11.1.1 Functional determinant and Jacobian

Let us start with a Dirac fermion field involved in a vector gauge theory with the following path integral,

$$Z \equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left(i \int d^4x \bar{\psi} (i\cancel{\partial} - \cancel{V} - m) \psi \right) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} = \det [i\cancel{\partial} - \cancel{V} - m], \quad (11.1)$$

with V a gauge field, element of $SU(N) \equiv G$. Let us consider an infinitesimal chiral reparametrisation of the fermionic field, of parameter $\theta(x) \in SU(N)$, under which the measure produces a Jacobian

$$\psi \rightarrow e^{i\theta(x)\gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\theta(x)\gamma_5}, \quad \mathcal{D}\psi \mathcal{D}\bar{\psi} \rightarrow J[\theta] \mathcal{D}\psi \mathcal{D}\bar{\psi}. \quad (11.2)$$

Since the path integral is unchanged under a reparametrisation of its integration variables we obtain

$$\begin{aligned} Z &= \int J[\theta] \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left(iS - i \int d^4x \bar{\psi} \left[2im\theta\gamma_5 + (\cancel{D}\theta)\gamma_5 \right] \psi \right) \\ &= J[\theta] \det [i\cancel{\partial} - \cancel{V} - m - 2im\theta\gamma_5 - (\cancel{D}\theta)\gamma_5]. \end{aligned} \quad (11.3)$$

with $(\cancel{D}\theta) = (\cancel{\partial}\theta) + i[\cancel{V}, \theta]$ and the parenthesis indicates the local derivative. In the second line, we used the fact that the Jacobian is fully determined by the structure of the gauge groups of the

theory, hence does not depend on the fermionic field and can be commuted out of the integral. Using (11.1) and (11.3) we obtain the Jacobian expressed as the difference between two EFTs

$$\log J[\theta] = \log \det [i\mathcal{D} - m] - \log \det [i\mathcal{D} - m + i\{\theta\gamma_5, i\mathcal{D} - m\}] \equiv \int d^4x \mathcal{A}(x) . \quad (11.4)$$

11.1.2 The ABJ anomaly from the Covariant Derivative Expansion

For the ABJ anomaly, we take $\theta(x)$ to be abelian, and the ABJ anomaly thus reads

$$\int d^4x \mathcal{A}(x) = -\text{Tr} \log (i\mathcal{D} - m - 2im\theta\gamma_5 - (\not{\partial}\theta)\gamma_5) + \text{Tr} \log (i\mathcal{D} - m) . \quad (11.5)$$

For clarity, we will first present the evaluation of the first functional trace in (11.5) that we label \mathcal{A}_θ , before combining both needed to compute the ABJ anomaly \mathcal{A} . Using the CDE presented in Part II, we straightforwardly obtain

$$\mathcal{A}_\theta = \int \frac{d^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left[\frac{-1}{\not{q} + m} (-i\mathcal{D} + 2im\theta\gamma_5 + (\not{\partial}\theta)\gamma_5) \right]^n . \quad (11.6)$$

If we now apply the very same treatment to the other contribution, $\text{Tr} \log(i\mathcal{D} - m)$ of (11.5), in order to evaluate the anomaly, we find that the terms which do not involve the θ parameter cancel with each other

$$\mathcal{A} = \int \frac{d^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left[\frac{-1}{\not{q} + m} (-i\mathcal{D} + 2im\theta\gamma_5 + (\not{\partial}\theta)\gamma_5) \right]^n \Bigg|_{\text{carrying } \theta \text{ dependence}} . \quad (11.7)$$

Alternatively, it can be written as ¹

$$\mathcal{A} = \int \frac{d^d q}{(2\pi)^d} \text{tr} (2im\theta\gamma_5 + (\not{\partial}\theta)\gamma_5) \sum_{n=0}^{\infty} \left[\frac{-1}{\not{q} + m} (-i\mathcal{D}) \right]^n \frac{-1}{\not{q} + m} . \quad (11.8)$$

Since we are only interested in the terms linear in θ in Eq. (11.7), the anomaly can be expressed as $\mathcal{A} = \mathcal{A}^{m\gamma_5} + \mathcal{A}^{\not{\partial}\gamma_5}$ with

$$\begin{aligned} \mathcal{A}^{m\gamma_5} &= \int \frac{d^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left[\frac{-1}{\not{q} + m} (-i\mathcal{D} + 2im\theta\gamma_5) \right]^n \Bigg|_{\mathcal{O}(\theta)} , \\ \mathcal{A}^{\not{\partial}\gamma_5} &= \int \frac{d^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left[\frac{-1}{\not{q} + m} (-i\mathcal{D} + (\not{\partial}\theta)\gamma_5) \right]^n \Bigg|_{\mathcal{O}(\theta)} . \end{aligned} \quad (11.9)$$

The terms which contribute to \mathcal{A} involve, here, exactly one γ_5 matrix and there can be no contribution from orders greater than $n = 5$, within the CDE approach, since they would carry a mass dependence. An explicit proof would rely on the fact that mass dependent terms, both coming from $\mathcal{A}^{m\gamma_5}$ and $\mathcal{A}^{\not{\partial}\gamma_5}$, are all finite.

As expected from the computation of anomalies, some of the integrals in (11.9) are divergent. They are regularised using dimensional regularisation $d = 4 - \epsilon$. The Dirac traces have to be performed in $d = 4 - \epsilon$ as well, and they will yield ϵ factors, that will cancel the poles $1/\epsilon$ from

¹This follows from using the cyclicity trace, which remains under control even in ambiguous computation, thanks to our procedure for γ_5 in dimensional regularisation, as will see later on.

the integrals to produce the expected finite anomaly. Note that the treatment of γ_5 in dimensional regularisation is ambiguous [104], since it is an intrinsically 4-dimensional object. There exists different schemes that may produce different outcomes (see for examples Refs. [152, 153]). This will be the heart of the computation of the gauge anomalies, but for now we simply adopt a specific scheme, the Breitenlohner-Maison-'t Hooft-Veltman (BMHV) scheme, to treat γ_5 . This scheme is known to be compatible with the conservation of the vector current.

We follow through with the CDE presented in Part II. $\mathcal{A}^{\not{\partial}\gamma_5}$ contains both a finite

$$\mathcal{A}_{n=4,\text{fin}}^{\not{\partial}\gamma_5} = -\frac{1}{32\pi^2} \text{tr}(\not{D}\not{D}\not{D}(\not{\partial}\theta)\gamma_5) = \frac{i}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}(D_\mu D_\nu D_\rho (\partial_\sigma \theta)), \quad (11.10)$$

and a divergent contribution

$$\mathcal{A}_{n=4,\text{div}}^{\not{\partial}\gamma_5} = \frac{-i}{16\pi^2} \left(\frac{2}{\epsilon} + \mathcal{O}(\epsilon) \right) \epsilon^{\mu\nu\rho\sigma} \text{tr}(D_\mu D_\nu D_\rho (\partial_\sigma \theta)) \xrightarrow{\epsilon \rightarrow 0} -\frac{i}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}(D_\mu D_\nu D_\rho (\partial_\sigma \theta)), \quad (11.11)$$

such that it actually vanishes

$$\mathcal{A}_{n=4}^{\not{\partial}\gamma_5} = \mathcal{A}_{n=4,\text{div}}^{\not{\partial}\gamma_5} + \mathcal{A}_{n=4,\text{fin}}^{\not{\partial}\gamma_5} = 0. \quad (11.12)$$

Note that by finite and divergent, we mean that there are finite or divergent integrals, but indeed the result is finite.

On the other hand, $\mathcal{A}^{m\gamma_5}$ only has finite integrals and reads

$$\mathcal{A}_{n=5}^{m\gamma_5} = -\frac{i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \theta \text{tr}(F_{\mu\nu} F_{\rho\sigma}), \quad (11.13)$$

where the convention for the field strength is $F_{\mu\nu} = [D_\mu, D_\nu]$.

Within the CDE approach, this is the only surviving contribution, and it matches the well-known result for the axial current anomaly in a vector gauge field theory [116, 118, 130]

$$\mathcal{A} = \mathcal{A}^{m\gamma_5} + \mathcal{A}^{\not{\partial}\gamma_5} = \mathcal{A}_{n=5}^{m\gamma_5} = -\frac{i}{8\pi^2} \theta \text{tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}), \quad (11.14)$$

where the convention for the dual tensor is $\tilde{F}^{\mu\nu} = 1/2 \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, with the choice $\epsilon^{0123} = +1$.

One may be a bit surprised by the fact that the anomaly ends up extracted from a non-divergent integral, for which no regularisation is needed. Let us stress though that the crucial step was to show that the $\mathcal{A}^{\not{\partial}\gamma_5}$ term gives no contribution in that particular case at order m^0 .

Following a similar strategy, we will now discuss more generalities and details of the evaluation of the covariant and consistent anomalies in QFT based on a careful regularisation.

11.2 Anomalies in vector-axial gauge field theory

In this Section, we are interested in the computation of the covariant and consistent gauge anomalies. This time, we carefully deal with the ambiguity in the definition of γ_5 in dimensional regularisation, which we fix by enforcing the adequate symmetry.

11.2.1 Definiteness and regularisation

We consider the vector-axial theory

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - \not{V} - \not{A}\gamma_5 - m)\psi, \quad (11.15)$$

with V_μ and A_μ elements of $SU(N)$. Note that we have now introduced a mass for the chiral fermion, which is necessary for the CDE. The mass term is a hard breaking source of axial symmetries (local or global). In order to make manifest those symmetries at tree-level one can evidently implement their spontaneous breaking introducing then their associated Goldstone bosons. We chose to work within the unitary basis and loose manifest tree-level axial invariance in order to deal with simpler functional determinants. Consequently, one should not be surprised if we discuss an anomalous global symmetry which looks naively already broken at tree-level.²

Under the axial reparametrisation of the fermions with $\theta(x) \in SU(N)$, we obtain the Jacobian

$$J[\theta] = \frac{\det(i\not{D} - m)}{\det(i\not{D} - m - (\not{D}\theta)\gamma_5 - 2im\theta\gamma_5)}. \quad (11.16)$$

As emphasised in Sec. 10.2.1, depending on how this Jacobian is handled, it may lead to different results, depending on which symmetry is enforced. We may in principle have the anomaly in the vector or the axial current, and we may obtain a result that is covariant or respects the Wess-Zumino consistency conditions.

We will propose two methods of regularising the Jacobian (11.16). The first method consists in working with the formal determinant in dimensional regularisation and, throughout the computation, deal with the ambiguity related to γ_5 using free parameters [75, 154]. The second method consists in bosonising the determinant, making it finite, hence fixing the ambiguity before the calculation. The first method can be seen as more general (or maybe naïve and brutal) as one first regularises an ill-defined quantity inserting as much freedom as needed and secondly call for coherence (covariance, integrability). We believe that a remarkable advantage of this approach is that its derivation is smooth and self-consistent within the path integral formalism. The second method works the opposite way, as one firstly calls for a well defined theory (free of any ambiguity) and secondly perform the regularisation. As we will see, each have their own advantages and disadvantages and we find it illuminating to present them both. We should also notice that while we believe the first method is novel in its approach, the bosonisation method is well-known [20, 69, 140], however its combined used with the CDE to evaluate anomalies, is new and since this offers a powerful tool and interesting implications for EFTs related topics, it deserves to be duly studied here.

11.2.2 Ambiguities and free parameters

In d dimensions, γ_5 is ill-defined. One cannot maintain both the cyclicity of the trace and Clifford algebra. There exist many ways of defining γ_5 in d dimensions consistently [104, 105, 152–154], although they may yield different results. The ambiguity in the Jacobian of Eq. (11.16) lies in the dependence on the choice of the γ_5 regularisation scheme.

In a diagrammatic approach, the amplitude of a diagram is dictated by the Feynman rules. However, it does not specify by which vertex we should start writing the amplitude of the diagram, which results in different possible positions for γ_5 in the traces. Since in d dimensions, the different positions of γ_5 are not equivalent, we have an ambiguity in its position.

Nonetheless, it is possible to compute traces of γ_5 in d dimensions while keeping track of the ambiguity by introducing free parameters [154]. We outline the method in the following.

Consider the trace

$$\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma). \quad (11.17)$$

In 4 dimensions, one can use Clifford algebra to move the γ_5 at different positions,

$$\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = \text{tr}(\gamma^\mu \gamma^\nu \gamma_5 \gamma^\rho \gamma^\sigma) = \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5). \quad (11.18)$$

²A detailed discussion on the parametrisation of local and global anomalous symmetries can be find in Ref. [75].

However, this may not be true anymore in d dimensions. For example if we use BMHV scheme [105], we maintain the cyclicity of the trace but we have to abandon Clifford algebra. The trick presented in Ref. [154] consists in implementing all the positions for γ_5 that are equivalent in 4 dimensions, with a free parameter for each

$$\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) \rightarrow \alpha \text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) + \beta \text{tr}(\gamma^\mu \gamma^\nu \gamma_5 \gamma^\rho \gamma^\sigma) + \delta \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5), \quad (11.19)$$

with the condition $\alpha + \beta + \delta = 1$, so that we recover $\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)$ in 4 dimensions.

The introduction of those free parameters with all the equivalent positions (in 4 dimensions) of γ_5 makes the trace regularisation scheme independent. Therefore, we can choose a specific scheme to compute each separate trace. If the result depends on the free parameters in the end, it means that the initial trace itself is ambiguous.

For the example above, we compute each separate trace using the BMHV scheme to get

$$\begin{aligned} \text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) &\rightarrow \alpha \text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) + \beta \text{tr}(\gamma^\mu \gamma^\nu \gamma_5 \gamma^\rho \gamma^\sigma) + \delta \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) \\ &= (\alpha + \beta + \delta)(-4i\epsilon^{\mu\nu\rho\sigma}) = -4i\epsilon^{\mu\nu\rho\sigma}, \end{aligned} \quad (11.20)$$

where we have used the condition $\alpha + \beta + \delta = 1$ to match with the result in 4 dimensions. It turns out that this trace is non-ambiguous.

However, consider the following trace with one contraction among the Dirac matrices

$$\begin{aligned} &\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma^\sigma \gamma_a) \\ &\rightarrow \alpha \text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma^\sigma \gamma_a) + \beta \text{tr}(\gamma^\mu \gamma^\nu \gamma_5 \gamma^\alpha \gamma^\rho \gamma^\sigma \gamma_a) + \gamma \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_5 \gamma^\sigma \gamma_a) + \delta \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma^\sigma \gamma_a \gamma_5) \\ &= (-1 + 2\gamma) 4i(d-4)\epsilon^{\mu\nu\rho\sigma}. \end{aligned} \quad (11.21)$$

It is ambiguous because even after enforcing the condition $\alpha + \beta + \gamma + \delta = 1$, the result still depends on a free parameter. Actually, insofar as there is more than one contraction among the Dirac matrices, the trace will be ambiguous. As a consequence, when computing the anomaly, the final result depends on free parameters. Those free parameters are then fixed under physical constraints, for example by enforcing gauge invariance and vector current conservation.

Although the positions of γ_5 in the computation of the path integral Jacobian are not arbitrary, as opposed to the diagrammatic approach, it may still bear traces that depends on the choice of γ_5 scheme. Despite the absence of arbitrariness in the position of γ_5 we will still rely on the free parameters trick to compute the ambiguous Jacobian, since it allows us to compute the traces in a γ_5 scheme independent way.

11.2.3 A well-known treatment : the bosonisation

Before delving into the expansion of the determinant, it is possible to regularise it. One way of achieving a regularised Jacobian is to bosonise it. The Dirac operator for the vector-axial theory is

$$i\mathcal{D} - m = i\mathcal{D} - \mathcal{V} - \mathcal{A}\gamma_5 - m. \quad (11.22)$$

This operator does not have a well-defined eigenvalue problem, the presence of the axial field spoils the hermiticity. It is however crucial to have a well-defined eigenvalue problem to make sense of the determinant, which is the product of the eigenvalues of the operator. One way to obtain a hermitian operator is to use the following Laplace operators

$$\mathcal{D}^\dagger \mathcal{D} \text{ and } \mathcal{D} \mathcal{D}^\dagger. \quad (11.23)$$

which thus have a well-defined eigenvalue problem. They preserve the spectrum of the theory (see for example Ref. [20]), hence do not change the value of the determinants (aside squaring them). The Jacobian, or rather its modulus, can thus be expressed as

$$|J[\theta]|^2 = \frac{\det(-(i\mathcal{D})^\dagger i\mathcal{D} + m^2)}{\det(-(i\mathcal{D})^\dagger i\mathcal{D} + m^2 + f(\theta))}, \quad (11.24)$$

where

$$f(\theta) = 4im^2\theta\gamma_5 - i[\theta, -D^2]\gamma_5 - \frac{1}{2}[\sigma.F^V, \theta]\gamma_5 - \frac{1}{2}[\sigma.F^A\gamma_5, \theta]\gamma_5. \quad (11.25)$$

F^V and F^A are the Bardeen curvatures defined in (10.112).

In [4], we showed by direct computation that this bosonised determinant is finite and hence unambiguous, and leads to the covariant form of the anomaly. The fact that it is the phase and not the modulus of the determinant which is ambiguous is well-known [76].

On the other hand, if we want to compute the consistent anomaly, we can try to use the bosonisation as in the vector gauge theory. However, the operator $\mathcal{D}^2 + m^2$ is still not hermitian. We palliate this problem using the analytic continuation $A_\mu \rightarrow iA_\mu$ that restores the hermiticity of $i\mathcal{D}$, hence of $\mathcal{D}^2 + m^2$. The Jacobian will then be written as

$$J[\theta]^2 = \frac{\det(\mathcal{D}^2 + m^2)}{\det(\mathcal{D}^2 + m^2 + \{i\mathcal{D}, (\mathcal{D}\theta)\gamma_5\} + 4im^2\theta\gamma_5)}. \quad (11.26)$$

Unfortunately, this does not suffice to fix the ambiguity, and does not necessarily lead to the consistent anomaly.

We do not dwell further on the bosonisation and refer the interested reader to the article instead [4]. Similarly, we put the focus on anomalies with conserved vector symmetry and anomalous axial symmetry, even though the converse is treated in [4].

11.3 Covariant anomaly

The Jacobian produced by the axial reparametrisation in the vector-axial theory is

$$J[\theta] = \frac{\det(i\mathcal{D} - m)}{\det e^{i\theta(x)\gamma_5} (i\mathcal{D} - m) e^{i\theta(x)\gamma_5}} = \frac{\det(i\mathcal{D} - m)}{\det(i\mathcal{D} - m - 2im\theta\gamma_5 - (\mathcal{D}\theta)\gamma_5)}. \quad (11.27)$$

As emphasised in the previous sections, this Jacobian is ill-defined. The next step is to explicitly compute it, according to the methods proposed in section 11.2.1.

We are now in the situation where we are looking to evaluate an equivalent of (11.7) for a vector and axial gauge field theory

$$\mathcal{A} = \int \frac{d^d q}{(2\pi)^d} \text{tr} \left((\mathcal{D}\theta)\gamma_5 + 2im\theta\gamma_5 \right) \sum_{n \geq 0} \left[\frac{-1}{\not{q} - m} (-i\mathcal{D}) \right]^n \frac{-1}{\not{q} - m}, \quad (11.28)$$

where θ belongs to $SU(N)$ and the covariant derivative is $D_\mu = \partial_\mu + iV_\mu + iA_\mu\gamma_5$.

Let's start by computing contributions from the mass term $2im\theta\gamma_5$, which is finite and unambiguous. It turns out to contribute for exactly the Bardeen anomaly (with conserved vector current), that is to say the consistent anomaly

$$\begin{aligned} \mathcal{A}^{m\gamma_5} = \frac{-i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \theta^a T^a \left(F_{\mu\nu}^V F_{\rho\sigma}^V + \frac{1}{3} F_{\mu\nu}^A F_{\rho\sigma}^A + \frac{8}{3} (A_\mu A_\nu F_{\rho\sigma}^V + A_\mu F_{\nu\rho}^V A_\sigma + F_{\mu\nu}^V A_\rho A_\sigma) \right. \\ \left. + \frac{32}{3} A_\mu A_\nu A_\rho A_\sigma \right) = \mathcal{A}^{\text{Bardeen}}. \end{aligned} \quad (11.29)$$

Now let us focus on the derivative term $(\not{D}\theta)\gamma_5$

$$\mathcal{A}^{\not{D}\gamma_5} = \int \frac{d^d q}{(2\pi)^d} \text{tr} (\not{D}\theta)\gamma_5 \sum_{n \geq 0} \left[\frac{-1}{\not{q} - m} (-i\not{D}) \right]^n \frac{-1}{\not{q} - m}. \quad (11.30)$$

Among others, the following term is involved

$$i\mathcal{I}[q^4]^4 \text{tr} \left((\not{D}\theta)\gamma_5 [\gamma^a \not{D}\gamma^b \not{D}\gamma^c \not{D}\gamma^d g_{abcd}] \right), \quad \mathcal{I}[q^4]^4 g_{\mu\nu\rho\sigma} = \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu q_\nu q_\rho q_\sigma}{(q^2 - m^2)^4}, \quad (11.31)$$

and is divergent. $g_{\mu\nu\rho\sigma}$ is the fully symmetrised metric. Therefore, the associated trace is ambiguous. We use the trick described in section 11.2.2 to keep track of the ambiguity, and all the contributions from the derivative term may be expressed as a sum of operators with a free parameter for each (after integrating by parts). The result can thus be written fully in terms of free parameters associated with each possible operator (the finite contributions will just combine with a fixed parameter that can be absorbed in the free parameter). We may therefore write

$$\mathcal{A}^{\not{D}\gamma_5} = \frac{-i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \theta^a T^a \left(\sum_i a_i X_{i, \mu\nu\rho\sigma} \right), \quad (11.32)$$

where X_i are all the possible operators of the form $\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4$ with $\mathcal{O}_{1 \leq i \leq 4} \in \{V, A, \partial\}$ that can be formed, provided it has an even number of A fields (the number of γ_5 must be odd). Note that the operators with a partial derivative to the right vanish, and those with consecutive partial derivatives vanish due to the contraction with the ϵ tensor. This leaves us with 22 possible operators and as many free parameters a_i . We then fix them by enforcing the covariance of $\mathcal{A}^{m\gamma_5} + \mathcal{A}^{\not{D}\gamma_5}$ under the vector and axial gauge transformations

$$\begin{cases} V_\mu \rightarrow V_\mu + (D_\mu^V \epsilon_V) + i[A_\mu, \epsilon_A] \\ A_\mu \rightarrow A_\mu + i[A_\mu, \epsilon_V] + (D_\mu^V \epsilon_A) \end{cases}, \quad (11.33)$$

where $D^V = \partial + iV$. In fact it turns out that considering only the axial gauge transformation associated with ϵ_A will prove sufficient to fix the free parameters.

The constraints are easily obtained by enforcing that the operators must transform covariantly, that is to say as

$$\delta_{\epsilon_A}^A \mathcal{O} = [\epsilon_A, \mathcal{O}]. \quad (11.34)$$

In practice, we enforce that the terms with derivatives of ϵ_A vanish, and also that ϵ_A must appear either at the beginning or at the end of each operator. It turns out that enforcing these conditions fixes 21 free parameters out of 22. We label β the remaining free parameter. To fix β , we may consider the limit $A \rightarrow 0$. This scenario corresponds to having an axial global (non-abelian) symmetry in a vector-like theory, and the anomaly should then take the same form as the ABJ anomaly. Setting $A = 0$ we are left with

$$\frac{-i}{16\pi^2} (1 + \beta) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^V|_{A=0} F_{\rho\sigma}^V|_{A=0}, \quad (11.35)$$

with $F_{\mu\nu}^V|_{A=0} = i\partial_\mu V_\nu - i\partial_\nu V_\mu - [V_\mu, V_\nu]$. Eq. (11.35) is covariant regardless of the normalisation, this is why it needs to be compared with the anomaly in a vector-like theory to fix β . Note that this procedure amounts to enforcing the conservation of the vector current. We thus deduce that $\beta = 0$ and, restoring A , we obtain the covariant anomaly in the axial current

$$\mathcal{A} = \mathcal{A}^{m\gamma_5} + \mathcal{A}^{\not{D}\gamma_5} = \frac{-i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \theta^a T^a (F_{\mu\nu}^V F_{\rho\sigma}^V + F_{\mu\nu}^A F_{\rho\sigma}^A), \quad (11.36)$$

with F^V and F^A the Bardeen curvatures as defined in (10.112). Note that the relative coefficient between $F^V \tilde{F}^V$ and $F^A \tilde{F}^A$ was fixed by requiring the covariance of the result, since $F^V \tilde{F}^V + b F^A \tilde{F}^A$ is not covariant unless $b = 1$.

The ambiguity in the derivative term $\mathcal{A}^{\not{D}\gamma_5}$ was fixed by requiring that the mass term and derivative term together are gauge covariant, that is to say, that the gauge variation of the derivative term cancels exactly the gauge variation of the unambiguous mass term. Since the mass term coincides with the consistent anomaly, then the derivative term is by definition the divergence of the BZ polynomial introduced in Sec. 10.2.2

$$\begin{aligned} \mathcal{A}^{\not{D}\gamma_5} &= \theta^a (D_\mu \mathcal{P}^\mu)^a \\ &= \frac{-i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \theta \left(\frac{2}{3} F_{\mu\nu}^A F_{\rho\sigma}^A + \frac{8}{3} (A_\mu A_\nu F_{\rho\sigma}^V + A_\mu F_{\nu\rho}^V A_\sigma + F_{\mu\nu}^V A_\rho A_\sigma) - \frac{32}{3} A_\mu A_\nu A_\rho A_\sigma \right). \end{aligned} \quad (11.37)$$

11.4 Consistent anomaly

We now apply the same procedure, except instead of enforcing the covariance of the result, we enforce the WZcc. The Jacobian of the axial reparametrisation is given in (11.28), and we recall it here

$$\mathcal{A} = \int \frac{d^d q}{(2\pi)^d} \text{tr} (2im\theta\gamma_5 + (\not{D}\theta)\gamma_5) \sum_{n \geq 0} \left[\left(\frac{-1}{\not{q} + m} \right) (-i\not{D}) \right]^n \left(\frac{-1}{\not{q} + m} \right). \quad (11.38)$$

As we saw above, the mass term matches the Bardeen anomaly

$$\mathcal{A}^{m\gamma_5} = \mathcal{A}^{\text{Bardeen}}. \quad (11.39)$$

With the same arguments, the divergent contributions from $(\not{D}\theta)\gamma_5$ can be put under the form

$$\mathcal{A}^{\not{D}\gamma_5} = \frac{-1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \theta^a T^a \left(\sum_i a_i X_{i\mu\nu\rho\sigma} \right), \quad (11.40)$$

where X_i are all the possible operators of the form $\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4$ with $\mathcal{O}_{1 \leq i \leq 4} \in \{V, A, \partial\}$ that can be formed, provided it has an even number of A fields (the number of γ_5 must be odd). This leaves us with 22 possible operators, and their 22 associated free parameters a_i .

We outline the procedure here. We first take the limit $A \rightarrow 0$ as we did previously. In that case, we enforce the conservation of the vector gauge symmetry, that is to say we fix the parameters such that all the vector gauge fields are stowed inside curvatures $F_{\mu\nu}^V|_{A=0} = i \partial_\mu V_\nu - i \partial_\nu V_\mu - [V_\mu, V_\nu]$. We then restore the axial gauge field A . As emphasised in their original paper [127], the Wess-Zumino consistency conditions fix uniquely all the coefficients of the operators with respect to the coefficient of the $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^V F_{\rho\sigma}^V$ term. This allows us to directly conclude that upon enforcing the vector gauge invariance and the Wess-Zumino consistency conditions, we must have

$$\mathcal{A}_{\not{D}\gamma_5} = \alpha \mathcal{A}^{\text{Bardeen}}, \quad (11.41)$$

where we still have one free parameter left, α .

Now let's put together the contributions from the mass term and the derivative term

$$\mathcal{A} = (1 + \alpha) \mathcal{A}^{\text{Bardeen}}. \quad (11.42)$$

The Wess-Zumino consistency conditions allow us to fix the coefficients of all the operators with respect to the coefficient of the term $F_{\mu\nu}^V F_{\rho\sigma}^V$, this is why we still have a remaining freedom at

the end. The coefficient of $F_{\mu\nu}^V F_{\rho\sigma}^V$ can be fixed by comparing the result with the anomaly in a vector-like gauge theory as suggested in [127]. That is to say, we set $A \rightarrow 0$ again, and can therefore identify our result with the ABJ anomaly (with $\theta \in SU(N)$), which immediately sets α to zero, leaving the expected result.

11.5 Scale anomaly

Let us pave the way for the next Section and introduce the scale anomaly.

It is well-known that there are two main categories of symmetries which are broken by the quantisation of a theory. The first is the axial symmetry associated with Dirac's γ_5 , the chiral anomaly, that we have just treated in details. The other is the scale transformation, which changes the length scale of space-time, keeping local angles invariant [155–159]. In the next Section, we will detail the difference between the scale, conformal and Weyl transformations. Here, we propose to evaluate the scale anomaly following the prescription described in Section 11.1 and for pedagogical reasons we stick to the case of QED

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - \cancel{V} - m)\psi - \frac{1}{4e^2}F^2. \quad (11.43)$$

Scale invariance is classically broken by the fermion mass term, and the divergence of the Noether current J^μ associated to the scale transformation, i.e the trace of the symmetric energy-momentum tensor $\tilde{T}^\mu{}_\mu$, reads

$$\partial_\mu J^\mu = \tilde{T}^\mu{}_\mu = m\bar{\psi}\psi. \quad (11.44)$$

This relation is also broken at the quantum level by the renormalisation of the coupling e . The scale transformation $x_\mu \rightarrow x'_\mu = e^\sigma x_\mu$ induces

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &\rightarrow \frac{\partial}{\partial x'^\mu} = e^{-\sigma} \frac{\partial}{\partial x^\mu}, \\ d^d x &\rightarrow d^d x' = e^{d\sigma} d^d x, \end{aligned} \quad (11.45)$$

and the fields transformation

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x') = e^{-(d-1)\sigma/2} \psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x') = e^{-(d-1)\sigma/2} \bar{\psi}(x), \\ A_\mu(x) &\rightarrow A'_\mu(x') = A_\mu(x) = e^{-\sigma} A_\mu(x), \end{aligned} \quad (11.46)$$

where d is the dimension of space-time. Note that the gauge field does not transform by itself, it only transforms due to its dependence on x [118].

Using the invariance of the path integral under the relabelling of the path integral variables, and the invariance of the space-time integral under relabelling the spacetime variable, we may write

$$\int (\mathcal{D}\psi)'(\mathcal{D}\bar{\psi})' \exp\left(i \int d^d x' \mathcal{L}[x', \psi'(x'), A_\mu(x')]\right) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(i \int d^d x \mathcal{L}[x, \psi(x), A_\mu(x)]\right). \quad (11.47)$$

On the other hand we know how the action transforms, and we can assume that the transformation of the measure produces a Jacobian

$$\begin{aligned} &\int (\mathcal{D}\psi)'(\mathcal{D}\bar{\psi})' \exp\left(i \int d^d x' \mathcal{L}[x', \psi'(x'), A_\mu(x')]\right) \\ &= \int J[\sigma] \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(i \int d^d x \mathcal{L}[x, \psi(x), A_\mu(x)] + \int d^d x \left(\bar{\psi} \left(-i \frac{d-1}{2} (\cancel{\partial}\sigma) - m\sigma\right) \psi\right)\right). \end{aligned} \quad (11.48)$$

As previously, since the two Path Integrals are equal, we obtain

$$J[\sigma] = \frac{\det(i\mathcal{D} - m)}{\det(i\mathcal{D} - m - \sigma m - i\frac{d-1}{2}(\not{\partial}\sigma))}. \quad (11.49)$$

The term proportional to $(\not{\partial}\sigma)$ requires to be regularised but no γ_5 is involved, hence the computation remains unambiguous. At order m^0 , the divergent contribution from $-i\frac{d-1}{2}(\not{\partial}\sigma)$ vanishes, only the finite contribution from σm remains and yields the scale anomaly [118]

$$\mathcal{A}_{\text{scale}} = \frac{\sigma}{24\pi^2} \text{tr} F^2. \quad (11.50)$$

We omit the details of calculation that can be found in the article [4]. However, higher order terms (terms of order $1/m^k$, with $k > 0$) involve contributions from both σm and $\frac{d-1}{2}(\not{\partial}\sigma)$ which should cancel one another.

We may then relate the anomaly to the β -function. At tree level the coupling e does not transform. It however transforms at one loop level, and by definition of the β -function we have

$$e \rightarrow e + \sigma\beta(e), \quad (11.51)$$

in (11.43) at one-loop, i.e

$$S_{\text{tree+1loop}} = \int d^d x \left(\bar{\psi}(i\mathcal{D} - m)\psi - \frac{1}{4e^2} F^2 + \sigma \frac{\beta(e)}{2e^3} F^2 \right).$$

By identification with the term produced at one loop by the Jacobian, we can deduce the expression of the one loop β -function

$$\beta(e) = \frac{e^3}{12\pi^2}, \quad (11.52)$$

which corresponds to the well-known QED β -function.

Chapter 12

Gravitational Anomalies

We introduced QFT in curved spacetime in Part II, and we saw that it is associated with two local symmetries: the diffeomorphism and Lorentz groups. After realising that vector and axial symmetries, whether global or gauge, can be broken at the quantum level, it is legitimate to ask ourselves whether the same can occur to the symmetries of gravity. In this Chapter, we discuss gravitational anomalies, and in which theories they may occur. In addition, we will also discuss the Weyl symmetry of gravity, which is the generalisation of scale invariance to curved spacetime. In particular, we dedicate significant efforts to dealing with chiral fermions in gravity and how they impact the Weyl anomaly, thus helping solving a recent controversy in the literature. Finally, we pursue this work to address the question of parity violating Weyl anomalies in a model-independent manner.

12.1 Axial-gravitational anomaly

Before delving into the symmetries of spacetime, we may first have a look at matter field symmetries in curved spacetime. As was shown in Sec. 10.1.1, the axial transformation of a vector-like fermion is anomalous in the presence of a background gauge field. In fact, this also holds for a background gravitational field, and it is the so-called axial-gravitational anomaly [160, 161]. In our work [3], we provide a computation via the path integral, using the EFT interpretation. It can also be found in details in [20, 118] and we only display the result here

$$\langle \partial_\mu j_5^\mu \rangle = \frac{-i}{192\pi^2} R\tilde{R}, \quad (12.1)$$

where

$$R\tilde{R} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\mu\nu} R_{\alpha\beta\rho\sigma}, \quad (12.2)$$

is the topological Pontryagin density for gravity. It is the equivalent of $F\tilde{F}$ for gravity, and will be the subject of discussion in the following Sections, especially regarding its place in the Weyl anomaly (to be introduced below).

We do not label such an anomaly a gravitational anomaly since it is not associated with a gravitational symmetry per se, but it is worth mentioning and has many phenomenological implications.

12.2 Diffeomorphism and Lorentz anomalies

It was Álvarez-Gaumé and Witten who first noticed that the symmetries of General Relativity may be broken by quantum fermionic fields [76]. In particular they studied left-handed Dirac fermion¹

$$S_L[e, \psi, \bar{\psi}] = \int d^d x \sqrt{|g|} \bar{\psi} i \not{D} P_L \psi . \quad (12.3)$$

Classical invariance This theory is classically invariant under the (active) diffeomorphism transformation of infinitesimal parameter $\xi^\mu(x)$

$$\begin{aligned} \delta_\xi^d \psi &= \xi^\mu \partial_\mu \psi , & \delta_\xi^d \bar{\psi} &= \xi^\mu \partial_\mu \bar{\psi} , \\ \delta_\xi^d e^\mu_a &= \xi^\nu \partial_\nu e^\mu_a - e^\nu_a \partial_\nu \xi^\mu , & \delta_\xi^d e &= (\partial_\mu e \xi^\mu) = e(D_\mu \xi^\mu) , \\ \delta_\xi^d \omega_\mu &= \xi^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu \xi^\nu , \end{aligned} \quad (12.4)$$

and the Lorentz transformation of infinitesimal parameter $\alpha_{ab} = -\alpha_{ba}$

$$\begin{aligned} \delta_\alpha^L \psi &= -\frac{1}{2} \alpha_{ab} \Sigma^{ab} \psi , & \delta_\alpha^L \bar{\psi} &= \frac{1}{2} \alpha_{ab} \bar{\psi} \Sigma^{ab} , & \delta_\alpha^L e^\mu_a &= e^\mu_b \alpha^b_a , & \delta_\alpha^L e &= 0 , \\ \delta_\alpha^L \omega_\mu &= \frac{1}{2} [D_\mu, \alpha_{ab} \Sigma^{ab}] = \frac{1}{2} \Sigma^{ab} [D_\mu, \alpha_{ab}] , \end{aligned} \quad (12.5)$$

where $\omega_\mu = \frac{1}{2} \Sigma^{ab} \omega_{\mu ab}$ is the spin-connection introduced in Part II and $\Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]$. And when the fermion is solution to its EoM, we obtain the conserved quantities

$$\begin{aligned} \delta_\xi^d S_L[e, \psi, \bar{\psi}] &= \int d^d x \delta_\xi^d e^\mu_a \frac{\delta S}{\delta e^\mu_a} = \int d^d x \xi^\nu (D^\mu T_{\mu\nu} - \omega_{\nu ab} T^{ab}) = 0 , \\ \delta_\alpha^L S_L[e, \psi, \bar{\psi}] &= \int d^d x \delta_\alpha^L e^\mu_a \frac{\delta S}{\delta e^\mu_a} = - \int d^d x \alpha_{ab} T^{ab} = 0 , \end{aligned} \quad (12.6)$$

where the energy-momentum tensor (EMT) is

$$T^\mu_a = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta e^a_\mu} , \quad T^{\mu\nu} = g^{\nu\rho} e^a_\rho T^\mu_a . \quad (12.7)$$

Anomalies In [76], it is showed that these relations are broken at the quantum level in $d = 2 + 4k$ ($k \in \mathbb{N}$) dimensions, but not in $d = 4 + 4k$. In $d = 2 + 4k$, the effective action \mathcal{W} is a priori neither diffeomorphism nor Lorentz invariant, and we define the anomalies by

$$\begin{aligned} \int d^4 x e \xi_\mu \mathcal{A}^\mu_{\text{diffeo}} &= -\delta_\xi^d \mathcal{W} = \int d^4 x e \xi^\nu \langle \omega_{\nu ab} T^{ab} - D^\mu T_{\mu\nu} \rangle , \\ \int d^4 x e \alpha_{ab} \mathcal{A}^{ab}_{\text{Lorentz}} &= -\delta_\alpha^L \mathcal{W} = \int d^4 x e \alpha_{ab} \langle T^{ab} \rangle . \end{aligned} \quad (12.8)$$

The diffeomorphism anomaly takes on a different expression in the presence of a gauge field, which is not diffeomorphism invariant [1].

Note that, as the vector and axial gauge anomalies, the presence of a diffeomorphism or Lorentz anomaly implies that the theory is inconsistent, or at least that it should be understood as an EFT

¹In Sec. 13.1, we will make an important distinction between a left-handed Dirac fermion, and a Weyl fermion. This distinction is crucial in the definition of the path integral, but we postpone it for later.

following [141, 151]. In $d = 4$, theories involving quantum chiral fermions are free from diffeomorphism and Lorentz anomalies, which makes possible to couple the SM to a classical gravitational background.

Another similarity with the vector and axial gauge anomalies, is that the diffeomorphism and Lorentz anomalies exist both under a covariant and a consistent form, which are related by the Bardeen-Zumino (BZ) local polynomial $\mathcal{P}^{\mu\nu}$ [20, 150]. As for axial gauge anomalies, the BZ polynomial does not arise at the level of the renormalising counterterms, i.e at the level of the effective action, but at the level of the effective action variation, i.e the EMT. The anomalies written in (12.6) arise from the effective action variation, hence respect the Wess-Zumino consistency conditions. The covariant anomalies are obtained using the modified energy momentum tensor

$$\langle \tilde{T}^{\mu\nu} \rangle = \langle T^{\mu\nu} \rangle + \mathcal{P}^{\mu\nu} , \quad (12.9)$$

and the BZ polynomial is obtained by canceling the diffeo-variation of the diffeomorphism anomaly. An example in $d = 2$ can found in [20], for the consistent and covariant diffeomorphism anomaly in the absence of Lorentz anomaly

$$D^\mu \langle T_{\mu\nu} \rangle = \frac{1}{192\pi^2} \epsilon^{ab} \omega_{\nu ab} R , \quad D^\mu \langle \tilde{T}_{\mu\nu} \rangle = \frac{1}{96\pi^2} \epsilon^{\mu\nu} D_\mu R , \quad (12.10)$$

where R is the Ricci scalar, and $\epsilon^{\mu\nu}$ is the Levi-Civita tensor.

Finally, the Lorentz and diffeomorphism anomalies are also topological anomalies [76, 162], and enjoy the same interesting properties as the ABJ and the gauge anomalies, that is to say they are mass independent and one-loop exact.

Relation between diffeomorphism and Lorentz anomalies In [126, 150], a relation is found between the diffeomorphism and the Lorentz anomalies. They introduce a local counterterm to the effective action that yields a symmetric EMT, and results the vanishing of the Lorentz anomaly, but not the diffeomorphism anomaly. Likewise, this counterterm can be used to obtain a vanishing diffeomorphism anomaly but non-vanishing Lorentz anomaly. This term is obtained by introducing an interpolating vierbein

$$e_t = 1 + (t - 1)e , \quad t \in [0; 1] , \quad (12.11)$$

where e is the vierbein field, with components e^a_μ . The following counterterm is then added to the action

$$\mathcal{W}_{ct} = \int d^d x \int dt C_{2d+1}(\omega[e^t] + v^t) , \quad (12.12)$$

where $\omega[e^t]$ is the spin-connection built with the interpolating vierbein e^t and

$$v^t = (e^t)^{-1} \partial_t e^t . \quad (12.13)$$

C_{2d+1} is the Chern-Simons current, whose divergence is the Pontryagin density. We have encountered already in (10.89) for the gauge field and in $d = 4$. For gravity in $d = 4$ it is expressed as

$$C^\mu[\Gamma] = \epsilon^{\mu\nu\rho\sigma} \left(\Gamma_{\nu\beta}^\alpha \partial_\rho \Gamma_{\sigma\alpha}^\beta + \frac{2}{3} \Gamma_{\nu\beta}^\alpha \Gamma_{\rho\delta}^\beta \Gamma_{\sigma\alpha}^\delta \right) , \quad R\tilde{R} = 2D_\mu C^\mu[\Gamma] . \quad (12.14)$$

In (12.12), it is evaluated with connection $\omega[e^t] + v^t$. More details can be found in [20].

If we have built an effective action \mathcal{W} which is free from Lorentz anomaly but has a diffeomorphism anomaly, then $\mathcal{W} + \mathcal{W}_{ct}$ is free from diffeomorphism anomaly but has a Lorentz anomaly.

In [126, 150], the authors claim that this counterterm implies that Lorentz and diffeomorphism anomalies are in fact equivalent. It was however pointed out by Leutwyler [163, 164], the counterterm in (12.12) is in indeed local, in the sense that it has a finite power in the background fields, but also *non-polynomial* due to the integral over an extra dimension. This implies that it lies outside of the class of legal counterterms that can be introduced for renormalising the theory.

12.3 Path integral measure in curved spacetime

As advertised in Part II, the path integral measure requires a careful treatment in gravity. The reason is that the measure that we use in flat spacetime is not diffeo-invariant in curved spacetime. In this Section we show how to naturally obtain the correct path integral measure in curved spacetime.

12.3.1 Diffeo-invariant measure

Naively, we are tempted to write the fermion path integral as

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^4x \sqrt{|g|} \bar{\psi} (i\mathcal{D} - m) \psi} . \quad (12.15)$$

The measure $\mathcal{D}\psi \mathcal{D}\bar{\psi}$ is however not the correct one, as was pointed out by [118]. The correct construction of the measure was then understood in [165]. To obtain the correct measure, let us consider the simple case of a finite N dimensional vector space with some metric E , such that the inner product is

$$(u, v) = \sum_{i,j=1}^N u_i E_{ij} v_j . \quad (12.16)$$

In that case, the invariant volume element is defined by

$$d\mu(v) = (\det E)^{1/2} d^N v . \quad (12.17)$$

This is the same procedure that we follow to define the invariant spacetime measure $\sqrt{|g|} d^4x$ on a 4-dimensional manifold.

We then consider the case of field theory, which is an infinite (uncountable) dimensional vector space. The inner product of two scalar fields with internal indices i is defined as

$$(\phi, \varphi) = \int d^4x d^4y \phi_i(x) G_{ij}(x, y) \varphi_j(y) , \quad (12.18)$$

where the field-space metric is diagonal

$$G_{ij}(x, y) = \rho_{ij}(x) \delta(x - y) . \quad (12.19)$$

The associated invariant volume element is

$$d\mu(\phi) = \prod_x \left((\det_i \rho(x))^{1/2} \prod_i d\phi_i(x) \right) , \quad (12.20)$$

where \det_i is the determinant on internal indices only, since we left the product on x explicit.

Now let us apply this to the case at hand: a field in curved spacetime. The diffeomorphism invariant inner product of two scalar fields is

$$(\phi, \varphi) = \int d^4x \sqrt{|g|} \phi_i(x) \varphi_i(x) , \quad (12.21)$$

where $|g|$ is the modulus of the determinant of the metric which is a determinant on the Lorentz indices, and we can identify the field-space metric to be

$$G_{ij}(x, y) = |g(x)|^{1/2} \delta_{ij} \delta(x, y) . \quad (12.22)$$

This implies that the diffeomorphism invariant measure is

$$d\mu(\phi) = \prod_x \left(|g(x)|^{1/4} \prod_i \phi_i(x) \right) = \mathcal{D}\tilde{\phi} , \quad (12.23)$$

where we defined $\tilde{\phi} = |g|^{1/4}\phi$. The same can be applied to fermions to obtain the diffeo-invariant measure

$$d\mu(\psi) = \mathcal{D}\tilde{\psi} \mathcal{D}\tilde{\bar{\psi}} , \quad (12.24)$$

where

$$\tilde{\psi} = |g(x)|^{1/4}\psi , \quad \tilde{\bar{\psi}} = |g|^{1/4}\bar{\psi} . \quad (12.25)$$

Note that since $\tilde{\psi}$ has a different weight than ψ , the covariant derivative acts differently on them

$$D_\mu \tilde{\psi} = D_\mu |g|^{1/4}\psi = |g|^{1/4} D_\mu \psi = |g|^{1/4} (\partial_\mu + \omega_\mu) \psi , \quad (12.26)$$

where we used $[D_\mu, |g|] = 0$, and commuting the metric determinant to the left we obtain

$$D_\mu \tilde{\psi} = \left(\partial_\mu + \omega_\mu - \frac{\partial_\mu \sqrt{|e|}}{\sqrt{|e|}} \right) \tilde{\psi} , \quad (12.27)$$

where we used the vierbein determinant $e = \sqrt{|g|}$.

This choice of variables is in agreement with the path integral measure advocated by Fujikawa in [158], up to a small caveat. Fujikawa claims that the correct variables to use in the path integral should be the weight $-1/2$ fields (12.25), which lead to the measure (12.24). However, as emphasised by Toms in [165], we may take any weight for the path integral variables, e.g

$$\phi_w = |g|^{-w/2} \phi , \quad (12.28)$$

which transforms under the diffeomorphism $x \rightarrow x'$ as

$$\phi'(x') = \det \left| \frac{\partial x'^\mu}{\partial x^\nu} \right|^w \phi(x) , \quad (12.29)$$

the path integral measure will not be affected

$$d\mu(\phi_w) = \prod_x \left(|g(x)|^{\frac{w}{2} + \frac{1}{4}} \prod_i \phi_w(x) \right) = \prod_x \left(|g(x)|^{1/4} \prod_i \phi(x) \right) = d\mu(\phi) . \quad (12.30)$$

Note that for fields of higher spin, the invariant measure involves the determinant of the metric with powers different than $1/4$. More details can be found in [118].

12.3.2 Effective action

The effective action for fermions using the naive measure is

$$\mathcal{W} = -i \log \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} = -\frac{i}{2} \log \det \left(\sqrt{|g|} (i\mathcal{D} - m) \right), \quad (12.31)$$

whereas the one obtained using the diffeo-invariant measure is

$$\tilde{\mathcal{W}} = -i \log \int \mathcal{D}\tilde{\psi} \mathcal{D}\tilde{\bar{\psi}} e^{iS} = -\frac{i}{2} \log \det (i\mathcal{D} - m), \quad (12.32)$$

such that the difference between the two simply is

$$\tilde{\mathcal{W}} - \mathcal{W} = \frac{i}{2} \log \det \sqrt{|g|}. \quad (12.33)$$

This difference is relevant for gravitational anomalies since it is not invariant under a diffeomorphism

$$\delta_\xi^d \tilde{\mathcal{W}} \neq \delta_\xi^d \mathcal{W}, \quad (12.34)$$

nor under a Weyl transformation. This implies that gravitational anomalies are impacted by the choice of measure. In fact, it was showed in [118, 158] that the naive path integral measure for vector-like fermions yields a spurious diffeomorphism anomaly, and the wrong Weyl anomaly. Let us show explicitly that the naive measure is not diffeo-invariant, using the Fujikawa procedure.

The naive path integral variables transform under a diffeomorphism as

$$\delta_\xi^d \psi = \xi^\mu \partial_\mu \psi, \quad \delta_\xi^d \bar{\psi} = \xi^\mu \partial_\mu \bar{\psi}. \quad (12.35)$$

Following the same procedure as in Sec. 10.1.2, we obtain the Jacobian of the transformation

$$\log J[\xi] = - \sum_n \int d^4x \left(\phi_n^\dagger \xi^\mu \partial_\mu \phi_n + \xi^\mu (\partial_\mu \phi_n^\dagger) \phi_n \right) = \sum_n \int d^4x (\partial_\mu \xi^\mu) \phi_n^\dagger \phi_n \neq 0, \quad (12.36)$$

where we integrated by parts using $\xi^\mu(|x| \rightarrow \infty) = 0$. On the other hand, using the diffeomorphism invariant path integral variables with transformations

$$\delta_\xi^d \tilde{\psi} = \xi^\mu \partial_\mu \tilde{\psi} + \frac{1}{2} (\partial_\mu \xi^\mu) \tilde{\psi}, \quad \delta_\xi^d \tilde{\bar{\psi}} = \xi^\mu \partial_\mu \tilde{\bar{\psi}} + \frac{1}{2} (\partial_\mu \xi^\mu) \tilde{\bar{\psi}}, \quad (12.37)$$

we obtain the Jacobian

$$\log J[\xi] = - \sum_n \int d^4x \sqrt{|g|} \left(\phi_n^\dagger \left(\xi^\mu \partial_\mu + \frac{1}{2} (\partial_\mu \xi^\mu) \right) \phi_n + \xi^\mu \left(\partial_\mu \phi_n^\dagger + \frac{1}{2} (\partial_\mu \xi^\mu) \phi_n^\dagger \right) \phi_n \right) = 0, \quad (12.38)$$

after integrating by parts.

Note that since the metric determinant is Lorentz invariant, both the naive and the diffeo-invariant measures are Lorentz invariant. Only the diffeomorphism and the Weyl anomalies (introduced below in Sec. 12.4) are affected by the choice of measure.

12.3.3 Back to the covariant derivative expansion

One may legitimately remark that in the CDE in curved spacetime in Part II, the path integral was defined using the naive measure. This is correct, however, during the CDE procedure the $\log \det \sqrt{|g|}$ term is eliminated, which amounts to using the diffeomorphism invariant measure. The fermionic effective action that we used is

$$\mathcal{W} = -i \log \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^4x \sqrt{|g|} \bar{\psi} (i\mathcal{D} - m - Q) \psi}, \quad (12.39)$$

and the one-loop part can be expressed as

$$\mathcal{W}_{1\text{-loop}} = -i \int d^d x \frac{d^d q}{(2\pi)^d} \text{tr} \log \sqrt{|g|} (i\mathcal{D} - \not{q} - m - Q), \quad (12.40)$$

where $\Delta^{-1} = -(\not{q} + m)$. At this stage, the only difference with $\tilde{\mathcal{W}}_{1\text{-loop}}$ is the $\sqrt{|g|}$ factor. Then, since $[D_\mu, \Delta] \neq 0$, we cannot straightforwardly expand the logarithm, and we instead write

$$\mathcal{W}_{1\text{-loop}} = i \int d^d x \frac{d^d q}{(2\pi)^d} \int^m dm' \text{tr} \frac{\sqrt{|g|}}{\sqrt{|g|} (i\mathcal{D} - \not{q} - m - Q)}, \quad (12.41)$$

so that we may then use

$$\frac{1}{A^{-1}(1 - AB)} = \sum_{n \geq 0} A(AB)^n, \quad (12.42)$$

which holds for $[A, B] \neq 0$. We notice that in (12.41), the factor $\sqrt{|g|}$ cancels, and we obtain the same quantity as if we had used the diffeo-invariant measure.

In fact, we have been a hasty in writing the logarithm as the integral of the inverse function, and to be precise, from (12.40) we should use

$$\begin{aligned} \frac{d}{dm} \text{tr} \log \sqrt{|g|} (i\mathcal{D} - \not{q} - m - Q) &= - \text{tr} \frac{1}{i\mathcal{D} - \not{q} - m - Q} \\ \Leftrightarrow \text{tr} \log \sqrt{|g|} (i\mathcal{D} - \not{q} - m - Q) &= - \int_{m_0}^m dm' \text{tr} \frac{1}{i\mathcal{D} - \not{q} - m' - Q} - C \end{aligned} \quad (12.43)$$

$$= - \text{tr} \log \frac{i\mathcal{D} - \not{q} - m - Q}{i\mathcal{D} - \not{q} - m_0 - Q} - C \quad (12.44)$$

where C is independent of m . We see that the equality holds by taking

$$C = \text{tr} \log (i\mathcal{D} - \not{q} - m_0 - Q) + \text{tr} \log \sqrt{|g|}. \quad (12.45)$$

In practice, to derive the UOLEA in curved spacetime we used

$$C = \text{tr} \log (i\mathcal{D} - \not{q} - m_0 - Q), \quad (12.46)$$

which amounts to ignoring the factor $\sqrt{|g|}$, and is equivalent to using the diffeo-invariant measure.

12.4 Weyl symmetry

In Sec. 11.5, we encountered a symmetry of massless fields in flat spacetime that is the scale symmetry. It is the invariance of the theory under rescaling both the fields and the spacetime. One may wonder what happens to this symmetry in curved spacetime.

We may attempt to perform a similar transformation of the fields and of spacetime on a general background manifold, however the rescaling of the spacetime coordinates simply becomes a diffeomorphism. It thus appears that the scale symmetry is not well suited to curved spacetime since it is a mix of a diffeomorphism and a rescaling of the fields.

Instead of rescaling the spacetime coordinates, we could rescale the metric, which would result in transforming fields only. This is how the Weyl transformation is obtained, and it can be seen as the generalisation of the scale transformation to curved spacetime.

12.4.1 Weyl vs conformal transformations

In this paragraph we would like to outline the difference between the Weyl and conformal symmetries that are often confused with one another in the literature.

In flat spacetime, a theory may exhibit an invariance under the scale transformation defined in Sec. 11.5. This transformation is however a subgroup of a more general group known as the conformal group. In flat spacetime, and only in flat spacetime, a conformal transformation can be decomposed in a combination of a Lorentz transformation, a translation, the scale transformation, and the special conformal transformation.

As mentioned above, in curved spacetime, the scale transformation becomes a mix of a diffeomorphism and a rescaling of the matter fields. It is instead generalised as a Weyl transformation denoted δ_σ^W , where $\sigma(x)$ is the infinitesimal parameter of the transformation, under which the metric is rescaled

$$\delta_\sigma^W g_{\mu\nu} = 2\sigma g_{\mu\nu} . \quad \delta_\sigma^W g^{\mu\nu} = -2\sigma g^{\mu\nu} . \quad (12.47)$$

Let us then consider a diffeomorphism transformation along a vector field ξ^μ such that the metric is transformed as

$$\delta_\xi^d g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu = \frac{2}{d} (D_\lambda \xi^\lambda) g_{\mu\nu} . \quad (12.48)$$

Although this is a diffeomorphism, the metric is effectively rescaled, as for a Weyl transformation. Vectors that respect the condition (12.48) are called Conformal Killing Vector Fields (CKVF), and they produce the conformal transformations in curved spacetime. An important caveat is that most manifolds do not admit any CKVF. It is known that two-dimensional manifolds always exhibit CKVF, however in $d \neq 2$, only manifolds of constant curvature admits CKVF, which is very restrictive. One should keep in mind that although conformal and Weyl transformations both rescale the metric, the conformal transformation is a diffeomorphism, whereas the Weyl transformation is a change of metric, i.e a transformation of the manifold.

12.4.2 Classical Weyl invariance

The Weyl transformation of the metric is given in (12.47). The metric determinant transforms as

$$\delta_\sigma^W \sqrt{|g|} = d\sigma \sqrt{|g|} , \quad (12.49)$$

with d the dimension of the manifold. Matter fields transform according to their canonical mass dimension w

$$\delta_\sigma^W \phi = w\sigma\phi , \quad (12.50)$$

except for the gauge fields which are Weyl invariant [118].

Scalar field Let us consider first a real scalar field

$$S = -\frac{1}{2} \int d^d x \sqrt{|g|} \varphi (\square + \alpha R) \varphi , \quad (12.51)$$

where $\square = D^2$ and the covariant derivative may include a gauge field. We also introduced a coupling to the Ricci scalar since it is allowed dimensionally. Scalars have mass dimension 1 and transform as

$$\delta_\sigma^W \varphi = \sigma \varphi . \quad (12.52)$$

The scalar theory is Weyl invariant only for $\alpha = \alpha_d = -\frac{d-2}{4(d-1)}$.

Spin-1 field The action for a spin-1 field A_μ is given by

$$S = \int d^d x \sqrt{|g|} g^{\mu\rho} g^{\nu\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma} , \quad (12.53)$$

with $F_{\mu\nu} = i \partial_\mu A_\nu - i \partial_\nu A_\mu - [A_\mu, A_\nu]$. Since $\delta_\sigma^W A_\mu = 0$, the Weyl variation of the action reads

$$\delta_\sigma^W S = (d-4) \int d^d x \sqrt{|g|} \text{tr} F^2 , \quad (12.54)$$

and is only invariant in $d = 4$. This also applies to abelian fields.

Fermion field Let us turn to the case of a fermionic theory

$$S = \int d^d x \bar{\psi} \gamma^a e_a^\mu i D_\mu \psi , \quad (12.55)$$

where the covariant derivative includes the spin-connection and possibly gauge fields, and $e = \sqrt{|g|}$ is the vierbein determinant. This theory is invariant under the Weyl transformation

$$\begin{aligned} \delta_\sigma^W e_a^\mu &= -\sigma e_a^\mu , & \delta_\sigma^W e &= d\sigma e , & \delta_\sigma^W \omega_\mu &= \frac{d-1}{2} (\partial_\mu \sigma) , \\ \delta_\sigma^W \psi &= -\frac{d-1}{2} \sigma \psi , & \delta_\sigma^W \bar{\psi} &= -\frac{d-1}{2} \sigma \bar{\psi} . \end{aligned} \quad (12.56)$$

Conserved current When the matter fields are solution to their EoM, we obtain the conserved current associated with the Weyl transformation

$$\delta_\sigma^W S[g_{\mu\nu}, \phi] = \int d^d x \delta_\sigma^W g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} = \int d^d x \sqrt{|g|} T^\mu{}_\mu , \quad (12.57)$$

$$\delta_\sigma^W S[e^a{}_\mu, \phi] = \int d^d x \delta_\sigma^W e^a{}_\mu \frac{\delta S}{\delta e^a{}_\mu} = \int d^d x \sqrt{|g|} T^\mu{}_\mu , \quad (12.58)$$

where ϕ denotes a generic matter field.

Note that massive matter fields spoil Weyl invariance since the mass introduces an intrinsic scale.

12.4.3 Weyl anomaly

It was Capper and Duff that first discovered the anomalous behavior of the Weyl symmetry [166]. Its expression was then worked out in dimensional regularisation in [167], and importantly they discovered that the anomaly is not $\langle T^\mu{}_\mu \rangle$, which vanishes for a classically Weyl invariant theory, but is

$$\int d^4x \sqrt{|g|} \sigma \mathcal{A}_{\text{Weyl}} = \delta_\sigma^W \mathcal{W} = \int d^4x \sqrt{|g|} \sigma g_{\mu\nu} \langle T^{\mu\nu} \rangle . \quad (12.59)$$

This anomaly is also called trace anomaly in the literature. This formula has to be amended in the presence of explicit breaking, for example when fields are massive. We will dwell further on that point in Sec. 14.1.

In Sec. 11.5, we derived the scale anomaly in flat spacetime, which is the same as the Weyl anomaly in the flat spacetime limit. In this limit, the Weyl anomaly in a theory of a vector-like fermion (11.43) takes the form

$$\mathcal{A}_{\text{Weyl}} = \frac{\sigma}{24\pi^2} \text{tr} F^2 = \frac{\beta(e_g)}{2e_g^3} \text{tr} F^2 , \quad (12.60)$$

where e_g is the coupling between the fermions and the gauge field, and $\beta(e_g)$ the one-loop β -function that describes its running with the renormalisation scale. In general, β -functions are not one-loop exact, and receive corrections from higher loops up to any order. This is the first hint that the Weyl anomaly, contrary to all the other anomalies we have encountered so far, is not topological.

Let us confirm our intuition with an analysis à la Fujikawa. We consider the quantum theory

$$Z = \int \mathcal{D}\tilde{\psi} \mathcal{D}\tilde{\bar{\psi}} e^{i \int d^4x \tilde{\bar{\psi}} i \mathcal{D} \tilde{\psi}} , \quad (12.61)$$

where the Dirac operator is

$$i\mathcal{D} = i\cancel{\partial} + i\cancel{\psi} - i \frac{\cancel{\phi} \sqrt{e}}{\sqrt{e}} - \cancel{V} , \quad (12.62)$$

and we used the diffeo-invariant measure introduced in Sec. 12.3.1. As emphasised earlier, it was showed in [118, 158] that the naive measure leads to the wrong value for the Weyl anomaly, and to a spurious diffeomorphism anomaly. Indeed, the improved variables have a different Weyl transformation than the naives ones, and can be worked out from (12.56)

$$\delta_\sigma^W \tilde{\psi} = \frac{1}{2} \sigma \tilde{\psi} , \quad \delta_\sigma^W \tilde{\bar{\psi}} = \frac{1}{2} \sigma \tilde{\bar{\psi}} . \quad (12.63)$$

The orthonormal eigenvectors of the Dirac operator are

$$i\mathcal{D} \tilde{\phi}_n = \lambda_n \tilde{\phi}_n , \quad \lambda_n \in \mathbb{R} , \quad (12.64)$$

where we may include the factor $|g|^{1/4}$ in the eigenvectors. We then decompose the path integral variables on the eigenbasis

$$\tilde{\psi} = \sum_n a_n \tilde{\phi}_n , \quad \tilde{\bar{\psi}} = \sum_n b_n \tilde{\phi}_n^\dagger , \quad (12.65)$$

and likewise for the Weyl transformed variables. We then deduce the transformation matrices of the coefficients a_n and b_n

$$A_{mn} = \delta_{mn} + \int d^4x \frac{\sigma}{2} \tilde{\phi}_m^\dagger \tilde{\phi}_n , \quad B_{mn} = \delta_{mn} + \int d^4x \frac{\sigma}{2} \tilde{\phi}_n^\dagger \tilde{\phi}_m , \quad (12.66)$$

and finally obtain the Jacobian of the transformation

$$\log J_{\text{Weyl}}[\sigma] = \log \det A^{-1} B^{-1} = \sum_n \int d^4x \sigma(x) \tilde{\phi}_n^\dagger \tilde{\phi}_n + \mathcal{O}(\sigma^2). \quad (12.67)$$

To understand why this is not a topological invariant, let us recall that the Jacobian from the ABJ anomaly² is similar except for the presence of the γ_5 matrix

$$\log J_{\text{ABJ}}[\theta] = \sum_n \int d^4x \theta(x) \tilde{\phi}_n^\dagger \gamma_5 \tilde{\phi}_n + \mathcal{O}(\theta^2). \quad (12.68)$$

The argument was the following: since the eigenvectors are orthonormal, and since $\gamma_5 \phi_n$ has the eigenvalue $-\lambda_n$ whereas ϕ_n has eigenvalue λ_n , then, unless $\lambda_n = 0$, $\gamma_5 \phi_n$ and ϕ_n have a different eigenvalues are thus orthogonal. As a result, only the zero-modes remain in (12.68), which leaves a sum over a *finite* number of eigenvalues

$$\log J_{\text{ABJ}}[\theta] = \theta(n_+ - n_-), \quad (12.69)$$

in the limit θ constant, with n_\pm the number of (independent) zero-modes with chirality \pm . Let us now return to the Weyl anomaly. In the limit σ constant we obtain

$$\log J_{\text{Weyl}}[\sigma] = \sigma \sum_n \int d^4x \phi_n^\dagger \phi_n = \sigma \sum_n 1 = \sigma \sum_n N(\lambda_n), \quad (12.70)$$

where $N(\lambda_n)$ is the number of (independent) eigenvectors with eigenvalue λ_n . The Weyl anomaly involves a sum over all the eigenvalues, up to arbitrary large values. As a result it is not finite, and hence will be renormalised. This implies that it will receive corrections from any loop order, as opposed to a topological anomaly.

The Weyl anomaly is obtained at lowest order by cutting the non-zero eigenvalues using the Fujikawa regulator

$$\log J_{\text{Weyl}}[\sigma] \longrightarrow \sigma \lim_{\Lambda \rightarrow \infty} \sum_n f\left(\frac{\lambda_n}{\Lambda}\right) \int d^4x \phi_n^\dagger \phi_n = \sigma \sum_n N(\lambda_n) f\left(\frac{\lambda_n}{\Lambda}\right), \quad (12.71)$$

where we are now working with a convergent series

$$\sum_n N(\lambda_n) f\left(\frac{\lambda_n}{\Lambda}\right) < \infty. \quad (12.72)$$

In practice, we compute the anomaly perturbatively (i.e $\sigma(x)$ vanishing at infinity) using

$$\log J_{\text{Weyl}}[\sigma] = \lim_{\Lambda \rightarrow \infty} \sum_n \int d^4x \sigma \phi_n^\dagger f\left(\frac{i\not{D}}{\Lambda}\right) \phi_n = \text{Tr} \sigma f\left(\frac{i\not{D}}{\Lambda}\right). \quad (12.73)$$

We may then choose a specific function f , e.g $f(x) = 1/(1+x^2)$ is suitable, and use the CDE in gravity to obtain the Weyl anomaly of a vector-like fermion with gauge sector

$$\mathcal{A}_{\text{Weyl}} = \frac{1}{16\pi^2} \left(\frac{1}{72} R^2 - \frac{1}{45} R_{\mu\nu} R^{\mu\nu} - \frac{7}{360} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{30} \square R + \frac{2}{3} \text{tr} F^2 \right). \quad (12.74)$$

Note that the $\square R$ sign depends on the convention for the metric, we adopt the metric convention from [168]. The Weyl anomaly still arises from terms of the form “ $0 \times \infty$ ”, e.g in dimensional regularisation $d = 4 - \epsilon$, terms of the form $\epsilon \frac{1}{\epsilon}$. This is because the regulator f truly cuts off some eigenmodes, whereas for topological anomaly its cuts off non-zero eigenmodes that do not contribute anyway. The Weyl anomaly in (12.74) is thus a perturbative incomplete result. The full anomaly would require a non-perturbative computation that includes all the eigenmodes and up to any loop.

²In fact in gravity this is the axial-gravitational anomaly mentioned in Sec. 12.1.

12.4.4 Generic form of the Weyl anomaly

Since several operators may occur in the Weyl anomaly, as in (12.74), it is interesting to study its the generic structure. On dimensional grounds, and omitting a gauge sector for now, we may express the anomaly in $d = 4$ as

$$\mathcal{A}_{\text{Weyl}} = a E + b R^2 + c W^2 + d \square R, \quad (12.75)$$

where we chose a different basis of operators than in (12.74)

$$\begin{aligned} E &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \\ W^2 &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3}R^2. \end{aligned} \quad (12.76)$$

These operators have interesting properties. E is called the Gauss-Bonnet term, and matches with the Euler density in $d = 4$. It is a topological operator, and as the Pontryagin density was associated with the Pontryagin number, the Euler density is associated with the Euler number χ , also called Euler characteristic

$$\int_{\mathcal{M}} d^4x \sqrt{|g|} E = 8\pi^2 \chi(\mathcal{M}). \quad (12.77)$$

If \mathcal{M} is a polyhedron with n_v vertices, n_e edges and n_f faces, then

$$\chi(\mathcal{M}) = n_v - n_e + n_f. \quad (12.78)$$

If \mathcal{M} is a topological space, any polyhedron that is homeomorphic³ to \mathcal{M} has the same Euler number as \mathcal{M} , making it a topological invariant (see Fig. 12.1 for an example). The other operator

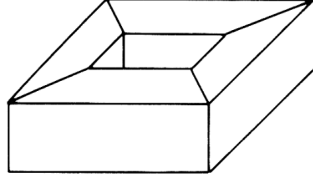


Figure 12.1: A polyhedron that is homeomorphic to a torus. It has a Euler characteristic of $\chi = 16 - 32 + 16 = 0$, and so does the torus.

is the Weyl tensor squared W^2 , where the Weyl tensor in $d > 2$ dimensions is

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{d-2}(R_{\mu\sigma}g_{\nu\rho} - R_{\mu\rho}g_{\nu\sigma} + R_{\nu\rho}g_{\mu\sigma} - R_{\nu\sigma}g_{\mu\rho}) + \frac{1}{(d-1)(d-2)}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \quad (12.79)$$

Note that both the Euler density, the Weyl tensor squared and $\square R$ are Weyl invariant

$$\delta_{\sigma}^W W_{\mu\nu\rho\sigma} = \delta_{\sigma}^W E = \delta_{\sigma}^W \square R = 0, \quad (12.80)$$

contrary to the Ricci scalar.

Let us return to the study of the generic form of the anomaly (12.75). In a classic paper [169], Duff shows using dimensional regularisation that in a classically Weyl invariant theory, the coefficients are in fact not independent but obey

$$b = 0, \quad \frac{2}{3}c = d. \quad (12.81)$$

³Two topological spaces are homeomorphic if there exist a bijective continuous map between them, and its inverse is continuous as well.

These relations are important when one considers the running of the Weyl anomaly with the renormalisation scale, which pertains to QCD for example.

Note that the constraint $b = 0$ implies that the R^2 is absent in (12.75), and therefore that the Weyl anomaly is Weyl invariant. Since the Weyl anomaly is itself the Weyl variation of the effective action, this implies

$$\delta_{\sigma_1}^W \delta_{\sigma_2}^W \mathcal{W} = 0 , \tag{12.82}$$

which is none other than the WZcc for the Weyl transformation. It takes a simpler form than the WZcc for the non-abelian axial gauge group because the Weyl group is abelian.

One may notice that an operator that we have encountered several times now is missing in (12.75), although it has the correct mass dimension to be included and was shown to respect the WZcc [147]. This operator is the P -violating Pontryagin density $R\tilde{R}$. In principle there is no reason to exclude it. The presence of the Pontryagin density has been the subject of debates in the recent literature, especially when considering the Weyl anomaly of a Weyl fermion in gravity. This will be the subject of the following Sections.

Chapter 13

Weyl Anomaly for Weyl Fermions

As emphasised at the end of the last Section, the most generic form of the Weyl anomaly include the Pontryagin density

$$g^{\alpha\beta}\langle T_{\alpha\beta}\rangle = a E_4 + b R^2 + c W^2 + d \square R + e R\tilde{R}. \quad (13.1)$$

The Pontryagin density has been subject to debates in the recent literature. It has been found to be non-vanishing in [170] and is known to satisfy the Wess-Zumino consistency condition [147]. Over the last few years a controversy has spanned in the literature since some authors have obtained a non-vanishing coefficient e [170–177] whereas others have found it to be vanishing [178–182]. Weyl fermions are subtle and probe spacetime in their own way as in $d = 2 \pmod{4}$ they give rise to gravitational (diffeomorphism and Lorentz) anomalies [76], or more definitely to Lorentz anomalies [163, 164]. In the specific determinations of $e \neq 0$ there is something similarly unsettling in that the authors found it to be purely imaginary in Lorentzian signature. This in fact implies that its contribution is CPT -violating since $T \circ i = -i$ which would indicate a CPT anomaly. Whereas it was noted that an imaginary e would violate unitarity [170], the CPT -violation itself seems to have been overlooked. This would either mean that such theories have to be discarded [170] or supplemented by new particles such as three right-handed neutrinos in the Standard Model. This, together with the fact that an $R\tilde{R}$ -term with real coefficient in the context P - and or CP -violation is of importance for phenomenological reasons (see for example the discussion in [183]), e.g. baryogenesis or gravitational waves as reviewed in [184], and could be observed experimentally [185], we consider it important to clarify the nature of this anomaly.

In this Section, we present our work [2], in which we show that $e = 0$ for Weyl fermions, from the path integral. The definition of the Weyl determinant is carefully worked out, building on earlier work by Leutwyler [163, 164, 186, 187]. The key idea is that it is the variation of the determinant, which is well-defined, that enters both of our approaches i) proper time regularisation and ii) the Fujikawa method adapted to Weyl fermions. In both cases this is combined with the CDE in curved spacetime presented in Part II, which has already proven useful to derive covariant and consistent anomalies in the context of effective field theories. The diffeomorphism, the Lorentz and the Weyl anomalies are entangled by regularisation, since they can be related by local counterterms (see end of Sec. 13.1 for further details); we therefore consider it essential to explicitly evaluate all three quantities.

13.1 Technical preliminaries

13.1.1 The determinant of the Weyl operator

There are several challenges in defining the determinant for a Weyl fermion. For example if one starts with a Dirac fermion with only left-handed components then its associated Dirac operator $i\not{D}P_L$, where P_L projects on left-handed fermions, cannot be inverted. This problem can be avoided if one starts directly with a two-component Weyl fermion ψ_L . The Weyl operator \mathcal{D} , which is the Dirac operator acting on a Weyl fermion, is given by

$$\mathcal{D}\psi_L = i\bar{\sigma}^\mu(\partial + \omega_L)_\mu\psi_L, \quad (13.2)$$

where $\bar{\sigma}^\mu = e^\mu_a\bar{\sigma}^a$, $\bar{\sigma}^a = (1, -\vec{\sigma})$ the Pauli matrices, and ω_L is the spin-connection for Weyl fermions (D.6); more precise definitions can be found in the App. D. Hereafter we work in Euclidean space as it is technically more convenient. The determinant of the Weyl operator appears formally in the effective action

$$\mathcal{W} = -\log \det \mathcal{D}, \quad (13.3)$$

after performing the Gaussian path integral. Unfortunately, $\det \mathcal{D}$ is ill-defined, as emphasised by Álvarez-Gaumé and Witten [76], since it maps left onto right handed fermions which have different Hilbert space, i.e. $\mathcal{D} : (\frac{1}{2}, 0) \rightarrow (0, \frac{1}{2})$ and vice versa. This makes the phase of the functional determinant ambiguous, whereas the modulus is unaffected (and is also gauge invariant). However, the determinant itself is not an observable. Leutwyler and Mallik [163, 164, 186, 187] pointed out that the variation of (13.3)

$$\delta \log \det \mathcal{D} = \text{Tr} \delta \mathcal{D} \mathcal{D}^{-1}, \quad (13.4)$$

which formally holds for any operator, is well-defined since it maps fermions to fermions of the same chirality: $\delta \mathcal{D} \mathcal{D}^{-1} : (0, \frac{1}{2}) \rightarrow (0, \frac{1}{2})$. This is in line with the observation that the relative phase between two operators is well-defined [76]. In both the proper time regularisation and the Fujikawa method adapted for Weyl fermion it will be the formula (13.4) and not the determinant itself which will form the starting point of the evaluation.

The zero modes of \mathcal{D} consist in another problem for the definition of the determinant. In what follows we will assume that there are no zero modes, as is often done [187, 188], which is believed to be true in the realm of perturbation theory. Note that in the absence of zero modes, the topological anomalies become trivial, i.e the Pontryagin density $\int d^4x R\tilde{R} = 0$. As mentioned in Sec. 10.1.4, this is not an issue since the x dependence of the parameter of the transformation, e.g $\theta(x)$, ensures that $\int d^4x \theta(x) R\tilde{R} \neq 0$.

On top of this, the operator \mathcal{D}^{-1} is singular, as apparent from perturbation theory at short distances, but can be regularised, which we will turn to in Secs. 13.2 and 13.3 respectively.

13.1.2 Path integral formulation

The path integral measure is defined following Sec. 12.3.1, which is free from spurious gravitational anomalies (i.e that can be removed by local counterterms) in any even dimension. It has been shown for Dirac fermions [118, 158] and in this paper we show that it equally holds for Weyl fermions.

The Weyl fermion invariant measure is therefore $\mathcal{D}\tilde{\psi}_L\mathcal{D}\bar{\psi}_L$ where

$$\tilde{\psi}_L = g^{1/4}\psi_L, \quad \bar{\psi}_L = g^{1/4}\bar{\psi}_L, \quad (13.5)$$

and similarly as for the Dirac fermion (12.27), the Weyl operator reads

$$\mathcal{D}\tilde{\psi}_L = i\bar{\sigma}^\mu \left(\partial_\mu + \omega_\mu^L - \frac{\partial_\mu \sqrt{e}}{\sqrt{e}} \right) \tilde{\psi}_L. \quad (13.6)$$

The corresponding path integral assumes the form

$$Z_{\text{Weyl}} = \int \mathcal{D}\tilde{\psi}_L \mathcal{D}\tilde{\bar{\psi}}_L e^{-\int d^4x \tilde{\psi}_L \mathcal{D}\tilde{\bar{\psi}}_L}, \quad (13.7)$$

where the $\tilde{\psi}_L$ and $\tilde{\bar{\psi}}_L$ variables are to be treated as vierbein-independent.

As seen above, the gravitational anomalies follow from a variation of the effective quantum action (13.3) with respect to a symmetry. Concretely, applying $\delta_{\alpha(x)}$ to (13.3), using (13.4), the associated anomaly \mathcal{A} is formally defined by

$$\delta_{\alpha}\mathcal{W} = -\text{Tr} \delta_{\alpha}\mathcal{D}\mathcal{D}^{-1} = -\int d^4x e \alpha(x)\mathcal{A}. \quad (13.8)$$

We are interested in the Weyl (trace), diffeomorphism and Lorentz anomalies which we recall here

$$\begin{aligned} \int d^4x e \sigma \mathcal{A}_{\text{trace}} &= -\delta_{\sigma}^W \mathcal{W} = \int d^4x e \sigma e^a_{\mu} \langle T^{\mu}_a \rangle, \\ \int d^4x e \xi_{\mu} \mathcal{A}_{\text{diffeo}}^{\mu} &= -\delta_{\xi}^d \mathcal{W} = \int d^4x e \xi^{\nu} \langle \omega_{\nu}{}^{ab} T_{ab} - D^{\mu} T_{\mu\nu} \rangle, \\ \int d^4x e \alpha_{ab} \mathcal{A}_{\text{Lorentz}}^{ab} &= -\delta_{\alpha}^L \mathcal{W} = \int d^4x e \alpha_{ab} \langle T^{ab} \rangle, \end{aligned} \quad (13.9)$$

with the EMT $e T^{\mu}_a = \delta S / \delta e^a_{\mu}$, obtained by treating $\tilde{\psi}$, $\tilde{\bar{\psi}}$ as vierbein-independent.

As mentioned in Sec. 12.2, the diffeomorphism and Lorentz anomalies are absent in $d = 4$. However, we are not safe from spurious diffeomorphism or Lorentz anomalies as the consequence of our regularisation procedure. In light of the controversy, we evaluate all three anomalies.

13.2 Proper time regularisation

In order to perform a concrete computation the singular operator \mathcal{D}^{-1} , in (13.4), has to be regularised. Following [164] we use the proper time regularisation¹

$$\mathcal{D}^{-1}|_{\Lambda} = \int_{\Lambda^{-2}}^{\infty} dt \mathcal{D}^{\dagger} e^{-t\mathcal{D}\mathcal{D}^{\dagger}}, \quad (13.10)$$

which is convergent in the infrared, that is $t \rightarrow \infty$, since $\mathcal{D}\mathcal{D}^{\dagger}$ is a manifestly positive operator, but still requires an ultraviolet regulator Λ . Inserting this expression above, the integral easily evaluates to

$$\delta \log \det \mathcal{D}|_{\Lambda} = \text{Tr} \delta \mathcal{D} \int_{\Lambda^{-2}}^{\infty} dt \mathcal{D}^{\dagger} e^{-t\mathcal{D}\mathcal{D}^{\dagger}} = \text{Tr} \delta \mathcal{D} \mathcal{D}^{-1} e^{-\frac{\mathcal{D}\mathcal{D}^{\dagger}}{\Lambda^2}}. \quad (13.11)$$

It is well-known that the anomalies are exactly marginal, that is Λ^0 -terms. The divergences in Λ^4 and Λ^2 are of no special interest and will therefore not be discussed any further. In what follows we will apply this regularisation to a Dirac and a Weyl fermion. The Dirac case is beyond doubt in the literature but serves to test and illustrate the method.

¹Alternatively, $\mathcal{D}^{-1}|_{\Lambda} = \int_{\Lambda^{-2}}^{\infty} dt e^{-t\mathcal{D}^{\dagger}\mathcal{D}} \mathcal{D}^{\dagger}$ could have been chosen.

13.2.1 Dirac fermion

For a Dirac fermion the operator \mathcal{D} assumes the form

$$\mathcal{D} \rightarrow ie^\mu{}_a \gamma^a D_\mu = i\not{D}, \quad (13.12)$$

where D_μ is given by (12.27), when acting on $\tilde{\psi}$. The fact that it is hermitian $\mathcal{D}^\dagger = \mathcal{D}$, makes this case particularly simple since the regulator $e^{-t\mathcal{D}\mathcal{D}^\dagger} = e^{-t\mathcal{D}^2}$ then commutes with both \mathcal{D} and \mathcal{D}^{-1} . The Weyl variation reads

$$\delta_\sigma^W \mathcal{D} = -\sigma \mathcal{D} - \frac{1}{2}[\mathcal{D}, \sigma]. \quad (13.13)$$

Therefore the trace anomaly is given by

$$\begin{aligned} \delta_\sigma^W \mathcal{W} &= \lim_{\Lambda \rightarrow \infty} \text{Tr} \left(\sigma \mathcal{D} + \frac{1}{2}[\mathcal{D}, \sigma] \right) \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}} \\ &= \lim_{\Lambda \rightarrow \infty} \text{Tr} \left(\sigma e^{-\frac{\mathcal{D}^2}{\Lambda^2}} - \frac{1}{2} \sigma [\mathcal{D}, \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}}] \right) = \lim_{\Lambda \rightarrow \infty} \text{Tr} \sigma e^{-\frac{(i\not{D})^2}{\Lambda^2}}, \end{aligned} \quad (13.14)$$

where the cyclicity of the functional trace has been used. To evaluate the last term we use the CDE and obtain

$$\mathcal{A}_{\text{trace}}^{\text{Dirac}} = \frac{1}{16\pi^2} \left(\frac{1}{72} R^2 - \frac{1}{45} R_{\mu\nu} R^{\mu\nu} - \frac{7}{360} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{30} \square R \right), \quad (13.15)$$

which is in agreement with our Fujikawa computation from Sec. 12.4.3. It remains to verify that there are no spurious Lorentz and diffeomorphism anomalies. Using $\delta_\alpha^L \mathcal{D} = \frac{1}{2}[\mathcal{D}, \alpha_{ab} \Sigma^{ab}]$, the Lorentz-variation reads

$$\delta_\alpha^L \mathcal{W}|_\Lambda = -\text{Tr} \delta_\alpha^L \mathcal{D} \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}} = \text{Tr} \frac{1}{2} \alpha_{ab} \Sigma^{ab} [\mathcal{D}, \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}}] = 0, \quad (13.16)$$

and thus $\mathcal{A}_{\text{Lorentz}}^{ab} = 0$ follows. The diffeo-variation of \mathcal{D} reads

$$\delta_\xi^d \mathcal{D} = -[\mathcal{D}, \xi^\mu] \nabla_\mu - \xi^\mu [\mathcal{D}, \nabla_\mu] - \frac{1}{2}[\mathcal{D}, (D_\mu \xi^\mu)], \quad (13.17)$$

where ∇ is the covariant derivative deprived of spin-connection²

$$\nabla_\mu = D_\mu - \omega_\mu. \quad (13.18)$$

The diffeo-variation reads

$$\begin{aligned} \delta_\xi^d \mathcal{W}|_\Lambda &= -\text{Tr} \delta_\xi^d \mathcal{D} \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}} \\ &= -\text{Tr} \left\{ \xi^\mu [\mathcal{D}, \nabla_\mu \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}}] - \xi^\mu [\mathcal{D}, \nabla_\mu] \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}} + \frac{1}{2} (\nabla_\mu \xi^\mu) [\mathcal{D}, \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}}] \right\} \\ &= -\text{Tr} \left\{ \xi^\mu \mathcal{D} \nabla_\mu \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}} - \xi^\mu \nabla_\mu e^{-\frac{\mathcal{D}^2}{\Lambda^2}} - \xi^\mu \mathcal{D} \nabla_\mu \mathcal{D}^{-1} e^{-\frac{\mathcal{D}^2}{\Lambda^2}} + \xi^\mu \nabla_\mu e^{-\frac{\mathcal{D}^2}{\Lambda^2}} \right\} = 0. \end{aligned} \quad (13.19)$$

It is noted that both the Lorentz and diffeomorphism anomalies are vanishing prior to taking the limit $\Lambda \rightarrow \infty$. That the Lorentz and diffeomorphism symmetry are not anomalous for Dirac fermions in any even dimension is a known result and further validates the method. For example, had we used the standard path integral measure $\mathcal{D}\psi \mathcal{D}\bar{\psi}$, instead of $\mathcal{D}\tilde{\psi} \mathcal{D}\tilde{\bar{\psi}}$, we would have obtained a non-vanishing but spurious diffeomorphism anomaly and the wrong trace anomaly [118]. The correct one would then be obtained by adding a local counterterm that removes the spurious diffeomorphism anomaly and leads to the correct trace anomaly. Note that no spurious Lorentz anomaly would arise since the determinant of the vierbein is Lorentz-invariant.

² ∇ only contracts indices in the tangent and base spaces, e.g. $D_\mu \xi^\nu = \nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma_{\mu\rho}^\nu \xi^\rho$ and $D_\mu \xi^a = \nabla_\mu \xi^a = \partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b$, but $\nabla_\mu \psi = \partial_\mu \psi \neq D_\mu \psi$.

13.2.2 Weyl fermion

In this section and the remaining part of the paper, \mathcal{D} is the Weyl operator and is given by (13.6). In order to compute the variation of the Weyl operator $\delta\mathcal{D}$ it is easier to evaluate its Dirac counterpart $\delta i\mathcal{D}$ as in the previous section and then project it in the left-right basis using Eq. (D.8). We obtain

$$\begin{aligned}\delta_\sigma^W \mathcal{D} &= -\sigma\mathcal{D} - \frac{1}{2}[\mathcal{D}, \sigma], \\ \delta_\xi^d \mathcal{D} &= -[\mathcal{D}, \xi^\mu]\nabla_\mu - \xi^\mu[\mathcal{D}, \nabla_\mu] - \frac{1}{2}[\mathcal{D}, (D_\mu\xi^\mu)], \\ \delta_\alpha^L \mathcal{D} &= [\mathcal{D}, \frac{1}{2}\alpha_{ab}\mu^{ab}] + \frac{1}{2}\alpha_{ab}(\mu^{ab} - \lambda^{ab})\mathcal{D},\end{aligned}\tag{13.20}$$

where μ^{ab} and λ^{ab} are defined in App.D, and we note that their form is similar to the Dirac case except for the Lorentz transformation.

Let us first verify that the diffeomorphism and Lorentz anomalies vanish, such that the regularisation induces no spurious gravitational anomaly. The diffeomorphism anomaly is given by

$$\delta_\xi^d \mathcal{W} = -\lim_{\Lambda \rightarrow \infty} \text{Tr} \delta_\xi^d \mathcal{D} \mathcal{D}^{-1} e^{-\frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2}} = -\lim_{\Lambda \rightarrow \infty} \text{Tr} \left(\xi^\mu \nabla_\mu + \frac{1}{2}(D_\mu\xi^\mu) \right) \left(e^{-\frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2}} - e^{-\frac{\mathcal{D}^\dagger\mathcal{D}}{\Lambda^2}} \right),\tag{13.21}$$

where the cyclicity of the trace, Eq. (D.7) and

$$\mathcal{D}^{-1} e^{-\frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2}} \mathcal{D} = e^{-\frac{\mathcal{D}^\dagger\mathcal{D}}{\Lambda^2}},\tag{13.22}$$

have been used. Noting that

$$i\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix}, \quad e^{-\frac{(i\mathcal{D})^2}{\Lambda^2}} = \begin{pmatrix} e^{-\frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2}} & 0 \\ 0 & e^{-\frac{\mathcal{D}^\dagger\mathcal{D}}{\Lambda^2}} \end{pmatrix},\tag{13.23}$$

we can recast Eq. (13.21) in Dirac space

$$\delta_\xi^d \mathcal{W} = -\lim_{\Lambda \rightarrow \infty} \text{Tr} \left(\gamma_5 \left(\xi^\mu \nabla_\mu + \frac{1}{2}(D_\mu\xi^\mu) \right) e^{-\frac{(i\mathcal{D})^2}{\Lambda^2}} \right).\tag{13.24}$$

A direct computation using the CDE in curved spacetime shows that it vanishes. Importantly, the diffeomorphism anomaly may a priori not be covariant since ∇ in (13.24) is not the total covariant derivative. This is where our CDE [3] presented in Part II is useful since it easily allows for an expansion that is not manifestly covariant; with more details in App. E. It is noted that the heat kernel with Seeley-DeWitt coefficients [25, 86, 87] and former CDE in curved spacetime approaches [77, 78] are designed to compute traces involving a quadratic operator, such as $\text{Tr} a(x)e^{-D^2}$ where a is not a differential operator. With some work a trace of the form (13.24) can be brought to this form, but it is not straightforward and involves lengthy manipulations [164].

The Lorentz anomaly is given by

$$\delta_\alpha^L \mathcal{W} = -\lim_{\Lambda \rightarrow \infty} \text{Tr} \frac{1}{2}\alpha_{ab} \left(\mu^{ab} e^{-\frac{\mathcal{D}^\dagger\mathcal{D}}{\Lambda^2}} - \lambda^{ab} e^{-\frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2}} \right),\tag{13.25}$$

and once again it can be rewritten as a trace in Dirac space

$$\delta_\alpha^L \mathcal{W} = \lim_{\Lambda \rightarrow \infty} \text{Tr} \frac{1}{2}\alpha_{ab} \Sigma^{ab} \gamma_5 e^{-\frac{(i\mathcal{D})^2}{\Lambda^2}}.\tag{13.26}$$

The direct computation using the CDE shows that it equally vanishes.

We now turn to the trace anomaly. Using (13.22) one obtains

$$\delta_\sigma^W \mathcal{W} = \frac{1}{2} \lim_{\Lambda \rightarrow \infty} \text{Tr} \sigma \left(e^{-\frac{\mathcal{D}^\dagger \mathcal{D}}{\Lambda^2}} + e^{-\frac{\mathcal{D} \mathcal{D}^\dagger}{\Lambda^2}} \right) = \frac{1}{2} \lim_{\Lambda \rightarrow \infty} \text{Tr} \sigma e^{-\frac{(i\mathcal{D})^2}{\Lambda^2}}, \quad (13.27)$$

which is half the trace anomaly of a Dirac fermion

$$\mathcal{A}_{\text{trace}}^{\text{Weyl}} = \frac{1}{2} \mathcal{A}_{\text{trace}}^{\text{Dirac}}, \quad (13.28)$$

without spurious gravitational anomalies

$$\mathcal{A}_{\text{diffeo}}^\mu = \mathcal{A}_{\text{Lorentz}}^{ab} = 0. \quad (13.29)$$

In particular there is no Pontryagin density $R\tilde{R}$. We wish to emphasise that each term in (13.27) has an $R\tilde{R}$ -component, but it cancels in the sum of the two. This is in agreement with the computation of the heat kernel coefficient b_4 of the representation $(1/2, 0)$ of the Lorentz group [162], and we just showed that the trace anomaly of a Weyl fermion is determined by $b_4(1/2, 0) + b_4(0, 1/2)$. We further note that the second term in (13.27) originates from the second one in Eq. (13.20) which in turn is due to the spin-connection.

We note that this result has previously been obtained in a similar setting by Leutwyler and Mallik [164]. The main difference is in the evaluation of the expression in (13.24) for which they use the heat kernel which is rather laborious. In addition they use the fact that the Lorentz anomaly is not present in $d = 4$ and do not proceed to evaluate the corresponding term. Hence we improve on their work in verifying the vanishing of the Lorentz anomaly explicitly and can do so in an economic manner.

13.3 Fujikawa method adapted to Weyl fermions

The same results can be obtained adapting the path integral derivation of anomalies by Fujikawa [116, 158] (cf. also [20, 118]) to two-components Weyl fermions. As far as we know, only Dirac fermions with a projector P_L are considered in the literature, which suffer from an ill-defined path integral due to the non-invertibility of $i\mathcal{D}P_L$. In the path integral, the anomaly arises from a non-trivial Jacobian which can be written as a fraction of determinants [4]³

$$\begin{aligned} J[\alpha] &= e^{-\int d^4x e \alpha(x) \mathcal{A}(x)} \\ &= \frac{\det(\mathcal{D})}{\det(\mathcal{D} - \delta_\alpha \mathcal{D})} = \frac{1}{\det(\mathbb{1} - \delta_\alpha \mathcal{D} \mathcal{D}^{-1})} = \exp(\text{Tr}[\delta_\alpha \mathcal{D} \mathcal{D}^{-1}] + \mathcal{O}(\alpha^2)), \end{aligned} \quad (13.30)$$

and when expanded assumes the same form as in (13.4). This guarantees that it is well-defined, that is to say the global phase of the determinant cancels in this expression and the operator $\delta_\alpha \mathcal{D} \mathcal{D}^{-1}$ maps into the same Hilbert space as mentioned previously.

We proceed to construct the path integral measure. The Weyl operator (13.6) is not hermitian and does not have a well-defined eigenvalue problem. However, since $i\mathcal{D}$ is hermitian and $i\mathcal{D} : (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \rightarrow (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, the eigenvalue problem $i\mathcal{D}\phi_n = \lambda_n \phi_n$ is well-posed. In particular

³Let us comment on the sign in $\det(\mathcal{D} - \delta_\alpha \mathcal{D})$. In the Fujikawa approach one transforms the path integral variables only and since $\delta_\alpha S[\tilde{\psi}, \tilde{\bar{\psi}}, e] = \delta_\alpha \tilde{\psi} \frac{\delta S}{\delta \tilde{\psi}} + \delta_\alpha \tilde{\bar{\psi}} \frac{\delta S}{\delta \tilde{\bar{\psi}}} + \delta_\alpha e^\mu_a \frac{\delta S}{\delta e^\mu_a} = 0$, this then implies that it is $-\delta_\alpha \mathcal{D}$, which stems from $\delta_\alpha e^\mu_a \frac{\delta S}{\delta e^\mu_a}$, that appears under the α -variation.

$\lambda_n \in \mathbb{R}$ and the eigenfunctions $\{\phi_n\}$ define a complete orthonormal basis. These functions can be decomposed as follows

$$\phi_n = \begin{pmatrix} \phi_n^R \\ \phi_n^L \end{pmatrix}, \quad (13.31)$$

where $\{\phi_n^{R,L}\}$ define orthonormal eigenbases of right- and left-handed Weyl fermions respectively. The eigenvalue equations become

$$\mathcal{D}\phi_n^L = i\bar{\sigma}^\mu(\partial + \omega_L)_\mu\phi_n^L = \lambda_n\phi_n^R, \quad \mathcal{D}^\dagger\phi_n^R = i\sigma^\mu(\partial + \omega_R)_\mu\phi_n^R = \lambda_n\phi_n^L. \quad (13.32)$$

We thus decompose the path integral measure into these eigenbases

$$\tilde{\psi}_L = \sqrt{e} \sum_n a_n \phi_n^L, \quad \tilde{\bar{\psi}}_L = \sqrt{e} \sum_n \bar{b}_n (\phi_n^R)^\dagger, \quad (13.33)$$

for which the measure assumes the form⁴

$$\mathcal{D}\tilde{\psi}_L \mathcal{D}\tilde{\bar{\psi}}_L = \prod_n da_n d\bar{b}_n. \quad (13.34)$$

In order to evaluate the Jacobian, we need to determine how a variation δ_α acts on the Weyl fermion $\tilde{\psi}'_L \equiv \tilde{\psi}_L + \delta_\alpha \tilde{\psi}_L$ and its barred counterpart. We expand $\tilde{\psi}'_L$ and $\tilde{\bar{\psi}}'_L$ into the eigenbases as in (13.33) to obtain

$$a'_m = \sum_n (\delta_{mn} + A_{mn}) a_n, \quad \bar{b}'_m = \sum_n (\delta_{mn} + B_{mn}) \bar{b}_n. \quad (13.35)$$

where A and B are of $\mathcal{O}(\alpha)$, and the resulting Jacobian of the Grassmann variables reads

$$\log J = -\text{Tr} A - \text{Tr} B + \mathcal{O}(\alpha^2). \quad (13.36)$$

Using the orthonormality of the eigenbasis, we obtain the Jacobians of the transformations δ_σ^W , δ_ξ^d and δ_α^L ,

$$\begin{aligned} \log J_{\text{trace}}[\sigma] &= - \int d^4x e \frac{\sigma}{2} \sum_n \left((\phi_n^R)^\dagger \phi_n^R + \{R \leftrightarrow L\} \right), \\ \log J_{\text{diffeo}}[\xi^\mu] &= - \int d^4x e \sum_n \left((\phi_n^R)^\dagger (\xi^\mu \nabla_\mu + \frac{1}{2} (D_\mu \xi^\mu)) \phi_n^R - \{R \leftrightarrow L\} \right), \\ \log J_{\text{Lorentz}}[\alpha_{ab}] &= - \int d^4x e \frac{1}{2} \alpha_{ab} \sum_n \left((\phi_n^R)^\dagger \lambda^{ab} \phi_n^R - \{R, \lambda \leftrightarrow L, \mu\} \right). \end{aligned} \quad (13.37)$$

These expressions are ultraviolet divergent. They can be regularised by introducing $e^{-\frac{\lambda_n^2}{\Lambda^2}}$ into the sums, following Fujikawa. Note that J_{diffeo} and J_{Lorentz} are fully determined by the zero modes whereas J_{trace} is determined by the zero modes only at lowest order of the loop expansion [118]. This means that the diffeomorphism and Lorentz anomalies are of topological nature and determined at the one-loop level, as also argued by [76].

Using the eigenvalue equation Eq. (13.32) the following replacement can be made

$$e^{-\frac{\lambda_n^2}{\Lambda^2}} \phi_n^L = e^{-\frac{\mathcal{D}^\dagger \mathcal{D}}{\Lambda^2}} \phi_n^L, \quad e^{-\frac{\lambda_n^2}{\Lambda^2}} \phi_n^R = e^{-\frac{\mathcal{D} \mathcal{D}^\dagger}{\Lambda^2}} \phi_n^R, \quad (13.38)$$

⁴This change of variable is defined up to a phase. As emphasised in Sec. 13.1.1, this phase is irrelevant when dealing with the covariant form of anomalies which is the case here [118].

where \mathcal{D} is the Weyl operator (13.6). As in the previous section the Jacobians can be recast in Dirac space using Eq. (13.23)

$$\begin{aligned}\log J_{\text{trace}}[\sigma] &= -\delta_{\sigma}^W \mathcal{W} = -\frac{1}{2} \lim_{\Lambda \rightarrow \infty} \text{Tr} \sigma e^{-\frac{(i\mathcal{D})^2}{\Lambda^2}}, \\ \log J_{\text{diffeo}}[\xi] &= -\delta_{\xi}^d \mathcal{W} = +\frac{1}{2} \lim_{\Lambda \rightarrow \infty} \text{Tr} \gamma_5 \left(\xi^{\mu} \nabla_{\mu} + \frac{1}{2} (D_{\mu} \xi^{\mu}) \right) e^{-\frac{(i\mathcal{D})^2}{\Lambda^2}} = 0, \\ \log J_{\text{Lorentz}}[\alpha] &= -\delta_{\alpha}^L \mathcal{W} = -\frac{1}{2} \lim_{\Lambda \rightarrow \infty} \text{Tr} \alpha_{ab} \Sigma^{ab} \gamma_5 e^{-\frac{(i\mathcal{D})^2}{\Lambda^2}} = 0.\end{aligned}\tag{13.39}$$

These results are identical to Eqs. (13.24), (13.26) and (13.27) found in the proper time regularisation and can thus be seen as a confirmation.

13.4 Dimensions other than 4

In $d = 4$ we concluded that for free Weyl fermions the $R\tilde{R}$ -term is absent in the trace anomaly. It is therefore natural to ask whether P - and or CP -odd terms could be present in any other even dimension.⁵ In fact up to Eqs. (13.24), (13.26) and (13.27) the expressions are independent of the dimension in our approach. In particular the factor $\frac{1}{2}$ in (13.27) makes it clear that the trace anomaly of a Weyl fermion remains half the trace anomaly of a Dirac fermion in any even dimensional spacetime. Even in the absence of concrete computations one can make interesting observations and come to the same conclusion:

- It is important to distinguish $d = 2 \pmod{4}$ and $d = 4 \pmod{4}$ as their (Euclidean) Weyl representation are complex and real respectively [76]. For example, Lorentz and diffeomorphism anomalies cannot arise in $d = 4 \pmod{4}$ dimensions since the reality of the representation allows for a Pauli-Villars regulator mass term which is symmetry-preserving and this means that no diffeomorphism and Lorentz anomaly can appear [76]. Another way to look at it is that in $d = 4 \pmod{4}$, unlike in $d = 2 \pmod{4}$, the CPT operation flips the chirality such that the Weyl fermions effectively look vector-like [76] (and [190] for more detail).
- In $d = 2 \pmod{4}$ a P - and CP -odd term should not violate CPT since the appearance of the factor of i is dimension-dependent. For example in Minkowski space one has $\text{Tr}[\gamma_{\alpha}\gamma_{\beta}\gamma_5] = 2\epsilon_{\alpha\beta}$ in $d = 2$ whereas $\text{Tr}[\gamma_{\alpha}\gamma_{\beta}\gamma_{\gamma}\gamma_{\delta}\gamma_5] = 4i\epsilon_{\alpha\beta\gamma\delta}$ in $d = 4$, and this is related to the complex and real representations mentioned above. Hence in $d = 2 \pmod{4}$ we would not expect an imaginary prefactor and thus no CPT -violation.
- However, one can argue that in $d = 2 \pmod{4}$ one cannot write down a P - or CP -odd diffeoinvariant scalar of mass dimension d (which is relevant for the trace anomaly). Firstly, let us note that due to the Bianchi identity, $\epsilon^{\dots\mu\nu\rho} R_{\alpha\mu\nu\rho} = 0$ holds in any dimension, that is to say the Levi-Civita tensor cannot contract more than two indices of a Riemann tensor. For concreteness let us first focus on $d = 6$. Using the Bianchi identities one can show that a parity-odd operator of mass dimension 6 can only be of the form,

$$\epsilon^{\alpha_1 \dots \alpha_6} R_{\alpha_1 \alpha_2 \dots} R_{\alpha_3 \alpha_4 \dots} R_{\alpha_5 \alpha_6 \dots} \tag{13.40}$$

The only way to contract these Riemann tensors together is

$$\epsilon^{\alpha_1 \dots \alpha_6} R_{\alpha_1 \alpha_2 \mu}{}^{\nu} R_{\alpha_3 \alpha_4 \rho \nu} R_{\alpha_5 \alpha_6}{}^{\rho \mu} = 0, \tag{13.41}$$

⁵Odd dimensional spaces are beyond the scope of this paper and we refer the reader to [189] for intricate relations with the even dimensional case.

which vanishes since $\epsilon^{\alpha_1 \dots \alpha_6} R_{\alpha_1 \alpha_2 \mu}{}^\nu R_{\alpha_3 \alpha_4 \rho \nu}$ is symmetric under $\mu \leftrightarrow \rho$ whereas $R_{\alpha_5 \alpha_6}{}^{\rho \mu}$ is antisymmetric. On the other hand, in $d = 8$ for example, there is an even number of Riemann tensors, which can then be contracted in pairs

$$\epsilon^{\alpha_1 \dots \alpha_8} R_{\alpha_1 \alpha_2 \mu \nu} R_{\alpha_3 \alpha_4}{}^{\mu \nu} R_{\alpha_5 \alpha_6 \rho \sigma} R_{\alpha_7 \alpha_8}{}^{\rho \sigma} \neq 0. \quad (13.42)$$

This generalises straightforwardly to any even dimension

$$\epsilon^{\alpha_1 \dots \alpha_{2n}} R_{\alpha_1 \alpha_2 \dots} \dots R_{\alpha_{2n-1} \alpha_{2n} \dots} \begin{cases} = 0, & d = 2 \pmod{4} \\ \neq 0, & d = 4 \pmod{4} \end{cases}, \quad (13.43)$$

since it involves an odd number of Riemann tensors in $d = 2 \pmod{4}$ but an even number in $d = 4 \pmod{4}$. In fact, the absence of such a term in $d = 6$ has been inferred from a cohomology-type argument given a long time ago [147].

In summary the absence of parity-odd terms can be established in $d = 2 \pmod{4}$ without explicit computation whereas in $d = 4 \pmod{4}$ a computation is required.

Chapter 14

Generic approach to the Weyl anomaly

In the previous Section, we computed the Weyl anomaly specifically for a Weyl fermion, and concluded that the Pontryagin density is absent.

In this Section, we present our work [1] in which we investigate in a model-independent manner the generic form of the Weyl anomaly, including P -odd operators

$$g^{\alpha\beta}\langle T_{\alpha\beta}\rangle = a E + b R^2 + c W^2 + d \square R + e R\tilde{R} + f F^2 + h F\tilde{F} . \quad (14.1)$$

As seen in Sec. 12.4.4, some constraints on these coefficients were worked out in [169], but only concern the P -even operators, and classically Weyl invariant theories.

In our work [1], we derive the vanishing $e = 0$ and $h = 0$ from the decomposition of the EMT-counterterms and our results hold for all theories whose symmetries are compatible with dimensional regularisation, which includes the spin-3/2 case (and also spin-2) [83, 162, 191–198]. In addition we consider the case of explicit breaking of Weyl symmetry. Constraints are implied by the *finiteness* of the diffeomorphism (diffeo) and Lorentz anomalies.

In the following, we demonstrate the commonly used expressions for the Weyl anomaly with and without explicit breaking, hence filling a gap in the literature concerning the expression of Weyl anomaly with explicit breaking. We then describe how to use the finiteness of the diffeomorphism and Lorentz anomalies to constrain the form of the Weyl anomaly and conclude on the absence of the $R\tilde{R}$ term. The same procedure can be applied in the presence of a gauge sector to conclude on the absence of the $F\tilde{F}$ term, although we do not detail it here. We refer the interested reader to the article [1] for more details.

14.1 The Weyl anomaly with and without explicit breaking

In Sec. 12.4.3 we claimed that in dimensional regularisation, the Weyl anomaly in a classically Weyl invariant theory is given by [167]

$$\mathcal{A}_{\text{Weyl}} = g_{\mu\nu}\langle T^{\mu\nu}\rangle . \quad (14.2)$$

In this Section, we provide a different proof of this formula.

In a theory with explicit breaking of the Weyl symmetry, for example a theory of a massive field, this formula has to be amended to isolate the quantum breaking from the explicit breaking and reads [199–202]

$$\mathcal{A}_{\text{Weyl}} = g_{\mu\nu}\langle T^{\mu\nu}\rangle - \langle T^\mu{}_\mu \rangle . \quad (14.3)$$

To the best of our knowledge, this generally accepted formula was never proved. We proceed to demonstrate it in a model-independent manner.

14.1.1 Preliminary definitions

Let us consider a theory where quantum fields are collectively denoted by ϕ , and the metric is considered as an external field. In a space with Euclidean signature the quantum effective action \mathcal{W} is given by

$$\mathcal{W}[g_{\mu\nu}] = -\log \int \mathcal{D}\phi e^{-S[g_{\mu\nu}, \phi]} . \quad (14.4)$$

It is a divergent quantity that we regularise using dimensional regularisation, and by introducing renormalising counterterms. The renormalised effective action is

$$\mathcal{W}_{\text{ren}}(d) = \mathcal{W}(d) + \mathcal{W}_{\text{ct}}(d) , \quad (14.5)$$

and is finite when the regulator is removed, i.e. $d \rightarrow 4$. The separate pieces $\mathcal{W}(d)$ and $\mathcal{W}_{\text{ct}}(d)$ contain $\frac{1}{d-4}$ -terms at leading order (LO) [169]

$$\mathcal{W}_{\text{ct}}(d) = \frac{1}{d-4} \int d^d x \sqrt{|g|} (a E(d) + b R^2(d) + c C(d)) , \quad (14.6)$$

and the integrand is a local polynomial in the background fields. Note that no $\square R$ -term is included at the level of the action since it is a total derivative. The possibility of adding a P and CP -violating term $R\tilde{R}$ will be discussed in Sec. 14.4. For a classically Weyl invariant theory, the Weyl consistency conditions imply $b = 0$ [169]. The remaining two terms E and C are given by

$$\begin{aligned} E(d) &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 , \\ C(d) &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 , \end{aligned} \quad (14.7)$$

where $E(4)$ reduces to the topological Euler density and $C(4)$ to the Weyl tensor squared that we have encountered in Sec. 12.4.4. The curvature invariants in non-integer dimension are understood as power series in the metric [167].

As previously, the (unrenormalised) EMT and its VEV are given by

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} , \quad \langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{|g|}} \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}} . \quad (14.8)$$

The Weyl variation of the action yields the (classical) trace of the EMT (TEMT) when ϕ is on-shell

$$\delta_\sigma^W S = \int d^4 x \sqrt{|g|} \sigma T^\rho_\rho . \quad (14.9)$$

Our goal is to demonstrate the form of the anomaly with explicit breaking¹

$$\mathcal{A}_{\text{Weyl}} = g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle - \langle g_{\mu\nu}^{(d)} T^{\mu\nu} \rangle , \quad (14.10)$$

within dimensional regularisation.²

¹The superscript denotes the dimension in which the metric lives and we have $g_{\mu\nu}^{(d)} = g_{\mu\nu}^{(4)} + \mathcal{O}(d-4)$. When the superscript is omitted d -dimensional tensors are assumed.

²A new definition of the anomaly beyond dimensional regularisation has very recently been proposed in [203].

14.2 Weyl anomaly with classical Weyl invariance

Let us first consider the case of a theory which is classically Weyl invariant: $\delta_\sigma^W S = 0$. To define the anomaly from the unrenormalised EMT bears an ambiguity in that the Weyl variation can be performed in four or d dimension: $g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle \neq \langle T^\mu{}_\mu \rangle$. This ambiguity is absent if one directly defines the Weyl anomaly from the renormalised effective action

$$\int d^4x \sqrt{|g|} \sigma \mathcal{A}_{\text{Weyl}} \equiv \delta_\sigma^W \lim_{d \rightarrow 4} \mathcal{W}_{\text{ren}}(d) = \lim_{d \rightarrow 4} \delta_\sigma^W \mathcal{W}_{\text{ren}}(d), \quad (14.11)$$

since the finiteness of \mathcal{W}_{ren} guarantees that the Weyl variation and the $d \rightarrow 4$ limit commute. Applying the first definition results in³

$$\delta_\sigma^W \lim_{d \rightarrow 4} \mathcal{W}_{\text{ren}}(d) \equiv \int d^4x \sigma g_{\mu\nu}^{(4)} \left[\frac{\delta \mathcal{W}_{\text{ren}}(d)}{\delta g_{\mu\nu}} \right]_{d=4} = \int d^4x \sigma \sqrt{|g|} g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle, \quad (14.12)$$

since classical Weyl invariance implies [169]

$$\left[g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}}{\delta g_{\mu\nu}} \right]_{d=4} = 0. \quad (14.13)$$

If we apply the second definition we obtain, the expression used in [169],

$$\lim_{d \rightarrow 4} \delta_\sigma^W \mathcal{W}_{\text{ren}}(d) \equiv \int d^4x \sigma \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ren}}(d)}{\delta g_{\mu\nu}} \right]_{d=4} = \int d^4x \sigma \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{d=4}, \quad (14.14)$$

since

$$\left[g_{\mu\nu}^{(d)} \langle T^{\mu\nu} \rangle \right]_{d=4} = \left[\langle T^\mu{}_\mu \rangle \right]_{d=4} = 0, \quad (14.15)$$

by classical Weyl invariance, with more details in App. F. This demonstrates formula (14.10) for a classically Weyl invariant theory and clarifies why the Weyl anomaly can be obtained in different ways. Similar reasonings have been applied in [202] (cf. Sec. 1.2.3.) to show that the anomaly can be obtained in both ways.

Note that the form of the anomaly is constrained by the Wess-Zumino consistency conditions [127] in a classically Weyl invariant theory, since it is given by the variation of a functional (14.11).⁴ This aspect will be exploited in Sec. 14.4.2.

Anomalies are obtained by the variation of the bare effective action \mathcal{W} and since they are finite, the variation of \mathcal{W}_{ct} does not enter in general. We will use this invariance of \mathcal{W}_{ct} under the diffeo, Lorentz and gauge transformations later on. For the Weyl anomaly, it manifests itself in (14.13). On the other hand, the Weyl anomaly is known to be related to the renormalisation of the theory (e.g [118, 204]). This is apparent in the second expression of the Weyl anomaly (14.14), which is finite since the d -dimensional variation produces a term proportional to $d - 4$.

14.3 Weyl anomaly without classical Weyl invariance

We turn to the case of a theory with explicit Weyl symmetry breaking: $\delta_\sigma^W S \neq 0$. In that case, formula (14.11) is amended⁵

$$\int d^4x \sqrt{|g|} \sigma (\mathcal{A}_{\text{Weyl}} + \mathcal{E}_{\text{Weyl}}) = \delta_\sigma^W \lim_{d \rightarrow 4} \mathcal{W}_{\text{ren}}(d) = \lim_{d \rightarrow 4} \delta_\sigma^W \mathcal{W}_{\text{ren}}(d), \quad (14.16)$$

³This expression of the anomaly was found in [167] in $d = 2 - \epsilon$ using a Taylor expansion in ϵ . This assumes continuity of the tensors which may not always hold, as we will discuss in Sec. 14.4.

⁴The Wess-Zumino consistency conditions for the Weyl transformation have been investigated for a Weyl invariant theory in full generality some time ago [204].

⁵Note that (14.16) equals to $\left[g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle \right]_{\text{fin}}$ with “fin” referring to the finite part.

to include a term that parametrises the explicit breaking. It is worthwhile to emphasise that both $\mathcal{A}_{\text{Weyl}}$ and $\mathcal{E}_{\text{Weyl}}$ are finite quantities. The results in the previous section, that is Eqs. (14.13) and (14.15), will not apply as \mathcal{W}_{ct} does not need to consist of Weyl invariant terms. Applying the two definitions one gets

$$\begin{aligned} \delta_\sigma^W \lim_{d \rightarrow 4} \mathcal{W}_{\text{ren}}(d) &= \int d^4x \sigma \left[\sqrt{|g|} g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle + g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{d=4}, \\ \lim_{d \rightarrow 4} \delta_\sigma^W \mathcal{W}_{\text{ren}}(d) &= \int d^4x \sigma \left[\sqrt{|g|} \langle T^\mu{}_\mu \rangle + g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{d=4}, \end{aligned} \quad (14.17)$$

upon using Eq. (14.5). In each line, both terms are separately divergent but finite in their sum. The main idea is to define $\mathcal{E}_{\text{Weyl}}$ from the finite part of $\langle T^\mu{}_\mu \rangle$ and, using the second line of (14.17), this further implies⁶

$$\mathcal{E}_{\text{Weyl}} \equiv \langle T^\mu{}_\mu \rangle_{\text{fin}}, \quad \mathcal{A}_{\text{Weyl}} = \frac{1}{\sqrt{|g|}} \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{\text{fin}}, \quad (14.18)$$

where the subscript fin (div) indicates that only the finite (divergent) part is taken. Combining with the first line of (14.17) one gets

$$\mathcal{A}_{\text{Weyl}} = g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle - \langle T^\mu{}_\mu \rangle_{\text{fin}} + \frac{g_{\mu\nu}^{(4)}}{\sqrt{|g|}} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}}. \quad (14.19)$$

If we further use

$$g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} = -\langle T^\mu{}_\mu \rangle_{\text{div}} \quad (14.20)$$

which follows from the fact that the left-hand side vanishes in a classically Weyl invariant theory, we then get

$$\mathcal{A}_{\text{Weyl}} = g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle - \langle T^\mu{}_\mu \rangle, \quad (14.21)$$

the formula (14.10) which we promised to obtain by derivation. More details are provided in App. F. This formula is valid to all order in perturbation theory in dimensional regularisation and can be expected to be valid non-perturbatively. Note that now the Weyl anomaly is not the variation of a functional anymore, due to the explicit breaking in (14.16). In that case, the Weyl anomaly does not generally respect the Wess-Zumino consistency conditions.

14.4 Curved-space anomaly constraints on the Weyl anomaly

The goal of this Section is to extend the results in [169] to theories without classical Weyl invariance and to consider the possibility of adding the P - and CP -odd term

$$R\tilde{R} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\mu\nu} R_{\alpha\beta\mu\nu}, \quad (14.22)$$

to the counterterm action given in (14.6).⁷ Unlike the other operators the extension of $R\tilde{R}$ to d -dimension is ambiguous since the Levi-Civita tensor is intrinsically tied to $d = 4$. We may however

⁶ Note that we assumed a purely divergent \mathcal{W}_{ct} (i.e. minimal subtraction scheme). (14.18) has to be amended if we include a finite piece in \mathcal{W}_{ct} , but the anomaly remains unaffected (see App. F).

⁷In another approach, the parity-odd term is investigated in a conformal field theory [205] and it is found that it could appear in 3-point correlators with a marginal operator. We stress that this is a consistency constraint and not a proof of its existence.

proceed formally, without committing to a specific extension, as follows

$$\mathcal{W}_{\text{ct,odd}} = \frac{1}{d-4} \int d^d x \sqrt{|g|} \mathcal{L}_{\text{odd}}(d), \quad (14.23)$$

with $\mathcal{L}_{\text{odd}}(4) \propto R\tilde{R}$ as this is the only possible parity-odd term in $d=4$ (cf. Ref.[2]). Owing to its topological nature one has

$$\frac{\delta}{\delta g_{\alpha\beta}^{(4)}} \int d^4 x \sqrt{g^{(4)}} \mathcal{L}_{\text{odd}}(4) = 0. \quad (14.24)$$

Note that (14.24) does not imply that $\mathcal{L}_{\text{odd}}(d)$ is of $\mathcal{O}(d-4)$. For example, the extension of the P -even topological Euler density to d dimensions is discontinuous around $d=4$ [206, 207]. However, we may parametrise the metric variation of $\mathcal{W}_{\text{ct,odd}}$ in terms of a continuous $(d-4) \times \mathcal{V}^{\alpha\beta}$ and a discontinuous piece $\mathcal{U}^{\alpha\beta}$

$$\frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\alpha\beta}} \int d^d x \sqrt{|g|} \mathcal{L}_{\text{odd}}(d) = (d-4) \mathcal{V}^{\alpha\beta}(4) + \mathcal{U}^{\alpha\beta}(d) + \mathcal{O}((d-4)^2). \quad (14.25)$$

The 2-tensors $\mathcal{V}^{\alpha\beta}(4)$ and $\mathcal{U}^{\alpha\beta}(d)$ are parity-odd, of mass dimension four, and one has

$$\mathcal{U}^{\alpha\beta}(4) = 0, \quad \mathcal{U}^{\alpha\beta}(4-2\epsilon) \neq \mathcal{O}(\epsilon), \quad (14.26)$$

owing to (14.24). Using Bianchi identities, algebra, and intrinsically 4-dimensional identities [208–210] one can show that the only possible operator (cf. App. G) at $d=4$ is

$$\mathcal{V}^{\alpha\beta}(4) = e g^{\alpha\beta} R\tilde{R}, \quad (14.27)$$

where e is a constant to be determined. For our purposes it is not necessary to specify the extension of the ϵ -tensor to d dimensions in parametrising $\mathcal{U}^{\alpha\beta}$ (this is similar to [4, 154] whereby one uses free parameters in order to remain independent of a specific γ_5 -scheme) Using Bianchi identities, which are d -independent, we may write

$$\mathcal{U}^{\mu\nu}(d) = e_1 g^{\mu\nu} R\tilde{R}(d) + e_2 P^{\mu\nu}(d) + e_3 Q^{(\mu\nu)}(d) + e_4 S^{(\mu\nu)}(d), \quad (14.28)$$

where round brackets denote symmetrisation $t^{(\mu\nu)} = \frac{1}{2}(t^{\mu\nu} + t^{\nu\mu})$ and

$$\begin{aligned} g_{\alpha\beta} R\tilde{R}(d) &= g_{\alpha\beta} \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R_{\gamma\delta\mu\nu} R_{\rho\sigma}^{\gamma\delta}, & P_{\alpha\beta}(d) &= \epsilon^{\mu\nu\rho\sigma} R_{\alpha\lambda\mu\nu} R_{\beta}^{\lambda}{}_{\rho\sigma}, \\ Q_{\alpha\beta}(d) &= \epsilon_{\alpha}{}^{\nu\rho\sigma} R_{\beta\nu\gamma\delta} R_{\rho\sigma}^{\gamma\delta}, & S_{\alpha\beta}(d) &= \epsilon_{\alpha}{}^{\nu\rho\sigma} R_{\beta\lambda\rho\sigma} R_{\nu}^{\lambda}, \end{aligned}$$

and any other P -odd symmetric 2-tensor is related to these by algebra and Bianchi identities (see App. G). At $d=4$, the Schouten identity reduces all these tensors to

$$P_{\mu\nu}(4) = Q_{\mu\nu}(4) = \frac{1}{2} R\tilde{R}(4), \quad S_{\mu\nu}(4) = 0, \quad (14.29)$$

and thus $\mathcal{U}^{\alpha\beta}(4) = 0$ implies the constraint

$$2e_1 + e_2 + e_3 = 0. \quad (14.30)$$

As expected, this defines a tensor that vanishes at $d=4$, but *does not* scale as $\mathcal{O}(d-4)$.

Similarly, we can choose a basis of independent P -even operators, and then write a general ansatz for the metric variation of \mathcal{W}_{ct} as follows

$$\begin{aligned} \mathcal{T}_{\text{ct}}^{\alpha\beta} = \frac{1}{\sqrt{|g|}} \frac{\delta\mathcal{W}_{\text{ct}}}{\delta g_{\alpha\beta}} = & e g^{\alpha\beta} R\tilde{R} + \frac{1}{d-4} \left\{ e_1 g^{\alpha\beta} R\tilde{R} + e_2 P^{\alpha\beta} + e_3 Q^{(\alpha\beta)} + e_4 S^{(\alpha\beta)} \right. \\ & + a_1 g^{\alpha\beta} R^2 + a_2 g^{\alpha\beta} R_{\mu\nu} R^{\mu\nu} + a_3 g^{\alpha\beta} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + a_4 g^{\alpha\beta} \square R \\ & + b_1 R R^{\alpha\beta} + b_2 R^{\alpha\lambda} R^\beta{}_\lambda + b_3 R_{\mu\nu} R^{\mu\alpha\nu\beta} + b_4 R^{\alpha\lambda\mu\nu} R^\beta{}_{\lambda\mu\nu} \\ & \left. + c_1 D^\alpha D^\beta R + c_2 \square R^{\alpha\beta} \right\}, \end{aligned} \quad (14.31)$$

We would like to add the following remarks:

- The coefficients $\{a_i, b_i, c_i\}$ are function of the parameters in \mathcal{W}_{ct} and can depend on d , such that the parity-even part $\mathcal{T}_{\text{ct}}^{\alpha\beta}$ can contain finite pieces despite the global $1/(d-4)$ -factor.
- The coefficients $\{e, e_i\}$ depend on the parameters in \mathcal{W}_{ct} and the scheme chosen for the ϵ -tensor. For convenience, the e_i are taken to be independent of d , since this would only amount to relabel e .
- It is possible to write $\delta\mathcal{W}_{\text{ct}}/\delta g_{\alpha\beta}$ under this form because the counterterms \mathcal{W}_{ct} allowed by renormalisation are local polynomials. This is not the case of the effective action \mathcal{W} which is non-local, e.g. [167].
- The ansatz is valid for both theories with and without classical Weyl invariance.
- We only included tensors in \mathcal{T}_{ct} , i.e covariant quantities. This does not amount to the absence of diffeomorphism and Lorentz anomalies, but restricts them to their covariant form [20].

We may trace \mathcal{T}_{ct} in (14.31), using $g_{\mu\nu}^{(d)} g^{(d)\mu\nu} = d$, to obtain

$$\begin{aligned} g_{\alpha\beta}^{(d)} \mathcal{T}_{\text{ct}}^{\alpha\beta} = & (4e + e_1) R\tilde{R} + a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + a_4 \square R \\ & + \frac{1}{d-4} \left\{ (4a_1 + b_1) R^2 + (4a_2 + b_2 + b_3) R_{\mu\nu} R^{\mu\nu} \right. \\ & \left. + (4a_3 + b_4) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + (4a_4 + c_1 + c_2) \square R \right\}, \end{aligned} \quad (14.32)$$

where we used

$$g_{\mu\nu}^{(d)} P^{\mu\nu}(d) = g_{\mu\nu}^{(d)} Q^{\mu\nu}(d) = 2R\tilde{R}(d), \quad g_{\mu\nu}^{(d)} S^{\mu\nu}(d) = 0, \quad (14.33)$$

upon using the Bianchi identities only. Incidentally, we see that there remains no P -odd term in the divergent part upon using (14.30). In other words, the topological nature of the P -odd term in \mathcal{W}_{ct} (Eq. (14.24)), forbids the presence of P -odd divergent terms in the trace of \mathcal{T}_{ct} .

From (14.32) we can compute the Weyl anomaly. That is

$$\begin{aligned} \mathcal{A}_{\text{Weyl}} = [g_{\alpha\beta}^{(d)} \mathcal{T}_{\text{ct}}^{\alpha\beta}]_{d=4} & \quad \text{classically Weyl invariant,} \\ \mathcal{A}_{\text{Weyl}} = [g_{\alpha\beta}^{(d)} \mathcal{T}_{\text{ct}}^{\alpha\beta}]_{\text{fin}, d=4} & \quad \text{classically Weyl non-invariant,} \end{aligned} \quad (14.34)$$

which follow from Eqs. (14.14) and (14.18) respectively. In Sec. 14.4.1 and Sec. 14.4.2 the parameters $\{e, e_i\}$ and $\{a_i, b_i, c_i\}$ will be subjected to diffeomorphism constraints and the Wess-Zumino consistency conditions.

14.4.1 Constraints from the diffeomorphism anomaly

The diffeo-variation of the effective action is given by

$$\delta_\xi^d \mathcal{W} = \int d^4x \delta_\xi^d g_{\mu\nu} \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}} = - \int d^4x \sqrt{|g|} \xi_\mu D_\nu \langle T^{\mu\nu} \rangle, \quad (14.35)$$

where δ_ξ^d is the (active) diffeomorphism transformation, and we used $\delta_\xi^d g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu$. Note that, Eq. (14.35) may be non-zero but remains finite and one-loop exact (it is non-zero in theories in $d = 2 + 4k$ with Weyl fermions for example [76]). Since it is finite, it should not be altered by \mathcal{W}_{ct} as there is no divergence to remove, and therefore

$$\delta_\xi^d \mathcal{W}_{\text{ct}} = \int d^d x \delta_\xi^d g_{\mu\nu} \frac{\delta \mathcal{W}_{\text{ct}}}{\delta g_{\mu\nu}} = - \int d^d x \sqrt{|g|} \xi_\mu D_\nu \mathcal{T}_{\text{ct}}^{\mu\nu} = 0, \quad (14.36)$$

must hold. We thus compute the divergence of \mathcal{T}_{ct} and then enforce

$$D_\nu \mathcal{T}_{\text{ct}}^{\mu\nu} = 0. \quad (14.37)$$

The P -odd and -even pieces are independent of each other. Enforcing (14.37) on the P -even sector yields seven constraints

$$\begin{aligned} 4a_1 + b_1 &= 0, & 4a_2 + b_2 + b_3 &= 0, & 4a_3 + b_4 &= 0, \\ 4a_1 + a_2 + a_4 &= 0, & 8a_3 - b_2 &= 0, & 2a_2 + 8a_3 + c_2 &= 0, \\ 4a_4 + c_1 + c_2 &= -12a_1 - 4a_2 - 4a_3. \end{aligned} \quad (14.38)$$

Remarkably, the same procedure applied to the P -odd sector only admits the trivial solution

$$e = e_1 = e_2 = e_3 = e_4 = 0, \quad (14.39)$$

which implies the absence of P -odd operators in both the counterterms to the EMT, and the Weyl anomaly, for both classically Weyl invariant and non-invariant theories. We stress that this result is independent of both, the underlying theory and the ϵ -tensor scheme. In our previous work [2] we have obtained the same result specifically for a theory with a Weyl fermion only. Inserting these constraints in (14.32) we obtain

$$\begin{aligned} g_{\alpha\beta}^{(d)} \mathcal{T}_{\text{ct}}^{\alpha\beta} &= \frac{a_2 + 4a_3}{2} W^2 - \frac{a_2 + 2a_3}{2} E + \frac{3a_1 + a_2 + a_3}{3} R^2 + \\ &\left(-(4a_1 + a_2) - \frac{4}{d-4} (3a_1 + a_2 + a_3) \right) \square R, \end{aligned} \quad (14.40)$$

an expression in terms of the three coefficients $a_{1,2,3}$. The $1/(d-4)$ -term is addressed just below.

14.4.2 Constraints from classical (non-)invariance

As noted earlier anomalies are finite and thus the $1/(d-4)$ -term in (14.40) ought to vanish.

Classical Weyl invariance (2 anomaly coefficients) The Weyl anomaly for a classically Weyl invariant theory is given by Eq. (14.34) and must respect the Wess-Zumino consistency condition. In Eq. (14.40) it is only the R^2 -term which does not respect the consistency conditions [144, 147, 204, 211, 212] and therefore

$$3a_1 + a_2 + a_3 = 0, \quad (14.41)$$

must hold which indeed removes the divergent $\square R$ -term. The result assumes the form

$$\mathcal{A}_{\text{Weyl}} = a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} - (3a_1 + a_2) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - (4a_1 + a_2) \square R, \quad (14.42)$$

which depends on the two parameters a_1 and a_2 which are model-dependent. This is the result obtained by Duff in [169], supplemented by the additional constraint that $R\tilde{R}$ is absent.

The question that poses itself is whether the results apply beyond LO. We would think that the answer is affirmative to all orders in perturbation theory for theories whose non-anomalous symmetries are compatible with dimensional regularisation.⁸ Beyond LO, \mathcal{W}_{ct} and the ansatz (14.31) contains higher order poles in $d - 4$. However, since the Weyl anomaly is finite, their contributions have to cancel in analogy to the computation of an anomalous dimension of a parameter or an operator (e.g. [215]).

Broken classical Weyl symmetry (3 anomaly coefficients) In a theory that explicitly breaks Weyl invariance, the Wess-Zumino conditions do generally not apply. However, since the Weyl anomaly can be obtained as in Eq. (14.34), it is automatically given by the finite part and thus

$$\mathcal{A}_{\text{Weyl}} = a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - (4a_1 + a_2) \square R, \quad (14.43)$$

the Weyl anomaly depends on the three parameters $a_{1,2,3}$ which are again model-dependent. Note that the $\square R$ -term is fixed in terms of others which we believe to be a new observation in the presence of explicit breaking. With regards to the validity beyond LO the same remarks apply as in the previous section.

14.4.3 Constraints from the Lorentz anomaly

In the ansatz (14.31), the assumption is made that the EMT is symmetric in its indices. To remain the most generic, we should assume that the theory may exhibit a Lorentz anomaly, which is the breaking of rotational symmetry at the quantum level and manifests itself in the antisymmetry of $\langle T^{\mu\nu} \rangle$. In practice, the antisymmetry can arise in the presence of fermions since then the vierbein $e^a{}_\nu$ replace the metric, and the EMT is not automatically symmetric by metric-variation but follows from

$$\sqrt{|g|} \langle T^{\mu\nu} \rangle = e^{a\nu} \frac{\delta \mathcal{W}}{\delta e^a{}_\mu} [e^a{}_\mu], \quad (14.44)$$

where $e^{a\alpha} = g^{\alpha\beta} e^a{}_\beta$ with latin indices referring to the tangent frame. The Lorentz anomaly is then defined by

$$\int d^4x \sqrt{|g|} \alpha_{ab} \mathcal{A}_{\text{Lorentz}}^{ab} = \delta_\alpha^L \mathcal{W} [e^a{}_\mu] = \int d^4x \sqrt{|g|} \alpha_{ab} \langle T^{ab} \rangle, \quad (14.45)$$

where $\alpha_{ab}(x) = -\alpha_{ba}(x)$ is the Lorentz transformation parameter, and $\delta_\alpha^L e^a{}_\mu = -\alpha^a{}_b e^b{}_\mu$ [20] was made use of. Note that since the vierbein is not diffeomorphism invariant, the diffeomorphism anomaly becomes

$$\delta_\xi^d \mathcal{W} [e^a{}_\nu] = - \int d^4x \sqrt{|g|} \xi_\nu \left(D_\mu \langle T^{\mu\nu} \rangle - \omega^\nu{}_{ab} \langle T^{ab} \rangle \right), \quad (14.46)$$

where $\omega_{\mu ab} = -\omega_{\mu ba}$ is the spin-connection.

⁸Supersymmetry might be such a counter-example since it is well-known to be incompatible with dimensional regularisation and also its supersymmetry-improved version of dimensional reduction [213] (despite ongoing developments [214]).

The same argument as for the diffeomorphism anomaly applies: since the Lorentz anomaly is finite, the counterterms must not contribute (i.e. $\delta_\alpha^L \mathcal{W}_{\text{ct}} = 0$) yielding the Lorentz constraint

$$\mathcal{T}_{\text{ct}}^{\alpha\beta} = \mathcal{T}_{\text{ct}}^{\beta\alpha} . \quad (14.47)$$

Enforcing the Lorentz constraint is equivalent to considering directly the metric variation as was done in this Section and Sec. 14.1, since the vierbein variation can be split into its symmetric and antisymmetric parts $\delta e^a{}_\nu = \frac{1}{2}\delta g_{\mu\nu}e^{a\mu} - \alpha^a{}_b e^b{}_\nu$ [164]. Besides, the diffeomorphism constraint (14.37) is unchanged when the Lorentz constraint is verified.

14.4.4 The non-removable $\square R$ -term

The constraints on the Weyl anomaly in the case of a classically Weyl invariant theory imply that the coefficient of the $\square R$ is fixed with respect to the coefficients of R^2 , $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (14.42) which has been known since a long time [169]. We believe that it is a new result that the same holds true for a theory which is not classically Weyl invariant (14.43). Further notice that the usual ambiguity in $\square R$, due to the possibility of adding a local term

$$S_{R^2} = \int d^4x \sqrt{|g|} \alpha R^2 , \quad T^\mu{}_\mu \supset \frac{2}{\sqrt{|g|}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} S_{R^2} = -\frac{\alpha}{3} \square R , \quad (14.48)$$

in the action, which shifts $\square R$, is not present in the case at hand since this term cancels in the formula (14.21). This is the case since the term in (14.48) enters the explicit breaking $\mathcal{E}_{\text{Weyl}}$ and not the anomalous breaking in (14.8). That is (14.48) is intrinsically 4-dimensional and independent of the quantum fields, hence only affects the EMT at tree level.

The finding that the term (14.48) does not alter the anomalous part is related to the possibility that a $\square R$ flow theorem may exist [216]. However, for the latter there is a problem when one considers the flow of QCD. Pions cannot be coupled conformally, that is a quadratic term $\mathcal{L} \supset \frac{1}{2}(D_\alpha \pi D^\alpha \pi + \xi R \pi^2)$ with $\xi = \frac{1}{6}$ (or any $\xi \neq 0$) is not allowed by the shift symmetry for Goldstone bosons. However $\xi \neq \frac{1}{6}$ renders the integral for the flow term infrared divergent. In the purely anomalous part the non-conformal terms would drop out and suggest that it might be worthwhile to consider whether one can formulate flow theorems in terms of the anomalous part only.⁹ Effectively the infrared divergence is then shifted into the explicit breaking part which has no relevance for the definition of the (pure) Weyl anomaly.

Since we believe that the $\square R$ -anomaly is calculable, the constraints on $\square R$ in (14.42) and (14.43) have to hold when computed with other regularisations provided they respect the non-anomalous symmetries of the theory. For example, the ζ -function regularisation employed in [118, 158] satisfies the constraint for a spin-1/2 fermion and a spin-1 vector in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ Lorentz representations respectively. This is sometimes difficult to see when it is the total breaking $\mathcal{A}_{\text{Weyl}} + \mathcal{E}_{\text{Weyl}}$ that is given. That is the case for example for the spin 0 scalar given in [83] (Tab. 1 in chapter 5). However, since $\xi = \frac{1}{6}$ removes the explicit breaking it is readily verified that in this case the constraint is satisfied. In [200], uniquely $\mathcal{A}_{\text{Weyl}}$ is determined, and the constraint is satisfied for any ξ . Similarly, it was pointed in [202] that the heat kernel leads to a different definition to (14.10) in the presence of explicit breaking (the second term is replaced by $\langle g_{\mu\nu}^{(4)} T^{\mu\nu} \rangle$), and a different value for the $\square R$. According to our derivation in Sec. 14.3, this implies that some explicit breaking is included in the heat kernel anomaly definition. Furthermore, since ghosts and gauge fixing may

⁹Another alternative is that the theory has another Goldstone boson, the dilaton due to conformal symmetry breaking, in which case the pions can be coupled conformally [217]. Whether or not this is the case for low energy QCD is an open question and not generally believed to be the case.

affect explicit breaking they require careful assessment as well. Hence, care must be taken when comparing the literature. Note also that the sign of $\square R$ is dependent on the sign convention of the metric.

Chapter 15

Conclusion

This final Part was dedicated to quantum anomalies and their role in Quantum Field Theory (QFT) and Particle Physics. Both global and gauge and gravitational anomalies were introduced, as well as their ambiguities and topological interpretation.

After some classical diagrammatic and path integral computations, we presented our novel computational method [4] that relies on Effective Field Theory methods. The ambiguities inherent to anomalies are carefully treated, so as to obtain either the consistent or the covariant form of the gauge anomalies. We also encounter a new anomaly: the quantum breaking of scale invariance.

We then returned to QFT in curved spacetime and introduced the gravitational anomalies. We particularly dwelt on the Weyl symmetry, which is the curved spacetime generalisation of scale invariance. This symmetry is also anomalous and, in response with a recent controversy in the literature, we dedicated significant effort in determining the role of the Pontryagin density $R\tilde{R}$ in the Weyl anomaly. We first dealt with the specific and non-trivial case of the Weyl anomaly for Weyl fermions via the path integral following our work [2], which require careful definition and manipulations of the path integral. We were then able to help solve the controversy by proving that the Pontryagin density is absent from the Weyl anomaly of a Weyl fermion.

We then extended this result with a model-independent approach, in which the generic form of the Weyl anomaly is constrained using the finiteness of the Lorentz and diffeomorphism anomalies, based upon our work [1]. We were able to conclude on the absence of any parity violation in the Weyl anomaly in any theory which is compatible with dimensional regularisation. We also argue that there is a misconception in the literature regarding the role of the $\square R$ term. Our claim is that it is non-removable in the Weyl anomaly, when carefully taking only the quantum breaking into account. In the article, we showed that we arrive at the same conclusion when including a gauge sector.

Part IV

Summary, conclusion and outlook

The Part I of this thesis was dedicated to introducing the abstract object that is the path integral. Efforts were to make it seem less abstract and unravel the physical insight that it provides. We showed how to extract physical Quantum Field Theory (QFT) observables from the path integral, and the most common perturbative methods used to compute it.

Part II was dedicated to introducing and motivating the concept of Effective Field Theory (EFT). We showed the different approaches, and how well fit is the path integral to build EFTs. After an introduction to differential geometry and QFT in curved spacetime, we presented our novel method to perform path integral computations in gravity [3], and explained its advantages compared with existing methods.

In Part III, we provided an overall look at quantum anomalies. We started with the case of global anomalies, and detailed some computations, and how to understand their topological origin. Ambiguities intrinsic to anomalies were investigated, in particular for the gauge anomalies. We then presented our work [4] in which we rely on powerful EFT methods to compute various types of anomalies, while keeping non-anomalous symmetries under control. We then turn to the case of gravitational anomalies, introducing the diffeomorphism, Lorentz, and Weyl anomalies, and how to properly deal with them in the path integral. In light of the recent controversy concerning the presence of the parity violating Pontryagin density in the Weyl anomaly of a Weyl fermion, we presented our work [2] in which we prove the absence of such a term. Our computation relies on a careful definition of the anomaly via the path integral, while preserving the diffeomorphism and Lorentz symmetries. In a following paper [1], we were able to conclude on the absence of parity violation in the Weyl anomaly for any theory that is compatible with dimensional regularisation. This was achieved by considering the most generic form of the Weyl anomaly, and then constraining by enforcing the *finiteness* of the diffeomorphism and Lorentz anomalies.

The take away of this thesis is probably the impossibility to have a parity violating Weyl anomaly, in models that are compatible with dimensional regularisation, and when we define the anomaly by carefully excluding any explicit breaking contributions.

The consequences of this result should be investigated further, both on a formal and phenomenological side. Our path integral computation methods can surely find powerful applications in other domain of Physics, such as in condensed matter Physics where anomalies and even gravitational anomalies play a role. I believe that condensed matter systems are promising since experimental realisations are much more attainable than in high energy Physics.

Part V
Appendices

Appendix A

Two-loop β -function in ϕ^4 theory

In this Appendix, we detail the computation of the β -function of λ in the following theory

$$S[\phi] = \int d^4x \frac{1}{2} \phi \left[-\partial^2 - m^2 - \frac{\lambda}{12} \phi^2 \right] \phi. \quad (\text{A.1})$$

At one-loop, β -functions can be extracted from $W_{1\text{-loop}}[0]$ following [68]. Our aim is to extend the method at two-loops. This calculation was performed in collaboration with Baptiste Filoche and Wanda Isnard, from IP2I Lyon.

The bare coupling λ and the renormalised coupling λ_r are related by $\lambda = \mu^\epsilon \lambda_r(\mu)$, where μ is the renormalisation scale. λ_r is the coupling associated to the ϕ^4 operator in the two-loop effective action. Since λ is independent from the renormalisation scale μ we must have

$$0 = \frac{d}{d\mu} \mu^\epsilon \lambda_r(\mu) \Rightarrow \beta_\lambda(\mu) \equiv \mu \frac{d}{d\mu} \lambda_r = -\epsilon \lambda_r. \quad (\text{A.2})$$

We can therefore compute β_λ via the order one pole $1/\epsilon$ in λ_r . We are thus interested in the 2-loops corrections to the ϕ^4 operator in the Effective Theory. The corrections to the kinetic term will also come into play to normalise the β -function.

The Effective Action reads

$$W[0] = \Gamma[\langle \hat{\phi} \rangle] = -i \log \int \mathcal{D}\phi e^{iS}, \quad (\text{A.3})$$

and the propagator of the theory is

$$\Pi(x, y) = (-\delta^2 S)^{-1}(x, y) = \left[\frac{1}{\partial^2 + m^2 + \lambda\phi^2/2} \delta(x - y) \right], \quad (\text{A.4})$$

where the derivatives act within the brackets. Following the CDE method, it can be expressed in momentum space as

$$\Pi(x, y) = \int \frac{d^4q}{(2\pi)^4} e^{iq \cdot (x-y)} \mathcal{O}_q, \quad (\text{A.5})$$

where

$$\mathcal{O}_q = \frac{1}{(\partial + iq)^2 + m^2 + \lambda\phi^2/2} = - \sum_{n \geq 0} \left(\delta \left(\partial^2 + 2iq \cdot \partial + \frac{\lambda}{2} \phi^2 \right) \right)^n \delta, \quad (\text{A.6})$$

with $\delta^{-1} = q^2 - m^2$ (note that \mathcal{O}_q acts on 1 here, after doing the expansion there remains no open derivatives). $\Pi(x, x)$ is divergent and we regularise it by performing the momentum integration in

$d = 4 - \epsilon$ dimensions. Since it is essentially the CDE at one-loop, we give directly the relevant contributions in terms of one-loop master integral

$$\Pi(x, x) = - \left(\mathcal{I}_1 + \mathcal{I}_2 \frac{\lambda}{2} \phi^2 + \mathcal{I}_3 \left(\frac{\lambda}{2} \phi^2 \right)^2 + \dots \right), \quad (\text{A.7})$$

where the \mathcal{I}_i are the one-loop master integral given in A.1. The higher derivatives of the action read

$$\begin{aligned} \frac{\delta^3 S}{\delta\phi(x_1) \dots \delta\phi(x_3)} &= -\lambda \delta(x_1 - x_2) \delta(x_2 - x_3) \phi(x_1) \\ \frac{\delta^4 S}{\delta\phi(x_1) \dots \delta\phi(x_4)} &= -\lambda \delta(x_1 - x_2) \delta(x_2 - x_3) \delta(x_3 - x_4) \\ \delta^n S &= 0, \quad \forall n \geq 5. \end{aligned} \quad (\text{A.8})$$

The two-loop contributions (in Euclidean signature) are the following

$$e^{-W_{2\text{-loop}}[0]} = \int \mathcal{D}\chi e^{-\frac{1}{2}\chi^2} \left(\frac{-1}{1!4!} (\Pi^{1/2} \cdot \chi)^4 \cdot \delta^4 S + \frac{(-1)^2}{2!(3!)^2} \left((\Pi^{1/2} \cdot \chi)^3 \cdot \delta^3 S \right)^2 \right), \quad (\text{A.9})$$

where the derivatives of the action are evaluated at $\phi_0[J=0] = \phi_{cl}$ which is solution to the EoM without source

$$\frac{\delta S}{\delta\phi}[\phi_{cl}] = 0. \quad (\text{A.10})$$

We split the action in the two following terms

$$Z_1 = \frac{-1}{4!} \int \mathcal{D}\chi e^{-\frac{1}{2}\chi^2} (\Pi^{1/2} \cdot \chi)^4 \cdot \delta^4 S \quad (\text{A.11})$$

$$Z_2 = \frac{1}{2!(3!)^2} \int \mathcal{D}\chi e^{-\frac{1}{2}\chi^2} \left((\Pi^{1/2} \cdot \chi)^3 \cdot \delta^3 S \right)^2, \quad (\text{A.12})$$

Such that the two-loop effective action reads

$$e^{-W_{2\text{-loop}}[0]} \supset Z_1 + Z_2. \quad (\text{A.13})$$

Computation of Z_1

Let us write explicitly the integrals in Z_1 and insert the expression of $\delta^4 S$

$$\begin{aligned} Z_1 &= \frac{-1}{4!} \int dx_1 \dots dx_4 dy_1 \dots dy_4 \Pi^{1/2}(x_1, y_1) \dots \Pi^{1/2}(x_4, y_4) \delta^4 S(x_1, \dots, x_4) \int \mathcal{D}\chi e^{-\frac{1}{2}\chi^2} \chi(y_1) \dots \chi(y_4) \\ &= \frac{\lambda}{4!} \int dx dy_1 \dots dy_4 \Pi^{1/2}(x, y_1) \dots \Pi^{1/2}(x, y_4) \int \mathcal{D}\chi e^{-\frac{1}{2}\chi^2} \chi(y_1) \dots \chi(y_4), \end{aligned} \quad (\text{A.14})$$

where $dx \equiv d^4x$. We then perform the integral over χ using (2.56). This will contract the propagators together and using $\Pi^{1/2}(x, y) = \Pi^{1/2}(y, x)$ along with

$$(\Pi^{1/2} \cdot \Pi^{1/2})(x, z) = \int dy \Pi^{1/2}(x, y) \Pi^{1/2}(y, z) = \Pi(x, z), \quad (\text{A.15})$$

we obtain

$$Z_1 = \frac{3\lambda}{4!} \int dx \Pi(x, x)^2. \quad (\text{A.16})$$

This term does not contribute to the kinetic term but only to the ϕ^4 operator. Using (A.7) we obtain

$$Z_1 \supset \frac{\lambda^3}{32} \int dx \phi_{cl}^4 (\mathcal{I}_2^2 + 2\mathcal{I}_1\mathcal{I}_3) \supset -\kappa^2 \frac{\lambda^3}{16\epsilon} \left(1 + 2 \log \left(\frac{m^2}{\mu^2} \right) \right) \int dx \phi_{cl}^4(x), \quad (\text{A.17})$$

where we extracted the piece proportional to $1/\epsilon$ that is relevant for the β -function, and $\kappa = -1/(16\pi^2)$.

Diagrammatically, the piece proportional to \mathcal{I}_2^2 corresponds to first diagram of Fig. A.1, whereas the piece proportional to $\mathcal{I}_1\mathcal{I}_3$ corresponds to the second diagram.

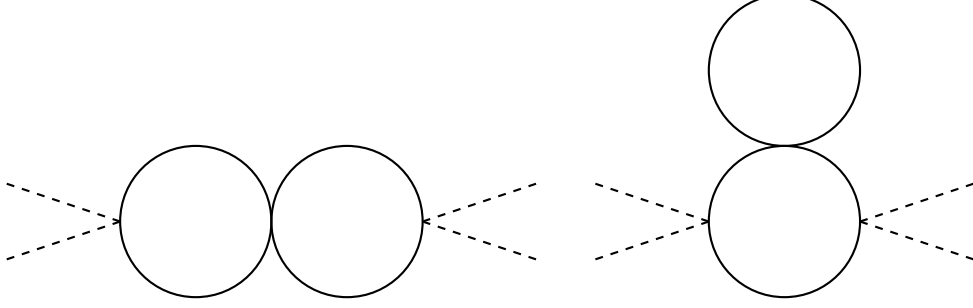


Figure A.1: Diagrams corresponding to Z_1 . The solid lines correspond to ϕ loops with propagator δ , whereas the dashed lines correspond to an insertion of ϕ_{cl} .

Computation of Z_2

Whereas Z_1 essentially looks like 1-loop squared, Z_2 is slightly more intricate. As we will see it gives quantum corrections to both the ϕ^4 and the kinetic terms. Using the expression of $\delta^3 S$ we can reduce Z_2 to

$$Z_2 = \frac{\lambda^2}{72} \int dx dy \phi_{cl}(x) \phi_{cl}(y) dy_1 \dots dy_6 \Pi^{1/2}(x, y_1) \dots \Pi^{1/2}(x, y_3) \Pi^{1/2}(y, y_4) \dots \Pi^{1/2}(y, y_6) \\ \times \int \mathcal{D}\chi e^{-\frac{1}{2}\chi^2} \chi(y_1) \dots \chi(y_6). \quad (\text{A.18})$$

And performing the path integration we obtain

$$Z_2 = \frac{\lambda^2}{72} \int dx dy \phi_{cl}(x) \phi_{cl}(y) (6 \Pi(x, y)^3 + 9 \Pi(x, x) \Pi(x, y) \Pi(y, y)). \quad (\text{A.19})$$

In terms of diagrams without external legs, the first term in $\Pi(x, y)^3$ corresponds to a sunset diagram (see Fig. A.2 left), whereas the term in $\Pi(x, x) \Pi(x, y) \Pi(y, y)$ corresponds to the one in Fig. A.2 right. This second diagram is not 1PI and will cancel in the effective action due to the Legendre transform [28], which corresponds to taking the logarithm when $J = 0$ (A.13). We thus drop the second piece and are left with

$$Z_2 \supset \frac{\lambda^2}{12} \int dx dy \phi_{cl}(x) \phi_{cl}(y) \Pi(x, y)^3. \quad (\text{A.20})$$

We now face a difficulty that does not arise in one-loop computations: Z_2 seems to be delocalised due to the integration over both x and y . There are two ways this can be handled to extract to the

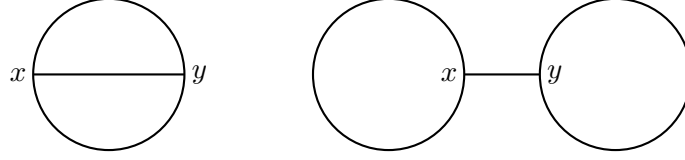


Figure A.2: Diagrams corresponding to Z_2 . The solid lines represent the propagator Π . The diagram on the right is not 1PI and will cancel due to the Legendre transform.

contributions to ϕ^4 and to the kinetic term. The first possibility is to Taylor expand $\phi_{cl}(y)$ around x . The other equivalent possibility that we follow is to use (A.4) and perform an integration by parts to release a Dirac

$$Z_2 \supset \frac{\lambda^2}{12} \int dx dy \phi_{cl}(x) \phi_{cl}(y) \left[\frac{1}{\partial_x^2 + m^2 + \lambda \phi_{cl}^2/2} \delta(x-y) \right] \Pi(x,y)^2 \quad (\text{A.21})$$

$$= \frac{\lambda^2}{12} \int dx dy \phi_{cl}(y) \delta(x-y) \left[\frac{1}{\partial_x^2 + m^2 + \lambda \phi_{cl}^2/2} \phi_{cl}(x) \Pi(x,y)^2 \right], \quad (\text{A.22})$$

where ∂_x in the inverse function acts on everything inside the brackets.¹ Using $dq \equiv d^d q / (2\pi)^d$ as a shorthand we obtain

$$\begin{aligned} Z_2 &= \frac{\lambda^2}{12} \int dx \phi_{cl}(x) dy \delta(x-y) \int dp dq e^{i(q+p) \cdot (x-y)} \frac{1}{(\partial + iq + ip)^2 + m^2 + \lambda \phi_b^2/2} \phi_b(y) [\mathcal{O}_q] [\mathcal{O}_p] \\ &= \frac{\lambda^2}{12} \int dx \phi_{cl}(x) \int dq dp \mathcal{O}_{q+p} \phi_{cl}(x) [\mathcal{O}_q] [\mathcal{O}_p], \end{aligned} \quad (\text{A.24})$$

where \mathcal{O}_q is defined in (A.6). The relevant contributions finally are

$$\begin{aligned} Z_2 &\supset \frac{\lambda^2}{12} \int dx \left(-\lambda \phi_{cl}^4/2 (\mathcal{J}_{100} + \mathcal{J}_{010} + \mathcal{J}_{001}) + \phi_{cl} (\partial^2 \phi_{cl}) \left(\frac{4}{d} \mathcal{J}[002]_{002} - \mathcal{J}_{001} \right) \right) \\ &\supset -\kappa^2 \frac{\lambda^3}{4\epsilon} \left(1 + \log \left(\frac{m^2}{\mu^2} \right) \right) \int dx \phi_{cl}^4 + \kappa^2 \frac{\lambda^2}{12\epsilon} \int dx \phi_{cl}(x) \partial^2 \phi_{cl}(x), \end{aligned} \quad (\text{A.25})$$

where we scalarised the momentum in the numerator by replacing $(q+p)^\mu (q+p)^\nu$ by $(q+p)^2 \eta^{\mu\nu} / d$, and displayed in the last line the $1/\epsilon$ terms only. The two-loop master integrals are defined and computed in Sec. A.2. Remarkably, all the two-loop integrals that appear in the two-loops effective action can easily be computed analytically.

In terms of diagrams, the ϕ^4 and kinetic terms are represented in Fig. A.3.

One-loop counter-terms

In order to properly obtain renormalisation of λ at two-loop, one must account for the contribution of the one-loop counter-terms at order one-loop.

¹Note that Π can be written as

$$\Pi(x,y) = \frac{1}{\partial^2 + m^2 + \lambda \phi_{cl}^2/2} \delta(x-y) = \frac{1}{m^2} \sum_{n \geq 0} (-1)^n \left(\frac{\partial^2 + \lambda \phi_{cl}^2/2}{m^2} \right)^n \delta(x-y), \quad (\text{A.23})$$

and then integrated by parts order by order up to infinity, or at least up to the required order for the operators we wish to compute.

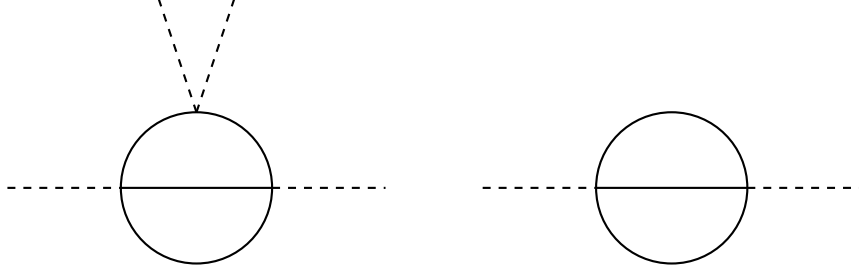


Figure A.3: Z_2 contribution to the ϕ^4 operator on the left and to the kinetic term on the right. The solid lines represent a ϕ loop with propagator δ , the dashed lines represent an insertion of ϕ_{cl} .

The one-loop effective action is given by the UOLEA [70]. Both the mass and λ receive corrections

$$W_{1\text{-loop}}[J=0] \supset \frac{\kappa}{2} \int d^4x \left\{ m^2 \left(1 + \frac{2}{\epsilon} - \log \left(\frac{m^2}{\mu^2} \right) \right) \frac{\lambda}{2} \phi_{cl}^2 + \frac{1}{2} \left(\frac{2}{\epsilon} - \log \left(\frac{m^2}{\mu^2} \right) \right) \left(\frac{\lambda}{2} \phi_{cl}^2 \right)^2 \right\}. \quad (\text{A.26})$$

The poles must be absorbed in the counter-terms at tree level such that

$$S_{\text{tree}} = \int d^4x \frac{1}{2} \phi_{cl} \left(-\partial^2 - (m + \delta m)^2 - \frac{\lambda + \delta \lambda}{12} \phi_{cl}^2 \right) \phi_{cl} \supset \int d^4x \left(-m \delta m \phi_{cl}^2 - \frac{\delta \lambda}{24} \phi_{cl}^4 \right), \quad (\text{A.27})$$

where λ , m are the bare couplings and $\delta \lambda$, δm are the one-loop counterterms. They are defined such that $S_{\text{tree}} + W_{1\text{-loop}}$ is finite.² We thus readily obtain

$$\delta m = \kappa \frac{m \lambda}{2\epsilon}, \quad \delta \lambda = \kappa \frac{3\lambda^2}{\epsilon}. \quad (\text{A.28})$$

In order to obtain the contribution of the one-loop counterterms at one-loop, we just replace the bare couplings in Eq. (A.26) by the renormalised couplings: $\lambda \rightarrow \lambda + \delta \lambda$ and $m \rightarrow m + \delta m$. After expanding the \log to extract δm we obtain at order $\mathcal{O}(1/\epsilon)$

$$W_{ct} = \kappa^2 \frac{1}{16} \int d^4x \phi_{cl}^4 \left(-\frac{\lambda^3}{\epsilon} - 6 \frac{\lambda^3}{\epsilon} \log \left(\frac{m^2}{\mu^2} \right) \right). \quad (\text{A.29})$$

The first term comes from the renormalisation of λ and can be represented by the first diagram of Fig. A.4, whereas the second term comes from the renormalisation of the mass and corresponds to the second diagram.

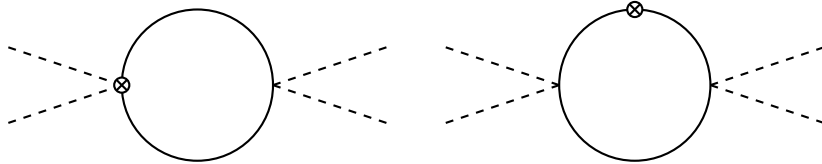


Figure A.4: One-loop contribution with insertions of one-loop counter-terms that contribute to the ϕ^4 operator at two-loops.

²In fact we follow the \overline{MS} scheme where the counterterms also absorb the $\gamma_E - \log 4\pi$ pieces. It plays no role in the computation hence we decided to ignore it for simplicity.

Result

We combine all the above contributions to have the two-loop corrections to the ϕ^4 and kinetic terms at order $1/\epsilon$

$$\begin{aligned} W_{2\text{-loop}} + W_{ct} &\supset \kappa^2 \frac{\lambda^3}{\epsilon} \left[\left(\frac{1}{16} + \frac{1}{4} - \frac{1}{16} \right) + \left(\frac{2}{16} + \frac{1}{4} - \frac{6}{16} \right) \log \left(\frac{m^2}{\mu^2} \right) \right] - \kappa^2 \frac{\lambda^2}{12\epsilon} \int d^4x \phi_{cl} \partial^2 \phi_{cl} \\ &= \kappa^2 \frac{\lambda^3}{4\epsilon} \int d^4x \phi_{cl}^4 - \kappa^2 \frac{\lambda^2}{12\epsilon} \int d^4x \phi_{cl} \partial^2 \phi_{cl}, \end{aligned} \quad (\text{A.30})$$

Notice that the terms of the form $\epsilon^{-1} \log \left(\frac{m^2}{\mu^2} \right)$ cancel as expected in dimensional regularisation, otherwise it would mean that the two-loop counterterms would depend on $\log \left(\frac{m^2}{\mu^2} \right)$. This result was obtained via Feynman diagrams in [218] in agreement with us. The two-loops β -function can then be straightforwardly derived from this result to obtain

$$\beta_\lambda^{(2)} = -\frac{1}{(16\pi^2)^2} \frac{17}{3\kappa^2} \lambda^3. \quad (\text{A.31})$$

The procedure is described in great details in [218].

A.1 One-loop Master Integrals

$\tilde{\mathcal{I}}[q^{2n_c}]_i^{n_i}$	$n_c = 0$	$n_c = 1$	$n_c = 2$
$n_i = 1$	$M_i^2 \left(1 - \log \frac{M_i^2}{\mu^2} \right)$	$\frac{M_i^4}{4} \left(\frac{3}{2} - \log \frac{M_i^2}{\mu^2} \right)$	$\frac{M_i^6}{24} \left(\frac{11}{6} - \log \frac{M_i^2}{\mu^2} \right)$
$n_i = 2$	$-\log \frac{M_i^2}{\mu^2}$	$\frac{M_i^2}{2} \left(1 - \log \frac{M_i^2}{\mu^2} \right)$	$\frac{M_i^4}{8} \left(\frac{3}{2} - \log \frac{M_i^2}{\mu^2} \right)$
$n_i = 3$	$-\frac{1}{2M_i^2}$	$-\frac{1}{4} \log \frac{M_i^2}{\mu^2}$	$\frac{M_i^2}{8} \left(1 - \log \frac{M_i^2}{\mu^2} \right)$
$n_i = 4$	$\frac{1}{6M_i^4}$	$-\frac{1}{12M_i^2}$	$-\frac{1}{24} \log \frac{M_i^2}{\mu^2}$
$n_i = 5$	$-\frac{1}{12M_i^6}$	$\frac{1}{48M_i^4}$	$-\frac{1}{96M_i^2}$

Table A.1: Commonly-used master integrals with degenerate heavy particle masses. $\tilde{\mathcal{I}} = \mathcal{I}/\frac{i}{16\pi^2}$ and the $\frac{2}{\epsilon} - \gamma + \log 4\pi$ contributions are dropped.

In this appendix, we discuss the master integrals and tabulate some of them that are useful in practice. In this paper our results are written in terms of master integrals \mathcal{I} , defined by

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^{\mu_1} \dots q^{\mu_{2n_c}}}{(q^2 - M_i^2)^{n_i} (q^2 - M_j^2)^{n_j} \dots} = g^{\mu_1 \dots \mu_{2n_c}} \mathcal{I}[q^{2n_c}]_{ij\dots}^{n_i n_j \dots} \quad (\text{A.32})$$

In the mass degenerate case, the master integrals, $\mathcal{I}[q^{2n_c}]_{ij\dots}^{n_i n_j \dots}$, reduce to the form $\mathcal{I}[q^{2n_c}]_i^{n_i}$, for which the general expression reads,

$$\mathcal{I}[q^{2n_c}]_i^{n_i} = \frac{i}{16\pi^2} (-M_i^2)^{2+n_c-n_i} \frac{1}{2^{n_c} (n_i - 1)!} \frac{\Gamma(\frac{\epsilon}{2} - 2 - n_c + n_i)}{\Gamma(\frac{\epsilon}{2})} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \frac{M_i^2}{\mu^2} \right), \quad (\text{A.33})$$

where $d = 4 - \epsilon$ is the space-time dimension, and μ is the renormalisation scale. In the $\overline{\text{MS}}$ scheme, we replace, $\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \frac{M_i^2}{\mu^2}\right)$ by $\left(-\log \frac{M_i^2}{\mu^2}\right)$ in the final result. We factor out the common prefactor, $\mathcal{I} = \frac{i}{16\pi^2} \tilde{\mathcal{I}}$ ($\mathcal{I} = \frac{-1}{16\pi^2} \tilde{\mathcal{I}}$ in Euclidean) and present a table of $\tilde{\mathcal{I}}[q^{2n_c}]_i^{n_i}$ for various n_c and n_i , needed in our computations, in Table B.1.

A.2 Two-loop Master Integrals

In this appendix, we discuss the master integrals at 2-loop order and tabulate some of them that are useful in practice. The conventions we follow are the same as in App. A.1. The master integrals \mathcal{J} are defined by:

$$\mathcal{J}_{ijk} = \int \frac{d^d k_1 d^d k_2}{(2\pi)^4} \frac{1}{(k_1^2 - m^2)^{i+1}} \frac{1}{(k_2^2 - m^2)^{j+1}} \frac{1}{((k_1 + k_2)^2 - m^2)^{k+1}}, \quad (\text{A.34})$$

with $d = 4 - \epsilon$. These integrals can be computed from

$$I(x, z, y) = \frac{(\mu^2)^{2\epsilon}}{(2\pi)^d} \int \frac{d^d k d^d q}{(k^2 + x)(q^2 + y)((k + q)^2 + z)}, \quad (\text{A.35})$$

by taking derivatives with respect to x , y or z . $I(x, y, z)$ can be found analytically in [219] (they use $d = 4 - 2\epsilon$).

The integrals with momentum in the numerator are denoted,

$$\mathcal{J}[2l_i, 2l_j, 2l_k]_{ijk} = \int \frac{d^4 k_1 d^4 k_2}{(2\pi)^4} \frac{k_1^{2l_i}}{(k_1^2 - m^2)^{i+1}} \frac{k_2^{2l_j}}{(k_2^2 - m^2)^{j+1}} \frac{(k_1 + k_2)^{2l_k}}{((k_1 + k_2)^2 - m^2)^{k+1}}. \quad (\text{A.36})$$

They can be computed from previous integrals. From example, in the calculation of the kinetic term the following integral occurs

$$\mathcal{J}[002]_{002} = \mathcal{J}_{001} + m^2 \mathcal{J}_{002}. \quad (\text{A.37})$$

$\tilde{\mathcal{J}}_{ij0}$	$i = 1$	$i = 2$
$j = 0$	$\frac{2}{\epsilon} \left(1 + \gamma_E + \log \frac{m^2}{\mu^2} - \log 4\pi\right) + c_1$	$\frac{1}{\epsilon m^2} + 1 - \gamma_E - \log \frac{m^2}{\mu^2} + \log 4\pi$
$j = 1$	$\frac{1}{2m^2}$	$\frac{1}{4m^4}$

Table A.2: Master integrals useful for 2-loop order computations. We note $\tilde{\mathcal{J}}_{ijk} = \mathcal{J}_{ijk} / \frac{i}{16\pi^2}$ and $c_1 = -\left(\log \frac{m^2}{\mu^2}\right)^2 + (1 - 2\gamma_E + 2 \log 4\pi) \log \frac{m^2}{\mu^2} + 3 + \gamma_E - \gamma_E^2 + 2\gamma_E \log 4\pi - \log 4\pi(1 + \log 4\pi)$. Those master integrals are symmetric in i, j, k , explicitly $\tilde{\mathcal{J}}_{100} = \tilde{\mathcal{J}}_{010} = \tilde{\mathcal{J}}_{001}$, $\tilde{\mathcal{J}}_{200} = \tilde{\mathcal{J}}_{020} = \tilde{\mathcal{J}}_{002}$ and $\tilde{\mathcal{J}}_{110} = \tilde{\mathcal{J}}_{101} = \tilde{\mathcal{J}}_{011}$.

Appendix B

Master integrals in Curved Spacetime

Master integrals in curved spacetime

In flat spacetime (latin indices), we define the master integrals \mathcal{I} , \mathcal{J} and \mathcal{K} as

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^{a_1} \dots q^{a_{2n_c}}}{(q^2 - m^2)^n} = \eta^{a_1 \dots a_{2n_c}} \mathcal{I}[q^{2n_c}]^n \quad (\text{B.1})$$

$$\int \frac{d^d q}{(2\pi)^d} \int^{m^2} dm'^2 \frac{q^{a_1} \dots q^{a_{2n_c}}}{(q^2 - m'^2)^n} = \eta^{a_1 \dots a_{2n_c}} \mathcal{J}[q^{2n_c}]^n \quad (\text{B.2})$$

$$\int \frac{d^d q}{(2\pi)^d} \int^m dm' m'^k \frac{q^{a_1} \dots q^{a_{2n_c}}}{(q^2 - m'^2)^n} = \eta^{a_1 \dots a_{2n_c}} \mathcal{K}[q^{2n_c}]^k_n, \quad (\text{B.3})$$

where in general the integral over the mass must be performed before the integral over momentum. $\eta^{a_1 \dots a_{2n_c}}$ is the fully symmetrised Minkowski metric.

The master integrals, \mathcal{I} are defined by the general expression

$$\mathcal{I}[q^{2n_c}]^n = \frac{i}{16\pi^2} (-m^2)^{2+n_c-n} \frac{1}{2^{n_c}(n-1)!} \frac{\Gamma(\frac{\epsilon}{2} - 2 - n_c + n)}{\Gamma(\frac{\epsilon}{2})} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \frac{m^2}{\mu^2} \right), \quad (\text{B.4})$$

where $d = 4 - \epsilon$ is the spacetime dimension, and μ is the renormalization scale. In the $\overline{\text{MS}}$ scheme, we replace, $\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \frac{m^2}{\mu^2} \right)$ by $\left(-\log \frac{m^2}{\mu^2} \right)$ in the final result. We factor out the common prefactor, $\mathcal{I} = \frac{i}{16\pi^2} \tilde{\mathcal{I}}$ and present a table of $\tilde{\mathcal{I}}[q^{2n_c}]^n$ for various n_c and n , needed in our computations, in Table B.1.

\mathcal{I} and \mathcal{J} are related by integrating the mass

$$\mathcal{J}[q^{2n_c}]^n = \frac{1}{n-1} \mathcal{I}[q^{2n_c}]^{n-1}. \quad (\text{B.5})$$

The fermionic master integrals \mathcal{K} are trickier to compute. The dimensionful integrals are computed in Eq. (B.15). The dimensionless integrals (i.e $\propto m^0$) can however be obtained using Eqs. (B.19) and Eq. (B.20).

We can relate the integrals in curved spacetime to those in flat spacetime using a tangent frame that is orthonormal everywhere in the whole manifold. We relate the flat metric η and the metric g using the vierbein

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \quad (\text{B.6})$$

$\tilde{\mathcal{I}}[q^{2n_c}]^n$	$n_c = 0$	$n_c = 1$	$n_c = 2$
$n = 1$	$m^2(1 - \log \frac{m^2}{\mu^2})$	$\frac{m^4}{4}(\frac{3}{2} - \log \frac{m^2}{\mu^2})$	$\frac{m^6}{24}(\frac{11}{6} - \log \frac{m^2}{\mu^2})$
$n = 2$	$-\log \frac{m^2}{\mu^2}$	$\frac{m^2}{2}(1 - \log \frac{m^2}{\mu^2})$	$\frac{m^4}{8}(\frac{3}{2} - \log \frac{m^2}{\mu^2})$
$n = 3$	$-\frac{1}{2m^2}$	$-\frac{1}{4} \log \frac{m^2}{\mu^2}$	$\frac{m^2}{8}(1 - \log \frac{m^2}{\mu^2})$
$n = 4$	$\frac{1}{6m^4}$	$-\frac{1}{12m^2}$	$-\frac{1}{24} \log \frac{m^2}{\mu^2}$
$n_i = 5$	$-\frac{1}{12m^6}$	$\frac{1}{48m^4}$	$-\frac{1}{96m^2}$

Table B.1: Commonly-used master integrals with degenerate heavy particle masses. $\tilde{\mathcal{I}} = \mathcal{I}/\frac{i}{16\pi^2}$ and the $\frac{2}{\epsilon} - \gamma + \log 4\pi$ contributions are dropped.

The latin indices refer to the orthonormal frame, while the greek indices refer to the initial frame. The momenta are expressed in the orthonormal frame using the vierbein and its inverse E

$$p_\mu = e_\mu^a q_a \quad q_a = E_a^\mu p_\mu. \quad (\text{B.7})$$

We can now relate the master integrals with momenta p^μ to the master integrals in flat spacetime by doing the change of variable $p_\mu = e_\mu^a q_a$. The momentum space measure is defined with the covariant vector p_μ , thus the jacobian of the change of variable is $\det(e) = \sqrt{|g|}$. We thus have

$$\begin{aligned} & \int \frac{d^d p}{(2\pi)^4} \int^{m^2} dm'^2 \frac{p^{\mu_1} \dots p^{\mu_{2n_c}}}{(p^2 - m'^2)^n} \\ &= \int \det(e) \frac{d^d q}{(2\pi)^4} \int^{m^2} dm'^2 E^{\mu_1}_{a_1} \dots E^{\mu_{2n_c}}_{a_{2n_c}} \frac{q^{a_1} \dots q^{a_{2n_c}}}{(q^2 - m'^2)^n} \\ &= \sqrt{|g|} E^{\mu_1}_{a_1} \dots E^{\mu_{2n_c}}_{a_{2n_c}} \eta^{a_1 \dots a_{2n_c}} \mathcal{J}[q^{2n_c}]^n \\ &= \sqrt{|g|} g^{\mu_1 \dots \mu_{2n_c}} \mathcal{J}[q^{2n_c}]^n, \end{aligned} \quad (\text{B.8})$$

and likewise for \mathcal{K} and \mathcal{I} .

Momentum and mass integration

The master integrals with integration over the mass should be computed by integrating over the mass first, and then over momentum. In general commuting the integration is not true for divergent integrals. However, we will show that it stands true for dimensionful integrals.

The discussion below is not very relevant for the bosonic integrals \mathcal{J} since we are able to perform the integration over the mass and then over momentum, as it should be done. However, it is not so simple for the fermionic integrals, therefore commuting the integrals will prove useful.

Let's reason on the bosonic integrals which are simpler to compute explicitly. Without commuting the integrals, we can perform the integration over the mass to obtain the correct result

$$\mathcal{J}[q^{2l}]^n = \frac{1}{n-1} \mathcal{I}[q^{2l}]^{n-1}. \quad (\text{B.9})$$

If we commute the integrals, then we obtain for $2 + l - (n - 1) \neq 0$

$$\begin{aligned} \mathcal{J}'[q^{2l}]^n &= \int^{m^2} dm'^2 \mathcal{I}[q^{2l}]^n(m') \\ &= \frac{i}{16\pi^2} \frac{(m^2)^{2+l-(n-1)}}{2+l-(n-1)} \frac{(-1)^{l-n}}{2^l(n-1)!} \frac{\Gamma(\frac{\epsilon}{2} - 2 - l + n)}{\Gamma(\frac{\epsilon}{2})} \left(\frac{1}{2+l-(n-1)} + \frac{2}{\bar{\epsilon}} - \log\left(\frac{m^2}{\mu^2}\right) \right). \end{aligned} \quad (\text{B.10})$$

Note that the integration constant vanishes by dimensional analysis since by definition it must be independent of the mass, but the integral is dimensionful for $2 + l - (n - 1) \neq 0$. We defined $2/\bar{\epsilon} = 2/\epsilon - \gamma + \log 4\pi$.

If $(n - 1) - l - 2 > 0$, then both $\mathcal{I}[q^{2l}]^{n-1}$ and $\mathcal{J}[q^{2l}]$ are finite in $d = 4$ dimensions. The commutation of the integral is thus correct, and we can indeed verify that Eqs. (B.9) and (B.10) are equal.

However, we can show that the commutation of the integrals remains true if $(n - 1) - l - 2 < 0$. Using for $N \geq 0$

$$\frac{\Gamma(\frac{\epsilon}{2} - N)}{\Gamma(\frac{\epsilon}{2})} = \frac{(-1)^N}{N!} \left(1 + \frac{\epsilon}{2} \sum_{k=1}^{N-1} 1/k + \mathcal{O}(\epsilon^2) \right), \quad (\text{B.11})$$

and the expression of \mathcal{I} in Eq. (B.4), we can show that Eqs. (B.9) and (B.10) remain equal.

A discrepancy between $\mathcal{J}[q^{2l}]^n$ and $\mathcal{J}'[q^{2l}]^n$ however happens when $(n - 1) - l - 2 = 0$ since we have

$$\mathcal{J}[q^{2l}]^n = \frac{i}{16\pi^2} \frac{1}{2^l(n-1)!} \left(\frac{2}{\bar{\epsilon}} - \log\left(\frac{m^2}{\mu^2}\right) \right), \quad (\text{B.12})$$

whereas

$$\mathcal{J}'[q^{2l}]^n = \frac{i}{16\pi^2} \int^{m^2} dm'^2 \frac{-1}{2^l(n-1)!m'^2} + \mathcal{O}(\epsilon) = \frac{i}{16\pi^2} \frac{-1}{2^l(n-1)!} \log\left(\frac{m^2}{\mu^2}\right) + F, \quad (\text{B.13})$$

where F is an integration constant that cannot be ruled out by dimensional analysis like before.

Now Eqs. (B.12) and (B.13) are not equal. Note however, that the coefficient of the logarithm in Eq. (B.13) is correct, and the difference lies in the undetermined integration constant.

As explained earlier, commuting the integrals is not useful for the bosonic integrals since we are able to easily perform the integration over the mass then over momentum. However, some fermionic integrals are much easier to compute if we are allowed to commute the integrals.

The fermionic integrals \mathcal{K} are more cumbersome to compute in general because the integration is over the mass instead of the mass square. For example, after Wick rotation, the integration over the mass of $\mathcal{K}[q^{2l}]_1^0$ yields

$$\mathcal{K}[q^{0l}]_1^0 \propto \int \frac{d^d q}{(2\pi)^d} q^{2l} \int^m dm' \frac{1}{q^2 - m'^2} = -i \int \frac{d^d q}{(2\pi)^d} q^{2l-1} \text{Arctan}\left(\frac{m}{q}\right), \quad (\text{B.14})$$

where $q = \sqrt{q^2}$ is well-defined since $q^2 \geq 0$ in Euclidean. The integration over momentum is then not trivial to perform in dimensional regularisation.

However, as shown for the bosonic integrals, the integration over mass and momentum can be commuted provided \mathcal{K} is dimensionful ¹.

¹We do not prove it for the fermionic integrals, we assume it behaves similarly.

Hence, for $k + 2l - 2n + 5 \neq 0$ (i.e for dimensionful integrals), we have

$$\begin{aligned} \mathcal{K}[q^{2l}]_n^k &= \int^m dm' m'^k \mathcal{I}[q^{2l}]^n \\ &= \frac{i}{16\pi^2} \frac{(-1)^{l-n} m^{5+k+2l-2n}}{2^l (n-1)! (5+k+2l-2n)} \frac{\Gamma(\frac{\epsilon}{2} - 2 - l + n)}{\Gamma(\frac{\epsilon}{2})} \left(\frac{2}{5+k+2l-2n} + \frac{2}{\epsilon} - \log\left(\frac{m^2}{\mu^2}\right) \right). \end{aligned} \quad (\text{B.15})$$

The dimensionless integrals can be computed using recursion formulae which can be obtained by integration by parts over the mass integration

$$\mathcal{K}[q^{2l}]_n^k = \frac{m^{k-1}}{2(n-1)} \mathcal{I}[q^{2l}]^{n-1} - \frac{k-1}{2(n-1)} \mathcal{K}[q^{2l}]_{n-1}^{k-2} \quad (\text{B.16})$$

$$\mathcal{K}[q^{2l}]_n^k = \frac{m^{k-1}}{2n-k-1} \mathcal{I}[q^{2l}]^{n-1} - \frac{k-1}{2n-k-1} (d+2l) \mathcal{K}[q^{2(l+1)}]_n^{k-2} \quad (\text{B.17})$$

$$\mathcal{K}[q^{2l}]_n^k = \frac{2n}{2n-k-1} (d+2l) \mathcal{K}[q^{2(l+1)}]_{n+1}^k - \frac{m^{k+1}}{2n-k-1} \mathcal{I}[q^{2l}]^n, \quad (\text{B.18})$$

where d is the dimension of spacetime ².

Using repeatedly Eq. (B.16), we obtain

$$\begin{aligned} \mathcal{K}[q^{2l}]_n^{2k+1} &= \frac{1}{2} \sum_{i=1}^k (-1)^{k-i} \frac{k!}{i!} \frac{(n-2-k+i)!}{(n-1)!} m^{2i} \mathcal{I}[q^{2l}]^{n-1-k+i} \\ &\quad + (-1)^k k! \frac{(n-k-1)!}{(n-1)!} \mathcal{K}[q^{2l}]_{n-k}^1, \end{aligned} \quad (\text{B.19})$$

and for $n-k > 1$ ³, we have,

$$\mathcal{K}[q^{2l}]_{n-k}^1 = \frac{1}{2(n-k-1)} \mathcal{I}[q^{2l}]^{n-k-1}. \quad (\text{B.20})$$

Note however, that integrals of the form $\mathcal{K}[q^{2l}]_n^{2k}$ remain troublesome to compute even using the iterative formulae. For example, using Eqs. (B.17) and (B.18) repeatedly, $\mathcal{K}[q^{2l}]_n^{2k}$ can be related to $\mathcal{K}[q^{2l}]_1^0$ from Eq. (B.14), which is not trivial to compute.

The dimensionless integrals only occur in our expansion in the term

$$\int \frac{d^d q}{(2\pi)^d} \int^m dm' [\Delta(-i\mathcal{D} + Q)]^4 \Delta. \quad (\text{B.21})$$

Since the number of propagators is odd, and the power in q in the numerator must be even lest the integral vanishes, then the power in m' in the numerator is odd. Therefore, these integrals can be computed using Eqs. (B.19) and (B.20).

²It arises due to the scalarisation of the vectors q . In the momentum integrals, $q_{\mu_1} \dots q_{\mu_{2l}}$ is traded for $c_l q^{2l} g_{\mu_1 \dots \mu_{2l}}$ where $c_l^{-1} = d(d+2) \dots (d+2(l-1))$.

³In our expansion, the integrals $\mathcal{K}[q^{2kl}]_n^p$ are such that $n \geq p$, so if $p = 2k + 1$, $n - k = 1$ can only be realised for $n = 1$ which contributes as a tadpole to the effective action.

Appendix C

Riemann Normal Coordinates and Fock-Schwinger gauge

In this Appendix, we provide the expansion of the metric, the Christoffel symbols and the spin-connection in RNC, as well as the expansion of the gauge fields in FS gauge, around x_0 , with $x = x_0 + y$. Note that in this Appendix, \mathcal{D} denotes the total covariant derivative.

Fock-Schwinger gauge In the FS gauge around x_0 , the gauge fields are expressed as follows

$$V_\mu(x_0) = - \sum_{n \geq 0} \frac{1}{n!(n+2)} y^\nu y^{\alpha_1} \dots y^{\alpha_n} \left[D_{\alpha_1}, \left[D_{\alpha_2}, \left[\dots [D_{\alpha_n}, F_{\mu\nu}] \dots \right] \right] \right] (x_0). \quad (\text{C.1})$$

We must then upgrade the covariant derivatives D to the general covariant derivatives \mathcal{D} to have a diffeomorphism covariant expression. For example, in RNC and FS gauge we have

$$\begin{aligned} [D_\mu, [D_\nu, F_{\rho\sigma}]] y^\mu y^\nu y^\sigma &= [D_\mu, \left([\mathcal{D}_\nu, F_{\rho\sigma}] + \Gamma_{\nu\rho}^\lambda F_{\lambda\sigma} + \Gamma_{\nu\sigma}^\lambda F_{\rho\lambda} \right)] y^\mu y^\nu y^\sigma \\ &= \left([\mathcal{D}_\mu, [\mathcal{D}_\nu, F_{\rho\sigma}]] + (\partial_\mu \Gamma_{\nu\rho}^\lambda) F_{\lambda\sigma} + (\partial_\mu \Gamma_{\nu\sigma}^\lambda) F_{\rho\lambda} \right) y^\mu y^\nu y^\sigma \\ &= \left([\mathcal{D}_\mu, [\mathcal{D}_\nu, F_{\rho\sigma}]] + \frac{1}{3} R_{\mu\rho}{}^\lambda{}_\nu F_{\lambda\sigma} \right) y^\mu y^\nu y^\sigma. \end{aligned} \quad (\text{C.2})$$

We thus obtain for the first orders the combination of FS gauge and RNC as in [220, 221]

$$V_\mu = -\frac{1}{2} F_{\mu\nu}(x_0) y^\nu - \frac{1}{3} (\mathcal{D}_\alpha F_{\mu\nu})(x_0) y^\nu y^\alpha - \frac{1}{8} \left((\mathcal{D}_{\alpha\beta} F_{\mu\nu}) + \frac{1}{3} R_{\alpha\mu}{}^\lambda{}_\beta F_{\lambda\nu} \right) (x_0) y^\alpha y^\beta y^\nu + \mathcal{O}(y^4). \quad (\text{C.3})$$

We do not consider the case of an axial gauge field in the covariant derivative as throughout the computation we put the axial gauge field in Q_o rather than in \mathcal{D} . More crucially, we expect the axial gauge invariance to be broken depending on the choice of scheme for γ_5 in dimensional regularisation, therefore it makes no sense to choose a gauge for the axial gauge field.

Riemann Normal Coordinates The metric and the Christoffel symbols are expressed around x_0 as

$$\begin{aligned} g_{\mu\nu}(x) &= \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}(x_0) y^\alpha y^\beta - \frac{1}{6} R_{\mu\alpha\nu\beta;\gamma}(x_0) y^\alpha y^\beta y^\gamma \\ &+ \left(-\frac{1}{20} R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45} R_{\alpha\mu\beta\lambda} R^\lambda{}_{\gamma\nu\delta} \right) (x_0) y^\alpha y^\beta y^\gamma y^\delta + \mathcal{O}(y^5), \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned}
 g^{\mu\nu}(x) = & \eta^{\mu\nu} + \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta(x_0)y^\alpha y^\beta + \frac{1}{6}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma}(x_0)y^\alpha y^\beta y^\gamma \\
 & + \left(\frac{1}{20}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma\delta} + \frac{1}{15}R^\mu{}_{\alpha\beta\lambda}R^\lambda{}_{\gamma\delta} \right) (x_0)y^\alpha y^\beta y^\gamma y^\delta + \mathcal{O}(y^5),
 \end{aligned} \tag{C.5}$$

and

$$\begin{aligned}
 \Gamma_{\nu\rho}^\mu = & -\frac{1}{3}(R^\mu{}_{\nu\rho\alpha} + R^\mu{}_{\rho\nu\alpha})(x_0)y^\alpha \\
 & -\frac{1}{12}\left(2R^\mu{}_{\nu\rho\alpha;\beta} + 2R^\mu{}_{\rho\nu\alpha;\beta} + R^\mu{}_{\alpha\rho\beta;\nu} + R^\mu{}_{\alpha\nu\beta;\rho} - R^\mu{}_{\nu\alpha\rho\beta}{}^\mu\right)(x_0)y^\alpha y^\beta \\
 & + \left[\frac{1}{18}R^\mu{}_\alpha{}^\lambda{}_\beta y^\alpha y^\beta \left(-R_{\rho\gamma\lambda\delta}(\partial_\nu y^\gamma y^\delta) - R_{\nu\gamma\lambda\delta}(\partial_\rho y^\gamma y^\delta) + R_{\nu\gamma\rho\delta}(\partial_\lambda y^\gamma y^\delta) \right) \right. \\
 & + \left. \left(\frac{1}{45}R_{\alpha\rho\beta\lambda}R^\lambda{}_{\gamma\delta}{}^\mu - \frac{1}{40}R_{\alpha\rho\beta}{}^\mu{}_{;\gamma\delta} \right) (\partial_\nu y^\alpha y^\beta y^\gamma y^\delta) \Big|_{\text{sym } \mu\leftrightarrow\nu} \right. \\
 & \left. - \left(\frac{1}{45}R_{\alpha\nu\beta\lambda}R^\lambda{}_{\rho\delta} - \frac{1}{40}R_{\alpha\nu\beta\rho;\gamma\delta} \right) g^{\mu\chi}(\partial_\chi y^\alpha y^\beta y^\gamma y^\delta) \right] (x_0) + \mathcal{O}(y^4)
 \end{aligned} \tag{C.6}$$

Higher order expansions in RNC can be found in [222].

For example, in the limit $y \rightarrow 0$, we have

$$\begin{aligned}
 (\partial_\mu g^{\alpha\beta}) & = 0 \\
 (\partial_{\rho\sigma} g^{\mu\nu}) & = \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta \left(g^\alpha{}_\rho g^\beta{}_\sigma + g^\alpha{}_\sigma g^\beta{}_\rho \right) = \frac{1}{3} \left(R^\mu{}_\rho{}^\nu{}_\sigma + R^\mu{}_\sigma{}^\nu{}_\rho \right).
 \end{aligned} \tag{C.7}$$

It is also possible to apply the RNC to the spin-connection since it depends on the vierbeins and the Christoffel connection. It boils down to using a FS gauge for the spin-connection as if it were a regular gauge field

$$\begin{aligned}
 \omega_\mu = & -\frac{1}{8}\gamma^{\alpha_1}(x_0)\gamma^{\alpha_2}(x_0)R_{\mu\nu\alpha_1\alpha_2}(x_0)y^\nu - \frac{1}{12}\gamma^{\alpha_1}(x_0)\gamma^{\alpha_2}(x_0)(\nabla_\alpha R_{\mu\nu\alpha_1\alpha_2})(x_0)y^\nu y^\alpha \\
 & - \frac{1}{32}\gamma^{\alpha_1}(x_0)\gamma^{\alpha_2}(x_0)\left((\nabla_{\alpha\beta} R_{\mu\nu\alpha_1\alpha_2}) + \frac{1}{3}R_{\alpha\mu}{}^\lambda{}_\beta R_{\lambda\nu\alpha_1\alpha_2} \right) (x_0)y^\alpha y^\beta y^\nu + \mathcal{O}(y^4).
 \end{aligned} \tag{C.8}$$

Note that the partial derivatives of ω only apply on y , since the Dirac matrices are at x_0 they commute with $\partial = \frac{\partial}{\partial y}$.

Example: $\mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}_\rho \mathcal{D}_\sigma$ Expanding open covariant derivatives in terms of the Christoffel connection, spin-connection and gauge connection and their derivatives can become a heavy computation very quickly. Then forming the covariant quantities adds to the complexity of the task. Using the RNC and the FS gauge drastically simplify this task.

Using Eq. (C.3), we can express the partial derivatives of V in terms of field strengths, and likewise for the spin-connection using Eq. (C.8).

If we denote $X = V + \omega$ for simplicity, we have

$$\begin{aligned}
 \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}_\rho \mathcal{D}_\sigma & = \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}_\rho X_\sigma \\
 & = \mathcal{D}_\mu \mathcal{D}_\nu \left((\partial_\rho X_\sigma) - \Gamma_{\rho\sigma}^\lambda X_\lambda + X_\rho X_\sigma \right) \\
 & = (\partial_{\mu\nu\rho} X_\sigma) + (\partial_\mu X_\nu)(\partial_\rho X_\sigma) - (\partial_\mu \Gamma_{\nu\rho}^\lambda)(\partial_\lambda X_\sigma) - (\partial_\mu \Gamma_{\nu\sigma}^\lambda)(\partial_\rho X_\lambda) \\
 & + \left[(\partial_\mu X_\rho)(\partial_\nu X_\sigma) - (\partial_\mu \Gamma_{\rho\sigma}^\lambda)(\partial_\nu X_\lambda) \right] \Big|_{\text{sym } \mu\leftrightarrow\nu}.
 \end{aligned} \tag{C.9}$$

We can then explicit $X = V + \omega$ and use Eqs. (C.3), (C.6) and (C.8) to form the covariant quantities.

Finally we obtain the 4 open covariant derivatives $\mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}_\rho \mathcal{D}_\sigma$ in a covariant form. Obviously, $\mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}_\rho \mathcal{D}_\sigma$ is not covariant by itself, but all the contributions to this operators that arise in the expansion combine together so that the result is covariant. The use of RNC and FS gauge is merely a shortcut to get to the final covariant form.

Appendix D

Weyl Fermions in Curved Spacetime

The Dirac matrices can be expressed as

$$\gamma^a = \begin{pmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad (\text{D.1})$$

where $\sigma^a = (\mathbb{1}_2, \sigma^i)$, $\bar{\sigma}^a = (\mathbb{1}_2, -\sigma^i)$, and $\sigma^{i=1,2,3}$ are the Pauli matrices. As for the Dirac matrices, we have $\sigma^\mu = e^\mu_a \sigma^a$. In an Euclidean space, it is possible to choose a representation of the σ -matrices such that $\bar{\sigma}^\mu = (\sigma^\mu)^\dagger$, i.e. $(\gamma^\mu)^\dagger = \gamma^\mu$ which is the convention used throughout. From $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{1}_4$ one deduces

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu} \mathbb{1}_2. \quad (\text{D.2})$$

The generator of rotations can be written as

$$\Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b] = \begin{pmatrix} \lambda^{ab} & 0 \\ 0 & \mu^{ab} \end{pmatrix}, \quad (\text{D.3})$$

where

$$\lambda^{ab} = \frac{1}{4} (\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a), \quad \mu^{ab} = \frac{1}{4} (\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a). \quad (\text{D.4})$$

Using $D_\mu \psi = (\partial_\mu + \frac{1}{2} \omega_{\mu,ab} \Sigma^{ab}) \psi$ and decomposing ψ and D_μ as¹

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad D_\mu = \begin{pmatrix} D_\mu^R & 0 \\ 0 & D_\mu^L \end{pmatrix}, \quad (\text{D.5})$$

one obtains

$$\begin{aligned} D_\mu^R \psi_R &= (\partial_\mu + \omega_\mu^R) \psi_R = (\partial_\mu + \frac{1}{2} \omega_{\mu,ab} \lambda^{ab}) \psi_R, \\ D_\mu^L \psi_L &= (\partial_\mu + \omega_\mu^L) \psi_L = (\partial_\mu + \frac{1}{2} \omega_{\mu,ab} \mu^{ab}) \psi_L. \end{aligned} \quad (\text{D.6})$$

The Weyl operator, used in the main text, can be identified from the second line. Note that when acting on a scalar with Lorentz index such as ξ^μ one has $(D_\mu^R \xi^\nu) = (D_\mu^L \xi^\nu) = (D_\mu \xi^\nu) = (\nabla_\mu \xi^\nu) = (\partial_\mu \xi^\nu) + \Gamma_{\mu\rho}^\nu \xi^\rho$.

¹ D_μ is diagonal since it does not change the chirality of fermions.

From the compatibility with Dirac matrices $[D_\mu, \gamma^\nu] = 0$ one obtains

$$D_\mu^R \bar{\sigma}^\nu = \bar{\sigma}^\nu D_\mu^L, \quad D_\mu^L \sigma^\nu = \sigma^\nu D_\mu^R . \quad (\text{D.7})$$

Finally, since

$$i\mathcal{D} = \begin{pmatrix} 0 & i\bar{\sigma}^\mu D_\mu^L \\ i\sigma^\mu D_\mu^R & 0 \end{pmatrix} , \quad (\text{D.8})$$

is hermitian in Euclidean space, it follows that

$$(i\bar{\sigma}^\mu D_\mu^L)^\dagger = i\sigma^\mu D_\mu^R . \quad (\text{D.9})$$

Appendix E

Covariant Derivative Expansion – Gravitational Anomaly Computations

In this Appendix, we outline the computation of the anomalies of a Weyl fermion using the CDE in curved spacetime [3]. In this Appendix, \mathcal{D} denotes the Weyl operator, and D is the total covariant derivative.

Let us first note that we regularised the functional traces using the function e^{-x} in both sections. The result is however independent of this choice. In fact, any smooth function f such that

$$f(0) = 1, \quad f(x) \xrightarrow{x \rightarrow \infty} 0, \quad x f'(x)|_{x=0} = x f'(x)|_{x \rightarrow \infty} = 0. \quad (\text{E.1})$$

This is well-known for Fujikawa's approach [118] and we establish it here for Leutwyler's approach. For a function f satisfying the criteria above the following equation holds

$$\mathcal{D}^{-1} = \mathcal{D}^\dagger \int_0^\infty dt f'(t \mathcal{D} \mathcal{D}^\dagger), \quad (\text{E.2})$$

for which (13.10) is a special case.

Having established this universality in the function f we turn to the CDE for which it is convenient to use $f(x) = 1/(1+x)$. We thus consider the following functional traces¹

$$\begin{aligned} T_1[a] &= \lim_{\Lambda \rightarrow \infty} \text{Tr} \left[a(x) f \left(\frac{(i\mathcal{D})^2}{\Lambda^2} \right) \right] = \lim_{\Lambda \rightarrow \infty} \text{Tr} \left[a(x) \frac{\Lambda^2}{-\mathcal{D}^2 + \Lambda^2} \right], \\ T_2[b] &= \lim_{\Lambda \rightarrow \infty} \text{Tr} \left[b^\mu(x) \nabla_\mu f \left(\frac{(i\mathcal{D})^2}{\Lambda^2} \right) \right] = \lim_{\Lambda \rightarrow \infty} \text{Tr} \left[b^\mu(x) \nabla_\mu \frac{\Lambda^2}{-\mathcal{D}^2 + \Lambda^2} \right], \end{aligned} \quad (\text{E.3})$$

where $a(x)$ and $b^\mu(x)$ are local functions. As in the main text it is understood that the Λ^2 - and Λ^4 -divergences are subtracted.

Computation of the trace and Lorentz anomalies

From T_1 the trace and the Lorentz anomalies of a Weyl fermion follow

$$\delta_\sigma^W W = \frac{1}{2} T_1[\sigma], \quad \delta_\alpha^L W = \frac{1}{2} T_1[\alpha_{ab} \Sigma^{ab} \gamma_5]. \quad (\text{E.4})$$

¹Note that if we had chosen a representation of the Dirac matrices such that $(\gamma^\mu)^\dagger = -\gamma^\mu$ in Euclidean space, we would obtain $f\left(\frac{\mathcal{D}^2}{\Lambda^2}\right)$ instead of $f\left(\frac{(i\mathcal{D})^2}{\Lambda^2}\right) = f\left(\frac{\mathcal{D}^2}{-\Lambda^2}\right)$. The anomaly being of order Λ^0 does not depend on that choice.

The functional trace can be recast in momentum space as

$$\begin{aligned} T_1[a] &= \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4q}{(2\pi)^4} \text{tr} a(x) \Lambda^2 \frac{1}{-(\not{D} + i\not{q})^2 + \Lambda^2} \\ &= - \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4q}{(2\pi)^4} \text{tr} a(x) (-\Lambda^2) \sum_{n \geq 0} \left[\Delta (D^2 + g^{\mu\nu} \{D_\mu, i q_\nu\} + \Sigma \cdot F) \right]^n \Delta, \end{aligned} \quad (\text{E.5})$$

where $\Delta = 1/(q^2 + \Lambda^2)$, $\Sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/4$, and tr denotes the trace in internal space. Note that we can maintain $\partial_\mu q_\nu = 0$, but contrary to the CDE in flat spacetime we have $D_\mu q_\nu = -\Gamma_{\mu\nu}^\rho q_\rho \neq 0$ and $[D_\mu, \Delta] \neq 0$ [3]. F is the fermion field strength due to the spin-connection such that for any fermion $\tilde{\psi}$ we have $[D_\mu, D_\nu] \tilde{\psi} = F_{\mu\nu} \tilde{\psi}$ with²

$$F_{\mu\nu} = \frac{1}{4} \gamma^\rho \gamma^\sigma R_{\mu\nu\rho\sigma}, \quad \Sigma \cdot F = -\frac{R}{4} \mathbb{1}_{\text{Dirac}}. \quad (\text{E.6})$$

The expansion is then carried out with the help of Mathematica and the package xAct [223]. The result reads

$$\begin{aligned} T_1[a] &= \frac{1}{16\pi^2} \int d^4x \text{tr} a(x) \left\{ -\frac{1}{6} \square (\Sigma \cdot F) - \frac{1}{12} F^2 - \frac{1}{72} R^2 + \frac{1}{180} R_{\mu\nu} R^{\mu\nu} \right. \\ &\quad \left. - \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{30} \square R - \frac{1}{6} R \Sigma \cdot F - \frac{1}{2} (\Sigma \cdot F)^2 \right\}, \end{aligned} \quad (\text{E.7})$$

from which, by using (E.4) and (E.6), the trace anomaly and the vanishing of the Lorentz anomaly follows.

In fact once we know, from (E.7), that the Lorentz anomaly is covariant we can infer its vanishing in yet another way in $d = 4$. It can be shown, using intrinsically 4-dimensional identities [208–210], that the only parity-odd (covariant) 2-tensor of mass dimension 4 is $g^{\mu\nu} R \tilde{R}$, which is symmetric in its indices and thus vanishes when contracted with the Lorentz parameter $\alpha_{\mu\nu}$. In particular, one has in $d = 4$

$$\frac{1}{2} g^{\alpha\beta} R \tilde{R} = \epsilon^{\alpha\nu\rho\sigma} R^\beta{}_\nu{}^{\lambda\chi} R_{\rho\sigma\lambda\chi} = \epsilon^{\mu\nu\rho\sigma} R^{\alpha\lambda}{}_{\mu\nu} R^\beta{}_{\lambda\rho\sigma}, \quad (\text{E.8})$$

and

$$\epsilon^{\alpha\nu\rho\sigma} R^{\beta\lambda}{}_{\rho\sigma} R_{\nu\lambda} = 0, \quad (\text{E.9})$$

for the Ricci-tensor contraction. Every other parity-odd 2-tensor of mass dimension 4 is related to these by Bianchi identities and algebra.

Computation of the diffeomorphism anomaly

The diffeomorphism anomaly is given by

$$\delta_\xi^d W = -T_2[\xi\gamma_5] - \frac{1}{2} T_1[(D_\mu \xi^\mu)\gamma_5], \quad (\text{E.10})$$

a sum of a T_1 - and a T_2 -term (E.3). Let us focus on the T_1 -term first. Using (E.6) the only term that is non-vanishing under the Dirac trace in $T_1[(D_\mu \xi^\mu)\gamma_5]$ is

$$\begin{aligned} T_1[(D_\mu \xi^\mu)\gamma_5] &= \frac{-1}{16\pi^2} \int d^4x e (D_\mu \xi^\mu) \text{tr} \gamma_5 \left(-\frac{1}{12} F^2 \right) \\ &= \frac{-1}{16\pi^2} \int d^4x e \xi^\mu \text{tr} \gamma_5 \left(\frac{1}{12} [D_\mu, F^2] \right), \end{aligned} \quad (\text{E.11})$$

²Note that ψ and $\tilde{\psi}$ have the same field strength.

where integration by parts was applied using the fact that $\xi^\mu(x)$ vanishes at infinity.

Finally, let us turn to the T_2 -term which is less straightforward. In Eq. (E.3), it is convenient to rewrite $\nabla = D - \omega$, such that

$$T_2[b] = \lim_{\Lambda \rightarrow \infty} \text{Tr} \left[b^\mu(x) D_\mu \frac{\Lambda^2}{-\not{D}^2 + \Lambda^2} \right] - \lim_{\Lambda \rightarrow \infty} \text{Tr} \left[b^\mu(x) \omega_\mu \frac{\Lambda^2}{-\not{D}^2 + \Lambda^2} \right]. \quad (\text{E.12})$$

We can notice that the second term is the Lorentz anomaly with $\alpha_{ab} = b^\mu \omega_{\mu ab}$ (with $b^\mu \propto \gamma_5$), hence vanishes as we just verified. We are left with

$$\begin{aligned} T_2[b] &= \lim_{\Lambda \rightarrow \infty} \text{Tr} \left[b^\mu(x) D_\mu \frac{\Lambda^2}{-\not{D}^2 + \Lambda^2} \right] \\ &= \lim_{\Lambda \rightarrow \infty} \int d^4x \frac{d^4q}{(2\pi)^4} \text{tr} b^\mu(x) (D_\mu + i q_\mu) (-\Lambda^2) \sum_{n \geq 0} \left[\Delta(D^2 + g^{\mu\nu} \{D_\mu, i q_\nu\} + \Sigma \cdot F) \right]^n \Delta. \end{aligned} \quad (\text{E.13})$$

The computation must not be carried out in a manifestly covariant manner, and one cannot use a specific choice of coordinate (for example Riemann normal coordinates). Since the diffeomorphism anomaly involves $b^\mu(x) = \xi^\mu(x) \gamma_5$, the computation can be simplified using $\text{Tr} \gamma_5 = \text{Tr} \gamma^\mu \gamma^\nu \gamma_5 = 0$, and noticing that the only source of Dirac matrices are in the covariant derivatives via the spin-connection. We finally obtain a very compact result

$$T_2[\xi \gamma_5] = \frac{-1}{16\pi^2} \int d^4x e \xi^\mu \text{tr} \gamma_5 \left(-\frac{1}{24} [D_\mu, F^2] \right), \quad (\text{E.14})$$

as every other term vanishes by lack of Dirac matrices.

Using (E.10), (E.11) and (E.14) we see the canceling of terms and finally conclude that the diffeomorphism anomaly of a Weyl fermion vanishes.

Appendix F

Weyl Anomaly Definition

In Appendix, we add precisions about the derivation of the expressions for the Weyl anomaly with and without explicit breaking. We also address questions about explicit breaking, counterterms, and the role of $\square R$. D denotes the total covariant derivative.

The Weyl variation of $\mathcal{W}_{\text{ren}}(d)$ is defined by (14.16) with explicit symmetry breaking, and (14.11) without. Either way, if we take the limit before the Weyl variation we obtain

$$\delta_{\sigma}^W \lim_{d \rightarrow 4} \mathcal{W}_{\text{ren}}(d) = \int d^4x \sigma g_{\mu\nu}^{(4)} \frac{\delta}{\delta g_{\mu\nu}} [\mathcal{W}_{\text{ren}}(d)]_{d=4} = \int d^4x \sigma \left[g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}(d)}{\delta g_{\mu\nu}} + g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{d=4},$$

where we used the fact that $\mathcal{W}_{\text{ren}}(4)$ is finite to commute the metric variation and the metric contraction inside the limit. Alternatively, the Weyl anomaly can be written by taking the limit after the Weyl variation

$$\lim_{d \rightarrow 4} \delta_{\sigma}^W \mathcal{W}_{\text{ren}}(d) = \int d^4x \sigma \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ren}}(d)}{\delta g_{\mu\nu}} \right]_{d=4} = \int d^4x \sigma \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}(d)}{\delta g_{\mu\nu}} + g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{d=4}.$$

In the following we prove some identities on the counterterms that are used in the derivation of Eqs. (14.12), (14.14), (14.18) and (14.21).

Classical Weyl invariance

The most generic form of the counterterms in a classically Weyl invariant theory reads [169]

$$\mathcal{W}_{\text{ct}}(d) = \frac{1}{d-4} \int d^d x \sqrt{|g|} (a E(d) + c C(d) + f \text{tr} F(d)^2), \quad (\text{F.1})$$

where we included only P -even terms. We omit P -odd operator since we have showed that they must be absent at the level of the metric variation of \mathcal{W}_{ct} . One can show that [169]

$$\begin{aligned} \frac{2}{\sqrt{|g|}} g_{\mu\nu}^{(d)} \frac{\delta}{\delta g_{\mu\nu}^{(d)}} \int d^d x \sqrt{|g|} C &= (d-4) \left(C + \frac{2}{3} \square R \right), \\ \frac{2}{\sqrt{|g|}} g_{\mu\nu}^{(d)} \frac{\delta}{\delta g_{\mu\nu}^{(d)}} \int d^d x \sqrt{|g|} E &= (d-4) E, \\ \frac{2}{\sqrt{|g|}} g_{\mu\nu}^{(d)} \frac{\delta}{\delta g_{\mu\nu}^{(d)}} \int d^d x \sqrt{|g|} F^2 &= (d-4) F^2. \end{aligned} \quad (\text{F.2})$$

Whereas contracting with the metric in $d = 4$ one obtains [169]

$$\begin{aligned} \frac{2}{\sqrt{|g|}} g_{\mu\nu}^{(4)} \frac{\delta}{\delta g_{\mu\nu}^{(d)}} \int d^d x \sqrt{|g|} C &= 0, \\ \frac{2}{\sqrt{|g|}} g_{\mu\nu}^{(4)} \frac{\delta}{\delta g_{\mu\nu}^{(d)}} \int d^d x \sqrt{|g|} E &= 0, \\ \frac{2}{\sqrt{|g|}} g_{\mu\nu}^{(4)} \frac{\delta}{\delta g_{\mu\nu}^{(d)}} \int d^d x \sqrt{|g|} F^2 &= 0, \end{aligned} \quad (\text{F.3})$$

which then can be used to deduce that

$$\left[g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}}{\delta g_{\mu\nu}} \right]_{d=4} = 0, \quad (\text{F.4})$$

holds in a classically Weyl invariant theory. For example for the Euler density we find

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}^{(d)}} \int d^d x \sqrt{|g|} E &= 8 R^{\mu\alpha} R^\nu{}_\alpha - 4 R^{\mu\nu} R + 8 R^{\alpha\beta} R^\mu{}_\alpha{}^\nu{}_\beta - 4 R^{\mu\alpha\beta\delta} R^\nu{}_{\alpha\beta\delta} \\ &\quad + g^{\mu\nu} (R^2 - 4 R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}), \end{aligned} \quad (\text{F.5})$$

where all the tensors are in d . We then obtain the above result for E upon using $g^{(d),\mu\nu} g_{\mu\nu}^{(d)} = d$ and $g^{(4),\mu\nu} g_{\mu\nu}^{(d)} = 4$.

Classical Weyl non-invariance

The Weyl variation after and before taking the $d \rightarrow 4$ limit read

$$\begin{aligned} \delta_\sigma^W \lim_{d \rightarrow 4} \mathcal{W}_{\text{ren}}(d) &= \int d^4 x \sigma \left[\sqrt{|g|} g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle + g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{d=4} \\ \lim_{d \rightarrow 4} \delta_\sigma^W \mathcal{W}_{\text{ren}}(d) &= \int d^4 x \sigma \left[\sqrt{|g|} \langle T^\mu{}_\mu \rangle + g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{d=4}, \end{aligned} \quad (\text{F.6})$$

where in each line, the sum is finite but each term is separately divergent. Let us separate

$$\begin{aligned} \langle T^\mu{}_\mu \rangle &= \langle T^\mu{}_\mu \rangle_{\text{div}} + \langle T^\mu{}_\mu \rangle_{\text{fin}} \\ g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} &= \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{\text{div}} + \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{\text{fin}}, \end{aligned} \quad (\text{F.7})$$

into their finite and divergent pieces. We then identify the explicit breaking with

$$\mathcal{E}_{\text{Weyl}} \equiv \langle T^\mu{}_\mu \rangle_{\text{fin}}, \quad (\text{F.8})$$

from which we can deduce the expressions for the anomaly in terms of \mathcal{W}_{ct}

$$\mathcal{A}_{\text{Weyl}} = \frac{1}{\sqrt{|g|}} \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{\text{fin}}, \quad (\text{F.9})$$

or equivalently in terms of the effective action \mathcal{W}

$$\mathcal{A}_{\text{Weyl}} = g_{\mu\nu}^{(4)} \langle T^{\mu\nu} \rangle - \langle T^\mu{}_\mu \rangle. \quad (\text{F.10})$$

Finite piece in \mathcal{W}_{ct} Let us first provide more details about the comment in footnote 6. One may argue regarding (F.9) that introducing a finite piece in \mathcal{W}_{ct} will alter the expression of the anomaly. In fact (F.9) and (F.8) have to be amended if we include a finite piece in \mathcal{W}_{ct} . In the derivation of Eqs. (F.9) and (F.8) we assumed that the counterterms are in minimal subtraction form

$$\mathcal{W}_{\text{ct}}(d) = \frac{1}{d-4} \int d^d x A(d). \quad (\text{F.11})$$

We may parametrise finite counterterms by

$$\mathcal{W}'_{\text{ct}}(d) = \mathcal{W}_{\text{ct}}(d) + \int d^4 x B(4), \quad (\text{F.12})$$

leading to

$$\mathcal{W}'_{\text{ren}}(d) = \mathcal{W}(d) + \mathcal{W}'_{\text{ct}}(d), \quad (\text{F.13})$$

where the effective action is defined from

$$\mathcal{W}(d) = -\log \int \mathcal{D}\phi e^{-S[\phi]}. \quad (\text{F.14})$$

Firstly, the addition of the 4-dimensional term $B(4)$ only amounts to introducing a classical (i.e independent of ϕ) term in the UV action

$$\mathcal{W}'_{\text{ren}}(d) = \mathcal{W}'(d) + \mathcal{W}_{\text{ct}}(d), \quad (\text{F.15})$$

with

$$\mathcal{W}'(d) = -\log \int \mathcal{D}\phi e^{-S[\phi] - \int d^4 x B(4)}, \quad (\text{F.16})$$

hence cannot alter the anomaly, even if it is not Weyl invariant (it would then just be an explicit breaking).

If we were to proceed with the counterterms \mathcal{W}'_{ct} instead of \mathcal{W}_{ct} , the explicit breaking and anomaly formulae would need to be amended to

$$\mathcal{E}_{\text{Weyl}} \equiv \langle T^\mu{}_\mu \rangle_{\text{fin}} + \frac{1}{\sqrt{|g|}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4 x B(4), \quad \mathcal{A}_{\text{Weyl}} = \frac{1}{\sqrt{|g|}} \left[g_{\mu\nu}^{(d)} \frac{\delta}{\delta g_{\mu\nu}} \frac{1}{d-4} \int d^d x A(d) \right]_{\text{fin}},$$

which are exactly the same as in Eqs. (F.9) and (F.8). The anomaly given in terms of the EMT (F.10) is unchanged whether we use $\langle T^{\mu\nu} \rangle = \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{W}}{\delta g_{\mu\nu}}$ or $\langle T'^{\mu\nu} \rangle = \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{W}'}{\delta g_{\mu\nu}}$.

The irrelevance of finite terms of \mathcal{W}_{ct} regarding the anomaly is the reason why we may use the tensor

$$C(d) = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \quad (\text{F.17})$$

in (14.6), which reduces to the Weyl tensor squared $C(4) = W^2(4)$ in four dimensions, instead of the natural extension of the Weyl tensor squared to d dimensions

$$W^2(d) = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{4}{d-2} R_{\mu\nu} R^{\mu\nu} + \frac{2}{(d-2)(d-1)} R^2. \quad (\text{F.18})$$

Indeed, this second choice of analytic continuation will only amount to finite terms in \mathcal{W}_{ct} since

$$\frac{1}{d-4} W^2(d) = \frac{1}{d-4} C(d) + \mathcal{O}(1). \quad (\text{F.19})$$

Explicit breaking counterterms Let us prove a relation that we use regarding the contribution from \mathcal{W}_{ct} that cancels the divergence of the explicit breaking. Since (F.6) is finite we have

$$\sqrt{|g|}\langle T^\mu{}_\mu \rangle_{\text{div}} = - \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} \right]_{\text{div}} . \quad (\text{F.20})$$

On the other hand, we also use (14.20)

$$g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} = -\sqrt{|g|}\langle T^\mu{}_\mu \rangle_{\text{div}} , \quad (\text{F.21})$$

in the derivation of (14.21). This relation follows from the fact that $g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} = 0$ in a classically Weyl invariant theory. Therefore, in an explicitly broken theory this is exactly the contribution from \mathcal{W}_{ct} that cancels the divergence of the explicit breaking $\langle T^\rho{}_\rho \rangle$.

When combined, these last two equations imply

$$g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} = \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}}{\delta g_{\mu\nu}} \right]_{\text{div}} . \quad (\text{F.22})$$

Let us show this equality explicitly. The most generic form of the counterterms, including again only P -even operators, is

$$\mathcal{W}_{\text{ct}}(d) = \frac{1}{d-4} \int d^d x \sqrt{|g|} (a E(d) + b R(d)^2 + c W(d)^2 + f \text{tr} F(d)^2) , \quad (\text{F.23})$$

where the R^2 -term is introduced since Weyl symmetry is explicitly broken. As shown above, in a classically Weyl invariant theory we have $g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} = 0$ (F.3) and $g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}}{\delta g_{\mu\nu}}$ is finite (F.2). Hence, only the R^2 -term may contribute in (F.22). Let us consider its variation under a generic metric variation $\delta g_{\mu\nu}$

$$\frac{1}{\sqrt{|g|}} \delta \sqrt{|g|} R^2 = \frac{1}{2} g^{\mu\nu} R^2 \delta g_{\mu\nu} - 2R R^{\mu\nu} \delta g_{\mu\nu} + 2R g^{\mu\nu} (D_\alpha \delta \Gamma_{\mu\nu}^\alpha - D_\mu \delta \Gamma_{\nu\alpha}^\alpha) , \quad (\text{F.24})$$

where

$$\delta \Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (D_\nu \delta g_{\mu\lambda} + D_\mu \delta g_{\nu\lambda} - D_\lambda \delta g_{\mu\nu}) . \quad (\text{F.25})$$

In order to perform the metric differentiation the last term of (F.24) is integrated by parts, using $\delta g_{\mu\nu}(x) \xrightarrow{x \rightarrow \infty} 0$. We obtain

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^d x \sqrt{|g|} R^2 = \frac{1}{2} \sqrt{|g|} R (g^{\mu\nu} R - 4R^{\mu\nu}) + 2\sqrt{|g|} (D^\mu D^\nu R - g^{\mu\nu} \square R) . \quad (\text{F.26})$$

Finally, using $g_{\mu\nu}^{(d)} g^{(d),\mu\nu} = d$ and $g_{\mu\nu}^{(4)} g^{(d),\mu\nu} = 4$, as well as (F.2) and (F.3) we obtain

$$\begin{aligned} \frac{1}{\sqrt{|g|}} g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} &= -\frac{6b}{d-4} \square R \\ \frac{1}{\sqrt{|g|}} g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} &= X(a, c, f) + \frac{b}{2} R^2 - 2b \square R - \frac{6b}{d-4} \square R , \end{aligned} \quad (\text{F.27})$$

where $X(a, c, f)$ denotes the contributions from the E , C and F^2 terms which are finite (F.2). As expected, we confirm (F.22)

$$g_{\mu\nu}^{(4)} \frac{\delta \mathcal{W}_{\text{ct}}(d)}{\delta g_{\mu\nu}} = -\frac{6b}{d-4} \square R = \left[g_{\mu\nu}^{(d)} \frac{\delta \mathcal{W}_{\text{ct}}}{\delta g_{\mu\nu}} \right]_{\text{div}} . \quad (\text{F.28})$$

This is the contribution from \mathcal{W}_{ct} that cancels the divergence of the explicit breaking $\langle T^\rho{}_\rho \rangle$. Once again, this relation still holds when P -odd operators are included in \mathcal{W}_{ct} since they vanish at the level of $\delta \mathcal{W}_{\text{ct}}/\delta g_{\mu\nu}$.

Appendix G

Constraints on the Energy-momentum Tensor

In this appendix, we discuss the basis of P -odd operators, and we detail how to enforce the diffeomorphism and Lorentz constraints. D denotes the total covariant derivative.

Let us recall the generic ansatz (14.31)

$$\begin{aligned} \mathcal{T}_{\text{ct}}^{\alpha\beta} = \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{W}_{\text{ct}}}{\delta g_{\alpha\beta}} = & e g^{\alpha\beta} R \tilde{R} + \frac{1}{d-4} \left\{ e_1 g^{\alpha\beta} R \tilde{R} + e_2 P^{\alpha\beta} + e_3 Q^{(\alpha\beta)} + e_4 S^{(\alpha\beta)} \right. \\ & + a_1 g^{\alpha\beta} R^2 + a_2 g^{\alpha\beta} R_{\mu\nu} R^{\mu\nu} + a_3 g^{\alpha\beta} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + a_4 g^{\alpha\beta} \square R \\ & + b_1 R R^{\alpha\beta} + b_2 R^{\alpha\lambda} R^\beta{}_\lambda + b_3 R_{\mu\nu} R^{\mu\alpha\nu\beta} + b_4 R^{\alpha\lambda\mu\nu} R^\beta{}_{\lambda\mu\nu} \\ & \left. + c_1 D^\alpha D^\beta R + c_2 \square R^{\alpha\beta} \right\}, \end{aligned} \quad (\text{G.1})$$

with the symmetrisation of indices $t^{(\mu\nu)} = \frac{1}{2}(t^{\mu\nu} + t^{\nu\mu})$. The P -odd coefficients are constrained by (14.24)

$$2e_1 + e_2 + e_3 = 0. \quad (\text{G.2})$$

Parity-odd operators basis

Let us try to write all possible parity-odd 2-tensors of mass dimension four depending only on curvature invariants. Firstly, by enumeration it is possible to show that there exists no P -even antisymmetric 2-tensor, therefore we have $\epsilon^{\alpha\beta\mu\nu} \mathcal{O}_{\mu\nu} = 0$ where \mathcal{O} is a P -even 2-tensor. Secondly, due to the Bianchi identities one has $\epsilon^{\alpha\nu\rho\sigma} R_{\beta\nu\rho\sigma} = 0$ and $\epsilon^{\mu\nu\rho\sigma} D_\nu R_{\alpha\beta\rho\sigma} = 0$. Using this result, any operator can be related to these by Bianchi identities and algebra

$$\begin{aligned} g_{\alpha\beta} R \tilde{R} &= g_{\alpha\beta} \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R_{\gamma\delta\mu\nu} R^{\gamma\delta}{}_{\rho\sigma}, \\ P_{\alpha\beta} &= \epsilon^{\mu\nu\rho\sigma} R_{\alpha\lambda\mu\nu} R_{\beta\rho\sigma}{}^\lambda, \\ Q_{\alpha\beta} &= \epsilon_\alpha{}^{\nu\rho\sigma} R_{\beta\nu\gamma\delta} R_{\rho\sigma}{}^{\gamma\delta}, \\ S_{\alpha\beta} &= \epsilon_\alpha{}^{\nu\rho\sigma} R_{\beta\lambda\rho\sigma} R_\nu{}^\lambda = -\epsilon_\alpha{}^{\nu\rho\sigma} D_\nu D_\gamma R_{\beta\rho\sigma}{}^\gamma. \end{aligned} \quad (\text{G.3})$$

However, upon using the intrinsically four-dimensional Schouten identity [208], we obtain

$$P_{\alpha\beta} = Q_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} R \tilde{R}, \quad S_{\alpha\beta} = 0, \quad (\text{G.4})$$

showing that these operators are not independent in $d = 4$. These relations can also directly be obtained from Ref.[210], or from useful applications of the Schouten identities (e.g. [209]).

Diffeomorphism anomaly constraint

The finiteness of diffeomorphism anomaly is ensured by

$$D_\alpha \mathcal{T}_{\text{ct}}^{\alpha\beta} = 0 . \quad (\text{G.5})$$

Using Bianchi identities and algebra, it is possible to write the P -even part of the divergence of \mathcal{T}_{ct} in terms of independent operators as

$$\begin{aligned} D_\alpha \mathcal{T}_{\text{ct}}^{\alpha\beta} \Big|_{P\text{-even}} &= (a_4 + c_1 + \frac{1}{2}c_2) \square D^\beta R + (-a_4 + b_1 + \frac{1}{2}b_2 + \frac{1}{2}c_2) R^{\beta\mu} D_\mu R \\ &+ (2a_2 + b_3 - c_2) R^{\mu\nu} D^\beta R_{\mu\nu} + (2a_1 + \frac{1}{2}b_1) R D^\beta R \\ &+ (2a_3 + \frac{1}{2}b_4) R^{\mu\nu\rho\sigma} D^\beta R_{\mu\nu\rho\sigma} + (b_2 - b_3 + 2c_2) R^{\mu\nu} D_\mu R^\beta{}_\nu \\ &+ (-b_3 - 2b_4 + 2c_2) R^\beta{}_{\nu\rho\sigma} D^\sigma R^{\nu\rho} . \end{aligned} \quad (\text{G.6})$$

Note that it is not allowed to use integrations by parts at this level (besides $D_\alpha \mathcal{T}_{\text{ct}}^{\alpha\beta}$ is itself a boundary term). Imposing $D_\alpha \mathcal{T}_{\text{ct}}^{\alpha\beta} \Big|_{P\text{-even}} = 0$ yields seven constraints on the ten parameters associated to the P -even operators. They can be recast to obtain (14.38).

Concerning the P -odd part, it is possible to write the divergence as

$$\begin{aligned} D_\nu \mathcal{T}_{\text{ct}}^{\mu\nu} \Big|_{P\text{-odd}} &= 2e W^\mu(4) + \frac{1}{d-4} \left\{ 2e_1 W^\mu + e_2 (2V^\mu + W^\mu) \right. \\ &\left. + \frac{e_3}{2} (2X^\mu + Y^\mu + W^\mu) + \frac{e_4}{2} (X^\mu + V^\mu + 2Z^\mu) \right\} = 0 , \end{aligned} \quad (\text{G.7})$$

where

$$\begin{aligned} V^\mu &= \epsilon^{\alpha\beta\gamma\delta} R^{\mu\lambda}{}_{\alpha\beta} D_\gamma R_{\lambda\delta} , & W^\mu &= \epsilon^{\alpha\beta\gamma\delta} R^{\omega\lambda}{}_{\gamma\delta} D_\omega R^\mu{}_{\lambda\alpha\beta} , \\ X^\mu &= \epsilon^{\mu\lambda\alpha\beta} R_{\alpha\beta}{}^{\gamma\delta} D_\gamma R_{\delta\lambda} , & Y^\mu &= \epsilon^{\mu\lambda\alpha\beta} R^\omega{}_{\lambda\gamma\delta} D_\omega R^{\gamma\delta}{}_{\alpha\beta} , & Z^\mu &= \epsilon^{\mu\lambda\alpha\beta} R_{\gamma\lambda} D_\alpha R^\gamma{}_\beta . \end{aligned}$$

Some of these operators may be related by the Schouten identity in $d = 4$ only. They are not related by Bianchi identities either and are thus independent. Additionally, the pole and the finite parts in (G.7) are independent. Contrary to the P -even part, the system (G.7) is overconstrained leaving the sole solution

$$e = e_1 = e_2 = e_3 = e_4 = 0 . \quad (\text{G.8})$$

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