Action des sous-groupes finis de SL2(C) sur la variété de carquois de Nakajima du carquois de Jordan et fibrés de Procesi

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Action des sous-groupes finis de $\mathrm{SL}_2(\mathbb{C})$ sur la variété de Nakajima du carquois de Jordan et Fibrés de Procesi

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Introduction en français

Contexte

Commençons par les singularités de Klein. Cette famille de singularités deux dimensionnelles est construite à partir d'un sous-groupe fini Γ de $\mathrm{SL}_2(\mathbb{C})$. Les sous-groupes finis de $\mathrm{SL}_2(\mathbb{C})$ sont classifiés, à l'aide des multigraphes de McKay, par les diagrammes de Dynkin affine de type ADE. Pour un tel choix de sous-groupe Γ, en laissant ce groupe agir naturellement sur $\mathbb{C}^2$, on peut considérer la variété affine $\mathbb{C}^2/\Gamma$. Le seul point singulier de cette variété normale est 0. Par résolution des singularités d'une variété algébrique X, on entend la donnée d'une variété lisse Y ainsi qu'une application birationnelle propre de Y vers X. La variété $\mathbb{C}^2/\Gamma$ étant deux dimensionnelle, il existe une résolution S_Γ telle que toute résolution de $\mathbb{C}^2/\Gamma$ se factorise par S_Γ. Une telle résolution est unique à isomorphisme près et est appelée résolution minimale de $\mathbb{C}^2/\Gamma$. De plus, S_Γ peut être construite explicitement par des éclatements successifs du point singulier. Cette famille de singularités est un exemple de singularités symplectiques. Ces singularités ont été définies et étudiées en premier par Arnaud Beauville [Beau, Définition 1.1]. Une manière de généraliser les singularités de Klein à des singularités symplectiques de plus grande dimension est de considérer le produit en couronne de Γ avec le groupe symétrique. Si n est un entier plus grand que 1, notons alors Γ_n le groupe $\Gamma^n \rtimes \mathfrak{S}_n$ où $\mathfrak{S}_n$ est le groupe symétrique des n premiers entiers. Si on laisse agir Γ naturellement sur $\mathbb{C}^2$, l'on obtient une action de Γ^n sur $(\mathbb{C}^2)^n$ et si l'on fait agir $\mathfrak{S}_n$ sur $(\mathbb{C}^2)^n$ par permutation des n copies de $(\mathbb{C}^2)$, l'on obtient alors une action de Γ_n sur $(\mathbb{C}^2)^n$. Cette action nous permet de construire une singularité symplectique de dimension $2n$ que l'on notera $\mathcal{Y}_\Gamma^n := (\mathbb{C}^2)^n/\Gamma_n$. Une résolution (Y, ρ) de singularités de X est dite projective si ρ est un morphisme projectif. Une résolution projective et symplectique bien connue ([Kuz, Théorème 4.9]) de $\mathcal{Y}_\Gamma^n$ est le schéma de Hilbert de n points dans S_Γ que l'on notera $\mathrm{Hilb}_n(S_\Gamma)$ et qui, ensemblistement, peut être apprécié comme l'ensemble des idéaux I de $\mathbb{C}[S_\Gamma]$ tels que le $\mathbb{C}$-espace vectoriel $\mathbb{C}[S_\Gamma]/I$ est de dimension n. Considérons le cas le plus simple i.e. lorsque $\Gamma = 1$ est le groupe trivial. Dans ce cas $\mathcal{Y}_1^n = (\mathbb{C}^2)^n/\mathfrak{S}_n$ et la résolution projective, symplectique $\mathrm{Hilb}_n(S_\Gamma)$ devient $\mathcal{H}_n := \mathrm{Hilb}_n(\mathbb{C}^2)$ avec le morphisme de Hilbert-Chow, cf. section 7.1 pour une définition précise de ce morphisme. L'action naturelle de Γ sur $\mathbb{C}^2$ induit une action algébrique de Γ sur $\mathcal{H}_n$. En 2018, Gwyn Bellamy et Alastair Craw [BC20, Corollaire 1.3] ont classifié toutes les résolutions projectives et symplectiques de $\mathcal{Y}_\Gamma^n$.

Étude des composantes irréductibles de $\mathcal{H}_n^\Gamma$ et résolutions projectives et symplectiques de $\mathcal{Y}_\Gamma^n$

Le premier volet de cette thèse est dédié à l'étude des composantes irréductibles du lieu des Γ-points fixes du schéma de Hilbert ponctuel de $\mathbb{C}^2$ et du lien avec les résolutions projectives, symplectiques de $\mathcal{Y}_\Gamma^n$. Débutons par les composantes irréductibles de $\mathcal{H}_n^\Gamma$. Pour rentrer un peu dans les détails, nous avons besoin d'introduire les variétés de carquois de Nakajima. Cette classe de variétés algébriques symplectiques a été définie ([Nak94, Section 2]) et étudiée par Hiraku Nakajima. Elles ont été initialement construites pour l'élaboration de solutions de certaines équations de Yang-Mills sur des espaces ALE. Aujourd'hui, les variétés de carquois ont trouvé des applications dans de nombreuses branches grâce à leurs bonnes propriétés et le fait qu'elles paramètrent certaines représentations d'algèbres préprojectives construites à partir de carquois. Parmi ces bonnes propriétés, on peut noter que si l'on choisit de bons paramètres de stabilité, alors la variété de carquois est une résolution projective, symplectique de la même variété de carquois sans condition de stabilité. Le carquois de Jordan est le carquois avec un sommet et une flèche. C'est un fait bien connu [Nak99, Théorème 1.9] que la variété de carquois associée au carquois de Jordan, avec le paramètre de stabilité non nul et avec le paramètre de dimension égal à n est isomorphe au schéma de Hilbert de n points dans $\mathbb{C}^2$. De plus, la variété de carquois associée au carquois de Jordan avec le paramètre de stabilité égal à zéro et le paramètre de déformation non nul est l'espace de Calogero-Moser, qui est lisse [Wil, Section 1]. Fixons une orientation Ω du multigraphe non-orienté de McKay associé à Γ. Ce multigraphe munit de cette orientation est un carquois appelé carquois de McKay et noté Q_Γ. Iain Gordon [Gor08, Lemme 7.8] a identifié les composantes irréductibles du lieu des Γ-points fixes du schéma de Hilbert ponctuel à l'aide de variétés de carquois de Q_Γ lorsque Γ est de type A. De plus, Alexander Kirillov Jr. étudie le lieu des Γ-points fixes du schéma de Hilbert ponctuel en utilisant des variétés de carquois et démontre avec peu de détails et par des arguments non-constructivistes [Kir, Théorème 12.13], qu'il existe un isomorphisme entre ces composantes irréductibles et certaines variétés de carquois. Enfin, Cédric Bonnafé et Ruslan Maksimau [BM21, Théorème 2.11] ont décrit les composantes irréductibles du lieu des Γ-points fixes d'espaces lisses de Calogero-Moser généralisés en se servant de variétés de carquois associées à Q_Γ lorsque Γ est de type A. Notons $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)$ la variété de carquois du carquois de Jordan, où θ est le paramètre de stabilité, λ est le paramètre de déformation et n est le paramètre de dimension. En se basant sur ces travaux, l'étude des composantes irréductibles du lieu des Γ-points fixes de $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)$ avec $(\theta,\lambda) \neq (0,0)$ a été menée dans le Chapitre 2. De manière plus précise, le théorème suivant a été démontré dans cette thèse.

Théorème I. *Soit $(\theta,\lambda) \in \mathbb{Q}\times\mathbb{C}$ tel que $(\theta,\lambda) \neq (0,0)$. Pour chaque entier n et chaque sous-groupe fini Γ de $\mathrm{SL}_2(\mathbb{C})$, la variété $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$ se décompose en composantes irréductibles de la manière suivante :*

$$\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma = \coprod_{d\in\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}} \mathcal{M}_d^\Gamma$$

où $\mathcal{M}_d^\Gamma$ est isomorphe à $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda\delta^\Gamma}(M^\sigma)$, une variété de carquois de McKay (i.e. variété de carquois de Q_Γ). Lors de la construction d'une preuve du Théorème I, le besoin d'un meilleur cadre technique pour travailler avec des variétés de McKay c'est fait sentir, particulièrement lorsque Γ est de type D ou E. L'élaboration de ce cadre technique

est inspirée de travaux de George Lusztig [L92, Section 2], de Michela Varagnolo et Eric Vasserot [VV99, Section 2], ainsi que de Weiqiang Wang [Wang, Théorème 5.1]. Si l'on note par $\mathbf{Rep}(\overline{Q_{\Gamma^f}})$ la catégorie des représentations du carquois de McKay doublé et encadré $\overline{Q_{\Gamma^f}}$ et par $\mathbf{McK}(\Gamma)$ une catégorie définie par des Γ-modules, alors pour chaque orientation Ω du carquois de McKay Q_Γ, on peut construire deux foncteurs F_Ω et G_Ω entre ces deux catégories. On a alors montré, dans le Chapitre 1, que ces foncteurs définissent une équivalence de catégories. De plus, dans le même chapitre, on a montré que la variété de carquois de McKay notée $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(d, d^f)$ et l'analogue naturel de la variété de carquois noté $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(M, M^f)$ venant de la catégorie $\mathbf{McK}(\Gamma)$ sont isomorphes. Le troisième chapitre de cette thèse est consacré à la décomposition du lieu des Γ-points fixes du schéma de Hilbert ponctuel et à une description combinatoire de l'ensemble $\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$ indexant les composantes irréductibles, apparu dans le Théorème I. En utilisant la correspondance de McKay, on peut associer à Γ une algèbre de Lie affine. Fixons une base de racines simples de cette algèbre de Lie affine donnée par la chambre fondamentale de Cartan. Notons $Q(\tilde{T}_\Gamma)$ le réseau de racines de cette algèbre de Lie affine. La taille d'un élément $\alpha \in Q(\tilde{T}_\Gamma)$ est la moyenne des coefficients de α dans la base de racine simple pondérée par les dimensions des représentations irréductibles de Γ (cf. Définition 0.1). En généralisant [BM21, Lemme 2.6] au type D et E, on peut définir le poids d'un élément de $Q(\tilde{T}_\Gamma)$ (cf. Definition 3.8). On a alors montré le théorème qui suit dans le Chapitre 3.

Théorème II. *Pour chaque entier n et pour chaque sous-groupe fini* Γ *de* $\mathrm{SL}_2(\mathbb{C})$*, l'ensemble indexant les composantes irréductibles de* $\mathcal{H}^\Gamma_n$ *et l'ensemble indexant les composantes irréductibles du lieu des* Γ*-points fixes du n-ième espace de Calogero-Moser sont les mêmes et sont égaux à l'ensemble des combinaisons linéaires positives de racines simples de taille n et de poids positif que l'on note par* $\mathcal{A}^n_\Gamma$*.*

Enfin, dans le Chapitre 4, nous avons décrit toutes les résolutions projectives, symplectiques de $\mathcal{Y}^n_\Gamma$ comme composantes irréductibles du lieu des Γ-points fixes du schéma ponctuel de Hilbert. Plus précisément, si d'un côté, on note $\mathcal{IC}^\Gamma_k$ l'ensemble des composantes irréductibles de $\mathcal{H}^\Gamma_k$ et $IC^\Gamma := \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathcal{IC}^\Gamma_k$ et d'un autre côté, on note $\mathscr{R}^\Gamma_k$, l'ensemble des classes d'isomorphismes de résolutions projectives, symplectiques de $\mathcal{Y}^\Gamma_k$ et $\mathscr{R}^\Gamma := \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathscr{R}^\Gamma_k$, alors on a démontré le théorème suivant :

Théorème III. *Il existe une application surjective de* $\mathcal{IC}^\Gamma$ *vers* $\mathscr{R}^\Gamma$*.*

Combinatoire en type D

Le second volet de cette thèse est consacré à l'élaboration d'un modèle combinatoire de $\mathcal{A}^n_\Gamma$ lorsque Γ est de type D. Lorsque Γ est de type A, un modèle en termes de partitions a déjà été obtenu [Gor08, Lemme 7.8]. Le même résultat peut être obtenu en combinant le Théorème II et [BM21, Lemma 4.9] dans le cas spécial où $m = l$. Si Γ est de type D, alors Γ n'est plus un sous-groupe de $\mathbb{C}^*$, où $\mathbb{C}^*$ est identifié ici avec le tore maximal diagonal de $\mathrm{SL}_2(\mathbb{C})$. Afin de construire un modèle combinatoire basé sur des partitions d'entiers, on a besoin de se restreindre au sous-ensemble des composantes irréductibles qui contiennent un $\mathbb{C}^*$-point fixe. Si $\lambda = (\lambda_1, ..., \lambda_r)$ est une partition, on note par $\mathcal{Y}(\lambda) := \{(i,j) \in \mathbb{Z}^2_{\geq 0} | i < \lambda_1, j < r\}$ son diagramme de Young et $|\lambda| := \sum_{i=1}^r \lambda_i$ sa taille. De plus, si $(a,b) \in \mathcal{Y}(\lambda)$ et l est un entier positif, alors on définit le l-contenu de (a,b) comme le reste de la division Euclidienne de $a - b$ par l. Le l-résidu

de λ, noté par r_λ^l, est l'élément de $\mathbb{Z}^l$ tel que pour chaque $i \in [\![0, l-1]\!]$, $r_\lambda^l(i)$ est égal au nombre de boîtes de $\mathcal{Y}(\lambda)$ avec un l-contenu égal à i. Le crochet d'une boîte (i,j) de $\mathcal{Y}(\lambda)$ est $H_{(i,j)}(\lambda) := \{(a,b) \in \mathcal{Y}(\lambda) | (a = i \text{ et } b \geq j) \text{ ou } (a \geq i \text{ et } b = j)\}$ et la longueur de $H_{(i,j)}(\lambda)$ est le cardinal de $H_{(i,j)}(\lambda)$. Si $l \in \mathbb{Z}_{\geq 1}$, notons μ_l le groupe cyclique d'ordre l de $\mathrm{SL}_2(\mathbb{C})$ engendré par la matrice diagonale $\mathrm{diag}(\zeta_l, \zeta_l^{-1})$ notée ω_l où ζ_l est la l-ième racine primitive de l'unité $e^{\frac{2i\pi}{l}}$. Prenons $s \in \mathrm{SL}_2(\mathbb{C})$ telle que ses coefficients soient entiers et telle que ces coefficients diagonaux soient nuls. Considérons maintenant le cas où Γ est égal à $\widetilde{BD}_{2l}$ le groupe binaire diédral d'ordre $4l$ (de type D) engendré par ω_{2l} et s. Dans ce cas, les points fixes de $\mathcal{H}_n$ sous le sous-groupe de $\mathrm{SL}_2(\mathbb{C})$ engendré par Γ et $\mathbb{C}^*$ sont indexés par des partitions qui sont égales à leurs conjuguées et que l'on appellera partitions symétriques. Dans le Chapitre 5, la notion de l-résidu est étendue à la notion de l-résidu de type D. Pour λ une partition d'un entier n, notons par I_λ l'idéal de $\mathbb{C}[x,y]$ généré par $x^i y^j$ pour chaque paire $(i,j) \in \mathbb{Z}^2_{\geq 0}$ qui n'est pas dans $\mathcal{Y}(\lambda)$. L'idéal I_λ est un élément de $\mathcal{H}_n$. La propriété du l-résidu que l'on a généralisée est : pour chaque partition λ, le l-résidu est le vecteur des multiplicités du caractère de μ_l de la représentation $\mathbb{C}[x,y]/I_\lambda$. De manière plus précise, le caractère de la représentation $\mathbb{C}[x,y]/I_\lambda$ de μ_l est égal à $\sum_{i=0}^{l-1} r_\lambda^l(i)\tau_l^i$ où τ_l est le caractère de μ_l qui envoie ω_l sur ζ_l. En utilisant la notion de l-résidu de type D, on a réussi à montrer le théorème suivant.

Théorème IV. *Pour chaque $(l,n) \in \mathbb{Z}^2_{\geq 0}$, si $\Gamma = \widetilde{BD}_{2l}$, alors l'ensemble des composantes irréductibles de $\mathcal{H}_n^\Gamma$ contenant un $\mathbb{C}^*$-point fixe est en bijection avec l'ensemble des partitions symétriques λ ne contenant aucun crochet de longueur $2l$ et telle que $|\lambda| \equiv n\ [2l]$ et $|\lambda| \leq n$.*

Dans le cas où Γ est un sous-groupe binaire tétraédral de $\mathrm{SL}_2(\mathbb{C})$, on a montré dans la section 5.3 de cette thèse que le sous-groupe de $\mathrm{SL}_2(\mathbb{C})$ engendré par Γ et $\mathbb{C}^*$ est $\mathrm{SL}_2(\mathbb{C})$. Ce résultat réduit alors l'étude des composantes irréductibles de $\mathcal{H}_n^\Gamma$ contenant un $\mathbb{C}^*$-point fixe, à l'étude des points fixes de $\mathrm{SL}_2(\mathbb{C})$ dans le schéma de Hilbert de n points dans $\mathbb{C}^2$. Ces points fixes sont indexés par les partitions en escalier, de la forme $(m, m-1, m-2, ..., 1)$ pour $m \in \mathbb{Z}_{\geq 1}$. Enfin, si Γ est un sous-groupe binaire octaédral ou un sous-groupe binaire icosaédral de $\mathrm{SL}_2(\mathbb{C})$, alors on obtient la même conclusion que dans le cas du sous-groupe binaire tétraédral, car ces deux sous-groupes contiennent un groupe binaire tétraédral.

Fibré tautologique et Fibrés de Procesi

Le troisième volet de cette thèse s'emploie à l'étude de la restriction à $\mathcal{H}_n^\Gamma$ de deux fibrés vectoriels sur $\mathcal{H}_n$. Le premier fibré vectoriel est le fibré tautologique. C'est un fibré vectoriel de rang n sur $\mathcal{H}_n$ noté $\mathcal{T}_n$, cf. Proposition 6.1 pour une définition précise. Hiraku Nakajima a aussi défini [Nak00, section 2.9] des fibrés vectoriels tautologiques sur les variétés de carquois. On peut alors se demander s'il est possible de décomposer la restriction de $\mathcal{T}_n$ à une composante irréductible de $\mathcal{H}_n^\Gamma$ en termes de fibrés vectoriels de la variété de carquois associée à la composante irréductible. C'est le résultat principal du Chapitre 6 de cette thèse. Le second fibré vectoriel est le fibré de Procesi cf. section 7.1 pour une définition précise. L'existence de ce fibré vectoriel a été démontrée par Marc Haiman [H01, Théorème 5.2.1] et il a été utilisé pour prouver la conjecture $n!$ sur la positivité des coefficients généralisés de Kostka-Macdonald qui sont les coefficients des fonctions symétriques de Haiman-Macdonald dans la base des fonctions de Schur. En laissant le groupe $\mathfrak{S}_n$ agir trivialement sur $\mathcal{H}_n$, le fibré de Procesi noté

$\mathscr{P}^n$ est par construction un fibré vectoriel $\big(\mathfrak{S}_n \times \mathrm{GL}_2(\mathbb{C})\big)$-équivariant sur $\mathcal{H}_n$. Pour chaque idéal $I \in \mathcal{H}_n$, la fibre de $\mathscr{P}^n$ en I, notée $\mathscr{P}^n_{|I}$, est isomorphe à la représentation régulière en tant que $\mathfrak{S}_n$-module. Cela implique que $\mathscr{P}^n$ est un fibré de rang $n!$. Dans le Chapitre 7, les fibres du fibré de Procesi sont étudiées, en tant que $(\mathfrak{S}_n \times \Gamma)$-modules, au-dessus de chaque composante irréductible de $\mathcal{H}_n^\Gamma$. Soit δ^Γ la racine ayant comme coefficients, dans la base de racine simple, les dimensions des représentations irréductibles de Γ. Pour chaque $d \in \mathcal{A}_\Gamma^n$, notons r_d le poids le d qui est un entier positif. Soit d_0 l'élément de $Q(\tilde{T}_\Gamma)$ égal à $d - r_d\delta^\Gamma$. Par construction, d_0 a pour poids 0. Notons g_Γ la taille de d_0 et I_{d_0} l'unique idéal de la composante irréductible de $\mathcal{H}^\Gamma_{g_\Gamma}$ associée à d_0. Soit $(p_1, ..., p_{r_d}) \in (\mathbb{C}^2)^{r_d}$ tel que les Γ-orbites des p_i soient libres et distinctes. Notons $p = (0, \Gamma p_1, ..., \Gamma p_{r_d}) \in (\mathbb{C}^2)^n$ et S_p le stabilisateur de p dans $\mathfrak{S}_n \times \Gamma$. La première partie du Chapitre 7 est dédiée à la preuve du théorème suivant qui est un travail commun avec Gwyn Bellamy.

Théorème V. *Soit n un entier positif, Γ un sous-groupe fini de $\mathrm{SL}_2(\mathbb{C})$ et $d \in \mathcal{A}_\Gamma^n$. Pour chaque idéal I de la composante irréductible $\mathcal{H}_n^\Gamma$ indexée par d, il existe un isomorphisme en tant que $(\mathfrak{S}_n \times \Gamma)$-module :*

$$\mathscr{P}^n_{|I} \simeq \mathrm{Ind}_{S_p}^{\mathfrak{S}_n \times \Gamma}(\mathscr{P}^{g_\Gamma}_{|I_{d_0}})$$

Un résultat analogue a été obtenu par Gwyn Bellamy dans le cadre des espaces de Calogero-Moser généralisés. En effet, Pavel Etingof et Victor Ginzburg ont construit, sur les espaces de Calogero-Moser généralisés, un faisceau cohérent qui est similaire au fibré de Procesi. En utilisant les travaux de Roman Bezrukavnikov et de Pavel Etingof [BE09], Gwyn Bellamy [Bell09, Theorem 3.5] a montré que l'étude du faisceau cohérent d'Etingof-Ginzburg sur les espaces généralisés de Calogero-Moser d'un groupe de réflexions complexe W se réduit à l'étude du faisceau cohérent d'Etingof-Ginzburg sur l'espace généralisé de Calogero-Moser d'un sous-groupe parabolique de W. Dans la deuxième partie du Chapitre 7, nous nous restreignons au cas où $\Gamma = \mu_l$ pour $l \geq 1$. Dans ce cas, nous avons aussi démontré en utilisant uniquement des arguments de théorie des représentations deux cas particuliers du Théorème V. Ces deux preuves illustrent d'autres approches que l'on peut avoir vis-à-vis du Théorème V. En particulier si $\lambda = (n)$ et si $l = n$, alors le Théorème V implique un résultat d'Hideaki Morita et de Tatsuhiro Nakajima concernant l'algèbre des coinvariants de $\mathfrak{S}_n$ [MN, Theorem 8]. Si l'on considère le cas où $\Gamma = \widetilde{BD}_{2l}$ avec $l \geq 1$, des formules de réductions ont également été obtenus à partir du cas cyclique. On notera aussi que le fibré de Procesi a été généralisé par Ivan Losev [**Los14**]. Il a montré qu'il existe des fibrés de Procesi en couronnes au-dessus des résolutions projectives, symplectiques de $\mathcal{Y}_\Gamma^n$ [**Los14**]. Ces fibrés ont été utilisés pour étendre le théorème $n!$ aux polynômes de Macdonald en couronnes dont l'existence a été conjecturée par Haiman [H01, section 7.2] et prouvée par Roman Bezrukavnikov et Michael Finkelberg [BF12, Corollaire 1.2]. On peut se demander si des théorèmes analogues au Théorème V peuvent être démontrés dans ce cadre généralisé.

Travaux en cours

Si l'on revient au cas où Γ est égal à μ_l, le Théorème V ramène l'étude de la structure de $(\mathfrak{S}_n \times \mu_l)$-module des fibres du fibré de Procesi, à l'étude des fibres au-dessus des idéaux monomiaux associés à des partitions qui ne contiennent pas de crochet

de longueur l et que l'on appelle des l-cœurs. Dans le Chapitre 8, une première approche de l'étude de ces fibres est proposée en conjecturant une connexion avec les représentations de Fock de l'algèbre de Lie affine de type A_l. En effet, il semble que la décomposition de la représentation régulière obtenu via la décomposition selon les caractères irréductibles de μ_l de $\mathscr{P}^n_{|I_\lambda}$ où λ est un l-cœur, peut être raffinée en laissant agir les générateurs de Chevalley f_i de l'algèbre de Lie affine de type A_l sur l'élément $|\emptyset\rangle$ de la représentation de Fock (cf. Conjecture 8.5 pour une formulation plus précise). Un exemple est également donné après la Conjecture 8.5. En annexe, la lectrice ou le lecteur pourra trouver un code SAGE qui permet de calculer pour une valeur de l et de n la décomposition de la fibre de $\mathscr{P}^n$ au-dessus d'un l-cœur (s'il en existe) et de la décomposition "en paquets" de l'action des f_i sur $|\emptyset\rangle$. Enfin, en annexe, on pourra également trouver des tables qui donnent les décompositions, en tant que $(\mathfrak{S}_n \times \mu_l)$-modules, des fibres du fibré de Procesi au-dessus d'idéaux monomiaux associés à des l-cœurs .

Index des notations

Afin de faciliter la lecture de cette thèse, un index rassemblant toutes les notations est disponible à la fin de ce document, juste avant la Bibliographie.

Introduction in english

Context

Let us start with the Kleinian singularities. This family of two-dimensional singularities is constructed out of a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$. Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are classified, thanks to McKay multigraphs, by the affine Dynkin diagrams of type ADE. For such a choice of Γ, let it act naturally on $\mathbb{C}^2$ and consider the affine variety $\mathbb{C}^2/\Gamma$. In this normal variety, the only singular point is 0. A resolution of singularities of a variety X is the data of a smooth variety Y and a proper birational map $Y \to X$. Being two dimensional the variety $\mathbb{C}^2/\Gamma$ has a resolution S_Γ such that any other resolution of $\mathbb{C}^2/\Gamma$ factors through S_Γ. Such a resolution is unique up to isomorphism and is referred to as the minimal resolution of $\mathbb{C}^2/\Gamma$. Moreover, S_Γ can be explicitly constructed as successive blow-ups of the singular point. This family of singularities is an example of symplectic singularities. These singularities have been defined and studied first by Arnaud Beauville [Beau, Definition 1.1]. A way to generalize the Kleinian singularities to higher dimensional symplectic singularities is to consider the wreath product of Γ with the symmetric group. More precisely, if n is an integer greater or equal to 1, then Γ_n, which denotes the group $\Gamma^n \rtimes \mathfrak{S}_n$, acts on $(\mathbb{C}^2)^n$ where $\mathfrak{S}_n$ is the symmetric group on n letters. Indeed Γ^n acts on $(\mathbb{C}^2)^n$ using the natural action of Γ on $\mathbb{C}^2$ and $\mathfrak{S}_n$ acts by permuting the n copies of $\mathbb{C}^2$. One can now consider the $2n$ dimensional symplectic singularity $\mathcal{Y}_\Gamma^n := (\mathbb{C}^2)^n/\Gamma_n$. A resolution (Y, ρ) of singularities in X is said to be projective if ρ is a projective morphism. A well-known ([Kuz, Theorem 4.9]) projective symplectic resolution of $\mathcal{Y}_\Gamma^n$ is the Hilbert scheme of n points in S_Γ denoted by $\mathrm{Hilb}_n(S_\Gamma)$ which, set-theoretically, is the set of all ideals I of $\mathbb{C}[S_\Gamma]$ such that the dimension of the $\mathbb{C}$-vector space $\mathbb{C}[S_\Gamma]/I$ is equal to n. Take the simplest case i.e. when $\Gamma = 1$ is the trivial group. In this case $\mathcal{Y}_1^n = (\mathbb{C}^2)^n/\mathfrak{S}_n$ and the projective, symplectic resolution $\mathrm{Hilb}_n(S_\Gamma)$ becomes $\mathcal{H}_n := \mathrm{Hilb}_n(\mathbb{C}^2)$ with the Hilbert-Chow morphism, cf. section 7.1 for a precise definition of this morphism. The natural action of Γ on $\mathbb{C}^2$ induces an algebraic action on the Hilbert scheme of points of $\mathbb{C}^2$. In 2018, Gwyn Bellamy and Alastair Craw [BC20, Corollary 1.3] have classified all projective, symplectic resolutions of $\mathcal{Y}_\Gamma^n$.

Irreducible components of $\mathcal{H}_n^\Gamma$ and projective, symplectic resolutions of $\mathcal{Y}_\Gamma^n$

The first facet of this thesis is dedicated to the study of $\mathcal{H}_n^\Gamma$ and to the study of all projective, symplectic resolutions of $\mathcal{Y}_\Gamma^n$ as irreducible components of Γ-fixed point loci of the Hilbert schemes of points in $\mathbb{C}^2$. Let us begin with the irreducible components of $\mathcal{H}_n^\Gamma$. To dive a bit more into details, one needs to introduce Nakajima's quiver varieties.

This class of symplectic varieties has initially been defined ([Nak94, Section 2]) and studied by Hiraku Nakajima. They were originally constructed to study solutions of some Yang-Mills equations on ALE spaces. Today, quiver varieties have infused many different branches thanks to their good properties and the fact that they parametrize some representations of preprojective algebras based on quivers. Among other properties, if one chooses good stability parameters, then a quiver variety is a projective, symplectic resolution of singularities of the same quiver variety without the stability condition. The Jordan quiver is the quiver with one vertex and one arrow. It is well known [Nak99, Theorem 1.9] that the quiver variety of the Jordan quiver with nonzero stability parameter and dimension parameter equal to n is isomorphic to the Hilbert scheme of n points in $\mathbb{C}^2$. Moreover, it turns out that if one takes the stability parameter to be zero and takes the deformation parameter to be nonzero, then the quiver variety, which is the Calogero-Moser space, is smooth [Wil, Section 1]. Fix an orientation Ω of the undirected McKay multigraph of Γ. This oriented multigraph is then a quiver. It is called the McKay quiver and is denoted by Q_Γ. Iain Gordon [Gor08, Lemma 7.8] has identified the irreducible components of the Γ-fixed point locus of the Hilbert scheme of point using quiver varieties of Q_Γ when Γ is of type A. Moreover, Alexander Kirillov Jr. has studied the Γ-fixed point locus of the punctual Hilbert scheme using quiver varieties. He showed, in a concise proof and based on nonconstructive arguments [Kir, Theorem 12.13], that there exists an isomorphism between the irreducible components of $\mathcal{H}_n^\Gamma$ and certain quiver varieties. Finally, Cédric Bonnafé and Ruslan Maksimau [BM21, Theorem 2.11] have described the irreducible components of Γ-fixed point loci of smooth generalized Calogero-Moser spaces using quiver varieties of Q_Γ also when Γ is of type A. Based on this work, the irreducible components of Γ-fixed point loci of the Jordan quiver variety denoted by $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)$ for $(\theta,\lambda) \neq (0,0)$ where θ is the stability parameter and λ the deformation parameter have been described in this thesis using quiver varieties of the McKay quiver of Γ. From now on, let us refer to quiver varieties of Q_Γ as McKay quiver varieties. More precisely the following theorem has been proven in Chapter 2.

Theorem I. *Let $(\theta,\lambda) \in \mathbb{Q}\times\mathbb{C}$ such that $(\theta,\lambda) \neq (0,0)$. For each integer n and each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, the variety $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$ decomposes into irreducible components*

$$\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma = \coprod_{d\in\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}} \mathcal{M}^\Gamma_d$$

where $\mathcal{M}^\Gamma_d$ is isomorphic to $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda\delta^\Gamma}(M^\sigma)$ a McKay quiver variety. In the process of proving Theorem I, the need for a technically better setting to work with McKay quiver varieties has been felt, especially when Γ is of type D and E. The development of this technical setting is inspired by the work of George Lusztig [L92, Section 2], of Michela Varagnolo and Eric Vasserot [VV99, Section 2], as well as the one of Weiqiang Wang [Wang, Theorem 5.1]. To be more precise, if one denotes by $\mathbf{Rep}(\overline{Q_{\Gamma^f}})$ the category of representations of the double framed McKay quiver $\overline{Q_{\Gamma^f}}$ and by $\mathbf{McK}(\Gamma)$ a category defined with Γ-modules, then for every orientation Ω of Q_Γ two functors F_Ω and G_Ω between these categories have been constructed in Chapter 1. The fact that these two functors define an equivalence of categories has been proven. Furthermore, in the same chapter, it has been proven that McKay quiver varieties denoted by $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(d,d^f)$ and the analog of quiver varieties, denoted by $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(M,M^f)$, coming naturally out of the category $\mathbf{McK}(\Gamma)$ are isomorphic as algebraic varieties. The third chapter of

this thesis is dedicated to the decomposition of the Γ-fixed point locus of the Hilbert scheme of points on $\mathbb{C}^2$ into irreducible components and to the description of a combinatorial model of the indexing set $\mathcal{A}^n_{\Gamma,\theta,\lambda}$, appearing in Theorem I. Thanks to McKay's correspondence, one can associate to Γ an affine Lie algebra. Fix a base of simple roots of this affine Lie algebra, associated with the fundamental Cartan chamber. Denote by $Q(\tilde{T}_\Gamma)$ the root lattice of this affine Lie algebra. The size of an element $\alpha \in Q(\tilde{T}_\Gamma)$ is the average of the coefficients of α in a chosen set of simple roots weighted by the dimensions of the irreducible representations of Γ (cf. Definition 0.1). Generalizing [BM21, Lemma 2.6] to type D and E, another statistic on $Q(\tilde{T}_\Gamma)$ can be defined called the weight (cf. Definition 3.8). The following theorem is proven in Chapter 3.

Theorem II. *For each integer n and each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, the indexing set of the irreducible components of $\mathcal{H}^\Gamma_n$ and of the irreducible components of the Γ-fixed point locus of the Calogero-Moser n space are the same and are equal to the set of all positive linear combinations of simple roots of size n and positive weight. Denote by $\mathcal{A}^n_\Gamma$ this set.*

Finally, in Chapter 4 one can find the description of all projective, symplectic resolutions of $\mathcal{Y}^n_\Gamma$ in terms of irreducible components of the Γ-fixed point locus of the punctual Hilbert scheme of $\mathbb{C}^2$. More precisely, if on one side, one denotes by $\mathcal{IC}^\Gamma_k$ the set of all irreducible components of $\mathcal{H}^\Gamma_k$ and $\mathcal{IC}^\Gamma := \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathcal{IC}^\Gamma_k$ and on the other side, one denotes by $\mathscr{R}^\Gamma_k$, the set of all isomorphism classes of projective symplectic resolutions of $\mathcal{Y}^\Gamma_k$ and $\mathscr{R}^\Gamma := \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathscr{R}^\Gamma_k$, then the theorem underneath has been demonstrated.

Theorem III. *There exists a surjective map denoted by $\mathcal{BC}$ from $\mathcal{IC}^\Gamma$ to $\mathscr{R}^\Gamma$.*

Combinatorics in type D

The second facet of this thesis is devoted to the conception of a combinatorial model of $\mathcal{A}^n_\Gamma$ when Γ is of type D. When Γ is of type A this has already been done [Gor08, Lemma 7.8]. The same result can be obtained by combining Theorem II and [BM21, Lemma 4.9] in the special case $m = l$. If Γ is of type D or E, then Γ is no more a subgroup of $\mathbb{C}^*$, where $\mathbb{C}^*$ is here identified with the maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$. To construct a combinatorial model based on partitions of integers, one needs to restrict $\mathcal{A}^n_\Gamma$ to the subset of irreducible components that contain a $\mathbb{C}^*$-fixed point. For $\lambda = (\lambda_1, ..., \lambda_r)$ a partition, denote by $\mathcal{Y}(\lambda) := \{(i,j) \in \mathbb{Z}^2_{\geq 0} | i < \lambda_1, j < r\}$ its associated Young diagram and by $|\lambda| := \sum_{i=1}^r \lambda_i$ its length. Moreover, if $(a,b) \in \mathcal{Y}(\lambda)$ and l is a nonnegative integer, then the remainder of a - b divided by l is called the l-content of the box (a,b). The l-residue of λ, denoted by r^l_λ, is the element of $\mathbb{Z}^l$, such that for all $i \in [\![0, l-1]\!]$, $r^l_\lambda(i)$ is equal to the number of boxes of the Young diagram of λ with l-content equal to i. Let $H_{(i,j)}(\lambda) := \{(a,b) \in \mathcal{Y}(\lambda) | (a = i \text{ and } b \geq j) \text{ or } (a \geq i \text{ and } b = j)\}$ be the hook of a box (i,j) of $\mathcal{Y}(\lambda)$ and the cardinal of $H_{(i,j)}(\lambda)$ be the length of $H_{(i,j)}(\lambda)$. If $l \in \mathbb{Z}_{\geq 1}$, let μ_l denote the cyclic group with l elements in $\mathrm{SL}_2(\mathbb{C})$ generated by the diagonal matrix $\mathrm{diag}(\zeta_l, \zeta_l^{-1})$ denoted by ω_l where ζ_l denotes the primitive l^{th} root of unity $e^{\frac{2i\pi}{l}}$. Take $s \in \mathrm{SL}_2(\mathbb{C})$ with integer coefficients and diagonal coefficients equal to 0. Consider the case where Γ is $\widetilde{BD}_{2l}$, the binary dihedral group of order $4l$ (of type D) generated by ω_{2l} and s. In that case, the fixed points of $\mathcal{H}_n$ under the subgroup of $\mathrm{SL}_2(C)$ generated by Γ and $\mathbb{C}^*$ are indexed by partitions that are equal to their conjugate, called symmetric partitions. In Chapter 5, the notion of l-residue has been extended to the notion of l-residue of type D. For λ a partition of an integer n, denote by I_λ the ideal of

$\mathbb{C}[x,y]$ generated by $x^i y^j$ for all pairs $(i,j) \in \mathbb{Z}^2_{\geq 0}$ that are not in $\mathcal{Y}(\lambda)$. The ideal I_λ is an element of $\mathcal{H}_n$. The property of the l-residue to be generalized is that for λ a partition, the l-residue is the multiplicity vector of the character of μ_l of the representation $\mathbb{C}[x,y]/I_\lambda$. More precisely, the character of the representation $\mathbb{C}[x,y]/I_\lambda$ of μ_l is equal to $\sum_{i=0}^{l-1} r_\lambda^l(i)\tau_l^i$ where τ_l is the character of μ_l that maps ω_l to ζ_l. Using this notion of l-residue of type D, the following theorem has been proven.

Theorem IV. *For each $(l,n) \in \mathbb{Z}^2_{\geq 0}$, if $\Gamma = \widetilde{BD}_{2l}$, then the set of irreducible components of $\mathcal{H}_n^\Gamma$ that contain a $\mathbb{C}^*$-fixed point is in bijection with the set of symmetric partitions λ such that λ does not contain any hook of length $2l$ and such that $|\lambda| \equiv n\ [2l]$ and $|\lambda| \leq n$.*

If Γ is a binary tetrahedral subgroup of $\mathrm{SL}_2(\mathbb{C})$, then it is proven in section 5.3 of this thesis that the subgroup of $\mathrm{SL}_2(\mathbb{C})$ generated by Γ and $\mathbb{C}^*$ is $\mathrm{SL}_2(\mathbb{C})$. This result reduces the study of the irreducible components of $\mathcal{H}_n^\Gamma$ containing a $\mathbb{C}^*$-fixed point to the study of $\mathrm{SL}_2(\mathbb{C})$-fixed point in the Hilbert scheme of n points in $\mathbb{C}^2$. These fixed points are known to be indexed by staircase partitions, partitions of the form $(m, m-1, m-2, ..., 1)$ for $m \in \mathbb{Z}_{\geq 1}$. If Γ is a binary octahedral or a binary icosahedral subgroup of $\mathrm{SL}_2(\mathbb{C})$, then the same conclusion as for the binary tetrahedral case can be drawn since these groups contain a binary tetrahedral group.

Tautological vector bundle and Procesi vector bundles

The third facet of this thesis is dedicated to the study of the restriction to $\mathcal{H}_n^\Gamma$ of two vector bundles on $\mathcal{H}_n$. The first vector bundle is the tautological vector bundle. It is a rank n vector bundle over $\mathcal{H}_n$ denoted by $\mathcal{T}_n$, cf. Proposition 6.1 for a precise definition. Hiraku Nakajima has also defined [Nak00, Section 2.9] tautological vector bundles on quiver varieties. One can wonder if it is possible to decompose the restriction of $\mathcal{T}_n$ over a given irreducible component of $\mathcal{H}_n^\Gamma$, which is also a connected component since $\mathcal{H}_n^\Gamma$ is a smooth variety in virtue of Γ being a finite group, in terms of tautological vector bundles of the quiver variety associated with the irreducible component. This is the main result of the concise Chapter 6 of this thesis. The second vector bundle is the Procesi vector bundle cf. section 7.1 for a precise definition. The existence of this vector bundle has been proven by Marc Haiman [H01, Theorem 5.2.1] and has been used to prove the $n!$ conjecture on the positivity of the generalized Kostka-Macdonald coefficients which are the coefficients of Haiman-Macdonald symmetric functions in the base of Schur functions. Let the group $\mathfrak{S}_n$ act trivially on the Hilbert scheme of n points in $\mathbb{C}^2$. The Procesi bundle denoted by $\mathscr{P}^n$ is by construction an $(\mathfrak{S}_n \times \mathrm{GL}_2(\mathbb{C}))$-equivariant vector bundle over $\mathcal{H}_n$. For each element $I \in \mathcal{H}_n$, the fiber of $\mathscr{P}^n$ at I, denoted by $\mathscr{P}^n_{|I}$ is isomorphic, as an $\mathfrak{S}_n$-module, to the regular representation which implies that the Procesi bundle is of rank $n!$. In Chapter 7 of this thesis, the fibers of the Procesi bundle, as $(\mathfrak{S}_n \times \Gamma)$-modules, are studied over each connected component of $\mathcal{H}_n^\Gamma$. Let δ^Γ be the root with coefficients in the base of simple roots given by the McKay correspondence equal to the dimension of the irreducible representations. Now, for $d \in \mathcal{A}_\Gamma^n$, let r_d denote the weight of d, which is a nonnegative integer. Let d_0 be the element of $Q(\tilde{T}_\Gamma)$ equal to $d - r_d\delta^\Gamma$. By construction, d_0 has weight 0. Denote by g_Γ the size of d_0 and by I_{d_0} the unique ideal of the connected component of $\mathcal{H}^\Gamma_{g_\Gamma}$ attached to d_0. Take $(p_1, ..., p_{r_d}) \in (\mathbb{C}^2)^{r_d}$ such that the Γ-orbits of the p_i are disctincts and free. Denote by $p = (0, \Gamma.p_1, ..., \Gamma.p_{r_d}) \in (\mathbb{C}^2)^n$ and by S_p the stabilizer of p in $\mathfrak{S}_n \times \Gamma$. The first part

of Chapter 7 is dedicated to the proof of the following theorem which is a joint work with Gwyn Bellamy.

Theorem V. *Let n be a nonnegative integer, Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ and $d \in \mathcal{A}_\Gamma^n$. For each ideal I in the connected component of $\mathcal{H}_n^\Gamma$ indexed by d, there exists an isomorphism as $\mathfrak{S}_n \times \Gamma$-modules*

$$\mathscr{P}^n_{|I} \simeq \mathrm{Ind}_{S_p}^{\mathfrak{S}_n \times \Gamma}(\mathscr{P}^{\mathrm{g}_\Gamma}_{|I_{d_0}})$$

A similar result has been obtained by Gwyn Bellamy in the context of generalized Calogero-Moser spaces. Pavel Etingof and Victor Ginzburg have constructed, on generalized Calogero-Moser spaces, a coherent sheaf that is alike the Procesi bundle. Using the work of Roman Bezrukavnikov and Pavel Etingof [BE09], Gwyn Bellamy [Bell09, Theorem 3.5] has proven that the study of Etingof-Ginzburg's coherent sheaf on the generalized Calogero-Moser space of a complex reflection group W reduces to the study of Etingof-Ginzburg's sheaf on the generalized Calogero-Moser space of a parabolic subgroup of W. In the second part of Chapter 7, Γ is taken to be equal to μ_l for $l \geq 1$. In that case, Theorem V is proven in two edge cases. The proofs use only representation theory and show the diversity of techniques that one can use regarding the proof of this theorem. Note that, if $\lambda = (n)$ and $l = n$, then Theorem V implies the result of Hideaki Morita and Tatsuhiro Nakajima on the coinvariant algebra of $\mathfrak{S}_n$ [MN, Theorem 8]. When Γ is equal to $\widetilde{BD_2}l$ with $l \geq 1$, reduction formulas have also been derived from the cyclic case. Note that the Procesi bundle has been generalized by Ivan Losev [**Los14**]. He has proven that wreath Procesi bundles exist over resolutions of $\mathcal{Y}_\Gamma^n$ [**Los14**]. This was then of great use to extend the $n!$ theorem to the wreath Macdonald polynomials whose existence was conjectured by Haiman [H01, Section 7.2] and proved by Roman Bezrukavnikov and Michael Finkelberg [BF12, Corollary 1.2]. One can then wonder if analogous theorems to Theorem V can be proven in these generalized settings.

Work in progress

When Γ is equal to μ_l, Theorem V reduces the study of the $(\mathfrak{S}_n \times \mu_l)$-structure of the fibers of the Procesi bundle to the study of the fibers on monomial ideals associated with partitions that do not contain hooks of length l. Such partitions are called l-cores. In Chapter 8, a first approach to study these fibers suggests that there might be connections with the Fock representation of the affine algebra of type A_l. It seems that the decomposition of the regular representation obtained via the decomposition along irreducible characters of μ_l of $\mathscr{P}^n_{|I_\lambda}$ where λ is an l-core, can be refined from the action of Chevalley's generators f_i of the affine Lie algebra of type A_l on the element $|\emptyset\rangle$ of the Fock representation (cf. Conjecture 8.5 for a precise formulation). An example is given after Conjecture 8.5. In the appendix, the reader will find a SAGE code that computes for a given value of l and n the decomposition of the fiber of $\mathscr{P}^n$ above the monomial ideal associated with an l-core (if it exists) and the decomposition "in packets" of the action of the f_i on $|\emptyset\rangle$. Finally, in the appendix, tables are given that give decompositions of the Procesi bundle on monomial ideals associated with l-cores, as $(\mathfrak{S}_n \times \mu_l)$-modules.

Index of notation

In order to facilitate the reading of this thesis, an index bringing together all the notation is available at the end of this document, just before the Bibliography.

Notation

Here is a good place to introduce common notation on Kac-Moody Lie algebras. The type Y denotes indifferently finite or affine complex Kac-Moody Lie algebra types. Take a finite or affine Kac-Moody Lie algebra $\mathfrak{g}$ of type Y and a Cartan subalgebra $\mathfrak{h}$. If $\mathfrak{g}$ is affine of type $\tilde{T}$, then it contains $(\mathfrak{g}^0, \mathfrak{h}^0)$ a finite simple complex Lie algebra of finite type T and a Cartan subalgebra. Denote by $\mathrm{Dyn}(Y)$ the Dynkin diagram associated with the type Y, by $\Phi^+(Y) \subset \Phi(Y) \subset \mathfrak{h}^*$ the positive roots and root system associated with the data $(\mathfrak{g}, \mathfrak{h})$ and by $\Phi^\vee(Y) \subset \mathfrak{h}$ the coroot system. If $\mathfrak{g}$ is of finite type, let $\Delta(T)$ be the set of simple roots associated with the fundamental Cartan chamber and $\Delta^\vee(T)$ the set of simple coroots. If $\mathfrak{g}$ is of affine type $\tilde{T}$ let α_0 and $\alpha_0^\vee$ be respectively the root and coroot defined in [Kac, §6.4]. Then $\Delta(\tilde{T}) := \{\alpha_0\} \coprod \Delta(T)$ and $\Delta^\vee(\tilde{T}) := \{\alpha_0^\vee\} \coprod \Delta^\vee(T)$. Moreover let $\mathbf{d}$ be the element of $\mathfrak{h}$ defined in [Kac, §6.2]. Note that $\Delta^\vee(\tilde{T}) \cup \{\mathbf{d}\}$ is a basis of $\mathfrak{h}$. Consider the generalized Cartan matrix denoted by $A(Y)$ associated with the data $\big(\Delta(Y), \Delta^\vee(Y)\big)$. Note that ${}^tA(Y)$ is the generalized Cartan matrix associated with the data $\big(\Delta^\vee(Y), \Delta(Y)\big)$. Denote by $Q(Y)$ and $Q^\vee(Y)$ respectively the root and coroot lattice of type Y. Let $W(Y)$ be the Weyl group associated with the root system of type Y. In the affine case, one root and one coroot will have a special role. Let us denote by $\delta(\tilde{T}) := \sum_{\alpha \in \Delta(\tilde{T})} \delta_\alpha \alpha \in \Phi(\tilde{T})$ the affine root such that $A(\tilde{T})\delta(\tilde{T}) = 0$ and such that the integral coefficients $(\delta_\alpha)_{\alpha \in \Delta(\tilde{T})} \in \mathbb{Z}_{\geq 0}$ are the smallest possible. In the same way, one can define $\delta^\vee(\tilde{T}) := \sum_{\alpha^\vee \in \Delta^\vee(\tilde{T})} \delta^\vee_{\alpha^\vee} \alpha^\vee \in \Phi^\vee(\tilde{T})$ the affine coroot such that ${}^tA(\tilde{T})\delta^\vee(\tilde{T}) = 0$ and the integral coefficients $(\delta^\vee_{\alpha^\vee})_{\alpha^\vee \in \Delta^\vee(\tilde{T})} \in \mathbb{Z}_{\geq 0}$ are the smallest possible. Let the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$ be denoted by $\langle , \rangle$. Finally, the fundamental weights and coweights will be needed. For each $\alpha \in \Delta(\tilde{T})$, consider $\Lambda_\alpha \in \mathfrak{h}^*$ such that:

$$\forall \alpha^\vee \in \Delta^\vee(\tilde{T}), \langle \Lambda_\alpha, \alpha^\vee \rangle = \begin{cases} 1 & \text{if } \langle \alpha, \alpha^\vee \rangle = 2 \\ 0 & \text{else} \end{cases} \text{ and } \langle \Lambda_\alpha, \mathbf{d} \rangle = 0$$

Similarly one can define the fundamental coweights. For each $\alpha^\vee \in \Delta^\vee(\tilde{T})$, consider $\Lambda^\vee_{\alpha^\vee} \in \mathfrak{h}$ such that:

$$\forall \alpha \in \Delta(\tilde{T}), \langle \alpha, \Lambda^\vee_{\alpha^\vee} \rangle = \begin{cases} 1 & \text{if } \langle \alpha, \alpha^\vee \rangle = 2 \\ 0 & \text{else} \end{cases} \text{ and } \langle \Lambda_{\alpha_0}, \Lambda^\vee_{\alpha^\vee} \rangle = 0$$

Let us denote by $P(Y)$ and $P^\vee(Y)$ respectively the weight and coweight lattice of type Y respectively generated by the fundamental weights and fundamental coweights.

Definition 0.1. If $\alpha \in Q(\tilde{T})$, then the size of α is defined as $|\alpha|_{\tilde{T}} := \langle \alpha, \sum_{\alpha^\vee \in \Delta^\vee(\tilde{T})} \delta^\vee_{\alpha^\vee} \Lambda^\vee_{\alpha^\vee} \rangle$ and if $\alpha^\vee \in Q^\vee(\tilde{T})$, define $|\alpha^\vee|_{\tilde{T}} := \langle \sum_{\alpha \in \Delta(\tilde{T})} \delta_\alpha \Lambda_\alpha, \alpha^\vee \rangle$.

The group $W(\tilde{T})$ is the affine Weyl group of the finite type T. One has the following isomorphism ([BLie02, Chap. VI, §2, Prop. 1])

$$W(T) \ltimes Q(T) \xrightarrow{\sim} W(\tilde{T})$$

For $a \in Q(T)$, one will denote by $t_a \in W(\tilde{T})$ the corresponding element.
All algebraic varieties will be over $\mathbb{C}$ and by algebraic variety one means a reduced scheme such that the structure morphism is separated and of finite type. If S is an

algebraic variety and $s \in S$, denote by $\kappa_S(s)$ the residue field of the local ring $\mathcal{O}_{S,s}$. A reductive algebraic group is a connected linear algebraic group on $\mathbb{C}$ with trivial unipotent radical. From now on and in the rest of this thesis, fix an integer $n \geq 1$ and a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$.

CHAPTER 1

QUIVER VARIETIES OVER MCKAY QUIVERS

This chapter is here to set up the quiver theoretical background needed in Chapter 2. It is inspired by the work of George Lusztig [L92, Section 2], of Michela Varagnolo and Eric Vasserot [VV99, Section 2], as well as the one of Weiqiang Wang [Wang, Theorem 5.1]. It is decomposed into four parts. Firstly, we recall the construction of the representation space of a double framed quiver and of its natural symplectic structure. Then, if one starts with the double framed McKay quiver of a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, one can construct the representation space of this quiver in terms of Γ-modules. This is the new setting to work with McKay quiver varieties. This setting will be used to prove the main theorem in Chapter 2. In the third part, we show that over the double framed McKay quiver the two representation spaces are isomorphic and that moreover, the symplectic forms coincide as well as the momentum maps. In the third section of this chapter we will, for McKay quivers, define an analog of Nakajima's quiver varieties that does not involve the choice of an orientation of the underlying quiver and thus will provide a more intrinsic point of view. We will show that this variety is isomorphic to Nakajima's quiver variety. Finally, we will give a couple of results that are necessary for the following chapters.

1.1 Representations of double framed quivers

Let us begin this section with general notation for quivers. Take an undirected multigraph $G := (I_G, E_G)$ where I_G is the set of vertices and E_G the multiset of undirected edges and choose an orientation $\Omega : E_G \to I_G \times I_G$ for G.
The framed undirected multigraph associated with G denoted by G^f is the multigraph with one extra vertex for each vertex in I_G so that $I_{G^f} = I_G \coprod I_G$ is the set of vertices and the multiset of edges is defined as $E_{G^f} := E_G \coprod \{\{i, j_i\} | i \in I_G\}$ where j_i denotes the new vertex associated with $i \in I_G$. Let us denote by $Q_G(\Omega)$ the quiver associated with the undirected multigraph G and the orientation Ω. Let $E_G^{\Omega+}$ be the multiset of arrows of $Q_G(\Omega)$ oriented with Ω. The source and target maps of $Q_G(\Omega)$ are given by Ω and will be respectively denoted by h' and h'' for $h \in E_G^{\Omega+}$. Let us impose the convention that the orientation for the framing arrows is the "outgoing" orientation. This implies that choosing an orientation of a framed quiver $\Omega^f : E_{G^f} \to I_{G^f} \times I_{G^f}$ is the same data as choosing an orientation Ω of the associated unframed undirected multigraph. Consider the following orientation

$$\Omega^{\mathrm{op}} : \begin{array}{ccc} E_G & \to & I_G \times I_G \\ h & \mapsto & (h'', h') \end{array}$$

then $Q_G(\Omega^{\mathrm{op}})$ is called the opposite quiver of $Q_G(\Omega)$ i.e. the quiver which has the same underlying undirected multigraph G but reversed orientation. The multiset of arrows of $Q_G^{\mathrm{op}}(\Omega)$ will be denoted by $E_G^{\Omega-}$. Finally denote by $\overline{Q_G}$ the double quiver associated with G which has the same set of vertices as G and the multiset of arrows $\overline{E_G} := E_G^{\Omega+} \coprod E_G^{\Omega-}$ so that $\overline{Q_G}$ does not depend on the choice of Ω. Let $\bar{h}$ be the involution of $\overline{E_G}$ sending an edge $h \in \overline{E_G}$ to its reversed edge. Let us denote by Δ_G the free abelian group associated with the set I_G. Let $\Delta_G^+ \subset \Delta_G$ be the free monoid associated with I_G. The set Δ_G^+ will be referred to as the set of dimension parameters. A dimension parameter d will often be defined by giving nonnegative integers $(d_\nu)_{\nu \in I_G}$ which gives $d := \sum_{\nu \in I_G} d_\nu \nu \in \Delta_G^+$. Let us moreover consider the set of stability parameters denoted by $\Theta_G := \mathrm{Hom}_{\mathbb{Z}}(\Delta_G, \mathbb{Q})$ and the set of deformation parameters denoted by $\Lambda_G := \mathrm{Hom}_{\mathbb{Z}}(\Delta_G, \mathbb{C})$. If $\theta \in \mathbb{Q}$, let us denote by θ the stability parameter defined as $\theta(\nu) := \theta$ for all $\nu \in I_G$. If $\lambda \in \mathbb{C}$, the same notation will be used for the constant deformation parameter equal to λ. Denote $\Theta_G^+ := \{\theta \in \Theta_G | \forall \nu \in I_G, \theta(\nu) \geq 0\}$, $\Theta_G^{++} := \{\theta \in \Theta_G | \forall \nu \in I_G, \theta(\nu) > 0\}$ and $\Lambda_G^+ := \{\lambda \in \Lambda_G | \forall \nu \in I_G, \lambda(\nu) \in \mathbb{Q}_{\geq 0}\}$, $\Lambda_G^{++} := \{\lambda \in \Lambda_G | \forall \nu \in I_G, \lambda(\nu) \in \mathbb{Q}_{>0}\}$. Let us denote $\mathbf{Rep}(\overline{Q_{G^f}})$ the category of representations of the quiver $\overline{Q_{G^f}}$ in which objects are tuples $(V, V^f, (x_h)_{h \in \overline{E_G}}, (v_i^1, v_i^2)_{i \in I_G})$ where

- $V = \oplus_{i \in I_G} V_i$ is an I_G-graded complex vector space.
- $V^f = \oplus_{i \in I_G} V_i^f$ is an I_G-graded complex vector space.
- $\forall h \in \overline{E_G}, x_h : V_{h'} \to V_{h''}$ is a linear map.
- $\forall i \in I_G, v_i^1 : V_i^f \to V_i$ is a linear map.
- $\forall i \in I_G, v_i^2 : V_i \to V_i^f$ is a linear map.

Note that v_i^1 and v_i^2 correspond to the representation of the framing. A morphism between (V, V^f, x, v^1, v^2) and $(\tilde{V}, \tilde{V}^f, \tilde{x}, \tilde{v}^1, \tilde{v}^2)$ is given by the following data. For all $i \in I_G$ we need to provide a morphism $\phi_i : V_i \to \tilde{V}_i$ and a morphism $\psi_i : V_i^f \to \tilde{V}_i^f$ such that for all $h \in \overline{E_G}$ and $i \in I_G$ the following diagrams commute

$$\begin{array}{ccc} V_{h'} & \xrightarrow{\phi_{h'}} & \tilde{V}_{h'} \\ {\scriptstyle x_h}\downarrow & & \downarrow{\scriptstyle \tilde{x}_h} \\ V_{h''} & \xrightarrow[\phi_{h''}]{} & \tilde{V}_{h''} \end{array}$$

$$\begin{array}{ccccc} V_i & \xrightarrow{v_i^2} & V_i^f & \xrightarrow{v_i^1} & V_i \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \phi_i} \\ \tilde{V}_i & \xrightarrow[\tilde{v}_i^2]{} & \tilde{V}_i^f & \xrightarrow[\tilde{v}_i^1]{} & \tilde{V}_i \end{array}$$

Definition 1.1. Given two I_G-graded complex vector spaces V, V^f define the representation space of $\overline{Q_{G^f}}$ with fixed I_G-graded complex vector spaces V, V^f as

$$\mathcal{R}^G_{V,V^f} := \bigoplus_{h\in\overline{E_G}} \mathrm{Hom}(V_{h'}, V_{h''}) \oplus \bigoplus_{i\in I_G} \mathrm{Hom}(V^f_i, V_i) \oplus \bigoplus_{i\in I_\Gamma} \mathrm{Hom}(V_i, V^f_i)$$

Remark 1.2. Take a vertex $i_0 \in I_G$ and define V^{f,i_0} such that $\forall i \in I_G, V^{f,i_0}_i = \mathbb{C}^{\delta^{i_0}_i}$. In the following, let us shorten $\mathcal{R}^G_{V,V^{f,i_0}}$ to just $\mathcal{R}^G_{V,i_0}$. Note that in that case v^1 is just a linear map from $\mathbb{C}$ to V_{i_0} i.e. can be identified with an element of V_{i_0}. We will refer to $\mathcal{R}^G_{V,i_0}$ as the representation space of the double, framed at i_0, quiver $Q_G(\Omega)$.

The construction of a symplectic structure on $\mathcal{R}^G_{V,V^f}$ will depend on the choice of an orientation Ω. Recall that we have chosen an orientation Ω of G. The sign function associated with Ω will be denoted by $\epsilon_\Omega : \overline{E_G} \to \{-1, 1\}$. This map is such that $\forall h \in E^{\Omega+}_G, \epsilon_\Omega(h) := 1$ and $\forall h \in E^{\Omega-}_G, \epsilon_\Omega(h) := -1$. Define the symplectic form

$$\omega^{\epsilon_\Omega}_G : \begin{array}{ccc} \mathcal{R}^G_{V,V^f} \times \mathcal{R}^G_{V,V^f} & \to & \mathbb{C} \\ ((x,v^1,v^2),(\tilde{x},\tilde{v}^1,\tilde{v}^2)) & \mapsto & \sum_{h\in\overline{E_G}} \epsilon_\Omega(h)\mathrm{Tr}(x_h\tilde{x}_{\bar{h}}) + \sum_{i\in I_G} \mathrm{Tr}(v^1_i\tilde{v}^2_i - \tilde{v}^1_i v^2_i) \end{array}$$

This structure comes from the natural isomorphism with the cotangent bundle of the representation space of the framed quiver. Consider the group $\mathrm{G}(V) := \prod_{i\in I_G} \mathrm{GL}(V_i)$ and its action on $\mathcal{R}^G_{V,V^f}$ given by

$$g.(x,v^1,v^2) := (g.x, (g_i v^1_i)_{i\in I_G}, (v^2_i g_i^{-1})_{i\in I_G})$$

for $g \in \mathrm{G}(V), (x,v^1,v^2) \in \mathcal{R}^G_{V,V^f}$ and where $\forall h \in \overline{E_G}, (g.x)_h := g_{h''} x_h g^{-1}_{h'}$.

Remark 1.3. The symplectic form $\omega^{\epsilon_\Omega}_G$ is $\mathrm{G}(V)$-invariant.

For $\theta \in \Theta_G$ such that $\mathrm{Im}(\theta) \subset \mathbb{Z}$, define the character of $\mathrm{G}(V)$ associated with $\theta \in \Theta_G$ as

$$\chi_\theta : \begin{array}{ccc} \mathrm{G}(V) & \to & \mathbb{C}^* \\ (g_i)_{i\in I_G} & \mapsto & \prod_{i\in I_G} \det(g_i)^{\theta(i)} \end{array} .$$

Remark 1.4. Note that the main difference between the framed and unframed setting is that here we let $\mathrm{G}(V)$ act and not $\mathrm{G}(V) \times \mathrm{G}(V^f)$.

Consider now the momentum map attached to that action [Kir, Definition 9.43]

$$\mu^{\epsilon_\Omega}_G : \begin{array}{ccc} \mathcal{R}^G_{V,V^f} & \to & \bigoplus_{i\in I_G} \mathrm{End}(V_i) \\ (x,v^1,v^2) & \mapsto & \sum_{h\in\overline{E_G}, h''=i} \epsilon_\Omega(h) x_h x_{\bar{h}} + \sum_{j\in I_G} v^1_j v^2_j \end{array}$$

For $\lambda \in \Lambda_G$, let us still denote by $\lambda := \sum_{i\in I_G} \lambda(i)\mathrm{id}_{V_i} \in \bigoplus_{i\in I_G} \mathrm{End}(V_i)$.

1.2 Representation space of double framed McKay quivers

Take $k \in \mathbb{Z}_{\geq 0}$ and recall that we have fixed Γ a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. By a Γ-module, we mean a $\mathbb{C}[\Gamma]$-module of finite dimension. This subsection is devoted to exposing the representation space of the double framed McKay quiver of Γ in terms

of Γ-modules. Denote by $\mathrm{Rep}_{\Gamma,k} := \mathrm{Hom}_{\mathrm{gp}}(\Gamma, \mathrm{GL}_k(\mathbb{C}))$ and by $\mathrm{Char}_{\Gamma,k}$ the set of all characters of elements in $\mathrm{Rep}_{\Gamma,k}$. Consider $\mathrm{Rep}_\Gamma := \bigcup_{k\geq 0} \mathrm{Rep}_{\Gamma,k}$ and Char_Γ the set of all characters of representations in Rep_Γ. For $\chi \in \mathrm{Char}_\Gamma$, let us denote $k_\chi := \chi(\mathrm{id})$ and choose $\rho_\chi \in \mathrm{Rep}_{\Gamma,k_\chi}$ such that the character of ρ_χ denoted by $\mathrm{Tr}(\rho_\chi)$ is χ. Note that ρ_χ is determined, up to conjugation by an element of $\mathrm{GL}_{k_\chi}(\mathbb{C})$. Denote the representation space of ρ_χ by X_χ which is endowed with the Γ-action given by ρ_χ. Let Irr_Γ be the set of all characters of irreducible representations of Γ. It is finite since Γ is finite. Denote by $\chi_0 \in \mathrm{Irr}_\Gamma$ the trivial character. The group Γ being a subgroup of $\mathrm{SL}_2(\mathbb{C})$, it has a natural representation of dimension 2 called the standard representation and denoted by ρ_{std}. In the following, the character of the standard representation, which is irreducible whenever Γ is not a cyclic group, and its associated representation space will be respectively denoted by χ_{std} and X_{std}. If A and B are two Γ-modules, then $A \to_\Gamma B$ denotes a Γ-equivariant morphism and $\mathrm{Hom}_\Gamma(A, B)$ will denote the set of all Γ-equivariant morphisms.
Consider $\mathbf{McK}(\Gamma)$ the category in which objects are tuples of the form (M, M^f, Δ, Z) where

- M is a Γ-module
- M^f is a Γ-module
- $\Delta : X_{\mathrm{std}} \otimes M \to_\Gamma M$
- $Z_1 : M^f \to_\Gamma M$
- $Z_2 : M \to_\Gamma M^f$

Morphisms of this category, between $(M, M^f, \Delta, Z_1, Z_2)$ and $(\tilde{M}, \tilde{M}^f, \tilde{\Delta}, \tilde{Z}_1, \tilde{Z}_2)$, are pairs (Φ, Ψ) where $\Phi : M \to_\Gamma \tilde{M}$ and $\Psi : M^f \to_\Gamma \tilde{M}^f$ are such that the following diagrams commute

$$\begin{array}{ccc} X_{\mathrm{std}} \otimes M & \xrightarrow{\Delta} & M \\ {\scriptstyle \mathrm{id}\otimes\Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ X_{\mathrm{std}} \otimes \tilde{M} & \xrightarrow[\tilde{\Delta}]{} & \tilde{M} \end{array}$$

$$\begin{array}{ccccc} M & \xrightarrow{Z_2} & M^f & \xrightarrow{Z_1} & M \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Phi} \\ \tilde{M} & \xrightarrow[\tilde{Z}_2]{} & \tilde{M}^f & \xrightarrow[\tilde{Z}_1]{} & \tilde{M} \end{array}$$

The universal property of the tensor product gives

$$\mathrm{Hom}(X_{\mathrm{std}} \otimes M, M) \simeq \mathrm{Hom}(X_{\mathrm{std}}, \mathrm{End}(M))$$

Moreover, if we let Γ act on $\mathrm{End}(M)$ by conjugacy, then we have

$$\mathrm{Hom}_\Gamma(X_{\mathrm{std}} \otimes M, M) \simeq \mathrm{Hom}_\Gamma(X_{\mathrm{std}}, \mathrm{End}(M))$$

For $\Delta \in \mathrm{Hom}_\Gamma(X_{\mathrm{std}} \otimes M, M)$ and $x \in X_{\mathrm{std}}$, let us denote Δ_x the obtained endomorphism of M. Concretely, for all $m \in M$, $\Delta_x(m)$ is equal to $\Delta(x \otimes m)$.

Definition 1.5. Take two Γ-modules M, M^f and define the following complex vector space attached to M, M^f

$$\mathcal{R}^{\Gamma}_{M,M^f} := \mathrm{Hom}_{\Gamma}(X_{\mathrm{std}} \otimes M, M) \oplus \mathrm{Hom}_{\Gamma}(M^f, M) \oplus \mathrm{Hom}_{\Gamma}(M, M^f)$$

Definition 1.6. Let $\mathrm{Aut}_{\Gamma}(M)$ be the algebraic group of linear automorphisms of M commuting with the Γ-action.

We can now consider an action of $\mathrm{Aut}_{\Gamma}(M)$ on $\mathcal{R}^{\Gamma}_{M,M^f}$ given by

$$g.(\Delta, Z_1, Z_2) := (g.\Delta, gZ_1, Z_2 g^{-1})$$

for $g \in \mathrm{Aut}_{\Gamma}(M)$, $(\Delta, Z_1, Z_2) \in \mathcal{R}^{\Gamma}_{M,M^f}$ and where

$$\forall (x, m) \in X_{\mathrm{std}} \times M, g.\Delta(x \otimes m) := g\Delta(x \otimes g^{-1}m)$$

Let us denote by (e_1, e_2) the canonical basis of X_{std}.

Remark 1.7. One can check that $\Delta_{e_1}\Delta_{e_2} - \Delta_{e_2}\Delta_{e_1} \in \mathrm{End}_{\Gamma}(M)$, using the following fact

$$\forall g \in \mathrm{GL}_2(\mathbb{C}), \Delta_{g.e_1}\Delta_{g.e_2} - \Delta_{g.e_2}\Delta_{g.e_1} = \det(g)(\Delta_{e_1}\Delta_{e_2} - \Delta_{e_2}\Delta_{e_1})$$

In the special case where $M^f = X_\chi$ for some $\chi \in I_\Gamma$, let us denote $\mathcal{R}^{\Gamma}_{M,M^f}$ by $\mathcal{R}^{\Gamma}_{M,\chi}$ and $\mathcal{R}^{\Gamma}_{M} := \mathcal{R}^{\Gamma}_{M,\chi_0}$.

Define an $\mathrm{Aut}_{\Gamma}(M)$-invariant symplectic form on $\mathcal{R}^{\Gamma}_{M,M^f}$

$$\omega_{\Gamma} : \begin{array}{ccc} \mathcal{R}^{\Gamma}_{M,M^f} \times \mathcal{R}^{\Gamma}_{M,M^f} & \to & \mathbb{C} \\ \left((\Delta, Z_1, Z_2), (\tilde{\Delta}, \tilde{Z}_1, \tilde{Z}_2)\right) & \mapsto & \mathrm{Tr}\left(\Delta_{e_1}\tilde{\Delta}_{e_2} - \Delta_{e_2}\tilde{\Delta}_{e_1}\right) + \mathrm{Tr}\left(Z_1\tilde{Z}_2 - \tilde{Z}_1 Z_2\right) \end{array}$$

Remark 1.8. Note that ω_{Γ} is independent of the choice of basis of X_{std} as long as one picks a basis (e, f) such that $\det(e, f) = 1$.

The momentum map attached to ω_{Γ} and the action of $\mathrm{Aut}_{\Gamma}(M)$ on $\mathcal{R}^{\Gamma}_{M,M^f}$ is

$$\mu_{\Gamma} : \begin{array}{ccc} \mathcal{R}^{\Gamma}_{M,M^f} & \to & \mathrm{End}_{\Gamma}(M) \\ (\Delta, Z_1, Z_2) & \mapsto & \Delta_{e_1}\Delta_{e_2} - \Delta_{e_2}\Delta_{e_1} + Z_1 Z_2 \end{array}$$

Remark 1.9. Note that the symplectic form and the momentum map only depend on the finite group Γ.

Definition 1.10. Define the McKay undirected multigraph G_{Γ} associated with Γ the following way. The set of vertices, denoted I_{Γ}, is Irr_{Γ} and there is an edge between a pair of irreducible characters (χ, χ') if and only if $\langle \chi\chi_{\mathrm{std}} | \chi' \rangle \neq 0$ with multiplicity $\langle \chi\chi_{\mathrm{std}} | \chi' \rangle$.

Remark 1.11. Note that G_{Γ} is indeed undirected because Γ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$. By choosing an orientation of the McKay multigraph of Γ, we define the McKay quiver denoted by Q_{Γ}. The choice of orientation will not be relevant to this section.

For the sake of clarity, we will shorten the notation and when no confusion is possible, write Γ instead of G_{Γ}. For example, we will shorten $\Delta_{G_{\Gamma}}$ to Δ_{Γ}. Let us finish this subsection by presenting a way to formulate the McKay Correspondence. This will be useful in Chapter 5.

Proposition 1.12 (McKay Correspondence, [McKay]). *For each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, if V_Γ denotes the vector space $\Delta_\Gamma \otimes \mathbb{C}$, then I_Γ defines an affine root system in V_Γ of type $\tilde{T}_\Gamma$ where $\tilde{T}_\Gamma$ is the type of the Dynkin diagram G_Γ.*

Remark 1.13. To be a bit more precise and use the language developed by Kac in [Kac, Chapter 1]. Let $\tilde{\Pi}_\Gamma^\vee := I_\Gamma$, which is then a base of V_Γ. We can now construct a realization of the generalized Cartan matrix of type $\tilde{T}_\Gamma$ out of V_Γ and $\tilde{\Pi}_\Gamma^\vee$. Consider the following linear map defined on the base I_Γ

$$\phi : \begin{array}{ccc} V_\Gamma & \to & V_\Gamma^* \\ \chi & \mapsto & (\psi \mapsto 2\langle \chi, \psi \rangle - \langle \chi \chi_{\mathrm{std}} | \psi \rangle) \end{array}$$

Note that the element $\phi(\chi) \in V_\Gamma^*$ is defined on the base I_Γ of V_Γ. By construction, we have that $\phi(\chi)(\psi)$ is exactly the corresponding coefficient of the generalized Cartan matrix of type $\tilde{T}_\Gamma$. Let us denote this coefficient by $a_{\chi,\psi}$. Take $\mathfrak{t}_0$ to be a one-dimensional complex vector space. Consider $\mathfrak{t}_\Gamma := \mathfrak{t}_0 \oplus V_\Gamma$. For each $\chi \in I_\Gamma, \tilde{\alpha}_\chi \in \mathfrak{t}_\Gamma^*$ define

$$\forall (t_0, (c_\psi)_\psi) \in \mathfrak{t}_0 \times \mathbb{C}^{|I_\Gamma|}, \tilde{\alpha}_\chi(t_0 + \sum_{\psi \in I_\Gamma} c_\psi \psi) = t_0 + \sum_{\psi \in I_\Gamma} c_\psi a_{\psi,\chi}$$

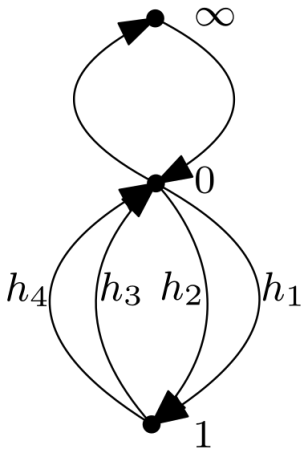
The set $\tilde{\Pi}_\Gamma := \{\tilde{\alpha}_\chi | \chi \in I_\Gamma\}$ is linearly independant. The triple $(\mathfrak{t}_\Gamma, \tilde{\Pi}_\Gamma, \tilde{\Pi}_\Gamma^\vee)$ is a realization of the generalized Cartan matrix of type $\tilde{T}_\Gamma$. Consider $(\mathfrak{h}_\Gamma, \Pi_\Gamma, \Pi_\Gamma^\vee) := (\mathfrak{t}_\Gamma^*, \tilde{\Pi}_\Gamma^{\vee **}, \tilde{\Pi}_\Gamma)$ where

$$^{**} : \begin{array}{ccc} \mathfrak{t}_\Gamma & \to & \mathfrak{h}_\Gamma^* = \mathfrak{t}_\Gamma^{**} \\ t & \mapsto & (\phi \mapsto \phi(t)) \end{array}$$

For $\chi \in I_\Gamma$, let us denote $\alpha_\chi := \chi^{**} \in \Pi_\Gamma$. The type $\tilde{T}_\Gamma$ being simply laced, $(\mathfrak{h}_\Gamma, \Pi_\Gamma, \Pi_\Gamma^\vee)$ is also a realization of the generalized Cartan matrix of type $\tilde{T}_\Gamma$ and we will referred to it as the realization given by the McKay correspondence.

1.3 Equivalence of categories between $\mathbf{Rep}(\overline{Q_{\Gamma f}})$ and $\mathbf{McK}(\Gamma)$

In this section, we will establish an equivalence between the two previously defined categories. To be more precise, let us show that $\mathbf{Rep}(\overline{Q_{\Gamma f}})$ and $\mathbf{McK}(\Gamma)$ are equivalent and that this equivalence is compatible with the momentum maps. To do so, we have to make choices. Thanks to the fact that Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, we know that for all $h \in \overline{E_\Gamma}$, $T_h^\Gamma := \mathrm{Hom}_\Gamma(X_{\mathrm{std}} \otimes X_{h'}, X_{h''})$ is of dimension either $0, 1$ or 2 (it is only of dimension 2 for $\Gamma = \mu_2$, the cyclic group with two elements). For each edge $h \in \overline{E_\Gamma}$, choose a nonzero element $y_h^0 \in \mathrm{Hom}_\Gamma(X_{\mathrm{std}} \otimes X_{h'}, X_{h''})$. The irreducibility of $X_{h''}$ implies that y_h^0 is surjective. Since representations of finite groups are semisimple, we can consider a Γ-equivariant section of y_h^0 denoted by $\tilde{y}_h^0 \in \mathrm{Hom}_\Gamma(X_{h''}, X_{\mathrm{std}} \otimes X_{h'})$. For μ_2, we can construct everything explicitly. Take the following labeling of the double, framed at χ_0 McKay quiver of μ_2



such that $\overline{h_1} = h_4$ and $\overline{h_3} = h_2$. Let χ_1 denote the nontrivial irreducible character of μ_2 and let $X_1 := X_{\chi_1}$. With this notation, $X_{\text{std}} = X_1 \oplus X_1$. Take these maps

- $y^0_{h_1} : \begin{array}{ccc} (X_1 \oplus X_1) \otimes X_0 & \to & X_1 \\ (a,b) \otimes 1 & \mapsto & a \end{array}$
- $\tilde{y}^0_{h_1} : \begin{array}{ccc} X_1 & \to & (X_1 \oplus X_1) \otimes X_0 \\ 1 & \mapsto & (1,0) \otimes 1 \end{array}$
- $y^0_{h_2} : \begin{array}{ccc} (X_1 \oplus X_1) \otimes X_0 & \to & X_1 \\ (a,b) \otimes 1 & \mapsto & b \end{array}$
- $\tilde{y}^0_{h_2} : \begin{array}{ccc} X_1 & \to & (X_1 \oplus X_1) \otimes X_0 \\ 1 & \mapsto & (0,1) \otimes 1 \end{array}$
- $y^0_{h_3} : \begin{array}{ccc} (X_1 \oplus X_1) \otimes X_1 & \to & X_0 \\ (a,b) \otimes c & \mapsto & ac \end{array}$
- $\tilde{y}^0_{h_3} : \begin{array}{ccc} X_0 & \to & (X_1 \oplus X_1) \otimes X_1 \\ 1 & \mapsto & (1,0) \otimes 1 \end{array}$
- $y^0_{h_4} : \begin{array}{ccc} (X_1 \oplus X_1) \otimes X_1 & \to & X_0 \\ (a,b) \otimes c & \mapsto & bc \end{array}$
- $\tilde{y}^0_{h_4} : \begin{array}{ccc} X_0 & \to & (X_1 \oplus X_1) \otimes X_1 \\ 1 & \mapsto & (0,1) \otimes 1 \end{array}$

Lemma 1.14. *For each $\chi \in I_\Gamma$, $\sum_{h \in \overline{E_\Gamma}, h' = \chi} \tilde{y}^0_h \circ y^0_h = \mathrm{id}_{X_{\text{std}} \otimes X_\chi}$*

Proof. For $\Gamma = \mu_2$ it is an easy computation. Now take $\Gamma \neq \mu_2$. It is then clear that $\sum_{h \in \overline{E_\Gamma}, h' = \chi} y^0_h \in \mathrm{Hom}_\Gamma(X_{\text{std}} \otimes X_\chi, \bigoplus_{h \in \overline{E_\Gamma}, h' = \chi} X_{h''})$ is surjective. By construction of the McKay graph, we have

$$\dim(X_{\text{std}} \otimes X_\chi) = \sum_{h \in \overline{E_\Gamma}} \dim(X_{h''})$$

In particular, this implies that $\sum_{h \in \overline{E_\Gamma}, h' = \chi} y^0_h$ is an isomorphism. It is then enough to prove that

$$\Big(\sum_{l \in \overline{E_\Gamma}, l' = \chi} y^0_l \Big) \circ \Big(\sum_{h \in \overline{E_\Gamma}, h' = \chi} \tilde{y}^0_h \circ y^0_h \Big) = \sum_{h \in \overline{E_\Gamma}, h' = \chi} y^0_h$$

For each pair of edge $(l, h) \in \overline{E_\Gamma}^2$ such that $l' = h'$, using Schur's Lemma, we have that

$$y^0_l \circ \tilde{y}^0_h = \begin{cases} \mathrm{id}_{X_{h''}} & \text{if } l = h \\ 0 & \text{otherwise} \end{cases}$$

□

Consider a special dimension parameter $\delta^\Gamma \in \Delta_\Gamma$ defined as $\delta^\Gamma_\chi := \dim(X_\chi)$ for all $\chi \in I_\Gamma$. With the preceding choice of orientation and maps for μ_2, everything that will follow will also work.

Define two functors $F_{y^0} : \mathbf{Rep}(\overline{Q_{\Gamma^f}}) \to \mathbf{McK}(\Gamma)$ and $G_{\tilde{y}^0} : \mathbf{McK}(\Gamma) \to \mathbf{Rep}(\overline{Q_{\Gamma^f}})$. First, let

$$F_{y^0} : \begin{array}{ccc} \mathbf{Rep}(\overline{Q_{\Gamma^f}}) & \to & \mathbf{McK}(\Gamma) \\ (V, V^f, x, v^1, v^2) & \rightsquigarrow & (M_V, M_{V^f}, \Delta^x, Z_1^v, Z_2^v) \end{array}$$

where

- $M_V := \bigoplus_{\chi \in I_\Gamma} V_\chi \otimes X_\chi$
- $M_{V^f} := \bigoplus_{\chi \in I_\Gamma} V_\chi^f \otimes X_\chi$
- Δ^x is defined as the following composition

$$\begin{array}{ccc} X_{\mathrm{std}} \otimes \left(\bigoplus_{\chi \in I_\Gamma} V_\chi \otimes X_\chi \right) & \xrightarrow{\Delta^x} & \bigoplus_{\chi' \in I_\Gamma} V_{\chi'} \otimes X_{\chi'} \\ \wr \downarrow & \nearrow_{\sum_{h \in \overline{E_\Gamma}} y_h^0 \otimes x_h} & \\ \bigoplus_{\chi \in I_\Gamma} X_{\mathrm{std}} \otimes V_\chi \otimes X_\chi & & \end{array}$$

- $Z_1^v := (\sum_{\chi \in I_\Gamma} v_\chi^1 \otimes \frac{1}{\delta_\chi^\Gamma} \mathrm{id}_{X_\chi}) : \bigoplus_{\chi \in I_\Gamma} V_\chi^f \otimes X_\chi \to \bigoplus_{\chi \in I_\Gamma} V_\chi \otimes X_\chi$
- $Z_2^v := (\sum_{\chi \in I_\Gamma} v_\chi^2 \otimes \mathrm{id}_{X_\chi}) : \bigoplus_{\chi \in I_\Gamma} V_\chi \otimes X_\chi \to \bigoplus_{\chi \in I_\Gamma} V_\chi^f \otimes X_\chi$

Moreover F_{y^0} sends a morphism (ϕ_χ, ψ_χ) to (Φ, Ψ) where

- $\Phi := \bigoplus_{\chi \in I_\Gamma} \phi_\chi \otimes \mathrm{id}_{X_\chi}$
- $\Psi := \bigoplus_{\chi \in I_\Gamma} \psi_\chi \otimes \mathrm{id}_{X_\chi}$

Now, let

$$G_{\tilde{y}^0} : \begin{array}{ccc} \mathbf{McK}(\Gamma) & \to & \mathbf{Rep}(\overline{Q_{\Gamma^f}}) \\ (M, M^f, \Delta, Z_1, Z_2) & \rightsquigarrow & (V_M, V_{M^f}, x^\Delta, v_{Z_1}, v_{Z_2}) \end{array}$$

where

- $V_M = \bigoplus_{\chi \in I_\Gamma} \mathrm{Hom}_\Gamma(X_\chi, M)$
- $V_{M^f} = \bigoplus_{\chi \in I_\Gamma} \mathrm{Hom}_\Gamma(X_\chi, M^f)$.
- For all $h \in \overline{E_\Gamma}$, define x_h^Δ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_\Gamma(X_{h'}, M) & \xrightarrow{x_h^\Delta} & \mathrm{Hom}_\Gamma(X_{h''}, M) \\ {\scriptstyle -\otimes \mathrm{id}_{X_{\mathrm{std}}}} \downarrow & & \uparrow {\scriptstyle \Delta \circ -} \\ \mathrm{Hom}_\Gamma(X_{\mathrm{std}} \otimes X_{h'}, X_{\mathrm{std}} \otimes M) & \xrightarrow[-\circ \tilde{y}_h^0]{} & \mathrm{Hom}_\Gamma(X_{h''}, X_{\mathrm{std}} \otimes M) \end{array}$$

- $\forall \chi \in I_\Gamma$

$$(v_{Z_1})_\chi : \begin{array}{ccc} \mathrm{Hom}_\Gamma(X_\chi, M^f) & \to & \mathrm{Hom}_\Gamma(X_\chi, M) \\ f & \mapsto & \delta_\chi^\Gamma Z_1 \circ f \end{array}$$

- $\forall \chi \in I_\Gamma$

$$(v_{Z_2})_\chi : \begin{array}{ccc} \mathrm{Hom}_\Gamma(X_\chi, M) & \to & \mathrm{Hom}_\Gamma(X_\chi, M^f) \\ f & \mapsto & Z_2 \circ f \end{array}$$

Moreover $G_{\tilde{y}^0}$ sends a morphism (Φ, Ψ) to $(\phi_\chi, \psi_\chi)_{\chi \in I_\Gamma}$ where

- $\phi_\chi := \Phi \circ -$
- $\psi_\chi := \Psi \circ -$

Theorem 1.15. *F_{y^0} and $G_{\tilde{y}^0}$ defines an equivalence of categories.*

Proof. Define two natural transformations of functors

- $\epsilon : F_{y^0} G_{\tilde{y}^0} \to \mathrm{Id}_{\mathbf{McK}(\Gamma)}$
- $\eta : \mathrm{Id}_{\mathbf{Rep}(\overline{Q_{\Gamma^f}})} \to G_{\tilde{y}^0} F_{y^0}$

Start with ϵ. Let us fix $(M, M^f, \Delta, Z_1, Z_2) \in \mathbf{McK}(\Gamma)$, and consider

- $\epsilon_M : \bigoplus_{\chi \in I_\Gamma} \mathrm{Hom}_\Gamma(X_\chi, M) \otimes X_\chi \xrightarrow{\sim} M$
- $\epsilon_{M^f} : \bigoplus_{\chi \in I_\Gamma} \mathrm{Hom}_\Gamma(X_\chi, M^f) \otimes X_\chi \xrightarrow{\sim} M^f$
- $\epsilon_\Delta : F_{y^0}(G_{\tilde{y}^0}(\Delta)) \mapsto \epsilon_M \circ F_{y^0}(G_{\tilde{y}^0}(\Delta)) \circ (\mathrm{id}_{X_{\mathrm{std}}} \otimes \epsilon_M^{-1})$
- $\epsilon_{Z_1} : F_{y^0}(G_{\tilde{y}^0}(Z_1) \mapsto \epsilon_M \circ F_{y^0}(G_{\tilde{y}^0}(Z_1)) \circ \epsilon_{M^f}^{-1}$
- $\epsilon_{Z_2} : F_{y^0}(G_{\tilde{y}^0}(Z_2) \mapsto \epsilon_{M^f} \circ F_{y^0}(G_{\tilde{y}^0}(Z_2)) \circ \epsilon_M^{-1}$

We need to show that $\epsilon_\Delta(F_{y^0}(G_{\tilde{y}^0}(\Delta))) = \Delta$ which can be reformulated as the following equality $\epsilon_M \circ F_{y^0}(G_{\tilde{y}^0}(\Delta)) = \Delta \circ (\mathrm{id}_{X_{\mathrm{std}}} \otimes \epsilon_M)$.
Let us fix $\chi \in I_\Gamma$ and $(a, f, z) \in X_{\mathrm{std}} \times \mathrm{Hom}_\Gamma(X_\chi, M) \times X_\chi$, then

$$\epsilon_M(F_{y^0}(G_{\tilde{y}^0}(\Delta))(a \otimes f \otimes z)) = \Delta(f(\sum_{h \in \overline{E_\Gamma}, h' = \chi} \tilde{y}_h(y_h(a \otimes z))))$$

Thanks to Lemma 1.14, we have that $\Delta(f(\sum_{h \in \overline{E_\Gamma}, h' = \chi} \tilde{y}_h(y_h(a \otimes z)))) = \Delta(f(a \otimes z))$ which is equal to $\big(\Delta \circ (\mathrm{id}_{X_{\mathrm{std}}} \otimes \epsilon_M)\big)(a \otimes f \otimes z)$.
Furthermore, we have by construction that

$$\epsilon_M \circ F_{y^0}(G_{\tilde{y}^0}(Z_1)) = Z_1 \circ \epsilon_{M^f}$$

$$\epsilon_{M^f} \circ F_{y^0}(G_{\tilde{y}^0}(Z_2)) = Z_2 \circ \epsilon_M$$

Now let us give $(V, V^f, x, v^1, v^2) \in \mathbf{Rep}(\overline{Q_{\Gamma^f}})$. Define η

• $\eta_V : V \xrightarrow{\sim} \bigoplus_{\chi \in I_\Gamma} \mathrm{Hom}_\Gamma(X_\chi, \bigoplus_{\xi \in I_\Gamma} V_\xi \otimes X_\xi)$ as the composition of the following natural isomorphisms

$$\begin{array}{ccc} \bigoplus_{\chi \in I_\Gamma} V_\chi = V & \xrightarrow{\eta_V} & \bigoplus_{\chi \in I_\Gamma} \mathrm{Hom}_\Gamma(X_\chi, \bigoplus_{\xi \in I_\Gamma} V_\xi \otimes X_\xi) \\ {\scriptstyle \sum_{\chi \in I_\Gamma} \mathrm{id}_{V_\chi} \otimes \mathrm{id}_{X_\chi}} \downarrow & \nearrow_{\sim} & \\ \bigoplus_{(\chi, \xi) \in I_\Gamma^2} V_\xi \otimes \mathrm{Hom}_\Gamma(X_\chi, X_\xi) & & \end{array}$$

- $\eta_{V^f} : V^f \xrightarrow{\sim} \bigoplus_{\chi\in I_\Gamma} \mathrm{Hom}_\Gamma(X_\chi, \bigoplus_{\xi\in I_\Gamma} V^f_\xi \otimes X_\xi)$ defined in the same way as η_V
- $\eta_x : x \mapsto \eta_V \circ x \circ \eta_V^{-1}$
- $\eta_{v^1} : v^1 \mapsto \eta_V \circ v^1 \circ \eta_{V^f}^{-1}$
- $\eta_{v^2} : v^2 \mapsto \eta_{V^f} \circ v^2 \circ \eta_V^{-1}$

Let us show that $G_{\tilde{y}^0}(F_{y^0}(x)) \circ \eta_V = \eta_V \circ x$. If $\chi \in I_\Gamma$ and $v \in V_\chi$, then $\eta_V(v) = v \otimes \mathrm{id}_{X_\chi}$. If we take $h \in \overline{E_\Gamma}$ such that $h' = \chi$, then we have

$$G_{\tilde{y}^0}(F_{y^0}(x))_h(\eta_V(v)) = [(y_h \otimes x_h) \otimes (v \otimes \mathrm{id}_{X_\chi} \otimes \mathrm{id}_{X_{\mathrm{std}}}) \circ \tilde{y}_h] \in \mathrm{Hom}_\Gamma(X_{h''}, \bigoplus_{\xi\in I_\Gamma} V_\xi \otimes X_\xi)$$

Using that $\tilde{y}_h$ is a section of y_h, we have $(y_h \otimes x_h) \otimes (v \otimes \mathrm{id}_{X_\chi} \otimes \mathrm{id}_{X_{\mathrm{std}}}) \circ \tilde{y}_h = \eta_V(x(v))$. Furthermore, we have by construction that

$$\eta_V \circ v^1 = (G_{\tilde{y}^0}(F_{y^0}(v^1))) \circ \eta_{V^f}$$

and that

$$\eta_{V^f} \circ v^2 = (G_{\tilde{y}^0}(F_{y^0}(v^2))) \circ \eta_V$$

□

Remark 1.16. If M, M^f are two Γ-modules, then it is clear that the functor $G_{\tilde{y}^0}$ induces a morphism $\mathcal{R}^\Gamma_{M,M^f} \to \mathcal{R}^\Gamma_{G_{\tilde{y}^0}(M),G_{\tilde{y}^0}(M^f)}$. Moreover, if V and V^f are two I_Γ-graded complex vector spaces, it is clear that F_{y^0} induces a morphism $\mathcal{R}^\Gamma_{V,V^f} \to \mathcal{R}^\Gamma_{F_{y^0}(V),F_{y^0}(V^f)}$.

We can now wonder what happens to the symplectic structures.

Proposition 1.17. *For every orientation Ω of G_Γ , there exists a family $(\lrcorner^\Omega_h)_{h\in\overline{E_\Gamma}} \in (\mathbb{C}^*)^{|\overline{E_\Gamma}|}$ such that $\forall\big((x, v^1, v^2), (\tilde{x}, \tilde{v}^1, \tilde{v}^2)\big) \in (\mathcal{R}^\Gamma_{V,V^f})^2$*

$$\omega_\Gamma^{\epsilon_\Omega}\big((x, v^1, v^2), (\tilde{x}, \tilde{v}^1, \tilde{v}^2)\big) = \omega_\Gamma(F_{y^\Omega}\big(x, v^1, v^2), F_{y^\Omega}(\tilde{x}, \tilde{v}^1, \tilde{v}^2)\big)$$

and $\forall\big((\Delta, Z_1, Z_2), (\tilde{\Delta}, \tilde{Z}_1, \tilde{Z}_2)\big) \in (\mathcal{R}^\Gamma_{M,M^f})^2$

$$\omega_\Gamma^{\epsilon_\Omega}\big(G_{\tilde{y}^\Omega}(\Delta, Z_1, Z_2), G_{\tilde{y}^\Omega}(\tilde{\Delta}, \tilde{Z}_1, \tilde{Z}_2)\big) = \omega_\Gamma\big((\Delta, Z_1, Z_2), (\tilde{\Delta}, \tilde{Z}_1, \tilde{Z}_2)\big)$$

where $\forall h \in \overline{E_\Gamma}, y^\Omega_h := \lrcorner^\Omega_h y^0_h, \tilde{y}^\Omega_h := \lrcorner^{\Omega^{-1}}_h \tilde{y}^0_h$

Proof. Expliciting the right-hand side of the first equality gives

$$\sum_{h\in\overline{E_\Gamma}} \lrcorner^\Omega_h \lrcorner^\Omega_{\bar{h}} \mathrm{Tr}(x_{\bar{h}} \otimes [y^0_{\bar{h}}]_1 \circ \tilde{x}_h \otimes [y^0_h]_2 - x_{\bar{h}} \otimes [y^0_{\bar{h}}]_2 \circ \tilde{x}_h \otimes [y^0_h]_1) + \mathrm{Tr}(\sum_{\chi\in I_\Gamma} v^1_\chi \tilde{v}^2_\chi - \tilde{v}^1_\chi v^2_\chi)$$
$$= \sum_{h\in\overline{E_\Gamma}} \lrcorner^\Omega_h \lrcorner^\Omega_{\bar{h}} \mathrm{Tr}([y^0_{\bar{h}}]_1 [y^0_h]_2 - [y^0_{\bar{h}}]_2 [y^0_h]_1)\mathrm{Tr}(x_{\bar{h}} \tilde{x}_h) + \mathrm{Tr}(\sum_{\chi\in I_\Gamma} v^1_\chi \tilde{v}^2_\chi - \tilde{v}^1_\chi v^2_\chi)$$

where we denote $[y_h^0]_k \in \mathrm{Hom}(X_{h'}, X_{h''})$ the morphism $[y_h^0]_{e_k}$ which comes from the canonical identification of T_h^Γ with $\mathrm{Hom}_\Gamma(X_{\mathrm{std}}, \mathrm{Hom}(X_{h'}, X_{h''}))$ and taking the morphism corresponding to $e_k \in X_{\mathrm{std}}$. Take $h \in \overline{E_\Gamma}$. The term $\mathrm{Tr}([y_{\bar{h}}^0]_1[y_h^0]_2 - [y_{\bar{h}}^0]_2[y_h^0]_1)$ is nonzero thanks to the nondegenerescence of ω^Γ and Theorem 1.15. Moreover

$$\mathrm{Tr}\left([y_{\bar{h}}^0]_1[y_h^0]_2 - [y_{\bar{h}}^0]_2[y_h^0]_1\right) + \mathrm{Tr}\left([y_h^0]_1[y_{\bar{h}}^0]_2 - [y_h^0]_2[y_{\bar{h}}^0]_1\right) = 0$$

thanks to the well-known fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Now, we can choose $\lrcorner_h^\Omega$ and $\lrcorner_{\bar{h}}^\Omega$ in such a way that $\lrcorner_h^\Omega \lrcorner_{\bar{h}}^\Omega \mathrm{Tr}([y_{\bar{h}}^0]_1[y_h^0]_2 - [y_{\bar{h}}^0]_2[y_h^0]_1) = \epsilon_\Omega(\bar{h})$. Note that the previous equation gives that the pair $(\lrcorner_h^\Omega, \lrcorner_{\bar{h}}^\Omega)$ is unique up to a non-zero scalar. To prove the second statement, we use Theorem 1.15 to obtain that

$$\begin{aligned}
\epsilon_M \circ F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(\Delta)\right) \circ \left(\mathrm{id}_{X_{\mathrm{std}}} \otimes \epsilon_M^{-1}\right) &= \Delta \\
\epsilon_M \circ F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(Z_1)\right) \circ \epsilon_{M^f}^{-1} &= Z_1 \\
\epsilon_{M^f} \circ F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(Z_2)\right) \circ \epsilon_M^{-1} &= Z_2
\end{aligned}$$

Use what has been proven to get

$$\omega_\Gamma^{\epsilon_\Omega}\left(G_{\tilde{y}^\Omega}(\Delta, Z_1, Z_2), G_{\tilde{y}^\Omega}(\tilde{\Delta}, \tilde{Z}_1, \tilde{Z}_2)\right) = \omega_\Gamma\left(F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(\Delta, Z_1, Z_2)\right), F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(\tilde{\Delta}, \tilde{Z}_1, \tilde{Z}_2)\right)\right)$$

Using the invariance of the trace by conjugacy we get the desired equality. □

Fix an orientation Ω of the McKay multigraph G_Γ. We now have a family y^Ω which will be used later on. The functors F_{y^Ω} and $G_{\tilde{y}^\Omega}$ intertwine the actions of $\mathrm{Aut}_\Gamma(M)$ on $\mathcal{R}^\Gamma_{M,M^f}$ and of $\mathrm{G}(V)$ on $\mathcal{R}^\Gamma_{V,V^f}$. Consider the morphism

$$\tilde{p}_V : \begin{array}{ccc} \bigoplus_{\chi\in I_\Gamma} \mathrm{End}(V_\chi) & \to & \mathrm{End}_\Gamma(\bigoplus_{\chi\in I_\Gamma} V_\chi \otimes X_\chi) \\ (f_\chi)_{\chi\in I_\Gamma} & \mapsto & (\sum_{\chi\in I_\Gamma} v_\chi \otimes x_\chi \mapsto \sum_{\chi\in I_\Gamma} f_\chi(v_\chi) \otimes x_\chi) \end{array}$$

where V is an I_Γ-graded complex vector space. Using Schur's Lemma it is clear that $\tilde{p}_V$ is an isomorphism and by construction it restricts to $p_V : G(V) \xrightarrow{\sim} \mathrm{Aut}_\Gamma\left(F_{y^\Omega}(V)\right)$ which is an isomorphism of algebraic groups.
If M is a Γ-module, consider $\tilde{p}_M : \mathrm{End}_\Gamma(M) \xrightarrow{\sim} \bigoplus_{\chi\in I_\Gamma} \mathrm{End}(\mathrm{Hom}_\Gamma(X_\chi, M))$ the conjugation by the isomorphism ϵ_M^{-1} (the isotypical decomposition of M). The morphism $\tilde{p}_M$ restricts to an isomorphism of algebraic groups $p_M : \mathrm{Aut}_\Gamma(M) \xrightarrow{\sim} \mathrm{G}\left(G_{\tilde{y}^\Omega}(M)\right)$.

Proposition 1.18. *Let V and V^f be two I_Γ-graded complex vector spaces. If one lets the algebraic group $\mathrm{G}(V)$ act on $\mathcal{R}^\Gamma_{F_{y^\Omega}(V),F_{y^\Omega}(V^f)}$ though p_V, then the map $\mathcal{R}^\Gamma_{V,V^f} \to \mathcal{R}^\Gamma_{F_{y^\Omega}(V),F_{y^\Omega}(V^f)}$ is $\mathrm{G}(V)$-equivariant.*
Let M and M^f be two Γ-modules. If one lets the algebraic group $\mathrm{Aut}_\Gamma(M)$ act on $\mathcal{R}^\Gamma_{G_{\tilde{y}^\Omega}(M),G_{\tilde{y}^\Omega}(M^f)}$ though p_M, then the map $\mathcal{R}^\Gamma_{M,M^f} \to \mathcal{R}^\Gamma_{G_{\tilde{y}^\Omega}(M),G_{\tilde{y}^\Omega}(M^f)}$ is $\mathrm{Aut}_\Gamma(M)$-equivariant.

Proof. The result follows directly from the definitions of the actions on $\mathcal{R}^\Gamma_{V,V^f}$ and $\mathcal{R}^\Gamma_{M,M^f}$ and from the definition of the functors F_{y^Ω} and $G_{\tilde{y}^\Omega}$. □

We can also link the two momentum maps.

Proposition 1.19. *For any two I_Γ-graded complex vector spaces V and V^f, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{R}^\Gamma_{V,V^f} & \xrightarrow{F_{y^\Omega}} & \mathcal{R}^\Gamma_{F_{y^\Omega}(V),F_{y^\Omega}(V^f)} \\ \Big\downarrow{\scriptstyle \mu_\Gamma^{\epsilon_\Omega}} & & \Big\downarrow{\scriptstyle \mu_\Gamma} \\ \bigoplus_{\chi\in I_\Gamma} \mathrm{End}(V_\chi) & \xrightarrow[\sum_{\chi\in I_\Gamma} -\otimes \frac{1}{\delta^\Gamma_\chi}\mathrm{id}_{X_\chi}]{\sim} & \mathrm{End}_\Gamma\left(F_{y^\Omega}(V)\right) \end{array}$$

For any two Γ-modules M and M^f, the following diagram commutes

$$\begin{array}{ccc} \mathcal{R}^\Gamma_{M,M^f} & \xrightarrow{G_{\tilde{y}^\Omega}} & \mathcal{R}^\Gamma_{G_{\tilde{y}^\Omega}(M),G_{\tilde{y}^\Omega}(M^f)} \\ \Big\downarrow{\scriptstyle \mu_\Gamma} & & \Big\downarrow{\scriptstyle \mu_\Gamma^{\epsilon_\Omega}} \\ \mathrm{End}_\Gamma(M) & \xleftarrow[\epsilon_M\circ\left(\sum_{\chi\in I_\Gamma} -\otimes \frac{1}{\delta^\Gamma_\chi}\mathrm{id}_{X_\chi}\right)\circ\epsilon_M^{-1}]{\sim} & \bigoplus_{\chi\in I_\Gamma} \mathrm{End}\left(G_{\tilde{y}^\Omega}(M)_\chi\right) \end{array}$$

Proof. Take $(x, v^1, v^2) \in \mathcal{R}^\Gamma_{V,V^f}$. We want to show that

$$\sum_{\chi\in I_\Gamma} \mu_\Gamma^{\epsilon_\Omega}(x, v^1, v^2)_\chi \otimes \frac{1}{\delta^\Gamma_\chi}\mathrm{id}_{X_\chi} = \mu_\Gamma\left(F_{y^\Omega}(x, v^1, v^2)\right)$$

Expanding the right-hand side, we get

$$\begin{aligned} &\sum_{h\in\overline{E_\Gamma}} \lrcorner^\Omega_h \lrcorner^\Omega_{\bar{h}} x_{\bar{h}} \otimes [y^0_{\bar{h}}]_1 \circ x_h \otimes [y^0_h]_2 - x_{\bar{h}} \otimes [y^0_{\bar{h}}]_2 \circ x_h \otimes [y^0_h]_1 + \sum_{\chi\in I_\Gamma} v^1_\chi v^2_\chi \otimes \frac{1}{\delta^\Gamma_\chi}\mathrm{id}_{X_\chi} \\ &= \sum_{h\in\overline{E_\Gamma}} \lrcorner^\Omega_h \lrcorner^\Omega_{\bar{h}} x_{\bar{h}} x_h \otimes \left([y^0_{\bar{h}}]_1[y^0_h]_2 - [y^0_{\bar{h}}]_2[y^0_h]_1\right) + \sum_{\chi\in I_\Gamma} v^1_\chi v^2_\chi \otimes \frac{1}{\delta^\Gamma_\chi}\mathrm{id}_{X_\chi} \end{aligned}$$

Using the fact that ω^Γ is nondegenerate, there exists $\lrcorner^0_h \in \mathbb{C}^*$ for each $h \in \overline{E_\Gamma}$, such that

$$[y^0_{\bar{h}}]_1[y^0_h]_2 - [y^0_{\bar{h}}]_2[y^0_h]_1 = \lrcorner^0_h \mathrm{id}_{X_{h'}}$$

By construction of the constants $\lrcorner^\Omega_h$, we have the following relation for all $h \in \overline{E_\Gamma}$

$$\epsilon_\Omega(h) = \lrcorner^\Omega_h \lrcorner^\Omega_{\bar{h}} \lrcorner^0_{\bar{h}} \delta^\Gamma_{h''}$$

Summing it all up, the right-hand side gives

$$\sum_{h\in\overline{E_\Gamma}} \epsilon_\Omega(h) x_h x_{\bar{h}} \otimes \frac{1}{\delta^\Gamma_{h''}}\mathrm{id}_{X_{h''}} + \sum_{\chi\in I_\Gamma} v^1_\chi v^2_\chi \otimes \frac{1}{\delta^\Gamma_\chi}\mathrm{id}_{X_\chi}$$

Which is exactly what we wanted to show. For the other square, recall that

$$\begin{aligned} \epsilon_M \circ F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(\Delta)\right) \circ \left(\mathrm{id}_{X_{\mathrm{std}}} \otimes \epsilon_M^{-1}\right) &= \Delta \\ \epsilon_M \circ F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(Z_1)\right) \circ \epsilon_{M^f}^{-1} &= Z_1 \\ \epsilon_{M^f} \circ F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(Z_2)\right) \circ \epsilon_M^{-1} &= Z_2 \end{aligned}$$

Using what we just proved we have

$$\sum_{\chi\in I_\Gamma} \mu_\Gamma^{\epsilon_\Omega}\left(G_{\tilde{y}^\Omega}(\Delta,Z_1,Z_2)_\chi\right)\otimes\frac{1}{\delta_\chi^\Gamma}\mathrm{id}_{X_\chi}=\mu_\Gamma\left(F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(\Delta,Z_1,Z_2)\right)\right)$$

Since $\mu_\Gamma\left(F_{y^\Omega}\left(G_{\tilde{y}^\Omega}(\Delta,Z_1,Z_2)\right)\right)=\epsilon_M^{-1}\mu_\Gamma(\Delta,Z_1,Z_2)\epsilon_M$, we have shown that the second square also commutes. □

Remark 1.20. One might wonder why the map $\bigoplus_{\chi\in I_\Gamma}\mathrm{End}(V_\chi)\to\mathrm{End}_\Gamma\left(F_{y^\Omega}(V)\right)$ is not $\tilde{p}_V$. This comes from the fact that the identifications between $\bigoplus_{\chi\in I_\Gamma}\mathrm{End}(V_\chi)^*$ and $\bigoplus_{\chi\in I_\Gamma}\mathrm{End}(V_\chi)$ and between $\mathrm{End}_\Gamma\left(F_{y^\Omega}(V)\right)^*$ and $\mathrm{End}_\Gamma(F_{y^\Omega}(V))$ is not commuting with this map. Indeed, we need to define an isomorphism $\bigoplus_{\chi\in I_\Gamma}\mathrm{End}(V_\chi)\to\mathrm{End}_\Gamma\left(F_{y^\Omega}(V)\right)$ that makes the following diagram commute

$$\begin{array}{ccc}\mathrm{End}_\Gamma(F_{y^\Omega}(V)) & \xrightarrow{\mathrm{Tr}} & \mathrm{End}_\Gamma(F_{y^\Omega}(V))^*\\ \uparrow & & \downarrow\\ \bigoplus_{\chi\in I_\Gamma}\mathrm{End}(V_\chi) & \xrightarrow[\sum_{\chi\in I_\Gamma}\mathrm{Tr}]{} & \bigoplus_{\chi\in I_\Gamma}\mathrm{End}(V_\chi)^*\end{array}$$

1.4 Nakajima's quiver varieties

In this section, we will recall the general construction of Nakajima's quiver variety and use the setting of Γ-modules to introduce an analog. First, we introduce notation. Take G an undirected multigraph and $d\in\Delta_G^+$. Denote for all $\nu\in I_G$, $V_\nu^d:=\mathbb{C}^{d_\nu}$. Let $\mathrm{G}(d):=\mathrm{G}(V^d)$ and for an I_G-graded complex vector space V denote by $\dim(V)$ the dimension parameter $\sum_{\nu\in I_G}\dim(V_\nu)\nu\in\Delta_G^+$. Let us now recall the notion of semistability.

Definition 1.21. The following definition comes from the general Geometric Invariant Theory developed by David Mumford [FKM, Definition 1.7].
Let H be a reductive algebraic group over $\mathbb{C}$, let $\chi:H\to\mathbb{C}^*$ be a rational character and let X be a complex affine algebraic H-variety, then we define an action of H on $X\times\mathbb{C}$ by the formula $h.(x,z):=(h.x,\chi(h)^{-1}.z)$.
Now $x\in X$ is **χ-semistable** when $\forall z\in\mathbb{C}^*$, $\overline{H.(x,z)}\cap(X\times\{0\})=\emptyset$.
Denote by $X^{\chi-ss}$ the set of all χ-semistable points of X.
Define also the set of semi-invariants

$$\mathbb{C}[X]^\chi:=\{f\in\mathbb{C}[X]|\forall(h,x)\in H\times X, f(h.x)=\chi(h)f(x)\}$$

and denote $X/\!\!/_\chi H:=\mathrm{Proj}\left(\bigoplus_{n\geq 0}\mathbb{C}[X]^{\chi^n}\right)$.

For all $\theta\in\Theta_G$, there exists $N\in\mathbb{Z}_{\geq 1}$, such that $\mathrm{Im}(N\theta)\subset\mathbb{Z}$. If $\mathrm{G}(d)$ acts on an affine complex algebraic variety X, we will denote by $X^{\boldsymbol{\theta}-ss}$ the $\chi_{N\theta}$-semistable points of X and by $X/\!\!/_{\boldsymbol{\theta}}\mathrm{G}(d)$ the GIT quotient $X/\!\!/_{\chi_{N\theta}}\mathrm{G}(d)$. Note that [Kir, Corollary 9.15] assures that the notion of $\boldsymbol{\theta}$-semistability, does not depend on the choice of N.

Definition 1.22. Define **Nakajima's quiver variety** of $\overline{Q_{G^f}}$ attached to an orientation Ω of G, to a dimension parameter $(d, d^f) \in \Delta^+_{G^f}$, to a stability parameter $\theta \in \Theta_G$ and to a deformation parameter $\lambda \in \Lambda_G$ as follows

$$\mathcal{M}^G_{\boldsymbol{\theta},\lambda}\left(d, d^f\right) := \left(\mu_G^{\epsilon_\Omega}\right)^{-1}(\lambda) /\!\!/_{\boldsymbol{\theta}} \mathrm{G}(d) \simeq \left(\mu_G^{\epsilon_\Omega}\right)^{-1}(\lambda)^{\boldsymbol{\theta}-ss} /\!\!/ \mathrm{G}(d)$$

Remark 1.23. We will often need to work with $\left(\mu_G^{\epsilon_\Omega}\right)^{-1}(\lambda)^{\boldsymbol{\theta}-ss} \subset \mathcal{R}^G_{d,d^f} := \mathcal{R}^G_{V^d,V^{d^f}}$ which will be denoted by $\tilde{\mathcal{M}}^G_{\boldsymbol{\theta},\lambda}(d, d^f)$. In the special case where $d^{f,i_0} := \dim(V^{f,i_0})$ for some vertex $i_0 \in I_\Gamma$, we will denote $\mathcal{M}^G_{\boldsymbol{\theta},\lambda}(d, d^{f,i_0})$ by $\mathcal{M}^G_{\boldsymbol{\theta},\lambda}(d, i_0)$.

Let us define an analog of Nakajima's quiver variety. If M is a Γ-module and $\theta \in \Theta_\Gamma$, consider $\chi_\theta \circ p_M$ the rational character of $\mathrm{Aut}_\Gamma(M)$. If X is an $\mathrm{Aut}_\Gamma(M)$-variety, we will also shorten $X^{\chi_\theta \circ p_M - ss}$ to $X^{\boldsymbol{\theta}-ss}$. If $\lambda \in \Lambda_\Gamma$, recall that we still denote by λ the induced element of $\bigoplus_{\chi \in I_\Gamma} \mathrm{End}\left(G_{\tilde{y}^\Omega}(M)_\chi\right)$. Depending on the context, let us also denote by λ the element of $\mathrm{End}_\Gamma(M)$ defined as $\epsilon_M \circ \left(\sum_{\chi \in I_\Gamma} \lambda(\chi) \mathrm{id}_{G_{\tilde{y}^\Omega}(M)_\chi} \otimes \frac{1}{\delta^\Gamma_\chi} \mathrm{id}_{X_\chi}\right) \circ \epsilon_M^{-1}$.

Definition 1.24. Define **Nakajima's quiver variety** of $\overline{Q_{\Gamma^f}}$ attached to Γ-modules M, M^f, to a stability parameter $\theta \in \Theta_\Gamma$ and to a deformation parameter $\lambda \in \Lambda_\Gamma$ as follows

$$\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}\left(M, M^f\right) := \mu_\Gamma^{-1}(\lambda) /\!\!/_{\boldsymbol{\theta}} \mathrm{Aut}_\Gamma(M) \simeq \mu_\Gamma^{-1}(\lambda)^{\boldsymbol{\theta}-ss} /\!\!/ \mathrm{Aut}_\Gamma(M)$$

Let us also denote $\mu_\Gamma^{-1}(\lambda)^{\boldsymbol{\theta}-ss} \subset \mathcal{R}^\Gamma_{M,M^f}$ by $\tilde{\mathcal{M}}^\Gamma_{\boldsymbol{\theta},\lambda}(M, M^f)$. Denote by $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(M, \chi)$ the variety $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(M, X_\chi)$ for some $\chi \in I_\Gamma$ and $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(M) := \mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(M, \chi_0)$.

Remark 1.25. It is clear that if M and $\tilde{M}$ are two Γ-modules and are isomorphic as Γ-modules, then for every Γ-module M^f, stability parameter $\theta \in \Theta_\Gamma$ and deformation parameter $\lambda \in \Lambda_\Gamma$, we have $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}\left(M, M^f\right) \simeq \mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}\left(\tilde{M}, M^f\right)$.

To be able to compare the two preceding quiver varieties, we need to show that F_{y^Ω} is compatible with semistability. Let us introduce a handy characterization of this notion.

Definition 1.26. Let H be an algebraic group over $\mathbb{C}$, $\chi : H \to \mathbb{C}^*$ a rational character of H and $\mu : \mathbb{C}^* \to H$ a one-parameter subgroup. Since the morphism $\chi \circ \mu : \mathbb{C}^* \to \mathbb{C}^*$ is of algebraic groups, there exists then $k \in \mathbb{Z}$ such that $\forall t \in \mathbb{C}^*, \chi(\mu(t)) = t^k$. Define the pairing $\langle \chi, \mu \rangle$ to be equal to k.

Proposition 1.27. *Let us consider the same context as in Definition 1.21. A point $x \in X$ is χ-semistable if and only if for all algebraic group morphism $\mu : \mathbb{C}^* \to H$, such that $\lim_{t\to 0} \mu(t).x$ exists, we have $\langle \chi, \mu \rangle \geq 0$.*

Proof. The proof can be found here [FKM, Theorem 2.1]. □

We can use Proposition 1.27 to show that the functors F_{y^Ω} and $G_{\tilde{y}^\Omega}$ are compatible with semistability.

Lemma 1.28. *Take V, V^f two I_Γ-graded complex vector spaces and M, M^f two Γ-modules. The element $(x, v^1, v^2) \in \mathcal{R}^\Gamma_{V,V^f}$ is χ_θ-semistable if and only if $F_{y^\Omega}(x, v^1, v^2)$ is $\chi_\theta \circ p_V^{-1}$-semistable. Moreover $(\Delta, Z_1, Z_2) \in \mathcal{R}^\Gamma_{M,M^f}$ is $\chi_\theta \circ p_M$-semistable if and only if $G_{\tilde{y}^\Omega}(\Delta, Z_1, Z_2)$ is χ_θ-semistable.*

Proof. Take $\mu : \mathbb{C}^* \to \mathrm{Aut}_\Gamma\left(F_{y^\Omega}(V)\right)$ such that $\lim_{t\to 0}\mu(t).F_{y^\Omega}(x,v^1,v^2)$ exists. Using Proposition 1.18, we have for all $t\in\mathbb{C}^*$, $\mu(t).F_{y^\Omega}(x,v^1,v^2) = F_{y^\Omega}\left(p_V^{-1}(\mu(t)).(x,v^1,v^2)\right)$. Using the continuity of $\eta^{-1}\circ G_{\tilde{y}^\Omega}$ we have that $\lim_{t\to 0} p_V^{-1}(\mu(t)).(x,v^1,v^2)$ exists. Since (x,v^1,v^2) is χ_θ-semistable by hypothesis, $\langle\chi_\theta, p_V^{-1}\circ\mu\rangle = \langle\chi_\theta\circ p_V^{-1},\mu\rangle \geq 0$. Conversely we can use the equivariance and continuity of F_{y^Ω} to conclude. The same arguments apply if we start with $(\Delta,Z_1,Z_2)\in\mathcal{R}^\Gamma_{M,M^f}$ and replaces F_{y^Ω} with $G_{\tilde{y}^\Omega}$ and $\eta^{-1}\circ G_{\tilde{y}^\Omega}$ with $\epsilon\circ F_{y^\Omega}$. □

We can now compare the defined varieties using F_{y^Ω}.

Theorem 1.29. *For every orientation Ω of G_Γ and parameters $(d,d^f,\theta,\lambda)\in\Delta^+_{\Gamma^f}\times\Theta_\Gamma\times\Lambda_\Gamma$, F_{y^Ω} induces an isomorphism $\tau : \mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}\left(d,d^f\right) \xrightarrow{\sim} \mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}\left(F_{y^\Omega}\left(V^d\right), F_{y^\Omega}\left(V^{d^f}\right)\right)$.*

Proof. Consider the morphism $\hat{\tau} : \mathcal{R}^\Gamma_{V,V^f} \to \mathcal{R}^\Gamma_{F_{y^\Omega}(V),F_{y^\Omega}(V^f)}$ induced by F_{y^Ω}. This map is linear and computing dimensions gives that $\dim\left(\mathcal{R}^\Gamma_{V,V^f}\right) = \dim\left(\mathcal{R}^\Gamma_{F_{y^\Omega}(V),F_{y^\Omega}(V^f)}\right)$. Let us show that $\hat{\tau}$ is injective. Take $(x,v^1,v^2)\in\mathrm{Ker}(\hat{\tau})$. By construction, we have that $v^1=0$ and $v^2=0$. Moreover, $\Delta^x=0$ implies that $\forall\chi\in I_\Gamma, \sum_{h\in\overline{E_\Gamma},h''=\chi} y^\Omega_h\otimes x_h = 0$. Take $\chi\in I_\Gamma$ and $h\in\overline{E_\Gamma}$ such that $h''=\chi$, then $y^\Omega_h\neq 0$, and in particular $x_h=0$. The morphism $\hat{\tau}$ is then an isomorphism. Thanks to Proposition 1.19 and Lemma 1.28, we have $\tilde{\tau} : {\mu^\Gamma_{\epsilon_\Omega}}^{-1}(\lambda)^{\boldsymbol{\theta}-ss} \xrightarrow{\sim} \mu_\Gamma^{-1}(\lambda)^{\boldsymbol{\theta}-ss}$ which is the restriction of the isomorphism $\hat{\tau}$ to ${\mu^\Gamma_{\epsilon_\Omega}}^{-1}(\lambda)^{\boldsymbol{\theta}-ss}$. Finally, Proposition 1.18 shows that $\tilde{\tau}$ induces the desired isomorphism. □

When not specified, the deformation parameter is taken to be $0\in\Lambda_\Gamma$ and if the stability parameter is not specified it is also taken to be $0\in\Theta_\Gamma$.

1.5 Important results for McKay quiver varieties

In this section, we recall important results obtained for Nakajima's quiver varieties that will be used in the following chapters. The next Proposition is a reformulation of [Nak94, Theorem 2.8] and of [C-B01, Section 1].

Proposition 1.30. *Let Ω be an orientation of G_Γ and $(d,d^f)\in\Delta^+_{\Gamma^f}$ be dimension parameters. If $(\theta,\lambda)\in\mathbb{Q}\times\Lambda_\Gamma^{++}\setminus\{0,0\}$, then the variety $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(d,d^f)$ is smooth and irreducible.*

Proof. If $\lambda\neq 0$, [Gin09, Theorem 5.2.2] gives that $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda}(d,d^f)$ is smooth and connected. Moreover, if $\theta\neq 0$, we have that $\mathcal{M}^\Gamma_{\boldsymbol{\theta}}(d,d^f)$ is smooth and connected so irreducible thanks to [Kir, Example 10.36] and [Kir, Theorem 10.35, 10.37]. □

One can easily compute the dimension of quiver varieties over the McKay quiver of Γ when the dimension parameter has a particularly nice form.

Proposition 1.31. *If $\theta\in\Theta_\Gamma^{++}$ and if r is a positive integer, then the symplectic variety $\mathcal{M}^\Gamma_{\boldsymbol{\theta}}(r\delta^\Gamma)$ is smooth and has dimension $2r$.*

Proof. Let us apply [Kir, Theorem 10.35]. We have by definition that $A(\tilde{T}_\Gamma)\delta(\tilde{T}_\Gamma)=0$ so that the dimension of $\mathcal{M}^\Gamma_{\boldsymbol{\theta}}(r\delta^\Gamma)$ is $2r\delta^\Gamma_{\chi_0}=2r$. □

Semistability becomes a simpler condition for $\theta \in \Theta_\Gamma^{++}$.

Lemma 1.32. *Take θ in Θ_Γ^{++}.*
$(x, v^1, v^2) \in \mathcal{R}^\Gamma_{d,d^f}$ is $\boldsymbol{\theta}$-semistable if and only if for all $\chi \in I_\Gamma$ and for all $S_\chi \subset V^d_\chi$, if $\left(\forall h \in \overline{E_\Gamma}, x_h(S_{h'}) \subset S_{h''} \text{ and } \forall \chi \in I_\Gamma, \mathrm{Im}(v^1_\chi) \subset S_\chi\right)$, then $\bigoplus_{\chi \in I_\Gamma} S_\chi = V^d$.
Moreover $(\Delta, Z_1, Z_2) \in \mathcal{R}^\Gamma_{M,M^f}$ is $\boldsymbol{\theta}$-semistable if and only if for all Γ-submodules M' of M, if $\Delta(X_{\mathrm{std}} \otimes M') \subset M'$ and $\mathrm{Im}(Z_1) \subset M'$, then $M' = M$.

Proof. Let us proceed by contraposition. If we have that for all $\chi \in I_\Gamma, S_\chi \subset V^d_\chi$ such that it is stable by x, that $\mathrm{Im}(v^1_\chi) \subset S_\chi$ and that $\bigoplus_{\chi \in I_\Gamma} S_\chi \subsetneq V^d$. Let us construct $\mu : \mathbb{C}^* \to \mathrm{GL}(d)$ such that $\lim_{t\to 0} \lambda(t).(x, v^1, v^2)$ exists and such that $\langle \chi_\theta, \mu \rangle < 0$.
For each $\xi \in I_\Gamma$, consider $S^\perp_\xi$ a supplementary subspace of S_ξ in V^d_ξ. Define for each $\xi \in I_\Gamma$ and each $\forall t \in \mathbb{C}^*$

$$(\mu(t))_\xi := \left(\begin{array}{c|c} \mathrm{id}_{S_\xi} & 0 \\ \hline 0 & t^{-1}\mathrm{id}_{S^\perp_\xi} \end{array}\right) \in \mathrm{GL}(V^d_\xi)$$

Now, we use the fact that $\forall \xi \in I_\Gamma, S_\xi$ is stable under x and $\mathrm{Im}(v^1_\xi) \subset S_\xi$, to have the existence of $\lim_{t\to 0} \mu(t).(x, v^1, v^2)$. Moreover there exists $\chi \in I_\Gamma$ such that $\dim(S^\perp_\chi) \geq 1$ so we have that $\langle \chi_\theta, \mu \rangle < 0$ which contradicts Proposition 1.27 .
Conversely, take a one-parameter subgroup μ such that $\lim_{t\to 0} \mu(t).(x, v^1, v^2)$ exists. For all $\chi \in I_\Gamma$, consider the eigenspace decomposition of μ acting on $V^d_\chi = \bigoplus_{k\in\mathbb{Z}} V^d_{(\chi,k)}$ meaning that $\forall t \in \mathbb{C}^*, \mu(t)|_{V^d_{(\chi,k)}} = t^k \mathrm{id}_{V^d_{(\chi,k)}}$. For all $k \in \mathbb{Z}$ denote by $V^d_k := \bigoplus_{\chi\in I_\Gamma} V^d_{\chi,k}$ so that $V^d = \bigoplus_{k\in\mathbb{Z}} V^d_k$. Moreover, for $j \in \mathbb{Z}$, denote by $V^d_{\geq j} := \bigoplus_{k\geq j} \bigoplus_{\chi\in I_\Gamma} V^d_{(\chi,k)}$. Then we have $V^d_{\geq j} = \bigoplus_{\chi\in I_\Gamma} V^d_{(\chi,\geq j)}$.
Let us prove that for all $k \in \mathbb{Z}$, $V^d_{\geq k}$ is stable by x. Take $h \in \overline{E_\Gamma}$, then $x_h : V^d_{h'} \to V^d_{h''}$ can be decomposed into the direct sum $\bigoplus_{(r,s)\in\mathbb{Z}^2} x_h^{(r,s)}$ where $x_h^{(r,s)} : V^d_{(h',r)} \to V^d_{(h'',s)}$. From there, for $t \in \mathbb{C}^*$, $\mu(t).x_h = \sum_{(r,s)\in\mathbb{Z}^2} t^{s-r} x_h^{(r,s)}$. The limit of $\mu(t).x_h$, when t tends to 0, exists if and only if $x_h^{(r,s)} = 0$ for all pairs (r, s) such that $s < r$. This gives that $V^d_{\geq r}$ is stable by x_h for all $h \in \overline{E_\Gamma}$. The same argument shows that the existence of the limit $\mu(t)_\chi.v^1_\chi$ implies that $\mathrm{Im}(v^1_\chi) \in V^d_{\chi,\geq 0}$ for all $\chi \in I_\Gamma$. To resume, for each $\chi \in I_\Gamma$ we have an x-stable subspace $V^d_{\chi,\geq 0} \subset V^d_\chi$ and $\mathrm{Im}(v^1_\chi) \in V^d_{\chi,\geq 0}$. Then by hypothesis, $V^d_{\geq 0} = V^d$. The conclusion follows, $\langle \chi_\theta, \mu \rangle \geq 0$.
For the second statement, take $(\Delta, Z_1, Z_2) \in \mathcal{R}^\Gamma_{M,M^f}$ which is $\boldsymbol{\theta}$-semistable. Take M' a Γ-submodule of M stable by Δ and containing $\mathrm{Im}(Z_1)$. Thanks to Lemma 1.28, we know that $G_{\tilde{y}^\Omega}(\Delta, Z_1, Z_2)$ is $\boldsymbol{\theta}$-semistable. For each $\chi \in I_\Gamma$, consider $S_\chi = \mathrm{Hom}_\Gamma(X_\chi, M')$ which is a subspace of $\mathrm{Hom}_\Gamma(X_\chi, M)$. Since $\Delta(X_{\mathrm{std}} \otimes M') \subset M'$, we have by construction of $G_{\tilde{y}^\Omega}$ that $\forall h \in \overline{E_\Gamma}$, $\left(G_{\tilde{y}^\Omega}(\Delta)_h\right)(S_{h'}) \subset S_{h''}$. In addition, we also have that $\forall \chi \in I_\Gamma, \mathrm{Im}(G_{\tilde{y}^\Omega}(Z_1)_\chi) \subset S_\chi$, since $\mathrm{Im}(Z_1) \subset M'$. Let us use the first equivalence of this Lemma to deduce that $\forall \chi \in I_\Gamma, S_\chi = \mathrm{Hom}_\Gamma(X_\chi, M)$, which implies that $M' = M$. Conversely, using Lemma 1.28 it is enough to show that $G_{\tilde{y}^\Omega}(\Delta, Z_1, Z_2)$ is $\boldsymbol{\theta}$-semistable. Denote $x^\Delta := G_{\tilde{y}^\Omega}(\Delta)$. Suppose that we have for each $\chi \in I_\Gamma$ a subspace S_χ of $\mathrm{Hom}_\Gamma(X_\chi, M)$ such that $\forall h \in \overline{E_\Gamma}, x^\Delta_h(S_{h'}) \subset S_{h''}$ and $\forall \chi \in I_\Gamma, \mathrm{Im}(G_{\tilde{y}^\Omega}(Z_1)_\chi) \subset S_\chi$.

Consider now $M' = \epsilon_M(\bigoplus_{\chi\in I_\Gamma} S_\chi \otimes X_\chi)$ which is a Γ-submodule of M. This submodule is stable by Δ. Take $\chi \in I_\Gamma$, and $h \in \overline{E_\Gamma}$ such that $h' = \chi$. By construction of x^Δ, we have

$$\forall f \in S_\chi, x_h^\Delta(f) = \Delta \circ (\mathrm{id}_{X_{\mathrm{std}}} \otimes f) \circ \tilde{y}_h^\Omega$$

Now

$$\begin{aligned}\sum_{h\in\overline{E_\Gamma},h'=\chi} x_h^\Delta(f)\circ y_h^\Omega &= \sum_{h\in\overline{E_\Gamma}} \Delta\circ(\mathrm{id}_{X_{\mathrm{std}}}\otimes f)\circ\tilde{y}_h^0\circ y_h^0\\ &= \Delta\circ(\mathrm{id}_{X_{\mathrm{std}}}\otimes f)\circ\sum_{h\in\overline{E_\Gamma},h'=\chi}\tilde{y}_h^0 y_h^0\\ &= \Delta\circ(\mathrm{id}_{X_{\mathrm{std}}}\otimes f)\end{aligned}$$

The last equality follows from Lemma 1.14.
Then using the hypothesis on x^Δ, we have for all $(t, f, z) \in X_{\mathrm{std}} \times S_\chi \times X_\chi$

$$\Delta(t\otimes f(z)) = \sum_{h\in\overline{E_\Gamma},h'=\chi} x_h^\Delta(f)(y_h^\Omega(t\otimes z)) \in M'$$

Finally, using the fact that $\forall\chi \in I_\Gamma, \mathrm{Im}(G_{\tilde{y}^\Omega}(Z_1)_\chi) \in S_\chi$ it is clear that $\mathrm{Im}(Z_1) \subset M'$. By hypothesis, we have $M' = M$ and in particular that $\forall\chi \in I_\Gamma, S_\chi = \mathrm{Hom}_\Gamma(X_\chi, M)$ which shows that $G_{\tilde{y}^\Omega}(\Delta, Z_1, Z_2)$ is $\boldsymbol{\theta}$-semistable. □

Let us finish this section by recalling an important isomorphism between quiver varieties over McKay quivers. The explicit realization described in Remark 1.13, gives geometric insights on the set of dimension parameters Δ_Γ, the set of stability parameters Θ_Γ and the set of deformation parameters Λ_Γ. In fact, $(\Delta_\Gamma)^{**} = Q(\tilde{T}_\Gamma)$. From now on, we will identify δ^Γ and $\delta(\tilde{T}_\Gamma)$ and let us also denote for $d \in \Delta_\Gamma$ $|d|_\Gamma := |d|_{\tilde{T}_\Gamma}$. Concerning Θ_Γ, we need to introduce $P_{\mathbb{Q}}^\vee(\tilde{T}_\Gamma) := \{h \in \mathfrak{h}_\Gamma | \forall\chi \in I_\Gamma, \langle\alpha_\chi, h\rangle \in \mathbb{Q}\}$. Let us shorten $\delta^\vee(\tilde{T}_\Gamma)$ to $\delta_\Gamma^\vee$. We can then identify Λ_Γ with $\mathfrak{h}_\Gamma^*/\mathbb{C}\delta^\Gamma$ and Θ_Γ with $P_{\mathbb{Q}}^\vee(\tilde{T}_\Gamma)/\mathbb{Q}\delta_\Gamma^\vee$ in the following way

$$\kappa^\vee : \begin{array}{ccc} P_{\mathbb{Q}}^\vee(\tilde{T}_\Gamma)/\mathbb{Q}\delta_\Gamma^\vee & \to & \Theta_\Gamma \\ \overline{\Lambda^\vee} & \mapsto & (\chi \mapsto \langle\alpha_\chi, \Lambda^\vee\rangle)\end{array}$$

$$\kappa : \begin{array}{ccc} \mathfrak{h}_\Gamma^*/\mathbb{C}\delta^\Gamma & \to & \Lambda_\Gamma \\ \overline{\Lambda} & \mapsto & (\chi \mapsto \langle\Lambda, \tilde{\alpha}_\chi\rangle)\end{array}$$

Since $\langle\alpha_\chi, \delta_\Gamma^\vee\rangle = 0 = \langle\delta^\Gamma, \tilde{\alpha}_\chi\rangle$ for all $\chi \in I_\Gamma$, the morphisms κ and $\kappa^\vee$ are well defined and it is easy to check that these are isomorphisms of $\mathbb{Z}$-modules.

Definition 1.33. Let us define a $W(\tilde{T}_\Gamma)$-action on Δ_Γ, Θ_Γ and Λ_Γ. Denote by $s_\chi \in W(\tilde{T}_\Gamma)$ for $\chi \in I_\Gamma$ the generators of this affine Weyl group. Take $d \in \Delta_\Gamma$, $\theta \in \Theta_\Gamma$ and $\lambda \in \Lambda_\Gamma$

$$(s_\chi.d)_\xi = \begin{cases} (\sum_{h\in\overline{E_\Gamma},h'=\chi} d_{h''}) - d_\chi & \text{if } \chi = \xi \neq \chi_0 \\ (\sum_{h\in\overline{E_\Gamma},h'=\chi} d_{h''}) - d_\chi + 1 & \text{if } \chi = \xi = \chi_0 \\ d_\xi & \text{else}\end{cases}$$

$$(s_\chi.\theta)(\xi) = \begin{cases} \theta(\chi) + \theta(\xi) & \text{if } \exists h \in \overline{E_\Gamma}, h' = \chi, h'' = \xi \\ -\theta(\chi) & \text{if } \chi = \xi \\ \theta(\xi) & \text{else}\end{cases}$$

$$(s_\chi.\lambda)(\xi) = \begin{cases} \lambda(\chi) + \lambda(\xi) & \text{if } \exists h \in \overline{E_\Gamma}, h' = \chi, h'' = \xi \\ -\lambda(\chi) & \text{if } \chi = \xi \\ \lambda(\xi) & \text{else} \end{cases}$$

Remark 1.34. The group $W(\tilde{T}_\Gamma)$ acts by reflections on $P_{\mathbb{Q}}^\vee(\tilde{T}_\Gamma)$ and on $\mathfrak{h}_\Gamma^*$. Moreover $\delta_\Gamma^\vee$ and δ^Γ are stabilized by $W(\tilde{T}_\Gamma)$. The actions defined on Θ_Γ and on Λ_Γ turn the isomorphisms $\kappa^\vee$ and κ into $W(\tilde{T}_\Gamma)$-equivariant isomorphisms. Moreover, the action on Δ_Γ corresponds to the one defined in [Nak03, Definition 2.3] in the special case of double, one vertex framed quivers and it is linked to the natural action by reflections on $\mathfrak{h}_\Gamma^*$ (denoted $*$) in the following way. Thanks to the remark at the end of [Nak03, Definition 2.3], we have

$$\forall(\omega, \alpha) \in W(\tilde{T}_\Gamma) \times Q(\tilde{T}_\Gamma), \omega * (\Lambda_0 - \alpha) = \Lambda_0 - \omega.\alpha \tag{1.1}$$

where Λ_0 denotes Λ_{α_0}.

One important isomorphism between Nakajima quiver varieties for dimension, stability and deformation parameters that are linked by the actions of $W(\tilde{T}_\Gamma)$ defined in Definition 1.33 was discovered by George Lusztig [L00, Corollary 3.6], Hiraku Nakajima [Nak03, Theorem 8.1] and Andrea Maffei [Maff, Proposition 40].

Proposition 1.35. *Let $(d, \theta, \lambda) \in \Delta_\Gamma^+ \times \Theta_\Gamma^{++} \times \Lambda_\Gamma$, and $\omega \in W(\tilde{T}_\Gamma)$, such that $\omega.d \in \Delta_\Gamma^+$. We then have an isomorphism*

$$\mathrm{Maff}_{d,\boldsymbol{\theta},\lambda,\omega}^\Gamma : \mathcal{M}_{\boldsymbol{\theta},\lambda}^\Gamma(d) \xrightarrow{\sim} \mathcal{M}_{\omega.\boldsymbol{\theta},\omega.\lambda}^\Gamma(\omega.d)$$

of algebraic varieties.

As for quiver varieties, when $\lambda = 0$, let us shorten $\mathrm{Maff}_{d,\boldsymbol{\theta},\lambda,\omega}^\Gamma$ to $\mathrm{Maff}_{d,\boldsymbol{\theta},\omega}^\Gamma$. Finally, Ivan Losev [L12, Lemma 6.4.2] has proved the following result.

Proposition 1.36. *Let $(d, \theta) \in \Delta_\Gamma^+ \times \Theta_\Gamma^{++}$, and $\omega \in W(\tilde{T}_\Gamma)$, such that $\omega.d = d$. There is an isomorphism*

$$\mathrm{Maff}_{d,\boldsymbol{\theta},\omega}^\Gamma : \mathcal{M}_{\boldsymbol{\theta}}^\Gamma(d) \xrightarrow{\sim} \mathcal{M}_{\omega.\boldsymbol{\theta}}^\Gamma(d)$$

of algebraic varieties over $\mathcal{M}_{\mathbf{0}}^\Gamma(d)$.

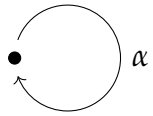
CHAPTER 2

DECOMPOSITION OF Γ-FIXED POINT LOCI

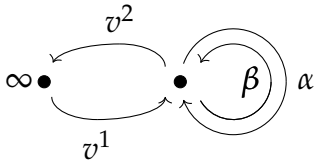
The Jordan quiver is the quiver with one vertex and one arrow. The name of this quiver comes from the fact that, over an algebraically closed field, the classification of the representations of this quiver is given by the Jordan normal form of a matrix. We will define an action of the group Γ on this quiver variety and describe the irreducible components of the Γ-fixed point locus of the Jordan quiver variety for nonzero stability or nonzero deformation parameter. The irreducible components will be identified with McKay quiver varieties. After introducing the Jordan quiver variety in the first section, the second section is dedicated to the construction of a morphism from the Γ-fixed point locus of the Jordan quiver variety to the representation space $\mathcal{R}_M^\Gamma$, for a Γ-module M built out of the Γ-fixed point. In the third section, we will build a morphism from a McKay quiver variety to the Γ-fixed point locus of the Jordan quiver variety. Finally, the last section binds these constructions together to prove the main theorem of this chapter. The setting developed in Chapter 1, will be of great use in this chapter.

2.1 McKay and Jordan quiver

From now on, we will be mainly interested in two types of double framed quivers. The first quiver is $\overline{Q_{\Gamma^f}}$, the double framed McKay quiver attached to the fixed finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$. To be more specific, the double quiver of Q_Γ framed at the vertex χ_0 will be the main player. The second quiver is the Jordan quiver that will be denoted $Q_\bullet$.



Let us denote $G_\bullet$ the underlying undirected graph. Consider $\overline{Q_{\bullet^f}}$ the double framed quiver of $G_\bullet$



Recall that we have fixed an integer $n \geq 1$. Let us denote $\mathcal{M}^{\bullet}_{\boldsymbol{\theta},\lambda}(n) := \mathcal{M}^{G,\bullet}_{\boldsymbol{\theta},\lambda}(n,1)$ and $\tilde{\mathcal{M}}^{\bullet}_{\boldsymbol{\theta},\lambda}(n) := \tilde{\mathcal{M}}^{G,\bullet}_{\boldsymbol{\theta},\lambda}(n,1)$ for $(\theta,\lambda) \in \mathbb{Q}\times\mathbb{C}$.

Lemma 2.1. *The group* $\mathrm{GL}(\mathbb{C}^n)$ *acts freely on* $\tilde{\mathcal{M}}^{\bullet}_{\mathbf{1}}(n)$ *and on* $\tilde{\mathcal{M}}^{\bullet}_{1}(n)$.

Proof. Using the result [Kir, Example 10.36], we have

$$\tilde{\mathcal{M}}^{\bullet}_{\mathbf{1}}(n) = \{(\alpha,\beta,v^1,0) \in M_n(\mathbb{C})^2 \times (\mathbb{C}^n)^2 \big| [\alpha,\beta] = 0, \mathbb{C}[\alpha,\beta]v^1 = \mathbb{C}^n\}$$

If $g \in \mathrm{GL}(\mathbb{C}^n)$ such that $g\alpha g^{-1} = \alpha$, $g\beta g^{-1} = \beta$ and $gv^1 = v^1$, then g is the identity morphism on $\mathbb{C}[\alpha,\beta]v^1$ which shows that $g = \mathrm{id}$. The action on $\tilde{\mathcal{M}}^{\bullet}_{\mathbf{1}}(n)$ is then free. The $\mathrm{GL}(\mathbb{C}^n)$-action on $\tilde{\mathcal{M}}^{\bullet}_{1}(n)$ is also free thanks to [Wil, Corollary 1.5]. □

Remark 2.2. If θ is nonzero, we have $\mathcal{M}^{\bullet}_{\boldsymbol{\theta}}(n) \simeq \mathcal{M}^{\bullet}_{\mathbf{1}}(n)$, thanks to [Kir, Lemma 10.29 & Theorem 11.5]. In addition, if λ is nonzero by rescaling we also have

$$\mathcal{M}^{\bullet}_{\lambda}(n) \simeq \mathcal{M}^{\bullet}_{1}(n)$$

Furthermore, thanks to Lemma 2.1 and the definition of semistability (Definition 1.21), it is clear that for each $\theta \in \mathbb{Q}^*$, $\tilde{\mathcal{M}}_1(n) = \tilde{\mathcal{M}}_{\boldsymbol{\theta},1}(n)$. We know [Kir, Theorem 11.5] that $\mathcal{M}^{\bullet}_{\mathbf{0},0}(n) \simeq (\mathbb{C}^2)^n/\mathfrak{S}_n$. In the end, for each $(\theta,\lambda) \in \mathbb{Q}\times\mathbb{C}$ we have

$$\mathcal{M}^{\bullet}_{\boldsymbol{\theta},\lambda}(n) \simeq \begin{cases} \mathcal{M}^{\bullet}_{1}(n) & \text{if } \lambda \neq 0 \\ \mathcal{M}^{\bullet}_{\mathbf{1}}(n) & \text{if } \lambda = 0 \text{ and } \theta \neq 0 \\ (\mathbb{C}^2)^n/\mathfrak{S}_n & \text{if } (\theta,\lambda) = (0,0) \end{cases}$$

Fix a couple $(\theta,\lambda) \in \mathbb{Q}\times\mathbb{C} \setminus \{(0,0)\}$. As a result of Remark 2.2 and [Kir, Lemma 10.29], if $\theta \neq 0$ we do not lose generality by assuming that $\theta > 0$.

Definition 2.3. Let us define a $\mathrm{GL}_2(\mathbb{C})$-action on $\tilde{\mathcal{M}}^{\bullet}_{\boldsymbol{\theta},\lambda}(n)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ and $(\alpha,\beta,v^1,v^2) \in \tilde{\mathcal{M}}^{\bullet}_{\boldsymbol{\theta},\lambda}(n)$

$$g.(\alpha,\beta,v^1,v^2) := (a\alpha + b\beta, c\alpha + d\beta, v^1, v^2)$$

Remark 2.4. It is easy to check that this action commutes with the $\mathrm{G}(n) = \mathrm{GL}(\mathbb{C}^n)$ action. The $\mathrm{GL}_2(\mathbb{C})$-action then descends to $\mathcal{M}^{\bullet}_{\boldsymbol{\theta},\lambda}(n)$.

2.2 Deconstruction

In this section, let us start with $\overline{(\alpha,\beta,v^1,v^2)} \in \mathcal{M}^{\bullet}_{\boldsymbol{\theta},\lambda}(n)^{\Gamma}$. Let $\tilde{A}_{\alpha,\beta}$ be $e_1 \otimes \alpha + e_2 \otimes \beta$ an element of $X_{\mathrm{std}} \otimes \mathrm{End}(\mathbb{C}^n)$ and denote by $A_{\alpha,\beta}$ the image of $\tilde{A}_{\alpha,\beta}$ though this chain of canonical isomorphisms

$$X_{\mathrm{std}} \otimes \mathrm{End}(\mathbb{C}^n) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}, X_{\mathrm{std}}) \otimes \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}^n, X_{\mathrm{std}} \otimes \mathbb{C}^n)$$

When the couple (α,β) is defined by the context, we just write A instead of $A_{\alpha,\beta}$. For each $\gamma \in \Gamma$, there exists a $g_\gamma \in \mathrm{GL}(\mathbb{C}^n)$ such that

$$\gamma.(\alpha,\beta,v^1,v^2) = g_\gamma.(\alpha,\beta,v^1,v^2) \tag{2.1}$$

Note that thanks to Lemma 2.1 the element g_γ is unique. Consider now the following group morphism

$$\sigma : \begin{array}{ccc} \Gamma & \to & \mathrm{GL}(\mathbb{C}^n) \\ \gamma & \mapsto & g_\gamma^{-1} \end{array}$$

This morphism equips $\mathbb{C}^n$ with a structure of Γ-module. Denote this Γ-module by M^σ. We can reformulate (2.1) as follows

$$\forall \gamma \in \Gamma, \gamma.(\alpha, \beta, v^1, v^2) = \sigma(\gamma^{-1}).(\alpha, \beta, v^1, v^2)$$

Lemma 2.5. *The morphism A is Γ-equivariant.*

Proof. Take $x \in M^\sigma$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, using equation (2.1), we have the following equalities

$$\begin{cases} \alpha(g_\gamma^{-1}(x)) = g_\gamma^{-1}(a\alpha(x) + b\beta(x)) \\ \beta(g_\gamma^{-1}(x)) = g_\gamma^{-1}(c\alpha(x) + d\beta(x)) \end{cases}$$

which then gives

$$\begin{cases} e_1 \otimes \alpha(g_\gamma^{-1}(x)) = e_1 \otimes g_\gamma^{-1}(a\alpha(x) + b\beta(x)) \\ e_2 \otimes \beta(g_\gamma^{-1}(x)) = e_2 \otimes g_\gamma^{-1}(c\alpha(x) + d\beta(x)) \end{cases}$$

Summing these two equations provides exactly that A is Γ-equivariant. □

Consider the morphism

$$\det : \begin{array}{ccc} X_{\mathrm{std}} \otimes X_{\mathrm{std}} & \to_\Gamma & X_{\chi_0} \\ \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \otimes \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} & \mapsto & r_1 s_2 - r_2 s_1 \end{array}$$

We now have everything, let us define $\Delta^A : X_{\mathrm{std}} \otimes M^\sigma \to M^\sigma$ as the composition of the two following maps

$$X_{\mathrm{std}} \otimes M^\sigma \xrightarrow{\mathrm{Id} \otimes A} X_{\mathrm{std}} \otimes X_{\mathrm{std}} \otimes M^\sigma \xrightarrow{\det \otimes \mathrm{Id}} M^\sigma$$

Lemma 2.6. *The morphism Δ^A is Γ-equivariant.*

Proof. This follows from the equivariance of A and of det. □

Finally, v^1 defines an element Z_1^v of $\mathrm{Hom}_\Gamma(X_{\chi_0}, M^\sigma)$ since $\forall \gamma \in \Gamma, v^1 = \sigma(\gamma^{-1})v^1$. In the same way, v^2 defines an element Z_2^v of $\mathrm{Hom}_\Gamma(M^\sigma, X_{\chi_0})$. Bringing everything together gives the following Proposition.

Proposition 2.7. *For each $\overline{(\alpha, \beta, v^1, v^2)} \in \mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$, $(\Delta^{A_{\alpha,\beta}}, Z_1^v, Z_2^v)$ is an element of $\mathcal{R}^\Gamma_{M^\sigma}$.*

2.3 Reconstruction

Let us turn it the other way around. Take a Γ-module M of dimension n. Consider the morphism

$$\text{ted} : \begin{array}{ccc} X_{\chi_0} & \to_\Gamma & X_{\text{std}} \otimes X_{\text{std}} \\ 1 & \mapsto & e_2 \otimes e_1 - e_1 \otimes e_2 \end{array}$$

Take $(\Delta, Z_1, Z_2) \in \mathcal{R}_M^\Gamma$ and let A_Δ be the composition

$$M \xrightarrow{\text{ted}\otimes\text{Id}} X_{\text{std}} \otimes X_{\text{std}} \otimes M \xrightarrow{\text{Id}\otimes\Delta} X_{\text{std}} \otimes M.$$

Lemma 2.8. *The morphism A_Δ is Γ-equivariant.*

Proof. Since ted and Δ are Γ-equivariant, A_Δ also is Γ-equivariant. □

Define $(\alpha_\Delta, \beta_\Delta) \in \text{End}(M)^2$ such that for all $m \in M$

$$A_\Delta(m) = e_1 \otimes \alpha_\Delta(m) + e_2 \otimes \beta_\Delta(m)$$

Let us denote $\mathcal{R}_M^\bullet := \mathcal{R}_{M,\chi_0}^{G\bullet}$. We can now consider the following linear map

$$\tilde{\imath}_M^\Gamma : \begin{array}{ccc} \mathcal{R}_M^\Gamma & \to & \mathcal{R}_M^\bullet \\ (\Delta, Z_1, Z_2) & \mapsto & (\alpha_\Delta, \beta_\Delta, Z_1, Z_2) \end{array}$$

Lemma 2.9. *The map $\tilde{\imath}_M^\Gamma$ is $\text{Aut}_\Gamma(M)$-equivariant.*

Proof. Take $g \in \text{Aut}_\Gamma(M)$ and $(\Delta, Z_1, Z_2) \in \mathcal{R}_M^\Gamma$. By definition of the action of $\text{Aut}_\Gamma(M)$ on $\mathcal{R}_M^\Gamma$, we have that $(g.\Delta)_{e_2} = g\Delta_{e_2}g^{-1}$ and $(g.\Delta)_{e_1} = g\Delta_{e_1}g^{-1}$. Since $\alpha_\Delta = -\Delta_{e_2}$ and $\beta_\Delta = \Delta_{e_1}$, it is clear that, by definition of the $\text{GL}(M)$-action on $\mathcal{R}_M^\bullet$, the map $\tilde{\imath}_M^\Gamma$ is $\text{Aut}_\Gamma(M)$-equivariant. □

Proposition 2.10. *The map $\tilde{\imath}_M^\Gamma$ induces a map $\imath_{\boldsymbol{\theta},\lambda,M}^\Gamma : \mathcal{M}_{\boldsymbol{\theta},\lambda\delta^\Gamma}^\Gamma(M) \to \mathcal{M}_{\boldsymbol{\theta},\lambda}^\bullet(n)^\Gamma$.*

Proof. First let us explain why $\tilde{\imath}_M^\Gamma(\tilde{\mathcal{M}}_{\lambda\delta^\Gamma}^\Gamma(M)) \subset \tilde{\mathcal{M}}_\lambda^\bullet(n)$.
Take $(\Delta, Z_1, Z_2) \in \mathcal{R}_M^\Gamma$. By construction α_Δ and β_Δ are respectively the morphisms $-\Delta_{e_2}$ and Δ_{e_1}. We then have

$$\alpha_\Delta\beta_\Delta - \beta_\Delta\alpha_\Delta + Z_1Z_2 = -\Delta_{e_2}\Delta_{e_1} + \Delta_{e_1}\Delta_{e_2} + Z_1Z_2$$

This computation shows that if $(\Delta, Z_1, Z_2) \in \tilde{\mathcal{M}}_{\lambda\delta^\Gamma}^\Gamma(M)$, then $(\alpha_\Delta, \beta_\Delta, Z_1, Z_2) \in \tilde{\mathcal{M}}_\lambda^\bullet(n)$. Let (Δ, Z_1, Z_2) be an element of $\tilde{\mathcal{M}}_{\boldsymbol{\theta},\lambda\delta^\Gamma}^\Gamma(M)$. If $\lambda \neq 0$ then Remark 2.2 gives directly that $\tilde{\imath}_M^\Gamma(\Delta, Z_1, Z_2)$ is $\boldsymbol{\theta}$-semistable. Take $(\Delta, Z_1, Z_2) \in \tilde{\mathcal{M}}_{\boldsymbol{\theta}}^\Gamma(M)$. If $\theta = 0$ there is nothing to show. If $\theta \neq 0$, we assume that $\theta > 0$. The Lemma 1.32 implies that $Z_1 \neq 0$ and using [Kir, Lemma 11.6] we have that α_Δ and β_Δ commute. To show that $(\alpha_\Delta, \beta_\Delta, Z_1, Z_2)$ is $\boldsymbol{\theta}$-semistable, using [Kir, Example 10.36], it is enough to show that $M = \mathbb{C}[\alpha_\Delta, \beta_\Delta]\text{Im}(Z_1)$. Let $S = \mathbb{C}[\alpha, \beta]\text{Im}(Z_1)$, the construction of α_Δ and β_Δ and the fact that $\text{Im}(Z_1)$ is a Γ-submodule of M, makes it clear that S is a Γ-submodule of M. Moreover, if $x = x_1e_1 + x_2e_2 \in X_{\text{std}}$ and $s \in S$, then $\Delta(x \otimes s) = x_1\beta_\Delta(s) - x_2\alpha_\Delta(s) \in S$. So, using Lemma 1.32, we have that $S = M$. The morphism $\tilde{\imath}_M^\Gamma$ is $\text{Aut}_\Gamma(M)$-equivariant (Lemma 2.9) . It then induces $\imath_{\boldsymbol{\theta},\lambda,M}^\Gamma : \mathcal{M}_{\boldsymbol{\theta},\lambda\delta^\Gamma}^\Gamma(M) \to \mathcal{M}_{\boldsymbol{\theta},\lambda}^\bullet(n)$. There remains to show

that $\iota^{\Gamma}_{\boldsymbol{\theta},\lambda,M}(\mathcal{M}^{\Gamma}_{\boldsymbol{\theta},\lambda\delta^{\Gamma}}(M)) \subset \mathcal{M}^{\bullet}_{\boldsymbol{\theta},\lambda}(n)^{\Gamma}$. Indeed, if $(\Delta, Z_1, Z_2) \in \mathcal{R}^{\Gamma}_M$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $m \in M$, then Lemma 2.8 implies that

$$\gamma.(e_1 \otimes \alpha_\Delta(m) + e_2 \otimes \beta_\Delta(m)) = e_1 \otimes \alpha_\Delta(\gamma.m) + e_2 \otimes \beta_\Delta(\gamma.m)$$

From there, we have

$$\begin{cases} \gamma.(a\alpha_\Delta(m) + b\beta_\Delta(m)) = \alpha_\Delta(\gamma.m) \\ \gamma.(c\alpha_\Delta(m) + d\beta_\Delta(m)) = \beta_\Delta(\gamma.m) \end{cases} \Rightarrow \begin{cases} a\alpha_\Delta(m) + b\beta_\Delta(m) = \gamma^{-1}.\alpha_\Delta(\gamma.m) \\ c\alpha_\Delta(m) + d\beta_\Delta(m) = \gamma^{-1}.\beta_\Delta(\gamma.m) \end{cases}$$

□

2.4 Synthesis

Let us now connect the last two sections. Let $L^{\Gamma}_{\boldsymbol{\theta},\lambda}$ be the following algebraic variety

$$\left\{(\alpha, \beta, v^1, v^2, \sigma) \in \tilde{\mathcal{M}}^{\bullet}_{\boldsymbol{\theta},\lambda}(n) \times \mathrm{Rep}_{\Gamma,n} \mid \forall \gamma \in \Gamma, \gamma.(\alpha, \beta, v^1, v^2) = \sigma(\gamma^{-1}).(\alpha, \beta, v^1, v^2)\right\}$$

and define

- $\tilde{p}_{\boldsymbol{\theta},\lambda} : L^{\Gamma}_{\boldsymbol{\theta},\lambda} \to \tilde{\mathcal{M}}^{\bullet}_{\boldsymbol{\theta},\lambda}(n)$
- $p_{\boldsymbol{\theta},\lambda} : L^{\Gamma}_{\boldsymbol{\theta},\lambda} \to \mathcal{M}^{\bullet}_{\boldsymbol{\theta},\lambda}(n) = \pi \circ \tilde{p}_{\boldsymbol{\theta},\lambda}$
- $q_{\boldsymbol{\theta},\lambda} : L^{\Gamma}_{\boldsymbol{\theta},\lambda} \to \mathrm{Rep}_{\Gamma,n}$

Let the group $\mathrm{GL}_n(\mathbb{C})$ act by conjugacy on $\mathrm{Rep}_{\Gamma,n}$ and diagonally on $L^{\Gamma}_{\boldsymbol{\theta},\lambda}$. The maps $\tilde{p}_{\boldsymbol{\theta},\lambda}$ and $q_{\boldsymbol{\theta},\lambda}$ are $\mathrm{GL}_n(\mathbb{C})$-equivariant maps. Moreover, note that $p_{\boldsymbol{\theta},\lambda}(L^{\Gamma}_{\boldsymbol{\theta},\lambda}) = \mathcal{M}^{\bullet}_{\boldsymbol{\theta},\lambda}(n)^{\Gamma}$.

Lemma 2.11. *If $(\alpha, \beta, v^1, v^2, \sigma)$ and $(\alpha, \beta, v^1, v^2, \sigma')$ are both in $L^{\Gamma}_{\boldsymbol{\theta},\lambda}$, then $\sigma = \sigma'$*

Proof. From the assumption, we have $\forall \gamma \in \Gamma\ \sigma(\gamma).(\alpha, \beta, v^1, v^2) = \sigma'(\gamma).(\alpha, \beta, v^1, v^2)$. We can then use Lemma 2.1 and Remark 2.2 to deduce that $\sigma = \sigma'$. □

If $\sigma \in \mathrm{Rep}_{\Gamma,n}$ recall that we denote by $M^\sigma := \mathbb{C}^n$ the Γ-module induced by σ.

Definition 2.12. Let $\sigma \in \mathrm{Rep}_{\Gamma,n}$ be such that $q^{-1}_{\boldsymbol{\theta},\lambda}(\sigma) \neq \emptyset$ and define

$$\tilde{\kappa}^{\Gamma}_{\boldsymbol{\theta},\lambda,\sigma} : \begin{array}{ccc} q^{-1}_{\boldsymbol{\theta},\lambda}(\sigma) & \to & \mathcal{R}^{\Gamma}_{M^\sigma} \\ (\alpha, \beta, v^1, v^2, \sigma) & \mapsto & (\Delta^{A_{\alpha,\beta}}, Z^v_1, Z^v_2) \end{array}$$

where $(\Delta^{A_{\alpha,\beta}}, Z^v_1, Z^v_2)$ is as in Proposition 2.7.

Proposition 2.13. *If $\sigma \in \mathrm{Rep}_{\Gamma,n}$ such that $q^{-1}_{\boldsymbol{\theta},\lambda}(\sigma) \neq \emptyset$, then $\tilde{\kappa}^{\Gamma}_{\boldsymbol{\theta},\lambda,\sigma}\left(q^{-1}_{\boldsymbol{\theta},\lambda}(\sigma)\right) = \tilde{\mathcal{M}}^{\Gamma}_{\boldsymbol{\theta},\lambda\delta^{\Gamma}}(M^\sigma)$.*

Proof. Take $(\alpha, \beta, v^1, v^2, \sigma) \in q^{-1}_{\boldsymbol{\theta},\lambda}(\sigma)$, by construction $\Delta^A_{e_1} = \beta$ and $\Delta^A_{e_2} = -\alpha$. We then have

$$\Delta^A_{e_1}\Delta^A_{e_2} - \Delta^A_{e_2}\Delta^A_{e_1} + Z^v_1 Z^v_2 = -\beta\alpha + \alpha\beta + v^1 v^2$$

This proves that if $(\alpha, \beta, v^1, v^2, \sigma) \in q^{-1}_{\boldsymbol{\theta},\lambda}(\sigma)$ then $\tilde{\kappa}^{\Gamma}_{\boldsymbol{\theta},\lambda,\sigma}(\alpha, \beta, v^1, v^2, \sigma) \in \tilde{\mathcal{M}}^{\Gamma}_{\lambda\delta^{\Gamma}}(M^\sigma)$. If $\theta = 0$ it is clear that $\tilde{\kappa}^{\Gamma}_{\boldsymbol{\theta},\lambda,\sigma}(\alpha, \beta, v^1, v^2, \sigma) \in \tilde{\mathcal{M}}^{\Gamma}_{\boldsymbol{\theta},\lambda\delta^{\Gamma}}(M^\sigma)$. If now $\theta \neq 0$, thanks to

Remark 2.2, we do not lose generality by assuming that $\lambda = 0$. Moreover, recall that in the case $\theta \neq 0$, we can reduce to the case where $\theta > 0$ so that Lemma 1.32 can be used to show that (Δ^A, Z_1^v, Z_2^v) is $\boldsymbol{\theta}$-semistable. Take M' a Γ-submodule of M^σ such that $\Delta^A(X_{\text{std}} \otimes M') \subset M'$ and $\text{Im}(Z_1^v) \subset M'$. Let us show that $M^\sigma \subset M'$. Since the element $(\alpha, \beta, v^1, v^2)$ is in $\tilde{\mathcal{M}}^\bullet_{\boldsymbol{\theta}}(n)$, we have that $[\alpha, \beta] = 0$ and $\mathbb{C}[\alpha, \beta]v^1 = M^\sigma$. Take $m \in M^\sigma$, we then have a polynomial $P \in \mathbb{C}[x, y]$ such that $P(\alpha, \beta)v^1 = m$. Since $\alpha = -\Delta^A_{e_2}$ and $\beta = \Delta^A_{e_1}$, $m = P(-\Delta^A_{e_2}, \Delta^A_{e_1})v$. By hypothesis $v^1 \in M'$ and using the stability of M' by Δ we can conclude that $m \in M'$.
To finish, let us prove the other inclusion i.e. if $(\Delta, Z_1, Z_2) \in \tilde{\mathcal{M}}^\Gamma_{\boldsymbol{\theta}, \lambda\delta^\Gamma}(M^\sigma)$ we need to show that $(-\Delta_{e_2}, \Delta_{e_1}, Z_1, Z_2, \sigma) \in L^\Gamma_{\boldsymbol{\theta}, \lambda}$. Take $\gamma \in \Gamma$, then a quick computation gives

$$\begin{aligned}\gamma.(-\Delta_{e_2}, \Delta_{e_1}, Z_1, Z_2) &= (-\Delta_{\gamma^{-1}e_2}, \Delta_{\gamma^{-1}e_1}, Z_1, Z_2) \\ &= (-\sigma(\gamma)^{-1}\Delta_{e_2}\sigma(\gamma), \sigma(\gamma)^{-1}\Delta_{e_1}\sigma(\gamma), \sigma(\gamma)^{-1}Z_1, Z_2\sigma(\gamma))\end{aligned}$$

The last equality comes from the Γ-equivariance of Δ, Z_1 and Z_2. □

Consider now $\kappa^\Gamma_{\boldsymbol{\theta}, \lambda, \sigma} : q^{-1}_{\boldsymbol{\theta}, \lambda}(\sigma) \to \mathcal{M}^\Gamma_{\boldsymbol{\theta}, \lambda\delta^\Gamma}(M^\sigma)$ which is onto thanks to Proposition 2.13.

Proposition 2.14. *For each $\sigma \in \text{Rep}_{\Gamma, n}$ such that $q^{-1}_{\boldsymbol{\theta}, \lambda}(\sigma) \neq \emptyset$, the following diagram commutes*

$$\begin{array}{ccc} q^{-1}_{\boldsymbol{\theta}, \lambda}(\sigma) & \xrightarrow{\kappa^\Gamma_{\boldsymbol{\theta}, \lambda, \sigma}} & \mathcal{M}^\Gamma_{\boldsymbol{\theta}, \lambda\delta^\Gamma}(M^\sigma) \\ \downarrow p_{\boldsymbol{\theta}, \lambda} & \swarrow \iota^\Gamma_{\boldsymbol{\theta}, \lambda, M^\sigma} & \\ \mathcal{M}^\bullet_{\boldsymbol{\theta}, \lambda}(n)^\Gamma & & \end{array}$$

Proof. Take $(\alpha, \beta, v^1, v^2, \sigma) \in q^{-1}_{\boldsymbol{\theta}, \lambda}(\sigma)$. By construction one has

$$\tilde{\iota}^\Gamma_{\boldsymbol{\theta}, \lambda, M^\sigma}\left(\tilde{\kappa}^\Gamma_{\boldsymbol{\theta}, \lambda, \sigma}(\alpha, \beta, v^1, v^2, \sigma)\right) = (\alpha, \beta, v^1, v^2)$$

which shows that the diagram commutes. □

For $\sigma \in \text{Rep}_{\Gamma, n}$ and for $\chi \in I_\Gamma$, let $d^\sigma_\chi := \dim\left(\text{Hom}_\Gamma(X_\chi, M^\sigma)\right)$. Denote the character map by

$$\text{char}_\Gamma : \begin{array}{ccc} \text{Rep}_{\Gamma, n} & \to & \Delta_\Gamma \\ \sigma & \mapsto & d^\sigma \end{array}$$

Let $\tilde{\mathcal{A}}^n_\Gamma := \text{char}_\Gamma(\text{Rep}_{\Gamma, n})$ which is just a combinatorial way to encode $\text{Char}_{\Gamma, n}$ the set of all characters associated to n-dimensional representations of Γ. Moreover for $d \in \tilde{\mathcal{A}}^n_\Gamma$, denote by $\mathcal{C}_d := \text{char}^{-1}_\Gamma(d)$ the set of all n-dimensional representations that have character $\sum_{\chi \in I_\Gamma} d_\chi \chi$. Note that if we take $\sigma \in \mathcal{C}_d$, then $\mathcal{C}_d$ is just the $\text{GL}_n(\mathbb{C})$-conjugacy class of σ. With that notation, we have

$$\text{Rep}_{\Gamma, n} = \coprod_{d \in \tilde{\mathcal{A}}^n_\Gamma} \mathcal{C}_d$$

Remark 2.15. The set $\tilde{\mathcal{A}}^n_\Gamma$ is equal to $\left\{d \in \Delta^+_\Gamma \middle| |d|_\Gamma = n\right\}$.

Denote by $\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda} := \left\{ d \in \tilde{\mathcal{A}}^n_\Gamma | q^{-1}_{\boldsymbol{\theta},\lambda}(\mathcal{C}_d) \neq \emptyset \right\}$. For $d \in \mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$, recall that we can associate an I_Γ-graded vector space $V^d := \bigoplus_{\chi \in I_\Gamma} \mathbb{C}^{d_\chi}$. Denote by $M^d := \bigoplus_{\chi \in I_\Gamma} V^d_\chi \otimes X_\chi$ the Γ-module associated to d.

Definition 2.16. Take $d \in \mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$. Let us define the variety $\mathcal{M}^\Gamma_d := p_{\boldsymbol{\theta},\lambda}\left(q^{-1}_{\boldsymbol{\theta},\lambda}(\mathcal{C}_d)\right)$.

Remark 2.17. Note that $\mathcal{M}^\Gamma_d = p_{\boldsymbol{\theta},\lambda}\left(q^{-1}_{\boldsymbol{\theta},\lambda}(\sigma)\right)$ for any $\sigma \in \mathcal{C}_d$ since $\tilde{p}_{\boldsymbol{\theta},\lambda}$ and $q_{\boldsymbol{\theta},\lambda}$ are $\mathrm{GL}_n(\mathbb{C})$-equivariant.

Theorem 2.18. *Let $(\theta,\lambda) \in \mathbb{Q} \times \mathbb{C} \setminus \{(0,0)\}$. For each integer n and each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, we have the following decomposition into irreducible components*

$$\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma = \coprod_{d \in \mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}} \mathcal{M}^\Gamma_d$$

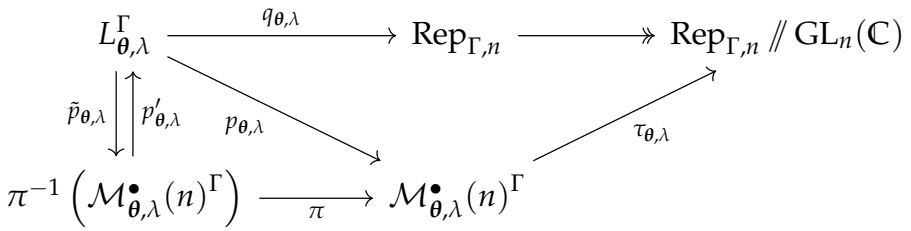
Proof. Take $d \in \mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$ and let us first show that $\mathcal{M}^\Gamma_d$ is an irreducible and closed set of $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$. Take $\sigma \in \mathcal{C}_d$. Proposition 1.30 gives that $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda\delta^\Gamma}(M^\sigma)$ is irreducible and since $\mathcal{M}^\Gamma_d = \iota^\Gamma_{\boldsymbol{\theta},\lambda,M^\sigma}\left(\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda\delta^\Gamma}(M^\sigma)\right)$, we have that $\mathcal{M}^\Gamma_d$ is irreducible. To show that $\mathcal{M}^\Gamma_d$ is a closed set of $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$, note that $L^\Gamma_{\boldsymbol{\theta},\lambda}$ is connected and that $\tilde{p}_{\boldsymbol{\theta},\lambda}$ is injective thanks to Lemma 2.11. The image of $p_{\boldsymbol{\theta},\lambda}$ being $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$, this implies that

$$\tilde{p}_{\boldsymbol{\theta},\lambda}\left(L^\Gamma_{\boldsymbol{\theta},\lambda}\right) = \pi^{-1}\left(\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma\right)$$

The group $\mathrm{GL}_n(\mathbb{C})$ acts freely on $\tilde{\mathcal{M}}^\bullet_{\boldsymbol{\theta},\lambda}(n)$ (Lemma 2.1), which gives that π is a smooth morphism. The group Γ being a finite group, we know that $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$ is smooth and in particular $\pi^{-1}\left(\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma\right)$ is smooth. Thus, the morphism $\tilde{p}_{\boldsymbol{\theta},\lambda}$ becomes an isomorphism of algebraic varieties between $L^\Gamma_{\boldsymbol{\theta},\lambda}$ and $\pi^{-1}\left(\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma\right)$. Let us denote by $p'_{\boldsymbol{\theta},\lambda} : \pi^{-1}\left(\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma\right) \to L^\Gamma_{\boldsymbol{\theta},\lambda}$ its inverse. Consider now

$$\tilde{\tau}_{\boldsymbol{\theta},\lambda} := q_{\boldsymbol{\theta},\lambda} \circ p'_{\boldsymbol{\theta},\lambda} : \pi^{-1}\left(\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma\right) \to \mathrm{Rep}_{\Gamma,n}.$$

Since $\tilde{\tau}_{\boldsymbol{\theta},\lambda}$ is $\mathrm{GL}_n(\mathbb{C})$-equivariant, we have $\tau_{\boldsymbol{\theta},\lambda} : \mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma \to \mathrm{Rep}_{\Gamma,n} /\!\!/ \mathrm{GL}_n(\mathbb{C})$.
Here is the big picture



It is then clear that

$$\tau^{-1}_{\boldsymbol{\theta},\lambda}(\mathcal{C}_d) = p_{\boldsymbol{\theta},\lambda}\left(q^{-1}_{\boldsymbol{\theta},\lambda}(\mathcal{C}_d)\right) = \mathcal{M}^\Gamma_d$$

which proves that $\mathcal{M}_d^\Gamma$ is a closed set. Indeed $\mathcal{C}_d$ is closed because all representations of Γ are semisimple since Γ is a finite group.
Finally, we have to show that $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma = \bigcup_{d\in\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}} \mathcal{M}_d^\Gamma$ which comes for free

$$\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma = \bigcup_{d\in\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}} \tau^{-1}_{\boldsymbol{\theta},\lambda}(\mathcal{C}_d) = \bigcup_{d\in\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}} \mathcal{M}_d^\Gamma$$

□

Proposition 2.19. *For each $d \in \mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$ and for each $\sigma \in \mathcal{C}_d$*

$$\iota^\Gamma_{\boldsymbol{\theta},\lambda,M^\sigma} : \mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda\delta^\Gamma}(M^\sigma) \to \mathcal{M}_d^\Gamma$$

is an isomorphism of algebraic varieties.

Proof. The morphism $\iota^\Gamma_{\boldsymbol{\theta},\lambda,M}$ is injective. Using Proposition 2.13 and 2.14, we have

$$\mathcal{M}_d^\Gamma = \iota^\Gamma_{\boldsymbol{\theta},\lambda,M^\sigma}\big(\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda\delta^\Gamma}(M^\sigma)\big)$$

Moreover, the variety $\mathcal{M}_d^\Gamma$ is smooth. Indeed, thanks to Theorem 2.18, we have that $\mathcal{M}_d^\Gamma$ is an irreducible component of the smooth variety $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$. Furthermore, since $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)^\Gamma$ has a finite number of irreducible components, this implies that $\mathcal{M}_d^\Gamma$ is open. Finally, using Theorem 1.29 and Proposition 1.30, we have that $\mathcal{M}^\Gamma_{\boldsymbol{\theta},\lambda\delta^\Gamma}(M^\sigma)$ is connected. Summing it all up, we can now conclude that $\iota^\Gamma_{\boldsymbol{\theta},\lambda,M^\sigma}$ is an isomorphism of algebraic varieties. □

CHAPTER 3

IRREDUCIBLE COMPONENTS OF $\mathcal{H}_n^\Gamma$ AND $\mathscr{X}_n^\Gamma$

In the previous chapter, we have studied the Γ-fixed point locus of the Jordan quiver variety $\mathcal{M}^\bullet_{\boldsymbol{\theta},\lambda}(n)$ when $(\theta,\lambda) \neq (0,0)$. In this chapter we will first use Theorem 2.18 to retrieve the irreducible components of the Hilbert scheme of n points in $\mathbb{C}^2$ and of the n^{th} Calogero-Moser space. In a second part, a combinatorial description of the indexing set $\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$ will be given. To do so, we will work with the McKay realization (Remark 1.13) and a new statistic on the root lattice $Q(\tilde{T}_\Gamma)$ will be introduced.

3.1 Hilbert scheme of n points in $\mathbb{C}^2$

On the one side, denote the Hilbert scheme of n points on $\mathbb{C}^2$ by $\mathcal{H}_n$ which is

$$\{I \subset \mathbb{C}[x,y] | I \text{ is an ideal and } \dim(\mathbb{C}[x,y]/I) = n\}$$

John Fogarty showed [Fo68, Proposition 2.2 & Theorem 2.9] that $\mathcal{H}_n$ is a smooth connected $2n$ dimensional algebraic variety. The algebraic group $\mathrm{GL}_2(\mathbb{C})$ acts naturally on $\mathbb{C}^2$, thus on the coordinate ring $\mathbb{C}[x,y]$. This action induces a $\mathrm{GL}_2(\mathbb{C})$-action on $\mathcal{H}_n$. Our fixed group Γ being a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, $\mathcal{H}_n^\Gamma$, the locus of Γ-fixed elements of $\mathcal{H}_n$ is also a smooth algebraic variety. For $I \in \mathcal{H}_n^\Gamma$, the n-dimensional vector space $\mathbb{C}[x,y]/I$ is a representation of Γ. For $\chi \in \mathrm{Char}_{\Gamma,n}$, let us denote by $\mathcal{H}_n^{\Gamma,\chi} := \{I \in \mathcal{H}_n^\Gamma | \mathrm{Tr}(\mathbb{C}[x,y]/I) = \chi\}$. On the level of sets, we have the following decomposition

$$\mathcal{H}_n^\Gamma = \coprod_{\chi \in \mathrm{Char}_{\Gamma,n}} \mathcal{H}_n^{\Gamma,\chi}$$

In what follows, we will show that this decomposition lifts to the category of algebraic varieties and that every nonempty $\mathcal{H}_n^{\Gamma,\chi}$ is an irreducible component of $\mathcal{H}_n^\Gamma$. Let us construct a morphism between $\mathcal{H}_n$ and $\mathcal{M}^\bullet_{\mathbf{1}}(n)$. Given an ideal $I \in \mathcal{H}_n$ consider the following two linear maps

- $m_x^I : \begin{array}{ccc} \mathbb{C}[x,y]/I & \to & \mathbb{C}[x,y]/I \\ \bar{P} & \mapsto & \overline{xP} \end{array}$
- $m_y^I : \begin{array}{ccc} \mathbb{C}[x,y]/I & \to & \mathbb{C}[x,y]/I \\ \bar{P} & \mapsto & \overline{yP} \end{array}$

Denote $v^I = \bar{1} \in \mathbb{C}[x,y]/I$ and define

$$H : \begin{array}{ccc} \mathcal{H}_n & \to & \mathcal{M}^\bullet_{\mathbf{1}}(n) \\ I & \mapsto & (m_x^I, m_y^I, v^I, 0) \end{array}$$

Proposition 3.1. *The morphism H is a $\mathrm{GL}_2(\mathbb{C})$-equivariant isomorphism.*

Proof. The fact that H is an isomorphism is proven in [Kir, Theorem 11.5]. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{GL}_2(\mathbb{C})$ and $I \in \mathcal{H}_n$. Then

$$\nu : \begin{array}{ccc} \mathbb{C}[x,y]/I & \to & \mathbb{C}[x,y]/g.I \\ \bar{P} & \mapsto & \overline{g.P} \end{array}$$

is an isomorphism. It is clear that $\nu^{-1}(v^{g.I}) = v^I$. A simple computation gives

$$\nu^{-1} \circ m_x^{g.I} \circ \nu = a m_x^I + b m_y^I$$

$$\nu^{-1} \circ m_y^{g.I} \circ \nu = c m_x^I + d m_y^I$$

which shows that $H(g.I) = g.H(I)$.

□

Remark 3.2. If $d \in \tilde{A}_\Gamma^n$, note that $\mathcal{M}_d^\Gamma$ is equal to $H(\mathcal{H}_n^{\Gamma,\xi_d})$ where $\xi_d := \sum_{\chi \in I_\Gamma} d_\chi \chi$ is an element of $\mathrm{Char}_{\Gamma,n}$. We also have $|d|_\Gamma = k_{\xi_d} = n$.

Corollary 3.3. *For each integer n and each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, we can decompose the Γ-fixed point locus of the Hilbert scheme of n points in $\mathbb{C}^2$ into irreducible components*

$$\mathcal{H}_n^\Gamma = \coprod_{d \in \mathcal{A}_{\Gamma,\mathbf{1}}^n} \mathcal{H}_n^{\Gamma,\xi_d}$$

Proof. Applying H^{-1} to Theorem 2.18 and using Proposition 3.1, we get this decomposition into irreducible components. □

3.2 Calogero-Moser space

On the other side, i.e. when $(\theta, \lambda) = (0, 1)$, let us introduce the Calogero-Moser space. Denote the n^{th} Calogero-Moser space by $\mathscr{X}_n$ which can be defined as

$$\{(X,Y) \in \mathrm{M}_n(\mathbb{C})^2 | XY - YX + I_n \text{ is a rank 1 matrix}\} /\!\!/ \mathrm{GL}_n(\mathbb{C})$$

where $\mathrm{GL}_n(\mathbb{C})$ acts by base change on X and Y. George Wilson showed that $\mathscr{X}_n$ is a smooth, connected affine algebraic variety of dimension $2n$ [Wil, Section 1]. To describe the $\mathrm{GL}_2(\mathbb{C})$ action on $\mathscr{X}_n$ in a natural way, we need a slightly different model. Let $V_n := \mathbb{C}^n$ be the reflection representation of $\mathfrak{S}_n$, $S := \{s \in \mathfrak{S}_n | \dim(\mathrm{Im}(s - \mathrm{id}_{V_n})) = 1\}$ the set of reflections in $\mathfrak{S}_n$. Let us denote by $T(V_n \oplus V_n^*)$ the tensor algebra of $V_n \oplus V_n^*$. For each reflection $s \in S$, let us choose two nonzero elements $\alpha_s^\vee \in \mathrm{Im}(s - \mathrm{id}_{V_n})$ and $\alpha_s \in \mathrm{Im}(s - \mathrm{id}_{V_n^*})$. Let Z_n be the center of the algebra $\left(T(V_n \oplus V_n^*) \rtimes \mathfrak{S}_n\right)/I_n$ where I_n is the ideal generated by

$$\{[x,x'] | (x,x') \in (V_n^*)^2\} \cup \{[y,y'] | (y,y') \in V_n^2\} \cup \{[x,y] + \sum_{s \in S} \alpha_s(y) x (\alpha_s^\vee) s | (x,y) \in V_n^* \times V_n\}$$

Thanks to [EG, Theorem 1.23], we have an isomorphism $\mathrm{EG} : \mathrm{Spec}(Z_n) \xrightarrow{\sim} \mathscr{X}_n$. Let the group $\mathrm{GL}_2(\mathbb{C})$ act naturally on $\mathbb{C}^2$. Note that $V_n \simeq_{\mathfrak{S}_n} V_n^*$ which implies that $\mathbb{C}[V_n \oplus V_n^*] \rtimes \mathfrak{S}_n \simeq \mathbb{C}[\mathbb{C}^2 \otimes V_n] \rtimes \mathfrak{S}_n$. This gives a natural $\mathrm{GL}_2(\mathbb{C})$-action on Z_n and

on $\mathscr{X}_n$ using EG. The isomorphism [EG, (11.12)] applied to our special case gives that $C : \begin{array}{ccc} \mathcal{M}_1^\bullet(n) & \to & \mathscr{X}_n \\ (\alpha,\beta,v,w) & \mapsto & (\alpha,\beta) \end{array}$ is an isomorphism. By construction, we have the following Lemma.

Lemma 3.4. *The isomorphism C is an isomorphism of $\mathrm{GL}_2(\mathbb{C})$-varieties.*

For $d \in \tilde{\mathcal{A}}^n_\Gamma$, let us denote by $\mathscr{X}_n^{\Gamma,d} := C(\mathcal{M}_d^\Gamma)$.

Corollary 3.5. *For each integer n and each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, we can decompose the Γ-fixed point locus of the n^{th} Calogero-Moser space into irreducible components*

$$\mathscr{X}_n^\Gamma = \coprod_{d \in \mathcal{A}^n_{\Gamma,1}} \mathscr{X}_n^{\Gamma,d}$$

Proof. Apply C to Theorem 2.18 and use Lemma 3.4 to obtain this decomposition. □

3.3 On the parametrization sets $\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$

Fix $(\theta,\lambda) \neq (0,0)$ and let us now give a combinatorial model of the indexing set $\mathcal{A}^n_{\Gamma,\boldsymbol{\theta},\lambda}$. Denote by G_Γ^{-0} the McKay graph without the vertex χ_0 and all arrows going in or out of this vertex. The lattice $\Delta_\Gamma^{-0} := \Delta_{G_\Gamma^{-0}}$ is then identified with the root lattice $Q(T_\Gamma)$ where T_Γ is the type of the Dynkin diagram G_Γ^{-0}. Recall that for $a \in Q(T_\Gamma)$ we have denoted by $t_a \in W(\tilde{T}_\Gamma)$ the image of a by the isomorphism $W(T_\Gamma) \ltimes Q(T_\Gamma) \xrightarrow{\sim} W(\tilde{T}_\Gamma)$. In the following, we will denote the fact that there exists $k \in \mathbb{Z}$, such that $a - b = kc$ for $(a,b,c) \in (\mathfrak{h}^*_\Gamma)^3$, by $a \equiv b[c]$.

Lemma 3.6. *For all $(a,d) \in Q(T_\Gamma) \times Q(\tilde{T}_\Gamma)$, we have $t_a.d \equiv d - a\ [\delta^\Gamma]$.*

Proof. Thanks to relation (1.1) and [Kac, Formula 6.5.2], we have

$$t_a.d = \Lambda_0 - t_a * (\Lambda_0 - d) \equiv d - a + \langle d, \delta_\Gamma^\vee \rangle [\delta^\Gamma]$$

Since $d \in Q(\tilde{T}_\Gamma)$, $\langle d, \delta_\Gamma^\vee \rangle = 0$ by definition of $\delta_\Gamma^\vee$. □

Lemma 3.7. *For each $d \in Q(\tilde{T}_\Gamma)$, there exists a unique integer r such that d and $r\delta^\Gamma$ are in the same $W(\tilde{T}_\Gamma)$-orbit for the . action from Definition 1.33.*

Proof. Take $d \in Q(\tilde{T}_\Gamma)$, then $a := d - d_0\delta^\Gamma \in Q(T_\Gamma)$ and thanks to Lemma 3.6, $t_a.d$ is an element of the desired form. Now suppose that there are two integers r_1 and r_2 such that $r_1\delta^\Gamma$ and $r_2\delta^\Gamma$ are in the same $W(\tilde{T}_\Gamma)$-orbit. Since δ^Γ is in the kernel of the generalized Cartan matrix of type $\tilde{T}_\Gamma$, δ^Γ is fixed under the action of $W(T_\Gamma)$. This observation reduces the $W(\tilde{T}_\Gamma)$-orbit of $r_1\delta^\Gamma$ to the $Q(T_\Gamma)$-orbit. There must then exist $a \in Q(T_\Gamma)$ such that $t_a.r_1\delta^\Gamma = r_2\delta^\Gamma$. Using Lemma 3.6, $t_a.r_1\delta^\Gamma \equiv r_1\delta^\Gamma - a\ [\delta^\Gamma]$, we can conclude that $a = 0$ and that $r_1 = r_2$. □

Definition 3.8. For $d \in Q(\tilde{T}_\Gamma)$ let the **weight of** d be the unique integer r such that $r\delta^\Gamma$ and d are in the same $W(\tilde{T}_\Gamma)$-orbit. It will be denoted by $\mathrm{wt}(d)$.

The following proposition establishes a bridge between quiver varieties and combinatorics.

Proposition 3.9. *Let $(\theta, \lambda) \in \Theta_\Gamma^{++} \times \Lambda_\Gamma^{++}$. For each $d \in \Delta_\Gamma$*

$$\mathcal{M}_{\boldsymbol{\theta}}^\Gamma(d) \neq \emptyset \iff \mathrm{wt}(d) \geq 0 \iff \mathcal{M}_\lambda^\Gamma(d) \neq \emptyset$$

Proof. Using [Nak98, Theorem 10.2] (or [Kir, Theorem 13.19]), the variety $\mathcal{M}_{\boldsymbol{\theta}}^\Gamma(d)$ is nonempty if and only if $\Lambda_0 - d$ is a weight of the basic representation $L(\Lambda_0)$ of the Kac-Moody Lie algebra of type $\tilde{T}_\Gamma$. Moreover, using [Cart05, Theorem 20.23] we have that the set of weight of $L(\Lambda_0)$ is the following

$$\left\{ \Lambda_0 + \gamma - (\frac{1}{2}\langle \gamma, \gamma \rangle - k)\delta^\Gamma | \gamma \in Q(T_\Gamma), k \in \mathbb{Z}_{\geq 0} \right\}$$

Lemma 3.7, tells us that there is $\omega \in W(\tilde{T}_\Gamma)$ such that $d = \omega.(r\delta^\Gamma)$ with $r = \mathrm{wt}(d)$. Since δ^Γ is fixed by the action of $W(T_\Gamma)$, there is $a \in Q(T_\Gamma)$ such that $d = t_a.(r\delta^\Gamma)$. Thanks to [Kac, Formula 6.5.2] and the definitions of δ^Γ and Λ_0, we have that

$$t_a * (r\delta^\Gamma) = r\delta^\Gamma - a + \frac{1}{2}\langle a, a \rangle \delta^\Gamma$$

Using relation (1.1), we can conclude that d is a weight of $L(\Lambda_0)$ if and only if $r \geq 0$. In addition, we know that both $\mathcal{M}_{\mathbf{2\lambda}}^\Gamma(d)$ and $\mathcal{M}_\lambda^\Gamma(d)$ are hyper-Kähler reductions [King, Corollary 6.2] and [Nak94, Theorem 3.1]. Using the rotation map defined in [Gor08, Section 3.7], we have that these varieties are diffeomorphic. By the first equivalence, we have that $\mathcal{M}_\lambda^\Gamma(d)$ is nonempty if and only if $\mathrm{wt}(d) \geq 0$. □

Theorem 3.10. *For each integer n and each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, the set indexing the irreducible components of $\mathcal{H}_n^\Gamma$ and of $\mathscr{X}_n^\Gamma$ are equal*

$$\mathcal{A}_{\Gamma,\mathbf{1}}^n = \mathcal{A}_{\Gamma,1}^n = \left\{ d \in \Delta_\Gamma^+ \big| |d|_\Gamma = n, \mathrm{wt}(d) \geq 0 \right\}$$

Proof. Take $d \in \tilde{\mathcal{A}}_\Gamma^n \subset \Delta_\Gamma$ and let us reformulate combinatorially the definiton of $\mathcal{A}_\Gamma^n$. Thanks to the proof of Theorem 2.18, d is in $\mathcal{A}_{\Gamma,\boldsymbol{\theta},\lambda}^n$ if and only if $\mathcal{M}_d^\Gamma$ is nonempty. Since $\mathcal{M}_d^\Gamma = \iota_{\boldsymbol{\theta},\lambda,M^\sigma}^\Gamma \left(\mathcal{M}_{\boldsymbol{\theta},\lambda\delta^\Gamma}^\Gamma(M^\sigma) \right)$ for any $\sigma \in \mathcal{C}_d$, it is nonempty if and only if $\mathcal{M}_{\boldsymbol{\theta},\lambda\delta^\Gamma}^\Gamma(d)$ is nonempty. Applying Proposition 3.9 first for $\theta = 1$ and then for $\lambda = \delta^\Gamma$, gives the result. □

From now on let us denote $\mathcal{A}_{\Gamma,\mathbf{1}}^n = \mathcal{A}_{\Gamma,1}^n$ by $\mathcal{A}_\Gamma^n$. To finish, let us give a simple expression of the dimension of the irreducible components $\mathcal{M}_d^\Gamma$ for each $d \in \mathcal{A}_\Gamma^n$.

Proposition 3.11. *For each $d \in \mathcal{A}_\Gamma^n$, the variety $\mathcal{M}_d^\Gamma$ has dimension $2\mathrm{wt}(d)$.*

Proof. Take $d \in \mathcal{A}_\Gamma^n$. There exists $\omega_d \in W(\tilde{T}_\Gamma)$ and $r_d \in \mathbb{Z}_{\geq 0}$ such that $\omega_d.d = r_d\delta^\Gamma$. If $r_d = 0$, it is clear that the dimension of $\mathcal{M}_d^\Gamma$ is 0. Now if $r_d > 0$, Proposition 2.19 gives that $\mathcal{M}_d^\Gamma$ is isomorphic to $\mathcal{M}_{\mathbf{1}}^\Gamma(M^\sigma)$ for $\sigma \in \mathcal{C}_d$. Moreover, Theorem 1.29 gives that $\mathcal{M}_{\mathbf{1}}^\Gamma(M^\sigma)$ is isomorphic to $\mathcal{M}_{\mathbf{1}}^\Gamma(d) := \mathcal{M}_{\mathbf{1}}^\Gamma(d, \chi_0)$. We can now use Proposition 1.35 and Proposition 1.31, to conclude that the dimension is $2r_d$. □

CHAPTER 4

SYMPLECTIC RESOLUTIONS OF $\mathbb{C}^{2n}/\Gamma_n$ AND $\pi_0(\mathcal{H}_k^\Gamma)$

Recall from introduction that with Γ and n we have build a group $\Gamma_n = \mathfrak{S}_n \ltimes \Gamma^n$ and a symplectic singularity $\mathcal{Y}_n^\Gamma = \mathbb{C}^{2n}/\Gamma_n$. Gwyn Bellamy and Alastair Craw have classified all projective, symplectic resolutions of $\mathcal{Y}_n^\Gamma$ in terms of quiver varieties [BC20, Corollary 1.3]. Moreover we have shown, in Chapter 3, that for each integer $k \geq 1$, the irreducible components of $\mathcal{H}_k^\Gamma$ can be described in terms of quiver varieties. A natural question then arises. Given $X \to \mathcal{Y}_n^\Gamma$ a projective, symplectic resolution, is it possible to find an integer p_X and an irreducible component of $\mathcal{H}_{p_X}^\Gamma$ that is isomorphic to X?
To answer this question, this chapter is decomposed into two sections. In the first section, we recall the description of all projective, symplectic resolutions of $\mathbb{C}^{2n}/\Gamma_n$ done by Gwyn Bellamy and Alastair Craw. In the second section, we will explain how to describe these resolutions as irreducible components of the Γ-fixed point locus of the Hilbert scheme of points in $\mathbb{C}^2$.

4.1 Chamber decomposition inside Θ_Γ

In this section, we will make use of the $\mathbb{R}$-vector space $\Theta_\Gamma^\mathbb{R} := \mathrm{Hom}_\mathbb{Z}(\Delta_\Gamma, \mathbb{R})$. Let us first recall the notation used in [BC20]. Let F be the following simplicial cone in $\Theta_\Gamma^\mathbb{R}$

$$\{\theta \in \Theta_\Gamma^\mathbb{R} | \theta(\delta^\Gamma) \geq 0, \forall \chi \in I_\Gamma \setminus \{\chi_0\}, \theta(\chi) \geq 0\}$$

For $d \in \Delta_\Gamma$, let us denote $d^\perp := \{\theta \in \Theta_\Gamma^\mathbb{R} | \theta(d) = 0\}$. Consider the following set of walls in $\Theta_\Gamma^\mathbb{R}$

$$\mathcal{W}_n := \{{\delta^\Gamma}^\perp\} \cup \{(m\delta^\Gamma + \alpha)^\perp | \alpha \in \Phi(T_\Gamma), -n < m < n\}$$

Definition 4.1. A connected component $\mathfrak{C}$ of $\left(\Theta_\Gamma^\mathbb{R}\right) \setminus \bigcup_{c^\perp \in \mathcal{W}_n} c^\perp$ will be called a GIT chamber of Θ_Γ. Let us denote by $\Theta_\Gamma^{\mathrm{reg}}$ the union of all GIT chambers of $\Theta_\Gamma^\mathbb{R}$.

Remark 4.2. Let $F^{\mathrm{reg}} := \Theta_\Gamma^{\mathrm{reg}} \cap F$ and note that by construction of F, F^{reg} is a union of GIT chambers.

We can now reformulate the main result [BC20, Corollary 6.4] as follows.

Theorem 4.3 ([BC20])**.** *For each projective, symplectic resolution $X \to \mathcal{Y}_n^\Gamma$ there exists a unique GIT chamber $\mathfrak{C}$ in F such that X is isomorphic to the quiver variety $\mathcal{M}_\theta^\Gamma(n\delta^\Gamma)$ for any $\theta \in \mathfrak{C} \cap \Theta_\Gamma$.*

Let us now come back to what has been done in Chapter 3 and to the realization described at Remark 1.13. This is inspired by [Kac, Section 6.6]. Let us fix a real form $\mathfrak{h}_\Gamma^{\mathbb{R}}$ of $\mathfrak{h}_\Gamma$ that contains $\{\Lambda_{\alpha_0}^\vee\} \cup \{\tilde{\alpha}_\chi | \chi \in I_\Gamma\}$ and such that $\forall \chi \in I_\Gamma, \alpha_\chi(\mathfrak{h}_\Gamma^{\mathbb{R}}) \subset \mathbb{R}$. Denote by $\mathcal{V}$ the quotient space $\mathfrak{h}_\Gamma^{\mathbb{R}}/\mathbb{R}\delta_\Gamma^\vee$. By definition of $\delta_\Gamma^\vee$, $\langle \delta^\Gamma, \delta_\Gamma^\vee \rangle$ is equal to 0. We can then consider $E := \{h \in \mathcal{V} | \langle \delta^\Gamma, h \rangle = 1\}$. Let $E^0 := \{h \in \mathcal{V} | \langle \delta^\Gamma, h \rangle = 0\}$. Then (E, E^0) is an affine space in $\mathcal{V}$. It is clear that $E^0 = \mathrm{Vect}(\{\tilde{\alpha}_\chi | \chi \in I_\Gamma\})$ but the family $\{\tilde{\alpha}_\chi | \chi \in I_\Gamma\}$ is no longer linearly independant since $\sum_{\chi \in I_\Gamma} \delta_\chi^\Gamma \tilde{\alpha}_\chi = 0$. By definition, we have $\Lambda_0^\vee := \Lambda_{\alpha_0}^\vee \in E$. Denote $\tau \in Q(T_\Gamma)$ and $\tau^\vee \in Q^\vee(T_\Gamma)$ respectively the highest root and coroot of the finite type T_Γ. Let $Q^\vee(T_\Gamma)_{\mathbb{R}} := Q^\vee(T_\Gamma) \otimes \mathbb{R}$ and consider the following linear map

$$\mathrm{Aff}^0 : \begin{array}{ccc} E^0 & \to & Q^\vee(T_\Gamma)_{\mathbb{R}} \\ \tilde{\alpha}_\chi & \mapsto & \begin{cases} \tilde{\alpha}_\chi & \text{if } \chi \neq \chi_0 \\ -\tau^\vee & \text{if } \chi = \chi_0 \end{cases} \end{array}$$

This linear map is well-defined since $\delta_\Gamma^\vee = \tilde{\alpha}_{\chi_0} + \tau^\vee$ and it then induces an affine map $\mathrm{Aff} : (E, E^0) \to (Q^\vee(T_\Gamma)_{\mathbb{R}}, Q^\vee(T_\Gamma)_{\mathbb{R}})$ such that $\mathrm{Aff}(\Lambda_0^\vee) = 0$. The linear map Aff^0 is surjective and by dimension is then an isomorphism. This implies that Aff is an isomorphism. Recall that in Chapter 3, the natural $W(\tilde{T}_\Gamma)$-action by reflections on $\mathfrak{h}_\Gamma$ has been denoted $*$. Since $\forall \chi \in I_\Gamma, s_\chi * \delta_\Gamma^\vee = \delta_\Gamma^\vee$, let us equip $\mathcal{V}$ with this $W(\tilde{T}_\Gamma)$-action.

Lemma 4.4. *The set E is $W(\tilde{T}_\Gamma)$-stable.*

Proof. Let us take $v \in E$. It is then enough to show that $\forall \chi \in I_\Gamma, \langle \delta^\Gamma, s_\chi * v \rangle = 1$. By definition of δ^Γ: $\forall \chi \in I_\Gamma, \langle \delta^\Gamma, \tilde{\alpha}_\chi \rangle = 0$. We then have that $s_\chi * v \in E$. □

Proposition 4.5. *The induced action of $W(\tilde{T}_\Gamma)$ on $Q^\vee(T_\Gamma)_{\mathbb{R}}$ via* Aff *is the usual action of the affine Weyl group defined in [BLie02, Chap. VI, §2, no. 1].*

Proof. It is enough to check that s_{χ_0} acts on $Q^\vee(T_\Gamma)_{\mathbb{R}}$ as $t_{\tau^\vee} s_\tau$. Indeed, the element s_χ acts naturally as an element of the finite Weyl group $W(T_\Gamma)$, for each $\chi \in I_\Gamma \setminus \{\chi_0\}$. Take $\chi \in I_\Gamma$, we have

$$\begin{aligned} \mathrm{Aff}(s_0 * (\Lambda_0^\vee + \tilde{\alpha}_\chi)) =& \mathrm{Aff}(\Lambda_0^\vee + \tilde{\alpha}_\chi - \tilde{\alpha}_{\chi_0} - \langle \alpha_{\chi_0}, \tilde{\alpha}_\chi \rangle \tilde{\alpha}_{\chi_0}) \\ =& \tilde{\alpha}_\chi + \tau^\vee + \langle \alpha_{\chi_0}, \tilde{\alpha}_\chi \rangle \tau^\vee \\ =& \tilde{\alpha}_\chi - \langle \tau, \tilde{\alpha}_\chi \rangle \tau^\vee + \tau^\vee \end{aligned}$$

The last equality comes from the fact that $\alpha_{\chi_0} = \delta^\Gamma - \tau$. □

For each $\alpha \in \Phi(T_\Gamma)$ and each $k \in \mathbb{Z}$, $L_{\alpha,k} := \{x \in Q^\vee(T_\Gamma)_{\mathbb{R}} | \langle \alpha, x \rangle = k\}$ defines an hyperplane in $Q^\vee(T_\Gamma)_{\mathbb{R}}$. Let us denote by $\mathcal{W}_{\mathrm{aff}} := \{\mathrm{Aff}^{-1}(L_{\alpha,k}) | (\alpha, k) \in \Phi(T_\Gamma) \times \mathbb{Z}\}$. We get an affine hyperplane arrangement in E. Denote by $\kappa_{\mathbb{R}}^\vee : \begin{array}{ccc} \mathcal{V} & \to & \Theta_\Gamma^{\mathbb{R}} \\ x & \mapsto & (\chi \mapsto \langle \alpha_\chi, x \rangle) \end{array}$.

Note that $E \subset {\kappa_{\mathbb{R}}^\vee}^{-1}(\Theta_\Gamma^{\mathbb{R}}) = \mathcal{V}$.

Proposition 4.6. *For each hyperplane $\gamma^\perp \in \mathcal{W}_n$, ${\kappa_{\mathbb{R}}^\vee}^{-1}(\gamma^\perp) \cap E \in \mathcal{W}_{\mathrm{aff}}$.*

Proof. The set ${\kappa_{\mathbb{R}}^\vee}^{-1}(\delta^{\Gamma\perp})$ is equal to E^0, which implies that ${\kappa_{\mathbb{R}}^\vee}^{-1}(\delta^{\Gamma\perp}) \cap E = \emptyset$. If $\alpha \in \Phi(T_\Gamma)$ and $m \in [\![1-n, n-1]\!]$, then

$${\kappa_{\mathbb{R}}^\vee}^{-1}((m\delta^\Gamma + \alpha)^\perp) \cap E = \{v \in E | \langle \alpha, v \rangle = -m\} = \mathrm{Aff}^{-1}(L_{\alpha,-m}) \in \mathcal{W}_{\mathrm{aff}}$$

□

Remark 4.7. This Proposition implies that for each GIT chamber $\mathfrak{C} \subset \Theta_\Gamma^{\mathrm{reg}}$, ${\kappa_{\mathbb{R}}^\vee}^{-1}(\mathfrak{C}) \cap E$ is a union of alcoves.

Let us now restrict our attention to the cone F.

Proposition 4.8. *(i) If* $\mathcal{C}_f := \{v \in Q^\vee(T_\Gamma)_{\mathbb{R}} | \forall \chi \in I_\Gamma \setminus \{\chi_0\}, \langle \alpha_\chi, v\rangle > 0\}$ *denotes the fundamental Weyl chamber in* $Q^\vee(T_\Gamma)_{\mathbb{R}}$*, then* $\mathrm{Aff}({\kappa_{\mathbb{R}}^\vee}^{-1}(F) \cap E) = \overline{\mathcal{C}_f}$.
(ii) If $\mathfrak{C} \subset F^{\mathrm{reg}}$ *is a GIT chamber such that* ${\kappa_{\mathbb{R}}^\vee}^{-1}(\mathfrak{C}) \cap E$ *is bounded then* ${\kappa_{\mathbb{R}}^\vee}^{-1}(\mathfrak{C}) \cap E$ *is an alcove in* E.

Proof. The first statement follows directly from the definition of $\mathcal{C}_f$ and F. Let us now prove the second statement. Take $\mathfrak{C}$ such a chamber. Use Proposition 4.6 to prove that ${\kappa_{\mathbb{R}}^\vee}^{-1}(\mathfrak{C}) \cap E$ is a union of alcoves. The definition of $\mathcal{W}_{\mathrm{aff}}$ and the fact that ${\kappa_{\mathbb{R}}^\vee}^{-1}(\mathfrak{C}) \cap E$ is bounded imply that ${\kappa_{\mathbb{R}}^\vee}^{-1}(\mathfrak{C}) \cap E$ is equal to exactly one alcove of E. □

4.2 Resolutions as irreducible components

Recall that in [BC20, Example 2.1], authors have introduced the following GIT chamber

$$\mathfrak{C}_+ = \{\theta \in \Theta_\Gamma^{\mathbb{R}} \big| \forall \chi \in I_\Gamma, \theta(\chi) > 0\}$$

Remark 4.9. It is clear that $\mathfrak{C}_+ \subset F$. Note moreover that ${\kappa_{\mathbb{R}}^\vee}^{-1}(\mathfrak{C}_+) \cap E$ is bounded and is the fundamental alcove $\mathfrak{A}_+ := \{h \in E | \forall \chi \in I_\Gamma \setminus \{\chi_0\}, \langle \alpha_\chi, h\rangle > 0 \text{ and } \langle \tau, h\rangle < 1\}$.

Lemma 4.10. *The vector* $1 = \kappa^\vee\big(\overline{\sum_{\chi \in I_\Gamma} \Lambda_\chi^\vee}\big) \in \Theta_\Gamma$ *is in* $\mathfrak{C}_+$.

Proof. By definition $\forall \chi \in I_\Gamma, 1(\chi) = 1 > 0$. □

The following proposition gives an isomorphism between quiver varieties for stability parameters in the GIT chamber $\mathfrak{C}_+$.

Proposition 4.11. *For each* $(\theta, d) \in \mathfrak{C}_+ \times \Delta_\Gamma^+$, $\mathcal{M}_{\mathbf{1}}^\Gamma(d) \simeq \mathcal{M}_{\boldsymbol{\theta}}^\Gamma(d)$.

Proof. Using Lemma 4.10, we know that $1 \in \mathfrak{C}_+$. Use now [DH98, Theorem 3.3.2] to obtain the result. □

Remark 4.12. Note that $\overline{\sum_{\chi \in I_\Gamma} \Lambda_\chi^\vee}$ is not in general in E. From now on, let us fix $\theta_0 \in \mathfrak{C}_+ \cap \Theta_\Gamma$ such that ${\kappa^\vee}^{-1}(\theta_0)$ is in an alcove $\mathfrak{A}_0 \subset E$. Thanks to Proposition 4.11, we have $\mathcal{M}_{\mathbf{1}}^\Gamma(d) \simeq \mathcal{M}_{\boldsymbol{\theta_0}}^\Gamma(d)$ for each $d \in \Delta_\Gamma^+$.

For a given integer k, let us denote the set of all isomorphism classes of projective symplectic resolutions of $\mathcal{Y}_k^\Gamma$ by $\mathscr{R}_k^\Gamma$ and by $\mathcal{IC}_k^\Gamma$ the set of all irreducible components of $\mathcal{H}_k^\Gamma$. Moreover, let $\mathscr{R}^\Gamma := \bigcup_{k \in \mathbb{N}} \mathscr{R}_k^\Gamma$ and $\mathcal{IC}^\Gamma := \bigcup_{k \in \mathbb{N}} \mathcal{IC}_k^\Gamma$. Take now an integer k and $I \in \mathcal{IC}_k^\Gamma$. Using Corollary 3.3, we know that there is an element $d \in \mathcal{A}_\Gamma^k$ such that $I = \mathcal{H}_k^{\Gamma,\zeta_d}$ and using Theorem 3.10, we have a unique element $r_d \in \mathbb{Z}_{\geq 0}$ and an element ω_d of $W(\tilde{T}_\Gamma)$ such that $\omega_d.d = r_d\delta^\Gamma$. The image by $\kappa_{\mathbb{R}}^\vee$ of the alcove $\omega_d * \mathfrak{A}_0$ is then contained in a GIT chamber $\mathfrak{C}_d$. Thus $\mathcal{M}_{\boldsymbol{\omega_d.\theta_0}}^\Gamma(r_d\delta^\Gamma)$ is a projective symplectic resolution of $\mathcal{Y}_{r_d}^\Gamma$ (Theorem 4.3). Thanks to the two isomorphisms $\mathrm{Maff}_{d,\mathbf{1},\omega_d}^\Gamma$ and $\mathrm{Maff}_{d,\boldsymbol{\theta_0},\omega_d}^\Gamma$ and Remark 4.12, we have that $\mathcal{M}_{\boldsymbol{\omega_d.\theta_0}}^\Gamma(r_d\delta^\Gamma) \simeq \mathcal{M}_{\boldsymbol{\omega_d.1}}^\Gamma(r_d\delta^\Gamma)$. The following map is then well defined

$$\mathcal{BC}: \begin{array}{ccc} \mathcal{IC}^\Gamma & \to & \mathscr{R}^\Gamma \\ I(d) := \mathcal{H}^{\Gamma,\zeta_d}_{|d|_\Gamma} & \mapsto & \mathcal{M}^\Gamma_{\boldsymbol{\omega_d}.\mathbf{1}}(r_d\delta^\Gamma) \end{array}$$

Remark 4.13. Note that $\mathcal{BC}$ does not depend on the choice of ω_d. Indeed, if $\omega'_d \in W(\tilde{T}_\Gamma)$ such that $\omega'_d.d = r_d\delta^\Gamma$, then there exists $\omega_0 \in W(T_\Gamma)$ such that $\omega'_d = \omega_0\omega_d$. Using Proposition 1.36, we have $\mathrm{Maff}^\Gamma_{d,\boldsymbol{\omega_d}.\mathbf{1},\omega_0} : \mathcal{M}^\Gamma_{\boldsymbol{\omega_d}.\mathbf{1}}(r_d\delta^\Gamma) \xrightarrow{\sim} \mathcal{M}^\Gamma_{\boldsymbol{\omega'_d}.\mathbf{1}}(r_d\delta^\Gamma)$ over $\mathcal{Y}^\Gamma_{r_d}$.

Theorem 4.14. *The map $\mathcal{BC}$ is surjective.*

Proof. Take a GIT chamber $\mathfrak{C} \in F^{\mathrm{reg}}$. Using Remark 4.7, we can choose an alcove $\mathfrak{A}_\mathfrak{C} \subset E$ such that $\kappa^\vee_\mathbb{R}(\mathfrak{A}_\mathfrak{C}) \subset \mathfrak{C}$. For each $\theta \in \kappa^\vee_\mathbb{R}(\mathfrak{A}_\mathfrak{C}) \cap \Theta_\Gamma$, the variety $\mathcal{M}^\Gamma_{\boldsymbol{\theta}}(n\delta^\Gamma)$ is then a projective, symplectic resolution of $\mathcal{Y}^\Gamma_n$. Thanks to Theorem 4.3, it is enough to show that there exists an irreducible component $\mathcal{I}_{\mathfrak{C},n} \in \mathcal{IC}^\Gamma$ such that $\mathcal{BC}(\mathcal{I}_{\mathfrak{C},n}) = \mathcal{M}^\Gamma_{\boldsymbol{\theta}}(n\delta^\Gamma)$ for an element $\theta \in \kappa^\vee_\mathbb{R}(\mathfrak{A}_\mathfrak{C}) \cap \Theta_\Gamma$. The action of $W(\tilde{T}_\Gamma)$ on alcoves in E is transitive [BLie02, Chap. V, §3, no. 2, Th 1]. There exists then $\omega_\mathfrak{C} \in W(\tilde{T}_\Gamma)$ such that $\omega_\mathfrak{C} * \kappa^{\vee -1}(\theta_0) \in \mathfrak{A}_\mathfrak{C}$. The map $\kappa^\vee$ being $W(\tilde{T}_\Gamma)$-equivariant (Remark 1.34), $\omega_\mathfrak{C}.\theta_0 \in \kappa^\vee_\mathbb{R}(\mathfrak{A}_\mathfrak{C}) \cap \Theta_\Gamma$. Using the isomorphism $\mathrm{Maff}^\Gamma_{n\delta^\Gamma,\boldsymbol{\omega_\mathfrak{C}}.\boldsymbol{\theta_0},\omega_\mathfrak{C}^{-1}}$, we have

$$\mathcal{M}^\Gamma_{\boldsymbol{\omega_\mathfrak{C}}.\boldsymbol{\theta_0}}(n\delta^\Gamma) \simeq \mathcal{M}^\Gamma_{\boldsymbol{\theta_0}}(\omega_\mathfrak{C}^{-1}.(n\delta^\Gamma)) \simeq \mathcal{M}^\Gamma_{\mathbf{1}}(\omega_\mathfrak{C}^{-1}.(n\delta^\Gamma))$$

The second isomorphism comes from Proposition 4.11. Let $p_{n,\omega_\mathfrak{C}}$ denote the integer $|\omega_\mathfrak{C}^{-1}.(n\delta^\Gamma)|_\Gamma$ and take $\mathcal{I}_{\mathfrak{C},n} := H^{-1}(\iota^\Gamma_{\mathbf{1},0,M^\sigma}(\mathcal{M}^\Gamma_{\mathbf{1}}(M^\sigma))) \subset \mathcal{H}^\Gamma_{p_{n,\omega_\mathfrak{C}}}$ which is an irreducible component for any $\sigma \in \mathcal{C}_{\omega_\mathfrak{C}^{-1}.(n\delta^\Gamma)}$. □

Let us finally see on an example that $\mathcal{BC}$ is not injective. Take n to be equal to 2 and Γ to be μ_3, the cyclic group of order 3. This group is generated by ω_3 the diagonal matrix $\mathrm{diag}(\zeta_3, \zeta_3^{-1}) \in \mathrm{SL}_2(\mathbb{C})$ where ζ_3 is the primitive root of unity $e^{\frac{2i\pi}{3}}$. In that case, there will be 5 GIT chambers in F. For each $i \in [\![0,2]\!]$, denote by s_i the generator s_{τ^i} of $W(\tilde{T}_{\mu_3})$ where τ is the character of μ_3 that maps ω_3 to ζ_3. Consider now $\omega_1 = s_0s_2s_1s_2$ and $\omega_2 = \omega_1 s_0$ two elements of $W(\tilde{T}_\Gamma)$. We have that $\omega_1.\theta_0$ and $\omega_2.\theta_0$ are in the same GIT chamber

$$\mathfrak{C}_- := \{\theta \in \Theta^\mathbb{R}_\Gamma | \theta(\delta^\Gamma) > 0, \forall (k,\alpha) \in [\![1-n, n-1]\!] \times \Phi^+(T_\Gamma), \theta(\alpha + m\delta^\Gamma) > 0\}$$

Moreover since $\omega_1^{-1}.(2\delta^\Gamma) \neq \omega_2^{-1}.(2\delta^\Gamma)$, this shows that $I(\omega_1^{-1}.(2\delta^\Gamma)) \neq I(\omega_2^{-1}.(2\delta^\Gamma))$ but these two irreducible components have the same images under $\mathcal{BC}$.

CHAPTER 5

COMBINATORICS IN TYPE D

Let us now describe in a bit more detail the combinatorics of the parametrization set of $\pi_0(\mathcal{H}_n^\Gamma)$. For $l \geq 1$, let μ_l denote the cyclic subgroup of $\mathrm{SL}_2(\mathbb{C})$ generated by the diagonal matrix $\mathrm{diag}(\zeta_l, \zeta_l^{-1})$, where $\zeta_l = e^{\frac{2i\pi}{l}}$. When Γ is equal to μ_l a combinatorial model using partitions has already been constructed by Iain Gordon [Gor08, Lemma 7.8] and by Cédric Bonnafé and Ruslan Maksimau [BM21, Lemma 4.9]. In this chapter, the type D and then the type E case will be studied. Type D corresponds to the class of $\mathrm{SL}_2(\mathbb{C})$ finite subgroups called the binary dihedral subgroups. Fix $l \geq 2$ and consider $\widetilde{BD}_{2l} :=< \omega_{2l}, s >$ with

- $\omega_{2l} = \begin{pmatrix} \zeta_{2l} & 0 \\ 0 & \zeta_{2l}^{-1} \end{pmatrix}$ where $\zeta_{2l} = e^{\frac{i\pi}{l}}$
- $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

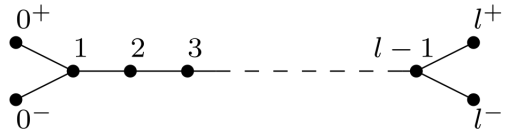
The group $\widetilde{BD}_{2l}$ is of order $4l$. Let τ_{2l} be the character of μ_{2l} that maps ω_{2l} to ζ_{2l}. For $i \in \mathbb{Z}$, consider $\chi_i := \mathrm{Ind}_{\mu_{2l}}^{\widetilde{BD}_{2l}} \tau_{2l}^i$. Note that χ_i is irreducible if and only if i is not congruent to 0 or l modulo $2l$. If l is even, the character table of $\widetilde{BD}_{2l}$ is

cardinality	1	1	2	l	l
classes	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\omega_{2l}^p (0 < p < l)$	s	$s\omega_{2l}$
χ_{0^+}	1	1	1	1	1
χ_{0^-}	1	1	1	-1	-1
χ_{l^+}	1	1	$(-1)^p$	-1	1
χ_{l^-}	1	1	$(-1)^p$	1	-1
χ_k $(0 < k < l)$	2	$(-1)^k 2$	$2\cos(\frac{kp\pi}{l})$	0	0

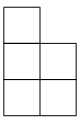
If l is odd, the character table of $\widetilde{BD}_{2l}$ is

cardinality	1	1	2	l	l
classes	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\omega_{2l}^p (0<p<l)$	s	$s\omega_{2l}$
χ_{0^+}	1	1	1	1	1
χ_{0^-}	1	1	1	-1	-1
χ_{l^+}	1	-1	$(-1)^p$	ζ_4	$-\zeta_4$
χ_{l^-}	1	-1	$(-1)^p$	$-\zeta_4$	ζ_4
χ_k $(0<k<l)$	2	$(-1)^k 2$	$2\cos(\frac{kp\pi}{l})$	0	0

The McKay graph of $\widetilde{BD}_{2l}$ is a Dynkin diagram of affine type $\tilde{D}_{l+2}$



The irreducible characters are labeled by their index in the McKay graph. A partition λ of n is a tuple $(\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0)$ such that $|\lambda| := \sum_{i=1}^r \lambda_i = n$. Denote by $\mathcal{P}_n$ the set of all partitions of n and by $\mathcal{P}$ the set of all partitions of integers. For $\lambda = (\lambda_1, ..., \lambda_r) \in \mathcal{P}$, denote by $\mathcal{Y}(\lambda) := \{(i,j) \in \mathbb{Z}_{\geq 0}^2 | i < \lambda_1, j < r\}$ its associated Young diagram. The conjugate partition of a partition λ of n, denoted by λ^*, is the partition associated with the reflection of $\mathcal{Y}(\lambda)$ along the diagonal (which is again a Young diagram of a partition of n). For example, consider $\lambda = (2,2,1)$. Its associated Young diagram is as follows



In that case $\lambda^* = (3,2)$. A partition λ will be called symmetric if it is equal to its conjugate. Let us denote by $\mathcal{P}^s$ the set of all symmetric partitions and by $\mathcal{P}_n^s := \mathcal{P}^s \cap \mathcal{P}_n$. A hook of a partition λ in position $(i,j) \in \mathcal{Y}(\lambda)$ denoted by $H_{(i,j)}(\lambda)$ is

$$\{(a,b) \in \mathcal{Y}(\lambda) | a = i \text{ and } b \geq j \text{ or } a > i \text{ and } b = j\}$$

Define the length of a hook $H_{(i,j)}(\lambda)$ as its cardinal and denote it by $h_{(i,j)}(\lambda)$. A box $c = (i,j) \in \mathcal{Y}(\lambda)$ is in the border of $\mathcal{Y}(\lambda)$ if either $i = 0$ or $j = 0$.

Definition 5.1. For a given integer $r \geq 1$, a partition λ is said to be an r-core if $\mathcal{Y}(\lambda)$ does not contain any hook of length r. Let us denote by $\mathfrak{C}_r$ the set of all r-cores and by $\mathfrak{C}_r^s := \mathfrak{C}_r \cap \mathcal{P}^s$.

Note also that in this context $\delta_{\chi_i}^{\widetilde{BD}_{2l}} := \begin{cases} 1 & \text{if } i = 0^+, 0^-, l^+, l^- \\ 2 & \text{otherwise} \end{cases}$.

Recall finally the result of Theorem 3.10. For each finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ we have indexed the irreducible components of $\mathcal{H}_n^\Gamma$ with the following set

$$\mathcal{A}_\Gamma^n := \left\{ d \in \Delta_\Gamma^+ \middle| |d|_\Gamma = n \text{ and } \mathrm{wt}(d) \geq 0 \right\}$$

We want to give a combinatorial description of $\mathcal{A}^n_{\widetilde{BD}_{2l}}$. Let $\mathbb{T}_2$ be the maximal diagonal torus of $\mathrm{GL}_2(\mathbb{C})$ and $\mathbb{T}_1 := \mathbb{T}_2 \cap \mathrm{SL}_2(\mathbb{C})$ be the one of $\mathrm{SL}_2(\mathbb{C})$. In this chapter, we will give a combinatorial description using partitions of the irreducible components of $\mathcal{H}_n^{\widetilde{BD}_{2l}}$. To do so, restrict $\mathcal{A}^n_{\widetilde{BD}_{2l}}$ to the irreducible components of $\mathcal{H}_n^{\widetilde{BD}_{2l}}$ that contain a $\mathbb{T}_1$-fixed point. Let us show that this set is exactly the following subset of $\mathcal{A}^n_{\widetilde{BD}_{2l}}$

$$\mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}} := \left\{ d \in \mathcal{A}^n_{\widetilde{BD}_{2l}} \middle| 0 \leq d_{\chi_{0^+}} - d_{\chi_{0^-}} \leq 1 \text{ and } d_{\chi_{l^+}} = d_{\chi_{l^-}} \right\}$$

The goal here is to find a combinatorial model of $\mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$. The central object of study will be the affine root lattice of type $\tilde{D}_{l+2}$ (which is the same object as the coroot lattice of type $\tilde{D}_{l+2}$ since it is a simply laced type) that we have denoted by $Q(\tilde{D}_{l+2}) \subset \mathfrak{h}^*_{\widetilde{BD}_{2l}}$. Denote by $\tau_l := \alpha_{\chi_{0^-}} + \alpha_{\chi_{l^+}} + \alpha_{\chi_{l^-}} + \sum_{i=1}^{l-1} 2\alpha_{\chi_i}$, the highest root of the finite root system of type D_{l+2}.

Definition 5.2. Define a bijection from the set $I_{\widetilde{BD}_{2l}}$ to itself

$$\sigma_{0^-} : \begin{array}{ccc} I_{\widetilde{BD}_{2l}} & \to & I_{\widetilde{BD}_{2l}} \\ \chi & \mapsto & \chi_{0^-} \cdot \chi \end{array}$$

Define an automorphism of $\mathrm{Dyn}(\tilde{D}_{l+2})$

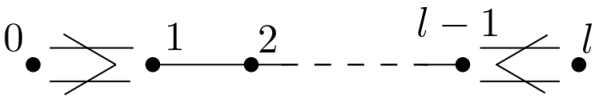
$$\sigma : \begin{array}{ccc} \Pi_{\widetilde{BD}_{2l}} & \to & \Pi_{\widetilde{BD}_{2l}} \\ \alpha_\chi & \mapsto & \alpha_{\sigma_{0^-}(\chi)} \end{array}$$

This automorphism swaps the first two vertices (the one with the label 0^+ and 0^-) and the last two (with the label l^+ and l^-) and fixes all the others.

We can apply Stembridge's construction [Stem] to the root system of type $\tilde{D}_{l+2}$ and to the automorphism σ. Denote the simple roots $(\beta_i)_{i \in [\![0,l]\!]}$ and $(\beta_i^\vee)_{i \in [\![0,l]\!]}$ the simple coroots associated with the root system $\Phi(\tilde{D}^\sigma_{l+2})$. By construction

- $\beta_0 = \alpha_{\chi_{0^+}} + \alpha_{\chi_{0^-}}$
- $\forall i \in [\![1, l\text{ - }1]\!]$, $\beta_i = \alpha_{\chi_i}$
- $\beta_l = \alpha_{\chi_{l^+}} + \alpha_{\chi_{l^-}}$
- $\beta_0^\vee = \frac{\tilde{\alpha}_{\chi_{0^+}} + \tilde{\alpha}_{\chi_{0^-}}}{2}$
- $\forall i \in [\![1, l\text{ - }1]\!]$, $\beta_i^\vee = \tilde{\alpha}_{\chi_i}$
- $\beta_l^\vee = \frac{\tilde{\alpha}_{\chi_{l^+}} + \tilde{\alpha}_{\chi_{l^-}}}{2}$.

If $A = (a_{ij})$ is a generalized Cartan matrix, recall that in the associated Dynkin diagram, if two vertices (i, j) are connected by more than one edge, then these edges are equipped with an arrow pointing toward i if $|a_{ij}| > 1$. With those conventions, the root system $\Phi(\tilde{D}^\sigma_{l+2})$ has the following Dynkin diagram



Proposition 5.3. *$\Phi(\tilde{D}^\sigma_{l+2})$ is a crystallographic root system of type $\tilde{C}_l$.*

Definition 5.4. Let $Q(\tilde{D}^{\sigma}_{l+2})[0^+] := Q(\tilde{D}^{\sigma}_{l+2}) \coprod \left(Q(\tilde{D}^{\sigma}_{l+2}) + \alpha_{\chi_{0^+}}\right) \subset Q(\tilde{D}_{l+2})$. Written more explicitly

$$Q(\tilde{D}^{\sigma}_{l+2})[0^+] = \{ \sum_{\chi \in I_{\widetilde{BD}_{2l}}} a_\chi \alpha_\chi \in Q(\tilde{D}_{l+2}) \big| 0 \leq a_{\chi_0^+} - a_{\chi_0^-} \leq 1 \text{ and } a_{\chi_l^+} = a_{\chi_l^-} \}$$

Definition 5.5. Define the following map

$$\mathcal{T} : \begin{array}{ccc} Q(\tilde{D}^{\sigma}_{l+2})[0^+] & \to & Q^\vee(\tilde{D}^{\sigma}_{l+2}) = Q^\vee(\tilde{C}_l) \\ \sum_{i=0}^{l} a_i\beta_i + q\alpha_{\chi_{0^+}} & \mapsto & (2a_0+q)\beta_0^\vee + \sum_{i=1}^{l-1} a_i\beta_i^\vee + 2a_l\beta_l^\vee \end{array}$$

with $q \in \{0,1\}$.

In type $\tilde{C}_l$, we have $\delta(\tilde{C}_l) := \beta_0 + \sum_{i=1}^{l-1} 2\beta_i + \beta_l \in Q(\tilde{C}_l)$ and $\delta^\vee(\tilde{C}_l) := \sum_{i=0}^{l} \beta_i^\vee \in Q^\vee(\tilde{C}_l)$.

Definition 5.6. For each $\chi \in I_{\widetilde{BD}_{2l}}$, let $\sigma.s_\chi := s_{\sigma_{0^-}(\chi)}$ and extend this action to $W(\tilde{D}_{l+2})$. Let $W(\tilde{D}_{l+2})^\sigma := \{\omega \in W(\tilde{D}_{l+2}) | \sigma.\omega = \omega\}$ be a subgroup of $W(\tilde{D}_{l+2})$.

Remark 5.7. The set $\{s_0 := s_{\chi_{0^+}} s_{\chi_{0^-}}, s_1 := s_{\chi_1}, \ldots, s_{l-1} := s_{\chi_{l-1}}, s_l := s_{\chi_{l^+}} s_{\chi_{l^-}}\}$ is a set of generators of $W(\tilde{D}_{l+2})^\sigma$. Applying [Stem, Claim 3] to our situation, gives a group isomorphism from $W(\tilde{D}^{\sigma}_{l+2})$ to $W(\tilde{D}_{l+2})^\sigma$. Let us, from now on, identify these two groups and refer to them as $W(\tilde{C}_l)$. This group acts naturally by reflection on $Q(\tilde{D}^{\sigma}_{l+2})[0^+]$ and $Q^\vee(\tilde{C}_l)$. Denote this action by $*$.

Definition 5.8. Define a $W(\tilde{C}_l)$-action on $Q(\tilde{D}^{\sigma}_{l+2})[0^+]$ the following way

$$\forall i \in [\![0,l]\!], \forall \alpha \in Q(\tilde{D}^{\sigma}_{l+2})[0^+], s_i.\alpha := s_i * \alpha + \delta_i^0 \alpha_{\chi_{0^+}}$$

and a $W(\tilde{C}_l)$-action on $Q^\vee(\tilde{D}^{\sigma}_{l+2})$ similarly

$$\forall i \in [\![0,l]\!], \forall \beta^\vee \in Q^\vee(\tilde{C}_l), s_i.\beta^\vee := s_i * \beta^\vee + \delta_i^0 \beta_0^\vee$$

A simple computation shows the equivariance of $\mathcal{T}$ with respect to the former actions.

Proposition 5.9. *The map $\mathcal{T}$ is $W(\tilde{C}_l)$-equivariant.*

Remark 5.10. Note also that $\mathcal{T}$ preserves sizes

$$\forall \alpha \in Q(\tilde{D}^{\sigma}_{l+2})[0^+], |\alpha|_{\tilde{D}_{l+2}} = |\mathcal{T}(\alpha)|_{\tilde{C}_l}$$

Lemma 5.11. *If $\beta^\vee \in \overline{0}^{W(\tilde{C}_l)} \subset Q^\vee(\tilde{C}_l)$, and $k \in \mathbb{Z}$, then $(\beta^\vee + k\delta^\vee(\tilde{C}_l)) \in \overline{k\delta^\vee(\tilde{C}_l)}^{W(\tilde{C}_l)}$.*

Proof. It is enough to check this on the set of generators $\{s_i | i \in [\![0,l]\!]\}$. For $i \in [\![1,l]\!]$, the action is the action by reflection. It is then linear and s_i stabilizes $k\delta^\vee(\tilde{C}_l)$. For $i = 0$, we can combine this fact

$$\forall (\beta_1^\vee, \beta_2^\vee) \in Q^\vee(\tilde{C}_l), s_0.(\beta_1^\vee + \beta_2^\vee) = s_0.\beta_1^\vee + s_0.\beta_2^\vee - \beta_0^\vee$$

with the fact that $s_0.\delta^\vee(\tilde{C}_l) = \delta^\vee(\tilde{C}_l) + \beta_0^\vee$ to conclude that

$$s_0.(\beta^\vee + k\delta^\vee(\tilde{C}_l)) = s_0.\beta^\vee + k\delta^\vee(\tilde{C}_l)$$

□

5.1 $\widetilde{BD_{2l}}$-Residue

The $\mathbb{T}_1$-fixed points in $\mathcal{H}_n$ are the ideals I_λ generated by $\{x^i y^j | (i,j) \in \mathbb{N}^2 \setminus \mathcal{Y}(\lambda)\}$ for λ a partition of n. These ideals are called monomial ideals. Among these ideals, the ideals fixed by $s \in \widetilde{BD_{2l}}$ are exactly the monomial ideals parametrized by symmetric partitions of n. In this subsection, our goal is to generalize the residue "of type A" i.e. the usual residue of partitions to a residue of type D. The property that we want to generalize is that in type A the residue of a partition λ is exactly the multiplicity vector of the character of the representation $\mathbb{C}[x,y]/I_\lambda$ for the isotypic decomposition. We then want to construct a map Res_D between $\mathcal{P}_n^s$ and $Q(\tilde{D}_{l+2})$. To do so let us first introduce for $k \in [\![0,l]\!]$ the functions $d_k : \mathcal{P}_n^s \to \mathbb{Z}_{\geq 0}$.
Let $\mathcal{Y}(\lambda)_k := \{(i,j) \in \mathcal{Y}(\lambda) | i - j \equiv k \ [2l]\}$ for $k \in [\![0, 2l-1]\!]$.

Definition 5.12. For $k \in [\![1,l]\!]$ define $d_k(\lambda) := \#\big(\mathcal{Y}(\lambda)_k \cup \mathcal{Y}(\lambda)_{2l-k}\big)$. When $k = 0$, consider $\tilde{d}_0(\lambda) := \#\{(i,j) \in \mathcal{Y}(\lambda) | i = j\}$ and $d_0(\lambda) := \#\mathcal{Y}(\lambda)_0 - \tilde{d}_0(\lambda)$.

Denote

- $d_0'(\lambda) := \frac{d_0(\lambda)}{2} + \tilde{d}_0(\lambda) - \lfloor \frac{\tilde{d}_0(\lambda)}{2} \rfloor$
- $d_0''(\lambda) := \frac{d_0(\lambda)}{2} + \lfloor \frac{\tilde{d}_0(\lambda)}{2} \rfloor$.

We are now able to define the residue in type D.

Definition 5.13. Let the residue of type D be this map

$$\mathrm{Res}_D : \begin{array}{ccc} \mathcal{P}_n^s & \to & Q(\tilde{D}_{l+2}) \\ \lambda & \mapsto & d_0'(\lambda)\alpha_{\chi_{0^+}} + d_0''(\lambda)\alpha_{\chi_{0^-}} + \sum_{i=1}^{l-1} \frac{d_i(\lambda)}{2}\alpha_{\chi_i} + \frac{d_l(\lambda)}{2}(\alpha_{\chi_{l^+}} + \alpha_{\chi_{l^-}}) \end{array}$$

Remark 5.14. Using the fact that the partition is symmetric, it is easy to see that the image of Res_D is indeed in the $\mathbb{Z}$-span of the $\{\alpha_\chi | \chi \in I_{\widetilde{BD_{2l}}}\}$. Note moreover, that

$$\forall \lambda \in \mathcal{P}_n^s, |\mathrm{Res}_D(\lambda)|_{\tilde{D}_{l+2}} = |\lambda| = n$$

Example 5.15. Take $l = 2$ and consider $\lambda = (4,4,3,2)$ which is symmetric and has the following Young diagram

1	2		
2	1	0^+	
1	0^-	1	2
0^+	1	2	1

then $\mathrm{Res}_D(\lambda) = (2,1,3,2,2)$.

Proposition 5.16. *For any $\lambda \in \mathcal{P}_n^s$, $\mathrm{Res}_D(\lambda)$ is the dimension vector of the $\widetilde{BD_{2l}}$-representation $\mathbb{C}[x,y]/I_\lambda$.*

Proof. Consider $(\overline{x^i y^j})_{(i,j)\in\mathcal{Y}(\lambda)}$ a base of the representation $\mathbb{C}[x,y]/I_\lambda$. Since λ is symmetric, restrict the attention to $\mathcal{Y}^-(\lambda) := \{(i,j) \in \mathcal{Y}(\lambda) | i > j\}$ and to the diagonal $\{(i,j) \in \mathcal{Y}(\lambda) | i = j\}$. Take first $(i,j) \in \mathcal{Y}^-(\lambda)$ and consider $V_{(i,j)} = \mathrm{Vect}(\overline{x^i y^j}, \overline{x^j y^i})$ a

subspace of $\mathbb{C}[x,y]/I_\lambda$. Let k be an element of $[\![1, l-1]\!]$. For each $(i,j) \in \mathcal{Y}^-(\lambda)$ such that $i-j \equiv k[2l]$, we have $V_{(i,j)} \simeq_{\widetilde{BD}_{2l}} X_{\chi_k}$ (recall that X_{χ_k} is the irreducible representation of $\widetilde{BD}_{2l}$ associated with the irreducible character χ_k). Moreover when $i-j \equiv 2l-k[2l]$, we have $V_{(i,j)} \simeq_{\widetilde{BD}_{2l}} X_{\chi_k}$. If $k=l$, then for each pair $(i,j) \in \mathcal{Y}^-(\lambda)$ such that $i-j \equiv l[2l]$, we have $V_{(i,j)} \simeq_{\widetilde{BD}_{2l}} X_{\chi_{l^+}} \oplus X_{\chi_{l^-}}$. In the same way if $(i,j) \in \mathcal{Y}^-(\lambda)$ such that $i \equiv j[2l]$, $V_{(i,j)} \simeq_{\widetilde{BD}_{2l}} X_{\chi_{0^+}} \oplus X_{\chi_{0^-}}$. It remains to understand the action of $\widetilde{BD}_{2l}$ on the diagonal. For each $i \in \mathbb{Z}_{\geq 0}, \omega_{2l}.\overline{x^i y^i} = \overline{x^i y^i}$ and $s.\overline{x^i y^i} = (-1)^i\, \overline{x^i y^i}$. These two computations show that if $i \equiv 0[2]$, then $V_i := V_{(i,i)} \simeq_{\widetilde{BD}_{2l}} X_{\chi_{0^+}}$ and that if $i \equiv 1[2]$, then $V_i \simeq_{\widetilde{BD}_{2l}} X_{\chi_{0^-}}$. To sum it all up, the vector of multiplicities of the isotypic components of $\mathbb{C}[x,y]/I_\lambda$ correspond to $\mathrm{Res}_D(\lambda)$. □

By construction Res_D factors though $Q(\tilde{D}^\sigma_{l+2})[0^+]$. For $(a,b) \in \mathbb{Z}^2$, let $\mathrm{rem}(a,b)$ denote the remainder of the Euclidian division of a by b. Thanks to the work of Christopher R.H. Hanusa and Brant C. Jones [HJ12, Theoreom 5.8] we can endow $\mathfrak{C}^s_{2l}$ with a $W(\tilde{C}_l)$-action. Let us quickly recall how this action is constructed.

Definition 5.17. For a symmetric $2l$-core λ define the C-residue of a box positioned at line i and column j in the Young diagram of λ as

$$\begin{cases} \mathrm{rem}(j-i, 2l) & \text{if } 0 \leq \mathrm{rem}(j-i,2l) \leq l \\ 2l - \mathrm{rem}(j-i,2l) & \text{if } l < \mathrm{rem}(j-i,2l) < 2l \end{cases}$$

Example 5.18. Take $l=2$ and the same symmetric 4-core $(4,4,3,2)$. The Young diagram filled with the C-residue of each box gives

1	2		
2	1	0	
1	0	1	2
0	1	2	1

Remark 5.19. Note that for each symmetric $2l$-core λ, the C-residue of each box of λ is always an integer between 0 and l.

Definition 5.20. The action of $W(\tilde{C}_l)$ on $\mathfrak{C}^s_{2l}$ is defined on generators. Take $s_i \in W(\tilde{C}_l)$ and $\lambda \in \mathfrak{C}^s_{2l}$. Note that there are only three disjoint cases. Either we can add boxes with C-residue i, or we can remove such boxes or there are no such boxes. Define $s_i.\lambda$ as the partition obtained from λ in either adding all boxes of $\mathcal{Y}(\lambda)$ with C-residue i so that $s_i.\lambda$ remains a partition or removing all boxes of $\mathcal{Y}(\lambda)$ with C-residue i so that $s_i.\lambda$ remains a partition.

Definition 5.21. The C-region of index $k \in \mathbb{Z}$ of a symmetric $2l$-core is the following subset of $\mathcal{Y}(\lambda)$

$$\mathcal{R}_k := \{(i,j) \in \mathcal{Y}(\lambda) | (i-j) \in \{2kl, ..., 2(k+1)l - 1\}\}$$

More generally, we can define a shifted C-region. Let $(k,h) \in \mathbb{Z}^2$ and define the h-shifted C-region of index k

$$\mathcal{R}_{k,h} := \{(i,j) \in \mathcal{Y}(\lambda) | (i-j) \in \{2kl + h, ..., 2(k+1)l - 1 + h\}\}$$

Proposition 5.22. $\mathrm{Res}_D : \mathfrak{C}^s_{2l} \to Q(\tilde{D}^\sigma_{l+2})[0^+]$ *is* $W(\tilde{C}_l)$*-equivariant.*

Proof. Take $\lambda \in \mathcal{C}^s_{2l}$ and denote $\mathrm{Res}_D(\lambda) = \sum_{\chi \in I_{\widetilde{BD}_{2l}}} d^\lambda_\chi \alpha_\chi$. Take $i \in [\![2, l\text{ - }2]\!]$ and let us show that the number of addable boxes with C-residue i (counted negatively if they are removable boxes) is exactly $d^\lambda_{\chi_{i+1}} + d^\lambda_{\chi_{i-1}} - 2d^\lambda_{\chi_i}$.
To prove this, we can proceed region by region. Using the fact that λ is a symmetric partition, we can restrict the study to half of the Young diagram. Fix an integer $k \in \mathbb{Z}_{\geq 0}$. For each $j \in [\![0, l]\!]$, denote by $d^j_{\mathcal{R}_k}$ the number of boxes in $\mathcal{R}_k$ with C-residue i. Recall that $\{(0, b) \in \mathcal{Y}(\lambda) | b \in \mathbb{Z}_{\geq 0}\} \cup \{(a, 0) \in \mathcal{Y}(\lambda) | a \in \mathbb{Z}_{\geq 0}\}$ is the border of the Young diagram $\mathcal{Y}(\lambda)$. There are three different cases. The first case is when there is just one addable box in $\mathcal{R}_k$ which is in the border of $\mathcal{Y}(\lambda)$. This means that either $d^{i+1}_{\mathcal{R}_k} = d^i_{\mathcal{R}_k} + 1$ and $d^{i-1}_{\mathcal{R}_k} = d^i_{\mathcal{R}_k}$ or $d^{i-1}_{\mathcal{R}_k} = d^i_{\mathcal{R}_k} + 1$ and $d^{i+1}_{\mathcal{R}_k} = d^i_{\mathcal{R}_k}$. We then get that $d^{i+1}_{\mathcal{R}_k} + d^{i-1}_{\mathcal{R}_k} - 2d^i_{\mathcal{R}_k} = 1$. The next case is when there is exactly one addable box in the region $\mathcal{R}_k$ and it is not in the border. This implies that $d^{i-1}_{\mathcal{R}_k} = d^{i+1}_{\mathcal{R}_k} + 1$ and $d^{i+1}_{\mathcal{R}_k} = d^i_{\mathcal{R}_k}$. Suming it up gives $d^{i+1}_{\mathcal{R}_k} + d^{i-1}_{\mathcal{R}_k} - 2d^i_{\mathcal{R}_k} = 1$. The last case is when there are 2 addable boxes in $\mathcal{R}_k$. One box must be in the border and we must be out of the border of $\mathcal{Y}(\lambda)$. We then have that $d^{i-1}_{\mathcal{R}_k} + d^{i+1}_{\mathcal{R}_k} - 2d^i_{\mathcal{R}_k} = 2$. Coming back to the definition of $\mathrm{Res}_D(\lambda)$, and suming, for each $j \in [\![0, l]\!]$, all $d^j_{\mathcal{R}_k}$ on all regions $\mathcal{R}_k$ for $k \geq 0$ up gives the result. The previous disjunction of cases will be referred to as the three-cases argument.
There remains to prove the equivariance for $i \in \{0, 1, l\text{ - }1, l\}$. Apply first the three-cases argument for $i \in \{1, l\text{ - }1\}$. We then have that the number of addable boxes with C-residue 1 and l - 1 is respectively $d^\lambda_0 + d^\lambda_{\chi_2} - 2d^\lambda_{\chi_1}$ and $d^\lambda_l + d^\lambda_{\chi_{l-2}} - 2d^\lambda_{\chi_{l-1}}$ where $d^\lambda_l = d^\lambda_{\chi_{l^+}} + d^\lambda_{\chi_{l^-}}$ and $d^\lambda_0 = d^\lambda_{\chi_{0^+}} + d^\lambda_{\chi_{0^-}}$. The three-cases argument can also be applied when $i = l$ and gives that the number of addable boxes with C-residue l is $d^\lambda_{\chi_{l-1}} - 2d^\lambda_l$. Since the image of Res_D is in $Q(\tilde{D}^\sigma_{l+2})[0^+]$, we have that $d^\lambda_{\chi_{l^+}} = d^\lambda_{\chi_{l^-}} = \frac{d^\lambda_l}{2}$.
It remains to consider the cases $i = 0$. Take $k > 0$ and instead of working in the region $\mathcal{R}_k$, apply the same arguments as in the three-cases argument in the region $\mathcal{R}_{k,-1}$ for a positive integer k. For the region containing the diagonal (i.e. $k = 0$), we can apply the three-cases argument, but since $s_0.\emptyset = (1) \vdash 1$, there are $d^1_{\mathcal{R}_{0,-1}} - 2d^0_{\mathcal{R}_{0,-1}} + 1$ addable boxes. Suming it all up gives that there are $d^\lambda_{\chi_1} - 2d^\lambda_0 + 1$ addable boxes with C-residue 0. Finally we need to split d^λ_0 back into $d^\lambda_{\chi_{0^+}}$ and $d^\lambda_{\chi_{0^-}}$. To do that, consider the parity of d^λ_0. Indeed if d^λ_0 is even then $d^\lambda_{\chi_{0^+}} = d^\lambda_{\chi_{0^-}}$ and if d^λ_0 is odd, then $d^\lambda_{\chi_{0^+}} = d^\lambda_{\chi_{0^-}} + 1$. □

Proposition 5.23. $\mathcal{T} \circ \mathrm{Res}_D : \mathfrak{C}^s_{2l} \to \overline{0}^{W(\tilde{C}_l)} \subset Q^\vee(\tilde{C}_l)$ *is a bijection.*

Proof. By definition, we have $\mathcal{T}(Res_D(\emptyset)) = 0$ and the stabilizer of $\emptyset \in \mathfrak{C}^s_{2l}$ in $W(\tilde{C}_l)$ denoted by $\mathrm{Stab}_{W(\tilde{C}_l)}(\emptyset)$ is equal to $W(C_l)$ which is equal to $\mathrm{Stab}_{W(\tilde{C}_l)}(0)$. Moreover, using Proposition 5.9 and Proposition 5.22, we know that $\mathcal{T} \circ \mathrm{Res}_D$ is $W(\tilde{C}_l)$-equivariant. To conclude, it is enough to show that the $W(\tilde{C}_l)$-action defined on $\mathfrak{C}^s_{2l}$ (Definition 5.20) is transitive. This has been proven in [HJ12, Proposition 6.2]. □

Proposition 5.24. *The following chain of maps*

$$\varphi : \mathfrak{C}^s_{2l} \xrightarrow{\mathcal{T} \circ \mathrm{Res}_D} Q^\vee(\tilde{C}_l) \xrightarrow{\pi} Q^\vee(\tilde{C}_l)/\mathbb{Z}\delta^\vee(\tilde{C}_l)$$

is a bijection.

Proof. Consider the bijection $\overline{0}^{W(\tilde{C}_l)} \xrightarrow{\sim} Q^\vee(C_l)$ which is the composition of these two bijections

$$\overline{0}^{W(\tilde{C}_l)} \xrightarrow{\sim} W(\tilde{C}_l)/W(C_l) \xrightarrow{\sim} Q^\vee(C_l)$$

The second bijection boils down to the choice of a representative with coordinate 0 along $\beta_0^\vee$. Moreover, consider the bijection

$$\begin{array}{ccc} Q^\vee(\tilde{C}_l)/\mathbb{Z}\delta^\vee(\tilde{C}_l) & \xrightarrow{\sim} & Q^\vee(C_l) \\ \beta^\vee & \mapsto & \beta^\vee - \beta_0^\vee \delta^\vee(\tilde{C}_l) \end{array}.$$

We then have the following commutative diagram

$$\begin{array}{ccc} \overline{0}^{W(\tilde{C}_l)} & \xrightarrow{\quad\pi\quad} & Q^\vee(\tilde{C}_l)/\mathbb{Z}\delta^\vee(\tilde{C}_l) \\ & \searrow^{\sim} \quad \swarrow_{\sim} & \\ & Q^\vee(C_l) & \end{array}$$

From there, we can use Proposition 5.23 to prove that φ is a bijection. □

5.2 Combinatorial description in type D

We now have everything to give a combinatorial description of the set $\mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$. Note that, thanks to Proposition 5.16, $\mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}} = \mathcal{A}^{n}_{\widetilde{BD}_{2l}} \cap Q(\tilde{D}^\sigma_{l+2})[0^+]$.
Consider the following map

$$\begin{array}{ccc} \epsilon : Q(\tilde{D}^\sigma_{l+2})[0^+] & \to & \mathfrak{C}^s_{2l} \\ d & \mapsto & (\varphi^{-1} \circ \pi \circ \mathcal{T})(d) \end{array}$$

Theorem 5.25. *The map ϵ defines a bijection between $\mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$ and the $2l$-symmetric cores λ, such that $|\lambda| \equiv n\ [2l]$ and $|\lambda| \leq n$.*

Proof. First let us show that if $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$ then $|\epsilon(d)| \equiv n\ [2l]$. Denote $\lambda := \epsilon(d)$, then

$$\exists k \in \mathbb{Z}, \mathcal{T}(d) = \mathcal{T}(\mathrm{Res}_D(\lambda)) + k\delta^\vee(\tilde{C}_l)$$

In particular $|\mathcal{T}(d)|_{\tilde{C}_l} = |\mathcal{T}(\mathrm{Res}_D(\lambda))|_{\tilde{C}_l} + 2kl$. Now since $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$, $|d|_{\tilde{D}_{l+2}} = n$ and using Remark 5.10 we have that $n = |\lambda| + 2kl$. Moreover, let us show that if $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$, then $|\epsilon(d)| \leq n$. Thanks to Lemma 5.11 and to the fact that $\mathcal{T}(\mathrm{Res}_D(\lambda)) \in \overline{0}^{W(\tilde{C}_l)}$, we have that $\mathcal{T}(d) \in \overline{k\delta^\vee(\tilde{C}_l)}^{W(\tilde{C}_l)}$. Since $\mathrm{wt}(d) \geq 0$, there exists $k' \in \mathbb{Z}_{\geq 0}$ such that $d \in \overline{k'\delta(\tilde{D}_{l+2})}^{W(\tilde{D}_{l+2})}$. In fact $d \in \overline{k'\delta(\tilde{D}_{l+2})}^{W(\tilde{C}_l)}$ since $d \in Q(\tilde{D}^\sigma_{l+2})[0^+]$. The map $\mathcal{T}$ sends $\delta(\tilde{D}_{l+2})$ to $2\delta^\vee(\tilde{C}_l)$, which then gives that $\mathcal{T}(d) \in \overline{2k'\delta^\vee(\tilde{C}_l)}^{W(\tilde{C}_l)}$ and so $k = 2k'$, thanks to Lemma 3.7. Since $n = |\lambda| + 2kl$, we have that $k \geq 0 \iff |\lambda| \leq n$. The map $\epsilon : \mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}} \to \{\lambda \in \mathfrak{C}^s_{2l} \,|\, |\lambda| \equiv n\ [2l], |\lambda| \leq n\}$ has now been proven to be well defined. By construction, ϵ is the converse map of Res_D and establishes a bijection between $\mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$ and $\{\lambda \in \mathfrak{C}^s_{2l} \,|\, |\lambda| \equiv n\ [2l], |\lambda| \leq n\}$. □

Remark 5.26. Take $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$ and $\lambda \in \mathcal{P}^s_n$ such that $I_\lambda \in \mathcal{H}^{\widetilde{BD}_{2l},\xi_d}_n$. Let γ_{2l} denote the $2l$-core of λ. We have, as a by-product of the proof of Theorem 5.25, that $\frac{n-|\gamma_{2l}|}{2l}$ which is the number of $2l$-hooks that we need to remove from λ to obtain its $2l$-core is equal to $2\mathrm{wt}(d)$.

Example 5.27. The set $\mathcal{A}^{n,\mathbb{T}_1}_{\widetilde{BD}_{2l}}$ is a proper subset of $\mathcal{A}^{n}_{\widetilde{BD}_{2l}}$. If $l = 2$, we can find for each $r \in \mathbb{Z}_{>0}$ an irreducible component of $\mathcal{H}^{\widetilde{BD}_4}_{8r+4}$ of dimension $2r$ that is parametrized by an element of $\mathcal{A}^{8r+4}_{\widetilde{BD}_4} \setminus \mathcal{A}^{8r+4,\mathbb{T}_1}_{\widetilde{BD}_4}$. Let $\omega = s_{\chi_{2^+}} s_{\chi_1} s_{\chi_{0^+}} \in W(\tilde{T}_{\widetilde{BD}_4})$ and consider $\omega.r\delta^{\widetilde{BD}_4}$. We have that $(\omega.r\delta^{\widetilde{BD}_4})_{\chi_{2^+}} = (\omega.r\delta^{\widetilde{BD}_4})_{\chi_{2^-}} + 1$, which implies that this element is not in $\mathcal{A}^{8r+4,\mathbb{T}_1}_{\widetilde{BD}_4}$ thanks to Proposition 5.16.

5.3 Absence of combinatorics in type E

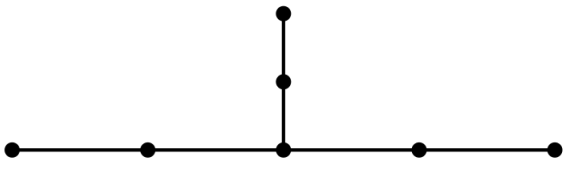
The binary tetrahedral group $\tilde{A}_4$ is a finite group of order 24 and has the following presentation

$$< a, b, c \mid a^2 = b^3 = c^3 = abc >$$

Let us denote by $z := abc$ which is a central element of $\tilde{A}_4$. Note that z has order 2. Take $\tilde{G}_4$ a finite group of $\mathrm{SL}_2(\mathbb{C})$ isomorphic to $\tilde{A}_4$. The character table of $\tilde{G}_4$ is

cardinality	1	1	6	4	4	4	4
classes	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	z	a	b	c	b^2	c^2
χ_0	1	1	1	1	1	1	1
ψ	1	1	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3
ψ^2	1	1	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2
χ_1	3	3	-1	0	0	0	0
χ_{std}	2	-2	0	1	1	-1	-1
$\psi\chi_{\mathrm{std}}$	2	-2	0	ζ_3	ζ_3^2	$-\zeta_3^2$	$-\zeta_3$
$\psi^2\chi_{\mathrm{std}}$	2	-2	0	ζ_3^2	ζ_3	$-\zeta_3$	$-\zeta_3^2$

The group $\tilde{G}_4$ has the following McKay graph of affine type $\tilde{E}_6$



The goal is here to study the combinatorics of the irreducible components of $\mathcal{H}^{\tilde{G}_4}_n$. Let us show that the irreducible components that contain a monomial ideal are fixed under $\mathrm{SL}_2(\mathbb{C})$. Let X_{std} denote the standard representation of $\mathrm{SL}_2(\mathbb{C})$ with its canonical basis (e_1, e_2) and denote by B_1, respectively B_2 the stabilizer of e_1 respectively e_2 in X_{std}. The subgroups B_1 and B_2 are the two Borel subgroups of $\mathrm{SL}_2(\mathbb{C})$ containing $\mathbb{T}_1$.

Lemma 5.28. *The finite group $\tilde{G}_4$ is not conjugate to any subgroup of the normalizer of $\mathbb{T}_1$ in $\mathrm{SL}_2(\mathbb{C})$ denoted by $N_{\mathrm{SL}_2(\mathbb{C})}(\mathbb{T}_1)$. Furthermore, the group $\tilde{G}_4$ is neither a subgroup of B_1 nor of B_2, the two Borel subgroups of $\mathrm{SL}_2(\mathbb{C})$ which contain the maximal torus $\mathbb{T}_1$.*

Proof. The representation $X_{\text{std}} \otimes X_{\text{std}}^*$ is isomorphic to the direct sum of the trivial representation (generated by $e_1 \otimes e_1^* + e_2 \otimes e_2^*$) and of the adjoint representation of $\mathrm{SL}_2(\mathbb{C})$. On the one hand, note that for the character χ_{std} of $\tilde{A}_4$, we have that $\langle \chi^2, \chi^2 \rangle = 2$ which implies that the restriction of the adjoint representation to $\tilde{G}_4$ is irreducible. On the other hand, the restriction of the adjoint representation to $N_{\mathrm{SL}_2(\mathbb{C})}(\mathbb{T}_1)$ is not irreducible since the one-dimensional subspace of $X_{\text{std}} \otimes X_{\text{std}}^*$ generated by $e_1 \otimes e_1^* - e_2 \otimes e_2^*$ is $N_{\mathrm{SL}_2(\mathbb{C})}(\mathbb{T}_1)$-stable. Moreover, the one-dimensional subspace of $X_{\text{std}} \otimes X_{\text{std}}^*$ generated by $e_1 \otimes e_2^*$ is B_1-stable and the one-dimensional subspace of $X_{\text{std}} \otimes X_{\text{std}}^*$ generated by $e_2 \otimes e_1^*$ is B_2-stable. □

Proposition 5.29. *The subgroup G of $\mathrm{SL}_2(\mathbb{C})$ generated by $\mathbb{T}_1$ and $\tilde{G}_4$ is equal to $\mathrm{SL}_2(\mathbb{C})$.*

Proof. Thanks to Lemma 5.28, there exists $x \in \tilde{G}_4$ such that $\mathbb{T}_1 \neq x\mathbb{T}_1x^{-1}$. We then have that the two subgroups $\mathbb{T}_1$ and $x\mathbb{T}_1x^{-1}$ are both irreducible and connected subgroups of $\mathrm{SL}_2(\mathbb{C})$. Let us denote by H the subgroup of $\mathrm{SL}_2(\mathbb{C})$ generated by these two one-dimensional tori. Thanks to [Hump, section 7.5], we know that H is a closed connected subgroup of $\mathrm{SL}_2(\mathbb{C})$. Since H is not equal to $\mathrm{SL}_2(\mathbb{C})$, and is of dimension at least two, H is of dimension 2. Using [Bor12, Corollary 11.6] we know that H is solvable. The algebraic group H is then a Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$ containing $\mathbb{T}_1$ and contained in G. Moreover, thanks to the Bruhat decomposition [Bor12, Theorem 14.12], we know that $\mathrm{SL}_2(\mathbb{C}) = B_1sB_1 \coprod B_1 = B_2sB_2 \coprod B_2$. Combining the Bruhat decomposition with Lemma 5.28, this gives that $s \in G$. Thanks to [Bor12, Proposition 11.19], we know that all Borel subgroups containing $\mathbb{T}_1$ are conjugated by the Weyl group of $\mathbb{T}_1$ denoted by $W(\mathbb{T}_1)$ which is by construction, the group generated by $\bar{s} \in W(\mathbb{T}_1)$. This implies that all Borel subgroups containing $\mathbb{T}_1$ are in G. Finally, using [Bor12, Proposition 13.7], we have that $G = \mathrm{SL}_2(\mathbb{C})$.

□

Definition 5.30. A partition λ is called a staircase partition if there exists a certain integer m such that $\lambda = (m, m-1, ..., 1) \vdash \frac{m(m+1)}{2}$.

Proposition 5.31. *The only $\mathrm{SL}_2(\mathbb{C})$ fixed points of $\mathcal{H}_n$ are the monomial ideals associated with staircase partitions of size n.*

Proof. We already know that $\mathbb{T}_1$-fixed points are exactly monomial ideals. Moreover, thanks to [KT, Lemma 12], we have that the fixed points under the subgroup $\mathbb{B}_2$ of $\mathrm{GL}_2(\mathbb{C})$ consisting of all upper triangular matrices are parametrized by staircase partitions. Since $\mathbb{B}_2 = \mathbb{T}_2\mathbb{B}_1$, we get that $\mathbb{B}_1$-fixed points of $\mathcal{H}_n$ are also parametrized by staircase partitions and the result follows. □

Finally, the binary octahedral group (type $\tilde{E}_7$) and the binary icosahedral group (type $\tilde{E}_8$) both contain a subgroup isomorphic to $\tilde{A}_4$ which then implies that the combinatorics of fixed points which are also $\mathbb{C}^*$-fixed is the same as the one of $\mathrm{SL}_2(\mathbb{C})$.

CHAPTER 6

TAUTOLOGICAL VECTOR BUNDLE

After studying the irreducible components of $\mathcal{H}_n^\Gamma$ which also are the connected components since $\mathcal{H}_n^\Gamma$ is a smooth variety and after building a combinatorial model using partitions for the indexing set of these components, we are now interested in the study of the restriction to $\mathcal{H}_n^\Gamma$ of two vector bundles over $\mathcal{H}_n$. Let S be an algebraic variety. A geometric vector bundle over S of rank n is the data of

- an algebraic variety X
- a morphism $\pi : X \to S$
- a covering $(U_i)_{i\in I}$ of open affine sets of S
- for each $i \in I$, an isomorphism ψ_i from $\pi^{-1}(U_i)$ to the affine n-space on U_i

such that for all $(i,j) \in I^2$ and for all $V = \mathrm{Spec}(A) \subset U_i \cap U_j$, the automorphism $\psi_j \circ \psi_i^{-1}$ is given by a linear automorphism of $A[x_1, ..., x_n]$. A morphism $\phi : X_1 \to X_2$ is an isomorphism between two geometric vector bundles $(X_1, \pi_1, \{U_i^1\}, \{\psi_i^1\})$ and $(X_2, \pi_2, \{U_i^2\}, \{\psi_i^2\})$ of rank n if $(X_1, \pi_1, \{U_i^1\} \coprod \{U_j^2\}, \{\psi_i^1\} \coprod \{\psi_j^2 \circ \phi\})$ is a geometric vector bundle on S and if ϕ is an isomorphism over S. We have a one-to-one correspondence between isomorphism classes of locally free sheaves of rank n on S and isomorphism classes of geometric vector bundles of rank n over S [Hart, Exercice 5.18]. Note that if $\mathcal{F}$ is a locally free sheaf of rank n on S, then the stalk of $\mathcal{F}$ at a point $s \in S$ is linked to the fiber of the associated geometric vector bundle $V(\mathcal{F})$. Indeed, we have an isomorphism of $\kappa_S(s)$-vector spaces

$$V(\mathcal{F})_s \simeq \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \kappa_S(s)$$

In this concise chapter, we will decompose the tautological vector bundle of $\mathcal{H}_n$ over a connected component of $\mathcal{H}_n^\Gamma$ in terms of tautological vector bundles on Nakajima's quiver varieties.

6.1 Tautological vector bundle

Let us start by constructing the tautological vector bundle over the Hilbert scheme of n points in $\mathbb{C}^2$. The variety $\mathcal{H}_n$ is isomorphic as a $\mathrm{GL}_2(\mathbb{C})$-algebraic variety to $\mathcal{M}_{\mathbf{1}}^\bullet(n)$, the Nakajima quiver variety of the double framed Jordan quiver with stability parameter 1 and dimension parameter n (Proposition 3.1). Let $V_n := \mathbb{C}^n$ be the standard representation of $\mathrm{G}(n) = \mathrm{GL}_n(\mathbb{C})$ and consider the trivial vector bundle of rank n on $\tilde{\mathcal{M}}_{\mathbf{1}}^\bullet(n)$

$$\pi : \tilde{\mathcal{M}}_{\mathbf{1}}^{\bullet}(n) \times V_n \to \tilde{\mathcal{M}}_{\mathbf{1}}^{\bullet}(n)$$

Denote this trivial vector bundle by $\tilde{\mathcal{T}}_n$. Moreover, let the group $\mathrm{G}(n)$ act diagonally on $\tilde{\mathcal{M}}_{\mathbf{1}}^{\bullet}(n) \times V_n$ and consider $\mathcal{T}_n := \left(\tilde{\mathcal{M}}_{\mathbf{1}}^{\bullet}(n) \times V_n\right) /\!\!/ \mathrm{G}(n)$. Recall that we have defined an isomorphism $H : \mathcal{H}_n \to \mathcal{M}_{\mathbf{1}}^{\bullet}(n)$ in Chapter 3.

Proposition 6.1. *The algebraic variety $\mathcal{T}_n$ is a vector bundle of rank n on $\mathcal{M}_{\mathbf{1}}^{\bullet}(n)$ and $H^*\mathcal{T}_n$ is called the tautological bundle of $\mathcal{H}_n$.*

Proof. Using [FKM, Theorem 1.10], we get that the map $\tilde{\mathcal{M}}_{\mathbf{1}}^{\bullet}(n) \to \mathcal{M}_{\mathbf{1}}^{\bullet}(n)$ is by construction affine. Recall that $\mathrm{G}(n)$ acts freely on $\tilde{\mathcal{M}}_{\mathbf{1}}^{\bullet}(n)$ (Lemma 2.1). We can now apply [Pot81, Proposition 4] to finish the proof. □

Note that this vector bundle can also be constructed thanks to the universal family lying over the punctual Hilbert scheme in $\mathbb{C}^2$.

Let us now construct tautological vector bundles on quiver varieties associated with quivers without loops. Take an undirected multigraph $\mathcal{G} = (I_{\mathcal{G}}, E_{\mathcal{G}})$ without loops, a vertex $v \in I_{\mathcal{G}}$ and $d \in \Delta_{\mathcal{G}}^{+}$ a dimension vector such that $\mathcal{M}_{\mathbf{1}}^{\mathcal{G}}(d, v)$, the quiver variety associated with the double quiver of a quiver on $\mathcal{G}$ framed at v with dimension parameter d and stability parameter 1, is nonempty. Recall that the group $\mathrm{G}(d) := \prod_{i \in I_{\mathcal{G}}} \mathrm{GL}_{d_i}(\mathbb{C})$ acts by base change on the space of representations of the double and v-framed quiver associated with $\mathcal{G}$. Fix a vertex $i \in I_{\mathcal{G}}$. Let us construct the tautological vector bundle at vertex i on $\mathcal{M}_{\mathbf{1}}^{\mathcal{G}}(d, v)$. To do so, let us first recall some general results on contracted products. Let G be a reductive algebraic group over $\mathbb{C}$.

Definition 6.2. Let B be an algebraic variety and let $\tilde{\pi} : E \to B$ be a fiber bundle. For $(E, \tilde{\pi})$ to be a G-principal bundle, the variety E needs to be equipped with a right G-action such that

- the fibers are preserved i.e. $\tilde{\pi}$ is G-invariant.
- there exists a G-equivariant trivializing cover of B i.e. there exists a cover of B, $(U_j)_{j \in J}$ by open sets, and for all $j \in J$, there exists G-equivariant isomorphisms $\phi_j : \tilde{\pi}^{-1}(U_j) \to U_j \times G$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{\pi}^{-1}(U_j) & \xrightarrow{\phi_j} & U_j \times G \\ {\scriptstyle \tilde{\pi}}\downarrow & \swarrow_{p_1} & \\ U_j & & \end{array}$$

Note that G acts on $U_j \times G$ via $\forall (u, g, h) \in U_j \times G^2, (u, g).h := (u, gh)$.

Remark 6.3. Let $\tilde{\pi} : E \to B$ be a G-principal bundle. Then a G-equivariant trivializing cover of B assures that the G-action on the fibers is free and transitive.

Definition 6.4. Let $\tilde{\pi} : E \to B$ be a G-principal bundle and V be a rational representation of G. Let $E \times V$ be the G-space with the action given by $g.(e, v) := (e.g^{-1}, g.v)$ for all $(g, e, v) \in G \times E \times V$. We can then define the contracted product of E and V, $E \times^G V := (E \times V) /\!\!/ G$.

Proposition 6.5. *If $\tilde{\pi} : E \to B$ is a G-principal bundle and V is a rational representation of G, then $E \times^G V$ is a vector bundle over B with fiber V.*

Proof. We have an open cover of $B := \bigcup_{j\in J} U_j$ such that there exists G-equivariant isomorphisms $\phi_j : \tilde{\pi}^{-1}(U_j) \to U_j \times G$ for all $j \in J$. Consider the following diagram

$$\begin{array}{ccc} E \times V & \xrightarrow{\tilde{q}} & E \times^G V \\ {\scriptstyle p_1}\downarrow & & \vdots{\scriptstyle \pi} \\ E & \xrightarrow[\tilde{\pi}]{} & B \simeq E /\!\!/ G \end{array}$$

Using the universal property of the quotient, we have $\pi : E \times^G V \to B$. Let us denote by $[e, v] \in E \times^G V$ the orbit of $(e, v) \in E \times V$. For each $b \in B$, $\pi^{-1}(b)$ has a natural structure of vector space. It is given by $\lambda[e, v] + [e, v'] := [e, \lambda v + v']$, for each $\lambda \in \mathbb{C}$ and for each $([e, v], [e, v']) \in (\pi^{-1}(b))^2$. Let us show that for all $j \in J$, $\pi^{-1}(U_j) \simeq U_j \times V$. Consider the following morphisms

$$\Phi_j : \begin{array}{ccc} U_j \times V & \to & \pi^{-1}(U_j) \\ (b, v) & \mapsto & [\phi_j^{-1}(b, 1), v] \end{array}$$

$$\tilde{\Psi}_j : \begin{array}{ccccc} \tilde{\pi}^{-1}(U_j) \times V & \to & U_j \times G \times V & \to & U_j \times V \\ (e, v) & \mapsto & (\phi_j(e), v) & & \\ & & (b, g, v) & \mapsto & (b, g.v) \end{array}$$

$\tilde{\Psi}_j$ is G-invariant and induces a morphism $\Psi_j : \pi^{-1}(U_j) \to U_j \times V$. By construction, we have that Φ_j and Ψ_j are mutually inverse isomorphisms. Finally, for each $j \in J$ and each $b \in U_j$, the morphism $\begin{array}{ccc} V & \to & \pi^{-1}(b) \\ v & \mapsto & [\phi_j^{-1}(b, 1), v] \end{array}$ is, by definition of the vector space structure on $\pi^{-1}(b)$, a linear isomorphism. □

Let us come back to the quiver variety setting. Since the stability parameter 1 is positive, thanks to [Nak00, Proposition 2.3.2] (which can be reformulated as [Kir, Theorem 10.34]), we have that $\tilde{\mathcal{M}}_{\mathbf{1}}^{\mathcal{G}}(d, v) \to \mathcal{M}_{\mathbf{1}}^{\mathcal{G}}(d, v)$ is a $\mathrm{G}(d)$-principal bundle, .

Definition 6.6. Consider the rational representation $V_i^d := \mathbb{C}^{d_i}$ of $\mathrm{G}(d)$ which acts by $\mathrm{GL}_{d_i}(\mathbb{C})$. Proposition 6.5 gives that $\mathcal{T}_d(i) := \tilde{\mathcal{M}}_{\mathbf{1}}^{\mathcal{G}}(d, v) \times^{\mathrm{G}(d)} V_i^d$ is a vector bundle of rank d_i on $\mathcal{M}_{\mathbf{1}}^{\mathcal{G}}(d, v)$. It is called the tautological vector bundle over $\mathcal{M}_{\mathbf{1}}^{\mathcal{G}}(d, v)$ associated with the vertex i. Let us also denote by $\tilde{\mathcal{T}}_d(i) := \tilde{\mathcal{M}}_{\mathbf{1}}^{\mathcal{G}}(d, v) \times V_i^d$.

Let us decompose $H^*\mathcal{T}_n$ over an irreducible component of $\mathcal{H}_n^\Gamma$ using the tautological vector bundles of quiver varieties. Fix an irreducible component of $\mathcal{H}_n^\Gamma$. This is the same data as a dimension parameter $d \in \mathcal{A}_\Gamma^n$ (Corollary 3.3). Let the group $\mathrm{G}(d)$ act on $V^{d,\delta^\Gamma} := \bigoplus_{\chi\in I_\Gamma} V_\chi^{d \oplus \delta_\chi^\Gamma}$ by $\mathrm{GL}(V_\chi^d)$ on each summand $V_\chi^{d\oplus\delta_\chi^\Gamma}$. Since $|d|_\Gamma = n$, we have an isomorphism

$$\phi_d : V^{d,\delta^\Gamma} \xrightarrow{\sim} V_n$$

Endow V_n with a $\mathrm{G}(d)$-module structure such that ϕ_d is a $\mathrm{G}(d)$-equivariant isomorphism. Take $\sigma \in \mathcal{C}_d$. Denote by $\mathcal{T}_n^\Gamma := \mathcal{T}_{n_{|\mathcal{M}_{\mathbf{1}}^\bullet(n)^\Gamma}}$. Moreover, using the isomorphism τ built in Theorem 1.29 denote by $\mathcal{T}_n^d := (\tau \circ \iota_{\mathbf{1},M^\sigma}^\Gamma)^* \mathcal{T}_n^\Gamma$ which is a vector bundle over

$\mathcal{M}_{\mathbf{1}}^{\Gamma}(d)$. We need also to consider the situation before taking quotients. Denote therefore $\tilde{\mathcal{T}}_n^d := (\tilde{\tau} \circ \tilde{\imath}_{M^{\sigma_d}}^{\Gamma})^* \tilde{\mathcal{T}}_n^{\Gamma}$. Since the McKay unoriented multigraph G_Γ has no loops, for all irreducible character $\chi \in I_\Gamma$, we can consider the tautological vector bundle $\mathcal{T}_d(\chi)$ over $\mathcal{M}_{\mathbf{1}}^{\Gamma}(d)$. Define

$$\begin{array}{rccc} \tilde{\mathcal{D}}_{\mathcal{T}} : & \bigoplus_{\chi \in I_\Gamma} \tilde{\mathcal{T}}_d(\chi)^{\oplus \delta_\chi^\Gamma} & \to & \tilde{\mathcal{T}}_n^d \\ & (x, v^1, v^2, z_\chi)_{\chi \in I_\Gamma} & \mapsto & (x, v^1, v^2, \phi_d(\sum_{\chi \in I_\Gamma} z_\chi)) \end{array}$$

Theorem 6.7. *The map $\tilde{\mathcal{D}}_{\mathcal{T}}$ induces a map $\mathcal{D}_{\mathcal{T}} : \bigoplus_{\chi \in I_\Gamma} \mathcal{T}_d(\chi)^{\oplus \delta_\chi^\Gamma} \to \mathcal{T}_n^d$ which is an isomorphism of vector bundles over $\mathcal{M}_{\mathbf{1}}^{\Gamma}(d)$.*

Proof. The map $\tilde{\mathcal{D}}_{\mathcal{T}}$ is $\mathrm{G}(d)$-equivariant since ϕ_d is. It then induces a morphism of vector bundles $\mathcal{D}_{\mathcal{T}} : \bigoplus_{\chi \in I_\Gamma} \mathcal{T}_d(\chi)^{\oplus \delta_\chi^\Gamma} \to \mathcal{T}_n^d$. Moreover, using the fact that ϕ_d is an isomorphism it is clear that $\tilde{\mathcal{D}}_{\mathcal{T}}$ is fiberwise a linear isomorphism. This implies that $\mathcal{D}_{\mathcal{T}}$ is also fiberwise a linear isomorphism and then is an isomorphism of vector bundles over $\mathcal{M}_{\mathbf{1}}^{\Gamma}(d)$. □

CHAPTER 7

PROCESI VECTOR BUNDLE

The Procesi bundle is another important vector bundle on the Hilbert scheme of n points in $\mathbb{C}^2$. It has played a key role in the proof of the $n!$ theorem [H01, Theorem 5.2.1]. The Procesi bundle induces an equivalence between the bounded derived category of $\mathbb{T}_2$-equivariant coherent sheaves on $\mathcal{H}_n$ and the bounded derived category of $(\mathfrak{S}_n \times \mathbb{T}_2)$-equivariant coherent sheaves on $(\mathbb{C}^2)^n$ [H01, Corollary 5.3.3]. This equivalence of bounded derived categories has been generalized by Roman Bezrukavnikov and Dmitry Kaledin [BK04, Theorem 1.1] to projective symplectic resolutions of $\mathcal{Y}_n^\Gamma$. In this chapter, let us first present the construction of the Procesi vector bundle. In the second section, a reduction theorem of the fibers of the Procesi bundle over a connected component of $\mathcal{H}_n^\Gamma$ as a $(\mathfrak{S}_n \times \Gamma)$-module is proven. This is a joint work with Gwyn Bellamy. The initial inspiration for this theorem is [BLM06, Theorem 4.6] and the intriguing fact that when Γ is the cyclic group with l elements, the dimensions of the isotypical components of the fiber of the Procesi bundle as a μ_l-module are constant over the monomial ideal I_λ if λ is not an l-core. In the third section, a corollary of the reduction theorem when Γ is μ_l is shown. Furthermore, in the fourth section, proofs of edge cases of the reduction theorem are given when Γ is the cyclic group. These proofs don't involve the same arguments used in section 2 and are interesting because they present the variety of approaches that we could use to understand the $(\mathfrak{S}_n \times \mu_l)$-module structure of the fibers of the Procesi bundle over $\mathcal{H}_n^\Gamma$. Finally, the last section presents also reduction formulas when Γ is the binary dihedral group.

7.1 Procesi vector bundle

The Procesi vector bundle is a $GL_2(\mathbb{C})$-equivariant vector bundle on $\mathcal{H}_n$. To construct the Procesi bundle, we need first to introduce the isospectral Hilbert scheme. Let k be an integer greater or equal to 1, and let $\mathfrak{S}_k$, the symmetric group on k letters, act on $(\mathbb{C}^2)^k$ by permuting the different copies of $\mathbb{C}^2$. The isospectral Hilbert scheme, denoted by $\mathcal{X}_k$, is the fiber product of $\mathcal{H}_k$ with $(\mathbb{C}^2)^k$ over $(\mathbb{C}^2)^k/\mathfrak{S}_k$ endowed with the reduced induced scheme structure

$$\begin{array}{ccc} \mathcal{H}_k \times_{(\mathbb{C}^2)^k/\mathfrak{S}_k} (\mathbb{C}^2)^k & \xrightarrow{f_k} & (\mathbb{C}^2)^k \\ \downarrow{\scriptstyle \rho_k} & \lrcorner & \downarrow{\scriptstyle \pi_k} \\ \mathcal{H}_k & \xrightarrow[\sigma_k]{} & (\mathbb{C}^2)^k/\mathfrak{S}_k \end{array}$$

The morphism σ_k is the Hilbert-Chow morphism. It is defined as follows

$$\sigma_k : \begin{array}{ccc} \mathcal{H}_k & \to & \frac{(\mathbb{C}^2)^k/\mathfrak{S}_k}{\sum_{p\in V(I)} \dim\big((\mathbb{C}[x,y]/I)_p\big)p} \\ I & \mapsto & \end{array}$$

where the sum is formal. The morphism π_k is the quotient map. The scheme $\mathcal{X}_k$ is an algebraic variety. Marc Haiman [H03, Theorem 5.2.1] has proven that ρ_k is a flat morphism. This implies that the sheaf $\mathscr{P}^k := \rho_{k*}\mathcal{O}_{\mathcal{X}_k}$ is locally free and thus defines a vector bundle on $\mathcal{H}_k$. This vector bundle is the k^{th}-Procesi bundle. Note that by construction the fibers of $\mathscr{P}^k$ are, as $\mathfrak{S}_k$-modules, isomorphic to the regular representation of $\mathfrak{S}_k$ and in particular are of dimension $k!$. Let the group $\mathrm{GL}_2(\mathbb{C})$ act naturally on $\mathbb{C}^2$. Recall that this gives a $\mathrm{GL}_2(\mathbb{C})$-action on the Hilbert scheme $\mathcal{H}_k$ and on $(\mathbb{C}^2)^k$. We then obtain a $\mathrm{GL}_2(\mathbb{C})$-action on $\mathcal{X}_k$ which turns $\mathscr{P}^k$ into a $\mathrm{GL}_2(\mathbb{C})$-equivariant vector bundle. Moreover by letting $\mathfrak{S}_k$ act trivially on $\mathcal{H}_k$, we have that all morphisms ρ_k, σ_k, π_k and f_k are $(\mathfrak{S}_k \times \mathrm{GL}_2(\mathbb{C}))$-equivariant. Finally, note that for each connected component of $\mathcal{H}_k^\Gamma$, the restriction of $\mathscr{P}^k$ defines a vector bundle. The fibers of this vector bundle are then $(\mathfrak{S}_k \times \Gamma)$-modules.
Let us finish this section by introducing common notation on representations of finite groups. Let G be a finite group. Denote by $\mathcal{R}(G)$ the Grothendieck ring of the category of finite-dimensional $\mathbb{C}G$-modules and $\mathcal{R}^{\mathrm{gr}}(G)$ the Grothendieck ring of the category of graded finite-dimensional $\mathbb{C}G$-modules. For V a given $\mathbb{C}G$-module (graded $\mathbb{C}G$-module), let $[V]_G$ ($[V]_G^{\mathrm{gr}}$), or just $[V]$ ($[V]^{\mathrm{gr}}$) when there is no possible confusion on G denote the element in $\mathcal{R}(G)$ ($\mathcal{R}^{\mathrm{gr}}(G)$) associated with V.

7.2 Reduction theorem

Let us fix $d \in \mathcal{A}_\Gamma^n$. To improve readability, denote by $r_d := \mathrm{wt}(d)$. Consider $d_0 = d - r_d\delta^\Gamma$. Denote by $\mathrm{g}_\Gamma := |d_0|_\Gamma$. By construction $\mathrm{wt}(d_0) = 0$. The connected component $\mathcal{H}_{\mathrm{g}_\Gamma}^{\Gamma,\xi_{d_0}}$ is of dimension 0 thanks to Proposition 3.11. Let us denote by I_{d_0} the unique ideal of $\mathbb{C}[x,y]$ that is in $\mathcal{H}_{\mathrm{g}_\Gamma}^{\Gamma,\xi_{d_0}} \subset \mathcal{H}_{\mathrm{g}_\Gamma}^\Gamma$. Denote by U^f the following open subset of $(\mathbb{C}^2)^{r_d}$

$$\{(p_1,...,p_{r_d}) \in (\mathbb{C}^2 \setminus \{(0,0)\})^{r_d} \big| \forall (i,j) \in [\![1,r_d]\!]^2, i \neq j \Rightarrow \Gamma p_i \cap \Gamma p_j = \emptyset\}$$

Let $D_{d_0} := \{I_{d_0} \cap \bigcap_{j=1}^{r_d} I(\Gamma p_j) \subset \mathbb{C}[x,y] \big| (p_1,...p_{r_d}) \in U^f\}$. For a commutative ring R, if M is a R-module, denote by $\mathrm{Ann}_R(M) := \{r \in R | \forall m \in M, r.m = 0\}$ and by $\mathrm{Supp}_R(M) = \{p \in \mathrm{Spec}(R) | M_p \neq 0\}$.

Lemma 7.1. *The image of I_{d_0} under $\sigma_{\mathrm{g}_\Gamma}$ is the point $\overline{0} \in (\mathbb{C}^2)^{\mathrm{g}_\Gamma}/\mathfrak{S}_{\mathrm{g}_\Gamma}$.*

Proof. Consider the diagonal $\mathbb{C}^*$-action on $\mathbb{C}^2$

$$\forall (t,(x,y)) \in \mathbb{C}^* \times \mathbb{C}^2, t.(x,y) := (tx,ty)$$

This action induces a $\mathbb{C}^*$-action on $\mathcal{H}_{\mathrm{g}_\Gamma}$ which commutes with the Γ-action. The Hilbert-Chow morphism $\sigma_{\mathrm{g}_\Gamma}$ becomes $\mathbb{C}^*$-equivariant. The fact that $\mathbb{C}^*$ is connected and that the irreducible component $\mathcal{H}_{\mathrm{g}_\Gamma}^{\Gamma,\xi_{d_0}}$ equals $\{I_{d_0}\}$ implies that I_{d_0} is a $\mathbb{C}^*$-fixed point. This ideal must then be mapped by $\sigma_{\mathrm{g}_\Gamma}$ to a $\mathbb{C}^*$-fixed point of $(\mathbb{C}^2)^{\mathrm{g}_\Gamma}/\mathfrak{S}_{\mathrm{g}_\Gamma}$ which gives the result. □

Lemma 7.2. *The set D_{d_0} is a dense open subset of $\mathcal{H}_n^{\Gamma,\xi_d}$.*

Proof. Take $I = I_{d_0} \cap \bigcap_{j=1}^{r_d} I(\Gamma p_j) \in D_{d_0}$. Lemma 7.1 implies that $V(I_{d_0}) = 0$. We then have that $\forall j \in [\![1, r_d]\!]$, $V(I_{d_0}) \cap \Gamma p_i = \emptyset$, which gives an isomorphism of Γ-modules

$$\mathbb{C}[x,y]/I \simeq \mathbb{C}[x,y]/I_{d_0} \oplus \bigoplus_{j=1}^{r_d} \mathbb{C}[x,y]/I(\Gamma p_j)$$

This isomorphism shows that I is of codimension $g_\Gamma + r_d|\Gamma| = n$ and that the character of the Γ-module $\mathbb{C}[x,y]/I$ is ξ_d. We then have that $D_{d_0} \subset \mathcal{H}_n^{\Gamma,\xi_d}$. Consider now the open subset of $(\mathbb{C}^2)^{r_d|\Gamma|}$

$$\{(\Gamma p_1, ..., \Gamma p_{r_d}) \in \mathbb{C}^{2r_d|\Gamma|} \big| (p_1, ..., p_{r_d}) \in U^f\}$$

It is connected since U^f is connected. This implies that D_{d_0} is a connected open subset of $\mathcal{H}_n^{\Gamma,\xi_d}$. Moreover D_{d_0} has dimension $2r_d$ which implies that $\overline{D_{d_0}} = \mathcal{H}_n^{\Gamma,\xi_d}$. □

Fix $(p_1, ..., p_{r_d}) \in U^f$. Denote by J the ideal $\bigcap_{j=1}^{r_d} I(\Gamma p_j)$ and by $I_d := I_{d_0} \cap J \in \mathcal{H}_n^{\Gamma,\xi_d}$. Let $q := (\Gamma p_1, ..., \Gamma p_{r_d}) \in (\mathbb{C}^2)^{r_d|\Gamma|}$ and $p := (0, q)$ which is in $\pi_n^{-1}(\sigma_n(I_d)) \subset (\mathbb{C}^2)^n$. Denote by S_p the stabilizer of p in $\mathfrak{S}_n \times \Gamma$. To improve readability, denote

- $x^0 := (I_{d_0}, 0) \in \mathcal{X}_{g_\Gamma}$
- $x^q := (J, q) \in \mathcal{X}_{r_d|\Gamma|}$
- $x^p := (I_d, p) \in \mathcal{X}_n$
- $x^{(0,q)} := ((I_{d_0}, 0), (J, q)) \in \mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|}$

Fix moreover $I_d \in U^d \subset \mathcal{H}_n$ and $I_{d_0} \in U^{d^0} \subset \mathcal{H}_{g_\Gamma}$ two affine open subsets. Since S_p is a finite group and I_d is fixed by S_p, we can suppose that U^d is S_p-stable. Denote by $A^d := \mathcal{O}_{\mathcal{H}_n}(U^d)$ and by $A^{d_0} := \mathcal{O}_{\mathcal{H}_{g_\Gamma}}(U^{d_0})$. Moreover ρ_n and ρ_{g_Γ} are finite morphisms and in particular, these two morphisms are affine. Denote by $B^d := \mathcal{O}_{\mathcal{X}_n}(\rho_n^{-1}(U^d))$ and by $B^{d_0} := \mathcal{O}_{\mathcal{X}_{g_\Gamma}}(\rho_{g_\Gamma}^{-1}(U^{d_0}))$.

Remark 7.3. By construction, J is an element of $\mathcal{H}_{r_d|\Gamma|}^{\Gamma}$.

Lemma 7.4. *Let G be a finite group acting on an affine variety V over $\mathbb{C}$. Let M be a finite dimensional $\mathbb{C}[V] \rtimes G$-module such that $\mathrm{Supp}_{\mathbb{C}[V]}(M)$ is a single G-orbit. Let $q \in \mathrm{Supp}_{\mathbb{C}[V]}(M)$ and denote by G_q the stabilizer of q in G. We then have an isomorphism of $(\mathbb{C}[V] \rtimes G)$-modules*

$$M \simeq \mathrm{Ind}_{\mathbb{C}[V]\rtimes G_q}^{\mathbb{C}[V]\rtimes G}(M_q)$$

Proof. The module M being of finite dimension, we can apply [Eis95, Theorem 2.13] to obtain $\tilde{\phi} : M \xrightarrow{\sim} \bigoplus_{p \in \mathrm{Supp}_{\mathbb{C}[V]}(M)} M_p$ an isomorphism of $\mathbb{C}[V]$-modules. For each $g \in G$, since $\mathrm{Supp}_{\mathbb{C}[V]}(M) = G.q$, consider the following isomorphism of $\mathbb{C}[V]$-modules

$$\begin{array}{ccc} M_q & \xrightarrow{\sim} & M_{g.q} \\ \frac{m}{s} & \mapsto & g.\frac{m}{s} := \frac{g.m}{g.s} \end{array}$$

We then have

$$\phi : M \xrightarrow{\sim} \bigoplus_{\bar{g} \in G/G_q} M_{\bar{g}.q}$$

Extend ϕ to an isomorphism of $(\mathbb{C}[V] \rtimes G)$-modules. Finally, define

$$\psi : \begin{array}{ccc} (\mathbb{C}[V] \rtimes G) \otimes_{\mathbb{C}[V] \rtimes G_q} M_q & \to & \bigoplus_{\bar{g} \in G/G_q} M_{\bar{g}.q} \\ f_g g \otimes \frac{m}{s} & \mapsto & f_g g . \frac{m}{s} = \frac{f_g g.m}{g.s} \end{array}$$

The morphism ψ is a well-defined isomorphism of $(\mathbb{C}[V] \rtimes G)$-modules, which gives the result by combining ψ and ϕ. □

Lemma 7.5. *Let R and S be two local noetherian $\mathbb{C}$-algebras. If $f : R \to S$ is an unramified morphism of local rings and M is an S-module that is R-semisimple, then M is S-semisimple.*

Proof. Let $\chi : R \to \mathbb{C}$ be the algebra morphism defined by the maximal ideal m_R of R. The module M being R-semisimple, $M = \{m \in M | \forall r \in R, r.m = \chi(r)m\}$. The action of S on M factors though $S/m_R S$. Thanks to [Stacks, Tag 02GF], we have that $S/m_R S$ is equal to the residue field of S, since the morphism f is unramified. The ring $S/m_R S$ is thus a semisimple ring which implies that M is S-semisimple. □

Denote by $(\mathbb{C}^2)^{r_d|\Gamma|^\circ} := \{(x_1, ..., x_{r_d|\Gamma|}) \in (\mathbb{C}^2)^{r_d|\Gamma|} | x_i \neq x_j\}$, the complement to the big diagonal. We then have

$$\begin{array}{ccc} \mathcal{X}^\circ_{r_d|\Gamma|} := f^{-1}_{r_d|\Gamma|}\big((\mathbb{C}^2)^{r_d|\Gamma|^\circ}\big) & \xrightarrow[f^\circ_{r_d|\Gamma|}]{\sim} & (\mathbb{C}^2)^{r_d|\Gamma|^\circ} \\ \rho^\circ_{r_d|\Gamma|} \big\downarrow & & \big\downarrow \pi^\circ_{r_d|\Gamma|} \\ \mathcal{H}^\circ_{r_d|\Gamma|} := \sigma^{-1}_{r_d|\Gamma|}\big((\mathbb{C}^2)^{r_d|\Gamma|^\circ}\big) & \xrightarrow[\sigma^\circ_{r_d|\Gamma|}]{\sim} & (\mathbb{C}^2)^{r_d|\Gamma|^\circ}/\mathfrak{S}_{r_d|\Gamma|} \end{array}$$

The Hilbert-Chow morphism $\sigma_{r_d|\Gamma|}$ is a resolution of singularities which implies that on the smooth locus the morphism $\sigma^\circ_{r_d|\Gamma|}$ is an isomorphism. This gives that $f^\circ_{r_d|\Gamma|}$ is also an isomorphism. Consider now

$$\tilde{h} : \begin{array}{ccc} (\mathbb{C}^2)^{r_d|\Gamma|} & \to & (\mathbb{C}^2)^n/\mathfrak{S}_n \\ x & \mapsto & \overline{(0, x)} \end{array}$$

The morphism $\tilde{h}$ is finite since it is the composition of the closed immersion $(\mathbb{C}^2)^{r_d|\Gamma|} \to (\mathbb{C}^2)^n$ with the finite morphism $(\mathbb{C}^2)^n \to (\mathbb{C}^2)^n/\mathfrak{S}_n$. In particular, $\tilde{h}$ is closed. We can then consider $h : (\mathbb{C}^2)^{r_d|\Gamma|} \to \mathrm{Im}(\tilde{h})$.

Lemma 7.6. *The morphism h is étale when restricted to $(\mathbb{C}^2)^{r_d|\Gamma|^\circ}$.*

Proof. The fact that the morphism $\tilde{h}$ is finite implies that h is finite. It is then enough to prove that h is smooth. Denote by $\mathbb{Z}^+_n := \{(k_1, k_2) \in (\mathbb{Z}_{\geq 0})^2 | 0 \leq k_1 + k_2 \leq n\}$. For $(k_1, k_2) \in \mathbb{Z}^+_n$, let

$$f_{k_1,k_2}(X_1, .., X_n, Y_1, ..., Y_n) := \sum_{i=1}^n X_i^{k_1} Y_i^{k_2} \in \mathbb{C}[(\mathbb{C}^2)^n]^{\mathfrak{S}_n}$$

$$P_{k_1,k_2} := \sum_{i=\mathrm{g}_\Gamma+1}^{n} Z_i^{k_1} T_i^{k_2} \in \mathbb{C}[(\mathbb{C}^2)^n]^{\mathfrak{S}_n}[Z_{\mathrm{g}_\Gamma+1},...,Z_n,T_{\mathrm{g}_\Gamma+1},...,T_n]$$

Thanks to [Weyl, Chapter II, section 3], the set $\{f_{k_1,k_2}|(k_1,k_2)\in\mathbb{Z}_n^+\}$ is a set of generators of $\mathbb{C}[(\mathbb{C}^2)^n]^{\mathfrak{S}_n}$. Moreover, the set $\{P_{k_1,k_2}|(k_1,k_2)\in\mathbb{Z}_n^+\}$ is a set of generators of $\mathbb{C}[(\mathbb{C}^2)^{r_d|\Gamma|}]^{\mathfrak{S}_{r_d|\Gamma|}}$. By definition

$$\tilde{h}^\sharp : \begin{array}{ccc} \mathbb{C}[(\mathbb{C}^2)^n]^{\mathfrak{S}_n} & \to & \mathbb{C}[(\mathbb{C}^2)^{r_d|\Gamma|}] \\ f_{k_1,k_2} & \mapsto & P_{k_1,k_2} \end{array}$$

This gives that $\mathbb{C}[\mathrm{Im}(\tilde{h})] = \mathbb{C}[Z_{\mathrm{g}_\Gamma+1},...,Z_n,T_{\mathrm{g}_\Gamma+1},...,T_n]^{\mathfrak{S}_{r_d|\Gamma|}}$ and in particular that

$$\mathrm{Im}(\tilde{h}) \simeq (\mathbb{C}^2)^{r_d|\Gamma|}/\mathfrak{S}_{r_d|\Gamma|} \subset (\mathbb{C}^2)^n/\mathfrak{S}_n$$

This gives that the morphism h is the quotient morphism $(\mathbb{C}^2)^{r_d|\Gamma|} \to (\mathbb{C}^2)^{r_d|\Gamma|}/\mathfrak{S}_{r_d|\Gamma|}$ and that the restriction of h to $(\mathbb{C}^2)^{r_d|\Gamma|^\circ}$ is smooth. Indeed, h is finite and the $\mathfrak{S}_{r_d|\Gamma|}$-action on $(\mathbb{C}^2)^{r_d|\Gamma|^\circ}$ is free, which implies that $(\mathbb{C}^2)^{r_d|\Gamma|^\circ}/\mathfrak{S}_{r_d|\Gamma|}$ is smooth.

□

The isospectral Hilbert scheme $\mathcal{X}_n$ is a variety over $\mathcal{H}_n \times (\mathbb{C}^2)^n$. This implies that the $n!$-dimensional fiber of the vector bundle $\mathscr{P}^n_{|I_d} := \mathscr{P}^n_{I_d} \otimes_{\mathcal{O}_{\mathcal{H},I_d}} \kappa_{\mathcal{H}_n}(I_d)$ is a $\mathbb{C}[(\mathbb{C}^2)^n]$-module. It is moreover an $\mathfrak{S}_n \times \Gamma$-module. This endows $\mathscr{P}^n_{|I_d}$ with a structure of $\big(\mathbb{C}[(\mathbb{C}^2)^n] \rtimes (\mathfrak{S}_n \times \Gamma)\big)$-module. Let us now construct $\Phi : \mathcal{O}_{\mathcal{X}_n,x^p} \twoheadrightarrow (\mathscr{P}^n_{|I_d})_p$ a surjective morphism of rings. This can be done locally around I_d. Let $m_{I_d} \in \mathrm{Spec}(A^d)$ be the maximal ideal of A^d corresponding to I_d and $m_{x^p} \in \mathrm{Spec}(B^d)$ be the maximal ideal of B^d corresponding to x^p. Now the stalk $\mathscr{P}^n_{I_d} \simeq B^d \otimes_{A^d} A^d_{m_{I_d}}$. Moreover, the fiber of the associated vector bundle is isomorphic to $\mathscr{P}^n_{|I_d} \simeq \mathscr{P}^n_{I_d} \otimes_{A^d_{m_{I_d}}} A^d_{m_{I_d}}/m_{I_d}A^d_{m_{I_d}}$ which is then isomorphic to $B^d/m_{I_d}B^d$. The localization of $(\mathscr{P}^n_{|I_d})$ at the maximal ideal associated with p in $\mathbb{C}[(\mathbb{C}^2)^n]$ is isomorphic to $B^d/m_{I_d}B^d \otimes_{B^d} B^d_{x^p} \simeq B^d_{x^p}/m_{I_d}B^d_{x^p}$. We finally have $\mathcal{O}_{\mathcal{X}_n,x^p} \simeq B^d_{x^p}$ which makes the construction of the desired morphism very natural. Indeed, it is the quotient map $B^d_{x^p} \twoheadrightarrow B^d_{x^p}/m_{I_d}B^d_{x^p}$.
Furthermore, let us denote by V the following open set of $(\mathbb{C}^2)^n$

$$\{(s_1,...,s_{\mathrm{g}_\Gamma},\Gamma t_1,...,\Gamma t_{r_d}) \in (\mathbb{C}^2)^n \big| \forall (i,j,\gamma) \in [\![1,\mathrm{g}_\Gamma]\!] \times [\![1,r_d]\!] \times \Gamma, s_i \neq \gamma.t_j\}$$

Applying [H01, Lemma 3.3.1], we have

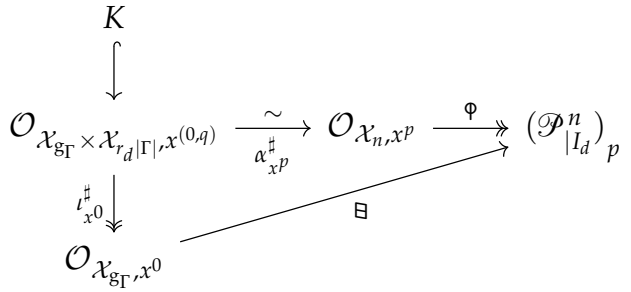
$$\beta : \begin{array}{ccc} (f_{\mathrm{g}_\Gamma} \times f_{r_d|\Gamma|})^{-1}(V) & \to & f_n^{-1}(V) \\ ((I,u),(I',u')) & \mapsto & (I \cap I',(u,u')) \end{array}$$

which is an isomorphism over $(\mathbb{C}^2)^n$. Denote by $\alpha : f_n^{-1}(V) \xrightarrow{\sim} (f_{\mathrm{g}_\Gamma} \times f_{r_d|\Gamma|})^{-1}(V)$ the inverse morphism of β. By construction, $p \in V$. The isomorphism α induces an isomorphism of local rings

$$\alpha^\sharp_{x^p} : \mathcal{O}_{\mathcal{X}_{\mathrm{g}_\Gamma} \times \mathcal{X}_{r_d|\Gamma|},x^{(0,q)}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_n,x^p}$$

Denote $\iota_{g_\Gamma} : \mathcal{X}_{g_\Gamma} \to \mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|}$ the morphism that, set theoretically, maps $(I,u) \in \mathcal{X}_{g_\Gamma}$ to $((I,u),(J,q)) \in \mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|}$. The morphism ι is a closed immersion. On the level of stalks, we have $\iota^\sharp_{x^0} : \mathcal{O}_{\mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|}, x^{(0,q)}} \twoheadrightarrow \mathcal{O}_{\mathcal{X}_{g_\Gamma}, x^0}$. Let us denote by K the kernel of this map.

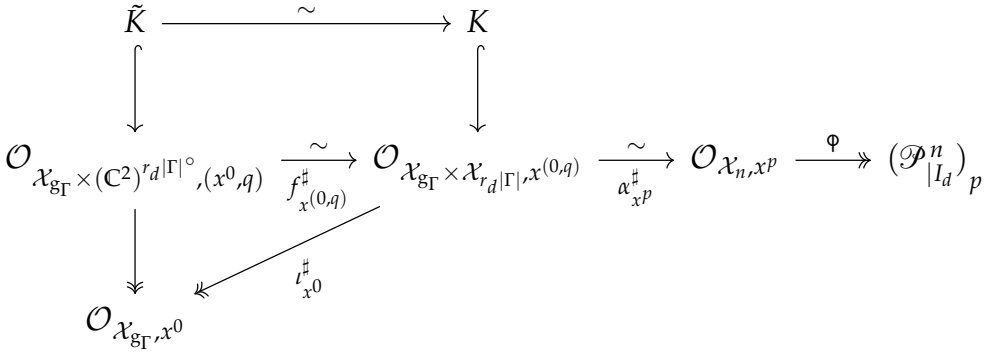
Proposition 7.7. *There exists a morphism $\boxminus : \mathcal{O}_{\mathcal{X}_{g_\Gamma}, x^0} \to \left(\mathscr{P}^n_{|I_d}\right)_p$ such that the following diagram commutes*



Proof. It is enough to show that $\varphi(\alpha^\sharp_{x^p}(K)) = 0$. Since the point q is a collection of r_d free and distinct Γ-orbits, $q \in (\mathbb{C}^2)^{r_d|\Gamma|^\circ}$. We then have the following isomorphism of local rings

$$(\mathrm{id}_{\mathcal{X}_{g_\Gamma}} \times f_{r_d|\Gamma|})^\sharp_{x^{(0,q)}} : \mathcal{O}_{\mathcal{X}_{g_\Gamma} \times (\mathbb{C}^2)^{r_d|\Gamma|^\circ}, (x^0,q)} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|}, x^{(0,q)}}$$

Note that $(\mathrm{id}_{\mathcal{X}_{g_\Gamma}} \times f_{r_d|\Gamma|})$ is a morphism over $(\mathbb{C}^2)^n$. To keep the notation concise, we will denote the preceding isomorphism by $f^\sharp_{x^{(0,q)}}$. This new piece of information gives



If we denote for all $i \in [\![1,n]\!]$, X_i and Y_i the coordinate functions on $(\mathbb{C}^2)^n$, then

$$\tilde{K} = \langle X_{g_\Gamma+1} - X_{g_\Gamma+1}(q), Y_{g_\Gamma+1} - Y_{g_\Gamma+1}(q), ..., X_n - X_n(q), Y_n - Y_n(q) \rangle$$

Let us denote $X_i - X_i(q)$ by $\tilde{X}_i$ and $Y_i - Y_i(q)$ by $\tilde{Y}_i$. Now the kernel K is equal to

$$\langle f^\sharp_{r_d|\Gamma|}(\tilde{X}_{g_\Gamma+1}), f^\sharp_{r_d|\Gamma|}(\tilde{Y}_{g_\Gamma+1}), ..., f^\sharp_{r_d|\Gamma|}(\tilde{X}_n), f^\sharp_{r_d|\Gamma|}(\tilde{Y}_n) \rangle$$

Proving that $\varphi(\alpha^\sharp_{x^p}(K)) = 0$ amounts to showing that for all $i \in [\![g_\Gamma + 1, n]\!]$

$$\varphi\left(\alpha^\sharp_{x^p}(f^\sharp_{r_d|\Gamma|}(X_i))\right) = X_i(q)$$
$$\varphi\left(\alpha^\sharp_{x^p}(f^\sharp_{r_d|\Gamma|}(Y_i))\right) = Y_i(q)$$

Let us focus on the previous picture. Zooming in the left part gives

$$\begin{array}{ccc}
\mathcal{O}_{\mathcal{X}_{\mathrm{g}_\Gamma}\times(\mathbb{C}^2)^{r_d|\Gamma|^\circ},(x^0,q)} & \longleftarrow & \mathbb{C}[(\mathbb{C}^2)^{r_d|\Gamma|^\circ}]_q \\
\wr\downarrow & & \uparrow \\
\mathcal{O}_{\mathcal{X}_n,x^p} & \xleftarrow[f^\sharp_{x^p}]{} & \mathbb{C}[(\mathbb{C}^2)^n]_p \\
\uparrow{\scriptstyle \rho^\sharp_{x^p}} & & \pi^\sharp_p\uparrow \\
\mathcal{O}_{\mathcal{H}_n,I_d} & \xleftarrow{\sigma^\sharp_{I_d}} & \mathbb{C}[(\mathbb{C}^2)^n]_p^{\mathfrak{S}_n}
\end{array}$$

The upper square commutes because the isomorphism is coming from an isomorphism over $(\mathbb{C}^2)^n$. Now zooming on the right gives

$$\begin{array}{ccccc}
\mathcal{O}_{\mathcal{H}_n,I_d} & \xrightarrow{\rho^\sharp_{x^p}} & \mathcal{O}_{\mathcal{X}_n,x^p} & \xrightarrow{\varphi} & \left(\mathscr{P}^n_{|I_d}\right)_p \\
& \searrow & & \nearrow & \\
& & \kappa_{\mathcal{H}_n}(I_d) & &
\end{array}$$

The fact that the preceding diagram commutes is clear once we come back to the description in terms of affine open subsets

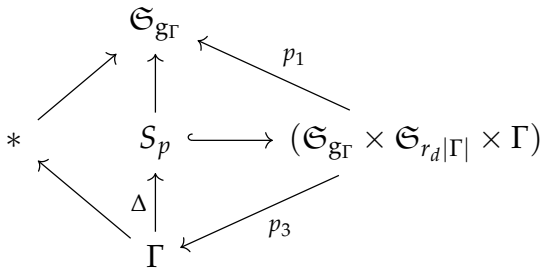
$$\begin{array}{ccccc}
A^d_{m_{I_d}} & \xrightarrow{\rho^\sharp_{x^p}} & B^d_{x^p} & \xrightarrow{\varphi} & B^d_{x^p}/m_{I_d}B^d_{x^p} \\
& \searrow & & \nearrow & \\
& & A^d_{m_{I_d}}/m_{I_d}A^d_{m_{I_d}} & &
\end{array}$$

The ring $\mathcal{O}_{\mathcal{H}_n,I_d}$ then acts on $\left(\mathscr{P}^n_{|I_d}\right)_p$ via $\kappa_{\mathcal{H}_n}(I_d) \simeq A^d_{m_{I_d}}/m_{I_d}A^d_{m_{I_d}}$. In particular $\left(\mathscr{P}^n_{|I_d}\right)_p$ is a semisimple $\mathbb{C}[(\mathbb{C}^2)^n]_p^{\mathfrak{S}_n}$-module since the action of the ring $\mathbb{C}[(\mathbb{C}^2)^n]_p^{\mathfrak{S}_n}$ is defined using $\sigma^\sharp_{I_d}$. Thanks to Lemma 7.6, we know that the restriction of h to $(\mathbb{C}^2)^{r_d|\Gamma|^\circ}$ is étale which in particular implies that this morphism is unramified. Now Lemma 7.5 with $R = \mathbb{C}[(\mathbb{C}^2)^n]_p$ and $S = \mathbb{C}[(\mathbb{C}^2)^{r_d|\Gamma|^\circ}]_q$ implies that $\left(\mathscr{P}^n_{|I_d}\right)_p$ is a $\mathbb{C}[(\mathbb{C}^2)^{r_d|\Gamma|^\circ}]_q$-semisimple module. Finally, since $\left(\mathscr{P}^n_{|I_d}\right)_p$ is a finite dimensional $\mathbb{C}[(\mathbb{C}^2)^n]$-module supported at p, using [BComAl, II, §4, no. 4, Corollary 1], we have that for all $i \in [\![1,n]\!]$ the endomorphisms of $\left(\mathscr{P}^n_{|I_d}\right)_p$ given by the action of $(X_i - X_i(p))$ and of $(Y_i - Y_i(p))$ are nilpotent. In particular, we have for all $i \in [\![1,n]\!]$ that the endomorphisms of $\left(\mathscr{P}^n_{|I_d}\right)_p$ given by the action of $(f_n^\sharp(X_i) - X_i(p))$ and of $(f_n^\sharp(Y_i) - Y_i(p))$ are nilpotent. Combining semisimplicity with nilpotency gives the result. □

Remark 7.8. Note that ⊟ is automatically surjective.

Lemma 7.9. *The morphism* $\boxminus : \mathcal{O}_{\mathcal{X}_{\mathrm{g}_\Gamma},x^0} \to \left(\mathscr{P}^n_{|I_d}\right)_p$ *is* S_p*-equivariant.*

Proof. By definition of p we know that S_p, the stabilizer of p in $\mathfrak{S}_n \times \Gamma$, is a subgroup of $\mathfrak{S}_{\mathrm{g}_\Gamma} \times \mathfrak{S}_{r_d|\Gamma|} \times \Gamma$. Let $\Delta : \Gamma \to S_p$ be the morphism of groups such that the following diagram commutes



where $*$ is the trivial group. Note that this diagram implies that $\Delta(\Gamma)$ is a subgroup of $(\mathfrak{S}_{r_d|\Gamma|} \times \Gamma)$. We then have a group isomorphism

$$\begin{array}{rccc} & \mathfrak{S}_{g_\Gamma} \times \Gamma & \xrightarrow{\sim} & S_p \\ \mp : & (\sigma, \gamma) & \mapsto & \sigma\Delta(\gamma) \\ & (x_1, \gamma) & \hookleftarrow & (x_1, x_2, \gamma) \end{array}$$

For each $(\sigma, \gamma) \in \mathfrak{S}_{g_\Gamma} \times \Gamma$ and for each $((I, u), (J, u')) \in \mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|}$, define

$$(\sigma, \gamma).((I, u), (I', u')) := ((\gamma.I, \sigma\gamma u), (\gamma.I', \Delta(\gamma)u'))$$

This endows the variety $\mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|}$ with an $(\mathfrak{S}_{g_\Gamma} \times \Gamma)$-action. The morphism ι_{g_Γ} is naturally S_p-equivariant since $J \in \mathcal{H}^\Gamma_{r_d|\Gamma|}$ and q is $\Delta(\Gamma)$-invariant. By construction, we have that the open set V of $(\mathbb{C}^2)^n$ is S_p-stable. By definition of Δ, β is S_p-equivariant which implies that $\alpha^\sharp_{x^p}$ is S_p-equivariant. Finally, the fact that U^d has been taken S_p-stable and the fact that I_d is $(\mathfrak{S}_n \times \Gamma)$-fixed, implies that Φ is S_p-equivariant. In the end, we have that $\boxminus : \mathcal{O}_{\mathcal{X}_{g_\Gamma}, x^0} \twoheadrightarrow (\mathscr{P}^n_{|I_d})_p$ is S_p-equivariant. □

Denote by $m_{I_{d_0}} \in \mathrm{Spec}(A^{d_0})$ the maximal ideal corresponding to I_{d_0}.

Lemma 7.10. *If $(\mathscr{P}^n_{|I_d})_p{}^{\mathfrak{S}_{g_\Gamma}}$ is a 1-dimensional vector space, then the ideal $m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma}, x^0}$ is contained in $\mathrm{Ann}_{\mathcal{O}_{\mathcal{X}_{g_\Gamma}, x^0}}\big((\mathscr{P}^n_{|I_d})_p\big)$.*

Proof. To show that $m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma}, x^0} \subset \mathrm{Ann}_{\mathcal{O}_{\mathcal{X}_{g_\Gamma}, x^0}}\big((\mathscr{P}^n_{|I_d})_p\big)$, it is enough to show that the ideal $\mathrm{Ann}_{A^{d_0}}\big((\mathscr{P}^n_{|I_d})_p\big)$ is maximal since the A^{d_0}-module $(\mathscr{P}^n_{|I_d})_p$ is supported at I_{d_0}. Denote by $e \in (\mathscr{P}^n_{|I_d})_p$ the identity element of this ring. By hypothesis, we have that $(\mathscr{P}^n_{|I_d})_p{}^{\mathfrak{S}_{g_\Gamma}}$ is equal to $\mathbb{C}.e$. Moreover, $(\mathscr{P}^n_{|I_d})_p{}^{\mathfrak{S}_{g_\Gamma}}$ is an A^{d_0}-submodule of $(\mathscr{P}^n_{|I_d})_p$ since the group $\mathfrak{S}_{g_\Gamma}$ acts trivially on A^{d_0}. We then have that $\mathrm{Ann}_{A^{d_0}}\big((\mathscr{P}^n_{|I_d})_p\big) = \mathrm{Ann}_{A^{d_0}}(\mathbb{C}.e)$. Finally, this gives that $\mathrm{Ann}_{A^{d_0}}\big((\mathscr{P}^n_{|I_d})_p\big)$ is maximal since the annihilator of a simple module is always maximal. □

Theorem 7.11. *For each $I \in \mathcal{H}^{\Gamma, \xi_d}_n$*

$$[\mathscr{P}^n_{|I}]_{\mathfrak{S}_n \times \Gamma} = [\mathrm{Ind}^{\mathfrak{S}_n \times \Gamma}_{S_p}(\mathscr{P}^{g_\Gamma}_{|I_{d_0}})]_{\mathfrak{S}_n \times \Gamma}$$

Proof. The variety $\mathcal{H}^{\Gamma, \xi_d}_n$ being an irreducible component of $\mathcal{H}^\Gamma_n$ on which $\mathfrak{S}_n \times \Gamma$ acts trivially, it is enough to prove this equality for $I = I_d$. The support of $\mathscr{P}^n_{|I_d}$ as a $\mathcal{O}_{\mathcal{X}_n}$-module is $\{(I_d, x) \in \mathcal{X}_n | \pi_n(x) = \sigma_n(I_d)\} = \rho_n^{-1}(I_d)\}$. Using [BComAl, II, §4, no. 4, Proposition 19], we have $\mathrm{Supp}_{\mathbb{C}[(\mathbb{C}^2)^n]}(\mathscr{P}^n_{|I_d}) = f_n(\rho^{-1}(I_d))$. The support of $\mathscr{P}^n_{|I_d}$ as a

$\mathbb{C}[(\mathbb{C}^2)^n]$-module is in particular an $\mathfrak{S}_n$-orbit which is Γ-stable, thus an $(\mathfrak{S}_n \times \Gamma)$-orbit. Thanks to Lemma 7.4, we have

$$[\mathscr{P}^n_{|I_d}]_{\mathfrak{S}_n \times \Gamma} = [\mathrm{Ind}_{S_p}^{\mathfrak{S}_n \times \Gamma}\big((\mathscr{P}^n_{|I_d})_p\big)]_{\mathfrak{S}_n \times \Gamma} \tag{7.1}$$

It remains to show that $[(\mathscr{P}^n_{|I_d})_p]_{S_p} = [\mathscr{P}^{g_\Gamma}_{|I_{d_0}}]_{S_p}$. Combining Proposition 7.7 and Lemma 7.9, we have $\boxminus : \mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0} \twoheadrightarrow (\mathscr{P}^n_{|I_d})_p$ an S_p-equivariant surjective morphism such that this diagram commutes

$$\begin{array}{ccccc} \mathcal{O}_{\mathcal{X}_{g_\Gamma} \times \mathcal{X}_{r_d|\Gamma|},x^{(0,q)}} & \xrightarrow[\alpha^\sharp_{x^p}]{\sim} & \mathcal{O}_{\mathcal{X}_n,x^p} & \xrightarrow{\varphi} & (\mathscr{P}^n_{|I_d})_p \\ \downarrow{\scriptstyle \iota^\sharp_{x^0}} & & & \nearrow{\scriptstyle \boxminus} & \\ \mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0} & & & & \end{array}$$

Equality (7.1) gives that $\mathrm{Ind}_{\mathfrak{S}_{g_\Gamma}}^{\mathfrak{S}_n}\big((\mathscr{P}^n_{|I_d})_p\big)$ is isomorphic to the regular representation of $\mathfrak{S}_n$. In particular we have that $(\mathscr{P}^n_{|I_d})_p{}^{\mathfrak{S}_{g_\Gamma}}$ is 1-dimensionnal. Now thanks to Lemma 7.10, we have $m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0} \subset \mathrm{Ann}_{\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}}\big((\mathscr{P}^n_{|I_d})_p\big)$. Consider the morphism $\hat{\boxminus}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0} & \xrightarrow{\boxminus} & (\mathscr{P}^n_{|I_d})_p \\ \downarrow & \nearrow{\scriptstyle \hat{\boxminus}} & \\ \mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}/\mathrm{Ann}_{\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}}\big((\mathscr{P}^n_{|I_d})_p\big) & & \end{array}$$

The definition of $\mathrm{Ann}_{\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}}\big((\mathscr{P}^n_{|I_d})_p\big)$ directly gives that $\hat{\boxminus}$ is an isomorphism of S_p-modules. Moreover, the ring $\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}/m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}$ is isomorphic to $(\mathscr{P}^{g_\Gamma}_{|I_{d_0}})_0$. If we let S_p act on $\mathbb{C}[(\mathbb{C}^2)^{g_\Gamma}]$ using $\mp$, we can use [Eis95, Theorem 2.13] to obtain an isomorphism of $(\mathbb{C}[(\mathbb{C}^2)^{g_\Gamma}] \rtimes S_p)$-modules between $\mathscr{P}^{g_\Gamma}_{|I_{d_0}}$ and $(\mathscr{P}^{g_\Gamma}_{|I_{d_0}})_0$, since the $\mathbb{C}[(\mathbb{C}^2)^{g_\Gamma}]$-module $\mathscr{P}^{g_\Gamma}_{|I_{d_0}}$ is supported at 0. In particular, we have that $\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}/m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}$ is of dimension g_Γ!. Moreover, thanks to equality (7.1) we have that the dimension of $(\mathscr{P}^n_{|I_d})_p$ is also g_Γ!. Now, the fact that $m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0} \subset \mathrm{Ann}_{\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}}\big((\mathscr{P}^n_{|I_d})_p\big)$ and that $(\mathscr{P}^n_{|I_d})_p$ and $\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}/m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}$ have the same dimension, implies that

$$m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0} = \mathrm{Ann}_{\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}}\big((\mathscr{P}^n_{|I_d})_p\big)$$

In the end, we have that $\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}/m_{I_{d_0}}\mathcal{O}_{\mathcal{X}_{g_\Gamma},x^0}$ and $(\mathscr{P}^n_{|I_d})_p$ are isomorphic as S_p-modules, which concludes the proof. □

7.3 Type A study

In this section, let us explore what Theorem 7.11 gives when Γ is of type A. Fix an integer $l \geq 1$. Recall that ζ_l denotes the primitive l^{th} root of unity $e^{\frac{2i\pi}{l}}$, and that $\omega_l \in \mathrm{SL}_2(\mathbb{C})$ is the diagonal matrix $\mathrm{diag}(\zeta_l, \zeta_l^{-1})$. The cyclic subgroup of order l in

$\mathrm{SL}_2(\mathbb{C})$ is $\mu_l = <\omega_l>$. Assume, in this section, that $\Gamma = \mu_l$. For a partition $\lambda \in \mathcal{P}$, denote by $\gamma_l(\lambda)$ the l-core associated with λ. Denote by $\mathrm{g}_l(\lambda) := |\gamma_l(\lambda)|$ and by $r_l(\lambda) := \frac{|\lambda| - \mathrm{g}_l(\lambda)}{l}$ the number of hooks of length l that we need to remove from λ to obtain $\mathrm{g}_l(\lambda)$. In addition, let $n(\lambda)$ denote the following quantity $\sum_{i=1}^{|\lambda|} (i - 1)\lambda_i$. Recall that $h_c(\lambda)$ denotes the length of the hook $H_c(\lambda)$, for $c \in \mathcal{Y}(\lambda)$. Denote by τ_l the character of μ_l such that $\tau_l(\omega_l) = \zeta_l$. Denote by $w_{l,n}(\lambda) \in \mathfrak{S}_n$ the product of the $r_l(\lambda)$ cycles of length l

$$(\mathrm{g}_l(\lambda) + 1, ..., \mathrm{g}_l(\lambda) + l)...(n - l + 1, ..., n)$$

and by $C_{l,n}(\lambda)$ the cyclic subgroup of $\mathfrak{S}_{r_l l}$ generated by $w_{l,n}(\lambda)$. Consider also the subgroup of $\mathfrak{S}_n$, $W_{l,n}^{\mathrm{g}_l} := \mathfrak{S}_{\mathrm{g}_l} \times C_{l,n}(\lambda)$. Denote by θ_l the character of $C_{l,n}(\lambda)$ such that $\theta_l(w_{l,n}(\lambda)) = \zeta_l$. If λ is clear from context, we will shorten $\gamma_l(\lambda)$, $\mathrm{g}_l(\lambda)$, $r_l(\lambda)$, $w_{l,n}(\lambda)$ and $C_{l,n}(\lambda)$ to γ_l, g_l, r_l, $w_{l,n}$ and $C_{l,n}$. Let us also use the following notation. For V a given $(\mathfrak{S}_{\mathrm{g}_l} \times C_{l,n})$-module let

$$[V]_{\mathfrak{S}_{\mathrm{g}_l} \times C_{l,n}} = \sum_{j=0}^{l-1} [V_j^l \boxtimes \theta_l^j]_{\mathfrak{S}_{\mathrm{g}_l} \times C_{l,n}}$$

where V_j^l is an $\mathfrak{S}_{\mathrm{g}_l}$-module for each $j \in [\![0, l - 1]\!]$ and $\boxtimes$ denotes the external tensor product. Moreover, if λ is a partition of n, let us shorten $\mathscr{P}^n_{|I_\lambda}$, the fiber of the n^{th}-Procesi bundle at the monomial ideal I_λ to $\mathscr{P}^n_\lambda$. To state the main result of this section, we need two lemmas.

Lemma 7.12. *Let C_1 and C_2 be two groups isomorphic to μ_l. Take $c_1 \in C_1$ and $c_2 \in C_2$ generators of C_1 and C_2. If we denote respectively by τ_1, τ_2 and τ_3 the characters of respectively C_1, C_2 and $\langle (c_1, c_2) \rangle < C_1 \times C_2$ that respectively map c_1, c_2 and (c_1, c_2) to ζ_l, then*

$$\forall j \in [\![0, l - 1]\!], \mathrm{Ind}_{\langle (c_1,c_2) \rangle}^{C_1 \times C_2}(\tau_3^j) = \sum_{i=0}^{l-1} \tau_1^{j-i} \boxtimes \tau_2^i$$

Proof. Take $(p, q) \in [\![0, l - 1]\!]^2$. On the one hand, Frobenius reciprocity theorem gives

$$\begin{aligned} \langle \tau_1^p \boxtimes \tau_2^q, \mathrm{Ind}_{\langle (c_1,c_2) \rangle}^{C_1 \times C_2}(\tau_3^j) \rangle &= \langle \mathrm{Res}_{\langle (c_1,c_2) \rangle}^{C_1 \times C_2}(\tau_1^p \boxtimes \tau_2^q), \tau_3^j \rangle \\ &= \langle \tau_3^{p+q}, \tau_3^j \rangle \\ &= \delta_{p+q}^j \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{i=0}^{l-1} \langle \tau_1^p \boxtimes \tau_2^q, \tau_1^{j-i} \boxtimes \tau_2^i \rangle &= \sum_{i=0}^{l-1} \delta_{j-i}^p \delta_q^i \\ &= \delta_{j-q}^p \end{aligned}$$

□

Lemma 7.13. *Let Δ be the cyclic subgroup of $\mathfrak{S}_{r_l l} \times \mu_l$ generated by the element $(w_{l,n}, \omega_l)$. If $\hat{\theta}_l$ denotes the character of Δ such that $\hat{\theta}_l\big((w_{l,n}, \omega_l)\big) = \zeta_l$ then*

$$\forall j \in [\![0, l - 1]\!], \mathrm{Ind}_{\Delta}^{\mathfrak{S}_{r_l l} \times \mu_l}(\hat{\theta}_l^j) = \sum_{i=0}^{l-1} \mathrm{Ind}_{C_{l,n}}^{\mathfrak{S}_{r_l l}}(\theta_l^{j-i}) \boxtimes \tau_l^i$$

Proof. We have $\mathrm{Ind}_{\Delta}^{\mathfrak{S}_{r_l l}\times\mu_l}(\hat{\theta}_l^j) = \mathrm{Ind}_{C_{l,n}\times\mu_l}^{\mathfrak{S}_{r_l l}\times\mu_l}\big(\mathrm{Ind}_{\Delta}^{C_{l,n}\times\mu_l}(\hat{\theta}_l^j)\big)$. Using Lemma 7.12

$$\begin{aligned}\mathrm{Ind}_{C_{l,n}\times\mu_l}^{\mathfrak{S}_{r_l l}\times\mu_l}\big(\mathrm{Ind}_{\Delta}^{C_{l,n}\times\mu_l}(\hat{\theta}_l^j)\big) &= \sum_{i=0}^{l-1}\mathrm{Ind}_{C_{l,n}\times\mu_l}^{\mathfrak{S}_{r_l l}\times\mu_l}(\theta_l^{j-i}\boxtimes\tau_l^i)\\ &= \sum_{i=0}^{l-1}\mathrm{Ind}_{C_{l,n}}^{\mathfrak{S}_{r_l l}}(\theta_l^{j-i})\boxtimes\tau_l^i\end{aligned}$$

□

We can now state the main result of this section.

Corollary 7.14. *For each partition λ of n, we have the following decomposition of $\mathscr{P}_\lambda^n$*

$$[\mathscr{P}_\lambda^n]_{\mathfrak{S}_n\times\mu_l} = \big[\sum_{i=0}^{l-1}\mathrm{Ind}_{W_{l,n}^{\mathfrak{g}_l}}^{\mathfrak{S}_n}\big(\sum_{j=0}^{l-1}(\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\theta_l^{i-j}\big)\boxtimes\tau_l^i\big]_{\mathfrak{S}_n\times\mu_l}$$

Proof. With the notation established at the beginning of this section, the group S_p introduced in section 7.2 is equal to $\mathfrak{S}_{\mathfrak{g}_l}\times\Delta$. Thanks to Theorem 7.11, it is enough to show that

$$[\mathrm{Ind}_{S_p}^{\mathfrak{S}_n\times\mu_l}(\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})]_{\mathfrak{S}_n\times\mu_l} = \big[\sum_{i=0}^{l-1}\mathrm{Ind}_{W_{l,n}^{\mathfrak{g}_l}}^{\mathfrak{S}_n}\big(\sum_{j=0}^{l-1}(\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\theta_l^{i-j}\big)\boxtimes\tau_l^i\big]_{\mathfrak{S}_n\times\mu_l}$$

We have

$$\big[\mathrm{Ind}_{S_p}^{\mathfrak{S}_n\times\mu_l}(\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})\big]_{\mathfrak{S}_n\times\mu_l} = \big[\sum_{j=0}^{l-1}\mathrm{Ind}_{\mathfrak{S}_{\mathfrak{g}_l}\times\Delta}^{\mathfrak{S}_n\times\mu_l}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\hat{\theta}_l^j\big)\big]_{\mathfrak{S}_n\times\mu_l}$$

Moreover

$$\begin{aligned}\sum_{j=0}^{l-1}\mathrm{Ind}_{\mathfrak{S}_{\mathfrak{g}_l}\times\Delta}^{\mathfrak{S}_n\times\mu_l}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\hat{\theta}_l^j\big) &= \sum_{j=0}^{l-1}\mathrm{Ind}_{\mathfrak{S}_{\mathfrak{g}_l}\times\mathfrak{S}_{r_l l}\times\mu_l}^{\mathfrak{S}_n\times\mu_l}\big(\mathrm{Ind}_{\mathfrak{S}_{\mathfrak{g}_l}\times\Delta}^{\mathfrak{S}_{\mathfrak{g}_l}\times\mathfrak{S}_{r_l l}\times\mu_l}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\hat{\theta}_l^j\big)\big)\\ &= \sum_{j=0}^{l-1}\mathrm{Ind}_{\mathfrak{S}_{\mathfrak{g}_l}\times\mathfrak{S}_{r_l l}\times\mu_l}^{\mathfrak{S}_n\times\mu_l}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\mathrm{Ind}_{\Delta}^{\mathfrak{S}_{r_l l}\times\mu_l}(\hat{\theta}_l^j)\big)\\ &= \sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\mathrm{Ind}_{\mathfrak{S}_{\mathfrak{g}_l}\times\mathfrak{S}_{r_l l}\times\mu_l}^{\mathfrak{S}_n\times\mu_l}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\mathrm{Ind}_{C_{l,n}}^{\mathfrak{S}_{r_l l}}(\theta_l^{j-i})\boxtimes\tau_l^i\big)\end{aligned}$$

The last equality comes from Lemma 7.13. By gathering, we have

$$\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\mathrm{Ind}_{\mathfrak{S}_{\mathfrak{g}_l}\times\mathfrak{S}_{r_l l}\times\mu_l}^{\mathfrak{S}_n\times\mu_l}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\mathrm{Ind}_{C_{l,n}}^{\mathfrak{S}_{r_l l}}(\theta_l^{j-i})\boxtimes\tau_l^i\big) = \sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\mathrm{Ind}_{W_{l,n}^{\mathfrak{g}_l}}^{\mathfrak{S}_n}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\theta_l^{j-i}\big)\boxtimes\tau_l^i\big)$$

Finally, since every representation of $\mathfrak{S}_n$ is isomorphic, as $\mathfrak{S}_n$-module, to its dual, for each $(i,j)\in[\![0,l\text{-}1]\!]^2$, we have

$$[\mathrm{Ind}_{W_{l,n}^{\mathfrak{g}_l}}^{\mathfrak{S}_n}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\theta_l^{j-i}\big)]_{\mathfrak{S}_n} = [\mathrm{Ind}_{W_{l,n}^{\mathfrak{g}_l}}^{\mathfrak{S}_n}\big((\mathscr{P}_{\gamma_l}^{\mathfrak{g}_l})_j^l\boxtimes\theta_l^{i-j}\big)]_{\mathfrak{S}_n}$$

□

Remark 7.15. If one takes $\lambda = \gamma_l$, then $r_l = 0$ and $W_{l,n}^{g_l} = \mathfrak{S}_n$. In that case, Corollary 7.14 is trivially true but isn't furnishing any interesting information. Note also that Corollary 7.14 implies [BLM06, Theorem 4.6] when the complex reflection group is taken to be $\mathfrak{S}_n$ and Γ is taken to be $C_{l,n}$.

The goal of what follows, is to prove Corollary 7.14 in two special cases without using Theorem 7.11. To prove Corollary 7.14 in these two edge cases, let us use the representation theory of the symmetric group and symmetric functions. We will in particular use [BLM06, Theorem 4.6]. The irreducible representations of $\mathfrak{S}_n$ are parametrized by partitions of n. Denote respectively by V_λ and χ_λ the representation space and the character of the irreducible representation of $\mathfrak{S}_n$ associated with $\lambda \vdash n$.

Definition 7.16. Let R be any finitely generated $\mathbb{Z}$-algebra. For a given integer k, define the ring of symmetric polynomials over R as $\Lambda_R^k := R[z_1, ..., z_k]^{\mathfrak{S}_k}$. It is clear that using the degree, Λ_R^k is a graded ring. Let us denote $\Lambda_R^k = \bigoplus_{d\geq 0} \Lambda_{R,d}^k$. We have moreover a ring morphism $\pi^k : \Lambda_R^{k+1} \to \Lambda_R^k$ by mapping x_{k+1} to 0. For each integer d, the morphism π^k restricts to a morphism $\pi_d^k : \Lambda_{R,d}^{k+1} \to \Lambda_{R,d}^k$ of R-modules. We can now define the graded R-algebra of symmetric functions.

$$\Lambda_R := \bigoplus_{d\geq 0} \varprojlim \Lambda_{R,d}^k$$

In the following, let us shorten $\Lambda_{\mathbb{Z}}$ to Λ. Let us recall the notation concerning symmetric functions. For $\mu \in \mathcal{P}$ a given partition, denote by p_μ and s_μ respectively the power symmetric function and the Schur function associated with μ. Define now the plethystic substitution. We know that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated as a free $\mathbb{Q}$-algebra by the family $\{p_k | k \in \mathbb{Z}_{\geq 0}\}$.

Definition 7.17. Take K a, finitely generated, field extension of $\mathbb{Q}$. Take $\{s_1, \ldots, s_m\}$ a set of generators of K i.e. $K = \mathbb{Q}(s_1, \ldots, s_m)$. For $A \in \Lambda_K := \Lambda \otimes_{\mathbb{Z}} K$, and $k \in \mathbb{Z}_{\geq 0}$ define $p_k[A]$ to be the symmetric function A in the indeterminates $s_1^k, \ldots s_m^k, z_1^k, z_2^k, \ldots$. We can now extend the plethystic substitution to the following endomorphism

$$[A] : \begin{array}{ccc} \Lambda_K & \to & \Lambda_K \\ f & \mapsto & f[A] \end{array} .$$

Remark 7.18. We will often do plethystic substitutions using $Z := p_1 = \sum_{k\geq 1} z_k \in \Lambda$. Note that for all $k \geq 1$, $p_k[Z] = p_k$ and so for all $f \in \Lambda_K$, $f[Z] = f$.

If λ is a partition of n, recall that $I_\lambda \in \mathcal{H}_n$ is the monomial ideal associated with λ. The fiber $\mathscr{P}_\lambda^n$ is a bigraded $\mathfrak{S}_n$-module. Haiman introduced the transformed Macdonald symmetric functions $\tilde{H}_\lambda(z; q, t)$ [H03, Definition 3.5.2]. For $([V], [W]) \in \mathcal{R}(\mathfrak{S}_{k_1}) \times \mathcal{R}(\mathfrak{S}_{k_2})$ define the induced product $[V].[W] := [\mathrm{Ind}_{\mathfrak{S}_{k_1}\times\mathfrak{S}_{k_2}}^{\mathfrak{S}_{k_1+k_2}} (V \otimes W)]$. This product endows $\mathcal{R}(\mathfrak{S}) := \bigoplus_{k\geq 0} \mathcal{R}(\mathfrak{S}_k)$ and $\mathcal{R}^{\mathrm{gr}}(\mathfrak{S}) := \bigoplus_{k\geq 0} \mathcal{R}^{\mathrm{gr}}(\mathfrak{S}_k)$ with a structure of graded rings. Let us denote by $\mathrm{Fr} : \mathcal{R}(\mathfrak{S}) \xrightarrow{\sim} \Lambda$ the Frobenius characteristic map which is an isomorphism of graded rings. If $A := \bigoplus_{(r,s)\in\mathbb{Z}^2} A_{r,s}$ is a bigraded $\mathfrak{S}_n$-module, denote by $\mathrm{Fr}(A)$ the following element $\sum_{(r,s)\in\mathbb{Z}^2} \mathrm{Fr}(A_{r,s}) q^r t^s \in \Lambda[q^{\pm 1}, t^{\pm 1}]$.

Remark 7.19. Graded $\mathfrak{S}_n$-modules will be considered bigraded with trivial t-graduation.

Definition 7.20. Take $(F, G) \in \Lambda^2$ and note $[V] = \mathrm{Fr}^{-1}(F)$, $[W] = \mathrm{Fr}^{-1}(G)$. Define the Kronecker product as

$$F \otimes G := \mathrm{Fr}([V] \otimes [W])$$

The $n!$ theorem ([H03, Theorem 4.1.5]) can be reformulated the following way.

Proposition 7.21. *For each partition λ of n, we have* $\mathrm{Fr}([\mathscr{P}_\lambda^n]) = \tilde{H}_\lambda(z;q,t)$.

Definition 7.22. Let V be a finite-dimensional complex vector space and $G < GL(V)$ be a complex reflection group. The group G acts then also on $S(V)$ the symmetric algebra of V which is naturally graded $S(V) = \bigoplus_{i\geq 0} S^i(V)$. Let $\mathfrak{M}$ be the graded maximal ideal of $S(V)^G$. Define $S(V)^{\mathrm{co}(G)} := S(V)/\mathfrak{M}S(V)$ the coinvariant algebra of G which is then also graded. Note that as a G-module it is isomorphic to the regular representation of G by Chevalley-Shephard-Todd's theorem.

If $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is a graded vector space, then let $\dim^{\mathrm{gr}}(V) := \sum_{i\in\mathbb{Z}} \dim(V_i)q^i \in \mathbb{Z}[q^{\pm 1}]$ be the graded dimension of V. In this section, let us denote by $V_n = \mathbb{C}^n$ the reflection representation of $\mathfrak{S}_n$.

Definition 7.23. For λ a partition of n, denoted by $\lambda \vdash n$, define

$$F_\lambda(q) := \dim^{\mathrm{gr}}\big((S(V_n)^{\mathrm{co}(\mathfrak{S}_n)} \otimes V_\lambda^*)^{\mathfrak{S}_n}\big)$$

the fake degree associated with the irreducible representation V_λ of $\mathfrak{S}_n$.

Lemma 7.24. *If $\lambda \vdash n$, then the fake degree $F_\lambda(q)$ is equal to* $q^{n(\lambda)}\frac{\prod_{i=1}^n(1-q^i)}{\prod_{c\in\mathcal{Y}(\lambda)}(1-q^{h_c(\lambda)})}$.

Proof. To prove this equality we can use [St, Proposition 4.11] and [FS, Corollary 7.21.5]. □

Let us first study $[\mathscr{P}_\lambda^n]$ as a $(\mathfrak{S}_n \times \mathbb{T}_1)$-module. Using [H03, Proposition 3.5.10], we have

$$\tilde{H}_\lambda(z;q,q^{-1}) = \frac{\prod_{c\in\mathcal{Y}(\lambda)}(1-q^{h_c(\lambda)})}{q^{n(\lambda)}} s_\lambda\big[\frac{Z}{1-q}\big] \qquad (\star)$$

Lemma 7.25. *We have the following equality in $\Lambda_{\mathbb{Q}(q)}$*

$$s_\lambda\big[\frac{Z}{1-q}\big] = \frac{\mathrm{Fr}\big(S(V_n)^{\mathrm{co}(\mathfrak{S}_n)} \otimes V_\lambda\big)}{\prod_{i=1}^n(1-q^i)}$$

Proof. Let us start rewriting the plethysm

$$\begin{aligned} s_\lambda\big[\frac{Z}{1-q}\big] &= \sum_{i=0}^{+\infty} \mathrm{Fr}\big([S^i(V_n)\otimes V_\lambda]\big)q^i \\ &= \sum_{i=0}^{+\infty} \mathrm{Fr}\big([S^i(V_n)]\big)\otimes \mathrm{Fr}([V_\lambda])q^i \\ &= \sum_{i=0}^{+\infty} \mathrm{Fr}\big([S^i(V_n)]\big)q^i \otimes \mathrm{Fr}([V_\lambda]) \\ &= s_n\big[\frac{Z}{1-q}\big]\otimes \mathrm{Fr}([V_\lambda]) \end{aligned}$$

The first and the last equalities come from [H03, Proposition 3.3.1]. Proposition 7.21 for $\lambda = (n)$ gives

$$\tilde{H}_n(z;q,t) = \tilde{H}_n(z;q,q^{-1}) = \mathrm{Fr}\big([S(V_n)^{\mathrm{co}(\mathfrak{S}_n)}]\big)$$

Moreover using the equation $(\star)$, we have

$$\tilde{H}_n(z;q,q^{-1}) = \prod_{i=1}^{n}(1-q^i)s_n\big[\frac{Z}{1-q}\big]$$

Summing it up, we get

$$\begin{aligned} s_\lambda\big[\frac{Z}{1-q}\big] &= \frac{\mathrm{Fr}\big([S(V_n)^{\mathrm{co}(\mathfrak{S}_n)}]\big)}{\prod_{i=1}^{n}(1-q^i)} \otimes \mathrm{Fr}([V_\lambda]) \\ &= \frac{\mathrm{Fr}\big([S(V_n)^{\mathrm{co}(\mathfrak{S}_n)} \otimes V_\lambda]\big)}{\prod_{i=1}^{n}(1-q^i)} \end{aligned}$$

□

Proposition 7.26. *Take $\lambda \in \mathcal{P}_n$. The following two elements of $\mathcal{R}(\mathfrak{S}_n)^{\mathrm{gr}}$ are equal*

$$F_\lambda[\mathscr{P}_\lambda^n]^{\mathrm{gr}}_{\mathfrak{S}_n} = \big[S(V_n)^{\mathrm{co}(\mathfrak{S}_n)} \otimes V_\lambda\big]^{\mathrm{gr}}_{\mathfrak{S}_n}$$

If, by abuse of notation, we denote by τ_l the irreducible character $\chi_{(n)} \boxtimes \tau_l$, we then have the following equality in the $\mathbb{Z}$-algebra $\mathcal{R}(\mathfrak{S}_n) \boxtimes \mathcal{R}(\mu_l)$

$$F_\lambda(\tau_l)[\mathscr{P}_\lambda^n]_{\mathfrak{S}_n\times\mu_l} = \big[S(V_n)^{\mathrm{co}(\mathfrak{S}_n)} \otimes V_\lambda\big]_{\mathfrak{S}_n\times\mu_l}$$

Proof. Let us use Lemma 7.25 with $(\star)$ to obtain

$$\begin{aligned} \frac{q^{n(\lambda)}}{\prod_{c\in\mathcal{Y}(\lambda)}(1-q^{|h_c(\lambda)|})}\tilde{H}_\lambda(z;q,q^{-1}) &= s_\lambda\big[\frac{Z}{1-q}\big] \\ &= \frac{\mathrm{Fr}\big([S(V_n)^{\mathrm{co}(\mathfrak{S}_n)} \otimes V_\lambda]\big)}{\prod_{i=1}^{n}(1-q^i)} \end{aligned}$$

Finally, by combining Lemma 7.24 and Proposition 7.21 we have

$$F_\lambda\,\mathrm{Fr}([\mathscr{P}_\lambda^n]^{\mathrm{gr}}) = \mathrm{Fr}\big([S(V_n)^{\mathrm{co}(\mathfrak{S}_n)} \otimes V_\lambda]^{\mathrm{gr}}\big)$$

Taking the inverse Frobenius characteristic map gives the result.
For the second equality, since graded modules are $\mathbb{C}^*$-modules, we can take the pull-back by $\tau_l : \mu_l \to \mathbb{C}^*$ of the first equality. □

Let us focus now on the structure of $\mathscr{P}_\lambda^n$ as a $(\mathfrak{S}_n \times \mu_l)$-module.

7.3.1 When γ_l is very small

Denote by $\mathcal{P}^o_{n,l}$ the set of all partitions of n with l-core either empty or equal to $(1) \vdash 1$. Let us show that Corollary 7.14 holds for all $\lambda \in \mathcal{P}^o_{n,l}$.

Lemma 7.27. *For each divisor j of l, the j-core of λ is equal to the j-core of the l-core of λ.*

Proof. We can use the link between partitions and abacuses [Ol, Proposition 3.2]. Consider the j-abacus of λ. Thanks to [Ol, Proposition 1.8] we know that to obtain the j-core of λ, we need to move, in each runner, all the beads as high as possible. Notice now that with the j-abacus we can also obtain the l-core. Let $l = kj$. Again using the

result of [Ol, Proposition 1.8], let us describe a procedure to obtain the l-core out of the j-abacus of λ. If $i \in [\![0, j\text{ - }1]\!]$, then the level of a position in the j-abacus $aj+i$ is defined to be the integer a and the length of a movement of a bead from a position $a_1 j + i$ to a position $a_2 j + i$ is defined to be $a_1 - a_2$. Now the l-core of λ is obtained by moving all beads, in each runner, as high as possible only with movements of length k. We then have that the j-core of λ is equal to the j-core of the l-core of λ. □

Lemma 7.28. *For each $\lambda \in \mathcal{P}^o_{n,l}$, and each $k \in [\![0, l\text{ - }1]\!]$, $F_\lambda(\zeta_l^k) \neq 0$.*

Proof. Take j a divisor of l and denote by Φ_j the jth cyclotomic polynomial. It is then enough to show that $v_{\Phi_j}(F_\lambda) = 0$ where $v_{\Phi_j} : \mathbb{Q}(q) \to \mathbb{Z}$ is the Φ_j-valuation. With Lemma 7.24, we have

$$v_{\Phi_j}(F_\lambda) = \#\{i \in [\![1, n]\!] \big| i \equiv 0[j]\} - \#\{c \in \mathcal{Y}(\lambda) \big| |h_c(\lambda)| \equiv 0[j]\}$$

Now, [Ol, Proposition 3.6] gives the result for $j = l$. If j is a divisor of l, we can again apply of [Ol, Proposition 3.6] to the j-core of λ which is just the j-core of γ_l via Lemma 7.27. □

Proposition 7.29. *If $\lambda \in \mathcal{P}^o_{n,l}$, then $[\mathrm{Res}^{\mathfrak{S}_n}_{C_{l,n}}(V_\lambda)] = [F_\lambda(\theta_l^{-1})]$.*

Proof. Consider $v_{l,n} := \begin{cases} (\zeta_l^{l-1}, ..., \zeta_l, 1, 2\zeta_l^{l-1}, ..., 2, ..., r_l\zeta_l^{l-1}, ..., r_l) \in V_n & \text{if } \gamma_l = \emptyset \\ (0, \zeta_l^{l-1}, ..., \zeta_l, 1, 2\zeta_l^{l-1}, ..., 2, ..., r_l\zeta_l^{l-1}, ..., r_l) \in V_n & \text{if } \gamma_l = (1) \end{cases}$. It is then clear that the stabilizer of $v_{l,n}$ in $\mathfrak{S}_n$ is the trivial group. Moreover $v_{l,n}$ is an eigenvector of $w_{l,n}$ with eigenvalue ζ_l. We can now apply [Spr, Proposition 4.5] to obtain the result. □

We are now able to prove Corollary 7.14 for all $\lambda \in \mathcal{P}^o_{n,l}$.

Proposition 7.30. *For each partition $\lambda \in \mathcal{P}^o_{n,l}$*

$$[\mathscr{P}^n_\lambda]_{\mathfrak{S}_n \times \mu_l} = [\sum_{i=0}^{l-1} \mathrm{Ind}^{\mathfrak{S}_n}_{C_{l,n}}(\theta_l^i) \boxtimes \tau_l^i]_{\mathfrak{S}_n \times \mu_l}$$

Proof. Let us start with [MN, Theorem 8] which can be reformulated in the following way

$$[S(V_n)^{\mathrm{co}(\mathfrak{S}_n)}]_{\mathfrak{S}_n \times \mu_l} = [\mathscr{P}^n_{(n)}]_{\mathfrak{S}_n \times \mu_l} = [\sum_{i=0}^{l-1} \mathrm{Ind}^{\mathfrak{S}_n}_{C_{l,n}}(\theta_l^i) \boxtimes \tau_l^i]_{\mathfrak{S}_n \times \mu_l}$$

Using the second equality of Proposition 7.26 for $\lambda = (n)$ and Proposition 7.29 we obtain

$$F_\lambda(\tau_l)[\mathscr{P}^n_\lambda]_{\mathfrak{S}_n \times \mu_l} = [\sum_{i=0}^{l-1} \mathrm{Ind}^{\mathfrak{S}_n}_{C_{l,n}}(\theta_l^i F_\lambda(\theta_l^{-1})) \boxtimes \tau_l^i]_{\mathfrak{S}_n \times \mu_l}$$

Let us decompose $F_\lambda(\theta) = \sum_{j=0}^{l-1} a_j \theta_l^j$ with $a_j \in \mathbb{Z}_{\geq 0}$ and rearrange the two sums

$$\begin{aligned} F_\lambda(\tau_l)[\mathscr{P}^n_\lambda]_{\mathfrak{S}_n \times \mu_l} &= [\sum_{i=0}^{l-1}\sum_{j=0}^{l-1} \mathrm{Ind}^{\mathfrak{S}_n}_{C_{l,n}}(a_j \theta_l^{i-j}) \boxtimes \tau_l^i]_{\mathfrak{S}_n \times \mu_l} \\ &= [\sum_{i=0}^{l-1}\sum_{j=0}^{l-1} \mathrm{Ind}^{\mathfrak{S}_n}_{C_{l,n}}(\theta_l^i) \boxtimes a_j \tau_l^{i+j}]_{\mathfrak{S}_n \times \mu_l} \\ &= F_\lambda(\tau_l)[\sum_{i=0}^{l-1} \mathrm{Ind}^{\mathfrak{S}_n}_{C_{l,n}}(\theta_l^i) \boxtimes \tau_l^i]_{\mathfrak{S}_n \times \mu_l} \end{aligned}$$

We can conclude thanks to Lemma 7.28. □

7.3.2 When γ_l is small and l is prime

Denote by $\mathscr{P}_n^{<l}$ the set of all partitions μ of n such that $\mathrm{g}_l(\mu) < l$. Let us show that Corollary 7.14 also holds for all partitions of $\mathscr{P}_n^{<l}$ with l a prime number.

Proposition 7.31. *For each partition λ of n, and each integer $l \geq 1$, we have the following equality of $(\mathfrak{S}_n \times \mu_l)$-modules*

$$[S(V_n)^{\mathrm{co}(\mathfrak{S}_n)}] = [\sum_{i=0}^{l-1} \mathrm{Ind}_{W_{l,n}^{\mathrm{g}_l}}^{\mathfrak{S}_n} (\sum_{j=0}^{l-1} (S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})})_j^l \boxtimes \theta_l^{i-j}) \boxtimes \tau_l^i]$$

Proof. Let us denote $\gamma_l = (\gamma_{l,1}, ..., \gamma_{l,t}) \vdash \mathrm{g}_l$. This result is a special case of [BLM06, Theorem 4.6]. Take

$$v = (1, .., 1, 2, .., 2, ..., t, ..., t, (t+1)\zeta_l^{l-1}, ..., t+1, ..., (t+r_l)\zeta_l^{l-1}, ..., (t+r_l)) \in V_n$$

where 1 is repeated $\gamma_{l,1}$ times, 2 is repeated $\gamma_{l,2}$ times and so on until t. The stabilizer of v in $\mathfrak{S}_n$ is exactly $\mathfrak{S}_{\mathrm{g}_l}$ and $w_{l,n}v = \zeta_l v$. □

For $\lambda \vdash n$, let us denote $a_{\mu,j}^l(\lambda) := \langle \mathrm{Res}_{W_{l,n}^{\mathrm{g}_l}}^{\mathfrak{S}_n}(V_\lambda), V_\mu \boxtimes \theta_l^j \rangle$ where $\mu \vdash \mathrm{g}_l(\lambda)$, $j \in [\![0, l-1]\!]$.

Proposition 7.32. *For all partitions λ of n*

$$F_\lambda(\tau_l) = \sum_{\mu \vdash \mathrm{g}_l} \sum_{j=0}^{l-1} a_{\mu,j}^l(\lambda) F_\mu(\tau_l) \tau_l^{-j}$$

Proof. Let us start this proof with

$$F_\lambda(\tau_l) = \sum_{i=0}^{l-1} \langle V_{(n)}, V_\lambda \otimes \left(S(V_n)^{\mathrm{co}(\mathfrak{S}_n)}\right)_i^l \rangle \tau_l^i$$

Using Proposition 7.31, we have

$$\begin{aligned} F_\lambda(\tau_l) &= \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \langle V_{(n)}, V_\lambda \otimes \mathrm{Ind}_{W_{l,n}^{\mathrm{g}_l}}^{\mathfrak{S}_n} \left(\left(S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})}\right)_j^l \boxtimes \theta_l^{i-j} \right) \rangle \tau_l^i \\ &= \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \sum_{k=0}^{l-1} \sum_{\mu \vdash \mathrm{g}_l} a_{\mu,k}^l(\lambda) \langle V_{(n)}, \mathrm{Ind}_{W_{l,n}^{\mathrm{g}_l}}^{\mathfrak{S}_n} \left((V_\mu \otimes \left(S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})}\right)_j^l) \boxtimes \theta_l^{i-j+k} \right) \rangle \tau_l^i \end{aligned}$$

We can now use Frobenius reciprocity theorem

$$\begin{aligned}
F_\lambda(\tau_l) &= \sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1}\sum_{\mu\vdash \mathrm{g}_l} a^l_{\mu,k}(\lambda)\langle V_{(\mathrm{g}_l)}\boxtimes\theta_l^0, (V_\mu\otimes\left(S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})}\right)^l_j)\boxtimes\theta_l^{i-j+k}\rangle\tau_l^i\\
&= \sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1}\sum_{\mu\vdash \mathrm{g}_l} a^l_{\mu,k}(\lambda)\delta_0^{i-j+k}\langle V_{(\mathrm{g}_l)}, V_\mu\otimes\left(S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})}\right)^l_j\rangle\tau_l^i\\
&= \sum_{j=0}^{l-1}\sum_{k=0}^{l-1}\sum_{\mu\vdash \mathrm{g}_l} a^l_{\mu,k}(\lambda)\langle V_{(\mathrm{g}_l)}, V_\mu\otimes\left(S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})}\right)^l_j\rangle\tau_l^{j-k}\\
&= \sum_{\mu\vdash \mathrm{g}_l}\sum_{k=0}^{l-1} a^l_{\mu,k}(\lambda)\sum_{j=0}^{l-1}\langle V_{(\mathrm{g}_l)}, V_\mu\otimes\left(S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})}\right)^l_j\rangle\tau_l^j\tau_l^{-k}\\
&= \sum_{\mu\vdash \mathrm{g}_l}\sum_{k=0}^{l-1} a^l_{\mu,k}(\lambda)F_\mu(\tau_l)\tau_l^{-k}
\end{aligned}$$

□

From now on and in the rest of this subsection, let us suppose that the fixed integer l is prime.

Lemma 7.33. *For each $\lambda\in\mathcal{P}_n^{<l}$ and each $k\in[\![0,l\text{-}1]\!]$, $F_\lambda(\zeta_l^k)\neq 0$.*

Proof. Since l is prime it is enough to show that $v_{\Phi_l}(F_\lambda)=0$. By using the fact that $\mathrm{g}_l<l$, we can now use the same argument as in Lemma 7.28. □

Lemma 7.34. *Take λ a partition of n. We then have that for all $\mu\in\mathcal{P}_{\mathrm{g}_l}\setminus\{\gamma_l\}$*

$$\forall j\in[\![0,l\text{-}1]\!], a^l_{\mu,j}(\lambda)=a^l_{\mu,0}(\lambda)$$

Proof. Murnaghan–Nakayama recursive formula gives the following result

$$\exists a\in\mathbb{Z},\forall i\in[\![1,l\text{-}1]\!],\forall x\in\mathfrak{S}_{\mathrm{g}_l},\chi_\lambda(xw^i_{l,n})=a\chi_{\gamma_l}(x)$$

In terms of multiplicities, this result can be rephrased as

$$\forall\mu\in\mathcal{P}_{\mathrm{g}_l}\setminus\{\gamma_l\},\sum_{j=0}^{l-1}a^l_{\mu,j}(\lambda)\zeta_l^j=0$$

Indeed,

$$\begin{aligned}
\sum_{j=0}^{l-1}a^l_{\mu,j}(\lambda)\zeta_l^j &= \frac{1}{|W^{\mathrm{g}_l}_{l,n}|}\sum_{j=0}^{l-1}\sum_{i=0}^{l-1}\sum_{x\in\mathfrak{S}_{\mathrm{g}_l}}\chi_\lambda(xw^i_{l,n})\chi_\mu(x)\theta_l^{-ij+j}(w_{l,n})\\
&= \frac{1}{|W^{\mathrm{g}_l}_{l,n}|}\Big(\sum_{j=0}^{l-1}\sum_{x\in\mathfrak{S}_{\mathrm{g}_l}}\chi_\lambda(x)\chi_\mu(x)\theta_l^j(w_{l,n})+\sum_{j=0}^{l-1}\sum_{i=1}^{l-1}\sum_{x\in\mathfrak{S}_{\mathrm{g}_l}}\chi_\lambda(xw^i_{l,n})\chi_\mu(x)\theta_l^{-ij+j}(w_{l,n})\Big)\\
&= \langle\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{\mathrm{g}_l}}(\chi_\lambda),\chi_\mu\rangle\frac{1}{l}\sum_{j=0}^{l-1}\theta_l^j(w_{l,n})+\frac{a}{|W^{\mathrm{g}_l}_{l,n}|}\sum_{j=0}^{l-1}\sum_{i=1}^{l-1}\sum_{x\in\mathfrak{S}_{\mathrm{g}_l}}\chi_{\gamma_l}(x)\chi_\mu(x)\theta_l^{-ij+j}(w_{l,n})\\
&= \langle\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{\mathrm{g}_l}}(\chi_\lambda),\chi_\mu\rangle\frac{1}{l}\sum_{j=0}^{l-1}\theta_l^j(w_{l,n})+a\frac{\langle\chi_{\gamma_l},\chi_\mu\rangle}{l}\sum_{j=0}^{l-1}\sum_{i=1}^{l-1}\theta_l^{-ij+j}(w_{l,n})
\end{aligned}$$

The first term is equal to 0 since $\theta_l(w_{l,n}) = \zeta_l$ and since $\mu \neq \gamma_l$, we have $\langle \chi_{\gamma_l}, \chi_\mu \rangle = 0$. Thus $\sum_{j=1}^{l-1} (a^l_{\mu,j}(\lambda) - a^l_{\mu,0}(\lambda))\zeta_l^j = 0$ which then gives the result since l is prime. □

For a given finite group G, let $\mathrm{Reg}(G)$ denote the regular representation of G. We are now able to prove Corollary 7.14 in this case.

Proposition 7.35. *For each $\lambda \in \mathcal{P}_n^{<l}$, Corollary 7.14 holds.*

Proof. We want to show the following equality of $(\mathfrak{S}_n \times \mu_l)$-modules

$$[\mathscr{P}^n_\lambda] = \sum_{i=0}^{l-1}\sum_{j=0}^{l-1} \left[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n} \left((\mathscr{P}^{\mathrm{g}_l}_{\gamma_l})^l_j \boxtimes \theta_l^{i-j} \right) \boxtimes \tau_l^i \right]$$

Using Proposition 7.31, the right-hand side of the second equality of Proposition 7.26 can be rewritten as

$$\sum_{\mu \vdash \mathrm{g}_l} \sum_{k=0}^{l-1}\sum_{j=0}^{l-1}\sum_{i=0}^{l-1} a^l_{\mu,k}(\lambda) \left[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n} \left(V_\mu \otimes \left(S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})}\right)^l_j \boxtimes \theta_l^{i-j+k} \right) \boxtimes \tau_l^i \right] \tag{7.2}$$

Let us fix $\mu \in \mathcal{P}_{\mathrm{g}_l} \setminus \{\gamma_l\}$ and consider the associated term in (7.2). Using Lemma 7.34, this term is equal to

$$a^l_{\mu,0}(\lambda) \sum_{j=0}^{l-1} \left[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n} \left(\left(V_\mu \otimes (S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})})^l_j\right) \boxtimes \mathrm{Reg}(C_{l,n}) \right) \boxtimes \mathrm{Reg}(\mu_l) \right]$$

Denote for all $\nu \vdash \mathrm{g}_l$, $F_\nu(\tau_l) = \sum_{k=0}^{l-1} f_{\nu,k}\tau_l^k$. Applying the second equality of Proposition 7.26 for μ, gives us

$$a^l_{\mu,0}(\lambda) \sum_{k=0}^{l-1}\sum_{j=0}^{l-1} f_{\mu,k} \left[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n} \left((\mathscr{P}^{\mathrm{g}_l}_{\mu})^l_{j-k} \boxtimes \mathrm{Reg}(C_{l,n}) \right) \boxtimes \mathrm{Reg}(\mu_l) \right]$$

By construction of the Procesi bundle $\forall \nu \vdash \mathrm{g}_l, \sum_{j=0}^{l-1} [(\mathscr{P}^{\mathrm{g}_l}_\nu)^l_j] = [\mathrm{Reg}(\mathfrak{S}_{\mathrm{g}_l})]$ and by definition of the fake degree $\sum_{j=0}^{l-1} f_{\nu,k} = \dim(V_\nu)$. Summing everything up leads to

$$a^l_{\mu,0}(\lambda) \sum_{k=0}^{l-1}\sum_{j=0}^{l-1} f_{\mu,k} \left[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n} \left((\mathscr{P}^{\mathrm{g}_l}_{\mu})^l_{j-k} \right) \boxtimes \mathrm{Reg}(C_{l,n})) \boxtimes \mathrm{Reg}(\mu_l) \right] = a^l_{\mu,0}(\lambda) \dim(V_\mu) \mathrm{Reg}(\mathfrak{S}_n \times \mu_l)$$

The last equality is then true for the fiber of the Procesi bundle over I_μ for any partition μ of g_l. In particular, it holds for I_{γ_l}. We get that the term

$$\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1} a^l_{\mu,k}(\lambda) \left[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n} \left(\left(V_\mu \otimes (S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})})^l_j\right) \boxtimes \theta_l^{i-j+k} \right) \boxtimes \tau_l^i \right]$$

is equal to

$$a^l_{\mu,0}(\lambda) \sum_{k=0}^{l-1}\sum_{j=0}^{l-1} f_{\mu,k} \left[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n} \left((\mathscr{P}^{\mathrm{g}_l}_{\gamma_l})^l_{j-k} \boxtimes \mathrm{Reg}(C_{l,n}) \right) \boxtimes \mathrm{Reg}(\mu_l) \right]$$

which can be rewritten as

$$\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1} a^l_{\mu,k}(\lambda)[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n}\left((\mathscr{P}^{\mathrm{g}_l}_{\gamma_l})^l_j \boxtimes \left(\theta_l^{i-j}F_\mu(\theta_l^{-1})\theta_l^k\right)\right)\boxtimes \tau_l^i]$$

Finally, for $\mu = \gamma_l$

$$\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1} a^l_{\gamma_l,k}(\lambda)[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n}\left(\left(V_{\gamma_l}\otimes (S(V_{\mathrm{g}_l})^{\mathrm{co}(\mathfrak{S}_{\mathrm{g}_l})})^l_j\right)\boxtimes \theta_l^{i-j+k}\right)\boxtimes \tau_l^i]$$

is equal to

$$\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\sum_{k=0}^{l-1} a^l_{\gamma_l,k}(\lambda)[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n}\left((\mathscr{P}^{\mathrm{g}_l}_{\gamma_l})^l_j \boxtimes \left(\theta_l^{i-j}F_{\gamma_l}(\theta_l^{-1})\theta_l^k\right)\right)\boxtimes \tau_l^i]$$

By putting the pieces back together, and using Proposition 7.32, we get

$$[\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n}\left((\mathscr{P}^{\mathrm{g}_l}_{\gamma_l})^l_j \boxtimes \left(\theta_l^{i-j}F_\lambda(\theta_l^{-1})\right)\right)\boxtimes \tau_l^i] = F_\lambda(\tau_l)\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}[\mathrm{Ind}_{W^{\mathrm{g}_l}_{l,n}}^{\mathfrak{S}_n}\left((\mathscr{P}^{\mathrm{g}_l}_{\gamma_l})^l_j \boxtimes \theta_l^{i-j}\right)\boxtimes \tau_l^i]$$

Which ends the proof using Lemma 7.33. □

7.4 Type D study

In this last section, let us consider the case when Γ is of type D. Let us fix an integer $l \geq 1$. Recall that in Chapter 5, we have parametrized the irreducible components of $\mathcal{H}_n^{\widetilde{BD}_{2l}}$ that contain a $\mathbb{T}_1$-fixed point. The fixed points of $\mathcal{H}_n$ under $<\mathbb{T}_1, \widetilde{BD}_{2l}>$ are the monomial ideals parametrized by symmetric partitions of n. When λ is a symmetric partition of n, the fiber of the Procesi bundle over I_λ is then an $(\mathfrak{S}_n \times \widetilde{BD}_{2l})$-module. Consider the following decomposition

$$[\mathscr{P}^n_\lambda]_{\mathfrak{S}_n\times\widetilde{BD}_{2l}} = \sum_{\chi\in I_{\widetilde{BD}_{2l}}} [D^{2l}_{n,\chi}(\lambda)\boxtimes\chi]_{\mathfrak{S}_n\times\widetilde{BD}_{2l}}$$

The goal here will be to describe the $\mathfrak{S}_n$-modules $D^{2l}_{n,\chi}(\lambda)$ for each $\chi \in I_{\widetilde{BD}_{2l}}$. To do so, we will use the Corollary 7.14.

Lemma 7.36. *If λ is a symmetric partition of n, then the number $r_{D,2l}(\lambda) := \frac{n-\mathrm{g}_{2l}(\lambda)}{2l}$ is a multiple of 2.*

Proof. To prove this we can use [Ol, Lemma 2.2] and the link between abacuses and β-sets to see that the $2l$-abacus of λ^* is equal to the horizontal reflection of the $2l$-abacus of λ. When λ is symmetric, we have that, $r_{D,2l}$ which is the number of $2l$-hooks that needs to be removed to go from λ to g_{2l}, is a multiple of 2. □

Lemma 7.37. *The restrictions of the irreducible characters of $\widetilde{BD}_{2l}$ to μ_{2l} are*

- $\mathrm{Res}^{\widetilde{BD}_{2l}}_{\mu_{2l}}(\chi_{0^+}) = \tau^0_{2l}$
- $\mathrm{Res}^{\widetilde{BD}_{2l}}_{\mu_{2l}}(\chi_{l^+}) = \tau^l_{2l}$
- $\mathrm{Res}^{\widetilde{BD}_{2l}}_{\mu_{2l}}(\chi_{0^-}) = \tau^0_{2l}$
- $\mathrm{Res}^{\widetilde{BD}_{2l}}_{\mu_{2l}}(\chi_{l^-}) = \tau^l_{2l}$
- $\forall i \in [\![1, l\text{ - }1]\!], \mathrm{Res}^{\widetilde{BD}_{2l}}_{\mu_{2l}}(\chi_i) = \tau^i_{2l} + \tau^{-i}_{2l}$

From there we can deduce the following information on the $D^{2l}_{n,\chi}(\lambda)$ modules.

Proposition 7.38. *For each symmetric partition λ of n, we have the following equalities in $\mathcal{R}(\mathfrak{S}_n)$*

(i) $[D^{2l}_{n,\chi_{0^+}}(\lambda) + D^{2l}_{n,\chi_{0^-}}(\lambda)] = [(\mathscr{P}^n_\lambda)^{2l}_0]$

(ii) $[D^{2l}_{n,\chi_{l^+}}(\lambda) + D^{2l}_{n,\chi_{l^-}}(\lambda)] = [(\mathscr{P}^n_\lambda)^{2l}_l]$

(iii) $[D^{2l}_{n,\chi_{l^+}}(\lambda)] = [D^{2l}_{n,\chi_{l^-}}(\lambda)]$

(iv) $\forall i \in [\![1, l\text{ - }1]\!], [D^{2l}_{n,\chi_i}(\lambda)] = [(\mathscr{P}^n_\lambda)^{2l}_i] = [(\mathscr{P}^n_\lambda)^{2l}_{2l-i}]$

Proof. Equality (i) and (ii) comes directly from Lemma 7.37. Concerning equality (iii), note that $\widetilde{BD}_{2l} \triangleleft \widetilde{BD}_{4l}$ and that ω_{4l} acts nontrivially on $I_{\widetilde{BD}_{2l}}$. It swaps χ_{l^+} and χ_{l^-} and fixes all other irreducible characters of $\widetilde{BD}_{2l}$. Since $[\mathscr{P}^n_\lambda]$ is a bigraded $\mathfrak{S}_n$-module, one automatically has the following equality

$$\omega_{4l}.[\mathscr{P}^n_\lambda]_{\mathfrak{S}_n \times \widetilde{BD}_{4l}} = [\mathscr{P}^n_\lambda]_{\mathfrak{S}_n \times \widetilde{BD}_{4l}}$$

Now, applying the restriction from $\mathfrak{S}_n \times \widetilde{BD}_{4l}$ to $\mathfrak{S}_n \times \widetilde{BD}_{2l}$, gives

$$[D^{2l}_{n,\chi_{l^+}}(\lambda)] = [D^{2l}_{n,\chi_{l^-}}(\lambda)]$$

Combining [H03, Proposition 3.5.11] with Lemma 7.37 gives

$$2[D^{2l}_{n,\chi_i}(\lambda)] = [(\mathscr{P}^n_\lambda)^{2l}_i + (\mathscr{P}^n_\lambda)^{2l}_{2l-i}] = 2[(\mathscr{P}^n_\lambda)^{2l}_i]$$

□

Since $n - \mathrm{g}_{2l}$ is a multiple of $4l$ thanks to Lemma 7.36, we can choose $s_{2l} \in \mathfrak{S}_{n-\mathrm{g}_{2l}}$ such that $< w^{\mathrm{g}_{2l}}_{2l,n}, s_{2l} > \simeq \widetilde{BD}_{2l}$.

Example 7.39. When $l = 2$, λ is a symmetric partition of 8, $r_{D,4}(\lambda) = 2$ then $\mathrm{g}_{2l} = \emptyset$. In that case $w^{\emptyset}_{2l,n} = (1234)(5678) \in \mathfrak{S}_8$ and we can take $s_{2l} = (1836)(2745) \in \mathfrak{S}_8$.

Proposition 7.40. *For each symmetric partition λ of n and for each $i \in [\![1, l\text{ - }1]\!]$*

$$[D^{2l}_{n,\chi_i}]_{\mathfrak{S}_n} = \sum_{j=0}^{2l-1} [\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{\mathrm{g}_{2l}} \times < w^{\mathrm{g}_{2l}}_{2l,n}, s_{2l} >} \left((\mathscr{P}^{\mathrm{g}_{2l}}_{\gamma_{2l}})^{2l}_j \boxtimes \tilde{\chi}_{i-j} \right)]_{\mathfrak{S}_n}$$

Moreover

$$[D^{2l}_{n,\chi_{l^+}}]_{\mathfrak{S}_n} = \frac{1}{2} \sum_{j=0}^{2l-1} [\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{\mathrm{g}_{2l}} \times < w^{\mathrm{g}_{2l}}_{2l,n}, s_{2l} >} \left((\mathscr{P}^{\mathrm{g}_{2l}}_{\gamma_{2l}})^{2l}_j \boxtimes \tilde{\chi}_{l-j} \right)]_{\mathfrak{S}_n}$$

Proof. Fix $i \in [\![1, l-1]\!]$. Thanks to Proposition 7.38 and Corollary 7.14, we have

$$\begin{aligned}
[D^{2l}_{n,\chi_i}]_{\mathfrak{S}_n} &= \sum_{j=0}^{2l-1} [\mathrm{Ind}^{\mathfrak{S}_n}_{W^{\mathfrak{g}_{2l}}_{2l,n}} \big((\mathscr{P}^{\mathfrak{g}_{2l}}_{\gamma_{2l}})^{2l}_j \boxtimes \theta^{i-j}_{2l}\big)]_{\mathfrak{S}_n} \\
&= \sum_{j=0}^{2l-1} [\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{\mathfrak{g}_{2l}} \times <w^{\mathfrak{g}_{2l}}_{2l,n}, s_{2l}>} \big(\mathrm{Ind}^{\mathfrak{S}_{\mathfrak{g}_{2l}} \times <w^{\mathfrak{g}_{2l}}_{2l,n}, s_{2l}>}_{W^{\mathfrak{g}_{2l}}_{2l,n}} \big((\mathscr{P}^{\mathfrak{g}_{2l}}_{\gamma_{2l}})^{2l}_j \boxtimes \theta^{i-j}_{2l}\big)\big)]_{\mathfrak{S}_n} \\
&= \sum_{j=0}^{2l-1} [\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{\mathfrak{g}_{2l}} \times <w^{\mathfrak{g}_{2l}}_{2l,n}, s_{2l}>} \big((\mathscr{P}^{\mathfrak{g}_{2l}}_{\gamma_{2l}})^{2l}_j \boxtimes \tilde{\chi}_{i-j}\big)]_{\mathfrak{S}_n}
\end{aligned}$$

The same computation gives us the second formula for $[D^{2l}_{n,\chi^+_l}]_{\mathfrak{S}_n}$. Thanks to Proposition 7.38 it is equal to $\frac{1}{2}[(\mathscr{P}^n_\lambda)^{2l}_l]_{\mathfrak{S}_n}$. □

Remark 7.41. Note that Proposition 7.40 can be rewritten in terms of the $\mathfrak{S}_{\mathfrak{g}_{2l}}$-modules $([D^{2l}_{\mathfrak{g}_{2l},\chi}])_{\chi \in I_{\widetilde{BD}_{2l}}}$ thanks to Proposition 7.38. There remains to understand how $[(\mathscr{P}^n_\lambda)^{2l}_0]$ splits in two.

CHAPTER 8

WORK IN PROGRESS

In Chapter 7, we have proven Theorem 7.11. If Γ is the cyclic group with l elements, this theorem reduces the study of the $(\mathfrak{S}_n \times \mu_l)$-module structure of the fiber of the Procesi bundle at a monomial ideal I_λ to the one of the fiber of the Procesi bundle over the monomial ideal $I_{\gamma_l(\lambda)}$ where $\gamma_l(\lambda)$ is the l-core of λ, i.e. the partition obtained from λ by removing all hooks of length l. The next question is then to study the $(\mathfrak{S}_n \times \mu_l)$-module $\mathscr{P}_\lambda^n$ when $\lambda \vdash n$ is an l-core. In this last chapter, we would like to present a conjecture that establishes a link with the Fock representation of the affine Lie algebra of type $\tilde{A}_l$. Let us fix $l, n \in \mathbb{Z}_{\geq 0}^2$ and take $\Gamma = \mu_l$. Recall that

$$\tau_l : \begin{array}{ccc} \mu_l & \to & \mathbb{C} \\ \omega_l & \mapsto & \zeta_l \end{array}$$

To improve readability, let us shorten Π_{μ_l}, the set of simple positive roots of the Kac-Moody Lie algebra of type $\tilde{A}_l$, to Π_l. Let us moreover, for each $i \in [\![0, l-1]\!]$, shorten $\alpha_{\tau_l^i} \in \Pi_l$ to α_i. Denote by $\hat{\mathfrak{sl}}_l$ the Kac-Moody Lie algebra of type $\tilde{A}_l$. Thanks to [Kac, Theorem 7.4], we have a set of Chevalley generators of $\hat{\mathfrak{sl}}_l$, that we will denote by $\{e_\alpha, f_\alpha | \alpha \in \Pi_l\}$. The set $\{e_\alpha, f_\alpha, \alpha^\vee | \alpha \in \Pi_l\} \cup \{\mathbf{d}\}$ generates the Lie algebra $\hat{\mathfrak{sl}}_l$. For a partition λ and an integer $i \in [\![0, l-1]\!]$, denote by $T(\lambda, i)$ the set of partitions μ obtained from λ by adding a box with residue i and by $R(\lambda, i)$ set of partitions μ obtained from λ by removing a box with residue i. Let $N(\lambda, i) := |T(\lambda, i)| - |R(\lambda, i)|$ and $N_0(\lambda)$ be the number of boxes of λ with residue 0. Denote by $\mathfrak{T}_l^n := [\![0, l-1]\!]^n$.

Definition 8.1. For each $k \in \mathbb{Z}_{\geq 0}$, let $\mathcal{F}^k$ be a $\mathbb{C}$-vector space with basis $\{|\lambda\rangle | \lambda \in \mathcal{P}_k\}$. Let $\mathcal{F}$ be the $\mathbb{C}$-vector space $\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathcal{F}^k$.

Remark 8.2. Note that we can easily switch from partitions to infinite wedges [Lec, Section 2.2.1].

Definition 8.3. Define an $\hat{\mathfrak{sl}}_l$-action on $\mathcal{F}$. For each $i \in [\![0, l-1]\!]$

- $e_{\alpha_i}|\lambda\rangle := \sum_{\mu \in T(\lambda, i)} |\mu\rangle$
- $f_{\alpha_i}|\lambda\rangle := \sum_{\mu \in R(\lambda, i)} |\mu\rangle$
- $\alpha_i^\vee |\lambda\rangle := N(\lambda, i)|\lambda\rangle$
- $\mathbf{d}|\lambda\rangle := N_0(\lambda)|\lambda\rangle$

[Lec, Theorem 1] gives the following proposition.

Proposition 8.4. *The vector space $\mathcal{F}$ is an $\hat{\mathfrak{sl}}_l$-module.*

Let us now state the conjecture. Consider the map

$$f_l^n : \begin{array}{ccc} \mathfrak{T}_l^n & \to & \mathcal{F}^n \\ (i_1, ..., i_n) & \mapsto & f_{i_1} ... f_{i_n} |\emptyset\rangle \end{array}$$

Consider also the following injective map

$$g_l^n : \begin{array}{ccc} K_0(\mathfrak{S}_n) & \to & \mathcal{F}^n \\ [V_\mu] & \mapsto & |\mu\rangle \end{array}$$

Since $\mathrm{Im}(f_l^n) \subset \mathrm{Im}(g_l^n)$, define

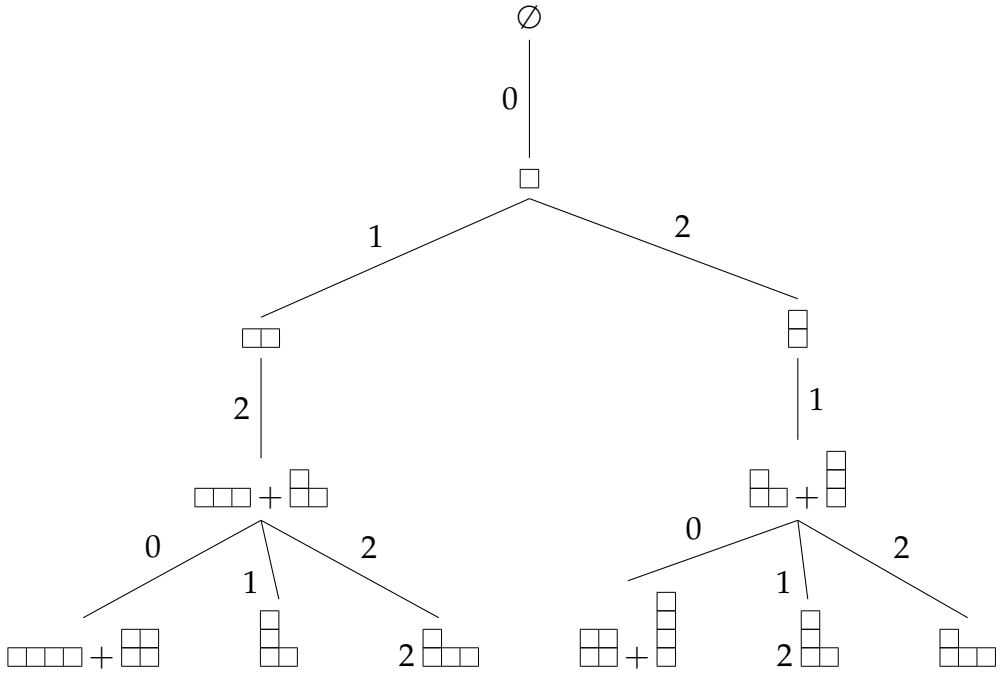
$$\varphi : \begin{array}{ccc} \mathfrak{T}_l^n & \to & K_0(\mathfrak{S}_n) \\ x & \mapsto & (g_l^n)^{-1}(f_l^n(x)) \end{array}$$

Conjecture 8.5. *For each partition λ of n that is an l-core, there exists $\psi_\lambda : \mathfrak{T}_l^n \to [\![0, l-1]\!]$ such that*

$$\Big[\sum_{i=0}^{l-1} \Big(\sum_{c \in {\psi_\lambda}^{-1}(i)} \varphi(c)\Big) \boxtimes \tau_i\Big]_{\mathfrak{S}_n \times \mu_l} = [\mathscr{P}_\lambda^n]_{\mathfrak{S}_n \times \mu_l}$$

Remark 8.6. Denote by triv, the trivial character of the trivial group $\mathfrak{S}_0$.
Since $[\sum_{c \in \mathfrak{T}_l^n} \varphi(c)]_{\mathfrak{S}_n} = [\mathrm{Ind}_{\mathfrak{S}_0}^{\mathfrak{S}_n}(\mathrm{triv})]_{\mathfrak{S}_n}$, it is clear that $[\sum_{c \in \mathfrak{T}_l^n} \varphi(c)]_{\mathfrak{S}_n} = [\mathscr{P}^n_{|I_\lambda}]_{\mathfrak{S}_n}$. But it is not clear, why we could be able to find a way to split $\mathfrak{T}_l^n$ into l parts such that the equality holds as $(\mathfrak{S}_n \times \mu_l)$-modules.

Example 8.7. Let us give an example when l equals 3.



If we take $\lambda = \ydiagram{2,1}$ (which is not a 3-core), then

$$[\mathscr{P}^3_{|I_\lambda}]_{\mathfrak{S}_3 \times \mu_3} = [(V_{\ydiagram{3}} + V_{\ydiagram{1,1,1}}) \boxtimes \tau_0 + V_{\ydiagram{2,1}} \boxtimes \tau_1 + V_{\ydiagram{2,1}} \boxtimes \tau_2]_{\mathfrak{S}_3 \times \mu_3}$$

This decomposition makes it impossible to find a ψ_λ satisfying Conjecture 8.5.
Consider now the case when $\mu = \ydiagram{3,1}$. We now have

$$[\mathscr{P}^4_{|I_\mu}]_{\mathfrak{S}_4\times\mu_3} = [(V_{\ydiagram{4}} + V_{\ydiagram{2,2}} + 2V_{\ydiagram{2,1,1}}) \boxtimes \tau_0 +$$
$$(V_{\ydiagram{3,1}} + V_{\ydiagram{2,1,1}}) \boxtimes \tau_1 +$$
$$(2V_{\ydiagram{3,1}} + V_{\ydiagram{2,2}} + V_{\ydiagram{1,1,1,1}}) \boxtimes \tau_2]_{\mathfrak{S}_3\times\mu_3}$$

Finally for $\mu' = \ydiagram{3,1}$, we have

$$[\mathscr{P}^4_{|I_{\mu'}}]_{\mathfrak{S}_4\times\mu_3} = [(V_{\ydiagram{4}} + V_{\ydiagram{2,2}} + 2V_{\ydiagram{2,1,1}}) \boxtimes \tau_0 +$$
$$(2V_{\ydiagram{3,1}} + V_{\ydiagram{2,2}} + V_{\ydiagram{1,1,1,1}}) \boxtimes \tau_1 +$$
$$(V_{\ydiagram{3,1}} + V_{\ydiagram{2,1,1}}) \boxtimes \tau_2]_{\mathfrak{S}_3\times\mu_3}$$

In the annexes, there is a SAGE code to test Conjecture 8.5 for a given values of l and n. Moreover, we have also added tables containing decompositions of fibers of Procesi bundles at some cores.

Annexes

8.1 Sage code for Conjecture 8.5

The following code tests Conjecture 8.5 for $n \in [\![3,5]\!]$ and $l \in [\![3,14]\!]$. These are arbitrary values and one can change these by changing the variables $nmin$, $nmax$, $lmin$ and $lmax$.

```
### This code tests Conjecture 7.5 for a given l and n range

## 1. General algorithms to test matchings

# Removes the subdictionary sd from the dictionary d
def remove_sub(sd,d):
  res = {}
  dkeys = d.keys()
  for k in dkeys:
     if not(k in sd):
        res[k] = d[k]
     elif sd[k] < d[k]:
        res[k] = d[k]-sd[k]
  return res

# Merges two dictionaries d1 and d2
def merge_dict(d1,d2):
  d3 = {**d1, **d2}
  for key in d3.keys():
    if (key in d1) and (key in d2):
      d3[key] = d1[key]+d2[key]
  return d3

# Tests if sd is a subdictionary of d
def test_sub(sd,d):
  return all(key in d and sd[key] <= d[key] for key in sd)

# Removes empty entries from the list of dictionaries
def clean_dic(T):
  res = []
  for d in T:
    if d != {}:
      res.append(d)
  return res
```

```
# Tests if T2 is a sublist of dictionaries of T1
def remove_nochoice(T1,T2):
  lT2 = len(T2)
  for i in range(lT2):
    temp = []
    for j in range(len(T1)):
      if test_sub(T2[i],T1[j]):
        temp.append(j)
    if temp == []:
       return False
    if len(temp) == 1:
      newdic = remove_sub(T2[i],T1[temp[0]])
      if not(newdic == {}):
        T1[temp[0]] = newdic
      else:
        T1.pop(temp[0])
      T2[i] = {}
  return True

# Using backtracking, tries to find a way to
# match T2 as a sublist of dictionaries of T1
def find_a_sol(T1,T2):
  def backtrack(T1,T2,matchings,pos):
    if pos == len(T2):
      return True
    for i in range(len(T1)):
      if test_sub(T2[pos],T1[i]):
        matchings[pos] = i
        T1[i] = remove_sub(T2[pos],T1[i])
        if backtrack(T1,T2,matchings,pos+1):
          return True
        matchings[pos] = -1
        T1[i] = merge_dict(T1[i],T2[pos])
    return False
  matchings = [-1]*len(T2)
  return backtrack(T1,T2,matchings,0)
```

```
# Using backtracking, find all ways to match
# T2 as a sublist of dictionnaries of T1
def find_all_sol(T1,T2):
  def backtrack_all(T1,T2,matchings,pos):
    if pos == len(T2):
      yield [x for x in matchings]
      return
    for i in range(len(T1)):
      if test_sub(T2[pos],T1[i]):
        matchings[pos] = i
        T1[i] = remove_sub(T2[pos],T1[i])
        yield from backtrack_all(T1,T2,matchings,pos+1)
        matchings[pos] = -1
        T1[i] = merge_dict(T1[i],T2[pos])
  matchings = [-1]*len(T2)
  res = []
  for sol in backtrack_all(T1,T2,matchings,0):
    res.append(sol)
  return res

# Wrapper around find_all_sol
def is_sub(T1,T2):
  if not(remove_nochoice(T1,T2)):
     return False
  T2 = clean_dic(T2)
  list_all = find_all_sol(T1,T2)
  return len(list_all)>0

#################################################################
# Now it is time to add the progs to obtain the decompositions
# that one wants to compare.
#################################################################

## 2. Decomposition from the Fock space

class Node(object):
    def __init__(self, data):
        self.data = data
        self.children = []
        self.parent = []

    def add_parent(self, obj):
        self.parent.append(obj)

    def add_child(self, obj):
        self.children.append(obj)
        obj.add_parent(self)
```

```
# Tests if node is a leaf
def is_a_leaf(tree):
  return tree.children == []


# Returns the leaves of a Tree
def leaves(tree):
  res = []
  def rec(res,tree):
    for c in tree.children:
      if is_a_leaf(c):
        res.append(c.data)
      else:
        rec(res,c)
  rec(res,tree)
  return res

# Returns the path to root of a Tree from a node
def path_to_root(node):
   res=[node.data]
   def rec(res,node):
      if node.parent!=[]:
        res.append(node.parent[0].data)
        rec(res,node.parent[0])
   rec(res,node)
   return res[::-1]

# Returns the leaves and paths to root of a Tree
def leaves_with_path(tree):
  res = []
  def rec(res,tree):
    for c in tree.children:
      if is_a_leaf(c):
        res.append(path_to_root(c))
      else:
        rec(res,c)
  rec(res,tree)
  return res

# Adds v to dictionary d with key k
def add_to_dic(d,v,k):
  if k in d:
     d[k] = d[k] + v
  else:
    d[k] = v
  return d
```

```
# Constructs decomposition Tree of Fock space
# Returns the leaves of the Tree, but can also
# return the leaves with paths to root of each leaf
def dec_fock(l,n):
  rootl = {}
  rootl[Partition([])] = 1
  root = Node(rootl)
  def children(node,l,n):
   if n > 0:
     for i in range(l):
       templ = {}
       for (lb,mult) in node.data.items():
           addable_cells_res_i = lb.addable_cells_residue(i,l)
           for (r,c) in addable_cells_res_i:
              templ = add_to_dic(templ,mult,lb.add_cell(r))
       if templ != {}:
         temp = Node(templ)
         temp.add_parent(node)
         node.add_child(temp)
         children(temp,l,n-1)
  children(root,l,n)
  return leaves(root)
  #return leaves_with_path(root)

#################################################################

## 3. Decomposition from the Procesi bundle

Pol.<q,t>=QQ['q,t']
Sym = SymmetricFunctions(Pol)
p = SymmetricFunctions(QQ).p()
s = Sym.schur()

# Returns the characteristic function of the fiber of the
# Procesi bundle at the ideal indexed by the partition mu
def sym_fibre_proc(n,mu):
    Ht = Sym.macdonald().Ht()
    return Ht(mu)

# Returns the Hilbert serie of the characteristic function
def dim_fibre_proc(n,mu):
    Ht_mu = sym_fibre_proc(n,mu)
    return s(Ht_mu).scalar(p([1]*n))

# Decomposition of the fibre along the multiplicative
# group and also along the symmetric group
def dim_fibre_proc_sn(n,mu,lb):
    Ht_mu = sym_fibre_proc(n,mu)
    return s(Ht_mu).scalar(s(lb))
```

```
# Returns the list of dimension of
# each part along the cyclic group
# for a fixed irrep of the symmetric
# group
def dim_fibre_proc_list_sn(n,mu,l,lb):
    P = dim_fibre_proc_sn(n,mu,lb)
    d = P.degree()
    res = [0]*l
    for k in range(d+1):
        for r in range(d+1):
            res[(k-r)%l] += P.coefficient({q:k,t:r})
    return res

# Returns the dimensions of each irrep
# of the symmetric group on n letters
def dim_irrep_sn(n,lb):
    return s(lb).scalar(p([1]*n))

# Returns the decomposition along the
# cyclic times symmetric group action
def list_dim_fibre_decomp_sn(lb,l,n):
  res = []
  for mu in Partitions(n):
    dim_irrep = int(dim_irrep_sn(n,mu))
    dim_fibre_temp = dim_fibre_proc_list_sn(n,lb,l,mu)
    ltemp = [elem for elem in dim_fibre_temp]
    if ltemp != [0]*l:
      res.append([mu,ltemp])
  return  res

# Removes useless data
def restruct_dim_fibre(lb,l,n):
  ld = list_dim_fibre_decomp_sn(lb,l,n)
  res = [{} for _ in range(l)]
  for item in ld:
    for i in range(l):
      if item[1][i] != 0 and item[0] in res[i]:
        res[i][item[0]] += item[1][i]
      elif item[1][i] != 0:
        res[i][item[0]] = item[1][i]
  return res
```

```
## 4. Tests

def list_cores(l,n):
  res = []
  lcores=Cores(l,size=n).list()
  for c in lcores:
    if not(c in res) and not(c.to_partition().conjugate() in res):
       res.append(c.to_partition())
  return res

nmin = 3
nmax = 5
lmin = 3
lmax = 14

def test_conj(lmin,lmax,nmin,nmax):
  res = True
  n = nmin
  while res and n <= nmax:
    l = lmin
    while res and  l <= lmax:
      print("This is (l,n): " + str((l,n)))
      for c in list_cores(l,n):
        print(c)
        dfock = dec_fock(l,n)
        res = res and is_sub(restruct_dim_fibre(c,l,n),dfock)
      l += 1
    n += 1
  return res

print(test_conj(lmin,lmax,nmin,nmax))
```

8.2 Decomposition tables of fibers of $\mathscr{P}^n$ at cores

This section is dedicated to exposing decomposition tables of $\mathscr{P}^n_\lambda$ as $(\mathfrak{S}_n \times \mu_l)$-module for different values of n and l such that there exists at least one partition λ of n that is an l-core. The first line is the l-core. The second line of the tables gives the dimension vector. For each $i \in [\![0, l-1]\!]$, the entry i of this vector, is the dimension of the $\mathfrak{S}_n$-module $(\mathscr{P}^n_\lambda)^l_i$. Recall that one has denoted by $(\mathscr{P}^n_\lambda)^l_i$ the $\mathfrak{S}_n$-modules such that

$$[\mathscr{P}^n_\lambda]_{\mathfrak{S}_n\times\mu_l} = [\sum_{i=0}^{l-1} (\mathscr{P}^n_\lambda)^l_i \boxtimes \tau^i_l]_{\mathfrak{S}_n\times\mu_l}$$

Finally, the entry i of a vector on a line which starts with partition μ, at colunm with l-core λ is equal to

$$\dim\big(\mathrm{Hom}_{\mathfrak{S}_n\times\mu_l}(V_\mu \boxtimes \tau^i_l, \mathscr{P}^n_\lambda)\big)$$

$n = 2, l$	3	4	5	6	7	8
l-cores	(2)	(2)	(2)	(2)	(2)	(2)
μ_l dcp	[1, 1, 0]	[1, 1, 0, 0]	[1, 1, 0, 0, 0]	[1, 1, 0, 0, 0, 0]	[1, 1, 0, 0, 0, 0, 0]	[1, 1, 0, 0, 0, 0, 0, 0]
(2)	[1, 0, 0]	[1, 0, 0, 0]	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0, 0, 0]
(1, 1)	[0, 1, 0]	[0, 1, 0, 0]	[0, 1, 0, 0, 0]	[0, 1, 0, 0, 0, 0]	[0, 1, 0, 0, 0, 0, 0]	[0, 1, 0, 0, 0, 0, 0, 0]

$n = 3, l$	2	4	4	5	5	6	6
l-cores	(2, 1)	(3)	(2, 1)	(3)	(2, 1)	(3)	(2, 1)
μ_l dcp	[2, 4]	[1, 2, 2, 1]	[2, 2, 0, 2]	[1, 2, 2, 1, 0]	[2, 2, 0, 0, 2]	[1, 2, 2, 1, 0, 0]	[2, 2, 0, 0, 0, 2]
(3)	[1, 0]	[1, 0, 0, 0]	[1, 0, 0, 0]	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]
(2, 1)	[0, 2]	[0, 1, 1, 0]	[0, 1, 0, 1]	[0, 1, 1, 0, 0]	[0, 1, 0, 0, 1]	[0, 1, 1, 0, 0, 0]	[0, 1, 0, 0, 0, 1]
(1, 1, 1)	[1, 0]	[0, 0, 0, 1]	[1, 0, 0, 0]	[0, 0, 0, 1, 0]	[1, 0, 0, 0, 0]	[0, 0, 0, 1, 0, 0]	[1, 0, 0, 0, 0, 0]

$n = 4, l$	3	4	5	5	5
l-cores	(3, 1)	(2, 2)	(4)	(3, 1)	(2, 2)
μ_l dcp	[9, 6, 9]	[8, 6, 4, 6]	[4, 4, 5, 6, 5]	[6, 6, 6, 3, 3]	[8, 6, 2, 2, 6]
(4)	[1, 0, 0]	[1, 0, 0, 0]	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0]
(3, 1)	[0, 1, 2]	[1, 1, 0, 1]	[0, 1, 1, 1, 0]	[0, 1, 1, 0, 1]	[1, 1, 0, 0, 1]
(2, 2)	[1, 0, 1]	[0, 0, 2, 0]	[0, 0, 1, 0, 1]	[1, 0, 1, 0, 0]	[0, 0, 1, 1, 0]
(2, 1, 1)	[2, 1, 0]	[1, 1, 0, 1]	[1, 0, 0, 1, 1]	[1, 1, 0, 1, 0]	[1, 1, 0, 0, 1]
(1, 1, 1, 1)	[0, 0, 1]	[1, 0, 0, 0]	[0, 1, 0, 0, 0]	[0, 0, 1, 0, 0]	[1, 0, 0, 0, 0]

$n=5, l$	3	4	4	5
l-cores	$(3,1,1)$	$(4,1)$	$(3,1,1)$	$(3,2)$
μ_l dcp	$[36,42,42]$	$[32,32,28,28]$	$[24,30,36,30]$	$[25,30,25,20,20]$
(5)	$[1,0,0]$	$[1,0,0,0]$	$[1,0,0,0]$	$[1,0,0,0,0]$
$(4,1)$	$[0,2,2]$	$[0,1,1,2]$	$[0,1,2,1]$	$[1,1,1,0,1]$
$(3,2)$	$[1,2,2]$	$[2,1,1,1]$	$[1,1,2,1]$	$[1,1,1,2,0]$
$(3,1,1)$	$[4,1,1]$	$[2,2,1,1]$	$[2,2,0,2]$	$[1,2,1,1,1]$
$(2,2,1)$	$[1,2,2]$	$[1,2,1,1]$	$[1,1,2,1]$	$[1,1,1,0,2]$
$(2,1,1,1)$	$[0,2,2]$	$[1,0,2,1]$	$[0,1,2,1]$	$[1,1,1,1,0]$
$\{5\}$	$[1,0,0]$	$[0,1,0,0]$	$[1,0,0,0]$	$[0,0,1,0,0]$

$n=6, l$	2	3	4	4
l-cores	$(3,2,1)$	$(4,2)$	$(4,1,1)$	$(3,2,1)$
μ_l dcp	$[368,352]$	$[243,234,243]$	$[190,170,170,190]$	$[176,176,192,176]$
(6)	$[1,0]$	$[1,0,0]$	$[1,0,0,0]$	$[1,0,0,0]$
$(5,1)$	$[3,2]$	$[2,1,2]$	$[0,1,2,2]$	$[1,1,2,1]$
$(4,2)$	$[3,6]$	$[2,5,2]$	$[3,1,3,2]$	$[1,3,2,3]$
$(4,1,1)$	$[4,6]$	$[3,3,4]$	$[3,4,1,2]$	$[2,3,2,3]$
$(3,3)$	$[1,4]$	$[1,1,3]$	$[1,1,0,3]$	$[1,2,0,2]$
$(3,2,1)$	$[12,4]$	$[6,4,6]$	$[4,4,4,4]$	$[6,2,6,2]$
$(3,1,1,1)$	$[4,6]$	$[4,3,3]$	$[2,1,4,3]$	$[2,3,2,3]$
$(2,2,2)$	$[1,4]$	$[3,1,1]$	$[3,0,1,1]$	$[1,2,0,2]$
$(2,2,1,1)$	$[3,6]$	$[2,5,2]$	$[2,3,1,3]$	$[1,3,2,3]$
$\{4,1\}$	$[3,2]$	$[2,1,2]$	$[2,2,1,0]$	$[1,1,2,1]$
$\{6\}$	$[1,0]$	$[0,0,1]$	$[0,0,0,1]$	$[1,0,0,0]$

$n = 6, l$	5	5	5
l-cores	(5, 1)	(4, 1, 1)	(3, 3)
μ_l dcp	[140, 145, 150, 145, 140]	[150, 130, 130, 150, 160]	[140, 155, 155, 140, 130]
(6)	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0]
(5, 1)	[0, 1, 1, 1, 2]	[0, 1, 1, 2, 1]	[1, 2, 1, 0, 1]
(4, 2)	[2, 2, 2, 1, 2]	[2, 1, 2, 2, 2]	[2, 1, 2, 2, 2]
(4, 1, 1)	[3, 2, 2, 2, 1]	[3, 2, 2, 1, 2]	[2, 3, 2, 2, 1]
(3, 3)	[1, 2, 0, 2, 0]	[1, 1, 0, 2, 1]	[0, 1, 2, 2, 0]
(3, 2, 1)	[3, 3, 4, 3, 3]	[3, 3, 3, 3, 4]	[3, 3, 3, 3, 4]
(3, 1, 1, 1)	[1, 2, 2, 2, 3]	[1, 2, 2, 3, 2]	[2, 2, 3, 2, 1]
(2, 2, 2)	[0, 2, 0, 2, 1]	[2, 0, 1, 1, 1]	[2, 2, 1, 0, 0]
(2, 2, 1, 1)	[2, 1, 2, 2, 2]	[2, 2, 1, 2, 2]	[2, 2, 1, 2, 2]
{4, 1}	[2, 1, 1, 1, 0]	[2, 1, 1, 0, 1]	[0, 1, 2, 1, 1]
{6}	[0, 0, 0, 0, 1]	[0, 0, 0, 1, 0]	[0, 0, 0, 1, 0]

$n = 7, l$	4	4
l-cores	(5, 2)	(4, 1, 1, 1)
μ_l dcp	[1274, 1274, 1246, 1246]	[1240, 1260, 1280, 1260]
(7)	[1, 0, 0, 0]	[1, 0, 0, 0]
(6, 1)	[2, 1, 1, 2]	[0, 2, 2, 2]
(5, 2)	[3, 4, 5, 2]	[4, 3, 4, 3]
(5, 1, 1)	[3, 4, 3, 5]	[5, 4, 2, 4]
(4, 3)	[3, 2, 4, 5]	[2, 4, 4, 4]
(4, 2, 1)	[10, 8, 8, 9]	[9, 9, 8, 9]
(4, 1, 1, 1)	[5, 5, 5, 5]	[4, 4, 8, 4]
(3, 3, 1)	[5, 7, 4, 5]	[5, 5, 6, 5]
(3, 2, 2)	[7, 5, 5, 4]	[5, 5, 6, 5]
(3, 2, 1, 1)	[8, 10, 9, 8]	[9, 9, 8, 9]
{4, 0, 1}	[4, 3, 5, 3]	[5, 4, 2, 4]
(2, 2, 2, 1)	[2, 3, 5, 4]	[2, 4, 4, 4]
{3, 2}	[4, 3, 2, 5]	[4, 3, 4, 3]
{5, 1}	[1, 2, 2, 1]	[0, 2, 2, 2]
{7}	[0, 1, 0, 0]	[1, 0, 0, 0]

$n=7, l$	5	5	5
l-cores	(5, 1, 1)	(4, 2, 1)	(4, 1, 1, 1)
μ_l dcp	[1020, 1050, 1020, 975, 975]	[1015, 980, 980, 1015, 1050]	[920, 980, 1080, 1080, 980]
(7)	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0]	[1, 0, 0, 0, 0]
(6, 1)	[0, 1, 1, 2, 2]	[1, 1, 1, 2, 1]	[0, 1, 2, 2, 1]
(5, 2)	[3, 3, 2, 3, 3]	[2, 2, 4, 2, 4]	[2, 3, 3, 3, 3]
(5, 1, 1)	[4, 3, 4, 2, 2]	[3, 3, 3, 3, 3]	[5, 3, 2, 2, 3]
(4, 3)	[4, 3, 2, 2, 3]	[4, 2, 3, 3, 2]	[2, 2, 4, 4, 2]
(4, 2, 1)	[7, 7, 7, 7, 7]	[7, 8, 5, 8, 7]	[7, 7, 7, 7, 7]
(4, 1, 1, 1)	[3, 4, 3, 5, 5]	[4, 4, 4, 4, 4]	[2, 4, 5, 5, 4]
(3, 3, 1)	[4, 5, 5, 3, 4]	[3, 5, 4, 4, 5]	[3, 4, 5, 5, 4]
(3, 2, 2)	[5, 5, 4, 4, 3]	[4, 4, 5, 3, 5]	[3, 4, 5, 5, 4]
(3, 2, 1, 1)	[7, 7, 7, 7, 7]	[8, 5, 8, 7, 7]	[7, 7, 7, 7, 7]
{4, 0, 1}	[4, 3, 4, 2, 2]	[3, 3, 3, 3, 3]	[5, 3, 2, 2, 3]
(2, 2, 2, 1)	[2, 3, 4, 3, 2]	[3, 3, 2, 4, 2]	[2, 2, 4, 4, 2]
{3, 2}	[2, 3, 3, 3, 3]	[2, 4, 2, 2, 4]	[2, 3, 3, 3, 3]
{5, 1}	[1, 1, 0, 2, 2]	[2, 1, 1, 1, 1]	[0, 1, 2, 2, 1]
{7}	[0, 0, 1, 0, 0]	[0, 0, 0, 1, 0]	[1, 0, 0, 0, 0]

$n=7, l$	6	6	6	6
l-cores	(6, 1)	(5, 1, 1)	(4, 3)	(4, 1, 1, 1)
μ_l dcp	[834, 828, 834, 846, 852, 846]	[885, 885, 840, 795, 795, 840]	[784, 812, 868, 896, 868, 812]	[760, 800, 880, 920, 880, 800]
(7)	[1, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]
(6, 1)	[0, 1, 1, 1, 1, 2]	[0, 1, 1, 1, 2, 1]	[1, 2, 1, 1, 0, 1]	[0, 1, 1, 2, 1, 1]
(5, 2)	[3, 2, 3, 2, 2, 2]	[3, 2, 2, 2, 3, 2]	[2, 2, 3, 2, 3, 2]	[2, 2, 3, 2, 3, 2]
(5, 1, 1)	[3, 3, 2, 3, 2, 2]	[3, 3, 2, 3, 1, 3]	[2, 3, 3, 3, 2, 2]	[3, 3, 2, 2, 2, 3]
(4, 3)	[2, 3, 2, 3, 2, 2]	[3, 2, 2, 2, 2, 3]	[2, 2, 2, 4, 2, 2]	[2, 2, 2, 4, 2, 2]
(4, 2, 1)	[5, 6, 6, 6, 6, 6]	[6, 6, 6, 5, 6, 6]	[5, 6, 6, 6, 6, 6]	[5, 6, 6, 6, 6, 6]
(4, 1, 1, 1)	[4, 2, 4, 3, 4, 3]	[3, 3, 3, 4, 4, 3]	[4, 2, 4, 4, 4, 2]	[4, 2, 4, 4, 4, 2]
(3, 3, 1)	[3, 3, 4, 3, 4, 4]	[3, 5, 3, 3, 3, 4]	[4, 3, 3, 3, 4, 4]	[3, 3, 4, 4, 4, 3]
(3, 2, 2)	[4, 3, 3, 4, 4, 3]	[5, 3, 4, 3, 3, 3]	[4, 4, 4, 3, 3, 3]	[3, 3, 4, 4, 4, 3]
(3, 2, 1, 1)	[6, 6, 5, 6, 6, 6]	[6, 6, 6, 6, 5, 6]	[5, 6, 6, 6, 6, 6]	[5, 6, 6, 6, 6, 6]
{4, 0, 1}	[2, 3, 3, 2, 2, 3]	[3, 3, 3, 1, 3, 2]	[2, 2, 2, 3, 3, 3]	[3, 3, 2, 2, 2, 3]
(2, 2, 2, 1)	[2, 3, 2, 2, 2, 3]	[2, 3, 3, 2, 2, 2]	[2, 2, 2, 4, 2, 2]	[2, 2, 2, 4, 2, 2]
{3, 2}	[3, 2, 3, 2, 2, 2]	[2, 3, 2, 3, 2, 2]	[2, 2, 3, 2, 3, 2]	[2, 2, 3, 2, 3, 2]
{5, 1}	[1, 1, 0, 2, 1, 1]	[1, 0, 1, 2, 1, 1]	[1, 1, 0, 1, 1, 2]	[0, 1, 1, 2, 1, 1]
{7}	[0, 0, 1, 0, 0, 0]	[0, 1, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0]

$n=7, l$	6	7
l-cores	(3, 3, 1)	(5, 2)
μ_l dcp	[903, 903, 840, 777, 777, 840]	[686, 672, 686, 728, 770, 770, 728]
(7)	[1, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0, 0]
(6, 1)	[1, 2, 1, 0, 1, 1]	[1, 1, 1, 1, 1, 0, 1]
(5, 2)	[3, 2, 2, 2, 3, 2]	[1, 2, 3, 2, 2, 3, 1]
(5, 1, 1)	[3, 3, 2, 3, 1, 3]	[2, 2, 2, 3, 2, 2, 2]
(4, 3)	[2, 2, 3, 3, 2, 2]	[2, 1, 2, 3, 2, 1, 3]
(4, 2, 1)	[6, 6, 6, 5, 6, 6]	[5, 5, 4, 5, 6, 5, 5]
(4, 1, 1, 1)	[4, 4, 3, 3, 3, 3]	[3, 2, 3, 3, 3, 3, 3]
(3, 3, 1)	[3, 5, 3, 3, 3, 4]	[3, 3, 3, 3, 3, 4, 2]
(3, 2, 2)	[5, 3, 4, 3, 3, 3]	[3, 3, 3, 2, 4, 3, 3]
(3, 2, 1, 1)	[6, 6, 6, 6, 5, 6]	[4, 5, 5, 5, 5, 6, 5]
{4, 0, 1}	[3, 3, 3, 1, 3, 2]	[2, 2, 2, 2, 2, 2, 3]
(2, 2, 2, 1)	[2, 2, 2, 2, 3, 3]	[2, 1, 2, 3, 1, 2, 3]
{3, 2}	[2, 3, 2, 3, 2, 2]	[3, 2, 1, 1, 3, 2, 2]
{5, 1}	[2, 1, 1, 1, 0, 1]	[1, 1, 1, 1, 0, 1, 1]
{7}	[0, 1, 0, 0, 0, 0]	[0, 0, 1, 0, 0, 0, 0]

$n=7, l$	7	7	7
l-cores	(4, 3)	(4, 2, 1)	(3, 3, 1)
μ_l dcp	[588, 686, 812, 868, 812, 686, 588]	[735, 805, 805, 735, 665, 630, 665]	[882, 882, 756, 609, 546, 609, 756]
(7)	[1, 0, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0, 0]	[1, 0, 0, 0, 0, 0, 0]
(6, 1)	[1, 2, 1, 1, 0, 0, 1]	[1, 1, 1, 1, 0, 1, 1]	[1, 2, 1, 0, 0, 1, 1]
(5, 2)	[2, 2, 3, 2, 2, 2, 1]	[2, 2, 3, 1, 3, 1, 2]	[3, 2, 2, 1, 2, 2, 2]
(5, 1, 1)	[2, 3, 3, 3, 2, 1, 1]	[2, 3, 2, 2, 2, 2, 2]	[3, 3, 2, 2, 1, 1, 3]
(4, 3)	[1, 2, 2, 3, 3, 1, 2]	[3, 2, 2, 2, 2, 2, 1]	[2, 2, 2, 3, 2, 1, 2]
(4, 2, 1)	[4, 5, 6, 6, 5, 5, 4]	[5, 6, 5, 6, 3, 5, 5]	[6, 6, 5, 4, 4, 5, 5]
(4, 1, 1, 1)	[2, 2, 4, 4, 4, 2, 2]	[3, 3, 3, 3, 3, 2, 3]	[4, 4, 3, 2, 2, 2, 3]
(3, 3, 1)	[3, 2, 3, 3, 3, 4, 3]	[2, 4, 3, 3, 3, 2, 4]	[3, 4, 4, 2, 2, 3, 3]
(3, 2, 2)	[3, 4, 3, 3, 3, 2, 3]	[3, 3, 4, 2, 4, 2, 3]	[4, 3, 3, 3, 2, 2, 4]
(3, 2, 1, 1)	[4, 5, 5, 6, 6, 5, 4]	[6, 5, 6, 5, 5, 5, 3]	[6, 6, 5, 5, 4, 4, 5]
{4, 0, 1}	[1, 1, 2, 3, 3, 3, 2]	[2, 2, 3, 2, 2, 2, 2]	[3, 3, 3, 1, 1, 2, 2]
(2, 2, 2, 1)	[2, 1, 3, 3, 2, 2, 1]	[2, 2, 2, 3, 1, 2, 2]	[2, 2, 2, 1, 2, 3, 2]
{3, 2}	[1, 2, 2, 2, 3, 2, 2]	[1, 3, 2, 2, 2, 1, 3]	[2, 3, 2, 2, 2, 1, 2]
{5, 1}	[1, 0, 0, 1, 1, 2, 1]	[1, 1, 1, 1, 1, 1, 0]	[2, 1, 1, 1, 0, 0, 1]
{7}	[0, 0, 0, 0, 0, 0, 1]	[0, 0, 0, 1, 0, 0, 0]	[0, 1, 0, 0, 0, 0, 0]

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Résumé de thèse vulgarisé

Si Γ est un sous-groupe fini de $\mathrm{SL}_2(\mathbb{C})$, un des points de départ de ce travail de thèse est l'étude géométrique et combinatoire de l'action de Γ sur le schéma ponctuel de Hilbert. Ce travail passe par la description des composantes irréductibles du lieu des Γ-points fixes en termes de variétés de carquois. Il a mené à la construction d'un modèle de l'ensemble d'indexation des composantes irréductibles pour Γ de type D en termes de partitions d'entiers symétriques ainsi qu'à l'exhibition d'un lien avec les résolutions projectives et symplectiques de singularités en couronnes. Dans une autre direction, la structure de représentation du groupe symétrique et de Γ des fibres de la restriction du fibré de Procesi aux composantes irréductibles du lieu des Γ-points fixes du schéma ponctuel de Hilbert a été exploré et a aboutit à un théorème de réduction. En type A, il fait intervenir la combinatoire des cœurs et il généralise des résultats sur l'algèbre des coinvariants du groupe symétrique.

Lay summary of the thesis

If Γ denotes a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, one starting point of this thesis is the geometrical and combinatorial study of the action Γ on the punctual Hilbert scheme. This work uses quiver varieties to describe the irreducible components of the Γ-fixed points locus. This led to the construction of a model of the indexing set of the irreducible components when Γ is of type D in terms of symmetric partitions of integers as well as the exhibition of a link with projectives and symplectic resolutions of wreath singularities. In another direction, the structure of representation of the symmetric group and of Γ of the fibers of the restriction of the Procesi bundle to the irreducibles components of the Γ-fixed point locus of the punctual Hilbert scheme has been explored and resulted in a reduction theorem. In type A, it involves the combinatorics of cores and it generalizes results of the coinvariant algebra of the symmetric group.