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# The ontological problem of quantum physics and the emergence of a backgroundless spacetime in the light of the theory of decoherence

Antoine Soulas

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# THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES

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*Mathématiques, Télécommunications, Informatique, Signal, Systèmes,  
Électronique*

*Spécialité : Mathématiques et leurs interactions*

Par

**Antoine SOULAS**

**Le problème ontologique en physique quantique et l'émergence  
d'un espace-temps relationnel à la lumière de la théorie de la dé-  
cohérence**

Thèse présentée et soutenue à Rennes, le 20 septembre 2024

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The ontological problem of quantum physics  
and the emergence of a backgroundless spacetime  
in the light of the theory of decoherence

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# Contents

<b>1 Introduction</b>	<b>7</b>
<b>2 Usual mathematical objects and notations in physics</b>	<b>21</b>
<b>3 Two models for the theory of decoherence</b>	<b>27</b>
3.1 The basics of decoherence	27
3.2 First model: decoherence as a high-dimensional geometrical phenomenon	30
3.2.1 Convergence to the uniform measure	30
3.2.2 Most vectors of $\mathbb{S}^n$ are almost orthogonal	32
3.2.3 Direct study of $\eta$	33
3.2.4 Comparison with simulation and consequences	39
3.2.5 Discussing the hypotheses	40
3.3 Alternative measures of decoherence	41
3.3.1 Fraction of interfering histories	41
3.3.2 Expectation of the scalar products	42
3.3.3 Distance between orthonormal bases	47
3.3.4 Axioms for measures of decoherence	50
3.4 Second model: why is the universe not frozen by the quantum Zeno effect?	52
3.4.1 The Zeno effect	52
3.4.2 The model: free evolution <i>vs.</i> decoherence	54
3.4.3 Analytic study of the criterion	58
3.4.4 Physical considerations concerning $\eta_n$	59
3.4.5 Discussion	61
<b>4 The ontological problem of quantum physics</b>	<b>63</b>
4.1 Blurring the border between ontology and epistemology	64
4.2 The preferred-basis problem	67
4.3 Collapse-realist interpretations	68
4.3.1 The Copenhagen interpretation	68
4.3.2 Collapse models	70
4.4 Bohmian mechanics	73
4.5 Unitary QM interpretations	74
4.5.1 Many-worlds interpretation	74

4.5.2	Relational quantum mechanics	76
<b>5</b>	<b>Quantum spacetime</b>	<b>79</b>
5.1	Is QM compatible with special relativity?	79
5.1.1	The conditions (C) and (MC)	79
5.1.2	(F) implies (C)	84
5.1.3	Previous works	86
5.1.4	Locality	87
5.1.5	Microcausality	91
5.1.6	Summary	95
5.2	A toy model: how the shape of the laws of physics promote a backgroundless preferred-basis	96
5.2.1	Motivation and preliminaries	96
5.2.2	Description of the model	97
5.2.3	Analytic resolution	98
5.2.4	Numerical simulation and discussion	99
<b>6</b>	<b>Conclusion</b>	<b>103</b>
<b>A</b>	<b>Some philosophical notions</b>	<b>109</b>
A.1	Ontology and epistemology	109
A.2	The Wigner's friend thought experiment	110
	<b>Bibliography</b>	<b>110</b>

# Chapter 1

## Introduction

*Que signifie la mécanique quantique<sup>1</sup> ?*

La mécanique quantique : 100 ans d’ambiguïté philosophique et de débats sur son interprétation. Une théorie basée sur des mathématiques relativement simples (du moins en dimension finie), jamais mise en défaut expérimentalement, mais qui déchire la communauté dès qu’il s’agit de comprendre ce qu’elle dit de l’univers et de ce qui existe. Dès les travaux pionniers ( $\sim 1920 - 1940$ ), une tension métaphysique est repérée, que des physiciens comme Einstein [54], Bohr [26], Schrödinger [122], Heisenberg [67], Born [30] et bien d’autres tentent d’éclaircir sans y parvenir réellement. Néanmoins, une interprétation dominante, aux contours flous et recouvrant des positions complexes et changeantes, semble émerger : l’interprétation de Copenhague [72]. Insatisfaisante à bien des égards, le pragmatisme anti-philosophique du célèbre « *Shut up and calculate !* » finit alors par s’imposer durant la deuxième moitié du XX<sup>e</sup> siècle. Les débats concernant le sens physique de la théorie sont perçus comme stériles, la théorie fonctionne pour toutes les applications pratiques, et c’est tout ce qui lui est demandé. Mais quelques audacieux-ses refuseront de se taire.

En 1952, Bohm, développant une idée déjà explorée deux décennies plus tôt par de Broglie, fonde la mécanique bohémienne [25] : une ré-organisation de la théorie qui semble la rendre compatible avec une interprétation intuitive ; les particules y redeviennent des entités concrètes, dotées d’une position et d’une vitesse bien définies à tout instant. En 1964, Bell démontre son célèbre théorème [18], reposant sur des observations mathématiques simples mais profondes, aux implications retentissantes : l’interprétation n’est pas seulement une affaire de choix métaphysique car elle peut aussi être contrainte par certains faits physiques. En réalisant des expériences bien choisies mettant en jeu le phénomène d’intrication, il est possible (i) soit de mettre en défaut la mécanique quantique (ii) soit de conclure que l’univers présente une forme de non-localité fondamentale, avec pour corollaire qu’il est en principe impossible pour l’esprit humain de bâtir une théorie physique entièrement déterministe (car alors la non-localité permettrait de communiquer plus vite que la lumière, ce qui induirait des paradoxes causaux). L’article de Bell passe presque inaperçu pendant 20 ans, cumulant

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<sup>1</sup>Les paragraphes qui suivent constituent un résumé très partiel d’une réalité historique complexe, où bien d’autres travaux auraient pu être mentionnés.



à peine quelques dizaines de citations par an.

Parmi les rares qui s'intéressent à ces sujets durant la seconde moitié du XX<sup>e</sup> siècle, la recherche prend même parfois une dimension politique, dans un contexte de guerre froide et idéologique. En URSS, la mécanique bohémienne apparaît bien plus compatible avec le matérialisme marxiste que l'interprétation de Copenhague, aux saveurs d'idéalisme bourgeois [106, 93]. La possibilité ouverte par Bell de mettre en défaut expérimentalement la théorie orthodoxe gagne l'intérêt, dès la fin des années 60, de jeunes socialistes italien·nes en quête de nouvelles manières de penser la recherche scientifique et la société [11]. Dans les États-Unis des années 70, alors que la physique dominante connaît une hyper-mathématisation croissante et un conformisme accentué par les coupes budgétaires liées à la guerre du Vietnam, quelques hippies (selon l'expression de Kaiser [79]) de Berkeley versé·es dans l'esprit New Age s'intéressent à la non-localité de Bell comme possible pont avec les spiritualités orientales et leurs expériences psychédéliques. La réalisation concrète de l'expérience proposée par Bell tient à l'époque de la prouesse technique, freinée d'autant plus par le manque général d'intérêt et de financements. Ce sont finalement les expériences de Freedman et Clauser en 1972 [60], et surtout celle d'Aspect et son équipe en 1982 [9], qui permettent de trancher : les prédictions de la mécanique quantique sont confirmées, et par conséquent la non-localité de Bell est avérée.

Peu à peu, le sujet gagne l'intérêt de la communauté physicienne. Ces recherches de l'ombre ont préfiguré le domaine naissant de la théorie de l'information quantique, qui connaît son essor à partir des années 2000, tandis que les approches conventionnelles à la physique des particules s'essouffent. Il profite également de l'afflux de financements publics et privés dans la course à l'ordinateur quantique. Le nombre de citations des articles de Bell [18] et d'Einstein *et al.* [54] explosent, au point qu'ils comptent aujourd'hui parmi les articles les plus cités de l'histoire de la physique. Le prix Nobel 2022 sera décerné à Clauser, Aspect et Zeilinger « pour [leurs] expériences avec des photons intriqués, établissant les violations des inégalités de Bell et ouvrant une voie pionnière vers l'informatique quantique », plus de 30 ans après la mort de Bell.

Toutefois, même si les concepts de la théorie de l'information peuvent éclairer la question de l'interprétation de la mécanique quantique, le débat divise toujours la communauté, comme en témoignent les réponses extrêmement divergentes de plusieurs sondages récents réalisés auprès de physicien·nes ou de philosophes de la physique [120, 125]. Et pour cause, le problème est particulièrement difficile à délimiter ou même à exprimer clairement. Il prend des formes variées selon l'interprétation défendue, et la frontière entre fait expérimental et posture philosophique est loin d'être évidente, comme l'ont montré Bohm et Bell. Dans cette thèse, nous tenterons d'exposer le plus clairement possible cette énigme que tentent de résoudre toutes les interprétations, communément désignée *problème de la mesure* (nous critiquerons cette expression), mais que nous appellerons le *problème ontologique de la physique quantique*.

Une autre raison probable du regain d'intérêt pour les questions de fondements est liée à l'échec de l'unification tant attendue entre la physique des particules (la théorie quantique des champs, notamment le modèle standard) et la gravitation (la relativité générale), et surtout la faillite de la théorie des cordes. Une idée de plus en plus partagée aujourd'hui, en particulier parmi les philosophes de la physique, est l'importance d'éclaircir le problème ontologique pour être en mesure d'aborder correctement le projet de la gravitation quantique [41, 3]. Un principe conducteur dans cette quête, qui n'est encore qu'une intuition vague plus qu'un principe rigoureusement formulé, est la notion de *background-independence* [115, 92]. Elle tente de prendre au sérieux une leçon essentielle de la relativité générale : la co-émergence de l'espace-temps et de la matière. C'est en effet l'un des apports majeurs d'Einstein d'avoir montré que l'espace-temps n'est pas un arrière-plan absolu dans lequel se meuvent les corps, mais plutôt une entité dynamique qui ne pré-existe pas à la matière, et qui n'a même pas de sens en l'absence de cette dernière (voir en particulier le fameux *hole argument*, qu'Einstein a commenté dans ces termes : « *There is no such thing as an empty space* » [53]).

Par souci de rigueur terminologique, nous préférons dans cette thèse le qualificatif *backgroundless* plutôt que l'usuel *background-independent*, car ce qui est recherché, c'est bien une théorie sans aucun arrière-plan, et non simplement indépendante de celui-ci. Sur quoi, dès lors, fonder ou contraindre cette co-émergence de l'espace-temps et de la matière ? Peut-être sur la notion de causalité, comme le proposent les approches de type *causal sets* [28, 50, 132], intuition soutenue par plusieurs théorèmes de géométrie Lorentzienne [150, 66, 90]. On pourrait formuler la difficulté essentielle de la gravitation quantique, raison possible de quatre décennies de recherches infructueuses, comme suit : *comment bâtir une théorie de la matière quantique (et donc soumise au problème ontologique) sans même disposer d'un espace-temps préexistant dans laquelle la concevoir ?*

### *La théorie de la décohérence*

Au début des années 80, en parallèle de la théorie de l'information quantique, un autre programme de recherche voit le jour avec la ferme intention de reconsidérer les problèmes fondamentaux de la mécanique quantique : la *théorie de la décohérence*. Des premières intuitions [97, 58, 151] aux articles fondateurs de Zurek [153, 154], le champ se développe rapidement et devient vite un élément incontournable à la compréhension de la mécanique quantique, connu au moins dans les grandes lignes par toute la communauté [158]. Une confirmation importante de la théorie a été permise par les expériences d'Haroche et son équipe [37], qui lui a valu le prix Nobel en 2012. Vraisemblablement, il s'agit d'une des (de la ?) plus grandes avancées de la physique fondamentale depuis l'achèvement du modèle standard au milieu des années 70.

Le but de la théorie de la décohérence est d'expliquer le passage du quantique au classique. Plus précisément, il s'agit de comprendre pourquoi le phénomène de superposition quantique — source du problème ontologique — est observé en laboratoire dans certaines expériences

spécifiques, mais imperceptible à nos échelles ; autrement dit, pourquoi il n'existe pas de chats de Schrödinger. Un des atouts de la théorie est qu'elle ne nécessite aucun nouveau postulat, n'ajoute ni n'enlève rien à la mécanique quantique, mais en est une conséquence. Elle est fondée sur l'idée décisive qu'un système physique, en règle générale, n'est pas isolé, mais baigne dans un environnement avec lequel il interagit continuellement et s'intrique. Ce faisant, l'environnement acquiert de l'information sur l'état du système, et la décohérence (*i.e.* la suppression des superpositions) est d'autant plus forte que cette information est complète. C'est pourquoi il est possible de réaliser des expériences d'interférences avec des photons ou des électrons, mais pas avec des objets macroscopiques. Les premiers, non chargés, interagissent peu, et les seconds sont assez petits pour ne laisser aucune trace de leur passage, tandis que la trajectoire d'objets plus imposants est rapidement enregistrée par les molécules d'air (même dans un vide de laboratoire) ou par les photons ambiants qu'ils rencontrent [77, Tableau 3.1].

Voici quelques modèles importants de décohérence. Dans [154], un spin (le système) interagit avec un grand nombre d'autres spins (l'environnement) *via* des constantes de couplages aléatoires. Un second modèle fait apparaître que les interférences disparaissent d'autant plus efficacement que les niveaux d'énergie d'interaction entre le système et l'environnement sont distincts. Intuitivement, plus l'environnement « ressent » la différence entre deux états possibles du système, plus ces derniers décohèrent. Dans [78] est notamment estimé le rayon typique de cohérence d'une particule non-localisée lorsqu'elle interagit avec une autre selon un potentiel de type coulombien, en fonction du temps et de leur distance. Les auteurs calculent aussi le taux de localisation d'un système plongé dans un bain thermique de particules (typiquement des photons) qu'il diffuse. Rapidement, son rayon de cohérence devient de l'ordre de sa longueur d'onde de de Broglie. Le mouvement brownien quantique [38] est un modèle qui met en jeu un grand nombre d'oscillateurs harmoniques couplés ; l'équation qui gouverne l'évolution d'un de ces oscillateurs permet d'étudier le niveau de décohérence induit par les autres [159, §V.B.] et confirme les conclusions des précédents modèles. L'analyse numérique d'un système quantique chaotique [64] révèle que son comportement dans l'espace des phases présente d'importantes caractéristiques quantiques lorsqu'il est isolé, mais reproduit précisément la distribution classique dès qu'une décohérence, même modeste, est imposée. Bien d'autres modèles existent, souvent adaptés à des contextes plus spécifiques [77, 23, 32, 159].

Plusieurs leçons principales peuvent être tirées de la théorie de la décohérence. Quand un système est en interaction avec un environnement extérieur, les superpositions quantiques disparaissent exponentiellement vite dans la plupart des situations réalistes. Cette suppression, cependant, a lieu dans une base particulière de l'espace de Hilbert (*i.e.* relativement à une manière de mesurer le système, par exemple en position, en impulsion, en énergie...), nommée *pointer basis*, qui dépend de la nature de l'interaction avec l'environnement à travers le phénomène d'*einselection* (termes introduits par Zurek) [159]. D'un point de vue philosophique, la décohérence nous apprend que ce qu'est un système n'est pas indépendant de ce que son environnement sait de lui. Ce sont les relations des systèmes entre eux qui les font exister mutuellement, une conclusion étonnamment *backgroundless*. Elle donne aussi le bon cadre pour définir précisément le problème ontologique quantique, qui s'estompe

après décohérence, raison pour laquelle il semble absent du monde que nous percevons, où rien n'est jamais isolé. Cependant, comme nous le verrons, elle ne résout pas le problème ontologique, puisque la décohérence n'est ni absolue, ni irrémédiable, ni même présente au niveau des systèmes isolés.

Le programme initié avec la théorie de la décohérence paraît avoir porté ses fruits, et certain-es considèrent qu'il s'agit à présent d'un champ de recherche éteint. Pourtant, il semble que plusieurs questions cruciales restent en suspens. Premièrement, quels sont les systèmes, et quels sont les environnements ? Quel *coarse-graining* de l'espace de Hilbert abstrait du formalisme quantique correspond au monde que nous expérimentons ? En outre, comment penser la décohérence lorsqu'aucun environnement extérieur ne peut être envisagé, comme en cosmologie par exemple, où l'objet d'étude est l'univers dans son ensemble ? Deuxièmement, on observe que la matière est en général bien décohérée dans la base de la position ; ce constat est habituellement expliqué par le fait que les lois de la physique s'écrivent souvent en termes de variables de position (interactions coulombienne, gravitationnelle, etc. . . ) [119]. Mais *n'y a-t-il pas quelque chose de circulaire à invoquer la notion de position avant même d'avoir une notion stable de celle-ci*, puisque les différentes potentialités spatiales n'ont pas encore été fixées ontologiquement par décohérence ? Le problème devient encore plus sérieux dans la perspective de construire une théorie quantique *backgroundless*. *Quel rôle est amenée à jouer la décohérence dans la co-émergence de l'espace-temps et de la matière ?*



# Introduction

*What does quantum mechanics mean<sup>2</sup>?*

Quantum mechanics: 100 years of philosophical ambiguity and debates on its interpretation. A theory based on relatively simple mathematics (at least in finite dimension), never experimentally disproven, but one that divides the community as soon as the question of what it says about the universe and existence arises. Since the pioneering works ( $\sim$  1920 - 1940), a metaphysical tension was identified, that physicists such as Einstein [54], Bohr [26], Schrödinger [122], Heisenberg [67], Born [30] and many others have tried to clarify without truly succeeding. Nevertheless, a dominant interpretation, with vague contours covering complex and changing positions, seemed to emerge: the Copenhagen interpretation [72]. Unsatisfactory in many ways, the anti-philosophical pragmatism of the famous "Shut up and calculate!" eventually prevailed during the second half of the 20th century. Debates concerning the physical meaning of the theory were perceived as sterile; the theory worked for all practical purposes, and that is all it was asked to. But a few bold ones refused to shut up.

In 1952, Bohm, developing an idea already explored two decades earlier by de Broglie, founded Bohmian mechanics [25]: a re-organization of the theory that seems to make it compatible with an intuitive interpretation; particles become concrete entities again, with well-defined positions and velocities at all times. In 1964, Bell proved his famous theorem [18], based on simple but deep mathematical observations with tremendous implications: interpretation is not merely a matter of metaphysical choice since it can also be constrained by certain physical facts. By conducting well-chosen experiments involving the phenomenon of entanglement, it is possible (i) either to disprove quantum mechanics (ii) or to conclude that the universe exhibits a form of fundamental non-locality, with the corollary that it is in principle impossible for the human mind to build an entirely deterministic physical theory (since non-locality would then allow faster-than-light communication, inducing causal paradoxes). Bell's article went almost unnoticed for 20 years, accumulating barely a few dozens of citations per year.

Among the few interested in these subjects during the second half of the 20th century, research sometimes even took on a political dimension, in a cold war context. In the USSR, Bohmian mechanics appeared much more compatible with Marxist materialism than the

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<sup>2</sup>The following paragraphs are a very incomplete historical summary, in which many other works could have been mentioned.

Copenhagen interpretation and its flavors of bourgeois idealism [106, 93]. The possibility opened by Bell to experimentally disprove the orthodox theory gained interest, from the late 60s, among young Italian socialists seeking new ways to organize scientific research and society [11]. In the United States of the 70s, while mainstream physics was undergoing an increasing hyper-mathematization and conformity accentuated by budget cuts related to the Vietnam War, some Berkeley hippies (as Kaiser puts it [79]) versed in New Age spirit were interested in Bell's non-locality as a possible bridge with Eastern spiritualities and their psychedelic experiences. The concrete realization of Bell's proposed experiment was a technical feat at the time, further hindered by the general lack of interest and funding. It was finally the experiments of Freedman and Clauser in 1972 [60], and even more those of Aspect and his team in 1982 [9], which settled the case: the predictions of quantum mechanics were confirmed and, consequently, Bell's non-locality was established.

Gradually, the subject gained the interest of the physics community. This pioneering research have foreshadowed the emerging field of quantum information theory, which began to flourish in the 2000s, at a time when conventional approaches to particle physics were losing steam. It also benefited from the influx of public and private funding in the race for the quantum computer. The number of citations of Bell's [18] and Einstein *et al.*'s [54] articles skyrocketed, so that they are now among the most cited papers in the history of physics. The 2022 Nobel Prize was awarded to Clauser, Aspect, and Zeilinger "for [their] experiments with entangled photons, establishing the violations of Bell inequalities and pioneering quantum information science", more than 30 years after Bell's death.

However, even though the concepts of information theory can shed light on the question of the interpretation of quantum mechanics, the debate still divides the community, as evidenced by the extremely divergent answers of several recent surveys conducted among physicists and philosophers of physics [120, 125]. One reason is that the problem is particularly difficult to delimitate or even to express clearly. It takes various forms depending on the interpretation defended, and the boundary between experimental fact and philosophical stance is far from obvious, as Bohm and Bell have shown. In this thesis, we shall attempt to formulate as clearly as possible this puzzle that all interpretations are trying to solve, usually designated as the *measurement problem* (we will criticize this expression), but which we will call the *ontological problem of quantum physics*.

### *A guiding principle for quantum gravity*

Another probable reason for the renewed interest in quantum foundations is related to the failure of the long-awaited unification between particle physics (quantum field theory, in particular the standard model) and gravitation (general relativity), and notably that of string theory. An increasingly shared idea today, particularly among philosophers of physics, is the importance of clarifying the ontological problem in order to properly address the project of quantum gravity [41, 3]. A guiding principle in this quest, which is still more a vague intuition than a rigorously formulated principle, is the notion of *background-independence* [115, 92]. Its aim is to draw seriously an essential lesson from general relativity: the co-emergence

of spacetime and matter. Indeed, one of Einstein's major contributions was to show that spacetime is not an absolute background in which bodies move, but rather a dynamic entity that does not pre-exist to matter, and does not even make sense without it (see particularly the famous hole argument, which Einstein commented on in these terms: "There is no such thing as an empty space" [53]).

For the sake of terminological rigour, we will prefer in this thesis the qualifier *backgroundless* rather than the usual background-independent, because what is sought is indeed a theory without any background, not just independent of it. On what, then, to base or constrain this co-emergence of spacetime and matter? Perhaps on the notion of causality, as proposed by causal set approaches [28, 50, 132], an intuition supported by several theorems of Lorentzian geometry [150, 66, 90]. The essential difficulty of quantum gravity, possibly explaining four decades of unsuccessful research, could be formulated as follows: *how to build a theory of quantum matter (and thus subject to the ontological problem) without even having a pre-existing space-time in which to conceive it?*

### *The theory of decoherence*

In the early 80s, alongside quantum information theory, another research program emerged with the firm intention of reconsidering the fundamental problems of quantum mechanics: the theory of decoherence. From the first intuitions [97, 58, 151] to Zurek's seminal papers [153, 154], the field has developed rapidly and quickly became a crucial element in understanding quantum mechanics, known at least broadly by the entire community [158]. An important confirmation of the theory was enabled by the experiments of Haroche and his team [37], for which he was awarded the Nobel Prize in 2012. It is arguably one of the (if not the) greatest advances in fundamental physics since the completion of the standard model in the mid-70s.

The goal of the theory of decoherence is to explain the transition from quantum to classical. More precisely, it aims to understand why the phenomenon of quantum superposition—the source of the ontological problem—is observed in the laboratory in certain specific experiments but is imperceptible at our scales; in other words, why Schrödinger's cats do not exist. One of the strengths of the theory is that it requires no new postulates, adds or removes nothing from quantum mechanics, but is a consequence of it. It is based on the decisive idea that a physical system, in general, is not isolated but bathes in an environment with which it interacts continuously and becomes entangled. In doing so, the environment acquires information about the system's state, and decoherence (*i.e.* the suppression of superpositions) is stronger as this information becomes complete. This is why it is possible to conduct interference experiments with photons or electrons, but not with macroscopic objects. The former, uncharged, interact little, and the seconds are small enough to leave no trace of their passage, while the trajectory of larger objects is quickly recorded by air molecules (even in a laboratory vacuum) or by ambient photons they encounter [77, Table 3.1].

Here are some important models of decoherence. In [154], a spin (the system) interacts with a large number of other spins (the environment) through random coupling constants. A second model reveals that interferences disappear all the more efficiently as the energy



interaction levels between the system and the environment are distinct. Intuitively, the more the environment "feels" the difference between two possible states of the system, the more these states decohere. In [78], the typical coherence radius of a non-localized particle is estimated when it interacts with another according to a Coulomb-like potential, depending on time and their distance. The authors also calculate the localization rate of a system immersed in a thermal bath of particles (typically photons) it scatters. Quickly, its coherence radius becomes of the order of its de Broglie wavelength. Quantum Brownian motion [38] is a model involving a large number of coupled harmonic oscillators; the equation governing the evolution of one of these oscillators allows to study the level of decoherence induced by the others [159, §V.B.] and confirms the conclusions of previous models. Numerical analysis of a chaotic quantum system [64] reveals that its behavior in phase space presents significant quantum characteristics when isolated, but precisely reproduces the classical distribution as soon as even modest decoherence is imposed. Many other models exist, often adapted to more specific contexts [77, 23, 32, 159].

Several main lessons can be drawn from the theory of decoherence. When a system interacts with an external environment, quantum superpositions disappear exponentially fast in most realistic situations. This suppression, however, occurs in a particular basis of Hilbert space (*i.e.* relative to a way of measuring the system, such as in position, momentum, energy...), called the *pointer basis*, which depends on the nature of the interaction with the environment through the phenomenon of *einselection* (terms introduced by Zurek) [159]. Philosophically, decoherence teaches us that what a system is is not independent of what its environment knows about it. It is the relationships between systems that make them mutually exist, an astonishingly backgroundless conclusion. It also provides the right framework to precisely define the quantum ontological problem, which fades after decoherence, explaining why it seems absent from the world we perceive, where nothing is ever isolated. However, as we shall see, it does not solve the ontological problem, since decoherence is neither absolute, nor irremediable, nor even present at the level of isolated systems.

The program initiated with the theory of decoherence seems to have been fruitful, and some consider that it not an active research field anymore. Yet, it appears that several crucial questions remain unanswered. First, what are the systems, and what are the environments? Which coarse-graining of the abstract Hilbert space of the quantum formalism corresponds to the world we experience? Furthermore, how to think about decoherence when no external environment can be considered, such as in cosmology, where the object of study is the universe as a whole? Secondly, it is observed that matter is generally well decohered in the position basis; this is usually explained by the fact that the laws of physics are often written in terms of position variables (Coulomb or gravitational interaction, etc.) [119]. But *isn't there something circular in invoking the notion of position before even having a stable notion of it, since the different spatial potentialities have not yet been ontologically fixed by decoherence?* The problem becomes even more serious in the perspective of building a backgroundless quantum theory. *What is the role of decoherence in the co-emergence of spacetime and matter?*

# Organization of the thesis

- In order for this thesis to be accessible to mathematicians as well as to physicists and philosophers, we first recap in Chapter 2 the basic concepts and notations of the quantum formalism, as well as some common abbreviations.
- In Chapter 3, we present two models designed to feel intuitively the general process of decoherence and to reveal the mathematical mechanisms that make it so universal. Contrary to the majority of the already existing models, built to understand decoherence in different specific contexts, we try to embrace a more general point of view *without specifying any Hamiltonian*. The whole chapter is restricted to the finite dimensional case. We start by introducing general quantities and notations to present as concisely as possible the idea underlying the theory of decoherence (§3.1).
- The first model, developed in §3.2, makes decoherence naturally appear as a geometrical phenomenon between reservoirs of dimensions. We recover in 3.2.2 the well-known typical decay of the non-diagonal terms of the density matrix in  $n^{-\frac{1}{2}}$ , with  $n$  the dimension of the Hilbert space describing the environment. The most important result is Theorem 3.2.5, proved in §3.2.3, giving estimates for the level of decoherence induced by a random environment on a system of given sizes. We conclude in §3.2.4 that even very small environments (of typical size at least  $N_{\mathcal{E}} = \ln(N_{\mathcal{S}})$  with  $N_{\mathcal{S}}$  the size of the system) suffice, under (strong) assumptions discussed in §3.2.5. Possible improvements lead to interesting directions of research, in particular trying to describe the submanifold of physically reasonable states in the Hilbert space, a question that may have some links with tensor networks theory and the area law for entanglement entropy, as well as with quantum information theory and quantum Darwinism.
- We propose other ways to quantify decoherence in §3.3, and derive some of their properties. The proposal of §3.3.2 can be generalized in infinite dimension and is related to the entanglement entropy. It also gives a physical meaning to the purity of a state, namely the approximate level of decoherence in most bases, according to Proposition 3.3.2. Section §3.3.3 introduces a distance between orthonormal bases that seems well adapted to the theory of decoherence, and will turn out to be a crucial tool in §5.2. We attempt to unify these proposals in §3.3.4, in which we formulate a list of properties that should satisfy any suitable measure of decoherence.
- The second model is presented in §3.4. It specifically tackles the question of why,

in a universe where decoherence is ubiquitous, things are not frozen by the quantum Zeno effect. After some recaps on the Zeno effect and on the previous works about this topic (§3.4.1), we present a discrete model that formalizes the competition between free evolution and decoherence (§3.4.2), yielding a criterion for Zeno freezing that depends on the level of short time decoherence. By testing this criterion on several plausible classes of functions (§3.4.3), and by assessing the physical situation they represent (§3.4.4), we conclude that in the most generic case, the system will not be affected by the Zeno effect. The result and the assumptions are discussed in §3.4.5.

- In Chapter 4, we discuss the ontological problem of quantum physics, often referred to as the ‘measurement problem’ (although it does not have so much to do with measurements, as we will argue). The reader unfamiliar with the philosophy of quantum mechanics is invited to read the preliminaries given in the annex A beforehand. Our aim in this chapter is not to give a solution, but rather to formulate the problem independently of any interpretation of quantum mechanics, so as to have a common starting point from which to discuss. We first identify the common root of the puzzle, determine what decoherence solves and what is still calling for an explanation. We propose that the notion of observers is best defined in this context as systems granted with subjectivity, on which the ontological problem is most severe (§4.1). The often related ‘preferred-basis problem’, which we divide into two distinct sub-problems, is also discussed in §4.2. We then study how these issues manifest themselves in five of the most common interpretations of quantum theory, and how they attempt to solve it: the Copenhagen interpretation (§4.3.1) and the objective collapse models (§4.3.2); Bohmian mechanics (§4.4); and finally the many-worlds interpretation (§4.5.1) and relational quantum mechanics (§4.5.2). As shown by Bell, the border between physical facts (experimental results) and possible interpretations (often deemed a mere metaphysical choice) can be particularly difficult to draw. In this respect, for all of these interpretations, we propose concrete experiments (be it thought experiments) that would allow to falsify them.
- In Chapter 5, a new ingredient is introduced in the discussion: spacetime. In §5.1, we focus on the apparent incompatibility between quantum mechanics and special relativity, due to the disappearance of the notion of instantaneity in the latter. This leads us to investigate, from a logician perspective, the interrelationships between fundamental properties in relativistic quantum theories. In §5.1.1, two possible sources of instantaneities in quantum mechanics are exhibited but, in the light of the previous Chapter 4, we argue that only one of them is *a priori* problematic. To deal with the latter, a consistency condition called (C), constraining the statistics of measurements outcomes in any relativistic quantum theory, is formulated and is shown to be necessary to prevent faster-than-light-communication and non-covariance. This condition takes a precise mathematical form, which can be derived under some (generally accepted) assumption on the unitary evolution operator of two isolated systems (§5.1.2). This result is known as the no-communication theorem. We extensively retrace the history

of this topic and compare our treatment with the existing literature in §5.1.3. At this point, we try to go further than the previous works by embracing a more axiomatic point of view. Rather than considering it as a theorem, we argue that (C) is the truly fundamental postulate of relativistic quantum theories because it is required by special relativity. We then look for the complete logical structure between (C) and the previously introduced properties, as well as with the locality of physics on one hand (§5.1.4) and with microcausality and the spin-statistics theorem in the context of quantum field theory on the other hand (§5.1.5). An important result is Theorem 5.1.4, which gives a new proof of a previously known result. Our study also reveals unexpectedly that locality and microcausality are actually two redundant hypotheses of quantum field theory, and provides an original proof of the fact that the Dirac field can in no sense be measured. Results are summarized in §5.1.6.

- In the last Section §5.2, we build a toy model designed to address the preferred-basis problem from a backgroundless perspective. This first contribution is still simplistic, but the long-term aim is to shed light on the link between the shape of the laws of physics and the emergence of a common preferred basis. The motivations are presented in §5.2.1 from a wide perspective. The toy model, restricted to the simplest possible case, is introduced in §5.2.2 and analysed mathematically in §5.2.3. The numerical simulations are then discussed: much more work is needed to interpret the results, but this is the first step of a long-term program whose possible next stages are listed (§5.2.4).
- In the concluding Chapter 6, we summarize our investigations and discuss their implications for quantum gravity. We finish with several general remarks on the notion of backgroundless co-emergence.



# Chapter 2

## Usual mathematical objects and notations in physics

**Hilbert space** *Quantum mechanics is a geometrical way to compute probabilities.* Physical states of the system  $\mathcal{S}$  under study are represented by vectors on the unit-sphere  $\mathbb{S}$  of some separable complex Hilbert space  $\mathcal{H}$ . Most quantum physicists use Dirac's notations:

- a vector of  $\mathcal{H}$  is denoted  $|\Psi\rangle$  and called a **ket**;
- the corresponding linear form in  $\mathcal{H}^*$  given by Riesz' theorem is denoted  $\langle\Psi|$  and called a **bra**;
- the scalar product between  $|\Psi\rangle$  and  $|\Phi\rangle$  is therefore naturally denoted  $\langle\Psi|\Phi\rangle \in \mathbb{C}$  and called a **braket**;
- finally, the rank-one linear operator mapping  $|\Phi\rangle$  to  $|\Psi\rangle$  is naturally denoted  $|\Psi\rangle\langle\Phi| \in \mathcal{L}(\mathcal{H})$ .

The set of bounded operators in  $\mathcal{L}(\mathcal{H})$  is equipped with the usual operator norm:

$$\|M\| = \sup_{\|\Psi\|=1} \|M|\Psi\rangle\|.$$

**Born rule and observables** Probabilistic events concerning  $\mathcal{S}$  correspond to subspaces of  $\mathcal{H}$ , and the probabilities are computed according to the **Born rule**. If  $F \subset \mathcal{H}$  is a subspace and  $\Pi_F$  its associated orthogonal projector, the probability of the event  $F$  when the system is in state  $|\Psi\rangle$  is:

$$\mathbb{P}_{|\Psi\rangle}(F) = \|\Pi_F(|\Psi\rangle)\|^2. \quad (2.1)$$

Consequently, the possible final states of a measurement must be orthogonal (since they should be mutually incompatible) and form a basis of  $\mathcal{H}$  (so that, for any initial state, the probabilities of the final states indeed sum to 1, using Pythagoras' or Parseval's theorem). If moreover one associates, to every final state, the real number corresponding to the physical outcome displayed by the measurement apparatus, we see that (at least in finite dimension)

*physical observables correspond to Hermitian operators, i.e.* diagonalizable in an orthonormal basis with real eigenvalues. Thus, in quantum mechanics, distinguishability corresponds to orthogonality, insofar as two states can be discriminated through a measurement with 0 probability of error if and only if they are orthogonal. The Hermitian operator associated to the physical variable  $A$  will be denoted  $\hat{A}$  and its eigenbasis  $\mathcal{B}_{\hat{A}}$ . This definition could be ambiguous, since the eigenbasis may not be unique if the spectrum are degenerate. **In this thesis, whenever the notation  $\mathcal{B}_{\hat{A}}$  is employed and if the spectrum is degenerate, the statement must be understood as being true for any eigenbasis of  $\hat{A}$ .** Finally, we denote  $\mathfrak{B}(\mathcal{H})$  the set of orthonormal bases of  $\mathcal{H}$ .

**Rigged Hilbert space** In infinite dimension, it is customary to still identify observables with Hermitian operators, although the latter are not necessarily diagonalizable in an orthonormal basis. In particular, the crucial operators  $\hat{X}$  and  $\hat{P}$  on  $\mathcal{L}^2(\mathbb{R}^3)$  corresponding to the position and momentum variables (which are undoubtedly physical observables) can be built so as to be Hermitian, but they don't have any bases of eigenvectors (actually, they have no eigenvectors in  $\mathcal{L}^2(\mathbb{R}^3)$ ). Fortunately, there exist a construction called a **rigging** of the Hilbert space, in which  $\mathcal{H}$  is sandwiched as  $H \subset \mathcal{H} \subset H^\times$  where  $H^\times$  is the dual of  $H$ , a locally convex topological vector subspace of  $\mathcal{H}$ , endowed with a locally convex topology finer than the norm topology and such that  $H$  is dense in  $\mathcal{H}$ , allowing to write for any Hermitian operator  $\hat{A}$  on  $\mathcal{H}$ :

$$\forall \Psi, \Phi \in H, \quad \langle \Psi | \hat{A} \Phi \rangle = \int_{\text{spec}(\hat{A})} \sum_{k=1}^{N(x)} x \langle \Psi | \Gamma_{x,k} \rangle \overline{\langle \Phi | \Gamma_{x,k} \rangle} \mu(dx), \quad (2.2)$$

where  $\mu$  is a measure on  $\text{spec}(\hat{A})$ ,  $N(x)$  is an integer interpreted as the multiplicity  $x \in \text{spec}(\hat{A})$  and the  $|\Gamma_{x,k}\rangle \in H^\times$  are called the **generalized eigenvectors** of  $\hat{A}$  [61]. This way, we can formally write  $\hat{A} = \int_{\text{spec}(\hat{A})} \sum_{k=1}^{N(x)} x |\Gamma_{x,k}\rangle \langle \Gamma_{x,k}| \mu(dx)$  and work with this generalized spectral decomposition. The appropriate choice of  $H$  is problem-dependent, but a typical construction for  $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$  involves the Schwartz space  $H = \mathcal{S}(\mathbb{R}^3)$  of rapidly decreasing functions, and its dual the distribution space  $H^\times = \mathcal{S}'(\mathbb{R}^3)$ .

**Entanglement and density matrices** When two systems  $\mathcal{S}_1$  and  $\mathcal{S}_2$  interact, they are jointly described by the tensor product  $\mathcal{H}_1 \otimes \mathcal{H}_2$  of their respective Hilbert spaces, equipped with the canonical extension of the scalar product. If they are in a state of the form  $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$  (a pure tensor), we say that they are in a **product state**; otherwise, they are called **entangled**. It is customary to omit the tensor product symbol whenever the context is clear, and write for instance  $|\Psi_1\rangle |\Psi_2\rangle$  or even  $|\Psi_1 \Psi_2\rangle$ . Now, if we are concerned with the probabilities of only one of the two systems, we would like to construct the minimal object, pertaining to  $\mathcal{H}_1$  only, that contains all the relevant information for  $\mathcal{S}_1$ . The events we consider are therefore the subspaces of the form  $F \otimes \mathcal{H}_2$  with  $F$  a subspace of  $\mathcal{H}_1$ , and their associated projectors are the  $\Pi_F \otimes \mathbb{1}_2$  where  $\mathbb{1}_2$  is the identity operator on  $\mathcal{H}_2$ . Hence:

$$\mathbb{P}_{|\Psi\rangle}(F \otimes \mathcal{H}_2) = \|\Pi_F \otimes \mathbb{1}_2(|\Psi\rangle)\|^2$$

$$\begin{aligned}
&= \langle \Psi | \Pi_F \otimes \mathbb{1}_2 | \Psi \rangle \\
&= \text{tr}(|\Psi\rangle \langle \Psi| \Pi_F \otimes \mathbb{1}_2) \\
&= \text{tr}(\rho_1 \Pi_F),
\end{aligned}$$

where  $\rho_1 \in \mathcal{L}(\mathcal{H}_1)$  is the **partial trace** of the rank-one projector  $|\Psi\rangle \langle \Psi| \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ , which can be defined on the grounds of the following proposition.

**Proposition 2.0.1** (Partial trace). *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two finite dimensional Hilbert spaces. There exists a unique linear map  $\text{tr}_2 : \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{L}(\mathcal{H}_1)$  such that:*

$$\forall \rho_1 \in \mathcal{L}(\mathcal{H}_1), \forall \rho_2 \in \mathcal{L}(\mathcal{H}_2), \quad \text{tr}_2(\rho_1 \otimes \rho_2) = \text{tr}(\rho_2)\rho_1.$$

*It satisfies the following property:*

$$\forall \rho_{12} \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2), \forall A_1 \in \mathcal{L}(\mathcal{H}_1), \quad \text{tr}(\rho_{12} A_1 \otimes \mathbb{1}_2) = \text{tr}(\text{tr}_2(\rho_{12}) A_1). \quad (2.3)$$

*The operator  $\text{tr}_2(\rho_{12})$  is called the partial trace of  $\rho_{12}$  with respect to  $\mathcal{H}_2$ .*

The result is also true in infinite dimension, provided one replaces the spaces  $\mathcal{L}(\mathcal{H})$  by  $\mathcal{B}_1(\mathcal{H})$ , the space of trace-class operators in  $\mathcal{H}$  [10].

*Proof.* Let  $(|e_i\rangle)_i$  and  $(|f_k\rangle)_k$  be some bases of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , so that  $(E_{ij})_{ij} = (|e_i\rangle \langle e_j|)_{ij}$  and  $(F_{kl})_{kl} = (|f_k\rangle \langle f_l|)_{kl}$  are bases of  $\mathcal{L}(\mathcal{H}_1)$  and  $\mathcal{L}(\mathcal{H}_2)$  respectively. Using the defining property given in the proposition, the linear map  $\text{tr}_2$  is uniquely defined by its image on the  $(E_{ij} \otimes F_{kl})_{ijkl}$ , which form a basis of  $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ .

To prove the second statement, it is enough by linearity to check it for pure tensors  $\rho_{12} = \rho_1 \otimes \rho_2$ . It then simply reads:  $\text{tr}(\text{tr}_2(\rho_1 \otimes \rho_2) A_1) = \text{tr}(\text{tr}(\rho_2)\rho_1 A_1) = \text{tr}(\rho_2) \text{tr}(\rho_1 A_1) = \text{tr}(\rho_1 A_1 \otimes \rho_2) = \text{tr}((\rho_1 \otimes \rho_2)(A_1 \otimes \mathbb{1}_2))$ .  $\square$

A useful characterisation of  $\text{tr}_2(\rho_{12})$  is the following. In any orthonormal bases  $(|e_i\rangle)_i$  and  $(|f_k\rangle)_k$  of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , its matrix elements in the  $(|e_i\rangle)_i$  basis are:

$$[\text{tr}_2(\rho_{12})]_{ij} = \langle e_i | \text{tr}_2(\rho_{12}) | e_j \rangle = \sum_k \langle e_i f_k | \rho_{12} | e_j f_k \rangle. \quad (2.4)$$

From a physical point of view, we see that the probabilities obeyed by an entangled system may not be given by the Born rule (2.1) applied to some ket  $|\Psi\rangle \in \mathcal{H}_1$ , but the most general physical state for a quantum system is some operator  $\rho_1 \in \mathcal{L}(\mathcal{H}_1)$  called its **density matrix**, obtained by tracing over the other systems it may be entangled with. It is the equivalent, in classical probability theory, of the marginal over  $\mathcal{S}_2$  of the joint probability law of  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . The probabilities are then computed using the **generalized Born rule**:

$$\mathbb{P}_{\rho_1}(F) = \text{tr}(\rho_1 \Pi_F),$$

which coincides with (2.1) when  $\rho_1$  is a rank-one orthogonal projector  $|\Psi_1\rangle \langle \Psi_1|$ . In this latter case, the system is said to be in a **pure state**, otherwise it is in a **mixed state**. One



can show that density matrices must be Hermitian, positive and have trace 1. The set of such operators over  $\mathcal{H}$  is denoted  $\mathcal{S}(\mathcal{H})$ .

In finite dimension, they are diagonalizable in an orthonormal basis  $\mathcal{B}_\rho$  with eigenvalues interpreted as probabilities (non negative real numbers summing to 1). In particular, the latter correspond to a Dirac measure when  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are not entangled (because  $\rho_1$  is then a rank-one orthogonal projector), and to a uniform measure when they are in a **maximally entangled state**  $|\Psi\rangle = \frac{1}{\sqrt{n}} \sum_i |e_i\rangle |f_i\rangle$  for some orthonormal bases  $(|e_i\rangle)_i$  and  $(|f_k\rangle)_k$  of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  (in which case  $\rho_1 = \frac{1}{n} \mathbb{1}_1$ ). It is then natural to associate to any density matrix an entropy which quantifies how entangled the system is. Here are two possibilities (among many others):

**Definition 2.0.2** (Entanglement entropies). Let  $\rho \in \mathcal{S}(\mathcal{H})$ . Define the **von Neumann entropy** of  $\rho$  as  $S(\rho) = -\text{tr}(\rho \ln(\rho))$  and its **linear entropy**  $S_{\text{lin}}(\rho) = 1 - \text{tr}(\rho^2)$ , where  $\text{tr}(\rho^2)$  is called the **purity** of  $\rho$ .

**Unitary evolution and Schrödinger equation** We now want to build some dynamics in order to know how our systems evolve in time to make predictions. *The key ingredient is the requirement that the laws of physics remain the same over time*, which is arguably one of the most basic defining properties for calling something ‘laws of physics’. Let’s first consider only pure states. We are therefore looking for a family of maps  $U_t : \mathcal{H} \rightarrow \mathcal{H}$  indexed by a real parameter representing time, mapping any initial state to the final state obtained by applying the laws of physics during  $t$ , such that:

$$\forall t, t', \quad U(t + t') = U(t)U(t').$$

To keep the probabilities normalized, we also know that the  $(U_t)_{t \in \mathbb{R}}$  must preserve the norm. Adding the additional postulate that they are linear (or simply that they preserve the scalar product, as done by Wigner [139]) entails that each  $U_t \in \mathcal{U}(\mathcal{H})$  is unitary. If one moreover makes the physically reasonable assumption that  $t \mapsto U_t$  is (strongly) continuous, then the  $(U_t)_{t \in \mathbb{R}}$  form a one-parameter Lie group which admits an infinitesimal generator from its Lie algebra. In this case, it ensures the existence of a Hermitian operator  $\hat{H}$  such that  $\forall t \in \mathbb{R}, U_t = e^{-it\hat{H}}$ . Equivalently, the time evolution of a ket  $|\Psi(t)\rangle$  in quantum mechanics is governed by the following **Schrödinger equation**:

$$i \frac{d|\Psi(t)\rangle}{dt} = \hat{H} |\Psi(t)\rangle.$$

What is the physical meaning of the observable<sup>1</sup>  $\hat{H}$ ? Necessarily, this quantity must be conserved on average, because for a system in state  $|\Psi\rangle$ , its mean value is  $\langle \hat{A} \rangle_{|\Psi\rangle} = \langle \Psi | \hat{A} | \Psi \rangle$  and satisfies:

$$\frac{d\langle \hat{H} \rangle_{|\Psi(t)\rangle}}{dt} = \langle -i\hat{H} |\Psi(t)\rangle | \hat{H} |\Psi(t)\rangle \rangle + \langle \Psi(t) | \hat{H} | -i\hat{H} |\Psi(t)\rangle \rangle = 0$$

---

<sup>1</sup>Isn’t it a miracle that the Lie algebra of  $\mathcal{U}(\mathcal{H})$ , up to a factor  $i$ , is precisely composed of the Hermitian operators, so that the infinitesimal generator of quantum mechanical time evolution can also make sense as an observable?

since  $\hat{H}$  is Hermitian and using the (anti-)linearity of the complex scalar product. This was expected due to Noether's theorem: any symmetry of the theory (here, the invariance of the laws over time) has an associated conserved quantity (here, the quantity  $H$ ). From other areas of physics, it is well-known that the conserved quantity associated to the time translation symmetry is the energy. Alternatively, this can be taken as the most fundamental definition of what we call energy. Finally, for a general mixed state  $\rho \in \mathcal{S}(\mathcal{H})$ , the time evolution (when no interaction with external systems occurs) takes the form  $\rho \mapsto U_t \rho U_t^\dagger$ , and the Schrödinger equation reads  $i \frac{d\rho(t)}{dt} = [\hat{H}, \rho(t)]$ .

## Abbreviations

- QM: quantum mechanics
- QFT: quantum field theory
- BM : Bohmian mechanics
- RQM: relational quantum mechanics
- SR : special relativity
- GR : general relativity
- FTL : faster-than-light



# Chapter 3

## Two models for the theory of decoherence

This chapter is a transcription of the published article [128] and of the preprint [129].

### 3.1 The basics of decoherence

The theory of decoherence sheds light on the reason why quantum interferences disappear when a system gets entangled with a macroscopic one, for example an electron in a double-slit experiment that doesn't interfere anymore when entangled with a detector. According to Di Biagio and Rovelli [46], the deep difference between classical and quantum is the way probabilities behave: all classical phenomena satisfy the total probability formula

$$\mathbb{P}(B = y) = \sum_{x \in \text{Im}(A)} \mathbb{P}(A = x) \mathbb{P}(B = y | A = x), \quad (3.1)$$

thanks to which, *even though the actual value of the variable  $A$  is not known, one can still assume that it has a definite value among the possible ones*. This, however, is not correct for quantum systems (the philosophical consequences of this are the topic of Chapter 4). The diagonal elements of the density matrix (written in the eigenbasis  $\mathcal{B}_{\hat{A}}$  of the observable on which the probabilities are conditioned) account for the classical behavior of a system because they correspond to the terms of the total probability formula, while the non-diagonal terms are the additional quantum interferences. This is because the probability to obtain an outcome  $x$  while being in state  $\rho$  is:

$$\text{tr}(\rho |x\rangle \langle x|) = \sum_{i,j=1}^n \rho_{ij} \langle j|x\rangle \langle x|i\rangle = \sum_{i=1}^n \underbrace{\rho_{ii} |\langle x|i\rangle|^2}_{\mathbb{P}(i)\mathbb{P}(x|i)} + \sum_{1 \leq i < j \leq n} \underbrace{2\text{Re}(\rho_{ij} \langle j|x\rangle \langle x|i\rangle)}_{\text{interferences}}.$$

Here is the typical situation encountered in decoherence studies. Consider a system  $\mathcal{S}$ , described by a Hilbert space  $\mathcal{H}_{\mathcal{S}}$  of dimension  $d$ , that interacts with an environment  $\mathcal{E}$  described by a space  $\mathcal{H}_{\mathcal{E}}$  of dimension  $n$ , and let  $\mathcal{B} = (|i\rangle)_{1 \leq i \leq d}$  be an orthonormal basis of  $\mathcal{H}_{\mathcal{S}}$ .

In the sequel, we will say that each  $|i\rangle$  corresponds to a **possible history** or **potentiality** of the system (this expression will be given its full meaning in §4.1). Let's also assume that  $\mathcal{B}$  is a **conserved basis** (an idealization of the concept of pointer basis, often invoked in the decoherence literature [157]), composed of states of  $\mathcal{S}$  that remain perfectly unchanged during the interaction with  $\mathcal{E}$ . When  $\mathcal{E}$  is a measurement apparatus for the observable  $A$ , the eigenbasis of  $\hat{A}$  is clearly a conserved basis; in general, the eigenbasis of any observable such that  $\hat{A} \otimes \mathbb{1}$  commutes with the interaction Hamiltonian is suitable (but the existence of such an observable is not guaranteed, unless  $\hat{H}_{int}$  takes the form  $\sum_i \hat{\Pi}_i^S \otimes \hat{H}_i^E$ , where  $(\hat{\Pi}_i^S)_{1 \leq i \leq d}$  is a family of commuting orthogonal projectors).

We further suppose that  $\mathcal{S}$  and  $\mathcal{E}$  are initially non-entangled, allowing to write  $|\Psi\rangle = \left(\sum_{i=1}^d c_i |i\rangle\right) \otimes |\mathcal{E}_0\rangle$  as the initial state before interaction. After a time  $t$ , due to its Schrödinger evolution in the conserved basis, the total state becomes  $|\Psi(t)\rangle = \sum_{i=1}^d c_i |i\rangle \otimes |\mathcal{E}_i(t)\rangle$  for some unit vectors  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$ . From this expression, the system's state  $\rho_S(t)$  can be obtained after tracing out the environment according to [2.4]. If  $(|e_k\rangle)_{1 \leq k \leq n}$  denotes an orthonormal basis of  $\mathcal{H}_E$ :

$$\begin{aligned}
\rho_S(t) &= \text{tr}_E |\Psi(t)\rangle \langle \Psi(t)| \\
&= \sum_{k=1}^n \left( \sum_{i=1}^d |c_i|^2 |\langle e_k | \mathcal{E}_i(t) \rangle|^2 |i\rangle \langle i| + \sum_{1 \leq i \neq j \leq d} c_i \bar{c}_j \langle e_k | \mathcal{E}_i(t) \rangle \langle \mathcal{E}_j(t) | e_k \rangle |i\rangle \langle j| \right) \\
&= \sum_{i=1}^d |c_i|^2 \underbrace{\sum_{k=1}^n |\langle e_k | \mathcal{E}_i(t) \rangle|^2}_{=1} |i\rangle \langle i| + \sum_{1 \leq i \neq j \leq d} c_i \bar{c}_j \langle \mathcal{E}_j(t) | \underbrace{\left( \sum_{k=1}^n |e_k\rangle \langle e_k| \right)}_{=1} | \mathcal{E}_i(t) \rangle |i\rangle \langle j| \\
&= \sum_{i=1}^d |c_i|^2 |i\rangle \langle i| + \sum_{1 \leq i \neq j \leq d} c_i \bar{c}_j \langle \mathcal{E}_j(t) | \mathcal{E}_i(t) \rangle |i\rangle \langle j| \\
&\equiv D_{\mathcal{B}}[\rho_S(t)] + Q_{\mathcal{B}}[\rho_S(t)],
\end{aligned}$$

where  $D_{\mathcal{B}}[\rho_S(t)]$  stands for the (time-independent) diagonal part of  $\rho_S(t)$  in  $\mathcal{B}$  (which corresponds to the total probability formula), and  $Q_{\mathcal{B}}[\rho_S(t)]$  for the remaining non-diagonal terms (responsible for the quantum interferences between the possible histories). Our main object of study in the next section will be the following quantity.

**Definition 3.1.1.** Let  $\eta_B(\rho_S(t)) = \max_{i \neq j} |\langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle|$ .

Note that this definition actually makes sense in any (potentially non-conserved) orthonormal basis. Indeed, in another basis  $\mathcal{B}' = (|i'\rangle)_{1 \leq i \leq d}$  of  $\mathcal{H}_S$ , one can still write  $|\Psi(t)\rangle = \sum_{i=1}^d c'_i(t) |i'\rangle \otimes |\mathcal{E}'_i(t)\rangle$ . If the  $(|\mathcal{E}'_i(t)\rangle)_{1 \leq i \leq d}$  are only required to be unit vectors, they are defined up to a phase, but this does not affect the above definition. Most of the time, the context will be clear, hence we will drop the arguments and simply write  $\eta$ . The reason for  $\eta$  to be is the following proposition, stating that  $\eta$  measures how close the system is from being classical.

**Proposition 3.1.2.** *We have  $\|Q_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]\| \leq \eta_{\mathcal{B}}(t)$  and for all subspaces  $F \subset \mathcal{H}_{\mathcal{S}}$ ,*

$$\left| \underbrace{\text{tr}(\rho_{\mathcal{S}}(t)\Pi_F)}_{\text{quantum probability}} - \underbrace{\text{tr}(D_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]\Pi_F)}_{\text{classical probability}} \right| \leq \dim(F) \eta_{\mathcal{B}}(t).$$

*In a nutshell:*  $\mathbb{P}_{\text{quantum}} = \mathbb{P}_{\text{classical}} + \mathcal{O}(\eta)$ .

*Proof.* For all vectors  $|\Psi\rangle = \sum_k \alpha_k |k\rangle \in \mathcal{H}_{\mathcal{S}}$  of norm 1,

$$\begin{aligned} \|Q_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]|\Psi\rangle\|^2 &= \left\| \sum_{1 \leq i \neq j \leq d} c_i \bar{c}_j \langle \mathcal{E}_j(t) | \mathcal{E}_i(t) \rangle \alpha_j |i\rangle \right\|^2 \\ &= \sum_{i=1}^d |c_i|^2 \left| \sum_{\substack{j=1 \\ j \neq i}}^d \bar{c}_j \langle \mathcal{E}_j(t) | \mathcal{E}_i(t) \rangle \alpha_j \right|^2 \\ &\leq \eta(t)^2 \sum_{i=1}^d |c_i|^2 \left( \sum_{j=1}^d |c_j| |\alpha_j| \right)^2 \\ &\stackrel{\text{C.-S.}}{\leq} \eta(t)^2 \sum_{i=1}^d |c_i|^2 \sum_{j=1}^d |c_j|^2 \sum_{k=1}^d |\alpha_j|^2 \\ &\leq \eta(t)^2, \end{aligned}$$

hence  $\|Q_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]\| \leq \eta(t)$ . Now, if  $F$  is a subspace of  $\mathcal{H}_{\mathcal{S}}$  (*i.e.* a probabilistic event), let  $(|\pi_k\rangle)_k$  be an orthonormal basis of  $F$  and  $\Pi_F$  the orthogonal projector of  $F$ . Then:

$$\begin{aligned} \text{tr}(\rho_{\mathcal{S}}(t)\Pi_F) - \text{tr}(D_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]\Pi_F) &= \text{tr}(Q_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]\Pi_F) = \sum_{k=1}^{\dim(F)} \langle \pi_k | Q_{\mathcal{B}}[\rho_{\mathcal{S}}(t)] \pi_k \rangle \\ \Rightarrow \left| \text{tr}(\rho_{\mathcal{S}}(t)\Pi_F) - \text{tr}(D_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]\Pi_F) \right| &\leq \sum_{k=1}^{\dim(F)} \|Q_{\mathcal{B}}[\rho_{\mathcal{S}}(t)]\| \leq \dim(F)\eta(t). \end{aligned}$$

□

In other words,  $\eta$  bounds the level of quantum interferences, *i.e.* it estimates how decohered the system is. Notice well that it is only during an interaction between  $\mathcal{S}$  and  $\mathcal{E}$  that decoherence can occur; any future internal evolution  $U$  of  $\mathcal{E}$  lets  $\eta$  unchanged since  $\langle U\mathcal{E}_j | U\mathcal{E}_i \rangle = \langle \mathcal{E}_j | \mathcal{E}_i \rangle$ . Also, a more precise definition for  $\eta$ , as we shall see in §3.3.4, could be  $\max_{i \neq j} |c_i| |c_j| |\langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle|$ , so that  $\rho_{\mathcal{S}}$  being diagonal in a basis becomes equivalent to  $\eta = 0$  in this basis. This way  $\eta$  really quantifies the interferences between *possible* histories (of non-zero probability) 1.

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<sup>1</sup>This is not true with the definition above, as is clear for example for the trivial interaction  $|\Psi(t)\rangle = \sum_{i=1}^d c_i |i\rangle \otimes |\mathcal{E}_0\rangle$ : here  $\rho_{\mathcal{S}}$  is diagonal (*i.e.* no interferences) in any orthonormal basis containing the vector  $\sum_{i=1}^d c_i |i\rangle$ , while the simpler definition yields  $\eta = 1$  in any basis.

The aim of the theory of decoherence is to explain why  $\eta_{\mathcal{B}}(\rho_{\mathcal{S}}(t))$  rapidly goes to zero in the conserved basis when  $n$  is large, so that the state of the system quickly<sup>2</sup> evolves from  $\rho_{\mathcal{S}}$  to  $D_{\mathcal{B}}[\rho_{\mathcal{S}}]$ . As recalled in the introduction, a wide variety of models already explain this phenomenon for some specific contexts and Hamiltonians. The model we shall build in the next section aims at grasping the fundamental mathematical reason why decoherence is so universal and ubiquitous, without specifying any Hamiltonian. This will allow us to determine the typical size of an environment needed to entail proper decoherence on a system.

## 3.2 First model: decoherence as a high-dimensional geometrical phenomenon

When no particular assumption is made to specify the type of environment under study, the only reasonable behavior to assume for  $|\mathcal{E}_i(t)\rangle$  is that of a Brownian motion on the sphere  $\mathbb{S}^n = \{|\Psi\rangle \in \mathcal{H}_{\mathcal{E}} \mid \|\Psi\| = 1\} \subset \mathcal{H}_{\mathcal{E}} \simeq \mathbb{C}^n \simeq \mathbb{R}^{2n}$ . It boils down to representing the environment as a purely random system with no preferred direction of evolution. This choice will be discussed in §3.2.5. Another bold assumption would be the independence of the  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$ ; we will dare to make this assumption anyway.

### 3.2.1 Convergence to the uniform measure

We will first show that the probabilistic law of each  $|\mathcal{E}_i(t)\rangle$  converges exponentially fast to the uniform probability measure on  $\mathbb{S}^n$ . To make things precise, endow  $\mathbb{S}^n$  with its Borel  $\sigma$ -algebra  $\mathcal{B}$  and with the canonical Riemannian metric  $g$ , which induces the uniform measure  $\mu$  that we suppose normalized to a probability measure. Let  $\nu_t$  be the law of the random variable  $|\mathcal{E}_i(t)\rangle$ , that is  $\nu_t(B) = \mathbb{P}(|\mathcal{E}_i(t)\rangle \in B)$  for all  $B \in \mathcal{B}$ . Denote  $\Delta f = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j f)$  the Laplacian operator on  $\mathcal{C}^\infty(\mathbb{S}^n)$  which can be extended to  $L^2(\mathbb{S}^n)$ , the completion of  $\mathcal{C}^\infty(\mathbb{S}^n)$  for the scalar product  $(f, h) = \int_{\mathbb{S}^n} f(x)h(x)d\mu$ . Finally, recall that the total variation norm of a measure defined on  $\mathcal{B}$  is given by  $\|\sigma\|_{TV} = \sup_{B \in \mathcal{B}} |\sigma(B)|$ .

**Proposition 3.2.1.** *We have  $\|\nu_t - \mu\|_{TV} \xrightarrow{t \rightarrow +\infty} 0$  exponentially fast. Moreover, if  $T(\mathbb{S}^n) = \inf\{t > 0 \mid \|\nu_t - \mu\|_{TV} \leq \frac{1}{e}\}$  denotes the characteristic time to equilibrium for the Brownian diffusion on  $\mathbb{S}^n$ , then  $T(\mathbb{S}^n) \underset{n \rightarrow +\infty}{\sim} \frac{\ln(n)}{4n}$ .*

*Proof.* We follow the ideas of [17], in which the eigenvalues of  $\Delta$  are recalled. Since we are working on the sphere of real dimension  $2n$ , they take the form  $\lambda_k = -k(k + 2n - 2)$  for  $k \in \mathbb{N}$  with multiplicity  $d_k = \frac{(k+2n-3)!}{(2n-2)!k!} (2k + 2n - 2)$ . Denote  $(f_{k,l})_{\substack{k \in \mathbb{N} \\ 1 \leq l \leq d_k}}$  an orthonormal Hilbert basis in  $L^2(\mathbb{S}^n)$  of eigenfunctions of  $\Delta$ , where  $f_{k,l}$  is associated with the eigenvalue

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<sup>2</sup>It is actually very important that the decoherence process (even a measurement) is not instantaneous. Otherwise, it would be impossible to explain why an unstable nucleus continuously measured by a Geiger counter is not frozen due to the quantum Zeno effect, see §3.4.

$\lambda_k$ . Note that  $d_0 = 1$  and that  $f_{0,1}$  is constant (as any harmonic function on a manifold without boundary, due to the maximum principle) so it is the density of a uniform measure. The law  $\nu_0$  of the deterministic variable  $|\mathcal{E}_i(0)\rangle = |\mathcal{E}_0\rangle$  corresponds to a Dirac distribution, which is not strictly speaking in  $L^2(\mathbb{S}^n)$ , so we rather consider it as given by a sharply peaked density (with respect to  $\mu$ ) denoted  $p_0 \in L^2(\mathbb{S}^n)$ ; the latter can be decomposed in the basis  $(f_{k,l})_{k,l}$  as  $p_0 = \sum_{k=0}^{+\infty} \sum_{l=1}^{d_k} a_{k,l} f_{k,l}$ . The fact that  $\|p_0\|_{L^2}^2 = \sum_{k=0}^{+\infty} \sum_{l=1}^{d_k} |a_{k,l}|^2$  yields  $|a_{k,l}| \leq \|p_0\|_{L^2}$ . Denote also  $p_t$  the density after a time  $t$ , *i.e.*  $\nu_t(dx) = p_t(x)\mu(dx)$ . The Hille-Yosida theory [147] allows to define the Brownian motion on the sphere as the Markov semigroup of stochastic kernels generated by  $\Delta$ ; in particular, this implies  $p_t = e^{t\Delta}p_0 = \sum_{k=0}^{+\infty} e^{-k(k+2n-2)t} \sum_{l=1}^{d_k} a_{k,l} f_{k,l}$ . Note that  $a_{0,1}f_{0,1}$  is a probability density because for all  $k \geq 1$ ,  $\int_{\mathbb{S}^n} f_{k,l} = \int_{\mathbb{S}^n} \frac{\Delta f_{k,l}}{\lambda_k} = 0$  due to Stokes' theorem, thus  $\int_{\mathbb{S}^n} a_{0,1}f_{0,1} = \int_{\mathbb{S}^n} a_{0,1}f_{0,1} + \sum_{k=1}^{+\infty} \sum_{l=1}^{d_k} a_{k,l} \int_{\mathbb{S}^n} f_{k,l} = \int_{\mathbb{S}^n} p_0 = 1$ . Hence  $a_{0,1}f_{0,1} = 1$ , and therefore:

$$\begin{aligned} \|\nu_t - \mu\|_{TV} &= \frac{1}{2} \int_{\mathbb{S}^n} |p_t(x) - 1| d\mu && \text{(classical result on the total variation norm)} \\ &\leq \frac{1}{2} \sum_{k=1}^{+\infty} e^{-k(k+2n-2)t} \sum_{l=1}^{d_k} |a_{k,l}| \underbrace{\|f_{k,l}\|_{L^1}}_{\leq 1 (*)} \\ &\leq \frac{1}{2} \|p_0\|_{L^2} \sum_{k=1}^{+\infty} e^{-k(k+2n-2)t} d_k \end{aligned}$$

where  $(*)$  relies on Hölder's inequality  $\|f_{k,l}\|_{L^1} \leq \mu(\mathbb{S}^n)^{1/2} \|f_{k,l}\|_{L^2} = 1$ . It remains to find a characteristic time after which the above series is efficiently bounded. The precise argument can be found in [117], but a first upper bound on  $T(\mathbb{S}^n)$  can be obtained as follows. Set  $u_k = e^{-k(k+2n-2)t} d_k$  so that  $u_1 = 2ne^{-(2n-1)t}$  and:

$$\frac{u_{k+1}}{u_k} = \frac{k+2n-2}{k+1} \frac{2k+2n}{2k+2n-2} e^{-(2n-2k-1)t}.$$

If  $n \geq 2$  and  $t \geq t_n = \frac{\ln(2n-1)}{2n-1}$ , then  $\frac{u_{k+1}}{u_k} \leq \frac{2n-1}{2} \frac{2n+2}{2n} \frac{1}{2n-1} \leq \frac{3}{4}$ , which implies:

$$\|\nu_t - \mu\|_{TV} \leq \frac{1}{2} \|p_0\|_{L^2} u_1 \sum_{k=1}^{+\infty} \left(\frac{3}{4}\right)^k \leq 3 \|p_0\|_{L^2} n e^{-(2n-1)t}.$$

Interestingly enough, the convergence is faster as  $n$  increases since the characteristic time to equilibrium satisfies  $T(\mathbb{S}^n) \xrightarrow{n \rightarrow \infty} 0$  and the exponential is sharper.  $\square$

**Remark 3.2.2.** Physically speaking, there is no reason for the  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$  to follow the canonical 'unit speed' Brownian motion used in the proof. Coupling constants  $g_i$  should be introduced to replace  $\Delta$  by  $g_i \Delta$  for the diffusion of  $|\mathcal{E}_i(t)\rangle$ . But thanks to the exponential convergence — all the more efficient that the dimension  $n$  is high, which will certainly be the case for macroscopic environments — the conclusion remains unchanged: the  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$  almost immediately follow the uniform law.



### 3.2.2 Most vectors of $\mathbb{S}^n$ are almost orthogonal

Consequently, we are now interested in the behavior of the scalar products between random vectors uniformly distributed on the complex  $n$ -sphere  $\mathbb{S}^n$ . The first thing to understand is that, *in high dimension, most pairs of unit vectors are almost orthogonal*.

**Proposition 3.2.3.** *Denote by  $S = \langle \mathcal{E}_1 | \mathcal{E}_2 \rangle \in \mathbb{C}$  the random variable where  $|\mathcal{E}_1\rangle$  and  $|\mathcal{E}_2\rangle$  are two independent uniform random variables on  $\mathbb{S}^n$ . Then  $\mathbb{E}(S) = 0$  and  $\mathbb{V}(S) = \mathbb{E}(|S|^2) = \frac{1}{n}$ .*

*Proof.* Clearly,  $|\mathcal{E}_1\rangle$  and  $-|\mathcal{E}_1\rangle$  have the same law, hence  $\mathbb{E}(S) = \mathbb{E}(-S) = 0$ . What about its variance? One can rotate the sphere to impose for example  $|\mathcal{E}_1\rangle = (1, 0, \dots, 0)$ , and by independence  $|\mathcal{E}_2\rangle$  still follows a uniform law. Such a uniform law can be achieved by generating  $2n$  independent normal random variables  $(X_i)_{1 \leq i \leq 2n}$  following  $\mathcal{N}(0, 1)$ , and by considering the random vector  $|\mathcal{E}_2\rangle = \left( \frac{X_1 + iX_2}{\sqrt{X_1^2 + \dots + X_{2n}^2}}, \dots, \frac{X_{2n-1} + iX_{2n}}{\sqrt{X_1^2 + \dots + X_{2n}^2}} \right)$ . Indeed, for any continuous function  $f : \mathbb{S}^n \rightarrow \mathbb{R}$  (with  $d\sigma^n$  denoting the measure induced by Lebesgue's on  $\mathbb{S}^n$ ):

$$\begin{aligned} \mathbb{E}[f(|\mathcal{E}_2\rangle)] &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} f\left(\frac{x_1 + ix_2}{\sqrt{x_1^2 + \dots + x_{2n}^2}}, \dots, \frac{x_{2n-1} + ix_{2n}}{\sqrt{x_1^2 + \dots + x_{2n}^2}}\right) e^{-(x_1^2 + \dots + x_{2n}^2)/2} dx_1 \dots dx_{2n} \\ &= \frac{1}{(2\pi)^n} \int_0^\infty \left[ \int_{\mathbb{S}^n} f(u) d\sigma^n(u) \right] e^{-\frac{r^2}{2}} r^{2n-1} dr \\ &= \omega_n \int_{\mathbb{S}^n} f(u) d\sigma^n(u), \end{aligned}$$

which means that  $|\mathcal{E}_2\rangle$  defined this way follows indeed the uniform law.

In these notations,  $|S|^2 = \frac{X_1^2 + X_2^2}{X_1^2 + \dots + X_{2n}^2}$ . Since each  $X_i^2$  follows a  $\chi^2$  law, it is then a classical lemma to show that  $|S|^2$  follows a  $\beta_{1, n-1}$  distribution, whose mean equals  $\frac{1}{n}$ . For a more elementary argument, note that, up to relabelling the variables, we have  $\forall k \in \llbracket 1, n \rrbracket$ ,  $\mathbb{E}\left(\frac{X_1^2 + X_2^2}{X_1^2 + \dots + X_{2n}^2}\right) = \mathbb{E}\left(\frac{X_{2k-1}^2 + X_{2k}^2}{X_1^2 + \dots + X_{2n}^2}\right)$  and so:

$$\mathbb{V}(S) = \mathbb{E}\left(\frac{X_1^2 + X_2^2}{X_1^2 + \dots + X_{2n}^2}\right) = \frac{1}{n} \sum_{k=1}^n \mathbb{E}\left(\frac{X_{2k-1}^2 + X_{2k}^2}{X_1^2 + \dots + X_{2n}^2}\right) = \frac{1}{n} \mathbb{E}(1) = \frac{1}{n}.$$

Alternatively, had we worked on the real sphere  $\subset \mathbb{R}^{2n}$  endowed with the real scalar product, the variance would have been  $\frac{1}{2n}$ . This highlights the fact that the real and complex spheres are indeed isomorphic as topological or differential manifolds, but not as Riemannian manifolds.

The same result would have been recovered if, instead of picking randomly a pair of vectors, we had chosen uniformly the unitary evolution operators  $(U^{(i)}(t))_{1 \leq i \leq d}$  such that  $|\mathcal{E}_i(t)\rangle = U^{(i)}(t) |\mathcal{E}_0\rangle$ , resulting from the interaction Hamiltonian. Again, if no direction of evolution is preferred, it is reasonable to consider the law of each  $U^{(i)}(t)$  to be given by the Haar measure  $dU$  on the unitary group  $\mathcal{U}_n$ . If moreover they are independent, then  $U^{(i)}(t)^\dagger U^{(j)}(t)$  also follows the Haar measure for all  $i, j$  so that, using [130], (112)]:

$$\mathbb{V}(\langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle) = \int_{\mathcal{U}_n} |\langle \mathcal{E}_0 | U \mathcal{E}_0 \rangle|^2 dU = \prod_{i=2}^n \frac{i-1}{i} = \frac{1}{n}.$$

□

Therefore,  $|\langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle|$  is, after a very short time, of order  $\sqrt{\mathbb{V}(S)} = \frac{1}{\sqrt{\dim(\mathcal{H}_{\mathcal{E}})}}$ , which is a well-known estimate already obtained by Zurek in [154]. When  $d = 2$ , if  $\mathcal{E}$  is composed of  $N_{\mathcal{E}}$  particles and each of them is described by a  $p$ -dimensional Hilbert space, then very rapidly:

$$\eta = |\langle \mathcal{E}_1(t) | \mathcal{E}_2(t) \rangle| \sim p^{-N_{\mathcal{E}}/2}$$

which is virtually zero for macroscopic environments, therefore decoherence is guaranteed. Of course, this is not true anymore if  $d$  is large, because there will be so many pairs that some of them will inevitably become non-negligible, and so will  $\eta$ . We would like to determine a condition between  $n$  and  $d$  under which proper decoherence is to be expected. In other words, what is the minimal size of an environment needed to decohere a given system?

**Remark 3.2.4.** In fact, the assumption of the existence of a conserved basis made in §3.1 is practical but not really necessary. In another (possibly non-conserved) basis, one can still decompose  $|\Psi(t)\rangle = \sum_{i=1}^d c'_i(t) |i'\rangle \otimes |\mathcal{E}'_i(t)\rangle$ . The Brownian character and the independence of the  $(|\mathcal{E}'_i(t)\rangle)_{1 \leq i \leq d}$  in  $\mathcal{B}$  then suffices to lead to strong decoherence in  $\mathcal{B}$ . Seeking to characterize the basis in which decoherence mainly happens, or the so-called ‘pointer basis’, is an important and challenging aspect of the theory of decoherence. Several criteria have been proposed [154] [157], but what precedes hints at a novel one (although physically very impractical): the pointer basis could be defined as the basis in which the  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$  are independent, *i.e.* the eigenbasis of their covariance matrix.

### 3.2.3 Direct study of $\eta$

To answer the above question, we should be more precise and consider directly the random variable  $\eta_{n,d} = \max_{i \neq j} |\langle \mathcal{E}_i | \mathcal{E}_j \rangle|$  where the  $(|\mathcal{E}_i\rangle)_{1 \leq i \leq d}$  are  $d$  random vectors uniformly distributed on the complex  $n$ -sphere  $\mathbb{S}^n$ . In the following, we fix  $\varepsilon \in ]0, 1[$  as well as a threshold  $s \in [0, 1[$  close to 1, and define  $d_{max}^{\varepsilon,s}(n) = \min\{d \in \mathbb{N} \mid \mathbb{P}(\eta_{n,d} \geq \varepsilon) \geq s\}$ , so that if  $d_{max}^{\varepsilon,s}(n)$  points or more are placed randomly on  $\mathbb{S}^n$ , it is very likely (with probability  $\geq s$ ) that at least one of the scalar products will be greater than  $\varepsilon$ .

**Theorem 3.2.5.** *The following asymptotic estimates hold:*

1.  $d_{max}^{\varepsilon,s}(n) \underset{n \rightarrow \infty}{\sim} \sqrt{-2 \ln(1-s)} \left( \frac{1}{1-\varepsilon^2} \right)^{\frac{n-1}{2}}$
2.  $\boxed{\eta_{n,d} - \sqrt{1 - d^{-\frac{2}{n}}} \xrightarrow[n \text{ or } d \rightarrow \infty]{\mathbb{P}} 0}$ , so that we can write informally:  $\eta_{n,d} \simeq \sqrt{1 - d^{-\frac{2}{n}}}$ .

To derive these formulas, we first need the following geometrical lemma.

**Lemma 3.2.6.** *Let  $A_n = |\mathbb{S}^n|$  be the area of the complex  $n$ -sphere for  $d\sigma^n$  (induced by Lebesgue's measure),  $C_n^\varepsilon(x) = \{u \in \mathbb{S}^n \mid |\langle u|x \rangle| \geq \varepsilon\}$  the 'spherical cap'<sup>3</sup> centered in  $x$  of parameter  $\varepsilon$ , and  $A_n^\varepsilon = |C_n^\varepsilon|$  the area of any spherical cap of parameter  $\varepsilon$ . Then for all  $n \geq 1$ :*

$$\frac{A_n^\varepsilon}{A_n} = (1 - \varepsilon^2)^{n-1}.$$

*Proof of Lemma.* This result can be directly obtained from the fact that, as noticed in the proof of Proposition 3.2.3,  $|\langle \mathcal{E}_1 | \mathcal{E}_2 \rangle|^2$  follows a  $\beta_{1,n-1}$  distribution when  $|\mathcal{E}_1\rangle$  and  $|\mathcal{E}_2\rangle$  are chosen uniformly and independently on  $\mathbb{S}^n$ . We can then write:

$$\frac{A_n^\varepsilon}{A_n} = \mathbb{P}(|\langle u|x \rangle|^2 \geq \varepsilon^2) = \int_{\varepsilon^2}^1 \frac{\Gamma(n)}{\Gamma(n-1)} (1-x)^{n-2} dx = [(1-x)^{n-1}]_{\varepsilon^2}^1 = (1 - \varepsilon^2)^{n-1}.$$

A more 'physicist-friendly' proof can also be given, based on an appropriate choice of coordinates on the  $n$ -sphere. Recall that  $\mathbb{S}^n \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$  can be seen as a real manifold of dimension  $2n - 1$ . Consider the set of coordinates  $(r, \theta, \varphi_1, \dots, \varphi_{2n-3})$  on  $\mathbb{S}^n$  defined by the chart

$$\begin{aligned} F : [0, 1] \times [0, 2\pi] \times [0, \pi]^{2n-4} \times [0, 2\pi] &\longrightarrow \mathbb{S}^n \\ (r, \theta, \varphi_1, \dots, \varphi_{2n-3}) &\longmapsto (x_1 + ix_2, \dots, x_{2n-1} + ix_{2n}) \simeq (x_1, \dots, x_{2n}) = \\ &\quad (r \cos(\theta), r \sin(\theta), \sqrt{1-r^2} \cos(\varphi_1), \sqrt{1-r^2} \sin(\varphi_1) \cos(\varphi_2), \dots, \\ &\quad \sqrt{1-r^2} \sin(\varphi_1) \dots \cos(\varphi_{2n-3}), \sqrt{1-r^2} \sin(\varphi_1) \dots \sin(\varphi_{2n-3})). \end{aligned}$$

This amounts to choose the modulus  $r$  and the argument  $\theta$  of  $x_1 + ix_2$ , and then describe the remaining parameters using the standard spherical coordinates on  $\mathbb{S}^{n-1}$ , seen as a sphere of real dimension  $2n - 3$ , including a radius factor  $\sqrt{1 - r^2}$ . The advantage of these coordinates is that  $C_n^\varepsilon(1, 0, \dots, 0)$  simply corresponds to the set of points for which  $r \geq \varepsilon$ .

Let's determine the metric  $g$ .

- $\vec{e}_r = \partial_r F(r, \theta, \varphi_1, \dots, \varphi_{2n-3}) = (\cos(\theta), \sin(\theta), \frac{-r}{\sqrt{1-r^2}} [\vec{u}])$ , where  $[\vec{u}]$  stands for the standard expression of the spherical coordinates on  $\mathbb{S}^{n-1}$
- $\vec{e}_\theta = \partial_\theta F(r, \theta, \varphi_1, \dots, \varphi_{2n-3}) = (-r \sin(\theta), r \cos(\theta), 0, \dots, 0)$
- $\vec{e}_{\varphi_i} = \partial_{\varphi_i} F(r, \theta, \varphi_1, \dots, \varphi_{2n-3}) = (0, 0, \sqrt{1-r^2} [e_{\varphi_i}^-])$  where  $[e_{\varphi_i}^-]$  stands for the tangent vector on  $\mathbb{S}^{n-1}$

---

<sup>3</sup>We use the quotation marks because, on  $\mathbb{S}^n$  equipped with its complex scalar product, this set doesn't look like a cap as it does in the real case. As stated in Chapter 2, QM is indeed a geometrical way of calculating probabilities, but the geometry in use is quite different from the intuitive one given by the familiar real scalar product. It is noteworthy to remark that the universe, through its quantum statistics, obeys very precisely the geometry of the *complex* scalar product, and more generally the geometry induced by its canonical extension on tensor products of Hilbert spaces.

Obviously,  $\vec{e}_r \perp \vec{e}_\theta$  and  $e_{\varphi_i} \perp \vec{e}_\theta$ , as well as  $e_{\varphi_i} \perp e_{\varphi_j}$  for  $i \neq j$  as is the case in the standard spherical coordinates. Moreover, since  $[\vec{u}]$  is radial and  $[e_{\varphi_i}^\rightarrow]$  tangent to  $\mathbb{S}^{n-1}$ , we also have  $\vec{e}_r \perp e_{\varphi_i}^\rightarrow$ , therefore  $g$  is diagonal in these coordinates. Its components are given by:

- $g_{rr} = \langle \vec{e}_r | \vec{e}_r \rangle = 1 + \frac{r^2}{1-r^2}$
- $g_{\theta\theta} = \langle \vec{e}_\theta | \vec{e}_\theta \rangle = r^2$
- $g_{\varphi_i\varphi_i} = (1-r^2)[g_{\varphi_i\varphi_i}]$  with  $[g]$  the metric corresponding to the spherical coordinates on  $\mathbb{S}^{n-1}$ .

It is now easy to compute the desired quantity:

$$\begin{aligned} A_n^\varepsilon &= \int_\varepsilon^1 \sqrt{1 + \frac{r^2}{1-r^2}} dr \int_{[0,2\pi[ \times [0,\pi[^{2n-4} \times [0,2\pi[} r d\theta \sqrt{1-r^2}^{2n-3} \sqrt{[g]} d\varphi_1 \dots d\varphi_{2n-3} \\ &= 2\pi A_{n-1} \int_\varepsilon^1 r(1-r^2)^{n-2} dr \\ &= \frac{\pi A_{n-1}}{n-1} \int_\varepsilon^1 2(n-1)r(1-r^2)^{n-2} dr \\ &= \frac{\pi A_{n-1}}{n-1} (1-\varepsilon^2)^{n-1} \end{aligned}$$

and, finally,

$$\frac{A_n^\varepsilon}{A_n} = \frac{A_n^\varepsilon}{A_n^0} = (1-\varepsilon^2)^{n-1}.$$

□

We are now ready to prove the theorem.

*Proof of Theorem [3.2.5](#).* For this proof, we find some inspiration in [\[96\]](#), but eventually obtain sharper bounds with simpler arguments. Another major reference concerning spherical caps is [\[111\]](#). We say that a set of vectors on a sphere are  $\varepsilon$ -separated if all scalar products between any pairs among them are not greater than  $\varepsilon$  in modulus. Denote  $d\bar{\sigma}^n$  the normalized Lebesgue's measure on  $\mathbb{S}^n$ , that is  $d\bar{\sigma}^n = \frac{d\sigma^n}{A_n}$ , and consider the following events:

- $A : \forall k \in \llbracket 1, d-1 \rrbracket, |\langle \mathcal{E}_d | \mathcal{E}_k \rangle| \leq \varepsilon$
- $B : (|\mathcal{E}_k\rangle)_{1 \leq k \leq d-1}$  are  $\varepsilon$ -separated

so as to write  $\mathbb{P}(\eta_{n,d} \leq \varepsilon) = \mathbb{P}(A | B)\mathbb{P}(B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}\mathbb{P}(\eta_{n,d-1} \leq \varepsilon)$ , with:

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\int_{(\mathbb{S}^n)^{d-1}} d\bar{\sigma}^n(x_1) \dots d\bar{\sigma}^n(x_{d-1}) \mathbb{1}_{\{x_1, \dots, x_{d-1} \text{ are } \varepsilon\text{-separated}\}} \left( 1 - \frac{\left| \bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k) \right|}{A_n} \right)}{\int_{(\mathbb{S}^n)^{d-1}} d\bar{\sigma}^n(x_1) \dots d\bar{\sigma}^n(x_{d-1}) \mathbb{1}_{\{x_1, \dots, x_{d-1} \text{ are } \varepsilon\text{-separated}\}}}$$

$$= 1 - \mathbb{E} \left( \frac{\left| \bigcup_{k=1}^{d-1} C_n^\varepsilon(|\mathcal{E}_k|) \right|}{A_n} \middle| B \right).$$

We need to find bounds on the latter quantity. Obviously,  $\mathbb{E} \left( \frac{\left| \bigcup_{k=1}^{d-1} C_n^\varepsilon(|\mathcal{E}_k|) \right|}{A_n} \middle| B \right) \leq (d-1) \frac{A_n^\varepsilon}{A_n}$ , corresponding to the case when all the caps are disjoint. For the lower bound, define the sequence  $u_d = \mathbb{E} \left( \frac{\left| \bigcup_{k=1}^d C_n^\varepsilon(|\mathcal{E}_k|) \right|}{A_n} \right)$ , which clearly satisfies  $u_d \leq \mathbb{E} \left( \frac{\left| \bigcup_{k=1}^d C_n^\varepsilon(|\mathcal{E}_k|) \right|}{A_n} \middle| B \right)$ , because conditioning on the vectors being separated can only decrease the overlap between the different caps. First observe that  $u_1 = \frac{A_n^\varepsilon}{A_n} \equiv \alpha$ , and compute:

$$\begin{aligned} u_d &= u_{d-1} + \mathbb{E} \left( \frac{\left| C_n^\varepsilon(|\mathcal{E}_d|) \setminus \bigcup_{k=1}^{d-1} C_n^\varepsilon(|\mathcal{E}_k|) \right|}{A_n} \right) \\ &= u_{d-1} + \int_{(\mathbb{S}^n)^d} d\bar{\sigma}^n(x_1) \dots d\bar{\sigma}^n(x_d) \int_{C_n^\varepsilon(x_d)} \mathbb{1}_{\{y \notin \bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k)\}} d\bar{\sigma}^n(y) \\ &= u_{d-1} + \int_{(\mathbb{S}^n)^{d-1}} d\bar{\sigma}^n(x_1) \dots d\bar{\sigma}^n(x_{d-1}) \int_{\mathbb{S}^n} \frac{|C_n^\varepsilon(y)|}{A_n} \mathbb{1}_{\{y \notin \bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k)\}} d\bar{\sigma}^n(y) \\ &= u_{d-1} + \frac{A_n^\varepsilon}{A_n} \int_{(\mathbb{S}^n)^{d-1}} d\bar{\sigma}^n(x_1) \dots d\bar{\sigma}^n(x_{d-1}) \left( 1 - \frac{\left| \bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k) \right|}{A_n} \right) \\ &= u_{d-1} + \frac{A_n^\varepsilon}{A_n} (1 - u_{d-1}) \\ &= (1 - \alpha)u_{d-1} + \alpha, \end{aligned}$$

where the main trick was to invert the integrals on  $x_d$  and on  $y$ . This result is actually quite intuitive: it states that when adding a new cap, only a fraction  $1 - u_{d-1}$  of it on average will be outside the previous caps and contribute to the new total area covered by the caps. Hence  $u_d = 1 - (1 - \alpha)^d$ , and the recurrence relation becomes:

$$\left( 1 - (d-1) \frac{A_n^\varepsilon}{A_n} \right) \mathbb{P}(\eta_{n,d-1} \leq \varepsilon) \leq \mathbb{P}(\eta_{n,d} \leq \varepsilon) \leq \left( 1 - \frac{A_n^\varepsilon}{A_n} \right)^{d-1} \mathbb{P}(\eta_{n,d-1} \leq \varepsilon).$$

Applying the lemma, we get by induction:

$$\prod_{k=1}^{d-1} (1 - k(1 - \varepsilon^2)^{n-1}) \leq \mathbb{P}(\eta_{n,d} \leq \varepsilon) \leq (1 - (1 - \varepsilon^2)^{n-1})^{\frac{d(d-1)}{2}}.$$

Note that the left inequality is valid only as long as  $d \leq \left( \frac{1}{1 - \varepsilon^2} \right)^{n-1}$ , but when  $d$  is larger than this critical value, the right hand side becomes very small (of order  $e^{-1/2(1 - \varepsilon^2)^{n-1}}$ ), so

we may take 0 as a good lower bound in this case. The two bounds are in fact extremely close to each other, and get closer as  $n$  or  $d$  goes larger. To quantify this precisely, let's denote  $f_{n,d}(\varepsilon) = (1 - (1 - \varepsilon^2)^{n-1})^{\frac{d(d-1)}{2}}$ ,  $g_{n,d}(\varepsilon) = \prod_{k=1}^{d-1} (1 - k(1 - \varepsilon^2)^{n-1})$ , and let's show that  $|f_{n,d}(\varepsilon) - g_{n,d}(\varepsilon)| \xrightarrow{n \text{ or } d \rightarrow \infty} 0$ . Two cases have to be considered.

- *First case:* if  $d \geq d_c \equiv \left(\frac{1}{1-\varepsilon^2}\right)^{\frac{3}{5}(n-1)}$ , then  $f_{n,d}(\varepsilon)$  is small so we can write:

$$\begin{aligned} |f_{n,d}(\varepsilon) - g_{n,d}(\varepsilon)| &\leq f_{n,d}(\varepsilon) = e^{\frac{d(d-1)}{2} \ln(1 - (1 - \varepsilon^2)^{n-1})} \\ &\leq e^{-\frac{(d-1)^2}{2} (1 - \varepsilon^2)^{n-1}} \quad (\text{since } \ln(1 - x) \leq -x) \\ &\leq e^{-\frac{1+o(1)}{2(1-\varepsilon^2)^{\frac{n-1}{5}}}} \quad (\text{using } d \geq d_c) \\ &\leq e^{-\frac{d^{1/3}}{2}(1+o(1))}, \end{aligned}$$

where  $1 + o(1) = \left(\frac{d-1}{d}\right)^2 \xrightarrow{n \text{ or } d \rightarrow \infty} 1$ .

- *Second case:* if  $d \leq d_c$ , first note that  $\forall k \in [1, d], \forall x \in [0, \frac{1}{d^{5/3}}]$ ,

$$1 \leq \frac{(1-x)^k}{1-kx} \leq \frac{1-kx + \frac{k(k-1)}{2}x^2}{1-kx} \leq 1 + \frac{k(k-1)}{2} \frac{x^2}{(1-x^{2/5})}.$$

Therefore,

$$\begin{aligned} |\ln(f_{n,d}(\varepsilon)) - \ln(g_{n,d}(\varepsilon))| &= \left| \sum_{k=1}^{d-1} k \ln(1 - (1 - \varepsilon^2)^{n-1}) - \ln(1 - k(1 - \varepsilon^2)^{n-1}) \right| \\ &\leq \sum_{k=1}^{d-1} \left| \ln \left( 1 + \frac{k(k-1)}{2} \frac{(1 - \varepsilon^2)^{2(n-1)}}{1 - (1 - \varepsilon^2)^{\frac{2}{5}(n-1)}} \right) \right| \quad (\text{applying the inequality for } x=(1-\varepsilon^2)^{n-1}) \\ &\leq \frac{(1 - \varepsilon^2)^{2(n-1)}}{1 - (1 - \varepsilon^2)^{\frac{2}{5}(n-1)}} \underbrace{\sum_{k=1}^{d-1} \frac{k(k-1)}{2}}_{=\frac{d^3}{6} - \frac{d^2}{2} + \frac{d}{3} \leq \frac{d^3}{6}} \\ &\leq \frac{(1 - \varepsilon^2)^{\frac{n-1}{5}}}{6(1 - (1 - \varepsilon^2)^{\frac{2}{5}(n-1)})} \quad (\text{using } d \leq d_c) \\ &\leq \frac{d^{-\frac{1}{3}}}{6(1 - d^{-\frac{2}{3}})}. \end{aligned}$$

Hence:

$$\frac{g_{n,d}(\varepsilon)}{f_{n,d}(\varepsilon)} \in \left[ \exp \left( -\frac{(1 - \varepsilon^2)^{\frac{n-1}{5}}}{6(1 - (1 - \varepsilon^2)^{\frac{2}{5}(n-1)})} \right), 1 \right]$$

$$\begin{aligned}
\Rightarrow |f_{n,d}(\varepsilon) - g_{n,d}(\varepsilon)| &\leq \left( 1 - \exp \left( - \frac{(1 - \varepsilon^2)^{\frac{n-1}{5}}}{6(1 - (1 - \varepsilon^2)^{\frac{2}{5}(n-1)})} \right) \right) f_{n,d}(\varepsilon) \\
&\leq \frac{(1 - \varepsilon^2)^{\frac{n-1}{5}}}{6(1 - (1 - \varepsilon^2)^{\frac{2}{5}(n-1)})} \quad (\text{since } 1 - e^{-x} \leq x \text{ and } f_{n,d}(\varepsilon) \leq 1) \\
&\leq \frac{d^{-\frac{1}{3}}}{6(1 - d^{-\frac{2}{3}})}.
\end{aligned}$$

We have thus shown that the difference between the two bounds  $f_{n,d}(\varepsilon)$  and  $g_{n,d}(\varepsilon)$  can be controlled by a quantity that can be expressed solely in terms of either  $n$  or  $d$  but that anyway vanishes when either  $n$  or  $d$  tend to infinity. If we call  $\xi$  this vanishing term, it is straightforward to see that:

$$\min\{d \in \mathbb{N} \mid 1 - f_{n,d}(\varepsilon) + \xi \geq s\} \leq d_{max}^{\varepsilon,s}(n) \leq \min\{d \in \mathbb{N} \mid 1 - f_{n,d}(\varepsilon) \geq s\}.$$

Now, setting  $d_- = \left\lceil \sqrt{-2 \ln(1 - s + \xi)} \sqrt{\frac{1}{(1-\varepsilon)^{n-1}} - 1} \right\rceil$  and using the fact that  $-\frac{1}{x} + 1 \geq \frac{1}{\ln(1-x)} \geq -\frac{1}{x}$ , we get:

$$\begin{aligned}
d_-(d_- - 1) &\leq d_-^2 \leq 2 \ln(1 - s + \xi) \left( -\frac{1}{(1 - \varepsilon^2)^{n-1}} + 1 \right) \leq \frac{2 \ln(1 - s + \xi)}{\ln(1 - (1 - \varepsilon^2)^{n-1})} \\
\Rightarrow 1 - f_{n,d_-}(\varepsilon) + \xi &\leq s \quad \Rightarrow \quad d_{max}^{\varepsilon,s}(n) \geq d_-.
\end{aligned}$$

Similarly, setting  $d_+ = \left\lceil \frac{\sqrt{-2 \ln(1-s)}}{(1-\varepsilon)^{\frac{n-1}{2}}} \right\rceil + 1$  yields:

$$\begin{aligned}
d_+(d_+ - 1) &\geq (d_+ - 1)^2 \geq \frac{-2 \ln(1 - s)}{(1 - \varepsilon^2)^{n-1}} \geq \frac{2 \ln(1 - s)}{\ln(1 - (1 - \varepsilon^2)^{n-1})} \\
\Rightarrow 1 - f_{n,d_+}(\varepsilon) &\geq s \quad \Rightarrow \quad d_{max}^{\varepsilon,s}(n) \leq d_+.
\end{aligned}$$

Thus  $d_{max}^{\varepsilon,s}(n) \underset{n \rightarrow \infty}{\sim} \sqrt{-2 \ln(1 - s)} \left( \frac{1}{1 - \varepsilon^2} \right)^{\frac{n-1}{2}}$ , which is the first part of the theorem.

The intuition concerning the second statement comes from the following observation. We know that  $\mathbb{P}(\eta_{n,d} \leq \varepsilon) \simeq f_{n,d}(\varepsilon)$ , and this function happens to be almost constant equal to 0 in the vicinity of  $\varepsilon = 0$ , almost 1 in the vicinity of  $\varepsilon = 1$ , and to have a very sharp step between the two; this step sharpens as  $n$  or  $d$  grows larger. This explains why the mass of probability is highly peaked around a critical value  $\varepsilon_c$ , so that  $\eta_{n,d} \simeq \mathbb{E}(\eta_{n,d})$  converges to a deterministic variable when  $n$  or  $d \rightarrow \infty$ . This is certainly due to the averaging effect of considering the maximum of a set of  $\frac{d(d+1)}{2}$  scalar products. The critical value  $\varepsilon_c$  satisfies:

$$(1 - (1 - \varepsilon_c^2)^{n-1})^{\frac{d(d-1)}{2}} = \frac{1}{2} \Leftrightarrow \varepsilon_c = \sqrt{1 - (1 - 2^{-2/d(d-1)})^{1/n-1}} \simeq \sqrt{1 - d^{-2/n}}.$$

Now, the precise proof of the convergence of  $\eta_{n,d}$  in probability goes as follows. We have to show that for all  $\delta > 0$ ,  $\mathbb{P} \left( \left| \eta_{n,d} - \sqrt{1 - d^{-\frac{2}{n}}} \right| \leq \delta \right) \underset{n \text{ or } d \rightarrow \infty}{\rightarrow} 1$ . It is equivalent but easier

to show that for all  $\delta > 0$ ,  $\mathbb{P}\left(\sqrt{1 - d^{-\frac{2}{n}} - \delta} \leq \eta_{n,d} \leq \sqrt{1 - d^{-\frac{2}{n}} + \delta}\right) \xrightarrow{n \text{ or } d \rightarrow \infty} 1$ . Taking  $f_{n,d}$  as an approximation for the distribution function of  $\eta_{n,d}$ , we can write:

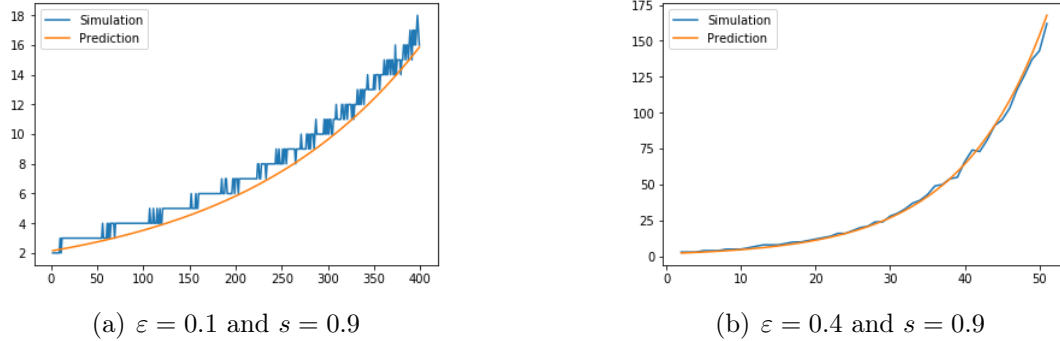
$$\mathbb{P}\left(\eta_{n,d} \leq \sqrt{1 - d^{-\frac{2}{n}} + \delta}\right) = \left(1 - \max(0, d^{-2/n} - \delta)^{n-1}\right)^{\frac{d(d-1)}{2}} + o(1),$$

where  $o(1)$  stands for a quantity that goes to zero when either  $n$  or  $d$  goes to infinity (bounded by  $\xi$ ), and where the max appears because if  $d^{-2/n} \leq \delta$ ,  $\mathbb{P}\left(\eta_{n,d} \leq \sqrt{1 - d^{-\frac{2}{n}} + \delta}\right)$  is simply equal to 1. Clearly,  $\mathbb{P}\left(\eta_{n,d} \leq \sqrt{1 - d^{-\frac{2}{n}} + \delta}\right) \xrightarrow{n \text{ or } d \rightarrow \infty} 1$ , and similarly, one shows that  $\mathbb{P}\left(\eta_{n,d} \leq \sqrt{1 - d^{-\frac{2}{n}} - \delta}\right) \xrightarrow{n \text{ or } d \rightarrow \infty} 0$ , which completes the proof.  $\square$

After establishing the theorem, we discovered that the formula  $\sqrt{1 - d^{-\frac{2}{n}}}$  had already been obtained in [152]. However, this latter work only deals with the maximum of the  $d$  scalar products between say the north pole and a set of  $d$  independent random vectors. This situation is easier to treat, in particular because the  $d$  scalar products are then independent random variables, which is certainly not the case for our  $\frac{d(d+1)}{2}$  scalar products.

### 3.2.4 Comparison with simulation and consequences

The above expressions actually give very good estimates for  $d_{max}^{\varepsilon,s}(n)$  and  $\eta_{n,d}$ , as shown in Figures 3.1 and 3.2.

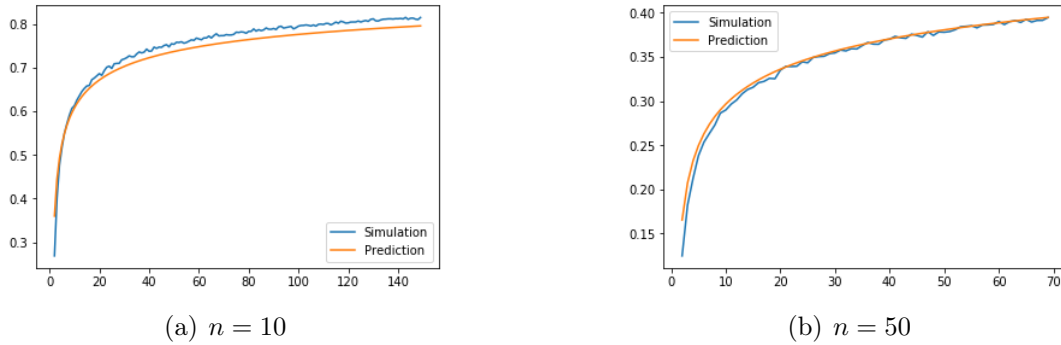


**Figure 3.1:** Simulation *vs* prediction for  $d_{max}^{\varepsilon,s}(n)$

Theorem 3.2.5 has a strong physical consequence. Indeed,  $\mathcal{E}$  induces proper decoherence on  $\mathcal{S}$  as long as  $\eta_{n,d} \ll 1$ , that is when  $d^{-2/n}$  is very close to 1, *i.e.* when  $d \ll e^{n/2}$ . Going back to physically meaningful quantities, we write as previously  $n = p^{N_{\mathcal{E}}}$  and  $d = p^{N_{\mathcal{S}}}$  where  $N_{\mathcal{E}}$  and  $N_{\mathcal{S}}$  stand for the number of particles composing  $\mathcal{E}$  and  $\mathcal{S}$ . The condition becomes:  $2 \ln(p) N_{\mathcal{S}} \ll p^{N_{\mathcal{E}}}$  or simply:

$$\frac{\ln(N_{\mathcal{S}})}{\ln(p)} \ll N_{\mathcal{E}}.$$





**Figure 3.2:** Simulation *vs* prediction for  $d \mapsto \mathbb{E}(\eta_{n,d})$  at fixed  $n$

A more precise condition can be obtained using  $d_{max}$ , because  $\mathcal{E}$  fails to induce proper decoherence on  $\mathcal{S}$  (with high probability  $s$ ) as soon as  $d \geq d_{max}^{\varepsilon,s}(n)$  for an arbitrary choice of  $\varepsilon$  close to 0 and  $s$  close to 1. This rewrites:  $\ln(p)N_{\mathcal{S}} \geq \ln(\sqrt{-2 \ln(1-s)}) + \frac{1}{2} \ln\left(\frac{1}{1-\varepsilon^2}\right) p^{N_{\mathcal{E}}} \simeq \frac{\varepsilon^2}{2} p^{N_{\mathcal{E}}}$  or simply:  $\ln(N_{\mathcal{S}}) \gtrsim 2 \ln(\varepsilon) + \ln(p)N_{\mathcal{E}}$ . Thus, for instance, a gas composed of thousands of particles will lose most of its coherence if it interacts with only a few external particles. This result is certainly not good news for quantum computing. It is rather surprising that so many points can be placed randomly on a  $n$ -sphere before having the maximum of the scalar products becoming non-negligible. *It is this property that makes decoherence an extremely efficient high-dimensional geometrical phenomenon.*

A way to think of Theorem 3.2.5 is the following: one can find at most  $n$  perfectly orthogonal vectors on  $\mathbb{S}^n$ , but  $\sim e^{n/2}$  almost orthogonal vectors. In particular,  $\mathbb{S}^n$  is a small reservoir of dimensions to store the information of an ideal measurement, but a much wider one if one allows for approximate measurements.

### 3.2.5 Discussing the hypotheses

On the one hand, this result could be seen as a worst case scenario for decoherence, since realistic Hamiltonians are far from random and actually discriminate even better the different possible histories. This is especially true if  $\mathcal{E}$  is a measurement apparatus for example, whose Hamiltonian is by construction such that the  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$  evolve quickly and deterministically towards orthogonal points of the sphere.

On the other hand, pursuing such a high level of generality led us to abstract and unphysical assumptions. First, realistic dynamics are not isotropic on the  $n$ -sphere (some transitions are more probable than others). Then, the assumption that each  $|\mathcal{E}_i(t)\rangle$  can explore indistinctly all the states of  $\mathcal{H}_{\mathcal{E}}$  is very criticizable. As explained in [110]:

‘... the set of quantum states that can be reached from a product state with a polynomial-time evolution of an arbitrary time-dependent quantum Hamiltonian is an exponentially small fraction of the Hilbert space. This means that the vast majority of quantum states in a many-body system are unphysical, as they

cannot be reached in any reasonable time. As a consequence, all physical states live on a tiny submanifold.’

It would then be more accurate in our model to replace  $\mathbb{S}^n$  by this submanifold. But how does it look like geometrically and what is its dimension? If it were a subsphere of  $\mathbb{S}^n$  of exponentially smaller dimension, then  $n$  should be replaced everywhere by something like  $\ln(n)$  in what precedes, so the condition would rather be  $N_S \ll N_E$  which is a completely different conclusion. Some clues to better grasp the submanifold are found in [102] §3.4]:

‘...one can prove that low-energy eigenstates of gapped Hamiltonians with local interactions obey the so-called area-law for the entanglement entropy. This means that the entanglement entropy of a region of space tends to scale, for large enough regions, as the size of the boundary of the region and not as the volume. (...) In other words, low-energy states of realistic Hamiltonians are not just “any” state in the Hilbert space: they are heavily constrained by locality so that they must obey the entanglement area-law.’

Another possible way to constrain the set of physically reachable states is given by the theory of quantum Darwinism [112]:

‘... for a randomly selected pure state in the global Hilbert space, an observer typically cannot learn anything about a system without sampling at least half of its environment. States that deviate (even by exponentially small amounts) from this property [which is in general the case for realistic systems] occupy an exponentially small volume in Hilbert space.’

More work is needed in order to draw precise conclusions taking these physical remarks into account.

### 3.3 Alternative measures of decoherence

Because it bounds the depart from the total probability formula, as shown in Proposition 3.1.2, our Definition 3.1.1 was a reasonable choice to quantify decoherence. Here are three alternative ideas, though.

#### 3.3.1 Fraction of interfering histories

Lemma 3.2.6 allows for another way to quantify decoherence, which could be to ask, for each possible history  $|i\rangle$  in  $\mathcal{B}$ , what is the fraction  $F_{\mathcal{B}}^{\varepsilon}(\rho_S)$  of the other possible histories with which it interferes significantly, that is how many indices  $j$  are there such that  $|\langle \mathcal{E}_i | \mathcal{E}_j \rangle| \geq \varepsilon$ ? As remarked in [146], this quantity is simply given by:

$$F_{\mathcal{B}}^{\varepsilon}(\rho_S) = \frac{1}{d-1} \sum_{\substack{j=1 \\ j \neq i}}^d h_{ij} \quad \text{with } h_{ij} = \begin{cases} 1 & \text{if } |\mathcal{E}_j\rangle \in C_n^{\varepsilon}(|\mathcal{E}_i\rangle) \\ 0 & \text{otherwise} \end{cases}$$

By the law of large numbers, we immediately deduce that  $F_{\mathcal{B}}^{\varepsilon}(\rho_S) \xrightarrow[d \rightarrow \infty]{\text{a.s.}} \mathbb{P}(h_{ij} = 1) = (1 - \varepsilon^2)^{n-1}$ , and we recover once again in this expression that the typical level of decoherence is  $\varepsilon \sim \frac{1}{\sqrt{n}}$ .

### 3.3.2 Expectation of the scalar products

*Definition*

More interesting, perhaps, could be to measure decoherence using the ‘expectation’ of the scalar products  $|\langle \mathcal{E}_i | \mathcal{E}_j \rangle|^2$  for  $i \neq j$ , weighted by their quantum probabilities  $\frac{|c_i|^2 |c_j|^2}{\sum_{i \neq j} |c_i|^2 |c_j|^2} = \frac{|c_i|^2 |c_j|^2}{1 - \sum_i |c_i|^4}$ . One can therefore define:

**Definition 3.3.1.** Denote:

$$\tilde{\eta}_{\mathcal{B}}^2(\rho_S) = \frac{1}{1 - \sum_i |c_i|^4} \sum_{i \neq j} |c_i|^2 |c_j|^2 |\langle \mathcal{E}_i | \mathcal{E}_j \rangle|^2 = \frac{1}{1 - \sum_i |c_i|^4} \left( \text{tr}(\rho_S^2) - \sum_i |c_i|^4 \right),$$

with, by convention,  $\tilde{\eta}_{\mathcal{B}}^2(\rho_S) = 1$  if  $\sum_i |c_i|^4 = 1$ , *i.e.* if  $\rho_S = |i\rangle \langle i|$  for some  $i$ .

As always, decoherence depends on the basis considered. It is clear in the above formula, including a basis-independent (‘covariant’) part  $\text{tr}(\rho_S^2)$ , as well as the term  $\sum_i |c_i|^4 = \text{tr}(D_{\mathcal{B}}[\rho_S]^2)$  which is basis-dependent.

*Advantages*

Definition [3.3.1](#) has three advantages.

1. It can naturally be extended to the infinite dimensional case, unlike  $\max_{i \neq j} |\langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle|$  (if the scalar products vary continuously, their supremum is necessarily 1). A proposal for quantifying decoherence in infinite dimension could therefore be:

$$\tilde{\eta}_{\mathcal{B}}^2(\rho_S) = \frac{1}{1 - \int |c_x|^4 dx} \left( \text{tr}(\rho_S^2) - \int |c_x|^4 dx \right).$$

2. It links decoherence with the notion of information. Indeed, understanding the mechanisms of decoherence suggests the following intuition: the smaller  $\eta$  or  $\tilde{\eta}$  is, the more information the environment has stored about the system because the more distinguishable (*i.e.* close to being orthogonal) the  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$  are; on the other hand, the smaller  $\eta$  is, the fewer quantum interferences occur. This motivates the search for a general relationship between entanglement entropy and the level of classicality of a system. Such results have already been derived for specific environments [\[77\]](#) (3.76) [\[86\]](#) [\[94\]](#). Here, recalling Definition [2.0.2](#) for the linear entropy, we can write:

$$\tilde{\eta}_{\mathcal{B}}^2(\rho_S) = 1 - \frac{S_{\text{lin}}(\rho_S)}{S_{\text{lin}}(D_{\mathcal{B}}[\rho_S])}.$$

If the system is initially in a pure state, the quantity  $\frac{S_{\text{lin}}(\rho_S(t))}{S_{\text{lin}}(D_{\mathcal{B}}[\rho_S(t)])}$  goes from 0 at  $t = 0$  to 1 when  $t \rightarrow +\infty$  in the basis of decoherence. It measures the ratio of entropy (*i.e.* of information) that has already been shared with the environment compared to its final value.

3. It allows to give a physical meaning to the purity of a state:  $\text{tr}(\rho_S^2)$  can be seen as the approximate level of decoherence in most bases, according to the following proposition.

**Proposition 3.3.2.** *Let  $\mathcal{B}$  be a random orthonormal basis of  $\mathcal{H}_S$ , defined as  $\mathcal{B} = (U|e_i\rangle)_{1 \leq i \leq d}$  where  $(|e_i\rangle)_{1 \leq i \leq d}$  is a reference orthonormal basis and  $U$  a random unitary following the Haar measure on  $\mathcal{U}(\mathcal{H}_S)$ . Then the level of decoherence in  $\mathcal{B}$  is a random variable satisfying:*

$$\mathbb{E} (|\tilde{\eta}_{\mathcal{B}}^2(\rho_S) - \text{tr}(\rho_S^2)|) \xrightarrow{d \rightarrow +\infty} 0.$$

For the sake of rigour, we do not write  $\tilde{\eta}_{\mathcal{B}}^2(\rho_S) \xrightarrow{d \rightarrow +\infty} \text{tr}(\rho_S^2)$  because the space  $\mathcal{H}_S$  on which the randomness is defined changes with  $d$ .

*Proof.* The result is immediate when  $\rho_S$  is a pure state, so let's now assume that  $\rho_S$  is not pure. Let  $\mathcal{B}_{\rho_S} = (|e_i\rangle)_{1 \leq i \leq d}$  be  $\rho_S$ 's eigenbasis, in which  $\rho_S = \sum_k \lambda_k |e_k\rangle \langle e_k|$  and  $\tilde{\eta}_{\mathcal{B}_{\rho_S}}^2 = 0$ . Without loss of generality, we can take  $\mathcal{B}_{\rho_S}$  as the reference basis of the proposition, since the Haar measure is invariant under composition with a fixed unitary. Expressed in  $\mathcal{B}$ , the  $i^{\text{th}}$  diagonal coefficient of  $\rho_S$  is:  $|c'_i|^2 = \langle Ue_i | \rho_S | Ue_i \rangle = \langle Ue_i | \sum_k \lambda_k |e_k\rangle \langle e_k| Ue_i \rangle = \sum_k \lambda_k |\langle e_i | Ue_k \rangle|^2$ . As  $U$  follows a uniform law on  $\mathcal{U}(\mathcal{H}_S)$ , so does  $U|e_k\rangle$  on  $\mathbb{S}^d$ . Define the following random variables:  $X_{ik} = |\langle e_i | Ue_k \rangle|^2$ ,  $Y_i = \sum_k \lambda_k X_{ik} = |c'_i|^2$ , and  $T = \sum_i Y_i^2 = \sum_i |c'_i|^4$ .

To prove the proposition, it suffices to show that  $\mathbb{E}(T) \xrightarrow{d \rightarrow +\infty} 0$ . Indeed,  $0 \leq T \leq \max_k \lambda_k = \Lambda$ , because  $|c'_i|^2 = \sum_k \lambda_k |\langle e_i | Ue_k \rangle|^2 \leq \Lambda \|U^\dagger e_i\|_2^2 \leq \Lambda$ , hence  $T = \sum_i |c'_i|^4 \leq \Lambda \sum_i |c'_i|^2 \leq \Lambda$ . Since  $\Lambda < 1$  (as  $\rho_S$  is not pure) and considering that the map  $x \mapsto \frac{x}{1-x}$  is continuous on  $[0, \Lambda]$ , this implies:

$$\mathbb{E} (|\tilde{\eta}_{\mathcal{B}}^2(\rho_S) - \text{tr}(\rho_S^2)|) = |\text{tr}(\rho_S^2) - 1| \mathbb{E} \left( \frac{T}{1-T} \right) \xrightarrow{d \rightarrow +\infty} 0.$$

An efficient way to achieve this is to use the following formula [81, Theorem 34] [70, Theorem 29.9]: for  $\mathbf{a} = (a_1, \dots, a_d)$  such that  $a_1 + \dots + a_d = m$ , we have

$$\int_{\mathcal{U}(d)} \prod_{k=1}^d |u_{1k}|^{2a_k} dU = \frac{(d-1)!}{(m+d-1)!} \prod_{k=1}^d a_k!,$$

where  $dU$  is the Haar measure on the unitary group in dimension  $d$  and  $u_{ik} = \langle e_i | Ue_k \rangle$  is the  $(i, k)$  coefficient of the unitary matrix  $U$ . Of course, one can replace the line index 1

by any index  $i$ , since the Haar measure remains unchanged when composed with a given permutation matrix. Applied to  $m = 2$  and  $\mathbf{a} = (2, 0 \dots, 0)$  or  $\mathbf{a} = (1, 1 \dots, 0)$ , we get:

$$\mathbb{E}(X_{ik}X_{il}) = \int_{\mathcal{U}(\mathcal{H}_S)} |u_{ik}|^2 |u_{il}|^2 dU = \begin{cases} \frac{2(d-1)!}{(2+d-1)!} = \frac{2}{d(d+1)} & \text{if } k = l \\ \frac{(d-1)!}{(2+d-1)!} = \frac{1}{d(d+1)} & \text{if } k \neq l \end{cases} = \frac{1}{d(d+1)}(\delta_{kl} + 1).$$

Thus:

$$\mathbb{E}(Y_i^2) = \sum_{1 \leq k, l \leq n} \lambda_k \lambda_l \mathbb{E}(X_{ik}X_{il}) = \frac{1}{d(d+1)} \left( \sum_{k=1}^n \lambda_k^2 + \sum_{1 \leq k, l \leq n} \lambda_k \lambda_l \right) = \frac{\text{tr}(\rho_S^2) + 1}{d(d+1)}.$$

Finally:

$$\mathbb{E}(T) = \sum_{i=1}^d \mathbb{E}(Y_i^2) = \frac{\text{tr}(\rho_S^2) + 1}{d+1} \xrightarrow{d \rightarrow +\infty} 0,$$

which completes the proof.

Here is a more elementary argument which does not rely on the above formula. As explained in the proof of Proposition [3.2.3](#), each  $X_{ik}$  follows a  $\beta_{1,d-1}$  distribution, whose mean and variance are  $\frac{1}{d}$  and  $V_d = \frac{d-1}{d^2(d+1)}$  respectively. Then, notice that:

$$\begin{aligned} \mathbb{P} \left( \left| \sum_{k=1}^d \lambda_k X_{ik} - \frac{1}{d} \right| \geq \varepsilon \right) &= \mathbb{P} \left( \left| \sum_{k=1}^d \lambda_k \left[ X_{ik} - \frac{1}{d} \right] \right| \geq \varepsilon \right) && \text{(since the } (\lambda_k)_{1 \leq k \leq d} \text{ sum to 1)} \\ &\leq \mathbb{P} \left( \exists k \in \llbracket 1, d \rrbracket : \left| X_{ik} - \frac{1}{d} \right| \geq \varepsilon \right) \\ &\leq \sum_{k=1}^d \mathbb{P} \left( \left| X_{ik} - \frac{1}{d} \right| \geq \varepsilon \right), \\ &\leq \frac{dV_d}{\varepsilon^2} \end{aligned}$$

where we have used Chebyshev's inequality in the last step. Define the event

$$A_d = \left\{ \omega \in \Omega \mid \left| \sum_{k=1}^d \lambda_k X_{ik} - \frac{1}{d} \right| \leq \frac{1}{d^{1/3}} \right\}$$

and denote  $\mathbb{1}_{A_d}$  the indicator function of  $A_d$ . We can compute:

$$\mathbb{E} \left( \left| T - \frac{1}{d} \right| \right) = \mathbb{E} \left( \left| \sum_{i=1}^d \left( \sum_{k=1}^d \lambda_k X_{ik} \right)^2 - \frac{1}{d} \right| \right)$$

$$\begin{aligned}
&= \mathbb{E} \left( \left| \sum_{i=1}^d \left[ \left( \sum_{k=1}^d \lambda_k X_{ik} \right)^2 - \frac{1}{d^2} \right] \right| \right) \\
&\leq \sum_{i=1}^d \mathbb{E} \left( \left| \left( \sum_{k=1}^d \lambda_k X_{ik} \right)^2 - \frac{1}{d^2} \right| \right) \\
&\leq \sum_{i=1}^d \mathbb{E} \left( \left| \left( \sum_{k=1}^d \lambda_k X_{ik} \right)^2 - \frac{1}{d^2} \right| \mathbb{1}_{A_d} \right) + \mathbb{E} \left( \left| \left( \sum_{k=1}^d \lambda_k X_{ik} \right)^2 - \frac{1}{d^2} \right| \mathbb{1}_{\bar{A}_d} \right).
\end{aligned}$$

On  $A_d$ , we have  $\sum_{k=1}^d \lambda_k X_{ik} \in [0, \frac{1}{d} + \frac{1}{d^{1/3}}]$ , an interval on which the map  $x \mapsto x^2$  is  $2(\frac{1}{d} + \frac{1}{d^{1/3}})$ -Lipschitz; on  $\bar{A}_d$ , we know at least that it is 2-Lipschitz. Thus:

$$\begin{aligned}
\mathbb{E} \left( \left| T - \frac{1}{d} \right| \right) &\leq \sum_{i=1}^d 2 \left( \frac{1}{d} + \frac{1}{d^{1/3}} \right) \mathbb{E} \left( \left| \sum_{k=1}^d \lambda_k X_{ik} - \frac{1}{d} \right| \right) + 2 \mathbb{E} \left( \left| \sum_{k=1}^d \lambda_k X_{ik} - \frac{1}{d} \right| \mathbb{1}_{\bar{A}_d} \right) \\
&\leq \sum_{i=1}^d \sum_{k=1}^d 2 \lambda_k \left( \frac{1}{d} + \frac{1}{d^{1/3}} \right) \mathbb{E} \left( \left| X_{ik} - \frac{1}{d} \right| \right) + 2 \lambda_k \mathbb{E} \left( \left| X_{ik} - \frac{1}{d} \right| \mathbb{1}_{\bar{A}_d} \right) \\
&\stackrel{\text{H\"older}}{\leq} \sum_{i=1}^d \sum_{k=1}^d 2 \lambda_k \left( \frac{1}{d} + \frac{1}{d^{1/3}} \right) \mathbb{E} \left( \left| X_{ik} - \frac{1}{d} \right|^2 \right)^{1/2} + 2 \lambda_k \mathbb{E} \left( \left| X_{ik} - \frac{1}{d} \right|^2 \right)^{1/2} \mathbb{P}(\bar{A}_d)^{1/2} \\
&\leq \left[ 2 \left( \frac{1}{d} + \frac{1}{d^{1/3}} \right) V_d^{1/2} + 2 V_d^{1/2} (d V_d d^{2/3})^{1/2} \right] \underbrace{\sum_{i=1}^d \sum_{k=1}^d \lambda_k}_{=d} \quad (\text{previous result with } \varepsilon = d^{-1/3}) \\
&\xrightarrow{d \rightarrow +\infty} 0 \quad \text{because } V_d \underset{d \rightarrow +\infty}{\sim} \frac{1}{d^2}
\end{aligned}$$

Finally,  $\mathbb{E}(T) \leq \mathbb{E} \left( \left| T - \frac{1}{d} \right| \right) + \frac{1}{d} \xrightarrow{d \rightarrow +\infty} 0$ . □

**Remark 3.3.3.** This is actually a strong result. Without having made any assumption on the particular shape of the Hamiltonian nor on the precise form of the  $(|\mathcal{E}_i(t)\rangle)_{1 \leq i \leq d}$ , we are still able to infer, for any subsystem of the universe (provided it is of dimension  $d \gg 1$ ), the approximate level of decoherence in most bases. Said differently, when performing experiments on any (sufficiently large) system, the typical deviation from the total probability formula will be approximately the most for all observables, and close to  $\text{tr}(\rho_S^2)$ .

One might even suspect a link with the rise of entropy. As time flows, entropy is likely to increase with our statistical ignorance, therefore  $\rho_S$  is expected to get closer to the maximally entangled state and the purity to  $\frac{1}{d}$ , which goes to 0 as  $d \rightarrow +\infty$ . Thus, the total probability formula tends to be satisfied in all bases because of the rise of our ignorance.

*Case of a purely random environment*

In the spirit of Theorem [3.2.5](#), we would like to assume again the existence of a conserved basis  $\mathcal{B}$  with respect to which the  $(|\mathcal{E}_i\rangle)_{1 \leq i \leq d}$  are uniformly distributed unit vectors (for example if their evolution is Brownian in virtue of Proposition [3.2.1](#)), and study the properties of the random variable  $\tilde{\eta}_{\mathcal{B}}^2$  in this case. Its expectation and variance are given by the following proposition.

**Proposition 3.3.4.** *Let  $\mathcal{B} = (|i\rangle)_{1 \leq i \leq d}$  be an orthonormal basis of  $\mathcal{H}_S$ ,  $(|\mathcal{E}_i\rangle)_{1 \leq i \leq d}$  some independent random vectors uniformly distributed on the complex  $n$ -sphere  $\mathbb{S}^n$ ,  $|\Psi\rangle = \sum_{i=1}^d c_i |i\rangle \otimes |\mathcal{E}_i\rangle$  and  $\rho_S = \text{tr}_{\mathcal{E}} |\Psi\rangle \langle \Psi|$ . Then:*

$$\mathbb{E}(\tilde{\eta}_{\mathcal{B}}^2(\rho_S)) = \frac{1}{n} \quad \text{and} \quad \mathbb{V}(\tilde{\eta}_{\mathcal{B}}^2(\rho_S)) = 2 \frac{(\sum_i |c_i|^4)^2 - \sum_i |c_i|^8}{(1 - \sum_i |c_i|^4)^2} \frac{n-1}{n^2(n+1)}.$$

*Proof.* Note that the  $(c_i)_{1 \leq i \leq d}$  are now fixed, so that the randomness of  $\tilde{\eta}_{\mathcal{B}}^2$  is entirely contained in the  $(|\mathcal{E}_i\rangle)_{1 \leq i \leq d}$ . According to Proposition [3.2.3](#), each variable  $X_{ij} = |\langle \mathcal{E}_i | \mathcal{E}_j \rangle|^2$  for  $i \neq j$  have mean  $\frac{1}{n}$ . Thus  $\tilde{\eta}_{\mathcal{B}}^2$  is a weighted mean of random variables of mean  $\frac{1}{n}$ , so  $\mathbb{E}(\tilde{\eta}_{\mathcal{B}}^2) = \frac{1}{n}$  as well. To compute the variance:

$$\begin{aligned} \mathbb{V}(\tilde{\eta}_{\mathcal{B}}^2) &= \mathbb{E}(\tilde{\eta}_{\mathcal{B}}^4) - \mathbb{E}(\tilde{\eta}_{\mathcal{B}}^2)^2 \\ &= \frac{1}{(1 - \sum_i |c_i|^4)^2} \sum_{i \neq j, k \neq l} |c_i|^2 |c_j|^2 |c_k|^2 |c_l|^2 \mathbb{E}(X_{ij} X_{kl}) - \frac{1}{n^2} \\ &= \frac{1}{(1 - \sum_i |c_i|^4)^2} \sum_{i \neq j, k \neq l} |c_i|^2 |c_j|^2 |c_k|^2 |c_l|^2 \left[ \mathbb{E}(X_{ij} X_{kl}) - \frac{1}{n^2} \right] \\ &= \frac{1}{(1 - \sum_i |c_i|^4)^2} \left( 2 \sum_{i \neq j} |c_i|^4 |c_j|^4 \left[ \mathbb{E}(X_{ij}^2) - \frac{1}{n^2} \right] \right. \\ &\quad \left. + 4 \sum_{\substack{i \neq j \\ k \notin \{i, j\}}} |c_i|^2 |c_j|^4 |c_k|^2 \left[ \mathbb{E}(X_{ij} X_{kj}) - \frac{1}{n^2} \right] \right. \\ &\quad \left. + \sum_{\substack{i \neq j, k \neq l \\ \{i, j\} \cap \{k, l\} = \emptyset}} |c_i|^2 |c_j|^2 |c_k|^2 |c_l|^2 \left[ \mathbb{E}(X_{ij} X_{kl}) - \frac{1}{n^2} \right] \right), \end{aligned}$$

where we have split the sum depending on the cardinality of  $\{i, j\} \cap \{k, l\}$ . The multiplicative factors count the different possible pairings (either  $[i = k \text{ and } j = l]$  or  $[i = l \text{ and } j = k]$  for the first sum, either  $i = k$  or  $i = l$  or  $j = k$  or  $j = l$  for the second sum).

Now, whenever  $\{i, j\} \cap \{k, l\} = \emptyset$ ,  $X_{ij}$  and  $X_{kl}$  are independent, so  $\mathbb{E}(X_{ij} X_{kl}) = \mathbb{E}(X_{ij}) \mathbb{E}(X_{kl}) = \frac{1}{n^2}$ . Concerning  $\mathbb{E}(X_{ij} X_{kj})$ , by rotational invariance, one can assume without loss of generality that  $|\mathcal{E}_j\rangle$  is the north pole  $|N\rangle$  of  $\mathbb{S}^n$ , while  $|\mathcal{E}_i\rangle$  and  $|\mathcal{E}_k\rangle$  are independent

and uniformly distributed. Hence  $\mathbb{E}(X_{ij}X_{kj}) = \mathbb{E}(|\langle \mathcal{E}_i | N \rangle|^2) \mathbb{E}(|\langle \mathcal{E}_k | N \rangle|^2) = \frac{1}{n^2}$  as well. Finally, using again the formula from [81, Theorem 34] [70, Theorem 29.9],  $\mathbb{E}(X_{ij}^2) = \frac{2}{n(n+1)}$ . Hence:

$$\begin{aligned} \mathbb{V}(\tilde{\eta}_{\mathcal{B}}^2) &= \frac{2}{(1 - \sum_i |c_i|^4)^2} \sum_{i \neq j} |c_i|^4 |c_j|^4 \left[ \frac{2}{n(n+1)} - \frac{1}{n^2} \right] \\ &= 2 \frac{(\sum_i |c_i|^4)^2 - \sum_i |c_i|^8}{(1 - \sum_i |c_i|^4)^2} \frac{n-1}{n^2(n+1)}. \end{aligned}$$

□

### 3.3.3 Distance between orthonormal bases

A quantum system is always perfectly decohered in exactly one basis (up to degeneracy): its eigenbasis. We would like to equip the set  $\mathfrak{B}(\mathcal{H})$  of orthonormal bases of  $\mathcal{H}$  with an appropriate distance  $d$ , so that it would make sense to quantify the level of decoherence of  $\rho_S$  in  $\mathcal{B}$  as  $d(\mathcal{B}_{\rho_S}, \mathcal{B})$ .

A first naive idea would be to consider  $d(\mathcal{B}_i, \mathcal{B}_j) = \|U_{\mathcal{B}_i, \mathcal{B}_j} - \mathbb{1}\|$ , where  $U_{\mathcal{B}_i, \mathcal{B}_j}$  is the unique unitary mapping  $\mathcal{B}_i$  to  $\mathcal{B}_j$ . Before even checking whether it satisfies the triangular inequality, this quantity is not suitable for our purposes because (i) if the two bases differ only by a mere relabelling of the vectors, they are physically equivalent (the same observable is perfectly decohered). Even by setting  $d(\mathcal{B}_i, \mathcal{B}_j) = \min_{\sigma \in \mathfrak{S}_n} \|U_{\mathcal{B}_i, \mathcal{B}_j^\sigma} - \mathbb{1}\|$  where the minimum is taken over all the permutations in the vectors of  $\mathcal{B}_j$ , this definition is still not suitable since (ii) the operator norm  $\|M\|$  being defined as a supremum of  $\|M|x\rangle\|$  when  $|x\rangle$  spans the unit sphere, it poorly takes into account the behaviour of  $M$  when applied to most vectors on this sphere. In our case, we may have a large  $d(\mathcal{B}_i, \mathcal{B}_j)$  even though most vectors of  $\mathcal{B}_i$  coincide very well with those of  $\mathcal{B}_j$ .

Arguably, the best distance between orthonormal bases for the theory of decoherence is the one developed in [22] (in the context of the search for mutually unbiased bases [52]). It has a deep geometrical meaning, and can be written in the following simple form: if  $\mathcal{B}_i = (|e_i\rangle)_i$  and  $\mathcal{B}_j = (|f_j\rangle)_j$ ,

$$d(\mathcal{B}_i, \mathcal{B}_j)^2 = \sum_{i,j} |\langle e_i | f_j \rangle|^2 (1 - |\langle e_i | f_j \rangle|^2) = n - \sum_{i,j} |\langle e_i | f_j \rangle|^4.$$

It is clear in this expression that the distance is 0 if and only if  $\mathcal{B}_i$  is just a relabelling of  $\mathcal{B}_j$ . One can also show that, in dimension  $n$ , the distance lies in  $[0, \sqrt{n-1}]$  and is maximal if and only if the two bases are mutually unbiased (*i.e.* such that  $\forall i, j, |\langle e_i | f_j \rangle|^2 = \frac{1}{n}$ ), as are for instance the eigenbases of the spin operators  $\hat{S}_X, \hat{S}_Y$  and  $\hat{S}_Z$  or the eigenbases of  $\hat{X}$  and  $\hat{P}$ .

We expect that two observables with close eigenbases should almost commute and, conversely, that two almost-commuting observables should have close eigenbases. This is true, according to the following inequalities.



**Proposition 3.3.5.** *Let  $\hat{A}$  and  $\hat{B}$  be two Hermitian operators in finite dimension  $n$  with respective spectra  $(a_i)_{1 \leq i \leq n}$  and  $(b_j)_{1 \leq j \leq n}$ . Then:*

$$\|[\hat{A}, \hat{B}]\| \leq \frac{\sqrt{n}}{2} C_{\hat{A}, \hat{B}} d(\mathcal{B}_{\hat{A}}, \mathcal{B}_{\hat{B}}),$$

where  $C_{\hat{A}, \hat{B}} = \max_{i,j} |a_i - a_j| \max_{k,l} |b_k - b_l|$ . If moreover their spectra are non-degenerate, we have:

$$d(\mathcal{B}_{\hat{A}}, \mathcal{B}_{\hat{B}}) \leq \frac{\sqrt{2n}}{c_{\hat{A}, \hat{B}}} \|[\hat{A}, \hat{B}]\|,$$

where  $c_{\hat{A}, \hat{B}} = \min_{i \neq j} |a_i - a_j| \min_{k \neq l} |b_k - b_l|$ .

To derive these inequalities, we start with a preliminary lemma.

**Lemma 3.3.6** (Almost equality in quadratic Jensen's inequality). *Let  $(x_i)_{1 \leq i \leq n}$  be a collection of  $n$  distinct real numbers, and  $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{R}_+^n$  be non-negative weights such that  $\sum_i \lambda_i = 1$ . By Jensen's inequality, we know that  $\sum_i \lambda_i x_i^2 \geq (\sum_i \lambda_i x_i)^2$ . If the bound is almost tight, i.e. if  $\sum_i \lambda_i x_i^2 - (\sum_i \lambda_i x_i)^2 \leq \varepsilon$ , then:*

$$\sum_i \lambda_i (1 - \lambda_i) \leq \frac{2\varepsilon}{\min_{i \neq j} |x_i - x_j|^2}.$$

*Proof of Lemma.* Define  $\Delta_{ij} = x_i - x_j \neq 0$ , and observe that:

$$\begin{aligned} \sum_i \lambda_i x_i^2 - \left( \sum_i \lambda_i x_i \right)^2 &= \sum_i \lambda_i \underbrace{(1 - \lambda_i)}_{=\sum_{j \neq i} \lambda_j} x_i^2 - \sum_{i \neq j} \lambda_i \lambda_j x_i x_j \\ &= \sum_i \lambda_i x_i \sum_{j \neq i} \lambda_j (x_i - x_j) \\ &= \sum_{i \neq j} \lambda_i \lambda_j x_i \Delta_{ij} \\ &= \sum_{i < j} \lambda_i \lambda_j x_i \Delta_{ij} + \sum_{i > j} \lambda_i \lambda_j x_i \Delta_{ij} \\ &= \sum_{i < j} \lambda_i \lambda_j x_i \Delta_{ij} - \sum_{i < j} \lambda_i \lambda_j x_j \Delta_{ij} \quad (\text{relabelling the second sum}) \\ &= \sum_{i < j} \lambda_i \lambda_j \Delta_{ij}^2. \end{aligned}$$

Now, because all the terms are non-negative,

$$\sum_{i < j} \lambda_i \lambda_j \Delta_{ij}^2 \leq \varepsilon \Rightarrow \sum_{i < j} \lambda_i \lambda_j \leq \frac{\varepsilon}{\min_{i \neq j} \Delta_{ij}^2}$$

$$\Rightarrow \sum_i \lambda_i(1 - \lambda_i) = \sum_{i \neq j} \lambda_i \lambda_j \leq \frac{2\varepsilon}{\min_{i \neq j} \Delta_{ij}^2}.$$

□

*Proof of Proposition 3.3.5. First inequality.* Denote  $\mathcal{B}_{\hat{A}} = (|e_i\rangle)_{1 \leq i \leq n}$  and  $\mathcal{B}_{\hat{B}} = (|f_j\rangle)_{1 \leq j \leq n}$  the eigenbases of  $\hat{A}$  and  $\hat{B}$  properly associated with the eigenvalues  $(a_i)_{1 \leq i \leq n}$  and  $(b_j)_{1 \leq j \leq n}$ , so that in particular  $\hat{B} = \sum_j b_j |f_j\rangle \langle f_j|$ , and let  $\tilde{C}_{\hat{A}, \hat{B}} = \max_{i,j} |a_i - a_j| \max_k |b_k|$ . For all  $k \in \llbracket 1, n \rrbracket$ ,

$$\hat{B} |e_k\rangle = \sum_{i=1}^n \langle e_i | \hat{B} e_k \rangle |e_i\rangle = \sum_{1 \leq i, j \leq n} b_j \langle e_i | f_j \rangle \langle f_j | e_k \rangle |e_i\rangle.$$

Hence:

$$\begin{aligned} \left\| (\hat{A}\hat{B} - \hat{B}\hat{A}) |e_k\rangle \right\|^2 &= \left\| \sum_{1 \leq i, j \leq n} (a_i - a_k) b_j \langle e_i | f_j \rangle \langle f_j | e_k \rangle |e_i\rangle \right\|^2 \\ &= \sum_{\substack{i=1 \\ i \neq k}}^n \left| (a_i - a_k) \sum_{j=1}^n b_j \langle e_i | f_j \rangle \langle f_j | e_k \rangle \right|^2 \\ &\leq \sum_{\substack{i=1 \\ i \neq k}}^n |a_i - a_k|^2 \left( \sum_{j=1}^n |b_j| |\langle e_i | f_j \rangle \langle f_j | e_k \rangle| \right)^2 \\ &\stackrel{\text{C.-S.}}{\leq} \tilde{C}_{\hat{A}, \hat{B}}^2 \sum_{\substack{i=1 \\ i \neq k}}^n \sum_{j=1}^n |\langle e_i | f_j \rangle|^2 |\langle e_k | f_j \rangle|^2 \sum_{l=1}^n 1^2 \\ &\leq n \tilde{C}_{\hat{A}, \hat{B}}^2 \sum_{j=1}^n |\langle e_k | f_j \rangle|^2 \sum_{\substack{i=1 \\ i \neq k}}^n |\langle e_i | f_j \rangle|^2 \\ &\leq n \tilde{C}_{\hat{A}, \hat{B}}^2 \sum_{j=1}^n |\langle e_k | f_j \rangle|^2 (1 - |\langle e_k | f_j \rangle|^2). \end{aligned}$$

Consequently, for all  $|\Psi\rangle = \sum_{k=1}^n \alpha_k |e_k\rangle$  of norm 1,

$$\left\| [\hat{A}, \hat{B}] |\Psi\rangle \right\|^2 \leq \sum_{k=1}^n \underbrace{|\alpha_k|^2}_{\leq 1} \left\| [\hat{A}, \hat{B}] |e_k\rangle \right\|^2 \leq n \tilde{C}_{\hat{A}, \hat{B}}^2 \sum_{1 \leq j, k \leq n} |\langle e_k | f_j \rangle|^2 (1 - |\langle e_k | f_j \rangle|^2) = n \tilde{C}_{\hat{A}, \hat{B}}^2 d(\mathcal{B}_{\hat{A}}, \mathcal{B}_{\hat{B}})^2.$$

Taking the supremum over  $|\Psi\rangle \in \mathbb{S}^n$  and a square root yields the first inequality with  $\tilde{C}$  instead of  $C$  and  $\sqrt{n}$  instead of  $\frac{\sqrt{n}}{2}$ . To improve the constant, notice that  $\|[\hat{A}, \hat{B}]\|$  and  $d(\mathcal{B}_{\hat{A}}, \mathcal{B}_{\hat{B}})$  are unchanged if one replaces  $\hat{B}$  by  $\hat{B} - \lambda \mathbb{1}$ . Hence, for all  $\lambda \in \mathbb{R}$ ,

$$\|[\hat{A}, \hat{B}]\| \leq \sqrt{n} \max_{i,j} |a_i - a_j| \max_k |b_k - \lambda| d(\mathcal{B}_{\hat{A}}, \mathcal{B}_{\hat{B}}).$$

The optimal choice of  $\lambda$  corresponds to having  $\min \text{spec}(\hat{B} - \lambda \mathbb{1}) = -\max \text{spec}(\hat{B} - \lambda \mathbb{1})$ , in which case  $\max_k |b_k - \lambda| = \frac{1}{2} \max_{k,l} |b_k - b_l|$ , yielding the announced result.

*Second inequality.* We now suppose that the  $(a_i)_{1 \leq i \leq n}$  are distincts, as well as the  $(b_j)_{1 \leq j \leq n}$ . Based on the computations above, we know that for all  $k \in \llbracket 1, n \rrbracket$ ,

$$\begin{aligned} \left\| (\hat{A}\hat{B} - \hat{B}\hat{A}) |e_k\rangle \right\|^2 &= \sum_{\substack{i=1 \\ i \neq k}}^n \left| (a_i - a_k) \langle e_i | \hat{B} e_k \rangle \right|^2 \\ \Rightarrow \sum_{\substack{i=1 \\ i \neq k}}^n \left| \langle e_i | \hat{B} e_k \rangle \right|^2 &\leq \frac{\|[\hat{A}, \hat{B}]\|^2}{\min_{i \neq j} |a_i - a_j|^2}. \end{aligned}$$

Furthermore, recalling that  $\langle e_k | \hat{B} e_k \rangle \in \mathbb{R}$ ,

$$\begin{aligned} \sum_{\substack{i=1 \\ i \neq k}}^n \left| \langle e_i | \hat{B} e_k \rangle \right|^2 &= \left\| \hat{B} |e_k\rangle \right\|^2 - \langle e_k | \hat{B} e_k \rangle^2 \\ &= \left( \sum_{i=1}^n b_i \langle e_k | f_i \rangle \langle f_i | \right) \cdot \left( \sum_{j=1}^n b_j |f_j\rangle \langle f_j | e_k \rangle \right) - \left( \sum_{i=1}^n b_i |\langle f_i | e_k \rangle|^2 \right)^2 \\ &= \sum_{i=1}^n b_i^2 |\langle e_k | f_i \rangle|^2 - \left( \sum_{i=1}^n b_i |\langle e_k | f_i \rangle|^2 \right)^2. \end{aligned}$$

From this, the lemma immediately implies that:

$$\sum_{i=1}^n |\langle e_k | f_i \rangle|^2 (1 - |\langle e_k | f_i \rangle|^2) \leq \frac{2 \|[\hat{A}, \hat{B}]\|^2}{c_{\hat{A}, \hat{B}}^2}.$$

Summing over all  $k$  and taking a square root yields the second inequality.  $\square$

### 3.3.4 Axioms for measures of decoherence

Is there a common feature among all the previous proposals  $\eta_{\mathcal{B}}(\rho_S)$ ,  $F_{\mathcal{B}}^{\varepsilon}(\rho_S)$ ,  $\tilde{\eta}_{\mathcal{B}}(\rho_S)$ ,  $d(\mathcal{B}_{\rho_S}, \mathcal{B})$ ? Which properties should a map satisfy in order to be a good measure of decoherence?

This question has been addressed in [15], in which the authors identify a property called ‘monotonicity under incoherent operations’ as the key ingredient. Their perspective is that of resource theory and quantum technologies<sup>4</sup>. As will become clear in Chapter 4, our aim, though, is ontological *i.e.* to understand what QM tells us about reality. The crucial property for this purpose, we argue, is to bound the deviation from the total probability formula (shown in §4.1 to be the root of the ontological problem of quantum physics), as was done for  $\eta$  in Proposition 3.1.2.

<sup>4</sup>As a matter of fact, they quantify quantum coherence, rather than decoherence. In particular, they fix once and for all a basis in which to quantify coherence, playing the role of the computational basis.

**Definition 3.3.7** (Measure of decoherence). A map

$$\begin{aligned} \eta : \mathcal{S}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H}) &\longrightarrow \mathbb{R}_+ \\ (\rho, \mathcal{B}) &\longmapsto \eta_{\mathcal{B}}(\rho) \end{aligned}$$

is called a *measure of decoherence* if it satisfies the following properties 1 and 2 below.

1.  $\lim_{\mathcal{B} \xrightarrow{d} \mathcal{B}_\rho} \eta_{\mathcal{B}}(\rho) = \eta_{\mathcal{B}_\rho}(\rho) = 0$ ;
2. for all  $\rho \in \mathcal{S}(\mathcal{H})$  and all subspaces  $F \subset \mathcal{H}$ ,

$$|\mathrm{tr}(\rho \Pi_F) - \mathrm{tr}(D_{\mathcal{B}}[\rho] \Pi_F)| \leq \dim(F) \eta_{\mathcal{B}}(\rho);$$

It is pointless to require  $\eta_{\mathcal{B}}(\rho) = 0 \Leftrightarrow \mathcal{B} = \mathcal{B}_\rho$  because the direct implication stems from 2. Without the continuity property in  $\mathcal{B}_\rho$ , being a measure of decoherence would be a very weak statement (any map such that  $\eta \geq 1$  for all  $\mathcal{B} \neq \mathcal{B}_\rho$  would work). Our main result is the following theorem.

**Theorem 3.3.8.** Denote  $\mathcal{B} = (|e_i\rangle)_{1 \leq i \leq d}$ . The following maps:

- $\eta_2 : (\rho, \mathcal{B}) \longmapsto \|Q_{\mathcal{B}}[\rho]\|_2 = \left( \sum_{i \neq j} |\langle e_i | \rho e_j \rangle|^2 \right)^{1/2}$
- $\eta_\infty : (\rho, \mathcal{B}) \longmapsto d \|Q_{\mathcal{B}}[\rho]\|_\infty = d \max_{i \neq j} |\langle e_i | \rho e_j \rangle|$
- $\delta : (\rho, \mathcal{B}) \longmapsto d(\mathcal{B}_\rho, \mathcal{B})$

are measures of decoherence.

Here,  $\|Q_{\mathcal{B}}[\rho]\|_p$  stands for the  $p$ -norm of the non-diagonal elements of  $\rho$  (seen as a  $d^2$ -uple) as written in  $\mathcal{B}$ . For  $p = +\infty$ , we almost recover  $\eta$  from §3.1 (actually, the more precise definition evoked below Proposition 3.1.2, along with a factor  $d$ ). The case  $p = 2$  corresponds to  $\tilde{\eta}$  introduced in §3.3.2 (up to a multiplying factor, that was introduced so as to fit better our probabilistic aims). It is easy to see, however, that  $F_{\mathcal{B}}^\varepsilon(\rho_{\mathcal{S}})$  as defined in §3.3.1 is not a measure of decoherence: the fraction of interfering histories may be arbitrarily low while still having a pair of histories on which the total probability formula completely fails.

*Proof. Property 1.* Clearly, for  $p = 2$  or  $p = \infty$ ,  $\eta_p(\rho, \mathcal{B}_\rho) = 0$  since  $\|0\|_p = 0$ ; and  $d(\mathcal{B}_\rho, \mathcal{B}_\rho) = 0$ . In addition, diagonalizing  $\rho = \sum_k \lambda_k |f_k\rangle \langle f_k|$  in its eigenbasis  $\mathcal{B}_\rho = (|f_k\rangle)_{1 \leq k \leq d}$  yields:

$$\begin{aligned} \frac{1}{d^2} \eta_\infty(\rho, \mathcal{B})^2 &\leq \eta_2(\rho, \mathcal{B})^2 \\ &\leq \sum_{i \neq j} \left| \langle e_i | \left( \sum_k \lambda_k |f_k\rangle \langle f_k| \right) | e_j \rangle \right|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \underset{\text{Jensen}}{\sum_{i,k}} \underbrace{\lambda_k}_{\in[0,1]} |\langle e_i | f_k \rangle|^2 \sum_{j \neq i} |\langle f_k | e_j \rangle|^2 \\
&\leq \sum_{i,k} |\langle e_i | f_k \rangle|^2 (1 - |\langle e_i | f_k \rangle|^2) = d(\mathcal{B}_\rho, \mathcal{B})^2 \xrightarrow{\mathcal{B} \rightarrow \mathcal{B}_\rho} 0.
\end{aligned}$$

Thus property 1 is verified.

*Property 2.* Since we have just proved that  $\eta_2 \leq \delta$ , it suffices to show property 2 for  $\eta_2$  and  $\eta_\infty$ . Repeating the argument of Proposition [3.1.2](#), property 2 is ensured for  $\eta_p$  if we can show that  $\|Q_{\mathcal{B}}[\rho]\| \leq \eta_p(\rho, \mathcal{B})$ . This is true, because for all vectors  $|\Psi\rangle = \sum_k \alpha_k |e_k\rangle \in \mathcal{H}$  of norm 1,

$$\begin{aligned}
\|Q_{\mathcal{B}}[\rho]|\Psi\rangle\|^2 &= \left\| \sum_{i \neq j} \langle e_i | \rho e_j \rangle \alpha_j |e_i\rangle \right\|^2 \\
&= \sum_i \left| \sum_{j \neq i} \langle e_i | \rho e_j \rangle \alpha_j \right|^2 \\
&\leq \sum_i \left( \sum_{j \neq i} |\langle e_i | \rho e_j \rangle| |\alpha_j| \right)^2 \\
&\leq \underset{\text{C.-S.}}{\sum_i} \sum_{j \neq i} |\langle e_i | \rho e_j \rangle|^2 \underbrace{\sum_{k \neq i} |\alpha_k|^2}_{\leq 1} \\
&\leq \eta_2(\rho, \mathcal{B})^2.
\end{aligned}$$

And similarly:

$$\|Q_{\mathcal{B}}[\rho]|\Psi\rangle\|^2 \leq \sum_i \sum_{j \neq i} |\langle e_i | \rho e_j \rangle|^2 \leq \frac{1}{d^2} \eta_\infty(\rho, \mathcal{B})^2 \sum_i \sum_{j \neq i} 1 \leq \eta_\infty(\rho, \mathcal{B})^2.$$

□

## 3.4 Second model: why is the universe not frozen by the quantum Zeno effect?

### 3.4.1 The Zeno effect

The Zeno effect typically occurs when a quantum system is repeatedly measured: if the time interval between two successive measurements tends to 0, the evolution of the system gets frozen. The main reason is that, in quantum mechanics, the general short time evolution is quadratic, *i.e.*:

$$|\langle \Psi | \Psi(t) \rangle|^2 = |\langle \Psi | e^{-i\hat{H}t} | \Psi \rangle|^2 = 1 - Vt^2 + O(t^4),$$

where  $V \equiv \text{Var}_{|\Psi\rangle}(\hat{H}) = \langle \Psi | \hat{H}^2 | \Psi \rangle - \langle \Psi | \hat{H} | \Psi \rangle^2$  (we take  $\hbar = 1$ ). Hence, if  $n$  projective measurements along  $|\Psi\rangle$  are performed during a fixed time interval  $T$ , the probability  $p_n$  that all the measurements gave the outcome  $|\Psi\rangle$  is, at leading order:

$$p_n \simeq \left( 1 - V \left( \frac{T}{n} \right)^2 \right)^n \xrightarrow{n \rightarrow +\infty} 1.$$

Note that to obtain this limit, one has to neglect the higher order terms, as is usually done in the standard presentations of the Zeno effect [105, §3] [77, §3.3.1.1]. Rigourously speaking, this is an additional assumption, because when developing the expression  $\left( 1 - V \left( \frac{T}{n} \right)^2 + O \left( \frac{1}{n^4} \right) \right)^n$ , the number of  $O \left( \frac{1}{n^4} \right)$  actually depends on  $n$ .

In the spirit of the theory of decoherence, one might wish to get rid of the ill-defined notion of (ideal) projective measurement. Since a measurement is nothing but a particular case of interaction with an environment that entails strong decoherence of the system in the measured basis, it is tempting to ask what level of decoherence is required to freeze a system. For example, a particle in a gas is continuously monitored ( $\sim$  measured) by its neighbors, yet the gas manifestly has an internal evolution... and so does the universe in general. It is not obvious *a priori* whether quantum mechanics actually predicts that the universe is not frozen.

This question has already been addressed by examining the continuous dynamics of the pair system - environment for relatively generic Hamiltonian [76]. The Zeno limit is recovered for strong interaction, and Fermi's golden rule is recovered in the limit of small interaction. 'The model shows that the coupling to the environment leads to constant transition rates which are unaffected by the measurement if the coupling is "coarse enough" to discriminate only between macroscopic properties. This may in turn be used to define what qualifies a property as macroscopic: it must be robust against monitoring by the environment' [77, §3.3.2.1]. Similarly, the master equation for the motion of a mass point under continuous measurement indicates that the latter is not slowed down because the Ehrenfest theorems are still valid. 'This may be understood as a consequence of the fact that, for a continuous degree of freedom, any measurement with finite resolution necessarily is too coarse to invoke the Zeno effect' [77, §3.3.1.1].

Another interesting model is that of [59, §8.3 and §8.4]. As mentioned in note [2], it may at first sight seem puzzling that an unstable nucleus continuously measured by a Geiger counter can actually decay. Indeed, if the measurement is treated as an ideal projective one, the nucleus should continuously be projected onto a non-decayed state. But as soon as the decoherence process is not supposed immediate anymore (even as short as  $10^{-16}$ s, see equation (8.45) in [59]), the deviation from the expected exponential decay is shown to be negligible.

Although these models are already convincing, our aim is to give a new contribution to this topic of understanding why the vast majority of physics is not affected by the quantum Zeno effect, the latter being detectable only in some very specific experimental setups. Our model also formalizes the competition between free evolution (no information leaking to the

rest of the world) and decoherence (interaction with the environment), but differs from the previous ones in two respects: its mathematical structure is discrete and it does not assume anything about the form of the Hamiltonian, so as to be as universal as possible. The use of a discrete framework is consistent with the approach adopted in a lot of mathematical studies on the quantum Zeno effect (see [95] and references therein).

### 3.4.2 The model: free evolution vs. decoherence

Having in mind the fact that continuous degrees of freedom are less prone to the Zeno effect (recall the previous quote from [77]), in order to explain why the universe is not frozen, it may suffice to check it on a two-level system. Our system of interest will therefore be a qubit, initially in the state  $|0\rangle$  and monitored by an environment producing partial decoherence in the basis  $(|0\rangle, |1\rangle)$ . We consider a fixed time interval  $T$ , divided into  $n$  phases of length  $\delta = \frac{T}{n}$  dominated by the free evolution. This evolution takes the general form:

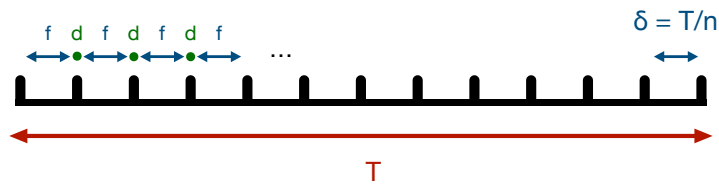
$$\begin{cases} U_\delta |0\rangle = c_{=}^0(\delta) |0\rangle + c_{\neq}^1(\delta) |1\rangle \\ U_\delta |1\rangle = c_{\neq}^0(\delta) |0\rangle + c_{=}^1(\delta) |1\rangle, \end{cases}$$

where the coefficients satisfy  $|c_{=}^0(\delta)|^2 = |c_{=}^1(\delta)|^2 = 1 - V\delta^2 + O(\delta^4)$  and  $|c_{\neq}^0(\delta)|^2 = |c_{\neq}^1(\delta)|^2 = V\delta^2 + O(\delta^4)$  (in the sequel, we will drop the argument  $\delta$  whenever the context is clear). As recalled in the introduction, to stick to the standard derivations of the Zeno effect, we need to neglect all the higher order terms, so that we actually suppose  $|c_{=}^0(\delta)|^2 = |c_{=}^1(\delta)|^2 = 1 - V\delta^2$  and  $|c_{\neq}^0(\delta)|^2 = |c_{\neq}^1(\delta)|^2 = V\delta^2$ .

After the  $i^{\text{th}}$  phase of free evolution, the system meets some neighboring environment  $\mathcal{E}^i$ , initially in the state  $|\mathcal{E}_{\text{init}}^i\rangle$ , and gets immediately entangled according to:

$$\begin{cases} |0\rangle \longrightarrow |0\rangle |\mathcal{E}_0^i\rangle \\ |1\rangle \longrightarrow |1\rangle |\mathcal{E}_1^i\rangle, \end{cases}$$

where  $|\langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle| \equiv \eta^i$  quantifies the level of decoherence induced by  $\mathcal{E}^i$ , *i.e.* how well the environment has recorded the system's state (recall from [3.1] that  $\eta^i = 1$  means no decoherence,  $\eta^i = 0$  perfect decoherence). See Figure [3.1].



**Figure 3.3:** Alternating steps of free evolution (f) and decoherence (d)

From now on, we suppose that  $\eta^i \equiv \eta$  does not depend on  $i$  (taken as a mean level of decoherence), which amounts to assuming that the strength of the interaction is more or less

constant over time. Finally, we also suppose that each environment  $\mathcal{E}^i$  is distinct from the others and non-entangled at the time it encounters the system.

Recalling that  $T = n\delta$ , the relevant quantity to compute is the probability  $p_n$  that, at the end of the time interval  $T$ , the system is still found in its initial state  $|0\rangle$  and that all the successive environments have recorded 0.

**Proposition 3.4.1.** *Neglecting all the higher order terms, we can write:*

$$p_n \simeq 1 - 2 \left[ \frac{n}{2} + (n-1)\eta + (n-2)\eta^2 + \dots + \eta^{n-1} \right] V\delta^2.$$

*Proof.* The cases  $n = 1$  or  $2$  are easy to treat. Indeed, the successive iterations go as follows ( $f$  stands for the free evolution and  $d$  for the decoherence step):

$$\begin{aligned} |0\rangle &\equiv |\Psi_0\rangle \xrightarrow{f} c_{=}^0 |0\rangle + c_{\neq}^1 |1\rangle \\ &\xrightarrow{d} c_{=}^0 |0\rangle |\mathcal{E}_0^1\rangle + c_{\neq}^1 |1\rangle |\mathcal{E}_1^1\rangle \equiv |\Psi_1\rangle \\ &\xrightarrow{f} c_{=}^0 \left[ c_{=}^0 |0\rangle + c_{\neq}^1 |1\rangle \right] |\mathcal{E}_0^1\rangle + c_{\neq}^1 \left[ c_{\neq}^0 |0\rangle + c_{=}^1 |1\rangle \right] |\mathcal{E}_1^1\rangle \\ &= |0\rangle \left[ c_{=}^0 c_{=}^0 |\mathcal{E}_0^1\rangle + c_{\neq}^0 c_{\neq}^1 |\mathcal{E}_1^1\rangle \right] + |1\rangle \left[ c_{\neq}^1 c_{=}^0 |\mathcal{E}_0^1\rangle + c_{=}^1 c_{\neq}^1 |\mathcal{E}_1^1\rangle \right] \\ &\xrightarrow{d} |0\rangle \left[ c_{=}^0 c_{=}^0 |\mathcal{E}_0^1\rangle + c_{\neq}^0 c_{\neq}^1 |\mathcal{E}_1^1\rangle \right] |\mathcal{E}_0^2\rangle + |1\rangle \left[ c_{\neq}^1 c_{=}^0 |\mathcal{E}_0^1\rangle + c_{=}^1 c_{\neq}^1 |\mathcal{E}_1^1\rangle \right] |\mathcal{E}_1^2\rangle \equiv |\Psi_2\rangle. \end{aligned}$$

The  $(|\Psi_n\rangle)_{n \in \mathbb{N}}$  seem to live in different Hilbert spaces only because we omit to write all the environments  $(\mathcal{E}^i)_{i \geq n+1}$  with which the system is not entangled yet. Consequently, neglecting all the higher order terms yields:

- $p_1 = |\langle 0 \mathcal{E}_0^1 | \Psi_1 \rangle|^2 = |c_{=}^0|^2 = 1 - V\delta^2$
- $p_2 = |\langle 0 \mathcal{E}_0^1 \mathcal{E}_0^2 | \Psi_2 \rangle|^2$   
 $= |c_{=}^0 c_{=}^0 + c_{\neq}^0 c_{\neq}^1 \langle \mathcal{E}_0^1 | \mathcal{E}_1^1 \rangle|^2$   
 $= (1 - V\delta^2)^2 + \eta^2 (V\delta^2)^2 + 2\Re \left( \overline{c_{=}^0} c_{\neq}^0 c_{\neq}^1 \langle \mathcal{E}_0^1 | \mathcal{E}_1^1 \rangle \right)$   
 $\simeq 1 - 2(1 + \eta)V\delta^2.$

The last step is not obvious and comes from the following argument. *A priori*, the quantity  $\Re \left( \overline{c_{=}^0} c_{\neq}^0 c_{\neq}^1 \langle \mathcal{E}_0^1 | \mathcal{E}_1^1 \rangle \right)$  lies in  $[-\eta V\delta^2, \eta V\delta^2]$  up to a  $O(\delta^4)$ , but the coefficients of the matrix  $U_\delta$  are not unrelated. Using the general parametrization of a  $2 \times 2$  unitary matrix,  $U_\delta = \begin{pmatrix} c_{=}^0 & c_{\neq}^0 \\ c_{\neq}^1 & c_{=}^1 \end{pmatrix}$  can be written  $\begin{pmatrix} a & b \\ -e^{i\varphi} \bar{b} & e^{i\varphi} \bar{a} \end{pmatrix}$ . Moreover, for small  $\delta$  (this approximation may be rough for the case  $n = 2$  but gets better as  $n$  increases),  $U_\delta \rightarrow \mathbb{1}$  hence  $\det(U_\delta) = e^{i\varphi} \rightarrow 1$  and  $\bar{a} \rightarrow 1$ . We also expect  $\langle \mathcal{E}_0^1 | \mathcal{E}_1^1 \rangle$  to be close to the real number 1 (infinitesimal decoherence). Combining all this,  $\overline{c_{=}^0} c_{\neq}^0 c_{\neq}^1 \langle \mathcal{E}_0^1 | \mathcal{E}_1^1 \rangle = -e^{i\varphi} \bar{a}^2 |b|^2 \langle \mathcal{E}_0^1 | \mathcal{E}_1^1 \rangle$  is close to



be a real negative number, therefore its real part is approximately the opposite of its modulus.

In general,  $p_n = |\langle 0 \mathcal{E}_0^1 \dots \mathcal{E}_0^n | \Psi_n \rangle|^2$  is the square modulus of a sum of terms of the form

$$z_\alpha^b = c_{\alpha_1}^{b_1} \dots c_{\alpha_n}^{b_n} \langle \mathcal{E}_0^1 | \mathcal{E}_{b_1}^1 \rangle \dots \langle \mathcal{E}_0^n | \mathcal{E}_{b_n}^n \rangle,$$

where  $\alpha = \alpha_1 \dots \alpha_n$  and  $b = b_1 \dots b_n$  are words on the alphabets  $\{=, \neq\}$  and  $\{0, 1\}$  respectively. The word  $b$  is entirely deduced from  $\alpha_1 \dots \alpha_i$  according to:

$$b_0 = 0 \quad ; \quad b_i = \begin{cases} b_{i-1} & \text{if } \alpha_i \text{ is } = \text{ (state preserved)} \\ b_{i-1} + 1 \pmod{2} & \text{if } \alpha_i \text{ is } \neq \text{ (state flipped),} \end{cases}$$

with the additional requirement that  $b_n = 0$  (system finally measured in state  $|0\rangle$ ), so that  $\alpha$  actually contains an even number of  $\neq$ . Note that only the indices  $i$  such that  $b_i = 1$  contribute non-trivially in the product of brackets, since  $\langle \mathcal{E}_0^i | \mathcal{E}_0^i \rangle = 1$ .

We now use the fact that  $|\sum_k z_k|^2 = \sum_k |z_k|^2 + \sum_{k < l} 2\Re(\bar{z}_k z_l)$  for all complex numbers  $(z_k)_k$ . In our case, the leading term is clearly  $|z_{\alpha_1 \dots \alpha_n}^0|^2 = |c_{\alpha_1 \dots \alpha_n}^0|^2 = (1 - V\delta^2)^n \simeq 1 - nV\delta^2$ , while all the other square moduli are of order  $\delta^4$  or less because they contain at least two factors  $|c_{\neq}^{b_i}|^2 = V\delta^2$ . Furthermore, repeating the above argument, the real parts can be approximately replaced by their opposite moduli (and this approximation is better as  $n$  gets larger). Therefore, only the cross-products of the form  $2\Re(\overline{c_{\alpha}^0} \times z_{\alpha}^b)$ , where  $\alpha$  contains exactly two  $\neq$ , contribute at order  $\delta^2$ . The power of  $\eta$  that appears in this cross-product (*i.e.* the number of non-trivial brackets  $\langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle$ ) is the number of indices  $i$  such that  $b_i = 1$ , that is the number of steps elapsed between the two  $\neq$ . For instance, if the two  $\neq$  happen at the  $i^{\text{th}}$  and  $j^{\text{th}}$  step, the contribution is:

$$\begin{aligned} & 2\Re(\overline{c_{\alpha}^0} \times c_{\alpha}^0) = 2\Re(\overline{c_{\neq}^0} \times c_{\neq}^0) = 2\Re(\overline{c_{\neq}^0} \times c_{\neq}^0) = 2\Re(\overline{c_{\neq}^0} \times c_{\neq}^0) \\ & \simeq 2|c_{\neq}^1 c_{\neq}^0 \langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle \dots \langle \mathcal{E}_0^{j-1} | \mathcal{E}_1^{j-1} \rangle| \\ & \simeq 2\eta^{j-i} V \delta^2. \end{aligned}$$

There are obviously  $n - k$  words  $\alpha$  with exactly two  $\neq$  separated by  $k$  steps, corresponding to the  $n - k$  possible choices for  $i$ , whose contribution is  $2\eta^k V \delta^2$ . Finally, the general expression for  $p_n$  when neglecting all the higher order terms is:

$$p_n = |\langle 0 \mathcal{E}_0^1 \dots \mathcal{E}_0^n | \Psi_n \rangle|^2 \simeq 1 - 2 \left[ \frac{n}{2} + (n-1)\eta + (n-2)\eta^2 + \dots + \eta^{n-1} \right] V \delta^2.$$

□

We can check the consistency of this result on two particular cases:

- if  $\eta = 1$ , no decoherence occurs, so we recover the free evolution case during a time interval  $n\delta$  instead of  $\delta$ , *i.e.*  $\mathbb{P}_n = 1 - V(n\delta)^2$ ;
- if  $\eta = 0$ , a perfect decoherence means that the environment acts as an ideal measuring device, so we recover the Zeno case recalled in the introduction, that is  $p_n = 1 - nV\delta^2 \simeq (1 - V\delta^2)^n$ .

Now, a Zeno effect will freeze the system in the limit of large  $n$  if and only if  $p_n \xrightarrow[n \rightarrow +\infty]{} 1$ , that is (using  $\delta = \frac{T}{n}$ ) if  $\frac{(n-1)\eta + (n-2)\eta^2 + \dots + \eta^{n-1}}{n^2} \xrightarrow[n \rightarrow +\infty]{} 0$ . After some algebra, this expression can be simplified and leads to the following criterion:

$$\boxed{\text{Zeno effect} \iff \frac{n\eta(1-\eta) + \eta(\eta^n - 1)}{n^2(1-\eta)^2} \xrightarrow[n \rightarrow +\infty]{} 0}$$

We immediately note that if  $\eta \in [0, 1[$  is a constant independent of  $n$ , the criterion is satisfied. This is natural because, as the duration of each free evolution phase goes to 0, a constant (even weak) decoherence is applied infinitely many times, so the system freezes.

From now on, we will suppose that the level of decoherence depends on  $n$ , with  $\eta_n \xrightarrow[n \rightarrow +\infty]{} 1$ . A global factor  $\eta$  can thus be dropped in the above criterion. Our task in the following sections will be (i) to check the criterion on some common classes of functions  $\eta_n$  (section §3.4.3) (ii) to estimate the level of decoherence really encountered in physical situations (section §3.4.4).

**Remark 3.4.2.** How finely should the time interval be divided so that the quadratic approximation be valid? Let's forget for a moment that our system is finite dimensional and consider the Hamiltonian of a free particle  $\frac{\hat{P}^2}{2m}$ , starting from the initial state  $|\Psi\rangle(p) = \sqrt{\frac{\sigma}{\sqrt{\pi}\hbar}} e^{-\frac{p^2\sigma^2}{2\hbar^2}}$  centred around  $x = 0$  and  $p = 0$ , and compute:

$$\begin{aligned} \text{Var}_{|\Psi\rangle}(\hat{H}) &= \frac{1}{4m^2} \left[ \langle \Psi | \hat{P}^4 | \Psi \rangle - \langle \Psi | \hat{P}^2 | \Psi \rangle^2 \right] \\ &= \frac{1}{4m^2} \left[ \int_{-\infty}^{+\infty} p^4 \frac{\sigma}{\sqrt{\pi}\hbar} e^{-\frac{p^2\sigma^2}{\hbar^2}} dp - \left( \int_{-\infty}^{+\infty} p^2 \frac{\sigma}{\sqrt{\pi}\hbar} e^{-\frac{p^2\sigma^2}{\hbar^2}} dp \right)^2 \right] \\ &= \frac{\hbar^4}{8m^2\sigma^4} \end{aligned}$$

Hence the quadratic approximation is valid for times shorter than  $t_c = \frac{\hbar}{\sqrt{\text{Var}_{|\Psi\rangle}(\hat{H})}} = \frac{2\sqrt{2}m\sigma^2}{\hbar}$ . Taking for instance  $m = 10^{-26}\text{kg}$  and  $\sigma = 10^{-10}\text{m}$ , we get  $t_c = 4.10^{-13}\text{s}$ . This is way shorter than the mean free time of a particle in a gas in standard conditions, which is of order  $10^{-10}\text{s}$ . So it seems at first sight that the decoherence steps could in practice be too separated in time for the quadratic approximation to be valid all along the free evolution step. However, decoherence doesn't need any actual interaction to take place (a 'null measurement' is still a measurement [83]). The fact that all the other surrounding particles do *not* interact with the particle of interest is still a gain of information for the environment, which suffices to suppress coherence with other possible histories in which they *would* have interacted. In this case, information is continually leaking to the environment, so it seems legitimate to divide the time interval  $T$  as finely as desired so that the quadratic approximation become valid, and the resulting behaviour is then determined by the intensity of infinitesimal decoherence only.

The philosophy of this argument is not specific to the infinite dimensional case, and may be applied to our two-level system. It relies, however, on the already mentioned assumption that the strength of the interaction is more or less constant over time. This will be discussed in Section §3.4.5.

### 3.4.3 Analytic study of the criterion

Whenever  $(n(1 - \eta_n))_{n \in \mathbb{N}}$  admits a limit in  $\overline{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{+\infty\}$ , the following lemma allows to check immediately the criterion of the previous Section.

**Lemma 3.4.3.** *Suppose  $n(1 - \eta_n) \xrightarrow[n \rightarrow +\infty]{} \alpha \in \overline{\mathbb{R}}_+$ . Then*

$$\frac{n(1 - \eta_n) + \eta_n^n - 1}{n^2(1 - \eta_n)^2} \xrightarrow[n \rightarrow +\infty]{} \begin{cases} \frac{1}{2} & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \\ \frac{1}{\alpha} + \frac{e^{-\alpha} - 1}{\alpha^2} & \text{otherwise.} \end{cases}$$

*Proof.* Let  $u_n \equiv n(1 - \eta_n)$ . If  $u_n \xrightarrow[n \rightarrow +\infty]{} +\infty$ , since  $\eta_n^n - 1$  is bounded, the result is immediate.

If  $0 < \alpha < +\infty$ , notice that  $\eta_n^n = e^{n \ln(1 - \frac{u_n}{n})} \xrightarrow[n \rightarrow +\infty]{} e^{-\alpha}$ , and rewrite:

$$\frac{n(1 - \eta_n) + \eta_n^n - 1}{n^2(1 - \eta_n)^2} = \frac{1}{u_n} + \frac{\eta_n^n - 1}{u_n^2} \xrightarrow[n \rightarrow +\infty]{} \frac{1}{\alpha} + \frac{e^{-\alpha} - 1}{\alpha^2}.$$

Finally, if  $\alpha = 0$ :

$$\begin{aligned} \eta_n^n &= e^{n \ln(1 - \frac{u_n}{n})} = e^{-u_n - \frac{u_n^2}{2n} + O(\frac{u_n^3}{n^2})} \\ &= 1 - u_n - \frac{u_n^2}{2n} + O\left(\frac{u_n^3}{n^2}\right) + \frac{1}{2} \left[ -u_n - \frac{u_n^2}{2n} + O\left(\frac{u_n^3}{n^2}\right) \right]^2 + O(u_n^3) \\ &= 1 - u_n + \frac{u_n^2}{2} + O\left(\frac{u_n^2}{n}\right). \end{aligned}$$

Consequently,  $\frac{n(1 - \eta_n) + \eta_n^n - 1}{n^2(1 - \eta_n)^2} = \frac{u_n^2/2 + O(u_n^2/n)}{u_n^2} \xrightarrow[n \rightarrow +\infty]{} \frac{1}{2}$ . □

Two natural candidates for the level of short-time decoherence are  $\eta_n = 1 - \frac{\alpha}{n^\beta}$  and  $\eta_n = 1 - \alpha e^{-\beta n}$  for  $\alpha, \beta > 0$ . These cases can be treated by the lemma, and the different possible situations are summarized in the following table.

$\eta_n$	Regime	$\lim_{n \rightarrow +\infty} p_n$
1	Free evolution	$1 - VT^2$
Constant $\in [0, 1[$	Zeno effect	1
$1 - \frac{\alpha}{n^\beta}$ with $\beta \in ]0, 1[$	Zeno effect	1
$1 - \frac{\alpha}{n^\beta}$ with $\beta > 1$	Free evolution	$1 - VT^2$
$1 - \frac{\alpha}{n}$	Intermediate	$1 - 2 \underbrace{\left( \frac{1}{\alpha} + \frac{e^{-\alpha} - 1}{\alpha^2} \right)}_{\substack{\xrightarrow{\alpha \rightarrow +\infty} 0 : \text{Zeno effect} \\ \xrightarrow{\alpha \rightarrow 0} 1 : \text{free evolution}}} VT^2$
$1 - \alpha e^{-\beta n}$	Free evolution	$1 - VT^2$

### 3.4.4 Physical considerations concerning $\eta_n$

1. As previously remarked, the constant case corresponds either to the absence of decoherence ( $\eta = 1$ ) or to infinite decoherence ( $\eta \in [0, 1[$ ): these are not physically expected, except in some particular experimental setups (perfectly isolated systems for the first, experiments specifically designed to probe the Zeno effect for the second).
2. For now, we have not yet introduced any duration for the decoherence step, which was considered immediate. Let's at present assume that the time evolution can be divided into alternating steps dominated by either the free Hamiltonian, or by the interaction Hamiltonian. The time of interaction between the system and each environment  $\mathcal{E}^i$ , governed by  $\hat{H}_{S\mathcal{E}^i}$  of variance  $\text{Var}(\hat{H}_{S\mathcal{E}^i}) \equiv V_{\text{int}}^i \equiv V_{\text{int}}$  (constant strength of interaction), is still taken proportional to  $\frac{T}{n}$ , say equal to  $\frac{cT}{n}$ . This is a new assumption we make: that the time increments of both steps scale as  $\frac{1}{n}$ , and that the two phases can be considered on an equal footing, means that the two Hamiltonians are of relatively comparable strength. Then the quadratic approximation recalled in the introduction can be applied to the whole {system + environment}. Thus  $|\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_0^i \rangle|^2 = |\langle 0_{\mathcal{E}_{\text{init}}}^i | 0_{\mathcal{E}_0}^i \rangle|^2 \simeq 1 - V_{\text{int}} \left( \frac{cT}{n} \right)^2$ . Since moreover  $\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_0^i \rangle$  is close to the real number 1 (infinitesimal decoherence),  $\Re(\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_0^i \rangle) \simeq |\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_0^i \rangle| \simeq 1 - \frac{1}{2} V_{\text{int}} \left( \frac{cT}{n} \right)^2$  is quadratic in time, and similarly for  $\Re(\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_1^i \rangle)$ . This will also be the case for  $\eta_n = |\langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle|$ , because:

$$\begin{aligned}
\sqrt{2 - 2|\langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle|} &\simeq \sqrt{2 - 2\Re(\langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle)} = \| |\mathcal{E}_0^i\rangle - |\mathcal{E}_1^i\rangle \| \\
&\leq \| |\mathcal{E}_{\text{init}}^i\rangle - |\mathcal{E}_0^i\rangle \| + \| |\mathcal{E}_{\text{init}}^i\rangle - |\mathcal{E}_1^i\rangle \| \\
&= \sqrt{2 - 2\Re(\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_0^i \rangle)} + \sqrt{2 - 2\Re(\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_1^i \rangle)} \\
&\simeq 2\sqrt{V_{\text{int}} \frac{cT}{n}},
\end{aligned}$$

hence  $\eta_n = |\langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle| \gtrsim 1 - 2V_{\text{int}} \left(\frac{cT}{n}\right)^2$  is also at least quadratic. Said differently, *because quantum mechanical short time evolutions are always quadratic, and this is true also for the environment's evolution, infinitesimal steps of decoherence induced on a system by its surrounding environment are likely to be of the form  $\eta_n = 1 - \frac{\alpha}{n^\beta}$  with  $\beta \gtrsim 2$ . This could constitute a universal reason why the universe is not frozen by the quantum Zeno effect.*

An example of such an interaction is the following. Consider that the system is a qbit in the state  $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ , and the environment is a particle initially centered around  $x = 0$  with momentum  $p_0$ , that is  $|\Psi_{0,p_0}\rangle = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} e^{ip_0x} e^{-\frac{x^2}{2\sigma^2}} \in L^2(\mathbb{R})$ . The system's state is recorded in the  $(|0\rangle, |1\rangle)$  basis due to the interaction  $\hat{H}_{S\mathcal{E}} = v\hat{\sigma}_z \otimes \hat{P}$  so that, after some time  $\delta \propto \frac{1}{n}$ :

$$\begin{cases} |0\rangle |\Psi_{0,p_0}\rangle \longrightarrow |0\rangle |\Psi_{v\delta,p_0}\rangle \\ |1\rangle |\Psi_{0,p_0}\rangle \longrightarrow |1\rangle |\Psi_{-v\delta,p_0}\rangle, \end{cases}$$

Here,

$$\eta_n = |\langle \Psi_{v\delta,p_0} | \Psi_{-v\delta,p_0} \rangle| = \left| \frac{e^{2ip_0v\delta}}{\sqrt{\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{x^2+(v\delta)^2}{\sigma^2}} dx \right| = e^{-\frac{(v\delta)^2}{\sigma^2}} \simeq 1 - \frac{v^2}{\sigma^2} \delta^2,$$

so this interaction induces indeed a short time quadratic decoherence as long as the increment of time satisfies  $\delta \ll \frac{v}{\sigma}$ .

3. What if the assumption of comparable strengths of the Hamiltonians fails, for instance if the free evolution term is negligible compared to the coupling with the environment? This amounts to taking  $c$  or  $V_{\text{int}} \rightarrow +\infty$ , hence to lift the quadratic approximation for the interaction Hamiltonian. A possibility then is to consider that  $|\mathcal{E}_0^i(t)\rangle$  and  $|\mathcal{E}_1^i(t)\rangle$  follow two independent Brownian motions starting in  $|\mathcal{E}_{\text{init}}^i\rangle$  on the sphere of all possible states in  $\mathcal{H}_{\mathcal{E}^i}$  during the duration  $\delta$  of the decoherence step. If the latter exceeds the typical time of diffusion on the sphere, we recover the case of a constant  $\eta \in [0, 1[$  (infinite decoherence, case n°1 above) with  $\eta \sim \frac{1}{\sqrt{\dim(\mathcal{H}_{\mathcal{E}^i})}}$  as shown in 3.2.2. If it is shorter than the diffusion time (but still longer than the quadratic regime),  $|\mathcal{E}_0^i(\delta)\rangle$  lies in the vicinity of  $|\mathcal{E}_{\text{init}}^i\rangle$  on the sphere, which is approximately a ball. It is well known that the typical length of diffusion goes as  $\| |\mathcal{E}_{\text{init}}^i\rangle - |\mathcal{E}_0^i(\delta)\rangle \| \simeq D\sqrt{\delta}$ , which implies  $|\langle \mathcal{E}_{\text{init}}^i | \mathcal{E}_0^i(\delta) \rangle| \simeq \sqrt{1 - (D\sqrt{\delta})^2} \simeq 1 - \frac{D^2}{2}\delta$ . If  $\delta$  is still taken  $\propto \frac{1}{n}$ , we are now in the intermediate regime studied above, with  $\beta = 1$  and  $\alpha \propto D^2$ . This corresponds to situations where the system's evolution is slowed down because of its monitoring by the environment. In the limit of strong interaction, the diffusion constant  $D$  will go to infinity and we recover the Zeno effect, whereas a weak interaction tends to the free evolution case.

### 3.4.5 Discussion

We have presented a model designed to check whether quantum mechanics indeed predicts that the universe should evolve. To remain as universal as possible, no specific form of Hamiltonian was assumed. It allowed to determine the level of decoherence (induced by a surrounding environment) needed to freeze a two-level quantum system, arguably the kind of system the most prone to the Zeno effect. We have found that if, during a time interval  $\frac{T}{n}$ , the environment distinguishes between the two states according to  $|\langle \mathcal{E}_0^i | \mathcal{E}_1^i \rangle| \simeq 1 - \frac{\alpha}{n^\beta}$  with  $\beta > 1$ , then free evolution wins over decoherence and the system is not frozen. In the most generic case, because quantum mechanical short time evolutions are always quadratic (and this is true for the system as well as for the pair {system + environment}), we find  $\beta \gtrsim 2$ , hence the universe is indeed not frozen.

The main weaknesses of the model, leading to possible improvements, are the following.

- Is the discrete setup legitimate? A succession of infinitesimal steps is not necessarily the same as a joint continuous evolution.
- What happens if the coupling with the environment is not supposed roughly constant anymore? Mathematically, this means that the  $\eta^i$  are not equal, and the infinitesimal decoherence rate (*i.e.* the flow of information) at time  $t$  could be modelled in the limit  $n \rightarrow +\infty$  as a continuous quantity  $1 - d\eta(t)$ . It is natural to ask for the set of such functions which entail a Zeno freezing. Besides, the durations of the steps could also be non-constant (like following a Poisson process, as done in [82]).
- Assuming the environments  $\mathcal{E}^i$  distinct and non-entangled is a very unphysical assumption. In some cases, previous entanglement among the environments can dramatically change the efficiency of decoherence. As an example, take an environment composed of two qbits called  $\mathcal{E}^1$  and  $\mathcal{E}^2$  initially maximally entangled; the system interacts with  $\mathcal{E}^1$  then with  $\mathcal{E}^2$  *via* a C-NOT gate:

$$\begin{aligned}
 \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{\mathcal{S}} \otimes \underbrace{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}_{\mathcal{E}^1 + \mathcal{E}^2} &\xrightarrow{C-NOT_{\mathcal{S}\mathcal{E}^1}} \underbrace{\frac{1}{2}(|000\rangle + |011\rangle + |110\rangle + |101\rangle)}_{\rho_{\mathcal{S}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} : \mathcal{S} \text{ is perfectly decohered}} \\
 &\xrightarrow{C-NOT_{\mathcal{S}\mathcal{E}^2}} \frac{1}{2}(|000\rangle + |011\rangle + |111\rangle + |100\rangle) \\
 &= \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{\rho_{\mathcal{S}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} : \text{coherence has revived}} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
 \end{aligned}$$



# Chapter 4

## The ontological problem of quantum physics

QM has puzzled everyone since the 1920s. The so-called measurement problem, sometimes deemed one of the greatest mysteries of modern physics, has been much debated by some, much repressed by others. The theory of decoherence has certainly provided an important step towards its elucidation, but has not ended the debates — quite the contrary. Which questions does decoherence solve, and what is still calling for an explanation? Why has the issue been called the ‘measurement problem’? Has it really something to do with consciousness and, if not, why has the latter so often been invoked (see [\[138\]](#) for a historical recap)?

One difficulty is that the problem takes very different forms depending on the interpretation of QM embraced. For this reason, it somehow lacks a precise accepted definition, a starting point from which to discuss. Furthermore, the way it was historically addressed in the first half of the 20th century has strongly crystallized the paradigms and the language at our disposal to think about it, at a time when we hadn’t developed the theory of decoherence yet. In this chapter, we propose to give a fresh look on the topic by identifying the common root of the puzzle in the light of the theory of decoherence, and only then to study how it manifests itself in five of the most famous interpretations of QM: how they propose to solve it and which new difficulties arise. This way can they be better compared. The price to pay may somehow be a feeling of anachronism, since for instance the Copenhagen interpretation was obviously not designed to answer the problem in the way we will formulate it.

At first glance, it may be tempting to believe in a sharp distinction between a physical theory and its interpretation. The former would be a mathematical structure which objectively predicts empirical facts, while the latter would consist in a presentation of those mathematics, and a linguistic structure to speak about them, which can be chosen according to different non-objective criteria (*e.g.* raising clear pictures in mind, being as little counter-intuitive as possible...), so that two competing interpretations could not be experimentally distinguished. Because of this prejudice, the measurement problem has often been considered as unsolvable, or even useless. However, as shown by Bell, the border between what is empirically constrained and what is really a metaphysical choice may be surprisingly



difficult to draw, and clarifying it can have deep philosophical consequences on the nature of the universe we live in. In this respect, we will also propose, for each analysed interpretation of QM, some experimental protocols that could falsify them.

Lastly, the problem is not a mathematical one. Although we tried to remain as clear as possible, the mathematician or physicist reader unfamiliar with the philosophy of QM must be warned that they might find some parts of this chapter difficult to read. Some philosophical preliminaries can be found in the annex [A](#).

## 4.1 Blurring the border between ontology and epistemology

*QM is a geometrical way of computing probabilities, in which the total probability formula does not hold.* The fact that probabilities do not fundamentally obey [\(3.1\)](#) in our universe is an experimental datum (just think of the double-slit experiment). This apparently harmless observation entails a huge conceptual difficulty, because the total probability formula is at the core of our most intuitive notion of an objective world. As already stated in [§3.1](#), [\(3.1\)](#) is necessary for us to consider that, even though the actual value of a variable is not known, it still has a definite value among the possible ones. Said differently, it guarantees that we can build an ontology distinct from epistemology, supporting the picture of an objective world independent of what is known about it. In the sequel, such an ontology will be called an **easy ontology**. *The ontological problem of quantum physics is the absence of easy ontology for QM.*

Indeed, the statistics of a quantum system in a state  $\rho$  satisfy the total probability formula only when conditioning on an observable whose eigenbasis corresponds to  $\rho$ 's eigenbasis  $\mathcal{B}_\rho$ . It is the basis in which the system is perfectly decohered or, equivalently, the basis about which the rest of the universe has stored complete information (recall item 2. in [§3.3.2](#)). In  $\mathcal{B}_\rho$ , the possible histories are easy to tell because they don't influence each other, as is expected when only one of them did effectively happen. With respect to any other basis, though, the system can not easily be thought of as being in one definite state, at least not in the way we are used to do: all potentialities seem to contribute to the statistics of subsequent experiments<sup>1</sup>, as if they were superposed.

The total probability formula is ubiquitous in the discussions on the interpretation of QM, although often unnoticed. The very definition of a hidden variable could be an unknown quantity  $\lambda \in \Lambda$  having probability distribution  $\mu(d\lambda)$ , with respect to which any quantum observable  $A$  can be conditioned so as to satisfy the total probability formula:

$$\mathbb{P}(A) = \int_{\Lambda} \mathbb{P}(A|\lambda)\mu(d\lambda),$$

---

<sup>1</sup>QM has first been formulated in terms of non-commuting operators in a Hilbert space possibly because it offers the simplest probabilistic structure violating [\(3.1\)](#). Feynman's sum-over-histories approach is another natural candidate of a probabilistic framework in which all potentialities contribute.

where  $\mathbb{P}(A|\lambda) = \delta(A = f(\lambda))$  is a Dirac distribution (or is at least less indeterministic than  $\mathbb{P}(A = a)$ ), meaning that  $\lambda$  can be thought of as an objective property of the system containing information about the values taken by the observables. This is already present in Bell’s seminal papers [18, equation (2)] [19, equation (12)], as well as in the more modern formulation of Bell’s theorem in terms of causal models [144, equation (13)]. The assumption of ‘macroscopic realism’ used to derive the Leggett-Garg inequalities also translates mathematically as a total probability formula [87, equation (1)]. Finally, the much debated hypothesis of ‘absoluteness of observed events’, at the core of the local friendliness no-go theorem [29] (see §4.5.2), is (among other things) the assumption a probability distribution satisfying a total probability formula when conditioning on observed events.

As Zurek states:

‘I strongly suspect that the ultimate message of quantum theory is that the separation between what exists and what is known to exist – between the epistemic and the ontic – must be abolished.’ [161]

This idea has been further developed by many philosophers.

‘In his *Dioptrique*, Descartes absorbed even the eye itself into his mechanistic account of nature; “[the eye] is no longer the end of the visual process, merely an arbitrary point of reference, an unprivileged station in the natural process”. This conceptual “detachment” of the physical process of image formation from the concept of seeing, and the abandonment of the latter in the new theories of optics, illustrates the [easy ontology] that lies at the heart of what was eventually to become known as classical physics, the classical physics that presupposes, as we discussed above, that nature is to be framed within a globally Boolean structure, and through this conceives of nature as something that exists in a particular way independently of how we interact with it. What quantum mechanics shows us is that as enormously useful as this strategy has been, and continues to be in non-fundamental physics, it eventually brings us to a point beyond which the usefulness of pursuing it diminishes. To make this worldview work for elementary phenomena we need to twist and contort nature, and hide, as if behind a curtain, the parts of it that are responsible for troubling the worldview. In hindsight it was inevitable, we maintain, that the classical worldview should eventually have found a limit to its usefulness.’ [75]

We routinely experience the validity of the total probability formula (and thereby the legitimacy of an easy ontology), to such an extent that no human ever had any reason to doubt of its universality before the first quantum experiments were conducted. Decoherence is sufficient to explain why. Indeed, the way we get information about a system  $\mathcal{S}$  is *via* physical interactions, which are governed by the same laws of physics as the interactions  $\mathcal{S}$  has with its environment. Therefore, the bases in which we are mainly able to measure  $\mathcal{S}$  are rightly the same as those continuously best recorded by its environment, that are the bases

such that  $\eta_{\mathcal{B}} \simeq 0$ . Said differently, we only distinguish between potentialities of a system that have already been rendered ontologically easy by the rest of the world.

Despite this, non-decohered quantum systems still need an ontology, albeit an uneasy one. Besides, decoherence is guaranteed in only one basis (up to degeneracy) hence, according to unitary QM, even a macroscopic system largely deviates from (3.1) in most bases. When conditioning the latter's statistics on a non-decohered observable, all potentialities interfere, *even all those which presumably did not happen*.

Now, many aspects of QM are not even testable at our scales, because we can neither perfectly isolate a large system, nor apply any unitary evolution we wish, nor prepare a precise macroscopic state (not to mention many copies of the same state), and we have access to a very small number of coarse measurements. Still today, we are able to detect a depart from (3.1) only on systems composed of few particles. So at this point, the philosophical problem is not immediately clear. For: do things happen for a particle? Does it have a point of view? Does it have facts? If not, there might simply not be any ontological problem for it. Yes, one can not assume that an unknown variable of the particle always has definite value among the possible ones, but is it so serious if nothing really happens for it?

However, the problem becomes absolutely unavoidable as soon as a human observer (or possibly a cat) is included in the quantum system (we are then limited to thought experiments), as famously captured by the Wigner's friend experiment [140] recalled in A.2. For the friend has necessarily *experienced* something, and only one thing. It is a fact for her, so how can the total probability formula possibly fail, even if Wigner does not know which outcome she has experienced? There is a conflict between the necessary unicity of a subjective experience [7], and the multiplicity of potentialities which all seem to have some kind of reality in contributing to the statistics of future experiments. *This is the reason why the role of consciousness has so often been questioned in QM.*

Because the ontological problem appears most severe on **observers**, defined as systems granted with subjectivity, QM urges us to specify which systems are to be considered as such. Crucially, it must be emphasized that this is not a scientific question, but a philosophical one. The domain of applicability of science is matter, that is everything that can be objectively measured and quantified. But there are no measurement apparatuses, nor even units of measurement, for subjective experiences. In [140], Wigner claimed that 'my friend [should have] the same types of impressions and sensations as I', and the question of which systems are we friendly with is becoming more and more pressing, especially in the context of the local friendliness no-go theorem [29]. It has been proposed that a human-level artificial intelligence (AI) running on a quantum computer could constitute a good candidate for a friend [141], based on the metaphysical assumption that 'If a system displays independent cognitive ability at least on par with my own, then (...) any thought they communicate is as real as any communicable thought of my own'. However, one might argue that a computer running such an AI should not be so fundamentally different, subjectively speaking, from the same computer running any other equally complex algorithm. If so, succeeding a Turing test might not be such a relevant criterion to assess the subjective experience of a system.

To summarize this section, *the main question raised by the ontological problem of quantum*

## 4.2 The preferred-basis problem

Often related to our discussion is the observation that, although the Hilbert space appears to be an intrinsic object independent of any particular choice of coordinates, some bases stand out. This issue is generally referred to as the preferred-basis problem. However, this expression is sometimes used to designate problems of different nature. We propose to divide them into two categories; only the second one seems to be directly linked to the ontological problem of QM. The precise nature of the link between the two remains to be explored.

1. The **objective preferred-basis problem**: as mentioned above, the shape of the laws of physics turns out to favour some bases over the others with regard to decoherence and to the redundancy of information [160, 31]. These ‘pointer bases’ certainly have something to do with the eigenbasis  $\mathcal{B}_{\hat{x}}$  of the position operator, because the laws of physics usually involve position variables, hence environments (including observers) accurately feel variations in position and record them [155, 5.] [119, III.E.2.]. Exchanging a few photons may suffice to roughly localize an object, explaining why it is so hard to build an interferometric experiment with macromolecules (the record to date involving oligoporphyrins,  $\sim 2,000$  atoms [57] [80]). For the very same reason the salient information we have about the world is also strongly linked with position, hence the observables of interest for us (‘is the cat dead or alive?’, ‘is my friend at home?’...) comply very well with (3.1)<sup>2</sup>. But how to properly characterize the common preferred-basis of decoherence (constraining the observables compatible with an easy ontology) using less vague arguments? Importantly, how to do this in a backgroundless manner, *i.e.* without relying on a pre-existing notion of space and position operator? Alternatively, could the preferred-basis selected by the shape of the laws of physics, first considered abstractly, be in turn used to *define* a backgroundless notion of position? This is the topic of Section §5.2.

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<sup>2</sup>This is a deep remark, but one should be very careful with its philosophical implications. I reckon that it would not be correct to state that the human body and mind (and presumably all living beings on Earth) evolved so as to develop perceptions strongly based on the notion of space and position because this evolutionary path allowed them to benefit from an easy and stable ontology entailed by decoherence on this preferred observable. Indeed, there was no ‘notion of space and position’ before something developed it, and even less any ‘laws of physics involving position variables’, since such laws only belong to the worlds of those sensitive beings whose minds are already appropriately shaped. I would be surprised to meet an alien, but I would not be surprised if it has absolutely nothing like a ‘notion of space and position’ even though its body, if it is made of matter, will certainly be perceived by us as obeying the laws of physics. This alien would probably not survive long on Earth, but maybe on its planet there is no heredity and therefore no Darwinian evolution; maybe an easy ontology, or the very notion of facts, don’t confer there any kind of advantage; maybe what we would deem to be its death would not be a relevant event in its world. If ever it has something like a physics, I would strive to understand it. When talking about the world from outside the earthly point of view, there is really nothing that can be expected. The earthly minds and bodies on one side, their perceived outer world and its laws on the other side, co-evolved backgroundlessly; it would be misleading but appreciably poetic to say that they emerged out of the amorphous void.

2. The **subjective preferred-basis problem**: decoherence alone can not suffice to characterize which one is the right basis corresponding to what has been subjectively experienced. When Wigner’s friend measures the spin along  $z$  and gets in the state  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z \mathcal{F}_{\uparrow_z}\rangle + |\downarrow_z \mathcal{F}_{\downarrow_z}\rangle)$ , it is required that  $|\mathcal{F}_{\uparrow_z}\rangle$  and  $|\mathcal{F}_{\downarrow_z}\rangle$  represent distinguishable states of perception, therefore  $\langle \mathcal{F}_{\uparrow_z} | \mathcal{F}_{\downarrow_z} \rangle = 0$  (spin perfectly decohered by  $\mathcal{F}$  along  $z$ ) [36]. Yet, one can rewrite, for any other direction  $u$ ,  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_u \mathcal{F}_{\uparrow_u}\rangle + |\downarrow_u \mathcal{F}_{\downarrow_u}\rangle)$ , with  $|\mathcal{F}_{\uparrow_u}\rangle$  and  $|\mathcal{F}_{\downarrow_u}\rangle$  some linear combinations of  $|\mathcal{F}_{\uparrow_z}\rangle$  and  $|\mathcal{F}_{\downarrow_z}\rangle$  still satisfying  $\langle \mathcal{F}_{\uparrow_u} | \mathcal{F}_{\downarrow_u} \rangle = 0$ <sup>3</sup> (spin perfectly decohered by  $\mathcal{F}$  along any axis). So QM ‘has nothing to say about which observable(s) of the system is (are) being recorded, via the formation of quantum correlations, by the apparatus (...) in obvious contrast to our experience of the workings of measuring devices that seem to be “designed” to measure certain quantities’ [119]. Should we add, then, an additional ingredient to the Hilbert space, like a set of ‘truly experienced states’, in which all observers having subjective experiences lie? Or is it just again linked with the position basis  $\mathcal{B}_{\hat{X}}$ ? Indeed, in our example,  $|\mathcal{F}_{\uparrow_z}\rangle$  and  $|\mathcal{F}_{\downarrow_z}\rangle$  seem to be (to a good approximation) position eigenvectors, whereas all the other  $|\mathcal{F}_{\uparrow_u}\rangle$  are spatial superpositions, and this could characterize the  $z$  axis. In the subsequent sections, we will see how the different interpretations of QM tackle this problem.

The two problems are not unrelated, though, for truly experienced states must (at least approximately) belong to the objective preferred-basis. Otherwise, decoherence would significantly alter the state of an observer after she has experienced something, hence the total probability formula fails when conditioning on what she has observed, and the ontological problem reappears.

## 4.3 Collapse-realist interpretations

### 4.3.1 The Copenhagen interpretation

The Copenhagen interpretation is a vague but widespread expression, hiding a plurality of stances on QM with a complex history [72]. We choose to present it as follows. It relies on a radical distinction between a world of inert matter that can display quantum properties *vs.* a world containing observers having subjectivities and absolute facts. In addition, the Schrödinger equation is not universally valid, due to an operation called ‘collapse’ of the quantum state which must be applied every time an observer acquires some new knowledge about a quantum system (this process being called a ‘measurement’). Actually, this operation is extremely natural: in any probabilistic model, an update of the probabilities is expected

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<sup>3</sup>The reason is that the two possible histories are equiprobable, so that the density matrix is a scalar matrix which is diagonal in any basis. Note that the problem arise rightly because of this degeneracy: in general, if all probabilities are distinct (which is a quite reasonable assumption in practice), there is only one basis such that  $\langle \mathcal{F}_{\uparrow_u} | \mathcal{F}_{\downarrow_u} \rangle = 0$ , which has to coincide with the subjectively experienced basis.

when one obtains information about which potentiality effectively happened [\[4\]](#)<sup>5</sup>. But in a theory in which ontology is not independent of epistemology, this leads to inextricable complications because all potentialities contribute to the statistics, so suppressing some of them by applying the collapse has very ontological flavour compared to the usual epistemic update in classical probability.

#### *Answer to the ontological problem*

In a way, the Copenhagen interpretation is the most economical answer to the ontological problem that one can think of. As explained in [§4.1](#), the issue with the violation of [\(3.1\)](#) appears most clearly only when entities granted with subjective experiences are included in the quantum system. Therefore, it suffices to deny any subjectivity to matter to alleviate the problem. When really an observer comes into play, whom can not be denied well-definite experiences, the probabilistic update is naturally applied to suppress all the histories which did not happen for her. And this is precisely what is needed to ensure that the statistics of any subsequent experiment performed by Wigner on his friend comply with [\(3.1\)](#) when conditioned on the observable ‘what has the friend experienced?’.

#### *Answer to the subjective preferred-basis problem*

By definition, the collapse occurs in the subjectively experienced basis each time something happens, so there is no need to precise explicitly the set of truly experienced states in the Hilbert space, as long as one merely supposes its existence, nor to invoke any link with  $\mathcal{B}_{\hat{X}}$ .

#### *Difficulties*

The main weakness is of course that the moment when a collapse should occur is ill-defined. Which systems are to be considered as observers with subjectivities, needing a collapse to enforce [\(3.1\)](#)? The latter may well be a natural ingredient of any probabilistic theory, but it normally has a purely epistemic meaning within an easy ontology. In our case, at which point exactly does ontology begins to be independent of epistemology? When do non-realized potentialities cease to contribute to statistics?

The historical primacy of the Copenhagen interpretation, the fact that its most obvious weakness lies in collapse, and that the latter was associated with the notion of measurement (physicists do come across a deep ontological problem every other day. . . ) is the reason why

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<sup>4</sup>As a trivial example, suppose for instance that you ask your computer to choose uniformly a number  $x_1$  in  $\{0; 1\}$ , and then to choose uniformly a second number  $x_2$  in  $\{0; 1\}$  if  $x_1 = 0$ , or to choose uniformly  $x_2$  in  $\{2; 3\}$  if  $x_1 = 1$ . If you don’t look at the result for  $x_1$ , you predict  $x_2$  to follow a uniform law in  $\{0; 1; 2; 3\}$ , but if you know the outcome for  $x_1$ , you obviously update your probabilities.

<sup>5</sup>The same observation was made in [\[34\]](#). It was referred to as the ‘small measurement problem’ and, precisely, not considered as a problem: ‘Not only quantum mechanics, but every probabilistic theory in which probabilities are taken to be irreducible “must have” the small measurement problem. (...) The lack of the small measurement problem in the probabilistic theories would contradict the very idea of having irreducible probabilities’.

the issue has been deemed the ‘measurement problem’, although it doesn’t have so much to do with measurements, and despite decoherence having taught us that, mathematically speaking, a measurement is nothing but a particular case of entanglement.

There is at least one other difficulty with this interpretation. Even when denying any subjectivity to most of the matter composing the universe, one can argue that an ontological problem remains. This is the stance adopted by all those who refused to shut up and stubbornly demanded through which slit has the electron passed. The long-standing primacy of the Copenhagen interpretation tells a lot about Western dominant ontology, in particular its concept of Nature as studied by Descola [44]. The naturalist ontology, a specificity of this civilization throughout human history, relies on a neat distinction between an objective and inert Nature on one side, and the human culture on the other side, sole realm of subjectivities (perhaps also including cats). Only thanks to naturalism could so many physicists not see any problem that this bump of inert matter be in superposition.

### *Falsifying experiments*

The distinctive physical process added by this interpretation to unitary QM is the collapse<sup>6</sup>. The latter is barely falsifiable even in principle for it is not rigorously defined. At least, one can say that the interpretation predicts that if Wigner could measure the spin-friend system along  $\mathcal{B} = (|\Psi\rangle, |\Phi\rangle, \dots) = \left( \frac{1}{\sqrt{2}} (|\uparrow_z \mathcal{F}_{\uparrow z}\rangle + |\downarrow_z \mathcal{F}_{\downarrow z}\rangle), \frac{1}{\sqrt{2}} (|\uparrow_z \mathcal{F}_{\uparrow z}\rangle - |\downarrow_z \mathcal{F}_{\downarrow z}\rangle), \dots \right)$ , he would obtain either  $|\Psi\rangle$  or  $|\Phi\rangle$  with probability  $\frac{1}{2}$ , and not  $|\Psi\rangle$  with certainty as predicted by unitary QM. This kind of measurements, performed on observers in a basis which does not contain the subjectively experienced states, is generally called a **supermeasurement**.

Some more realistic experiments have been performed, like the Hall *et al.* experiment [65], which has tested whether the conscious experience of an outcome is necessary to produce a collapse, and whether it can be used to communicate (expectedly, the result was negative).

## 4.3.2 Collapse models

Collapse models are the heirs of the Copenhagen interpretation: their aim is to properly define which deviation from the Schrödinger equation could lead to the collapse. The first model was proposed in 1986 by Ghirardi, Rimini, and Weber [63]. Roughly, the idea is to stipulate that random spontaneous micro-collapses occur all the time in any quantum system in the position basis so that, due to an amplification mechanism scaling with the number of particles, the effect on macroscopic systems is in practice indistinguishable from the Copenhagen collapse [14].

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<sup>6</sup>Some Copenhagen advocates consider that the collapse is not a physical process. We present it as such, however, because if one sticks to the textbook formulation of QM, the collapse indeed affects the quantum state and the statistics of subsequent experiments.

### *Answer to the ontological problem*

The answer is the same as the Copenhagen one. The ontological problem is not considered as regards particles, while systems which are known for sure to have subjective experiences are macroscopic enough to get (almost) fully collapsed. Since collapse models merely focus on giving a proper definition of the collapse, it seems that they still rely on a radical distinction between inert matter and subjective observers. It would be tempting to smooth this distinction, by postulating that the ability of physical systems to experience facts is gradually increasing with the intensity of the collapse (hence with their size). However, is it satisfactory to assert things like: ‘the system has mildly experienced one potentiality over the others, therefore all possible histories can mildly interfere’? Can the apparently binary opposition between factual *vs.* non-factual on one side, and the smoothly varying intensity of the collapse on the other side, be reconciled?

### *Answer to the subjective preferred-basis problem*

The preferred-basis is explicitly  $\mathcal{B}_{\hat{X}}$ , so that truly experienced states must be (almost) eigenvectors of  $\hat{X}$ .

### *Difficulties*

No collapse model have yet been made compatible with SR, despite numerous attempts [49, 134, 17]. The reason is that Bell’s non-locality forces the collapse mechanism to also bear some non-local features. Thanks to stochasticity, it can respect the no-FTL-communication constraint, but it still needs a preferred-reference frame to ground a notion of instantaneity and in which to write down the collapse equation. Lorenz-covariant formulations could be imagined, but without a radical change of paradigm they seem doomed to technical problems such as instability of the vacuum [99, 6.2.].

In addition, there are several suspicious facts to be remarked about collapse models. (i) Since the Copenhagen collapse is a naturally expected epistemic update to be found in any probabilistic theory, isn’t it strange to design a whole new fundamental process responsible for it? (ii) In some sense, in the Copenhagen interpretation, the collapse is used with parsimony, only when the presence of observers inevitably require it. But collapse models introduce the collapse everywhere, so that it competes with (experimentally confirmed) decoherence. What can well be the status of decoherence in a collapse model, apart from a curious redundant phenomenon which also happens to destroy very efficiently the quantum interferences?<sup>7</sup> (iii) Since the collapse is never perfect, superpositions always remain in  $\mathcal{B}_{\hat{X}}$ , although practically undetectable. How to interpret these?

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<sup>7</sup>It is instructive to observe that the master equations describing the spatial decoherence of objects by random scattering and the GRW collapse model *are the same* [77, §3.1.2.4.].

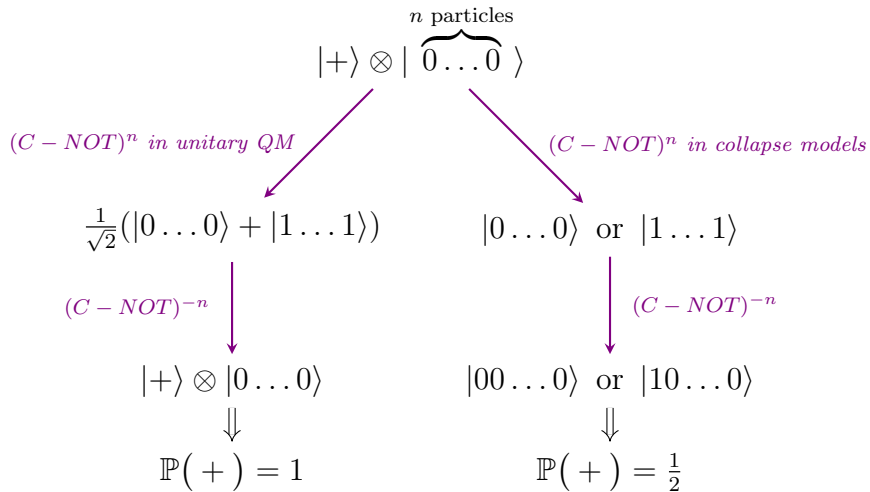


*Falsifying experiments*

Of course, any experiment falsifying the Copenhagen collapse would also falsify collapse models, but contrary to the former, the latter can be probed in realistic setups. Current experimental tests involve interferometry, cold atoms, optomechanical systems, X-ray detection, bulk heating, comparisons with cosmological observations. . . None of these have found any empirical signature of the collapse yet. Consequently, large portions of the set of possible parameters have already been ruled out [39].

Moreover, because of their incompatibility with SR, they fail to reproduce the results of many experiments involving QFT. According to Wallace, this includes ‘many of the most iconic successes of quantum mechanics [like] the quantitative account of Rayleigh scattering that explains the color of the sky’ [136].

Collapse models also predict that coherence revival is altered at some point, and becomes impossible for macroscopic objects. Therefore, entangling a particle with a large number of others would affect its statistics after reversing the operation. Schematically, if the number of particles is huge, we propose the following protocol, in which  $|0\rangle$  and  $|1\rangle$  do not necessarily denote spin states, but could also stand for a position degree of freedom, and  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ .



In practice, with fewer particles, even a partial collapse would lead to small but potentially detectable deviations from  $\mathbb{P}(+) = 1$ .

Finally, one may look for experimental situations in which the collapse would interfere with decoherence. For example, if the Hamiltonian of a system specifically entails decoherence in a basis  $\mathcal{B}$  different from  $\mathcal{B}_{\hat{X}}$  (for instance in the energy or momentum basis), the presumed collapse in  $\mathcal{B}_{\hat{X}}$  would compete with it and maintain some coherence in  $\mathcal{B}$ . It would be interesting to estimate quantitatively whether such effect can in principle be empirically detectable.

## 4.4 Bohmian mechanics

BM is the ultimate attempt to save easy ontology in quantum physics. It was proposed in 1952 by Bohm [25], pursuing an idea already explored by de Broglie two decades earlier [42]. They remarked that particles could be assigned well-defined positions and velocities after all. Provided their velocities follow the probability current defined by the wavefunction (*i.e.* the Schrödinger-evolving quantum state written in  $\mathcal{B}_{\hat{x}}$ ) instead of being given by classical mechanics, the probabilities predicted by QM are recovered. In this framework, the wavefunction can be thought of as a physically real field just as Maxwell's electromagnetic field.

### *Answer to the ontological problem*

At first sight, BM appears as the best possible solution to the ontological problem, because it restores easy ontology. The reason why (3.1) fails in general is that the wavefunction is a wave, therefore it does not have a position but instead spans the entire space. Although the electron passed through exactly one slit, the wavefunction did not, hence the probability currents emerging from both slits influence the subsequent path of the electron. If there is no ontological problem even at the level of particles, there is definitely none at larger scales either.

### *Answer to the subjective preferred-basis problem*

Quoting Bell (probably the most famous Bohmian advocate), BM relies on the crucial claim that 'in physics the only observations we must consider are position observations, if only the positions of instrument pointers. It is a great merit of the de Broglie-Bohm picture to force us to consider this fact. If you make axioms, rather than definitions and theorems, about the "measurement" of anything else, then you commit redundancy and risk inconsistency' [20]. The Hilbert space's objects should therefore exclusively be studied in  $\mathcal{B}_{\hat{x}}$ ; any other basis is physically meaningless. Truly experienced states are just spatial configurations of particles (certainly as in classical physics, although the question is not even raised in this context), whose trajectories are guided by the wavefunction. Notably, a particle has no degrees of freedom other than position; the spin is accounted for simply by treating the wavefunction as a multicomponent field.

### *Difficulties*

For the same reason as for collapse models, BM hasn't yet been made compatible with SR. In fact, the Bohmian formulation of QM is perhaps the best way to illustrate Bell's non-locality. When two particles are entangled, their trajectories are dictated by their common wavefunction in  $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{L}^2(\mathbb{R}^3)$ , so the probability current depends on two space variables. The current guiding one particle involves the actual position of the other, no matter how faraway. Thus, a preferred-reference frame is required [51] in which to apply the guiding equation. Concerning FTL-communication, it is again prevented by randomness, although

now an epistemic one (contrary to collapse-models): humans can not perfectly know initial conditions for particles. This claim can be justified based on arguments such as Heisenberg’s microscope (measuring involves interaction with the system, hence perturbs it), but these fail to explain why uncertainties precisely follow the distribution given by QM [33]. A undesirable supplementary axiom is therefore needed, referred to as the ‘quantum equilibrium hypothesis’.

Even more serious, the appealing trajectories of BM may not be as realistic as they seem [55]. In particular, a ‘detector in an interferometer (...) can become excited even when the electron passes along the other arm of the interferometer’ [45], while the detector through which the electron actually passes remains unexcited. Consequently, Bohmian trajectories do not actually represent the set of all positions where the particle would have been found, had one measured. What can they well mean, then? Besides, does the very notion of physics still makes sense if we can not trust our measurement devices? If BM indeed saves an easy ontology in the way we defined this term, it is still far from an intuitive one.

It is certainly a great merit of BM to throw away the vast majority of physically meaningless bases in  $\mathcal{H}$ . But what about  $\mathcal{B}_{\hat{p}}$ , the momentum basis? Is it really impossible to directly measure a velocity?

### *Falsifying experiments*

Wallace’s above argument is also directed to BM, which does not give good predictions as soon as QFT is needed.

We now propose another idea to test BM, based on an attempt to directly measure velocities. Consider Helling’s simple model [68] of two free particles in a box, prepared in an entangled state that is an energy eigenstate and such that the wavefunction is real. Since the formula for the probability current  $\vec{j} = \frac{\hbar}{m} \text{Im}(\bar{\Psi} \nabla \Psi)$  involves an imaginary part, it is in this case identically 0, hence the Bohmian velocities also vanish. After performing the measurement corresponding to the question ‘is particle 1 on the right side of the box?’, the wavefunction is no longer real, so the Bohmian velocities are non-zero. Helling concludes that ‘If one could observe velocities [of two entangled particles], this would violate No Signalling: by observing the velocity of particle 2, one would know if a measurement has been performed on particle 1.’ Now, if particle 2 is an excited atom about to emit a photon with well-defined frequency, measuring its Doppler shift would yield information particle 2’s velocity, wouldn’t it? If so, this experiment either falsify BM or the no-signalling constraint.

## 4.5 Unitary QM interpretations

### 4.5.1 Many-worlds interpretation

The many-worlds interpretation of QM originated in the late 1950s from the works of Everett [56]. The idea is to fully embrace unitary QM with no additional structure. The theory of decoherence, although it was developed much later, helps a lot in clarifying the interpre-

tation [118]. Ontologically, there is nothing but the quantum state, a magma of histories continuously mixing.

### *Answer to the ontological problem*

Humans, particles, or any other systems are treated on an equal footing. The reason why all potentialities interfere in the statistics of quantum experiments is because they *all happen* in some world<sup>8</sup>. When these potentialities are decohered, each specific fact concerning the system is a fact relative to an environment having recorded it [35] (which may be, for instance, an observer having experienced it). When coherence revives (like when the which-way information of an electron is erased, or when Wigner performs a supermeasurement on his friend), different branches previously split by decoherence are recombined, hence facts have a finite lifetime. At our scales, though, coherence never revives in practice because information quickly spreads into the environment where it remains recorded for long, thereby explaining why the collapse does not lead to wrong predictions for all practical purposes<sup>9</sup>.

### *Answer to the subjective preferred-basis problem*

If the many-worlds interpretation addresses very efficiently the ontological problem, it faces a thornier subjective preferred-basis problem than the previous ones. Indeed, choosing a basis  $\mathcal{B}$  of the Hilbert space amounts to choosing a particular way to tell the possible histories, and if  $\eta$  is a measure of decoherence, the value of  $\eta_{\mathcal{B}}$  quantifies to what extent these histories are split. Claiming that all potentialities happen exempts us to worry about the non-realized ones, but there is still an infinity of inequivalent ways to decompose the quantum state into a list of potentialities. Which one corresponds to what is experienced? For sure, it has to be a basis in which the observed system is perfectly decohered, because  $|\mathcal{F}_{\uparrow z}\rangle$  and  $|\mathcal{F}_{\downarrow z}\rangle$  represent distinguishable states of perception and must be orthogonal. Hence, decoherence is assigned an enormous ontological task, that of defining what truly happens in the universe.

However, as explained in 4.2, when probabilities are degenerate, it seems difficult not to rely on the additional axiom that experienced states (approximately) belong to  $\mathcal{B}_{\hat{X}}$  in order to characterize them. But this is surely an undesirable additional ingredient to QM for a many-worlds advocate. . .

### *Difficulties*

The most severe issue with this interpretation, already noticed by Everett [13], concerns the meaning of the probabilities in QM. If all potentialities happen, what does it mean to assign

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<sup>8</sup>Commonplace expressions such as ‘the particle is both here and there’, often informally used in introductory presentations of QM in the (by default) context of the Copenhagen interpretation reveal that, at the level of inert matter, the Copenhagen interpretation may tacitly rely on an unspecified, many-worlds-like ontology.

<sup>9</sup>In particular, following a measurement, a condition for the collapse to ultimately lead to wrong predictions is that nothing in the universe should remember the outcome (otherwise  $\eta$  remains 0)...but in this case the person who made the prediction is not here anymore willing to check it!

them probabilities? More precisely, Adlam has best formulated this as a problem of empirical confirmation: if one believes in the many-worlds interpretation, then no experimental result could be used to confirm or falsify QM, because we never learn anything except about the world we are in [1].

Two other weaknesses are the subjective preferred-basis problem mentioned above, and the difficulty to make sense of the unicity of the subjective experience: if all histories happen, why do I feel only one of them?

### *Falsifying experiments*

Because it adds nothing to QM, an experiment falsifies the many-world interpretation if and only if it falsifies QM. Any empirical deviation from the predictions of QM, in particular any positive result to the previously described experiments (for example, confirming the existence of an objective collapse) would do so.

## 4.5.2 Relational quantum mechanics

RQM was first proposed by Rovelli in 1996 [114], as an attempt to stick to unitary QM without postulating a plethora of worlds, which may seem very real in a interferometry experiment, but so ghostly and unscientific after decoherence. In RQM, no observable has a value in itself, but only relative to something else. Something happens every time an interaction takes place. The quantum state is the mathematical object encoding the relations (or correlations) between systems: the variable  $A$  of  $\mathcal{S}_1$  can be said in a definite state relative to  $\mathcal{S}_2$  when the latter stores the information about  $A$ , that is when  $\mathcal{S}_2$  has perfectly decohered  $\mathcal{S}_1$  in the  $\mathcal{B}_A$  basis. Here again, facts are relative and have a finite lifetime.

RQM requires a linguistic effort: it is meaningless to speak about the state of the spin, but one can speak of its state relative to the friend. Until Wigner enters the lab, the spin has no definite state relative to Wigner. Similarly, it is meaningless to ask through which slit electron has passed if it has not interacted with anything on its path.

### *Answer to the ontological problem*

Whenever someone detects a violation of (3.1), by definition the conditioning was made on an observable whose value is not a fact for anyone (otherwise it would be decohered). This is not permitted, though, for conditioning means distinguishing between possible states... but states relative to what? If a system's observable is not correlated with any other, it is normal that it can not be assumed to be in a definite state.

### *Answer to the subjective preferred-basis problem*

Like the many-worlds interpretation, RQM takes the stance of relying solely on the mathematical formalism to ground a notion of observers and facts, *via* correlations between systems in the quantum state. But, again, things are not so simple because decoherence alone can not sufficiently characterize what truly happens in the universe, at least not in the presence of

degeneracy. Moreover, in the case of low-dimensional systems, this definition is at odds with our intuitive notion of subjectivity. A qbit can indeed decohere another one with respect to any basis, but it can at most have two distinguishable (orthogonal) states of perception [36], so at most one observable of the observed qbit can be associated with states of the observing qbit having experienced a definite value for this observable.

### *Difficulties*

A major motivation for RQM is to avoid the pitfall of having multiple potentialities all realized in parallel, responsible for the problem of probabilities in the many-worlds interpretation. In RQM, when something happens relative to  $\mathcal{S}$ , it leads to a unique fact for  $\mathcal{S}$ . What could this mean, then, that the friend is in an indefinite state relative to Wigner when, in fact, she has experienced something (and only one)?

This problem can be made clearer in the light of the distinction proposed by Adlam in [4] between intrinsic condition and relative state, two different notions usually designated by the same word 'state'. The intrinsic condition stands for any objective description of a system which, in the case of an observer, determines her subjective experiences. The relative state is a mathematical object encoding the statistics of experiments that can be performed on a system at some time (deemed relative because it describes the dynamics of its interaction with another system measuring it). Adlam argues:

'it's not obvious what meaning claims about [the friend] being in an indefinite intrinsic condition "relative to [Wigner]" are supposed to have. (...) If there are "copies" of [the friend] in some other kind of condition in versions of reality defined relative to other observers, and those copies are in some sense physically real, why are those copies not also having conscious experiences of the different measurement outcomes occurring in those versions of reality? (...) Similarly, it's not straightforward to make sense of the claim that reality is composed entirely of interaction states without any intrinsic conditions at all, for interaction states are nothing more than sets of possibilities for future interactions, and it's certainly controversial to suggest that there could be a reality containing nothing other than possibilities. And it must be emphasized that all the relevant empirical content in this situation is already contained in the assertion that [the friend] has interaction state  $|\Psi\rangle$  relative to [Wigner] - further statements about what [the friend] has or has not seen relative to [Wigner] tell us nothing additional about what either [the friend] or [Wigner] will experience. So [in an approach with only one outcome per observer] this statement about [the friend] having a condition relative to [Wigner] which is different from her actual experiences seems to float free from reality, describing nothing but experiences that nobody can possibly have'. [4]

Another serious issue, which actually threatens any interpretation based on unitary QM only that denies the existence of observer-independent intrinsic conditions (as standard RQM

does), is that it does not allow for intersubjectivity (all macroscopic observers are not guaranteed to observe the same outcome), therefore it can not be scientifically confirmed or falsified [2].

Finally, RQM is subject to an infinite regress of relativisations, making it difficult to picture how anything can happen at all, and challenging the philosophical notion of relativism RQM is generally understood to be based on [113].

### *Falsifying experiments*

The statement made above for the many-worlds interpretation works also for RQM. Let's also mention another type of (thought) experiments. In the spirit of Bell's theorem, which constrains the possible hidden-variable theories if QM is universally valid, some authors have recently proposed experimental protocols that would either show a failure of QM, or constrain its admissible interpretations [121]. Unfortunately, these experiments generally involve supermeasurements, so they will almost certainly never be realized in practice. Bong *et al.* [29], for instance, show that if QM is universally valid, at least one of the three following properties must fail: (i) locality (ii) no superdeterminism (iii) absoluteness of observed events. Condition (i) is a no-FTL-communication constraint; (ii) ensures that Wigner's choice of measurement can be assumed independent of what the friend has experienced (no retrocausality); (iii) states that exactly one potentiality effectively happens, and that Wigner and his friend can communicate their experiences so as to share information. This no-go theorem is often taken as a clue that any viable interpretation which accepts the universal validity of QM, and in which only one potentiality is realized, must display some kind of radical relativism, because non-locality and superdeterminism are generally considered as very undesirable properties.

**Remark 4.5.1** (RQM + CPL). Being one of the most recent proposals for interpreting QM (also maybe one of the most philosophically demanding), and due to the renewed interest for quantum foundations, RQM is nourished by numerous critics that help clarifying it [108, 36, 47, 85].

Adlam and Rovelli's latest proposal [5], called RQM + CPL, is a substantial amendment to RQM in which an observer is now granted an intrinsic condition, along with an additional axiom (CPL for 'cross-perspective links') ensuring that observers can reach intersubjective agreement. In RQM + CPL, the potentiality experienced by the friend is an observer-independent information, not relative to Wigner, and the latter can have in principle access to it. In brief, RQM + CPL satisfies the aforementioned property of absoluteness of observed events.

This modification of RQM solves the three issues above (how can a unique subjective experience be relative to anything; the problem of scientific confirmation; the infinite regression of relativizations). In virtue of Bong *et al.*'s theorem, however, it can not satisfy both conditions (i) and (ii). Adlam shows that RQM + CPL actually violates (ii), hence features some kind of retrocausality, but in a way that does not entail any causal paradoxes, and which is best pictured as a network of events determined in an 'all-at-once' manner [4]. This is certainly a promising route to the solution of the ontological problem of QM.

# Chapter 5

## Quantum spacetime

Section §5.1 is a transcription of the submitted preprint [127].

### 5.1 Is QM compatible with special relativity?

The aim of this section is to clarify and present rigorously a lot of well-known, but sometimes poorly presented, facts about relativistic quantum theories. Most results are not new, but the perspective is. In so doing, unexpected questions and conclusions will emerge, as well as a better understanding of the interrelationships between the fundamental properties of these theories.

Standard QFT textbooks usually focus on constructing the quantum fields and the dynamics in the Fock space, but are rarely concerned with issues relative to measurements, as though it were a purely non-relativistic topic. More generally, most of them don't check the compatibility of QM with the disappearance of the notion of instantaneity in SR, and there have been surprisingly few works addressing this problem until the 2000s, when quantum information theory entered the game.

#### 5.1.1 The conditions (C) and (MC)

There are two different sources of instantaneity in QM that could cause troubles when trying to build a relativistic quantum theory. When two subsystems are entangled, they must be considered as a whole<sup>1</sup>, therefore:

- any physical evolution (for instance a measurement) performed on the first instantaneously affects the whole state no matter how far the other part may be,

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<sup>1</sup>It has been experimentally confirmed that non-local correlations are satisfied (almost) immediately: in [148], the authors present an incredibly precise experimental test which has allowed them to 'set a lower bound on the speed on quantum information to  $10^7c$ , *i.e.* seven orders of magnitude larger than the speed of light.'



- according to the standard Copenhagen interpretation (§4.3.1), any measurement performed on the first entails a collapse of the whole state, which is generally presented as an instantaneous update.

However, in view of the apparently unavoidable difficulties encountered by objective-collapse interpretations in the relativistic context, their absence of empirical confirmation, and recalling that the collapse is a naturally expected epistemic ingredient of any probabilistic theory (§4.3), the second item may not be problematic. The epistemic update of an observer having acquired new information has no reason to be constrained by the speed of light. As Bell wrote [21]: ‘When the Queen dies in London (may it long be delayed) the Prince of Wales, lecturing on modern architecture in Australia, becomes instantaneously King’, from the Queen and her entourage’s point of view. On the other hand, what has to be constrained by SR are the physically predictable effects, so that no experiment conducted on the King can determine FTL whether the Queen is alive. For this to be prevented in QM despite the non-locality of the entanglement phenomenon, the following consistency condition must hold:

**(C)** *For all quantum systems composed of two entangled subsystems, any physical evolution of the first must leave invariant **the statistical results** of any measurement on the second, if the two are spacelike separated.*

In the sequel, we will also consider the following more specific consistency condition (dealing only with measurements), that will subsequently be referred to as the condition (MC), because it is easier to manipulate mathematically and will turn out to be equivalent to (C):

**(MC)** *For all quantum systems composed of two entangled subsystems, any ideal projective measurement of the first must leave invariant **the statistical results** of any measurement on the second, if the two measurements are spacelike separated.*

The letter (C) may stand for ‘consistency condition’, but also ‘causality’ and ‘covariance’, which are two different things. Indeed, if (MC) were not satisfied (*a fortiori* (C)), the theory would face two types of inconsistencies:

- *Non-covariance* — Consider two entangled quantum systems that violate the condition (MC). Then there exists an experimental protocol concerning the second system that yields different statistical results depending on whether a certain measurement has been performed on the first system or not, such that the two measurements are spacelike separated. Thus one can find a reference frame in which the measurement on the first system happens *before* the other measurement, and another reference frame in which it happens *after*. Consequently, the statistical predictions of the theory depend on the reference frame.

- *Causal paradoxes* — Consider a reference frame in which Alice and Bob are apart, motionless, and share  $N$  entangled pairs that violate the condition (MC) as well as two synchronized clocks. At  $t = 0$ , Alice performs the suitable measurement on each of the  $N$  subsystems if she wants to communicate the bit 0, or does nothing if she wants to communicate the bit 1. At  $t = 0^+$ , Bob performs the corresponding measurement on each of the subsystems in his possession: the statistical distribution he obtains allows him to distinguish whether Alice has sent the bit 0 or 1, with an error margin arbitrarily small when  $N$  goes larger. It is well known that such FTL communications can induce causal loops like in the grandfather paradox.

The condition (MC) can be given a precise mathematical formulation. Let  $\mathcal{S}_1 + \mathcal{S}_2$  be two entangled systems described by a Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , prepared in a state  $\rho$ . Let also  $\hat{A}$  (resp.  $\hat{B}$ ) be an observable of  $\mathcal{H}_1$  (resp.  $\mathcal{H}_2$ ), and write its spectral decomposition  $\hat{A} = \sum_{x \in \text{spec}(\hat{A})} x \Pi_x^{(1)}$  (resp.  $\hat{B} = \sum_{y \in \text{spec}(\hat{B})} y \Pi_y^{(2)}$ ) where the  $\Pi_x^{(1)}$  (resp.  $\Pi_y^{(2)}$ ) are the spectral projectors of the observable. If the operators are not compact, this remains possible in the rigged Hilbert space formalism presented in Chapter 2. When an ideal measurement of  $A$  is performed on  $\mathcal{S}_1$ , the whole state evolves to  $\sum_{x \in \text{spec}(\hat{A})} (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2)$  (perfect decoherence in the eigenbasis of  $\hat{A}$ ). In this section, we will suppose for simplicity that the pair  $\mathcal{S}_1 + \mathcal{S}_2$  evolves after this interaction according to a unitary operator  $U$  (stronger conditions could be formulated for non-isolated pairs, but at least this weaker condition must hold). Since the state of  $\mathcal{S}_2$ , obtained by tracing over  $\mathcal{S}_1$ , fully encodes the probabilities of any measurement on  $\mathcal{S}_2$ , the invariance of the statistics of  $\mathcal{S}_2$  at a spacelike separated distance from the measurement on  $\mathcal{S}_1$  (as in Fig. 5.1) is expressed by:

### Mathematical formulation of (MC)

$$\text{(MC)} \quad \forall \rho, \forall U, \forall \hat{A}, \quad \text{tr}_1 \left( \sum_{x \in \text{spec}(\hat{A})} U (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2) U^\dagger \right) = \text{tr}_1 (U \rho U^\dagger).$$

Or, equivalently, due to the property (2.0.1) of the partial trace:

$$\begin{aligned} & \forall \rho, \forall U, \forall \hat{A}, \forall \hat{B}, \forall y_0 \in \text{spec} \hat{B}, \\ & \sum_{x \in \text{spec}(\hat{A})} \text{tr} \left( U (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2) U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) = \text{tr} \left( U \rho U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right). \end{aligned} \tag{5.1}$$

Concerning (C), the most general physical evolution that  $\mathcal{S}_1$  may undergo is a unitary  $V$  in  $\mathcal{H}_1 \otimes \mathcal{H}_\mathcal{E}$  (possibly non-trivial only on  $\mathcal{H}_1$ ), where  $\mathcal{E}$  stands for any external third system. It is not restrictive to consider them initially non-entangled, hence in an initial state  $\rho \otimes \rho_\mathcal{E} \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{S}(\mathcal{H}_\mathcal{E})$ , up to integrating some initial entanglement in the unitary  $V$ . Afterwards,  $\mathcal{S}_1 + \mathcal{S}_2$  evolves, as previously, according to a unitary  $U$  in  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . For the theory to be consistent, whether  $V$  has been applied or not must not modify the statistics of  $\mathcal{S}_2$  at a spacelike separated distance (as in Fig. 5.1), hence the following condition:

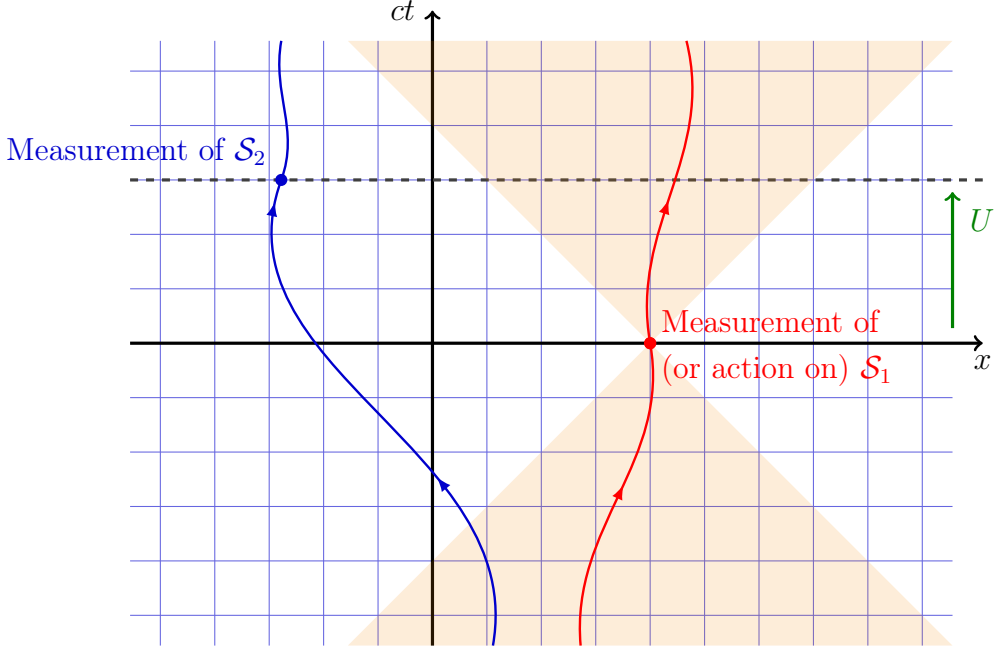


Figure 5.1: Spacelike measurements

### Mathematical formulation of (C)

$$(C) \quad \forall \rho, \forall \rho_{\mathcal{E}}, \forall U, \forall V, \quad \text{tr}_1 \left( U \text{tr}_{\mathcal{E}} \left[ (V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] U^\dagger \right) = \text{tr}_1 (U \rho U^\dagger).$$

Or, equivalently, due to the property [\(2.0.1\)](#) of the partial trace:

$$\forall \rho, \forall \rho_{\mathcal{E}}, \forall U, \forall V, \forall \hat{B}, \forall y_0 \in \text{spec} \hat{B},$$

$$\text{tr} \left( U \text{tr}_{\mathcal{E}} \left[ (V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) = \text{tr} (U \rho U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})).$$

Clearly, when  $\mathcal{E}$  is a measurement apparatus that causes perfect and immediate decoherence of  $\mathcal{S}_1$  in the eigenbasis of  $\hat{A}$ , then  $\text{tr}_{\mathcal{E}} \left[ (V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] = \sum_{x \in \text{spec}(\hat{A})} (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2)$  (recall that perfect decoherence in a given basis amounts, by definition, to extracting the diagonal part of the density matrix in this basis) and we recover the previous condition (MC). Therefore (C) **implies** (MC). As noticed in [\[16\]](#), it is equivalent to check (C) or (MC) only on initial states of the form  $\rho = \rho_1 \otimes \rho_2$  (product states).

**Remark 5.1.1. on the unavoidable probabilistic nature of QM.** In the literal formulations of (C) and (MC) given above, we have highlighted the fact that these conditions only constrain the statistical results of measurements. This is because SR only imposes the covariance of what is predictable by a given theory; and QM, being a probabilistic theory, only predicts the statistics. Similarly, the ability to transmit information depends on the

best theory available to the communicators. If they can't do better than probabilistic predictions, Alice has to be able to modify the statistical results of a repeated experiment on Bob's side in order to send him a bit.

Alternatively, if they had a deterministic (hidden-variable) theory superseding QM, only one run could suffice. Such a theory should then be constrained by a stronger version of (C) where the word 'statistical' is removed. But this is precisely forbidden by Bell's theorem [18] which implies that such a theory would necessarily display non-local features [98]. See the aforementioned [68] for a simple analytic example of non-locality in BM compared to the standard QM treatment, where it is made clear that no probabilistic knowledge other than the one given by the quantum state is compatible with SR. Quantum entanglement seemed spooky to Einstein because of the conviction that nothing in the physical universe could in principle be unpredictable, *i.e.* inaccessible to human physics<sup>2</sup> but it is actually the contrary: in a world where entanglement exists, QM is the only non-spooky theory! This remark proves that we will never be able to build a deterministic theory supplanting QM (at least not without a radical change in our physicists' paradigms). This is also why the terms added to the Schrödinger equation in collapse models have to be stochastic, 'because otherwise [they] would allow for FTL communication' [14], and why BM, despite being a hidden-variable theory reproducing QM's results, does not add any predictive power (remind §4.3.2 and §4.4).

There is another completely different reason why our physics will never be fully deterministic: the impossibility to access perfect initial conditions. First, of course, because the precision of our measurement apparatuses is limited. Second, even more importantly, because of the universal principle that *measuring a system requires to interact with it, and thereby perturbs it*. Recall that this is how Heisenberg initially justified his inequality before it became a theorem. Most people agree that the theorem formulation is much stronger but, in some sense, it is the contrary: the theorem is implied by the postulates of QM, whereas the principle is valid independently of the theory. In fact, Bohmian advocates rely precisely on this principle to justify the irreducible probabilistic nature of their theory.

Why, then, doesn't this principle affect classical physics? Until the 19th century, we hadn't empirical access to systems whose typical action is of order  $\hbar$ , for which a measurement necessarily perturbs the initial condition significantly, hence where indeterminism is present from the start. Of course, for larger systems, negligible initial uncertainties may propagate in time and stretch dramatically. But for systems like the solar system for instance, the uncertainty on the initial condition we have (much larger than  $\hbar$ !) remains for sufficiently long under the precision of our measurement apparatus, and is therefore practically undetectable. Moreover, when dealing with very chaotic systems (*i.e.* with large Lyapunov exponents, like in statistical physics) or composed of particles subject to a notable uncertainty (like photons in optics and electromagnetism), we usually restrict ourselves to well-chosen observables that become practically deterministic thanks to the law of large numbers. However, rigorously

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<sup>2</sup>This metaphysical stance is exemplified in his famous 'God does not play dice'. Note how positivist this remark is: it is not that God necessarily plays dice, but only that the human mind may not be able to access better than statistically the way God plays.

speaking, all chaotic systems amplify exponentially the initial quantum uncertainties so that their trajectories become in principle unpredictable after some time proportional to  $\log(\hbar)$  [162, 156].

One last word about the largely debated question of whether the brain has quantum properties or should be treated as a classical system [133]. Maybe this is after all of little philosophical interest compared to the question of knowing whether the brain can be considered as predictable: given the most precise initial conditions we can hope for a brain, how long will the indeterminacy remain negligible?

### 5.1.2 (F) implies (C)

Let's now check that the condition (C) (*a fortiori* (MC)) is indeed satisfied in any relativistic quantum theory provided it satisfies the following factorization property (F), usually accepted in QM:

**(F)** For all pairs of isolated systems<sup>3</sup>  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , the unitary evolution operator of  $\mathcal{S}_1 + \mathcal{S}_2$  takes the factorized form  $U = U_1 \otimes U_2$ .

**Proposition 5.1.2.** (F) implies (C).

*Proof.* Let's keep the notations introduced in §5.1.1. It is clear that the systems are isolated between the two measurements, since they are spacelike separated; therefore, assuming (F), we may write  $U = U_1 \otimes U_2$  and:

$$\begin{aligned} \mathbb{P}(B = y_0) &= \text{tr} \left( U \text{tr}_{\mathcal{E}} \left[ (V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) \\ &= \text{tr} \left( \text{tr}_{\mathcal{E}} \left[ (V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] (\mathbb{1}_1 \otimes U_2^\dagger \Pi_{y_0}^{(2)} U_2) \right) \\ &= \text{tr} \left( (V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)(\mathbb{1}_{\mathcal{E}} \otimes \mathbb{1}_1 \otimes U_2^\dagger \Pi_{y_0}^{(2)} U_2) \right) \\ &= \text{tr} \left( \rho_{\mathcal{E}} \otimes (\rho U_2^\dagger \Pi_{y_0}^{(2)} U_2) \right) \\ &= \text{tr} \left( U \rho U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) \end{aligned}$$

where we have used the fact that  $\text{tr}(\rho_{\mathcal{E}}) = 1$  in the last step, and the property (2.0.1) of the partial trace in the third step. Thus **(F) implies (C)**.  $\square$

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<sup>3</sup>We say that two systems are isolated if no particles of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  can meet each other. The possibility to write  $U$  as  $U_1 \otimes U_2$  is sometimes taken as a definition for being isolated, or we could have replaced 'isolated' by 'spacelike separated' in all this Section §5.1.1 but our choice will become clear when discussing locality in §5.1.4. Of course, the main difficulty with our definition is that the concept of particle is ill-defined. In particular, Unruh [135] has showed that the very presence of a particle is frame-dependent. Also, should we consider virtual particles? How to include gravitational interactions in our treatment?

We have therefore established that any relativistic quantum theory which satisfies (F) is consistent with an instantaneous effect of physical interactions on an entangled state, even though the notion of instantaneity is frame-dependent. Different observers with different knowledge may write different states for the entangled pair, but they agree on the statistics. What precedes proves that an entangled pair cannot be used to convey information in QM, a result known as the no-communication theorem.

**Remark 5.1.3.** In §5.1.1, we have motivated the importance of (C) by the fact that two kinds of inconsistencies appear if it were not satisfied. The computation given in 5.1.2 suffices to exclude the second kind (FTL-communication), but lacks an additional argument to get rid of the first kind (non-covariance). Indeed, it has assumed the choice of a fixed reference frame, in which the time evolutions of the systems between the instants  $t_{1i}$ ,  $t_{1f}$  and  $t_2$  (corresponding to the beginning and the end of the evolution undergone by  $\mathcal{S}_1$  and the measurement performed on  $\mathcal{S}_2$  respectively) are given by unitary operators  $V_{t_{1i}, t_{1f}}$  and  $U_{t_{1f}, t_2}$  that we have simply denoted  $V$  and  $U$ . In another reference frame, however, neither the temporal axis nor the unitary operators are conserved. Even the possible outcomes of the measurements may undergo a Lorentz transformation, if they correspond to the position or momentum observable for example. The invariance of the statistics between two reference frames  $\mathcal{R}$  and  $\mathcal{R}'$  may be written more precisely as follows (one assumes for simplicity  $t_{1i} = t_{1f} = t_1 > t_2$ , with  $t_2 < t_1$  in  $\mathcal{R}$  and denote  $t_0$  an earlier time at which the two entangled systems were separated):

$$\left\{ \begin{array}{ll} \bullet \operatorname{tr} (U_{t_0, t_2}^{(\mathcal{R})} \rho U_{t_0, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})) = \operatorname{tr} (U_{t'_0, t'_2}^{(\mathcal{R}')} \rho U_{t'_0, t'_2}^{(\mathcal{R}')\dagger} (\mathbb{1}_1 \otimes \Pi_{y'_0}^{(2)})) & \text{if } t'_2 < t'_1. \\ \bullet \operatorname{tr} (U_{t_0, t_2}^{(\mathcal{R})} \rho U_{t_0, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})) \\ = \operatorname{tr} (U_{t'_1, t'_2}^{(\mathcal{R}')} \operatorname{tr}_{\mathcal{E}} [(V_{t'_1}^{(\mathcal{R}')} \otimes \mathbb{1}_2) (\rho_{\mathcal{E}} \otimes U_{t'_0, t'_1}^{(\mathcal{R}')} \rho U_{t'_0, t'_1}^{(\mathcal{R}')\dagger}) (V_{t'_1}^{(\mathcal{R}')\dagger} \otimes \mathbb{1}_2)] U_{t'_1, t'_2}^{(\mathcal{R}')\dagger} (\mathbb{1}_1 \otimes \Pi_{y'_0}^{(2)})) & \text{if } t'_2 > t'_1 \end{array} \right.$$

The first line is simply the covariance of the theory in the absence of measurements: we suppose it already established. The only case to examine involves the changes of reference frames that reverse the temporal order of  $t_1$  and  $t_2$ . In particular, since the only potential discontinuity occur when the sign of  $t_2 - t_1$  flips, it suffices to write the second line above in the limit  $\varepsilon, \varepsilon' \rightarrow 0$  of an infinitesimal change of frame with  $t_2 = t_1 - \varepsilon$  and  $t'_2 = t'_1 + \varepsilon'$ . In that case,  $U_{t'_a, t'_b}^{(\mathcal{R}')} \rightarrow U_{t_a, t_b}^{(\mathcal{R})}$  for all instants  $t_a$  and  $t_b$ . Moreover, if  $y_0$  is a position or momentum variable,  $y'_0 \rightarrow y_0$  and the two are related by a (bijective) Lorentz transformation that does not change the multiplicity of the eigenvalues, which ensures the continuity of the spectral projectors. The condition to verify reads:

$$\begin{aligned} & \operatorname{tr} (U_{t_0, t_2}^{(\mathcal{R})} \rho U_{t_0, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})) \\ & = \operatorname{tr} (U_{t_1, t_2}^{(\mathcal{R})} \operatorname{tr}_{\mathcal{E}} [(V_{t_1}^{(\mathcal{R})} \otimes \mathbb{1}_2) (\rho_{\mathcal{E}} \otimes U_{t_0, t_1}^{(\mathcal{R})} \rho U_{t_0, t_1}^{(\mathcal{R})\dagger}) (V_{t_1}^{(\mathcal{R})\dagger} \otimes \mathbb{1}_2)] U_{t_1, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})) \end{aligned}$$

but since  $U_{t_0, t_2}^{(\mathcal{R})} = U_{t_1, t_2}^{(\mathcal{R})} U_{t_0, t_1}^{(\mathcal{R})}$ , this is nothing but the condition (C) that has been proved applied to  $U_{t_0, t_1}^{(\mathcal{R})} \rho U_{t_0, t_1}^{(\mathcal{R})\dagger}$  instead of  $\rho$ .

### 5.1.3 Previous works

The above verification of (C) is straightforward but essential to establish the compatibility between QM and SR. Of course, some previous works have already investigated these topics, although remarkably few. Bloch (1967) [24] pointed out some apparent inconsistencies and new intuitions, notably: ‘it appears that either causality or Lorentz covariance of wavefunctions must be sacrificed’. In reply, Hellwig and Kraus (1970) [69] published a paper in which they clarified Bloch’s ideas, checked a simple version (MC) (equation (6)) based on an assumption called locality (commutation of projectors associated to spacelike measurements), and then proposed that the effects of measurements should be implemented along the past light cone so that their description becomes covariant. A general proof of (MC), presented as a no-communication theorem, was detailed by Ghirardi, Rimini and Weber (1980) in [62]. The following year, Aharonov and Albert (1981) [6] explained how non-local measurements can actually be implemented based on local interactions only. As a consequence, they claim that ‘the proposal that the reduction be taken to occur covariantly along the backward light cone (...) or along *any* hypersurface other than  $t = 0$  will fail’ (although we must confess that we didn’t understand their argument), and that ‘the covariance of relativistic quantum theories (...) resides *exclusively* in the experimental probabilities, and not in the underlying quantum states. The states themselves make sense only within a given frame’. In a long footnote, Malament (1996) [91] actually proved the logical equivalence between a simple version of (MC) and the fact that spacelike measurements projectors commute, embracing a more logician perspective than his predecessors. Surprisingly enough, all these works were concerned with (MC), which is after all a very artificial statement compared to (C). The reason may be that these studies were sparked by foundational considerations on the wavefunction collapse and on the measurement problem; in addition, the theory of decoherence was still in its infancy, so that a measurement was less easily understood as a mere particular case of entangling interaction.

Then came the quantum information era. Beckman, Gottesman, Nielsen and Preskill’s article (2001) [16] was a milestone: the two systems were now called Alice and Bob’s parts, the evolution operators were renamed quantum operations, and a distinction between operations which are causal (not allowing communication) and localizable (realizable with local unitaries) was introduced. As explained in the introduction, the authors’ aim was to contribute to the long-standing problem of characterizing the set of observables in QFT, plagued by several issues such as Sorkin’s impossible measurements [126] already pointed out by Dirac [48]. The fact that (F) implies (C) is clearly stated from the beginning and the converse is also proved in finite dimension (Theorem 7), but they go well beyond and their framework is more general than ours since it allows for an ancilla system that can be sent between Alice to Bob. In particular, they solved a problem raised in [6] concerning the realizable non-local measurements. Schumacher and Westmoreland (2005) [123] exhibited three ways of expressing locality (Locality (III) being very similar to (C)), and showed their mathematical equivalence. Many works followed, and the condition (C) is now well-known in the quantum information literature (although the major reference in the domain [101] does not mention it), especially for those working on causal decomposition [88] and quantum causal

models [12]. Nonetheless, these works are not necessarily widespread among other research communities.

In the context of QFT, Polo-Gómez, Garay and Martín-Martínez (2022) [109] proved the condition (C) (equation (29)) based on the microcausality hypothesis, used to derive a relation not very different from (F) (equation (28)). They also argued that a collapse *à la* Copenhagen cannot be compatible with SR unless one accepts to define as many quantum states as there are observers, and that the update occurs along the future light cone. Contrary to the latter and in accordance with [6], we consider that, as soon as the collapse is understood as an epistemic update, hence the quantum state as observer-dependent (in particular, frame-dependent), it is pointless and even awkward to ask for frame-independent updates. Why would a given observer in her reference frame wait some time before using all the information she has, or before taking into account the evolution she knows has happened on  $\mathcal{S}_1$  to write down the whole state of  $\mathcal{S}_1 + \mathcal{S}_2$ ?

None of these works, however, have completely pursued the logician point of view that we now propose to explore in the sequel.

### 5.1.4 Locality

*(L') implies (L) and (F)*

In §5.1.2, we have seen that (F) ensures a theory to be consistent with SR. The property (F) is directly linked to a fundamental principle of physics, namely the locality hypothesis (L) which states the absence of interactions at a distance:

**(L)** *Two localized particles can interact<sup>4</sup> at a given time only if they are located at the same point in space.*

In QFT, this hypothesis has a more specific formulation:

**(L')** *The interaction Hamiltonian of any system can be written in the form  $\hat{H}_{int}(t) = \int_{\mathbb{R}^3} \hat{H}_{int}(\vec{x}, t) d^3\vec{x}$  where  $\hat{H}_{int}$  is a field of operators defined on the whole spacetime.*

Why is (L') called a locality hypothesis? Suppose one wants to compute the coefficient  $S_{\alpha\beta}$  of the  $S$ -matrix, where  $|\alpha\rangle$  is the state composed of two particles localized in  $\vec{x}_1$  and  $\vec{x}_2$ , *i.e.*

$$|\alpha\rangle = \int \frac{d^3\vec{p}_1}{(2\pi)^{3/2}} e^{-i\vec{p}_1 \cdot \vec{x}_1} a^\dagger(\vec{p}_1) \int \frac{d^3\vec{p}_2}{(2\pi)^{3/2}} e^{-i\vec{p}_2 \cdot \vec{x}_2} b^\dagger(\vec{p}_2) |0\rangle.$$

Under (L'), the Born approximation reads:  $S_{\alpha\beta} = -i \int d^4x^\mu \langle \alpha | \hat{H}_{int}(x^\mu) | \beta \rangle$ . Furthermore, even without specifying  $\hat{H}_{int}(x^\mu)$ , one knows by covariance that the latter can only be built

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<sup>4</sup>Again, as in note [3], this term can be confusing. One can think of possible interactions as allowed vertices in Feynman diagrams, but what about the gravitational interaction? If the latter is to be backgroundless, then there is no pre-existing spacetime in which to define a notion of locality...



from the different quantum fields of the theory, and that if an interaction is possible it necessarily contains two quantum fields of the type of the particles of the state  $|\alpha\rangle$ , and so includes at least an expression of the form  $\int \frac{d^3\vec{p}}{(2\pi)^{3/2}} e^{-ip_\mu x^\mu} u(\vec{p}) a^\dagger(\vec{p}) \int \frac{d^3\vec{p}'}{(2\pi)^{3/2}} e^{-ip'_\mu x^\mu} v(\vec{p}') b^\dagger(\vec{p}')$  where  $u$  and  $v$  are objects that depend on the nature of the quantum fields. Using the (anti-)commutation relation for the operators  $a$  and  $b$ , one finds:

$$\langle\alpha|\hat{H}_{int}(x^\mu)|\beta\rangle = \iint d^3\vec{p}_1 d^3\vec{p}_2 [\dots] e^{i\vec{p}_1 \cdot \vec{x}_1} e^{i\vec{p}_2 \cdot \vec{x}_2} e^{-i(p_1+p_2)_\mu x^\mu} \propto \delta^{(3)}(\vec{x}-\vec{x}_1) \delta^{(3)}(\vec{x}-\vec{x}_2) \propto \delta^{(3)}(\vec{x}_1-\vec{x}_2).$$

This means that two localized particles can interact at a given time only if they are located at the same point in space, namely (L). The electromagnetic force between two charged particles, for example, is an interaction ‘at a distance’ only because (virtual) photons are exchanged between them. Furthermore, under (L), two isolated<sup>5</sup> systems  $\mathcal{S}_1$  and  $\mathcal{S}_2$  can not interact, therefore the total Hamiltonian reads  $H = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$  with  $H_1$  and  $H_2$  the internal Hamiltonians of  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Therefore, the unitary evolution operator of  $\mathcal{S}_1 + \mathcal{S}_2$  is  $U = e^{itH} = e^{itH_1} \otimes e^{itH_2} = U_1 \otimes U_2$ . We have just shown that **(L’)** implies **(L)** which implies **(F)**.

*(F) implies (L) and justifies (L’)*

Conversely, suppose that (F) is true. Then, if  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are two isolated systems, the total Hamiltonian of  $\mathcal{S}_1 + \mathcal{S}_2$  reads:

$$\hat{H} = \frac{1}{i} \frac{dU}{dt} = \frac{1}{i} \frac{dU_1 \otimes U_2}{dt} \Big|_{t=0} = \frac{1}{i} \frac{dU_1}{dt} \Big|_{t=0} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \frac{1}{i} \frac{dU_2}{dt} \Big|_{t=0} = \hat{H}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{H}_2 \Rightarrow \hat{H}_{int} = 0.$$

In particular, if  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are two particles localized at different points in space (hence isolated, recall note [3](#)), they do not interact. Hence **(F)** implies **(L)**. Now, it is natural to imagine (this is not a proof!) that in a theory satisfying (L), the interaction Hamiltonian of an arbitrary system should be composed of a combination of local operators of the form  $\int_{\mathcal{D}} \hat{H}_{int}(\vec{x}, t)$  where  $\mathcal{D}$  is an integration domain *a priori* unknown. However, to preserve the translation invariance of the laws of physics,  $\mathcal{D}$  must be the whole space  $\mathbb{R}^3$ . Thus **(L)** justifies the locality hypothesis in QFT **(L’)**.

*(MC) implies (F)*

We need our theories to satisfy (C). In [§5.1.2](#), we have shown that (F)  $\Rightarrow$  (C) and remarked that (F) is generally postulated in QM. But, unlike (F), (C) is a consistency condition directly required by physical considerations, so it seems after all more natural to postulate the latter. One can then try to determine the set of unitaries compatible with (C), which would yield a constraint on any unitary evolution operator associated to a bipartite system  $\mathcal{S}_1 + \mathcal{S}_2$  with  $\mathcal{S}_1$  and  $\mathcal{S}_2$  spacelike separated. It turns out that the more restrictive condition (MC) suffices to

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<sup>5</sup>Recall that we defined ‘isolated’ as the fact that no particles of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  meet each other.

deduce (F), as shown in the following theorem. The same result was already proved in [I6, Theorem 7] using a completely different argument, also restricted to the finite dimensional case.

**Theorem 5.1.4** ((MC)  $\Rightarrow$  (F)). *Let  $U(t)$  be the unitary evolution operator, expressed in a fixed reference frame  $\mathcal{R}$ , of a quantum system  $\mathcal{S}_1 + \mathcal{S}_2$  composed of two isolated subsystems  $\mathcal{S}_1$  and  $\mathcal{S}_2$  of finite dimension, described in a consistent relativistic quantum theory. Then there exist two unitary operators  $U_1(t)$  and  $U_2(t)$  such that:*

$$U(t) = U_1(t) \otimes U_2(t)$$

*Proof.* The two subsystems being isolated, it is possible to divide the temporal axis of  $\mathcal{R}$  into small time intervals such that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are spacelike separated<sup>6</sup> during any of these time intervals. On each interval, the theory satisfies (in particular) the condition (MC) by consistency, and it suffices to show there the factorization result. From now on, for the sake of clarity, we will not write the parameter  $t$  which plays no role in the proof anymore. Let's denote  $\mathcal{H}_1$  and  $\mathcal{H}_2$  the Hilbert spaces associated with  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , and assume that they are finite dimensional with  $n_1 = \dim(\mathcal{H}_1)$  and  $n_2 = \dim(\mathcal{H}_2)$ . One can write  $U$  in the following generic form:

$$U = \sum_{\substack{i \\ 1 \leq k, l \leq n_2}} \alpha_{ikl} T_i \otimes |k\rangle \langle l| \quad (5.2)$$

with  $(|k\rangle \langle l|)_{1 \leq k, l \leq n_2}$  the canonical basis of  $\mathcal{L}(\mathcal{H}_2)$  associated with an orthonormal basis  $(|k\rangle)_{1 \leq k \leq n_2}$ , and  $(T_i)_i$  a basis of  $\mathcal{L}(\mathcal{H}_1)$ . When replacing  $U$  in the expression (5.1) for the condition (MC), one gets for all Hermitian operators  $\hat{A}$  and  $\hat{B}$  of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  and for all  $y_0 \in \text{spec}(\hat{B})$ :

$$\begin{aligned} & \forall \rho, \quad \text{tr} \left( \rho \sum_{\substack{i, k, l \\ j, k', l'}} \left[ \sum_{x \in \text{spec}(\hat{A})} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \alpha_{ikl} \overline{\alpha_{jk'l'}} |l'\rangle \langle k'| \Pi_{y_0}^{(2)} |k\rangle \langle l| \right) = 0 \\ \Rightarrow & \sum_{\substack{i, k, l \\ j, k', l'}} \left[ \sum_{x \in \text{spec}(\hat{A})} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \alpha_{ikl} \overline{\alpha_{jk'l'}} |l'\rangle \langle k'| \Pi_{y_0}^{(2)} |k\rangle \langle l| = 0. \end{aligned}$$

Note that  $\hat{B}$  may be chosen arbitrarily, and in particular  $\Pi_{y_0}^{(2)}$ . For any pair  $\{k_1, k_2\} \subset \llbracket 1, n_2 \rrbracket$  and  $\mu, \nu \in \mathbb{C}$  such that  $|\mu|^2 + |\nu|^2 = 1$ , one can define  $\Pi_{y_0}^{(2)}$  to be the projector on the vector  $\mu |k_1\rangle + \nu |k_2\rangle$ , that is  $|\mu|^2 |k_1\rangle \langle k_1| + \mu \bar{\nu} |k_1\rangle \langle k_2| + \bar{\mu} \nu |k_2\rangle \langle k_1| + |\nu|^2 |k_2\rangle \langle k_2|$ . Inserting into the previous equation divides it in four sums:

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<sup>6</sup>Meaning that every point of the spacetime region spanned by  $\mathcal{S}_1$  is spacelike separated to every point of the spacetime region spanned by  $\mathcal{S}_2$ .

$$\forall k_1, k_2, \sum_{\substack{i,l \\ j,l'}} \left[ \sum_{x \in \text{spec}(\hat{A})} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \\ \otimes \left[ |\mu|^2 \alpha_{ik_1 l} \overline{\alpha_{jk_1 l'}} + \mu \bar{\nu} \alpha_{ik_2 l} \overline{\alpha_{jk_1 l'}} + \bar{\mu} \nu \alpha_{ik_1 l} \overline{\alpha_{jk_2 l'}} + |\nu|^2 \alpha_{ik_2 l} \overline{\alpha_{jk_2 l'}} \right] |l'\rangle \langle l| = 0.$$

The particular cases  $\mu = 1, \nu = 0$  or  $\mu = 0, \nu = 1$  imply that the first and fourth terms actually always vanish. Setting  $\mu = \nu = \frac{1}{\sqrt{2}}$  or  $\mu = \frac{1}{\sqrt{2}}, \nu = \frac{i}{\sqrt{2}}$  leads to:

$$\left\{ \begin{array}{l} \sum_{\substack{i,l \\ j,l'}} \left[ \sum_{x \in \text{spec}(\hat{A})} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \left[ \frac{1}{2} \alpha_{ik_2 l} \overline{\alpha_{jk_1 l'}} + \frac{1}{2} \alpha_{ik_1 l} \overline{\alpha_{jk_2 l'}} \right] |l'\rangle \langle l| = 0 \\ \sum_{\substack{i,l \\ j,l'}} \left[ \sum_{x \in \text{spec}(\hat{A})} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \left[ -\frac{i}{2} \alpha_{ik_2 l} \overline{\alpha_{jk_1 l'}} + \frac{i}{2} \alpha_{ik_1 l} \overline{\alpha_{jk_2 l'}} \right] |l'\rangle \langle l| = 0 \end{array} \right.$$

and taking appropriate linear combinations of these shows that the second and third terms vanish as well. Therefore:

$$\forall k, k', \sum_{\substack{i,l \\ j,l'}} \alpha_{ikl} \overline{\alpha_{jk'l'}} \left[ \sum_{x \in \text{spec}(\hat{A})} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes |l'\rangle \langle l| = 0 \\ \Rightarrow \forall k, k', l, l', \sum_{i,j} \alpha_{ikl} \overline{\alpha_{jk'l'}} \left[ \sum_{x \in \text{spec}(\hat{A})} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] = 0$$

because  $(|l'\rangle \langle l|)_{1 \leq l, l' \leq n_2}$  is a basis of  $\mathcal{L}(\mathcal{H}_2)$ . This being true for all Hermitian operators  $\hat{A}$ , the operator  $\sum_{i,j} \alpha_{ikl} \overline{\alpha_{jk'l'}} T_j^\dagger T_i = (\sum_i \alpha_{ik'l'} T_i)^\dagger (\sum_i \alpha_{ikl} T_i)$  is diagonal in every orthonormal bases of  $\mathcal{H}_1$ , so it is a dilation:

$$\forall k, k', l, l', \exists \lambda_{kk'l'l'} \in \mathbb{C} : \left( \sum_i \alpha_{ik'l'} T_i \right)^\dagger \left( \sum_i \alpha_{ikl} T_i \right) = \lambda_{kk'l'l'} \mathbb{1}.$$

Now, there exist  $k_0$  and  $l_0$  such that  $\lambda_{k_0 k_0 l_0 l_0} \neq 0$ . Otherwise, for all  $k$  and  $l$  we would have  $\sum_i \alpha_{ikl} T_i = 0$  (since  $\mathcal{L}(\mathcal{H}_2)$  is a  $C^*$ -algebra satisfying  $\|X^\dagger X\| = \|X\|^2$  for all  $X \in \mathcal{L}(\mathcal{H}_2)$ ), and by linear independence of the  $(T_i)_i$  this would imply that all the  $\alpha_{ikl}$  vanish *i.e.*  $U = 0$ , which is not possible.

At present, pick any pair of indices  $k, l$ .

- If  $\lambda_{kk_0 l l_0} \neq 0$ , by unicity of the inverse in finite dimension, we have  $\frac{1}{\lambda_{kk_0 l l_0}} \sum_i \alpha_{ikl} T_i = \frac{1}{\lambda_{k_0 k_0 l_0 l_0}} \sum_i \alpha_{ik_0 l_0} T_i$  and since the  $(T_i)_i$  are linearly independent:  $\alpha_{ikl} = \frac{\lambda_{kk_0 l l_0}}{\lambda_{k_0 k_0 l_0 l_0}} \alpha_{ik_0 l_0}$ .
- If  $\lambda_{kk_0 l l_0} = 0$ , since  $\sum_i \alpha_{ik_0 l_0} T_i$  is invertible, we deduce that  $\sum_i \alpha_{ikl} T_i = 0$ , hence  $\alpha_{ikl} = 0$  for all  $i$ .

In any case:

$$\forall k, l, \exists \beta_{kl} \in \mathbb{C} : \forall i, \quad \alpha_{ikl} = \beta_{kl} \alpha_{ik_0l_0}.$$

It is now possible to factorize:

$$U = \sum_{\substack{i \\ 1 \leq k, l \leq n_2}} \beta_{kl} \alpha_{ik_0l_0} T_i \otimes |k\rangle \langle l| = \left( \sum_i \alpha_{ik_0l_0} T_i \right) \otimes \left( \sum_{1 \leq k, l \leq n_2} \beta_{kl} |k\rangle \langle l| \right) = U_1 \otimes U_2$$

where one can identify  $U_1$  and  $U_2$  with the evolution operators of  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , which are necessarily unitary since  $U = U_1 \otimes U_2$  is. □

A consequence of this theorem is that (F), (C) and (MC) imply themselves circularly, therefore **(F), (C) and (MC) are logically equivalent**. Another unexpected corollary is the following: while (C) and (MC) do not seem *a priori* to be symmetrical with respect to  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , their equivalence to (F) implies such a symmetry. Said differently,  $\mathcal{S}_1$  does not allow to communicate to  $\mathcal{S}_2$  if and only if the converse is true.

### 5.1.5 Microcausality

In addition to the locality hypothesis, another deep postulate of QFT is the microcausality hypothesis (M).

**(M)** *For all quantum fields  $\Phi$  and spacelike intervals  $x - y$ ,  $[\Phi(x), \Phi^\dagger(y)]_\pm = 0$  where  $[\ , ]_\pm$  stands for an anti-commutator or a commutator depending on the fermionic or bosonic nature of  $\Phi$ .*

Usually, standard QFT textbooks justify this hypothesis by invoking, for once, the concept of measurement [107, p.28] [74, p.106] [149, p.121], but they generally make do with the affirmation that two spacelike measurements must be independent, without more explanations. Not only is the argument too vague, but it is hard to see how the relation could differ according to the fermionic or bosonic nature of the field. Weinberg, on the other hand, makes an interesting remark:

‘The condition [(M)] is often described as a causality condition, because if  $x - y$  is spacelike then no signal can reach  $y$  from  $x$ , so that a measurement of  $\Phi$  at point  $x$  should not be able to interfere with a measurement of  $\Phi$  or  $\Phi^\dagger$  at point  $y$ . Such considerations of causality are plausible for the electromagnetic field, any one of whose components may be measured at a given spacetime point, as shown in a classic paper of Bohr and Rosenfeld [27]. However, we will be dealing here with fields like the Dirac field of the electron that do not seem in any sense measurable. The point of view taken here is that [(M)] is needed for the Lorentz invariance of the S-matrix, without any ancillary assumptions about measurability or causality.’ [137, p.198]

As Weinberg notes himself, microcausality is only a sufficient condition for the invariance of the  $S$ -matrix:

‘Theories of this class [satisfying (M)] are not the only ones that are Lorentz invariant, but the most general Lorentz invariant theories are not very different. In particular, there is always a commutation condition something like [(M)] that needs to be satisfied. This condition has no counterpart for non-relativistic systems, for which time-ordering is always Galilean-invariant. *It is this condition that makes the combination of Lorentz invariance and QM so restrictive.*’ [137, p.145]

However, Weinberg’s argument strongly relies on the use of normal-ordered fields. Arguably, this writing is only a computation convenience without physical meaning. Its main purpose is to get rid of the infinite constants that appear in certain computations, by making finite all the matrix elements of the operators manipulated. This operation is justified when the divergences have no influence in the considered context. Alternatively, the prediction of the Casimir effect or the Lamb shift by the vacuum energy is only possible *without* normal-ordering (one substantially uses the fact that  $\langle 0|\Phi^2|0\rangle \neq 0$  for a non-ordered  $\Phi$ , see [74, p.111]). It seems that a stronger argument is needed to justify the microcausality hypothesis (M). Note that (M) is especially crucial to prove the famous spin-statistics theorem (S):

(S) *Scalar and vector fields correspond to bosons, while Dirac fields correspond to fermions.*

Here is how most QFT textbooks proceed to establish (S). Since we don’t know the result yet, we have to compute both commutators and anti-commutators. A short calculation first shows that when  $\Phi$  is a Dirac field, we have:

$$[\Phi(x), \Phi^\dagger(y)]_\pm = \Delta_+(x - y) \mp \Delta_+(y - x) \quad (5.3)$$

depending on  $\Phi$  describing fermions or bosons, *i.e.*  $[a(\vec{p}), a^\dagger(\vec{p}')]_\pm = \delta^{(3)}(\vec{p} - \vec{p}')$ , and where  $\Delta_+(x) = \int \frac{d^3p}{(2\pi)^3 2p_0} e^{ip_\mu x^\mu}$  is shown to be a Lorentz invariant quantity. Likewise, when  $\Phi$  is a scalar or a vector field:

$$[\Phi(x), \Phi^\dagger(y)]_\pm = \Delta_+(x - y) \pm \Delta_+(y - x) \quad (5.4)$$

depending on  $\Phi$  describing fermions or bosons. It is quite easy to see that when  $x - y$  is spacelike,  $\Delta_+(x - y) - \Delta_+(y - x) = 0$ , while  $\Delta_+(x - y) + \Delta_+(y - x)$  is not identically zero [137, p.202]. All these remarks, in addition to (M), imply (S). Not the reverse: these computations generally don’t justify (M) but only the fermionic or bosonic nature of a field, whereas one can intuitively feel that they contain much more information.

We now present a simultaneous proof of the microcausality hypothesis and the spin-statistics theorem that relies only on the consistency of the theory and on the above (anti-)commutators computations.

**Theorem 5.1.5** ((MC)  $\Rightarrow$  (M) and (S)). *Let  $\Phi$  be a quantum field. Then, for all spacelike intervals  $x - y$ ,  $[\Phi(x), \Phi^\dagger(y)]_\pm = 0$  where  $[\ , \ ]_\pm$  stands for an anti-commutator or a commutator depending on  $\Phi$  being a Dirac field (and in that case it describes a fermion) or a scalar or vector field (and in that case it describes a boson).*

*Proof.* Let's first suppose that  $\Phi$  is effectively a measurable field, in other words  $\Phi(x)$  is Hermitian for all  $x$ . As stated by Weinberg in the previous quote, it is for example the case for the electromagnetic field that describes the photon. One can write its spectral decomposition  $\Phi(x) = \sum_{\lambda \in \text{spec } \Phi(x)} \lambda \Pi_\lambda^{(x)}$  if  $\Phi(x)$  is compact, or use in general the generalized spectral decomposition (2.2) of the rigged Hilbert space formalism. For the sake of clarity, we still denote the decomposition as in the compact case. Contrary to the previous sections, there is now only one Hilbert space, the Fock space  $\mathcal{H}_{Fock}$ , and the system's state is given by a density matrix  $\rho \in \mathcal{S}(\mathcal{H}_{Fock})$ . For all spacelike intervals  $x - y$ , a measurement of  $\Phi(x)$  doesn't affect the statistics of a measurement of  $\Phi(y)$  if and only if the following condition, variant of (MC), is satisfied:

$$\forall \rho, \forall \mu \in \text{spec } \Phi(y), \quad \text{tr} \left( \left( \sum_{\lambda \in \text{spec } \Phi(x)} \Pi_\lambda^{(x)} \rho \Pi_\lambda^{(x)} \right) \Pi_\mu^{(y)} \right) = \text{tr}(\rho \Pi_\mu^{(y)})$$

where we implicitly moved to a reference frame  $\mathcal{R}$  in which  $x^0 = y^0$ , so as to avoid to introduce the (non-covariant) unitary evolution operators<sup>7</sup>. It yields:

$$\begin{aligned} \forall \rho, \forall \mu \in \text{spec } \Phi(y), \quad & \text{tr} \left( \rho \left( \sum_{\lambda \in \text{spec } \Phi(x)} \Pi_\lambda^{(x)} \Pi_\mu^{(y)} \Pi_\lambda^{(x)} - \Pi_\mu^{(y)} \right) \right) = 0 \\ \Rightarrow \forall \mu \in \text{spec } \Phi(y), \quad & \sum_{\lambda \in \text{spec } \Phi(x)} \Pi_\lambda^{(x)} \Pi_\mu^{(y)} \Pi_\lambda^{(x)} = \Pi_\mu^{(y)} \\ \Rightarrow \sum_{\lambda \in \text{spec } \Phi(x)} \Pi_\lambda^{(x)} \Phi(y) \Pi_\lambda^{(x)} = \Phi(y). \end{aligned}$$

Thus,  $\Phi(y)$  is (block) diagonal in the eigenbasis of  $\Phi(x)$ , so they are codiagonalizable and  $[\Phi(x), \Phi(y)] = 0$ . This relation *a priori* holds in the frame  $\mathcal{R}$ , but when applying the appropriate representation of the Lorentz group under which  $\Phi$  transforms, one sees that  $\Phi(x)$  and  $\Phi(y)$  commute in all reference frames.

If now  $\Phi$  is not supposed to be Hermitian anymore, we still know that  $\Phi\Phi^\dagger$  is. Applying what precedes to  $\Phi\Phi^\dagger$  instead of  $\Phi$ , we obtain that for all  $x - y$  spacelike,  $[\Phi(x)\Phi^\dagger(x), \Phi(y)\Phi^\dagger(y)] = 0$ . Moreover, as  $\Phi$  is not Hermitian, one shows as usual that for all  $x$  and  $y$ ,  $[\Phi(x), \Phi(y)] =$

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<sup>7</sup>We are aware that there still doesn't exist a proper and consensual mathematical framework for the implementation of measurements in QFT [43]. This is probably why the same computation leads to oddities if one doesn't move to the special reference frame  $\mathcal{R}$  beforehand. Still, we reckon that this simple reasoning actually captures something of physical interest.

$[\Phi^\dagger(x), \Phi^\dagger(y)] = 0$  <sup>8</sup>. But on the other hand, it is also possible to compute the commutator  $[\Phi(x)\Phi^\dagger(x), \Phi(y)\Phi^\dagger(y)]$  using the commutation relations (5.3) and (5.4). Suppose for instance that  $\Phi$  is a Dirac field. Since we don't know yet if it is a boson or a fermion, let's distinguish the possible cases:

- if the particle described by  $\Phi$  is a fermion, we have (5.3) with the upper signs. Then:

$$\begin{aligned}
\Phi(x) \underbrace{\Phi^\dagger(x)\Phi(y)}_{\substack{=-\Phi(y)\Phi^\dagger(x) \\ -\Delta_+(x-y) \\ +\Delta_+(y-x)}} \Phi^\dagger(y) &= -\Phi(x)\Phi(y)\Phi^\dagger(x)\Phi^\dagger(y) + \left( \Delta_+(y-x) - \Delta_+(x-y) \right) \Phi(x)\Phi^\dagger(y) \\
&= -\Phi(y) \underbrace{\Phi(x)\Phi^\dagger(y)}_{\substack{=-\Phi^\dagger(y)\Phi(x) \\ +\Delta_+(x-y) \\ -\Delta_+(y-x)}} \Phi^\dagger(x) + \left( \Delta_+(y-x) - \Delta_+(x-y) \right) \Phi(x)\Phi^\dagger(y) \\
&= \Phi(y)\Phi^\dagger(y)\Phi(x)\Phi^\dagger(x) \\
&\quad + \underbrace{\left( \Delta_+(y-x) - \Delta_+(x-y) \right)}_{=0 \text{ if } x-y \text{ spacelike}} (\Phi(y)\Phi^\dagger(x) + \Phi(x)\Phi^\dagger(y)),
\end{aligned}$$

- if the particle described by  $\Phi$  is a boson, we have (5.3) with the lower signs. Then

$$\begin{aligned}
\Phi(x) \underbrace{\Phi^\dagger(x)\Phi(y)}_{\substack{=\Phi(y)\Phi^\dagger(x) \\ -\Delta_+(x-y) \\ -\Delta_+(y-x)}} \Phi^\dagger(y) &= \Phi(x)\Phi(y)\Phi^\dagger(x)\Phi^\dagger(y) - \left( \Delta_+(x-y) + \Delta_+(y-x) \right) \Phi(x)\Phi^\dagger(y) \\
&= \Phi(y) \underbrace{\Phi(x)\Phi^\dagger(y)}_{\substack{=\Phi^\dagger(y)\Phi(x) \\ +\Delta_+(x-y) \\ +\Delta_+(y-x)}} \Phi^\dagger(x) - \left( \Delta_+(x-y) + \Delta_+(y-x) \right) \Phi(x)\Phi^\dagger(y) \\
&= \Phi(y)\Phi^\dagger(y)\Phi(x)\Phi^\dagger(x) \\
&\quad + \underbrace{\left( \Delta_+(x-y) + \Delta_+(y-x) \right)}_{\text{non-identically zero if } x-y \text{ spacelike}} (\Phi(y)\Phi^\dagger(x) - \Phi(x)\Phi^\dagger(y)).
\end{aligned}$$

As a consequence, one recovers the commutation relation  $[\Phi(x)\Phi^\dagger(x), \Phi(y)\Phi^\dagger(y)] = 0$  imposed by the condition (MC) if, and only if,  $\Phi$  is a fermionic field and in this case, the relation (5.3) implies that  $\{\Phi(x), \Phi^\dagger(y)\} = 0$  for all spacelike intervals  $x-y$ . These are indeed the statements (M) and (S). When  $\Phi$  is a scalar or vector field, the proof is similar.  $\square$

---

<sup>8</sup>This is because the only commutators that can appear between a creation and an annihilation operator are  $[a(\vec{p}), a^{c\dagger}(\vec{p}')] ]$  or  $[a^c(\vec{p}), a^\dagger(\vec{p}')] ]$  (with the label  $c$  standing for the antiparticle) which are zero if  $a(\vec{p}) \neq a(\vec{p})^c$ .

**Remark 5.1.6.** Note that this proof makes use, as usually in physics, of the identification between the notion of observable in the mathematical sense (Hermitian operator) and in the physical sense (quantity measurable by a concrete experimental protocol). However, the second is far more restrictive, since in practice we can only measure a few very specific observables. Rigorously speaking, only the physical notion of observable is constrained by (MC), since the latter must ensure the absence of inconsistencies between *actual* measurements. Therefore, in the above proof, although  $\Phi\Phi^\dagger$  is Hermitian, one could question the legitimacy of imposing it (MC). Of course, any mathematical observable could be in principle measured by applying a suitable unitary evolution to the system that would map its eigenbasis to the eigenbasis of a physically measurable observable. But is it satisfactory to rely on the idea that all unitary evolutions are *a priori* feasible, even though we will never be able to implement them? Nonetheless, it is still possible to adapt the proof by replacing  $\Phi\Phi^\dagger$  by an undoubtedly physical observable, such that a function of the components of  $\hat{T}_{\mu\nu}$  or even the charge  $\hat{Q}$ , that one can express in terms of  $\Phi$ . For example, for a Dirac field,  $\hat{T}_\mu^\mu = m\Phi^\dagger\gamma^0\Phi$ , would allow a quite similar proof.

What precedes has an unexpected consequence, expressed in the corollary below.

**Corollary 5.1.7.** *A fermionic field is not measurable.*

*Proof.* In the previous proof, we have seen that if a field is an observable (*i.e.*  $\Phi = \Phi^\dagger$ ), then the condition (MC) implies that for all  $x - y$  spacelike,  $[\Phi(x), \Phi(y)] = 0$ . But if  $\Phi$  is a fermionic field, it also satisfies  $\{\Phi(x), \Phi(y)\} = 0$ . Adding these two relations yields for all  $x - y$  spacelike,  $\Phi(x)\Phi(y) = 0$ , which is too strong a constraint for a quantum field. In particular,  $\Phi(x)\Phi(y)|0\rangle$  is the state containing two localized particles at  $x$  and  $y$ ; in any case it is a non-zero vector of the Fock space<sup>9</sup>.  $\square$

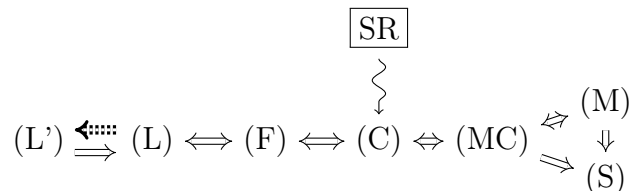
## 5.1.6 Summary

The starting point of this section was the apparent incompatibility between SR and two kinds of instantaneities that seem to appear in QM. By the time it was historically developed, quantum theory was not built to integrate SR, so that it is always surprising to contemplate their ‘peaceful coexistence’ (expression coined by Shimony [124]). Exploring this coexistence led us to investigate the logical interrelationships between fundamental properties in relativistic QM. They are summarized in the following diagram:

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<sup>9</sup>This result is not new, and can also be derived, for instance, by the fact that the equation  $\Phi = \Phi^\dagger$  is neither covariant nor independent of the representation of the gamma matrices [104], so it can’t be linked to a physically meaningful property such as being measurable. But it is interesting to see that it can be directly derived from considerations about measurements. Again, our argument relies on the widespread identification between the notions of mathematical and physical observable. Here,  $\Phi$  may indeed be Hermitian, but is it a physically measurable quantity subject to the condition (MC)?





where the dotted arrow stands for ‘justifies’ rather than ‘implies’. The implication (M)  $\Rightarrow$  (MC) is not proved in this paper but in the aforementioned [109]. Let’s emphasize again that (C) (*a fortiori* (MC)) is not an arbitrary postulate but a mere consistency criterion that must be valid for any relativistic quantum theory: it ‘costs nothing’ to be assumed because it stems from the constraints of SR. Surprisingly, it becomes obvious on this diagram that the two fundamental postulates (L’) and (M) of QFT are actually redundant, since *locality implies microcausality*.

## 5.2 A toy model: how the shape of the laws of physics promote a backgroundless preferred-basis

### 5.2.1 Motivation and preliminaries

In §4.2, we have explained how the eigenbasis  $\mathcal{B}_{\hat{X}}$  of the position operator seems to be a preferred-basis for decoherence, especially when the systems considered grow in size, constraining the observables compatible with an easy ontology. The explanation usually given in the decoherence literature is that the laws of physics, and therefore the interaction Hamiltonians, often involve position variables [155, 5.] [119, III.E.2.]. But in view of building a backgroundless quantum theory, one may wish to better understand which properties of the Hamiltonian are required to see the emergence of a decoherence basis, common to all systems, in the full abstract Hilbert space. The hope, in the end, is to be able to identify it to a position basis, thereby defining a backgroundless notion of position, or at least some backgroundless common structure linked to spacetime properties.

Here is a proposal to formalize this. Our starting point will be a total Hilbert space of the form  $\mathcal{H}_{tot} = \otimes_{i=1}^N \mathcal{H}$ , where all copies of  $\mathcal{H}$  stand for the same type of degree of freedom, but relative to different systems (*e.g.* particles)<sup>10</sup>. For any total state  $|\Psi\rangle \in \mathcal{H}_{tot}$ , one can compute the density matrix  $\rho_i = \text{tr}_{[1,N]\setminus\{i\}}(|\Psi\rangle\langle\Psi|)$  of the  $i^{\text{th}}$  system obtained by tracing out all the other ones. This state is perfectly decohered by the others in the eigenbasis  $\mathcal{B}_i$  of  $\rho_i$ . The question becomes: at some given instant, what can be said about the set

$$\{|\Psi\rangle \in \mathcal{H}_{tot} \mid \text{the } \mathcal{B}_i \text{'s are ‘close’ to each other}\}$$

<sup>10</sup>Of course, this factorization of the Hilbert space relates to one of the great unsolved questions of the theory of decoherence: what are the systems in QM? As Zurek writes: ‘the problem of measurement cannot be stated without a recognition of the existence of systems. Therefore, our appeal to the same assumption for its resolution is no sin. However, a compelling explanation of what are the systems — how to define them given, say, the overall Hamiltonian in some suitably large Hilbert space — would be undoubtedly most useful.’

and what is required in the form of the Hamiltonian governing the evolution of  $|\Psi\rangle$  to remain in this set? Determining this is perhaps reminiscent of the question, raised in §3.2.5, of grasping the set of reachable (or physically reasonable) states.

One difficulty immediately arises. How to give some meaning to the  $\mathcal{B}_i$  being close to each other? Indeed, if the  $\mathcal{H}$ 's are really considered abstractly, nothing allows to relate  $\mathcal{B}_i$  and  $\mathcal{B}_j$  and claim, for instance, that  $\mathcal{B}_i$  and  $\mathcal{B}_j$  are (or are not) the same basis. As an example, von Neumann has indeed proved that all the possible realizations of a separable Hilbert space equipped with two operators such that  $[\hat{X}, \hat{P}] = i\hbar\mathbb{1}$  are unitarily equivalent. But when several particles are considered, we actually assume the existence of a common background space  $\mathbb{R}^3$  and tacitly endow the Hilbert space  $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{L}^2(\mathbb{R}^3)$  with some additional structure, so as to say that  $|x\rangle_1$  and  $|x\rangle_2$  represent the *same* point in space. The problem is already present when considering finite dimensional spaces of spin, where we must be able to say that  $|\uparrow_x\rangle_1$  and  $|\uparrow_x\rangle_2$  are oriented along the same axis in the physical space (see [34, footnotes 7 and 8]). This issue of a required additional background structure in the Hilbert space of QM may be a direction of inquiry in the search of a backgroundless quantum theory. For now, however, we will do as always, and assume tacitly that our blocks  $\mathcal{H}$ 's are synchronized so that they can be compared.

This way, we can naturally use the distance  $d(\mathcal{B}_i, \mathcal{B}_j)$  between orthonormal bases, previously introduced in §3.3.3, in order to give a meaning to  $\mathcal{B}_i$  being close to  $\mathcal{B}_j$ . Notably, the map  $B : |\Psi\rangle \mapsto (\mathcal{B}_1, \dots, \mathcal{B}_N)$  is well behaved with respect to this distance.

**Proposition 5.2.1.** *Suppose that  $\mathcal{H}_{tot}$  is finite dimensional. Let  $\mathcal{U} = \{|\Psi\rangle \in \mathcal{H}_{tot} \mid \forall i \in \llbracket 1, N \rrbracket, \rho_i \text{ has a non-degenerate spectrum}\}$  an open dense subset of  $\mathcal{H}_{tot}$  on which the map*

$$\begin{aligned} B : \mathcal{U} &\longrightarrow \mathfrak{B}(\mathcal{H})^N \\ |\Psi\rangle &\longmapsto (\mathcal{B}_1, \dots, \mathcal{B}_N) \end{aligned}$$

*is unambiguously defined.  $B$  is continuous on  $\mathcal{U}$  for the topology induced by  $d$ .*

*Proof.* We have  $\mathcal{U} = \bigcap_{i=1}^N \mathcal{U}_i$ , where  $\mathcal{U}_i = \{|\Psi\rangle \in \mathcal{H}_{tot} \mid \rho_i \text{ has a non-degenerate spectrum}\}$ . In short, each  $\mathcal{U}_i$  is an open dense subset because it is defined by an algebraic equation in the coefficients of  $|\Psi\rangle \in \mathcal{H}_{tot}$  (the discriminant of the characteristic polynomial of  $\rho_i$  must not vanish). Therefore,  $\mathcal{U}$  is open dense as well.

Let  $(|\Psi_n\rangle)_{n \in \mathbb{N}}$  a sequence of vectors such that  $|\Psi_n\rangle \xrightarrow{n \rightarrow +\infty} |\Psi\rangle$ . By continuity of the partial trace, each  $\rho_{i,n} = \text{tr}_{\llbracket 1, N \rrbracket \setminus \{i\}}(|\Psi_n\rangle \langle \Psi_n|)$  converges to  $\rho_i$  when  $n \rightarrow +\infty$ , hence  $[\rho_i, \rho_{i,n}] \xrightarrow{n \rightarrow +\infty} 0$ . Indeed,  $\|\rho_i \rho_{i,n} - \rho_{i,n} \rho_i\| = \|\rho_i(\rho_{i,n} - \rho_i) - (\rho_{i,n} - \rho_i)\rho_i\| \leq 2\|\rho_i\| \|\rho_{i,n} - \rho_i\| \xrightarrow{n \rightarrow +\infty} 0$ . Finally, using the second inequality of Proposition 3.3.5,  $d(\mathcal{B}_i, \mathcal{B}_{\rho_{i,n}}) \xrightarrow{n \rightarrow +\infty} 0$ .  $\square$

## 5.2.2 Description of the model

We are now in position to build the model. We will modestly restrict ourselves to  $N = 2$  (only two blocks in  $\mathcal{H}_{tot}$  *i.e.* two interacting particles) and consider the simplest possible

case  $\mathcal{H} = \mathbb{C}^2$ , so that the model is analytically solvable. Consider a Hamiltonian of the form:

$$\hat{H} = \left( \alpha \hat{S}_X + \beta \hat{S}_Y \right) \otimes \hat{S}_Z,$$

for some positive constants  $\alpha$  and  $\beta$  to be modulated later. The spin operators don't have to really represent a spin; they are actually taken as a toy model for any set of conjugate operators on some abstract degree of freedom (although, for the sake of clarity, we will still use the spin and particles vocabulary). Denote  $\mathcal{B}_x$ ,  $\mathcal{B}_y$ , and  $\mathcal{B}_z$  their respective eigenbases.

Our aim is to observe the competition between two decohering processes. The effect of the part weighted by  $\alpha$  is to rotate particle 2 along the  $z$  axis clockwise or counterclockwise depending on the spin of particle 1 along  $x$  being  $\uparrow$  or  $\downarrow$ . Qualitatively, particle 2 registers the  $x$ -spin of particle 1, entailing decoherence of the latter in  $\mathcal{B}_x$ . The part weighted by  $\beta$  has a similar effect in  $\mathcal{B}_y$ . For some generic initial state of the pair, we will compute the evolving state  $|\Psi(t)\rangle$ , then the first particle's density matrix  $\rho_1(t)$  and its eigenbasis  $\mathcal{B}_1(t)$ , and finally the distances  $d(\mathcal{B}_1(t), \mathcal{B}_x)$ ,  $d(\mathcal{B}_1(t), \mathcal{B}_y)$  and  $d(\mathcal{B}_1(t), \mathcal{B}_z)$ , in order to check how far particle 1 is to be decohered in the  $x$ ,  $y$  and  $z$  bases. The main question we would like to elucidate is how the values of  $\alpha$  and  $\beta$  (the strength of the recording of  $S_X$  and  $S_Y$ , standing for 'the form of the laws of physics') influence  $\mathcal{B}_1(t)$ . In particular, we would expect to have  $\mathcal{B}_1(t) \xrightarrow{\alpha/\beta \rightarrow +\infty} \mathcal{B}_x$  and  $\mathcal{B}_1(t) \xrightarrow{\alpha/\beta \rightarrow 0} \mathcal{B}_y$ .

### 5.2.3 Analytic resolution

The model is simple enough to derive an analytic expression for  $\rho_1(t)$ . By default,  $|\uparrow\rangle$  and  $|\downarrow\rangle$  stand for the eigenvectors of  $\hat{S}_Z$ . A subscript is added to denote an eigenvector of another spin operator, for instance  $|\uparrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ . The factors  $\frac{\hbar}{2}$  can be absorbed in  $\alpha$  and  $\beta$ , so that  $\hat{S}_X$ ,  $\hat{S}_Y$  and  $\hat{S}_Z$  are nothing but the Pauli matrices. It seems reasonable to start from an initial product state, which we suppose of the form  $|\Psi(0)\rangle = (c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle) \otimes |\uparrow\rangle_x$ . This choice is generic in particle 1 (the particle of interest), whereas we are not really concerned for now with the influence of the initial state of particle 2 (the recording particle), as long as the latter is not  $|\uparrow\rangle$  or  $|\downarrow\rangle$ , in which case not entanglement ever occur.

**Proposition 5.2.2.** *Starting from an initial state of the form  $|\Psi(0)\rangle = (c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle) \otimes |\uparrow\rangle_x$ , particle 1's state at time  $t$  is given by:*

$$\rho_1(t) = \begin{pmatrix} \frac{1}{2} [1 - (|c_\downarrow|^2 - |c_\uparrow|^2) \cos(2\Omega t)] & \frac{1}{\alpha+i\beta} [\Re(z) + i\Im(z) \cos(2\Omega t)] \\ \frac{1}{\alpha+i\beta} [\Re(z) - i\Im(z) \cos(2\Omega t)] & \frac{1}{2} [1 + (|c_\downarrow|^2 - |c_\uparrow|^2) \cos(2\Omega t)] \end{pmatrix}$$

where  $\Omega = \sqrt{\alpha^2 + \beta^2}$  and  $z = \overline{c_\downarrow} c_\uparrow (\alpha + i\beta)$ .

*Proof.* Let's first diagonalize  $\hat{H}$ . In the  $(|\uparrow\rangle, |\downarrow\rangle)$  basis,  $\alpha \hat{S}_X + \beta \hat{S}_Y = \begin{pmatrix} 0 & \alpha - i\beta \\ \alpha + i\beta & 0 \end{pmatrix}$ , whose eigenvectors are

$$|E_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{-\Omega}{\alpha+i\beta} \\ 1 \end{pmatrix} \quad \text{and} \quad |E_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\Omega}{\alpha+i\beta} \\ 1 \end{pmatrix}$$

associated to the eigenvalues  $E_1 = -\Omega$  and  $E_2 = \Omega$ . The initial state takes the form:

$$|\Psi(0)\rangle = (c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle) \otimes |\uparrow\rangle_x = (\lambda_1 |E_1\rangle + \lambda_2 |E_2\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle),$$

where the coefficients in the new basis are given by  $\lambda_1 = \frac{1}{\sqrt{2}}(-\frac{\alpha+i\beta}{\Omega}c_\uparrow + c_\downarrow)$  and  $\lambda_2 = \frac{1}{\sqrt{2}}(\frac{\alpha+i\beta}{\Omega}c_\uparrow + c_\downarrow)$ . The Hamiltonian is diagonal in the  $(|E_1 \uparrow\rangle, |E_1 \downarrow\rangle, |E_2 \uparrow\rangle, |E_2 \downarrow\rangle)$  basis, in which the time evolution is simply:

$$\begin{aligned} |\Psi(t)\rangle &= \frac{\lambda_1}{\sqrt{2}}e^{i\Omega t} |E_1 \uparrow\rangle + \frac{\lambda_1}{\sqrt{2}}e^{-i\Omega t} |E_1 \downarrow\rangle + \frac{\lambda_2}{\sqrt{2}}e^{-i\Omega t} |E_2 \uparrow\rangle + \frac{\lambda_2}{\sqrt{2}}e^{i\Omega t} |E_2 \downarrow\rangle \\ &= \frac{\Omega}{\alpha + i\beta} \left( \frac{-\lambda_1}{2}e^{i\Omega t} + \frac{\lambda_2}{2}e^{-i\Omega t} \right) |\uparrow\uparrow\rangle + \left( \frac{\lambda_1}{2}e^{i\Omega t} + \frac{\lambda_2}{2}e^{-i\Omega t} \right) |\downarrow\downarrow\rangle \\ &\quad + \frac{\Omega}{\alpha + i\beta} \left( \frac{-\lambda_1}{2}e^{-i\Omega t} + \frac{\lambda_2}{2}e^{i\Omega t} \right) |\uparrow\downarrow\rangle + \left( \frac{\lambda_1}{2}e^{-i\Omega t} + \frac{\lambda_2}{2}e^{i\Omega t} \right) |\downarrow\uparrow\rangle. \end{aligned}$$

Now, to compute the partial trace of this state with respect to particle 2, we use the fact that if  $|\Phi\rangle = a|\uparrow\uparrow\rangle + b|\downarrow\uparrow\rangle + c|\uparrow\downarrow\rangle + d|\downarrow\downarrow\rangle$  for some  $a, b, c, d \in \mathbb{C}$ , then  $\text{tr}_2(|\Phi\rangle\langle\Phi|) = \begin{pmatrix} |a|^2 + |c|^2 & a\bar{b} + c\bar{d} \\ \bar{a}b + \bar{c}d & |b|^2 + |d|^2 \end{pmatrix}$ , which can be seen from the computation made in [3.1](#). Here, in brief,

$$\begin{aligned} |a|^2 + |c|^2 &= \left| \frac{-\lambda_1}{2}e^{i\Omega t} + \frac{\lambda_2}{2}e^{-i\Omega t} \right|^2 + \left| \frac{-\lambda_1}{2}e^{-i\Omega t} + \frac{\lambda_2}{2}e^{i\Omega t} \right|^2 \\ &= \frac{1}{2} (|\lambda_1|^2 + |\lambda_2|^2 - \bar{\lambda}_1\lambda_2 \cos(2\Omega t) - \lambda_1\bar{\lambda}_2 \cos(2\Omega t)) \\ &= \frac{1}{2} - \Re(\lambda_1\bar{\lambda}_2) \cos(2\Omega t) \\ &= \frac{1}{2} [1 - (|c_\downarrow|^2 - |c_\uparrow|^2) \cos(2\Omega t)] \end{aligned}$$

yields the first component of  $\rho_1(t)$ . Similarly,

$$\begin{aligned} a\bar{b} + c\bar{d} &= \frac{\Omega}{4(\alpha + i\beta)} (-\lambda_1 e^{i\Omega t} + \lambda_2 e^{-i\Omega t}) (\bar{\lambda}_1 e^{-i\Omega t} + \bar{\lambda}_2 e^{i\Omega t}) \\ &\quad + \frac{\Omega}{4(\alpha + i\beta)} (-\lambda_1 e^{-i\Omega t} + \lambda_2 e^{i\Omega t}) (\bar{\lambda}_1 e^{i\Omega t} + \bar{\lambda}_2 e^{-i\Omega t}) \\ &= \frac{\Omega}{2(\alpha + i\beta)} [|\lambda_2|^2 - |\lambda_1|^2 - 2i\Im(\lambda_1\bar{\lambda}_2) \cos(2\Omega t)] \\ &= \frac{1}{\alpha + i\beta} [\Re(\bar{c}_\downarrow c_\uparrow (\alpha + i\beta)) + i\Im(\bar{c}_\downarrow c_\uparrow (\alpha + i\beta)) \cos(2\Omega t)]. \end{aligned}$$

The two remaining components are deduced from the fact that  $\rho_1(t)$  is Hermitian and have trace 1.  $\square$

## 5.2.4 Numerical simulation and discussion

It would be extremely tedious to write down exact expressions for  $\mathcal{B}_1(t)$ , not to mention its distances to  $\mathcal{B}_x$ ,  $\mathcal{B}_y$ , and  $\mathcal{B}_z$ , but they can easily be computed numerically. Here are the behaviours for different values of  $c_\uparrow$  and  $c_\downarrow$  as well as, more importantly, of  $\alpha$  and  $\beta$ .

*The case  $\alpha = \beta = 1$*

Let's first check how the choice of the initial state influences the results. It turns out that if  $c_\uparrow$  or  $c_\downarrow$  equals 0,  $\mathcal{B}_1(t) = \mathcal{B}_z$  for all  $t$  (which was not obvious *a priori*), and that the simulation is very unstable for  $|c_\uparrow| = |c_\downarrow| = \frac{1}{\sqrt{2}}$  (certainly due to the non-continuity of the eigenvectors with respect to the coefficients of a matrix near some critical points). Keeping away from these extreme cases, here are a few examples (figure 5.2). Obviously, we observe a periodic behaviour. The period is not  $\frac{\pi}{\Omega}$  (as would suggest Proposition 5.2.2) but  $\frac{\pi}{2\Omega}$ , certainly because the cosine contribute only through its square in the final result.

*The case  $\frac{\alpha}{\beta}$  or  $\frac{\beta}{\alpha} \rightarrow +\infty$*

At present, let's keep  $\beta = 1$  and increase  $\alpha$ . As hoped, decoherence seems to prefer  $\mathcal{B}_x$  when the coupling in  $x$  increases. But even a small initial phase can alter this process. Sometimes,  $\mathcal{B}_x$  is not even preferred (figure 5.3). The conclusions are the same when  $\beta \rightarrow +\infty$  while  $\alpha = 1$ .

### Discussion

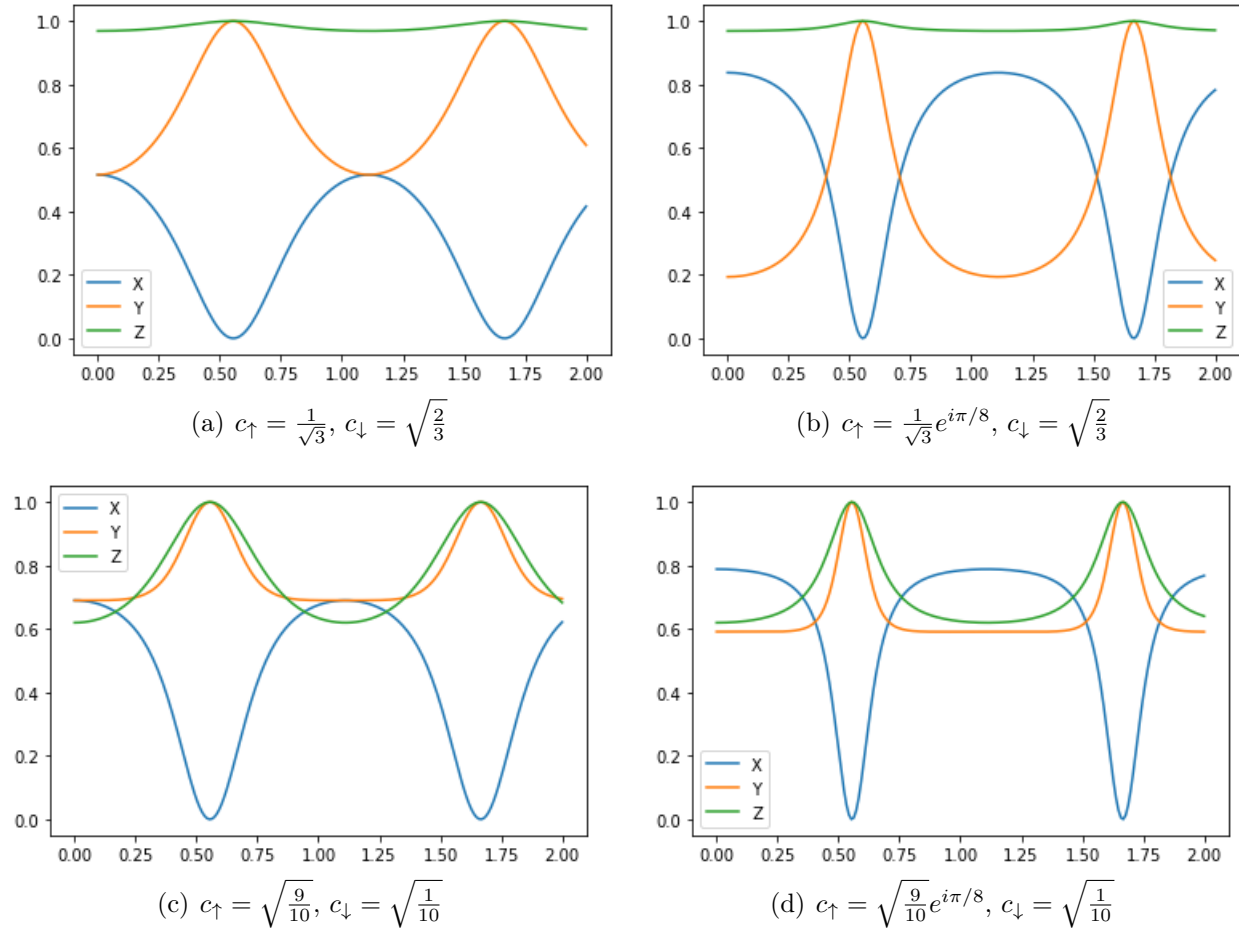
Clearly, more work is needed to understand the mechanisms at stake. However, a first glance at the simulations lead us to the formulate the following conjecture:

**Conjecture 5.2.3.** *For any orthonormal basis  $\mathcal{B}$ ,*

$$d(\mathcal{B}, \mathcal{B}_x)^2 + d(\mathcal{B}, \mathcal{B}_y)^2 + d(\mathcal{B}, \mathcal{B}_z)^2 = 2.$$

Although the model is still simplistic, simple extensions could considerably enhance its relevance. There are two main directions to explore:

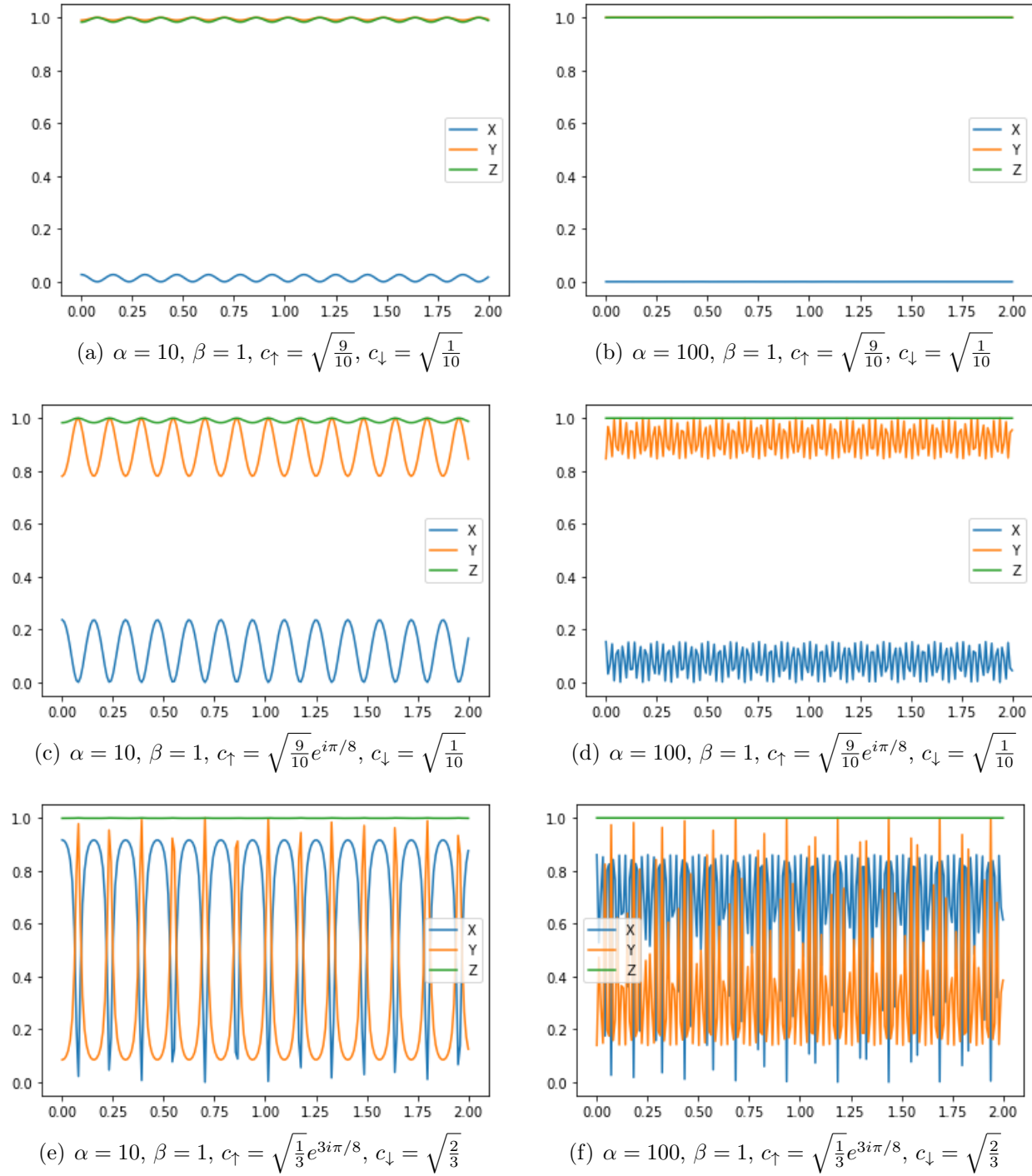
- Periodicity is a major weakness. It is unavoidable in dimension 2 with Pauli matrices because the latter are traceless, hence the two frequencies are the same ( $|E_1| = |E_2|$ ). Already in dimension 3 the problem would disappear. Considering an infinite dimensional space would be even more interesting physically. Indeed, the problem with spin operators is that the recording particle rotates clockwise or counterclockwise depending on the state of the other, but it soon arrives back in its initial angle so the recording quickly erases itself.
- Our initial motivation was to study the conditions of emergence of common preferred-basis. So far, we have only considered the basis  $\mathcal{B}_1(t)$  of the first particle. The next step is to compare  $\mathcal{B}_1(t)$  and  $\mathcal{B}_2(t)$  and, more generally, all the  $(\mathcal{B}_i(t))_{1 \leq i \leq N}$  for  $N$  interacting systems.



**Figure 5.2:** Distances to  $\mathcal{B}_x$ ,  $\mathcal{B}_y$ , and  $\mathcal{B}_z$  for  $\alpha = \beta = 1$  and different values of the initial state.

More difficult long-term goals include, among others:

- how to also recover the parameter  $t$  in a backgroundless manner?
- Which principle(s) should characterize the Hamiltonian in  $\mathcal{H}_{tot}$ ? Since we standardly use spacetime to define locality (as done in §5.1.4), could a backgroundless formulation of locality allow to operationally reconstruct spacetime? See for instance [8] for an abstract theorem linking causality, unitarity and locality.



**Figure 5.3:** Distances to  $\mathcal{B}_x, \mathcal{B}_y,$  and  $\mathcal{B}_z$  for different initial states,  $\beta = 1$  and  $\alpha \rightarrow +\infty$ .

# Chapter 6

## Conclusion

The aim of this thesis was to better understand the features of quantum theory that makes quantum gravity so hard to reach. How come Einstein was able to formulate a backgroundless theory of classical matter as soon as 1915, while no one has been able to do so for quantum matter in more than a century?

Einstein's theories of relativity are deemed relativistic because lengths and durations are relative to the observer. GR is in some sense even more relativistic — as opposed to absolute — than SR because it is a backgroundless theory in which spacetime is not independent of what happens, but co-emerges with matter. However, there is still some kind of residual absoluteness in GR. Mind that Einstein's principle of relativity is the requirement of the invariance of the laws of physics in all admissible reference frames: shouldn't it rather be called the principle of absoluteness? Indeed, the idea behind the use of differential geometry is that all physically meaningful geometrical objects (proper times, light cones, metric and tensor fields, curvature, spacetime itself. . .) are intrinsic. They 'are there', common to everyone, independently of how particular observers relate to it and label spacetime events by means of coordinates<sup>1</sup>. Under some additional assumptions, the Einstein-Hilbert action is then the only acceptable functional build from intrinsic quantities only; therefore, the matter side of Einstein's equation has to be an intrinsic geometrical object too. Said differently GR presupposes an easy ontology for the energy-momentum observable, as we defined this term in Chapter 4, but we have rightly shown that there can't be such an ontology for this quantity in QM, unless  $\hat{T}_{\mu,\nu}$  is perfectly decohered<sup>2</sup>.

A good understanding of the theory of decoherence is a crucial step to grasp the ontological problem of quantum physics. In Chapter 3, we have presented the main mathematical patterns of decoherence. Two models were designed to feel intuitively the general processes at stake with a high level of generality, in particular this was achieved without specifying

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<sup>1</sup>Provided there is at least some matter there, *cf.* the hole argument mentioned in the introduction.

<sup>2</sup>Another major challenge for quantum gravity is the following: when coherence revives in  $\mathcal{B}_{\hat{x}}$ , *i.e.* when several well-localized possible histories are decohered and then interfere again, how to stick together the different spacetimes associated to these potentialities? How to say that the event  $x$  in possible spacetime 1 corresponds to the event  $x'$  in possible spacetime 2?



any Hamiltonian. The main lesson of the first model (§3.2) is that decoherence is a high-dimensional geometrical phenomenon between reservoir of dimensions, whose efficiency stems from the fact that exponentially many points can be placed randomly on a  $n$ -sphere before having the maximum of the scalar products becoming non-negligible. The second model (§3.4) gave a quite universal reason why QM indeed predicts that the universe should not be frozen by the Zeno effect. Namely, since quantum systems' short time evolutions are always quadratic, and this being true for the pair system + environment as well, infinitesimal steps of decoherence induced on a system by its surrounding environment are likely to be of the form  $\eta_n = 1 - \frac{\alpha}{n^\beta}$  with  $\beta \gtrsim 2$ , which we have shown not to entail any Zeno freezing. An important lesson of the Zeno effect is that things are able to evolve because they are intermittently isolated<sup>3</sup>. More precisely, there is a competition between free evolution (typically governed by  $\hat{P}$ ) striving to escape decoherence in  $\mathcal{B}_{\hat{X}}$  induced by the interaction with the rest of the universe (which typically acquires information about  $\hat{X}$ ). Different candidates to quantify decoherence were then proposed and studied (§3.3); they all satisfy the key property of bounding the deviation from the total probability formula (Definition 3.3.4).

The (empirical) fact that our universe does not fundamentally obey the total probability formula is the root of the ontological problem raised by quantum physics, the so-called 'measurement problem'. In Chapter 4, we have characterized it as the impossibility to build an easy ontology for quantum systems, that is an ontology independent of epistemology, in which variables can be assumed to have a definite value among the possible ones, even when unknown. In QM, indeed, all potentialities seem to contribute to the statistics of subsequent experiments, except when the conditioning is made on a decohered observable. The theory of decoherence is sufficient to explain why we almost never experience any deviation from the total probability formula: the way we get information about a system is governed by the same laws of physics as the interactions it has with its environment, hence we only distinguish between potentialities that have already been rendered ontologically easy by the rest of the world. However, QM still needs an ontology, albeit an uneasy one. Facts, observers, consciousness... are notions often encountered in the debates on the interpretation of QM. Our investigation has shown how they enter the discussion; in particular, we have argued that observers are best defined as systems granted with subjectivity, because the ontological problem is most severe and unavoidable on the latter. After giving a precise formulation of the preferred-basis problem (§4.2), we then moved on to study how five of the most famous interpretations of QM attempt to solve these issues (§4.3 to §4.5). We believe that this novel approach (defining first the problem in an interpretation-independent way) might be a valuable way to grasp what is really at stake with each of these interpretations, in order to better compare them. According to Rovelli [16], just as nearly 150 years passed between Copernicus's *De Revolutionibus* and Newton's *Principia*, during which 'the old picture of reality was reshaped in depth (...) we are [again] in the middle of an unfinished scientific revolution', waiting in particular for science to incorporate the philosophical lessons of QM

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<sup>3</sup>Paradoxically, it is also because a system interacts with an environment that it experiences change and time. Indeed, its free Hamiltonian  $\hat{H} = \hat{\gamma}mc^2 \simeq mc^2\mathbb{1} + \frac{\hat{P}^2}{2m}$  becomes in its proper reference frame  $\hat{H} = mc^2\mathbb{1}$ , yielding no evolution.

and to overcome its attachments to easy ontology.

Equipped with a precise formulation of the ontological problem of QM in the light of the theory of decoherence, Chapter 5 could at last introduce the last character of this thesis: spacetime. A first topic of investigation, although not immediately related to the ontological problem, was the apparent incompatibility between QM and SR and their surprisingly ‘peaceful coexistence’. Adopting a logicist point of view has led us to derive several logical interrelationships between fundamental properties in relativistic QM, namely: the no-signalling/covariance constraint (and its restriction to the particular case of ideal measurements), the factorization property of the unitary evolution operator, locality, microcausality and the spin-statistics theorem (§5.1). Finally, we have presented a general idea for the quest of a backgroundless quantum theory. It actually turns the objective preferred-basis problem defined in §4.2 into a force. The aim is to determine what is required, in the shape of the laws of physics, to see the emergence of a common decoherence basis. Our hope, in the end, is to be able to identify it to a position basis, thereby defining a backgroundless notion of position, or at least some backgroundless common structure linked to spacetime properties. These ideas were illustrated by a toy model, for now simplistic, but whose further extensions will considerably enhance its relevance (§5.2). An essential ingredient in this program is the distance between orthonormal bases introduced in §3.3.3: this mathematical tool may be a step towards the (still mysterious) geometrical description of what takes place inside a tensor product of Hilbert spaces.

We close this thesis with some general considerations on the notion of backgroundless co-emergence. Many philosophers were opposed to Newton’s absolute space and time. Among them, Leibniz, who wrote in a letter to Clarke in 1716:

‘I have many demonstrations, to confute the fancy of those who take space to be a substance, or at least an absolute being. (...) I say then, that if space was an absolute being, there would something happen, for which it would be impossible there would be a sufficient reason. Which is against my axiom. And I prove it thus. Space is something absolutely uniform; and, without the things placed in it, one point of space does not absolutely differ in any respect whatsoever from another point of space. Now from hence it follows, (supposing space to be something in itself, besides the order of bodies among themselves,) that it is impossible there would be a reason, why God, preserving the same situations of bodies among themselves, would have placed them in space after one certain particular manner, and not otherwise; why everything was not placed the quite contrary way, for instance, by changing East into West. But if space is nothing else, but that order or relation; and is nothing at all without bodies, but the possibility of placing them; then those two states, the one such as it now is, the other supposed to be the quite contrary way, would not at all differ from one another. Their difference therefore is only to be found in our chimerical supposition of the reality of space in itself. But in truth the one would exactly be the same thing as the other, they being absolutely indiscernible; and consequently

there is no room to enquire after a reason of the preference of the one to the other.’  
[40, p.57-59]

Another famous opponent of the absoluteness of spacetime is Mach, whose works were a great source of inspiration for Einstein. Interestingly, Newton’s and Mach’s conceptions of spacetime are empirically distinguishable in principle (although the experiment is clearly not realizable in practice) [73]. Suppose you have a bucket of water at rest. Now, move the entire universe around (!) except the bucket. According to Newton, whatever happens to the rest of the universe does not change the fact that the bucket is still at rest (or in uniform translation) with respect to the absolute spacetime, so the surface of the water remains flat. But according to Mach and his generalized law of inertia defined with respect to the center of mass of the universe [89, Chapter II, Section VI], the surface of water would curve. It seems that our best current physical theories predict no such effect and therefore still have a quite Newtonian flavour.

Apart from GR, what other cases of backgroundless co-creation do we have, from which to gain some inspiration? What mechanisms can mutually constrain several things enough to give them a shape without any pre-established form? Here is a non-detailed and non-exhaustive list of examples.

- In physics, one can think of the Page-Wooters mechanism to define backgroundless times [103, 145];
- there is also mutual decoherence between all systems in the universe, in particular the objective preferred-basis mentioned in §4.2, backgroundlessly defining the observables compatible with an easy ontology;
- as argued in note 2 from Chapter 4, there have been a co-evolution between the modes of perception and the laws of physics. The same is true for the laws of mathematics or logic: life on Earth has evolved so as to grasp regularities that did not pre-exist independently of these beings.
- Darwinian evolution, in general, provides numerous examples of backgroundless co-evolution paths between species: it is meaningless to talk about the genetic evolution of a single species independently of its environment, and its selected mutations influence in turn the environment’s own evolution;
- philosophy of ecology often invites us to think in terms of dynamically retroacting ecosystems. See Naess’ Gestalt ontology [100], or the concept of earthy in the work of Latour [84] and his criticism of the ‘view from nowhere’ in science;
- anarchy is a backgroundless political organization without a State or a Constitution: decisions and rules continuously emerge from the collective discussions and are always subject to change.

- Bouddhist and Taoist canonical texts emphasize that each dualism is actually a backgroundless co-creation from the human intellect: a notion of good cannot exist without a notion of bad, nothing is long if nothing is short, just as the left defines the right and vice-versa;
- language co-emerges with the world: the former is a network of meanings supposed to fit the latter, which in turn shapes our very experience of it [143];
- finally, if there is no absolute self, gender (or in general any qualia associated with a person) emerges backgroundlessly at the interaction between two or more persons, possibly at the scale of a whole society, or even of humanity.

Is there something common to all of these examples?



# Appendix A

## Some philosophical notions

Here is some background intended to make Chapter [4](#) more accessible to readers unfamiliar with the philosophy of QM.

### A.1 Ontology and epistemology

The terms ‘ontology’ and ‘epistemology’ come from the Greek words *on* and *epistēmē*, which can be respectively translated as ‘being’ (or ‘that which is’) and ‘knowledge’ (or ‘understanding’). According to the Stanford Encyclopedia of Philosophy: ‘we have at least two parts to the overall philosophical project of ontology, on our preliminary understanding of it: first, say what there is, what exists, what the stuff of reality is made out of, secondly, say what the most general features and relations of these things are’ [\[71\]](#). One could propose that the aim of ontology is to give some meaning to the verb ‘to exist’ by including it into a proper language game [\[142\]](#).

On the other hand, epistemology lies at the interface between reality and its knowers. ‘In different parts of its extensive history, different facets of epistemology have attracted attention. Plato’s epistemology was an attempt to understand what it was to know, and how knowledge (unlike mere true opinion) is good for the knower. Locke’s epistemology was an attempt to understand the operations of human understanding, Kant’s epistemology was an attempt to understand the conditions of the possibility of human understanding, and Russell’s epistemology was an attempt to understand how modern science could be justified by appeal to sensory experience. (...) In all these cases, epistemology seeks to understand one or another kind of *cognitive success*’ [\[131\]](#).

A major topic in the philosophy of QM is to determine whether the quantum state is of ontological or epistemic nature. Is it something that should be granted existence, that ‘is there’ independently of what anyone knows or think, as is often done for particles or fields in classical physics? Or is it a mathematical tool advantageously used by a physicist, reflecting what she knows about the system, as is the case for a classical probability distribution?

## A.2 The Wigner’s friend thought experiment

In 1961, Wigner proposed the now famous Wigner’s friend thought experiment [140], which turns out to be crucial when discussing the ontological problem of QM. The setup goes as follows. A quantum system  $\mathcal{S}$ , say a spin  $\frac{1}{2}$ , is initially prepared in the state  $\frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$ , where  $|\uparrow_z\rangle$  and  $|\downarrow_z\rangle$  denote the eigenvectors of the spin operator along the  $z$ -axis  $\hat{S}_Z$ . In the lab, Wigner’s friend  $\mathcal{F}$  has at her disposal a measurement device for the observable  $\hat{S}_Z$ . In addition, this lab is supposed to be perfectly isolated from the rest of the universe. Wigner stands outside, and wishes to make predictions about the system  $\mathcal{S} + \mathcal{F}$  (the measurement apparatus, as well as any other constituent of the lab distinct from  $\mathcal{S}$ , is included in  $\mathcal{F}$ ).

In Chapter 4, we will sometimes call **unitary QM**, instead of simply QM, the usual quantum theory that we have summarized in Chapter 2 in which states always evolve unitarily according to the Schrödinger equation. This way, we emphasize that we consider the theory without adding any additional postulate or yet-to-be-confirmed physical phenomenon (*e.g.* no collapse, no guiding field), assuming that the theory is valid at all scales.

For Wigner applying unitary QM, after the friend has measured the spin, the content of the lab is in a state  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z \mathcal{F}_{\uparrow_z}\rangle + |\downarrow_z \mathcal{F}_{\downarrow_z}\rangle)$ . The key feature of the thought experiment is that the quantum system modelled by Wigner contains an observer. This is exactly the situation in which the ontological problem of QM appears most clearly, as we shall see in §4.1.

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**Titre :** Le problème ontologique en physique quantique et l'émergence d'un espace-temps relationnel à la lumière de la théorie de la décohérence

**Mot clés :** Fondements de la mécanique quantique, théorie de la décohérence, physique mathématique, background independence

**Résumé :** L'objectif de cette thèse est de mieux comprendre les caractéristiques de la matière quantique qui rendent la gravitation quantique si difficile à atteindre. Dans le Chapitre 3, nous présentons deux modèles généraux de décohérence, sans spécifier aucun Hamiltonien. Le premier modèle fait apparaître naturellement la décohérence comme un phénomène géométrique entre deux réservoirs de dimensions. Le second modèle tente d'expliquer pourquoi, dans un univers où la décohérence est omniprésente, la matière n'est pas figée par l'effet Zénon quantique. Le Chapitre 4 est consacré au problème ontologique de la physique quantique, communé-

ment appelé « problème de la mesure ». Notre objectif n'est pas d'en fournir une solution, mais plutôt de formuler le problème indépendamment de toute interprétation de la mécanique quantique, à la lumière de la théorie de la décohérence. Dans le Chapitre 5, un nouvel ingrédient est introduit dans la discussion : l'espace-temps. Après avoir éclairci l'incompatibilité apparente entre la mécanique quantique et la relativité restreinte, nous présentons un programme de recherche et un modèle jouet en vue de définir une notion *backgroundless* de position au sein du formalisme quantique.

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**Title:** The ontological problem of quantum physics and the emergence of a backgroundless spacetime in the light of the theory of decoherence

**Keywords:** Quantum foundations, theory of decoherence, mathematical physics, background independence

**Abstract:** The aim of this thesis is to better understand the features of quantum matter that makes quantum gravity so hard to reach. In Chapter 3, we present two general models of decoherence, without specifying any Hamiltonian. The first one makes decoherence naturally appear as a geometrical phenomenon between reservoirs of dimensions. The second one specifically tackles the question of why, in a universe where decoherence is ubiquitous, things are not frozen by the quantum Zeno effect. In Chapter 4, we discuss the ontological problem of quantum physics, usually

referred to as the 'measurement problem'. Our aim in this chapter is not to give a solution, but rather to formulate the problem independently of any interpretation of quantum mechanics, in the light of the theory of decoherence. In Chapter 5, a new ingredient is introduced in the discussion: spacetime. After investigating the apparent incompatibility between quantum mechanics and special relativity, we present a program of research and a toy model designed to build a backgroundless notion of position within the quantum formalism.