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The mathematical and epistemological works of Georges Bouligand

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Rossana was offering a doctoral research topic on the mathematician Georges Bouligand, who I had encountered briefly in my masters thesis. Together with Anouk and others, Rossana had started some initial research on Bouligand and they had identified him as a figure of interwar French mathematics who, although rich in terms of mathematics, his more philosophical reflections and his connections within the mathematical community of his time, was not yet studied in any depth.

I was immediately interested in the topic but was not keen to go back to studying full time, so I first found a job near Paris and then came back to Rossana and Anouk some months later to enroll with the intention of carrying out the thesis part time, while continuing my full time job.

It has been a long and enriching journey through a number of years, a number of jobs and plenty of interesting encounters and learning experiences. I am particularly grateful to my thesis committee advisors, June Barrow-Green and Mark van Atten - leaders in their fields who have been generous in providing me with valuable advice and encouragement, always grounded in common sense and good humour.

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While my thesis is just a starting point, I would like to highlight that Georges Bouligand has been a thoroughly enjoyable figure to study, not only due to his ideas and academic output but also due to the nature of his personal character which begins to shine through in the available written correspondence - a charming, modest and humble family man with diverse interests and talents, who carried out his own research with great enthusiasm while thoroughly supporting students, colleagues and friends alike. I believe that my thesis shows that he is a figure to be remembered, with many promising avenues of further study possible.

Introduction

The intellectual output of Georges Bouligand during the interwar period represents a rich corpus consisting not only of mathematical articles and texts in divers areas of the discipline but also his published ideas on the epistemology of mathematics - on his views regarding mathematical methods, how mathematical theories should be formulated and how mathematics should be taught. In addition, as a member of the mathematical community in France who was well connected with some of the most prominent mathematicians not only of his own generation but of the academic generation before and after, a study of Bouligand and his works enables us to add a modest brush stroke to the currently available picture of the French mathematical scene of the day. Our study may shed some further light on these mathematicians and on certain ideas and trends - such themes that will arise include, for example, debates on rigour and intuition and the emergence of Cantor's set theory in France; looking at Bouligand's contributions enables us to bring to light further examples of the manifestation and influence of these trends during the interwar period.

We will elaborate below on these key points and we will present the chosen approach and scope of our study. Before proceeding, we emphasise first that, despite the clear interest of a somewhat in depth study of Bouligand, relatively little work exists today giving us an insight into this figure of 20th century French mathematics.

A note on existing works on Bouligand.

The most detailed accounts of Bouligand's contributions are to be found in his own published notes, one published towards the end of his career in 1961 [101] and an earlier summary covering the period 1912 to 1937 [95], shortly before he moved from Poitiers to Paris. Besides Bouligand's own accounts, perhaps the most detailed historical elements are to be found in the impressive thesis of Juliette Leloup [170], which studies interwar mathematics in France through the lens of all doctoral theses defended during this period. A section of twelve pages [170, 119-131] is dedicated to Bouligand's influence in Poitiers through the doctoral students guided by him during the interwar period. While this account provides precious insights upon which we will draw here regarding Bouligand's role in the mathematical community and regarding

his influence on doctoral students, Leloup's goal was clearly not to provide a more detailed presentation specifically of Bouligand and his works. A further available source focussing on Bouligand is Henri Poncin's obituary of five pages in honour of Bouligand [202]. Although Poncin's article provides valuable insights into Bouligand as a person, his role in the mathematical community in Poitiers and to his mathematical contributions, it by no means represents an in depth study. Roland Brasseur's fascinating online *Dictionnaire des professeurs de mathématiques spéciales* [106] contains many precious gems of information on Bouligand which have been extracted from relevant archives, although we stress that Brasseur's goal was not to provide a formalised and elaborated account.

The 1999 doctoral thesis of Dorothée Bessis [9] introduces in some detail the tools of Bouligand's theory of direct infinitesimal geometry (to which we dedicate two chapters below), as well as commenting on its motivations, possible influencing factors and translating concepts into more modern terms. However, Bessis' thesis belongs clearly to mathematics itself and introduces only one key area of Bouligand's work, whose exposition is not her main goal. Also from a modern mathematical perspective, Rockefeller and Wets refer quite extensively to Bouligand and highlight certain of his contributions in their text on variational analysis [204]. They focus specifically on concepts dealt with in Bouligand's direct infinitesimal geometry and their presentation of Bouligand's ideas remain as insightful references in the broader context of a modern text on variational analysis.

Finally we note that perhaps the most detailed exploration of Bouligand's ideas in the epistemology of mathematics is to be found in the work of Paolo Mancosu on mathematical explanation, in his 2001 article on the topic [176]. Here he draws on several articles of Bouligand to illustrate Bouligand's concepts of causal and non-causal proofs and alludes briefly to the concepts of domain of causality and direct versus non-direct methods. It is interesting to note that philosopher Albert Lautman commented briefly in the 1930s on Bouligand's ideas in the context of a discussion on axiomatics and generalisation [164]. We have not however found any thorough presentation of Bouligand's core concepts in the epistemology of mathematics and no work on the interaction between his mathematics and his epistemological reflections.

We note that quite extensive work exists on the Swiss mathematician and contemporary of Bouligand, Ferdinand Gonseth (1890-1975). While we do not attempt to go too far with the analogy here, Gonseth could perhaps be seen in a similar light as Bouligand in terms of the diversity of his academic output and the fact that he dealt both with mathematics and his philosophical reflections. Like Bouligand, Gonseth was

well connected in the mathematical and institutional community of his time. We see, in comparison to Gonseth, a clear lack of work presenting Bouligand and his mathematical and philosophical output.

0.1 Key motivating factors for the present account

0.1.1 Bouligand as a mathematician active in publishing his epistemological ideas in France

Far from being two distinct themes of research, Bouligand's mathematics and his ideas on the epistemology of mathematics evolved, to a certain extent, hand-in-hand. Bouligand appears as a mathematician who was particularly active in publishing his epistemological ideas about mathematics. While it is beyond the scope of the present account to provide a survey of the work of French mathematicians during the 1920s and 30s on the epistemology of mathematics, we can gain at least some rough measure of Bouligand's activity by considering the two main journals in which he published his ideas, namely the *Revue générale des sciences pures et appliquées* and the *Revue scientifique*.

A brief survey of the *Revue Scientifique* in the interwar years in which Bouligand published approximately nine articles dealing with his epistemological reflections (each year from 1927 to 1933, excluding 1931), then we see that other contributions relating to mathematics are few and far between. In the 1927 volume, Emile Borel published an article on *Les lois physiques et les probabilités* and the other volumes collectively contain only a handful of other articles dealing with mathematics. For example, the Italian mathematician and astronomer Giovanni Boccardi (whose articles in this journal are published under the name of Jean Boccardi), a correspondent of the *Bureau des Longitudes* published a small number of articles providing a largely historical account on topics such as geodesics and probability. Other articles relating to mathematics include a paper on mathematical statistics by the mathematician Robert de Montessus de Ballore and an account by Madame Paul Tannery (the name under which Marie-Alexandrine, the wife of the mathematician Paul Tannery, published her work) of historical correspondence between mathematicians - both of these were of a predominantly historical nature.

In the volumes of the *Revue Générale des Sciences Pures et Appliquées* published between 1930 and 1936 (the years in which Bouligand published his ideas in approximately six articles in this journal) again, articles relating to mathematics are relatively few.

The authors of the several articles relating to mathematics include Boccardi and de Montessus de Ballore mentioned above each of whom publish once. Maurice d’Ocagne (1862-1938) - a French mathematician and engineer whose who is considered to be the founder of nomography, appears as a more regular contributor to this journal within the time period, publishing for example on geometrical constructions in two articles of 1933 and 1934. It is also worth noting a single publication during this time period by Nikolai Krylov (1879-1955) - an eminent Russian mathematician whose work focussed on the area of differential equations and non-linear mechanics. The article is co-authored by the Russian mathematician and theoretical physicist Nikolay Bogolyubov (1909-1992) and discusses problems in non-linear mechanics. Besides a short article of 1936 by Z. Rupeika on analytic geometry ¹, we mention an article by Jean-Louis Destouches of 1935 on *Les acquisitions récentes des mathématiques applicables à la physique* which draws significantly on Bouligand’s ideas regarding a concept he referred to as the stability of mathematical theorems, which will be explored in greater depth in the present account.

0.1.2 Bouligand as an actor embedded in the mathematical community

Bouligand was well connected with some of the most prominent mathematicians of his time. During his education at *lycée*, in his *mathématiques spéciales* class, Bouligand was taught by Paul Montel and Maurice Fréchet [101, p. 6]. Particularly with Fréchet, Bouligand maintained close contact and regular written correspondence throughout his career and which we will draw upon at various points in the present account. During his education at the Ecole Normale Supérieure, Bouligand was taught by many of the most eminent mathematicians, including Jules Tannery, Emile Borel, Edouard Goursat, Gaston Darboux, Jacques Hadamard, Emile Picard, Henri Lebesgue and Ernest Vessiot [101, p. 6]. It was perhaps Jacques Hadamard who had the greatest influence on Bouligand at the very start of his mathematical career by orienting him towards certain areas of research in potential theory. It is also worth noting that Bouligand participated in the famous seminars named after Jacques Hadamard, in which he was influenced by the ideas of prominent mathematicians.

In the chapter regarding Bouligand’s contribution to the Dirichlet problem, we will reveal an interaction between Bouligand and Lebesgue leading towards a generalisation of this problem, a generalisation to which Norbert Wiener first formulated a complete solution. It was in this context that Bouligand first came into contact with Wiener and

¹It seems likely that the author is Zigmas Rupeika (1898-1973) - a Lithuanian mathematician and teacher of mathematics.

subsequently collaborated on questions relating to the generalised Dirichlet problem. This collaboration is highlighted in the available written correspondence between Wiener and Bouligand, to which we will refer in the chapter below on Bouligand's contribution to the Dirichlet problem. It was also in the context of Bouligand's work on the Dirichlet problem and potential theory that he carried out a research visit to the University of Krakow in 1925, hosted by the Polish mathematician Stanislaw Zaremba. As a result of this research visit, Bouligand was strongly influenced by the Polish school of mathematicians, which represents a factor leading to the development of his direct infinitesimal geometry and the ideals it sets out to achieve.

Elie Cartan is a further figure who will appear in the present account in the context of Bouligand's direct infinitesimal geometry. While the available written correspondence between Cartan and Bouligand is limited, Elie Cartan nevertheless appears as one of the first prominent members of the French mathematical community to acknowledge and support Bouligand's work in this area.

Regarding mathematicians of the next academic generation, Gustave Choquet appears as a notable example of a mathematician who was influenced by Bouligand's ideas on direct infinitesimal geometry. While we do not explore this correspondence in the present account, we note that quite extensive correspondence is available between Bouligand and Marcel Brelot, relating to potential theory².

Bouligand's participation in certain key conferences represents a further angle from which we gain an insight into the broader mathematical and scientific community at the time. Major conferences at which Bouligand presented his work include the *Conférence internationale de topologie*, held in Geneva in 1935 and in which he presented ideas relating to his theory of direct infinitesimal geometry, the *Congrès international de philosophie scientifique* and the *Congrès international de philosophie* of 1937 both held in Paris and in which Bouligand presented certain ideas in the epistemology of mathematics.

0.1.3 Bouligand in the context of trends in mathematics and the epistemology of mathematics

As mentioned above, A study of Bouligand's work centered around his ideas on the epistemology of mathematics leads us to placing certain of his key ideas in the context of broader trends in mathematics, thus providing further examples of the manifestation

²Available at the time of writing at the Bibliothèque de Sorbonne Université, Section Mathématiques Informatique Recherche.

and influence of these trends during the interwar period. Bouligand's first articles relating to intuition in mathematics will be placed in the context of the debates on rigour and intuition taking place in France and abroad, for example the debates sparked by the reception of Peano's work on mathematical logic between 1904 and 1907. Perhaps the trend in mathematics which is of most central importance to the present account is that of the emergence of Cantor's set theory in France. As will be explored in the chapters dealing with Bouligand's theory of direct infinitesimal geometry, Bouligand's work is motivated partly by his desire to implement in differential geometry a set-based approach following the example of the theory of functions - the context in which set theory was first widely adopted in France.

0.2 The scope and focus of the present account

0.2.1 Bouligand's core reflections on the epistemology of mathematics as the connecting thread

The entire mathematical and epistemological works of Georges Bouligand represent a considerable corpus, spanning diverse areas of mathematics and its applications over approximately half a century, a corpus which we could not hope to address in the present thesis. The present account will therefore focus on a specific theme and on a time period, both chosen with the aim of bringing to light a significant body of Bouligand's main mathematical contributions while faithfully representing what could be considered as the key ideas and motivations of his lengthy mathematical career. The theme representing a common thread running throughout our account will be Bouligand's reflections on causality in geometry and in mathematical and physical theories as well as his closely related concept of direct methods. The approach we have chosen of bringing to light certain key areas of his mathematical contributions which interact closely with his epistemological views is in agreement with Bouligand's own motivations. The following comment of Bouligand illustrates this point:

Contrairement à une tendance fréquente, j'ai évité de m'abandonner à une spécialisation trop marquée. J'ai préféré chercher quelque unité de conception dans les régions assez vastes de la science mathématique, unité susceptible de favoriser mes propres travaux et de permettre à d'autres chercheurs de suivre mon sillage [95, p. 2].

[Contrary to a frequent trend, I have avoided devoting myself to an excessive specialisation. I have preferred to search for something of a unity of conception in quite broad regions of mathematical science - a unity which

is likely to favour my own works and to allow other researchers to follow in my footsteps].

We will aim to focus our account on the mathematical works of Bouligand which are most intimately connected with his epistemological views. On the other hand, we have chosen to focus on the concepts relating to causality and direct methods as these are precisely the themes belonging to Bouligand's epistemological reflections which tie in most closely with his mathematical output. This choice is one factor in leading us to focus on a specific time period during Bouligand's career, namely the interwar period, a choice upon which we expand below.

We emphasise that Bouligand's epistemological reflections relating to causality and direct methods do not represent an isolated or marginal area of his academic activity. On the contrary, these ideas are seen to be present both in purely mathematical works as well as in a significant corpus of a more epistemological nature. Regarding the interplay between his mathematics and his epistemology, the most striking example is that of his theory of direct infinitesimal geometry - which represents one of the areas of mathematical research to which he dedicated most efforts and which has as a central goal to apply the concepts of causality and direct methods.

0.2.2 Focussing on the interwar period (1918-1939)

The choice has been made to focus the present account mainly on the mathematical and epistemological works of Bouligand between the two World Wars, that is between 1918 and 1939. In addition to this being a period in the history of mathematics which still warrants further study, we outline a number of other factors motivating this choice. First, although it may not be reasonable to claim a causal relationship, we observe that the majority of Bouligand's mathematical output occurred during the interwar period. Furthermore, it is during this period that his mathematical output most intimately linked with the concepts of causality and direct methods was first introduced, including his work on a generalised version of the Dirichlet problem (starting from 1923) and his theory of direct infinitesimal geometry and its applications, an area of research which began in approximately 1927 and which was first expressed as a comprehensive theory in 1932. We note that Bouligand did not have a significant academic output, in terms of publications during the second World War. After this period, a much larger proportion of Bouligand's output is dedicated to the epistemology of mathematics. His articles in this area after 1945 relate less specifically to causality and direct methods and tend to deal with topics of a broader nature. To illustrate this, we can consider for example his article on *La nature des choses en mathématiques* [The nature of things in mathematics] [98], or his 1955 publication entitled *Perspectives mathématiques* [Mathematical perspectives] [100]. As we might speculate from the titles of these

works, the dynamic interaction here with his mathematical works is not so clear as it is during the interwar period in relation to his reflections on causality and direct methods.

The establishment of the Polish school of mathematics represents a factor directly linked to the ending of the First World War which reinforces our choice to focus on the interwar period. We see in the chapter dealing with direct infinitesimal geometry that Bouligand was greatly impressed and also influenced by the Polish school of mathematicians which emerged after the First World War, a school which focussed on topics in set theory, topology and the foundations of mathematics. Indeed, the founding of this school took place in the context of - and was connected to - the end of the war and the independence of Poland. After 123 years, Poland regained independence in 1918 and, as Murawski comments [187, p. 91], ‘The state and its institutions, in particular the entire scientific and educational system had to be restored’. Consequently, universities were re-established in Krakow, Lvov and Vilnius and other universities were opened or newly established for example in Warsaw and Lublin. Murawski comments that ‘The interwar period (1918-1939) was a time of intensive development for scientific research and education in Poland’ [187, p. 91]. It was in this context that the founding of the school of mathematicians specialising in set theory, topology and the foundations of mathematics took place, including the founding of a new mathematical journal dedicated to these topics, named *Fundamenta Mathematicae*, by Zygmunt Janiszewski, Waclaw Sierpiński and Stefan Mazurkiewicz. Janiszewski played a key role in establishing the new school of Polish mathematics in Warsaw after independence [187, p. 113] and in a 1917 article ‘On the needs of mathematics in Poland’ argued in favour of placing specific areas of expertise at the centre of Polish mathematics, such as set theory and related areas including topology and the theory of real functions [187, p. 113]. Murawski comments that this article ‘became a program for a whole generation of Polish mathematicians’ [187, p. 113]. In summary, the historical circumstances related to the end of World War I and Polish independence had an impact on Bouligand, his mathematics and his epistemological views, thanks notably to his 1925 visit to the University of Krakow.

0.2.3 Outlining the structure of the following account

The present account consists of five main chapters. In the first chapter, we introduce Bouligand’s work on and towards what he referred to as the generalised Dirichlet problem. As we will see in greater depth in this chapter, Bouligand was led to this topic through the research in the area of potential theory which he started at the beginning of his mathematical career, initially oriented in this field by one of his mentors, Jacques Hadamard. It is in the context of Bouligand’s mathematical work on the generalised

Dirichlet problem that certain key elements of his epistemological views first occur explicitly. Specifically, an approach involving organising or classifying mathematical results and concepts according to their level of generality features explicitly in this context and is clearly akin to the concept of a domain of causality, which he introduced and explored in depth in later epistemological, mathematical and pedagogical works. It was also in this context that he first considered the local dimensional properties of sets in Euclidean space - representing a later component of his direct infinitesimal geometry. A local and set-focussed approach (we will expand upon what this means later on) to the study of the geometry of sets represents a central pillar of Bouligand's direct infinitesimal geometry.

Our presentation of Bouligand's direct infinitesimal geometry will include a discussion on the factors influencing the emergence of this work, notably the broader trend of the emergence of set theory in France together with influence of the Polish school of mathematics and certain other individual mathematicians. In addition, we will discuss Bouligand's efforts to disseminate his ideas as well as presenting key examples of the uptake of his work.

In the subsequent chapter, we will introduce the first topic in the epistemology of mathematics upon which Bouligand published his reflections, namely the concept of intuition in mathematics. The main significance of Bouligand's ideas on intuition are of great interest in the present account as they help to illuminate certain views and preferences of Bouligand regarding the way in which mathematics should be practiced, formulated and taught. Exploring his views on intuition will help to understand his later concepts of causality and direct methods. As will be explored, for Bouligand, intuition was understood predominantly in terms of making connections with objects or concepts in the physical environment or with geometrical concepts. To place these ideas in context, we will explore debates occurring around the start of the twentieth century regarding rigour and logic versus intuition, involving eminent French mathematicians such as Henri Poincaré and of the following generation, Hadamard, Borel, Lebesgue and Baire. Also of relevance here was a national educational reform in France adopted in 1902.

While the generalised Dirichlet problem represents the context in which certain of his epistemological ideas about mathematics first occur, it was his later work on direct infinitesimal geometry which represents the most striking example of the implementation of such ideas to his mathematical work. Our account on this topic will be split into two chapters - the first focussing on Bouligand's motivations, influences and the mathematical formulation, while the second will discuss applications, dissemination

and uptake. Direct infinitesimal geometry was a term introduced by Bouligand for his approach to certain concepts and problems in differential geometry in which he sought to avoid the need for assumptions regarding, for example, the differentiability of functions representing curves and surfaces. Informally speaking Bouligand's approach, as we will see, was an approach focused more towards sets than functions. He relies on certain concepts of topology and favours a local study of the geometrical properties of sets in Euclidean space. He seeks to avoid parametric representation as far as possible - we therefore see why his theory was referred to as 'direct'. Bouligand's work on potential theory including in particular his work on the Dirichlet problem from a more mathematical point of view also represents a key factor leading him to formulate his direct infinitesimal geometry. In the context of the generalised Dirichlet problem, Bouligand was led to characterise certain types of sets of points on the boundary in question from a local geometrical point of view. Specifically, this involved investigating the dimensional properties of sets in Euclidean space - an approach which was later incorporated into his direct infinitesimal geometry. Before Bouligand gave a comprehensive formulation of his direct infinitesimal geometry, he published his broader reflections regarding the need for a new approach in infinitesimal geometry and outlined what he saw as the main tools and ingredients of such an approach. Our later discussion of direct infinitesimal geometry will therefore also place the theory in the context of this programme of research founded upon Bouligand's epistemological views on geometry. Notably, these principles include reducing the number of assumptions made to a minimum, in particular avoiding assuming differentiability and continuity properties of functions representing the curves and surfaces being studied. In addition, Bouligand seeks to *restore causality* to differential geometry by emphasising the study of concepts in the context of groups of transformations under which they are invariant - a group which he refers to as a domain of causality.

The final chapter will explore the development of Bouligand's epistemological ideas after the introduction of his theory of direct infinitesimal geometry. We will see that the concepts of causality and direct methods initially introduced in the context of geometry were later considered and developed in the broader setting of mathematical and physical theories. The key ideas explored will include direct methods and their unity, from Bouligand's point of view, the idea of a causal proof and finally the idea of a domain of causality and related concepts regarding what Bouligand referred to as the stability of mathematical propositions - here no longer exclusively in the context of geometry but in mathematical and physical theories more generally. We will investigate the way in which these ideas were circulated by Bouligand both in a scientific and a pedagogical context and we will briefly situate Bouligand's work in relation to a few major landmarks in the epistemology of mathematics during the

interwar period, namely in relation to the work of Emile Borel, Albert Lautman and Jean Cavailles.

0.2.4 A running theme - Bouligand's focus on pedagogy and mathematics education

While Bouligand's core concepts of causality and direct methods represent the common thread of this thesis, we highlight a further recurring theme which adds to the interest in studying Bouligand in greater depth - namely the strong connection between his mathematical and epistemological output and the teaching of mathematics as well as a pedagogical approach to presenting mathematical research. Indeed, each chapter contains at least one short section dealing with the pedagogical aspect of the topic at hand. One of the clearest examples of the interaction between Bouligand's research and teaching will arise in the context of his work on direct infinitesimal geometry - a research programme in which he initiated doctoral students to whom he also transmitted the epistemological considerations motivating his theory. In the chapter dealing with Bouligand's work on the Dirichlet problem, we note the importance of a lecture course he gave in Krakow in 1925, in which key new concepts in his research were presented. In addition, we come across certain key publications relating to the Dirichlet problem and potential theory in which pedagogical considerations are central. Regarding his ideas in the epistemology of mathematics, again pedagogical considerations are prominent: for example, his concepts of causality and direct methods are included in his mathematical textbooks and he comments explicitly on his opinion of the relevance of these ideas for mathematics education.

0.2.5 Areas of Bouligand's mathematical output not included in this account

In Bouligand's 1961 *Titres et travaux scientifiques de M. Georges Bouligand*, he presents and categorises the academic output of his lengthy career, highlighting three broad domains, namely *analyse et physique mathématique* [analysis and mathematical physics], *géométrie différentielle* [differential geometry] and *principes d'enseignement, méthodes et heuristique* [principles of teaching, methods and heuristics]. Regarding the first area of analysis and mathematical physics, if we consider Bouligand's articles relating to the generalised Dirichlet problem together with work which later fed into this topic and articles which were inspired by work on the generalised Dirichlet problem, then we have one of the most substantial - if not the most substantial - subset of his published work within the broader category mentioned during the interwar period. Regarding the second theme of differential geometry, this area is clearly dominated during the

interwar period, in terms of the volume of published articles, by Bouligand's work on direct infinitesimal geometry, including applications of these concepts and work which was later integrated into Bouligand's theory.

Which areas belonging to the two broader domains of mathematics and mathematical physics are not dealt with in this account? To answer this question, it is useful to refer to Bouligand's earlier 1937 *Notice sur les travaux scientifiques de M. Georges Bouligand*, in which he details and classifies his mathematical output between 1912 and 1937, therefore covering most of the interwar period. The chapters corresponding to the main areas of his research here are, in the order presented by Bouligand:

1. *Fonctions harmoniques, problème généralisé de Dirichlet, équations du type elliptique* [harmonic functions, generalised Dirichlet problem, elliptic equations]
2. *Application de la notion de flux et problèmes divers* [application of the concept of flux and divers problems]
3. *Recherches de géométrie des ensembles suggérées par la théorie du potentiel* [research into the geometry of sets suggested by potential theory]
4. *Systématisation: la création de méthodes directes en géométrie infinitésimale* [systematisation: the creation of direct methods in infinitesimal geometry]
5. *Recherches sur la méthodologie directe* [research into the direct methodology]
6. *Applications de la géométrie infinitésimale directe aux équations différentielles, ordinaire ou partielles* [applications of infinitesimal geometry to ordinary and partial differential equations]
7. *Recherches d'hydrodynamique et sujets connexes* [research into hydrodynamics and related topics].

First we note that points 3, 4 and 6 all related to Bouligand's direct infinitesimal geometry, representing the most substantial chapter of the present account. In addition, point 6 seems to represent applications of direct infinitesimal geometry to analysis. We cover elements of Bouligand's work in point 3, which is a precursor to his formalised theory, such as the concept studying the dimensional properties of sets in Euclidean space. However, our emphasis is in presenting the main ideas of Bouligand's structured theory in point 4, focussing on how his theory puts into practice key ideas in his epistemology of mathematics. While we do touch on applications of Bouligand's theory as well as the connection of these significance of these applications in relation to his epistemological work, we do not enter into any great depth in point 6.

We can view points 1, 2 and 7 above as constituting, more or less, Bouligand's output in the broader theme of analysis and mathematical physics (mentioned above) during the interwar period. We have chosen to focus, for all of the reasons outlined already, on the key topic of the generalised Dirichlet problem, mentioned in point 1, which includes considering certain concepts and articles in the areas of harmonic functions. We do not enter into the topic of elliptic equations. Perhaps the most substantial area (in terms of volume of published articles) not covered in any detail within the broader theme of analysis and mathematical physics is that of fluid mechanics, including points 2 and 7 above. That said, we will encounter briefly in the present account, interactions and influences between these topics and the themes selected for more detailed investigation.

Finally, regarding point 5 above, this can be seen as a natural progression in Bouligand's ideas in the epistemology of mathematics relating to direct methods as seen in his work on direct infinitesimal geometry. We will consider briefly his ideas here in the context of the last chapter on the evolution of Bouligand's epistemological work.

Additional topics of research are present in Bouligand's published work which are perhaps less prominent in terms of volume of published articles and which we do not deal with in this account. For example, Bouligand's applied work during the interwar period is more varied than the topics mentioned in points 1 to 7 above. For example, on the topic of percussion, see [27]. In terms of textbooks on applied topics, we refer the reader to, for example, Bouligand's *Précis de mécanique rationnelle* [31]. In the broader theme of differential geometry, highlighted in [101], Bouligand lists his published articles under the heading *Géométrie différentielle et thèmes divers* [differential geometry and divers themes] and includes a small number of papers dealing with vector geometry (see for example [21], [22]). To this theme, we associate Bouligand's textbook *Leçons de géométrie vectorielle, préliminaires à l'étude de la théorie d'Einstein* [Lessons in vector geometry preliminary to the study of Einstein's theory] [26]. Another example of a more minor theme identified by Bouligand [95, p. 27] corresponding to a small set of articles is that of the theory of functions of a complex variable (see for example [84]).

0.2.6 The main objectives of the present account

The main objective of the present account is to bring to light key areas of Bouligand's work during the interwar period with an emphasis on the interaction between his epistemological ideals and his mathematical work. In addition we wish to bring Bouligand himself to life, as it were, in the context of interwar French mathematics. This will be

achieved through the biographical elements of this account, by presenting his efforts to communicate his ideas in the mathematical and broader scientific community, as well as - in the case of his theory of direct infinitesimal geometry - presenting key examples of the initial reception of his work. In certain cases we situate his ideas in relation to major known landmarks in the mathematics or its philosophy, however we highlight that we have opted for an approach clearly centered around Bouligand. Given the lack of literature currently available on Bouligand, an account of this nature aims to facilitate the work of future researchers who wish go to further in situating and interpreting Bouligand in the context of key trends or figures in mathematics and its philosophy.

Chapter 1

Georges Bouligand, a brief biography

Before launching into our account as outlined above, we first paint a picture of Georges Bouligand through a short biography, giving an insight into his life, education and career. I believe that this will help to bring Bouligand to life for the reader and will enrich and contextualise the subsequent account of his mathematical and epistemological contributions.

1.1 Early years and family

Georges Louis Bouligand¹ was born on 13 October 1889 in the town of Lorient, in the Morbihan, at 23 rue de la Corderie. Georges was an only child and his father was Louis Marie François Bouligand, born on 7 May 1860 in Cordemais, where he worked as a veal merchant. By 1889, he was a *garde-stagiaire d'artillerie* in the navy and subsequently *garde*. He died on 16 August 1896 in Madagascar, after having contracted an endemic disease, while Georges was only six years old. Georges was therefore brought up by his mother, Adolphine Marie Armelle Bonard, who was born on 10 March 1863 in Lorient, where her father was a caulker at the city's port. She lived until the age of 78 - she died on the 3 May 1941, in her son's home in Paris. According to Poncin [202], Georges also benefited from the guidance of his uncle during his early years.

In November 1920, Georges married Jeanne Augustine Marie Glain (17 December 1890 - 7 August 1981), who was a childhood friend who he had met through his uncle. Her father was Pierre Génulphe Glain and her mother Céline Marie Josseau, a teacher. Little information is available regarding Jeanne Glain, besides Poncin's comments [202] where he described the *simplicité charmante* with which Madame

¹We acknowledge here the impressive *Dictionnaire des professeurs de mathématiques spéciales* of Roland Brasseur [106] which contains a considerable amount of information about Georges Bouligand. This rich resource led me on to the original sources often cited below

Bouligand received him when he was invited to their home to work on mathematics problems. Georges and Jeanne had nine children, all of whom were born in Poitiers:

- Anne Marie (21 March 1922 - 26 April 2011), a journalist for the *Dauphine Libéré*
- Jeannette Radegonde, born on 22 May 1924, who died two days later on 24 May.
- Pierre Adolphe Marie Louis (20 May 1925 - 23 January 2007), a jesuit.
- Hélène Marie Thérèse (6 January 1927) - a lawyer and later a magistrate
- Paulette Marie Louise Radegonde (12 January 1928 - 13 April 2007) - a social worker
- Marie Thérèse Augustine (22 août 1929 - 10/09/2016); *Professeur de Lettres*
- Georges Marie Marcel (9 October 1930) - PhD in the physical sciences, engineer at the *Centre d'Énergie Atomique*
- Marcelle Josèphe Marie (31 October 1932). A benedictine nun (mère Maria Assumpta)
- Yves Jean Marie (8 May 1935 - 21 January 2011) - PhD in natural sciences, Director of the laboratoire d'histo- et cytophysique at the Ecole Pratique des Hautes Etudes from 1972 to 2001

The following short passage [202] was prepared by Bouligand's children and gives an insight into his personality and role as a family man:

Son affection à l'égard de son épouse et de sa famille était évidente, mais toujours réservée et discrète...il était sévère à notre égard, et en particulier, exigeait le silence le plus absolu pendant son travail ; mais, en dehors de ce travail, il devenait un bon vivant, créant autour de lui une atmosphère très détendue. Il chantait souvent en s'accompagnant de quelques notes sur le piano et récitait des vers. Il possédait aussi un réel talent de chansonnier et les divers événements de sa famille donnait lieu à des couplets nouveaux. Il aimait la marche à pied dans la campagne, connaissait très bien les champignons et, parfois, pratiquait la pêche à la ligne. Il possédait une quantité de petits carnets et y notait ses réflexions à tout moment. Pour lui-même il se contentait de peu. A Paris, son bureau était la plus petite pièce de l'appartement et, dans sa propriété de Quiberon, il se réfugiait dans une petite cabane de 2 m x 2 m construite au fond du jardin. Il confiait toutes les questions d'argent à son épouse, se contentant de très peu d'argent de poche (il arrivait parfois qu'il

n'en avait plus assez pour acheter un journal!)...

[His affection for his wife and family was obvious, but always reserved and discreet...he was strict with us, and in particular, demanded absolute silence while he was at work. But when he wasn't at work, he became a bon vivant, creating a very relaxed atmosphere around him. He often sang, accompanying himself with a few notes on the piano, and recited verses. He also had a real talent for song, and his family's various events gave rise to new verses. He enjoyed walking in the countryside, was very knowledgeable about mushrooms and sometimes went angling. He had a number of little notebooks in which he jotted down his thoughts at any given moment. For himself, he was content with very little. In Paris, his office was the smallest room in his apartment, and on his property in Quiberon, he took refuge in a little hut measuring 2 m x 2 m built at the bottom of the garden. He entrusted all money matters to his wife, making do with very little pocket money (it sometimes happened that he didn't have enough to buy a newspaper!)].

1.2 Lycée and higher education

Georges studied at the Lycée de Lorient until 1906 and he took his Baccalauréat in 1905 and 1906 in Rennes. He later took his *classes spéciales* at the Lycée de Nantes from 1906 to 1909, where one of his teachers was the mathematician Maurice Fréchet (during the academic year 1907/1908) - an influential figure for Bouligand who would also become a long-term friend and correspondent. Georges showed promising academic results coming in first place in the entrance examination for the Ecole Polytechnique but preferred to attend the Ecole Normale Supérieure. Poncin [202] quotes Bouligand sharing his thoughts on this decision:

Moins brillamment admis à l'Ecole...j'optai cependant pour elle car je savais que je pourrais y travailler plus à ma guise. C'est ainsi que, en seconde année, ignorant tout encore de l'hydrodynamique, j'accueillis avec joie une proposition de J. Hadamard: faire un diplôme sur les ondes liquides et approfondir les résultats qu'il venait de donner à ce sujet...

[Although I was less brilliantly admitted to the Ecole... I nevertheless opted for it because I knew that I would be able to work there more in my own style. So, in my second year, still knowing nothing about hydrodynamics, I was delighted to receive a proposal from J. Hadamard: to do a degree

on fluid waves and expand on the results he had recently given on this subject].

Bouligand's contemporary at the *Ecole Normale Supérieure*, Maurice Janet gives us an interesting glimpse into Bouligand's character [202]:

En dehors de son travail, Bouligand manifestait une certaine fantaisie, tant pour les mathématiques que pour le chant. Je me souviens qu'il essayait, à partir d'un dessin simple, de trouver une équation pouvant le représenter....A sa sortie de l'Ecole...il avait organisé, avec quelques camarades, leurs sœurs et leurs cousines, une chorale qui n'hésitait pas à s'attaquer à des chœurs assez difficiles.

[Outside his work, Bouligand showed a certain sense of fantasy, both for mathematics and for singing. I remember him trying to find, based on a simple drawing, an equation to try to represent it.... When he left school... he organised a choir with a few friends, their sisters and their cousins, who didn't hesitate to tackle some fairly difficult choruses].

Georges passed his *agrégation* in mathematics in 1912 (an examination qualifying him to teach mathematics as a civil servant within certain institutions including the *lycées* and in universities), coming in third place. He became *Docteur ès sciences* on 13 June 1914, having defended his thesis entitled *Sur les fonctions de Green et de Neumann du cylindre*. The jury was composed of Emile Picard (presiding member) Emile Borel and Jacques Hadamard. In the report on Bouligand's thesis, Picard notes that '*Ce mémoire témoigne d'une grande ingéniosité, en même temps que d'une connaissance approfondie des travaux récents sur les équations aux dérivées partielles de la physique mathématique*' [195]. Given their prominence in the mathematical community at the time and their direct influence on Bouligand, it is worth briefly introducing the mathematicians mentioned above². Emile Picard (1894-1937), made contributions to differential equations (including introducing the famous method of successive approximations), the theory of functions, analytic geometry and analysis as well as dealing with applications such as heat and elasticity. He was also influential in the circulation of ideas regarding the theory of relativity and it is believed he was responsible for training more than 10000 engineers in mechanics. A mathematician and politician, Emile Borel (1871-1956) made fundamental contributions to measure and probability theory as well as contributing to many other areas of mathematics,

²We note that these brief introductions to mathematicians, as well as those featuring later in this thesis, draw frequently (though not exclusively) from the University of St Andrews MacTutor History of Mathematics Archive

from the theory of functions to mathematical physics. He also published ideas on topics of a more pedagogical and philosophical nature, such as the role of intuition and rigour in mathematics. He founded the Institut Henri Poincaré in 1928. Jacques Hadamard - a prolific contributor to mathematics who worked in diverse fields including number theory - in which he proved the famous prime number theorem, partial differential equations (including the Dirichlet problem), mechanics, functional analysis, differential geometry. Hadamard also famously published his ideas on the psychology of invention in mathematics.

1.3 Bouligand's early career as a mathematics teacher (1914-1920)

Bouligand was exempt from military service in 1910 and was again exempt at the start of the First World War in 1914. His exemption was maintained in 1915 and again in 1917, due to *faiblesse générale* [general weakness]. Throughout the war, Bouligand worked as a mathematics teacher, first at the Lycée Descartes, in Tours and then at the Lycée de garçons de Rennes (today known as the Lycée Emile-Zola de Rennes). In Tours, as of July 1914, he taught a class of *mathématiques spéciales* and a *première* class (the second of the three years of *lycée*). From 1915, at the same *lycée* he taught a preparatory class for students aiming to be admitted to the Saint-Cyr military school. From August 1916 and for the rest of the war, Bouligand taught the *spéciales* mathematics class at the Lycée de garçons de Rennes. He seems to have been well appreciated by his colleagues and students alike for his professional and personal qualities. For example:

I.G. Niewenglowski, March 1917 : "M. Bouligand est un professeur d'avenir"

IA, April 1917 "*Très calme et maître de soi, M. Bouligand est en même temps un homme du monde que la société rennaise a eu vite fait d'apprécier.*"

Provisieur, March 1918: "*respecté pour son savoir, aimé pour son dévouement et son attachement à ses élèves*"

IA March 1918: "*Homme poli, délicat, attentionné, sans la moindre morgue*"

Provisieur, mars 1919 "*Jeune maître très posé ; très estimé pour son savoir et pour son dévouement et pour son caractère... Brillant professeur de spéciales*"

1.4 Bouligand at the *Faculté des Sciences de Poitiers* (1920-1938)

After a brief period, between March 1920 and April 1921, working as a *Maître de Conférences* at the *Faculté des Sciences de Rennes*, Bouligand started a position as *Professeur de Mécanique Rationnelle et Appliquée* replacing René Garnier. He was also responsible for teaching *mathématiques générales* for two lessons per week for one year, starting from November 1921. Nearly ten years after his arrival at the *Faculté des Sciences de Poitiers*, Bouligand became *Professeur de calcul différentiel et intégral*, as of 16 November 1931, a position which he held until he left Poitiers in 1938.

1.4.1 Bouligand as a teacher

Leloup presents a comprehensive study of all doctoral theses in mathematics defended between 1914 and 1945 in France [170] and includes a significant section on the influence of Georges Bouligand at the *Faculté des Sciences de Poitiers*. Leloup notes that Bouligand's appears to be a particularly active figure in terms of guiding students in their doctoral studies. She highlights the significant role Bouligand played in orienting his student's research - specifically by guiding them in pursuing questions arising from his own research and also points out the personal nature of the acknowledgements included in his students' manuscripts. With Leloup, we cite the acknowledgement included in the 1934 thesis of Jean Capoulade [110] :

Je tiens à remercier M. Le Professeur Bouligand des utiles directions qu'il m'a données et qui m'ont permis d'aboutir dans ce travail ; mais je lui suis surtout reconnaissant de l'affectueuse sympathie avec laquelle il a encouragé mes efforts, de son rôle d'animateur toujours bienveillant. Je suis particulièrement fier d'être son Élève et d'avoir eu le plaisir et l'honneur de développer une question dont il avait le premier aperçu toute l'importance.

[I would like to thank Professor Bouligand for the useful orientation and advice he gave me, which enabled me to succeed in this work. But I am especially grateful to him for the affectionate sympathy with which he encouraged my efforts, and for his ever benevolent role as a guide. I am particularly proud to be his pupil and to have had the pleasure and honour of elaborating on a topic whose importance he first recognised].

Bouligand himself expresses the convergence of his teaching and his research in Poitiers in his *NOTICE sur les recherches faites, inspirées ou dirigées de 1919 à 1931*, submitted

to the *Comité de la Section des Sciences Mathématiques de la Caisse nationale des Sciences* on 13 May 1931 and available at the Archives Nationales [61]. The following quotation illustrates this and the nature of the relationship between Bouligand and his students:

J'ai dû, en raison des nécessités locales, consacrer beaucoup de temps à l'enseignement. Mais j'ai toujours fait en sorte de rapprocher le plus étroitement possible mes occupations didactiques et scientifiques. Peut-être cette manière de voir m'a t-elle justement assuré les nombreux disciples dont je parlerai tout à l'heure.

[Because of local needs, I had to devote a lot of time to teaching. But I have always tried to bring together as closely as possible my teaching and scientific activities as. Perhaps this approach has given me the many disciples I will talk about in a moment].

The convergence of Bouligand's teaching and research is also reflected by his significant output while in Poitiers in terms of mathematics textbooks covering themes in his own research. Bouligand highlights six different texts in the period 1919-1931 [61], listed hereafter, commenting that, '*En rédigeant ces livres, une fois pour toutes, j'ai gagné du temps pour moi-même et j'ai orienté mes élèves dans la voie de l'effort personnel*'. These are: *Cours de géométrie analytique* (1919); *Leçons de géométrie vectorielle, préliminaires à l'étude de la théorie d'Einstein* (1924); *Précis de mécanique rationnelle à l'usage des élèves des Facultés des sciences* (1925); *Initiation aux méthodes vectorielles et aux applications géométriques de l'analyse* (by G. Bouligand and G. Rabaté) (1926); *Notions sur la géométrie réglée et le complexe quadratique* (1929); *Compléments et exercices sur la mécanique des solides : cinématique: frottement, systèmes non holonomes ; choc et percussions avec frottement, liaisons unilatérales* (1929).

Bouligand was proactive in offering extra learning opportunities for his students, for example, Poncin reports [202] that Bouligand voluntarily organised preparation sessions for students who aimed to take the *agrégation*, for which no formal preparation was offered at the *Faculté* at that time. In addition, he organised working sessions followed by discussion about certain questions relating to geometry.

1.4.2 In institutional life

There were seven professors at the *Faculté des Sciences de Poitiers* [202] (Poncin uses the term *professeur*) in the early 1920s, among which two mathematicians who had no assistant or secretary to support them with administrative tasks. Bouligand evidently

was instrumental in gaining additional resources and recognition for the mathematics department:

Il n'était pas question d'obtenir quoi que ce fût de l'administration centrale; aussi essaya-t-il de convaincre les assemblées municipales et départementales de l'intérêt que présentait le développement de la Faculté de Poitiers. Il y parvint et obtint la création d'un emploi d'assistant et de deux maîtres de conférences ainsi que des crédits complémentaires permettant de rétribuer un certain nombre de chargés de cours. Enfin il obtint que Poitiers figurât sur la liste des centres d'enseignement de mécanique des fluides créés par le ministère de l'Air aux environs de 1930.

[There was no question of obtaining anything from the central administration, so he tried to convince the municipal and departmental assemblies of the importance of developing the Faculté de Poitiers. He succeeded and obtained the creation of an assistant position and two *maîtres de conférences*, as well as additional funds to pay a certain number of *chargés de cours*. Finally, he got Poitiers included on the list of fluid mechanics teaching centres created by the Ministry of Air around 1930].

During the 18 years Bouligand spent at the *Faculté des Sciences de Poitiers*, he was presented with opportunities to move his mathematical career elsewhere. For example, in two letters of 1927, Bouligand declines for the second time an invitation made by Maurice Fréchet to join the *Université de Strasbourg*. Bouligand had to decline the offer due to family reasons, more specifically relating to the health of his mother who lived with him at that time as well as for financial reasons, his mother's pension amounting to only 5000 francs and the cost of living in Strasbourg being higher than in Poitiers. According to Markovitch, an official member of the International Education Board and as reported by Reinhard Siegmund-Schulze [206], Bouligand was considered as a possible replacement for Goursat, after the latter's retirement from the *Faculté de Paris* in 1931 but it was decided that Bouligand was to remain in Poitiers as a founding figure of a pole of high quality mathematical research [170]. We see in a number of letters addressed from Bouligand to René Garnier, that Bouligand aspired to move to the *Sorbonne* well before his appointment at the *Faculté des Sciences de Paris* as of 1 October 1938.

Bouligand seems to have been a central figure in scientific life at the *Faculté des Sciences de Poitiers*. For example, in addition to the working sessions mentioned above, he organised research seminars which attracted numerous academics from France and from abroad, including G.D. Birkhoff, O.D. Kellogg, Norbert Wiener, D. Pompeiu and F.

Vasilesco [61].

Bouligand became a member of a number of academic institutions in France and abroad. We note specifically that he was made *Correspondant* of the *Académie Polonaise des Sciences et Lettres* in 1928 and in 1933 he was awarded the same title within the *Société Royale des Sciences de Liège* and the *Académie des Sciences de Paris* in 1937.

1.5 At the Sorbonne (1938-1961)

Bouligand was appointed *Maître de Conférences* at the *Faculté des Sciences de Paris* in 1938, *honoris causa* due to his long service and status as professor at Poitiers. He was later appointed as *professeur sans chaire* in 1939. He was appointed *professeur titulaire d'application de l'analyse à la géométrie* in 1947, a position he held until his retirement in 1961, almost 50 years after his *agrégation*. Poncin comments on the significance of this position in the context of Bouligand's lengthy career [202]:

...ce fut certainement une très grande satisfaction pour lui, car c'était l'aboutissement de ses efforts qu'il avait entrepris dès son entrée dans l'enseignement supérieur.

[...it was certainly a source of great satisfaction for him as it was the fruit of the efforts which he had made since the time he first entered into higher education].

His teaching responsibilities during this time included for example classes in *mathématiques complémentaires* and classes and conferences in *mathématiques supérieures* and *géométrie supérieure*. The author of this work had the good fortune of interviewing Rudolf Bkouche (1934-2016) - mathematician, historian of mathematics and former student of Bouligand during the 1950s, who commented,

J'ai suivi ses cours en 54. Donc il avait un cours de géométrie différentielle, dans lequel il racontait ce qu'il raconte dans les leçons de géométrie vectorielle. Et puis... il avait un cours où il parlait de tout...Mais en fait, un des grands thèmes, c'était les fonctions implicites, les théorèmes des fonctions implicites...il racontait sa vision des mathématiques.

[I followed his classes in '54. So he had classes in differential geometry in which he talks about what he discussed in the *leçons de géométrie vectorielle*. And then...he had a class where he spoke about everything. But actually, one of the main themes was that of implicit functions, the theorems on implicit functions...he spoke about his vision of mathematics].

Dr Bkouche highlighted the rich nature of the content of Bouligand's teaching, while noting that it could however become somewhat difficult to follow.

Working at the hub of intellectual life in France, Bouligand's time at the *Sorbonne* provided him with opportunities to interact with high-level academics from different disciplines and to pursue his epistemological ideas about mathematics from different angles. A prominent example of this is to be found in the work of the physicists and philosophers Jean-Louis Destouches (1909-1980) and Paulette Février-Destouches (1914-2013), at the time students of the eminent physicist Louis de Broglie (1892-1987). For example, Jean-Louis Destouches, under the inspiration of Bouligand, generalises a concept of Bouligand referred to as the *stability* of mathematical propositions and examines its application in the context of quantum physics [127].

Bouligand played an active role in academic, scientific and cultural life while in Paris. He became a member of the *Comité national de l'Histoire et de Philosophie des Sciences (Académie des Sciences de Paris)* from 1938 and later, from 1956, a member of the *Commission des Mathématiques* for the *Palais de la Découverte*. He was instrumental in the organisation of the *jubilé scientifique* of Paul Montel, which took place in 1947. In terms of honours and distinctions received during this period, he was made *Officier de la Légion d'Honneur* in January 1954 and received an honorary doctorate from the *Université de Louvain* in February 1956.

We note that Bouligand participated in a number of conferences in the philosophy of science, namely the *Congrès international de philosophie des sciences* in Paris in 1949 and in numerous meetings of the *Société Française de Philosophie* between 1950 and 1968, which covered a highly diverse range of topics, both related and unrelated to mathematics.

1.6 A brief overview of the development of Bouligand's academic output (1919-1939)

We base the aspects of the present summary relating to Bouligand's mathematical output partly on his own summary note [61] covering the period 1919 to 1931. While we are identifying predominant trends, we emphasise that these trends may not necessarily cover all topics dealt with during the periods outlined. From 1919 to 1926, Bouligand's mathematical activity focused almost exclusively on the interrelated areas of harmonic functions, the Dirichlet problem and potential theory. These efforts culminated in significant contributions to formulating and characterising the solution to a

generalised version of the Dirichlet problem, which was first introduced by Norbert Wiener. This topic will be discussed in depth below, we note here that Bouligand used notably an extended notion of Green's function, provided in three articles of 1919 in the *Comptes Rendus* ([18], [19] and [20]) as well as the concept he referred to as *prolongement fonctionnel* (extending the domain of a functional as far as possible while preserving certain continuity conditions) discussed for example in two 1923 articles in the *Bulletin des Sciences Mathématiques* ([24], [25]) and a study of the dimensional properties of certain sets of points on the boundary (see for example [33]). As will be discussed later, Bouligand's results regarding a more general approach to the Dirichlet problem in 1924 coincided with closely related but more complete results of Norbert Wiener. Initially unaware of each other's work, a rapprochement between French and American efforts occurred, as well as a personal friendship between Bouligand and Wiener.

Between 1926 and 1928, Bouligand focused predominantly on problems in fluid mechanics which had originally motivated his doctoral thesis. We note that Bouligand's work in hydrodynamics interrelates in a number of publications with his research on the Dirichlet problem. Between 1928 and 1931, Bouligand focuses his mathematical output around a new approach to differential geometry which he eventually called *la géométrie infinitésimale directe* [direct infinitesimal geometry] - a theory aiming to rid infinitesimal geometry of hypotheses introduced for convenience for certain methods to be employed. This theory aims to put into practice the main ideals laid out by Bouligand in his epistemological reflections and relies on the mathematical concepts of *contingent* and *paratingent*, which will be explored below. Bouligand comments on the motivation behind this new area of research as well as its connection to his previous work on the Dirichlet problem and his epistemological views about mathematics [61]:

Mes recherches sur le problème de Dirichlet m'ont convaincu de la nécessité d'une refonte de la géométrie infinitésimale, ayant pour effet de la débarrasser des hypothèses de commodité, et par là même d'y remettre la causalité en pleine lumière.

[My research into the Dirichlet problem convinced me of the necessity to reformulate infinitesimal geometry in a way that rids it of hypotheses introduced for convenience thereby bringing causality fully to light].

Bouligand initiated a number of his students into his programme of research in *géométrie infinitésimale directe*, as will be explored in depth below. He also began, in this period, to apply his new theory to problems in applied mathematics and mathe-

mathematical physics (see for example [50]).

The most predominant themes of research between 1932 and 1937 were those of *géométrie infinitésimale directe* and mathematical and epistemological considerations relating to direct methods more generally. The mathematical works included creating new methods in *géométrie infinitésimale directe* and applying them for example in the context of differential equations and problems in mathematical physics. Notably, in 1932, Bouligand published his textbook *Introduction à la Géométrie Infinitésimale Directe* [75], presenting in a systematic and didactic way the methods developed thus far. We also see, during this period, a greater focus on direct methods themselves. For example, in an essay of 88 pages entitled *Sur l'unité des méthodes directes* [79] Bouligand examines what he interprets to be direct methods in other areas of mathematics, notably in the calculus of variations, and argues that these methods are unified by certain common characteristics. Bouligand maintained a similar focus for the remainder of the period in question (up until 1939), perhaps with a broadening of topics dealt with in the epistemology of mathematics.

1.6.1 Focussing in on the development of Bouligand's reflections on the epistemology of mathematics

As of 1927, Bouligand began publishing his reflections on topics in the epistemology of mathematics, focusing on the concepts including intuition, methods and teaching of geometry, and notions he referred to as direct methods and causality. These articles were often published in the *Revue Scientifique* and the *Revue Générale des Sciences*, while a small number of articles with a pedagogical focus were published for example in *Enseignement Mathématique*. Bouligand's reflections were occasionally published abroad, for example in the *Rivista hispano-americana di matematica* (Madrid), *Scientia* (Bologna) and, notably, in the Belgian journals *Bulletin de l'Académie royale de Belgique* and *Bulletin de la Société royale des Sciences de Liège*. As we will see below, his epistemological reflections interrelated with his mathematical works. His reflections were inspired by trends in the development of mathematics during and prior to his time and by his own prior mathematical contributions. In turn, his methodological reflections, notably regarding infinitesimal geometry, led him to outline and pursue, together with a significant number of students, a programme of mathematical research.

We are able to identify predominant trends in Bouligand's work on the epistemology of mathematics during different time periods, although we should emphasise that these trends do not necessarily encompass all of his work. From 1927 to 1930, Bouligand's reflections focused equally on two distinct topics of intuition and methodology in

geometry (particularly differential geometry), the latter of which went hand-in-hand with the development of his mathematical works on *géométrie infinitésimale directe*. From 1931 to 1933, Bouligand focused on illustrating the applicability of the founding concepts of his *géométrie infinitésimale directe* to areas of mathematical physics (for example relating to hydrodynamics and the uncertainty principle) and he reflected on the philosophical significance of these applications from his point of view. After 1933 and for the remainder of the period in question (up until 1939), Bouligand focused on both compiling and going some way towards formalising his concepts of causality and direct methods, which had played a central role in his approach to differential geometry, examining these concepts in a more general setting of mathematical and physical theories, as well as in relation to logic (through the a concept he called the stability of mathematical propositions). As mentioned above, after 1937 we also see a broadening of topics dealt with by Bouligand including for example reflections on the use of a finitist approach without relying on irrational numbers [93] and works in which his ideas are presented in a broader setting. Regarding the latter, we take for example his 1937 volume on the *Structure des théories, problèmes infinis* [96] or his work in which he connects his reflections with his views on the teaching of mathematics [91].

Chapter 2

Bouligand's work in relation to the generalised Dirichlet problem

2.1 Introduction

The Dirichlet problem - and specifically a generalisation of this problem - represents an area of mathematics to which Bouligand made a significant contribution which has been recognized by influential mathematicians in this field such as Norbert Wiener [220] as well as in historical accounts such as Landkof's in [162]. This area of research is central to the present account of Bouligand's mathematical and epistemological work as it is in this context that he first implemented certain tools and concepts which would subsequently have a decisive influence on the development of both his epistemological views about mathematics and mathematical practice and on his later theory of direct infinitesimal geometry. Regarding his work in the epistemology of mathematics, key ingredients of Bouligand's later ideas on causality and direct methods first appear - specifically the concepts of *prolongement fonctionnel* and that of structuring a mathematical theory by investigating its propositions at an appropriate level of generality. In addition, Bouligand indicates [95] that it was his work on the Dirichlet problem which was pivotal in inspiring him to formulate his theory of direct infinitesimal geometry - which perhaps most fully and intentionally embodies his ideals of causality and direct methods. These points will be explored in greater detail in the following chapters. We note here that, in the context of his work on the Dirichlet problem, Bouligand is led to investigating the geometry of sets, for example by examining their dimensional properties, and through a mathematical object called the *contingent* - both of these ideas are key mathematical tools of his later theory of geometry.

In the following section, we will see that Bouligand became involved in potential theory (a broader area of mathematics to which the Dirichlet problem belongs) at the

beginning of his mathematical career - unsurprisingly given the prominence of this area of research at the time. The Dirichlet problem is introduced formally below.

2.1.1 The problem and the context of Bouligand's work

Peter Gustav Lejeune Dirichlet (1805-1859) made profound contributions to various areas of mathematics - in addition to being considered as one of the first mathematicians to formally define the concept of a function, he is considered as the founder of the study of Fourier series and made contributions to analytical number theory, potential theory and mechanics. In his lectures on potential theory in 1856/7, he was interested in the question below [210, p. 233], which is intimately linked with broad classes of applications, for example in electrostatics, studying the electrical flow through a conductor, or in thermodynamics, studying the distribution of heat for a given surface.

Formal definition of the Dirichlet problem. Given a domain D , for example in three-dimensional Euclidean space and a continuous function f defined on the boundary δD of our domain, the Dirichlet problem consists of finding a function V defined on $D \cup \delta D$ which is harmonic in D and which takes on the same values as f on the boundary δD . We recall that a function U is harmonic in D if U is a solution of Laplace's equation in D , i.e., in three dimensions this is given by:

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

In this chapter, we will refer to the above problem as the Dirichlet problem or the classical Dirichlet problem, so as to highlight the distinction with the generalised Dirichlet problem, which will be introduced below. The Dirichlet problem is, as we can see above, a problem which depends on the boundary conditions, given by the function f and on the shape of the boundary itself - its complexity or regularity.

The context of Bouligand's work. The Dirichlet problem and related questions in potential theory represented an active field of research at the start of Bouligand's mathematical career. The solution of this problem in the general case was one of Hilbert's famous 23 unsolved mathematical problems, published in 1900. Indeed, problem 20 is entitled '*Dirichlet problem in the general case*' and is formulated as follows¹ [150, p. 46]:

Does each regular problem in the calculus of variations have a solution, provided that certain hypotheses are satisfied in relation to the given boundary

¹The translation from French to English is my own.

conditions, in relation for example to continuity and the possibility of differentiating the functions in question successively a greater or lesser number of times, and also necessarily provided that the notion of solution be suitably generalised?

With this in mind, together with the fact that potential theory represented an active area of research for Bouligand's own mentors such as Emile Picard and Jacques Hadamard, it is perhaps unsurprising that he became active in this area at the start of his academic career and subsequently took an interest in extending the Dirichlet problem. The starting point of his research in potential theory came even before the publication of his doctoral thesis in 1914. He notes [95, p. 2] that it was Jacques Hadamard who guided him towards a problem in fluid dynamics, which involved harmonic functions and which led to his first published paper in 1912 [16]. Specifically the problem in question related to the small oscillations of a heavy non-viscous liquid in a container with fixed walls - a problem to which Hadamard had previously dedicated his own efforts. In Bouligand's own words,

M. Hadamard m'avait primitivement orienté vers un problème d'hydrodynamique, l'étude des petites oscillations d'un liquide pesant, non visqueux, dans une auge à parois fixe, problème dont il avait réussi, le premier, la mise en équations ; par là, mon attention s'était en même temps trouvée sollicitée vers divers problèmes de théorie du potentiel [95, p. 2].

[Mr Hadamard had originally oriented me towards a problem in hydrodynamics, the study of the small oscillations of a heavy non-viscous liquid in a container with fixed walls - a problem which he had been the first to have put into equations. By this route, my attention was at the same time drawn towards various problems of potential theory].

Subsequently, Bouligand's thesis entitled *Sur les fonctions de Green et de Neumann du cylindre* [17], focuses on concepts in potential theory including the Dirichlet problem - specifically he studies Green's function and Neumann's function for an infinite cylindrical domain. As it is a central component of Bouligand's contribution to the Dirichlet problem, we note that Green's function, developed by George Green in 1828, was the key ingredient in a direct constructive approach to solving certain cases of the Dirichlet problem. This function is defined with reference to a specific point P of the domain in question and has certain key properties, namely it is equal to zero on the boundary and becomes infinite at the point P . Neumann's function (*fonction de Neumann*), a term used by Hadamard for example in [145] is a harmonic function which plays a role analogous to that of Green's function [145, p. 307] for the Neumann problem².

²Neumann problem (*problème de Neumann*) is a term used for example by Bouligand in [17]

Instead of finding a harmonic function V coinciding with a given continuous boundary function f as with the Dirichlet problem, the Neumann problem involves finding a solution V whose normal derivative is equal to f on the boundary.

Having introduced the Dirichlet problem and the context of Bouligand's interest in this area of research, the following sections will explore the theme of generalising the Dirichlet problem. As we will see below, it was discovered towards the end of the first decade of the twentieth century that the Dirichlet problem does, in fact, not always have a solution. As such, it is perhaps natural to wonder if the problem can be extended in some meaningful way so that a unique and appropriate solution does always exist. Bouligand was led to work towards the generalisation of the Dirichlet problem via a concept he referred to as *prolongement fonctionnel*.

2.2 Bouligand on prolongement fonctionnel and the need to generalise the Dirichlet problem

The method introduced in this section, referred to by Bouligand as *prolongement fonctionnel* [functional extension] plays a significant role in the present account of his mathematical and epistemological work. Below, we consider this concept as one of the factors leading Bouligand to seek a generalisation of the Dirichlet problem. In addition to this, *prolongement fonctionnel* will appear later in this chapter as a key method in his approach to the generalised Dirichlet problem. Finally and of central importance to our account, *prolongement fonctionnel* is later considered by Bouligand as a means of successfully studying a problem in its appropriate domain, informally speaking, or its appropriate level of generality - a concept he later formulated under the name of a 'domain of causality'. This theme, and the influence of Bouligand's work on the generalised Dirichlet problem on his epistemological views more generally, will be explored in greater depth in the following chapters. Below, we first present in brief the context behind the generalisation of the Dirichlet problem, namely the discovery that the classical Dirichlet problem is not always solvable.

2.2.1 The context - examples of the impossibility of solving the Dirichlet problem

According to O.D. Kellogg in [157, p.603], it was generally believed until the end of the first decade of the twentieth century that the classical Dirichlet problem was always solvable - mathematicians were thus focused on finding sufficient methods of solving the problem in different settings. We note that Oliver Kellogg (1878-1932) was an American mathematician most active in the field of potential theory, including the Dirichlet problem³. This changed with specific examples of cases for which there is no solution. Notably, as reported by Kellogg [157, p.603], Zaremba showed in an article of 1911 [223], which expanded upon ideas presented at the *Congrès international de mathématiciens* in Rome in 1908, that the Dirichlet problem for a domain whose boundary contains an isolated point has no solution. Henri Lebesgue went on to provide a further example in 1913 [167], of a domain with a yet more *well-behaved* nature, for which the Dirichlet problem cannot be solved. Lebesgue's example involved

³Kellogg was a major contributor to the generalised Dirichlet problem as formulated by Wiener. Landkof, in his 1972 textbook on potential theory [162], identifies Kellogg as one of the main contributors to the characterisation of irregular points in the generalised Dirichlet problem, together with Bouligand and primarily Wiener himself. In this chapter, we will also refer to Kellogg's insights of a more historical nature regarding the development of the generalised Dirichlet problem. These are present notably in his 1926 article [157]. Kellogg also published a textbook on potential theory in 1929 [158] to which we will refer in order to describe certain mathematical concepts used by Bouligand from a perspective which is representative of mathematics at the time.

a domain bounded by a surface [214, p. 89], whose boundary contained an inward projecting spine at the point of which any potential solution of the Dirichlet problem could not fulfil the boundary conditions [157, p.603]. The fact that the Dirichlet problem does not always have a solution may naturally give rise to the question as to whether the problem may be reformulated or extended in such a way that a unique solution does always exist and which would coincide with the solution to the classical problem whenever the latter is solvable. As is outlined below, Bouligand became interested in the generalisation of the classical Dirichlet problem through his prior research in this field and through his work on the concept of *prolongement fonctionnel*.

2.2.2 Prolongement fonctionnel and Bouligand's motivation for extending the Dirichlet problem

The noun 'functional', or *fonctionnelle* in French, was first proposed by Hadamard in 1904 or 1905, according to Taylor in [209, p. 250] - Hadamard comments in a letter to his student Fréchet that he had decided to use this term to refer to functions of functions (*fonctions de fonctions*) or functions of lines (*fonctions de lignes*). As also noted by Taylor, functionals, in this sense, were first systematically studied by Vito Volterra in 1887, notably in [215] and [216]. The eminent mathematician Vito Volterra (1860-1940) in addition to contributing to the early formation of the study of functionals, worked in areas including partial differential equations and integral equations. In accordance with the above, Bouligand uses the term *fonctionnelle* to refer to a function whose variables are themselves functions.

Bouligand comments in [24] that his investigations into the concept of *prolongement fonctionnel* - the extension of functionals - were inspired by the 1922 textbook of Paul Lévy on functional analysis [174]. Paul Lévy (1886-1971) was also a former student of Hadamard and made important contributions to the areas of functional analysis and probability theory. In his textbook [174], Lévy sets out to generalise certain concepts of differential calculus, the theory of partial differential equations and the concept of a multiple integral [174, p. 8]. The main area of application subsequently considered relates to functionals involved in the Dirichlet problem for the Laplace equation [174, p. 8], such as, for example, Green's function.

Lévy's work inspired Bouligand to consider the following question: given a functional defined over a certain domain and with certain continuity properties, what continuity will the functional and its derivatives have if we extend the domain over which it is defined? Bouligand referred to the process of extending the domain of functionals while ensuring that certain continuity properties are retained as *prolongement fonc-*

tionnel. He considers the generalisation of the integral by Lebesgue as an example of a *prolongement fonctionnel* with respect to the Riemann integral, since the Lebesgue integral generalises the concept of area under a curve enabling it to be defined for a broader class of functions than the Riemann integral.

Bouligand first studies the concept of *prolongement fonctionnel* from a theoretical point of view, publishing his results notably in [24]. Subsequently, in [25], Bouligand applies his results to the area of potential theory and indicates for example how a *prolongement fonctionnel* can be applied so as to generalise Green's function. He argues in this paper against overly restrictive definitions of certain functionals in potential theory and speculates that a *prolongement fonctionnel* of the Dirichlet problem may well be possible and of interest. The arguments given by Bouligand here for such an extension are twofold - first in relation to physical applications, he expresses that such a generalisation would seem to be necessary:

Il n'est pas douteux qu'on puisse (plus ou moins commodément) prolonger le problème de Dirichlet comme MM. Lebesgue et Denjoy ont prolongé l'intégrale. Sans préjuger l'intérêt de cette généralisation, on peut remarquer cependant qu'elle apparaît, au point de vue physique, comme une nécessité. [25, p. 396]

[It is not doubtful that we may (more or less conveniently) extend the Dirichlet problem as Lebesgue and Denjoy extended the integral. Without prejudging the interest of this generalisation, we can see that it appears, however, as a necessity from a physical point of view].

He illustrates this point by referring to the example of the problem of finding the thermal equilibrium of a homogeneous domain - a problem which can be directly translated as a Dirichlet problem. He comments that this problem always has a solution from the point of view of the physical reality, even if the boundary surface is such that the corresponding Dirichlet problem has no solution from a mathematical point of view.

Secondly, Bouligand expresses that the extension of the Dirichlet problem seems to be a necessity in order to bring greater coherence to the results of this area of mathematical investigation:

Si cette recherche semble offrir, a priori un intérêt relatif, tout au moins serait-elle nécessaire pour donner aux études sur la dépendance entre une fonction harmonique et les données de Dirichlet dont elle provient, une forme plus synthétique et plus facilement maniable [25, p. 397].

[If this research seems to offer, a priori, a relative interest, at least it would be necessary to give to the studies on the dependence between a harmonic function and the Dirichlet data from which it comes, a more synthetic and more easily manageable form].

In order to establish an extended version of the Dirichlet problem, Bouligand comments that it is necessary to find a way of characterising a set of points on the boundary which can be considered acceptable in this more extended setting, even if they would exclude the possibility of a solution in the classical sense [25, p. 397]. We will see later on that Bouligand responded to this requirement through his concept of improper sets.

Having expressed a need to generalise the Dirichlet problem, Bouligand goes on to attempt such a generalisation the following year, in 1924. As we will see below, his generalisation nevertheless assumed certain assumptions regarding the boundary and therefore did not apply to an arbitrary domain. Simultaneously, it was the American mathematician Norbert Wiener who first formulated the generalised Dirichlet problem in a somewhat definitive form, without restrictions regarding the boundary - Wiener's ideas will be explored later in this chapter.

2.3 Bouligand's initial efforts towards a generalised Dirichlet problem

Having identified a programme of research to extend the Dirichlet problem, it would appear that Bouligand communicated on this topic with Henri Lebesgue. As explored in this section, Lebesgue subsequently provided Bouligand with references to prior work on the classical Dirichlet problem in which a generalisation could be said to be present implicitly. In 1924, Lebesgue highlights this point in his own work and brings out the concepts of regular and irregular boundary points for the Dirichlet problem. As explained below, a mutual influence between Lebesgue and Bouligand towards the generalisation of the Dirichlet problem can be identified. We will briefly explore all of these ideas before introducing Bouligand's first attempt at the generalisation of the Dirichlet problem.

2.3.1 The generalised Dirichlet problem in seed form

In 1924, in [168] Lebesgue clearly distinguishes between two separate stages in the resolution of the Dirichlet problem present in well-established methods to date, namely first finding a harmonic function V associated with the boundary function f and then investigating the behaviour of V in the neighbourhood of the boundary:

Dans toutes les méthodes de résolution du problème de Dirichlet, celle que j'ai donnée dans les Rendiconti du Cercle de Palerme exceptée, on peut distinguer deux parties: la construction d'une fonction V harmonique dans le domain considéré D , attachée à une fonction continue f donnée sur la frontière F de D , et l'étude de la continuité de V au voisinage de cette frontière F [168, p. 349].

[In all the methods for solving the Dirichlet problem, except the one I gave in the Rendiconti du Cercle de Palerme, we can distinguish two parts: the construction of a harmonic function V in the domain D , attached to a continuous function f given on the boundary F of D , and the study of the continuity of V in the neighbourhood of this boundary F].

According to Florin Vasilescu (1897-1958)⁴ - a Romanian mathematician who spent much of his mathematical career in France and who contributed to the generalised Dirichlet problem after Wiener and Bouligand - in his 1936 article [214, p. 93], which gives a partly historical account of the development of the generalised Dirichlet problem, Lebesgue was the first to draw attention to this distinction. In the context of the

⁴For more information regarding Florin Vasilescu, see [206, p. 68-72].

second of these two steps, Lebesgue goes on to explicitly define two types of boundary points: regular and irregular points. [168, p. 350]: a regular point is one at which the function V is continuous and equal to f , for any continuous boundary function f , otherwise this point is said to be irregular. We note that if all points on the boundary are regular, then V is a solution to the classical Dirichlet problem. If the boundary contains an irregular point, then V cannot be a solution to the classical Dirichlet problem. Lebesgue emphasises that in investigating the regularity or irregularity of points on the boundary, conditions can be formulated which do not require considering the boundary function f , but only the shape of the boundary itself in the neighbourhood of the boundary point in question [169, p. 1053] thereby highlighting the *local* nature of regularity and irregularity.

In the same vein as Lebesgue's comment above, in [214] Vasilescu presents a number of earlier methods for solving the classical Dirichlet problem, including methods due to Poincaré, Zaremba and Lebesgue, commenting that each of these methods involved, even if implicitly, these two main steps but that the harmonic function obtained in the first step was not studied in its own right as a solution to a generalised version of the Dirichlet problem:

Quelques-uns des procédés que l'on avait donnés pour résoudre le problème de Dirichlet définissent bien une fonction harmonique V , attachée à des valeurs frontières données f , et cela dans le cas d'un domaine général - défini simplement comme un ensemble ouvert. Mais, soit que leurs auteurs eussent pour objectif simplement la résolution du problème de Dirichlet classique, soit que les critères de régularité ou d'irrégularité que l'on donnait ne permettent pas de connaître le comportement de la fonction V en tous les points de la frontière...une telle fonction V n'a jamais été désignée pour être solution d'un problème de Dirichlet plus étendu que le problème classique, devant remplacer celui-ci dans les cas où il est impossible [214, p. 94].

[Some of the procedures we used to solve the Dirichlet problem define a harmonic function V , attached to given boundary values f , and this in the case of a domain defined simply as an open set. However, either the authors' objective was simply to solve the classical Dirichlet problem, or the criteria of regularity or irregularity given did not allow the behaviour of the function V to be known at all points on the boundary...such a function V has never been designated as the solution to a Dirichlet problem which is more general than the classical problem, and which is intended to replace the latter in cases where it is impossible to solve it].

After Bouligand took an interest in extending the Dirichlet problem, it would appear that Lebesgue subsequently took an interest in Bouligand's research in this direction and provided him with certain references in which the situation described by Vasilesco above is present. This can be seen in Bouligand's comments in [25]:

M.H. Lebesgue a bien voulu s'intéresser à mes recherches et me signaler dans cet ordre d'idées certains travaux⁵ de MM. B. Levi, Fubini...et Zarembo... [25, p. 387]

[Mr H. Lebesgue took an interest in my research and indicated to me certain works along these lines due to B. Levi, Fubini...and Zarembo...].

It is interesting to note that the mutual influence between Bouligand and Lebesgue regarding their work on the Dirichlet problem is again seen in the 1924 article of Lebesgue referred to above [168], in which Lebesgue comments that it was Bouligand's ideas in [29] which provided him with an opportunity to publish his own results regarding conditions for regularity, irregularity and impossibility of the classical Dirichlet problem:

L'occasion de faire connaître ces diverses conditions m'est fourni par une Note fort intéressante dans laquelle M. Bouligand retrouve une condition de régularité, obtenue par M. Zarembo, puis la généralise considérablement [168, p. 350].

[The opportunity to make these divers conditions known is provided to me by a highly interesting Note in which Mr Bouligand finds a condition for regularity obtained by Mr Zarembo, then generalises it considerably].

2.3.2 Bouligand's first attempt to extend the Dirichlet problem - *Sur le problème de Dirichlet harmonique (January 1924)*

In January 1924 [29], Bouligand sets out to apply the concept of *prolongement fonctionnel* to formulate an extended version of the Dirichlet problem. His approach centered around the extended version of Green's function which he had first considered in [18]. Given the central role of Bouligand's extended Green's function in the present paper and in his subsequent work on the generalised Dirichlet problem, it is worth elaborating briefly on this point. As can be seen from the results of Lebesgue in [167] and as reported by Tazzioli in [210, p. 236], Green's function does not exist for all domains

⁵The works of Levi and Fubini mentioned above can be found for example in [172] and [139]. Zarembo's article referred to above by Bouligand is [222].

and boundaries and was defined by Green with reference to regular domains [210, p. 235], that is bounded by surfaces which do not contain any non-differentiable lines. Bouligand shows in [18] how a generalisation of Green's function can be obtained for a domain D without such restrictions imposed on the boundary. This function is the limit of the Green's functions of a sequence of nested regular domains whose union is D .

Bouligand sets out the main goal of his paper - that is applying the method of *prolongement fonctionnel* to extend the Dirichlet problem as follows in the opening lines of [29]:

L'idée d'envisager une fonctionnelle dans le champ le plus large, où la continuité permet de l'étendre, s'applique au problème de Dirichlet, et permet dans les cas d'impossibilité signalés par M. Henri Lebesgue, de définir une solution, susceptible, en restant bornée, d'enfreindre parfois la condition aux limites en certains points de la frontière [29, p. 55].

[The idea of considering a functional in the broadest field, where continuity allows for extension, applies to the Dirichlet problem, and makes it possible, in the cases of impossibility pointed out by Mr Henri Lebesgue, to define a solution which, while remaining bounded, may sometimes violate the boundary condition at certain points of the boundary].

We note however that Bouligand's results in [29] involved certain assumptions regarding the nature of the boundary, in particular he assumes that the boundary is such that Fredholm's method - a method for solving integral equations seen as functional equations, applicable to the Dirichlet problem⁶ - can be applied at everywhere except at isolated points or at a finite number of line segments of the boundary [29, p. 55]. The specific form of Bouligand's generalised solution is given below and is a slight rewording of his results in [29, p. 56]. However, the precise form is not of central importance to the present account and the reader is as such invited to skip the details.

Theorem. *Let D be a domain with boundary δD . Let M be a regular point of the boundary. Then Bouligand argues that:*

1. *The normal derivative of the extended Green's function for the point M , denoted by $\frac{dG_A^M}{dn_M} dS_M$ is well-defined and equal to 4π for any point $A \in D$.*

⁶For more information on this method, see [2].

2. The integral, I of the following function is harmonic in D and bounded⁷:

$$\frac{f(M)}{4\pi} \frac{dG_A^M}{dn_M} dS_M$$

3. If M_0 is a regular boundary point, then I tends towards $f(M_0)$ as A tends towards M_0 .

The integral I is defined by Bouligand to be the generalised solution of the Dirichlet problem and it can easily be seen from the above to coincide with the solution to the classical Dirichlet problem whenever such a solution exists. Building on results of his own previous work and of Zaremba, Bouligand goes on to give conditions for an irregular boundary point to be *prohibitive* [29, p. 56], that is, a boundary point at which Bouligand's generalised solution I is not continuous. The geometrical characterisation provided is, Bouligand highlights, of a purely *local* nature. He subsequently uses his results to provide a condition for the solubility of the classical Dirichlet problem. We omit the details of this theorem and refer the reader to [29, p. 57].

As mentioned above, at the same time as Bouligand's generalisation, Norbert Wiener formulated a more complete generalisation of the Dirichlet problem which did not assume extra restriction on the boundary. It is worth exploring briefly Wiener's formulation of the generalised Dirichlet problem as it was to this theory that Bouligand subsequently made significant contributions - once he had become aware of Wiener's work, incorporating both new ideas and those brought out in his previous publications.

⁷Bouligand does not explicitly define the function $f(M)$ here but it would seem reasonable to speculate that it denotes the given continuous boundary function for the Dirichlet problem for the domain D .

2.4 Wiener's formulation of the generalised Dirichlet problem (January 1924)

Norbert Wiener (1894-1964) was an American mathematician who is known for his contributions to potential theory and probability. He was also active in areas outside of mathematics having made, for example, key contributions to the development of cybernetics. Marcel Brelot (1903-1987), who himself later made significant contributions to potential theory, comments in [108] that it was Norbert Wiener who gave a '*definitive form of the generalized solution of the Dirichlet problem for a continuous given boundary function*' [108, p. 39]. This is echoed by Vasilesco who comments further in [214, p. 95] that Wiener's procedure for finding a harmonic function V to be considered the generalised solution was independent of the idea of solving the classical Dirichlet problem. Wiener showed that the function he considered a solution always existed and was unique. No extra assumptions were placed on the boundary and Wiener's solution coincided with the classical solution whenever the latter exists. These results were published [218] in the same month of 1924 as Bouligand's work discussed above, however, as discussed below, the two were at that point unaware of each other's work.

2.4.1 Wiener's theorem

As described by Marcel Brelot in [108], Wiener's solution was the limit of the classical solutions for an increasing nested sequence of open sets $D_n \subset D$ (where $\cup D_n = D$) for which the Dirichlet problem is solvable. The boundary function taken for each D_n is the restriction to the boundary of D_n of a continuous extension of the initial boundary function f . We express Wiener's theorem formally below. This is a slight rewording of the theorem as expressed by Wiener in [218].

Theorem. *Let D be an open domain (i.e. an open connected set) in n -dimensional Euclidean space with boundary δD . Let f be a continuous function defined on δD . Then we have:*

1. *There is a continuous function F defined on D and δD , which when restricted to the boundary δD coincides with f .*
2. *There exists a sequence of nested domains $D_1 \subseteq D_2 \subseteq D_3 \subseteq \dots$ which converges towards D and for which the Dirichlet problem is solvable in the classical sense.*
3. *For each $k \geq 1$ construct the function V_k which is the solution to the Dirichlet problem associated with the domain D_k . and the continuous function f_k , where f_k is the restriction of f to the boundary of D_k . Then the sequence of functions V_k converges uniformly to a harmonic function V over any closed subset of D .*

4. *The existence of this function V is independent of the choice of function F and of the choice of domains and boundaries D_k and δD_k .*

This function V is defined to be the solution to the generalised Dirichlet problem. One of the natural questions that arise at this point is - in the context of the generalised Dirichlet problem - how do we characterise the conditions under which the classical Dirichlet problem is soluble? As we see below, Wiener used his generalised solution to introduce the notion of the capacity of closed sets and used this concept to characterise, in a somewhat geometrical way, the regularity of boundary points - that is, the property of a boundary point whereby it does not exclude the possibility of a solution to the classical Dirichlet problem.

Before moving on, we first note here that the term 'generalised Dirichlet problem' has also been used to refer to the Dirichlet problem for general equations of which the Laplace equation is a specific case. For example, Archibald and Tazzioli in [2] refer to the generalised Dirichlet problem for elliptic equations of the following form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = f,$$

a , b , and c being functions of x and y . As noted by Archibald and Tazzioli, this problem was studied near the beginning of the 20th century, for example by Picard in [194] and by Italian mathematicians such as Almansi, Lauricella, Boggio and Marcolongo. The topic is dealt with for example by Marcolongo in the context of elasticity theory in [179].

2.4.2 Wiener's concept of capacity for sets

In addition to proving the existence and uniqueness of a solution to a generalised Dirichlet problem, in [218], Wiener generalises the concept of the electrostatic capacity of a conductor to define the *capacity* of an arbitrary subset of \mathbb{R}^n . For simplicity, we express Wiener's concept of capacity in the form provided by Kellogg in [157]:

Let B be a bounded set. Let B' be the set consisting of B and its limit points. Kellogg comments that B' will certainly contain the entire boundary δD of a domain D extending to infinity. We denote the solution of the generalised Dirichlet problem for D and the boundary values 1 on δD by ν , referred to by Kellogg as the conductor potential of B . The capacity of B is obtained by taking the following integral over any smooth simple surface containing B :

$$\frac{1}{4\pi} \iint \frac{d\nu}{dn} dS$$

Wiener used this concept of capacity to study the regularity of boundary points in this more general setting of the generalised Dirichlet problem. As indicated for example, in [219, p. 1052], Wiener uses capacity to provide a new, quasi-geometric necessary and sufficient condition for the regularity of a boundary point [214, p. 95] - thus providing a necessary and sufficient condition for the solubility of the classical Dirichlet problem:

Theorem. *Let O be a point of the boundary C of a an open set D in three-dimensional space. Let λ be a positive number smaller than 1. Let γ_n be the capacity of the set of all points Q which do not belong to D , such that*

$$\lambda^n \leq \overline{OQ} \leq \lambda^{n-1}.$$

Then O is regular or irregular according to whether

$$\frac{\gamma_1}{\lambda} + \frac{\gamma_2}{\lambda^2} + \dots + \frac{\gamma_n}{\lambda^n} + \dots$$

converges or diverges' [219, p. 1052].

In [219, p. 1052], Wiener compares the above result to a necessary and sufficient condition for regularity previously provided by Lebesgue in [168, p. 353], which was of a less geometrical nature. He comments, regarding Lebesgue's result:

Quoique cette condition soit nécessaire et suffisante, comme elle manque cependant de caractérisation géométrique, on ne peut pas la considérer comme une solution définitive du problème de Dirichlet [219, p. 1052].

[Although this condition is necessary and sufficient, since it lacks however a geometric characterisation, it cannot be considered as a definitive solution to the Dirichlet problem].

We will see below that Bouligand also contributed to the geometrical characterisation of boundary points in the context of the generalised Dirichlet problem, through his concepts of dimension and improper sets.

2.4.3 Concluding remarks

Wiener was the first to the give what can be viewed as a complete and definitive form of the problem towards which Bouligand had been working - a generalised version of the Dirichlet problem which has a solution for any domain and corresponding continuous

boundary conditions, the key idea being to define the solution in terms of the solutions to a sequence of nested domains for which the problem is soluble in the classical sense. Wiener's formulation of the concept of electrostatic capacity for sets provided a tool for studying the nature of boundary points in a somewhat geometrical manner. This tool provided a method applicable from this new standpoint of the generalised problem which helped to characterise the cases in which the classical Dirichlet problem has a solution.

Having presented Wiener's formulation of the generalised Dirichlet problem, we will go on to present a number of Bouligand's contributions to this theory. In what way was Bouligand led to contribute to Wiener's theory, having himself worked towards his own generalisation? Wiener and Bouligand were initially mutually unaware of each other's work on the generalisation of the Dirichlet problem, published around the same time. However, shortly after the publications of 1924 explored above, a rapprochement between the French and American protagonists took place which led to collaborative exchanges between Wiener and Bouligand and to Bouligand's contribution to the elaboration of the theory formulated by Wiener. After first presenting Bouligand's work, this rapprochement will be explored in more depth below.

Before considering the above questions, we will first place our account in context by introducing Bouligand's 1925 visit to the University of Krakow, in Poland. We will draw significantly from the paper which arose from this visit to present Bouligand's contributions to Wiener's theory. The relevance of this context goes further: Bouligand's visit influenced his work on potential theory as well as the later development of his direct infinitesimal geometry and the epistemological views motivating it.

2.5 Bouligand's visit to Poland in 1925 and his connection with Polish mathematics

Bouligand was invited in 1925 to the University of Krakow by the eminent Polish mathematician Stanisław Zaremba. Zaremba's research included notably partial differential equations, potential theory, including the Dirichlet problem, and more applied considerations in mathematical physics. Zaremba had strong connections with the French mathematical community, within which he was highly respected, and spent 11 years teaching and researching in France. During Bouligand's stay in Poland, he taught a course from mid October to mid December (the first trimester of the academic year 1925/1926) on recent advances in the Dirichlet problem, including notably the work of Norbert Wiener and Bouligand's own original results. The subject matter of Bouligand's course and the new results generated by him around this time led to his 1926 publication in the *Annales de la Société Polonaise de Mathématique* entitled *Sur le problème de Dirichlet* [36]. He was subsequently made *Correspondant de l'Académie Polonaise des Sciences et Lettres* and he was highly interested in cultivating scientific collaborations between France and Poland, as indicated in his letter to René Garnier dated 10 November 1925:

...Pour ma part, je souhaite de voir se développer ces relations intellectuelles franco-polonaises... [For my part, I hope to see the development of these Franco-Polish intellectual relations] [136].

Bouligand's interest and involvement in Franco-Polish scientific relations later led to his involvement in French initiative in 1945 to re-establish cultural relations with Poland. Poncin comments [202],

...la notoriété acquise en Pologne grâce à ses leçons de Cracovie, restées célèbres, devait, en 1945, le désigner pour participer à un haut niveau à la mission chargée de rétablir des relations culturelles avec ce pays...

[the fame acquired in Poland thanks to his lessons in Krakow, which have remained famous, meant that in 1945 he was chosen to take part at a high level in the mission for re-establishing cultural relations with that country].

We also note that Bouligand identified his encounter with the Polish school of mathematics as an influential event in spurring his epistemological reflections. This is seen in a letter addressed to Maurice Fréchet dated 24 November 1938 [135] - a letter in which Bouligand retraces the development of his ideas relating to the concept he refers to as causality. He comments in the opening lines, '*Quand je suis allé à Cracovie, fin*

1925, j'ai été très frappé par l'esprit de l'Ecole polonaise'. We will further explore this influence in the following chapter.

2.6 Bouligand's contributions in the context of the generalised Dirichlet problem as formulated by Wiener

In his 1925 paper [220], Norbert Wiener comments that, from his point of view at the time, the generalised Dirichlet problem had reached a high level of 'completeness and definitiveness'. He identifies three main stages of progress which led to this situation as follows and indicates the significant contribution of Bouligand:

In this development there have been three principal stages or moments: (1) the envisagement of the first boundary value problem (which, as Lebesgue has shown, need not always admit a solution) as a special case of a more general linear problem which is always soluble for continuous boundary conditions; (2) the precise determination of the types of boundaries for which the solution of this more general problem for any continuous boundary condition is ipso facto a solution of the first boundary-value problem in the classical sense; (3) the extension of the solution to very general cases of discontinuous boundary conditions. Stage (1) has been clearly outlined by Lebesgue, and has been developed in detail by Bouligand and Wiener, who have likewise obtained what may be regarded as a substantially complete theory of stage (2). In stage (3), the furthest developments thus far obtained are those of Evans⁸ and of Wiener... [220, p. 21]

Thus we see that Wiener identifies Bouligand as a major contributor to the formulation of the generalised Dirichlet problem and to the characterisation - in the setting of the generalised problem - of the boundaries for which the classical Dirichlet problem always has a solution. Further, Landkof's comments in [162] reinforce Bouligand's recognized contribution to this problem:

At the beginning of the 20th century the work of S. Zaremba and especially of H. Lebesgue attracted the attention of mathematicians to the unsolvable cases of the classical Dirichlet problem. Through the efforts of O. Kellogg, G. Bouligand, and primarily N. Wiener, by the middle of the 20th century the problem of characterizing the so-called irregular points of the boundary of a region (i. e. the points at which the continuity of the solution of the Dirichlet

⁸We note that Griffith Evans (1887-1973) was an American mathematician whose areas of contribution included potential theory, functional analysis and integral equations. He was an important institutional figure in mathematics in America as chairman of mathematics at Berkeley during the 1930s and 1940s and for some time as the vice-president of the American Mathematical Association and of the Mathematical Association of America.

problem may be violated) was completely solved and a procedure to obtain a generalized solution to the Dirichlet problem was described.

In this section, we will focus on three main aspects of Bouligand's contribution drawing predominantly from papers between 1924 and 1928. We will first present briefly certain of his efforts towards the characterisation of regular and irregular boundary points in the context of the generalised Dirichlet problem. This theme of research can be seen as belonging to the second stage in the development of the generalised Dirichlet problem outlined by Wiener above and in the context of which he identifies Bouligand as a main contributor - namely characterising the types of boundaries for which the solution to the generalised Dirichlet problem is necessarily a solution of the classical problem. Indeed, if the boundary has no irregular points, then the classical Dirichlet problem has a solution. Secondly, we will explore a concept referred to by Bouligand as an improper boundary set - an idea he introduced in an effort to study the distribution of regular and irregular points. In addition to the fact that this represents an original contribution by Bouligand, this concept is central to our account as he uses it to classify different types of result in the study of the generalised Dirichlet problem according to their level of generality. This relates closely to aspects of his epistemological ideas he subsequently expressed in the following years, starting with two articles of 1927 and 1930 explored in the following chapter. Next, we will explore Bouligand's approach to characterising improper sets through the concept of dimension. As is explained in the corresponding subsection, these ideas again relate closely to Bouligand's later epistemological and mathematical theories as well as representing an area of his work broadly cited today.

We have chosen to present Bouligand's contributions from the single standpoint of the generalised Dirichlet problem. This includes the presentation of certain ideas which were initially published by Bouligand in the context of the classical Dirichlet problem and which were later incorporated into the generalised setting. We have chosen to proceed in this manner as the unifying theme of this thesis is to explore the interrelation between the development of Bouligand's ideas on the epistemology of mathematics and his mathematical works. As we will see in the following chapters, the development of the generalised Dirichlet problem and Bouligand's work in this area had a decisive influence on his epistemological reflections.

The presentation below draws considerably from elements of Bouligand's 1926 article entitled *Sur le problème de Dirichlet* [36], which represents, as mentioned above, a major part of the contents of the lecture course he taught at the University of Krakow during the first trimester of the academic year 1925-1926. Bouligand refers at various

points in this article to his own prior work on the Dirichlet problem between 1919 and 1924, thus situating his own contributions in the context of the new theory. In addition, he presents a number of new noteworthy results. Again, in this article, the concept of *prolongement fonctionnel* appears in the foreground:

Dans le problème de Dirichlet, notre méthode sera une méthode de prolongement fonctionnel. Cette Méthode, telle que nous allons l'exposer, peut être regardée comme la résultante des travaux de M. Norbert Wiener...et des nôtres... [36, p. 60]

[In the Dirichlet problem, our Method will be a method of prolongement fonctionnel. This method, as we will present it here, may be regarded as resulting from the work of Norbert Wiener...and from our own work...]

As is seen below, another key tool in Bouligand's results was his extension of Green's function introduced above. In addition to the results mentioned below, Bouligand comments in [95, p. 9] this tool enabled him to provide a simplification of the proof of Wiener's theorem ([36, p. 74-76]).

2.6.1 The characterisation of regular and irregular points of the boundary in the generalised Dirichlet problem

In the setting of the generalised solution to the Dirichlet problem, a regular point is a point Q on the boundary Σ for which the solution V of the generalised Dirichlet problem tends towards $f(P)$ when $P \rightarrow Q$ in whatever manner [36, p. 88]. Bouligand used his generalised version of Green's function to introduce a new way of characterising regular points in the context of the generalised Dirichlet problem. Specifically, he showed first in [28, p. 1054], (once already aware of Wiener's work) that the following theorem holds:

'In order that a point Q of Σ be regular, it is necessary and sufficient that the Green's function $G(A, P)$ tends to zero as P tends in whatever manner to the point Q .' [36, p. 88]

The following result gives a different necessary and sufficient condition which, Bouligand states, brings to light in a more immediate manner the *local* nature of the property of the regularity of a boundary point:

'In order that a point Q_0 of Σ be regular, it is necessary and sufficient that one can find, in the portion Ω_ρ of Ω inside a sphere of centre Q_0 and radius ρ a

harmonic function $H(P)$ which is positive or zero (but not $\equiv 0$) in Ω_ρ , and which tends to zero as the point P tends to Q_0 in an arbitrary manner.' [36, p. 88]

Bouligand also provided results helping to characterise irregular points in the context of the generalised Dirichlet problem. To give one example considered as noteworthy by Bouligand in his later note on his own works [95, p. 10] we reproduce the following necessary condition for irregularity:

'In order that a point Q_0 of Σ be irregular, it is necessary that we have

$$\lim_{\rho \rightarrow 0} \frac{\sigma}{4\pi\rho^2} = 1$$

where σ denotes the area of the open domain consisting of the points of Ω which are on the surface of the sphere with centre Q_0 and radius ρ .'

Bouligand's proof of the above result in [36, p. 89], relies on Wiener's theorem regarding the existence of the solution to the generalised Dirichlet problem.

A further method for characterising irregular points. Regarding criteria for the irregularity of boundary points in the generalised Dirichlet problem, Bouligand comments in [61, p. 3]

...rejetant à l'infini, par la transformation de Lord Kelvin, le point frontière dont on veut éprouver l'irrégularité, je réussis à obtenir un critère offrant aussi un grand degré de généralité...

[...sending to infinity, by means of Lord Kelvin's transformation, the boundary point for which we wish to verify the irregularity, I succeeded in obtaining a criterion which also offers a great degree of generality...]

In order to convey a basic understanding of Bouligand's work in this direction, it is useful to briefly outline the concepts of inversion and the Kelvin transform, from the point of view of mathematics during the 1920s. We stress that the details are not central to our account. The Kelvin transform is a commonly used tool of potential theory, used in the investigation of the Dirichlet problem, and is defined in terms of the concept of an inversion. Kellogg states in [158, p. 231] that two points, for example in three-dimensional space, *'are said to be inverse in a sphere...if they are on the same ray from the center, and if the radius of the sphere is a mean proportional between their distances from the center.'* As such, an inversion is a function which sends

every point within a sphere to its inverse, in the above sense of the word. The Kelvin transform then, is a transformation which sends a harmonic function which is defined for a domain d and boundary δd to another harmonic function defined on the inverse domain and boundary D and δD . If O is a boundary point in δd and is the centre of the inversion in question, then the Kelvin transform sends O to infinity. For further details regarding the Kelvin transform, in relation to the time period in question, we refer the reader to Kellogg's account in [158, p. 231-233]. The result mentioned by Bouligand would seem to be the following, provided in [28, p. 1054]

...selon que O est ou non point irrégulier (ou prohibitif) de d , le domaine inverse D , qui s'étend à l'infini, est ou non capable d'une fonction harmonique bornée et préférentiellement nulle.

[...depending on whether or not O is an irregular (or prohibitive) point of d , the inverse domain D , which extends to infinity, is or is not capable of having a harmonic function which is bounded and equal to zero on the boundary].

Bouligand calls inverse domains for which such a function does exist *exceptional domains* and refers to inverse domains for which there is no such function *normal domains*. He goes on in [28] to provide a number of criteria for a domain to be normal or exceptional. Kellogg highlights such results of Bouligand, obtained via the Kelvin transform, commenting in [157, p. 615] that '*Bouligand has given special attention to the character of the infinite point as a boundary point.*'. Results in this direction can also be found in [30] and in [36, p. 97-101].

2.6.2 Bouligand's improper sets and capacity

As seen above, Bouligand indicates in [29, p. 55] that, in order to pursue the study of a generalised version of the Dirichlet problem, it is necessary to find a way of characterising sets of points on the boundary which may well exclude the possibility of a solution for the classical Dirichlet problem but which we could consider acceptable in the context of the generalised problem. To this end, In 1925 in [36, p. 78] he introduces the concept of improper sets and shows that his improper sets are the same as Wiener's sets of capacity zero [36, p. 82]. Bouligand defines improper sets in [37, p. 250] as follows as those consisting entirely of points for which the limit inferior of Green's function is greater than zero.

Definition. Let D be a domain with boundary δD and let S be a subset of δD . If for all

(fixed) points A in S we have that $\liminf_{P \rightarrow A} G(A, P) > 0$, ⁹then we say that S is an improper set. Here $G(A, P)$ is the Green's function corresponding to the point A and P in D is variable.

We note that Bouligand emphasises that the concept of improper set is intrinsic:

Il faut observer que la notion d'ensemble impropre est une notion intrinsèque: autrement dit, le fait que σ_1 est une portion impropre de la frontière Σ n'est nullement spécial à celle-ci [36, p. 79].

[It should be noted that the notion of an improper set is an intrinsic notion: in other words, the fact that σ_1 is an improper portion of the boundary Σ is in no way specific to that portion].

The basic idea of Bouligand's improper sets is that of a set of points on the boundary which can be removed from the boundary without altering the solution to the generalised Dirichlet problem [37, p. 250]. As such, Bouligand emphasises that improper sets, or sets of zero capacity, play the same role in the Dirichlet problem as do zero measure sets in the theory of integration [36, p. 82], that is, '*...les informations portées par de tels ensembles peuvent être arbitrairement altérés sans que la solution s'en trouve modifiée (en maintenant ces données bornées)...*' [36, p. 82]

A new emphasis regarding sets of zero capacity. It would appear that Bouligand was the first to emphasise the analogy between zero capacity sets and sets of zero measure in the theory of integration, and the first to highlight the concept of a *frontière réduite* when studying the distribution of regular and irregular points on the boundary - that is a boundary for which all improper sets have been removed. Indeed, he comments in [95, p. 10],

En abordant pour la première fois le difficile problème de la distribution des points réguliers et irréguliers sur la frontière d'un domaine, j'ai attiré le premier l'attention sur la nécessité d'éliminer les ensembles impropres pour aboutir à la définition d'une frontière réduite.

[When I first tackled the difficult problem of the distribution of regular and irregular points on the boundary of a domain, I was the first to draw attention to the need to eliminate improper sets in order to arrive at the definition of a reduced boundary].

⁹In fact Bouligand writes $\liminf_{P \rightarrow A} G(A, P) > 0$. The above appears to be a reasonable interpretation.

This is reinforced by Vasilescu's comments in [213, p. 83]:

M. Bouligand s'étant proposé d'étudier la solution du problème de Dirichlet généralisé à la frontière du domaine, et plus particulièrement la fonction de Green généralisée, qu'il avait définie dès 1919....a été conduit à l'idée de débarrasser la frontière du domaine de certains ensembles de points, en lesquels la fonction de Green n'avait pas de singularités. Ces ensembles fermés et de capacité nulle, M. Bouligand les appelait impropres...

[Mr Bouligand, having set out to study the solution of the generalised Dirichlet problem at the boundary of the domain and in particular the generalised Green's function that he had defined as early as 1919...was led to the idea of ridding the boundary of the domain of certain sets of points at which Green's function had no singularities. Mr Bouligand called these closed sets of zero capacity improper sets...]

Using improper sets to categorise results according to their level of generality.

While the ideas included under this heading do not necessarily represent a key mathematical contribution of Bouligand to the generalised Dirichlet problem, they certainly do shed light on his views later expressed on the epistemology of mathematics, views which were also implemented in his approach to mathematics. He highlights in [36, p. 77] two categories of mathematical results in the context of the generalised Dirichlet problem. Results which are of the first type (*première espèce*) are defined as those which hold for an arbitrary boundary, or in Bouligand's words, *sans restriction à la généralité de la frontière*). Results of the first type would include, for example, Wiener's theorem regarding the existence of the solution to the generalised Dirichlet problem, its uniqueness and the various general properties of this solution that can be established. Bouligand goes on to suggest that, if we wish to investigate the behaviour of the solution V of the generalised Dirichlet problem in the neighbourhood of the boundary and in relation to the boundary conditions given by the boundary function f , then it is necessary to operate in a less general setting. For Bouligand, these propositions of the second type (*de seconde espèce*), are defined as those which involve assuming that the boundary of the domain in question is a reduced boundary (one which contains no improper sets). For example, he identifies the following result regarding Green's function as a theorem of the second kind:

Theorem. *Assume that Σ contains no improper sets. Then there exists a unique function $G(A, P)$ such that the function $\frac{1}{AP} - G(A, P)$ is harmonic in Ω and such that for any point Q on Σ , the smallest limit of $G(A, P)$ at Q is equal to zero. This function is the Green's function of the generalised Dirichlet problem. [36, p. 82]*

We note that the distinction between results of the first and second type by Bouligand plays no active mathematical role in [36]. As such, this distinction seems to appear purely as a means of organising mathematical ideas into categories based on their level of generality. This theme will be expanded upon in the following chapter, in the context of Bouligand's views on studying geometrical concepts at the appropriate level of generality, or within the correct domain of causality.

2.6.3 Dimensional properties of sets

As we have seen above, the idea of improper sets represent one of the central contributions of Bouligand in the study of the generalised Dirichlet problem. Here, we elaborate on a way in which Bouligand went about characterising these sets, namely by studying the dimensional properties of the boundary. Here we outline in brief the details of the mathematical formulation of these ideas. In addition to their role in relation to improper sets, it seems relevant to explore these ideas for three other main reasons. First, Bouligand's work on the concept of dimension is today broadly known in the context of the modern study of fractal geometry. Secondly, as we will see below, Bouligand highlights that his concept of dimension is invariant under a particular type of transformation and therefore labels it as belonging to the area which he refers to as *topologie restreinte du premier ordre*, which we will define below. The point we wish to highlight here is that this is an example of an effort on Bouligand's part to identify a setting in which a concept or problem should be studied - a theme upon which he later reflects in his epistemological articles under the heading of *domains of causality* or *natural domains of existence*. Finally, Bouligand's ideas regarding the dimensional properties of sets represent one factor which inspired certain methods in his later theory of direct infinitesimal geometry.

In order to characterise improper sets in the generalised Dirichlet problem, Bouligand adopted a geometrical approach which relied on studying the dimensional properties of the boundary. Bouligand published his first results on this topic in [33], initially submitted to the *Comptes Rendus* in a sealed envelope¹⁰ in 1924, later opened and published the following year. Further results appear in a more elaborated form in the following articles published in consecutive years between 1926 and 1929: [36],

¹⁰The sealed envelope (*pli cacheté*) was traditionally a means of proving precedence, for example for work which was not yet necessarily fully polished. It would seem that Bouligand does not explain his motivation for submitting this article in a sealed envelope. However, Wiener states in his autobiographical work [221] published much later in 1956 that '*...Bouligand obtained some extremely important results which he had not yet had time to polish up. He took council with Lebesgue, who advised him to submit these results to the academy in a sealed envelope, after a custom sanctioned by centuries of academy tradition*'. However, as will be seen later in this chapter, Wiener's account of events taking place much earlier is potentially partially inaccurate and it is not clear whether he is indeed referring to Bouligand's work in [33] or not.

[37] (in which a more pedagogical presentation is given), [43] and [45]. Bouligand comments regarding his motivation:

Il est important d'étudier l'allure d'un potentiel d'après la structure de l'ensemble potential et les caractères de la répartition des masses sur cet ensemble [37, p. 240].

[It is important to study the allure of a potential according to the structure of the potential set and the characteristics of the distribution of masses on this set].

Here we note that an example of the potential described above could be the solution to the generalised Dirichlet problem, where the *ensemble potential* is the boundary of the domain in question. In addition, Bouligand comments regarding improper sets:

C'est précisément pour discriminer ces ensembles que j'ai songé à introduire le nombre dimensionnel [It is precisely in order to distinguish these sets that I thought to introduce the dimension number] [37, p. 250].

While considering the origins and motivations of Bouligand's study of the dimensional properties of the boundary, we note that Bouligand recounts in [36, p. 106] that it was an example provided by Kellogg in [156] which inspired him to explore this approach¹¹.

Following Bouligand's account of his result in [37], we note that he first obtained results which enabled him to characterise improper sets in terms of mass distributions on that set. He then introduces the dimension number, allowing him to translate these theorems in such a way that brings to light the geometrical properties of the sets with which they deal [33, p. 39]. We will omit the details of the theorems formulated in terms of mass distribution and focus rather on the characterisation of improper sets in terms of dimension.

The essential ingredient of Bouligand's approach to dimension was that of comparing the way in which functions tend to zero. Informally, he considered a natural volume function at scale ρ and examined how this measurement behaves as $\rho \rightarrow 0$. Bouligand's approach leads to the possibility of non-integer dimensions. He provided both the concept of the global dimension of a closed set as a whole and the local dimension

¹¹Kellogg's 1923 article [156], published a year prior to Wiener's formulation of the generalised Dirichlet problem, focuses on an example of a domain for which the then established methods of solving the classical Dirichlet problem did not apply. While Bouligand states that he was inspired by this example, we note that Kellogg was not working in the context of the generalised Dirichlet problem and his work does not represent a somewhat systematic study of the dimensional properties of sets as did Bouligand's work cited above.

of a closed set at a point. These definitions rely on what Bouligand referred to as the Cantor-Minkowsky construction, as defined below. The presentation of Bouligand's concepts below borrows significantly from the present author's work on this topic in [115], which focuses on Bouligand's ideas in [43]. As commented in [115], Bouligand's motivation in these two articles is predominantly related to potential theory and the Dirichlet problem. As such, his ideas on dimension do not appear as a fully elaborated theory. A certain degree of interpretation was therefore necessary.

In [43], Bouligand's splits his work into three main sections, the first dealing with the dimensional properties of closed sets, the second dealing with the application of these ideas to the study of improper sets in the generalised Dirichlet problem and the third section dealing with extending these applications to a broader class of problems in the calculus of variations. Our account below focusses on the first of these three sections, relating to the dimensional properties of sets, although we cite two theorems proved by Bouligand from the second section relating to the characterisation of improper sets. We briefly outline three main themes contained in the first section, starting with what Bouligand refers to as the *construction de Cantor-Minkowsky* [Cantor-Minkowsky construction], which represents the natural volume function which we referred above. Secondly, we introduce a tool presented by Bouligand for obtaining an upper bound for the dimension of a set. We use the term 'global dimensional order' below, although Bouligand did not name the concept as such. Thirdly, we introduce Bouligand's method for evaluating the dimension of a set at a point, which he described as a local method - we have classified these ideas under the heading 'local dimensional order'. We note that we have largely followed Bouligand's notation below.

The Cantor-Minkowsky construction. Let E be a closed set in three-dimensional Euclidean space. From each point Q of the set E , draw a sphere of radius ρ . By $E(\rho)$, we denote the ρ -neighbourhood of E , i.e. the set of points contained in these spheres. By $f(\rho)$ we denote the volume of the set $E(\rho)$. Bouligand refers to this procedure as the Cantor-Minkowsky construction of the set $E(\rho)$ starting from E . Examining how $f(\rho)$ behaves as $\rho \rightarrow 0$ is the key idea in Bouligand's approach to dimension.

Global dimensional order. Bouligand comments in [43, p. 322] that a lack of homogeneity of a set may mean that it is not possible to ascribe a single dimension number to that set. To see this point, we can think of the set which is the union of a circle and a line for example. The following tool introduced by Bouligand enables us to say in what way the local dimension of a set is bounded above (in the same way as the dimension of a set consisting of a line and a circle is bounded above by the number 2).

Suppose there is a function $f_1(\rho)$ such that, for sufficiently small values of ρ , we have

$$\left| \frac{f(\rho)}{f_1(\rho)} \right| \leq c$$

for some real number c . Then Bouligand states 'If the function $f_1(\rho)$ has a well defined infinitesimal order α_1 , one can say that the dimensional order of the set is [less than or equal to]¹² $3 - \alpha_1$ '. In addition, 'in the special case when the above ratio tends to zero we will say that the dimensional order is $< 3 - \alpha_1$ ' [43, p. 323].

The infinitesimal order described above is not defined explicitly by Bouligand. We interpret the infinitesimal order of a function $f(\rho)$ in the following way. We say that $f(\rho)$ has infinitesimal order α (where α is a real number) whenever we have, for all $\epsilon > 0$:

$$\lim_{\rho \rightarrow 0} \frac{f(\rho)}{\rho^{\alpha - \epsilon}} = \infty$$

and

$$\lim_{\rho \rightarrow 0} \frac{f(\rho)}{\rho^{\alpha + \epsilon}} = 0$$

With the above definition in hand, the goal is to find the best possible upper bound for the dimensional order of the set, through the wisest possible choice of the comparison function f_1 . We note that the above definition, which implicitly formulates dimension in terms of a function on the real numbers, allows for the possibility of non-integer dimensions.

Local dimensional order. Bouligand introduces a local dimensional order of a set at a specific point. This tool can be used to express in what way the dimension of a set is bounded below. Bouligand achieves this by defining the dimensional order of the set E at a point Q at scale ϵ (where ϵ is a positive real number), before letting ϵ tend to zero. We omit the details here and simply note that the key concept is similar to the case of the global dimensional order - that is examining the way in which a natural volume function (in this case defined at a point Q and at scale ϵ) tends towards zero. His definition enables us to say that the dimensional order of E at the point Q is less than, or greater than the infinitesimal order of chosen comparison functions.

Characterising improper sets by their dimensional properties. The following two

¹²Here Bouligand write $< 3 - \alpha_1$

theorems [43, p. 333] illustrate how Bouligand was able to characterise improper sets and reduced boundaries in the generalised Dirichlet problem in terms of dimension:

Theorem. *‘Soit un ensemble fermé E , de l'espace euclidien à n dimensions si son nombre dimensionnel est $\leq n - 2$...cet ensemble E est impropre.’*

[Let E be a closed set, in Euclidean space of n dimensions. If the dimension number of E is $\leq n - 2$...then the set E is improper.]

Theorem. *‘Si h est un nombre positif fixe, d'ailleurs quelconque, et si le nombre dimensionnel de E surpasse en chaque point $n - 2 + h$, alors l'ensemble E supposé dépourvu de points intérieurs forme une frontière réduite.’*

‘If h is a fixed positive number, which is arbitrary, and if the dimension number of E is at every point greater than $n - 2 + h$, then the set E assumed to have no interior points forms a reduced boundary’.

Regarding these results Bouligand concludes as follows,

‘...la considération du nombre dimensionnel d'un ensemble permet, d'une manière incomplète, mais suffisante dans beaucoup d'applications, de caractériser les ensembles impropres du problème de Dirichlet’ [37, p. 242].

[...the consideration of the dimensional number of a set allows, in a way which is incomplete but sufficient in many applications, to characterise improper sets in the Dirichlet problem].

We note that further results relating to the characterisation of improper sets and reduced boundaries are provided by Bouligand in [43], where he also goes some way in generalising his ideas in a broader setting within the calculus of variations than that of the generalised Dirichlet problem. Given that the above two theorems are presented by Bouligand as two of his main results on the topic, and as the focus of this chapter is on the generalised Dirichlet problem, we omit the details.

Classifying the concept of dimension in terms of invariance. As it will become a significant recurring theme, we note that Bouligand emphasises the nature of his concept of dimension as an invariant under certain types of transformation. In particular, he comments that the dimensional order is not a topological invariant (i.e. invariant under homeomorphisms) but is always invariant under homeomorphisms with continuous first derivative. Bouligand refers to concepts which are invariant under this type of transformation as belonging to *topologie restreinte du premier ordre*

(restricted topology of the first order) [43, p. 321]. Bouligand highlights this point, at least in part, to distinguish between his concept of dimension and the *type de dimension* of Maurice Fréchet, which is a topological invariant [43, p. 321]. The following comment of Bouligand in [95] sheds further light on the origins of this classification with reference to previous ideas of René Baire:

'...prolongeant le point de vue de Baire isolant les propriétés purement topologique (dénomés par lui descriptives¹³), je me suis appliqué à classer les propriétés infinitésimales suivant leurs champs d'invariance respectifs en retenant, avec les transformations ponctuelles correspondantes, ceux de ces champs auxquels s'attachent les dénominations que voici: topologie restreinte du premier ordre, topologie restreinte du second ordre' [95, p. 3].

[...extending the point of view of Baire, isolating purely topological properties (which he named descriptive properties), I set about classifying infinitesimal properties according to their respective fields of invariance, retaining the fields with the following denominations and their corresponding point transformations: first order restricted topology, second order restricted topology].

Secondly, we noted above that Wiener provided a quasi-geometric characterisation of the regularity of boundary points by means of his concept of capacity. In light of the relationship between capacity and Bouligand's improper noted above, Bouligand's ideas can be seen as a step further in the geometric characterisation of boundary points. To this effect, in [157, p. 625] Kellogg refers to Bouligand's ideas in [33] in answer to the question of '*whether it is possible to express conditions for regularity in some even more purely geometric form than those in terms of capacity*'.

¹³It is not clear to which source Bouligand is referring implicitly here. We speculate that he may be referring to Baire's *ensembles de première catégorie* as defined in [4, p. 78]. Lebesgue distinguished in [166, p. 99-100] between properties he called *descriptives* and those he called *constructives*.

2.7 A rapprochement between Bouligand and Wiener in the context of the generalised Dirichlet problem

In order to place the rapprochement between Bouligand and Wiener in context, it is interesting to consider briefly the American mathematical scene, from an institutional point of view. We note that the firm establishment of American universities took place much later than academic institutions in European countries such as Germany, France and Great Britain. As Parshall and Rowe comment in [189],

The major institutional structures and research tradition of American mathematics stood firmly in place by the end of the first decade of the twentieth century, when the United States began to make its presence felt on the international mathematical scene [189, p. 431].

Further, Parshall and Rowe highlight Wiener's own role in making Cambridge Massachusetts a main hub of American mathematical research during the period in question:

Sparked by Birkhoff and Wiener, the Cambridge area emerged as America's foremost center for mathematical research by the mid-1920s [189, p. 448].

In light of the later institutional establishment of American mathematics, it is perhaps unsurprising that it was common for promising American mathematicians to travel to Europe to collaborate and receive instruction. Accordingly, Wiener, after the completion of his PhD, visited Göttingen, where he studied differential equations with David Hilbert, and Cambridge, England, where he was a student of Bertrand Russell and G.H. Hardy. In 1920, he attended the International Congress of Mathematicians in Strasbourg, and collaborated with Maurice Fréchet.

Although Wiener's formulation of the generalised Dirichlet problem was published in [218] in the same month of 1924 in which Bouligand's attempt to extend the Dirichlet problem was published in the *Comptes Rendus* [29], it would appear that Bouligand and Lebesgue on one side and Wiener on the other were mutually unaware of each other's work until approximately March 1924. Lebesgue commented on this in a short note regarding Wiener's article [219], in which Wiener summarises his own results, situating them in relation to the results of Lebesgue published earlier the same year in [168]. Lebesgue comments:

En janvier dernier, un article de M. G. Bouligand m'a fourni l'occasion de donner quelques résultats obtenus au cours de recherches sur les cas d'impossibilité

du problème de Dirichlet, recherches qui remontent à 1913. M. Wiener parle longuement de ma publication ; ce qu'il ne dit pas et que je tiens à dire, c'est que, dans ces dernières années, le problème de Dirichlet avait été, à mon insu, l'objet de recherches fécondes, due à divers savants américains et en particulier à M. Wiener. Leurs théorèmes dépassent souvent les miens et ils ont si bien abordé les diverses questions qui m'ont occupées que, si j'avais connu leurs travaux, j'aurais sans doute jugé inutile de revenir sur mes résultats de 1913... Les travaux des Savants américains avaient aussi échappé à M. Bouligand ; il convient d'ajouter que, par contre, l'importante Note publiée par M. Bouligand, en 1919, semble être restée inconnue des Américains. Maintenant que ces efforts parallèles ne s'ignorent plus, on peut espérer qu'ils seront plus fructueux encore... [169, p. 1052]

[Last January, an article by Mr G. Bouligand gave me the opportunity to give some results obtained in the course of research into the impossible cases of the Dirichlet problem, research which goes back to 1913. Mr Wiener speaks at length about my publication; what he does not say and what I would like to say is that, in recent years, the Dirichlet problem had been, unbeknownst to me, the subject of fruitful research, due to various American scientists and in particular to Mr Wiener. Their theorems often surpassed mine and they dealt so well with the various questions that occupied me that, had I known of their work, I would probably have considered it unnecessary to go back to my results of 1913... The work of the American scientists had also escaped Mr. Bouligand; it should be added that, on the other hand, the important Note published by Mr. Bouligand in 1919 seems to have remained unknown to the Americans. Now that these parallel efforts are no longer unknown to each other, we can hope that they will be even more fruitful].

It is interesting to consider Wiener's later account in his autobiographical work of 1956 [221], in which he recounts the simultaneous efforts of his and of Bouligand. While this account may not be completely accurate, it illustrates well Wiener's point of view regarding the work of Bouligand in relation to his own and regarding the simultaneous nature of the efforts of Bouligand and Lebesgue in the same direction as his own.

...I added a considerable number of new and sharply definable concepts to the armament of potential theory. When I applied these to the old problem of Zaremba, which still had not reached a final solution, I found that they fitted. This was at about the time at which Comptes Rendus of the French Academy of Sciences began to be filled with papers on the theme of the

Zaremba theory. These were written by Lebesgue himself and by a young pupil of his, G. Bouligand. In many scientific subject there comes a time when the sharpness and definiteness of the new papers indicate that an important goal is about to be achieved. So it was with the work of Lebesgue and Bouligand. I knew that if I did not put forth a maximum effort, the whole topic was soon to be crossed off the account books of mathematics as one finally resolved. Accordingly I put forth a maximum effort, employing those new tools which I had made my own, and I was delighted to find that I had achieved what was from the standpoint of research at that time a complete solution to the problem....What followed is a coincidence of a sort much commoner in the history of discovery and invention than one might suppose. While my letter was crossing the ocean, Bouligand obtained some extremely important results which he had not yet had time to polish up. He took council with Lebesgue, who advised him to submit these results to the academy in a sealed envelope, after a custom sanctioned by centuries of academy tradition. The very next day, my paper came in and Bouligand's sealed envelope was opened. The two papers appeared side by side in the next number of the Comptes Rendus, with a preface by Lebesgue covering both of them¹⁴. While they were expressed in different mathematical language, the main idea of the two was identical. However, the logic of Bouligand's paper was not as complete as mine, owing to the fact that what he had sent in was only a preliminary communication for purposes of record, and not a polished and finished job. This was even more of a dead heat than my previous double discovery of Banach space. Both the Banach space competition and that with Bouligand turned out to be extremely friendly. Bouligand was even more ready than I was to admit the somewhat greater completeness of my paper, and we made arrangements to meet when I should come to France and visit.

Following the rapprochement mentioned above, a number of letters and postcards were exchanged between Wiener and Bouligand between 1924 and 1930 (and predominantly between 1924 and 1925). The letters sent by Bouligand, in French, are available in the Norbert Wiener papers, kept at the Massachusetts Institute of Technology Libraries [122]. It is worth emphasising certain main recurring themes in this correspondence. First, Bouligand clearly expresses recognition of the importance and

¹⁴It is possible that Wiener's account of events having taken place more than 30 years earlier was not completely accurate. The simultaneous and less complete results of Bouligand to which Wiener is referring are most probably those in [29]. This paper of Bouligand it seems was not submitted as a sealed envelope - in fact the later publication of Bouligand [33] was submitted as a sealed envelope but was written after Wiener's formulation of the generalised Dirichlet problem, to which Bouligand refers in this article. In addition, Bouligand's paper [29] did appear in the same volume of the *Comptes Rendus* as Wiener's article [219] but did not appear side by side. Bouligand's paper [28] appeared side by side with Wiener's article, with Lebesgue's comment in between.

completeness of Wiener's results and seems to have made efforts to communicate this to colleagues and students. This is seen in the following two extracts, the first of his letter of February 1924:

Votre théorème I, (du mémoire Certain Notions in pot. th.) établit l'existence de la solution au sens large du problème de Dirichlet, dans les condtions les plus générales, alors que M. Zaremba ne l'établissait que moyenant une restriction "Ad Fini". Je vous félicite du grand progrès que vous avez ainsi accompli.

[Your theorem I (of the article Certain Notions in pot. th.) established the existence of the solution in the broad sense of the Dirichlet problem in the most general conditions, while Mr Zaremba only established the solution by means of an "Ad Fini" restriction. I congratulate you on this great progress that you have accomplished].

In addition, in Bouligand's letter of 5 April 1928, Bouligand relates to Wiener how he emphasised the importance of Wiener's results to students during a course he taught at the University of Krakow in 1925, having been invited by Zaremba and in an article covering a part of the contents of his course [36]:

J'ajoute qu'appelé à Cracovie par M. Zaremba...j'ai eu l'occasion de vous rendre publiquement hommage, et que cela est d'ailleurs relaté dans la publication partielle de mon cours parue au tôme IV des Annales de la Société Polonaise de Math...

[I add that, having been called to Krakow by Mr Zaremba...I had the opportunity to pay tribute to you publicly and that this was recounted in the partial publication of my course published in volume IV of the Annales de la Société Polonaise de Math...]

The correspondence also clearly indicates an element of collaboration between Wiener and Bouligand on the generalised Dirichlet problem. For example, in his letter of November 1924, Bouligand sends his own work to Wiener and thanks him for having put him in contact with Kellogg:

Je vous adresse les tirés à part et mes deux derniers mémoires du Bulletin. Je vous remercie vivement de m'avoir mis en relation avec M. Kellogg, qui m'a adressé aussi ses beaux travaux.

[I am sending you the offprints and my last two memoirs from *Le Bulletin*. Thank you very much for putting me in contact with Mr Kellogg, who has also sent me his fine works].

This collaboration is seen yet more clearly in Bouligand's letter of December 1924:

En attendant que je trouve le temps de publier, sur le thème de cette note, un mémoire un peu détaillé, j'en livre le contenu à votre méditation. Dites-moi ce que vous en pensez! Si cela peut hâter la solution du problème que nous cherchons, j'en serais fort heureux!

[While I am waiting for the time to publish a detailed memoir on the subject of this note, I'll leave you to ponder its contents. Tell me what you think of it! If it can hasten the solution to the problem we are seeking, I would be delighted!]

Thus we see that Bouligand and Wiener were actively collaborating towards a common goal. We also note that Wiener visited Bouligand in Poitiers in 1924. This is reflected in the tone of Bouligand's letters, which becomes clearly amicable, personal and family-oriented as well as mathematical.

Having retraced the path leading Bouligand towards the generalisation of the Dirichlet problem and having seen a number of his contributions to Wiener's formulation of this problem, it is interesting to situate this area of Bouligand's research in the broader setting of his research in potential theory. We will address this below as well as briefly drawing attention to further themes of Bouligand's research in potential theory which developed later and which were linked to his work on the generalised Dirichlet problem. Due to the fairly diverse nature of Bouligand's research in potential theory and perhaps due to the nature of this topic, which bridges theoretical and applied considerations, we are naturally led to attempting to classify his work under broader headings. Was Bouligand mainly working in the setting of pure mathematics in the context of this area of research? Did some areas of his work belong more to mathematical physics? The inherent danger of such questions would be to impose our modern perspective and labels on ideas generated during the 1920s. As such, a brief discussion is included below on the classification of research topics which, we hope, provide a perspective which is somewhat faithful to the situation as it was during the time of Bouligand's research.

2.8 Further directions of Bouligand's research in potential theory

We note that Bouligand's work on the generalised Dirichlet problem together with his research leading him in this direction (i.e. studying the classical Dirichlet problem in relation to the boundary) [95, p. 9] represents one of the main themes in his research in the more general area of potential theory, and the predominant theme at least until around 1926, once his main ideas on this topic had been published. Another theme of research prior to this date included harmonic functions and their singularities more generally, for example in [23] and [32]. Bouligand comments in [47, p. 4] that his work on the topic of harmonic functions and the Dirichlet problem prior to his visit to the University of Krakow in 1925 had been undertaken '*indépendamment de toute préoccupation d'ordre concret*'. As we see below, certain other themes of research within the area of potential theory are associated to greater degrees with physical problems.

Influenced by his work on the generalised Dirichlet problem and his collaboration with Zaremba in Krakow, Bouligand later worked on extending the Neumann problem, which involved using an extended notion of flux in fluid mechanics. The extract below of Bouligand in his 1930 article [47, p. 4] highlights these ideas:

...j'ai échangé des idées, de la manière la plus fructueuse, avec un puissant géomètre polonais, M. Stanislas Zaremba. Pendant un séjour à l'Université de Cracovie où j'avais exposé, sur son invitation, les recherches que j'avais consacrées (indépendamment de toute préoccupation d'ordre concret) à la question du prolongement fonctionnel, aux, flux, au principe de Picard, au problème de Dirichlet, je rentrai en France avec la conviction que le problème de Neumann pouvait participer lui-même de cet esprit nouveau, qu'il admettait un prolongement fonctionnel, et qu'on devait le découvrir à partir de cette idée : le mouvement irrotationnel d'un liquide diffère peu dans deux vases dont les parois, de configuration peu différentes, ont des mouvements qui diffèrent peu...

[I had a most fruitful exchange of ideas with a powerful Polish geometer, Mr Stanislas Zaremba. During a stay at the University of Krakow where, at his invitation, I had presented the research I had devoted (independently of any concrete concerns) to the question of functional extension, flux, the Picard principle and the Dirichlet problem, I returned to France with the conviction that the Neumann problem could itself participate in this new approach, that it admitted a functional extension, and that it should be

discovered on the basis of this idea: the irrotational motion of a liquid differs little in two vases whose walls, with similar configuration, have motions that differ little].

His ideas on this topic are summarised in detail and in a pedagogical style in [47] and were also published for example in [35] - an article appearing side by side with a paper of Zaremba and which illustrates clearly their collaboration - as well as in [39]. Another strand of Bouligand's research relating to potential theory, which we will not explore in more detail here, deals directly with questions in the field of hydrodynamics. This can be seen in Bouligand's first mathematical publication [16] and for example later in [53].

A further direction of Bouligand's research that later developed combined potential theory and probability for chains of events. As Bouligand comments in [95, p. 10] these developments took place notably through the orientation of the 1937 doctoral thesis of his student A. Fouillade [137], which builds upon ideas of Bouligand in [38]. Finally, we note a theme of research focusing on the Dirichlet problem for more general equations than the Laplace equation, in particular certain classes of elliptical equations. Here Bouligand was able to extend for example his concept of improper sets and his results can be found notably in [56], [57], [70] and [78]. He comments in [95, p. 12] that it was his orientation of J. Capoulade's thesis which was the starting point for this line of investigation.

2.8.1 Classifying Bouligand's work; areas of mathematics and its applications.

It is interesting to situate Bouligand's work on the Dirichlet problem and potential theory in relation the different subdivisions of mathematics and its applications as they existed at the time Bouligand was publishing. To this end, we first refer to Leloup's account in [170, p. 88-95] where she uses notably key words extracted from the subdivision of the various chapters and sections of the *Jahrbuch über die Fortschritte der Mathematik*, during the interwar period, supporting these also with information contained in thesis reports for dissertations defended during the interwar period in France.

Leloup identifies four main headings for mathematical publications in France during that period, namely:

- **Arithmetic and algebra** - here Leloup identifies dissertations belonging to two main areas - first between 1914 and 1926, the study of forms (i.e. homogeneous

polynomials) and secondly between 1928 and 1944, the theory of ideals [170, p. 153].

- **Geometry** - including differential geometry, algebraic geometry, topology (a key word appearing from 1925) and analytical geometry
- **Analysis** - including the theory of functions and differential and integral calculus. Of relevance to our account here, we note that she identifies within the latter, the theory of ordinary and partial differential equations, elliptic differential equations and potential theory.
- **Applied mathematics** - comprising publications classed under the headings of mechanics and of mathematical physics.

Leloup notes that the classification of subtopics under either mechanics or mathematical physics is found to be frequently evolving back and forth throughout the interwar period. The two together include the topics of fluid mechanics, the mechanics of solids, kinematics, the theory of relativity, astronomy, geodesics and geophysics and the *théorie des quantas* (which preceded quantum physics). Leloup notes that the boundary between certain areas of analysis and applied mathematics is also somewhat porous in France during this period, particularly in the case of differential equations. Specifically, papers dealing with differential equations are sometimes placed within mechanics or mathematical physics, rather than within analysis.

We also note that Leloup identifies a further chapter of the *Jahrbuch* entitled '*Fondements des mathématiques. Théorie des ensembles*', which conserves this name until 1939 (i.e. the period of interest in the present account) before *théorie des ensembles* is later included within analysis, leaving a separate chapter entitled *fondements*. She notes, further, that probability and its applications are classed under analysis until 1939, before being included as a separate chapter in its own right [170, p. 91-92].

It is interesting to compare Leloup's analysis briefly outlined above with the structure of the *Encyclopédie des sciences mathématiques pures et appliquées* [185] - an enlarged version in French of the German encyclopedia edited by Felix Klein and others, first published in 1898. The French version was published in seven volumes between 1904 and 1916 and the numerous contributors included mathematicians such as Elie Cartan, Emile Borel, Jacques Hadamard, Paul Painlevé - members of the previous generation of mathematicians to Bouligand who certainly had an influence on his education and/or mathematical work. While the chapter structure of the encyclopedia provides an overview of mathematics and its application based on significantly less data than

Leloup's detailed analysis, it nevertheless provides an interesting perspective of a number of top mathematicians at a time which coincides with Bouligand's mathematical education. As we see below, a further advantage is the inclusion of a separate chapter on physics, giving a tentative insight into the boundary between physics and applied mathematics at the time. The seven volumes are as follows:

- **Arithmetic and Algebra** - including arithmetic, algebra, number theory, and probability, the theory of error and related applications
- **Analysis** - including functions of a real variable, functions of a complex variable, ordinary differential equations, partial differential equations series expansions and the calculus of variations
- **Geometry** - comprising the foundations of geometry and general geometry, descriptive geometry and elementary geometry, planar algebraic geometry, algebraic geometry in space
- **Mechanics** - including general mechanics, deformable systems (includes hydrodynamics), ballistics and hydraulics
- **Physics** - including thermodynamics, molecular physics, physical principals of electricity, physical principals of optics
- **Geodesics and geophysics**
- **Astronomy**

Based on Leloup's subdivision and on the above, we may say, using the appropriate terminology of the interwar period, that Bouligand's work on the Dirichlet problem and potential theory more generally belongs predominantly in the area of analysis - in particular his work on and towards the generalised Dirichlet problem. A number of his papers however are in the area of mechanics (within Leloup's broader area of applied mathematics), such as those dealing with hydrodynamics. It is reasonable to say that Bouligand's ideas interact to a certain extent with physics (in particular thermodynamics and electricity) - for example in his motivation to extend the Dirichlet problem based on a *physical necessity* (justified through an example relating to thermodynamics) as seen above, and in the context of mathematical generalisations of physical concepts such as electrostatic capacity.

2.8.2 Pedagogical efforts and contributions to the circulation of knowledge to the mathematical community

While the points presented under this heading do not represent a further direction of research as such, pedagogical efforts and those relating to the circulation of knowledge

in the mathematical community represents nevertheless a direction in which Bouligand's efforts and ideas evolved. He made efforts to communicate his ideas on the Dirichlet problem in a pedagogical manner, notably through his lecture course at the University of Krakow in 1925 and the subsequent publication mentioned above. We note in addition that [37], focusing on ideas relating to improper sets and dimension was published in *L'Enseignement Mathématique*, which placed a strong emphasis on mathematics education and which included in addition a number of book reviews. Further, as mentioned by Barbut, Locker and Mazliak in [175, p. 34] Bouligand's 1926 article [34] which represented the eleventh volume of the *Mémorial des sciences mathématiques*, provides a broad survey of progress to date in potential theory, including but not limited to the Dirichlet problem. As such, we note that, in this instance, Bouligand was proactive in the circulation of knowledge within the mathematical community beyond the publication of his own results. This point is perhaps reinforced by the fact, as mentioned in [61] that he organised research seminars in Poitiers which attracted high-level mathematicians including Wiener and Kellogg. It is reasonable to speculate that a research seminar organised by Bouligand in which these mathematicians participated was most likely related to the Dirichlet problem and potential theory.

Chapter 3

Bouligand's reflections on the concept of intuition in mathematics

Exploring Bouligand's ideas on intuition in mathematics will help to clarify certain key concepts of his epistemological reflections, namely those of direct methods and causality, which will be discussed later on. More broadly speaking, his ideas on intuition help us to understand Bouligand's preferences regarding the practice and teaching of mathematics. In addition, examining the nature of his ideas about intuition will later facilitate positioning and differentiating Bouligand with respect to relevant landmarks in the epistemology of mathematics during the interwar period.

3.1 The concept of intuition in mathematics in France at the start of the 20th century (some background)

Before exploring Bouligand's reflections regarding intuition in mathematics, it is instructive to first briefly present the context in which these ideas came about. Below is an introduction to certain discussions regarding intuition at the start of the 20th century in France, in the context of debates regarding the formalisation and foundations of mathematics and a major educational reform implemented between 1902 and 1905. We do not attempt a detailed and comprehensive exploration of debates regarding intuition at the turn of the 20th century but focus instead on some ideas and sources with which Bouligand was most likely to have come into contact.

3.1.1 The dichotomy between logic/rigour and intuition

Henri Poincaré on logic versus intuition

In his 1902 article on the role of intuition and logic in mathematics [197] Poincaré

explores the dichotomy between intuition and logic in mathematics and between intuitive mathematicians (such as Riemann) versus logicians (such as Weierstrass), acknowledging the necessary role of both and the insufficiency of each alone. Intuition, Poincaré states, cannot provide certainty and logic alone cannot allow the mathematician to grasp how each element relates to each other and to the whole:

Or, pour comprendre un plan, il faut en apercevoir à la fois toutes les parties, et le moyen de tout embrasser dans un coup d'oeil d'ensemble, c'est l'intuition seule qui peut nous le donner. Dans ces édifices compliqués élevés par les maîtres de la Science mathématique, il ne suffit pas de constater la solidité de chaque partie et d'admirer l'oeuvre du maçon, il faut comprendre le plan de l'architecte.

Or, pour comprendre un plan, il faut en apercevoir à la fois toutes les parties, et le moyen de tout embrasser dans un coup d'oeil d'ensemble, c'est l'intuition seule qui peut nous le donner [197, p. 125].

[In these complicated edifices constructed by the masters of mathematical Science, it is not enough to notice the solidity of each part and to admire the work of the builder, it is necessary to understand the architectural plan. Yet, to understand the plan, it is necessary to perceive simultaneously all of the parts and the way of encompassing everything in a single view. It is intuition alone that can provide us with this].

Logic and logicians are associated in this article to a certain extent with analysis, whereas intuition is associated more with geometry. Logic is associated with mathematical demonstration whereas intuition is associated with mathematical invention. Poincaré identifies different types of intuition - first that which calls upon the senses and the imagination, secondly that which is associated with generalisation by induction and finally that which he refers to as *l'intuition du nombre pur* [197, p. 122]. Do logicians also take recourse to intuition for mathematical invention? For Poincaré, there is a rare type of intuition which enables certain logicians to have a '*sentiment direct de ce qui fait l'unité d'un raisonnement, de ce qui en fait pour ainsi dire, l'âme et la vie intime*' [197, p. 128] - he associates this type of intuition with the *intuition du nombre pur*.

The debate on rigour and intuition in the context of the reception of Peano's mathematical logic (1904-1907)

Giuseppe Peano's *Formulario Mathematico* was a textbook first published in 1895 and a project which set out to formulate all known mathematical theorems and methods using the symbolic notation of mathematical logic. In [173], Erika Luciano gives a detailed account of heated debates on rigour and intuition ensuing from a presentation given by the philosopher Louis Couturat at the second International Congress of Philosophy in Geneva in 1904, *Sur l'utilité de la logique algorithmique*, advocating Peano's movement and emphasising the utility of logic in mathematics. Couturat was strongly opposed by the philosopher Pierre Boutroux, nephew of Henri Poincaré. Luciano comments that Boutroux,

'...expressed strong reservations on the possibility that logic could ground mathematics, and made harsh criticism on the expectation of reducing mathematics to logic, isolating the minimum number of axioms and eliminating (thanks to ideography) any resort whatsoever to intuition'. [173, p. 196]

These opposing views led to the involvement of a number of the most eminent mathematicians during this debate taking place during the period 1904-1907 and led to numerous publications on the topic notably in the *Revue de Métaphysique et de Morale*. Poincaré became centrally involved in the debates, defending the position of Emile Boutroux. For example in [199], while expressing that Hilbert is always preoccupied with preserving the role of intuition in his work, Poincaré strongly criticises certain aspects of logicist theories, strongly opposing the view that logical elements can be isolated from mathematical reasoning and that intuition can be removed from mathematics and mathematical reasoning. Hadamard, Borel, Baire and Lebesgue also became involved to varying degrees, as did Peano himself, together with other eminent Italian mathematicians, and Bertrand Russell. As these mathematicians have not yet been introduced, and as they are influential in relation to Bouligand, we note that **Henri Lebesgue** (1875-1941), another key figure in French mathematics during the period in question, was one of the founders of the theory measure and his contributions also included introducing the Lebesgue integral, which generalises the Riemann integral. **René Baire** (1874-1932) was also a highly prominent figure in French mathematics and made important contributions to the theory of functions and the concepts of limits and continuity. Luciano comments in [173] that the debate came to an end in 1908 when Poincaré and Russell finally came to an agreement that,

'...logic is an auxiliary of intuition and that, like mathematics, it is marked out by an inextricable weaving of intuition and rigour, induction and deduction.' [173, p. 199]

We refer the reader to [173] for a full account of the debates on rigour and intuition between 1904 and 1907 and simply underline here their high-profile and level of controversy.

3.1.2 Mathematical intuition in the context of an educational reform

A national reform of the lycées, first adopted in 1902, led to a major reorganisation of the structure and content of the education provided by these institutions. We base our brief introduction to this event on Hélène Gispert's account in [140]. The aim of the reform was to modernize the education provided by these institutions in line with the economic and industrial needs of the day, thus moving away from an exclusively abstract or theoretical training. This reform led to expanding the role of the sciences and of mathematics in particular and also led to revising the contents and methods in the teaching of these subjects. A number of eminent mathematicians were involved to varying degrees in designing and promoting the reform in relation to mathematics. For example, Gaston Darboux headed the commission responsible for revising the teaching of the sciences. Henri Poincaré and Emile Borel supported the reform in relation to mathematics and were invited to speak at the *Musée pédagogique* in Paris in 1904 and 1905, where the reform was presented to teachers.

Henri Poincaré's presentation, entitled *les définitions générales en mathématiques* [198], explores the question of mathematical definitions best suited to the needs of students. This presentation is based in part on the Poincaré's shorter publication of 1899 [196] and includes a discussion regarding the roles of logic versus that of intuition in mathematics education, through the lens of definitions. It is relevant in relation to the work of Bouligand discussed below to explore a few main points that arise in Poincaré's presentation on the topic of intuition.

Poincaré presents a trade-off between rigour and intuition in mathematics. Increased rigour and the developments in the foundations of mathematics allowed, on one hand, to progress beyond errors made by previous generations of mathematicians due to a reliance on intuition in the absence of rigour. For example, for Poincaré, intuition teaches us mistakenly that every curve has a tangent and that, therefore, every continuous function has a derivative [198, p. 262], whereas the rigorous approach to the concept of continuity achieved by that time prevented such mistakes. However, Poincaré expresses a sacrifice associated with the rigour gained in the mathematics of his day. He comments that,

Ce qu'elles ont gagné en rigueur, elles l'ont perdu en objectivité [198, p. 263].

[What has been gained in rigour has been lost in objectivity].

The extract below elaborates on this point in the context of definitions, comparing past and contemporary practices.

On possédait une notion vague, formé d'éléments disparates, les uns a priori, les autres provenant d'expériences plus ou moins digérées; on croyait en connaître, par l'intuition, les principales propriétés. Aujourd'hui on rejette les éléments empiriques en ne conservant que les éléments a priori C'est très bien, mais il reste à prouver que cette propriété, qui est devenue une définition, appartient bien aux objets réels que l'expérience nous avait fait connaître et d'où nous avons tiré notre vague notion intuitive. Pour le prouver, il faudra bien en appeler à l'expérience, ou faire un effort d'intuition, et si nous ne pouvons le prouver, nos théorèmes seraient parfaitement rigoureux, mais parfaitement inutiles [198, p. 263]

[We had a vague notion, made up of disparate elements, some a priori, others derived from more or less processed experiences; we thought we knew, through intuition, the main properties. Today we reject the empirical elements retaining only the a priori elements. This is all very well, but we still have to prove that this property, which has become a definition, does in fact belong to the real objects that experience had made known to us and from which we had derived our vague intuitive notion. To prove it, we will have to appeal to experience, or make an effort of intuition, and if we are not able to prove it, our theorems would be perfectly rigorous, but perfectly useless].

We note in addition from the above a connection made by Poincaré between intuition, the physical world and what he refers to as the objectivity of mathematics. The roles of logic and intuition are further discussed later in this article when Poincaré portrays logic as the aspect of mathematics that, for example, enables one to decompose a proof into a sequence of correct formal operations, whereas intuition enables the mathematician or student to be able to grasp the '*je ne sais quoi qui fait l'unité de la démonstration*' [198, p. 264]. He emphasises that,

...cette vue d'ensemble, la logique pure ne peut nous la donner; c'est à l'intuition qu'il faut la demander... [198, p. 264]

[...pure logic cannot give us this holistic view, it is to intuition that we must appeal...]

To illustrate this point, Poincaré refers again to the definition of a continuous function: the initial image is of a line drawn on a blackboard. The process of formalisation leads us to formulate an abstracted version of this intuition, ultimately leading to the formal definition of continuity. However, Poincaré recommends that the teacher should then re-establish the connection between the formal definition and the initial image. In the same vein, he presents logic as that which enables one to see if a particular combination is correct and intuition as that which enables one to choose among all of the possible combinations [198, p. 267], otherwise formulated,

...il faut voir le but de loin, et la faculté qui nous apprend à voir, c'est l'intuition.
[198, p. 267].

[...you have to see the goal from afar and the faculty which teaches us to see is that of intuition].

For Poincaré, logic is associated here with proof, and intuition with mathematical invention: *'c'est par la logique qu'on démontre, c'est par l'intuition qu'on invente'* [198, p. 267].

Regarding the role of intuition in the teaching of mathematics, the following illustrates well Poincaré's views while emphasising its connection with his views above on the objectivity of mathematics:

Le but principal de l'enseignement mathématique est de développer certaines facultés de l'esprit et parmi elles l'intuition n'est pas la moins précieuse. C'est par elle que le monde mathématique reste en contact avec le monde réel...
[198, p. 266].

[the main goal of mathematical education is to develop certain mental faculties and among these, intuition is not the least precious. It is through intuition that the mathematical world remains in contact with the real world...]

What practical measures does Poincaré suggest regarding intuition in mathematics education? He recommends that, for future engineers, not too much time should be spent on formal logical considerations (*'chercher la petite bête'*). For students considering later becoming teachers of mathematics, Poincaré recommends both an in depth and rigorous teaching of first principles but also stresses the importance of cultivating intuition. For those who aspire to become professional geometers (*le géomètre pur*), he emphasises intuition as essential given its role in mathematical invention: *'Savoir critiquer est bon, savoir créer est mieux'* [198, p. 266].

Emile Borel's presentation at the *Musée Pédagogique*, entitled *les exercices pratiques de mathématiques dans l'enseignement secondaire* [11] did not focus so explicitly on intuition. However, in his later article of 1907 entitled *la logique et l'intuition en mathématiques* [13], he reflects in part of this publication upon the role of intuition in the teaching of mathematics in light of the reform of 1902. In his concluding remarks, Borel suggests that both logic and intuition should be taught. However, to avoid confusion among students learning elementary mathematics, he suggests that some mathematical disciplines be taught more from a more logical point of view and others more from an intuitive point of view. Specifically, he recommends an intuitive approach to teaching geometry, thus calling for a reform in the teaching of geometry more radical than the recent reform of 1902 [13, p. 283]. How does Borel define intuition in mathematics? While no attempt is made to give a definition, an aspect of Borel's idea of intuition is reflected in his comments in this article regarding the development of the theory of functions of a complex variable. He comments that the difficulties of extending the rules of differential and integral calculus to this theory, once resolved, were formulated in a rigorous deductive form by Weierstrass. However, Borel expresses that it was Riemann's geometrical representation using manifolds that was essential in allowing intuition to come back into play:

...avec Riemann, l'intuition reprend ses droits; la représentation géométrique de la variable complexe joue un rôle essentiel; cette représentation est généralisée; au lieu d'un plan simple, on considère une sphère à plusieurs feuillets, réunis par des coupures, et les propriétés géométriques de cette surface compliquée dominent la théorie des fonctions algébriques... [13, p. 281]

[...with Riemann, intuition regained its rights; the geometric representation of the complex variable plays an essential role; this representation is generalised. Instead of a simple plane, we consider a sphere with several layers, joined by cuts, and the geometric properties of this complicated surface dominate the theory of algebraic functions...]

As such, Borel's view of intuition seems to be, in this case, connected with facilitating mathematical practice through the use of concrete geometrical representations.

We saw above Poincaré's views regarding logic as that which enables one to see if a certain combination is correct and intuition as that which enables one to choose fruitful combinations. In a similar vein, Borel emphasises the insufficiency of logic alone, particularly for '*questions dont l'intérêt réside dans le rapprochement établi entre deux ordres de recherche en apparence différents*', adding later that '*...l'on ne confond pas le tailleur de pierres avec l'architecte*'. [13, p. 279]. We note that, although intuition is not

explicitly mentioned here, the title of the article seems to be a reasonable justification for identifying intuition as the element in the above that provides, for Borel, what logic alone does not.

3.2 Bouligand on the concept of intuition in mathematics

Bouligand addressed the concept of intuition mainly in four articles published between 1927 and 1930 in *La Revue Scientifique* - a journal aimed at a scientific but non-specialist audience and covering a wide range of subjects, from mathematics and chemistry to the application of science to industry. The four articles in question are listed below, each with a short description of the main focus:

1. *Aperçus intuitifs sur les mathématiques usuelles* - 1927 [40]. This article was Bouligand's first non-mathematical publication and it presents his view on how existing mathematics can be presented and understood in a way that allows the student's intuition (in Bouligand's sense of the word which will be explored below) to come into play.
2. *L'intuition et le symbolisme mathématiques* - 1928 [44]. Here the interplay and opposition between algebraic/algorithmic methods and intuition is explored, highlighting notably the dangers according to Bouligand of an excessive reliance on method to the exclusion of intuition, as well as the rewards of allowing intuition to play a central role.
3. *L'intuition mathématique, son mécanisme, ses aspects variés* - 1929 [46]. Here Bouligand explores and exemplifies in greater depth the concept of intuition itself and its role in the mathematical discovery process, focusing on two different aspects of intuition, which he refers to as *intuition prolongée* and *contre-intuition*.
4. *Autour de l'intuition mathématique* - 1930 [54]. Here Bouligand explores the interplay between intuition and formalisation from a subjective viewpoint of the individual mathematician - thinking in terms of a space of mathematical concepts, facts and intuitions (*l'espace des réalités mathématiques et des intuitions*). In this article he attempts for the first time an explicit definition of intuition and of *intuition prolongée*. The concepts of objectivity and beauty in mathematics are very briefly explored.

For Bouligand, the concept of intuition in mathematics is, above all, identified in terms of **making connections with familiar concepts from one's perception of the physical environment or with geometrical concepts**. This main message is seen from different angles and elaborated on to varying degrees in the four articles listed above. We will take a thematic approach in understanding Bouligand's views on intuition.

3.2.1 Intuition from a pedagogical point of view

In Bouligand's 1927 article [40], he stresses the important role of intuition in mathematics education. This view is perhaps most clearly summarised in the following extract of this article:

...malgré les erreurs que peut faire naître son usage exclusif, l'intuition mérite-elle dans l'enseignement une large place. Dans une simple formation de culture générale, cette place peut, sans inconvénient, être prédominante. Dans la préparation des ingénieurs, il faut développer l'exercice de l'intuition et du calcul en les associant étroitement. Enfin, dans l'instruction des théoriciens, il y a lieu d'initier en outre aux procédés de construction scientifique, par voie purement déductive... [40, p. 132]

[...despite the errors that can arise from its exclusive use, intuition deserves a significant place in education. In a simple general education course, this role can, without disadvantage, be predominant. In the preparation of engineers, the exercise of intuition and calculation should be developed by closely associating them. Finally, in the training of theorists, there is also a need to initiate them into the processes of scientific construction, using a purely deductive approach].

In the above, we see that Bouligand seems to place intuition in the foreground of mathematics education. The addition of methods and formalisation are suggested to varying degrees based on the level of technical manipulation necessary.

As stated above, the focus of this article is on illustrating how intuition can help non-specialists gain a better grasp on existing mathematical concepts. In order to achieve this, Bouligand presents how a sequence of increasingly sophisticated mathematical concepts can be associated with familiar geometrical or physical concepts. For example, the concept of distance in mathematics can be associated naturally with a straight line, a taut wire or a ray of light [40, p. 132]. Sums and differences can be represented by juxtaposing line segments; products and quotients can be represented (Bouligand uses the word *concrétisées*) by the relationship between the area of a rectangle and the length of its sides. He emphasises the greater simplicity brought about through such associations:

Ces relations originelles sont associées dans le monde de figures, à des propriétés géométriques très simples [40, p. 133].

[In the world of figures, these original relationships are associated with very simple geometric properties].

In a similar way, the derivative can be associated with the slope of a curve and the integral in terms of the area under a curve. Given the focus of this section on the concept of intuition itself, we omit the details of the associations mentioned above as well as further examples given by Bouligand, including the concept of a function, Cartesian geometry and partial derivatives.

In addition to illustrating how mathematical concepts such as those mentioned above can be understood in terms of geometrical/physical concepts or images, Bouligand also expresses the usefulness of this approach for students in their practice of mathematics. A simple example provided is that of solving a cubic equation, $x^3 - 3x - 10 = 0$. This could, Bouligand comments, be approached by trying different values of x , however, the situation is made more 'palpable' by taking recourse to a graphical representation - that is sketching the equation in the Cartesian plane and seeing where the curve crosses the x axis so as to infer the roots [40, p. 133].

We have seen above that Bouligand is in favour of formulating mathematical concepts in terms of familiar geometrical/physical concepts and that he believes this will be of use for those learning mathematics. However, the concept of intuition is not explicitly defined in [40] but is implicitly understood to require a certain input in order to function, this input being in the form of mental associations with geometrical or physical concepts/objects. The concept of a necessary input for intuition and the nature of this input is perhaps most simply illustrated in the opening lines as follows:

Une des premières données de l'intuition est l'espace où sont situés et où se meuvent les corps [40, p. 132].

[One of the first inputs for intuition is the space in which bodies are located and move].

The concept of intuition and its mechanics are explored in greater depth when considering intuition in the context of mathematical discovery.

3.2.2 Intuition as a tool in the mathematical discovery process

In [46], Bouligand stresses the role of intuition as the essential tool in the process of mathematical discovery [46, p. 289]. He explores three aspects of this concept which he refers to as intuition, *intuition prolongée* (extended intuition) and *contre-intuition*

(counter-intuition). We will first introduce intuition and *intuition prolongée* through an example provided by Bouligand, namely the famous realisation by Henri Poincaré of the occurrence of non-Euclidean geometry in the theory of complex functions, specifically in the study of so-called Fuchsian functions. This example is used by Bouligand on more than one occasion and provides a well known context within which we may explore the idea of *intuition prolongée*.

Intuition and intuition prolongée

Fuchsian functions, which Poincaré had been studying in 1879 and had named after the German mathematician Lazarus Fuchs, had arisen in the context of differential equations. He had not yet succeeded in understanding them analytically and the key realisation came in the form of ‘unexpected geometrical inspiration’, borrowing Stillwell’s expression in [208, p. 493]. Poincaré later recounts his moment of realisation in [200, p. 8]:

...A ce moment, je quittai Caen, que j’habitais alors pour prendre part à une course géologique entreprise par l’Ecole des Mines. Les péripéties du voyage me firent oublier mes travaux mathématiques ; arrivés à Coutances, nous montâmes dans un omnibus pour je ne sais quelle promenade ; au moment où je mettais le pied sur le marche-pied, l’idée me vint, sans que rien dans mes pensées antérieures parût m’y avoir préparé, que les transformations dont j’avais fait usage pour définir les fonctions fuchsiennes étaient identiques à celles de la géométrie non euclidienne....

[..At that time, I left Caen, where I lived at the time, to take part in a geological excursion organised by the Ecole des Mines. The events of the journey made me forget my mathematical work; when we arrived in Coutances, we boarded an omnibus for I don’t know what ride. The moment I placed my foot on the step, the idea came to me, without anything in my previous thoughts seeming to have prepared me for it, that the transformations I had used to define Fuchsian functions were identical to those of non-Euclidean geometry...].

This realisation provided a familiar geometric reference - namely the non-Euclidean disk of Beltrami - enabling Poincaré to better explore and understand the new analytical concepts at hand. Bouligand comments that this analogy between non-Euclidean geometry and Fuchsian functions provided a ‘*support intuitif merveilleusement adapté à l’édification de sa mémorable théorie des groupes fuchsiens*’ [46, p. 290]. For a more

detailed and contextualised explanation of Poincaré's use of non-Euclidean geometry in the study of Fuchsian functions, we refer the reader to Jeremy Gray's account in [144, p.179].

Thus, *intuition prolongée* is in [46] described by Bouligand as a '*réédition spontanée de quelque intuition immédiate, mais sur un terrain beaucoup plus large*'. We infer here that the *intuition immédiate* is the association between the geometrical representation of Beltrami's disk and the theory of non-Euclidean geometry it represents. The *intuition prolongée* is the re-association of the geometrical framework of Beltrami's disk in a new context - that of Fuchsian functions. Bouligand goes on in [54], where intuition is considered from a subjective viewpoint, to attempt a definition of what he refers to above as *intuition immédiate* and of *intuition prolongée*:

Nous dirons maintenant que chaque connexion entre un groupe de réalités sensibles et un groupe de notions mathématiques est une intuition ; une connexion (très proche parente) établie entre un premier groupe de notions mathématiques et un second, sera pareillement une intuition prolongée [46, p. 290].

[We will now say that each connection between a group of tangible realities and a group of mathematical concepts is an intuition; a (very closely related) connection established between a first group of mathematical concepts and a second will similarly be an *intuition prolongée* (extended intuition)].

Therefore, intuition is the mental process of connecting or associating sensory experience with mathematical concepts; *intuition prolongée* is the mental process of connecting a first set of mathematical ideas with a second set. It is instructive to refer here to two further examples provided by Bouligand so as to clarify the definitions above. As an example of basic intuition provided in [46, p. 290], Bouligand considers a moving fluid with a steady flow (i.e. at any given point, all molecules passing through that point do so with the same velocity). If we consider transformations in the configuration of the fluid from a moment in time to a later moment in time then we note, with Bouligand, that these transformations are continuous and the set of these transformations is closed under composition. Therefore, Bouligand states that this situation corresponds to a continuous group.¹ The intuition here is the connection between the mental picture of a moving fluid with the mathematical concept of a

¹We deduce that, at the time of writing, that Bouligand's use of the word 'group' may not be equivalent to the modern use of this word. Specifically, not all of the four properties are explicitly of the operation in question are included in the definition, only the property of closure under composition.

continuous group. A further example, this time of *intuition prolongée*, is the use of higher dimensional geometries as a means of working with certain algebraic systems, such as systems of linear equations of n unknowns. Bouligand comments,

On a élaboré la géométrie à un nombre quelconque de dimensions et cette construction s'est montrée extrêmement féconde, en donnant un support imagé à des généralisations que leur aridité eût rendues impraticables [44, p. 261].

[Geometry of an arbitrary number of dimensions was developed and this construction proved itself to be extremely productive by providing a visual backdrop for generalisations whose aridity had made them unusable].

Here the *intuition prolongée* is the connection made between, for example, a system of linear equations with n variables and n dimensional space - a connection between two different mathematical constructs, which mirrors the connection that can be made between, for example, a system of linear equations in at most 3 unknowns with 3-dimensional Cartesian space.

Contre-intuition

In [46], the concept of *contre-intuition* is presented, which represents the '*possibilité, pour l'intuition, de rectifier, elle-même ses jugements hâtifs*' [54, p. 193]. For example, it may seem intuitive, based on one's familiar geometrical associations, to adopt the following reasoning: every continuous function represents a curve, every curve admits a tangent, therefore every continuous function has a derivative. To test the truth of this reasoning, one may appeal to *contre-intuition*. To see why the assumption that every curve admits a tangent is false, Bouligand invites the reader to imagine a jagged section of the coastline in Brittany where, no matter how powerful a microscope one uses, the contour always appears as jagged and thus would seem never to admit a tangent. He then outlines how this '*donnée intuitive*' can inspire the construction of a mathematical object with a similar property. In particular, Bouligand specifies the following iterative process: take a line AB and separate it into thirds. On the middle third, construct a square. If we repeat this process in a specific way for each of the new line segments, Bouligand shows how we can use this construction to deduce the existence of a continuous function without a derivative, with the characteristic of jaggedness as in the Breton coastline to which we referred above. While the details are not central to the question at hand, this example illustrates well the concept of *contre-intuition* as a mechanism whereby intuition, in Bouligand's sense of the word, is kept in check. As such, the mathematical discovery seems to involve for Bouligand an

interplay between intuition, *intuition prolongée* and *contre-intuition*.

Finally, to reinforce the importance attributed by Bouligand to intuition as the central tool in the mathematical discovery process, we quote this view expressed in [54] from the subjective point of view of what he refers to as the *espace de réalités mathématiques et des intuitions*:

Dans ce schème, la découverte apparaît comme une liaison brusquement établie, par des intuitions nouvelles, entre des domaines primitivement distants dans l'espace des réalités mathématiques et des intuitions [54, p. 194].

[In this framework, discovery appears as an abruptly established connection, established through new intuitions, between primitively distant domains and the space of mathematical realities and intuitions].

Therefore, not only does Bouligand consider intuition as central for discovery in mathematics but goes one step further by actually defining discovery in terms of intuition.

3.2.3 Intuition versus formalisation and methods

While in [44], Bouligand explores the interplay and opposition between algebraic/algorithmic methods and intuition, in [54], he explores the roles of intuition and formalisation and does so from the point of view of the mental processes of the individual mathematician. These two articles have been grouped under the same heading as certain similar ideas appear from a different angles.

Intuition versus algebraic and algorithmic methods

Although in [44] Bouligand raises diverse points including interesting asides which we will return to later, we will highlight two main messages here. First, Bouligand seems to express a negative view on what he saw as an excessive reliance on methods or symbolism² which did not allow intuition, in his sense of the word, to come into play. The second point we will explore is that, even when mathematics develops predominantly in the form of methods, Bouligand gives examples to show that intuition nevertheless can and does have a profound influence, through geometrical or physical associations, on the formal side of the area of mathematics in question.

²In [44], Bouligand refers variously to *éléments de nature combinatoire*, *symbolisme mathématique*, *algèbre* and *calcul*.

Perhaps the example on which Bouligand places the most emphasis for the first of these main points is to be found in his comments regarding Cartesian geometry:

...il y a eu sans interruption contact et rivalité entre intuitifs et partisans du calcul. Descartes, par sa méthode analytique, avait assuré aux uns une victoire éclatante; mais il s'en trouva d'autres pour récriminer contre le recours aux axes, et les intermédiaires trop longs qui masquaient les véritables liens entre les prémisses et la conclusion. D'où une scission entre la géométrie élémentaire, ou tout était clair, et ce qu'on appelait par opposition, encore tout récemment, les mathématiques supérieures: d'une manière plus ou moins consciente, on désignait par là une sorte de champ touffu, où l'on marchait souvent à l'aveugle [44, p. 258].

[...there has been constant contact and rivalry between those in favour of intuition and proponents of calculation. Descartes, with his analytical method, had ensured a resounding victory for the former but there were others who recriminated against the use of axes, and the excessively long intermediaries that masked the true links between premises and conclusion. This led to a split between elementary geometry, where everything was clear, and what until quite recently was called, by contrast, higher mathematics: more or less consciously, this was used to designate a sort of confused field where we often walked blindly].

Although the views expressed above regarding Cartesian geometry are not in themselves unusual or original for the time, implicit here are some new insights into Bouligand's views of intuition. First, he sees taking recourse to axes to solve certain problems in geometry as being opposed to his notion of intuition. We will see in the section below on geometry and causality that Bouligand sees this approach as being extrinsic rather intrinsic, dealing directly with the objects being studied. We see in the above therefore that an extrinsic approach is in conflict with Bouligand's concept of intuition. This is a new dimension, even if expressed implicitly: we have seen that geometrical and physical associations are at the heart of his interpretation of intuition and clearly, the use of axes in Cartesian geometry *does* have a geometrical representation. As such, for Bouligand, the choice of geometrical/physical associations with which intuition can work matters and that dealing with mathematical objects directly, or in an intrinsic manner, is one factor for determining this choice. In addition, seeing the connection between the initial definitions/conditions and the conclusions of a mathematical argument appear above as being connected to intuition. Again this is a theme which will be revisited in the next section. Finally, in the above, intuition is related to a broader area of mathematics (such as elementary geometry) having the

quality of being somehow *clear* to the mathematician or student.

Bouligand's second main message which we highlight is the influence of intuition on the development areas of mathematics where methods appear to be predominant. An example to which Bouligand attributes particular importance in [44] relates to imaginary numbers. He recalls the context in which imaginary numbers were first introduced, namely that of finding real solutions to polynomial equations by radicals. He considers the example of the equation $x^3 = 15x + 4$ which, when applying the formula for solving cubic equations, leads to the consideration of the square root of -121 . This is an intermediate step in the application of the formula and we subsequently obtain the real roots of the polynomial. Bouligand comments that this sort of approach led to working with complex numbers of the form $a + bi$ and applying familiar rules of algebra when manipulating them. Bouligand comments that this led to valuable mathematical results but that, without a concrete representation, the introduction of imaginary numbers appeared mysterious and based on uncertain foundations:

...on était donc assuré de suivre une route propice aux applications; mais on s'engageait sur un terrain où semblait compromise cette impression de sécurité qui régnait ailleurs. L'imaginaire apparut sous un jour mystérieux, tant qu'on ne fut pas à même d'en donner une représentation concrète... [44, p.259]

[...we were thus assured of following a path that was conducive to applications but we were entering a field where the impression of security that prevailed elsewhere seemed to be compromised. The imaginary appeared in a mysterious light until we were able to give it a concrete representation...]

Bouligand emphasises that the geometrical representation of complex numbers in the plane, today known as the Argand diagram, allowed the mathematician to see why the formal algebraic rules already adopted, for example in the addition and multiplication of complex numbers, were indeed justified.

Bouligand raises a further point through this example of the imaginary numbers which, although not so directly related to the topic of intuition, is part of a theme which later becomes central in his epistemological views about mathematics. Specifically, he goes on to present the following example of a key difference between real numbers and complex numbers, seen through the lens of the geometrical representations of these sets of numbers - the real line and the complex plane respectively. The key difference

in question is the property of the real line whereby a point on the line can be said to be situated between two other given points. This is, Bouligand notes, related to the notion of inequalities, which is not present in the same way in the complex plane. He comments that the greater generality of the complex plane, in which this concept of inequalities is not present, allows for the validity of the fundamental theorem of algebra, which states that a polynomial of degree n with complex coefficients has n roots. He comments:

...dans le champ réel, le nombre des racines était déterminé par des conditions d'inégalité ; on conçoit que la disparition des inégalités, qui accompagne le passage au domaine complexe, ait pour conséquence naturelle de faire disparaître les effets variés qu'elles engendraient dans le champ réel, pour nous conduire d'un Algèbre disloquée à une Algèbre permanente. Autrement dit, on met en évidence le rôle causal des inégalités [44, p. 260].

[In the field of the reals, the number of roots was determined by conditions relating to inequalities. It is conceivable that the disappearance of inequalities, which accompanies the transition to the complex field, has the natural consequence of eliminating the varied effects that they would generate in the real field, leading us from a dislocated Algebra to a permanent Algebra. In other words, we highlight the causal role of inequalities].

It is of interest to draw attention to the use of the expression *causal role* in the above. This idea will be explored in detail in the following subsections but we note here that it seems to be the first occurrence of the word *causal* in Bouligand's work.

The following extract summarises best Bouligand's views regarding the roles of intuition and symbolism in mathematics:

En résumé, il y a en nous-même deux aptitudes bien distinctes, l'une à prélever sur le concret, par intuition, des éléments susceptibles d'acquiescer la forme mathématique, l'autre à combiner ces éléments. De ces tendances, l'intuition détient une priorité marquée, l'autre ne s'exerce qu'après coup pour organiser le terrain conquis, pour grouper les résultats et les faire rentrer dans le cadre du calcul...Par l'ordre qu'elle apporte, l'Algèbre régénère sans cesse l'intuition et facilite la conquête de vérités nouvelles... [44, p. 291].

[To sum up, there are two quite distinct aptitudes within us, one for intuitively extracting elements from the concrete that are likely to acquire mathematical form, and the other for combining these elements. Of these

tendencies, intuition has a marked priority, while the other is exercised only after the fact, to organise the ground we have conquered, to group the results together and make them fit into the framework of calculation... Through the order it brings, Algebra constantly regenerates intuition and facilitates the conquest of new truths].

Thus, while Bouligand seems to attribute greater importance to intuition as the primary element in creating new mathematics, he sees intuition and symbolism as two necessary and complementary aspects of mathematical practice.

Intuition and formalism in a subjective setting

In [46], the opposition and interplay between the formal and intuitive aspects of mathematics are explored in the context of what Bouligand refers to as the *espace des réalités mathématiques et des intuitions* [54, p. 194]. In this imagined mental space portrayed (but not exemplified) by Bouligand, mathematical concepts already acquired are considered as points in the space and mathematical results or theorems are represented as paths connecting these points. Acquired concepts are also connected in this space to intuitions (i.e. associated with concepts derived directly from sensory perception) and may be connected one to another by *intuitions prolongées*. The paths connecting different points in the space may be made up of a combination of logical deductions, intuitions and *intuitions prolongées*. In addition, Bouligand discusses a notion of distance between concepts in this space in terms of the subjective impression of the length of the paths connecting them. He emphasises a path perceived as short from an intuitive point of view may well involve lengthy and complex logical deductions. On the other hand, Bouligand expresses that certain mathematical results, such as fundamental theorems in the theory of numbers and the theory of functions, are often lengthy from both an intuitive and formal point of view.

Bouligand's views on the interplay between intuition and the formal aspect of mathematics are also brought out here in relation to mathematical proof. He states that,

...démontrer consiste à réduire un système comportant des liaisons intuitives à un autre ne mettant plus en jeu que des liaisons logiques... [proof consists of reducing a system comprising intuitive connections to another which brings into play only logical connections] [54, p. 194].

He adds that this reduction to a system of logical connections is necessary as intuitions only provide an incomplete knowledge of the *espace des réalités mathématiques* - the extent to which a judgement based on intuition is valid, Bouligand comments, is not

clear in advance and significant work involving both intuition and counter-intuition is often necessary before checking that these ideas satisfy the requirements of logic.

The following extract illustrates the same principle of removing intuitive connections but this time with regard to the formalisation and organisation of mathematical knowledge in field of geometry. In addition, Bouligand expresses here his views on the way in which this should be done in relation to both axiomatization and intuition:

Pour la synthèse de tout ensemble de faits géométriques, il faudra, non seulement, opérer la suppression de toute liaison intuitive, mais encore chercher, autant que possible, à simplifier la structure du système des liaisons logiques mettant en connexion les faits du système : l'idée se présente ainsi naturellement d'ordonner ces liaisons, de remonter à leurs racines, en un mot d'entreprendre une élaboration axiomatique : au cours de celle-ci, tenant compte de la faculté de l'intuition de s'adapter à toutes les formes de la réalité mathématique, on a grand avantage à choisir, si possible, des liaisons logiques susceptibles de se superposer à des liaisons intuitives. On a là un principe directeur pour orienter le goût, indispensable en mathématique, autant, sinon plus encore qu'en tout autre domaine [54, p. 198].

[For the synthesis of any set of geometrical facts, it will be necessary not only to eliminate all intuitive connections, but also to try, as far as possible, to simplify the structure of the system of logical connections between the facts of the system. In this process, taking into account the ability of intuition to adapt to all forms of mathematical reality, it is highly advantageous to choose, if possible, logical connections that can be superimposed on intuitive connections. This provides a guiding principle to orient our chosen style, which is as essential in mathematics as in any other field, if not more so].

Therefore, the process of formalisation should, for Bouligand, aim to simplify as far as possible the network of logical connections. The axiomatic approach is seen as the means of achieving the desired formalisation and organisation of mathematical ideas. We note that he encourages adopting an axiomatic approach in such a way that the chosen formalisation can be easily married with intuitive associations. An example of how this might be achieved is not provided at this stage. It is not made explicitly clear whether the *associations intuitives* above correspond to intuition (in Bouligand's sense of the word) or *intuitions prolongée* or perhaps both.

3.2.4 Further types of intuition

In his contribution to the 1935 book *Sur l'évolution des sciences physiques et mathématiques* [103], which collates the contributions of several speakers at a series of conferences co-organised by Bouligand at the *Faculté des Sciences de Poitiers*, Bouligand also briefly identifies another type of intuition, which he calls *intuition finitiste* [103, p. 135], although he does not explore this concept here. This type of intuition is based on the ideas first presented in his 1928 article entitled *Le finitisme et son efficacité dans la recherche mathématique* [42], although this article does not refer explicitly to *intuition finitiste*. Finitism, for Bouligand, seems to refer to an approach to infinite problems by looking at finite approximations. In [42], he discusses the advantages and limitations of such an approach, its connection with his concept of intuition and with his idea of Cantorian form in geometry, which will be explored in the following section.

The following is a slight rewording of an example given in [103] and illustrates the idea of a finitist approach and its connection with Bouligand's concept of intuition. Consider the following theorem: the surface S generated by the half-tangents to a skew curve C (one which does not lie in only one plane) is developable. Bouligand states that this well-known result is made intuitive if we consider, instead of C , a polygonal line L_c made by joining certain points lying on C . If we extend the lines making up L_c in a specific way, we can obtain a surface with polyhedral faces of infinite area. If we define L_c with points of C that are increasingly close together, thus approximating C with greater and greater precision, then the polyhedral surface mentioned can be seen to tend towards S . The fact that each polyhedral surface in the sequence is clearly developable (because it is made up of segments of the plane), for Bouligand, makes the theorem intuitive.

A question that may come to mind here is to understand the difference between Bouligand's *intuition prolongée* and what he later referred to as *intuition finitiste*. Considering the example above and the definition of *intuition prolongée*, could we not consider *intuition finitiste* as a special case of *intuition prolongée*? Bouligand does not address this question but does, in [103], seem to relate the two when commenting, *à côté de l'intuition prolongée, il y aurait beaucoup à dire de l'intuition finitiste...* [103, p. 133].

3.2.5 Bouligand's stance on intuition in relation to the debate on space and non-Euclidean geometry?

A reader familiar with the debates taking place from the end of the 19th century regarding the nature of space, in relation to the rise of non-Euclidean geometries, may wonder what Bouligand's position was in this regard, in the context of his ideas on intuition. We will see that Bouligand did not address these questions and we will suggest that this is due to the motivation behind his epistemological work, which was centered primarily around mathematical methods.

As Nabonnand comments in [188, p. 220] at the beginning of the 19th century, the concept of space was taken as a given. With Nabonnand, we cite Federigo Enriques (1871-1946) - an eminent Italian mathematician who also contributed his ideas to the philosophy and history of mathematics - who stated that at the end of the 19th century, the growing acceptance of non-Euclidean geometries led to,

'...cette idée nouvelle et remarquable que l'espace physique pourrait être différent de l'image que nous en fournit notre intuition habituelle' [185].

[...this new and remarkable idea that physical space could be different to the image provided by our usual intuition.]

In this context, explains Nabonnand, mathematicians became directly involved in philosophical debates regarding the axioms of geometry and the status and origins of concepts of space and geometry. Such debates included considerations of Emmanuel Kant's ideas - and of those who later followed and interpreted his ideas - for example regarding his concept of an a priori intuition of space³. In France, Henri Poincaré was centrally involved in these debates and, as noted by Nabonnand, he saw the axioms of geometry not in terms of a priori intuition, or as an experimental fact, but simply as a matter of convention. He considered that it is a matter of choosing the most convenient convention for representing the physical world. In [201, p. 67], Poincaré states,

la géométrie euclidienne est et restera la plus commode...Parce qu'elle est la plus simple; et elle n'est pas telle seulement par suite de nos habitudes de l'esprit ou de je ne sais quelle intuition directe que nous aurions de l'espace euclidien; elle est la plus simple en soi de même qu'un polynôme du premier degré est plus simple qu'un polynôme du second degré...

³For more details, we refer the reader to [151]

[Euclidean geometry is and will remain the most convenient...Because it is the most simple; and it is the most simple not only as a result of the habits of our mind or some sort of direct intuition we may have of Euclidean space; it is the most simple in itself, in the same way as a polynomial of the first degree is simpler than a polynomial of the second degree...]

As we see from the above discussion - intuition, and in particular intuition of space, was a key ingredient in the philosophical debates on non-Euclidean geometries and the nature of space. Did Bouligand take a stance in such debates in the context of his discussion of intuition? Despite the fact that he was almost certainly aware of these debates, it would seem that Bouligand did not explore these questions in his published work and did not make active efforts to put forward his own stance. As we have seen in this section, for Bouligand, intuition is considered first and foremost as an important tool in the process of mathematical discovery and progress and is also considered with reference to the teaching of mathematics. In Bouligand's later article of 1944 [97] - which includes an account of some of his ideas on intuition introduced above - he comments regarding different systems of geometry within mathematics,

La pluralité des systèmes de géométrie,...atteste qu'il n'y a pas en mathématiques de vérités absolues. Ce qu'on y peut trouver, ce sont des propositions qui sont vraies, sous la condition d'admettre tel ou tel faisceaux de prémisses, exempt de contradiction... [97, p. 98].

[The plurality of systems of geometry...shows that there are no absolute truths in mathematics. What we can find in mathematics are proposition which are true if we admit a certain collection of assumptions which is free from contradiction...]

Although the above comment is not made explicitly with reference to Euclidean versus non-Euclidean geometries and questions regarding the nature of space, this would seem to be compatible with Poincaré's view that no system of geometry is inherently more true than any other. However, Bouligand does not express any position here on what the best system of geometry may be for representing physical space, or on the role of intuition in this question. In particular he does not assert or imply Poincaré's view regarding Euclidean geometry being the most convenient system for representing space, nor does he comment on intuition in relation to the nature of space.

Bouligand's main focus on mathematical practice in the context of his ideas on intuition reflects a broader theme - that is, a focus on mathematical methods and progress within mathematics as a key motivation behind his works of a more philosophical nature. This theme will be brought out more fully later on in our account. We note

at this stage that this point is made explicit in Bouligand's 1961 account of his own works [101], in which this aspect of his work is included under the heading *principes d'enseignement, méthodes et heuristique* [principles of teaching, methods and heuristics] [101, p. 54].

Chapter 4

Bouligand's theory of Direct Infinitesimal Geometry: motivations, influences and mathematical formulation

Following his work on the generalised Dirichlet problem and potential theory, Bouligand began to focus as of 1928 on constructing and promoting an approach to differential geometry which he referred to, as of 1932, as *géométrie infinitésimale directe* [direct infinitesimal geometry]. This theory was inspired to a great extent by his encounter with the generalised Dirichlet problem and the methods he employed therein. Bouligand's theory will appear as a central element in this account as it is tied up intimately with his ideas in the epistemology of mathematics. Indeed, his theory can be viewed as the implementation of his reflections on certain faults, as he perceives them, in the differential geometry of the time and of the principles and methods he puts forward to attempt to remedy these faults. These reflections are published for example in his 1927 paper [41] and his article of 1930 [51], which will be introduced below. This area of Bouligand's work represents a conscious attempt at bringing his concept of causality into differential geometry and at implementing direct methods - both of these concepts will be explored in greater depth in the following chapter.

In addition to being a central element of the present account, Bouligand himself viewed this theory as representing a major part of his contribution to mathematics. In his 1937 note on his own mathematical works [95], three out of eight sections are related to direct infinitesimal geometry (*recherches de géométrie des ensembles suggérées par la théorie du potentiel; systématisation: la création de méthodes directes en géométrie infinitésimale; applications de la géométrie infinitésimale directe aux équations*

différentielles, ordinaires ou partielles). In addition, in his 1961 note on the works of his entire career, one of three main sections, '*géométrie différentielle et thèmes divers*' deals mainly with concepts belonging to this theory. Direct infinitesimal geometry was identified by other mathematicians as one of the main defining works of Bouligand. For example, Gustave Choquet comments:

'Le nom de Bouligand restera attaché à la géométrie infinitésimale directe, c'est-à-dire à l'étude des propriétés différentielles des ensembles quelconques de l'espace par des méthodes intrinsèques. C'est la théorie du potentiel et, plus précisément, l'étude de la régularité des points frontières dans le problème de Dirichlet qui le conduisit à créer deux outils importants, le contingent et le paratingent, pour aborder un cadre plus général que le cadre des fonctions différentiables...' [202, p. 38]

[Bouligand's name will remain attached to direct infinitesimal geometry, that is the study of differential properties of arbitrary sets in space by means of intrinsic methods. It is potential theory and more precisely the study of the regularity of boundary points in the Dirichlet problem which led him to create two important tools, the contingent and the paratingent, for dealing with a more general setting than that of differentiable functions...]

We note that Gustave Choquet (1915-2006) was a French mathematician who contributed to potential theory, functional analysis, topology and measure theory. He also identified direct infinitesimal geometry as one of his own areas of mathematical research, particularly between 1941 and 1946 [120, p. 158,160].

The ideas in this chapter will be split into five main sections as follows:

1. We will first consider Bouligand's epistemological reflections regarding geometry between 1927 and 1930 in which he attempts an initial presentation of the principles and main tools and methods of his approach to differential geometry. Here we will highlight the influence of Bouligand's prior work on the Dirichlet problem.
2. To place Bouligand's theory in context, we will also review the ideas of certain mathematicians which had a direct influence on the development of Bouligand's theory, or which are closely related and cited by Bouligand.
3. We will illustrate the contents of Bouligand's theory of direct infinitesimal geometry by exploring some key concepts and examples contained in his textbook

Introduction à la géométrie infinitésimale directe [Introduction to direct infinitesimal geometry], which represents the first comprehensive presentation of his theory. In this part, we will also connect the main ideas of Bouligand's theory with the principles brought to light in 1.

4. We will briefly introduce efforts to demonstrate applications of his theory.
5. Finally, we will explore the close connection between direct infinitesimal geometry and Bouligand's role as a teacher of mathematics. Along with these pedagogical aspects, this section will deal with Bouligand's efforts to circulate his ideas through various publications, contributions to conferences and through the orientation of his students in his own area of research.

We highlight that in the second section indicated above, we will comment on a particularly important broader trend in mathematics in which Bouligand's theory has to be placed, namely that of the uptake of concepts of set theory in France. We will see that Bouligand himself viewed his approach to differential geometry as an attempt at recreating the success, as he perceived it, of the application of sets in the theory of functions - arguably the first area of mathematics in which set theory penetrated in France. This success is seen, from Bouligand's point of view, in terms of his philosophical views regarding regarding causality in the context of geometry.

4.1 On the need for a new approach to infinitesimal geometry

In his 1927 article *Sur l'évolution des idées géométriques* [41] and his 1930 publication *Sur quelques points de méthodologie géométrique* [51], Bouligand outlines the need, from his point of view, for a new approach in the area of geometry. He presents both what he sees as the faults prevailing in infinitesimal geometry at the time and begins to outline new tools and methods to remedy these faults. It is in this context that he first brings out his ideas about causality in geometry - more precisely, what he refers to as causal relations between a mathematical result and that premises upon which it is based, and the concept of a domain of causality - these concepts will be introduced below. In later works, these concepts are considered by Bouligand in the more general settings of mathematical and physical theories, including mathematical proofs. Bouligand's ideas in [41] and [51] regarding the form and methods of geometry which he considers desirable combine a number of interrelated concepts, which we will list as follows:

- **Domains of causality** - an appropriate domain in which a geometrical concept or fact should in Bouligand's view be studied

- **Achieving maximum generality or removing unnecessary hypotheses**
- **Preferring a *direct* approach** over one whose methods are further dissociated with the concepts or objects being studied.

Conversely, the absence of the above, for Bouligand, corresponds as we will see to an undesirable approach. We will consider these points below one by one for maximum clarity and completeness but we emphasise that these are merely different angles from which we can understand the same ideas, rather than being separate ingredients of a theory or definition. Bouligand does not necessarily separate these ideas in this way in his own work. First we briefly introduce the two articles from which we construct our account in this section:

- **Sur l'évolution des idées géométriques (1927)** [41] - focusing in this instance on the three dimensional Euclidean setting, Bouligand outlines his views regarding the need for a new approach to differential geometry - one which formulates its problems and methods in what he refers to as *la forme Cantorienne* (Cantorian form). Although the term Cantorian form is later abandoned by Bouligand, this article outlines the start of a programme of research in geometry which later culminates in his theory of *géométrie infinitésimale directe* and also expresses in seed form certain of his main epistemological ideas about mathematics upon which he later elaborated.
- **Sur quelques points de méthodologie géométrique (1930)** [51] - this article can be seen as an elaboration of [41] and is Bouligand's first publication in which concepts relating to causality in geometry appear explicitly. In particular, Bouligand explores for the first time the concept of a domain of causality. This article deals extensively with mathematical tools and methods which aim to enable him to put into to practice the principles he had laid out - tools and methods which are later essential ingredients in his *géométrie infinitésimale directe*. The article appears in two parts - the first part being focussed more on principles and the second being focussed more on mathematical tools.

Before investigating Bouligand's ideas according to the three themes outlined above, we first give a broad summary of the main ideas. Bouligand's main objective here is to take an approach which allows the mathematician and the student to better understand geometry. In [51, p. 43], he highlights the possibility for an '*effort systématique POUR MIEUX COMPRENDRE LA GEOMETRIE*' [a systematic effort aimed at BETTER UNDERSTANDING GEOMETRY]. Put differently, he promotes adopting methods which allow us to appreciate the '*simplicité souvent cachée*' [the often hidden

simplicity] [41, p. 589] of a geometrical result or theory. Essential in achieving this, for Bouligand, is introducing a way of considering geometrical concepts in their proper setting and at the correct level of generality. This is achieved through the notion of a domain of causality which, as we will see below, is closely connected to the concept of a mathematical group. It is also achieved by seeking to avoid unnecessary hypotheses which can be seen as hiding the true cause of a given fact in geometry. Bouligand's notion of *prolongement fonctionnel* will also be presented below as a technique for departing from an overly restrictive set of initial hypotheses to a more general setting, although we note that this technique is not emphasised in his subsequent work in the context of his theory of direct infinitesimal geometry. Finally, Bouligand sees the use methods of a more algorithmic nature, involving lengthy calculations, or which are dissociated conceptually from the problem or object being studied as another factor that obscures the link between the initial hypotheses and the proven result.

4.1.1 The need to eliminate unnecessary hypotheses and achieve maximum generality

In [41], Bouligand raises certain objections to the analytical geometry of Descartes, notably that it leads to introducing what he considers as unnecessary assumptions present only to ensure the applicability of the Cartesian methods at hand. This would include for example the assumption that lines or surfaces being studied may be represented by functions which are differentiable up to a certain order. Bouligand comments in this regard,

'La commodité de ce point de vue le fit adopter universellement et bientôt, tout exposé géométrique commença par un préambule de cette nature : soit une ligne, ou une surface, définie par des formules dérivables jusqu'à un ordre qui sera précisé ultérieurement ; les conditions déterminantes qui fixaient cet ordre étaient de pure commodité et visaient à une application des méthodes cartésiennes' [41, p. 586-587].

[The convenience of this point of view led to its universal adoption and soon any reasoned argument in geometry would begin with a preamble of the following kind: consider a line or a surface defined by formulae which are differentiable up to a certain order which will be specified later. The conditions which determined this order were chosen purely according to convenience and aimed at enabling the application of Cartesian methods].

The effect of relying on assumptions introduced for convenience, in Bouligand's view,

is to obscure the connection between the essential principles upon which a result is based and the geometrical result itself. This is expressed well in [51, p. 39]:

*‘...Le résultat obtenu n’est donc plus le produit causale des hypothèses initiales.’
[...The result obtained is therefore no longer the causal consequence of the initial hypotheses].*

An example given by Bouligand of this situation relates to the work of Henri Lebesgue, in his doctoral thesis of 1902 [165]. It is useful to consider the following opening lines of Lebesgue’s thesis, which illustrate both the content of the thesis and the relevance of this work in relation to the present discussion about eliminating unnecessary hypotheses and achieving greater generality.

‘Dans ce travail j’essaie de donner des définitions aussi générales et précises que possible de quelques uns des nombres que l’on considère en Analyse: intégrale définie, longueur d’une courbe, aire d’une surface’ [165, p. 232]

[In this work I attempt to give the most general and precise definitions possible for certain numbers considered in Analysis: definite integral, length of a curve, area of a surface].

A section of Lebesgue’s thesis deals with developable surfaces. In this context, he comments that one common understanding of developable surfaces at the time was that of a surface which can be decomposed into cones, cylinders and surfaces generated by the tangents of a non-planar curve [165, p. 330]. He goes on, however, to show that these are not necessary or sufficient conditions for developability. Lebesgue provides precise necessary and sufficient geometric conditions for each of these three types of surface (conical, cylindrical and those generated by the tangents of a planar curve) to be developable. With regard to the common assumption prior to Lebesgue’s results, Bouligand comments,

En réalité, on n’a pas ainsi la solution complète, mais seulement une solution restreinte qui se sépare grâce à l’hypothèse de commodité de la continuité des premières dérivées [41, p. 587].

[In reality, we do not have in this way a complete solution but only a restricted solution that becomes apparent thanks to the hypothesis introduced for convenience that the first derivatives are continuous].

In other words, Bouligand’s view is that the assumption of continuous first derivatives - adopted to facilitate the application of Cartesian methods - led to the false impression

that a complete solution to the problem at hand was available. As such, Bouligand emphasises, in relation to this example, that the question of assumptions introduced for convenience are not merely a matter of aesthetics:

Le recours à des restrictions de commodité peut bouleverser complètement l'aspect d'un problème.

[Taking recourse to restrictions introduced for convenience can completely change the face of the problem].

The concept of a domain of causality

Unnecessary assumptions and natural domains of existence. We have seen above that Bouligand views the reliance on extra assumptions introduced to facilitate the application of methods as an undesirable feature of geometry. A lack of generality in a geometrical theory or argument, for Bouligand, is closely connected with the absence of what he refers to as a causal structure - an idea that will be explored below. The other side to the same coin is that he connects achieving the right level of generality by eliminating extra assumptions with reinstating a causal structure. The main concept through which Bouligand expresses these ideas is a *domain of causality*, although he initially uses the term *natural domain of existence* in [41], as we will see below. The connection between eliminating unnecessary assumptions and studying geometrical concepts or objects in an appropriate setting is perhaps most explicitly present in the following:

'Si l'on veut rompre avec les restrictions de commodité, c'est-à-dire étudier les faits géométriques dans leur domaine naturel d'existence...moyennant un minimum d'hypothèses, on se rallie à une tendance nouvelle, fréquente dans les travaux récents, et en faveur de laquelle Georges Cantor semble avoir opté l'un des premiers, en étudiant précisément la définition qu'il convient de donner d'une courbe' [41, p. 587].

[If we want to break away from restrictions introduced for convenience, that is study geometrical facts in their natural domain of existence...by means of a minimum set of hypotheses, we join a new trend, frequent in recent works and in favour of which Georg Cantor seems to be one of the first to have opted by studying the precise definition that should be given for a curve].

A discussion of the methods presented by Bouligand to achieve the ideals he lays out here will be presented later in this chapter, including this ‘new trend’ - which we interpret here as referring to an approach to differential geometry based on concepts originating in Cantor’s work on set theory (including concepts which we would today identify with the area of topology). The term *domaine naturel d’existence* [natural domain of existence], Bouligand comments, originates in an article of Pavel Urysohn [212] published posthumously in 1925. Pavel Urysohn (1898-1924) was a Soviet mathematician who made contributions to dimension theory and topology. Urysohn’s work here focusses on introducing in the context of topology certain familiar geometrical notions (such as lines and surfaces) in a purely geometrical manner, also focussing on intrinsic definitions. Urysohn also concentrates here on the possibility of generalising as far as possible such notions. When justifying the setting in which he chooses to operate, namely compact metric spaces and the corresponding assumptions made about these spaces, he comments:

‘...toute nouvelle restriction imposée à l’espace ne simplifierait d’aucune manière ni les énoncés ni les démonstrations: elle resterait tout simplement sans emploi. Or l’introduction d’une restriction qui n’intervient nulle part me semble assez mal motivée’ Urysohn adds on this point, *‘C’est là, il me semble, un fait général: toute théorie mathématique possède un domaine naturel d’existence, c. à d. une classe d’individus auxquels ses résultats s’étendent d’eux-mêmes tandis que toute généralisation ultérieure n’est que partielle et demande des recherches nouvelles et presque toujours bien compliquées’* [212, p. 41].

[‘...any new restriction imposed on the space would in no way simplify the propositions nor the proofs: it would quite simply be without use. The introduction of a restriction which plays no role seems quite ill justified...It is therein, it seems, that we find a general fact: any mathematical theory has a natural domain of existence, that is a class of individual objects to which its results extend for which any further generalisation is only partial and requires new and almost always highly complicated research’].

Bouligand’s comment regarding a need to study concepts in geometry in their natural domain of existence was brought up in [41] in the context of his views on an excessive reliance on extra assumptions in the analytical geometry of Descartes and in vector geometry. The following illustrates this view in relation to the teaching of vector geometry:

La géométrie vectorielle, telle que qu’on l’enseigne aujourd’hui, a conservé cet opportunisme. Qu’il s’agisse d’un point ou d’un vecteur, fonction géométrique

d'un ou deux paramètres, on se hâte de supposer à cette fonction des propriétés de dérivabilité, afin d'écartier, pour la suite, certaines difficultés...la géométrie vectorielle reste cartésienne dans sa manière de poser les problèmes ; elle les délimite, artificiellement parfois, pour permettre à un certain mécanisme opératoire de s'exercer [41, p. 587].

[Vector geometry as it is taught today maintains the same opportunistic quality. Whether it be a point or a vector, a geometric function of one or two parameters, we hasten to assume that this function has properties of differentiability in order to clear the way of later difficulties...vector geometry remains Cartesian in its manner of posing problems; it delimits the problem sometimes artificially so as to allow for a certain operational mechanism to be applied].

Bouligand's expression regarding the artificial delimitation of the problem at hand in vector geometry highlights the link between assumptions introduced for convenience with that of failing to consider the problem in its appropriate domain.

Groups, structure, domains of causality. Next, we note that, fundamental to Bouligand's ideas with regard to finding the appropriate domain mentioned above is the concept of a mathematical group. In [41], he comments regarding the new principles and methods in differential geometry which he is promoting,

Tout d'abord la construction systématique dont nous avons montré l'opportunité sera dominée par des considérations de groupes, qui en fixeront la structure dans ses grandes lignes. Chaque proposition sera classé d'après son degré de généralité, ou ce qui revient au même, suivant son groupe d'invariance [41, p. 588]. He adds that 'L'intérêt de cette classification est de constituer par elle-même un programme de recherches.'

[First of all the systematic construction for which we have illustrated the opportunity, will be dominated by considerations relating to groups which will fix the structure in its broad outlines. Each proposition will be classified according to its level of generality, or equivalently, according to its group of invariance...The interest in this classification is to constitute a programme of research in its own right].

Again, here we see the connection for Bouligand between the structure of a geometrical theory and that of the concept of a natural or appropriate domain in which we consider a given problem or concept - a setting expressed in terms of invariance under groups of

transformations. An example Bouligand provides in [41] of the classification of a mathematical proposition according to its level of generality and its group of invariance is as follows. Consider the theorem stating that, given a simple closed curve in the plane, this curve separates the plane into two distinct regions. This Bouligand considers as a topological statement, as it is invariant under the group of continuous bijections¹, i.e. we may apply any continuous bijection to the simple closed curve in question and the result will remain true. Bouligand also considers the theorem stating that a rectifiable curve has a tangent almost everywhere (by almost everywhere, we mean that the set of points for which this is not true has measure zero). This theorem is invariant under the group of continuous bijections which are also continuously differentiable. Bouligand refers to this setting as *topologie restreinte du premier ordre* and comments that we could, in a similar way, consider statements belonging to *topologie restreinte* of order 2, 3 and so on.

A further example provided by Bouligand in [51, p. 41] is that classification of different systems of geometry by Felix Klein in his *Erlanger Programm* according to invariance under groups of transformations. In addition to the connections he established between geometry and group theory, Felix Klein (1849-1925), a German mathematician and a key figure in the history of mathematics, contributed notably to non-Euclidean geometry and function theory. Klein's Erlanger Programm, first published as a booklet in 1872 [159] is widely considered as a major event in mathematics in the nineteenth century and provides a systematic way of organising various branches of geometry existing at the time. The main concept is that a system of geometry consists of what Klein then referred to as a manifold², a group of transformations defined on that set and a collection of properties which are invariant under the group of transformations. For example, in Euclidean geometry, given a shape in two-dimensional space, we can see that side-lengths and angles are invariant under translations and reflections. This way of formulating geometry leads for example to the possibility of organising different systems of geometry in relation to one another, for example, Euclidean geometry can be seen as being included within projective geometry. For a more comprehensive and detailed account of the Klein's Erlanger Program, we refer the reader to [211].

A final noteworthy example of studying a concept in relation to a certain field of invariance relates to *topologie restreinte du premier ordre*, discussed above. Bouligand notes [51, p. 42] that his concept of dimension used in the context of studying improper sets in the generalised Dirichlet problem, introduced above, belongs to this field of

¹Bouligand refers in [41, p. 588] to continuous bijections, although it is not clear whether he intended to refer to homeomorphisms - a type of function which today characterises topological invariance.

²For more details, see [211]

invariance.

Bouligand refers such fields of invariance **domains of causality**, commenting,

‘Tout système de propriétés invariantes (relatif à un certain groupe) est un DOMAINE DE CAUSALITE. Si l’on préfère: des causes intérieures à ce domaine entraînent aussi des effets intérieurs à ce domaine [51, p. 42].’

[‘Any system of invariant properties (in relation to a certain group) is a DOMAIN OF CAUSALITY. Or, if one prefers: causes within this domain lead to effects within this domain’].

In this line of thinking, Bouligand sees the application of the concept of a domain of causality as a means of achieving what he refers to as causal structure in geometry:

Pour restaurer la structure causale, il faut d’abord bien séparer les champs d’invariance [51, p. 41].

[In order to restore causal structure, it is first necessary to properly separate the fields of invariance].

4.1.2 Favours a direct approach

In his view of differential geometry, Bouligand expresses objections against an approach which involves methods which are detached in some sense from the concepts or objects being studied. He sees such methods and those involving unnecessarily lengthy calculations as a further factor in obscuring the connection between a fact in geometry and the basic principles which lead to that result. These ideas are perhaps most clearly expressed by Bouligand in relation to the use of axes in the analytical geometry of Descartes to deal with certain geometrical problems:

‘L’inauguration par Descartes de la méthode analytique a marqué une étape mémorable, en donnant des procédés permanents pour la résolution de questions jusqu’alors hasardeuses. Toutefois, le recours à des axes, étrangers au problème étudié, la longueur fréquente des calculs à effectuer, en même temps que leur absence d’apparemment logique avec le but poursuivi, ont suscité une réaction qui a fait naître le calcul géométrique, et plus récemment a conduit à l’édification d’une véritable doctrine à laquelle on peut donner le nom de géométrie vectorielle...’ [41, p. 586-587]

[‘The inauguration by Descartes of the analytical method marks a memorable step [in the evolution of geometry], by providing permanent procedures to solve questions which were hitherto problematic. However, the reliance on axes which are foreign to the problem being studied, the frequently lengthy calculations to be carried out and at the same time their lack of a logical relationship with the sought goal, led to a reaction which gave rise to the birth of geometric calculus and more recently has led to the elaboration of a veritable doctrine which we may name vectorial geometry..’]

Here Bouligand is clearly objecting to the ‘foreign’ nature of the method in relation to the geometrical problems and it is this aspect we wish to emphasise in relation to a preference for a direct approach. The quotation also illustrates how the different aspects of Bouligand’s views on differential geometry are closely connected and often considered together. For example, above, Bouligand combines his objection to the use of axes dissociated with the problem at hand with the lack of a logical connection between the methods and the concepts being studied. Regarding the latter, it seems we could equate this with Bouligand’s comment above in favour of studying geometrical facts in their natural domain of existence, or domain of causality.

We see in the above that Bouligand views the development of vector calculus as a step of progress away from the faults of the analytical geometry of Descartes, even if, as we noted earlier, he sees vector calculus as retaining certain faults of the Cartesian approach. We note here that this view regarding vector calculus as an improvement upon analytical geometry in terms of dealing with the concepts being studied in a more geometrical and intuitive way was certainly not unique to Bouligand. The following from Leveugle’s 1920 textbook illustrates a similar point of view:

L’Analyse ne met pas toujours suffisamment en lumière le sens géométrique des propriétés qu’elle étudie; au contraire, la méthode vectorielle réalise un double bénéfice, celui de simplifier considérablement l’écriture, et de resserrer le lien qui existe entre les propriétés synthétiques et analytiques. Algébriques par leurs symboles, les transformations du calcul vectoriel ont une signification géométrique intuitive qui permet d’interpréter constamment les opérations et leurs résultats. [171, p. V]

[Analysis does not always sufficiently bring to light the geometrical meaning of the properties studied. On the contrary, the vectorial method achieves a double benefit, that of considerably simplifying the writing and reinforcing the link that exists between synthetic and analytical properties. Algebraic in terms of their symbols, the transformations of vector calculus have an

intuitive geometrical meaning which allows for a constant interpretation of the operations and their results].

Leveugle goes on to describe geometric calculus as follows:

'méthode d'Algèbre générale qui permette de faire entrer directement dans le Calcul les éléments géométriques des figures, sans l'aide de nombres étrangers à la question que l'on a en vue, tels que les coordonnées cartésiennes ou autres.'
[171, p. VII]

[a method of general Algebra which enables bringing the geometric elements of figures directly into Calculus without the help of numbers which are foreign to the question at hand such as Cartesian coordinates or otherwise].

A further example provided by Bouligand in [51] relates to the study of geodesics (i.e. paths which give the shortest distance between two points). He comments in this regard,

...chaque fois qu'il est question des lignes géodésiques d'une surface (c'est-à-dire donnant le plus court chemin entre deux de leurs points, suffisamment proches) on pense presque infailliblement, à l'équation différentielle du second ordre définissant ces lignes : cet auxiliaire s'impose à l'esprit d'une manière tellement impérieuse qu'on éprouve de la difficulté à s'en passer : la vraie raison pour laquelle, au moins dans des cas étendus, il part, d'un point d'une surface des géodésiques dans toutes les directions issues de ce point, n'est pas aisée à apercevoir [51, p. 42].

[...whenever we are interested in the geodesic lines on a surface (that is, those giving the shortest path between two points on the surface, sufficiently close) we think almost invariably of the second order differential equation which defines these lines. This auxiliary is imposed on the mind in such a compelling way that it seems difficult to do without it. The real reason for which, at least in extended cases, geodesics go out from a point of a surface in all directions is not easy to see].

Bouligand's use of the word 'auxiliary' seems to imply taking recourse to a method which is somehow external or secondary in relation to the problem at hand - in this case the use of differential equations which, we note, do not represent a geometrical method. The consequence of this, Bouligand states above, is that the 'real reason' for the result is not brought to light in an obvious way.

4.1.3 Summary of Bouligand's motivations

We have explored the main principles and desiderata motivating Bouligand in later formulating his theory of direct infinitesimal geometry. Above we have extracted and introduced separately three interrelated key aspects of the approach to differential geometry desired by Bouligand, namely one in which extra hypotheses are avoided, where results are considered in an appropriate domain by means of the concepts of groups and invariance and finally in which a direct approach is favoured, for example by avoiding relying on coordinates or differential equations. The main tools and methods for putting into practice the principles Bouligand lays out in the two articles dealt with above are expressed in a comprehensive way in his *Introduction à la géométrie infinitésimale directe* - addressed later on in this chapter. First, beyond Bouligand's views and ideals which led him to express the need to formulate his theory, we explore below the external influences which had the greatest impact on his work in this direction as well as certain related works which Bouligand identified as having common elements in terms of either their goals or methods.

4.2 Factors influencing direct infinitesimal geometry - the emergence of set theory in France, the Polish School and other related works

It is interesting to consider the influences behind Bouligand's direct infinitesimal geometry and to ask whether other works of a similar nature existed and of which Bouligand was aware. Fortunately, Bouligand himself comments explicitly on the origins and influences of his theory as well as on works of a similar nature. These comments are to be found both in his notes regarding his own mathematical works, for example in [95] and in the introductions to the main expositions of his theory, notably in [75] and in [87]. The influences and related works emphasised the most can be split into three groups. First, we will consider the influence of French mathematicians of the previous academic generation, notably Lebesgue and Baire. We will situate these influences in the context of a broader trend in the history of mathematics, namely the adoption of set theory in France first in the theory of functions. Secondly, we will consider the influence of the Polish school of mathematicians established in the 1920s focussing on set theory, topology and foundational questions and finally we will introduce briefly the ideas of Christian Juel as presented by Paul Montel on what was referred to in France as *la géométrie finie*.

4.2.1 The emergence of set theory in France through the theory of functions

Cantor's set theory. Initial opposition in France. While other mathematicians had considered point sets in their work, it was famously the German mathematician Georg Cantor who first '*systematically developed the myriad implications of point sets in general, and who produced an entirely new field of mathematical research in the process*' [125, p. 6]. His work on sets and transfinite numbers was first published in a number of articles in 1870 and this research occurred in the context of a problem relating to trigonometric series. It was, according to Dauben, at least partially, a response to Riemann's research in this area and in the closely related study of discontinuous functions [125, p. 6]. It is well known that Cantor's work was met with contrasting reactions and that, in France, the criticism of Cantor's theory by Henri Poincaré was particularly strong. Dauben comments,

Intuitionists like Poincaré argued that most of the ideas of Cantorian set theory should be banished from mathematics once and for all...Transfinite set theory, Cantor's great contribution to mathematics, involved nothing in Poincaré's view but contradictory and therefore meaningless concepts [125, p. 266].

When discussing the rising adoption of concepts in Cantor's set theory in France, we have not focussed in our account on the controversies which occurred in France in 1905 around transfinite numbers and specifically a proof proposed by Zermelo that every set could be well-ordered. This theorem relied on and is equivalent to the axiom of choice in set theory. For a fuller account of these events, which involved a strong reaction in France, for example from mathematicians such as Borel, Baire, Lebesgue and Hadamard, we refer the reader to Dauben's account in [125, p. 253-259].

We can identify different factors later leading to the wider acceptance and adoption of set theory in France. Notably, Gispert identifies Poincaré's own encounter with a specific type of set studied by Cantor as a factor in leading to a transformation in the acceptance and use of set theory in France [141, p. 52]. Specifically, Poincaré encountered perfect sets which are dense in no interval and used their properties in the context of his research on curves defined by a differential equation [141, p. 52]. In [125], Dauben emphasises other major events leading to an increase in the interest in set theory, including in France. Notably, Dauben emphasises the talk given by Hurwitz during the First International Congress for Mathematicians in Zürich, 1897, in which Hurwitz highlighted the significance of Cantor's work [125, p. 247]. Further,

Dauben emphasises the address of Hilbert to the Second International Congress for Mathematicians in Paris in 1900, in which he included Cantor's continuum hypothesis as the first of the major unsolved problems in mathematics. Dauben associates such events with an increasing interest in set theory and connects this with the subsequent appearance of various books by eminent French mathematicians which used basic concepts of Cantor's theory:

Indicative of this growing interest, books soon began to appear which employed the basic ideas of Cantor's set theory in new ways. Here the French were particularly active, at least initially, and Couturat, Baire, Borel, and Lebesgue were representative of those who were influenced by, or who began to develop, certain aspects of Cantor's work [125, p. 161].

Another factor identified and evaluated by Gispert in [141, p. 48-51] which was influential in terms of the uptake of concepts in set theory in France is the second edition of the *cours d'analyse* taught by Jordan at the Ecole Polytechnique and published in 1893 [154], which included a chapter entitled *Ensembles*. We comment that this edition was likely to have inspired mathematicians such as Borel, Baire and Lebesgue (those who had perhaps the most direct influence on Bouligand) in their use of set theory.

The emergence of set theory in France in the theory of functions. In [141] Gispert investigates the way in which set theory was first adopted in France and also highlights the specificity of this adoption to French mathematics. Gispert highlights that set theory was first adopted in France in the theory of functions and it is this point which is of greatest relevance for the purposes of the present thesis. Gispert comments regarding the use of set theory in the theory of functions in France:

Loin d'être marginale, la théorie des ensembles devient à partir de la fin des années 1890 un outil nécessaire et privilégié de ces nouvelles recherches et s'insère ainsi dans l'activité mathématique classique du milieu français [141, p. 41].

[Far from being marginal, from the 1890s, set theory became a necessary and favoured tool of this new research and became part of classical French mathematical activity].

This phenomenon, of concepts of set theory first entering into the theory of functions in France, after an initial period of resistance, can also be found in the account of Nicolas Bourbaki in [104]:

Les idées de Cantor avaient d'abord rencontré une assez vive opposition...Du moins sa théorie des ensembles de points sur la droite et dans le plan fut-elle bientôt utilisées et largement répandue par les écoles françaises et allemandes de théorie des fonctions (Jordan, Poincaré, Klein, Mittag-Leffler, puis Hadamard, Borel, Baire, Lebesgue, etc.) : les premiers volumes de la collection Borel, en particulier, contiennent chacun un exposé élémentaire de cette théorie... [104, p. 178].

[Cantor's ideas had initially met with rather lively opposition...At least his theory of sets of points on a line and in the plane was soon used and widely disseminated by the French and German schools of function theory (Jordan, Poincaré, Klein, Mittag-Leffler, then Hadamard, Borel, Baire, Lebesgue, etc.): the first volumes of the Borel collection, in particular, each contain an elementary exposition of this theory].

The *Collection Borel* referred to above by Bourbaki constitutes a major example considered by Gispert of the adoption of concepts in set theory first in the theory of functions. This collection is the *Collection de monographies sur la théorie des fonctions*, created in 1898, of which the first volumes included Borel's *Leçons sur la théorie des fonctions* of 1898 [10], Baire's *Leçons sur les fonctions discontinues* of 1905 [4], which focusses on a part of the work presented in his thesis of 1899 and Lebesgue's *Leçons sur l'intégration et la recherche des fonctions primitives* of 1904 [166].

Borel's *Leçons* is focussed on the application of set theory to the theory of functions. With Gispert, we note that Borel expressed his goal as being '*d'exposer d'une manière élémentaire, certaines recherches qui, bien que relativement récentes, prennent chaque jour une importance considérable...De ce nombre, est la théorie des ensembles*' [to expose in an elementary manner certain areas of research which, although relatively recent, are with each day becoming increasingly important...Among these is set theory] [10, p. VII-VIII]. The goal of Baire's work is summarised as follows:

Me proposant de rechercher toutes les fonctions discontinues représentables par des séries de fonctions continues, j'étudie en détail, à mesure qu'elles se présentent, toutes les notions et les théories qui me sont utiles pour donner la solution de ce problème [4, p.VII].

[Proposing to search for all discontinuous functions that can be represented by series of continuous functions, I study in detail, as they arise, all the concepts and theories that are useful to me in order to give the solution of this problem].

In [141, p. 55], Gispert comments regarding the use of concepts in set theory in France prior to 1905 and in relation to Baire's work in [4] that Baire was '*...un de ceux - sinon celui - qui recourut le plus systématiquement et le plus complètement à la théorie des ensembles*' [one of the mathematicians if not the mathematician who took recourse to set theory in the most systematic and complete way]. Accordingly, a significant part of Baire's exposition in [4] serves to introduce the concepts in set theory useful in obtaining the desired results.

As indicated by Gispert [141, p. 62], in Lebesgue's *Leçons*, he includes a note entitled *Sur les nombres transfinis*, the first sections of which explores the concepts of set theory which he utilises in his work relating to integration and the measure of sets [166, p. 314-319].

The influence on Bouligand's theory. The adoption of concepts of set theory in the theory of functions in France appears to be a key theme influencing Bouligand's project to introduce a direct infinitesimal geometry. This is perhaps most clearly captured in the following passage of his 1931 article on *Les courants de pensée Cantorienne et l'hydrodynamique* [58]:

On sait que les idées de Cantor sur la théorie des ensembles ont pénétré d'une manière profonde dans le champ de la théorie des fonctions et l'ont considérablement enrichi. Ces idées ont également influencé la géométrie, surtout en ce qui concerne les problèmes consistant à définir en générale une ligne, une surface, un domaine, etc...., ou bien à évaluer les longueurs, les aires, les volumes. Mais on peut prévoir...que cette influence des courants de la pensée cantorienne sur la géométrie ne s'en tiendra pas là, car on aperçoit déjà, dans cette voie, la possibilité de se passer des représentations analytiques et des dérivées qu'elles mettent en jeu, autrement dit de constituer une GÉOMÉTRIE INFINITÉSIMALE DIRECTE [58, p. 103].

[We know that the ideas of Cantor on set theory have penetrated in a profound way into the field of function theory and have considerably enriched this field. These ideas have also influenced geometry, most notably in relation to problems which consist of giving a general definition of a line, a surface, a domain, etc. . . ., or consisting of evaluating lengths, areas or volumes. But we can predict...that this influence of the trends of Cantorian thought on geometry will not end here, as we can already glimpse, in this direction, the possibility of doing without analytical representations and the derivatives they involve. In other words, the possibility of constituting

a DIRECT INFINITESIMAL GEOMETRY].

We can speculate with relative certainty that Bouligand was familiar with the *Collection de Monographies* on function theory including the volumes contributed by Borel, Lebesgue and Baire. Indeed, Bouligand refers explicitly to Baire's *Leçons* when referring to a '*besoin de travaux géométriques d'un esprit nouveau analogue à celui de la théorie des fonctions*' [need for geometrical works taking a new perspective analogous to that of the theory of functions] [95, p. 17].

Although this point appears only implicitly in Bouligand's work, it would seem that he considered function theory as an example of an area of mathematics in which causality, in his sense of the word, had been achieved to a greater extent than in infinitesimal geometry. He comments for example in [51, p. 41], after presenting a number of examples including a discussion on the theory of functions:

Il y a donc, dans la Mathématique, des domaines très vastes où l'on peut regarder la crise de la causalité comme conjurée. Cependant, n'en est-il pas ainsi en géométrie infinitésimale?

[There is therefore, in Mathematics, vast areas in which we can consider the crisis of causality as being resolved. However, is it not the same for infinitesimal geometry?]

While the connection is not made explicitly by Bouligand, it seems highly reasonable based on the juxtaposition of these ideas in various works that he viewed the use of set theory in the theory of functions as a factor leading to resolving what he calls the 'crisis of causality' in this area of mathematics and that the adoption of such ideas in infinitesimal geometry is a likely candidate for helping to achieve his goals in that area of mathematics. A further key characteristic of the works mentioned above, such as the *Leçons* of Baire which seems to have influenced Bouligand in relation to his theory is the emphasis they place on eliminating unnecessary assumptions and achieving the greatest possible generality. For example, in [41, p. 40], where Bouligand first presents his programme of research towards what is later referred to as his direct infinitesimal geometry, he comments as follows in relation to the preface of Baire's *Leçons*:

...cette préface est-elle un programme orienté vers la recherche de généralité la plus grande possible et laissant déjà pressentir "l'avènement d'une Physique mathématique dans laquelle la part de l'hypothèse serait réduite au minimum [41, p. 40].

[...this preface is a programme oriented towards the search for the greatest possible generality, which heralds the “arrival of a new mathematical Physics in which the role of hypotheses would be reduced to a minimum.”]

The theme of minimising unnecessary hypotheses in the works of Lebesgue and Baire. In the above we see the idea of achieving greatest generality in a mathematical theory by eliminating unnecessary hypotheses - one of the key goals of Bouligand’s direct infinitesimal geometry. In Bouligand’s comments on this theme, the names of Lebesgue and Baire are the most prominent, with the work of Lebesgue appearing to have the greatest and most direct influence in relation to Bouligand’s theory.

Bouligand explicitly identifies the theme of getting rid of unnecessary hypotheses in the context of Baire’s contribution to the *Collection Borel*:

Au début de la célèbre monographie dont il a enrichi la Collection Borel, Baire aspire, à se débarrasser des restrictions, sans cesse introduites dans les cours d’analyse... [77, p. 266].

[At the start of the famous monograph with which he enriched the *Collection Borel*, Baire aspires to get rid of restrictions constantly introduced in texts on analysis...]

Since, as we have discussed above, Bouligand made efforts to translate and apply his theory to mathematical physics, it is also interesting to note in passing that he cites Baire’s ideas in relation to an approach to physics based on a minimal set of hypotheses. This comment is again made in relation to the preface of Baire’s contribution to the *Collection Borel*, as seen in the above citation [41, p. 40].

When discussing contributions of other mathematicians to the theme of avoiding unnecessary hypotheses, Bouligand places greatest emphasis on the doctoral thesis of Lebesgue. For example, in [87, p. 51], he comments,

...la méthode directe n’est pas l’apanage de telle ou telle partie des mathématiques, elle tend à les dominer à peu près toutes. Et si l’on cherche, en analyse infinitésimale, à préciser les origines de cette évolution, c’est sans nul doute à la Thèse de Lebesgue qu’il faut remonter.

[...the direct method does not belong exclusively to this or that part of mathematics, rather it tends to dominate more or less all parts. And if we seek to specify, in infinitesimal analysis, the origins of this evolution, it is without doubt to the thesis of Lebesgue that we must trace back].

The idea of direct methods being a common feature of various areas of mathematics will be expanded upon in the following chapter. Here we simply emphasise the influence of Lebesgue's ideas in relation to Bouligand's approach to direct infinitesimal geometry. Bouligand adds regarding this influence and in relation to Lebesgue's contribution to 'infinitesimal analysis',

La Thèse de Lebesgue est le premier ouvrage qui ait montré l'influence perturbatrice des hypothèses accessoires... [87, p. 51].

[The Thesis of Lebesgue is the first work which demonstrated the distorting influence of unnecessary hypotheses].

While Bouligand does not explicitly exemplify this comment in [87], in [75] he connects the idea that Lebesgue's thesis illustrates the influence of unnecessary assumptions with the example explored above regarding developable surfaces. We recall that, in this example, Bouligand stressed that the assumption of continuous first derivatives, in his view, had led to a distorted view of the solution, namely that a developable surface is one which can be decomposed into cones, cylinders and surfaces generated by the tangents of a non-planar curve. Lebesgue's thesis brought this error to light and gave necessary and sufficient conditions for each of these three objects to be a developable surface. The connection between this aspect of Lebesgue's work and the motivation for Bouligand's direct infinitesimal geometry is perhaps most explicitly expressed in [95, p. 2]:

Il me semble que la pointe hardie poussée par M. Henri Lebesgue dans le problème de la détermination des surfaces applicables sur le plan révélait la nécessité de restaurer tout l'édifice classique de la géométrie infinitésimale en lui donnant un nouveau style.

[It seems to me that the bold progress achieved by Mr Henri Lebesgue in the problem of the determination of surfaces which are developable on the plane revealed the necessity of restoring the entire classical edifice of infinitesimal geometry by formulating it in a new style].

4.2.2 The influence of topologists and the Polish school

In his letter to Fréchet dated 24 November 1938 [135], Bouligand elaborates in detail on the origins and development of his reflections on the concept of causality, including factors which influenced him in his goal of formulating ideas in differential geometry in a way which brings to light causality and avoids unnecessary hypotheses. He recalls, regarding his visit to the University of Krakow and to Zaremba in 1925,

Quand je suis allé à Cracovie, fin 1925, j'ai été très frappé par l'esprit de l'Ecole polonaise, et je me suis abonné à Fundamenta, l'un des mémoires qui m'atteignirent le plus fût celui d'Urysohn sur les Multiplicités Cantoriennes (t. VII, 1925).

[When I went to Krakow at the end of 1925, I was very struck by the approach of the Polish school and I subscribed to Fundamenta. One of the articles which made the greatest impression on me was that of Urysohn on Cantorian multiplicities].

Here Bouligand is referring to Urysohn's paper discussed above [212]. Fundamenta Mathematicae was a journal founded in Poland in 1920 by Zygmunt Janiszewski, Stefan Mazurkiewicz and Waclaw Sierpinski [161, p. 1] and focussed exclusively on set theory and related problems and applications. Early contributors included mathematicians such as Stefan Banach, Janiszewski, Kazimierz Kuratowski, Mazurkiewicz and Sierpinski - mathematicians who are most likely included in what Bouligand referred to as the Polish School, with whose approach he was so impressed. For further details regarding this journal, we refer the reader to [161], upon which the brief introduction here is based.

It is worthwhile, at this point, introducing Janiszewski, one of the founders of Fundamenta Mathematicae, and his works as they are frequently cited by Bouligand in relation to his direct infinitesimal geometry. Zygmunt Janiszewski (1888-1920) was a Polish mathematician whose mathematical contribution focussed on topology and the foundations of mathematics. He is considered as one of the founding members of the Polish school of mathematics established after the First World War, specialising in set theory, topology and the foundations of mathematics. He completed his higher education in various locations in Europe and studied under a number of eminent mathematicians. His doctoral thesis of 1911 *Sur les continus irréductibles entre deux points* was completed in Paris and supervised by Lebesgue. Favouring an intrinsic approach, Janiszewski's thesis studies, for example in n -dimensional Euclidean space, the concept of a curve from a set-theoretic and topological point of view. The following extract paints a picture of the context and motivations of his work:

Les notions fondamentales de la Géométrie pure ne sont pas encore toutes rigoureusement définies et analysées. Seule l'étude de la notion de la droite est amenée à un degré de perfection dans les célèbres Grundlagen der Geometrie de M. Hilbert...Quelles propriétés doit posséder un ensemble de points pour mériter d'être appelé courbe, surface, etc...C'est le premier de ces problèmes

que je traite dans cette thèse. [153, p. 79].

[The fundamental concepts of pure Geometry are not yet all rigorously defined and analysed. Only the study of the concept of a straight line has been brought to a degree of perfection in the famous *Grundlagen der Geometrie* of Mr Hilbert...Which properties must a set of points possess to warrant being called a *curve*, a *surface*, etc...It is the first of these problems that I address in this thesis].

What influence did Janiszewski's thesis have upon Bouligand's direct infinitesimal geometry? The most explicit influence is the frequent use by Bouligand in [75] of certain concepts in topology, namely the *ensemble limite* [limit set]³ and the *ensemble d'accumulation* [accumulation set]⁴. The connection between Janiszewski's thesis and Bouligand's knowledge of these concepts is clearly stated by Bouligand in [75, p. 154]. In addition, Bouligand uses certain results provided by Janiszewski in order to obtain the desired results in his direct infinitesimal geometry. For example, the following theorem is used a number of times:

Une collection infinie de continus dont l'ensemble limite contient deux points distincts A et B a pour ensemble d'accumulation un continu. [75, p. 157].

[An infinite collection of continua of which the limit set contains two distinct points A and B has an accumulation set which is a continuum].

While these are the most explicitly visible influences, we might also speculate that Bouligand's lengthy discussion in [75] regarding the appropriate definitions to use in relation to curves and surfaces, in the context of his theory may also have been inspired by Janiszewski's work. We might also speculate that Janiszewski's preference for an intrinsic and local approach which focusses, in addition, on rigorous definitions without extra assumptions may also have reinforced the choice of style in which Bouligand formulated his own ideas.

³In [75, p. 155], Bouligand defines the *ensemble limite* as follows: 'Nous dirons qu'un point L fait partie de l'ensemble limite J d'une collection infinie d'ensembles ponctuels s'il n'y a qu'un nombre fini d'ensembles de la collection dont la distance à L dépasse ϵ , et cela, si petite soit la longueur ϵ' [We will say that a point L belongs to the limit set J of an infinite collection of point sets if there is only a finite number of sets of the collection for which the distance to L is greater than ϵ , provided the length ϵ is sufficiently small].

⁴In [75, p. 155], Bouligand defines the accumulation set as follows: 'Nous dirons qu'un point H fait partie de l'ensemble d'accumulation d'une collection infinie d'ensembles ponctuels s'il existe une infinité d'ensembles de la collection dont le point H soit à une distance moindre que ϵ , et cela, si petite soit la longueur ϵ' . [We will say that a point H belongs to the accumulation set of an infinite collection of point sets if there exists infinitely many sets of the collection for which the point H is at a distance of less than ϵ if ϵ is sufficiently small.

Regarding Janiszewski's thesis and the ideas of the Polish school more generally, having emphasised their importance in relation to his own goals with regard to differential geometry, Bouligand distinguished between their goals and his own as follows:

...cette école s'attache spécialement à l'Analysis situs elle-même, c'est-à-dire à l'étude des invariants des déformations continues et biunivoques, plutôt qu'à la géométrie dans son ensemble [51, p. 40].

[...this school is associated specifically with Analysis situs itself, that is the study of invariants of continuous and bijective deformations, rather than with geometry as a whole].

However, Bouligand goes on to cite an exception to the above by referring to Janiszewski's 1910 article regarding the geometry of planar curves [152]. The following two main theorems provided in Janiszewski's paper are clearly concerned with geometrical properties. Informally speaking, they deal with curves in a local and non-parametric manner and therefore illustrate a qualitative similarity with results of Bouligand's direct infinitesimal geometry:

Theorem A. *Soit C une courbe plane, qui a au plus un nombre limité de points ou de segments communs avec chaque segment de longueur finie d'une droite arbitraire; je dis que C a au moins une tangente en chaque point [152, p. 606].*

[Let C be a planar curve which has at most a finite number of points and segments in common with each segment of finite length of an arbitrary straight line. Then we have that C has at most one tangent at each point].

Theorem B. *Une courbe C , n'ayant, dans un domaine fini quelconque, qu'un nombre fini K de points multiples et jamais plus de N (entier positif fixe) points communs avec une droite quelconque parallèle à l'une de deux directions fixes α et β , est rectifiable [152, p. 607]*

[A curve C which, in an arbitrary finite domain, has only a finite number K of multiple points and never more than N (fixed positive integer) points in common with an arbitrary straight line that is parallel to one of the two fixed directions α and β , is rectifiable].

4.2.3 A related theme - the finite geometry of Christian Juel and the work of Paul Montel

In Bouligand's 1937 note on his own mathematical works [61], when introducing the chapter entitled '*Systématisation: la création de méthode directes en géométrie infinitésimale*' [systematisation: the creation of direct methods in infinitesimal geometry], he notes that work in the desired direction in geometry existed already. Here he refers again to the ideas of Lebesgue, mentioned above, and adds to this reference the work of Christian Juel (1855-1935) on '*géométrie finie*' [finite geometry]. We note that Juel was a Danish mathematician and geometer who contributed, for example, to projective geometry. His ideas referred to by French mathematicians as *géométrie finie* were first published in 1899 in Danish [155]. In Bouligand's account here, it would appear that his attention was brought to these ideas through a presentation given by Paul Montel in 1923 at the *séminaire Hadamard* [95, p. 17]⁵, published subsequently in [186] in which Montel presented Juel's ideas. It is worth introducing briefly the French mathematician Paul Antoine Aristide Montel (1876-1975): an influential figure in the French mathematical scene during his time, whose work focussed for example on the theory of analytic functions of a complex variable and holomorphic functions. Given that Bouligand's attention was drawn to these ideas through the work of Montel, it is interesting and relevant to consider the following description by Montel of Juel's finite geometry:

La géométrie infinitésimale étudie les propriétés locales ou générales des courbes et des surfaces en admettant l'analyticité des fonctions introduites, ou au moins l'existence pour ces fonctions, de différentielles d'un ordre assez élevé. Les courbes et les surfaces algébriques ont été longtemps presque les seules à permettre un examen complet de leurs caractères locaux ou régionaux. Si l'on suppose seulement la continuité dans la variation de l'élément de contact, on est conduit à des recherches géométriques dont l'ensemble forme ce que Darboux a appelé la "Géométrie finie" [186, p. 109].

[Infinitesimal geometry studies the local or general properties of curves and surfaces assuming that the functions introduced are analytical or at least assuming the existence for these functions of derivatives of a sufficiently

⁵As described by Audin [3], the *séminaire Hadamard* was the first seminar of mathematics in France, starting in 1913 and continuing again after the war in 1920 until Hadamard's retirement in 1937. It took place at Collège de France in Paris and dealt with diverse topics in all areas of mathematics. With Audin, we refer to Fréchet's account of the seminar [138] in which he emphasises its success, eventually attracting researchers worldwide. The format seemed to involve the presentation of papers, the analysis of these papers by the participating audience and a final presentation given in summary by Hadamard himself. According to Fréchet, there was an element of selection for the auditors of the seminar.

high order. Algebraic curves and surfaces have long been the only curves and surfaces that allow a complete investigation of their local or regional characteristics. If we assume only the continuity in the variation of the contact element, we are led to geometrical research which, as a whole forms what Darboux referred to as “finite Geometry”].

It would seem, from Montel’s account, that an example of the continuity of the contact element referred to above would be ‘l’existence d’une tangente qui varie avec le point de contact d’une manière continue’ [the existence of a tangent which varies with the point of contact in a continuous manner] [186, p. 110]. Montel gives a number of examples of results in geometry known at the time which are shown by Juel to require less restrictive assumptions than commonly utilized at the time. We give just one example below and refer the reader to [186, p. 110] for further examples.

Une courbe plane du troisième degré a, en général, trois points d’inflexion réels : ce résultat est encore exact pour une courbe plane du troisième ordre, c’est-à-dire rencontrée en trois points au plus par toute droite du plan [186, p. 110].

[A planar curve of third degree has in general three real points of inflection: this result is still correct for a planar curve of third order, that is met at three points at most by any straight line in the plane].

As such, there is at least a clear similarity between Juel’s ideas, as recounted by Montel and Bouligand’s direct infinitesimal geometry in the sense that both seek to reduce the number of assumptions introduced so as to obtain certain results in a broader setting. The following comment of Montel with regard to the examples of which we quoted one above reinforce the similarity between the spirit of Juel’s work, at least as seen by Montel, and that of Bouligand:

Bien des propositions, établies par le calcul, demeurent liées dans notre esprit à leur support algébrique. Or, dans beaucoup de cas, le caractère algébrique ne sert qu’à faciliter la démonstration analytique : les hypothèses nécessaires pour l’exactitude du résultat ont un caractère synthétique beaucoup moins apparent [186, p. 110].

[Many propositions established through calculation remain connected, in our minds, with the algebraic medium [used to obtain the results]. However, in many cases, the algebraic character serves only to facilitate the analytical proof. The necessary hypotheses for the validity of the result have a synthetic character which is far less apparent].

We note here the similarity to Bouligand's criticism of the introduction of extra assumptions to facilitate the use of Cartesian methods. What then are the key differences between the ideas of Juel, as presented by Montel and Bouligand's theory? Bouligand reflects on this point in [95] and identifies two main differences - first the group of invariance involved in *géométrie finie* and secondly the means by which certain geometrical objects studied are characterised:

Au lieu d'adopter, comme le faisait Juel, une sélection a priori du matériel géométrique, j'ai opéré la sélection au moyen de conditions simples supposées remplies par le paratingent ordinaire, conditions qui conduisent à ce que j'ai appelées les lemmes d'univocité. A ce titre, la géométrie infinitésimale directe diffère déjà de la géométrie finie de Juel : la différence la plus essentielle réside en ce fait que le champ d'invariance de la géométrie finie se limite au groupe projectif [95, p. 4].

[Instead of adopting an *a priori* selection, as did Juel, of the geometric material, I carried out the selection by means of conditions assumed to be satisfied by the ordinary paratingent, conditions which lead to what I called the *uniqueness lemmas*. In this way, one can already see that direct infinitesimal geometry differs from the finite geometry of Juel: the most essential difference residing in the fact that the field of invariance of finite geometry is limited to the projective group].

4.2.4 A further related theme - the work of Karl Menger on distance geometry

It is worth noting that Bouligand identified his theory of direct infinitesimal geometry as belonging, in some sense, to the same trend as the work of Karl Menger on distance geometry. Karl Menger (1902-1985) was an Austrian born mathematician and son of the economist Carl Menger. He contributed to various areas of mathematics including dimension theory, the theory of curves, algebra and geometry. Beyond mathematics, Karl Menger also contributed to social sciences. Menger is considered as one of the influential figures in the founding of what is referred to today as distance geometry. His work in this direction includes for example his 1928 article on what he referred to as *allgemeine Metrik* [general metric] [180] and his 1931 article on the *New foundation of Euclidean geometry* [181]. In Menger's research, he was interested in the 'characterisation of the n -dimensional Euclidean space among general semi-metrical spaces in terms of relations between the distances of its points' [181, p. 721].

We note that in the 1935 *Conférences internationales de topologie* in Geneva, Bouligand

presented his theory of direct infinitesimal geometry, while Menger presented his work on '*La géométrie des distances et ses relations avec les autres branches des mathématiques*' [the geometry of distances and its relationship with other branches of mathematics]. In the following, we see that Bouligand viewed Menger's work as being a different approach to achieving direct methods in infinitesimal geometry. Bouligand comments regarding Menger:

Si ce géomètre a donné une autre conception de méthodes directes en géométrie infinitésimale, mes vues sur l'unité des méthodes directes...ne s'en trouvent que renforcées [95, p. 21].

[If this geometer gave a different conception of direct methods in infinitesimal geometry, my views on the unity of direct methods are found only to be reinforced].

Here Bouligand is referring in particular to Menger's ideas on *Allgemeine Metrik* as expressed in his article submitted in the context of the *Conférences internationales de topologie* of 1935 [182]. Here we focus on the first part of Bouligand's comment - identifying Menger's work as an alternative approach to direct methods. Menger describes the motivation behind his approach to differential geometry in the following passage, which exhibits certain clear similarities with Bouligand's views.

Malgré son importance historique et ses nombreux avantages on ne doit cependant pas oublier...que d'un point de vue purement géométrique l'étude des modèles arithmétiques au moyen de l'analyse n'est qu'un procédé entre plusieurs possible; ce procédé impose par ailleurs aux recherches des restrictions assez considérables qui ne sont pas inhérentes à la nature des figures spatiales. [182, p. 348].

[Despite its historical importance and its numerous advantages, we must however not forget. . . that from a purely geometrical point of view the study of arithmetic models by means of analysis is but one among many possible procedures. Besides, this procedure imposes considerable restrictions upon research which are not inherent to the nature of figures in space].

The above is not dissimilar to Bouligand's views explored already regarding the potential dangers of a purely analytic or algorithmic approach to geometry, including notably the introduction of extra unnecessary restrictions. Further, Bouligand's idea of a direct approach could potentially be compared with Menger's comment above regarding the inherent nature of figures in space. Menger goes on to describe his approach:

J'ai été ainsi conduit depuis quelques années à développer une géométrie qui se passe des modèles arithmétiques, tout en s'occupant de problèmes relatifs aux notions classiques: convexité, courbure, géodésiques, etc. Les points ne sont alors pas nécessairement définis par des coordonnées, ni les figures par des équations. La géométrie des distances ou géométrie métrique est basée sur la donnée d'un ensemble d'éléments de nature quelconque assujettis à la seule condition que deux d'entre eux corresponde toujours un certain nombre [182, p. 349].

[I have thus been led, for the past several years, to developing a geometry which forgoes arithmetic models, while still dealing with problems related to classical concepts: convexity, curvature, geodesics, etc. Points are not necessarily defined by coordinates, nor are figures necessarily defined by equations. The geometry of distances or metric geometry is based on a set of elements being given, which are of arbitrary nature subject only to the condition that two elements always correspond to a certain number].

Therefore, a further similarity between Menger's work and that of Bouligand is the goal of studying certain concepts in differential geometry while avoiding any reliance on coordinates or representation by means of differential equations. How may we describe the key differences between Bouligand's direct infinitesimal geometry and Menger's '*géométrie des distances*'? These differences are summarised as below by Christian Pauc (1911-1981), who completed a doctoral thesis dealing with both the direct infinitesimal geometry of Bouligand the work of Karl Menger. We introduce Pauc in greater detail later in this chapter.

La différence principale entre la géométrie infinitésimale directe de M. Bouligand et la géométrie des distances de M. Menger est que celle-ci étudie des espaces distanciés généraux et celle-là des ensembles euclidiens; la première suit parfois d'assez près la géométrie infinitésimale classique...alors que la seconde s'en écarte fréquemment, non seulement par les raisonnements mais aussi par les résultats. . . [191, p. 83].

[The main difference between the direct infinitesimal geometry of Mr. Bouligand and the geometry of distances of Mr. Menger is that the latter studies general distanced spaces whereas the former studies Euclidean sets. The first often follows classical infinitesimal geometry quite closely. . . while the second deviates quite frequently, not only in its reasoning but also in its results].

4.3 The formulation of Bouligand's theory of direct infinitesimal geometry

The first comprehensive presentation of Bouligand's ideas as a theory was in his 1932 textbook entitled *Introduction à la géométrie infinitésimale directe*. This book groups together a number of different tools, ideas and results published in several articles starting from 1928 while aiming to do this in a way which is accessible to students of mathematics and young researchers who might be encouraged to pursue their own mathematical research in the area of direct infinitesimal geometry. While the target audience of the book is not explicitly stated, the presence of numerous exercises as well as the fact that basic concepts of set theory and topology are introduced in the early chapters would seem to support this interpretation of the target readership. Elie Cartan's wish included at the end of his preface, '*Puisse ce livre susciter de nombreuses vocations !*' [p. VII][75] [*May this book inspire many vocations!*] would reinforce the assertion that Bouligand's book is intended to inspire new research into direct infinitesimal geometry.

The first eight of the 16 chapters are largely dedicated to introducing basic concepts and results necessary as background to the introduction of Bouligand's theory. These include, for example, basic operations on sets, basic concepts in topology, the Cantor-Minkowsky construction which we encountered previously and certain theorems employed later on. We will focus on the core concepts of Bouligand's theory, providing the necessary background understanding as and when necessary.

4.3.1 Selecting an appropriate setting (domain of causality) for direct infinitesimal geometry

After presenting these preliminary elements, Bouligand dedicates a chapter to the '*indications fournies par la théorie des groupes pour l'édification de la géométrie infinitésimale directe*' [indications provided by group theory for the construction of direct infinitesimal geometry] [75, p. 60]. In this chapter, Bouligand provides a detailed discussion regarding the appropriate setting in which we should be working in direct infinitesimal geometry. That is, he addresses the question of what group of transformations we should consider, under which certain key geometrical properties will remain invariant.

The setting in which Bouligand wishes to work in his direct infinitesimal geometry is that of a group of transformations closely related to that of *topologie restreinte du premier ordre*, encountered in the previous chapter. Specifically, he proposes including

one extra condition, namely that the transformations in question have a positive and non-zero Jacobian determinant⁶. As such, Bouligand chooses to consider the group of transformations in three-dimensional Euclidean space which are continuous bijections with non-zero Jacobian determinant. With Bouligand, we will refer to this set of transformations as γ . What is the reason for choosing this setting? We will see in the following subsections that certain key properties of the geometrical concepts with which Bouligand works are invariant under these transformations, thus shedding further light on this choice. In addition, he comments on reasons for not considering certain other sets of transformations as the causal domain for direct infinitesimal geometry. In particular, Bouligand views the set of all continuous bijections, which he refers to as the '*domaine causal de la topologie pure*' [*the causal domain of pure topology*]. However, he considers this class of transformations to be far too broad for the purpose at hand:

Au point de vue de la théorie des lignes ou des surfaces, nous serions ainsi conduits à faire l'étude de classes de variétés beaucoup trop étendues [75, p. 60].

[From the point of view of the theory of lines and surfaces, we would as such be led to carry out the study of a classes of manifolds which are far too broad].

Bouligand considers also the more restricted setting of *topologie restreinte du premier ordre* - that is continuous bijections with continuous first derivative. However, he comments that this class of transformations does not form a mathematical group. In particular, in order for a set of functions to form a group, it is necessary that for each function in the set, its inverse is also in the set - a condition which is not satisfied in the setting of *topologie restreinte du premier ordre*. The added condition that the transformations in question have a positive and non-zero Jacobian determinant remedies this problem, meaning that γ does form a mathematical group with respect to the operation of the composition of functions.

⁶In [75, p. 64] Bouligand in fact refers to the '*déterminant fonctionnel*'. A reasonable interpretation seems to be that this a historical term referring to the Jacobian determinant of a function.

4.3.2 Introducing two key tools of direct infinitesimal geometry - the contingent and the paratingent

The central tools of Bouligand's direct infinitesimal geometry are certain sets of lines defined at accumulation points of sets in Euclidean space. The two main tools of this type are referred to by Bouligand as the contingent and the paratingent, concepts which are closely related to the idea of a derivative. These tools allow for an approach which avoids parametric representations and enable us to operate in a setting where the assumption of the differentiability of functions representing curves or surfaces in Euclidean space is avoided. Informally speaking, as we will see below, the contingent, the paratingent and other related concepts enabled Bouligand to define certain familiar types of curves and surfaces in a more direct or intrinsic manner compared to parametric representation and they enabled him to undertake a geometric study of sets in Euclidean space.

Introducing the contingent As Bessis comments in [9, p. 1] the contingent is a generalisation of the idea of a tangent to a curve, or a planar tangent to a surface. The contingent proves to be of considerable use to Bouligand in the geometrical study of sets in Euclidean space. In order to define the contingent, Bouligand first defines a half-tangent to a set E at a point O of this set.

Une demi-droite OT , issue du point d'accumulation O de l'ensemble E , sera dite une demi-tangente au point O , si tout cône droit à base circulaire, de sommet O et d'axe OT , contient...un point de l'ensemble E distinct du point O [75, p. 66]

[A half-line OT , starting from an accumulation point O of the set E , will be called a half-tangent at the point O if every right circular cone with vertex O and axis OT , contains...a point of the set E distinct from the point O].

We note that in the setting in which Bouligand is working, an accumulation point O of a set E is a point for which any sphere centered around O , no matter how small the diameter, always contains another point of E other than the point O . Bouligand then goes on to define the contingent as follows:

L'ensemble de toutes les demi-tangentes à l'ensemble E en un même point d'accumulation sera appelé, moyennant une désignation abrégée conforme à l'étymologie, le contingent de l'ensemble E au point O . [75, p. 66]

[The set of all half-tangents of the set E at a particular accumulation point will be called, using an abbreviation which is in accordance with the etymology⁷, the contingent of the set E at the point O].

A key property of the contingent with respect to the group γ Bouligand emphasises an invariance property of the contingent with respect to the group of transformations γ which becomes a key tool in mathematical arguments in direct infinitesimal geometry. The covariance of the contingent is expressed through the following theorem:

Lorsqu'on effectue sur un ensemble ponctuel une transformation ponctuelle du groupe (γ), on obtient un nouvel ensemble ponctuel ; en des points correspondants, les contingents se correspondent par la transformation linéaire tangente. [75, p. 67]

[When we apply a point transformation of the group (γ) to a point set, we obtain a new point set. At corresponding points, the contingents correspond to one another via the linear tangent transformation].

As Bouligand notes that this property would not hold, for example, in the broader setting of all continuous bijections, this represents a further justification of the choice of γ as the causal domain of direct infinitesimal geometry.

Background and origins of Bouligand's use of the contingent. Bouligand reports in [95, p. 15-16] that he first used the concept of the contingent in the context of his work on harmonic functions, in [34, p. 20]. Specifically, he considered this concept in the context of the following theorem which he formulated during his visit to the University of Krakow in 1925 and upon which he commented in [87, p. 57] in relation to his later explicit formulation of the contingent:

Une suite de fonctions harmoniques dans le domaine D , bornées dans leur ensemble, converge dans D vers une fonction harmonique, si elle converge en une infinité de points de D , ayant un point O de D comme point d'accumulation pourvu qu'à l'intérieur d'un cône droit (limité) de sommet O , tout autre cône droit (si petites soient son ouverture et sa hauteur) contienne des points de convergence (ce qui revient à dire : il existe, dans le ctg, des rayons jouant le rôle d'éléments intérieurs).

⁷It does not seem completely clear what Bouligand intended by this reference to etymology. He comments in a footnote [75, p. 66] that the word contingent has both philosophical and military connotations. We note that, in philosophy and in language more generally, the adjective contingent is ascribed to something whose existence is not necessary. Perhaps this corresponds to the fact that the contingent of a set at a point may be empty.

[A sequence of harmonic functions in the domain D , all bounded, converge in D to a harmonic function if the sequence converges at infinitely many points of D having the point O in D as an accumulation point, provided that inside a right (bounded) cone with vertex O , every other right cone (if its opening and height are sufficiently small) contains points of convergence (which equates with saying: the ctg contains lines assuming the role of interior elements)].

He comments in [95, p. 4] that the contingent is a generalisation of the concept of the set of derivative numbers (*nombres dérivés*) and in [87] emphasises Arnaud Denjoy's 1915 paper on derivative numbers [126] as an important influence for his own ideas relating to the contingent. As explained by Bessis in [9, p. 1], the set of derivative numbers studied by Denjoy for a real-valued function f^8 at a point x_0 is the set of adherent points as $x \rightarrow x_0$ of

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

As Bessis comments, the contingent used by Bouligand can be seen as a generalisation of the geometric representation of the set of *nombres dérivés*. Specifically, if we let E denote the graph of the function f in two-dimensional Euclidean space, then the set of derivative numbers can be represented as the set of the directions of the half-tangents of E at the point $((x_0), f(x_0))$. Bouligand's contingent can be seen as a generalisation of this to an arbitrary set in Euclidean space. Bouligand focusses on the three-dimensional setting in [75] but this could easily be translated to n dimensions).

Finally, we note that Bouligand does not claim to be the first to consider this concept. He comments in [87, p. 57]:

L'importance des notions de ctg et de ptg est telle qu'on les retrouve sans doute d'une manière plus ou moins explicite et sous des dénominations diverses.

[The importance of the concepts of ctg and ptg is such that they can no doubt be found more or less explicitly and under various names].

To give just one example, Bouligand refers to the works of Francesco Severi, for example in [205] in which, like Bouligand, Severi considers differentiability in terms of mathematical objects comprising certain tangents. For further details and for an account containing additional historical information regarding the characterisation of differentiability in terms of tangency, we refer to the reader to [128].

⁸We note that in [126], Denjoy is interested in working with continuous functions.

Introducing the paratingent. Bouligand's use of the contingent inspired him to introduce a second key tool of direct infinitesimal geometry [95, p. 4] - the paratingent of a set E at an accumulation point O in E , defined in a similar manner to the contingent:

Nous dirons qu'une droite RS passant par un point d'accumulation O de l'ensemble ponctuel E appartient au paratingent de E en O ...si l'on peut trouver une suite de segments P_iQ_i (non nuls) dont les extrémités appartiennent à E , tendent vers O et dont les droites supports tendent vers la droite RS ou bien coïncident avec elles [75, p. 72].

[We will say that a straight line RS passing through an accumulation point O of the point set E belongs to the paratingent of E at O ...if we can find a sequence of (non-empty) segments P_iQ_i whose endpoints belong to E , tend towards O and for which the lines in which the segments are contained either tend towards the line RS or coincide with it].

We note that Bouligand refers to a line which belongs to the paratingent of the set E at the point O as a *paratingente*.

Two key properties of the paratingent. Like the contingent, Bouligand demonstrates the covariance of the paratingent with respect to the group of transformations γ , as such we can consider it as being associated with the causal domain chosen for direct infinitesimal geometry. In addition to the covariance of the paratingent, we will note with Bouligand one further property which he refers to as '*semi-continuité supérieure d'inclusion*' [upper semi-continuity with respect to inclusion]. We choose to note this property here since, first, it plays a central role in Bouligand's mathematical arguments in direct infinitesimal geometry. Secondly, it is worth noting as Bouligand is stated by later mathematicians as being one of the earliest mathematicians to study the concept of the semi-continuity of set-valued mappings, for example in [204, p. 192]. Bouligand defines the upper semi-continuity of the paratingent of the set E at a point M with respect to inclusion as follows:

...étant donné une suite quelconque de points M_i tendant vers le point M , toute droite d'accumulation de paratingentes en M_i est une paratingente en M [85, p. 249].

[...given an arbitrary sequence of points M_i tending towards the point M , any accumulation line of paratingentes at M_i is a paratingente at M].

While it is not explicitly stated by Bouligand, we assume that the above definition applies for all $i > k$ where k is some sufficiently large natural number. Bouligand does not define what is meant for a line belonging to the paratingent at a point P to be an accumulation line. It would seem reasonable to interpret this as follows: we will say that a line L_0 of the paratingent at P is an accumulation line if for all $\delta > 0$ there exists a line L belonging to the paratingent at P which is δ close to L_0 .

We also note that dealing with semi-continuity of functions is, for Bouligand, associated with bringing to light causality. He comments:

Pour bien mettre en lumière la causalité, il nous sera commode de considérer la continuité comme la coexistence de deux propriétés, la semi-continuité inférieure et la semi-continuité supérieure [75, p. 16].

[To effectively bring causality to light, it will be convenient to consider continuity as the coexistence of two properties, lower semi-continuity and upper semi-continuity].

4.3.3 Selecting lines and surfaces through criteria relating to the paratingent

Having introduced the above main tools of direct infinitesimal geometry, Bouligand illustrates how the paratingent can be used to characterise certain classes of lines and surfaces in Euclidean space which are of interest in the study of differential geometry. This approach avoids the need for parametric representation. For example, he establishes the following results [75, p. 79-83]

Consider a bounded continuum⁹ K in the plane such that for all accumulation points O in K , there is at least one straight line passing through O which is not in the paratingent of K at O . If K contains no endpoint, then Bouligand demonstrates that K is a **simple closed curve**¹⁰. On the other hand, if K contains at least one endpoint, then he shows that K is a **simple arc**. We note that a curve, for Bouligand in this context, is what he refers to as a *continu de Jordan* [Jordan continuum] which in two-dimensional Euclidean space is defined here as the image of an interval under a continuous function. In other words, a Jordan continuum in two dimensions can be represented by the equations [75, p. 55]:

⁹For Bouligand, a continuum is in this setting a closed set which cannot be represented as the union of two disjoint closed sets.

¹⁰In fact Bouligand uses the term 'cycle simple', which we have interpreted as above.

$$x = f(t), y = g(t)$$

where the functions f and g are continuous and where t runs through the interval $a \leq t \leq b$. For a surface, we can simply include an extra function $z = h(t)$ where h is also continuous¹¹.

Bouligand shows that a similar approach can be used to obtain familiar lines in three-dimensional space. If K is a bounded continuum in three-dimensional space such that for any accumulation point O of K , there exists a plane passing through O which contains no line belonging to the paratingent of K at O . The Bouligand considers the same two cases - either K has an endpoint or K has no endpoint. If K has no endpoint, then again it is a simple closed curve (this time in three-dimensional space). If K has an endpoint, then it is a simple arc. Bouligand provides the following interpretation of the significance of such results:

Il est bien entendu qu'en considérant les continus dont le paratingent en chaque point laisse échapper toutes les directions d'un plan, nous ne parvenons qu'à une classe particulière de lignes. Mais ces lignes sont définies par une condition infinitésimale directe (c'est-à-dire pouvant se formuler sans recourir à une représentation analytique préalable) [75, p. 81].

[We acknowledge that by considering continua for which the paratingent at each point excludes all directions in a plane, we obtain only a specific class of lines. But these lines are defined by a direct infinitesimal condition (that is, one which can be formulated without relying on a given analytical representation)...

Bouligand highlights the following advantages of this characterisation of the lines specified above:

La condition qui sert à les définir est invariante par les transformations du group γ ; et elle est telle qu'un pont se trouve immédiatement jeté entre le point de vue infinitésimal direct, dont procède la définition, et le point de vue de la représentation analytique.

[The condition which serves to define them is invariant under transformations of the group γ and is such that a bridge is immediately established

¹¹A detailed discussion is included in [75, p. 46-59] regarding the most suitable definition for a curve or a surface in the context of direct infinitesimal geometry. Ultimately, Bouligand chooses to focus predominantly on Jordan continua as he felt first they provided a sufficiently rich class of objects for direct infinitesimal geometry, including familiar geometrical objects and secondly that this class excludes certain undesirable cases presented by certain alternative definitions [75, p. 55]

connecting the direct infinitesimal point of view, from which the definition is introduced and the point of view of analytical representation].

Having shown how criteria involving the paratingent can be used to introduce certain simple classes of lines in two and three dimensions, Bouligand goes on to show how certain classes of surfaces in three dimensional space can be introduced by means of a direct infinitesimal approach. First, he shows that the following assumptions about a bounded continuum lead to a surface, that is a *continu de Jordan* in three dimensions, as introduced above:

1. *en chaque point O d'un tel continu K existe au moins une direction Oz exclue du paratingent en ce point.*
2. *toute parallèle à cette direction qui se rapproche indéfiniment du point O de K finit par contenir un point de K qui tend vers le point O .*

[1. at each point O of such a continuum K there exists at least one direction Oz excluded from the paratingent at this point.

2. every parallel to this direction which approaches indefinitely the point O of K ends up containing a point of K which tends towards the point O .]

More specific types of surfaces are constructed by Bouligand by modifying the above assumptions. For example, by modifying condition 2. above so that the condition is satisfied by all parallels except a finite number of continua whose paratingent at each point excludes every direction of a plane, Bouligand characterises a class of bounded portions of surfaces with a finite number of edges.

4.3.4 Introducing contingents and paratingents of different order

To the contingent and paratingent already introduced, Bouligand adds a number of related concepts. Together, all of these are referred to as the *méthodes infinitésimales directes* [direct infinitesimal methods] of Bouligand's theory [75, p. 109]. We cite his definitions below together with examples of results formulated in terms of some of these tools.

Introducing the planar contingent:

...nous dirons qu'un demi-plan issu de la droite portant OT appartient au contingent d'osculatation ou contingent planaire relatif au point O et à la demi-droite OT s'il est limite d'une suite de demi-plans déterminés par OT et les points M_i d'une suite extraite de E tendant vers O , dont aucun ne se trouve

sur OT ni son prolongement [75, p. 115].

[...we will say that a half-plane starting from the line containing OT belongs to the contingent of oscillation or the planar contingent with respect to the point O and the half-line OT if that half-plane is the limit of a series of half-planes determined by OT and the points M_i of a sequence extracted from E and tending towards O where no point M_i is situated on OT nor its extension].

The circular contingent.

Un cercle appartiendra de même au contingent circulaire relatif à O et à OT s'il est limite d'une suite de cercles tangents en O à OT , passant respectivement par les points M_i d'une suite extraite de E , tendant vers O , dont aucun ne se trouve sur OT ni son prolongement [75, p. 115].

[A circle will in the same way belong to the circular contingent with respect to O and OT if it is the limit of a sequence of circles tangent at O to OT , passing respectively through the points M_i of a sequence extracted from E , tending towards O where no point M_i is situated on OT nor its extension].

The spherical contingent.

Soit maintenant Γ_{OT} un cercle du contingent circulaire relatif à O et à OT . Nous dirons qu'une sphère appartient au contingent sphérique relatif aux éléments O , OT , Γ_{OT} si elle est limite d'une suite de sphères passant toutes par Γ_{OT} et respectivement par les points M_i d'une suite extraite de E , tendant vers O et dont aucun point ne soit sur le cercle Γ_{OT} [75, p. 115].

[Now let Γ_{OT} be a circle of the circular contingent with respect to O and OT . We will say that a sphere belongs to the spherical contingent with respect to the elements O , OT , Γ_{OT} if it is the limit of a sequence of spheres all passing through Γ_{OT} and passing respectively through the points M_i of a sequence extracted from E , tending towards O and where no point M_i is on the circle Γ_{OT}].

Applying the four contingents to the study of curves

The following results included in Bouligand's account illustrate the application of the four contingents to the study of curves¹². If we let OL be a Jordan continuum which

¹²Bouligand's results here apply to what he refers to as 'courbes de Jordan' [Jordan curves]. While it is not stated explicitly, it would seem that this refers to a Jordan continuum as defined above

is a simple half-arc with origin OL and if OT is an arbitrary half-tangent to OL at O , then we have:

Si le contingent planaire relatif à O et à OT contient au moins deux demi-plans P et Q , tout plan passant par OT et tel que P et Q soient de part et d'autre de ce plan coupe OL en une infinité de points dont le point O est un point d'accumulation [75, p. 116].

[If the planar contingent relative to O and OT contains at least two half-planes P and Q , any plane passing through OT such that P and Q are on either side of this plane intersects OL at an infinite number of points of which the point O is an accumulation point].

Bouligand goes on to formulate similar results in terms of the circular and spherical contingents. Below is the above result formulated in terms of the circular contingent:

Si le contingent circulaire relatif à O et à OT contient au moins deux cercles, toute sphère de rayon fini ou infini tangente en O à OT , et telle que les cercles lui sont l'un intérieur et l'autre extérieur, coupe OL en une infinité de points dont le point O est un point d'accumulation [75, p. 116].

[If the circular contingent with respect to O and OT contains at least two circles, then every sphere of finite or infinite radius which is tangent at O to OT and such that one of the circles is inside the sphere and the other is outside, cuts OL at infinitely many points of which the point O is an accumulation point].

4.3.5 Introducing paratingents of different rank

In addition to introducing the various concepts above inspired by the definition and application of the contingent, Bouligand also defined a concepts inspired by the paratingent. Below we introduce the paratingents of rank k (where k is a natural number, the usual paratingent introduced earlier being of rank 1). Again, we refer to examples provided by Bouligand of results which can be formulated in terms of these new concepts. The paratingent of rank k is defined as follows:

Soit un ensemble ponctuel E . Nous dirons qu'une droite D appartient au paratingent de rang k de cet ensemble au point M si l'on peut trouver une suite de divisions rectilignes, formées chacune de $K + 1$ points distincts de E , tendant simultanément vers M , en même temps que les droites supportant ces

divisions tendant vers D , si elles ne coïncident avec cette droite. [75, p. 127]

[Let E be a point set. We will say that a line D belongs to the paratingent of rank k of this set at the point M if we can find a sequence of rectilinear divisions, each formed by $K + 1$ distinct points of E which tend simultaneously towards M at the same time as the lines in which these divisions are contained tend towards D , if they do not coincide with this line].

The result below relates to a local geometrical property of a set based on the assumption that the paratingents of certain orders are either empty or not.

Supposons qu'en O le paratingent de rang k de l'ensemble E soit vide, celui de rang $k - 1$ contenant au moins une droite. On peut alors trouver une sphère de centre O et de rayon ρ suffisamment petit pour qu'à l'intérieur de cette sphère il se trouve au plus k points de E en ligne droite. [75, p. 128]

[Suppose that at O the paratingent of rank k of the set E is empty and that the paratingent of rank $k - 1$ contains at least one line. Then we can find a sphere with centre O and radius ρ sufficiently small such that within this sphere there are at most K points of E situated along a straight line].

A further example relates to a geometrical property of a set E whose paratingent of rank k is empty at every point of the derived set of E . Assuming E to be a bounded set with this property, Bouligand establishes:

...il existe un entier N tel que le nombre de points de E susceptibles de se trouver sur une droite quelconque soit inférieur ou égal à N . [75, p. 130]

[...there exists an integer N such that the number of point of E which may be situated on an arbitrary straight line is less than or equal to N .]

Bouligand notes that, while the contingent and paratingent are covariant (in the sense defined above) with respect to the group γ , the closely related concepts of planar, circular and spherical contingents as well as paratingents of higher rank do not retain this property [75, p. 109].

4.3.6 Dimensional properties of sets seen as part of direct infinitesimal geometry

A short chapter of [75] is dedicated to the topic of '*la recherche de caractères dimensionnels d'ensembles fermés*' [research into the dimensional properties of closed sets]. This

section of Bouligand's book presents an accessible and pedagogical introduction to the concepts of dimension discussed in the previous chapter on the Dirichlet problem, concepts which Bouligand first studied in the context of characterising improper sets in the generalised Dirichlet problem. Here, applications to the Dirichlet problem are not discussed, rather Bouligand focusses on introducing the *Cantor-Minkowsky order* of a closed set (described below) as well as certain variants of this concept. He also emphasises that the Cantor-Minkowsky order of a set is invariant under transformations of the group γ .

Defining the Cantor-Minkowsky order. As in the previous chapter, the main idea is that of studying the dimensional properties of a set by considering a natural volume function at scale ρ and then examining how that function behaves as ρ tends to zero. Let E be a closed set in three-dimensional Euclidean space and let $E(\rho)$ denote the set obtained by applying the Cantor-Minkowsky construction (introduced in the previous chapter) to the set E , with spheres of radius ρ . Let $f(\rho)$ denote the volume of the set $E(\rho)$. Then the Cantor-Minkowsky order of E is defined by Bouligand as the number $3 - \alpha$ where α is a real number such that

$$\frac{f(\rho)}{\rho^\alpha}$$

remains finite and non-zero. By this, we mean that there is a real number c such that for sufficiently small values of ρ , we have

$$0 < \left| \frac{f(\rho)}{\rho^\alpha} \right| \leq c$$

The Cantor-Minkowsky order, Bouligand shows has properties which would be expected of a function associated with dimensional properties, for example the number α can never be greater than 3. We note that what we referred to in the last chapter as the global dimensional order of a set can be seen as a generalisation of the Cantor-Minkowsky order above¹³ and still exists in cases where the Cantor-Minkowsky order does not.

4.3.7 Remarks regarding *prolongement fonctionnel* - a key concept not implemented in the context of direct infinitesimal geometry

A further technique highlighted by Bouligand [41] for achieving *Cantorian form* is that of *prolongement fonctionnel* - an idea we have encountered in the previous chapter

¹³To see this, replace the function f_1 in the global dimensional order by ρ^α .

in context of his work on the generalised Dirichlet problem. However, we note that *prolongement fonctionnel* does not appear explicitly in Bouligand's theory of direct infinitesimal geometry as it is introduced in [75] - the first comprehensive account of his theory. The concept does, on the other hand, figure in Bouligand's later epistemological ideas developed during the interwar period and after the publication of his direct infinitesimal geometry. The term *prolongement fonctionnel*, as we will see in the final chapter, is no longer used but we can clearly identify an elaboration of the same ideas in the context of what he refers to as the *stability* of mathematical results. It is worth noting at this stage the significance of *prolongement fonctionnel* from Bouligand's point of view as well as his opinion regarding the role of this concept in achieving Cantorian form.

Prolongement fonctionnel in the context of Bouligand's ideas on a new approach to geometry. In his later accounts of his own work and in his written correspondence to other mathematicians, Bouligand highlights strongly the concept of *prolongement fonctionnel* as a concept of great significance in mathematics. He considers the explicit emphasis he placed on this concept as a noteworthy contribution of his own. In [95, p. 9] he comments regarding his contribution to the generalised Dirichlet problem:

Si je n'ai su donner un résultat aussi complet que celui du M. Norbert Wiener (paru en 1924), cela m'a-t-il du moins fourni l'occasion, en examinant certains énoncés de M. Paul Lévy, d'attirer systématiquement l'attention sur la notion de prolongement d'une fonctionnelle par continuité. . . Peu de temps après, MM. Fréchet, Banach, Cacciopoli, Flamant, et d'autres géomètres publiaient des travaux systématiquement consacrés à divers problèmes de prolongement fonctionnel.

[If I was not able to give as complete a result as that of Mr Norbert Wiener (published in 1924), that [my attempt at extending the Dirichlet problem] at least provided me with a chance, by examining certain propositions of Mr Paul Lévy, to systematically draw attention to the concept of the extension of a functional by continuity..Shortly thereafter, Mr Fréchet, Banach, Cacciopoli, Flamant and other geometers published works aimed systematically at divers problems of functional extension].

In [41, p. 588], Bouligand identifies *prolongement fonctionnel* - a concept which emerged and played a central role in the context of his contribution to the generalised Dirichlet problem - as one means of achieving Cantorian form. He comments in this regard:

...la poursuite systématique du prolongement fonctionnel sera un moyen d'atteindre la forme cantorienne...étant donné un problème géométrique quelconque, supposons-le résolu dans un champ restreint, c'est-à-dire moyennant certaines hypothèses de commodité sur les données ; supposons encore que l'on sache étudier, dans ce champ, comment une variation des données affecte la solution. Si certaines conditions de continuité sont remplies, on pourra passer du champ restreint à un champ plus large. En somme, la solution est considérée ici comme une fonction des données du problème...c'est donc une fonction au sens de l'Analyse fonctionnelle, ou précisément encore, avec le degré de généralité donné par M. Maurice Fréchet à ce terme, dans ses profonds travaux sur les ensembles abstraits

[...the systematic pursuit of functional extension will be a means of achieving Cantorian form...given an arbitrary geometric problem, assume it to be solved in a restricted setting, that is by virtue of certain hypotheses introduced for convenience. Suppose in addition that we know how to study, within this setting, how a variation of the initial conditions affect the solution. If certain continuity conditions are fulfilled, we will be able to pass from the restricted setting into a broader setting. In summary, the solution is considered here as a function of the initial conditions of the problem...it is therefore a function in the sense of Functional Analysis or, more precisely, with the degree of generality given to this term by Mr Maurice Fréchet in his profound works on abstract sets].

Bouligand does not give an example of the principle above in [41] and we do not attempt to give a precise interpretation of the above as the ideas are elaborated and exemplified later in the context of his reflections on the *stability* of mathematical propositions.

4.3.8 Concluding remarks

In this section, we have explored the main ingredients of Bouligand's theory of direct infinitesimal geometry, as they are presented in his first comprehensive account of this topic, published in the form of a textbook [75]. We have seen that this theory puts into practice certain epistemological views, such as the need, from Bouligand's point of view, to deal with mathematical concepts and results in some sort of natural context, called a domain of causality, which is defined in terms of invariance under a group of transformations. Concretely, Bouligand justified the selection of a group of transformations which represent the domain of causality of his theory. Indeed, the key

tools of his direct infinitesimal geometry - the contingent and the paratingent - both possess a particular type of invariance under the transformations in questions. The contingent, which was first used by Bouligand in the context of his work on harmonic functions, generalises the idea of a tangent to a curve and, together with the closely related paratingent, allows for an approach to the study of sets in Euclidean space which does not require, for example, that the functions representing the surfaces being studied be differentiable. Therefore, these tools, together with the various contingents and paratingents of different order, help to put into practice the principle promoted by Bouligand of eliminating as far as possible extra assumptions in differential geometry.

Regarding the paratingent, we have seen how Bouligand used criteria relating to this set of lines to characterise certain familiar classes of lines and surfaces. A further ingredient of direct infinitesimal geometry is the study of the dimensional properties of sets by means of a concept of dimension similar to that first used by Bouligand in the study of improper sets in the context of the generalised Dirichlet problem. As was emphasised by Bouligand himself, his work on the Dirichlet problem - which involved a local geometrical study of sets of points in Euclidean space - inspired to a great extent the his theory of direct infinitesimal geometry.

Chapter 5

Bouligand's theory of Direct Infinitesimal Geometry: applications, dissemination and uptake

5.1 Direct infinitesimal geometry and classical mathematical physics (1935), the example of Meusnier's theorem

In a lengthy article of 1935 entitled *Géométrie infinitésimale directe et physique mathématique classique* [Direct infinitesimal geometry and classical mathematical physics] [87], Bouligand outlined the extension of the principles of his direct infinitesimal geometry to certain considerations in or related to mathematical physics (as this area of research was perceived at the time). Numerous mathematical and epistemological ideas are highlighted. Here, we wish simply to highlight the following three points: first that Bouligand extended and applied his direct infinitesimal geometry to the area he then termed as mathematical physics; secondly we wish to highlight the significance of these applications for Bouligand in terms of justifying or legitimising his theory and lastly, we will focus on a specific example, namely that of Meusnier's theorem. As will be explored below in greater depth, the example of Bouligand's formulation of Meusnier's theorem aims at illustrating how by removing assumptions and applying his direct infinitesimal approach, mathematical results can be viewed in a more general setting, or using Ursyohn's terms as borrowed by Bouligand, in their natural domain of existence.

In this article, he outlines two main themes - the first of these revolves around the concept of a group of transformations being seen as a domain of causality. He describes

this theme as follows:

Rôle primordial des groupes pour la restauration de la causalité logique en géométrie différentielle aussi bien qu'en physique mathématique [87, p. 3].

[The primordial role of groups for the restoration of logical causality in differential geometry as well as in physics].

This idea was encountered at length in [75] but here Bouligand considers these ideas not only in the context of differential geometry but also in the context of mathematical physics. Secondly, throughout his exposition, Bouligand seeks to illustrate that direct methods are present in different areas of mathematics and its applications and that these methods can be seen in a unified manner:

Communauté d'essence des méthodes directes, d'une part dans le calcul des variations, mode d'expression très courant des lois physiques, d'autre part, en géométrie différentielle [87, p. 3].

[The essential commonality of direct methods, on one hand in the calculus of variations, a very common mode of expression of physical laws and on the other hand in differential geometry].

Regarding this second objective, Bouligand aims to show throughout his work that it is by virtue of certain shared characteristics that direct methods in different areas can be seen as unified. Perhaps most central to these common features is the concept of distance. Here, we will focus predominantly on outlining the first of Bouligand's two objectives and on the extension and application of certain ideas introduced in the previous section. We will briefly discuss the unity of direct methods in the next chapter.

Bouligand's exposition is split into three main chapters. The first introduces a number of main concepts in direct infinitesimal geometry, including contingents and paratingents, their key properties and the characterisation of certain types of curves and surfaces by means of the paratingent. Examples of how direct infinitesimal can be applied are provided, for example in the context of extending Meusnier's theorem and Euler's theorem in differential geometry. The second chapter is predominantly dedicated to the idea of a group of transformations as a domain of causality and more specifically, to introducing the main groups of this type which are relevant to direct infinitesimal geometry and its applications. Also discussed at length in this chapter is the idea of what Bouligand refers to as a causal proof in mathematics and

mathematical physics - we will discuss this concept in greater depth in the next chapter. The third chapter of Bouligand's article sets up the necessary mathematical results in order to demonstrate an application of direct infinitesimal geometry to the study of wave propagation, based on a tool defined by Bouligand called the *intégrale contingente* [the contingent integral].

5.1.1 Examples of the extension of key ideas in direct infinitesimal geometry

As we have seen, a central theme or building block of Bouligand's direct infinitesimal geometry is the idea of a domain of causality, or a group of transformations under which certain objects or properties are invariant. As we illustrate below, Bouligand not only related this concept to applications but also saw this connection as a motivating factor for research in direct infinitesimal geometry. He comments for example:

La recherche des groupes intéressant la géométrie infinitésimale est inspirée par le souci d'appliquer le même principe de découverte dans des conditions plus larges [87, p. 29].

[The search for groups of interest in infinitesimal geometry is inspired by the concern of applying the same discovery principle in broader conditions].

As in [75], Bouligand includes a discussion on invariance under different collections of transformations, again emphasising the group of transformations referred to as γ in the previous section. Regarding the interest of this specific group in relation to applications, Bouligand notes,

Au point de vue de la physique mathématique, le groupe γ_1 a une importance capitale¹, car il coïncide avec celui des déformations envisagées dans la théorie classique de l'élasticité et plus généralement dans la mécanique de milieux continus [87, p. 29].

[From the point of view of mathematical physics, the group γ_1 is of vital importance as it coincides with that of the deformations envisaged in the classical theory of elasticity and more generally in continuum mechanics].

The above provides a brief indication of the way in which Bouligand perceived the connection between groups of invariance in mathematical physics and in direct infinitesimal geometry. A second central theme of Bouligand's theory for which he

¹We note that the group of transformations γ_1 here corresponds to the group γ in [75].

considered physical applications is the characterisation of certain geometrical objects, such as curves or surfaces, by means of criteria formulated in terms of the paratingent. Regarding this way of characterising surfaces, Bouligand exemplifies the relevance to physical applications by referring to the properties of the fracture of certain minerals:

Les surfaces qu'on est conduit à distinguer de la sorte ont d'ailleurs un intérêt physique; leurs propriétés expliquent l'aspect macroscopique, d'une certaine régularité relative, de la cassure des minéraux à grain très fin [87, p. 4].

[The surfaces that we are led to distinguish in this manner are of interest from a physical point of view. Their properties explain the macroscopic appearance of a certain relative regularity of the fracture of very fine-grained minerals].

We simply note here that, based on some basic assumptions regarding the nature of the fracture, namely that it is what Bouligand refers to as a '*cassure stable*' [stable fracture] (one for which the shape of the mineral is not altered when subjected to a shock less powerful than that which caused the initial fracture) he deduces basic geometric properties of the mineral. These properties imply that, at any point of the surface in question representing the fracture, there must be at least one line which does not belong to the paratingent. Here Bouligand does not elaborate in any depth - as such the example would seem to be intended simply to outline how the principle of the characterisation of a surface by means of the paratingent can be seen to be applicable in a physical context. Regarding the significance of this application from Bouligand's point of view, the following provides a particular clear insight:

Nous trouvons de la sorte, à propos d'une question de morphologie macroscopique, la classe de surfaces que la théorie des ensembles nous avait conduit à sélectionner, pour des raisons géométriques simples. En résumé, il y a donc un intérêt manifeste à poursuivre l'unification de la géométrie infinitésimale et la théorie des ensembles, si la seconde permet de mieux comprendre la première, celle-ci, par son incorporation à une doctrine, regardée jadis comme purement philosophique, lui assigne des objectifs de recherche qui la rapprochent de la physique [74, p. 492].

[We find in this way, in relation to a question in macroscopic morphology, the class of surfaces that the theory of sets had led us to select for simple geometrical reasons. In summary, there is therefore an evident interest in pursuing the unification of infinitesimal geometry and the theory of sets. The second enables a better understanding of the first which, through its

incorporation into a theory [the theory of sets] considered purely philosophical in the past, assigns to that theory objectives which bring it closer to physics].

As such, from Bouligand's point of view, this example of a physical application of direct infinitesimal geometry helps to justify taking a set-based approach to differential geometry.

5.1.2 The application of direct infinitesimal geometry to Meusnier's theorem

Bouligand comments that a main theme of his direct infinitesimal geometry is the unification of set theory with geometrical concepts usually considered, at that time, in the context of differential calculus [87, p. 2]. He goes on to describe the usefulness of this approach as follows:

...cette unification révèle en matière d'ensembles l'opportunité de notions nouvelles, grâce auxquelles des propositions (exemple : le théorème de Meusnier), indissolublement liées (semblait-il) à la théorie des surfaces, s'affirment valables dans un champ beaucoup plus vaste... [87, p. 3].

[...this unification reveals in terms of sets the opportunity for new concepts, thanks to which propositions (example: Meusnier's theorem) which were inextricably linked (it seemed) to the theory of surfaces now appear as valid in a much broader setting].

The new concepts to which Bouligand is referring include the contingent, the paratangent and the other tools belonging to direct infinitesimal methods introduced in the previous section. The above shows that the ability of direct infinitesimal geometry to reveal mathematical results as applicable in a more general setting justifies for Bouligand the usefulness of his theory. In relation to Meusnier's theorem, which we will introduce below, he states that, through direct infinitesimal approach, the 'véritable signification' [the true meaning] [87, p. 4] of this result becomes apparent. Below we introduce Meusnier's theorem and Bouligand's formulation of it using tools of direct infinitesimal geometry.

Meusnier's theorem in differential geometry states that for a surface S and a point x on S , all curves lying on S and passing through x which have the same tangent line, also have the same normal curvature at the point x . Moreover, the osculating circles of these curves form a sphere. In [83], Bouligand provides an equivalent form

of Meusnier's theorem as follows. Consider a surface S and a point O on S . Let OT be a tangent to S at the point O . Let OZ be the normal to S at the point O . We also consider a point M on S which tends towards O . Bouligand formulates Meusnier's theorem as below:

...toute position limite du cercle tangent en O à OT et passant par M (cercle qui est sur la sphère engendrée par C_M tournant autour de OZ) se trouve nécessairement sur la sphère engendrée par C tournant autour de OZ . [83, p. 32]

[...Every limit position² of the tangent circle at O of OT passing through M (a circle which is on the sphere generated by turning C_M about OZ) is necessarily situated on the sphere generated by turning C about OZ].

For clarity, we will refer to the statement above as Meusnier's theorem (*). Bouligand next states that if we make the following assumption, then we are able to deduce that Meusnier's theorem (*) holds.

... M tendant vers O sur la surface de manière que l'angle MOT tende vers zéro, il existe dans le plan ZOT une position limite unique C pour le demi-cercle C_M contenant le point M et dont le diamètre, porté par OZ , a une extrémité en O [83, p. 32].

[...if M is tending towards O on the surface such that the angle MOT tends towards zero, there exists in the plane ZOT a unique limit position C for the semi-circle C_M which contains the point M , and which has its diameter lying on OZ and has O as an endpoint].

In fact, in order to deduce Meusnier's theorem (*) from the assumption above, Bouligand states that it is not necessary for S to be a surface. We can apply the same reasoning if we replace S with an arbitrary set E in three-dimensional Euclidean space provided we assume the following two conditions are satisfied.

Au point O , la demi-droite OT sera une demi-tangente de cet ensemble, c'est-à-dire qu'il existe une suite de points M de l'ensemble tendant vers O de manière que l'angle MOT tende vers zéro (ce qui implique pour O le fait d'être point d'accumulation) [83, p. 33].

²Here we assume that the limit is with respect to M tending towards O .

[Pour une certaine direction OZ orthogonale à OT , il y a unicité, dans le plan ZOT , de la limite pour un demi-cercle C_M passant par M et dont le diamètre, porté par OZ , a une extrémité O].

The first of these assumptions means that OT is a line belonging to the contingent of E at O while the second assumption corresponds to that made above in order to deduce Meusnier's theorem (*). In this new setting, working with the set E , the statement of the theorem can be slightly reworded in terms of what Bouligand refers to as the *semi-circular contingent*, defined in the same way as the circular contingent introduced above but replacing circles with semi-circles³ [71, p. 482].

The semi-circular contingent of E with respect to O and OT is necessarily situated entirely on the sphere generated by turning C about OZ ⁴.

Regarding the significance of this result from Bouligand's perspective, it is interesting to consider the following comment:

Grâce à l'introduction de deux contingents formés de demi-cercles, le théorème de Meusnier, issu de la théorie des surfaces et prolongeable, comme il est connu, aux trajectoires orthogonales des champs de vecteurs, prend sa forme la plus simple et la plus large en théorie des ensembles [63, p. 138].

[Thanks to the introduction of two contingents formed of semi-circles, Meusnier's theorem, which originates in the theory of surfaces and which is extendable, as it is known, to orthogonal trajectories of vector fields, assumes its most simple and broad form in the theory of sets].

We might question Bouligand's comment above as the theorem stated in the context of sets depends on certain assumptions which are not required in order for Meusnier's theorem to hold for surfaces. It could be speculated that Bouligand considered the class of sets in Euclidean space which satisfy the required conditions as broad enough for the result to belong, in some sense, to this broader setting. In [83], he emphasises that considering Meusnier's theorem in the context of sets allows us to add substantially to the existing class of objects to which the result applies:

³Explicitly, the definition of the semi-circular contingent of a set E with respect to an accumulation point O and a half-line OS would be as follows: a semi-circle belongs to the semi-circular contingent with respect to O and a half-line OS if it is the limit of a sequence of semi-circles tangent at O to OS , passing respectively through the points M_i of a sequence extracted from E , tending towards O where no point M_i is situated on OS nor its extension

⁴The wording used here is not exactly Bouligand's. It is a combination of his wording of Meusnier's theorem above and the statement of the extended result in the context of sets as given in [71, p. 482]

Ici, nous avons supprimé l'hypothèse que l'ensemble est une surface. Et cela nous ramène à faire l'adjonction, aux surfaces de l'énoncé primitif, de certains ensembles ponctuels, répondant aux deux conditions ci-dessus [83, p. 33].

[Here, we have removed the assumption that the set is a surface. And that leads us to adjoining, to the surfaces to which the original statement applies, certain point sets satisfying the two conditions above].

5.1.3 A note on further applications

The applications outlined above serve to illustrate the fact that Bouligand envisaged and applied his direct infinitesimal geometry in different areas, including the theory of surfaces and in even more concrete physical applications. These examples have illustrated how Bouligand saw his theory and the epistemological concepts behind it as achieving their initial objectives and also how he saw physical applications as a factor justifying further research into direct infinitesimal geometry. The diverse applications of direct infinitesimal geometry investigated by Bouligand to problems in analysis and in more applied settings are too lengthy a topic to explore in the present account, therefore we simply outline a few further references below.

Further examples in [75] relate to the study of envelopes introduced by means of the Cantor-Minkowski construction [75, p. 102-103] as well as the study of convex surfaces [75, p. 88-106]. We could also cite, similar to the above example relating to Meusnier's theorem, Bouligand's formulation of Euler's theorem in differential geometry which, in the same way, serves to illustrate that taking a direct infinitesimal approach can allow us to view certain results in what he views as a broader setting⁵. In [95, p. 22-25], Bouligand dedicates a short section to *Applications de la géométrie infinitésimale directe aux équations différentielles, ordinaires ou partielles* [applications of direct infinitesimal geometry to ordinary and partial differential equations]. Here Bouligand seems to consider these applications as being predominately in the area of analysis. The following confirms this point and shows Bouligand's conceptual explanation behind the possibility for such applications:

A l'encontre de la notion de différentielle, valable pour certaines catégories de fonctions, le contingent ordinaire et le paratingent ordinaire s'appliquent à tous ensembles ponctuels d'un espace cartésien (ou même d'espaces vectoriels très généraux)...On comprend donc que l'introduction systématique du contingent et du paratingent appelle la révision de certains chapitres de l'Analyse

⁵Here we refer the reader to [75, 174] or [87, p. 23] for full details of this example.

[95, p. 22].

[Contrary to the concept of a derivative, valid for certain categories of functions, the ordinary contingent and the ordinary paratingent apply to all point sets of a cartesian space (or even to very general vector spaces)...We understand therefore that the systematic introduction of the contingent and of the paratingent call for the revision of certain chapters of Analysis].

For more details regarding a direct infinitesimal approach to certain problems related to ordinary differential equations, we refer the reader for example to the notes included at the end of [75], notably that of his student Georges Durand entitled '*Sur la recherche d'une condition de planéité d'un arc simple à partir du contingent*', which builds on a previous article by Bouligand. For an example of the application of direct infinitesimal geometry to partial differential equations, we refer the reader to [92]. For an account focussed on the application of direct infinitesimal geometry to mechanics, we refer the reader for example to [88] and [89].

In [80], Bouligand outlines some broad ideas, without entering into any mathematical details, regarding how an approach in the spirit of direct infinitesimal geometry⁶ might be applied in the characterisation of the recently introduced phenomenon of uncertainty, as encapsulated by Heisenberg's uncertainty principle. A main point of emphasis in this article is the potential usefulness of the application of geometrical methods based on the theory of sets to the physical phenomenon of uncertainty. As such, Bouligand comments that these reflections may interest mathematicians, philosophers and physicists alike [80, p. 737].

⁶Bouligand does not refer explicitly in the body of the article to direct infinitesimal geometry but the approach highlighted is a local geometrical one in which the concept of tangency is key. Concepts such as that of limit sets (also prevalent in his direct infinitesimal geometry) are at the forefront and [75] is cited.

5.2 Efforts to circulate and popularise the ideas of direct infinitesimal geometry

It is clear that Bouligand put a great deal of energy into publishing and popularising the methods and principals of his direct infinitesimal geometry, perhaps more so than any other area of research in his long mathematical career. These publications served different purposes, including communicating his theory to the mathematical research community, communicating the tools of his theory in a more pedagogical manner and popularising the principles of his theory to a broader scientific audience. Regarding his mathematical research papers on direct infinitesimal geometry, we will see below that these were published not only in French journals but also, notably, a significant number of papers were published in Polish journals. In addition to published articles, direct infinitesimal geometry represented a central part of Bouligand's teaching activities (we will expand upon this point below). He established a strong following of students who undertook research into direct infinitesimal geometry, although this point will be covered in the following section on the reception of Bouligand's work.

5.2.1 Research papers published in Polish journals

The year 1932 appears as the most intensive year for publishing mathematical papers on direct infinitesimal geometry and is the year in which Bouligand published his *Introduction*. In addition to this book, at least seven articles on various concepts and problems relating to direct infinitesimal geometry were published in French mathematical journals, including four short papers in the *Comptes Rendus*, submitted by Elie Cartan [67], [68], [69], [71], two in the *Journal de Mathématiques Pures et Appliquées* (see [63] and [64]) focussing on the application of direct infinitesimal geometry to Meusnier's theorem and one in the *Bulletin de la Société Mathématique de France* [66]. In addition to his output of articles in the most popular French journals such as these, it is interesting to highlight that Bouligand published quite a significant number of papers on direct infinitesimal geometry in Polish journals. Specifically, these include approximately seven articles relating to direct infinitesimal geometry published at the same time and shortly before the publication of Bouligand's *Introduction*, between 1930 and 1932. Two articles [52], [60] were published in *Fundamenta Mathematicae*, introduced above, two in the *Annales de la Société Polonaise de Mathématique* [48], [49] and three papers relating to direct infinitesimal geometry in the *Bulletin international de l'Académie polonaise des sciences et des Lettres* [55], [62], [76].

In addition to presenting the mathematical tools, results and applications of direct infinitesimal geometry, Bouligand also emphasises, for example in [55] the theme

of applying set theory to infinitesimal geometry, highlighting how this had already occurred in the theory of functions. In this article, Bouligand also emphasises the intrinsic nature of his approach:

...on voit s'ouvrir une voie où la Géométrie Infinitésimale pourrait se développer d'une manière complètement intrinsèque [55, p. 420].

[...we see a path opening up by which Infinitesimal Geometry could be developed in a completely intrinsic manner].

It is interesting to note that this article appears in the section of the journal entitled *théorie des ensembles* [set theory]. Bouligand emphasises the ideas of applying set theory to infinitesimal geometry and the possibility of taking an intrinsic approach here but does not emphasise certain other main principles of direct infinitesimal geometry, such as the concept of a domain of causality. We can speculate that this choice was made bearing in mind the ideas likely to have the greatest impact on the readership of Polish journals at the time.

In addition to publishing a significant number of articles related to direct infinitesimal geometry in Polish journals, it is worth noting that Bouligand published in a number of other foreign journals, perhaps the next most significant example being that of Belgian journals such as the *Bulletin de la Société Royale des Sciences de Liège* in which he published several papers during the time period in question (see for example [88], [89]). We note, in addition, a small number of articles published in Italian journals such as the *Annali della Scuola Normale Superiore di Pisa* (see for example [81]) and the *Rendiconti della reale Accademia dei Lincei* (see for example [102]) and finally an article published in the Romanian journal *Bulletin Scientifique de l'École polytechnique de Timișoara* (see [82]). This distribution of publications can be seen as being representative of Bouligand's efforts to communicate his mathematical work more generally in foreign journals during the interwar period.

5.2.2 Articles published in L'Enseignement Mathématique and Bouligand's participation in the international topological conference of 1935

In addition to the fact that the first comprehensive account of Bouligand's direct infinitesimal geometry appeared in the form of a textbook [75] aimed at a general mathematical audience, it is also worth noting that he published a handful of articles on topics relating to direct infinitesimal geometry in the journal *L'Enseignement Mathématique*. Three of these articles [59], [72], [73] appeared in 1931 and 1932 and

dealt with specific aspects of direct infinitesimal geometry, one focussing on the the concept of an *ensemble d'accumulation* [accumulation set] [59], as discussed above, including the utility of this concept for example in the context of the local geometrical study of sets in Euclidean space by means of tools in direct infinitesimal geometry such as the contingent and paratingent. A second [72] article deals with semi-continuity with respect to inclusion - a concept also considered in Bouligand's *Introduction*, which had been published earlier the same year. These articles focus first on the broader use and utility in mathematics of the concepts of accumulation set and of semi-continuity respectively before illustrating their application in terms of the tools of direct infinitesimal geometry. A third article [73] in *L'Enseignement Mathématique* deals briefly with the theme of characterising certain curves by means of conditions placed upon the paratingent.

While the above three articles in *L'Enseignement Mathématique* deal with specific aspects of direct infinitesimal geometry, a later article of 1937 [94] contains the ideas presented by Bouligand during the *Conférence Internationale de Topologie* held at the University of Geneva between 21 and 25 October 1935. The conference in Geneva was the second international topological conference, the first having been held at the University of Moscow in September of same year. Both conferences were attended by prominent mathematicians from different countries, for example in Geneva the attendees included Elie Cartan, the Polish mathematician Kazimierz Kuratowski, Karl Menger, Paul Finsler and two founding members of the Bourbaki group, André Weil and Charles Ehresmann. A larger representation of Polish and Soviet mathematicians can be seen in the first conference in Moscow, together with prominent American participants such as John von Neumann and Garrett Birkhoff.

Bouligand's article includes an introduction of the main tools and concepts of direct infinitesimal geometry and is particularly focussed on the invariance of these concepts with respect to certain groups of transformations, including but not restricted to the group γ introduced above, as well as the results that can be derived through these properties. Bouligand again considers Meusnier's theorem as an example of the applications of his theory but considers this theorem here not only in a Euclidean setting but in the setting of other metric spaces endowed with certain types of metric, namely a Riemannian metric and a Finsler metric. It is worth noting that Bouligand also highlights his epistemological views in this presentation:

...devant l'abondance des résultats mathématiques, une oeuvre de coordination se poursuit, pour préciser les hypothèses et dégager le pourquoi des faits...Pour les questions de causalité ainsi posées, l'idée de groupe donne un guide [94,

p. 7]

[...with the abundance of mathematical results, an effort of coordination is being pursued in order to specify the hypotheses and reveal the why behind the results...For questions of causality posed as such, the idea of a group provides a guide].

5.2.3 Direct infinitesimal geometry as a part of Bouligand's teaching activities

In addition to his publications in *Enseignement Mathématique*, direct infinitesimal geometry appears to have been a core part of Bouligand's teaching activities. While our account is focussed on the interwar period, it is instructive nevertheless to note that this theory still formed a central part of his teaching more than two decades after its inception. Bouligand notes in [101, p. 7]

...Arnaud Denjoy, quand il occupa la chaire de Géométrie supérieure, m'y demanda des conférences de géométrie infinitésimale directe. Je les continuai jusqu'en 1957, où cette branche devient l'unique objet de mon cours.

[...Arnaud Denjoy, when he held the chair for advanced Geometry⁷, asked me to give lectures on direct infinitesimal geometry. I continued these until 1957, when this branch became the sole subject of my course].

Did direct infinitesimal geometry also represent an active area of teaching during the interwar period including Bouligand's time at the *faculté des Sciences de Poitiers*? As will be explored in the following section, Bouligand certainly guided doctoral students in carrying out their own research in this area. We are not aware of direct infinitesimal geometry being a central part of Bouligand's teaching programme in Poitiers, where he held first the position of *Professeur de Mécanique Rationnelle et Appliquée* and subsequently that of *Professeur de calcul différentiel et intégral*. Given Bouligand's active role in the scientific life of the university, we can reasonably speculate that advanced seminars might have been organised on the topic but we have yet to find evidence of this.

⁷Here Bouligand is referring to Denjoy's position as chair for advanced geometry at the *faculté des Sciences de l'Université de Paris* in the 1940s.

5.2.4 Concluding remarks

We have seen that Bouligand made intensive efforts to popularise his direct infinitesimal geometry, efforts which were most concentrated around the time at which he published his *Introduction*. These efforts were targeted at the mathematical community not only through French journals but notably through Polish journals - in which his presentation seems to be adapted to the main themes of research of the Polish school. Direct infinitesimal geometry continued to represent a central teaching activity throughout his career. Finally, we highlight that the articles in which Bouligand first presented his programme of research which culminated in his direct infinitesimal geometry [41], [51] were published respectively in the *Revue Scientifique* and the *Revue Générale des Sciences Pures et Appliquées* - journals which were both targeted at a broad scientific audience. These articles were discussed in greater depth at the beginning of this chapter.

5.3 The uptake of Bouligand's ideas on direct infinitesimal geometry.

In what way was Bouligand's theory received by the mathematical community? It would seem that Bouligand's theory was not considered as part of the mainstream of mathematical research during the interwar period. This is implied, for example, by the introductory remarks of Christian Pauc's doctoral thesis of 1939, supervised by Fréchet and introduced below, in which Pauc describes Bouligand's ideas on direct infinitesimal geometry as '*encore insuffisamment connues*' [still insufficiently known] [191, p. 5]. In this section, we will survey the positive of reception Bouligand's work following the publication of his *Introduction* in 1932 and during the remainder of the interwar period, focussing on mathematicians who were directly influenced by Bouligand's own work.

A key figure to consider when evaluating the reception of Bouligand's ideas is Elie Cartan. Although Cartan was not active as a researcher in the area of direct infinitesimal geometry, we will establish below that he played an influential role in circulating Bouligand's ideas and took an active interest in this theory. Below, we will also explore the reception of Bouligand's work through the research of doctoral students guided by him in their research relating to direct infinitesimal geometry. We will see in this case that certain students adopted not only Bouligand's mathematical ideas but also echoed his more philosophical motivations regarding causality. Finally, we will explore the active uptake of direct infinitesimal geometry by mathematicians between 1932 and

1939, including notably Christian Pauc, Otto Haupt and André Marchaud. We include just one exception regarding the time period set of 1932-1939 by considering the work of Gustave Choquet on direct infinitesimal geometry. Choquet interacted directly with Bouligand and his work in this area represents perhaps one of the most prominent examples of the uptake of direct infinitesimal geometry.

5.3.1 Elie Cartan's interest in direct infinitesimal geometry and his role in circulating Bouligand's ideas

Perhaps the most striking indication of Elie Cartan's interest in Bouligand's theory and his most substantial contribution to its promotion came in the form of his preface to Bouligand's *Introduction à la géométrie infinitésimale directe* of 1932. Cartan reinforces both the mathematical relevance of the tools of Bouligand's theory, such as the contingent and the paratingent '*dont M. Bouligand montre l'importance fondamentale*' [of which M. Bouligand shows the fundamental importance] [75, p. VI] and emphasises the epistemological aspects of Bouligand's work, notably the belonging of the concepts of direct infinitesimal geometry to specific groups of invariance. Regarding the latter, Cartan comments for example that the contingent and paratingent belong to the area of '*topologie restreinte*' [75, p. VI-VII]. He highlights, furthermore, the aspect of Bouligand's theory which seeks to eliminate unnecessary hypotheses, describing Bouligand's *Introduction* as,

...un exposé didactique servant d'introduction à une nouvelle géométrie différentielle libérée des restrictions artificielles que l'ancienne s'imposait par raison de commodité [75, p. VI].

[...a didactic exposition serving as an introduction to a new differential geometry freed from the artificial restrictions that the former theory imposed upon itself for convenience].

This point of view is echoed in Cartan's 1937 article on the role of analytic geometry in the evolution of geometry, representing his contribution to the ninth *Congrès International de Philosophie* of 1937. Cartan comments,

Il existe maintenant tout un ensemble de recherches, constituant ce que M. Bouligand appelle la géométrie infinitésimale directe, dans lesquelles les géomètres cherchent à résoudre les problèmes sans y mettre plus que ce que contient leur énoncé. Ils s'écartent ainsi de la géométrie différentielle classique [112, p. 153].

[There now exists a whole body of research constituting what Mr Bouligand calls direct infinitesimal geometry in which geometers seek to solve problems without putting in more than their statement contains. This research differs therefore from classical differential geometry].

Coming back to Cartan's preface of Bouligand's *Introduction*, we note finally that he provides an informal historical account in which he places Bouligand's work. Notably, Cartan emphasises that the effect of the development of recent ideas in set theory and the theory of functions were slow to penetrate into infinitesimal geometry and portrays Bouligand's theory as being part of this trend [75, p. VI].

We also recall that it was Cartan who submitted Bouligand's articles on direct infinitesimal geometry to the *Comptes Rendus* journal in the period in question, for example the following articles of 1932 [67], [68], [69], [63]. We find however an example of Cartan's more direct involvement in Bouligand's direct infinitesimal geometry in his note [111] on a 1935 paper of Bouligand [90] regarding Meusnier's theorem from the point of view of direct infinitesimal geometry (more specifically certain properties of the sphere formed by the osculating featuring in this theorem). In Cartan's note, he provides an alternative proof for a result given by Bouligand.

Finally, as a minor point reinforcing the view that Cartan took a keen interest in direct infinitesimal geometry, we note that, in the Elie Cartan archives available at the archives of the *Académie des Sciences* [134], Cartan records in a rough notebook, certain elements under the heading 'géométrie différentielle directe'. These appear to be, potentially, rough notes taken by Cartan purely for his own reference, regarding recent developments in this area of mathematics. He includes brief notes on Bouligand's work and on the work of Pauc introduced briefly below.

5.3.2 Bouligand's influence on doctoral students

In Elie Cartan's preface to Bouligand's *Introduction*, he comments regarding direct infinitesimal geometry that Bouligand '...a su aiguiller dans cette voie toute une équipe de jeunes chercheurs...' [was able to guide a whole team of young researchers in this direction] [75, p. VI]. It is fair to say that the adoption of Bouligand's concepts and programme of research represent a significant part of doctoral research in mathematics taking place in Poitiers during the interwar period. To this effect, Leloup comments in [170, p. 126] regarding Bouligand's influence on doctoral theses during the interwar period:

Mise à part la thèse de Marie Charpentier, Bouligand apparaît comme le

dénominateur commun à tous les doctorats soutenus à Poitiers. Cette étude révèle en outre le rôle d'organisateur de la recherche joué par Bouligand. Il impose et fait travailler ses propres thématiques de recherche aux étudiants de Poitiers. De plus, cette influence dépasse le cadre de la seule faculté de Poitiers.

[Apart from the thesis of Marie Charpentier, Bouligand appears as the common denominator of all doctorates defended in Poitiers. This study reveals the role played by Bouligand as an organiser of research. He imposes his own themes of research and initiates work therein by students at Poitiers. In addition, this influence goes beyond the university of Poitiers alone].

Leloup subsequently refers to the doctoral theses completed between 1931 and 1938 of ten young mathematicians influenced by Bouligand, many of which were defended in Poitiers, others in Paris or Toulouse. A majority of these doctorates related to themes of direct infinitesimal geometry. The thesis of Georges Durand, defended in Paris in 1931 [129] and that of Gaston Rabaté [203], defended in Toulouse are clear examples of doctoral students being guided in Bouligand's own programme of mathematical research. Moreover, these two young mathematicians echo Bouligand's epistemological ideas in their work. The introduction to Durand's thesis *Sur une généralisation des surfaces convexes* cites several of the same mathematicians and concepts which influenced Bouligand in formulating his theory and which were considered above - these include Baire's reflections regarding the need for achieving greater generality, Lebesgue's doctoral thesis and the works of Janiszewski and Juel. The first section of Durand's introduction ends with the following conclusion regarding a perceived need to restore *causality* in geometry:

Beaucoup de théories géométriques nécessitent à l'heure actuelle une révision à la faveur de laquelle la causalité, masquée par les restrictions de commodité, soit remise en pleine lumière... [129, p. 2].

[Many geometrical theories require, at the present time, to be revised so that causality, which is masked by restrictions introduced for convenience, may be brought back to light fully].

In the same vein, the introduction of Rabaté's thesis *Sur les notions originelles de la géométrie infinitésimale directe* begins as follows:

Dans ses recherches récentes procédant du souci de restaurer la causalité en Géométrie Infinitésimale, M. G. Bouligand a introduit le contingent et le

paratingent d'un ensemble ponctuel en un point d'accumulation... [203, p. 1].

[In his recent research which seeks to restore causality in Infinitesimal Geometry, Mr G. Bouligand introduced the *contingent* and the *paratingent* of a point set at an accumulation point...]

While they do not reflect on the epistemological motivations behind the theory, the works of Lucien Chamard [116], Jean Mirguet [184], Charles Brunold [109], Show-Lien Chow [121] and Louis Pasqualini [190] are all centered around certain aspects of direct infinitesimal geometry or its applications. Of the students mentioned above, Bouligand was involved in the thesis defense jury (either as a member or as president) for Chamard, Brunold, Show-Lien Chow and Pasqualini. Mirguet, Brunold, Show-Lien Chow and Pasqualini all dedicate their work to Bouligand in the thesis acknowledgement. Regarding Durand and Rabaté, Bouligand is not a member of the jury but it is clear that Bouligand is the main guide and influence. For a more detailed account of these doctoral theses in relation to Bouligand's academic influence in Poitiers during the interwar period, we refer the reader to Leloup's account in [170, p. 122-131].

5.3.3 Notable examples of the uptake of direct infinitesimal geometry

In Bouligand's 1961 note on his mathematical and epistemological works [101, p. 6], he refers briefly to several notable examples of mathematicians who took an active interest in direct infinitesimal geometry or in tools introduced in the context of this theory. Among these mathematicians are notably Gustave Choquet, André Marchaud, Christian Pauc and Otto Haupt. We will briefly comment on the involvement of these mathematicians in direct infinitesimal geometry as well as exploring briefly the contributions of S.K. Zaremba. Given the connection from Bouligand's perspective between direct infinitesimal geometry and the work of Karl Menger, highlighted above, it is also interesting to explore briefly certain comments made by Menger regarding Bouligand's theory.

The work of Gustave Choquet. Gustave Choquet (1915-2006) was a French mathematician whose research interests included potential theory, topology, measure theory and functional analysis. In [131] Edwards comments,

His work in functional analysis and potential theory profoundly marked the development of mathematical analysis in the second half of the twentieth century. In particular, he created the theory of capacities, as well as that of integral representations in convex sets [131, p. 341].

In Choquet's account of his own mathematical works, he specifically identified direct infinitesimal geometry as one of his research interests [120, p. 10]. Here, Choquet identifies his interactions with Bouligand as one of the factors influencing his doctoral thesis *Application des propriétés descriptives de la fonction "contingent" à la théorie des fonctions de variable réelle et à la théorie différentielle des variétés cartésiennes*, defended in 1946 and published in 1947 [118]. As described by Edwards, Choquet's doctoral thesis 'is a study of the differentiability properties of subsets of Euclidean spaces, and is a pioneering contribution to non-smooth analysis which reveals profound relations between certain differentiable and topological structures'. Edwards comments that the key result of Choquet's thesis is the so-called contingent-paratingent theorem [131, p. 346], which is formulated in terms of generalisations of the tools introduced by Bouligand. For a detailed account of this result, we refer the reader to [131, p. 346] or [120, p. 13-18]. In [120, p. 13], Choquet identifies a specific problem of direct infinitesimal geometry posed by Bouligand:

En 1943, Georges Bouligand avait posé le problème de déterminer si, dans la classe des surfaces de R^3 qui sont C^1 (i.e. à plans tangents continus), les portions de sphère sont caractérisées par la propriété d'avoir en tout point un contingent sphérique réduit à une seule sphère. Je réussis, par une étude géométrique de la courbure, et grâce au fait que cette courbure est une fonction de 1ère classe de Baire, à donner une réponse positive au problème, tout en affaiblissant encore ses hypothèses...

[In 1943, Georges Bouligand had posed the problem of determining whether, in the class of surfaces in R^3 which are C^1 (i.e. which have continuous planar tangents), the portions of sphere are characterised by the property of having at every point a spherical contingent reduced to a single sphere. I succeeded, by a geometrical study of the curvature and thanks to the fact that the curvature is of the 1st class of Baire, in giving a positive response to the problem while further weakening the hypotheses].

Choquet's results were published in the *Revue Scientifique* [117], the same journal in which Bouligand published several of his articles focussed on his epistemological ideas. Choquet comments that his experience with Bouligand's problem provided him with inspiration [120, p. 13] in solving a further problem dealt with in his thesis which had been posed by Lebesgue in his *Leçons sur l'intégration*. Specifically, the problem was that of determining the real continuous function f given α and g where f and α are real continuous functions on the unit interval and g is the derivative of f with respect to α [131, p. 346]. Another example of a publication by Choquet which uses concepts of direct infinitesimal geometry is [119], which for example extends the

contingent-paratingent theorem of his doctoral thesis [120, p. 58]. For a more detailed account of Choquet's work on direct infinitesimal geometry including his generalisation of the concepts of contingent and paratingent, for example in the context of a general metric space, we refer the reader to the doctoral thesis of Bessis [9].

Christian Pauc's research relating to direct infinitesimal geometry.

Christian Pauc (1911-1981) was a French mathematician who was a student of Ernest Vessiot [107, p. 258] and of Maurice Fréchet, who was the advisor for Pauc's doctoral thesis [191]. According to Krickerberg in [160] Pauc came to Germany in the early 1940s as a prisoner of war. The Erlangen mathematician Otto Haupt managed to have Pauc released so as to collaborate with him in Erlangen. Pauc's doctoral thesis of 1939 [191], completed under Fréchet, dealt with *Les méthodes directes en calcul des variations et en géométrie différentielle*. The jury consisted of Arnaud Denjoy as presiding member and of Fréchet and Bouligand as examiners. Pauc's thesis was split into two parts: the first dealing with direct methods in the calculus of variations and the second part deals with direct methods in the area of direct infinitesimal geometry. In the first section, Pauc is influenced, notably, by the work of Karl Menger and Bouligand's ideas are strongly present in the second section. Pauc comments in the introduction,

On trouvera dans ce qui suit le résultat des études de l'auteur sur les méthodes directes en Calcul des Variations et en Géométrie Différentielle. Nous avons cru rendre service en encadrant celles-ci dans un exposé d'ensemble de très importantes recherches sur ces sujets dues principalement à MM. Bouligand et Menger et encore insuffisamment connues [191, p. V].

[We find in the following the results of the author's research on direct methods in the Calculus of Variations and in Differential Geometry. We believe we have done a service [to the mathematical community] by placing this research in the context of a comprehensive exposition of highly important research on these subjects due to Mr Bouligand and Mr Menger which are still not sufficiently known].

Pauc's results relate for example to continua in Euclidean space defined by certain criteria formulated in terms of the contingent and the paratingent. For example '*les continus euclidiens dont le contingent en chaque point est non connexe*' [Euclidean continua whose contingent at each point is non connected] or those whose paratingent at each point contains only one line [191, p. VIII].

The reception of Bouligand's ideas by Otto Haupt.

The German mathematician Otto Haupt (1887-1988) is a notable example of a mathematician outside of France who took up Bouligand's ideas on direct infinitesimal geometry. Haupt was based for a significant part of his career at the University of Erlangen and his research focusses notably on geometry and real analysis. He also produced a significant output in the form of mathematical textbooks. Haupt demonstrated a positive reception of Bouligand's work both through a review of Bouligand's *Introduction* and by incorporating tools of direct infinitesimal geometry into his own work. In Haupt's review in German of Bouligand's *Introduction à la géométrie infinitésimale directe*, he comments,

...führt das Buch in diejenigen Untersuchungen ein, bei welchen die moderne mengentheoretische Geometrie für die Differentialgeometrie nutzbar gemacht wird [146].

[...the book introduces those investigations whereby modern set-theoretic geometry is made usable for differential geometry].

Haupt expresses similar views to Bouligand regarding the classical approach to differential geometry relying heavily on calculation, thus reducing the generality of the results. Further, in addition to describing Bouligand's approach, Haupt cites results obtained by Bouligand in the theory of surfaces under more general assumptions, namely Meusnier's theorem and Euler's theorem. Haupt thus portrays Bouligand's work in a positive light and describes it as *'leicht verständlich geschrieben'* [written in an easily understandable way] as well as *'eine sehr anregende Einführung'* [a very stimulating introduction].

As mentioned, Haupt also employs tools of Bouligand's direct infinitesimal geometry and takes inspiration from research problems identified by Bouligand. For example, in a 1936 paper [147] *Über ebene Punktmengen mit überall unendlicher Krümmung* [On planar point sets with infinite curvature everywhere], Haupt investigates the structure of certain sets in two-dimensional Euclidean space which are noted by Bouligand in a section of [75] entitled *'sujets de recherche'* [research topics]. Haupt explicitly cites Bouligand in this respect; we refer the reader to [75, p. 221-222] and to [147] for further details. Furthermore, in a 1939 paper [148], Haupt extensively employs the paratingent, citing Bouligand with regard to this concept, in the context of his study of ruled surfaces in three-dimensional projective space.

The work S.K. Zaremba on paratingent equations. Stanisław Krystyn Zaremba (1903-1990), the Polish mathematician and son of the Stanisław Zaremba encountered earlier in this account, also conducted research, published in French, involving

concepts of direct infinitesimal geometry, citing Bouligand extensively. For example, his 1936 work [225] deals with a generalisation of the concept of a system of ordinary differential equations by means of the paratingent, a topic also dealt with more briefly in an earlier article of 1934 published in the *Comptes Rendus* and submitted by Elie Cartan [224]. While it is outside of the scope of the present account, it is interesting to note that quite a significant number of papers authored by Polish mathematicians were published in the 1960s dealing with Zaremba's *équations au paratingent* [paratingent equations]. While there may well be external factors leading to a renewed interest in these concepts, there is nonetheless a clear link to Zaremba's work. For example, Krzyzowa's 1963 article [226] establishes this connection via the work of A. Bielecki.

André Marchaud's use of the contingent in the study of differential equations.

André Marchaud (1887-1973) also undertook research using concepts of direct infinitesimal geometry, explicitly citing Bouligand. Marchaud was strongly influenced by Paul Montel [170, p. 236] and his research interests included for example differential geometry and *géométrie finie* (as introduced earlier in the present account). During the period in question, Marchaud authored a number of papers involving concepts of direct infinitesimal geometry. For example, in a 1934 article [177] Marchaud uses the contingent in the context of his research on first order differential equations of a real variable, explicitly citing Bouligand with regard to this tool. In a 1936 paper [178], he uses the contingent and the contingent integral, again citing Bouligand, in the context of his research on a concept of integration for certain types of convex sets, also comparing his results to those of Bouligand.

Karl Menger on the connections between direct infinitesimal geometry and the 'géométrie des distances'.

We noted above that Bouligand considered his direct infinitesimal geometry as belonging in some sense to the same trend - one of the emergence of direct methods in differential geometry. Menger himself comments in his contribution to the 1935 *Conférences internationales de topologie* on Bouligand's work, praising his efforts while also putting forward the view that the *géométrie des distances* perhaps goes even further than Bouligand's theory along the path towards a differential geometry dissociated from coordinates by operating in the setting of more general spaces:

Et même M. BOULIGAND qui a eu le mérite en créant sa Géométrie infinitésimale directe d'introduire l'analyse moderne, en particulier la théorie des fonctions de variable réelle dans l'étude des propriétés géométriques locales - se borne à l'étude d'espaces où chaque point est (ou pourrait être) caractérisé par un système de coordonnées. L'idée d'une géométrie différentielle

sans coordonnées semble encore aujourd'hui presque absurde à la plupart des géomètres ; cependant la géométrie des distances a déjà résolu le problème...
[182, p. 362].

[And even Mr. BOULIGAND who has the merit, by creating his direct infinitesimal Geometry, of introducing modern analysis and in particular the theory of functions of a real variable in the study of local geometric properties - limits himself to the study of spaces where each point is (or could be) characterised by a system of coordinates. The idea of a differential geometry without coordinates still seems, today, almost absurd to most geometers; however the geometry of distances has already resolved the problem. . .]

Although they took place after the time period in question, it is worth noting a series of presentations given by Menger at the Sorbonne in the Spring of 1951 [183], for which Bouligand and Fréchet, it would appear, participated in the organisation. In the corresponding article, Menger refers to Bouligand a handful of times and identifies for example the fundamental motivations behind his own work as being the same as those motivating Bouligand's direct infinitesimal geometry - namely the goal of taking a local approach to the study of objects in space which does not rely on coordinates [183, p. 15-16]. In other words, this shared goal was avoiding an approach to such a study where 'la fin n'est justifiée que par les moyens' [the end is justified only by the means] [183, p. 15-16].

Chapter 6

The development of Bouligand's epistemological ideas from 1932 to 1939

We saw in the previous chapter that Bouligand's theory of direct infinitesimal geometry was the culmination of a programme of research which was motivated substantially by his epistemological views regarding geometry. The following three key concepts were interrelated but again could be considered separately as follows. First, Bouligand promoted the idea of considering mathematical results in natural or appropriate domain, which for him is achieved by means of the concept he referred to as a domain of causality - in this case characterised in terms of invariance under a certain group of transformations. In addition, Bouligand emphasised eliminating unnecessary hypotheses to achieve greater generality and finally he promoted taking a more intrinsic or 'direct' approach, which in the case of differential geometry involved for example avoiding parametric representations. Certain elements of these ideals and the mathematical tools through which Bouligand sought to put them into practice were present earlier in his work on the generalised Dirichlet problem, in particular the idea of classifying mathematical results according to their level of generality as well as mathematical tools such as those used by Bouligand to study the dimensional properties of sets in Euclidean space.

The purpose of this final chapter is to consider the way in which Bouligand's epistemological ideas about mathematics evolved after the publication of his *Introduction à la géométrie infinitésimale directe* in 1932 and for the remainder of the interwar period. In keeping with the central theme of the present account - namely concentrating on Bouligand's mathematical and epistemological concepts which interrelate most closely, we will focus on a few key ideas. Specifically, we will focus on the concepts which directly evolved from the epistemological views already introduced in the context of

the previous two chapters. For the interested reader, we will briefly outline other questions considered by Bouligand during this period which are not quite so central to our account. It will become apparent in this chapter that certain views communicated earlier by Bouligand in the context of differential geometry were later considered in a broader setting. We will focus here on the following three main concepts: first what Bouligand referred to as causal demonstrations of mathematical results, secondly Bouligand's elaboration of the concept of a domain of causality and finally his reflections on direct methods. As well as introducing these concepts and providing examples, we will also bring to light the efforts made by Bouligand to communicate these ideas to the mathematical and scientific community as well as in a pedagogical setting. Connections between Bouligand's earliest epistemological reflections on intuition and his later ideas of causality and direct methods are also highlighted below, mainly in the context of direct methods.

Bouligand's published output expressing his ideas on the epistemology of mathematics between 1932 and 1939 includes approximately 15 publications on various topics, predominantly but not exclusively connected with causality, direct methods and related concepts. While his views regarding intuition and geometry considered in the earlier chapters were published in *La Revue Scientifique* and *La Revue Générale des Sciences Pures et Appliquées*, the channels used for communicating his later ideas were more varied. In addition to several articles published in these two journals during the time period in question, Bouligand's work was also published in the *Actualités Scientifiques et Industrielles* a series of books, published by Hermann and covering a wide range of topics in science, technology and industry. The *Actualités Scientifiques* included collections on different topics under the direction of an expert in the domain in question. For example, Bouligand's 1934 account on *La Causalité des Théories Mathématiques* [83] was part of the *exposés de philosophie* collection of Louis de Broglie. Bouligand's 1937 work on *Structure des Théories, problèmes infinis* was part of Abel Rey's collection on the history and philosophy of science. It is interesting to note that three publications of Bouligand's ideas during this period are based on presentations given at conferences and later published as part of a book. Two of these were published in the *Actualités Scientifiques et Industrielles* and related respectively to Bouligand's contributions to the *Congrès international de philosophie scientifique* in 1935 and the ninth *Congrès International de Philosophie* of 1937. The third example of this kind is to be found in [103] which was published by Flammarion as a book in the *Bibliothèque de Philosophie Scientifique* series and containing presentations given at the University of Poitiers in 1935 in the context of a conference which appears to have been organised by Bouligand and his former student Charles Brunold. Further channels used by Bouligand for communicating his ideas include the journal *Thalès*, which focusses on the history of science, the *Bulletin de l'Académie Royale des Sciences de Belgique* and

one article in each of the *Comptes Rendus* of the French *Académie des Sciences* and *L'Enseignement Mathématique*. In the following presentation of Bouligand's ideas, we will take a thematic approach, drawing from the most relevant sources for each of the main topics outlined: causal demonstrations, domains of causality and direct methods.

6.1 Bouligand on causal demonstrations

Bouligand's concept of a causal proof represents a component of his concept of causality which is elaborated on in several different articles during the period in question and interrelates with his concepts of domain of causality and direct method. He briefly mentions the idea of a *démonstration causale* of mathematical results prior to the time period of interest in the present chapter, specifically in a paper of 1930 [51, p. 369], although the concept is elaborated on in greater depth in later accounts, upon which we will base the present discussion.

6.1.1 Causal demonstrations defined in terms of their ability to reveal the 'reason why'

In [77], Bouligand gives an account of causal demonstrations based on an example which we will discuss below which emphasises, as the key defining feature, the explanatory nature of the proof. The example relates to the following fact in geometry. If we consider a parabolic object rolling without sliding on a flat surface, then the focus of the parabola traces out a curve known as a catenary. Bouligand states that this result could be proven using an analytic method but that this approach would not be enlightening with respect to understanding why the result holds. Bouligand comments regarding this problem:

En le soumettant à une méthode analytique permanente, l'on obtient une chaînette. Mais, le calcul ainsi réalisé ne nous apprend rien sur les relations de cause à effet, c'est-à-dire sur la manière dont la forme parabolique du profil et la position spéciale de la particule sur la plaque...interviennent pour imposer en fin de compte la trajectoire de cette particule de rentrer dans la catégorie particulière des chaînettes [77, p. 257].

[By applying to it a permanent analytic method, one obtains a catenary. But the calculation carried out as such teaches us nothing about the relationships of cause and effect, that is regarding the way in which the parabolic shape of the profile and the particular position of the particle on the rolling

plate...intervene so as ensure that the trajectory of the particle belongs to the specific category of catenaries].

What proof can be given then which, from Bouligand's point of view, does provide a more enlightening demonstration of the theorem in question? Bouligand emphasises selecting a specific definition of a catenary which can be seen, through an elementary geometric argument, to correspond to the geometrical nature of the trajectory of the parabola's focus. In Bouligand's own words:

...la trajectoire du foyer, on le montre par un raisonnement élémentaire, est une ligne dont la tangente reste à distance constante du pied de la verticale menée par son point de contact...On se ramène donc ainsi à l'une des définitions de la chaînette... [77, p. 257-258]

[...the trajectory of the focus, we show through elementary reasoning, is a line whose tangent remains at a constant distance from the base of the vertical led through its contact point...We therefore arrive in this way at one of the definitions of a catenary...]

Here, Bouligand goes on to emphasise the importance of selecting appropriate definitions which lend themselves well to establishing causal proofs:

Cet exemple est instructif. On y voit l'influence d'une option convenable parmi les définitions équivalentes...d'une même entité mathématique. La propriété de la chaînette par laquelle nous venons de la distinguer se présente le plus simplement possible, car elle revêt la forme d'une équation différentielle algébrique du premier ordre, c'est-à-dire d'ordre minimum, à laquelle satisfait cette courbe. Et cela nous confirme que nous avons bien atteint, dans cette voie, la cause profonde du théorème énoncé. [77, p. 258].

[This example is instructive. We see the influence of a suitable choice among the equivalent definitions...of the same entity. The property of the catenary by which we have distinguished it appears in the most simple way possible as it assumes the form of a first order algebraic differential equation, that is of minimum order. And this confirms that we have indeed reached, in this way, the underlying reason for the stated theorem].

Regarding Bouligand's above statement about having reached the underlying reason for the theorem, no further explanation is provided in this article. We speculate that Bouligand is emphasising that only first order differentiability is involved in the proof

and not extra assumptions regarding higher order differentiability of the function characterising the catenary. We see above that Bouligand discusses simplicity in the context of causal proofs, in the sense of a reliance on the minimum set of assumptions. However, he goes on to emphasise that a causal proof is not necessarily an easy proof:

Deux démonstrations d'un même théorème peuvent être également inattaquables, mais il peut se faire que l'une soit causale et l'autre non... [77, p. 258].

[Two proofs of the same theorem may be equally unassailable but it may be that one is causal and the other is not...]

6.1.2 Causal demonstrations characterised in terms of generality

While the above shows that Bouligand initially characterised causal proofs in terms of their ability to reveal the reason why behind a result, the most recent definition in the time period in question, in [103] defines a causal demonstration purely in terms of generality. His definition in [103, p. 170] is as follows:

Il peut arriver que l'énoncé d'un théorème aboutissant à une conclusion fixée par avance, ne contienne aucune hypothèse superflue, toute réduction des prémisses étant de nature à compromettre cette conclusion. C'est dans ces conditions, rarement réalisées, que la démonstration sera dite: causale.....C'est en cherchant les conditions les plus larges dans lesquelles un théorème déterminé est valable qu'on parvient à sa démonstration causale. Autrement dit, le souci de généralité va de pair avec celui de la causalité.

[It can happen that the statement of a theorem leading to a conclusion fixed in advance contains no superfluous assumptions. Any reduction of the assumptions would compromise the conclusion. It is in these conditions, rarely achieved, that the proof will be called *causal*...It is in searching for the broadest possible conditions in which a given theorem is valid that we reach a causal demonstration. In other words, the issue of generality goes hand in hand with that of causality].

As such, in the above account, Bouligand equates causal proofs with those which are based on the minimal set of initial assumptions. He goes on to provide examples of causal demonstrations, referring notably to his formulation of Meusnier's theorem taking recourse to concepts in direct infinitesimal geometry. As seen in the previous chapter, these concepts allow Bouligand to enlarge the class of objects for which

Meusnier's theorem holds by adding a certain class of sets to the familiar surfaces which lead to this result. We note that in [103], Bouligand does not attempt to justify his formulation of Meusnier's theorem discussed in the previous chapter being the most general formulation. It is not clear whether, by referring to this example, Bouligand is claiming that his formulation of Meusnier's theorem is the most general, or perhaps that the most general setting in which one can consider the theorem is a set-theoretic setting, rather than merely considering surfaces. Based on Bouligand's comments relating to Meusnier's theorem considered in the previous chapter, the author of the present account would speculate that the latter would perhaps be the best interpretation. It is also worth emphasising that this example was provided here in the context of a conference aimed at a general and multidisciplinary academic audience, as such the examples may have been chosen to illustrate intuitively the main ideas behind Bouligand's views on causal proofs.

A further example provided by Bouligand in [103] is that of Pythagoras' theorem. It can be easily seen that the familiar formulation of Pythagoras' theorem is equivalent to the following. Consider a right angle triangle ABC with hypotenuse BC . The area of the square constructed on BC is equal to the sum of the areas of the squares constructed on AB and AC . This result can be generalised to any set of three similar polygons constructed on the sides of the triangle ABC . Regarding this fact, Bouligand comments in relation to Pythagoras' theorem:

...et la vraie raison s'en dégage. Car si le théorème est vrai pour un type de figure tracé sur BC , il sera pour tous les autres types, et il suffit de le vérifier pour une configuration particulière, qu'on obtient en prenant le triangle ABC lui-même... [103, p. 172]

[...And the true reason [for the validity of of Pythagoras theorem] is revealed. Since, if the theorem is true for one type of figure drawn on BC , it will also be true for all other types of figure and it suffices to verify this for a particular configuration that is obtained by taking the triangle ABC itself].

A further aspect of Bouligand's views of causal proofs in relation to generality is seen by referring back to the example above regarding the trajectory of the focus of a rolling plate. He comments that, whether or not a particular proof is causal depends on the level of generality of the result in question. To clarify this point, he considers again the example of the rolling parabolic shape and states that, had we have posed a more general question to begin with, the analytic method which he did not consider

as providing a causal proof in the above example, may well have provided a causal demonstration:

...il suffit souvent de s'élever d'une question à une autre dont la première soit un cas particulier pour que l'axe de la causalité soit complètement déplacé...Lorsque qu'une plaque se meut dans son plan, le mode de calcul permanent qui détermine la trajectoire d'un point de la plaque revêt, ipso facto, un caractère causal, précisément en raison de la généralité du problème, où l'on abstrait telle ou telle spécialisation imposée au profil de la plaque. Mais dans notre exemple particulier, ce profil était une parabole...et nous devons expliquer comment ces circonstances impliquaient pour la trajectoire la forme chaînette. D'où un déplacement important de la causalité [77, p. 258].

[...it often suffices to rise from one question to another of which the first is a specific case in order for the axis of causality to be completely shifted...When a plate is moving in its plane, the permanent method of calculation which determines the trajectory of a point has, *ipso facto* a causal character, due precisely to the generality of the problem where we abstract a particular specificity imposed upon the plate. But in our particular example, the profile was a parabola...and we had to explain how these circumstances implied that the shape of the trajectory would be a catenary. From this, we see a significant shift of causality].

6.2 An elaboration of the concept of domain of causality and the related concept of the stability of a mathematical proposition

In the context of Bouligand's epistemological reflections on geometry leading up to his theory of direct infinitesimal geometry, and in the theory itself, he discusses the importance, from his point of view, of the concept of a domain of causality. We have seen that, for Bouligand, this concept represents, intuitively, an appropriate setting or domain in which a mathematical result should be placed. The domain of causality is, in addition, closely tied up with the goal of viewing a given mathematical result in a general setting, getting rid of unnecessary assumptions. Perhaps the most central concept in Bouligand's characterisation of domains of causality is that of a mathematical group. In his direct infinitesimal geometry, Bouligand equates the domain of causality with a group of transformations under which certain properties are invariant. During the interwar years following the publication of his *Introduction à*

la géométrie infinitésimale directe, Bouligand made considerable efforts in elaborating, abstracting and to some extent formalising his ideas about domains of causality. Before exploring his ideas in greater detail, we emphasise the centrality of the concept of a group, from Bouligand's point of view. This is conveyed strikingly in a letter from Bouligand to Maurice Fréchet dated 24 November 1938 [135], in which Bouligand outlines the development of some of his main ideas to date regarding causality:

Pour ce qui concerne les questions de causalité, Destouches¹ m'en avait dit un mot ces jours-ci, je crois intéressant de vous en retracer l'histoire...

[Regarding questions of causality, Destouches spoke to me about it recently; I believe it is of interest to retrace, for you, the history of these questions].

Bouligand emphasises, as the first step in the development of his ideas on causality, the influence of the Polish school and the articles featuring in the journal *Fundamenta Mathematicae*, including the ideas of Urysohn discussed in the previous chapter. However, he emphasises that in the work of Urysohn, as is also the case of the French mathematicians mentioned below, the emphasis is placed on eliminating unnecessary assumptions in order to view mathematical results in their *domaine naturel d'existence*, which is only one aspect of Bouligand's reflections regarding causality. Urysohn's account does not focus, according to Bouligand, on the concept of a mathematical group. While Bouligand notes the influence of the ideas of Urysohn and the Polish school on his own work (*L'influence de tout ce courant d'idées sur mes propres recherches est très nette*' [the influence of this whole trend of ideas on my own research is very clear]), he nevertheless emphasises that there is a key ingredient of his own ideas on causality which is not present in the work of Urysohn:

Voilà des remarques d'une profonde psychologie...mais où n'apparaît aucune trace du rôle de la NOTION DE GROUPE. Urysohn jette surtout un anathème contre les hypoth. parasites. C'est ce que vous avez déjà fait vous-même. C'est aussi ce que Baire avait fait dans la Préface de ses Leçons sur les Fonctions Discontinues. J. Tannery n'avait-il pas eu, d'une manière certes beaucoup moins marquée, des mouvements dans ce sens....?

[These remarks illustrate a profound psychology...but they show no trace of the role of the CONCEPT OF A GROUP. Urysohn denounces parasitic hypotheses. This is what you yourself have done. It is also what Baire did in the preface of his *Leçons sur les Fonctions Discontinues*. Did J.

¹Here Bouligand is referring to Jean-Louis Destouches.

Tannery not also make efforts, albeit in a much less pronounced way, in this direction....?]

As such, Bouligand appears to be emphasising to Fréchet the distinct component of his ideas on causality which relates to the characterisation of domains of causality in terms of mathematical groups. With this in mind, we will briefly present Bouligand's ideas together with examples.

6.2.1 Defining domains of causality

Having outlined informally the main ideas behind Bouligand's concept of domains of causality, we will now consider the way in which he went some way in elaborating and formalising this concept. In [96, p. 21-22], the following definition is provided.

Soit, dans un champ défini de prémisses, une proposition P dont l'énoncé a été préalablement formulé. Supposons P vraie pour un choix des objets qu'elle met en relation.

[Under circumstances which have been defined, let P be a proposition whose statement has been formulated. Assume P to be true for a choice of objects which P relates together].

To clarify the above, we note that in [94, p. 7], Bouligand states that these circumstances could for example be that we are working in Euclidean space. Next, Bouligand goes on to consider the modifications which could be performed on these objects and states that the collection of all such possible modifications form a group.

Les modifications auxquelles on peut soumettre ces objets, pour passer à un cas d'exactitude de P à un nouveau cas d'exactitude de P , forment une famille qui présente ces deux caractères importants:

1. *avec une modification, elle contient toujours la modification inverse*
2. *avec deux modifications, elle contient leur résultante.*

Une telle famille est donc un groupe, au sens classique de ce terme [96, . 22].

[The modifications to which we may subject these objects to go from one correct version of P to another correct version of P form a family which has the following two important characteristics:

1. for any given modification, the family always contains the inverse modification

2. for two modifications, the family contains their composition.

Such a family is therefore a group, in the classical sense of the term].

In [94, p. 7], Bouligand notes that it is by reducing the set of hypotheses of the proposition P (consisting of hypotheses and a conclusion) that we reach the above group. Based on the above, Bouligand goes on to justify why he refers to the family of modifications introduced above as the domain of causality of P :

Puisque c'est la recherche des conditions les plus larges de validité de P qui fait apparaître ce groupe, je l'ai dénommé le domaine de causalité de P [96, p. 22].

[Since it is the search for the broadest conditions in which P is valid that leads us to this group, I have called it the *domain of causality* of P].

Here Bouligand is referring to the idea previously encountered in the last chapter that by eliminating unnecessary hypotheses and considering a mathematical result in a more general setting, we thus are able to appreciate to a greater extent the relationship between the initial assumptions and the conclusion. As we have seen, he associates the ability to perceive this connection with the term causality.

It is worth noting that Bouligand considered the idea of a domain of causality not only in relation to individual mathematical theorems but also in relation to systems of mathematical results:

Ce qui vient d'être dit pour une proposition isolée s'applique sans changement à un système de propositions, les éléments communs à plusieurs groupes formant un groupe [96, p. 22].

[What has just been said about an isolated proposition can also be applied without modification to a system of propositions, as the elements common to multiple groups form a group].

What examples of domains of causality are provided by Bouligand? In [96, p. 22-23], he refers to the example of Bézout's theorem in the context of planar algebraic curves, which states that a curve of degree m intersects a curve of degree n in mn points [208, p. 109]. In order to establish this result without exceptions, it is not sufficient to operate within the real or the complex plane, whereas the result does become true in general in the complex projective plane, by enabling both intersection points at infinity (thanks to operating in a projective space), as well as, for example, intersection points

with the same location in the real plane but with a different imaginary part. Bouligand comments as follows regarding this example and regarding the connection between the idea of domains of causality and the structure of mathematical theories:

Dans cette question, le champ réel ne conduisait pas à des résultats permanents. Il a fallu d'abord modifier les prémisses par l'introduction des éléments imaginaires...Toutefois, ce résultat comportait-il des cas d'exception...Cet exemple me paraît propice à l'acquisition d'une idée capitale : celle de la structure d'une théorie, structure que les chercheurs sont obligés de respecter. Pour édifier d'une manière satisfaisante la théorie...des intersections, en géométrie algébrique...L'espace projectif complexe s'est-il donc imposé comme domaine de causalité [96, p. 22-23].

[In relation to this question, the field of the reals did not lead to permanent results. It was necessary first to modify the conditions by introducing imaginary elements...However, the result still allowed certain exceptions...This example seems to me to be conducive to acquiring a key idea: that of the *structure of a theory*, a structure which researchers are obliged to respect. To build a theory of intersections in algebraic geometry in a satisfactory way...the complex projective space was established as the domain of causality].

Bouligand does not include any detailed explanation of the connection between this example and the preceding definition. It is relevant to bear in mind in relation to this point that the *Actualités scientifiques* is likely to be targeted at a broad scientific readership.

A further example provided by Bouligand following the same definition of the domain of causality is available in [94, p. 8], where he again cites for example the group of transformations constituting the *topologie restreinte du premier ordre* introduced in the last chapter as the domain of causality in relation to certain properties of the tools of direct infinitesimal geometry, such as the invariance of the intersection of the paratingent of two sets under transformations of this group.

To gain a further insight into Bouligand's ideas regarding domains of causality, we note that the above example is situated in the context of a discussion regarding the search for domains of causality in geometry. Bouligand outlines two possible approaches that could be taken to arriving at a situation in which mathematical results in differential geometry are placed within their domains of causality and expresses a clear preference

for one of these approaches. He notes that one could attempt to consider each specific theorem and generalise it to the most general space possible. Here he alludes to certain geometrical results valid in Euclidean space which can be generalised to metric spaces endowed with certain types of metric [94, p. 8]. However, Bouligand does not advocate this approach:

Mais la méthodologie disjonctive ralentirait le travail. Il vaut mieux encadrer les résultats dans des groupes familiers, inclus dans le groupe topologique général G des transformations ponctuelles continues et biunivoques opérant entre portions d'espaces cartésiens (on se limite ici au point de vue local, ce qui dispense de distinguer G du groupe analogue extrait d'une variété de Riemann)...

[But the disjunctive method would slow the work. It is better to frame the results in familiar groups included in the *general topological group* G of continuous and bijective point transformations operating between portions of Cartesian spaces (we are limiting ourselves here to the local point of view, which removes the need to distinguish G from the analogous group extracted from a Riemannian manifold)...

Therefore, regarding the way in which we attempt to place results in differential geometry in their domains of causality, Bouligand favours the approach of searching first for the appropriate groups of invariance, potentially in a less general space such as Euclidean space. He justifies this approach by stating that the nature of this approach enables one to easily translate the domain of causality in the less general setting to an analogous domain of causality in a more general space.

6.2.2 The stability of mathematical propositions - a concept closely related to domains of causality

Closely related to Bouligand's ideas regarding domains of causality is his concept of the stability or instability of a mathematical proposition. Together with domains of causality, Bouligand views the concept of stability as a further tool in establishing and understanding the structure of a mathematical theory [96, p. 28]. While the domain of causality of a proposition equates with finding a setting in which a fixed conclusion is invariant under a family of modifications to the initial hypotheses, the stability of a proposition relates to the way in which a conclusion might vary when the hypotheses are varied in a certain way. In Bouligand's own words:

Les considérations qui précèdent sont relatives à l'équilibre d'un énoncé propositionnel donné [considérations relatives aux domaines de causalité], équilibre

maintenu pour certaines modifications des objets que cet énoncé met en relation. Au lieu d'étudier l'état permanent du dit énoncé devant ces modifications, on peut en rechercher les conditions de variance [en étudiant la stabilité ou instabilité d'une proposition] [91].

[The preceding considerations [considerations relating to domains of causality] relate to the equilibrium of a given proposition, an equilibrium which is maintained for certain modifications of the objects which this proposition relates together. Instead of studying these modifications, we can search for the variance conditions [by studying the stability or instability of a proposition].

To further clarify Bouligand's view on the distinction between the role of domains of causality, the following assessment provides a similar insight in different words:

...la notion de groupe offre...un cadre à la classification des théorèmes pris isolément, tandis que la stabilité donnera prise sur les propositions à conclusion variables [91, P. 588].

[...the concept of a group offers a framework for the classification of theorems taken in isolation, whereas stability will enable us to get a hold on propositions with a variable conclusion].

How does Bouligand go about defining the stability or instability of a mathematical proposition? The basic idea here is that of considering a theorem to be made up of assumptions and a conclusion and to view the conclusion as a function of the assumptions. Bouligand defines a proposition as stable if this function is in some sense continuous and also considers in detail how this continuity could be defined. As such, he comments regarding the concept of stability:

De la sorte, on introduit en logique le point de vue de la théorie des fonctions [91, p. 583].

[In this way, we introduce, in the field of logic *the point of view of the theory of functions*].

The following proposition is an example provided by Bouligand of a stable mathematical proposition: **zero slope implies constant** (in a given interval) [96, p. 26]. Here, Bouligand explains, continuity has been defined by means of a specific 'module de déplacement' (a unit of distance) for the assumptions and for the conclusion. In this

case, the unit of distance for the assumptions is the maximum of the absolute value of the slope of the function in question and the unit of distance for the conclusion is defined in terms of the oscillation of the function in question within the interval [96, p. 26]. He notes, further, that the converse of this theorem (constant function implies zero slope) is true but not stable. However, Bouligand emphasises that the stability of a mathematical proposition depends on the way in which we define units of distance.

As we have seen above, Bouligand defines stability in terms of continuity - the continuity of a function that maps initial assumptions onto a conclusion. In the above example, this is achieved by means of the definition of the continuity of a function in terms of the concept of distance. However, Bouligand states that we may also use a topological definition of the continuity of the function in question, that is involving the concept of the neighbourhoods of a point. As noted by Bouligand [91, p. 583] this point of view appears to have been developed by a young Jean-Louis Destouches under Bouligand's guidance and operating in the setting of the *espaces abstraits* [abstract spaces] of Maurice Fréchet [138]. We refer the reader to [91] for a far more detailed account by Bouligand of the concept of stability.

6.2.3 Connecting Bouligand ideas regarding functional extension to the later concept of stability

In the earlier chapter regarding Bouligand's efforts towards the generalised Dirichlet problem, we encountered the concept referred to by Bouligand as *prolongement fonctionnel* [functional extension]. We have seen that Bouligand's interest in such questions stemmed from his encounter with a 1923 textbook of Paul Lévy on functionals. In this setting, we recall that functional extension, for Bouligand, referred to the process of extending the domain of functionals while ensuring that certain continuity properties are retained. He applied his results on this problem, for example, to extending Green's function in potential theory. He was subsequently motivated to carry out what he considered a functional extension of the Dirichlet problem - a task which Bouligand considered achieved with Norbert Wiener's formulation of the generalised Dirichlet problem.

In the previous chapter on Bouligand's direct infinitesimal geometry, it was seen that Bouligand identified *prolongement fonctionnel*, in articles of 1927 [41] and 1930 [51] as one of the methods for achieving what he initially referred to as Cantorian form in geometry - a way of formulating differential geometry which brought causality, in Bouligand's sense of the word, to light. In what way does the concept of functional extension relate to Bouligand's later philosophical reflections, between 1932 and 1939? It would appear reasonable to identify Bouligand's ideas regarding the stability of

mathematical propositions as an effort to formalise the earlier concept of *prolongement fonctionnel* in the broader setting of mathematical propositions. Based on the above account of the idea of stability, this can be seen by recalling the following comments of Bouligand in his 1927 paper [41] which provides the initial programme of research later culminating in his direct infinitesimal geometry:

...la poursuite systématique du prolongement fonctionnel sera un moyen d'atteindre la forme cantorienne...étant donné un problème géométrique quelconque, supposons-le résolu dans un champ restreint, c'est-à-dire moyennant certaines hypothèses de commodité sur les données ; supposon encore que l'on sache étudier, dans ce champ comment une variation des données affecte la solution. Si certaines conditions de continuité sont remplies, on pourra passer du champ restreint à un champ plus large. En somme, la solution est considérée ici comme une fonction des données du problème, qui peuvent être très complexes : c'est donc une fonction au sens de l'Analyse fonctionnelle, ou plus précisément encore, avec le degré de généralité donné par M. Maurice Fréchet à ce terme, dans ses profonds travaux sur les ensembles abstraits...
[41, p. 588].

[...the systematic pursuit of functional extension will be a way of reaching Cantorian form...given any geometrical problem, assume it to be solved in within a restricted setting, that is by virtue of assumptions introduced for convenience regarding the inputs to the problem. Assume, furthermore, that we know how to study, within this setting, how a variation of the inputs to the problem affects the solution. If certain continuity conditions are satisfied, it will be possible to go from a restricted setting to a broader setting. In short, the solution is considered here as a function of the inputs to the problem: it is therefore a function in the sense of Functional Analysis, or more precisely yet, [it is a function] with the degree of generality given to this term by Mr Maurice Fréchet, in his profound works on abstract sets...]

The above citation provides us with at least two key insights. First, we see that the concept referred to by Bouligand as *prolongement fonctionnel* in the context of his programme of research to reformulate differential geometry was later considered and somewhat formalised in the context of mathematical propositions under the name of the stability or instability of mathematical results. Secondly, we gain a further insight into Bouligand's concept of stability - namely that it has its origins in the idea of extending the setting in which we consider a given result in order to reach its most

suitable domain - the latter concept being formalised through Bouligand's domains of causality.

6.2.4 The elaboration of Bouligand's ideas on direct methods

Bouligand qualified his approach to differential geometry as being direct, as this approach seeks to avoid parametric representation, favouring a more geometric study of the local properties of sets in Euclidean space. Between 1932 and 1939, Bouligand elaborated on, in a handful of published works, the concept of a direct method. He considers direct methods not only in differential geometry but in diverse areas of mathematics. This concept interrelates closely with his ideas regarding causality, causal proofs and with his earlier reflections on intuition in mathematics.

In contrast with Bouligand's reflections on domains of causality and the stability of mathematical results, he does not attempt to formalise the idea of a direct method. Bouligand's account of direct methods are of a more conceptual and discursive nature. There does not appear to be a definition, provided by Bouligand of direct methods, although we can explore the way in which he described and exemplified this concept. Given that no definition is provided, it would seem advisable not to interpret the following elements of Bouligand's discussion of direct methods as separate components which must all be present in order to qualify a method as being direct.

Consistently present in Bouligand's discussion of direct methods is the close connection with the concept of causality. In short, direct methods contribute to achieving causality in a mathematical theory. This basic idea is encapsulated well in the following:

Les méthodes directes, en géométrie ou ailleurs, visent à remettre en lumière la causalité, masquée par l'exclusif recours aux procédés analytiques [65, p. 499].

[Direct methods in geometry or elsewhere aim to bring causality back to light, which is hidden by the exclusive use of analytic methods].

The above highlights not only the connection between direct methods and causality but also indicates another main theme emphasised by Bouligand, namely the contrast between direct methods and analytic or algorithmic methods. As examples of algorithmic methods, Bouligand specifies Picard's method of successive approximations - an iterative method for solving first order differential equations by means of increasingly accurate approximations. In addition, Bouligand cites methods for solving problems in mathematical physics which rely on integral equations and the use of methods in

geometry and mechanics which rely on coordinates [79, p. 4]. Although Bouligand describes direct methods by contrasting them with algorithmic methods, he does not consider them as mutually exclusive:

Loin d'exclure les méthodes de calcul, les méthodes directes tendent plutôt à les discipliner en les orientant vers le meilleur rendement...L'agorithme sera dans telle catégorie de problèmes un a postieri dont l'examen direct aura préalablement révélé la meilleure adaptation [83, p. 7].

[Far from excluding methods of calculation, direct methods tend rather to discipline them by orienting them towards the most productive outcome...The algorithm will appear *a posteriori* in a given category of problems for which a direct examination will have already revealed the best adaptation].

Bouligand's idea of direct methods relates to his earlier reflections on intuition in mathematics. This principle is encapsulated clearly by Bouligand in the following:

...en présence de deux éléments extrêmes, l'intuition et la logique, l'attitude du mathématicien, en optant pour le méthodes directes, procède du souci de ne jamais rompre, au cours de ses déductions, le contact avec l'intuition [83, p. 13].

[...in the presence of two extreme elements, intuition and logic, the attitude of the mathematician in opting for direct methods stems from the concern of never losing contact, throughout the course of his or her logical deductions, with intuition].

Here Bouligand refers the reader to his articles on intuition discussed earlier and published in the *Revue Scientifique* [44], [46], [54], thereby clearly establishing the connection with his earlier reflections on this concept. We recall that, in Bouligand's characterisation of intuition, making connections with geometrical concepts or objects from the physical world is key. The link between this aspect of his characterisation of intuition and the concept of direct methods is present in the following:

...l'état d'esprit qui s'attache à la recherche de méthodes directes est inséparable d'une conviction profonde de l'objectivité des mathématiques [83, p. 12].

[...the mindset associated with the search for direct methods is inseparable in relation to a profound conviction of the objectivity of mathematics].

A natural question in considering Bouligand's idea of direct methods is that of the difference between this concept and his reflections on causal proofs. The answer would appear to be that a direct method is considered as an approach to a fairly broad area of mathematics (such as differential geometry or the calculus of variations) whereas a causal proof relates to a specific mathematical result. Bouligand advocates the use of causal proofs as a part of establishing direct methods, stating that the mathematician searching for direct methods,

...poursuivra sa tâche par la recherche d'une démonstration causale, mettant bien en évidence le rôle de chaque hypothèse et abandonnant les suppositions parasites [83, p. 13].

[...will pursue his or her task through the search for a causal demonstration, making apparent the role of each hypothesis and abandoning parasitic assumptions].

Having considered divers ways in which Bouligand described direct methods, it is instructive to consider a few examples of existing areas of mathematics in which Bouligand considered direct methods as being present. Given that he did not attempt to formalise the concept of a direct method itself, it is unsurprising that explicit efforts are not made to connect these examples back to the descriptions of direct methods above.

Three examples of direct methods in mathematics identified by Bouligand. Perhaps the origin of Bouligand's use of the term 'direct method' [65, p. 498] and also a key example he provides is the approach to problems in the calculus of variations based on what are commonly referred to as direct methods in the calculus of variations [65, p. 498]. The calculus of variations is an area of mathematics with diverse applications which deals with minimising (or maximising) functionals. For example, the brachistochrone problem in classical mechanics can be regarded as a problem in the calculus of variations: given a starting point A and an end point B in a vertical plane, determine the curve traced out by a point which is acted on only by gravity and which goes from A to B in the shortest time. Techniques of direct methods in the calculus of variations were introduced at the start of the 20th century, for example by Hilbert and Lebesgue. The previously existing methods, referred to as the classical approach, relied on introducing differential equations. In contrast, direct methods do not require the introduction of differential equations and deal directly with the functional we wish to minimise or maximise ².

²For further details regarding direct methods in the calculus of variations including historical

A further example given by Bouligand [79, p. 10] of an area of mathematics in which direct methods are present is the approach to studying the solubility of polynomial equations originating in the work of Evariste Galois (1811-1832), who is considered to be one of the founders of modern abstract algebra. Prior to the ideas of Galois, the classical approach to solving polynomial equations was that of searching for a certain type of formula known as a radical expression (otherwise referred to as solving by radicals). Formulas of this type exist for quadratic, cubic and quartic equations but no such general formula exists for quintic polynomial equations. Galois' ideas, which were published posthumously and which were later developed into a formal theory, provide a general method for determining whether or not a given polynomial equation can or cannot be solved by radicals without having to find any such formulas. In its more modern form, Galois theory is formulated in the setting of mathematical fields. While the mathematical terms are not central in the present account, we note that the key idea revolves around studying the group of automorphisms of a field extension obtained by adjoining the roots of the equation in question to the field in which the coefficients of the polynomial belong. If this group has certain properties, then we can deduce whether or not the equation is solvable by radicals. For an in depth account of Galois theory and information regarding its historical development, see [6], [207].

We cite one further example of what Bouligand considered to be a direct method, namely a technique in differential equations which he refers to as *intégration qualitative* [qualitative integration] and which belongs to what is referred to today as the qualitative theory of differential equations. Bouligand explicitly identifies this approach as a direct method [83, p. 32-39]. The qualitative approach was introduced around the end of the 19th century, for example in the work of Henri Poincaré, as reported by Petrovitch [193]. The basic premise motivating this approach is that, in the majority of all possible cases, differential equations cannot be integrated by means of known functions. The qualitative approach does not involve searching for a solution of the differential equation but rather involves studying directly the properties that a solution must have, for example its shape, its points of intersection and so forth. In addition to techniques in analysis, this approach incorporates topological and geometric methods.

Bouligand on the unity of direct methods. In the above examples, Bouligand

elements, see [124]. While the details of this method are not important in the context of the present account, we note that the basic idea of direct methods in the calculus of variations is that of constructing what is referred to as a minimizing sequence of functions - that is, if $J[y]$ is the functional we wish to minimize, then we must construct a sequence of functions y_n such that $\lim J[y_n] = \mu$ where μ is equal to the infimum of $J[y]$. If y_n has a limit function y^* and if our function $J[y]$ has the property of being lower semi-continuous at y^* , then the minimizing function we are looking for is indeed y^* , that is $J[y^*] = \mu$.

identified what he considered as direct methods in diverse areas of mathematics. Did he make any attempt at further justifying giving the same name of direct methods to diverse techniques? Indeed, Bouligand wrote extensively on what he referred to as the unity of direct methods in a lengthy essay published in 1933 entitled '*Essai sur l'unité des méthodes directes*' [79], whose content originated in three conferences given by Bouligand at the University of Liège in May that year. Bouligand's approach in this work was to consider methods in differential equations, the calculus of variations and differential geometry (specifically his own methods in direct infinitesimal geometry) and to show that these methods, which he considered as direct, shared certain common characteristics. These common characteristics consist, for example, of concepts in topology such as the limit sets and accumulation sets encountered briefly in the previous chapter, as well as certain metrics and the concept of semi-continuity.

6.3 Placing Bouligand's reflections in the context of three landmarks in French epistemology of mathematics: Borel, Cavailles and Lautman

In order to place Bouligand's epistemological reflections in better context, it is useful to briefly consider a few different major landmarks in the epistemology of mathematics in France. We have chosen to introduce some of the main ideas of Emile Borel, Jean Cavailles (1903-1944) and Albert Lautman (1908-1944). It is interesting to review the ideas of Borel as he represents, like Bouligand, a mathematician who took an interest in the epistemology of mathematics with a practical purpose related to the advancement of mathematics itself, and its applications. Borel was a prominent figure of the French mathematical community with whose ideas Bouligand certainly came into direct contact. Lautman and Cavailles, who we introduce briefly below, were philosophers of mathematics rather than mathematicians. However, both famously had a highly advanced grasp of the subject and their works represent landmarks in the epistemology of mathematics.

6.3.1 Emile Borel

Certain of Borel's epistemological reflections about mathematics have been encountered in earlier chapters. We have seen already that Borel reflected on the roles of logic versus intuition in mathematics, in particular in relation to the educational reform of 1902. It was seen that he promoted an intuitive approach to the teaching of geometry (although intuition was not defined) and emphasised geometrical representations as a means of facilitating mathematical practice by allowing intuition to operate. Regarding

set theory, we have seen that Borel, through his *Leçons sur la théorie des fonctions* and the related collection of publications for which he was scientific director, highlighted the usefulness of set theory in the theory of functions and other areas of mathematics. Caroline Ehrhardt explores in detail [132] the way in which Borel used the *Collection de monographies* as an 'intermediary between teaching and research' - therefore his views regarding the fruitfulness of set theory in the theory of functions is communicated in a pedagogical way, with a view to increasing the uptake of such ideas in a research setting.

In addition to the elements of Borel's views already encountered, we elaborate briefly below on his reflections regarding the status of mathematics in relation to physics and, very much within the same theme, his contributions to what is referred to today as the philosophy of approximation.

Borel on the role of mathematics in relation to physics. In Borel's reflections, he promotes a clearly empiricist and utilitarian view of mathematics which is closely tied to applications in physics. Borel explores this theme in some depth, for example, in his *Introduction géométrique à quelques théories physiques* [geometrical introduction to certain physical theories] [14], published in 1914. Here he emphasises for example that:

La Science mathématique toute entière doit son origine et la plupart de ses progrès à l'observation et à l'expérience; cette origine ne doit pas être méconnue [14, p. V].

[The entirety of mathematical Science owes its origin and most of its progress to observation and experiment; this origin must not be ignored].

Borel goes further than this in stating that the goal of mathematical theories is defined in relation to their applications:

...le but propre de la discipline mathématique est d'abstraire les éléments communs aux réalités diverses, de manière à créer des théories dont le champ d'application soit aussi large que possible [14, p. V].

[...the goal of the mathematical discipline is to abstract the elements common to diverse realities in such a way that leads to creating theories whose field of application is as broad as possible].

Therefore Borel sees the mathematician's role as that of serving physics:

Epurer les concepts mathématiques qui sont suggérés par les théories physiques nouvelles, les vider de leur contenu physique pour les étudier en eux-mêmes, voilà la tâche propre du mathématicien. Il travaille ainsi à fournir au physicien des instruments de travail adéquats à ses besoins futurs [14, p. VI].

[To refine the mathematical concepts that are suggested by new physical theories, emptying them of their physical content in order to study them in themselves, this is the task of the mathematician. He works in this way to provide working tools to the physicist which are adequate for their future needs].

Further to Borel's view that mathematicians should develop mathematical theories as tools for the physicist, he also emphasises the importance, from his point of view, of focussing sufficiently on newer, less well studied mathematical theories which may correspond to newly discovered physical theories or phenomena. Specifically, he emphasises the importance of elaborating not only classical mathematical theories such as differential and integral calculus applied to the study of physical phenomena for which continuity is assumed but also emphasises developing mathematical theories which lend themselves to areas of physics in which continuity is not assumed. These include for example settings in which we are dealing with atoms or molecules. Therefore, Borel promoted placing more attention on theories such as statistical mechanics, which can be applied in such settings, for example in the study of gases. This position promoted by Borel can be found in the final note of [14, p. 116-137] on *Les théories moléculaires et les mathématiques* [Molecular theories and mathematics], to which we refer the reader for a more complete account including Borel's interpretation of the context in the history of mathematics in which his views are being put forward.

Borel on the philosophy of approximation.

As a mathematician who contributed to the development of the theory of probability, Borel also wrote extensively in this context on his epistemological views regarding the approximative approach to knowledge. These views are expressed in relation to the interplay between mathematics and physics and are dealt with in depth by Barberousse in [5] to which we refer the reader for a detailed account in which Borel's views are compared for example with those of the physicist Pierre Duhem. Borel's reflections, Barberousse reports, are published as of 1906 and until the end of his life. These articles and books deal centrally with, for example, the theory of probability [15], [130] or focus on the application of probabilistic methods to mathematical physics, such as in [12].

In summary, Borel proposes a *'théorie originale de la connaissance, dans laquelle la notion de connaissance approchée est première, et celle de connaissance exacte seconde'* [an original theory of knowledge, in which the concept of approximated knowledge is primary and that of exact knowledge is secondary] [5, p. 56]. Borel criticises the concept of there being ideal, exact knowledge in physics, considering for example the idea of an exact value corresponding to physical entity as being purely a mathematical abstraction.

6.3.2 Albert Lautman

Albert Lautman (1908-1944) was a French philosopher of mathematics. Well known for his strong mathematical culture, he was influenced by mathematicians such as Jacques Herbrand and Gaston Julia and was friends with two of the founding members of the Bourbaki group, Charles Ehresman and Claude Chevalley. He was a student of the idealist philosopher Léon Brunschvicg (1869-1944) and was also influenced, for example, by the German philosopher Martin Heidegger.

Lautman defended his doctoral thesis in 1937, which included a main thesis and a secondary thesis - his main thesis was entitled *Essai sur les notions de structure et d'existence en mathématiques* [Essay on the concepts of structure and existence in mathematics] and his secondary thesis was entitled *Essai sur l'unité des sciences mathématiques dans leur développement actuel* [Essay on the unity of mathematical sciences in their current development]; both were later republished together in [163]. As the title of his primary thesis might suggest, Lautman had a structural conception of mathematics which relates to the same trend of thought as Hilbert's axiomatic view [192] and which is in the same style as the structuralism of Bourbaki [149]. Lautman was opposed to the reductionist aspect of logical empiricism [149]. To see this, it is helpful to consider, as does Petitot [192], the following explanation of Lautman regarding his view on the axiomatic approach which he states,

substitue à la méthode des définitions génétiques celle des définitions axiomatiques, et loin de vouloir reconstruire l'ensemble des mathématiques à partir de la logique, introduit au contraire...de nouvelles variables et de nouveaux axiomes qui élargissent à chaque fois le domaine des conséquences [163, p. 26].

[substitutes the method of genetic definitions by that of axiomatic definitions and, far from wanting to reconstruct all of mathematics starting from logic, introduces on the contrary...new variables and new axioms which expand the domain of consequences each time].

As well as having an axiomatic and structural view, Lautman also promoted a *dynamic* view of mathematics, the latter of which he associates with the evolution in time of mathematical theories. With Petitot [192], it is useful to consider the following explanation of Lautman to illustrate his view on the distinction between these two points of view both of which were integrated into his philosophy of mathematics:

...la conception structurale et la conception dynamique des mathématiques semblent de prime abord s'opposer: l'une tend en effet à considérer une théorie mathématique comme un tout achevé, indépendant du temps, l'autre au contraire ne la sépare pas des étapes temporelles de son élaboration... [163, p. 27].

[the structural conception and the dynamic conception of mathematics seem at first sight to be opposed to one another: one tends to consider a mathematical theory as a completed whole, independent of time, the other on the contrary does not separate the theory from the temporal stages of its elaboration].

Lautman characterises mathematical reality in terms of four different elements, or points of view: mathematical facts, mathematical beings (or entities), mathematical theories and the ideas behind these theories [192, p. 86]:

Loin de s'opposer, ces quatre conceptions s'intègrent naturellement les unes dans les autres: les faits consistent dans la découverte d'êtres nouveaux, ces êtres s'organisent en théories et le mouvement de ces théories incarne le schéma des liaisons de certaines Idées [163, p. 135].

[Far from being opposed to one another, these four conceptions integrate naturally with one another: facts consist of the discovery of new beings, these new beings are organised into theories and the movement of these theories embodies the pattern of connections of certain Ideas].

We will not attempt to explore in depth Lautman's views regarding the unity of mathematics - a topic which figures in the title of his secondary thesis. However, it is interesting in relation to our account of Bouligand's work to note that, in the context of the development of theories and connections between ideas, Lautman discusses the concept of unity. As quoted by Petitot [192, p. 86], Lautman states:

Des résultats partiels, des rapprochements arrêtés à mi-chemin, des essais qui ressemblent encore à des tâtonnements s'organisent sous l'unité d'un même

thème, et laissent apercevoir dans leur mouvement une liaison qui se dessine entre certaines idées abstraites... [163, p. 28].

[Partial results, rapprochements halted midway, and attempts that still resemble trial and error become organized in a unified way under a common theme. They reveal in their movement a connection that takes shape between certain abstract ideas...]

Regarding Lautman's broader vision of mathematics from the point of view of philosophy, as expressed by Zalamea [164]³,

Les théories mathématiques sont une matière et la tâche du philosophe est conçue comme effort pour dégager la réalité idéale à laquelle elles participent.

[Mathematical theories form a subject and the task of the philosopher is conceived as an effort to reveal the ideal reality in which they participate].

In addition, Zalamea identifies three broader themes with which Lautman's ideas could be identified. First, mathematical creation in terms of,

...approfondissements d'oppositions dialectiques au sens platonicien entre des contraires tels que le Continu et le Discontinu, le Même et l'Autre, l'Essence et l'Existence

[elaborations of dialectical oppositions in the Platonic sense between opposites such as the Continuous and the Discontinuous, Same and Other, Essence and Existence].

Secondly, Zalamea identifies the theme of the fundamental unity of mathematics and finally the theme of the profound harmony between mathematical theories and the emergence of quantum physics and relativity.

6.3.3 Jean Cavailles

The philosopher of mathematics Jean Cavailles, a friend of Lautman and fellow student of Léon Brunschvicg, is also known for his mathematical erudition - acquired perhaps in part due to his close connections with mathematicians such as Emmy Noether - the influential German algebraist who made fundamental contributions to the study of rings and with whom he collaborated during an extended stay in Germany. Cavailles

³See [164], back cover.

also completed two doctoral dissertations in 1938, the titles of which give a brief glimpse into his areas of research relating to the foundations of mathematics and the emergence of set theory: the primary thesis was entitled *Méthode axiomatique et formalisme, essai sur le problème du fondement des mathématiques* [Axiomatic method and formalism, essay on the problem of the foundation of mathematics] [113] and his secondary thesis was entitled *Remarques sur la formation de la théorie abstraite des ensembles* [Remarks on the formation of abstract set theory] [114].

As commented by Granger [142], Cavailles bases his work on an in depth study of mathematical examples or results in order to interpret them from a broader philosophical point of view, or as Benis-Sinaceur explains,

Il y a chez Jean Cavailles une solidarité étroite entre l'étude de l'activité mathématique et une philosophie d'ordre plus général qu'il qualifie lui-même de 'théorie de la raison' [7, p. 6].

[There is, in the work of Jean Cavailles, a strong solidarity between the study of mathematical activity and a philosophy of a more general order which he qualifies as the 'the theory of reason'].

In addition to the works of mathematicians such as Cantor, Dedekind and Hilbert on set theory and the axiomatic approach to mathematics, abstract algebra is also at the basis of the philosophical works of Jean Cavailles, inspired for example by his interactions with Emmy Noether. With Benis-Sinaceur, we note that Cavailles considered mathematics as done by algebrists as the '*modèle par excellence de l'activité de la raison*' [model par excellence for the activity of reason] [7, p. 6].

As emphasised by Benis-Sinaceur, two prominent themes in the epistemology of mathematics of Cavailles are those of structure and concept in mathematics. While we will not examine these ideas in depth, we note with Benis-Sinaceur that, for Cavailles, '*la structure préside à l'organisation du savoir*' [structure presides over the organisation of knowledge] and is therefore considered as a '*principe de mouvement, de progrès*' [a principle of movement, of progress] [7, p. 26]. Cavailles was in favour of creating a philosophy of mathematics centered around concepts. As Benis-Sinaceur details,

Le modèle mathématique de la philosophie du concept c'est la dialectique objective des contenus mathématiques qui agissent et réagissent les uns sur les autres spontanément, dans un mouvement dont la conscience mathématicienne n'est qu'une détermination extrinsèque, au même titre que les données de culture ou de société [7, p. 29].

[The mathematical model for the philosophy of concept is the objective dialectic regarding the contents of mathematics whose elements act and react on each other spontaneously, in the context of an evolution with respect to which the mathematician's consciousness is only an extrinsic factor, like the influence of culture or society].

Cavaillès was well informed regarding the intuitionist view of mathematics and, as we learn in the biography of Cavaillès by his sister Gabrielle Ferrières [133, p. 161], he even presented his thesis to the founder of intuitionism, the Dutch mathematician L.E.J. Brouwer and his student Arend Heyting, also one of the major proponents of this school of thought. However, Cavaillès was opposed to what he considered an overly dogmatic approach to intuitionism [7, p. 21], particularly regarding the role of axiomatisation and formalisation in the process of mathematical creation or discovery [7, p. 29].

As explained by Heinzmann [149], in Cavaillès' secondary thesis, he is interested in mathematical creation and the way in which this creation evolves:

...la construction mathématique (créations), qui, maintenant, se trouve placée dans son évolution autonome, c'est-à-dire indépendamment de la physique et de l'histoire de la philosophie... [149].

[mathematical construction (creations) which is placed within its own autonomous evolution, that is, independent with respect to physics and the history of philosophy].

6.3.4 Concluding remarks.

There are certain clear similarities between the subject matter, goals and target readership of Borel's epistemological reflections and those of Bouligand. The ideas of Borel encountered thus far have had either a predominantly pedagogical focus, or else seem to be aimed towards mathematicians with a view to promoting a particular focus in the development of mathematics (such as that of placing primary emphasis on the development of those theories in mathematics which are of greatest use to physics). In addition, as we have seen, intuition and its connection with a geometrical approach is present in the work of both mathematicians, as is an emphasis on the utility of set theory.

There are also clear differences between Borel and Bouligand. First, there are obvious differences in terms of the key themes with which they deal, to list a few: the philoso-

phy of approximation is not central for Bouligand as it is for Borel; Bouligand goes further in elaborating on his concept of intuition than Borel; geometry is at the heart and origin of Bouligand's reflections. We also highlight the distinct nature of Bouligand's work relating to causality. Principles initially expressed in a more qualitative way were later both concretised by Bouligand in terms of mathematical tools (such as the contingent, paratingent and specific groups of invariance) and to some extent formalised in the direction of mathematical logic, as was the case for Bouligand's concept of the stability of mathematical results.

Regarding Cavailles and Lautman, we find in their work several broad themes which are also found in the reflections of Bouligand, such as: the concept of structure in relation to mathematical theories; the unity of mathematics; set theory, axiomatics and the foundations of mathematics. However, Cavailles and Lautman were both philosophers of mathematics and their work was undertaken from a broader philosophical standpoint, not necessarily with a view to guiding mathematicians in their practice or helping to initiate new mathematicians into the discipline, which were central goals for Bouligand. While Lautman and Cavailles were both highly erudite with regard to the content of mathematics, they also qualified their views in relation to the thinking of other philosophers, such as for example Wittgenstein, Carnap or Heidegger and in relation to philosophical theories such as Platonism or intuitionism - such qualifications do not figure significantly in the work of Bouligand.

6.4 Bouligand's efforts to communicate and popularise his ideas

We have reviewed, in the introduction to this chapter, the various articles, journals and conferences through which Bouligand communicated his ideas on the themes in the epistemology of mathematics explored above. In this section, we introduce briefly Bouligand's contribution to two significant conferences during the period in question, namely the *Congrès international de philosophie scientifique* in 1935 and the ninth *Congrès International de Philosophie* of 1937. Secondly, we will discuss the way in which Bouligand locates his own ideas regarding causality, for example, in the context of major landmarks in the development of mathematics and physics - with a view, we could speculate, to further popularising his contribution. Finally we comment briefly on examples of the way in which Bouligand communicated his ideas in the epistemology of mathematics with a pedagogical goal.

6.4.1 Situating ideas in relation to landmarks in the development of mathematics

In addition to the channels through which Bouligand communicated his ideas on causality, direct methods and the related concepts discussed above, it is also worth commenting on the way in which he presented his ideas as part of an attempt, we could speculate, to enhance the reception of his work. We have seen already that Bouligand presented his epistemological ideas regarding causality and direct methods in a way that related his ideas to major trends in mathematics - such as the theory of Galois or more recent work of the previous generation of French mathematicians such as Lebesgue and Baire. A further recurring theme in certain of Bouligand's articles during the period in question dealing with epistemological reflections is that of the theory of relativity in relation to causality. Bouligand refers to the theory of relativity in relation to causality even earlier than 1932-1939 - in his earliest published reflections of 1927 [41]. He comments in the opening lines,

On a beaucoup insisté, et avec juste raison, sur l'influence profonde exercée récemment sur le cours de la pensée géométrique par les doctrines relativistes. Mais cet aspect de l'évolution scientifique ne peut en faire oublier certains autres, dont nous allons nous occuper ici [41, p. 586].

[The profound influence that the relativist doctrines have had recently on the development of thought in geometry has been greatly emphasised and

rightly so. But this aspect of scientific evolution should not lead us to forget certain other developments which we will address here].

We interpret the above as meaning that, while the theory of relativity may seem to perturb one's intuitive understanding of space and time, we may nevertheless retain a concept of causality that can continue to exist together with this theory. We do not interpret the theory of relativity as a primary motivating factor behind Bouligand's ideas regarding causality. Indeed, as already discussed, Bouligand identified his motivations rather in the context of trends and influences within mathematics itself: the influence of the Polish school of topologists and set theorist; the work of mathematicians of the previous generation in France such as Lebesgue; the utility of concepts from set theory and topology demonstrated in other disciplines such as the theory of functions. The view we put forward here is that the theory of relativity appears rather as part of the packaging, so to speak, of Bouligand's reflections on the epistemology of mathematics. This packaging reappears in greater detail in later articles, for example in his 1933 paper on causality in mathematics and physical theories [77] in which he discusses causality with reference to the concepts of group and invariance in the context of special relativity:

L'étroite liaison entre groupe et causalité semble avoir guidé le courant d'où est issue la Mécanique de la Relativité restreinte. Les aspirations à l'unité de la Physique venaient de se heurter à une constatation inattendue : le fait, pour les équations de la Dynamique newtonienne d'une part, et celles de l'électromagnétisme d'autre part d'appartenir à des champs d'invariance biens distincts, attachés à des groupes différents...D'où une discordance dans la causalité, qui faisait conclure à l'impossibilité de mettre, dans une même synthèse logique, l'électro-magnétisme en prolongement de la Mécanique. Pour faire cesser cette incompatibilité, et restaurer l'unité d'essence des lois physiques, il suffisait de retoucher les équations de la Dynamique...Cette retouche s'attachait à soumettre la Dynamique au groupe de l'électro-magnétisme...On considérait donc désormais un groupe comme un cadre délimitant la prétendue causalité...et on pouvait traduire cet état d'esprit, en disant: pour le physico-mathématicien, les groupes font partie du matériel causal. . . [77, p. 262-263].

[The close connection between group and causality seems to have guided the trend from which the Mechanics of Special Relativity emerged. Aspirations towards the unity of Physics had come up against an unexpected observation: the fact that the equations of Newtonian Dynamics on the one hand, and those of electromagnetism on the other hand, belong to quite distinct fields of invariance, attached to different groups...thus a

discordance of causality, which led to the conclusion that it is impossible to include, in the same logical synthesis, electromagnetism as an extension of Mechanics. To put an end to this incompatibility and to restore the essential unity of the laws of physics, it was enough to alter the equations of Dynamics...This alteration involved subjecting Dynamics to the group of electromagnetism...A group was henceforth considered as a framework which defines and delimits the alleged causality...and one could translate this way of thinking by saying: for the physico-mathematician, groups are part of the causal toolkit. . .]

Therefore we see that Bouligand emphasises the role of groups, from his point of view, as providing a framework within which causality is achieved in the context of the theory of special relativity. We note that Bouligand describes this causality as *prétendue* [alleged, claimed] and goes on to emphasise the provisional nature of this framework, which '*en matière d'analyse causale, ne vise pas à être complète*' [in the way of causal analysis, does not aim to be complete] [77, p. 263].

6.4.2 Bouligand's contribution to the eighth *Congrès international de philosophie*

The following background provided regarding this conference is based on the introduction to volumes 22 and 23 of *Philosophia Scientiæ* [105], dedicated entirely to various topics relating to the event. Preceded by the *huitième Congrès international de philosophie* in Prague, the ninth *Congrès international de philosophie* took place in Paris, from 15-23 September 1935 and aimed at addressing reflections on the sciences as a whole. The conference was organised under the auspices of a number of organisations, namely the *Institut international de coopération intellectuelle*, the organising committee of the *Encyclopédie française*, the *Cité des sciences*, the *Institut d'histoire des sciences et des techniques* and the *Centre international de synthèse*. Among the eminent figures from abroad taking part in the conference were Bertrand Russell and Federigo Enriques, both of whom gave an opening address. The proceedings of the conference were published in eight volumes by Hermann, reflecting the main topics of the conference:

- Philosophie scientifique et empirisme logique [philosophy of science and logical empiricism];
- Unité de la science [unity of science];
- Langage et pseudo-problèmes [language and pseudo-problems];
- Induction et probabilités [induction and probability];

- Logique et expérience [logic and experiment];
- Philosophie des mathématiques [philosophy of mathematics];
- Logistique [logistics];
- Histoire de la logique et de l'empirisme logique [history of logic and of logical empiricism].

Bouligand's contribution to the conference was published as part of the sixth volume on the philosophy of mathematics, together with for example contributions by the Swiss mathematician Ferdinand Gonseth (1890-1975) and by Albert Lautman. Bouligand's article, *Quelques aspects de l'étude des propositions mathématiques* [Certain aspects of the study of mathematical propositions] [1, p. 34-40], after an introduction to the main ideas regarding his reflections on causality as already explored, focusses on elaborating his concept of the stability of mathematical statements. He comments that the aim of his presentation is:

...montrer l'opportunité de recherches dans un secteur intermédiaire entre la logique et les mathématiques proprement dites, recherches dont l'enseignement pourrait bénéficier d'une manière assez immédiate [1, p. 34-40].

[...to demonstrate the research opportunities in an intermediate sector between logic and mathematics itself, research thanks to which education could benefit in quite an immediate manner].

6.4.3 Bouligand's efforts to communicate his epistemological ideas of 1932-1939 in a pedagogical setting

In the context of a textbook on group theory.

Bouligand dedicated considerable efforts to presenting his epistemological ideas in a pedagogical setting. Perhaps the most significant example of this, as we have seen already, is to be found in his *Introduction à la géométrie infinitésimale direct* - a textbook representing the first comprehensive presentation of his theory which included also his epistemological reflections on geometry motivating his work. A further example during the time period in question is to be found in Bouligand's 1935 textbook on group theory [86]. The first chapter on '*Généralités sur l'idée de groupe*' [generalities relating to the idea of a group], focusses on Bouligand's epistemological views on groups, which are centered around the concepts of domain of causality and causal proof - two subsections are included on '*le groupe, domaine de causalité*' [a group as a domain of causality] and '*démonstrations causales*' [causal proofs]. Here, he introduces

the idea of a domain of causality as follows, before going on to give the examples of different systems of geometry, such as metric geometry, projective geometry and so on:

...domaine de causalité, vu que des hypothèses (ou causes) invariantes par les modifications du groupe engendrent des conclusions (ou effets) qui se conservent aussi par ces modifications [86, p. 3].

[...domain of causality, given that hypotheses (or causes) invariant under modifications of the group lead to conclusions (or effects) which are also preserved under these modifications].

Again, Bouligand stresses the role, from his point of view, of groups seen as domains of causality as a means of better understanding geometry. He comments, regarding the concept of a domain of causality,

Une fois ces points bien compris, on pénètre d'une manière profonde la structure de l'édifice géométrique [86, p. 4].

[Once these points are well understood, we penetrate profoundly into the structure of the edifice of geometry].

In a subsection entitled '*Comment les groupes révèlent l'armature logique d'une théorie*' [how groups reveal the logical framework of a theory], Bouligand goes on to describe the more general conception of a domain of causality (which he sees as a group) in the context of mathematical propositions and systems of mathematical facts, as explored already in the present account. He also specifies that, although he always considers a domain of causality to be a group, the converse statement is not necessarily true, i.e. he does not consider that a group is always a domain of causality:

..à un groupe correspond toujours un domaine de causalité dont on peut faire à volonté une géométrie, mais une géométrie peut...constituer quelque chose de plus complexe que le domaine de causalité d'un groupe qu'on envisage isolément... [86, p. 4].

[...to a group there always corresponds a domain of causality with which we may choose to introduce a geometry but a geometry can...constitute something more complex than the domain of causality of a group that we envisage in a isolated manner...].

As mentioned above, Bouligand introduces briefly causal proofs, linking the concept to domains of causality. He describes them as those which are most '*éducatives*' and which

rely on no superfluous assumption before going on to provide a concrete example [86, p. 6].

Bouligand's views on the relevance of domains of causality and stability in the context of mathematics education.

Bouligand's 1936 article 'Sur la répercussion de quelques courants d'idées géométriques en matière de logique et d'enseignement' [on the repercussion of certain trends of geometrical thought in logic and teaching] [91] focusses on the concepts of domain of causality and a detailed exploration of the concept of stability. A final section is dedicated to exploring the importance of these concepts, from Bouligand's point of view, in the teaching of mathematics. He expresses his view on the change needed in mathematics educations as follows:

...refondre l'enseignement de l'Analyse mathématique, dans un esprit nouveau, qui soit à la fois substantiel, éducatif, soucieux des contacts entre l'Analyse et d'autres branches de la Science [91, p. 587].

[to redesign the teaching of mathematical Analysis in a new way which is both substantial and educative, concerned with the contacts between analysis and other branches of Science].

As we will see below, Bouligand views the concepts brought out earlier in the article, notably those of domains of causality and stability as a central part in moving towards such an approach. In his view, an ideal approach to the teaching of mathematics would start with an introduction to general concepts found throughout mathematics and logic:

Quelques leçons d'un caractère très général trouveront donc place utile au début du cours, leçons où l'on ne craindra pas de mettre en évidence l'idée de fonction sous sa forme la plus large...l'idée de groupe; toutes idées communes à la mathématique d'une part, à la logique d'autre part. [91, p. 587].

[A few lessons of a very general nature will usefully find their place at the start of the course - lessons in which we do not fear bringing to light the idea of a function in its most general form...the idea of a group; all ideas common to mathematics on one hand and logic on the other].

After the step of focussing on such general concepts, Bouligand highlights the role of the concepts of domains of causality and stability:

A la suite de ces préliminaires, les théories particulières pourront être développés en profondeur, avec une aisance plus grande. Leur mise en ordre sera facilitée. En même temps qu'elle prélude à la formation des algorithmes, la notion de groupe offre en effet un cadre à la classification des théorèmes pris isolément, tandis que la stabilité donnera prise sur les propositions à conclusion variable... [91, p. 588].

[Following these preliminaries, the specific theories can be developed in depth, with greater ease. Putting them in order will be facilitated. While it preludes the formation of algorithms, the concept of group also offers a framework for the classification of theorems taken first in isolation, while stability enables getting a grasp of propositions with variable conclusion...]⁴.

6.5 A brief note on further topics in the epistemology of mathematics dealt with by Bouligand between 1932 and 1939

In the above account, we have focussed on the concepts which interact most closely with Bouligand's mathematical output presented and which evolved from the ideas encountered in the earlier chapters. We have also briefly encountered Bouligand's reflections, from the time period in question, on the applications of direct infinitesimal geometry to the phenomenon of uncertainty including his speculations regarding the potential interest of these reflections for mathematicians, philosophers and physicists alike [80]. For the interested reader, we highlight briefly one further topic which featured in Bouligand's work in the epistemology of mathematics between 1932 and 1939.

The possibility and merits of the finitist approach in mathematics without irrational numbers

We have already briefly met Bouligand's views regarding the connection between a finitist approach and intuition. In the context of the ninth *Congrès International de Philosophie* in Paris in 1937, Bouligand presented a paper [93] in which he explores the consequences and merit of what he calls the '*style finitiste*' in mathematical analysis without the use of irrational numbers. For Bouligand, the *style finitiste* is an approach which,

⁴Variable conclusions or ambiguous propositions for Bouligand refer to propositions which are stable with respect to the cases for which the conclusion holds (in the sense introduced earlier on) and not stable with respect to the cases where the statement does not hold.

...répond à la préoccupation de substituer à un problème infini certaines collections de problèmes finis, par chacun desquels le problème infini soit serré de plus en plus près... [93, p. 175].

[...responds to the concern of substituting an infinite problem by certain collections of finite problems each of which approaches the infinite problem more and more closely...].

Bouligand contrasts this approach with the ‘*style totalitaire*’ which ‘*rappelle la tendance dominante de la théorie des ensembles à raisonner sur les totalités*’ [which recalls the dominant trend in the theory of sets to reason on totalities]. The context and motivation for Bouligand is captured in his own words:

Devant l’intuitionisme, on conçoit un cours d’analyse amputé du nombre irrationnel. Des théorèmes qui semblent liés à l’arithmétisation du continu subsistent pourtant, dans le style finitiste, remplaçant le style totalitaire. Cette idée révèle des domaines de non-contradiction et favorise les efforts de coordination... [93, p. 174].

[In the direction of intuitionism, we conceive of an exposition of analysis without irrational numbers. Theorems which seem connected with the arithmetisation of the continuum nevertheless subsist in the ‘*style finitiste*’, replacing the ‘*style totalitaire*’. This idea reveals domains of non-contradiction and facilitates efforts [towards greater] coordination...].

More concretely, Bouligand discusses the consequences of omitting the irrational numbers in the context of a finitist approach and the conditions under which familiar results can be translated into the *style finitiste*, for example in relation to finding the integral of a first order differential equation. He then goes somewhat in evaluating the resulting theory by means of his concept of domain of causality and that of the stability of a mathematical result in order, we could speculate, to assess the extent to which a greater level of ‘coordination’ is achieved. Bouligand explored the same topics as those presented here at greater length during the same year [96, p. 45-57]. Our objective here regarding this area of Bouligand’s reflections is merely to highlight the subject matter for the interested.

Conclusion

Looking predominantly at Bouligand's work between 1919 and 1926 (with a strong focus around 1924), we have seen, first, that he made a significant contribution to the formulation of generalisation of the Dirichlet problem as provided by Norbert Wiener as well as the characterisation, in this context, of the boundaries for which the classical Dirichlet problem always has a solution. Initially unaware of Wiener, he reports to have been motivated to work towards a generalisation by both physical applications and in order to bring greater coherence to the results in this area of research. We also have seen a mutual influence between Bouligand and Lebesgue in this direction. Published almost simultaneously with respect to Wiener's formulation, Bouligand's formulation of a generalised Dirichlet problem, which implemented the concept he referred to as *prolongement fonctionnel*, retained certain restrictions. After becoming aware of Wiener's more complete formulation through Lebesgue, we saw a rapprochement between Bouligand and Wiener including written correspondence, mutual support and an element of mathematical collaboration. We have seen that key elements of Bouligand's later ideas on causality and direct methods are first encountered - specifically the concepts of *prolongement fonctionnel* and that of structuring a mathematical theory by investigating its propositions at an appropriate level of generality. In addition, we have seen that Bouligand's work on the Dirichlet problem was influential in terms of bringing his attention to mathematical tools which would play a role in his later theory of direct infinitesimal geometry. Specifically, to give one example, he investigates the geometry of sets by looking at their dimensional properties.

Regarding Bouligand's reflections on intuition in mathematics between 1927 and 1930, we saw that he identified this concept in terms of making connections with familiar concepts from one's perception of the physical environment or with geometrical concepts. We identified three main themes: namely the importance of intuition from a pedagogical point of view; the interplay between algebraic/algorithmic methods and intuition and finally the role of intuition in the mathematical discovery process. Regarding the first of these themes, we learned that Bouligand saw intuition as a central element in mathematics education. In the context of the mathematical discovery

process, Bouligand identifies different types of intuition which come into play, namely intuition, extended intuition and counter-intuition and he goes somewhat in defining these concepts. Regarding the relationship between intuition and methods, he saw the two as complementary and necessary aspects of mathematical practice and promotes taking an axiomatic approach in such a way that the formalisation of the ideas at hand can be easily married with intuitive associations.

Drawing from sources between approximately 1928 and 1935 (with a strong focus around 1932), Bouligand's theory of direct infinitesimal geometry has been seen as a central theme of his research and as the area in which he seeks to implement his ideas in the epistemology of mathematics relating to geometry. He wrote extensively on his views on the need for a new approach, the three main goals being to avoid unnecessary hypotheses, to consider results and concepts in their suitable domain by means of the concepts of groups and invariance and finally favouring a direct approach (for example by avoiding relying on differential equations). We have explored the main ingredients of Bouligand's theory of direct infinitesimal geometry as presented in his comprehensive account of 1932: first he highlighted and justified a group of transformations representing what he considered the domain of causality of his theory. The key mathematical tools for his set-based approach to the study of sets in Euclidean space were the contingent and paratingent and closely related concepts, allowing for the study of sets in Euclidean space which does not require, for example, that the functions representing the surfaces being studied be differentiable. We have seen that Bouligand worked actively on exploring applications of his theory and that he viewed these applications as justifying or legitimising his work.

Two key influences leading to Bouligand's formulation of direct infinitesimal geometry were identified. First we identified the influence of the previous generation of French mathematicians, notably in the work of Lebesgue and Baire, which we placed within the broader theme of the emergence of set-theory in France, emerging first in the theory of functions. Secondly, the influence of the Polish school of mathematicians established in the 1920s appeared as a crucial influence both in terms of mathematical tools employed and in terms of the epistemological principles motivating his theory. We identified, in addition, links recognised by Bouligand between his theory and certain aspects of Juel's finite geometry and Menger's distance geometry. We have seen that Bouligand made intensive efforts to popularise his direct infinitesimal geometry: targeting the mathematical community through French and Polish journals; targeting students by including the topic as a central part of his teaching activities throughout his career and targeting a broader scientific readership through articles published in the *Revue Scientifique* and the *Revue Générale des Sciences Pures et Appliquées*. What

was the result of these efforts? While Bouligand's theory was not considered as part of the mainstream of mathematical research during the interwar period, we highlighted notable examples of the uptake of his theory: the key figure here was Elie Cartan, who played an influential role in disseminating Bouligand's ideas and who took an active interest in this theory. Moreover, we saw that a number of doctoral students pursued research in this field. Those who actively carried out research in direct infinitesimal geometry included Gustave Choquet, Christian Pauc, André Marchaud and - outside of France - Otto Haupt and S.K. Zaremba.

Our exploration of Bouligand's work in the epistemology of mathematics between 1932 and 1939 - after the publication of his comprehensive text on direct infinitesimal geometry in 1932 - was focussed around the concepts of causality (causal proofs and domains of causality) and direct methods, which were considered beyond the context of geometry in which they were originally discussed by Bouligand. Causal proofs were described by Bouligand as being able to reveal the reason why behind a mathematical result and were characterised in terms of relying on a the minimal set of assumptions. Bouligand also went further in defining domains of causality in a semi-formal manner, in terms of a group of transformations which are considered as modifications made to a given proposition. The concept of domain of causality and the related concept of the stability of a mathematical proposition (this stability being related to Bouligand's earlier idea of *prolongement fonctionnel*) are viewed by Bouligand as tools to help understand the structure of a mathematical theory. While the concept of direct methods were not formalised, Bouligand considers direct methods in different areas of mathematics, such as in the calculus of variations and in qualitative integration for example. Bouligand explicitly connects direct methods with intuition: direct methods do not allow for losing contact with intuition in formal mathematical reasoning. As well as the interplay between the formal aspect of mathematics and the intuitive aspect, his ideas on direct methods mirror another key aspect of his earlier work on intuition, namely the centrality of 'objectivity' or put differently, connections with geometrical concepts/objects from the physical world. Finally, we noted that Bouligand worked on explicitly arguing for the unity of direct methods as he identified them in different areas of mathematics, by means of certain common mathematical concepts belonging, for example, to topology.

We briefly situated Bouligand's ideas in the epistemology of mathematics with those of Borel, Cavailles and Lautman. While similarities did not extend far, common to Borel and Bouligand were certain key topics, target readership and goals. For example, the pedagogical emphasis and goal of orienting the development of mathematical work. While Bouligand reflects on certain themes of interest to Cavailles and Lautman (struc-

ture, unity, set theory,...), his work clearly did not belong to the field of philosophy and was not situated in relation to key landmarks of this discipline. His goals included guiding mathematicians in their practice or helping to initiate new mathematicians into the discipline - goals which we do not find in the work of Cavailles and Lautman.

Finally we emphasise the strong presence in our account of Bouligand's pedagogical efforts. In the context of his work in the generalised Dirichlet problem, he communicated his ideas during his 1925 lecture course at the University of Krakow, as well as publishing work with a strong pedagogical dimension in *L'Enseignement Mathématique*. In the context of intuition, as mentioned above, Bouligand viewed this aspect of mathematical practice as central for mathematics education. Regarding direct infinitesimal geometry, we reiterate Bouligand's pedagogical efforts in the context of teaching activities and in the presentation of his work in a comprehensive text. Bouligand made significant efforts to communicate his ideas in the epistemology of mathematics in a pedagogical way - through integrating these ideas in his books on group theory and direct infinitesimal geometry and by commenting explicitly on the relevance of his ideas for mathematics education.

Certain areas for future consideration

Bouligand in relation to modern mathematics. Certain themes often associated with the modernist transformation of mathematics have been encountered to a greater or lesser extent at certain points in our account on Bouligand, namely the themes of the use of concepts in set theory, the strong emphasis on the use of concepts in topology, the use of an axiomatic approach, the emphasis on dealing with mathematical objects in a more direct (to borrow Bouligand's terminology) or intrinsic manner and the concept of structure in mathematics.

First, the trend of the emergence of set theory was encountered as a key influencing factor behind the formulation of Bouligand's direct infinitesimal geometry, drawing inspiration from the example of the penetration of set theory into the theory of functions in France. Concepts in topology were strongly present in Bouligand's direct infinitesimal geometry and represented the backbone of his argument for the unity of direct methods. We recall that these direct methods show clearly Bouligand's preference for dealing with mathematical concepts or objects in themselves, without having to rely for example on parametric representation or differential equations.

To a certain extent, the axiomatic approach to mathematics has also arisen in our account. In the context of his reflections on intuition in mathematics, Bouligand advo-

cates taking an axiomatic approach in such a way that enables the chosen formalisation of facts in geometry can be easily married with intuitive associations. Furthermore, the idea of operating under a minimum number of assumptions is highly present in his concept of causal proofs and in his direct infinitesimal geometry, where for example a central point is that of avoiding unnecessary assumptions about the differentiability of functions.

The concept of structure is clearly present in our account of Bouligand's work, first in his organisation of ideas according to level generality in the context of his work on the Dirichlet problem but most prominently in his concept of a domain of causality. A key goal, through implementing the concept of domains of causality in directing infinitesimal geometry was to establish a structure for this theory in which mathematical facts and concepts are their appropriate setting or level of generality - in short, what Bouligand calls a causal structure. In addition, his concept of the stability of a mathematical fact is seen as a further tool in establishing and understanding the structure of a mathematical theory.

Jeremy Gray argues that mathematics underwent a modernist transformation between 1890 to 1930 [143], therefore overlapping with Bouligand's time. A question for further consideration could be to understand how we might situate Bouligand's ideas in relation to this much broader phenomenon. In addition, Leo Corry has researched the idea of mathematical structures in twentieth century mathematics, which he identifies as central during this period [123]. For example, Corry investigates structure in the work of Hilbert and Bourbaki, viewing it as an image of mathematics. In what way can we situate Bouligand's views and of structure and their implementation in his work in relation to, for example, to the work of Hilbert and Bourbaki? Do we view Bouligand's comments on structure as being part of an image of mathematics? Having mentioned Bourbaki, we highlight a further area of potential future investigation helping us to situate Bouligand in relation to modern mathematics: comparing Bouligand's ideas in the philosophy of mathematics with the views of the Bourbaki group during the interwar period. We comment that the founding members of Bourbaki belonged to the next generation of mathematicians in France after Bouligand. While we are not aware of Bourbaki having referred to Bouligand's views on mathematics, Bouligand did on certain occasions write actively about Bourbaki (see for example [99]).

Situating Bouligand in relation to Hermann Weyl. German mathematician Hermann Weyl (1885-1955) represents one of the major landmarks in 20th century mathematics and contributed to diverse areas of mathematics and mathematical physics including differential geometry, topology, space-time theory and the foundations of mathematics.

Weyl made major contributions to the mathematical foundations of relativity theory, including from the point of view of differential geometry and the close interaction between his philosophical reflections and his mathematical works in this area are well known, for example in his 1919 text *Raum-Zeit-Materie* [Space-Time-Matter] [217]. For an account on this interaction, we refer the reader to Julien Bernard's work (see for example [8]). A potential area for future research is therefore to address the question of how we situate Bouligand - as a mathematician who developed a theory in differential geometry based on epistemological ideals - in relation to Hermann Weyl.

Extending the present account around our central theme. It would be possible to extend our account of Bouligand remaining centered on the theme of his mathematical works which interacted closely with his concepts of causality and direct methods. First this could be done by investigating Bouligand's work on vector geometry. Although, as mentioned in the introduction, this does not represent an area of significant output in terms of published articles, it is potentially an area where Bouligand's later ideas in the epistemology of mathematics could be identified. For example, in his 1924 text *Leçons de géométrie vectorielle préliminaires à l'étude de la théorie d'Einstein* [Lessons in vector geometry preliminary to the study of Einstein's theory], he comments,

Ne pas limiter la portée des méthodes vectorielles à une simplification des calculs, plus étroitement soudés aux figures, tel fut à leur origine le but de ces Leçons. Les principes même de ces méthodes les font solidaires de l'axiomatique, dont les théories relativistes avaient révélé toute la puissance constructive. Or, cette puissance est aujourd'hui confirmée du calcul fonctionnel vers des méthodes se réclamant de l'intervention d'espaces auxiliaires. Dans la formation mathématique, la géométrie vectorielle devient à ce titre un rouage indispensable [26, p. IX].

[The original aim of these lessons was to not limit the scope of vectorial methods to a simplification of calculations, which were more closely related to figures. The very principles of these methods make them integral to axiomatics, the constructive power of which had been revealed by relativistic theories. Today, however, this power has been confirmed by functional calculus which is in favour of methods which avoid the involvement of auxiliary spaces. In this respect, vector geometry is becoming an indispensable component of mathematical training].

In the chapter on Bouligand's direct infinitesimal geometry, we touched on applications to mathematical physics and the significance of these applications from Bouligand's perspective in terms of legitimising and motivating his work in this area. A further area

of investigation which would enable us to expand on the present account would be to carry out a comprehensive survey of Bouligand's work more applied mathematical work which relates to or implements in an explicit way his ideas in the epistemology of mathematics. To give just two examples, we refer to Bouligand's on article on *Les courants de pensée Cantorienne et l'hydrodynamique* [Cantorian trends of thought and hydrodynamics] [58] and his 1933 text on *Relations d'incertitude en géométrie et en physique*, with a preface by Louis de Broglie.

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Les œuvres mathématiques et épistémologiques de Georges Bouligand

Résumé

La production intellectuelle de Georges Bouligand pendant l'entre-deux-guerres représente un riche corpus composé non seulement d'articles et de textes mathématiques dans divers domaines de la discipline, mais aussi de ses idées publiées sur l'épistémologie des mathématiques - sur ses opinions concernant les méthodes mathématiques, la manière dont les théories mathématiques devraient être formulées et la manière dont les mathématiques devraient être enseignées. En outre, en tant que membre de la communauté mathématique en France qui était en contact avec certains des mathématiciens les plus éminents, non seulement de sa propre génération mais aussi de la génération académique qui l'a précédée et suivie, l'étude de Bouligand et de ses œuvres nous permet d'ajouter une modeste contribution à la connaissance de l'activité mathématique en France pendant l'entre-deux-guerres. Malgré l'intérêt évident d'une étude plus approfondie de Bouligand, il existe aujourd'hui relativement peu permettant de mieux connaître cette figure des mathématiques françaises du XX^e siècle. Le présent exposé se concentrera sur un thème spécifique représentant un fil conducteur, à savoir les réflexions de Bouligand sur ce qu'il appelait la causalité en géométrie et dans les théories mathématiques et physiques, ainsi que sur son concept étroitement lié de méthodes directes. Les domaines de production mathématique au cœur du présent exposé sont ses travaux sur une généralisation du problème de Dirichlet et sa théorie de la géométrie infinitésimale directe.

Mots-clés : mathématiques ; histoire des mathématiques ; philosophie des mathématiques, épistémologie des mathématiques ; 20^{ème} siècle ; géométrie ; l'entre-deux-guerres

The mathematical and epistemological works of Georges Bouligand

Summary

The intellectual output of Georges Bouligand during the interwar period represents a rich corpus consisting not only of mathematical articles and texts in diverse areas of the discipline but also his published ideas on the epistemology of mathematics - on his views regarding mathematical methods, how mathematical theories should be formulated and how mathematics should be taught. In addition, as a member of the mathematical community in France who was well connected with some of the most prominent mathematicians not only of his own generation but of the academic generation before and after, a study of Bouligand and his works enables us to add a modest brush stroke to the currently available picture of the French mathematical scene of the day. Despite the clear interest of a somewhat in-depth study of Bouligand, relatively little work exists today giving us an insight into this figure of 20th century French mathematics. The present account will focus on a specific theme representing a common thread running, namely Bouligand's reflections on what he referred to as causality in geometry and in mathematical and physical theories as well as his closely related concept of direct methods. The areas of mathematical output which will figure centrally in the present account are his work on a generalisation of the Dirichlet problem and his theory of direct infinitesimal geometry.

Keywords : mathematics; history of mathematics; philosophy of mathematics; epistemology of mathematics; twentieth century; geometry; interwar period

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