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# The (Exceptional) Generalised Geometry of non-supersymmetric flux vacua

Vincent Menet

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DE SORBONNE UNIVERSITÉ**

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sous la direction de Michela Petrini

présentée par

**Vincent Menet**

pour obtenir le grade de :

**DOCTEUR DE SORBONNE UNIVERSITÉ**

Sujet de la thèse :

**Géométrie Généralisée (Exceptionnelle) des vides  
non-supersymétriques avec Flux**

**soutenue le 10 Septembre 2024**

devant le jury composé de :

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# Table of contents

<b>Acknowledgments</b>	<b>v</b>
<b>Publication list</b>	<b>vii</b>
<b>Abstract</b>	<b>ix</b>
<b>Résumé court en français</b>	<b>xi</b>
<b>I Introduction</b>	<b>1</b>
<b>II Supergravity flux backgrounds</b>	<b>7</b>
II.1 $G$ -structures . . . . .	8
II.1.1 Definition and examples . . . . .	9
II.1.2 Integrability and torsion . . . . .	12
II.2 Supersymmetric Flux Backgrounds . . . . .	17
II.2.1 Fluxless Compactifications . . . . .	17
II.2.2 Compactifications with Fluxes . . . . .	19
<b>III Generalised Geometry and Supersymmetry</b>	<b>23</b>
III.1 $O(6,6)$ Generalised Geometry . . . . .	24
III.1.1 Generalised Complex Geometry . . . . .	24
III.1.2 Generalised structures of supersymmetric backgrounds . . . . .	33
III.2 $E_{7(7)} \times \mathbb{R}^+$ Generalised Geometry . . . . .	35
III.2.1 Exceptional Generalised Geometry . . . . .	36
III.2.2 Generalised structures of supersymmetric backgrounds . . . . .	50
<b>IV Non-supersymmetric flux vacua and Generalised calibrations</b>	<b>59</b>
IV.1 Calibrations in generalised complex geometry . . . . .	60
IV.2 New non-supersymmetric flux vacua from generalised calibrations . . . . .	63
IV.2.1 $\mathcal{N} = 0$ flux vacua in generalised complex geometry . . . . .	65
IV.2.2 Effective potential and equations of motion from pure spinors . . . . .	74

IV.2.3	Examples of vacua with SSB supersymmetry breaking . . . . .	88
IV.2.4	Discussion . . . . .	106
IV.3	Generalised calibrations and D-term supersymmetry-breaking . . . . .	108
IV.3.1	Reformulating the gauge BPSness . . . . .	110
IV.3.2	4D structure and effective potential from pure spinors . . . . .	113
IV.3.3	D-terms in generalised complex geometry . . . . .	118
IV.3.4	The example of type IIB $SU(3)$ -backgrounds with BPS O5-planes .	125
IV.3.5	Discussion . . . . .	127
<b>V</b>	<b>Non-supersymmetric flux vacua in Exceptional Generalised Geometry</b>	<b>129</b>
V.1	The $SU(7)$ torsion . . . . .	130
V.1.1	Connection, torsion and Dorfman derivative . . . . .	131
V.1.2	The $SU(7)$ torsion from the adjoint bundle . . . . .	133
V.1.3	Backgrounds with non-vanishing $SU(7)$ intrinsic torsion . . . . .	136
V.2	The equations of motion . . . . .	144
V.3	Deformations of non-integrable $SU(7)$ structures . . . . .	148
<b>VI</b>	<b>Discussion</b>	<b>153</b>
VI.1	Future directions . . . . .	155
<b>A</b>	<b>Supergravity Conventions</b>	<b>159</b>
A.1	Notations . . . . .	159
A.2	Bosonic sector . . . . .	160
A.3	Gamma matrices . . . . .	162
<b>B</b>	<b>Algebra Conventions</b>	<b>163</b>
B.1	$O(6,6) \times \mathbb{R}^+$ Algebra . . . . .	163
B.2	$E_{7(7)} \times \mathbb{R}^+$ Algebra in type IIB . . . . .	164
B.2.1	$SL(7, \mathbb{C}) \subset SL(8, \mathbb{C}) \subset E_{7(7)\mathbb{C}}$ embedding for type IIB . . . . .	166
B.2.2	The $SU(3,3)$ parametrisation . . . . .	170
B.3	$SU(7)$ and $SU(8)$ bundles and projections . . . . .	172
<b>C</b>	<b>Supersymmetry breaking and pure spinors</b>	<b>175</b>
C.1	String-like supersymmetry breaking . . . . .	176
C.2	Pure spinors and D-term supersymmetry-breaking . . . . .	178
	<b>Bibliography</b>	<b>181</b>

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[1] Vincent Menet; “New non-supersymmetric flux vacua from generalised calibrations;” *JHEP* **05**, p. 100 (2024)

[2] Vincent Menet; “D-terms in Generalised Complex Geometry;” *JHEP* **07**, p. 071 (2024)



# Abstract

The construction of string theory relies on a symmetry relating bosons and fermions, called supersymmetry, which must be broken at low energies. Supersymmetric solutions of string theory are significantly simpler than their non-supersymmetric counterpart, and thus populate the vast majority of the literature. However, supersymmetry can be spontaneously broken at arbitrarily high energy. In this thesis, we therefore investigate solutions of type II supergravity, a classical low energy limit of string theory, that are non-supersymmetric. We do so within the framework of generalised complex geometry, a generalisation of differential geometry which unifies the spacetime coordinate transformations and the gauge transformations of one of string theory potentials, called the  $B$  field. We first construct new type II supergravity solutions, where the supersymmetry breaking mechanism is dictated by the generalised notion of stability for extended objects sourcing the supergravity fluxes. We then derive a generalised geometric expression for the non-supersymmetric supergravity fluxes, and use it to derive constraints that non-supersymmetric type II supergravity solutions should satisfy in order for their low energy effective theories to fall in the well known class of  $\mathcal{N} = 1$  four-dimensional supergravity. Finally, we describe non-supersymmetric solutions of type II supergravity within exceptional generalised geometry, a framework now unifying the spacetime coordinate transformations and the gauge transformations of all string theory potentials.



# Résumé court en français

Les travaux présentés dans cette thèse portent sur l'étude des solutions de la supergravité de type II, une limite classique à basse énergie de la théorie des cordes.

Dans ce contexte, nous nous intéressons à des solutions dix dimensionnelles de type produit : la géométrie qu'ils décrivent est le produit d'un espace quadridimensionnel non compact à symétrie maximale et d'une variété compacte à six dimensions  $M$ .

Dans les compactifications de cordes, une telle structure est utilisée pour expliquer pourquoi nous n'observons que quatre des dix dimensions prédites par la théorie des cordes : l'espace externe est l'espace que nous observons, tandis que les dimensions supplémentaires déterminent les caractéristiques de la théorie quadridimensionnelle obtenue en réduisant la supergravité de dimension supérieure.

Un aspect clé des compactifications est le fait que les modules associés à  $M$  (par exemple le volume de l'espace interne) sont associés à des champs scalaires sans masse dans les théories effectives à basse énergie correspondantes. La phénoménologie dicte donc la nécessité de stabiliser ces modules par le biais d'un potentiel, par exemple. Au niveau classique, cela se fait typiquement par l'introduction de flux : des champs décrits par des formes différentielles vivant le long de cycles de l'espace compact, de telle sorte qu'ils génèrent le potentiel approprié pour les modules.

En raison de la rétroaction des flux, les géométries compactes résultantes sont complexes et ont été étudiées avec succès dans le cadre de la géométrie généralisée. La géométrie généralisée est une généralisation de la géométrie différentielle ordinaire, où les potentiels des flux sont traités de manière géométrique. Le fibré tangent de  $M$  est étendu à un fibré tangent généralisé dont les fonctions de transition sont dictées par les difféomorphismes plus les transformations de jauge pour les potentiels des flux. Celles-ci incluent les transformations de jauge du champ  $B$  pour la géométrie complexe généralisée, et les transformations de jauge du champ  $B$  et des potentiels de Ramond-Ramond pour la géométrie généralisée exceptionnelle. Les travaux présentés dans cette thèse s'inscrivent dans le cadre des géométries généralisées, complexe et exceptionnelle.

L'exploration du paysage des compactifications quadridimensionnelles de la théorie des cordes s'est principalement concentrée sur les espaces préservant au moins la su-

persymétrie  $\mathcal{N} = 1$ . L'une des raisons est pratique : la résolution des conditions de supersymétrie, qui sont des équations différentielles du premier ordre, ainsi que celle des identités de Bianchi pour les flux, garantit d'avoir des solutions à l'ensemble des équations de mouvement de la théorie des cordes ou de la supergravité. Il est très difficile de traiter les équations du mouvement sans cette approche, même dans l'approximation de la supergravité, puisqu'il s'agit d'équations différentielles du second ordre compliquées.

Des considérations physiques motivent également l'étude des compactifications de cordes supersymétriques, notamment l'attente que la supersymétrie soit brisée à des énergies inférieures à l'échelle de compactification.

Même si la brisure de la supersymétrie à basse énergie est un scénario phénoménologiquement motivé, rien n'empêche en principe de briser spontanément la supersymétrie à des énergies arbitrairement élevées. Dans cette thèse, nous considérons cette possibilité, et nous nous concentrons sur cette région du paysage de la compactification des cordes, bien moins étudiée, et qui vaut la peine d'être exploré per se.

Une première contribution aux compactifications non-supersymétriques de la supergravité présentée dans cette thèse aborde la construction de nouvelles classes de solutions de supergravité de type II non-supersymétriques, où la supersymétrie est brisée de manière contrôlée: nous déformons les conditions de supersymétrie  $\mathcal{N} = 1$  en ajoutant des termes de brisure de supersymétrie.

Cette approche est motivée par la volonté de préserver certaines des caractéristiques pratiques des vides supersymétriques, principalement la possibilité de les caractériser via des équations différentielles du premier ordre. Comme la supersymétrie est brisée, nous devons nous assurer que les équations du mouvement sont satisfaites. L'objectif est de trouver des déformations spécifiques des équations BPS telles que les contraintes supplémentaires à imposer pour résoudre les équations du mouvement soient raisonnablement tractables.

Nous utiliserons le cadre de la géométrie complexe généralisée, où les conditions BPS  $\mathcal{N} = 1$  ont une interprétation en termes de conditions de calibration de différentes D-branes. Les conditions de supersymétrie  $\mathcal{N} = 1$  peuvent être reformulées en un ensemble de trois équations différentielles sur des polyformes définies uniquement sur l'espace de compactification interne [1]. Chacune de ces trois conditions peut être interprétée comme une condition de calibration pour des D-branes dans la géométrie [2] : des branes remplissant tout l'espace externe et des branes qui sont des murs de domaine ou des "D-cordes" dans l'espace externe.

Dans ce langage, on peut identifier différents termes de brisure de supersymétrie en fonction de la condition de calibration déformée.

Dans cette construction, nous souhaitons étendre l'étude des vides non-supersymétriques violant la condition de calibration des D-cordes. Plus précisément, nous construirons de

nouvelles solutions non-supersymétriques de type II, où le courant associé aux D-branes remplissant tout l'espace externe, présentes dans nos solution, servira d'élément constitutif pour le terme de brisure de supersymétrie violant la condition de calibrage des D-cordes.

La motivation derrière cette construction est double. La première est la simplicité : définir la brisure de supersymétrie en termes de courant des D-branes de la solution est un ansatz simple et naturel, qui à son tour réduit les équations du mouvement à un ensemble raisonnable de contraintes supplémentaires. La deuxième raison est la question de la stabilité de ces vides non-supersymétriques : en géométrie complexe généralisée, le courant des D-branes peut entrer dans le potentiel effectif associé à un vide de dix dimensions donné, ce qui est particulièrement utile car il permet d'utiliser de puissants arguments de positivité dérivés des bornes de calibration des branes dans l'étude du potentiel effectif.

Nous pouvons montrer que notre nouvelle classe de solutions partage une propriété intéressante avec les vides GKP [3], à savoir le fait qu'il existe une troncature naturelle de la théorie à dix dimensions, suggérée par la géométrie, telle que le potentiel effectif hors couche est semi-défini positif, et qu'il s'annule au niveau des solutions. Cette affirmation n'est cependant pas tout à fait équivalente à celle de la stabilité de ces nouveaux vides, puisque nous avons un contrôle limité sur cette troncature.

Par ailleurs, nous construisons également une nouvelle classe de vides généralisant les vides GKP, où les conditions de calibration des branes murs de domaine et des D-cordes sont toutes deux violées.

Une deuxième contribution aux compactifications non-supersymétriques de la supergravité présentée dans cette thèse aborde les vides dont la théorie effective à basse énergie est une solution de la supergravité quadridimensionnelle  $\mathcal{N} = 1$ , avec un superpotentiel et des F-termes non nuls, et des D-termes potentiellement non nuls.

Construire des solutions de supergravité avec des D-termes est difficile. En effet, l'exemple prototypique de la brisure de supersymétrie par un D-terme est le terme de Fayet-Iliopoulos, et la réalisation de son intégration dans la supergravité à une énergie paramétriquement inférieure à l'échelle de Planck s'avère être très compliqué [4, 5].

Nous étudions ici la possibilité d'avoir des solutions de supergravité avec des D-termes sous un angle différent, en utilisant le cadre de la géométrie complexe généralisée.

Nous nous concentrons sur une classe de solutions qui admet des sources BPS remplissant tout l'espace externe. Cela signifie que la condition BPS associée à la condition de calibration des D-branes remplissant l'espace externe, appelée condition de BPSité de jauge, est préservée, tandis que nous permettons que les conditions correspondant aux calibrations des D-branes de type D-cordes et murs de domaine soient violées.

Dans le cas supersymétrique, la condition de BPSité de jauge a été reformulée dans [6], en éliminant la dépendance explicite en la métrique, et en introduisant une version



généralisée de l'opérateur de Dolbeault. Nous généralisons cette dérivation au cas des vides non-supersymétriques violant les conditions de calibration des D-cordes et des murs de domaine.

Toujours dans le cas  $\mathcal{N} = 1$ , les conditions de BPSité de jauge et des murs de domaine ont été identifiées avec des conditions de F-termes, tandis que la BPSité des D-cordes a été interprétée comme une condition de D-terme pour la théorie effective quadridimensionnelle  $\mathcal{N} = 1$  dans [7, 8].

Dans notre classe de vides non-supersymétriques préservant la BPSité de jauge, nous nous concentrerons sur ceux qui peuvent être dimensionnellement réduits à des solutions de supergravité quadridimensionnelle  $\mathcal{N} = 1$  avec des F-termes non nuls, et éventuellement des D-termes. Pour ce faire, nous exigeons que notre ensemble d'équations BPS modifiées continue à avoir une interprétation en termes de (D-) F-termes ou de conditions d'annulation des (D-) F-termes. En particulier, la BPSité de jauge, nouvellement dérivée, doit continuer à être identifié à une condition de F-terme.

Il est intéressant de noter que cette procédure contraint certains termes de brisure de supersymétrie entrant dans la condition de calibration modifiée des D-cordes, et donc les D-termes possibles. Nous explorons ces contraintes pour certains exemples concrets de compactifications.

Nous donnerons le potentiel effectif et en déduirons les équations du mouvement pour notre classe de vides se réduisant à des solutions de supergravité quadridimensionnelle  $\mathcal{N} = 1$  avec des F-termes et des D-termes non nuls, et pour une sous-classe où les contributions des D-termes au potentiel effectif sont mises à zéro sur couche, restaurant la BPSité des D-cordes. Les solutions de ce type correspondent à des vides avec seulement des F-termes et incluent la classe amplement discutée des vides sans échelle.

Une troisième et dernière contribution aux compactifications non-supersymétriques de la supergravité présentée dans cette thèse traite du développement du formalisme de géométrie généralisée exceptionnelle pour les vides non-supersymétriques. Nous étudions en particulier la torsion associée à une certaines structure généralisée non-intégrable, dont l'obstruction à l'intégrabilité provient de la brisure de supersymétrie. Nous établissons notamment un dictionnaire entre les termes de brisure de supersymétrie en géométrie généralisée complexe et la torsion de la structure généralisée en géométrie généralisée exceptionnelle, que nous explicitons pour certains vides non-supersymétriques connus.

# Chapter I

## Introduction

The twentieth century has witnessed the birth of the two pillars of modern physics. On the one hand, quantum mechanics -or quantum field theory, its special relativistic version- describes matter and interactions at the microscopic level through the exchange of fundamental quanta [9]. On the other hand, general relativity, Einstein's theory of gravitation, spells the large scale behaviour of massive objects in terms of their interplay with the curvature of spacetime [10].

Crucially, the efforts to develop a microscopic understanding of the gravitational interaction has revealed the profound irreconcilability between these two frameworks. There are many clues and ideas pointing towards this incompatibility, let us just mention that gravity is non-renormalisable: applying the usual quantisation technique from quantum field theory [11, 12] to the gravitational interaction leads to drastically unphysical results [13].

A microscopic formulation of gravity must therefore go beyond this naive way of quantising. The most promising candidate to this enterprise is string theory. In string theory, elementary particles are replaced by excitations on vibrating strings, fundamental one-dimensional extended objects [14–21]. Not only does the spatial extension of the strings allows to avoid the usual problems one encounters when applying quantum field theory techniques to gravity [20], but also in the spectrum of string excitations one finds a particle with the properties of the mediator of gravity, the so called graviton [19].

String theory stands on an extremely simple postulate, from which spectacular consequences can be drawn:

- i) The cancellation of quantum anomalies on the string dictates spacetime to be ten-dimensional [22].
- ii) The construction of string theory relies on a symmetry relating bosons and fermions, called supersymmetry [23]. Supersymmetry arranges fundamental particles into multi-

plets, groups of bosons and fermions of the same mass, related with each other through supersymmetry transformations. However, such multiplets are not observed at low energies -below the electroweak scale-: if string theory describes our universe, supersymmetry must therefore be broken somewhere below the Planck scale.

iii) There are five different consistent superstring theories: type I, type IIA and type IIB, and  $SO(32)$  and  $E_8 \times E_8$  heterotic string theories. These are all connected through a web of dualities [24–28] and are regarded as different limits of a single eleven-dimensional theory called M-theory [29, 30].

Taken together, these facts have motivated the study of what are called ten-dimensional type II and heterotic supergravity theories<sup>1</sup>. Indeed, the string spectrum contains infinitely many massive states, but a finite number of massless states, which at low energy -much lower than the scale corresponding to the string’s extension- define an effective field theory: the theory of supergravity. In that sense, these theories of supergravity are nothing more than supersymmetric field theories, but their field content and sources are dictated by the corresponding high-energy superstring theory.

Studying solutions of type II supergravity is the main focus of this thesis<sup>2</sup>. More precisely, we are interested in ten dimensional backgrounds of type II supergravity of warped product type: the geometry they describe is the (warped) product of a four-dimensional maximally symmetric non-compact space, the external space, and a six-dimensional compact manifold  $M$ , the internal space.

Solutions of this type are called string compactifications, and such structures model the fact that we only observe four out of the ten dimensions predicted by string theory: the external space is the space we observe, while the extra dimensions determine the features of the four-dimensional theory obtained by reducing the higher-dimensional supergravity.

A key aspect of compactifications is the fact that the moduli associated to  $M$  (e.g. the volume of the internal space) are associated to massless scalar fields in the corresponding low-energy effective theories. Phenomenology thus dictates the need for these moduli to be stabilised via a potential, for instance. At the classical level this is typically done through the introduction of fluxes: non-vanishing  $p$ -form field-strengths along cycles of the compact space, such that they generate the appropriate potential for the moduli [32, 33]. Due to the back-reaction of the fluxes, the resulting compactified geometries are intricate and have been successfully studied within the framework of generalised complex geometry [34–36] and exceptional generalised geometry [37–42].

Generalised geometry is a generalisation of ordinary differential geometry, where the potentials for the fluxes are treated in a geometric way. The ordinary tangent bundle

<sup>1</sup>Supergravity is the theory of supersymmetric Einsteinian gravity [31], and is of interest on its own.

<sup>2</sup>A self-contained review of type II supergravity is presented in appendix A.

of  $M$  is extended to a generalised tangent bundle whose transitions functions are the diffeomorphisms plus the gauge transformations for the fluxes potentials. These include the  $B$ -field gauge transformations for generalised complex geometry, and both the  $B$ -field and RR potentials gauge transformations for exceptional generalised geometry. Throughout this thesis, our study of type II supergravity solutions will sit within the generalised geometry framework.

The exploration of the landscape of four-dimensional string compactifications has been mostly focused on vacua preserving at least some supersymmetry. One reason is practical: for the kind of backgrounds we are interested in, solving the supersymmetry conditions, or BPS conditions, which are first order differential equations, plus the Bianchi identities for the fluxes, guarantees to have solutions to the full set of string or supergravity equations of motion. Handling the equations of motion upfront is very hard, since they are cumbersome second order differential equations.

There are also physical considerations motivating the study of supersymmetric string compactifications, namely the expectation that supersymmetry should be broken at energies smaller than the compactification scale.

Even if low energy supersymmetry breaking is a phenomenologically motivated scenario, in principle nothing prevents supersymmetry from being spontaneously broken at arbitrarily high energies. In this thesis, we consider this possibility, and focus on this much less studied corner of the string compactification landscape, worth exploring per se.

A first contribution to non-supersymmetric compactifications of type II supergravity presented in this thesis consists in the construction of new classes of non-supersymmetric solutions. In these classes of backgrounds, supersymmetry is broken in a controlled way: we deform the supersymmetry conditions by adding supersymmetry breaking terms.

The motivation behind this approach is to preserve some of the convenient features of supersymmetric vacua, mainly the possibility to characterise them via first order differential equations. Since supersymmetry is broken, in order to find solutions we have to make sure that the equations of motion are satisfied. The goal is then to find specific deformations of the BPS equations such that the additional constraints to impose in order to solve the equations of motion are manageable.

We will use the framework of generalised complex geometry, where the BPS conditions have an interpretation in terms of stability conditions -called calibration conditions- for different probe D-branes, extended objects sourcing the supergravity fluxes<sup>3</sup> [43]. The supersymmetry conditions for warped compactifications can be recast in a set of three differential equations on polyforms defined only on the internal space [1]. Each of these

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<sup>3</sup>In the full stringy regime, D-branes are dynamical objects on which open strings end.

three conditions can be interpreted as the conditions for calibrated D-brane probes in the geometry [2]: branes filling all the external space and branes that are domain-wall or string-like.

In this language, one can identify different supersymmetry breaking terms depending on which calibration condition is modified. In these constructions we will always assume that space-filling branes are calibrated, while we will allow the calibrations of D-strings and domain-wall branes to be violated.

A famous example of non-supersymmetric type IIB solutions that violate the domain-wall calibration condition are the GKP solutions [3], describing flux compactifications to four-dimensional Minkowski space with D3 and O3 sources, where supersymmetry is broken by the  $H_{(0,3)}$  components of the NSNS-flux. The GKP backgrounds have been described within Generalised Complex Geometry in [44] as specific examples of a general framework to describe non-supersymmetric solutions.

The generalised complex geometry description of the GKP backgrounds also offers an insightful geometrical interpretation of the domain-wall supersymmetry breaking term: it is given by the current of the D-branes in the background, an internal polyform associated to the internal submanifold wrapped by the D-brane<sup>4</sup>

In the literature there is another example of non-supersymmetric solution, this time in type IIA, [45], which in the language of generalised complex geometry corresponds to the violation of the D-string calibration condition, where supersymmetry is again violated through additional NS flux components with respect to the supersymmetric case, but there is no further geometrical interpretation of the corresponding supersymmetry breaking term. Moreover, the question of stability of such backgrounds remains unaddressed.

The new non-supersymmetric type II solutions presented in this thesis extend the study of non-supersymmetric vacua violating the D-string calibration condition. More precisely, we will construct non-supersymmetric type II solutions where the current associated to the space-filling D-branes present in our backgrounds will serve as a building block for the supersymmetry breaking term violating the D-string calibration condition.

The motivation behind this construction is two-fold. The first one is simplicity: defining supersymmetry breaking in terms of the current of the background's D-branes is a natural and simple ansatz, which in turn reduces the equations of motion to a reasonable set of additional constraints. The second reason is that it can be useful to address the question of stability of these non-supersymmetric vacua: in generalised complex geometry, D-branes current can enter the effective potential associated to a given ten-dimensional background, and are particularly useful as they allow to use powerful positivity arguments

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<sup>4</sup>In the fluxless case, the current is simply the Poincaré dual of the corresponding submanifold. Its generalisation to the case of non-vanishing fluxes will be discussed at length in the text.

from some D-brane calibration bounds in the study of the effective potential.

We will also construct a new class of backgrounds generalising the GKP vacua, where both the domain-wall and D-string calibration conditions are violated.

Another approach to the study of string compactifications is the analysis of the associated low-energy effective theories. The results presented in section IV.3 will be devoted to studying a class of non-supersymmetric solutions of four-dimensional  $\mathcal{N} = 1$  supergravity.

The supersymmetry conditions in  $\mathcal{N} = 1$  supergravity can be split in F- and D-term conditions which come from the superpotential.

Interestingly, the gauge and domain-wall BPSness conditions have been identified with F-term conditions, while the string BPSness has been interpreted as a D-term condition in [7, 8].

In section IV.3 we will look for non-supersymmetric solutions of four-dimensional  $\mathcal{N} = 1$  supergravity, with non-vanishing superpotential and F-terms, and potentially non-vanishing D-terms.

Constructing supergravity solutions with D-terms is difficult. Indeed, the prototypical example of D-term supersymmetry breaking is the Fayet–Iliopoulos term, and realising its (field-dependent) embedding in supergravity at energy parametrically lower than the Planck scale turns out to be challenging [4, 5].

In this work, we investigate the possibility of having supergravity solutions with D-terms from a different angle, using the framework of generalised complex geometry.

We focus again on a class of vacua that still admit calibrated space-filling sources. This means that the BPS condition associated to the calibration condition of space-filling D-branes, dubbed the gauge BPSness condition in [44], is preserved, while we allow for the conditions corresponding to the calibrations of string-like and domain-wall probe D-branes to be violated.

In the supersymmetric case, the gauge BPSness condition has been reformulated in [6], eliminating the explicit metric dependence, and introducing a generalised version of the Dolbeault operator. We will generalise this derivation to the case of non-supersymmetric vacua violating the string and domain-wall BPSness conditions.

Within our class of non-supersymmetric backgrounds preserving the gauge BPSness, we will focus on those who can be dimensionally reduced to four-dimensional  $\mathcal{N} = 1$  supergravity solutions with non-vanishing F-terms, and possibly non-vanishing D-terms. To do so, we require that our set of modified supersymmetry conditions continues to have an interpretation in terms of either (D-) F-term or (D-) F-term conditions. In particular, the non-supersymmetric formulation of the gauge BPSness, newly derived, should still be identified with an F-term condition.

Interestingly, this procedure constrains some supersymmetry breaking terms entering the modified D-string calibration condition, and therefore the possible D-terms.

We will give the effective potential and derive the equations of motion for our class of backgrounds dimensionally reducing to non-supersymmetric solutions of four-dimensional  $\mathcal{N} = 1$  supergravity with non-vanishing F-terms and D-terms, and for a subclass where the D-term contributions to the effective potential are set to zero on-shell, restoring the D-string BPSness. Solutions of this kind correspond to vacua with only F-terms, and include the amply discussed class of no-scale vacua.

A third and last contribution to non-supersymmetric compactifications of type II supergravity presented in this thesis develops the exceptional generalised geometry formalism for non-supersymmetric type II backgrounds. An ambitious goal would be to define a notion of integrability of a given generalised geometric structure for non-supersymmetric type II supergravity solutions. This is still work in progress, and for the time being we investigate in this thesis the torsion associated to a specific non-integrable generalised structure, with the breaking of supersymmetry obstructing its integrability. We establish a dictionary between the supersymmetry breaking terms in generalised complex geometry and the torsion associated to the generalised structure in exceptional generalised geometry.

The outline of this thesis is as follows. In chapter II, we introduce the formalism of  $G$ -structures and apply it to the description of supergravity backgrounds. In chapter III, we develop the frameworks of both generalised complex geometry and exceptional generalised geometry, and use them to describe supergravity backgrounds, emphasising the geometrical interpretation associated to preserving supersymmetry. In chapter IV, we present our new classes of non-supersymmetric type II backgrounds, and our results on backgrounds reducing to non-supersymmetric solutions of four-dimensional  $\mathcal{N} = 1$  supergravity, notably the derivation of the generalisation of the gauge BPSness condition. Finally, in chapter V, we construct the exceptional generalised geometry formalism for non-supersymmetric type II backgrounds.

# Chapter II

## Supergravity flux backgrounds

In this chapter we introduce solutions of type II supergravity that are relevant for string compactifications.

We are interested in ten-dimensional solutions of (warped) product type: the geometry they describe is the (warped) product of a four-dimensional maximally symmetric non-compact space, the external space, and a six-dimensional compact manifold  $M$ , the internal space

$$ds_{10}^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{II.1})$$

with  $x^\mu$ ,  $\mu = 0, \dots, 3$  the external coordinates on  $X_4$ , and  $y^m$ ,  $m = 1, \dots, 6$  are the coordinates on  $M$ . The external space can be Minkowski or AdS<sup>1</sup>, but for the rest of this thesis we focus on compactifications to four-dimensional Minkowski space.

It has been shown that when such a background is supersymmetric, and when the fluxes satisfy their Bianchi identities, the background will automatically satisfy the equations of motion [44, 46–50]. One can therefore trade the complicated second order equations of motion of type II supergravity with the simpler first order supersymmetry conditions (and Bianchi identities), rendering the supersymmetric solutions of type II supergravity a favoured corner of the string compactification landscape. The physical scenario behind such constructions is that supersymmetry is then broken below the compactification scale, by some four-dimensional effects, such as gaugino condensation for example [51, 52]. We will discuss non-supersymmetric backgrounds, where supersymmetry is (spontaneously) broken at energies higher than the compactification scale in chapter IV.

Crucially, we must set all fermionic fields to zero in order to preserve the maximal symmetry of the external Minkowski spacetime. A background will be supersymmetric if all the supergravity fields are invariant under supersymmetry transformations, namely if their supersymmetry variations vanish. These depend on a choice of supersymmetry

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<sup>1</sup>Here we don't discuss the possibility of a dS external spacetime, as it doesn't allow for supersymmetric solutions.



parameter  $\epsilon$ , and schematically they take the following form

$$\delta_\epsilon(\text{boson}) = \epsilon(\text{fermion}) \quad \delta_\epsilon(\text{fermion}) = \epsilon(\text{boson}). \quad (\text{II.2})$$

The variations of the bosonic fields automatically vanish, given that they always contain a fermionic field, set to zero for maximally symmetric spacetime. The non-trivial conditions thus come from the variations of the fermionic fields.

As will be discussed in details later, the supersymmetry conditions in supergravity can be reformulated as differential conditions on the internal geometry. This is done in the language of  $G$ -structure, which we review in what follows.

## II.1 $G$ -structures

In this section we introduce the formalism of  $G$ -structures, a framework widely used in the literature to describe supergravity flux backgrounds [48, 53–59]. Most of the concepts introduced in this section will be generalised in the next chapter, within the Generalised Geometry framework.

We carry the discussion for a  $d$ -dimensional manifold  $M$ , before specialising to  $d = 6$  when discussing supergravity compactification manifolds in the next section.

At each point  $p$  of a  $d$ -dimensional manifold  $M$ , one can define a vector space tangent to the manifold, the *tangent space*  $T_p$ , whose elements are vectors. The collection of the tangent spaces at every point on the manifold defines the *tangent bundle*  $T$ . Picking out in a smooth way a specific vector at every point on the manifold defines a *vector field*, which is a section of the tangent bundle, denoted  $\Gamma(T)$ . At each point  $p$ , one can similarly define the space dual to the tangent space, the *cotangent space*  $T_p^*$ , whose elements are one-forms, and the corresponding *cotangent bundle*  $T^*$ .

For a given patch  $\mathcal{U}_i$  on the manifold, one can define a *local frame*  $e_m^i$  with  $m = 1, \dots, d$ , a collection of  $d$ -independent vectors spanning  $T$  at every point on  $\mathcal{U}_i$ <sup>2</sup>, forming a local basis<sup>3</sup>. One can thus locally expand a vector on such a basis  $v = v_i^m e_m^i$ , and its expressions on two overlapping patches are related by

$$v_i^m = (M_{ij})^m_n v_j^n \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j, \quad (\text{II.3})$$

where generically  $M_{ij} \in \text{GL}(d, \mathbb{R})$ . The  $M_{ij}$  are called the *transition functions* and contain all the topological information of the bundle  $T$ .

<sup>2</sup>One can similarly define a *local coframe*  $e_i^m$ , spanning  $T^*$ .

<sup>3</sup>The bundle having the set of all frames as fibres is called the *frame bundle* associated to the vector bundle  $T$ .

On a triple overlap they must satisfy the consistency condition

$$M_{ij}M_{jk} = M_{ik} \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k, \quad (\text{II.4})$$

together with

$$M_{ij}M_{ji} = \mathbb{1}. \quad (\text{II.5})$$

The set of transition functions therefore forms a group, called the *structure group*. For the tangent bundle, the structure group is  $\text{GL}(d, \mathbb{R})$ .

However one can repeat the above discussion for any vector space  $V$  and its associated vector bundle, and the transition functions will take values in another group acting on  $V$ . We will do so in the next chapter, introducing several physically interesting vector bundles.

### II.1.1 Definition and examples

Crucially, if a manifold admits a (or several) globally defined tensor  $\xi$  which is invariant under a group  $G$ , the structure group reduces to  $G \subset \text{GL}(d, \mathbb{R})$ , and is called the *G-structure* of the manifold. One can define a corresponding *principal G-bundle*  $P_G \rightarrow M$ , a bundle where the fibres are  $G$  itself<sup>4</sup>. In the literature and throughout this thesis, the *G-structure* also refers to the corresponding principal *G-bundle*.

Indeed, given that  $\xi$  is globally defined, one can choose a set of frames preserving its form on the whole manifold. The set of allowed transition functions is then restricted to the ones preserving said form, which is precisely  $G$ .

Note that the converse is also true: in general, tensors on a manifold are representations of  $\text{GL}(d, \mathbb{R})$ . When the manifold admits a *G-structure*, the tensors can be written as irreducible representations of  $G$ , and the trivial representations of  $G$  form globally defined trivial bundles, corresponding to the *G-invariant tensors*.

In supergravity, the physically relevant compactification manifolds will typically admit globally defined *G-invariant tensors*, reducing their structure group. We give here a few examples.

#### **SL( $d, \mathbb{R}$ )-structure**

The simplest example is the one of orientable manifolds, admitting a globally defined volume form, invariant under  $\text{SL}(d, \mathbb{R}) \subset \text{GL}(d, \mathbb{R})$ .

#### **O( $d$ )-structure**

Manifolds admitting a globally defined, symmetric, positive-definite two-tensor  $g$ , a *metric*, are called *Riemannian manifolds*. They admit local frames on every patch, to-

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<sup>4</sup>See [60] for formal details about principal *G-bundles*.

gether called a *vielbein* respecting

$$e_i^m e_j^n g_{mn} = \delta_{ij}. \quad (\text{II.6})$$

The structure group thus reduces to  $O(d)$ . If a Riemannian manifold also admits a globally defined volume form, its structure group further reduces to  $SO(d)$ .

### Almost complex structure

An *almost complex structure* is a globally defined endomorphism on the tangent bundle

$$I : T \rightarrow T, \quad (\text{II.7})$$

respecting

$$I^2 = -\mathbb{1}. \quad (\text{II.8})$$

Only manifolds of even dimensions can admit almost complex structures. A manifold admitting an almost complex structure is called a *almost complex manifold*. The presence of an almost complex structure reduces the structure group to  $GL(d/2, \mathbb{C})$ .

An almost complex structure induces the decomposition of the complexified tangent bundle into two maximal eigenspaces<sup>5</sup>

$$T \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} := L_1 \oplus L_{-1}, \quad (\text{II.9})$$

where  $L_{\pm 1}$  is the eigenbundle of eigenvalue  $\pm i$  under the action of  $I$ . These are complex conjugate  $\bar{L}_1 = L_{-1}$  and they respect  $L_1 \cap L_{-1} = 0$ . This decomposition allows for the introduction of holomorphic and anti-holomorphic components, not only on the complexified tangent bundle, but also on the complexified cotangent bundle

$$T^* \otimes \mathbb{C} = T^{*1,0} \oplus T^{*0,1}. \quad (\text{II.10})$$

This induces the decomposition of  $k$ -forms into holomorphic and anti-holomorphic components: we denote  $\Lambda^k T^*$  as the bundle of  $k$ -forms, and  $\Omega^k$  its global sections, and we have

$$\Lambda^k T^* = \bigoplus_i^k (\Lambda^i T^{*1,0} \otimes \Lambda^{k-i} T^{*0,1}) := \bigoplus_i^k \Lambda^{i,k-i} T^*. \quad (\text{II.11})$$

A section of  $\Lambda^{p,q} T^*$  is denoted as  $\Omega^{p,q}$ , and called a  $(p,q)$ -form. In particular,  $\Lambda^{d/2,0} T^*$  is called the *canonical line bundle*. One can define a local frame of  $d/2$  independent

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<sup>5</sup>A subspace is maximal if its dimension is half the dimension of the original space.

(1,0)-forms  $\theta^m \in \Omega^{1,0}$ , such that

$$\Omega = \theta^1 \wedge \dots \wedge \theta^{d/2} \quad (\text{II.12})$$

is a local section of the canonical line bundle, which is decomposable<sup>6</sup> and non-degenerate

$$\Omega \wedge \bar{\Omega} \neq 0. \quad (\text{II.13})$$

### **SL( $d/2, \mathbb{C}$ ) structure**

Given a  $\text{GL}(d/2, \mathbb{C})$  structure, if  $\Omega$  is globally defined, the structure group is further reduced to  $\text{SL}(d/2, \mathbb{C})$ .

### **Symplectic structure**

A manifold admits a *pre-symplectic structure* if there exists a globally defined non-degenerate real two-form  $J \in \Omega^2$ . Its non-degeneracy can be expressed as

$$J^{d/2} \neq 0. \quad (\text{II.14})$$

The existence of such a two-form reduces the structure group to  $\text{Sp}(d, \mathbb{R})$ . A manifold admitting a pre-symplectic structure is called *almost symplectic*.

### **U( $d/2$ ) structure**

A  $U(d/2)$  structure on an almost complex manifold is defined by admitting both a decomposable  $(d/2, 0)$ -form  $\Omega$  and a pre-symplectic form  $J$ , compatible with the almost complex structure, in the sense that

$$J \wedge \Omega = 0. \quad (\text{II.15})$$

The forms  $J$  and  $\Omega$  are then simultaneously invariant under the intersection  $\text{Sp}(d, \mathbb{R}) \cap \text{GL}(d/2, \mathbb{C}) = \text{U}(d/2)$ , reducing the structure group to  $\text{U}(d/2)$ . In terms of the local frame  $\theta^m$  introduced above, we have

$$\Omega = \theta^1 \wedge \dots \wedge \theta^{d/2} \quad (\text{II.16})$$

$$J = -\frac{i}{2} \sum_m \theta^m \wedge \bar{\theta}^m. \quad (\text{II.17})$$

Together, the pre-symplectic and almost complex structure define a metric. In indices, it is given by

$$g_{mn} = -J_{mp} I^p{}_n. \quad (\text{II.18})$$

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<sup>6</sup>A form is decomposable if it can be locally written as the wedge product of one-forms.

This metric is *hermitian*:

$$g_{pq}I^p{}_m I^q{}_n = g_{mn}. \quad (\text{II.19})$$

### SU( $d/2$ ) structure

If a manifold admits a  $U(d/2)$  structure and if the decomposable  $(d/2, 0)$ -form  $\Omega$  is globally defined, then the structure groups is further reduced to  $Sp(d, \mathbb{R}) \cap SL(d/2, \mathbb{C}) = SU(d/2)$ .

One can define an  $SU(d/2)$  structure in an alternative way. If the manifold admits a globally defined chiral pure spinor<sup>7</sup> invariant under  $SU(d/2)$ , the structure group is reduced to  $SU(d/2)$ . One can explicitly construct the pre-symplectic two-form and the  $(d/2, 0)$ -form associated with the  $SU(d/2)$  structure out of the spinor  $\eta$

$$J_{mn} = i\eta^\dagger \gamma_{mn} \eta \quad (\text{II.20})$$

$$\Omega_{mnp} = \eta^T C \gamma_{mnp} \eta, \quad (\text{II.21})$$

where the charge conjugation operator  $C$  is defined so that the complex conjugate  $C^{-1}\eta^*$  of a given spinor  $\eta$  transforms in the same way as  $\eta$  under the Clifford algebra.

### II.1.2 Integrability and torsion

Up until this point, we have organised the geometry of  $d$ -dimensional manifolds in terms of the invariant and non-degenerate tensors they admit, stressing the reduction of their structure group accordingly. In this subsection, we study the differential conditions that can be imposed on such tensors. These will be of the utmost physical importance, given that they will be identified with the conditions one has to impose in order to preserve some amount of supersymmetry, within the context of supergravity<sup>8</sup>, as will be made precise later on.

In order to do so, we introduce connections on the tangent bundle. A *connection*  $\nabla$  is a map

$$\nabla : \Gamma(T) \rightarrow \Gamma(T \otimes T^*) \quad (\text{II.22})$$

satisfying the following derivative property<sup>9</sup>

$$\nabla(fv) = f\nabla v + v \otimes df, \quad (\text{II.23})$$

with  $v$  a section of the tangent bundle and  $f$  a smooth function on the manifold.

<sup>7</sup>A spinor  $\eta$  is chiral if  $\gamma_{(d)}\eta = \eta$ , with  $\gamma_{(d)}$  the chirality operator, defined in Appendix A.3, together with the gamma matrices conventions. A spinor is pure if it is annihilated by half of the gamma matrices.

<sup>8</sup>The presence of non-trivial fluxes will spoil this correspondence, as discussed in the next section.

<sup>9</sup>Here  $d$  is the standard exterior derivative  $d : \Omega^n \rightarrow \Omega^{n+1}$

One can then define the *torsion* of a given connection

$$T_{\nabla}(v, w) = \nabla_v w - \nabla_w v - [v, w], \quad (\text{II.24})$$

where  $[, ]$  is the usual Lie bracket<sup>10</sup>.

For a given *G*-structure, let  $\xi$  be an invariant tensor under *G*. A connection is said to be *compatible* with the *G*-structure if

$$\nabla \xi = 0. \quad (\text{II.25})$$

A *G*-structure is then said to be *integrable*, or *torsion-free* if there exists a corresponding compatible connection with vanishing torsion.

One can define a notion of torsion associated to a *G*-structure without resorting to picking a specific connection. To do so, we start from a given compatible connection  $\nabla$ , and we write any compatible connection  $\nabla'$  as  $\nabla' = \nabla + \Sigma$  where

$$\Sigma = \nabla' - \nabla \in \Gamma(K_G) \quad \text{with} \quad K_G = T^* \otimes \text{ad}P_G, \quad (\text{II.26})$$

and with  $\text{ad}P_G$  the  $\text{ad}P_G$ -bundle with fibres belonging to the adjoint representation of *G*. The torsion of a generic connection will be a section of the bundle

$$T_{\nabla} \in \Gamma(W) \quad \text{with} \quad W = T \otimes \Lambda^2 T^*. \quad (\text{II.27})$$

We then define the map

$$\begin{aligned} \tau : K_G &\rightarrow W \\ \Sigma &\rightarrow \tau(\Sigma) = T_{\nabla'} - T_{\nabla}, \end{aligned} \quad (\text{II.28})$$

and denoting the vector bundle associated to the image of  $\tau$  by  $\text{Im}\tau = W_G$ , we can define

$$W_{\text{int}}^G = W/W_G. \quad (\text{II.29})$$

$W_{\text{int}}^G$  does not depend on the choice of compatible connection, it only depends on the *G*-structure.  $W_{\text{int}}^G$  is the *intrinsic torsion* of the *G*-structure.

It will prove to be useful throughout the thesis to decompose the intrinsic torsion of a given *G*-structure into irreducible representations of *G*

$$W_{\text{int}}^G = \bigoplus_i W_i. \quad (\text{II.30})$$

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<sup>10</sup>The Lie bracket is an antisymmetric bracket on *T* satisfying the Jacobi identity. In coordinates, with  $v = v^m \frac{\partial}{\partial y^m}$  and  $w = w^n \frac{\partial}{\partial y^n}$ , it reads  $[v, w] = \left( v^n \frac{\partial w^m}{\partial y^n} - w^n \frac{\partial v^m}{\partial y^n} \right) \frac{\partial}{\partial y^m}$ .

A torsion-free compatible connection exists if and only if the intrinsic torsion of the  $G$ -structure vanishes.

The requirement for a  $G$ -structure to be torsion-free can be formulated as differential conditions on the  $G$ -invariant objects, as we now illustrate for a few examples.

### a) Examples

#### Symplectic structure

For a given connection  $\nabla$  compatible with a pre-symplectic structure  $J$ :  $\nabla J = 0$ , one can show that<sup>11</sup>

$$dJ(v, w, u) = J(T_{\nabla}(v, w), u) + J(T_{\nabla}(u, v), w) + J(T_{\nabla}(w, u), v). \quad (\text{II.31})$$

A pre-symplectic structure  $J$  will therefore be integrable if

$$dJ = 0, \quad (\text{II.32})$$

such that its intrinsic torsion vanishes. This statement depends only on the intrinsic component of the torsion, since the left-hand side of (II.31) is clearly independent of the choice of connection. An integrable pre-symplectic structure is called a *symplectic structure*.

#### Complex structure

For a given almost complex structure  $I$ , and for any pair  $v, w$  of smooth vectors fields, we introduce the following tensor, the *Nijenhuis tensor*

$$N_I(v, w) = I[Iv, w] + I[v, Iw] - [Iv, Iw] + [v, w]. \quad (\text{II.33})$$

Then, introducing a connection  $\nabla$  compatible with the almost complex structure:  $\nabla I = 0$ , one can show that

$$N_I(v, w) = T_{\nabla}(v, w) - T_{\nabla}(Iv, Iw) + IT_{\nabla}(Iv, w) + IT_{\nabla}(v, Iw). \quad (\text{II.34})$$

The almost complex structure will thus be integrable if and only if

$$N_I(v, w) = 0 \quad \forall v, w \in \Gamma(T). \quad (\text{II.35})$$

Interestingly, this condition is equivalent to

$$[L_1, L_1] \subset L_1, \quad (\text{II.36})$$

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<sup>11</sup>Here the three-form and two-forms are taken as the maps  $T \otimes T \otimes T \rightarrow \mathbb{R}$  and  $T \otimes T \rightarrow \mathbb{R}$  respectively.

with again  $L_1$  the  $+i$ -eigenbundle of  $I$ . The integrability of the almost complex structure is hence equivalent to the stability of the  $L_1$  bundle under the Lie bracket<sup>12</sup>, in this case we say that the bundle  $L_1$  is *involutive*.

Spelling the integrability of an almost complex structure as the involutivity of its eigenbundles will turn out to be a natural formulation to generalise the notion of integrability in the case of two generalisations of complex structures, the generalised complex structure and the exceptional complex structure, both discussed in the next chapter.

It is then easy to show that the condition (II.36) is equivalent to

$$w \lrcorner v \lrcorner d\Omega = 0 \quad \forall v, w \in \Gamma(T^{1,0}), \quad (\text{II.37})$$

which is itself equivalent to

$$d\Omega = \bar{\theta} \wedge \Omega \quad \text{with} \quad \bar{\theta} \in \Gamma(T^{*0,1}). \quad (\text{II.38})$$

The almost complex structure, or  $\text{GL}(d/2, \mathbb{C})$  structure, is thus integrable if (II.38) is respected. It is then called a *complex structure*.

If we consider again the decomposition (II.11), acting with the exterior derivative on a  $(p, q)$ -form  $\phi^{p,q}$ , we find generically

$$d\phi^{p,q} \in \Omega^{p+2,q-1} \cup \Omega^{p+1,q} \cup \Omega^{p,q+1} \cup \Omega^{p-1,q+2}. \quad (\text{II.39})$$

However, if the complex structure is integrable, this decomposition reduces to

$$d\phi^{p,q} \in \Omega^{p+1,q} \cup \Omega^{p,q+1}, \quad (\text{II.40})$$

and the exterior derivative decomposes into the Dolbeault operators  $\partial$  and  $\bar{\partial}$

$$d = \partial + \bar{\partial}, \quad (\text{II.41})$$

with

$$\partial : \Omega^{p,q} \rightarrow \Omega^{p+1,q} \quad \bar{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q+1}. \quad (\text{II.42})$$

### **SL( $d/2, \mathbb{C}$ ) structure**

Given an integrable  $\text{GL}(d/2, \mathbb{C})$  structure, it is straightforward to see that the right-hand side of (II.38) actually belongs to the  $\text{SL}(d/2, \mathbb{C})$  intrinsic torsion. An  $\text{SL}(d/2, \mathbb{C})$  structure will therefore be integrable if and only if one imposes the stronger condition

$$d\Omega = 0. \quad (\text{II.43})$$

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<sup>12</sup>Or equivalently the stability of the  $L_{-1}$  bundle under the Lie bracket.



Just as the integrability of a complex structure can be formulated geometrically, as the involutivity of its subbundles, the further requirement for an  $\mathrm{SL}(d/2, \mathbb{C})$  to be integrable also has a geometrical interpretation, as the vanishing of a moment map<sup>13</sup>.

For an  $\mathrm{SL}(d/2, \mathbb{C})$  structure, a choice of  $\Omega$  at a point  $p \in M$  is equivalent to picking a point in the coset

$$\Omega|_p \in Q_{\mathrm{SL}(d/2, \mathbb{C})} = \frac{\mathrm{GL}(d/2, \mathbb{C})}{\mathrm{SL}(d/2, \mathbb{C})}. \quad (\text{II.45})$$

The choice of an  $\mathrm{SL}(d/2, \mathbb{C})$  structure on  $M$  therefore corresponds to a section of the fibre bundle

$$Q_{\mathrm{SL}(d/2, \mathbb{C})} \rightarrow \mathcal{Q}_{\mathrm{SL}(d/2, \mathbb{C})} \rightarrow M \quad (\text{II.46})$$

and we can identify the space of  $\mathrm{SL}(d/2, \mathbb{C})$  structures as  $\mathcal{Z} \simeq \Gamma(\mathcal{Q}_{\mathrm{SL}(d/2, \mathbb{C})})$ . One can also restrict to the space of  $\mathrm{SL}(d/2, \mathbb{C})$  structures with an integrable associated complex structure

$$\hat{\mathcal{Z}} = \{\Omega \in \mathcal{Z} \mid I \text{ is integrable}\}. \quad (\text{II.47})$$

Crucially, the space  $\hat{\mathcal{Z}}$  inherits a symplectic structure from the symplectic structure on the coset space  $Q_{\mathrm{SL}(d/2, \mathbb{C})}$ . One can therefore define the following moment map [61]

$$\mu : \hat{\mathcal{Z}} \rightarrow \mathfrak{diff}^* \quad (\text{II.48})$$

for the action of diffeomorphisms, where  $\mathfrak{diff}$  is the Lie algebra of diffeomorphisms, and where

$$\mu(V) = \int_M \mathcal{L}_V \Omega \wedge \bar{\Omega}, \quad (\text{II.49})$$

with  $V \in \Gamma(T)$  and  $\mathcal{L}_V$  the standard Lie derivative acting on a  $p$ -form  $\omega$  as  $\mathcal{L}_V \omega = \{d, V \lrcorner\} \omega$ . Using the integrability of the complex structure (II.38), one can show that imposing the vanishing of the moment map (II.49) results in  $\bar{\theta} = 0$  and thus

$$d\Omega = 0. \quad (\text{II.50})$$

The vanishing of the moment map hence imposes the final condition that promotes a complex structure to a torsion-free  $\mathrm{SL}(d/2, \mathbb{C})$  structure.

This interpretation of the integrability of an  $\mathrm{SL}(d/2, \mathbb{C})$  structure as the involutivity of the complex structure subbundles and the vanishing of a moment map for the diffeo-

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<sup>13</sup>Given a Lie group  $G$ , its algebra  $\mathfrak{g}$  and dual algebra  $\mathfrak{g}^*$ , with the action of  $G$  on a symplectic manifold  $M$  preserving the symplectic form  $J$ , the *moment map* for the  $G$ -action is a map

$$\mu : M \rightarrow \mathfrak{g}^* \quad (\text{II.44})$$

such that  $d\langle \mu, \xi \rangle = \rho(\xi) \lrcorner J$ , with  $\xi \in \mathfrak{g}$ ,  $\rho(\xi)$  a vector field on  $M$  induced by the infinitesimal action of  $\xi$ , and here  $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ .

morphisms action will also turn out to have a natural generalisation when discussing the integrability of the generalised versions of the complex and  $\mathrm{SL}(d/2, \mathbb{C})$  structures, as will be discussed in the next chapter.

### SU(3) structure

Given that this is a standard example widely used throughout this thesis, we specialise here to  $d = 6$  and discuss the case of  $\mathrm{SU}(3)$  structures.

The intrinsic torsion of an  $\mathrm{SU}(3)$  structure  $W_{\mathrm{int}}^G$  can be shown to decompose into  $\mathrm{SU}(3)$  irreducible representations as [62]

$$W_{\mathrm{int}}^G \sim (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus 2(\mathbf{3} \oplus \bar{\mathbf{3}}). \quad (\text{II.51})$$

The intrinsic torsion of an  $\mathrm{SU}(3)$  structure can thus be represented as 5 tensors  $W_1, \dots, W_5$ .  $W_1 \in \mathbf{1} \oplus \mathbf{1}$  is a complex scalar,  $W_2 \in \mathbf{8} \oplus \mathbf{8}$  is a complex *primitive*  $(1, 1)$ -form<sup>14</sup>,  $W_3 \in \mathbf{6} \oplus \bar{\mathbf{6}}$  is a real primitive  $(2, 1) + (1, 2)$  form, and  $W_4, W_5$  are real one-forms. One can show that they obey

$$dJ = \frac{3}{2} \mathrm{Im}(\bar{W}_1 \Omega) + W_4 \wedge J + W_3 \quad (\text{II.52})$$

$$d\Omega = W_1 J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega. \quad (\text{II.53})$$

Hence an  $\mathrm{SU}(3)$  structure is integrable if and only if

$$dJ = 0 \quad d\Omega = 0. \quad (\text{II.54})$$

More generally, table II.1 below describes the possible geometries arising for various torsion class configurations [63].

## II.2 Supersymmetric Flux Backgrounds

In this section, we show how supersymmetry conditions for string compactifications are described in terms of  $G$ -structure. We start with fluxless compactifications before treating flux compactifications, discussing the interplay between physical fluxes and the integrability of the structures on the compactification manifold.

### II.2.1 Fluxless Compactifications

We first consider type II backgrounds where the supergravity fluxes are set to zero. The only bosonic fields are thus the metric and the dilaton. As for the fermions, there are two

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<sup>14</sup>A primitive form  $\omega$  obeys  $\omega \wedge J = 0$ .

Geometry	Torsion Classes
Complex	$W_1 = W_2 = 0$
Symplectic	$W_1 = W_3 = W_4 = 0$
Half Flat	$\text{Im}W_1 = \text{Im}W_2 = W_4 = W_5 = 0$
Special hermitian	$W_1 = W_2 = W_4 = W_5 = 0$
Nearly Kähler	$W_2 = W_3 = W_4 = W_5 = 0$
Almost Kähler	$W_1 = W_3 = W_4 = W_5 = 0$
Kähler	$W_1 = W_2 = W_3 = W_4 = 0$
Calabi-Yau	$W_1 = W_2 = W_3 = W_4 = W_5 = 0$
Conformal Calabi-Yau	$W_1 = W_2 = W_3 = 3W_4 - 2W_5 = 0.$

Table II.1: Different geometries of six-dimensional manifolds with an  $SU(3)$  structure depending on their  $SU(3)$  torsion classes.

gravitinos  $\psi_M^i$  with  $M = 1, \dots, 10$  ten-dimensional indices and two dilatinos  $\lambda^i$  of opposite (the same) chirality for type IIA (IIB). Given two ten-dimensional Majorana-Weyl spinors  $\epsilon^i$  of opposite (the same) chirality for type IIA (IIB), the supersymmetry conditions are [64, 65]

$$\delta_\epsilon \psi_M^i = \hat{\nabla}_M \epsilon^i \quad (\text{II.55})$$

$$\delta_\epsilon \lambda^i = (\not{\nabla} \phi) \epsilon^i \quad (\text{II.56})$$

where  $\hat{\nabla}_M$  is the ten-dimensional Levi-Civita connection,  $\phi$  is the dilaton and where for a  $p$ -form  $\omega$  the slash symbol denotes

$$\not{\omega} = \frac{1}{p!} \omega_{M_1 \dots M_p} \Gamma^{M_1 \dots M_p}, \quad (\text{II.57})$$

with  $\Gamma$  the ten-dimensional gamma matrices.

We consider the ten-dimensional space-time to be the product of a four-dimensional Minkowski space  $X_4$  and a six-dimensional compact manifold  $M$ , with the following metric ansatz

$$ds_{10}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{II.58})$$

where  $x^\mu$ ,  $\mu = 0, \dots, 3$  are the external coordinates on  $X_4$ , and  $y^m$ ,  $m = 1, \dots, 6$  are the coordinates on  $M$ . The supersymmetry parameters decompose accordingly under  $\text{Spin}(9, 1) \rightarrow \text{Spin}(3, 1) \times \text{Spin}(6)$  as

$$\epsilon_1 = \zeta \otimes \eta_1 + c.c. \quad \epsilon_2 = \zeta \otimes \eta_2 + c.c. \quad (\text{II.59})$$

where  $\zeta$  is a Weyl spinor of positive chirality on  $X_4$ , and  $\eta_1$  and  $\eta_2$  are Weyl spinors on the six-dimensional internal space.  $\eta_1$  has positive chirality, while  $\eta_2$  has negative chirality in type IIA and positive chirality in type IIB.

The presence of nowhere vanishing spinor fields on  $M$  imposes a reduction of the structure group of  $M$  to a subgroup of  $\text{Spin}(6) \simeq \text{SU}(4)$ . Seen as four-dimensional Weyl spinors, each internal spinor has an  $\text{SU}(3)$  stabiliser group, reducing the structure group of  $M$  to  $\text{SU}(3)$ . The two internal spinors can be parallel, nowhere parallel or a mix of the two depending on the position on the manifold. If they are nowhere parallel, the intersection of their  $\text{SU}(3)$  structures defines an  $\text{SU}(2)$  structure. We will return to this case in chapter IV, and we focus here on the case where the spinors are parallel, and  $M$  has an  $\text{SU}(3)$  structure group. Without loss of generality we consider  $\eta_1 = \eta_2 := \eta$  in type IIB<sup>15</sup> and normalise it to one. We write the pre-symplectic two-form and the  $(3,0)$ -form associated with the  $\text{SU}(3)$  structure in terms of the spinor  $\eta$

$$J_{mn} = i\eta^\dagger \gamma_{mn} \eta \quad (\text{II.60})$$

$$\Omega_{mnp} = \eta^T \gamma_{mnp} \eta. \quad (\text{II.61})$$

The supersymmetry condition (II.55) then decomposes into

$$\tilde{\nabla} \zeta = 0 \quad \nabla \eta = 0 \quad (\text{II.62})$$

with  $\tilde{\nabla}$  the four-dimensional Levi-Civita and  $\nabla$  the six-dimensional one.  $\zeta$  is hence a constant spinor on  $X_4$  and  $\eta$  is constant on  $M$ . The constant external spinor  $\zeta$  generates the  $\mathcal{N} = 1$  rigid supersymmetry of the effective theory.

The second condition in (II.62) translates into

$$dJ = 0 \quad d\Omega = 0. \quad (\text{II.63})$$

Imposing a fluxless background with parallel internal spinors to be supersymmetric is therefore equivalent to requiring its  $\text{SU}(3)$  structure to be integrable.

## II.2.2 Compactifications with Fluxes

A key aspect of compactifications is the fact that the moduli associated to  $M$  (for example the volume of the internal space) are associated to massless scalar fields in the corresponding low-energy effective theories. Phenomenology thus dictates the need for these moduli to be stabilised via a potential, for instance. At the classical level this is typically done through the introduction of fluxes: non-vanishing  $p$ -form field-strengths

<sup>15</sup>In type IIA one would have  $\eta_1 = \eta_2^* := \eta$

along cycles of the compact space, such that they generate the appropriate potential for the moduli. We are thus led to the investigation of type II supergravity backgrounds with non-trivial fluxes.

It is important to recall that type II compactifications to Minkowski space with non-trivial fluxes must have some negative contributions to the energy momentum tensor [66, 67], which is typically realised through the introduction of Orientifold Planes [68]. In that context, it is important to note that the discussion below and throughout the thesis only holds to describe the geometry of the compactification manifold away from the sources.

We start by quickly reviewing the bosonic content of type II supergravity with fluxes. The bosonic sector of type II supergravity is composed of the NS sector and the RR sector, given that the fields originate from states in the String Theory that obey NS-NS or R-R boundary conditions. The NS sector contains the metric, the dilaton, and the NS three-form flux, which can locally be written

$$H = dB \tag{II.64}$$

away from the NS sources, with  $B$  its two-form potential.

The RR sector contains the RR field-strength: we use the democratic formulation of [69], where

$$F^{10} = \sum_q F_q^{10} \tag{II.65}$$

with  $q = 0, 2, \dots, 10$  for type IIA and  $q = 1, 3, \dots, 9$  for type IIB. These fields obey the following self-duality condition

$$F^{10} = \tilde{*}F^{10}, \tag{II.66}$$

with the Hodge operator  $\tilde{*}$  defined in Appendix A. Away from the RR sources, we write the RR fluxes from the RR potentials as

$$F^{10} = dC + H \wedge C \tag{II.67}$$

with  $C = \sum_q C_{q-1}$ .

We now specify the compactification ansatz: we consider type II solutions that are the warped product of four-dimensional Minkowski space  $X_4$  and a six-dimensional compact manifold  $M$ , with the following metric

$$ds_{10}^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \tag{II.68}$$

where again  $x^\mu$ ,  $\mu = 0, \dots, 3$  are the external coordinates on  $X_4$ , and  $y^m$ ,  $m = 1, \dots, 6$  are

the coordinates on  $M$ .

The Poincaré invariance of  $X_4$  constrains the NS and RR-fluxes: the NS-field-strength  $H$  can only have internal legs, and the ten-dimensional RR-field-strength must take the form

$$F^{10} = F + e^{4A} \text{vol}_4 \wedge \tilde{F}, \quad (\text{II.69})$$

with  $\text{vol}_4$  the volume form on  $X_4$  and where  $F$  and  $\tilde{F}$  are purely internal and are related by the self-duality of  $F^{10}$  (II.66) as

$$\tilde{F} = \tilde{*}_6 F, \quad (\text{II.70})$$

with  $\tilde{*}_6$  defined in Appendix A.

Preserving  $\mathcal{N} = 1$  supersymmetry amounts to the vanishing of the following gravitinos and dilatino variations

$$\delta_\epsilon \psi_M = \left( \nabla_M + \frac{1}{4} \iota_M \not{H} \sigma_3 + \frac{e^\phi}{16} \begin{pmatrix} 0 & \not{F}^{10} \\ -\sigma(\not{F}^{10}) & 0 \end{pmatrix} \Gamma_M \Gamma_{(10)} \right) \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad (\text{II.71})$$

$$\delta_\epsilon \lambda = \left( \not{\partial} \phi + \frac{1}{2} \not{H} \sigma_3 + \frac{e^\phi}{16} \Gamma^M \begin{pmatrix} 0 & \not{F}^{10} \\ -\sigma(\not{F}^{10}) & 0 \end{pmatrix} \Gamma_M \Gamma_{(10)} \right) \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad (\text{II.72})$$

with  $\sigma$  the reversal of all form indices, and where  $\epsilon_1$  and  $\epsilon_2$  are again ten-dimensional Majorana-Weyl spinors. The spinors  $\epsilon_1$  and  $\epsilon_2$  again split in the following way

$$\epsilon_1 = \zeta \otimes \eta_1 + c.c. \quad \epsilon_2 = \zeta \otimes \eta_2 + c.c. \quad (\text{II.73})$$

where  $\zeta$  is a Weyl spinor of positive chirality on  $X_4$ , and  $\eta_1$  and  $\eta_2$  are Weyl spinors on the six-dimensional internal space.  $\eta_1$  has again positive chirality, while  $\eta_2$  has negative chirality in type IIA and positive chirality in type IIB. We focus here on the case where the internal spinors are parallel, and  $M$  has an  $SU(3)$  structure group, taking  $\eta_1 = \eta_2 := \eta$  in type IIB<sup>16</sup> and normalising it to one.

We then split the supersymmetry conditions (II.71), (II.72) into internal and external components. The external spinor again has to be constant, while we get a complicated set of conditions on the internal spinor.

Crucially, the six-dimensional Levi-Civita connection is no longer compatible with the internal spinor, and the lack of compatibility precisely comes from the non-vanishing fluxes, which thus arrange themselves into the torsion classes of the  $SU(3)$  structure.

Defining again the pre-symplectic two-form and the  $(3,0)$ -form associated with the  $SU(3)$  structure in terms of the spinor  $\eta$ , as in (II.60), (II.61), recall from the discussion

<sup>16</sup>In type IIA one would have  $\eta_1 = \eta_2^* := \eta$

of  $SU(3)$  structures that we generically have

$$dJ = \frac{3}{2}\text{Im}(\bar{W}_1\Omega) + W_4 \wedge J + W_3 \quad (\text{II.74})$$

$$d\Omega = W_1 J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega. \quad (\text{II.75})$$

For a given background, the non-vanishing torsion classes are completely defined by the fluxes. Solutions to these supersymmetry conditions with different fluxes configurations have been studied in depth in the literature [56, 70–76], while a summary of all possible  $\mathcal{N} = 1$  Minkowski  $SU(3)$  flux backgrounds have been presented in [77]. We won't give here the different solutions and the expressions of their torsion classes in terms of the fluxes, since these are rather convoluted. We instead display a few solutions in terms of their vanishing torsion classes

<b>IIA</b>	<b>IIB</b>
$W_1 = W_2 = 0, \bar{W}_5 = 2W_4$	$W_1 = W_2 = 0, \bar{W}_5 = 2W_4$
$W_1 = W_3 = W_4 = 0$	$W_1 = W_2 = W_3 = 0$
	$W_1 = W_2 = W_3 = 0.$

Each line corresponds to the torsion classes configuration for a given class of solutions, where the non-vanishing torsion classes are defined by the fluxes. Recalling table II.1, we see that for type IIA, the compactification manifold is either complex or symplectic. For type IIB however, the compactification manifold is always complex but it might have a more refined structure depending on which other torsion classes vanish. It is important to note that the IIB cases are not exhaustive since there exist solutions that interpolate between the different cases [78].

# Chapter III

## Generalised Geometry and Supersymmetry

As we discussed in the previous chapter, for a given supergravity background, the presence of fluxes spoils the integrability of the corresponding  $G$ -structure on the compactification manifold. Studying non-integrable  $G$ -structures turns out to be a complicated task: classifying the torsion classes can be an involved process, and the moduli space of such structures is largely unknown except for some specific simple cases. The  $G$ -structure formalism thus suffers from important limitations to draw a complete picture of the geometry of supergravity backgrounds.

Fortunately, these limitations can be circumvented in the formalism of *generalised geometry*, which naturally generalises the concept of  $G$ -structures. In generalised geometry, instead of the tangent bundle, one defines structures on a more general bundle  $E \rightarrow M$ . The generalised geometry is then organised in terms of an enlarged structure group  $\mathrm{GL}(d, \mathbb{R}) \subset \mathcal{G} \subset \mathrm{GL}(\mathrm{rk} E, \mathbb{R})$ , and the generalised structures on  $E$  are subgroups of  $\mathcal{G}$ .

Crucially, for the appropriate generalised geometry, one can define a notion of integrability for a given generalised structure, even in the presence of fluxes. Roughly speaking, such generalised geometries "geometrise the fluxes", in a sense that will be made precise later on. Studying integrable generalised structures has significantly shed some light on the geometries of flux backgrounds [1, 61, 77, 79, 80].

Two specific generalised geometries are of particular importance to describe type II supergravity compactification manifolds:  $\mathcal{G} \equiv \mathrm{O}(d, d)$  [81] and  $\mathcal{G} \equiv E_{d+1(d+1)}$  [37].

$\mathrm{O}(d, d)$  generalised geometry geometrises the NS fluxes, while non-vanishing RR fluxes obstruct the involutivity of the corresponding generalised structure, and  $E_{d+1(d+1)}$  generalised geometry geometrises both the NS and RR sectors, it therefore has integrable generalised structures describing flux backgrounds. We introduce both generalised geome-



tries in this chapter, before discussing the interplay between the fluxes, supersymmetry, and the integrability of the generalised structures.

### III.1 $O(6,6)$ Generalised Geometry

In this section we introduce  $O(d, d)$  generalised geometry, also referred to as generalised complex geometry, before specialising to six-dimensional compactification manifolds in order to connect with supergravity flux backgrounds compactified to four-dimensional Minkowski spacetime. We then discuss the relationship between supersymmetry and integrability of the generalised structures, and comment on the geometries of non-supersymmetric flux backgrounds.

#### III.1.1 Generalised Complex Geometry

In Generalised Complex Geometry, as introduced in [34, 35], one replaces the tangent bundle of the internal manifold with a *generalised tangent bundle*, which is the sum of the tangent and cotangent bundle

$$E = T \oplus T^*, \quad (\text{III.1})$$

and whose sections are thus sums of vectors and one-forms

$$V = v + \xi \in \Gamma(E). \quad (\text{III.2})$$

There is a natural inner product defined on the generalised tangent bundle: for two generalised vectors  $V = v + \xi \in \Gamma(E)$  and  $W = w + \rho \in \Gamma(E)$ , it is

$$\eta(V, W) := \frac{1}{2}(v \lrcorner \rho + w \lrcorner \xi). \quad (\text{III.3})$$

Using a two-component notation for the generalised vectors

$$V = v + \xi \equiv \begin{pmatrix} v \\ \xi \end{pmatrix}, \quad (\text{III.4})$$

we can write it as

$$\eta(\cdot, \cdot) = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad (\text{III.5})$$

which is clearly invariant under  $O(d, d)$ , where  $d$  is the dimension of the manifold. The structure group on the generalised tangent bundle is therefore not generic, but reduces to  $O(d, d)$ <sup>1</sup>.

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<sup>1</sup>Actually the inner product  $\eta(\cdot, \cdot)$  defines a globally defined volume form on  $E$ , so the structure group further reduces to  $SO(d, d)$

Considering  $E$  as an  $O(d, d)$  frame bundle  $\tilde{F}$ , one can parametrise the corresponding adjoint representation of the  $\mathfrak{o}(d, d)$  algebra in terms of  $\mathfrak{gl}(d, \mathbb{R})$  representations as the following bundle

$$\text{ad } \tilde{F} = (T \otimes T^*) \oplus \Lambda^2 T \oplus \Lambda^2 T^*. \quad (\text{III.6})$$

It has a natural action on the generalised tangent bundle (and on itself) given in Appendix B.1.

### a) The Dorfman derivative

The bundle  $E$  admits a derivative operator, called the *Dorfman derivative* or *Generalised Lie derivative*, as it is the Generalised Geometry analogue of the standard Lie derivative. It acts on generalised vectors as

$$L_V W = \mathcal{L}_v w + \mathcal{L}_v \rho - w \lrcorner d\xi \quad (\text{III.7})$$

Defining a natural projection  $a : E \rightarrow T$ , commonly called the *anchor map*, the collection  $(E, \eta, L, a)$  is a *Leibniz Algebroid* [82].

One can define an antisymmetric bracket on  $E$  as

$$\llbracket V, W \rrbracket = \frac{1}{2}(L_V W - L_W V) \quad (\text{III.8})$$

$$= [v, w] + \mathcal{L}_v \rho - \mathcal{L}_w \xi - \frac{1}{2}d(v \lrcorner \rho - w \lrcorner \xi) \quad (\text{III.9})$$

called the *Courant bracket*. The collection  $(E, \eta, \llbracket, \rrbracket, a)$  is then a *Courant algebroid* [83].

Crucially, this is not a *Lie algebroid*, given that the Courant bracket fails to satisfy the Jacobi identity

$$\text{Jac}(X, Y, Z) := \llbracket X, \llbracket Y, Z \rrbracket \rrbracket + \text{cyclic perms} \quad (\text{III.10})$$

$$= \frac{1}{3}d(\eta(\llbracket X, Y \rrbracket), Z) + \text{cyclic perms}), \quad (\text{III.11})$$

and is therefore not a Lie bracket.

On another note, using the adjoint action defined in Appendix B.1, we can evaluate the following derivative

$$L_{e^B \cdot V}(e^B \cdot W) = e^B \cdot L_V W + w \lrcorner v \lrcorner dB, \quad (\text{III.12})$$

where  $B$  is a two-form and  $\cdot$  denotes the adjoint action.

The Dorfman derivative is therefore covariant under such a transformation if  $B$  is closed  $B \in \Omega_{\text{cl}}^2$ . Moreover, the Dorfman derivative is obviously covariant under diffeo-

morphism transformations. We call the set of symmetries of the Dorfman derivative the *generalised diffeomorphisms*. It is thus given by the following semi-direct product

$$\text{GDiff} = \text{Diff} \ltimes \Omega_{\text{cl}}^2 \sim \text{Diff} \ltimes \Omega_{\text{ex}}^2, \quad (\text{III.13})$$

where the second expression is only valid on local patches. This is exactly the set of transformations generated by  $L_V$  in (III.7). One can hence think about  $O(d, d)$  Generalised Geometry as a way to locally geometrize the two-form gauge transformations, through their inclusion in a natural generalisation of the diffeomorphism transformations.

The expression (III.12) motivates the definition of an alternative derivative, given a three-form  $H$

$$L_V^H W = L_V W + w \lrcorner v \lrcorner H. \quad (\text{III.14})$$

However  $H$  must be closed  $H \in \Omega_{\text{cl}}^3$  in order for this derivative to satisfy the required Leibniz property. We call  $L^H$  the *H-twisted Dorfman derivative*<sup>2</sup>. It has the same set of symmetries as the Dorfman derivative (III.13).

On a local patch  $\mathcal{U}_i \subset \mathcal{M}$ , one can always write the three-form  $H$  as  $H = dB_i$  with  $B_i \in \Omega^2(\mathcal{U}_i)$ . The H-twisted Dorfman derivative can then locally be written from the untwisted one as

$$L_V^H W = e^{-B_i} \cdot L_{e^{B_i} \cdot V} (e^{B_i} \cdot W). \quad (\text{III.15})$$

In order for this picture to be coherent globally we must have, on the overlap of two patches

$$V = e^{B_i}(v_i + \xi_i) = e^{B_j}(v_j + \xi_j) \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j, \quad (\text{III.16})$$

where the two-form potentials are related by an exact two-form  $B_j = B_i + d\Lambda_{ji}$ . This ensures that the sections of  $E$  are globally defined. This imposes

$$x_i = x_j \quad (\text{III.17})$$

$$\xi_i = \xi_j + x_j \lrcorner d\Lambda_{ji}. \quad (\text{III.18})$$

This means that the  $V_i = v_i + \xi_i$  are local sections of a bundle  $E_H$  defined as an extension of the tangent bundle by the cotangent bundle

$$0 \longrightarrow T^* \longrightarrow E_H \xrightarrow{a} T \longrightarrow 0. \quad (\text{III.19})$$

We therefore have two equivalent pictures, corresponding to the two equivalent Leibniz algebroid

$$(E, L^H) \longleftrightarrow (E_H, L), \quad (\text{III.20})$$

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<sup>2</sup>As in (III.8), one can define an alternative Courant bracket, the *H-twisted Courant bracket*  $[\cdot, \cdot]_H$ .

with the isomorphism between them being defined by the three-form flux  $H$ .

To be more precise, the two-form potential  $B$  actually has the interpretation of a connection on a *gerbe* [84], defined by the following condition on a triple overlap

$$\Lambda_{ji} + \Lambda_{ik} + \Lambda_{kj} = d\Lambda_{kji} \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k. \quad (\text{III.21})$$

### b) Generalised Complex Structure

One can define a structure on the generalised tangent bundle which naturally generalises the complex structure on the tangent bundle. It is called a *generalised complex structure*. First, an *almost generalised complex structure* is an endomorphism on  $E$

$$\mathcal{J} : E \rightarrow E, \quad (\text{III.22})$$

respecting

$$\mathcal{J}^2 = -\mathbb{1} \quad \eta(\mathcal{J} \cdot V, \mathcal{J} \cdot W) = \eta(V, W). \quad (\text{III.23})$$

It is thus simply an almost complex structure on  $E$  preserving the  $O(d, d)$  structure.

Unlike the ordinary complex structure case, the presence of an almost generalised complex structure reduces the structure group to  $U(\frac{d}{2}, \frac{d}{2}) \subset O(d, d)$ .

In the two components notation, (III.23) implies that we can always write the almost generalised complex structure as

$$\mathcal{J} = \begin{pmatrix} -I & P \\ L & I^T \end{pmatrix}, \quad (\text{III.24})$$

where  $P$  and  $L$  are a two-vector and a two-form respectively. An (almost) generalised complex structure induces the decomposition of the complexified generalised tangent space into two maximal eigenspaces

$$E \otimes \mathbb{C} = L_1 \oplus L_{-1}, \quad (\text{III.25})$$

where  $L_{\pm 1}$  is the eigenbundle of eigenvalue  $\pm i$  under the action of  $\mathcal{J}$ . They are complex conjugate  $\bar{L}_1 = L_{-1}$  and they respect  $L_1 \cap L_{-1} = 0$ . Moreover, it follows immediately from (III.23) that

$$\eta(L_1, L_1) = 0. \quad (\text{III.26})$$

The bundle  $L_1$  is therefore said to be *isotropic*.

It is important to note that the subbundle  $L_1$  (or likewise  $L_{-1}$ ) and the properties it respects equivalently define a generalised complex structure.

Indeed, a generalised complex structure is defined by a subbundle  $L \subset E_{\mathbb{C}}$  such that

- i)  $L$  is maximal and isotropic
- ii)  $L \cap \bar{L} = 0$ .

A structure satisfying only the first property is called a *Dirac structure* [81].

In complete analogy with the complex structure case, an almost generalised complex structure is said to be *integrable*, or simply a generalised complex structure, if the subbundle  $L_1$  is involutive under the Courant bracket

$$[[L_1, L_1]] \subset L_1. \quad (\text{III.27})$$

In that case, one can define the restriction of the Courant bracket on  $L_1$ , and given that  $L_1$  is an isotropic space, the Jacobiator (III.11) vanishes and the algebroid  $(L_1, \eta, [[, ]], a)$  is thus a Lie algebroid. One can then naturally define a notion of exterior derivative on the space of forms of  $L_1$ , as shown in [35].

Finally, let's mention that the usual complex and symplectic structures naturally embed as special cases of generalised complex structures. Indeed, an almost complex structure  $I$  defines an almost generalised complex structure as

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^T \end{pmatrix} \quad L_1 = T^{1,0} \oplus T^{*0,1}, \quad (\text{III.28})$$

and the almost generalised complex structure is only integrable if the almost complex structure is integrable.

A symplectic structure  $J$  defines a generalised complex structure as

$$\mathcal{J}_J = \begin{pmatrix} 0 & J^{-1} \\ J & 0 \end{pmatrix} \quad L_1 = e^{iJ} \cdot T_{\mathbb{C}} \quad (\text{III.29})$$

and the almost generalised complex structure is only integrable if the pre-symplectic structure is integrable.

### c) Spinors on $E_H$

One can naturally define a Clifford algebra on  $E_H$ , as

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}, \quad (\text{III.30})$$

where  $M = 1, \dots, 2d$  are  $O(d, d)$  indices,  $\{\Gamma_M\}$  is a basis on  $E_H$  and  $\eta_{MN} = \eta(\Gamma_M, \Gamma_N)$ . It is therefore a  $\text{Cliff}(d, d)$  algebra, and one can naturally define spinors on  $E_H$ , as the spinor representation of this algebra.

To do so, we introduce the following Clifford action of sections of  $E_H$  on sections of

$\Lambda^\bullet T^*$ , the exterior algebra of  $T^*$

$$\not{V}\Psi = v \lrcorner \Psi + \xi \wedge \Psi \quad V \in \Lambda(E_H), \Psi \in \Gamma(\Lambda^\bullet T^*). \quad (\text{III.31})$$

Here we denoted this action using the slashed notation in reference to the usual notation of contraction with gamma matrices.

It immediately follows that this action is compatible with the Clifford algebra

$$\not{[V, W]}\Psi = 2\eta(V, W)\Psi \quad \forall V, W \in \Lambda(E_H), \Psi \in \Gamma(\Lambda^\bullet T^*). \quad (\text{III.32})$$

It would therefore be natural to identify the spinors on  $E_H$  as being sections of  $\Lambda^\bullet T^*$ . However, the spinorial representation of  $\Lambda^\bullet T^*$  actually transforms under the  $\mathfrak{gl}(d, \mathbb{R})$  adjoint action as

$$\not{A}\Psi = A \cdot \Psi + \frac{1}{2} \text{Tr} A \Psi, \quad (\text{III.33})$$

where here  $\cdot$  denotes the standard  $\mathfrak{gl}(d, \mathbb{R})$  adjoint action on polyforms. As a representation of  $\text{Spin}(d, d)$ , the spinor bundle  $S(E_H)$  is thus isomorphic to  $\Lambda^\bullet T^* \otimes (\det T)^{1/2}$ . We can compensate this  $\det T$  factor by the inclusion of an appropriate  $\mathbb{R}^+$  factor in the structure group, which physically corresponds to the trombone symmetry of supergravity. The spinor bundle can hence take the form

$$S(E_H) \simeq \Lambda^\bullet T^*. \quad (\text{III.34})$$

The exterior algebra  $\Lambda^\bullet T^*$  can be decomposed into irreducible  $\text{Spin}(d, d)$  representations

$$S(E_H) := S^+(E_H) \oplus S^-(E_H) \simeq \Lambda^+ T^* \oplus \Lambda^- T^*, \quad (\text{III.35})$$

where  $\Lambda^\pm T^*$  are the bundles of even/odd forms. The bundles  $S^+(E_H)$  and  $S^-(E_H)$  are said to be of positive and negative chirality, respectively.

The  $\text{Spin}(d, d)$  spinor bundle are only locally defined: on the overlap of two patches  $\mathcal{U}_i$  and  $\mathcal{U}_j$ , where the  $B$ -field glues as  $B_j = B_i + d\Lambda_{ji}$ , the spinors satisfy the following gluing condition

$$\Psi_j = e^{d\Lambda_{ji}} \cdot \Psi_i \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j. \quad (\text{III.36})$$

One can define a bilinear form on the space of polyforms, taking value in  $\det T^*$ , the *Mukai pairing*

$$\langle \omega, \chi \rangle = [\omega \wedge \sigma(\chi)]_{\text{top}}, \quad (\text{III.37})$$

for  $\omega, \chi \in \Gamma(\Lambda^\bullet T^*)$ , where the top subscript means the projection onto the top form component, and where  $\sigma$  acts as the reversal of all form indices  $\sigma(\omega_p) = (-1)^{\text{Int}[p/2]} \omega_p$ .

The Mukai pairing is invariant under  $O(d, d)$  actions, and it respects

$$\langle \not{V}\Psi_1, \Psi_2 \rangle = (-1)^{d+1} \langle \Psi_1, \not{V}\Psi_2 \rangle \quad (\text{III.38})$$

$$\int_{\mathcal{M}} \langle d\Psi_1, \Psi_2 \rangle = (-1)^d \int_{\mathcal{M}} \langle \Psi_1, d\Psi_2 \rangle, \quad (\text{III.39})$$

where the derivative on the spinors is the usual exterior derivative on forms, which maps spinors with positive chirality to spinors with negative one and vice versa

$$d : S_{\pm}(E_H) \rightarrow S_{\mp}(E_H). \quad (\text{III.40})$$

One can similarly define spinors on the untwisted bundle  $E$ . If one locally writes the three-form  $H$  on a patch  $\mathcal{U}_i$  as  $H = dB_i$ , we define

$$\Psi(E_H)_i = e^{B_i} \cdot \Psi(E). \quad (\text{III.41})$$

We therefore also have

$$S(E) \simeq \Lambda^{\bullet} T^* \quad (\text{III.42})$$

and its decomposition into irreducible  $\text{Spin}(d, d)$  representations

$$S(E) := S^+(E) \oplus S^-(E) \simeq \Lambda^+ T^* \oplus \Lambda^- T^* \quad (\text{III.43})$$

As opposed to the twisted spinors, the spinors on  $E$  are globally defined. The definition of a derivative on the untwisted spinors naturally follows from (III.41)

$$d_H := d + H \wedge : S_{\pm}(E) \rightarrow S_{\mp}(E). \quad (\text{III.44})$$

While the exterior derivative  $d$  is the differential associated to the untwisted Courant bracket,  $d_H$  is the one associated to the H-twisted Courant bracket.

On another note, one can define a natural action of (almost) generalised complex structures on the space of differential forms<sup>3</sup>

$$\mathcal{J} \cdot = \frac{1}{2} (J_{mn} e^m \wedge e^n \wedge + 2I^n{}_n [e^n \wedge, e_m \lrcorner] + P^{mn} e_m \lrcorner e_n \lrcorner). \quad (\text{III.45})$$

An (almost) generalised complex structure determines an alternative grading of the spinor bundle

$$\Lambda^{\bullet} T^* \otimes \mathbb{C} = \bigoplus_{-\frac{d}{2} \leq k \leq \frac{d}{2}} S_k, \quad (\text{III.46})$$

with  $k$  the eigenvalues of the eigenspaces  $S_k$  of  $\mathcal{J}$ , which are representations of the  $U(\frac{d}{2}, \frac{d}{2})$

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<sup>3</sup>We introduce here a frame  $\{e_m\}$  of  $T$  and its dual coframe  $\{e^m\}$  on  $T^*$ .

structure group associated with  $\mathcal{J}$ .

For example, in the case of the (almost) generalised complex structure associated to an almost complex structure, its relation with the standard Hodge decomposition is

$$S_k = \bigoplus_p \Lambda^{\frac{d}{2}-p, \frac{d}{2}-k-p}, \quad (\text{III.47})$$

where  $\Lambda^{\frac{d}{2}-p, \frac{d}{2}-k-p}$  are sections of the  $(p, q)$ -forms defined by the standard associated (almost) complex structure.

Decomposing the action of the exterior derivatives on the twisted and untwisted spinor bundles gives

$$d : S_k(E_H) \rightarrow S_{k-3}(E_H) \oplus S_{k-1}(E_H) \oplus S_{k+1}(E_H) \oplus S_{k+3}(E_H) \quad (\text{III.48})$$

$$d_H : S_k(E) \rightarrow S_{k-3}(E) \oplus S_{k-1}(E) \oplus S_{k+1}(E) \oplus S_{k+3}(E). \quad (\text{III.49})$$

### c).1 Pure spinors

One can define a subset of the spinors on  $E_H$  (or  $E$ ), the *pure spinors* as the vacuum of the  $\text{Cliff}(d, d)$  algebra, in the sense that the pure spinors will be annihilated by  $d$  independent generators of  $\text{Cliff}(d, d)$ . We call the subbundle of  $E_H \otimes \mathbb{C}$  annihilating a given spinor its *annihilator space*

$$L_\Psi = \{V \in E_H \otimes \mathbb{C} : \Psi V = 0\}, \quad (\text{III.50})$$

which is thus  $d$  dimensional for pure spinors. Importantly, given that sections of  $E_H$  acting on spinors obey the Clifford algebra (III.32), the bundle  $L_\Psi$  is isotropic. Moreover, if a pure spinor is complex and obeys

$$\langle \Psi, \bar{\Psi} \rangle \neq 0, \quad (\text{III.51})$$

its annihilator space respects  $L_\Psi \cap \bar{L}_\Psi = 0$ . Such a pure spinor is called a non-degenerate pure spinor.

A non-degenerate pure spinor  $\Psi$  therefore defines a maximal isotropic bundle obeying  $L_\Psi \cap \bar{L}_\Psi = 0$ , it thus defines a generalised complex structure  $\mathcal{J}$  as

$$L_\Psi = L_1, \quad (\text{III.52})$$

with  $L_1$  the eigenbundle of charge  $+i$  with respect to  $\mathcal{J}$ .

Given that the normalisation of  $\Psi$  plays no role in defining  $L_\Psi$ , an (almost) generalised complex structure is hence in one-to-one correspondence with the line-bundle of a pure spinor.

It is insightful to reformulate the integrability condition of a given almost generalised



complex structure  $\mathcal{J}$  as a differential condition on its associated pure spinor  $\Psi(E)$ <sup>4</sup>. The condition (III.27) reads

$$\llbracket V, W \rrbracket_H \Psi = 0 \quad \forall V, W \in \Gamma(E). \quad (\text{III.53})$$

This is equivalent to

$$d_H \Psi = \not{V} \Psi \quad \text{for } V \in \Gamma(E). \quad (\text{III.54})$$

Hence the almost generalised complex structure associated to a pure spinor respecting (III.54) is integrable<sup>5</sup>.

The integrability of the generalised complex structure (III.54) implies that the H-twisted exterior derivative behaves as

$$d_H : S_k(E) \rightarrow S_{k-1}(E) \oplus S_{k+1}(E), \quad (\text{III.55})$$

as opposed to the case where the generalised complex structure is non-integrable (III.49). This allows for a decomposition of the exterior derivative in terms of the generalised analogues of the Dolbeault operators

$$d_H = \partial_H + \bar{\partial}_H \quad \partial_H : S_k(E) \rightarrow S_{k+1}(E) \quad \bar{\partial}_H : S_k(E) \rightarrow S_{k-1}(E). \quad (\text{III.56})$$

#### d) Generalised Calabi-Yau structure

As we mentioned above, a non-degenerate pure spinor  $\Psi$  reduces the structure group on  $E$  to  $U(d/2, d/2)$ . If  $\Psi$  is globally defined, the structure group is further reduced to  $SU(d/2, d/2)$ . An  $SU(d/2, d/2)$  structure is called a *generalised Calabi-Yau structure*. A generalised Calabi-Yau structure is integrable if

$$d_H \Psi = 0. \quad (\text{III.57})$$

The relationship between a generalised complex structure and a generalised Calabi-Yau structure is analogous to the relationship between a complex and an  $SL(d/2, \mathbb{C})$  structure, discussed in the previous chapter.

#### e) Compatible Generalised Complex Structures

One can consider the case where a manifold admits several generalised complex structures. In type II supergravity, a physically crucial case is the one where the manifold admits two

<sup>4</sup>A similar condition could be written on the twisted pure spinor  $\Psi(E_H)$ . We denote  $\Psi(E)$  as  $\Psi$  in the following.

<sup>5</sup>Note that (III.54) is independent of the specific choice of the local section  $\Psi$  of the line bundle defined by  $\mathcal{J}$ . The associated condition on the twisted spinor is simply  $d\Psi = \not{V}\Psi$  for  $V \in \Gamma(E_H)$ .

generalised complex structures, that are said to be *compatible*. Two generalised complex structures are compatible if they commute

$$[\mathcal{J}_1, \mathcal{J}_2] = 0. \quad (\text{III.58})$$

Each generalised complex structure determines a specific grading of the space of forms

$$\Lambda^\bullet T^* \otimes \mathbb{C} = \bigoplus_{-\frac{d}{2} \leq k \leq \frac{d}{2}} S_k^1 = \bigoplus_{-\frac{d}{2} \leq k \leq \frac{d}{2}} S_k^2. \quad (\text{III.59})$$

In terms of the associated pure spinors, the compatibility condition translates as

$$\Psi_1 \in \Gamma(S_0^2) \quad \Psi_2 \in \Gamma(S_0^1), \quad (\text{III.60})$$

and two pure spinors respecting (III.60) are *compatible*. The presence of two compatible pure spinors reduces the structure group to  $U(\frac{d}{2}) \times U(\frac{d}{2}) \subset O(d, d)$ , and if the pure spinors are nowhere vanishing and globally defined, it further reduces to an  $SU(\frac{d}{2}) \times SU(\frac{d}{2})$  structure.

The condition (III.60) implies that the pure spinors share  $\frac{d}{2}$  annihilators, and there is a double grading of the space of forms as

$$\Lambda^\bullet T^* \mathcal{M} \otimes \mathbb{C} = \bigoplus_{k=-\frac{d}{2}}^{\frac{d}{2}} \bigoplus_{l=-\frac{d}{2}}^{\frac{d}{2}} S_{k,l}, \quad (\text{III.61})$$

where  $k$  and  $l$  denote the eigenvalues with respect to  $\mathcal{J}_1$  and  $\mathcal{J}_2$ , and for a given value of  $k$  and  $l$ , the  $S_{k,l}$  are irreducible representations of the  $SU(\frac{d}{2}) \times SU(\frac{d}{2})$  structure.

### III.1.2 Generalised structures of supersymmetric backgrounds

In this subsection, we revisit type II solutions that are the warped product of four-dimensional Minkowski space  $X_4$  and a six-dimensional compact manifold  $M$ , with the following metric

$$ds_{10}^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{III.62})$$

where again  $x^\mu$ ,  $\mu = 0, \dots, 3$  are the external coordinates on  $X_4$ , and  $y^m$ ,  $m = 1, \dots, 6$  are the coordinates on  $M$ . We will describe such backgrounds using the formalism of  $O(6,6)$  generalised geometry, specialising the results of the above subsection to the six-dimensional compactification manifold. The framework of  $O(6,6)$  generalised geometry has been successfully used for example to classify [77, 85] and derive new type II flux backgrounds [44, 85, 86].

Recall that the Poincaré invariance of  $X_4$  constrains the NS and RR-fluxes: the NS-

field-strength  $H$  can only have internal legs, and the ten-dimensional RR-field-strength takes the form

$$F^{10} = F + e^{4A} \text{vol}_4 \wedge \tilde{F}, \quad (\text{III.63})$$

where  $F$  and  $\tilde{F}$  are purely internal and obey

$$\tilde{F} = \tilde{*}_6 F. \quad (\text{III.64})$$

We recall the split of the ten-dimensional spinors  $\epsilon_1$  and  $\epsilon_2$

$$\epsilon_1 = \zeta \otimes \eta_1 + c.c. \quad \epsilon_2 = \zeta \otimes \eta_2 + c.c. \quad (\text{III.65})$$

From now on we take the internal spinors to be globally defined, such that they each define an  $SU(3)$  structure on  $M$ . It is also important to note that the two six-dimensional internal spinors must have the same norm in order to admit supersymmetric sources<sup>6</sup> [2], which we denote  $\|\eta_{1,2}\|$ .

In order to make contact with the formalism of generalised complex geometry, we introduce the following objects

$$\Psi_1 = -\frac{8i}{\|\eta_{1,2}\|^2} \eta_1 \otimes \eta_2^\dagger \quad (\text{III.66})$$

$$\Psi_2 = -\frac{8i}{\|\eta_{1,2}\|^2} \eta_1 \otimes \eta_2^T. \quad (\text{III.67})$$

One should think of these tensor products in terms of the following Fierz identity

$$\eta \otimes \chi = \sum_{k=0}^6 \frac{1}{k!} \left( \chi^\dagger \gamma_{m_k \dots m_1} \eta \right) \gamma^{m_1 \dots m_k}, \quad (\text{III.68})$$

for two spinors  $\eta$  and  $\chi$ . The conventions for the internal gamma matrices are given in Appendix A. These tensor products are then isomorphic to polyforms through the Clifford map (A.34), so we treat them as such from now on. Through (III.42), we interpret these polyforms as spinors on  $E = T \oplus T^*$ , the generalised tangent bundle of the six-dimensional compactification manifold.

The spinors  $\Psi_1$  and  $\Psi_2$  are odd/even and even/odd in type IIA/IIB, respectively

$$\Psi_1 = \Psi_{\mp}, \quad \Psi_2 = \Psi_{\pm}, \quad (\text{III.69})$$

and they are normalised such that

$$\langle \Psi_1, \bar{\Psi}_1 \rangle = \langle \Psi_2, \bar{\Psi}_2 \rangle = -8i \text{vol}_6, \quad (\text{III.70})$$

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<sup>6</sup>Supersymmetric sources of type II supergravity will be discussed in the next chapter.

with  $\text{vol}_6$  the volume form on  $M$  in the string frame. Finally, it follows directly from (III.66) and (III.67) that the spinors  $\Psi_1$  and  $\Psi_2$  are pure<sup>7</sup>.

Given that they are non-degenerate pure spinors, they each define an almost generalised complex structure on  $E$ , denoted  $\mathcal{J}_1$  and  $\mathcal{J}_2$  respectively. Moreover, given that the internal spinors are globally defined, the pure spinors  $\Psi_1$  and  $\Psi_2$  are also globally defined, so they each define an  $\text{SU}(3,3) \subset \text{O}(6,6)$  structure on  $E$ . Finally,  $\Psi_1$  and  $\Psi_2$  are compatible, so as discussed in the previous subsection, they further define an  $\text{SU}(3) \times \text{SU}(3)$  structure on  $E$ .

Crucially, the vanishing of the ten-dimensional supersymmetry variations (II.71), (II.72) can be elegantly reformulated as the following set of differential equations on the pure spinors [1]

$$d_H(e^{3A-\phi}\Psi_2) = 0 \quad (\text{III.71})$$

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) = 0 \quad (\text{III.72})$$

$$d_H(e^{4A-\phi}\text{Re}\Psi_1) = e^{4A}\tilde{*}_6 F. \quad (\text{III.73})$$

Given that the first equation is of the form (III.54), imposing supersymmetry amounts to requiring the almost generalised complex structure  $\mathcal{J}_2$  to be integrable, while the last supersymmetry condition (III.73) tells us that the integrability of the almost generalised complex structure  $\mathcal{J}_1$  is precisely obstructed by the RR fluxes.

We illustrate how standard  $G$ -structures embed in this formalism with the example of an  $\text{SU}(3)$  structure in type IIB, defined by a pre-symplectic two form  $J$  and a  $(3,0)$ -form  $\Omega$ . If we parametrise the proportionality between the internal spinors as  $\eta_1 = \eta_2$ , the pure spinors take the simple form

$$\Psi_1 = -ie^{iJ} \quad \Psi_2 = \Omega. \quad (\text{III.74})$$

From the supersymmetry conditions (III.71) and (III.73), we immediately see that the almost complex structure defined by  $\Omega$  is integrable, while the integrability of the pre-symplectic structure defined by  $J$  is precisely spoiled by the RR fluxes.

## III.2 $E_{7(7)} \times \mathbb{R}^+$ Generalised Geometry

We have seen that the formalism of  $\text{O}(d,d)$  geometry treats the full NS sector in a geometric way. When doing so, the only obstructions to the integrability of the  $\text{O}(d,d)$  generalised structures are the remaining supergravity bosonic degrees of freedom, the RR fluxes. In this section we introduce another generalised geometry,  $E_{d+1(d+1)} \times \mathbb{R}^+$  gener-

<sup>7</sup>In six dimensions, every Weyl spinor is pure, so  $\eta_1$  and  $\eta_2$  are pure.

alised geometry, or exceptional generalised geometry, which treats both the NS and the RR sectors of type II supergravity geometrically.

We first introduce exceptional generalised geometry, before describing type II flux backgrounds within this framework, emphasising on the relationship between supersymmetry and the integrability of the relevant generalised structures.

### III.2.1 Exceptional Generalised Geometry

On a manifold  $M$  of dimension  $d = 6$ , the generalised tangent bundle for  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry is locally isomorphic to

$$E \simeq E_{O(6,6)} \oplus S^\pm \oplus (\Lambda^6 T^* \otimes E_{O(6,6)}), \quad (\text{III.75})$$

in type IIA/IIB, with  $S^\pm$  the spinor bundles (III.43) on  $E_{O(6,6)} \simeq T \oplus T^*$ . We write sections of this bundle as

$$V = X + \omega + \tilde{X}, \quad (\text{III.76})$$

with  $X \in \Gamma(E_{O(6,6)})$ ,  $\omega \in \Gamma(S^\pm)$ , and  $\tilde{X} \in \Gamma(\Lambda^6 T^* \otimes E_{O(6,6)})$ .

The isomorphism (III.75) is not unique, as the bundle  $E$  is actually defined as an extension, through the following exact sequence

$$0 \rightarrow S^\pm \rightarrow E' \rightarrow E_{O(6,6)} \rightarrow 0 \quad (\text{III.77})$$

$$0 \rightarrow \Lambda^6 T^* \otimes E_{O(6,6)} \rightarrow E' \rightarrow E \rightarrow 0. \quad (\text{III.78})$$

There is thus a non-trivial patching between local sections of the generalised tangent bundle  $E$ : on two overlapping patches  $\mathcal{U}_i \cap \mathcal{U}_j$ , the patching is defined by

$$V_i = e^{d\tilde{\Lambda}_{ij}} e^{d\Omega_{ij}} e^{d\Lambda_{ij}} \cdot V_j \quad \text{for } V_i \in \Gamma(\mathcal{U}_i, E), V_j \in \Gamma(\mathcal{U}_j, E) \quad (\text{III.79})$$

in type IIA, where  $\tilde{\Lambda}_{ij}$ ,  $\Omega_{ij}$  and  $\tilde{\Lambda}_{ij}$  are locally a five-form, a polyform of even degree and a one-form respectively. The action  $\cdot$  is the adjoint action, defined in Appendix B.2.

Defining an isomorphism as in (III.75) is equivalent to locally choosing a two-form, a polyform of odd degree, and a six form  $B$ ,  $C$  and  $\tilde{B}$  which are patched on overlaps  $\mathcal{U}_i \cap \mathcal{U}_j$  through<sup>8</sup>

$$B_i = B_j + d\Lambda_{ij} \quad (\text{III.80})$$

$$C_i = C_j + e^{B_j + d\Lambda_{ij}} \wedge d\Omega_{ij} \quad (\text{III.81})$$

$$\tilde{B}_i = \tilde{B}_j + d\tilde{\Lambda}_{ij} + \frac{1}{2} \langle d\Omega_{ij}, e^{B_j + d\Lambda_{ij}} \wedge C_j \rangle. \quad (\text{III.82})$$

<sup>8</sup>This is the patching for massless type IIA.

Then, for a section  $\tilde{V} \in \Gamma(E_{O(6,6)} \oplus S^\pm \oplus (\Lambda^6 T^* \otimes E_{O(6,6)}))$ , we can define a local section of  $E$  on the patch  $\mathcal{U}_i$  with the appropriate patching as

$$V = e^{\tilde{B}_i} e^{B_i} e^{C_i} \cdot \tilde{V}. \quad (\text{III.83})$$

From the supergravity point of view, the forms  $B$  and  $C$  correspond to the NS and RR gauge potentials, and the patching on  $E$  allows one to define the following globally defined forms

$$H = dB \quad F = d_H C, \quad (\text{III.84})$$

and from the supergravity point of view these correspond to the NS and RR fluxes. Different values of  $H$  and  $F$  define different isomorphism classes of  $E$ . We thus call the vectors  $\tilde{V} \in \Gamma(E_{O(6,6)} \oplus S^\pm \oplus (\Lambda^6 T^* \otimes E_{O(6,6)}))$  untwisted vectors, and we say that the vectors  $V \in \Gamma(E)$  are "twisted by the fluxes".

A priori, the bundle  $E$  has structure group  $\text{GL}(\text{rk } E, \mathbb{R})$ . We can introduce invariant tensors in order to reduce the structure group. We introduce the *symplectic* and *quartic invariants*

$$s : \Lambda^2 E \rightarrow \det T^* \quad q : \Lambda^4 E \rightarrow (\det T^*)^2. \quad (\text{III.85})$$

The symplectic invariant is given explicitly in Appendix B.2, and the presence of these invariants precisely reduces the structure group to  $E_{7(7)} \times \mathbb{R}^+$ . The generalised tangent bundle is then an  $E_{7(7)} \times \mathbb{R}^+$  vector bundle, and it transforms in the fundamental representation. We denote the  $\mathbb{R}^+$  value as the weight, and for example a section of the trivial representation of  $E_{7(7)} \times \mathbb{R}^+$  with weight  $p$  is a section of  $(\det T^*)^{p/2}$ .

The isomorphism (III.75) is a direct sum of irreducible representations of  $O(6,6)$ , it realises the embedding of  $O(6,6)$  in  $E_{7(7)} \times \mathbb{R}^+$ . Different isomorphisms of  $E$ , realising the embedding of different  $E_{7(7)} \times \mathbb{R}^+$  subgroups have been used in the exceptional generalised geometry literature [40, 79]. However, we solely use the embedding  $O(6,6) \subset E_{7(7)} \times \mathbb{R}^+$ , since throughout this thesis we will be concerned with the interplay between the frameworks of  $O(6,6) \times \mathbb{R}^+$  and  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry.

We recall that the generalised frame bundle  $\tilde{F}$  is an  $E_{7(7)} \times \mathbb{R}^+$  principal bundle constructed from frames of  $E$ . One can define generalised tensors as sections of vector bundles associated with different  $E_{7(7)} \times \mathbb{R}^+$  representations. Of particular interest is the adjoint bundle  $\text{ad } \tilde{F}$ , corresponding to the adjoint representation of  $E_{7(7)} \times \mathbb{R}^+$

$$\text{ad } \tilde{F} \simeq \mathbb{R} \oplus \mathbb{R} \oplus \text{ad } \tilde{F}_{O(6,6)} \oplus S^\mp \oplus (\Lambda^6 T \otimes S^\mp) \oplus \Lambda^6 T^* \oplus \Lambda^6 T. \quad (\text{III.86})$$

We write sections of this bundle as

$$R = q + l + \nu + s + \tilde{s} + a + \tilde{a}, \quad (\text{III.87})$$

with  $q \in \mathbb{R}$ ,  $l \in \mathbb{R}$ ,  $\nu \in \Gamma(\text{ad}\tilde{F}_{O(6,6)})$ , etc.

Given that the fibres of this bundle are isomorphic to the Lie algebra of  $E_{7(7)} \times \mathbb{R}^+$ , its sections have a natural action on  $E$  which is given in Appendix B.2. Let us mention that the NS and RR gauge potentials  $B$  and  $C$  are sections of this adjoint bundle.

Another important bundle is the space of generalised torsions. This will be relevant when we introduce generalised connections shortly. For now we will just state the properties of the bundle.

The torsion bundle  $K \subset E^* \otimes_{\text{ad}} \tilde{F}$  can locally be written in terms of  $O(6,6)$  irreducible representations as

$$\begin{aligned} K \simeq & E_{O(6,6)} \oplus (E_{O(6,6)} \otimes \Lambda^6 T^*) \oplus S^\pm \oplus (\Lambda^6 T \otimes S^\pm) \oplus (\Lambda^6 T^* \otimes S^\pm) \\ & \oplus [E_{O(6,6)} \otimes S^\pm]_0 \oplus \Lambda^3(E_{O(6,6)}) \oplus (\Lambda^6 T^* \otimes \Lambda^3(E_{O(6,6)})), \end{aligned} \quad (\text{III.88})$$

where  $\Phi^A \in \Gamma([E_{O(6,6)} \otimes S^\pm]_0)$  respects  $\Gamma_A \Phi^A = 0$ . The fibres of this bundle transforms in the  $\mathbf{912}_1$  representation of  $E_{7(7)} \times \mathbb{R}^+$ , where the subscript denotes the  $\mathbb{R}^+$  weight.

The NS and RR fluxes are sections of the torsion bundle  $H \in \Gamma(\Lambda^3(E_{O(6,6)})) \in \Gamma(K)$  and  $F \in \Gamma(S^\pm) \in \Gamma(K)$ .

Let us mention one more  $E_{7(7)} \times \mathbb{R}^+$  bundle  $N$ , a subbundle of the symmetric product  $S^2 E$ . The fibres of  $N$  belongs to the  $\mathbf{133}_2$  representation of  $E_{7(7)} \times \mathbb{R}^+$ , and  $N$  is isomorphic to

$$N \simeq \mathbb{R} \oplus \Lambda^6 T^* \oplus (\Lambda^6 T^* \otimes \Lambda^6 T^*) \oplus S^\mp \oplus (\Lambda^6 T^* \otimes S^\mp) \oplus (\text{ad}\tilde{F}_{O(6,6)} \otimes \Lambda^6 T^*). \quad (\text{III.89})$$

### a) The Dorfman derivative

As was the case in  $O(6,6)$  generalised geometry, the bundle  $E$  admits a derivative operator, generalising the Lie derivative. We call it the Dorfman or generalised Lie derivative, and on a local patch  $\mathcal{U}_i$  it acts on generalised vectors as

$$\begin{aligned} L_{V_i} V'_i &= L_{X_i} X'_i + (d(\tilde{X}_i \omega'_i) + \tilde{X}_i d\omega'_i - d\omega_i \cdot X'_i) \\ &+ (L_{X_i} \tilde{X}'_i + (\partial, \tilde{X}_i) \cdot X'_i - d\omega_i \cdot \omega'_i), \end{aligned} \quad (\text{III.90})$$

with  $V_i = X_i + \omega_i + \tilde{X}_i \in \Gamma(\mathcal{U}_i, E)$ , and similarly for  $V'$ . Here  $L_X X'$  is the standard  $O(6,6)$  Dorfman derivative (III.7). We also introduced

$$L_X \tilde{X}' = \mathcal{L}_v \tilde{X}' + j(\tilde{v}' \lrcorner \text{vol}) \wedge d\lambda \quad (\text{III.91})$$

$$(\partial, \tilde{X}) = d(\tilde{v} \lrcorner \text{vol}). \quad (\text{III.92})$$

with  $j$  defined in Appendix A.1 and with the following  $\mathrm{GL}(6)$  decomposition of sections of the  $\mathrm{O}(6,6)$  generalised bundles  $E_{\mathrm{O}(6,6)}$  and  $\Lambda^6 T^* \otimes E_{\mathrm{O}(6,6)}$

$$X = v + \lambda \tag{III.93}$$

$$\tilde{X} = (\tilde{v} + \tilde{\lambda}) \otimes \mathrm{vol}. \tag{III.94}$$

Defining an anchor map  $a : E \rightarrow T$  the collection  $(E, L, a)$  is a Leibniz Algebroid.

As was argued in the  $\mathrm{O}(6,6)$  generalised geometry case, the Dorfman derivative along  $X \in \Gamma(E_{\mathrm{O}(6,6)})$  generalises the diffeomorphisms to also include the NS gauge transformations, and from (III.90) we see that the another part of the action of the  $E_{7(7)}$  Dorfman derivative is generated by the action of the exact differential forms  $d\omega$ . These are precisely the RR gauge degrees of freedom of the type II backgrounds.

The Dorfman derivative thus generates the gauge transformations of the supergravity background.

One can locally define an antisymmetric bracket on  $\mathcal{U}_i$  patches of  $E$ , the Courant bracket

$$\llbracket V_i, W_i \rrbracket = \frac{1}{2}(L_{V_i} W_i - L_{W_i} V_i). \tag{III.95}$$

The collection  $(E, \llbracket, \rrbracket, a)$  is then a Courant algebroid.

Crucially, this is not a Lie algebroid, given that the Courant bracket fails to satisfy the Jacobi identity. Indeed, the Jacobiator reads<sup>9</sup>

$$\begin{aligned} \mathrm{Jac}(X, Y, Z) &= \llbracket X, \llbracket Y, Z \rrbracket \rrbracket + \text{cyclic perms} \\ &= d \times_E (\llbracket X, Y \rrbracket \times_N Z + \text{cyclic perms}), \end{aligned} \tag{III.98}$$

and doesn't vanish generically.

Let us stress here that without choosing an isomorphism (III.75), we cannot define a global expression for the Dorfman derivative and (III.90) is a local expression. Choosing an (III.75) picks out the gauge potentials  $B$  and  $C$  with field-strengths  $H$  and  $F$ . We can then define a global Dorfman derivative for untwisted generalised vectors, the *twisted Dorfman derivative*

$$L_V^{H+F} V' := L_V V' + (X \odot H + H \wedge \omega + \lrcorner F + \langle F, \omega \rangle) \cdot V', \tag{III.99}$$

<sup>9</sup>The projections  $\times_E$  and  $\times_N$  are the maps

$$\times_E : N \times E^* \rightarrow E \tag{III.96}$$

$$\times_N : E \times E \rightarrow N \tag{III.97}$$



where  $V = X + \omega + \tilde{X} \in \Gamma(E_{O(6,6)} \oplus S^\pm \oplus (\Lambda^6 T^* \otimes E_{O(6,6)}))$  is a global section, and similarly for  $V'$ . We introduced  $X \odot H := v \lrcorner H$ .

Here we see that the Dorfman derivative is twisted by the fluxes while the generalised tangent bundle remains untwisted. As already discussed in the  $O(6,6)$  generalised geometry framework, this is an equivalent formulation of generalised geometry and we will mainly make use of this approach in the following of this thesis.

The twisted Dorfman derivative (III.99) can be rearranged as

$$L_V^{H+F} V' = L_V^H V' + (\lrcorner F + \langle F, \omega \rangle) \cdot V', \quad (\text{III.100})$$

with

$$\begin{aligned} L_V^H V' &= L_X^H X' + (d_H(\lrcorner \omega') + \lrcorner d_H \omega' - \lrcorner' d_H \omega) \\ &\quad + (L_X \tilde{X}' + (\partial, \tilde{X}) \cdot X' - d_H \omega \cdot \omega'), \end{aligned}$$

where  $L_X^H X'$  is the twisted  $O(6,6)$  Dorfman derivative (III.14). Here we simply expanded the contributions involving the NS field-strength  $H$  and repackaged them into the twisted exterior derivative  $d_H$ . This form of the Dorfman derivative will turn out to be convenient to make contact with the  $O(6,6)$  formalism later on.

We can similarly define a global twisted Courant bracket

$$[[V, W]]^{H+F} = \frac{1}{2}(L_V^{H+F} W - L_W^{H+F} V), \quad (\text{III.101})$$

which of course also isn't a Lie bracket, as it also fails to satisfy the Jacobi identity.

### b) Generalised connection and generalised torsion

One can naturally generalise the notion of connection on the generalised tangent bundle.

A generalised connection  $D$  is a map

$$D : \Gamma(E) \rightarrow \Gamma(E^* \otimes E) \quad (\text{III.102})$$

such that

$$D(fV) = df \otimes V + fDV \quad (\text{III.103})$$

for all  $f \in \mathcal{C}^\infty(M)$ ,  $V \in \Gamma(E)$ . We used here the identification  $df \xrightarrow{a^*} \Gamma(E^*)$ . The generalised torsion of a generalised connection  $D$  is then some tensor  $T \in \Gamma(E^* \otimes \text{ad} \tilde{F})$  defined by

$$T(V) \cdot V' = L_V^D V' - L_{V'} V \quad (\text{III.104})$$

where  $L_V^D$  is the Dorfman derivative (III.90), where every instance of the exterior deriva-

tive  $d$  is replaced by the generalised connection  $D$ <sup>10</sup>.

From the torsion definition (III.104), we can see the torsion as the map

$$T : E \rightarrow \text{ad}\tilde{F}. \quad (\text{III.106})$$

One may think that the torsion fills out the whole of  $E^* \otimes \text{ad}\tilde{F}$ . However, due to the precise form of the Dorfman derivative, the torsion can only live in  $K \oplus E^* \subset E^* \otimes \text{ad}\tilde{F}$ , highlighting the importance of the bundle  $K$  defined above.

One can similarly define a generalised connection and its associated torsion in the untwisted picture. We denote the untwisted generalised tangent bundle as

$$\tilde{E} = E_{O(6,6)} \oplus S^\pm \oplus (\Lambda^6 T^* \otimes E_{O(6,6)}). \quad (\text{III.107})$$

A generalised connection  $\tilde{D}$  then defines a map

$$\tilde{D} : \Gamma(\tilde{E}) \rightarrow \Gamma(\tilde{E}^* \otimes \tilde{E}) \quad (\text{III.108})$$

which respects (III.103) for  $V \in \Gamma(\tilde{E})$ . Its corresponding generalised torsion is

$$T(V) \cdot V' = L_V^{\tilde{D}, H+F} V' - L_V^{H+F} V', \quad (\text{III.109})$$

where  $L_V^{\tilde{D}, H+F}$  is the twisted Dorfman derivative (III.99), where every instance of the exterior derivative  $d$  is replaced by the generalised connection  $\tilde{D}$ .

### c) Generalised G-structures

One can naturally generalise the notion of G-structures on the tangent bundle, to define generalised G-structures on the generalised tangent bundle. The generalised  $G$ -structure  $G$  is again a subgroup of the (now generalised) structure group  $E_{7(7)}$ , and defines a principal  $G$  bundle, with fibres transforming in the fundamental representation of  $G$ .

As in conventional geometry, the existence of a  $G$ -structure is equivalent to the existence of some globally defined non-vanishing generalised tensors that are preserved by  $G$ . We denote such a generalised tensor as  $\xi$ .

This allows us to naturally generalise the notion of integrability of the  $G$ -structure: for a given  $G$ -structure, we define the generalised connections  $D$  that are compatible with  $G$  as

$$D\xi = 0. \quad (\text{III.110})$$

---

<sup>10</sup>The generalised connection  $D$  has a natural action on any generalised tensor bundle  $\mathcal{X}$  induced by (III.102):

$$D : \mathcal{X} \rightarrow \Gamma(E^* \otimes \mathcal{X}). \quad (\text{III.105})$$

The  $G$  structure is then integrable if there exists a torsion free compatible generalised connection. From the generalised torsion definition (III.104), this corresponds to

$$L_V^D = L_V. \quad (\text{III.111})$$

One can similarly define generalised  $G$ -structures on the untwisted generalised tangent bundle  $\tilde{E}$ . Then, if  $\tilde{\xi}$  is some globally defined non-vanishing generalised tensor preserved by  $G$ , a generalised connection  $\tilde{D}$  is compatible with the  $G$ -structure if

$$(\tilde{D} + H + F)\tilde{\xi} = 0. \quad (\text{III.112})$$

Given (III.109), the integrability of the  $G$ -structure then translates into

$$L_V^{\tilde{D}, H+F} = L_V^{H+F}. \quad (\text{III.113})$$

Lets come back to the twisted picture and discuss further the integrability of the  $G$ -structure.

Whether or not a generalised connection  $D$  satisfying (III.111) exists depends on the intrinsic torsion associated to the  $G$ -structure. It is the component of the torsion which is independent of the specific choice of connection. Once again, it can be defined in complete analogy with the conventional geometry case.

To define the intrinsic torsion, we start from a given connection  $D$ , compatible with the  $G$ -structure, and we write any compatible connection  $D'$  as  $D' = D + \Sigma$  where

$$\Sigma = D' - D \in \Gamma(K_G) \quad \text{with} \quad K_G = E^* \otimes \text{ad}P_G, \quad (\text{III.114})$$

and with  $\text{ad}P_G$  the  $\text{ad}P_G$ -bundle with fibres belonging to the adjoint representation of  $G$ . As mentioned above, the torsion of a generic connection will be a section of the bundle

$$T_D \in \Gamma(W) \quad \text{with} \quad W \simeq K \oplus E^*. \quad (\text{III.115})$$

We then define the map

$$\begin{aligned} \tau : K_G &\rightarrow W \\ \Sigma &\rightarrow \tau(\Sigma) = T_{D'} - T_D, \end{aligned} \quad (\text{III.116})$$

and denoting the vector bundle associated to the image of  $\tau$  by  $\text{Im}\tau = W_G$ , we can define

$$W_{\text{int}}^G = W/W_G. \quad (\text{III.117})$$

$W_{\text{int}}^G$  does not depend on the choice of compatible connection, it only depends on the  $G$ -structure.  $W_{\text{int}}^G$  is the *intrinsic torsion* of the  $G$ -structure.

It will prove to be useful throughout the thesis to decompose the intrinsic torsion of a given  $G$ -structure into irreducible representations of  $G$

$$W_{\text{int}}^G = \bigoplus_i W_i. \quad (\text{III.118})$$

A torsion-free compatible connection exists if and only if the intrinsic torsion of the  $G$ -structure vanishes.

Indeed, given a compatible connection  $D$  with non-vanishing torsion  $T$ , we can consider its projection onto  $W_{\text{int}}^G$ , which corresponds to the intrinsic torsion  $T^{\text{int}} \in \Gamma(W_{\text{int}}^G)$ . As explained above, this part of the torsion is unchanged by the shift  $D \rightarrow D' = D + \Sigma$ . It is therefore possible to find a  $\Sigma$  such that the torsion  $T'$  of  $D'$  vanishes if and only if the intrinsic torsion  $T^{\text{int}}$  vanishes.

#### d) $\mathbb{R}^+ \times \mathbf{U}(7)$ and $\mathbf{SU}(7)$ structures

We introduce here two important generalised structures, the  $\mathbb{R}^+ \times \mathbf{U}(7)$  structure and the  $\mathbf{SU}(7)$  structure.

As we have done in the case of conventional geometry and  $\mathbf{O}(6, 6)$  generalised geometry, we define these structures through their corresponding invariant generalised tensors [61]

$$J \in \Gamma(\text{ad}\tilde{F}) : \text{stabilised by } G = \mathbb{C}^* \times \mathbf{SU}(7) = \mathbb{R}^+ \times \mathbf{U}(7) \quad (\text{III.119})$$

$$\psi \in \Gamma(\tilde{K}) : \text{stabilised by } G = \mathbf{SU}(7), \quad (\text{III.120})$$

with  $\tilde{K} = (\det T^*)^2 \otimes K$ . We denote  $J$  as the *almost exceptional complex structure* and  $\psi$  as the *generalised  $\mathbf{SU}(7)$  structure*.

They are stabilised by the same  $\mathbf{SU}(7)$ , but  $J$  is invariant under an extra  $\mathbb{C}^*$  action.

As we will see, these structures are the exceptional analogues of the complex  $(\mathbf{GL}(3, \mathbb{C}))$  and  $\mathbf{SL}(3, \mathbb{C})$  structures of conventional geometry introduced in II.1.1, or of the generalised complex  $(\mathbf{U}(3, 3))$  and generalised Calabi-Yau  $(\mathbf{SU}(3, 3))$  structures of  $\mathbf{O}(6, 6)$  generalised geometry discussed in III.1.1.

In order to define the exceptional complex structure, we introduce a globally defined nowhere vanishing  $\mathbf{SU}(8)$  spinor  $\eta$ , transforming in the fundamental representation  $\mathbf{8}$  of  $\mathbf{SU}(8)$ . The stabiliser of  $\eta$  is  $\mathbf{SU}(7)$ , and as such it defines a generalised  $\mathbf{SU}(7)$  structure.

There is a  $\mathbf{U}(1) \in \mathbf{SU}(8)/\mathbb{Z}_2$  which commutes with this  $\mathbf{SU}(7)$  subgroup. It is generated by an element of the  $\mathbf{SU}(8)$  Lie algebra conjugate to the diagonal matrix

$$\alpha = \text{diag}(-1/2, -1/2, \dots, 7/2) \in \mathbf{SU}(8) \subset E_{7(7)} \oplus \mathbb{R}. \quad (\text{III.121})$$

The normalisation is chosen so that  $\exp(i\theta J)$  with  $0 \leq \theta \leq 2\pi$  generates a  $U(1)$  subgroup of  $SU(8)/\mathbb{Z}_2$ . The commutant of this  $U(1)$  is then a  $\mathbb{R}^+ \times U(7)$  subgroup of  $E_{7(7)} \times \mathbb{R}^+$ . This leads us to the following definition.

A generalised  $\mathbb{R}^+ \times U(7)$  structure, or almost exceptional complex structure, is a section  $J \in \Gamma(\text{ad}\tilde{F})$  that is conjugate at each point  $p \in M$  to the element  $\alpha \in SU(8)$  in (III.121).

The exceptional generalised structure  $J$  lies within a particular orbit of the **133** representation space of  $E_{7(7)}$ . Decomposing the adjoint  $E_{7(7)}$  representation into irreducible  $SU(8)$  representations, we have

$$\mathbf{133} = \mathbf{63} \oplus \mathbf{70} \ni (\mu^\alpha{}_\beta, \mu_{\alpha\beta\gamma\delta}), \quad (\text{III.122})$$

where we introduced  $SU(8)$  indices.

We can then write the exceptional complex structure  $J$  in terms of the spinor  $\eta$  as

$$J^\alpha{}_\beta = 4\eta^\alpha\bar{\eta}_\beta - \frac{1}{2}(\bar{\eta}\eta)\delta^\alpha{}_\beta, \quad J_{\alpha\beta\gamma\delta} = 0, \quad (\text{III.123})$$

where we set  $\bar{\eta}\eta = 1$ . We can further decompose the **133**  $E_{7(7)}$  representation space into irreducible  $SU(7) \times U(1)$  representations

$$\mathbf{133} = \mathbf{1}_0 \oplus \mathbf{48}_0 \oplus (\mathbf{7}_{-4} \oplus \bar{\mathbf{7}}_4) \oplus (\mathbf{35}_2 \oplus \bar{\mathbf{35}}_{-2}), \quad (\text{III.124})$$

where now the subscripts denote the  $U(1)$  charge. The exceptional complex structure  $J$  lies in the singlet  $\mathbf{1}_0$  representation.

Given  $J$ , in analogy with a conventional almost complex structure, we can decompose the complexified generalised tangent bundle into  $J$  eigenbundles, through its adjoint action. The eigenbundles are irreducible representations of  $SU(7) \times U(1)$ , and we find

$$E_{\mathbb{C}} = L_3 \oplus L_{-1} \oplus L_1 \oplus L_{-3} \quad (\text{III.125})$$

$$\mathbf{56}_{\mathbb{C}} = \mathbf{7}_3 \oplus \mathbf{21}_{-1} \oplus \bar{\mathbf{21}}_1 \oplus \bar{\mathbf{7}}_{-3}, \quad (\text{III.126})$$

with  $L_{-3} \simeq \bar{L}_3$  and  $L_{-1} \simeq \bar{L}_1$ .

We can similarly decompose the untwisted complexified generalised tangent bundle

$$E_{\mathbb{C}} = \tilde{L}_3 \oplus \tilde{L}_{-1} \oplus \tilde{L}_1 \oplus \tilde{L}_{-3} \quad (\text{III.127})$$

$$\mathbf{56}_{\mathbb{C}} = \mathbf{7}_3 \oplus \mathbf{21}_{-1} \oplus \bar{\mathbf{21}}_1 \oplus \bar{\mathbf{7}}_{-3}, \quad (\text{III.128})$$

also respecting  $\tilde{L}_{-3} \simeq \bar{\tilde{L}}_3$  and  $\tilde{L}_{-1} \simeq \bar{\tilde{L}}_1$ .

As we will see,  $L_3$  can be seen as the analogue of  $T^{1,0}$  in conventional complex geometry

and of  $L_1$  in generalised complex geometry. This thus leads to the following alternative definition of the exceptional complex structure.

An almost exceptional complex structure is a subbundle  $L_3 \subset E_{\mathbb{C}}$  such that

i)  $\dim_{\mathbb{C}} L_3 = 7$

ii)  $L_3 \times_N L_3 = 0$ .

iii)  $L_3 \cap \bar{L}_{-3} = \{0\}$

iv) The map  $h : L_3 \times L_3 \rightarrow (\det T^*)_{\mathbb{C}}$  defined by  $h(V, W) = is(V, \bar{W})$  is a definite hermitian inner product valued in  $\det T^*$ .

The second condition is the analogue of the isotropy condition for a generalised complex structure.

Again in analogy with the generalised complex structure case, we call a subbundle  $L_3$  satisfying the first two conditions a (*complex*) *exceptional polarisation*.

The  $\tilde{L}_3$  bundle similarly defines an untwisted version of the almost exceptional complex structure.

We now define the generalised  $SU(7)$  structure. To do so, we write down the decomposition  $SU(7) \times U(1) \subset SU(8)/\mathbb{Z}_2 \subset E_{7(7)}$  of the **912**  $E_{7(7)}$  representation space

$$\mathbf{912} = \mathbf{36} \oplus \mathbf{420} \oplus c.c. \quad (\text{III.129})$$

$$= \mathbf{1}_7 \oplus \mathbf{7}_3 \oplus \mathbf{28}_{-1} \oplus \mathbf{21}_{-1} \oplus \mathbf{35}_{-5} \oplus \mathbf{140}_3 \oplus \mathbf{224}_{-1} \oplus c.c. \quad (\text{III.130})$$

From now on we consider the generalised tensor bundle  $\tilde{K}$  transforming in the **912**<sub>3</sub> representation of  $E_{7(7)} \times \mathbb{R}^+$ . The presence of an  $SU(7)$  singlet in the decomposition (III.130) implies that each almost exceptional complex structure  $J$  defines a unique line bundle  $\mathcal{U}_J \subset \tilde{K}_{\mathbb{C}}$ , satisfying

$$V \bullet \psi = 0 \quad \forall V \in \Gamma(L_3), \quad s(\psi, \bar{\psi}) \neq 0, \quad (\text{III.131})$$

where  $\psi$  is a local section of  $\mathcal{U}_J$ , the product  $V \bullet \psi$  is defined by the projection map  $E \otimes \tilde{K}_{\mathbb{C}}$  where  $C$  is the generalised tensor bundle transforming in the **8645**<sub>4</sub> representation of  $E_{7(7)} \times \mathbb{R}^+$ , and  $s$  is the symplectic invariant on the **912** bundle  $\tilde{K} \subset E \otimes E \otimes E$  induced from the symplectic invariant on  $E$ . We are thus led to define a generalised  $SU(7)$  structure in the following way.

Given an almost exceptional complex structure  $J$  with trivial line bundle  $\mathcal{U}_J$ , a generalised  $SU(7)$  structure is a global nowhere-vanishing section  $\psi \in \Gamma(\mathcal{U}_J)$ .

We stress again that this is in complete analogy with the almost complex and almost generalised complex cases: the  $SU(7)$  structure naturally generalises the holomorphic  $(3, 0)$ -form  $\Omega$  and the pure spinor  $\Psi$  defining the  $SL(3, \mathbb{C})$  and  $SU(3, 3)$  structures respectively. The second condition in (III.131) simply generalises the non-degeneracy condition

of the tensors  $\Omega$  and  $\Psi$ .

One can again define an untwisted version of the generalised  $SU(7)$  structure, from the untwisted almost exceptional complex structure  $\tilde{J}$ : it indeed defines a unique line bundle  $\mathcal{U}_{\tilde{J}} \subset \tilde{K}_{\mathbb{C}}$  as

$$V \bullet \tilde{\psi} = 0 \quad \forall V \in \Gamma(L_3), \quad s(\tilde{\psi}, \bar{\tilde{\psi}}) \neq 0, \quad (\text{III.132})$$

where  $\tilde{\psi}$  is a local section of  $\mathcal{U}_{\tilde{J}}$ .

The generalised  $SU(7)$  structure again lies within a particular orbit of the  $\mathbf{912}_3$  representation space. Concretely, decomposing the  $\mathbf{912}$  space in terms of irreducible representations of  $SU(8)$

$$\mathbf{912} = \mathbf{36} \oplus \mathbf{420} \oplus \overline{\mathbf{36}} \oplus \overline{\mathbf{420}} \ni (\kappa^{\alpha\beta}, \kappa^{\alpha\beta\gamma}_{\delta}, \bar{\kappa}_{\alpha\beta}, \bar{\kappa}_{\alpha\beta\gamma}^{\delta}), \quad (\text{III.133})$$

we can write the generalised  $SU(7)$  structure in terms of the spinor  $\eta$  as

$$\psi^{\alpha\beta} = \lambda(\text{vol}_G)^{3/2} \eta^{\alpha} \eta^{\beta} \quad \psi^{\alpha\beta\gamma}_{\delta} = \bar{\psi}_{\alpha\beta} = \bar{\psi}_{\alpha\beta\gamma}^{\delta} = 0, \quad (\text{III.134})$$

with  $\text{vol}_G$  the  $E_{7(7)}$ -invariant volume defined by the generalised metric  $G$  [36, 39] and  $\lambda$  a non-vanishing complex number.

#### d).1 $\mathbb{R}^+ \times U(7)$ and $SU(7)$ intrinsic torsions and integrability

We now discuss the integrability of the  $\mathbb{R}^+ \times U(7)$  and  $SU(7)$  structures.

As mentioned above, a generalised structure is integrable if its intrinsic torsion vanishes. Regarding the  $SU(7)$  structure, the  $\tau$  map (III.116) has been explicitly constructed and the corresponding vector bundle  $W_{SU(7)}$  has been given in [42, 87]. One can do the same for the  $\mathbb{R}^+ \times U(7)$  structure, and the resulting intrinsic torsions  $W_{\text{int}}^{SU(7)} = W/W_{SU(7)}$  and  $W_{\text{int}}^{\mathbb{R}^+ \times U(7)} = W/W_{\mathbb{R}^+ \times U(7)}$  are, in terms of  $SU(7)$  irreducible representations:

$$W_{\text{int}}^{SU(7)} : \mathbf{1}_{-7} \oplus \bar{\mathbf{7}}_{-3} \oplus \mathbf{21}_{-1} \oplus \mathbf{35}_{-5} \oplus c.c. \quad (\text{III.135})$$

$$W_{\text{int}}^{\mathbb{R}^+ \times U(7)} : \mathbf{1}_{-7} \oplus \mathbf{35}_{-5} \oplus c.c. \quad (\text{III.136})$$

where again, the subscript denotes that  $U(1)$  charge under the action of  $J$ .

Let us first investigate the integrability of the  $\mathbb{R}^+ \times U(7)$  structure. It is integrable if the intrinsic torsion components  $\mathbf{1}_{-7}$  and  $\mathbf{35}_{-5}$  vanishes. This can be reformulated in a convenient way, naturally generalising the integrability conditions of the complex and generalised complex structures.

To do so, we consider a connection compatible with such a structure  $J$ :  $DJ = 0$ .

Evaluating (III.104) with  $V, V' \in \Gamma(L_3)$ , we have

$$L_V V' = L_V^D V' - T(V) \cdot V' \quad \forall V, V' \in \Gamma(L_3). \quad (\text{III.137})$$

It is important to note that since the left-hand side of (III.137) doesn't depend on the choice of connection  $D$ , the non-vanishing sections of  $E$  in the right-hand side only involve the intrinsic  $\mathbb{R}^+ \times \text{U}(7)$  torsion.

The compatibility of the connection with  $J$  ensures  $L_V^D V' \subset L_3$ . The second term then generates the following sections of the generalised tangent bundle, in terms of  $\text{SU}(7) \times \text{U}(1)$  representations

$$\mathbf{1}_{-7} \oplus \mathbf{7}_3 \oplus \mathbf{7}_3 \supset \mathbf{21}_{-1} \quad (\text{III.138})$$

$$\mathbf{35}_{-5} \oplus \mathbf{7}_3 \oplus \mathbf{7}_3 \supset \overline{\mathbf{21}}_1. \quad (\text{III.139})$$

This is equivalent to

$$\forall V, V' \in \Gamma(L_3): \quad \mathbf{1}_{-7} \neq 0 \iff L_V V' \cap L_{-1} \neq 0 \quad (\text{III.140})$$

$$\mathbf{35}_{-5} \neq 0 \iff L_V V' \cap L_1 \neq 0. \quad (\text{III.141})$$

This motivates the following equivalent definition of the integrability of the exceptional complex structure.

A torsion-free  $\mathbb{R}^+ \times \text{U}(7)$  structure  $J$  or *exceptional complex structure* is one satisfying involutivity of  $L_3$  under the generalised Lie derivative:

$$L_V V' \subset L_3 \quad \forall V, V' \in \Gamma(L_3). \quad (\text{III.142})$$

One can then define the restriction of the Dorfman derivative on  $L_3$ . Given that the  $L_3$  bundle respects  $L_3 \times_N L_3 = 0$ , the Jacobiator (III.98) vanishes and the algebroid  $(L_3, L, a)$  is actually a Lie algebroid.

The integrability of the exceptional complex structure  $J$  can similarly be spelled in the untwisted picture:  $J$  is integrable if  $\tilde{L}_3$  is involutive under the twisted generalised Lie derivative

$$L_V^{H+F} V' \subset \tilde{L}_3 \quad \forall V, V' \in \Gamma(\tilde{L}_3). \quad (\text{III.143})$$

This definition of the integrability is in complete analogy with the complex and generalised complex structures cases, where the integrability of a (generalised) complex structure can be recast as involutivity of eigenspaces of the (generalised) complex structure under the (generalised) Lie bracket.

Furthermore, the analogy carries through to the integrability of the  $\text{SU}(7)$  structure.



Indeed, the complex geometry analogue of the  $SU(7)$  structure is the  $SL(3, \mathbb{C})$  structure. We have seen in section II.1 that the integrability of the  $SL(3, \mathbb{C})$  structure can be recast as the vanishing of a moment map for the diffeomorphisms action. We now discuss the generalisation of this reformulation of the integrability for the  $SU(7)$  structure, as the vanishing of a moment map for the generalised diffeomorphisms action.

For an  $SU(7)$  structure, a choice of  $\psi$  at a point  $p \in M$  is equivalent to picking a point in the coset

$$\psi|_p \in \mathcal{Q}_{SU(7)} = \frac{E_{7(7)} \times \mathbb{R}^+}{SU(7)}. \quad (\text{III.144})$$

The choice of an  $SU(7)$  structure on  $M$  therefore corresponds to a section of the fibre bundle

$$\mathcal{Q}_{SU(7)} \rightarrow \mathcal{Q}_{SU(7)} \rightarrow M \quad (\text{III.145})$$

and we can identify the space of  $SU(7)$  structures as  $\mathcal{Z} \simeq \Gamma(\mathcal{Q}_{SU(7)})$ . One can also restrict to the space of  $SU(7)$  structures with an integrable associated exceptional complex structure

$$\hat{\mathcal{Z}} = \{\Omega \in \mathcal{Z} \mid J \text{ is integrable}\}. \quad (\text{III.146})$$

Crucially, the space  $\hat{\mathcal{Z}}$  inherits a Kähler metric from the Kähler metric on the coset space  $\mathcal{Q}_{SU(7)}$ , picked out by supersymmetry, with the following corresponding Kähler potential

$$\mathcal{K} = \int_M (i s(\psi, \bar{\psi}))^{1/3}. \quad (\text{III.147})$$

From the symplectic structure defined by this Kähler potential, one can define the corresponding moment map [61]

$$\mu : \hat{\mathcal{Z}} \rightarrow \mathfrak{gdiff}^* \quad (\text{III.148})$$

for the action of generalised diffeomorphisms, where  $\mathfrak{gdiff}$  is the Lie algebra of generalised diffeomorphisms, and where

$$\begin{aligned} \mu(V) &= \frac{1}{3} \int_M s(L_V \phi, \bar{\phi}) \\ &= \frac{1}{3} \int_M s(L_V \psi, \bar{\psi}) (i s(\psi, \bar{\psi}))^{-2/3}, \end{aligned} \quad (\text{III.149})$$

with  $V \in \Gamma(E)$  and  $\phi = (i s(\psi, \bar{\psi}))^{-1/3} \psi$  which transforms in the  $\mathbf{912}_1$  representation. Going to the second line we used  $\int_M L_V(\dots) = 0$ .

Let us introduce a generalised connection  $D$  compatible with the  $SU(7)$  structure  $D\psi = D\phi = 0$ . Using the definition of the torsion (III.104), we have

$$\mu(V) = \frac{1}{3} \int_M s(L_V^D \phi, \bar{\phi}) - s(T(V)\phi, \bar{\phi})$$

$$= -\frac{1}{3} \int_M s(T(V)\phi, \bar{\phi}), \quad (\text{III.150})$$

where the first term in the right-hand side of the first line vanishes from the compatibility of the generalised connection  $D$ .

Since the definition of the moment map is independent of any choice of connection, only the  $SU(7)$  intrinsic torsion can contribute in the last expression.

Crucially, given that the vector  $V \in \Gamma(E)$  transforms in  $\mathbf{7} \oplus \mathbf{21} \oplus c.c.$  representations of  $SU(7)$ , and since  $\phi$  is an  $SU(7)$  singlet, only the  $\mathbf{7}$  and  $\mathbf{21}$   $SU(7)$  representations of the intrinsic  $SU(7)$  torsion enter the moment map (III.150).

From (III.135) and (III.136), we see that these are precisely the extra components of the  $SU(7)$  torsion, relative to the  $\mathbb{R}^+ \times U(7)$  torsion, which must vanish in order for the  $SU(7)$  torsion to be integrable.

This leads to the following reformulation of the integrability of the  $SU(7)$  structure.

A torsion-free generalised  $SU(7)$  structure is one where  $L_3$  is involutive and the moment map (III.149) vanishes.

The involutivity of  $L_3$  kills the  $\mathbf{1}$  and  $\mathbf{35}$  components of the  $SU(7)$  intrinsic torsion (III.135), while the vanishing of the moment map kills the remaining  $\mathbf{7}$  and  $\mathbf{21}$  irreducible representations.

One can define an untwisted version of the moment map, to define the integrability of the untwisted generalised  $SU(7)$  structure  $\tilde{\psi}$

$$\tilde{\mu}(\tilde{V}) = \frac{1}{3} \int_M s(L_{\tilde{V}}^{H+F} \tilde{\psi}, \bar{\tilde{\psi}}) (i s(\tilde{\psi}, \bar{\tilde{\psi}}))^{-2/3}, \quad (\text{III.151})$$

and the untwisted generalised  $SU(7)$  structure will be integrable if this moment map vanishes.

Finally, let us mention the moduli space of integrable generalised  $SU(7)$  structures. Two  $SU(7)$  structures that are related by generalised diffeomorphisms are equivalent, so the moduli space of  $SU(7)$  structures  $\mathcal{M}_\psi$  should be viewed as the space of torsion-free  $SU(7)$  structures quotiented by the action of these transformations. Given that  $\hat{\mathcal{Z}}$  admits both a symplectic structure and a Kähler structure, there are two ways to view this quotient, namely as a symplectic quotient by  $\text{GDiff}$  or as a standard quotient by the complexified group  $\text{GDiff}_{\mathbb{C}}$ :

$$\mathcal{M}_\psi = \{\psi \in \hat{\mathcal{Z}} \mid \mu = 0\} / \text{Gdiff} \equiv \hat{\mathcal{Z}} // \text{GDiff} \simeq \hat{\mathcal{Z}} / \text{GDiff}_{\mathbb{C}}. \quad (\text{III.152})$$

We will see in the next subsection how the  $SU(7)$  moduli space  $\mathcal{M}_\psi$  is related to the physical moduli space of  $\mathcal{N} = 1$  type II supergravity backgrounds compactified to four-dimensions.

### III.2.2 Generalised structures of supersymmetric backgrounds

In this subsection, we apply the exceptional generalised geometry formalism developed throughout this chapter to four-dimensional  $\mathcal{N} = 1$  flux backgrounds.

#### a) $\mathbb{R}^+ \times \mathbf{U}(7)$ and $\mathbf{SU}(7)$ structures of four-dimensional $\mathcal{N} = 1$ backgrounds

In this subsection we show how the  $\mathbb{R}^+ \times \mathbf{U}(7)$  and  $\mathbf{SU}(7)$  structures defined above are well suited to describe the geometry of  $\mathcal{N} = 1$  type II supergravity flux backgrounds compactified to four-dimensions.

We described the  $\mathbb{R}^+ \times \mathbf{U}(7)$  and  $\mathbf{SU}(7)$  structures in terms of a globally defined nowhere vanishing  $\mathbf{SU}(8)$  spinor  $\eta$ . In the context of type II supergravity, such a spinor can be naturally constructed from the two internal type II Killing spinors.

To do so explicitly, we recall our compactification ansatz: we consider here type II solutions that are the warped product of four-dimensional Minkowski space  $X_4$  and a six-dimensional compact manifold  $M$ , with the following metric

$$ds_{10}^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{III.153})$$

where again  $x^\mu$ ,  $\mu = 0, \dots, 3$  are the external coordinates on  $X_4$ , and  $y^m$ ,  $m = 1, \dots, 6$  are the coordinates on  $M$ .

The ten-dimensional type II Killing spinors are then two ten-dimensional Majorana-Weyl spinors  $\epsilon^i$  of opposite (the same) chirality for type IIA (IIB).

They decompose accordingly under  $\text{Spin}(9,1) \rightarrow \text{Spin}(3,1) \times \text{Spin}(6)$  as

$$\epsilon_1 = \zeta \otimes \eta_1 + c.c. \quad \epsilon_2 = \zeta \otimes \eta_2 + c.c. \quad (\text{III.154})$$

where  $\zeta$  is a Weyl spinor of positive chirality on  $X_4$ , and  $\eta_1$  and  $\eta_2$  are Weyl spinors on the six-dimensional internal space.  $\eta_1$  has positive chirality, while  $\eta_2$  has negative chirality in type IIA and positive chirality in type IIB.

Writing the  $\text{Spin}(6)$  spinors as  $\mathbf{SU}(4)$  spinors, we can construct the following  $\mathbf{SU}(8)$  spinor

$$\eta = e^{A/2 - \phi/6} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \text{ in type IIA,} \quad \eta = e^{A/2 - \phi/6} \begin{pmatrix} \eta_1 \\ \eta_2^* \end{pmatrix} \text{ in type IIB,} \quad (\text{III.155})$$

where  $\eta_{1,2}$  are seen here as four-dimensional Weyl-spinors, and  $\eta$  hence transforms in the  $\mathbf{8}$  representation of  $\mathbf{SU}(8)$ .

Even though  $\eta$  must be nowhere vanishing, the individual  $\text{Spin}(6)$  spinors  $\eta_{1,2}$  may vanish while still properly defining a generalised  $\mathbf{SU}(7)$  structure. Backgrounds with such Killing spinors are thus well defined in  $E_{7(7)}$  generalised geometry, even if they don't define

conventional or  $O(6,6)$  generalised (global) G-structures [87, 88]. This is for instance the case of type II backgrounds with NS5-branes wrapping a Calabi-Yau, described in  $E_{7(7)}$  generalised geometry in [79].

Throughout this thesis, we focus on the subclass of type II backgrounds having nowhere vanishing globally defined Killing spinors  $\eta_{1,2}$ . In the  $\mathcal{N} = 1$  case, this class of backgrounds, which we refer to as the GMPT family, has been studied within the context of  $O(6,6)$  generalised geometry in [1, 6, 85].

We first briefly introduce the key aspects of the GMPT class within the  $O(6,6)$  generalised geometry framework, before embedding it into the  $E_{7(7)}$  generalised geometry formalism, following [61].

As we will discuss in the next chapter, note that the two six-dimensional internal spinors must have the same norm in order to admit supersymmetric sources, which we denote  $\|\eta_{1,2}\|$ .

In this section we need a slightly different definition<sup>11</sup> of the pure spinors introduced in (III.66) and (III.67):

$$\Phi_1 = \frac{e^{3A-\phi}}{\|\eta_{1,2}\|^2} \eta_1 \otimes \eta_2^\dagger \quad (\text{III.156})$$

$$\Phi_2 = \frac{e^{3A-\phi}}{\|\eta_{1,2}\|^2} \eta_1 \otimes \eta_2^T. \quad (\text{III.157})$$

The pure spinors  $\Phi_1$  and  $\Phi_2$  are again odd/even and even/odd in type IIA/IIB, respectively

$$\Phi_1 = \Phi_\mp, \quad \Phi_2 = \Phi_\pm, \quad (\text{III.158})$$

and their normalisation is such that

$$\langle \Phi_1, \bar{\Phi}_1 \rangle = \langle \Phi_2, \bar{\Phi}_2 \rangle = \frac{i}{8} e^{6A-2\phi} \text{vol}_6, \quad (\text{III.159})$$

with  $\text{vol}_6$  the volume form on  $M$  in the string frame.

The resulting pure spinor equations, equivalent to preserving ten-dimensional  $\mathcal{N} = 1$  supersymmetry, read

$$d_H \Phi_2 = 0 \quad (\text{III.160})$$

$$d_H(e^{-A} \text{Re} \Phi_1) = 0 \quad (\text{III.161})$$

---

<sup>11</sup>The reason for these two set of conventions is that we use the two formalism to do different things. In  $O(6,6)$  generalised geometry, among other things, we will derive the equations of motion of new type II backgrounds in chapter IV, requiring the scalar fields to be factorised out of the pure spinors to vary the type II action. In  $E_{7(7)}$  generalised geometry, we will study the generalised structure and its associated generalised torsion for non-supersymmetric type II flux backgrounds in chapter V. To do so, it is most natural to incorporate some physical fields into the pure spinor definition, themselves embedded into an all encompassing generalised structure.

$$d_H^{\mathcal{J}_1}(e^{-3A}\text{Im}\Phi_1) = -\frac{F}{8}, \quad (\text{III.162})$$

where we wrote here an equivalent form of the pure spinor equations (III.71), (III.72) and (III.73), derived by Tomasiello [6], with  $d_H^{\mathcal{J}_1} = [d_H, \mathcal{J}_1]$ .

We now show how to embed these solutions into the framework of  $E_{7(7)}$  generalised geometry, first introducing an exceptional complex structure. We work here in the untwisted picture, and define

$$\tilde{L}_3 = e^{8ie^{-3A}\text{Im}\Phi_1} \cdot (L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2}). \quad (\text{III.163})$$

Here  $L_1^{\mathcal{J}_2} \subset E_{O(6,6)\mathbb{C}} \simeq (T \oplus T^*)_{\mathbb{C}}$  is the  $+i$ -eigenspace of  $\mathcal{J}_2$ , and  $\mathcal{U}_{\mathcal{J}_2}$  is the pure spinor line bundle defined by  $\mathcal{J}_2$ . It is relatively straightforward to check that this  $\tilde{L}_3$  satisfies the necessary and sufficient conditions to define an almost exceptional complex structure.

The corresponding untwisted generalised  $\text{SU}(7)$  structure is

$$\psi = e^{8ie^{-3A}\text{Im}\Phi_1} \cdot \Phi_2, \quad (\text{III.164})$$

where here  $\Phi_2$  is to be understood as a section of the untwisted  $\tilde{K}_{\mathbb{C}}$  bundle.

We now study the integrability condition of the almost exceptional complex structure, in order to see how it relates to the supersymmetry conditions (III.160), (III.161), and (III.162). Requiring the almost exceptional complex structure to be integrable amounts to imposing the involutivity of  $\tilde{L}_3$  under the twisted generalised Lie derivative

$$L_V^{H+F} V' \subset \tilde{L}_3 \quad \forall V, V' \in \Gamma(\tilde{L}_3). \quad (\text{III.165})$$

We write a section of  $\tilde{L}_3$  as  $e^{\Sigma} \cdot V \equiv e^{\Sigma} \cdot (W + \alpha\Phi_2)$ , with  $W \in \Gamma(L_1^{\mathcal{J}_2})$ ,  $\alpha$  a non-vanishing complex number, and  $\Sigma \equiv 8ie^{-3A}\text{Im}\Phi_1$ . We then evaluate the twisted Dorfman derivative on elements of the  $\tilde{L}_3$  bundle

$$L_{e^{\Sigma} \cdot V}^{H+F}(e^{\Sigma} \cdot V') = e^{\Sigma} \cdot [L_V^H V' + (W(F + d_H \Sigma) + \alpha \langle F + d_H \Sigma, \Phi_2 \rangle) \cdot V'], \quad (\text{III.166})$$

with  $V, V' \in \Gamma(L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2})$ . The involutivity of  $\tilde{L}_3$  then requires the term in brackets to be an element of  $L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2}$  for all  $V, V' \in \Gamma(L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2})$ . Given that the first term in (III.166) is differential, and the remaining terms are algebraic, involutivity requires that these are sections of  $L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2}$  independently:

$$L_V^H V' \in \Gamma(L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2}) \quad (\text{III.167})$$

$$(W(F + d_H \Sigma) + \alpha \langle F + d_H \Sigma, \Phi_2 \rangle) \cdot V' \in \Gamma(L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2}). \quad (\text{III.168})$$

We start with the first condition. Requiring

$$L_{W+\alpha\Phi_2}^H(W' + \alpha'\Phi_2) \in \Gamma(L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2}) \quad (\text{III.169})$$

implies first that

$$L_W^H W' \in \Gamma(L_1^{\mathcal{J}_2}) \quad \forall W, W' \in \Gamma(L_1^{\mathcal{J}_2}), \quad (\text{III.170})$$

that is, the generalised complex structure  $\mathcal{J}_2$  associated to  $\Phi_2$  must be integrable. Recall that this is equivalent to

$$d_H \Phi_2 = \mathcal{X} \Phi_2 \quad \text{for some } X \in \Gamma(L_{-1}^{\mathcal{J}_2}). \quad (\text{III.171})$$

From this condition, and from  $W\Phi_2 = W'\Phi_2 = 0$ , we immediately have

$$L_W^H(\alpha'\Phi_2) = \eta(W, d\alpha' + 2X)\Phi_2 \in \Gamma(\mathcal{U}_{\mathcal{J}_2}) \quad (\text{III.172})$$

$$L_{\alpha\Phi_2}^H W' = -\eta(W', d\alpha + 2X)\Phi_2 \in \Gamma(\mathcal{U}_{\mathcal{J}_2}), \quad (\text{III.173})$$

as required. Here  $\eta(\cdot, \cdot)$  is the  $O(6, 6)$  inner product (III.3). Finally, using (III.171) again, we have

$$L_{\alpha\Phi_2}^H(\alpha'\Phi_2) = -\alpha'[(d\alpha + \mathcal{X})\Phi_2] \cdot \Phi_2 = 0, \quad (\text{III.174})$$

identically, as can be seen by counting the  $\mathcal{J}_2$  charges.

The integrability of the generalised complex structure  $\mathcal{J}_2$  is therefore enough to ensure the condition (III.167). Turning to the second condition (III.168), it first imposes

$$W'W(F + d_H\Sigma) \in \mathcal{U}_{\mathcal{J}_2}, \quad (\text{III.175})$$

which is satisfied if

$$(F + d_H\Sigma)_{V_{-1}} = (F + d_H\Sigma)_{V_{-3}} = 0, \quad (\text{III.176})$$

where the subscript  $V_n$  denotes the projection onto the  $in$ -eigenspace of  $\mathcal{J}_2$ . Combining these conditions with their complex conjugates is equivalent to

$$d_H^{\mathcal{J}_1}(e^{-3A}\text{Im}\Phi_1) = -\frac{F}{8}, \quad (\text{III.177})$$

which precisely corresponds to the supersymmetry condition (III.162). Moreover, (III.176) implies

$$[W(F + d_H\Sigma)] \cdot \Phi_2 = 0 \quad (\text{III.178})$$

$$\langle F + d_H\Sigma, \Phi_2 \rangle = 0. \quad (\text{III.179})$$

Every other terms in the involutivity condition (III.168) therefore vanish identically.

The integrability of the exceptional complex structure is hence equivalent to

$$d_H \Phi_2 = \not{X} \Phi_2 \quad d_H^{\mathcal{J}_1}(e^{-3A} \text{Im} \Phi_1) = -\frac{F}{8}, \quad (\text{III.180})$$

which is not equivalent to the full set of supersymmetry conditions (III.160), (III.161), and (III.162).

Lets now investigate the integrability condition of the generalised SU(7) structure. We thus specify the untwisted version of the moment map (III.151) for the untwisted generalised SU(7) structure (III.164), which reads

$$\tilde{\mu}(e^\Sigma \cdot \tilde{V}) = \frac{1}{3} \int_M s(L_{\tilde{V}}^{H+F+d_H \Sigma} \Phi_2, e^{-2\Sigma} \cdot \bar{\Phi}_2) (i s(\tilde{\psi}, \bar{\tilde{\psi}}))^{-2/3}. \quad (\text{III.181})$$

The integrability of the exceptional complex structure ensures that  $F + d_H \Sigma$  stabilises  $L_1^{\mathcal{J}_2} \oplus \mathcal{U}_{\mathcal{J}_2}$ , and hence the singlet  $\Phi_2 \in \Gamma(\tilde{K}_{\mathbb{C}})$ . We thus have

$$L_{\tilde{V}}^{H+F+d_H \Sigma} \Phi_2 = L_{\tilde{V}}^H \Phi_2. \quad (\text{III.182})$$

The remaining terms in the moment map are then of the form

$$\tilde{\mu}(e^\Sigma \cdot \tilde{V}) \sim \text{const} \int_M \langle L_Z^H \Phi_2, \bar{\Phi}_2 \rangle + \text{const} \int_M \langle d\Lambda_\pm, e^{-A} \text{Re} \Phi_1 \rangle, \quad (\text{III.183})$$

Where we wrote  $\tilde{V} = Z + \Lambda_\pm + \tilde{Z}$ . This form of the moment map follows from keeping track of the  $U(1) \subset \text{SL}(2, \mathbb{R})$  charge in the  $O(6, 6) \times \text{SL}(2, \mathbb{R}) \subset E_{7(7)}$  decomposition, noting the  $\mathbb{R}_+$  weight to get the correct  $e^A$  factor [61].

The two terms in this moment map must vanish independently in order for the generalised SU(7) structure to be integrable. Plugging the involutivity condition  $d_H \Phi_2 = \not{X} \Phi_2$  into the first term, it can then only vanish if  $X = 0$ , yielding

$$d_H \Phi_2 = 0. \quad (\text{III.184})$$

Integrating by part the second term, it vanishes if

$$d_H(e^{-A} \text{Re} \Phi_1) = 0. \quad (\text{III.185})$$

The integrability of the generalised SU(7) structure (III.164) is therefore equivalent to

$$d_H \Phi_2 = 0 \quad (\text{III.186})$$

$$d_H(e^{-A} \text{Re} \Phi_1) = 0 \quad (\text{III.187})$$

$$d_H^{\mathcal{J}_1}(e^{-3A}\text{Im}\Phi_1) = -\frac{F}{8}, \quad (\text{III.188})$$

which precisely corresponds to the supersymmetry conditions (III.160), (III.161), and (III.162).

We can therefore conclude that a type II flux background compactified to four-dimensional Minkowski space and preserving  $\mathcal{N} = 1$  supersymmetry defines an integrable generalised  $\text{SU}(7)$  structure  $\psi$ , and the integrability conditions for  $\psi$  are equivalent to the supersymmetry conditions of said background.

### b) Equations of motion

We conclude this section by reformulating the equations of motion of  $\mathcal{N} = 1$  type II supergravity flux backgrounds compactified to four-dimensions, within the framework of exceptional generalised geometry.

To do so, we briefly introduce the notion of generalised metric. It is well known [89, 90] that the bosonic fields of reduced supergravity parametrise a coset  $(E_{d(d)} \times \mathbb{R}^+)/H_d$ , with  $H_d$  the maximal compact subgroup of  $E_{d(d)}$ . In the case of type II supergravity reduced to four-dimensions, the bosonic fields parametrise a coset  $(E_{7(7)} \times \mathbb{R}^+)/(\text{SU}(8)/\mathbb{Z}_2)$ , that is, at each point  $p \in M$

$$\{g, B, \tilde{B}, \phi, C^\pm, A\} \in \frac{E_{7(7)} \times \mathbb{R}^+}{\text{SU}(8)/\mathbb{Z}_2}, \quad (\text{III.189})$$

with  $\tilde{B}$  the six-form potential dual to  $B$  and  $C^\pm$  the RR potentials in type IIB/IIA.

The group  $\text{SU}(8)/\mathbb{Z}_2$  is the maximal compact subgroup of  $E_{7(7)}$ , and it is the analogue of the orthogonal group  $\text{O}(d) \subset \text{GL}(d, \mathbb{R})$  in standard Riemannian geometry. As such, it defines a generalised version of the metric  $G$  [39], invariant under  $\text{SU}(8)/\mathbb{Z}_2$ , and (III.189) means that giving the bosonic fields is thus equivalent to specifying a generalised metric  $G$ .

We now introduce a generalised connection  $D$  compatible with  $\text{SU}(8)/\mathbb{Z}_2$ , that is  $DG = 0$ . Crucially, it is always possible to find such connections that are torsion-free, but they are not unique in general [39] (unlike the case of ordinary Riemannian geometry which singles out the Levi-Civita connection).

Indeed, writing the bundle  $K_{\text{SU}(8)} = E^* \otimes \text{ad}P_{\text{SU}(8)}$ , in terms of  $\text{SU}(8)$  irreducible representations, we have

$$K_{\text{SU}(8)} = (\bar{\mathbf{28}} + \mathbf{28}) \times \mathbf{63} = \mathbf{28} + \mathbf{36} + \mathbf{420} + \mathbf{1280} + c.c. \quad (\text{III.190})$$

The map  $\tau$  in (III.116) then splits  $K_{\text{SU}(8)}$  into

$$\text{Im } \tau = \mathbf{28} + \mathbf{36} + \mathbf{420} + c.c. \equiv W_{\text{SU}(8)} \quad (\text{III.191})$$



$$\ker \tau = \mathbf{1280} + c.c. . \quad (\text{III.192})$$

Then, given that the  $SU(8)$  decomposition of the torsion bundle  $W$  reads

$$W = \mathbf{28} + \mathbf{36} + \mathbf{420} + c.c. , \quad (\text{III.193})$$

we are left with the following intrinsic torsion

$$W_{\text{int}}^{\text{SU}(8)} = W/W_{\text{SU}(8)} = 0, \quad (\text{III.194})$$

implying that every  $SU(8)/\mathbb{Z}_2$  structure is torsion-free, and the space of torsion-free, compatible connections is given by  $\mathbf{1280} + c.c.$ .

Turning to the equations of motion, it is remarkable that the dynamics are simply the generalised geometrical analogue of Einstein gravity: the bosonic action is given by [39]

$$S_B = \int \text{vol}_G R, \quad (\text{III.195})$$

where  $\text{vol}_G$  is the volume form associated to the generalised metric and  $R$  is the analogue of the Ricci scalar. The corresponding equations of motion are simply

$$R_{MN} = 0, \quad (\text{III.196})$$

where  $R_{MN}$  is the analogue of the Ricci tensor.

Remarkably, one can reformulate the generalised Ricci tensor naturally in terms of generalised objects. To do so, we introduce two real  $SU(8)$  bundles  $S$  and  $J$ , which we refer to as the ‘‘spinor’’ bundle and the ‘‘gravitino’’ bundle respectively, since the supersymmetry parameter and the physical gravitino field in supergravity are embedded in exceptional generalised geometry as sections of them [42]. These are

$$S = \mathbf{8} + \bar{\mathbf{8}} \equiv S^+ + S^- \quad J = \mathbf{56} + \bar{\mathbf{56}} \equiv J^+ + J^-. \quad (\text{III.197})$$

Then, the  $SU(8)$  spinor  $\eta$  defined by the two internal supersymmetry parameters (III.155) is naturally a section of  $S$ :  $\eta \in \Gamma(S)$ . A gravitino  $\psi$  would then be a section of  $J$ :  $\psi \in \Gamma(J)$ . One can then rewrite the generalised Ricci tensor as

$$D \times_J (D \times_J \eta) + 2D \times_J (D \times_S \eta) = R^0 \cdot \eta \quad (\text{III.198})$$

$$D \times_S (D \times_J \eta) + D \times_S (D \times_S \eta) = R \eta, \quad (\text{III.199})$$

where  $R$  and  $R_{MN}^0$  provide the scalar and non-scalar parts of  $R_{MN}$  respectively. The projections  $\times_S$  and  $\times_J$  are unique, and in terms of  $SU(8)$  indices, projections with the

generalised connection  $D \equiv (D^{[\alpha\beta]}, \bar{D}_{[\alpha\beta]})$  read

$$(D \times_J \eta)^{\alpha\beta\gamma} = D^{[\alpha\beta} \eta^{\gamma]} \in \Gamma(J^+) \quad (\text{III.200})$$

$$(D \times_S \eta)_\alpha = \bar{D}_{[\alpha\beta]} \eta^\beta \in \Gamma(S^-) \quad (\text{III.201})$$

$$(D \times_J \psi)_{\alpha\beta\gamma} = -\frac{1}{12} \epsilon_{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3} D^{[\delta\delta']} \psi^{\theta_1\theta_2\theta_3} \in \Gamma(J^-) \quad (\text{III.202})$$

$$(D \times_S \psi)^\alpha = -\frac{1}{2} \bar{D}_{[\beta\gamma]} \psi^{\alpha\beta\gamma} \in \Gamma(S^+). \quad (\text{III.203})$$

The existence of the expressions (III.198) and (III.199) is a non-trivial statement. It can be shown that the left-hand sides are linear in  $\eta$ , and since  $\eta$  and the left-hand sides are manifestly covariant, these expressions define a tensor. This generalised Ricci tensor has been calculated explicitly in the M-theory case in [39] for instance.

The resulting equations of motions are therefore

$$D \times_J (D \times_J \eta) + 2D \times_J (D \times_S \eta) = 0 \quad (\text{III.204})$$

$$D \times_S (D \times_J \eta) + D \times_S (D \times_S \eta) = 0. \quad (\text{III.205})$$

Interestingly, it is immediate in this formalism to read off the fact that preserving supersymmetry implies that the equations of motion are satisfied. Indeed, one can reformulate the supersymmetry conditions as

$$\delta\psi = D \times_J \eta = 0 \quad (\text{III.206})$$

$$\delta\eta = D \times_S \eta = 0. \quad (\text{III.207})$$

If these are satisfied, the equations of motion (III.204) and (III.205) are trivially obeyed.



# Chapter IV

## Non-supersymmetric flux vacua and Generalised calibrations

In this chapter, we will discuss the results derived in [91, 92]. These concern non-supersymmetric type II supergravity flux vacua. More precisely, a class of  $\mathcal{N} = 0$  flux vacua which are thought of as supersymmetric backgrounds deformed by some perturbations which break supersymmetry in a controllable way, in a sense that will be made precise throughout this chapter.

In section IV.2, we present a new class of non-supersymmetric flux vacua [91], generalising the GKP vacua [3] in a sense that will be made precise later on. We derive the corresponding equations of motion and solve them for various explicit examples. We also discuss the stability and the effective theories associated to this class of backgrounds.

On another note, the supersymmetry condition involving the RR fluxes (III.73) has been reformulated by Tomasiello [6], eliminating the explicit dependence on the metric. We generalise this derivation in section IV.3 for non-supersymmetric backgrounds violating the other supersymmetry conditions (III.71) and (III.72). We use this reformulation to derive constraints that the ten-dimensional solutions satisfying (III.73) must respect in order to dimensionally reduce to solutions of four-dimensional  $\mathcal{N} = 1$  supergravity with non-vanishing F-terms and potentially non-vanishing D-terms [92]. We give the equations of motion for the class of type II vacua satisfying these constraints in the language of pure spinors.

The physical motivation to study non-supersymmetric backgrounds which still respect (III.73) relies on the theory of generalised calibrations, which provides an interpretation of the supersymmetry conditions in terms of stability conditions for certain probe D-branes [2, 8, 93]. Moreover, regarding the class of solutions discussed in IV.2, both its general construction and the discussion of its stability heavily relies on the generalised calibration theory. We therefore start this chapter by briefly reviewing the theory of

generalised calibration in section IV.1, following mainly [2].

Finally, let us mention that throughout this whole chapter, we adopt the generalised complex geometry conventions introduced in subsection III.1.1.

## IV.1 Calibrations in generalised complex geometry

The  $\mathcal{N} = 1$  supersymmetry equations for type II flux backgrounds compactified to four-dimensional Minkowski space, written in the pure spinor formalism, (III.71), (III.72) and (III.73), have a clear interpretation as being calibration conditions in the generalised sense, for a certain type of D-branes in the geometry. This interpretation turns out to provide great tools to understand the geometry and discuss the stability of both supersymmetric and non-supersymmetric backgrounds.

Generalised calibrations are natural extensions of ordinary calibrations.<sup>1</sup> For supersymmetric compactification on a spin manifold  $M$ , where there are no non-trivial bulk and world-volume fluxes, there is a nice relation between branes wrapping cycles in  $M$  and ordinary calibrations. The calibration forms can be built as bilinears in the covariantly constant spinors on the manifold. As these spinors are the internal supersymmetry parameters, the closure of the calibration form follows supersymmetry. In this case, the energy of a brane wrapping a cycle in the manifold is given by its volume. Supersymmetric configurations are energy minimising, and therefore correspond to branes wrapping calibrated cycles in the spin manifold.

In flux compactifications, the energy of the static branes gets contribution both from the volume and the fluxes. Generalised complex geometry provides a natural extension of this construction to flux backgrounds, which takes into account the contribution to the energy of both RR background fluxes and world-volume degrees of freedom.

We consider D-branes in the warped geometry (III.62). They can wrap a cycle  $\Sigma$  in the internal manifold  $M$  and they can be string, domain-walls or space-filling in the external Minkowski space. As discussed in [2], one can show that a static brane wrapping a cycle  $\Sigma$  in the internal manifold of an  $\mathcal{N} = 1$  warped flux backgrounds is supersymmetric if it wraps a calibrated generalised submanifold.

To make these statements precise, we need to introduce the technology required to

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<sup>1</sup>A calibration form  $\omega$  is a  $p$ -form on  $M$  that satisfies an algebraic and a differential condition. At every point  $q \in M$  and for every  $p$ -dimensional oriented subspace  $\tau$  of the tangent space  $T_q$

$$\omega|_{\tau} \leq \sqrt{\det g|_{\tau}} d\tau \equiv \text{vol}_{\tau}, \quad (\text{IV.1})$$

where  $d\tau = t^1 \wedge \dots \wedge t^p$ , with  $t^{\alpha}$  a basis for  $\tau^*$  (the dual of  $\tau$ ) and  $\det g|_{\tau} := \det(g_{\alpha\beta})$ , with  $g_{\alpha\beta}$  the components of the pulled-back metric  $g|_{\tau}$  in the coframe  $t^{\alpha}$ . At every point there must exist a subspace  $\tau$  such that the above bound is saturated. Then the form  $\omega$  must be closed  $d\omega = 0$ .

describe D-branes in generalised geometry: a generalised submanifold  $(\Sigma, \mathcal{F})$  and a generalised calibration form  $\omega$ .

A generalised submanifold is a pair  $(\Sigma, \mathcal{F})$  with  $\Sigma \subset M$  a submanifold and  $\mathcal{F}$  a two-form, which for a D-brane is a two-form on its world-volume, such that

$$d\mathcal{F} = H|_{\Sigma}, \quad (\text{IV.2})$$

with  $H|_{\Sigma}$  the pullback of the NS-field-strength on  $\Sigma$ . The generalised submanifold is a generalised cycle if  $\partial\Sigma = \emptyset$ .

As shown in [2], one can construct polyforms of definite parity in terms of the pure spinors defining the  $SU(3) \times SU(3)$  structure of the  $\mathcal{N} = 1$  background

$$\omega^{\text{string}} = e^{2A-\phi} \text{Im}\Psi_1 \quad (\text{IV.3})$$

$$\omega^{\text{DW}} = e^{3A-\phi} \Psi_2 \quad (\text{IV.4})$$

$$\omega^{\text{sf}} = e^{4A-\phi} \text{Re}\Psi_1, \quad (\text{IV.5})$$

which satisfy the properties of a generalised calibration. They first satisfy an algebraic condition corresponding to the minimisation of the D-brane energy

$$\mathcal{E}(\Pi, \mathcal{R}) \geq (\omega|_{\Pi} \wedge e^{\mathcal{R}})(\Pi) \quad (\text{IV.6})$$

for any point  $p \in M$  and any generalised submanifold  $(\Pi, \mathcal{R})$ <sup>2</sup>. Here for  $\rho$  a form,  $\rho(\Pi)$  is the coefficient of the top form on  $\Pi$  of  $\rho|_{\Pi}$ . The energy density corresponds to the following DBI contribution

$$\mathcal{E}(\Pi, \mathcal{R}) = e^{qA-\phi} \sqrt{\det(g|_{\Pi} + \mathcal{R})}, \quad (\text{IV.7})$$

where  $q$  is the number of external dimensions. Moreover the differential conditions that must be respected by the above generalised calibration forms correspond to the supersymmetry conditions (III.71), (III.72), and (III.73)

$$d_H(\omega^{\text{DW}}) = 0 \quad \text{domain-wall BPSness} \quad (\text{IV.8})$$

$$d_H(\omega^{\text{string}}) = 0 \quad \text{D-string BPSness} \quad (\text{IV.9})$$

$$d_H(\omega^{\text{sf}}) = e^{4A} \tilde{F} \quad \text{gauge BPSness.} \quad (\text{IV.10})$$

A calibrated generalised cycle is a generalised cycle saturating the calibration bound

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<sup>2</sup>Strictly speaking for any point  $p \in M$  there must exist a generalised submanifold  $(\Pi, \mathcal{R})$  such that the above bound is saturated.

(IV.6). A D-brane in a  $\mathcal{N} = 1$  backgrounds is supersymmetric, or BPS, if it wraps a calibrated generalised cycle. This is why we refer to the  $\mathcal{N} = 1$  supersymmetry conditions as the domain-wall, D-string and gauge BPSness respectively. The above generalised calibration forms are associated to space-filling, domain-wall, and string-like D-branes, which wrap respectively four, three, and two non-compact dimensions.

Another useful characterisation of D-branes is in terms of their generalised current.

The generalised current  $j_{(\Sigma, \mathcal{F})}$  can be seen as the Poincaré dual of the generalised submanifold  $(\Sigma, \mathcal{F})$ :

$$\int_M \langle \phi, j_{(\Sigma, \mathcal{F})} \rangle = \int_\Sigma \phi|_\Sigma \wedge e^{\mathcal{F}} \quad (\text{IV.11})$$

with  $\phi$  any polyform on  $M$ . Loosely speaking, as a distribution  $j_{(\Sigma, \mathcal{F})}$  is a localised real pure spinor proportional to  $e^{-\mathcal{F}} \wedge \delta^{(d-k)}(\Sigma)$ , with  $\Sigma$  of rank  $k$  and with  $\delta^{(d-k)}(\Sigma)$  the standard Poincaré dual of the submanifold  $\Sigma$ . One can also consider the smeared version of this current, that we call  $j$ , proportional to  $e^{-\mathcal{F}} \wedge \text{vol}_\perp$  with  $\text{vol}_\perp$  the transverse volume to  $\Sigma$ .

We can define the generalised tangent bundle of the foliation associated to the generalised submanifold  $(\Sigma, \mathcal{F})$  as

$$T_{(\Sigma, \mathcal{F})} = \{V = v + \xi \in T \oplus T^* \mid V \cdot j = 0\} \quad (\text{IV.12})$$

$$= \{V = v + \xi \in T\Sigma \oplus T^* \mid \xi|_\Sigma = \iota_v \mathcal{F}\}. \quad (\text{IV.13})$$

This is a real maximally isotropic subbundle of the generalised tangent bundle  $E$ .

As discussed in [2], the calibration condition (III.73) implies that  $T_{(\Sigma, \mathcal{F})}$  is stable under the generalised complex structure  $\mathcal{J}_2$  and hence  $(\Sigma, \mathcal{F})$  is a generalised complex submanifold.

Finally, it is easy to prove that the generalised current associated to a generalised cycle  $(\Sigma, \mathcal{F})$  is  $d_H$ -closed. From (IV.11), we have:

$$\int_M \langle \phi, d_H j_{(\Sigma, \mathcal{F})} \rangle = \int_M \langle d_H \phi, j_{(\Sigma, \mathcal{F})} \rangle = \int_\Sigma d_H \phi|_\Sigma \wedge e^{\mathcal{F}} = \int_{\partial\Sigma} \phi|_{\partial\Sigma} \wedge e^{\mathcal{F}}, \quad (\text{IV.14})$$

where we used both the property of the Mukai pairing (B.13) and Stoke theorem. Therefore we have

$$d_H j_{(\Sigma, \mathcal{F})} = j_{(\partial\Sigma, \mathcal{F}|_{\partial\Sigma})}, \quad (\text{IV.15})$$

which reduces to

$$d_H j_{(\Sigma, \mathcal{F})} = 0 \quad (\text{IV.16})$$

if  $(\Sigma, \mathcal{F})$  is a generalised cycle.

Note that, as the generalised tangent bundle  $T_{(\Sigma, \mathcal{F})}$  is a real maximally isotropic

sub-bundle of  $T \oplus T^*$ , it also defines an almost Dirac structure [35]. From (IV.12) and Frobenius theorem it follows that  $d_H j_{(\Sigma, \mathcal{F})} = 0$  implies that the almost Dirac structure is actually integrable.

## IV.2 New non-supersymmetric flux vacua from generalised calibrations

The exploration of the landscape of four-dimensional string compactifications has been mostly focused on vacua preserving at least  $\mathcal{N} = 1$  supersymmetry. One reason is practical: solving the supersymmetry conditions, which are first order differential equations, plus the Bianchi identities for the fluxes, guarantees to have solutions to the full set of string or supergravity equations of motion. Without this way out, handling the equations of motion upfront is very hard, even in the supergravity approximation, since they are cumbersome second order differential equations.

There are also physical considerations motivating the study of supersymmetric string compactifications, namely the expectation that supersymmetry should be broken at energies smaller than the compactification scale.

Even if low energy supersymmetry breaking is a phenomenologically motivated scenario, in principle nothing prevents supersymmetry from being spontaneously broken at arbitrarily high energies. In this section, we consider this possibility, and focus on this much less studied corner of the string compactification landscape, worth exploring per se.

More precisely, we construct new classes of non-supersymmetric type II supergravity solutions by breaking supersymmetry in a controlled way. We deform the conditions for  $\mathcal{N} = 1$  supersymmetry by adding supersymmetry breaking terms, which are controlled by some parameters, whose vanishing would restore supersymmetry.

The motivation behind this approach is to preserve some of the convenient features of supersymmetric vacua, mainly the possibility to use first order differential equations. Since supersymmetry is broken, in order to find solutions we have to make sure that the equations of motion are satisfied. The goal is to find specific deformations of the BPS equations such that the additional constraints to impose in order to solve the equations of motion are manageable.

We will use the framework of generalised complex geometry, where the  $\mathcal{N} = 1$  BPS conditions have an interpretation in terms of calibration conditions of different probe D-branes. As we have discussed in the previous chapter, the  $\mathcal{N} = 1$  supersymmetry conditions for warped compactifications can be recast in a set of three differential equations on polyforms defined only on the internal compactification space (III.71), (III.72), and (III.73). Each of these three conditions can be interpreted as the conditions for calibrated



D-brane probes in the geometry [2]: branes filling all the external space and branes that are domain-wall or string-like.

In this language, one can identify different supersymmetry breaking terms depending on which calibration condition is modified. In this section we will always assume that space-filling branes are calibrated<sup>3</sup>, while we will allow the calibrations of D-strings and domain-wall branes to be violated.

A famous example of non-supersymmetric type IIB solutions that violate the domain-wall calibration condition are the GKP solutions [3], describing flux compactifications to four-dimensional Minkowski space with D3 and O3 sources, where supersymmetry is broken by the  $H_{(0,3)}$  components of the NS-flux. The GKP backgrounds have been described within generalised complex geometry in [44] as specific examples of a general framework to describe non-supersymmetric solutions.

The generalised complex geometry description of the GKP backgrounds also offers an insightful geometrical interpretation of the domain-wall supersymmetry breaking term: it is given by the current associated to the D3-branes in the background<sup>4</sup>.

In the literature there is another example of non-supersymmetric solution in type IIA, [45], which in the language of generalised complex geometry corresponds to the violation of the D-string calibration condition, where supersymmetry is again violated through additional NS-flux components with respect to the supersymmetric case, but there is no further geometrical interpretation of the corresponding supersymmetry breaking term. Moreover, the question of stability of such backgrounds remains unaddressed.

In this section, we want to extend the study of non-supersymmetric vacua violating the D-string calibration condition. More precisely, we will construct new non-supersymmetric type II solutions, where the current associated to the space-filling D-branes present in our backgrounds will serve as a building block for the supersymmetry breaking term violating the D-string calibration condition, in a sense that will be made precise later on.

The motivation behind this construction is two-fold. The first one is simplicity: defining supersymmetry breaking in terms of the current of the background's D-branes is a natural and simple ansatz, which in turn reduces the equations of motion to a reasonable set of additional constraints. The second reason is that it can be useful to address the question of stability of these non-supersymmetric vacua: in generalised complex geometry, D-branes current can enter the effective potential associated to a given ten-dimensional background, and are particularly useful as they allow to use powerful positivity arguments

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<sup>3</sup>Let us stress here that the interpretation of the BPS conditions in terms of D-brane calibrations doesn't mean that the backgrounds have to have D-string, domain-wall or space-filling D-branes, but we impose the presence of space-filling D-branes for model building considerations.

<sup>4</sup>Strictly speaking it is the smeared version of the generalised current associated to the D3-branes, as we will discuss in the text.

from the branes calibration bounds in the study of the effective potential.

We can show that our new class of vacua shares one interesting property with the GKP backgrounds, namely the fact that there is a natural truncation of the ten-dimensional theory, suggested by the geometry, such that the off-shell effective potential is positive semi-definite, and vanishes at the solutions. This statement is however not quite equivalent to claiming the stability of these new vacua, since we have limited control on the aforementioned truncation, as we will discuss at length.

On another note, we will also construct a new class of backgrounds generalising the GKP vacua, where both the domain-wall and D-string calibration conditions are violated.

The outline of this section is as follows. In subsection [IV.2.1](#) we review the generalised complex geometry description of the GKP-like backgrounds and we introduce our two new classes of non-supersymmetric backgrounds, with pure D-string supersymmetry breaking and mixed Domain-wall and D-string supersymmetry breaking. In subsection [IV.2.2](#), we write the effective potential for these compactifications and derive the equations of motion in the generalised complex geometry formalism, first for completely general D-string supersymmetry breaking, and then for the two new classes of backgrounds we found. We also address the question of the stability of these new solutions. Finally, in subsection [IV.2.3](#), which can be read (almost) independently, we present different explicit examples of new non-supersymmetric vacua with  $SU(2)$  and  $SU(3)$  structures.

### IV.2.1 $\mathcal{N} = 0$ flux vacua in generalised complex geometry

The goal of this subsection is to construct and study new non-supersymmetric backgrounds. To do so we will focus on situations where supersymmetry breaking occurs as a perturbation around some supersymmetric backgrounds and it is controlled by parameters whose vanishing would restore supersymmetry.

The idea is to modify the Killing spinor equations ([II.71](#)) and ([II.72](#)) while still assuming that the internal spinors  $\eta_1$  and  $\eta_2$  in ([II.59](#)) are globally defined. This means that the internal manifolds are still characterised by an  $SU(3)$  or  $SU(2)$  structure, and, in the generalised geometry language, by an  $SU(3) \times SU(3)$  structure. The modified Killing spinor equations are then equivalent to adding supersymmetry breaking terms to the right-hand side of the  $\mathcal{N} = 1$  pure spinor equations ([III.71](#)), ([III.72](#)), and ([III.73](#)). From now on, we will call these new equations modified pure spinor equations.

As discussed in section [IV.1](#), the supersymmetry conditions ([III.71](#)), ([III.72](#)), and ([III.73](#)) correspond to the calibration conditions for supersymmetric domain-wall, string-like and space-filling probe branes, respectively. Thus we will call the corresponding supersymmetry breaking terms, domain-wall (DWSB), string-like (SSB) and space-filling supersymmetry breaking.

DWSB non-supersymmetric backgrounds have been studied in the framework of GCG in [44], which also gave the general expression for the non-supersymmetric deformations of the Killing spinor equations and of the associated modified pure spinor equations.

In this section we will review a simple subclass of the DWSB discussed in [44] and we will discuss its geometrical properties, a discussion that we will then extend to our new classes of solutions describing SSB supersymmetry breaking, with and without DWSB breaking.

However, in contrast with the supersymmetric case, there is no reason to expect the solutions of the modified pure spinor equations and the Bianchi identities to be solutions of the equations of motion. So the supersymmetric breaking terms have to satisfy additional constraints in order to have a real vacuum, which we will discuss in subsection IV.2.2.

### a) The DWSB vacua

Non-supersymmetric solutions corresponding to DWSB have been studied in [44]. The parametrisation of the most general DWSB deformation can be found in Appendix B of [44]. As it is hard to find solution in such general context, [44] focuses on a subset of solutions that only depend on a single supersymmetry breaking parameter.

For the one-parameter DWSB class, the modified Killing spinor equations are<sup>5</sup>

$$\delta\psi_\mu^{(1)} = \frac{1}{2}e^A\hat{\gamma}_\mu\zeta \otimes (r\eta_1^*) + c.c. \quad \delta\psi_\mu^{(2)} = \frac{1}{2}e^A\hat{\gamma}_\mu\zeta \otimes (r\eta_2^*) + c.c. \quad (\text{IV.17})$$

$$\delta\psi_m^{(1)} = \zeta \otimes \left(-\frac{1}{2}r\Lambda^n{}_m\gamma_n\eta_1^*\right) + c.c. \quad \delta\psi_m^{(2)} = \zeta \otimes \left(-\frac{1}{2}r\Lambda_m{}^n\gamma_n\eta_2^*\right) + c.c. \quad (\text{IV.18})$$

$$\Delta\epsilon_1 = \zeta \otimes (-r\eta_1^*) + c.c. \quad \Delta\epsilon_2 = \zeta \otimes (-r\eta_2^*) + c.c. \quad (\text{IV.19})$$

As suggested by its name, the one-parameter DWSB subclass only depends on a single supersymmetry breaking parameter,  $r$ , as the  $O(6)$  matrix  $\Lambda$  is completely defined by the background geometry:

$$\eta_1 = iU\eta_2 \quad U\gamma_m U^{-1} = \Lambda^n{}_m\gamma_n, \quad (\text{IV.20})$$

where  $U$  is a unitary, point-dependent operator acting on six-dimensional spinors.

The corresponding domain-wall BPSness violation is then

$$d_H(e^{3A-\phi}\Psi_2) = ire^{3A-\phi}((-1)^{|\Psi_2|}\text{Im}\Psi_1 + \frac{1}{2}\Lambda_{mn}\gamma^m\text{Im}\Psi_1\gamma^n). \quad (\text{IV.21})$$

where  $\gamma^m$  are Cliff(6) gamma matrices acting on a form  $\omega$  as

$$\gamma^m\omega = (g^{mn}\iota_n + dy^m\wedge)\omega \quad \omega\gamma^m = (-1)^{|\omega|+1}(g^{mn}\iota_n - dy^m\wedge)\omega. \quad (\text{IV.22})$$

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<sup>5</sup>Here we write the modified dilatino variation defined in (C.4).

The modified domain-wall condition (IV.21) can be rewritten in a way that makes explicit its implications for the geometry of the internal background [44].

We suppose that the internal manifold  $M$  admits a generalised submanifold  $(\Sigma, \mathcal{F})$ , where  $\Sigma$  is a subbundle of odd/even dimension  $n$  in type IIA/IIB. Since the space-filling calibration condition (III.73) still holds for this class of backgrounds, we can choose  $(\Sigma, \mathcal{F})$  to be calibrated by  $\omega^{\text{sf}} = e^{4A-\phi}\text{Re}\Psi_1$ , such that the BPS space-filling branes of our backgrounds will wrap the calibrated generalised submanifold  $(\Sigma, \mathcal{F})$ .

We can then split the tangent bundle as

$$T = T\Sigma \oplus T\Sigma^\perp, \quad (\text{IV.23})$$

with  $T\Sigma^\perp$  the orthogonal completion of  $T\Sigma$ , and define a local vielbein  $\{e^a\}$  on  $T\Sigma \oplus T\Sigma^\perp$  and its associated gamma matrices

$$\hat{\gamma}^a \omega = (\delta^{ab} \iota_b + e^m \wedge) \omega \quad \omega \hat{\gamma}^a = (-1)^{|\omega|+1} (\delta^{ab} \iota_b - e^a \wedge) \omega. \quad (\text{IV.24})$$

One can then express the operator  $U$  in (IV.20) as

$$U = \gamma_{(6)}^n \sum_k \frac{\epsilon^{a_1 \dots a_{n-2k} b_1 \dots b_{2k}}}{(n-2k)! k! 2^k \sqrt{\det(g|_\Sigma + \mathcal{F})}} \hat{\gamma}_{a_1 \dots a_{n-2k}} \mathcal{F}_{b_1 b_2} \dots \mathcal{F}_{b_{2k-1} b_{2k}}, \quad (\text{IV.25})$$

and the corresponding  $O(6)$  matrix as

$$\hat{\Lambda} = \mathbb{1}_\perp - (g|_\Sigma + \mathcal{F})^{-1} (g|_\Sigma - \mathcal{F}), \quad (\text{IV.26})$$

where  $\mathbb{1}_\perp$  is the projection onto  $T\Sigma^\perp$ .

Then (IV.21) becomes

$$d_H(e^{3A-\phi}\Psi_2) = irj, \quad (\text{IV.27})$$

with

$$j = 4(-1)^{|\Psi_2|} e^{3A-\phi} \frac{\sqrt{\det g|_\Sigma}}{\sqrt{\det(g|_\Sigma + \mathcal{F})}} e^{-\mathcal{F}} \wedge \sigma(\text{vol}_\perp), \quad (\text{IV.28})$$

where  $\text{vol}_\perp$  is the volume form on the space orthogonal to the cycle  $\Sigma$  such that  $\text{vol}_6 = \text{vol}_\Sigma \wedge \text{vol}_\perp$ , and  $|\Psi_2|$  is the degree mod 2 of  $\Psi_2$ .

The polyform  $j$  is a smeared version of the Poincaré dual to the generalised cycle  $(\Sigma, \mathcal{F})$ , and therefore it is a (smeared) generalised current for  $(\Sigma, \mathcal{F})$ <sup>6</sup>. Moreover, the

<sup>6</sup>To illustrate, let's consider the simple cases where  $\mathcal{F} = 0$ . We have

$$j = 4(-1)^{|\Psi_2|} e^{3A-\phi} \sigma(\text{vol}_\perp), \quad (\text{IV.29})$$

which is the ordinary smeared Poincaré dual to the cycle  $\Sigma$ , and  $T_{(\Sigma,0)} \equiv T\Sigma \oplus T^*|_{\Sigma^\perp}$  is the null space of  $j$ , which is a defining property of the generalised current.

right-hand side of (IV.27) is  $d_H$ -exact and so  $d_H$ -closed and, by Frobenius's theorem, it follows that the generalised sub-bundle  $(\Sigma, \mathcal{F})$  is integrable and that  $d\mathcal{F} = H|_\Sigma$ . This means that the manifold  $M$  is a foliation with leaves  $(\Sigma, \mathcal{F})$ , which are calibrated generalised submanifolds, thanks to (III.73).

Note that solving (IV.27) for a given generalised foliation  $(\Sigma, \mathcal{F})$  constrains the possible choices for  $r$ . Indeed, it has to be chosen such that the right-hand side of (IV.27) is  $d_H$  closed. The supersymmetry parameter can therefore not be multiplied by arbitrary complex functions, and these backgrounds truly depend on one parameter only.

It is also important to note that the sources of these backgrounds are taken to be parallel, so their internal manifolds admit a unique generalised calibrated cycle  $(\Sigma, \mathcal{F})$  wrapped by all the sources.

The one-parameter DWSB class includes the GKP vacua [3] as well as vacua with D4, D5 or D6-brane sources that can be obtained by T-dualising the GKP solution. Non-supersymmetric GKP vacua [3] are solutions of type IIB compactifications, with D3-branes and O3-plane sources, and non trivial NS and RR three-form fluxes.

GKP-like vacua correspond to particularly simple representative of the one-parameter DWSB class of vacua, where we have

$$\Lambda = \mathbb{1} \tag{IV.30}$$

and  $(\Sigma, \mathcal{F})$  is the trivial foliation whose leaves are points of  $M$ . In this case, the D-branes sitting on such leaves are D3-branes, and the failure to calibrate the would-be domain-wall branes originates purely from the  $H_{(3,0)}$  components of the NS-flux<sup>7</sup>:

$$d_H(e^{3A-\phi}\Psi_2) = e^{3A-\phi}H \wedge \Psi_2 = 4ire^{3A-\phi}\text{vol}_6. \tag{IV.31}$$

Backgrounds with D5 and D6-branes have been explicitly constructed in [44], and we will revisit them when turning to the examples of our new backgrounds in section IV.2.3.

Let us consider again the generalised tangent bundle to the foliation with the generalised submanifold  $(\Sigma, \mathcal{F})$ . We saw that for supersymmetric backgrounds,  $(\Sigma, \mathcal{F})$  is a complex generalised submanifold due to the integrability of the complex structure. Thanks to this property it is possible to study deformation  $(\Sigma, \mathcal{F})$  and then of D-branes in the background [2].

For the one-parameter DWSB backgrounds  $\mathcal{J}_2$  is not integrable anymore and one might wonder what can be said about  $(\Sigma, \mathcal{F})$ . Recall from section IV.1 that the generalised tangent bundle  $T_{(\Sigma, \mathcal{F})}$  associated to the foliation with the generalised submanifold  $(\Sigma, \mathcal{F})$

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<sup>7</sup>with respect to the almost complex structure defined by the internal spinor, for which  $\Psi_2$  is a (3,0)-form.

is an almost Dirac structure. As pointed out in [44], for DWSB vacua (IV.27) implies the (conformal) closure of the associated generalised current and therefore  $T_{(\Sigma, \mathcal{F})}$  is still integrable: it remains closed under the twisted Courant bracket.

The integrability of the Dirac structure associated to  $(\Sigma, \mathcal{F})$  has the important consequence that one can define a differential  $d_{(\Sigma, \mathcal{F})}$  acting on the graded complex  $\bigoplus_{k=0}^6 \Lambda^k T_{(\Sigma, \mathcal{F})}^*$ . It also allows to preserve some notion of "generalised holomorphicity". Let  $L_2 \subset E$  be the subbundle with  $+i$ -eigenvalues with respect to the generalised almost complex structure  $\mathcal{J}_2$ .  $L_2$  defines an almost Dirac structure, which is not integrable as  $\mathcal{J}_2$  is not. However, we can consider the complex bundle

$$L_{(\Sigma, \mathcal{F})} = T_{(\Sigma, \mathcal{F})} \cap L_2 = \{V \in (T \oplus T^*) \otimes \mathbb{C} \mid V \cdot \Psi_2 = V \cdot j = 0\}, \quad (\text{IV.32})$$

which, in contrast to  $L_2$ , is stable under the twisted Courant bracket:

$$[[V, W]]_H \cdot j = -W \cdot V \cdot d_H j = 0 \quad (\text{IV.33})$$

$$[[V, W]]_H \cdot e^{3A-\phi} \Psi_2 = -W \cdot V \cdot d_H(e^{3A-\phi} \Psi_2) = -ir W \cdot V \cdot j = 0 \quad (\text{IV.34})$$

precisely because<sup>8</sup> of the modified pure spinor equation (IV.27) and because of the integrability of the Dirac structure associated to  $T_{(\Sigma, \mathcal{F})}$ . The collection  $(L_2, [[, ]_H, a)$ <sup>9</sup> is therefore a Lie algebroid and one can thus define a differential that we call  $\bar{\partial}_{(\Sigma, \mathcal{F})}$  on  $\bigoplus_{k=0}^3 \Lambda^k L_{(\Sigma, \mathcal{F})}^*$ <sup>10</sup>, and thus even though  $\mathcal{J}_2$  is not integrable, the structure of the one-parameter DWSB class allows one to have a notion of holomorphic differential  $\bar{\partial}_{(\Sigma, \mathcal{F})}$ , at least with respect to the foliation with  $(\Sigma, \mathcal{F})$ .

It has been speculated in [44] that the first cohomology group of this differential  $H_{\bar{\partial}_{(\Sigma, \mathcal{F})}}^1$  might define the moduli-space of the D-branes in the one-parameter DWSB backgrounds, much like in the case of supersymmetric compactifications [93]. The authors also postulated that the non-integrability of the structure  $\mathcal{J}_2$  might result in a closed string moduli space that is not a complex manifold, since it is what happens for the GKP construction. We refer the reader to [44] for more details on this matter.

## b) The SSB vacua

We consider here another class of non-supersymmetric backgrounds where supersymmetry is broken by deforming the D-string calibration condition (III.72). We refer to this way of breaking supersymmetry as SSB, D-string or string-like supersymmetry breaking.

An example of SSB solution has been discussed in type IIA [45]. It is obtained by

<sup>8</sup>Here we considered  $e^{3A-\phi} \Psi_2$  for convenience but of course one finds the same result for  $\Psi_2$ .

<sup>9</sup>We recall that  $a$  is the anchor map.

<sup>10</sup>The subbundle  $L_{(\Sigma, \mathcal{F})}$  is guaranteed to be three-dimensional from the compatibility of the pure spinors  $\Psi_2$  and  $j$ , see for instance [35].

adding to a supersymmetric solution new components to the NS-field-strength, carefully chosen for the background to keep on respecting the equations of motion. The modified pure spinor equations read

$$d_H(e^{3A-\phi}\Psi_2) = 0 \tag{IV.35}$$

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) = e^{2A-\phi}H \wedge \text{Im}\Psi_1 = c e^{6A-2\phi}\text{vol}_6 \tag{IV.36}$$

$$d_H(e^{4A-\phi}\text{Re}\Psi_1) = e^{4A}\tilde{*}_6 F, \tag{IV.37}$$

where  $c$  is a supersymmetry breaking parameter. We refer the reader to [45] for the explicit form of the solutions and the details of the construction.<sup>11</sup>

It is important to note that the supersymmetry breaking term  $c e^{6A-2\phi}\text{vol}_6$  has no given geometrical interpretation<sup>12</sup>, and then that no conclusion has been reached regarding the stability of these backgrounds.

This is in stark contrast with the situation of the one-parameter DWSB class, where the supersymmetry breaking term  $j$  is understood as the (smeared) generalised current associated to the background sources and is a key ingredient in the discussion of the stability of the one-parameter DWSB class.

It would therefore be interesting to construct non-supersymmetric backgrounds with string-like supersymmetry breaking, but this time with a supersymmetry breaking term that has a given geometrical interpretation.

In this section, we will introduce a new class of SSB backgrounds, which relies on the same ingredient as for one-parameter DWSB solutions, namely the foliation structure of  $M$  by a generalised calibrated submanifold  $(\Sigma, \mathcal{F})$ .

More precisely, its supersymmetry breaking term will depend again on the (smeared) generalised current associated to the calibrated space-filling D-branes present in the background, much like the one-parameter DWSB case. The main motivation is that we can then benefit from the same kind of arguments when addressing the question of stability.

Our starting point is the following assumption: the internal manifold admits a generalised foliation by the generalised cycle  $(\Sigma, \mathcal{F})$ . More precisely, we consider backgrounds that admit calibrated parallel space-filling sources and we introduce a polyform  $j$  which plays the role of a (smeared) generalised current for our sources, which therefore wrap

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<sup>11</sup>Note that our pure spinors, volume form and NS-flux conventions differ to the ones in [45]:  $\Psi_{\text{ours}} = -8i\Psi_{\text{theirs}}$ ,  $H_{\text{ours}} = -H_{\text{theirs}}$ ,  $\text{vol}_{6 \text{ ours}} = -\text{vol}_{6 \text{ theirs}}$ .

<sup>12</sup>The backgrounds considered in [45] are intersecting NS5-D6-D8 models: the geometrical understanding of intersecting branes in GCG is fairly limited, and the literature on the subject is scarce, with the exception of [94].

$(\Sigma, \mathcal{F})$ , as in the DWSB case

$$j = 4(-1)^{|\Psi_2|} e^{3A-\phi} \frac{\sqrt{\det g|_\Sigma}}{\sqrt{\det(g|_\Sigma + \mathcal{F})}} e^{-\mathcal{F}} \wedge \sigma(\text{vol}_\perp), \quad (\text{IV.38})$$

where  $\text{vol}_\perp$  is the transverse volume to the cycle  $\Sigma$ . The current  $j$  is by construction  $d_H$  closed as  $(\Sigma, \mathcal{F})$  is a generalised cycle (see section IV.1).

The decomposition of the generalised current  $j$  on the  $SU(3) \times SU(3)$  structure is again

$$j = e^{3A-\phi} ((-1)^{|\Psi_2|} \text{Im} \Psi_1 + \frac{1}{2} \Lambda_{mn} \gamma^m \text{Im} \Psi_1 \gamma^n) \quad (\text{IV.39})$$

with, as in the previous section,

$$\hat{\Lambda} = \mathbb{1}_\perp - (g|_\Sigma + \mathcal{F})^{-1} (g|_\Sigma - \mathcal{F}). \quad (\text{IV.40})$$

However, in contrast with the DWSB construction, the domain-wall and the gauge BPSness conditions are both obeyed

$$d_H(e^{3A-\phi} \Psi_2) = 0 \quad (\text{IV.41})$$

$$d_H(e^{4A-\phi} \text{Re} \Psi_1) = e^{4A} \tilde{*}_6 F, \quad (\text{IV.42})$$

and we consider the following violation of the D-string BPSness<sup>13</sup>

$$d_H(e^{2A-\phi} \text{Im} \Psi_1) = \alpha_m [\gamma^m j + (-1)^{|j|} j \gamma^m] \quad (\text{IV.45})$$

$$= 2\alpha_m dy^m \wedge j, \quad (\text{IV.46})$$

where the gamma matrices are defined in (IV.22) and  $dy^m$  span the directions transverse to the covolume of  $\Sigma$ . The coefficients  $\{\alpha_m\}$  are real and are the supersymmetry breaking parameters.

As for DWSB, imposing that the manifold is a generalised foliation constrains the possible choices for the  $\alpha_m$ . Indeed, it has to be chosen such that the right-hand side of (IV.46) is  $d_H$  closed. Therefore the  $\alpha_m$  can't be multiplied by arbitrary complex functions, and these backgrounds depend on  $\dim(\Sigma)$  parameters only.

The supersymmetry breaking term in (IV.45) can also be expanded on the  $SU(3) \times SU(3)$  structure defined by the two pure spinors  $\Psi_1$  and  $\Psi_2$ , it is given in (C.33). As it is not

<sup>13</sup>It is also useful to consider a local formulation in terms of the vielbein basis, which we will use when discussing concrete constructions of backgrounds:

$$d_H(e^{2A-\phi} \text{Im} \Psi_1) = \hat{\alpha}_a [\hat{\gamma}^a j + (-1)^{|j|} j \hat{\gamma}^a] \quad (\text{IV.43})$$

$$= 2\hat{\alpha}_a e^a \wedge j. \quad (\text{IV.44})$$



a particularly insightful expression we will keep on using the parametrisation in terms of the generalised current from then on.

Note also that this ansatz is simply one of the different possibilities to write down an SSB term in terms of the generalised current. Our choice is dictated by simplicity: it is natural to construct backgrounds with such a supersymmetry breaking term and it requires only relatively reasonable additional constraints in order to find solutions to the equations of motion, as we will discuss at length in subsections IV.2.2 and IV.2.3.

As for the one-parameter DWSB, the closure of the generalised current  $j$  implies that the generalised bundle  $T_{(\Sigma, \mathcal{F})}$  associated to the generalised submanifold  $(\Sigma, \mathcal{F})$  defines a Dirac structure which is integrable. Then, it is again possible to define the differential  $d_{(\Sigma, \mathcal{F})}$  acting on the graded complex  $\bigoplus_{k=0}^6 \Lambda^k T_{(\Sigma, \mathcal{F})}^*$ .

However, in contrast with the one-parameter DWSB situation, the generalised almost complex structure  $\mathcal{J}_2$  is integrable, because of the conformal closure of the pure spinor  $\Psi_2$  (IV.41). Therefore, one can define a differential  $\bar{\partial}_{\mathcal{J}_2}$  acting on the graded complex  $\bigoplus_{k=0}^6 \Lambda^k L_2^*$ , where  $L_2$  is the  $+i$  eigenbundle of  $\mathcal{J}_2$ . The differential  $\bar{\partial}_{\mathcal{J}_2}$  is the generalised Dolbeaut differential, introduced in III.1.1.

Moreover, as the foliation  $(\Sigma, \mathcal{F})$  is an almost generalised complex foliation, we can consider the three-dimensional complex sub-bundle

$$L_{(\Sigma, \mathcal{F})} = T_{(\Sigma, \mathcal{F})} \cap L_2 = \{V \in (T \oplus T^*) \otimes \mathbb{C} \mid V \cdot \Psi_2 = V \cdot j = 0\}, \quad (\text{IV.47})$$

which is stable under the twisted Courant bracket:

$$[[V, W]]_H \cdot j = -W \cdot V \cdot d_H j = 0 \quad (\text{IV.48})$$

$$[[V, W]]_H \cdot e^{3A-\phi} \Psi_2 = -W \cdot V \cdot d_H (e^{3A-\phi} \Psi_2) = 0. \quad (\text{IV.49})$$

As in the one-parameter DWSB case, one can therefore define a differential that we call  $\bar{\partial}_{(\Sigma, \mathcal{F})}$  on the graded complex  $\bigoplus_{k=0}^3 \Lambda^k L_{(\Sigma, \mathcal{F})}^*$ . It is then reasonable to postulate that the first cohomology group of this differential  $H_{\bar{\partial}_{(\Sigma, \mathcal{F})}}^1$  might define the moduli-space of the D-branes present in our backgrounds, like in the supersymmetric case. Note that the complex bundle  $L_{(\Sigma, \mathcal{F})}$  and thus the cohomology group  $H_{\bar{\partial}_{(\Sigma, \mathcal{F})}}^1$  clearly depend only on one of the pure spinors,  $\Psi_2$ . This fact has a clear interpretation for supersymmetric backgrounds, when described from the four-dimensional perspective, see [93], whereas we do not have access to an analogous explanation, since we have much less understanding of the four-dimensional theories associated to our backgrounds (see subsection IV.2.2).

Moreover, the fact that this way of breaking supersymmetry preserves the integrability of the generalised complex structure  $\mathcal{J}_2$  seems to suggest that the closed string moduli space could still be a complex manifold. It could be useful to turn to the exceptional

generalised geometry formalism [39] to address these questions, for instance by performing a non-supersymmetric analogue analysis of the one developed in [61]. However, even if we will further comment on such questions in chapter V, making precise statements about these problems is beyond the scope of the present work.

Finally, from now on, we consider generalised foliations with  $\mathcal{F} = 0$ . The form degree of the supersymmetry breaking term is therefore  $\text{codim}(\Sigma) + 1$ . Then (IV.43) can only be respected if

$$\text{type}(\Psi_1) \leq \text{codim}(\Sigma), \quad (\text{IV.50})$$

where the type of a pure spinor is its lower form degree. For instance backgrounds with an  $\text{SU}(3)$  structure in type IIA have  $\text{type}(\Psi_1) = 3$  so  $\text{dim}(\Sigma) \leq 3$ , which corresponds to D4 or D6 branes only. We will explicitly construct some backgrounds with such sources in section IV.2.3.

### c) Vacua with both DWSB and SSB contributions

The two classes of symmetry breaking described so far correspond each to the failure to respect one specific differential calibration condition, either the domain-wall or the string-one. Here, we would like to discuss more general cases where supersymmetry is broken by violating both the domain-wall and string-like calibration condition, while keeping calibrated space-filling sources in the background.

In the light of the previous discussion, it is natural to consider non-supersymmetric backgrounds which combine the specific DWSB and SSB contributions discussed in the previous sections

$$d_H(e^{3A-\phi}\Psi_2) = irj \quad (\text{IV.51})$$

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) = \alpha_m[\gamma^m j + (-1)^{|j|} j \gamma^m] \quad (\text{IV.52})$$

$$d_H(e^{4A-\phi}\text{Re}\Psi_1) = e^{4A} \tilde{*}_6 F. \quad (\text{IV.53})$$

Here again the internal space is taken to be a generalised foliation  $(\Sigma, \mathcal{F})$ , and  $j$  is its smeared generalised current, while the sources wrap the calibrated generalised cycle  $\Sigma$ .

The main geometric properties of the one-parameter DWSB class will be preserved by the additional SSB contribution: the subbundle  $T_{(\Sigma, \mathcal{F})}$  also defines an integrable Dirac structure, the generalised complex structure  $\mathcal{J}_2$  is not integrable, and the foliation  $(\Sigma, \mathcal{F})$  is an almost generalised complex foliation, so one can still define the following complex bundle

$$L_{(\Sigma, \mathcal{F})} = T_{(\Sigma, \mathcal{F})} \cap L_2 = \{V \in (T \oplus T^*) \otimes \mathbb{C} \mid V \cdot \Psi_2 = V \cdot j = 0\}, \quad (\text{IV.54})$$

which is stable under the Courant bracket. One can therefore again define the differential  $\bar{\partial}_{(\Sigma, \mathcal{F})}$  on  $\bigoplus_{k=0}^3 \Lambda^k L_{(\Sigma, \mathcal{F})}^*$ , which provides a notion of a holomorphic differential, at least with respect to  $(\Sigma, \mathcal{F})$ .

We will come back to this class of vacua later on, write down its equations of motion in subsection IV.2.2 and construct some explicit backgrounds solving these equations in subsection IV.2.3.

## IV.2.2 Effective potential and equations of motion from pure spinors

In the previous subsections, we presented different classes of non-supersymmetric backgrounds in terms of the modified pure spinor equations they obey. However, unlike the supersymmetric case, the solutions of the modified pure spinor equations (plus Bianchi identities) are not guaranteed to solve the full set of equations of motion and thus to describe true vacua of type II supergravity.

In order to check that the solutions of the modified pure spinor equations are actual supergravity backgrounds, we will follow the strategy of [44]: write the most general four-dimensional ‘effective potential’ from the ten-dimensional type II supergravity action, and use the fact that the extremisation of the four-dimensional action is equivalent to satisfying the ten-dimensional type II supergravity equations of motion.

The term ‘effective potential’ is a bit misleading since it does not come from a truncation of the ten-dimensional theory to a finite set of four-dimensional modes.

However, the reason behind this choice (instead of just tackling the ten-dimensional equations of motion upfront) is two-fold. First of all, the effective potential can be written as an integral over the internal space of expressions involving the pure spinors and, by varying it, one can express the equations of motion as some tractable differential equations on the pure spinors. Secondly, the effective potential allows us to discuss the potential presence of closed string tachyons from the four-dimensional perspective and thus (partially) address the question of stability of our different backgrounds. For our Minkowski backgrounds, this translates into the requirement that the effective potential must be positive semi-definite.

Indeed, even though we do not perform a complete reduction to the four-dimensional effective theories, we will be able to find natural ‘truncations’ suggested by the geometry for our new classes of backgrounds such that the effective potential will be positive semi-definite, therefore excluding the potential presence of closed string tachyons. The semi-definite positiveness will be argued through the use of calibration bounds such as (IV.6) rewritten in the pure spinor formalism.

The equations of motion in terms of pure spinors are hard to obtain in full generality, and they have not been derived yet. However, they have been written down in [44] for

the case of the one-parameter DWSB family, using the specific forms of the modified pure spinor equations discussed in IV.2.1.a). In these cases it has been shown that the equations of motion reduce to relatively mild additional constraints to add on top of the modified pure spinor equations.

We start this section by recalling the general expression of the effective potential in terms of the pure spinors derived in [44]. In subsection IV.2.2.b) we summarise the results for the one-parameter DWSB case, presenting its effective potential and equations of motion. The subsection IV.2.2.c) then contains our new results on the effective potential and equations of motion for both purely SSB constructions and mixed SSB and DWSB constructions. We first derive the equations of motion for the most general violation of the D-string calibration condition, and we then specify them to our two new classes of backgrounds.

### a) The type II effective potential from pure spinors

In this section, we recall the derivation of the ‘effective’ potential of [44]. We are interested in configurations where the space-time is warped

$$ds_{10}^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{IV.55})$$

with  $g_{\mu\nu}$  now a generic four-dimensional metric, and with non-trivial NS and RR-fluxes. We assume that the metric  $g_{\mu\nu}$  depends only on the external coordinates, while all the other fields, warp factor, internal metric and fluxes depend only on the internal coordinates.

The effective four-dimensional action for such configurations takes the form

$$S_{\text{eff}} = \int_{X_4} d^4x \sqrt{-g_4} \left( \frac{1}{2} \mathcal{N} R_4 - 2\pi \mathcal{V}_{\text{eff}} \right), \quad (\text{IV.56})$$

where  $R_4$  is the four-dimensional scalar curvature,  $\mathcal{N}$  is the warped volume of the internal space

$$\mathcal{N} = 4\pi \int_M e^{2A-2\phi} \text{vol}_6 \quad (\text{IV.57})$$

and the effective potential density is given by

$$\mathcal{V}_{\text{eff}} = \int_M \text{vol}_6 e^{4A} \left\{ e^{-2\phi} \left[ -\mathcal{R} + \frac{1}{2} H^2 - 4(d\phi)^2 + 8\nabla^2 A + 20(dA)^2 \right] - \frac{1}{2} \tilde{F}^2 \right\} \quad (\text{IV.58})$$

$$+ \sum_{i \in \text{loc. sources}} \tau_i \left( \int_{\Sigma_i} e^{4A-\phi} \sqrt{\det(g|_{\Sigma_i} + \mathcal{F}_i)} - \int_{\Sigma_i} \tilde{C}|_{\Sigma_i} \wedge e^{\mathcal{F}_i} \right), \quad (\text{IV.59})$$

with  $\mathcal{R}$  the six-dimensional scalar curvature.

The first line in  $\mathcal{V}_{\text{eff}}$  corresponds to the closed string sector, while the second line is the

contribution from localised sources. We follow the conventions of [44] where  $2\pi\sqrt{\alpha'} = 1$ , so that the tensions of all D-branes are equal  $\tau_{D_p} = 1$  and for O-planes  $\tau_{O_q} = -2^{q-5}$ . Notice that we are also omitting the internal field kinetic terms, since they are taken to be constant along the external directions<sup>14</sup>. The sources couple to the RR potentials defined by  $d_H\tilde{C} = e^{4A}\tilde{F}$ .

As argued in [44], the variations of the four dimensional action (IV.56) exactly reproduce the ten-dimensional equations of motion (see Appendix A for the expression of the equations of motion). Moreover, from the variation of the four-dimensional action with respect to  $g_{\mu\nu}$ <sup>15</sup>, one gets that the external space is Einstein, with

$$R_4 = 8\pi\mathcal{V}_{\text{eff}}/\mathcal{N}. \quad (\text{IV.60})$$

Notice also that the variation of the effective action with respect to the electric RR potentials reproduces the Bianchi identities

$$d_H F = -j_{\text{tot}} = -\sum_i \tau_i j_i, \quad (\text{IV.61})$$

where, as described in section IV.1,  $j_i$  are the (smeared) generalised current associated to the D-branes wrapping cycles on the internal manifold.

In the following discussions it will be convenient to consider the external component of the modified Einstein equations (A.29). This can be obtained by combined variations of (IV.56)

$$\frac{\delta S_{\text{eff}}}{\delta A} + 2\frac{\delta S_{\text{eff}}}{\delta \phi} = 0. \quad (\text{IV.62})$$

A central result of [44] is the fact that the effective potential (IV.58) can be expressed in term of the pure spinors. The general expression is

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \frac{1}{2} \int_M \langle \tilde{*}_6 [d_H(e^{2A-\phi}\text{Im}\Psi_1)], d_H(e^{2A-\phi}\text{Im}\Psi_1) \rangle \\ & + \frac{1}{2} \int_M e^{-2A} \langle \tilde{*}_6 [d_H(e^{3A-\phi}\Psi_2)], d_H(e^{3A-\phi}\bar{\Psi}_2) \rangle \\ & + \frac{1}{2} \int_M \text{vol}_6 e^{4A} |\tilde{*}_6 F - e^{-4A} d_H(e^{4A-\phi}\text{Re}\Psi_1)|^2 \\ & - \frac{1}{4} \int_M e^{-2A} \left( \frac{|\langle \Psi_1, d_H(e^{3A-\phi}\Psi_2) \rangle|^2}{\text{vol}_6} + \frac{|\langle \bar{\Psi}_1, d_H(e^{3A-\phi}\Psi_2) \rangle|^2}{\text{vol}_6} \right) \\ & - 4 \int_M \text{vol}_6 e^{4A-2\phi} [(u_R^1)^2 + (u_R^2)^2] \end{aligned}$$

<sup>14</sup>As in [44] we neglect anomalous curvature-like corrections to the sources contribution and, for O-planes, we take  $\mathcal{F} = 0$ , as they are not seen as dynamical objects in the compactification.

<sup>15</sup>which from the ten-dimensional perspective is equivalent to the internal space integral of the external ten-dimensional Einstein equation's trace.

$$\begin{aligned}
 & + \sum_{i \subset \text{D-branes}} \tau_i \int_M e^{4A-\phi} (\text{vol}_6 \rho_i^{\text{loc}} - \langle \text{Re}\Psi_1, j_i \rangle) \\
 & + \int_M \langle e^{4A-\phi} \text{Re}\Psi_1 - \tilde{C}, d_H F + j_{\text{tot}} \rangle, \tag{IV.63}
 \end{aligned}$$

where the square of a polyform is defined in Appendix A, and

$$u_R^{1,2} = u_{Rm}^{1,2} dy^m \equiv (u_m^{1,2} + u_m^{*1,2}) dy^m, \tag{IV.64}$$

with

$$u_m^1 = \frac{i \langle \gamma_m \bar{\Psi}_1, d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle}{e^{2A-\phi} \langle \Psi_1, \bar{\Psi}_1 \rangle} + \frac{\langle \gamma_m \bar{\Psi}_2, d_H(e^{3A-\phi} \Psi_2) \rangle}{2e^{3A-\phi} \langle \Psi_2, \bar{\Psi}_2 \rangle} \tag{IV.65}$$

$$u_m^2 = \frac{i(-1)^{|\Psi_2|} \langle \Psi_1 \gamma_m, d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle}{e^{2A-\phi} \langle \Psi_1, \bar{\Psi}_1 \rangle} + \frac{(-1)^{|\Psi_1|} \langle \bar{\Psi}_2 \gamma_m, d_H(e^{3A-\phi} \Psi_2) \rangle}{2e^{3A-\phi} \langle \Psi_2, \bar{\Psi}_2 \rangle}. \tag{IV.66}$$

We have also introduced the Born-Infeld density  $\rho_i^{\text{loc}}$  associated with a source wrapping a generalised submanifold  $(\Sigma_i, \mathcal{F}_i)$

$$\rho_i^{\text{loc}} = \frac{\sqrt{\det(g|_{\Sigma_i} + \mathcal{F}_i)}}{\sqrt{\det g}} \delta(\Sigma_i). \tag{IV.67}$$

With this definition, it's insightful to rewrite the algebraic inequality (IV.6) in terms of  $\rho^{\text{loc}}$ :

$$\rho_i^{\text{loc}} \geq \frac{\langle \text{Re}\Psi_1, j_i \rangle}{\text{vol}_6}, \tag{IV.68}$$

where the division by the volume form means that we remove the  $\text{vol}_6$  factor in the numerator. We take the sources to be calibrated as boundary conditions, which corresponds to the saturation of this bound. This bound also implies that the sixth line in the expression (IV.63) of  $\mathcal{V}_{\text{eff}}$  is always positive.

Varying the effective potential (IV.63) with respect to the dilaton, the NS and RR-fields, and the warp factor, one should obtain the ten-dimensional equations motion directly in terms of the pure spinors. However, so far this has not been done in the general case.

In the following, we derive the equations of motion for the most general violation of the D-string calibration, and we specify them for the cases of our DWSB and SSB constructions presented in subsections IV.2.1.b) and IV.2.1.c).

## b) DWSB effective potential and equations of motion

We can now turn to the case of the one-parameter DWSB construction, and extremise its effective potential to find out what are the additional constraints to impose on the

one-parameter DWSB solutions to promote them to actual supergravity vacua. We briefly present the results, since they have been derived in [44].

Given that the D-string and gauge BPSness conditions (III.72) and (III.73) are respected for the one-parameter DWSB class, the first and third line of the effective potential (IV.63) vanish. Then, combining D-string BPSness (III.72) and the modified DWSB condition (IV.27) one can show that the terms proportional to  $u^1$  and  $u^2$  in (IV.63) are also zero. Finally, the saturation of the calibration bound (IV.68) and the Bianchi identities (IV.61) imply that the last two lines of (IV.63) vanish, leaving

$$\mathcal{V}_{\text{eff}} = \frac{1}{2} \int_M e^{-2A} \langle \tilde{*}_6 [d_H(e^{3A-\phi}\Psi_2)], d_H(e^{3A-\phi}\bar{\Psi}_2) \rangle - \frac{1}{4} \int_M e^{-2A} \left( \frac{|\langle \Psi_1, d_H(e^{3A-\phi}\Psi_2) \rangle|^2}{\text{vol}_6} + \frac{|\langle \bar{\Psi}_1, d_H(e^{3A-\phi}\Psi_2) \rangle|^2}{\text{vol}_6} \right). \quad (\text{IV.69})$$

The effective potential (IV.69) is sufficiently simple to derive the corresponding equations of motion [44]. We simply give the main results of the analysis and we refer to [44] for more details.  $\mathcal{V}_{\text{eff}}$  depends explicitly on the warp factor, the dilaton and the NS-field-strength, and it depends implicitly on the internal metric through the Hodge operator, the pure spinors and the volume form. One should therefore vary it with respect to these fields to find the equations of motions.

The dilaton equation is obtained by varying  $\mathcal{V}_{\text{eff}}$  with respect to the dilaton and it is satisfied identically by imposing the modified domain-wall BPSness equation (IV.27).

As we will also discuss in the next subsection, the external modified Einstein equation can be written as

$$\langle e^{4A}\tilde{F} - d_H(e^{4A-\phi}\text{Re}\Psi_1), F + d_H\tilde{*}_6(e^{-\phi}\text{Re}\Psi_1) \rangle + e^{4A-\phi} \sum_{i \in \text{loc. sources}} \tau_i [\rho_i^{\text{loc}} \text{vol}_6 - \langle \text{Re}\Psi_1, j_i \rangle] = 0, \quad (\text{IV.70})$$

which is identically satisfied when the gauge BPSness is obeyed and the calibration bound (IV.68) is saturated. The external components of the modified Einstein equations are satisfied not only for the one-parameter DWSB class, but for any background preserving the D-string and gauge BPSness and violating the domain-wall calibration condition.

For the internal Einstein equation, the variation of the effective potential with respect to the internal metric gives

$$\text{Im} \left\{ \langle g_{k(m} dy^k \wedge \iota_{n)} \Psi_2, d_H [e^{A-\phi} r^* (3\text{Re}\Psi_1 + \frac{1}{2} (-1)^{|\Psi_2|} \Lambda_{pq} \gamma^p \text{Re}\Psi_1 \gamma^q)] \rangle \right\} = 0, \quad (\text{IV.71})$$

which imposes some non-trivial constraints the DWSB configurations must satisfy to be true supergravity solutions.

Another set of constraints comes from the variation of  $\mathcal{V}_{\text{eff}}$  with respect to the NS-field

$$d\left[e^{4A-2\phi}\langle\text{Im}(r^*\Psi_2), 3\text{Re}\Psi_1 + \frac{1}{2}(-1)^{|\Psi_2|}\Lambda_{mn}\gamma^m\text{Re}\Psi_1\gamma^n\rangle_3\right] = 0. \quad (\text{IV.72})$$

Interestingly, the NS-field equation of motion and the internal Einstein equation can be unified into the following condition:

$$\int_M e^{A-\phi}\text{Im}\{r^*\langle\delta_{g,B}[d_H(e^{3A-\phi}\Psi_2)], 3\text{Re}\Psi_1 + \frac{1}{2}(-1)^{|\Psi_2|}\Lambda_{mn}\gamma^m\text{Re}\Psi_1\gamma^n\rangle\} = 0, \quad (\text{IV.73})$$

where  $\delta_{g,B}$  denote a generic deformation of internal metric and  $B$ -field. Given that the one-parameter DWSB backgrounds satisfy

$$\langle d_H(e^{3A-\phi}\Psi_2), 3\text{Re}\Psi_1 + \frac{1}{2}(-1)^{|\Psi_2|}\Lambda_{mn}\gamma^m\text{Re}\Psi_1\gamma^n\rangle = 0, \quad (\text{IV.74})$$

the conditions (IV.73) can be seen as a stability condition of (IV.74) under deformations of  $d_H(e^{3A-\phi}\Psi_2)$ , thus providing a four-dimensional interpretation of these equations of motion, in terms of a stability condition for some F-flatness condition under such deformations, see [44] for more details.

Finally, note that the effective potential (IV.69) vanishes when evaluated on the one-parameter DWSB background. This can be seen by directly expanding the effective potential (IV.69) on the  $\text{SU}(3)\times\text{SU}(3)$  structure: we find that the terms in the two lines compensate exactly, as expected for backgrounds with  $\text{Minkowski}_4$  external spaces.

Alternatively we can also use (IV.27) to rewrite the effective potential in terms of the generalised current  $j$  associated to the generalised foliation  $(\Sigma, \mathcal{F})$

$$\mathcal{V}_{\text{eff}} = \frac{1}{2} \int_M e^{-2A}|r|^2 \left[ \langle \tilde{*}_6 j, j \rangle - \frac{|\langle \Psi_1, j \rangle|^2}{\text{vol}_6} \right], \quad (\text{IV.75})$$

which will turn out to be an insightful formulation when considering the off-shell one-parameter DWSB potential in subsection IV.2.2.d).

The vanishing of the one-parameter DWSB effective potential can here be interpreted as the saturation of a calibration bound. Indeed, if  $\tilde{j}_{(\Pi, \mathcal{R})}$  is the generalised current of a submanifold  $(\Pi, \mathcal{R})$ , which is not necessarily calibrated, the following bound can be derived from (IV.6)

$$\langle \tilde{*}_6 \tilde{j}_{(\Pi, \mathcal{R})}, \tilde{j}_{(\Pi, \mathcal{R})} \rangle \geq \frac{|\langle \Psi_1, \tilde{j}_{(\Pi, \mathcal{R})} \rangle|^2}{\text{vol}_6}. \quad (\text{IV.76})$$

This bound gets saturated if the generalised submanifold is calibrated, which is the case for the one-parameter DWSB backgrounds, and the effective potential therefore vanish.



### c) SSB effective potential and equations of motion

We consider now the SSB configurations introduced in subsection IV.2.1.b). Since SSB backgrounds haven't been studied in details in the literature, we discuss this case in more details.

We consider the most general violation of the D-string BPSness, without any non-calibrated sources in the backgrounds. We also assume that the Bianchi identities are respected.

We first discuss how the effective potential simplifies. The domain-wall and gauge BPS conditions, (III.71) and (III.73) set the second, third and fourth line in (IV.63) to zero. Also in this case, the last two lines of (IV.63) vanish due to the saturation of the bound (IV.68) and the Bianchi identities (IV.61). Then the effective potential is

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \frac{1}{2} \int_M \langle \tilde{*}_6 [d_H(e^{2A-\phi} \text{Im}\Psi_1)], d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle \\ & - 4 \int_M \text{vol}_6 e^{4A-2\phi} [(u_R^1)^2 + (u_R^2)^2], \end{aligned} \quad (\text{IV.77})$$

where  $\{u_m^{1,2}\}$  reduce to

$$u_m^1 = \frac{i \langle \gamma_m \bar{\Psi}_1, d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle}{e^{2A-\phi} \langle \Psi_1, \bar{\Psi}_1 \rangle} \quad (\text{IV.78})$$

$$u_m^2 = \frac{i(-1)^{|\Psi_2|} \langle \Psi_1 \gamma_m, d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle}{e^{2A-\phi} \langle \Psi_1, \bar{\Psi}_1 \rangle}. \quad (\text{IV.79})$$

Varying  $\mathcal{V}_{\text{eff}}$  with respect to the dilaton gives the dilaton equation of motion

$$\langle d_H(e^{2A-\phi} \text{Im}\Psi_1), \Xi \rangle = 0, \quad (\text{IV.80})$$

while the variation with respect to NS-field  $B$  is

$$d[e^{2A-\phi} \langle \text{Im}\Psi_1, \Xi \rangle_3] = 0. \quad (\text{IV.81})$$

The polyform  $\Xi$  is defined as

$$\Xi = \tilde{*}_6 d_H(e^{2A-\phi} \text{Im}\Psi_1) + 2e^{2A-\phi} u_{Rm}^1 \gamma^m \text{Re}\Psi_1 + 2(-1)^{|\Psi_2|} e^{2A-\phi} u_{Rm}^2 \text{Re}\Psi_1 \gamma^m. \quad (\text{IV.82})$$

We are left with the Einstein equations. To derive the internal component of the Einstein equations, one needs the following rules for the variations with respect to the internal metric

$$\delta \sqrt{\det g} = -\frac{1}{2} \delta g^{mn} g_{mn} \sqrt{\det g} \quad (\text{IV.83})$$

$$\delta \langle \tilde{*}_6 \chi_1, \chi_2 \rangle = \delta g^{mn} \left[ \langle \tilde{*}_6 \iota_m \chi_1, \iota_n \chi_2 \rangle - \frac{1}{2} g_{mn} \langle \tilde{*}_6 \chi_1, \chi_2 \rangle \right] \quad (\text{IV.84})$$

$$\delta \Psi_i = -\frac{1}{2} \delta g^{mn} g_{k(m} dy^k \wedge \iota_n) \Psi_i \quad i = 1, 2. \quad (\text{IV.85})$$

Then, we find that the internal Einstein equations read

$$\langle g_{k(m} dy^k \wedge \iota_n) (e^{2A-\phi} \text{Im} \Psi_1), d_H \Xi \rangle - \langle g_{k(m} dy^k \wedge \iota_n) d_H (e^{2A-\phi} \text{Im} \Psi_1), \Xi \rangle = 0. \quad (\text{IV.86})$$

It would be interesting to further develop  $\Xi$  by plugging in the general  $\text{SU}(3) \times \text{SU}(3)$  decomposition of  $d_H(e^{2A-\phi} \text{Im} \Psi_1)$  with only D-string BPSness violation (C.24). One could then use the fact that the  $\tilde{*}_6$  operator eigenstates are also eigenstates of the  $\text{SU}(3) \times \text{SU}(3)$  structure<sup>16</sup> to write these equations of motion on the generalised Hodge diamond, as first order differential equations on the supersymmetry breaking parameters introduced in Appendix C. One could then search for more general non-supersymmetric solutions of type II supergravity with SSB-like supersymmetry breaking. We won't do this here, as we will focus on our ansatz in IV.2.1.b).

To study the external component of the modified Einstein equation we will follow [44]. We first reduce the ten-dimensional equation (A.29) on our warped configurations

$$\nabla^m (e^{-2\phi} \nabla_m e^{4A}) = e^{4A} \tilde{F} \cdot \tilde{F} + e^{4A-\phi} \sum_{i \in \text{loc. sources}} \tau_i \rho_i^{\text{loc}}, \quad (\text{IV.87})$$

and rewrite it in terms of pure spinors as

$$\begin{aligned} -d(e^{-2\phi} *_6 de^{4A}) &= \langle \tilde{*}_6 \tilde{F}, e^{4A} \tilde{F} \rangle - \langle d_H \tilde{*}_6 \tilde{F}, e^{4A-\phi} \text{Re} \Psi_1 \rangle \\ &+ e^{4A-\phi} \sum_{i \in \text{loc. sources}} \tau_i \left[ \rho_i^{\text{loc}} \text{vol}_6 - \langle \text{Re} \Psi_1, j_i \rangle \right] \end{aligned} \quad (\text{IV.88})$$

by using the Bianchi identity (IV.61) together with the RR-field-strength self-duality (III.64).

At this point we see the difference between the DWSB configurations of [44] and the SSB ones. For DWSB, using D-string and gauge BPSness, one can prove the identity

$$d(e^{-2\phi} *_6 de^{4A}) = d \langle \tilde{*}_6 d_H (e^{4A-\phi} \text{Re} \Psi_1), e^{-\phi} \text{Re} \Psi_1 \rangle_5, \quad (\text{IV.89})$$

which in turn allows to rephrase (IV.88) as

$$\begin{aligned} &\langle e^{4A} \tilde{F} - d_H (e^{4A-\phi} \text{Re} \Psi_1), F + d_H \tilde{*}_6 (e^{-\phi} \text{Re} \Psi_1) \rangle \\ &+ e^{4A-\phi} \sum_{i \in \text{loc. sources}} \tau_i \left[ \rho_i^{\text{loc}} \text{vol}_6 - \langle \text{Re} \Psi_1, j_i \rangle \right] = 0. \end{aligned} \quad (\text{IV.90})$$

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<sup>16</sup>See for instance [6].

However, for generic SSB configurations the identity (IV.89) does not hold and therefore, neither does (IV.90).

For generic SSB configurations without non-calibrated sources, the external component of the Einstein equation therefore can't be reduced further than

$$-d(e^{-2\phi} *_6 de^{4A}) = \langle \tilde{*}_6 \tilde{F}, e^{4A} \tilde{F} \rangle - \langle d_H \tilde{*}_6 \tilde{F}, e^{4A-\phi} \text{Re}\Psi_1 \rangle. \quad (\text{IV.91})$$

We can now specify the equations of motion we found above to the SSB class introduced in subsection IV.2.1.b), where the pure spinor equations are (IV.41), (IV.42), and (IV.45).

Let's first discuss further the external components of the modified Einstein equations. Interestingly, using the gauge BPSness and our specific ansatz for the D-string BPSness violation (IV.45), we can show that once again the identity (IV.89) holds. The external modified Einstein equations are therefore again

$$\begin{aligned} \langle e^{4A} \tilde{F} - d_H(e^{4A-\phi} \text{Re}\Psi_1), F + d_H \tilde{*}_6(e^{-\phi} \text{Re}\Psi_1) \rangle \\ + e^{4A-\phi} \sum_{i \in \text{loc. sources}} \tau_i \left[ \rho_i^{\text{loc}} \text{vol}_6 - \langle \text{Re}\Psi_1, j_i \rangle \right] = 0, \end{aligned} \quad (\text{IV.92})$$

which are satisfied for our class of backgrounds, thanks to the gauge BPSness and the calibration of our space-filling sources.

Moving on to the other equations of motion, the  $\{u_{Rm}^{1,2}\}$  terms in  $\Xi$  reduce to<sup>17</sup>

$$u_{Rm}^1 = (-1)^{|\Psi_2|} e^A (\alpha_m - \Lambda_m^n \alpha_n) \quad (\text{IV.93})$$

$$u_{Rm}^2 = (-1)^{|\Psi_2|} e^A (\alpha_m - \Lambda_m^n \alpha_n). \quad (\text{IV.94})$$

Then, the dilaton equation of motion is automatically satisfied, once the specific form of the D-string BPSness violation (IV.45) is used. The NS-field equation of motion doesn't get simplified further, while the internal Einstein equations reduce to

$$\langle g_{k(m} dy^k \wedge \iota_n \text{Im}\Psi_1, d_H \Xi \rangle = 0. \quad (\text{IV.95})$$

We will construct different concrete examples of backgrounds in section IV.2.3, and we will see then that the equations of motion presented here can indeed be satisfied by such vacua.

Note also that, as in the one-parameter DWSB case, one can unify the equations of motion for the NS-field and the internal Einstein equations, into the following integrated

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<sup>17</sup>We could also further specify  $\Xi$  by expanding on the  $SU(3) \times SU(3)$  structure the specific ansatz (IV.45) in the  $\tilde{*}_6 d_H(e^{2A-\phi} \text{Im}\Psi_1)$  term, but the resulting expression is neither compact nor enlightening.

condition.

$$\int_M \langle \delta_{g,B} [d_H(e^{2A-\phi} \text{Im}\Psi_1)], \Xi \rangle = 0. \quad (\text{IV.96})$$

However, we don't have access to a four-dimensional interpretation of this condition.

Finally, it is important to realise that for our class of SSB backgrounds, the effective potential  $\mathcal{V}_{\text{eff}}$  can also be rewritten in terms of the generalised current associated to the generalised foliation. Indeed, it is straightforward to show that<sup>18</sup>

$$\langle \tilde{*}_6 [d_H(e^{2A-\phi} \text{Im}\Psi_1)], d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle = 4\hat{\alpha}_m \hat{\alpha}^m \langle \tilde{*}_6 j, j \rangle \quad (\text{IV.97})$$

$$(u_R^1)^2 = (u_R^2)^2 = 4\hat{\alpha}_m \hat{\alpha}^m e^{-(4A-2\phi)} \frac{|\langle \Psi_1, j \rangle|^2}{\text{vol}_6^2}. \quad (\text{IV.98})$$

Using these two equations, the effective potential of our SSB class can be brought to

$$\mathcal{V}_{\text{eff}} = 2 \int_M \hat{\alpha}_m \hat{\alpha}^m \left[ \langle \tilde{*}_6 j, j \rangle - \frac{|\langle \Psi_1, j \rangle|^2}{\text{vol}_6} \right]. \quad (\text{IV.99})$$

Given that the bound (IV.76) is saturated for our SSB class, the effective potential vanishes again, as expected for Minkowski backgrounds.

However, even though this expression is interesting to highlight a common structure between the one-parameter DWSB backgrounds, with effective potential (IV.75), and our SSB class, it is also important to remember that the two expressions come from different terms in the effective potential, and these have drastically different physical interpretations in terms of the effective theory, which we will make more precise later on.

We can now turn to configurations having both SSB and DWSB contributions, with the modified pure spinor equations (IV.51), (IV.52) and (IV.53).

The effective potential is

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \frac{1}{2} \int_M e^{-2A} \langle \tilde{*}_6 [d_H(e^{3A-\phi} \Psi_2)], d_H(e^{3A-\phi} \bar{\Psi}_2) \rangle \\ & - \frac{1}{4} \int_M e^{-2A} \left( \frac{|\langle \Psi_1, d_H(e^{3A-\phi} \Psi_2) \rangle|^2}{\text{vol}_6} + \frac{|\langle \bar{\Psi}_1, d_H(e^{3A-\phi} \Psi_2) \rangle|^2}{\text{vol}_6} \right) \\ & + \frac{1}{2} \int_M \langle \tilde{*}_6 [d_H(e^{2A-\phi} \text{Im}\Psi_1)], d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle \\ & - 4 \int_M \text{vol}_6 e^{4A-2\phi} [(u_R^1)^2 + (u_R^2)^2], \end{aligned} \quad (\text{IV.100})$$

where the first two lines are the contributions from the violation of the domain-wall calibration condition, while the last two lines correspond to the contributions from the

<sup>18</sup>We prove these equalities using vielbein, since it gives more compact expressions and allows to make the connection with the one-parameter DWSB class in the most natural way, but similar equalities can be derived in the coordinate basis.

D-string one.

The potential (IV.100) is nothing but the sum of the two effective potentials for the one-parameter DWSB backgrounds and our SSB ansatz from IV.2.1.b). It can thus be written as

$$\mathcal{V}_{\text{eff}} = \int_M \left( 2\hat{\alpha}_m \hat{\alpha}^m + \frac{1}{2} e^{-2A} |r|^2 \right) \left[ \langle \tilde{*}_6 j, j \rangle - \frac{|\langle \Psi_1, j \rangle|^2}{\text{vol}_6} \right] \quad (\text{IV.101})$$

where the two terms in the bracket still compensate through the saturation of (IV.76), such that these backgrounds again have vanishing effective potentials.

The equations of motion for these backgrounds simply bring together the contributions from the variations of the two effective potentials presented above.

The dilaton equation of motion and the external modified Einstein equation are therefore automatically obeyed given that the modified pure spinor equations are respected. Then the NS-field equation of motion is

$$d \left[ (-1)^{|\Psi_2|} e^{4A-2\phi} \langle \text{Im}(r^* \Psi_2), \Theta \rangle_3 - e^{2A-\phi} \langle \text{Im} \Psi_1, \Xi \rangle_3 \right] = 0, \quad (\text{IV.102})$$

and the internal Einstein equations are

$$\text{Im} \left\{ (-1)^{|\Psi_2|} e^A \langle g_{k(m} dy^k \wedge \iota_n \Psi_2, d_H [e^{A-\phi} r^* \Theta] \rangle \right\} - \langle g_{k(m} dy^k \wedge \iota_n \text{Im} \Psi_1, d_H \Xi \rangle = 0. \quad (\text{IV.103})$$

For compactness, we introduced here the polyform

$$\Theta = 3 \text{Re} \Psi_1 + \frac{1}{2} (-1)^{|\Psi_2|} \Lambda_{mn} \gamma^m \text{Re} \Psi_1 \gamma^n. \quad (\text{IV.104})$$

When discussing concrete background examples, the following relationship between the polyforms  $\Theta$  and  $\Xi$  will prove to be useful

$$\Xi = (-1)^{|\Psi_2|} e^{3A-\phi} \hat{\alpha}_a (\gamma^a \Theta + (-1)^{|\Psi_1|+1} \Theta \gamma^a) \quad (\text{IV.105})$$

$$= 2 (-1)^{|\Psi_2|} e^{3A-\phi} \hat{\alpha}^a \iota_a \Theta. \quad (\text{IV.106})$$

In contrast with the previous situations, both the NS-field equation of motion and the internal Einstein equations could in principle have non-vanishing DWSB and SSB contributions, that could cancel each other out. However, we won't explore this possibility and the background examples that we will present in section IV.2.3 will have both their DWSB and SSB contributions vanishing independently.

#### d) Stability and generalised calibrations

In the previous sections we showed how to find classes of non-supersymmetric backgrounds by solving modified supersymmetry variations and then considering the additional con-

straints the configurations must satisfy to be solutions of the equations of motion.

An important question to address is whether these non-supersymmetric backgrounds are stable. There are two kinds of instabilities one could face: under small perturbation or by quantum tunnelling.

We will try here to address the first, namely the potential presence of tachyonic directions. As discussed in [44], a possible way to answer this question is again looking at the four-dimensional effective potentials for the ‘off-shell’ fields of ten-dimensional supergravity around the given configurations.

Notice that the effective potentials obtained this way are not genuine potentials associated to four-dimensional theories, since we didn’t choose an appropriate truncation for the ten-dimensional modes and we didn’t perform the actual reduction. To do so would require the knowledge of the light modes of the theory, which is complicated to access for general flux vacua<sup>19</sup>. This is beyond the scope of this work. However we will see that there are still some interesting things one can say about our solutions from this approach.

We start by reviewing the analysis of [44] for the one-parameter DWSB backgrounds and then discuss how to extend it to SBB backgrounds.

The idea of [44] is to go off-shell, which is a way to look at fluctuations around a given solution, and to see whether, under minor constraint on the ten-dimensional supergravity fields, it is possible to derive an effective potential that is positive semi-definite.

From the general expression for  $\mathcal{V}_{\text{eff}}$ , it is clear that a first constraint to impose is the Bianchi identity (IV.61) so that the last line of (IV.63) vanishes. The second condition is to assume that the parameters  $\{u_m^{1,2}\}$  in the modified pure spinor equations are zero.<sup>20</sup> These terms correspond to vector-like modes of the  $SU(3) \times SU(3)$  structure group, and should correspond to massive modes from the perspective of the reduced  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  four-dimensional supergravity theories.

This is particularly clear for reductions to the  $\mathcal{N} = 2$  four-dimensional supergravity. In this case, the vector-like modes correspond to massive spin  $\frac{3}{2}$ -multiplet degrees of freedom for the four-dimensional theory, see for instance [38], and these are seen as non-physical degrees of freedom of  $\mathcal{N} = 2$  four-dimensional supergravity, that should be truncated away.

Anyway, these vector-like modes are not expected to give rise to light or tachyonic contributions to the effective theories. Therefore by ‘truncating’ them away one is not

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<sup>19</sup>See for instance [95] for a discussion of the dimensional reduction of general  $SU(3) \times SU(3)$  type II supergravity backgrounds to  $\mathcal{N} = 2$  gauged four-dimensional supergravity. It is also worth mentioning here recent work where the full Kaluza-Klein spectrum has been worked out for a variety of flux backgrounds using techniques from Exceptional Field Theory [96, 97].

<sup>20</sup>In terms of deformations of the ordinary supersymmetry variations these terms appear in the modified dilatino variations. This condition means that the only allowed deformations of this equation are  $SU(3) \times SU(3)$  singlets.

discarding potential instabilities of the reduced effective theory.

In order to go off-shell, we still consider an internal geometry that is a generalised foliation, but we do not require the associate generalised submanifold,  $(\Pi, \mathcal{R})$ , to be calibrated. This means that the violation of the domain-wall calibration condition now takes the more general form

$$d_H(e^{3A-\phi}\Psi_2) = i\tilde{r}\tilde{J}_{(\Pi, \mathcal{R})}, \quad (\text{IV.107})$$

where  $\tilde{J}_{(\Pi, \mathcal{R})}$  is now a generalised current associated to the submanifold  $(\Pi, \mathcal{R})$ , which isn't necessarily calibrated away from the solution, and  $\tilde{r}$  is just a parameter, eventually identified with the DWSB supersymmetry-breaking parameter mentioned above.

The 'off-shell' potential is then<sup>21</sup>

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \frac{1}{2} \int_M \text{vol}_6 |d_H(e^{2A-\phi}\text{Im}\Psi_1)|^2 \\ & + \frac{1}{2} \int_M \text{vol}_6 e^{4A} |\tilde{*}_6 F - e^{-4A} d_H(e^{4A-\phi}\text{Re}\Psi_1)|^2 \\ & + \frac{1}{2} \int_M e^{-2A} |\tilde{r}|^2 \left[ \langle \tilde{*}_6 \tilde{J}_{(\Pi, \mathcal{R})}, \tilde{J}_{(\Pi, \mathcal{R})} \rangle - \frac{|\langle \Psi_1, \tilde{J}_{(\Pi, \mathcal{R})} \rangle|^2}{\text{vol}_6} \right] \\ & + \sum_{i \in \text{D-branes}} \tau_i \int_M e^{4A-\phi} (\text{vol}_6 \rho_i^{\text{loc}} - \langle \text{Re}\Psi_1, j_i \rangle). \end{aligned} \quad (\text{IV.108})$$

The terms in the first two lines are trivially positive, and the last two lines are positive because of the calibration bounds (IV.68) and (IV.76). The potential is therefore positive semi-definite, and vanishes precisely for the one-parameter DWSB solutions.

As discussed at length in [44], notice that this implies that, under the previous assumptions, the effective potential can naturally be interpreted as being of the no-scale type. This is not a surprise, given that the one-parameter DWSB class contains the GKP solutions and all its T-duals, which are of the no-scale type [3].

Let us stress that this is an interesting property of the one-parameter DWSB class, but it is only an argument for the stability of this class with the caveat that we assumed a specific truncation, that we don't have precise control over.

A somewhat similar construction can be found for the backgrounds with SSB discussed in sections IV.2.1.b) and IV.2.1.c). Here we discuss the backgrounds with SSB and DWSB contributions of IV.2.1.c), but the backgrounds of IV.2.1.b) with only SSB contributions exhibit the same behaviour.

We also impose that the Bianchi identities are satisfied away from the solutions and that the internal manifold is still a generalised foliation. However we do not truncate away the vector-like modes, as they are fundamental for the SSB constructions. This

<sup>21</sup>Here we wrote the first line of (IV.63) as the square of a polyform, for clarity.

means that the off-shell violations of the D-string and domain-wall BPSness are

$$d_H(e^{3A-\phi}\Psi_2) = i\tilde{r}\tilde{j}_{(\Pi,\mathcal{R})} \quad (\text{IV.109})$$

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) = \tilde{\alpha}_m[\hat{\gamma}^m\tilde{j}_{(\Pi,\mathcal{R})} + (-1)^{|\tilde{j}_{(\Pi,\mathcal{R})}|}\tilde{j}_{(\Pi,\mathcal{R})}\hat{\gamma}^m]. \quad (\text{IV.110})$$

Here  $\tilde{j}_{(\Pi,\mathcal{R})}$  is again a generalised current associated to the submanifold  $(\Pi, \mathcal{R})$ , which isn't necessarily calibrated away from the solution, and  $\{\tilde{\alpha}_m\}$  and  $\tilde{r}$  are just parameters, eventually identified with the SSB and DWSB supersymmetry-breaking parameters.

The 'off-shell' potential is then

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \frac{1}{2} \int_M \text{vol}_6 e^{4A} |\tilde{*}_6 F - e^{-4A} d_H(e^{4A-\phi} \text{Re}\Psi_1)|^2 \\ & + \int_M \left( 2\tilde{\alpha}_m \tilde{\alpha}^m + \frac{1}{2} e^{-2A} |\tilde{r}|^2 \right) \left[ \langle \tilde{*}_6 \tilde{j}_{(\Pi,\mathcal{R})}, \tilde{j}_{(\Pi,\mathcal{R})} \rangle - \frac{|\langle \Psi_1, \tilde{j}_{(\Pi,\mathcal{R})} \rangle|^2}{\text{vol}_6} \right] \\ & + \sum_{i \in \text{D-branes}} \tau_i \int_M e^{4A-\phi} (\text{vol}_6 \rho_i^{\text{loc}} - \langle \text{Re}\Psi_1, j_i \rangle). \end{aligned} \quad (\text{IV.111})$$

The first line is again trivially positive, and the last two lines are positive because of the calibration bounds (IV.68) and (IV.76). The potential (IV.111) is therefore positive semi-definite, and vanishes precisely for the solutions introduced in IV.2.1.c).

However, the situation differs from the one-parameter DWSB case, by the fact that we purposely keep the vector-like modes  $\{u_m^{1,2}\}$ , that are believed to be massive modes from the effective perspective.

Therefore, we stress that we present this truncation as an interesting property of our backgrounds: there is a 'truncation' naturally suggested by the geometry such that the effective potential is positive semi-definite, but this doesn't constitute a proof of perturbative stability, since we have no way to conclude whether we can truncate the ten-dimensional modes in this way or not, and no way to reflect on the relative effective masses between the vector-like modes we kept, and the modes we discarded.

Moreover, it might not even be sensible to talk about effective theories associated to these ten-dimensional backgrounds, given the presence of these massive vector-like modes, or it could be that their effective theories are non-supersymmetric with the field content of  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  supergravity with additional massive multiplets, or non-supersymmetric solutions of four-dimensional supergravity theory with higher supersymmetry. Singling out one option among these scenarios would require a rigorous prescription to truncate and reduce the ten-dimensional theory, so we won't address further these questions.



### IV.2.3 Examples of vacua with SSB supersymmetry breaking

Up until here we made a rather abstract presentation of our non-supersymmetric backgrounds. The purpose of this section is to present concrete examples of our classes of vacua with SSB contributions. We will focus on internal geometries admitting an  $SU(3)$ - or a static  $SU(2)$ -structure, and we will discard the possibility to have a non-trivial two-form  $\mathcal{F}$  such that the generalised foliation  $(\Sigma, \mathcal{F})$  will be entirely defined by the cycle  $\Sigma$ .

Throughout this section, we revisit the examples of one-parameter DWSB vacua considered in [44], adding SSB contributions to the pure spinor equations and removing or keeping the DWSB one, to construct examples of the class of backgrounds introduced in IV.2.1.b) and IV.2.1.c) respectively. We will therefore specify the following set of modified pure spinor equations

$$d_H(e^{3A-\phi}\Psi_2) = 0 \quad \text{or} \quad irj \tag{IV.112}$$

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) = \alpha_m[\gamma^m j + (-1)^{|j|} j \gamma^m] \tag{IV.113}$$

$$d_H(e^{4A-\phi}\text{Re}\Psi_1) = e^{4A} \tilde{*}_6 F \tag{IV.114}$$

for our different concrete cases.

We consider the generalised foliations of our internal manifolds to be fibrations

$$\Sigma \hookrightarrow M \rightarrow \mathcal{B} \tag{IV.115}$$

with  $\mathcal{B}$  the base manifold and  $\Sigma$  the fibre.

As discussed in IV.2.1, the fibres will be calibrated by  $\omega^{\text{sf}}$  and will be wrapped by the space-filling sources.

#### a) Type IIB $SU(3)$ -structure backgrounds with D5-branes

The internal manifolds admitting an  $SU(3)$ -structure have parallel internal spinors  $\eta_1$  and  $\eta_2$ , and in type IIB they have the following pure spinors

$$\Psi_1 = e^{i\theta} e^{iJ} \quad \Psi_2 = e^{-i\theta} \Omega. \tag{IV.116}$$

We can introduce a local vielbein to write the Kähler form  $J$  and the  $(3,0)$  form  $\Omega$  as

$$J = - (e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6) \tag{IV.117}$$

$$\Omega = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6). \tag{IV.118}$$

Then, following [2, 8] one can show that for D5 space-filling branes the algebraic calibration condition (IV.6) imposes both that the fibre  $\Sigma$  is almost-complex with respect to the almost-complex structure defined by  $\Omega$ , and

$$\theta = -\frac{\pi}{2}. \quad (\text{IV.119})$$

We consider constructions with  $2^4$  O5-planes wrapping a two-cycle in the internal geometry, at the fixed point of the  $\mathbb{Z}_2$  involutions on the orthogonal four-dimensional space, and we allow for  $n_{D5}$  D5-branes, taken to be parallel to these orientifolds.

We will now specialise the discussion to the case of backgrounds with and without DWSB contributions.

### a).1 Backgrounds with only SSB contributions

We can here specify the pure spinor equations (IV.41), (IV.42) and (IV.44) satisfied by type IIB  $SU(3)$  backgrounds with space-filling D5-branes and with an SSB contribution of the type presented in IV.2.1.b). They yield

$$H = 0 \quad e^{2A-\phi} = \text{const.} \quad (\text{IV.120})$$

$$F_1 = F_5 = 0 \quad *_6 F_3 = -e^{-2\phi} d(e^\phi J) \quad (\text{IV.121})$$

$$d(e^A \Omega) = 0, \quad (\text{IV.122})$$

with the first line coming from the specific violation of the D-string calibration (IV.43), and the second and third line coming from gauge and domain-wall BPSness respectively. One condition from (IV.43) is missing here: for now these conditions are the same as the ones one would impose to have a supersymmetric background. If for instance we take the fibres to be along the directions  $e^1$  and  $e^4$ , we have  $\text{vol}_6 = \text{vol}_\Sigma \wedge \text{vol}_{\mathcal{B}_4}$  with  $\text{vol}_\Sigma = e^1 \wedge e^4$  and  $\text{vol}_{\mathcal{B}_4} = e^2 \wedge e^3 \wedge e^5 \wedge e^6$ , and we can write the last condition from (IV.43), which introduces the breaking of supersymmetry, as

$$e^{2A-\phi} d(J \wedge J) = 4(\hat{\alpha}_1 e^1 + \hat{\alpha}_4 e^4) \wedge j, \quad (\text{IV.123})$$

with

$$j = 4e^{3A-\phi} \text{vol}_{\mathcal{B}_4}. \quad (\text{IV.124})$$

Finally, we can specify the equations of motion. The NS-field, dilaton and external modified Einstein equations are trivially respected, and to rewrite the internal Einstein equations, it is useful to notice that in the case where  $\mathcal{F} = 0$ , we have

$$\Theta = 4(\text{Re}\Psi_1 - \text{Re}\Psi_1|_\Sigma). \quad (\text{IV.125})$$

Using this identity and (IV.106), the internal Einstein equations can be shown to reduce to

$$\langle g_{k(m} dy^k \wedge \iota_n) J \wedge J, d[\hat{\alpha}^a \iota_a (J \wedge J \wedge J)] \rangle = 0 \quad a = 1, 4 \quad (\text{IV.126})$$

which is identically satisfied. We conclude that this family of SSB SU(3) backgrounds with calibrated D5-branes automatically solves its equations of motion, without any further constraints.

We now turn to the construction of an explicit example of background from this class. We begin by choosing the following metric:

$$ds^2 = e^{2A} ds_{\mathbb{R}_{1,3}}^2 + ds_M^2 \quad (\text{IV.127})$$

$$ds_M^2 = \alpha' (2\pi)^2 \left\{ e^{2A} [R_1^2 (\eta^1)^2 + R_4^2 (\eta^4)^2] + e^{-2A} \sum_{j=2,3,5,6} R_j^2 (dy^j)^2 \right\}, \quad (\text{IV.128})$$

where the warp factor  $A$  only depends on the base direction  $\mathcal{B}_4 = \{y^2, y^3, y^5, y^6\}$  and  $\eta^1, \eta^4$  are non-closed one-forms satisfying

$$d\eta^a = f_{ij}^a dy^i \wedge dy^j \quad a = 1, 4 \quad i, j = 2, 3, 5, 6. \quad (\text{IV.129})$$

In the constant warp factor limit, this is nothing else then the geometry of a twisted torus. To ensure the compactness of  $M$  we take the structure constants  $\{f_{ij}^a\}$  to be integer constants, while the radii  $\{R_a, R_i\}$  can take any real value. The Kähler form  $J$  and the (3, 0) form  $\Omega$  are

$$J = -\alpha' (2\pi)^2 [e^{2A} R_1 R_4 \eta^1 \wedge \eta^4 + e^{-2A} (R_2 R_5 dy^2 \wedge dy^5 + R_3 R_6 dy^3 \wedge dy^6)] \quad (\text{IV.130})$$

$$\Omega = \alpha'^{3/2} (2\pi)^3 e^{-A} (R_1 \eta^1 + i R_4 \eta^4) \wedge (R_2 dy^2 + i R_5 dy^5) \wedge (R_3 dy^3 + i R_6 dy^6). \quad (\text{IV.131})$$

The domain-wall BPSness (IV.122) now takes the form

$$f_{26}^1 R_1 R_3 R_5 - f_{56}^4 R_2 R_3 R_4 - f_{35}^1 R_1 R_2 R_6 + f_{23}^4 R_4 R_5 R_6 = 0 \quad (\text{IV.132})$$

$$f_{26}^4 R_3 R_4 R_5 + f_{56}^1 R_1 R_2 R_3 - f_{35}^4 R_2 R_4 R_6 - f_{23}^1 R_1 R_5 R_6 = 0, \quad (\text{IV.133})$$

and the RR-fluxes (IV.121) read

$$F_3 = e^{2A-\phi} [*_{\mathcal{B}_4} de^{-4A} - \alpha' (2\pi)^2 (R_1^2 \eta^1 \wedge *_{\mathcal{B}} d\eta^1 + R_4^2 \eta^4 \wedge *_{\mathcal{B}} d\eta^4)], \quad (\text{IV.134})$$

$$\equiv e^{2A-\phi} *_{\mathcal{B}_4} de^{-4A} + \alpha' (2\pi)^2 F_3^{\text{bg}} \quad (\text{IV.135})$$

with  $*_{\mathcal{B}_4}$  the four-dimensional Hodge operator on the unwarped base. The first term comes from the back-reaction of the D5-branes and O5-planes, while the second term

should be thought as a properly quantised background flux, so it must be an integer valued three-form, which constrains the radii  $R_i$  and the value of  $e^{2A-\phi}$ .

The generalised current associated to the sources is

$$j = 4\alpha'^2(2\pi)^4 e^{-A-\phi} R_2 R_3 R_5 R_6 dy^2 \wedge dy^3 \wedge dy^5 \wedge dy^6, \quad (\text{IV.136})$$

and the D-string BPSness violation (IV.123) is

$$e^{2A-\phi} d(J \wedge J) = 8\pi\sqrt{\alpha'} e^A (R_1 \hat{\alpha}_1 \eta^1 + R_4 \hat{\alpha}_4 \eta^4) \wedge j, \quad (\text{IV.137})$$

which is satisfied for the following supersymmetry breaking parameters

$$\hat{\alpha}_1 = e^{2A} \frac{R_4}{16\pi\sqrt{\alpha'}} \left( \frac{f_{36}^4}{R_3 R_6} + \frac{f_{25}^4}{R_2 R_5} \right) \quad (\text{IV.138})$$

$$\hat{\alpha}_4 = -e^{2A} \frac{R_1}{16\pi\sqrt{\alpha'}} \left( \frac{f_{36}^1}{R_3 R_6} + \frac{f_{25}^1}{R_2 R_5} \right). \quad (\text{IV.139})$$

Finally, we can specify the Bianchi identities (IV.61), which in this case are only non trivial for  $F_3$ <sup>22</sup>

$$dF_3 = e^{2A-\phi} \left[ -\nabla_{\mathcal{B}_4}^2 e^{-4A} + \frac{Y}{4\alpha'(2\pi)^2} \right] \text{völ}_{\mathcal{B}_4} \quad (\text{IV.140})$$

$$= \frac{e^{2A-\phi}}{\alpha'(2\pi)^2} \left( \sum_{i=1}^{16} \delta_{\mathcal{B}_4}^4(y_i) - \sum_{j=1}^{n_{D5}} \delta_{\mathcal{B}_4}^4(y_j) \right) \text{völ}_{\mathcal{B}_4}, \quad (\text{IV.141})$$

where  $\text{völ}_{\mathcal{B}_4}$  is the unwarped volume form on  $\mathcal{B}_4$ , the charges of the O5 and D5 sources have been normalised to  $-1$  and  $1$  respectively, and with

$$Y = \frac{(f_{23}^1)^2}{(R_2 R_3)^2} + \frac{(f_{25}^1)^2}{(R_2 R_5)^2} + \frac{(f_{26}^1)^2}{(R_2 R_6)^2} + \frac{(f_{35}^1)^2}{(R_3 R_5)^2} + \frac{(f_{36}^1)^2}{(R_3 R_6)^2} + \frac{(f_{56}^1)^2}{(R_5 R_6)^2} \\ + \frac{(f_{23}^4)^2}{(R_2 R_3)^2} + \frac{(f_{25}^4)^2}{(R_2 R_5)^2} + \frac{(f_{26}^4)^2}{(R_2 R_6)^2} + \frac{(f_{35}^4)^2}{(R_3 R_5)^2} + \frac{(f_{36}^4)^2}{(R_3 R_6)^2} + \frac{(f_{56}^4)^2}{(R_5 R_6)^2}. \quad (\text{IV.142})$$

The corresponding tadpole condition connects the sources to the radii and structure constants:

$$n_{D5} + \frac{Y}{4} = 16. \quad (\text{IV.143})$$

### a).2 Backgrounds with both SSB and DWSB contributions

One can construct a similar class of SU(3) backgrounds with space-filling D5-branes, with an additional supersymmetry breaking contribution from the violation of the domain-wall BPSness. Considering again the fibres to be along the directions  $e^1$  and  $e^4$ , the pure spinor

<sup>22</sup>Note that the current (IV.137) and the one in (IV.11) differ by an overall factor.

equations (IV.51), (IV.52) and (IV.53) take the form

$$H = 0 \quad e^{2A-\phi} = \text{const.} \quad (\text{IV.144})$$

$$F_1 = F_5 = 0 \quad *_6 F_3 = -e^{-2\phi} d(e^\phi J) \quad (\text{IV.145})$$

and

$$e^{2A-\phi} d(J \wedge J) = 4(\hat{\alpha}_1 e^1 + \hat{\alpha}_4 e^4) \wedge j \quad (\text{IV.146})$$

$$e^{2A-\phi} d(e^A \Omega) = irj \quad (\text{IV.147})$$

with again

$$j = 4e^{3A-\phi} \text{vol}_{\mathcal{B}_4}. \quad (\text{IV.148})$$

Turning to the equations of motion, the NS-field equation is still trivially satisfied, such as the dilaton equation, and the external modified Einstein equations. Therefore only the internal Einstein equation has an additional non-trivial contribution. From (IV.125), we see that it reduces to

$$\text{Re} \langle g_{k(m} dy^k \wedge \iota_n \Omega, d(e^{-A} r^* J|_{\mathcal{B}_4}) \rangle = 0. \quad (\text{IV.149})$$

Since  $g_{k(m} dy^k \wedge \iota_n \Omega$  is either a (3, 0) or a primitive (2, 1) form, imposing

$$\left[ d(e^{-A} r J|_{\mathcal{B}_4}) \right]^{3,0} = \left[ d(e^{-A} r J|_{\mathcal{B}_4}) \right]_{\text{prim}}^{2,1} = 0 \quad (\text{IV.150})$$

is enough to satisfy (IV.149). If we consider  $r$  and the warp factor to be constant along the fibre, as is usual when the localised sources wrap the fibres, and if we have  $dJ|_{\mathcal{B}_4} = f \wedge J|_{\mathcal{B}_4}$  with  $f$  a real function on the base, then the conditions (IV.150) amounts to

$$\left[ d(e^{-A} r f) \right]^{1,0} = 0. \quad (\text{IV.151})$$

Let us construct an example of these backgrounds, which will respect this condition.

We start off with the same metric ansatz as for the case with only the SSB contribution

$$ds^2 = e^{2A} ds_{\mathbb{R}_{1,3}}^2 + ds_M^2 \quad (\text{IV.152})$$

$$ds_M^2 = \alpha' (2\pi)^2 \left\{ e^{2A} [R_1^2(\eta^1)^2 + R_4^2(\eta^4)^2] + e^{-2A} \sum_{j=2,3,5,6} R_j^2 (dy^j)^2 \right\}, \quad (\text{IV.153})$$

with the one forms  $\eta^1$  and  $\eta^4$  respecting again (IV.129). The Kähler form  $J$  and the (3, 0)

form  $\Omega$  are once more

$$J = -\alpha'(2\pi)^2[e^{2A}R_1R_4\eta^1 \wedge \eta^4 + e^{-2A}(R_2R_5dy^2 \wedge dy^5 + R_3R_6dy^3 \wedge dy^6)] \quad (\text{IV.154})$$

$$\Omega = \alpha'^{3/2}(2\pi)^3 e^{-A}(R_1\eta^1 + iR_4\eta^4) \wedge (R_2dy^2 + iR_5dy^5) \wedge (R_3dy^3 + iR_6dy^6). \quad (\text{IV.155})$$

The gauge BPSness (IV.145) yields the following RR-flux

$$F_3 = e^{2A-\phi}[*_{\mathcal{B}_4}de^{-4A} - \alpha'(2\pi)^2(R_1^2\eta^1 \wedge *_{\mathcal{B}}d\eta^1 + R_4^2\eta^4 \wedge *_{\mathcal{B}}d\eta^4)], \quad (\text{IV.156})$$

$$\equiv e^{2A-\phi} *_{\mathcal{B}_4}de^{-4A} + \alpha'(2\pi)^2 F_3^{\text{bg}} \quad (\text{IV.157})$$

Even though the form of the RR-flux is similar to the background with only an SSB contribution (IV.134), we stress that their components differ, since the structure constants  $\{f_{ij}^a\}$  are different. Indeed, here we don't respect the conditions (IV.132) and (IV.133) since we have a DWSB contribution. Besides, the values of  $e^{2A-\phi}$  and the radii  $R_i$  should be chosen appropriately such that the background fluxes are integer-valued.

The generalised current is again

$$j = 4\alpha'^2(2\pi)^4 e^{-A-\phi} R_2R_3R_5R_6 dy^2 \wedge dy^3 \wedge dy^5 \wedge dy^6, \quad (\text{IV.158})$$

and the D-string (IV.146) and domain-wall (IV.147) BPSness violations are

$$e^{2A-\phi}d(J \wedge J) = 8\pi\sqrt{\alpha'}e^A(R_1\hat{\alpha}_1\eta^1 + R_4\hat{\alpha}_4\eta^4) \wedge j \quad (\text{IV.159})$$

$$e^{2A-\phi}d(e^A\Omega) = irj \quad (\text{IV.160})$$

which are satisfied if

$$r = \frac{e^{3A}}{8\pi\sqrt{\alpha'}} \left( \frac{R_1f_{56}^1 + iR_4f_{56}^4}{R_5R_6} - \frac{R_1f_{23}^1 + iR_4f_{23}^4}{R_2R_3} \right) \quad (\text{IV.161})$$

$$+ \frac{iR_1f_{35}^1 - R_4f_{35}^4}{R_3R_5} - \frac{iR_1f_{26}^1 - R_4f_{26}^4}{R_2R_6} \quad (\text{IV.162})$$

$$\hat{\alpha}_1 = e^{2A} \frac{R_4}{16\pi\sqrt{\alpha'}} \left( \frac{f_{36}^4}{R_3R_6} + \frac{f_{25}^4}{R_2R_5} \right) \quad (\text{IV.163})$$

$$\hat{\alpha}_4 = -e^{2A} \frac{R_1}{16\pi\sqrt{\alpha'}} \left( \frac{f_{36}^1}{R_3R_6} + \frac{f_{25}^1}{R_2R_5} \right). \quad (\text{IV.164})$$

Turning to the equations of motion, we have

$$d(e^{-A}rJ|_{\mathcal{B}_4}) = 0 \quad (\text{IV.165})$$

so (IV.150) are obeyed and the internal Einstein equations are satisfied. As discussed above, every other equations of motion are satisfied.

Finally, we can write down the only non-trivial Bianchi identity, the one for the RR-flux

$$dF_3 = e^{2A-\phi} \left[ -\nabla_{\mathcal{B}_4}^2 e^{-4A} + \frac{Y}{4\alpha'(2\pi)^2} \right] \text{völ}_{\mathcal{B}_4} \quad (\text{IV.166})$$

$$= \frac{e^{2A-\phi}}{\alpha'(2\pi)^2} \left( \sum_{i=1}^{16} \delta_{\mathcal{B}_4}^4(y_i) - \sum_{j=1}^{n_{D5}} \delta_{\mathcal{B}_4}^4(y_j) \right) \text{völ}_{\mathcal{B}_4}, \quad (\text{IV.167})$$

with

$$Y = \frac{(f_{23}^1)^2}{(R_2 R_3)^2} + \frac{(f_{25}^1)^2}{(R_2 R_5)^2} + \frac{(f_{26}^1)^2}{(R_2 R_6)^2} + \frac{(f_{35}^1)^2}{(R_3 R_5)^2} + \frac{(f_{36}^1)^2}{(R_3 R_6)^2} + \frac{(f_{56}^1)^2}{(R_5 R_6)^2} \\ + \frac{(f_{23}^4)^2}{(R_2 R_3)^2} + \frac{(f_{25}^4)^2}{(R_2 R_5)^2} + \frac{(f_{26}^4)^2}{(R_2 R_6)^2} + \frac{(f_{35}^4)^2}{(R_3 R_5)^2} + \frac{(f_{36}^4)^2}{(R_3 R_6)^2} + \frac{(f_{56}^4)^2}{(R_5 R_6)^2}. \quad (\text{IV.168})$$

Once again, this expression is similar to the one for the background with only SSB contribution presented in a).1, but the addition of the DWSB contribution actually modifies the structure constants  $\{f_{ij}^a\}$  upon releasing the constraints (IV.132) and (IV.133), which also alters the corresponding tadpole condition

$$n_{D5} + \frac{Y}{4} = 16. \quad (\text{IV.169})$$

### b) Type IIA SU(3)-structure backgrounds with D6-branes

Type IIA SU(3) backgrounds have the following pure spinors

$$\Psi_1 = \Omega \quad \Psi_2 = e^{-i\theta} e^{iJ}. \quad (\text{IV.170})$$

Then, following [2, 8], one can show that the algebraic calibration condition (IV.6) for D6 space-filling branes wrapping an internal cycle  $\Sigma$  imposes

$$J|_{\Sigma} = 0 \quad \text{Im}\Omega|_{\Sigma} = 0. \quad (\text{IV.171})$$

We consider constructions with  $2^3$  O6-planes wrapping a two-cycle in the internal geometry, at the fixed point of the  $\mathbb{Z}_2$  involutions on the orthogonal three-dimensional space, and we allow for  $n_{D6}$  D6-branes, taken to be parallel to these orientifolds.

We again introduce a local vielbein and express the Kähler and (3, 0) form as

$$J = - (e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6) \quad (\text{IV.172})$$

$$\Omega = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6), \quad (\text{IV.173})$$

and we consider the fibres wrapped by the sources to be along the  $e^1$ ,  $e^2$  and  $e^3$  directions.

We then have  $\text{vol}_6 = \text{vol}_\Sigma \wedge \text{vol}_{\mathcal{B}_3}$  with  $\text{vol}_\Sigma = e^1 \wedge e^2 \wedge e^3$  and  $\text{vol}_{\mathcal{B}_3} = e^4 \wedge e^5 \wedge e^6$ . We can now write the pure spinor equations obeyed by these backgrounds, for the cases with and without a DWSB contribution.

### b).1 Backgrounds with only SSB contributions

The pure spinor equations (IV.41), (IV.42) and (IV.44), which correspond to having only an SSB contribution, read

$$e^{3A-\phi} = \text{const.} \quad e^{i\theta} = \text{const.} \quad (\text{IV.174})$$

$$H = 0 \quad dJ = 0 \quad (\text{IV.175})$$

for the domain-wall BPSness,

$$F_0 = F_4 = F_6 \quad (\text{IV.176})$$

$$*F_2 = -e^{-4A} d(e^{4A-\phi} \text{Re}\Omega) \quad (\text{IV.177})$$

for the gauge BPSness, and

$$e^{3A-\phi} d(e^{-A} \text{Im}\Omega) = 2(\hat{\alpha}_1 e^1 + \hat{\alpha}_2 e^2 + \hat{\alpha}_3 e^3) \wedge j \quad (\text{IV.178})$$

with

$$j = -4e^{3A-\phi} \text{vol}_{\mathcal{B}_3} \quad (\text{IV.179})$$

for the violation of the D-string BPSness.

Turning to the equations of motion, the internal Einstein equations are

$$\langle g_{k(m} dy^k \wedge \iota_n \text{Im}\Omega, d[\hat{\alpha}^a \iota_a (\text{Re}\Omega - \text{Re}\Omega|_\Sigma)] \rangle = 0 \quad a = 1, 2, 3. \quad (\text{IV.180})$$

It is natural to construct fibered backgrounds respecting

$$d[\hat{\alpha}^a \iota_a (\text{Re}\Omega - \text{Re}\Omega|_\Sigma)] = 0 \quad a = 1, 2, 3 \quad (\text{IV.181})$$

and thus satisfying the internal Einstein equations. We will now construct such a background.

We start with the following metric ansatz

$$ds^2 = e^{2A} ds_{\mathbb{R}_{1,3}}^2 + ds_M^2 \quad (\text{IV.182})$$

$$ds_M^2 = \alpha' (2\pi)^2 \left\{ e^{2A} \sum_{a=1,2,3} R_a^2 (\eta^a)^2 + e^{-2A} \sum_{j=4,5,6} R_j^2 (dy^j)^2 \right\} \quad (\text{IV.183})$$

where as in the type IIB case, the warp factor  $A$  only depends on the base direction



$\mathcal{B}_3 = \{y^4, y^5, y^6\}$  and  $\eta^1, \eta^2$  and  $\eta^3$  are non-closed one-forms satisfying

$$d\eta^a = f_{ij}^a dy^i \wedge dy^j \quad a = 1, 2, 3 \quad i, j = 4, 5, 6. \quad (\text{IV.184})$$

In the constant warp factor limit, this again corresponds to the geometry of a twisted torus. To ensure the compactness of  $M$ , the structure constants  $\{f_{ij}^a\}$  have to be integer constants, and the radii  $\{R_a, R_i\}$  can take any real value. The Kähler form  $J$  and the  $(3, 0)$  form  $\Omega$  are

$$J = -\alpha'(2\pi)^2 [R_1 R_4 \eta^1 \wedge dy^4 + R_2 R_5 \eta^2 \wedge dy^5 + R_3 R_6 \eta^3 \wedge dy^6] \quad (\text{IV.185})$$

$$\Omega = \alpha'^{3/2} (2\pi)^3 (e^A R_1 \eta^1 + i e^{-A} R_4 dy^4) \wedge (e^A R_2 \eta^2 + i e^{-A} R_5 dy^5) \quad (\text{IV.186})$$

$$\wedge (e^A R_3 \eta^3 + i e^{-A} R_6 dy^6). \quad (\text{IV.187})$$

The domain-wall BPSness (IV.175) reduces to

$$R_1 R_4 f_{56}^1 + R_2 R_5 f_{64}^2 + R_3 R_6 f_{45}^3 = 0, \quad (\text{IV.188})$$

and the RR-fluxes (IV.177) read

$$F_2 = e^{3A-\phi} [ *_{\mathcal{B}_3} de^{-4A} - \alpha'(2\pi)^2 \sum_{a=1,2,3} R_a^2 \eta^a \wedge *_{\mathcal{B}_3} d\eta^a ] \quad (\text{IV.189})$$

$$F_2 \equiv e^{3A-\phi} *_{\mathcal{B}_3} de^{-4A} + \alpha'(2\pi)^2 F_2^{\text{bg}}, \quad (\text{IV.190})$$

with  $*_{\mathcal{B}_3}$  the three-dimensional Hodge operator on the unwarped base. As in the type IIB case, the background RR-fluxes must be integer-valued, constraining the values of the radii  $R_i$  and of  $e^{3A-\phi}$ .

The generalised current is

$$j = -4\alpha'^{3/2} (2\pi)^3 e^{-\phi} R_4 R_5 R_6 dy^4 \wedge dy^5 \wedge dy^6, \quad (\text{IV.191})$$

and the D-string BPSness violation (IV.178) is

$$e^{3A-\phi} d(e^{-A} \text{Im}\Omega) = 4\pi \sqrt{\alpha'} e^A (R_1 \hat{\alpha}_1 \eta^1 + R_2 \hat{\alpha}_2 \eta^2 + R_3 \hat{\alpha}_3 \eta^3) \wedge j \quad (\text{IV.192})$$

which gives

$$\hat{\alpha}_1 = \frac{e^{2A}}{16\pi \sqrt{\alpha'} R_4} \left( \frac{R_3}{R_6} f_{46}^3 + \frac{R_2}{R_5} f_{45}^2 \right) \quad (\text{IV.193})$$

$$\hat{\alpha}_2 = \frac{e^{2A}}{16\pi \sqrt{\alpha'} R_5} \left( \frac{R_3}{R_6} f_{56}^3 - \frac{R_1}{R_4} f_{45}^1 \right) \quad (\text{IV.194})$$

$$\hat{\alpha}_3 = -\frac{e^{2A}}{16\pi\sqrt{\alpha'}R_6} \left( \frac{R_2}{R_5} f_{56}^2 + \frac{R_1}{R_4} f_{46}^1 \right). \quad (\text{IV.195})$$

With these  $\hat{\alpha}$ 's, one can show that

$$d[\hat{\alpha}^a \iota_a(\text{Re}\Omega - \text{Re}\Omega|_\Sigma)] = 0 \quad a = 1, 2, 3, \quad (\text{IV.196})$$

and the internal Einstein equations (IV.180) are therefore satisfied.

Finally, the Bianchi identities reduce to

$$\begin{aligned} dF_2 &= e^{3A-\phi} \left[ -\nabla_{\mathcal{B}_3}^2 e^{-4A} + \frac{Z}{2\alpha'(2\pi)^2} \right] \text{völ}_{\mathcal{B}_3} \\ &= \frac{e^{3A-\phi}}{\alpha'(2\pi)^2} \left( \sum_{i=1}^8 \delta_{\mathcal{B}_4}^4(y_i) - \sum_{j=1}^{n_{D6}} \delta_{\mathcal{B}_4}^4(y_j) \right) \text{völ}_{\mathcal{B}_4}, \end{aligned} \quad (\text{IV.197})$$

with  $\text{völ}_{\mathcal{B}_3}$  the unwarped volume form on  $\mathcal{B}_3$ , the charges of the O6 and D6 sources have been normalised to  $-1$  and  $1$  respectively, and with

$$\begin{aligned} Z &= \frac{(f_{45}^1)^2}{(R_4 R_5)^2} + \frac{(f_{46}^1)^2}{(R_4 R_6)^2} + \frac{(f_{56}^1)^2}{(R_5 R_6)^2} + \frac{(f_{45}^2)^2}{(R_4 R_5)^2} + \frac{(f_{46}^2)^2}{(R_4 R_6)^2} + \frac{(f_{56}^2)^2}{(R_5 R_6)^2} \\ &\quad + \frac{(f_{45}^3)^2}{(R_4 R_5)^2} + \frac{(f_{46}^3)^2}{(R_4 R_6)^2} + \frac{(f_{56}^3)^2}{(R_5 R_6)^2}. \end{aligned} \quad (\text{IV.198})$$

The corresponding tadpole condition connects the sources to the radii and structure constants:

$$n_{D6} + \frac{Z}{2} = 8. \quad (\text{IV.199})$$

### b).2 Backgrounds with both SSB and DWSB contributions

One can construct a similar class of SU(3) backgrounds that has space-filling D6-branes, with an additional supersymmetry breaking contribution from the violation of the domain-wall BPSness. In this case, the pure spinor equations (IV.51), (IV.52) and (IV.53) take the form

$$e^{3A-\phi} = \text{const.} \quad e^{i\theta} = \text{const.} \quad F_4 = F_6 = 0 \quad (\text{IV.200})$$

$$*F_2 = -e^{-4A} d(e^{4A-\phi} \text{Re}\Omega) \quad *F_0 = e^{-\phi} H \wedge \text{Re}\Omega \quad (\text{IV.201})$$

and

$$e^{3A-\phi} e^{-i\theta} (H + idJ) = irj \quad (\text{IV.202})$$

$$e^{3A-\phi} d(e^{-A} \text{Im}\Omega) = 2(\hat{\alpha}_1 e^1 + \hat{\alpha}_2 e^2 + \hat{\alpha}_3 e^3) \wedge j \quad (\text{IV.203})$$

with again

$$j = -4e^{3A-\phi}\text{vol}_{\mathcal{B}_3}. \quad (\text{IV.204})$$

Turning to the equations of motion, the internal Einstein equations are

$$\text{Re}\left\{e^A e^{-i\theta} \langle g_{k(m)} dy^k \wedge \iota_n J, d_H[e^{-2A} r^* \Theta] \rangle\right\} + \langle g_{k(m)} dy^k \wedge \iota_n \text{Im}\Omega, d_H(\hat{\alpha}^a \iota_a \Theta) \rangle = 0, \quad (\text{IV.205})$$

with  $a = 1, 2, 3$  and again

$$\Theta = 4(\text{Re}\Omega - \text{Re}\Omega|_{\Sigma}). \quad (\text{IV.206})$$

It is sufficient to obey

$$d_H[e^{-2A} r^* \Theta] = 0 \quad (\text{IV.207})$$

$$d_H(\hat{\alpha}^a \iota_a \Theta) = 0 \quad a = 1, 2, 3 \quad (\text{IV.208})$$

to satisfy the internal Einstein equations, and we will shortly turn to the construction of an example background respecting (IV.207) and (IV.208). As for the NS-field equation of motion, it reduces exactly to (IV.207).

We start the construction of an explicit background belonging to this class with considering the following NS ansatz

$$ds^2 = e^{2A} ds_{\mathbb{R}_{1,3}}^2 + ds_M^2 \quad (\text{IV.209})$$

$$ds_M^2 = \alpha'(2\pi)^2 \left\{ e^{2A} \sum_{a=1,2,3} R_a^2 (\eta^a)^2 + e^{-2A} \sum_{j=4,5,6} R_j^2 (dy^j)^2 \right\} \quad (\text{IV.210})$$

$$H = \alpha'(2\pi)^2 \left( N dy^4 \wedge dy^5 \wedge dy^6 + \sum_{a=1,2,3} B_a d\eta^a \wedge dy^{a+3} \right) \quad (\text{IV.211})$$

where  $N \in \mathbb{Z}$ ,  $B_a \in \mathbb{Z}$  and again the warp factor  $A$  only depends on the base direction  $\mathcal{B}_3 = \{y^4, y^5, y^6\}$  and  $\eta^1, \eta^2$  and  $\eta^3$  are non-closed one-forms satisfying (IV.184). The Kähler form  $J$  and the (3, 0) form  $\Omega$  are again

$$J = -\alpha'(2\pi)^2 [R_1 R_4 \eta^1 \wedge dy^4 + R_2 R_5 \eta^2 \wedge dy^5 + R_3 R_6 \eta^3 \wedge dy^6] \quad (\text{IV.212})$$

$$\Omega = \alpha'^{3/2} (2\pi)^3 (e^A R_1 \eta^1 + i e^{-A} R_4 dy^4) \wedge (e^A R_2 \eta^2 + i e^{-A} R_5 dy^5) \quad (\text{IV.213})$$

$$\wedge (e^A R_3 \eta^3 + i e^{-A} R_6 dy^6). \quad (\text{IV.214})$$

The RR-fluxes (IV.201) read

$$F_0 = -e^{3A-\phi} \frac{R_1 R_2 R_3}{2\pi \sqrt{\alpha'} \text{Vol}(M)} [N + B_1 f_{56}^1 - B_2 f_{46}^2 + B_3 f_{45}^3] \quad (\text{IV.215})$$

$$F_2 = e^{3A-\phi} [ *_{\mathcal{B}_3} de^{-4A} - \alpha'(2\pi)^2 \sum_{a=1,2,3} R_a^2 \eta^a \wedge *_{\mathcal{B}_3} d\eta^a ] \quad (\text{IV.216})$$

$$F_2 \equiv e^{3A-\phi} *_{\mathcal{B}_3} de^{-4A} + \alpha'(2\pi)^2 F_2^{\text{bg}}, \quad (\text{IV.217})$$

with the internal manifold volume  $\text{Vol}(M)$  normalised in  $\alpha'$  units. Both  $F_0$  and  $F_2^{\text{bg}}$  are background fluxes and must be integer-valued, constraining the radii and  $e^{3A-\phi}$  again. As before, the generalised current is

$$j = -4\alpha'^{3/2}(2\pi)^3 e^{-\phi} R_4 R_5 R_6 dy^4 \wedge dy^5 \wedge dy^6, \quad (\text{IV.218})$$

and the domain-wall and D-string BPSness violations are

$$e^{3A-\phi} e^{-i\theta} (H + idJ) = irj \quad (\text{IV.219})$$

$$e^{3A-\phi} d(e^{-A} \text{Im}\Omega) = 4\pi\sqrt{\alpha'} e^A (R_1 \hat{\alpha}_1 \eta^1 + R_2 \hat{\alpha}_2 \eta^2 + R_3 \hat{\alpha}_3 \eta^3) \wedge j \quad (\text{IV.220})$$

and are satisfied if

$$r = \frac{e^{3A} e^{-i\theta}}{8\pi\sqrt{\alpha'} R_4 R_5 R_6} \left[ i(N + B_1 f_{56}^1 - B_2 f_{46}^2 + B_3 f_{45}^3) \right] \quad (\text{IV.221})$$

$$+ f_{56}^1 R_1 R_4 - f_{246}^2 R_2 R_5 + f_{345}^3 R_3 R_6 \quad (\text{IV.222})$$

$$\hat{\alpha}_1 = \frac{e^{2A}}{16\pi\sqrt{\alpha'} R_4} \left( \frac{R_3}{R_6} f_{46}^3 + \frac{R_2}{R_5} f_{45}^2 \right) \quad (\text{IV.223})$$

$$\hat{\alpha}_2 = \frac{e^{2A}}{16\pi\sqrt{\alpha'} R_5} \left( \frac{R_3}{R_6} f_{56}^3 - \frac{R_1}{R_4} f_{45}^1 \right) \quad (\text{IV.224})$$

$$\hat{\alpha}_3 = -\frac{e^{2A}}{16\pi\sqrt{\alpha'} R_6} \left( \frac{R_2}{R_5} f_{56}^2 + \frac{R_1}{R_4} f_{46}^1 \right). \quad (\text{IV.225})$$

With these supersymmetry breaking parameters, one can show that (IV.207) and (IV.208) are satisfied, and therefore the NS-field and internal metric equations of motion are obeyed.

Finally, the Bianchi identities for the RR-fluxes read

$$\begin{aligned} dF_2 &= e^{3A-\phi} \left[ -\nabla_{\mathcal{B}_3}^2 e^{-4A} + \frac{Z}{2\alpha'(2\pi)^2} \right] \text{völ}_{\mathcal{B}_3} \\ &= \frac{e^{3A-\phi}}{\alpha'(2\pi)^2} \left( \sum_{i=1}^8 \delta_{\mathcal{B}_4}^4(y_i) - \sum_{j=1}^{n_{D6}} \delta_{\mathcal{B}_4}^4(y_j) \right) \text{völ}_{\mathcal{B}_4}, \end{aligned} \quad (\text{IV.226})$$

with again

$$\begin{aligned} Z &= \frac{(f_{45}^1)^2}{(R_4 R_5)^2} + \frac{(f_{46}^1)^2}{(R_4 R_6)^2} + \frac{(f_{56}^1)^2}{(R_5 R_6)^2} + \frac{(f_{45}^2)^2}{(R_4 R_5)^2} + \frac{(f_{46}^2)^2}{(R_4 R_6)^2} + \frac{(f_{56}^2)^2}{(R_5 R_6)^2} \\ &+ \frac{(f_{45}^3)^2}{(R_4 R_5)^2} + \frac{(f_{46}^3)^2}{(R_4 R_6)^2} + \frac{(f_{56}^3)^2}{(R_5 R_6)^2}. \end{aligned} \quad (\text{IV.227})$$

This expression is similar to the one for the background with only SSB contribution presented in b).1, but the addition of the DWSB contribution actually modifies the structure constants  $\{f_{ij}^a\}$  upon releasing the constraint (IV.188), which also alters the corresponding tadpole condition

$$n_{D6} + \frac{Z}{2} = 8. \quad (\text{IV.228})$$

### c) Type IIB SU(2)-structure backgrounds with D5-branes

We now turn to the discussion of type IIB backgrounds admitting a static SU(2) structure and having space-filling D5-branes. The two internal spinors  $\eta_1$  and  $\eta_2$  of such backgrounds are everywhere orthogonal, which means that one can specify a one-form  $\theta = \theta_m dy^m$  such that

$$\eta_2 = -\frac{i}{2}\theta_m \gamma^m \eta_1^*. \quad (\text{IV.229})$$

It is natural to parametrise the two SU(3) structures defined by  $\eta_1$  and  $\eta_2$  as

$$J_1 = -\frac{i}{2}\theta \wedge \bar{\theta} + j \quad \Omega_1 = -\theta \wedge w \quad (\text{IV.230})$$

$$J_2 = -\frac{i}{2}\theta \wedge \bar{\theta} - j \quad \Omega_2 = \theta \wedge \bar{w} \quad (\text{IV.231})$$

with  $\iota_{\theta}j = \iota_{\bar{\theta}}j = \iota_{\theta}w = \iota_{\bar{\theta}}w = 0$ . The corresponding pure spinors are

$$\Psi_1 = w \wedge e^{\frac{1}{2}\theta \wedge \bar{\theta}} \quad (\text{IV.232})$$

$$\Psi_2 = \theta \wedge e^{ij}. \quad (\text{IV.233})$$

Following [8], one can show that the calibration of the space-filling D5-branes wrapping a cycle  $\Sigma$  imposes

$$\theta|_{\Sigma} = 0 \quad j|_{\Sigma} \quad \mathcal{F} = 0 \quad \text{Im}w|_{\Sigma} = 0. \quad (\text{IV.234})$$

We will now specify the pure spinor equations and the equations of motion for both the cases with and without a DWSB contribution.

#### c).1 Backgrounds with only SSB contributions

We start by discussing the class of backgrounds having only an SSB contribution. The domain-wall BPSness (IV.41) first imposes

$$d(e^{3A-\phi}\theta) = 0. \quad (\text{IV.235})$$

This means that locally, one can introduce a complex coordinate  $z$  such that  $dz = e^{3A-\phi}\theta$ . Then, the hypersurface  $D$  defined by  $z = \text{constant}$  admits an SU(2) structure defined by

the pair  $(j|_D, w|_D)$ , and (IV.234) means that the fibre  $\Sigma$  defines a Slag fibration of the leaves  $D$ .

We can choose a local basis such that  $e^1, e^2, e^4, e^5$  are along  $D$ , and  $e^1, e^2$  are tangent to the fibre  $\Sigma$ . We also take

$$j = -(e^1 \wedge e^4 + e^2 \wedge e^5) \quad (\text{IV.236})$$

$$w = (e^1 + ie^4) \wedge (e^2 + ie^5) \quad (\text{IV.237})$$

$$\theta = e^3 + ie^6. \quad (\text{IV.238})$$

The domain-wall BPSness also imposes

$$dj|_D = 0 \quad H|_D = 0, \quad (\text{IV.239})$$

so one can expand  $dj$  and  $H$  as

$$dj = (f \wedge \bar{\theta} + c.c.) + \frac{i}{2} u \wedge \theta \wedge \bar{\theta} \quad (\text{IV.240})$$

$$H = (g \wedge \bar{\theta} + c.c.) + \frac{i}{2} h \wedge \theta \wedge \bar{\theta} \quad (\text{IV.241})$$

with  $f$  and  $g$  complex two-forms, and  $u$  and  $h$  real one-forms which can be decomposed purely along  $D$ . The last condition imposed by domain-wall BPSness is then

$$g + if = 0. \quad (\text{IV.242})$$

The gauge BPSness yields the following RR-fluxes

$$e^\phi * F_1 = H \wedge \text{Re}w - id(2A - \phi) \wedge \text{Im}w \wedge \theta \wedge \bar{\theta} \quad (\text{IV.243})$$

$$*F_3 = -e^{-4A} d(e^{4A-\phi} \text{Re}w) \quad (\text{IV.244})$$

$$F_5 = 0, \quad (\text{IV.245})$$

and the violation of the D-string BPSness takes the form

$$d(e^{2A-\phi} \text{Im}w) = 0 \quad (\text{IV.246})$$

$$e^{2A-\phi} [H \wedge \text{Im}w - \frac{i}{2} d(\text{Re}w \wedge \theta \wedge \bar{\theta})] = 4\pi\sqrt{\alpha'} e^A (R_1 \hat{\alpha}_1 dy^1 + R_2 \hat{\alpha}_2 dy^2) \wedge j \quad (\text{IV.247})$$

with

$$j = -4e^{3A-\phi} \text{vol}_{\mathcal{B}_4}. \quad (\text{IV.248})$$

Turning to the equations of motion, the NS-field equation is

$$d\left[e^{5A-2\phi}\langle\text{Im}w, \hat{\alpha}^a \iota_a(\text{Re}w - \text{Re}w|_{\Sigma})\rangle\right] = 0 \quad a = 1, 2 \quad (\text{IV.249})$$

while the internal Einstein equations reduce to

$$\langle g_{k(m} dy^k \wedge \iota_n) \text{Im}w, d(e^{3A-\phi} \hat{\alpha}^a \iota_a (i \text{Im}w \wedge \theta \wedge \bar{\theta})) \rangle = 0 \quad a = 1, 2. \quad (\text{IV.250})$$

We now turn to the construction of a concrete background satisfying these conditions.

To do so, we consider a factorisable warped six-torus

$$ds^2 = e^{2A} ds_{\mathbb{R}_{1,3}}^2 + ds_M^2 \quad (\text{IV.251})$$

$$ds_M^2 = \alpha' (2\pi)^2 \left\{ e^{2A} [R_1^2 (dy^1)^2 + R_2^2 (dy^2)^2] + e^{-2A} \sum_{j=3}^6 R_j^2 (dy^j)^2 \right\}, \quad (\text{IV.252})$$

and we take the two-torus spanned by  $y^1$  and  $y^2$  to be the fibre  $\Sigma$  over the four-torus base spanned by  $y^3, y^4, y^5, y^6$ . We also consider the fibre to be wrapped by  $2^4$  O5-planes and  $n_{D5}$  D5-branes. The  $SU(2)$  structure is then

$$j = -\alpha' (2\pi)^2 (R_1 R_4 dy^1 \wedge dy^4 + R_2 R_5 dy^2 \wedge dy^5) \quad (\text{IV.253})$$

$$w = \alpha' (2\pi)^2 (e^A R_1 dy^1 + i e^{-A} R_4 dy^4) \wedge (e^A R_2 dy^2 + i e^{-A} R_5 dy^5) \quad (\text{IV.254})$$

$$\theta = 2\pi \sqrt{\alpha'} e^{-A} (R_3 dy^3 + i R_6 dy^6). \quad (\text{IV.255})$$

Let us now see what the domain-wall BPSness imposes on the background. First of all (IV.235) sets

$$e^{2A-\phi} = \text{const.} \quad (\text{IV.256})$$

Then, if we consider the following NS-field ansatz

$$H = (2\pi)^2 \alpha' [N_{\text{NS1}} dy^3 \wedge dy^4 \wedge dy^6 + N_{\text{NS2}} dy^3 \wedge dy^5 \wedge dy^6] \quad N_{\text{NS1},2} \in \mathbb{Z}, \quad (\text{IV.257})$$

we see that (IV.239) and (IV.242) are satisfied, with  $g = f = 0$ , and we have  $u = 0$  and  $h = -\frac{e^{-2A}}{R_3 R_6} [N_{\text{NS1}} dy^4 + N_{\text{NS2}} dy^5]$ .

The generalised current is

$$j = -4\alpha'^2 (2\pi)^4 e^{-A-\phi} R_3 R_4 R_5 R_6 dy^3 \wedge dy^4 \wedge dy^5 \wedge dy^6. \quad (\text{IV.258})$$

The violation of the D-string BPSness (IV.246) is identically satisfied, while (IV.247) reduces to

$$e^{2A-\phi} H \wedge \text{Im}w = 4\pi \sqrt{\alpha'} e^A (R_1 \hat{\alpha}_1 dy^1 + R_2 \hat{\alpha}_2 dy^2) \wedge j \quad (\text{IV.259})$$

which is satisfied with

$$\hat{\alpha}_1 = -\frac{e^{2A}N_{\text{NS1}}}{16\pi\sqrt{\alpha'}R_3R_4R_6} \quad (\text{IV.260})$$

$$\hat{\alpha}_2 = -\frac{e^{2A}N_{\text{NS2}}}{16\pi\sqrt{\alpha'}R_3R_5R_6}. \quad (\text{IV.261})$$

The gauge BPSness constrains the RR-fluxes to be

$$F_1 = e^{2A-\phi} \left[ \frac{R_4N_{\text{NS2}}}{R_3R_5R_6} dy^4 - \frac{R_5N_{\text{NS1}}}{R_3R_4R_6} dy^5 \right] \quad (\text{IV.262})$$

$$F_3 = e^{2A-\phi} *_{\mathcal{B}_4} de^{-4A} \quad (\text{IV.263})$$

with  $*_{\mathcal{B}_4}$  the four-dimensional Hodge operator on the unwarped base.

The RR-flux quantisation implies that we must have

$$e^{2A-\phi} \frac{R_5N_{\text{NS1}}}{R_3R_4R_6} = N_{\text{R1}} \in \mathbb{Z} \quad e^{2A-\phi} \frac{R_4N_{\text{NS2}}}{R_3R_5R_6} = N_{\text{R2}} \in \mathbb{Z}, \quad (\text{IV.264})$$

so the RR Bianchi identities reduce to

$$-\hat{\nabla}_{\mathcal{B}_4}^2 e^{-4A} = \frac{1}{e^{2A-\phi}(2\pi)^2\alpha'\Pi_{a=3}^6 R_a} [N_{\text{NS1}}N_{\text{R1}} + N_{\text{NS2}}N_{\text{R2}} + \sum_{i \in \text{D5}'s, \text{O5}'s} q_i \delta_{\mathcal{B}_4}^4(y_i)] \quad (\text{IV.265})$$

with again  $q_{\text{D5}} = -q_{\text{O5}} = 1$ . Integrating this condition on the base yields the following tadpole condition

$$N_{\text{NS1}}N_{\text{R1}} + N_{\text{NS2}}N_{\text{R2}} + n_{\text{D5}} = 16. \quad (\text{IV.266})$$

Turning to the equations of motion, the NS-field equation (IV.249) is identically satisfied, and we have

$$d(e^{3A-\phi}\hat{\alpha}^a \iota_a(i\text{Im}w \wedge \theta \wedge \bar{\theta})) = 0 \quad (\text{IV.267})$$

so the internal Einstein equations are satisfied.

### c).2 Backgrounds with both SSB and DWSB contributions

We now consider  $\text{SU}(2)$  backgrounds with space-filling D5-branes with both SSB and DWSB contributions. The internal manifold geometry is similar to the one of the backgrounds with only an SSB contribution. Indeed, the domain-wall BPSness violation (IV.51) keeps on imposing

$$d(e^{3A-\phi}\theta) = 0, \quad (\text{IV.268})$$



so we can similarly define the hypersurface  $D$  along  $e^1, e^2, e^4, e^5$ , which admits a Slag fibration with fibres  $\Sigma$ . We keep the same parametrisation of the  $SU(2)$  structure

$$j = -(e^1 \wedge e^4 + e^2 \wedge e^5) \quad (\text{IV.269})$$

$$w = (e^1 + ie^4) \wedge (e^2 + ie^5) \quad (\text{IV.270})$$

$$\theta = e^3 + ie^6, \quad (\text{IV.271})$$

and (IV.51) also imposes

$$dj|_D = 0 \quad H|_D = 0. \quad (\text{IV.272})$$

However, using the same expansion of  $dj$  and  $H$  (IV.240), (IV.241), the domain-wall BPSness violation now imposes

$$g + if = -2re^4 \wedge e^5. \quad (\text{IV.273})$$

The gauge BPSness sets the RR-fluxes to be

$$e^\phi * F_1 = H \wedge \text{Re}w - id(2A - \phi) \wedge \text{Im}w \wedge \theta \wedge \bar{\theta} \quad (\text{IV.274})$$

$$*F_3 = -e^{-4A} d(e^{4A-\phi} \text{Re}w) \quad (\text{IV.275})$$

$$F_5 = 0, \quad (\text{IV.276})$$

and the violation of the D-string BPSness takes the form

$$d(e^{2A-\phi} \text{Im}w) = 0 \quad (\text{IV.277})$$

$$e^{2A-\phi} [H \wedge \text{Im}w - \frac{i}{2} d(\text{Re}w \wedge \theta \wedge \bar{\theta})] = 4\pi\sqrt{\alpha'} e^A (R_1 \hat{\alpha}_1 dy^1 + R_2 \hat{\alpha}_2 dy^2) \wedge j \quad (\text{IV.278})$$

with

$$j = -4e^{3A-\phi} \text{vol}_{\mathcal{B}_4}. \quad (\text{IV.279})$$

Turning to the equations of motion, the NS-field equation is

$$e^{3A-\phi} \text{Im} \langle \theta, d[e^{A-\phi} r^* \tilde{\Theta}] \rangle + 2d[e^{5A-2\phi} \langle \text{Im}w, \hat{\alpha}^a \iota_a \tilde{\Theta} \rangle] = 0 \quad a = 1, 2, \quad (\text{IV.280})$$

with  $\tilde{\Theta} = \text{Re}w - \text{Re}w|_\Sigma$ . Specifying the internal Einstein equations (IV.103) to this case gives a rather long and not particularly enlightening expression, so we just give here the following conditions

$$d[e^{3A-\phi} \hat{\alpha}^a \iota_a (i \text{Im}w \wedge \theta \wedge \bar{\theta})] = 0 \quad (\text{IV.281})$$

$$d[e^{A-\phi} r^* i \text{Im}w \wedge \theta \wedge \bar{\theta}] = 0 \quad (\text{IV.282})$$

$$d[e^{A-\phi}r^*\tilde{\Theta}] = 0 \quad (\text{IV.283})$$

which are stronger than the internal Einstein equations, but are reasonable conditions to impose on such SU(2) backgrounds, and which guarantee that the internal Einstein equations are satisfied. The example background that we construct below will satisfy these conditions.

We consider here again a factorisable six-torus

$$ds^2 = e^{2A} ds_{\mathbb{R}_{1,3}}^2 + ds_M^2 \quad (\text{IV.284})$$

$$ds_M^2 = \alpha'(2\pi)^2 \left\{ e^{2A} [R_1^2 (dy^1)^2 + R_2^2 (dy^2)^2] + e^{-2A} \sum_{j=3}^6 R_j^2 (dy^j)^2 \right\}, \quad (\text{IV.285})$$

and as before we take the two-torus spanned by  $y^1$  and  $y^2$  to be the fibre  $\Sigma$  over the four-torus base spanned by  $y^3, y^4, y^5, y^6$ . We also consider the fibre to be wrapped by  $n_{O5}$  O5-planes and  $n_{D5}$  D5-branes. The SU(2) structure is again

$$j = -\alpha'(2\pi)^2 (R_1 R_4 dy^1 \wedge dy^4 + R_2 R_5 dy^2 \wedge dy^5) \quad (\text{IV.286})$$

$$w = \alpha'(2\pi)^2 (e^A R_1 dy^1 + i e^{-A} R_4 dy^4) \wedge (e^A R_2 dy^2 + i e^{-A} R_5 dy^5) \quad (\text{IV.287})$$

$$\theta = 2\pi \sqrt{\alpha'} e^{-A} (R_3 dy^3 + i R_6 dy^6). \quad (\text{IV.288})$$

From (IV.268), we see that the domain-wall BPSness violation sets again

$$e^{2A-\phi} = \text{const.} \quad (\text{IV.289})$$

However, we now consider the following NS-field

$$H = (2\pi)^2 \alpha' [N_{\text{NS1}} dy^3 \wedge dy^4 \wedge dy^6 + N_{\text{NS2}} dy^3 \wedge dy^5 \wedge dy^6 + N_{\text{NS3}} dy^4 \wedge dy^5 \wedge dy^6] \quad N_{\text{NS1},2,3} \in \mathbb{Z}, \quad (\text{IV.290})$$

and we see that (IV.272) are still satisfied. We now have  $f = u = 0$  and  $g = \frac{i\pi\sqrt{\alpha'}}{R_6} e^A N_{\text{NS3}} dy^4 \wedge dy^5$ , and the domain-wall BPSness violation (IV.273) now reads

$$g + if = -4\pi r \sqrt{\alpha'} e^{-2A} R_4 R_5 dy^4 \wedge dy^5 \quad (\text{IV.291})$$

with

$$r = -\frac{ie^{3A} N_{\text{NS3}}}{8\pi \sqrt{\alpha'} R_4 R_5 R_6}. \quad (\text{IV.292})$$

The additional DWSB contribution, which happens through the new components of the NS-flux in (IV.290), doesn't modify the violation of the D-string BPSness: (IV.277) is

again identically satisfied, while (IV.278) reduces to

$$e^{2A-\phi} H \wedge \text{Im} w = 4\pi\sqrt{\alpha'} e^A (R_1 \hat{\alpha}_1 dy^1 + R_2 \hat{\alpha}_2 dy^2) \wedge j \quad (\text{IV.293})$$

which is satisfied with

$$\hat{\alpha}_1 = -\frac{e^{2A} N_{\text{NS1}}}{16\pi\sqrt{\alpha'} R_3 R_4 R_6} \quad (\text{IV.294})$$

$$\hat{\alpha}_2 = -\frac{e^{2A} N_{\text{NS2}}}{16\pi\sqrt{\alpha'} R_3 R_5 R_6}. \quad (\text{IV.295})$$

The gauge BPSness sets the RR-fluxes to

$$F_1 = e^{2A-\phi} \left[ \frac{R_4 N_{\text{NS2}}}{R_3 R_5 R_6} dy^4 - \frac{R_5 N_{\text{NS1}}}{R_3 R_4 R_6} dy^5 - \frac{R_3 N_{\text{NS3}}}{R_4 R_5 R_6} dy^3 \right] \quad (\text{IV.296})$$

$$F_3 = e^{2A-\phi} *_B de^{-4A} \quad (\text{IV.297})$$

The RR-flux quantisation implies that we must have

$$e^{2A-\phi} \frac{R_5 N_{\text{NS1}}}{R_3 R_4 R_6} = N_{\text{R1}} \quad e^{2A-\phi} \frac{R_4 N_{\text{NS2}}}{R_3 R_5 R_6} = N_{\text{R2}} \quad e^{2A-\phi} \frac{R_3 N_{\text{NS3}}}{R_4 R_5 R_6} = N_{\text{R3}} \quad (\text{IV.298})$$

with  $N_{\text{R1}}, N_{\text{R2}}, N_{\text{R3}} \in \mathbb{Z}$ . The RR Bianchi identities then reduce to

$$-\hat{\nabla}_{\mathcal{B}_4}^2 e^{-4A} = \frac{1}{e^{2A-\phi} (2\pi)^2 \alpha' \prod_{a=3}^6 R_a} \left[ \sum_{k=1}^3 N_{\text{NS}k} N_{\text{R}k} + \sum_{i \in \text{D5's, O5's}} q_i \delta_{\mathcal{B}_4}^4(y_i) \right] \quad (\text{IV.299})$$

with again  $q_{\text{D5}} = -q_{\text{O5}} = 1$ . Integrating this condition on the base yields the following tadpole condition

$$\sum_{k=1}^3 N_{\text{NS}k} N_{\text{R}k} + n_{\text{D5}} = 16. \quad (\text{IV.300})$$

Turning to the equations of motion, the conditions (IV.280)–(IV.283) are obeyed, provided that  $e^{2A-\phi} = \text{constant}$ . The NS-field equation and internal Einstein equations are therefore satisfied, as discussed above.

#### IV.2.4 Discussion

In this section, we investigated some corners of the landscape of non-supersymmetric flux vacua, in the light of generalised complex geometry. In generalised complex geometry the conditions for warped backgrounds with  $\mathcal{N} = 1$  supersymmetry can be expressed as differential equations on polyforms, called pure spinors, which, in turn, are seen as generalised calibration conditions for various types of D-branes wrapping cycles on the internal manifold: D-string, domain-wall, or space-filling branes, from the four-dimensional per-

spective.

This interpretation provides different controlled ways of breaking supersymmetry in ten-dimensions by violating the BPSness of D-string, domain-wall, or space-filling probe branes. The philosophy is to look at non-supersymmetric solutions via a two-step procedure: one first solves the deformed supersymmetry equations, which are still first order equations, and then look at the extra constraints that the equations of motion impose.

This approach was first used in [44], where non-supersymmetric solutions were found by deforming the domain-wall calibration condition. Our work extends [44] to new classes of solutions obtained by deforming the D-string BPS condition or both the domain-wall and D-string ones.

To solve the ten-dimensional equations of motion, it proves convenient to use an effective four-dimensional potential obtained by integrating the ten-dimensional supergravity action on the internal manifold. The effective potential can be expressed in terms of pure spinors as an integral over the compact space. Then the equations of motion are obtained by varying it with respect to the physical fields and are also given in terms of the pure spinors. The equations of motion are rather convoluted in the general case and are still to be studied. However, by choosing specific classes of supersymmetry breaking, they become tractable. We were able to solve them for a variety of concrete type II  $SU(2)$  and  $SU(3)$  backgrounds respecting (IV.45).

When studying the effective potential for the SSB backgrounds, we witnessed the presence of some terms belonging to vector representations of the  $SU(3) \times SU(3)$  structure, which are believed to be massive modes from the effective point of view. In that sense, our classes of backgrounds describe a set of fully ten-dimensional non-supersymmetric solutions of type II supergravity. It is not clear whether they admit a proper four-dimensional interpretation. It would be interesting to apply the recently developed exceptional generalised geometry techniques [80] to study potential consistent truncations of such backgrounds.

However, we showed that our class of background shares a property with the one-parameter DWSB class introduced in [44], namely the existence of a ‘truncation’ dictated by the geometry such that the ‘off-shell’ effective potential is positive semi-definite, and vanishes at the solutions. By ‘truncation’ we meant that we only consider the off-shell deformations of the potential which are compatible with the generalised foliation of the internal space, so it is by no means a rigorous truncation to a finite set of modes. We therefore present this as an interesting property of our class of backgrounds, but this is a weaker statement than arguing for the perturbative stability of our backgrounds, since we have no control over the modes we are ‘truncating’ away and keeping in our ‘off-shell’ potential.

We carried the same analysis, from the effective potential and the derivation of the

equations of motion, to the construction of concrete backgrounds, for another class of vacua, which have the same supersymmetry breaking term violating the D-string BPSness, but supplemented with the one-parameter DWSB contribution violating the domain-wall BPSness. Each supersymmetry breaking term brings different contributions to the effective potential and therefore to the equations of motion, which can be solved separately, which we did for several explicit  $SU(2)$  and  $SU(3)$  background constructions.

One obvious extension of this work would be to consider different patterns of the D-string BPSness violation, depending on the generalised current associated to the background D-branes or not, and to look for explicit solutions.

Another natural extension of the present work is to carry the constructions of the analogous Heterotic backgrounds, where the building block of the supersymmetry breaking term entering the modified D-string condition could now be, for instance, the base volume-form of some elliptically fibered internal manifolds like the ones discussed in [98].

Another interesting direction would be to consider solutions of ten-dimensional supergravity violating the D-string BPSness, but without any vector modes under the  $SU(3) \times SU(3)$  structure, for which it would probably be possible to get a deeper understanding of their associated effective theories. In the case of vacua with an external space being Minkowski, such a D-string BPSness violation should be accompanied by a domain-wall BPSness violation, otherwise there would only be positive contributions to the effective potential. One could then hope to interpret the associated effective theories as, for instance, solutions of four-dimensional  $\mathcal{N} = 1$  supergravity with non-vanishing D-terms, F-terms and superpotential, extending the supersymmetric analysis of [7]. We will come back to these questions in the next section.

Finally, the techniques from exceptional generalised geometry can shed a different light on the study of non-supersymmetric equations of motion: by classifying the possible supersymmetry breaking terms in terms of different representations of the torsion of an appropriate generalised  $SU(7)$  structure, one can reformulate the equations of motion as first order differential conditions on these torsion representations. We will come back to this idea in the next chapter.

### IV.3 Generalised calibrations and D-term supersymmetry-breaking

Constructing supergravity solutions with D-terms is difficult. Indeed, the prototypical example of D-term supersymmetry breaking is the Fayet–Iliopoulos term, and realising its (field-dependent) embedding in supergravity at energy parametrically lower than the Planck scale turns out to be challenging [4, 5].

In this section, we investigate the possibility of having supergravity solutions with D-terms from a different angle, using the framework of generalised complex geometry. We

consider again solutions that are warped products of four-dimensional Minkowski space and an internal six-dimensional compact manifold, and we again break supersymmetry in a sort of controlled way: we consider first order differential equations which correspond to deformations of the pure spinor equations via the introduction of supersymmetry breaking terms, and then we impose some further requirements from the equations of motion which guarantee that we have actual solutions of supergravity.

We keep on focusing on the class of vacua that still admits BPS space-filling sources. This means that the BPS condition associated to the calibration condition of space-filling D-branes is preserved, while we allow for the conditions corresponding to the calibrations of string-like and domain-wall probe D-branes to be violated.

In the supersymmetric case, the gauge BPSness condition has been reformulated in [6], eliminating the explicit metric dependence, and introducing a generalised version of the Dolbeault operator. We will generalise this derivation to the case of non-supersymmetric vacua violating the string and domain-wall BPSness conditions.

Still in the  $\mathcal{N} = 1$  case, the gauge and domain-wall BPSness conditions have been identified with F-term conditions, while the string BPSness has been interpreted as a D-term condition for the dimensionally reduced four-dimensional  $\mathcal{N} = 1$  theory in [7, 8].

Within our class of non-supersymmetric backgrounds preserving the gauge BPSness, we will focus on those who can be dimensionally reduced to four-dimensional  $\mathcal{N} = 1$  supergravity solutions with non-vanishing F-terms, and possibly non-vanishing D-terms. To do so, we require that our set of modified pure spinor equations continues to have an interpretation in terms of either (D-) F-term or (D-) F-term conditions. In particular, the gauge BPSness should still be identified with an F-term condition.

Interestingly, this procedure constrains some supersymmetry breaking terms entering the modified D-string calibration condition, and therefore the possible D-terms. More precisely, we will see that the supersymmetry breaking terms set to zero by this requirement belong to vector representations of the  $SU(3) \times SU(3)$  structure defined by the pure spinors, just like the massive spin  $\frac{3}{2}$ -multiplet degrees of freedom of the four-dimensional theory [38, 40].

On another note, we will give the effective potential for our class of backgrounds, written as an integral over the internal compactification space, in the language of pure spinors. We will see that the requirement to interpret the gauge BPSness condition as an F-term condition results in the vanishing of some negative semi-definite contributions to the effective potential, which naturally fits within the four-dimensional  $\mathcal{N} = 1$  picture, given that these contributions do not originate from the superpotential.

We derive the equations of motion for our class of backgrounds dimensionally reducing to four-dimensional  $\mathcal{N} = 1$  supergravity solutions with non-vanishing F-terms and D-terms, and for a subclass where the D-term contributions to the effective potential are set

to zero on-shell, restoring the D-string BPSness. Solutions of this kind correspond to vacua with only F-terms, and include the amply discussed class of no-scale vacua [99–108]. The relative simplicity of these equations of motion compared to the general ten-dimensional type II equations of motion could be used to find new non-supersymmetric flux vacua.

Finally, the requirement to interpret the gauge BPSness condition as an F-term condition turns out to be rather restrictive on the possible D-terms, and we investigate this in a more concrete way for a class of  $SU(3)$  backgrounds with parallel BPS O5-planes.

The outline of this section is as follows. In subsection IV.3.1, we reformulate the calibration condition associated to space-filling sources in the non-supersymmetric case. In subsection IV.3.2, we briefly review the four-dimensional  $\mathcal{N} = 1$  interpretation of the BPS conditions in the GCG framework, and we introduce the general type II effective potential in the language of pure spinors. In subsection IV.3.3, we derive the constraint that the gauge BPSness should obey in order to be interpreted as an F-term condition, and we derive the equations of motion for backgrounds satisfying this constraint. In subsection IV.3.4, we apply this constraint to  $SU(3)$  backgrounds with parallel BPS O5-planes, and study the restriction it imposes on their possible D-terms.

### IV.3.1 Reformulating the gauge BPSness

Throughout this section, we consider non-supersymmetric solutions of type II supergravity, and we use first order differential equations on the pure spinors to describe them. These equations can be thought as modifications of the pure spinor equations (III.71), (III.72) and (III.73), to which we add supersymmetry breaking terms, which is a standard procedure in the GCG literature [44, 45, 92]. Unlike the supersymmetric case, these modified pure spinor equations are not equivalent to the equations of motion, and one has to also consider additional constraints to have genuine  $\mathcal{N} = 0$  backgrounds.

In this section, we will again only consider the case of  $\mathcal{N} = 0$  backgrounds which still respect the gauge BPSness (III.73), therefore admitting stable (BPS) space-filling sources. We therefore consider generically

$$d_H(e^{3A-\phi}\Psi_2) \neq 0 \tag{IV.301}$$

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) \neq 0 \tag{IV.302}$$

$$d_H(e^{4A-\phi}\text{Re}\Psi_1) = e^{4A}\tilde{*}F. \tag{IV.303}$$

It is more convenient to consider a different reformulation of the space-filling calibration condition (IV.303), which eliminate its implicit dependence on the metric coming from the Hodge operator, and express the RR field-strength itself in terms of the pure spinors.

In the supersymmetric case, this reformulation was derived in [6]:

$$F = -d_H^{\mathcal{J}_2}(e^{-\phi}\text{Re}\Psi_1) \quad (\text{IV.304})$$

with the introduction of the following differential operator

$$d^{\mathcal{J}} = e^{-\frac{\pi}{2}\mathcal{J}} \cdot d e^{\frac{\pi}{2}\mathcal{J}}, \quad (\text{IV.305})$$

where  $\cdot$  denotes the action of the (almost) generalised complex structure on polyforms (III.45). When  $\mathcal{J}_2$  is integrable, which is the case for  $\mathcal{N} = 1$  backgrounds, this differential reduces to

$$d^{\mathcal{J}} = [d, \mathcal{J}]. \quad (\text{IV.306})$$

It is worth stressing that the derivation of (IV.304) relies on (III.72) being satisfied.

Still in the supersymmetric case, a second reformulation of the gauge BPSness has been presented for instance in [7], and requires the introduction of the following complexification of the RR field-strength:

$$G = F + id_H(e^{-\phi}\text{Re}\Psi_1). \quad (\text{IV.307})$$

Decomposing (III.73) on the generalised Hodge diamond, and using again (III.72), it can be shown that (III.73) is equivalent to

$$G|_{V_{-1}} = 0 \quad G|_{V_{-3}} = 0, \quad (\text{IV.308})$$

where the subscript  $V_n$  denotes the projection onto the  $in$ -eigenspace of  $\mathcal{J}_2$ . The second equation is actually just

$$F|_{V_{-3}} = 0, \quad (\text{IV.309})$$

since the second term in  $G$  is the derivative of a polyform of null charge under  $\mathcal{J}_2$ , which cannot have  $V_{-3}$  components, from (III.55). This reformulation is particularly useful to interpret (III.73) as an F-term condition in the four-dimensional theory, as discussed in [7].

Let us now derive similar reformulations of the gauge BPSness (IV.303) in the non-supersymmetric case, where the domain-wall and D-string BPSness conditions are relaxed.

We generalise the derivation of [6] in this case, and after some long but straightforward calculations we find

$$F = -d_H^{\mathcal{J}_2}(e^{-\phi}\text{Re}\Psi_1) + e^{-2A}\mathcal{J}_1 \cdot d_H^{\mathcal{J}_2}(e^{2A-\phi}\text{Im}\Psi_1). \quad (\text{IV.310})$$

The second term in this expression explicitly shows the relationship between the additional



RR flux components and the violation of the D-string BPSness (IV.302). Moreover, the supersymmetry breaking terms that appear in (IV.301), which are in the  $V_0$  bundle, prevent the generalised complex structure  $\mathcal{J}_2$  from being integrable. As a consequence, recall that the exterior derivative property (III.55) is generalised to

$$d_H: \Gamma(V_k) \rightarrow \Gamma(V_{k-3}) \oplus \Gamma(V_{k-1}) \oplus \Gamma(V_{k+1}) \oplus \Gamma(V_{k+3}), \quad (\text{IV.311})$$

which also results in additional RR field-strength components compared to the supersymmetric case.

From this reformulation of (IV.303), one can derive the non-supersymmetric analogue of (IV.308), by decomposing (IV.310) on the generalised Hodge diamond.

Looking first at the  $V|_{-3}$  component, we now have

$$F|_{V_{-3}} = id_H(e^{-\phi} \text{Re}\Psi_1)|_{V_{-3}}. \quad (\text{IV.312})$$

The  $V|_{-1}$  component is

$$F|_{V_{-1}} = -id_H(e^{-\phi} \text{Re}\Psi_1)|_{V_{-1}} + ie^{-2A} \mathcal{J}_1 \cdot d_H(e^{2A-\phi} \text{Im}\Psi_1)|_{V_{-1}}. \quad (\text{IV.313})$$

These can be rearranged into

$$G|_{V_{-1}} = ie^{-2A} \mathcal{J}_1 \cdot d_H(e^{2A-\phi} \text{Im}\Psi_1)|_{V_{-1}} \quad (\text{IV.314})$$

$$G|_{V_3} = 0, \quad (\text{IV.315})$$

which again is the most convenient form to make contact with the four-dimensional  $\mathcal{N} = 1$  effective theory. Indeed, an advantage of reformulating the gauge BPSness in this way for non-supersymmetric backgrounds is that it highlights the relationship between the complexified RR flux and the D-string BPSness violation (IV.302). This is particularly interesting to discuss solutions dimensionally reducing to solutions of four-dimensional  $\mathcal{N} = 1$  supergravity, since the former enters in the corresponding on-shell superpotential, and the latter has been identified as the D-term contribution in four-dimensional  $\mathcal{N} = 1$  supergravity [7]. We will address the four-dimensional  $\mathcal{N} = 1$  supergravity interpretation in more details in subsection IV.3.3.

Finally, these derivations of the non-supersymmetric version of the gauge BPSness will prove to be insightful when we will turn to the study of the intrinsic torsion of the generalised SU(7) structure associated to non-supersymmetric flux backgrounds. Indeed, these will highlight the interplay between different irreducible representations of the SU(7) intrinsic torsion, in a sense that will be made precise in chapter V.

### IV.3.2 4D structure and effective potential from pure spinors

In this section we briefly review  $\mathcal{N} = 1$  Minkowski solutions of type II supergravity discussed in [7]. More precisely, we discuss the interpretation of the pure spinor equations (III.71), (III.72) and (III.73) as the vanishing of some D-terms, F-terms and superpotential.

We then write again the most general four-dimensional ‘effective potential’ from the ten-dimensional type II supergravity action, following [44]. Recall that calling these scalar functions ‘effective potentials’ is a bit misleading, since we write them as integral over the internal space, without choosing a specific truncation for the ten-dimensional modes and performing the actual dimensional reduction to write down a genuine scalar potential for the associated effective theories. However, in doing so we are able to interpret the different terms in the closed string sector of this effective potential as contributions from F-terms, D-terms and a superpotential.

In the next section, a similar interpretation in the non-supersymmetric case will motivate some constraints that the modified pure spinor equations should obey in order to be compatible with a four-dimensional  $\mathcal{N} = 1$  supergravity description with non-vanishing F-terms and D-terms.

#### a) Four-dimensional $\mathcal{N} = 1$ supergravity

We briefly review here the work of [7], which provides ten-dimensional expressions for the four-dimensional  $\mathcal{N} = 1$  supergravity superpotential, D- and F-term conditions. The expression obtained in [7] for the superpotential can be thought as the generalisation of the well-known Gukov-Vafa-Witten superpotential [109] in the GCG formalism.

We review these notions in order to then move on to the description of the effective potential for non-supersymmetric solutions of four-dimensional  $\mathcal{N} = 1$  supergravity, with non-vanishing D- and F-terms. We introduce the following rescaled pure spinors

$$t = e^{-\phi}\Psi_1 \quad \mathcal{Z} = e^{3A-\phi}\Psi_2, \quad (\text{IV.316})$$

as well as

$$\mathcal{T} = \text{Re}t - iC, \quad (\text{IV.317})$$

with  $F = F^{\text{bg}} + d_H C$  and  $F^{\text{bg}}$  some fixed non-trivial background flux. Both  $\mathcal{Z}$  and  $\mathcal{T}$  are chiral fields of the associated four-dimensional  $\mathcal{N} = 1$  description. The polyform  $\mathcal{T}$  is defined such that the complexified flux  $G$  is its field-strength  $G = id_H \mathcal{T}$ .

We can now define the following superpotential and conformal Kähler potential den-

sities

$$W = \pi(-1)^{|\mathcal{Z}|+1} \langle \mathcal{Z}, G \rangle \quad N = \frac{i\pi}{2} \langle \mathcal{Z}, \bar{\mathcal{Z}} \rangle^{1/3} \langle t, \bar{t} \rangle^{2/3}, \quad (\text{IV.318})$$

where  $|\mathcal{Z}|$  is the degree (mod 2) of  $\mathcal{Z}$ , which depend on both chiral fields  $\mathcal{Z}$  and  $\mathcal{T}$ , and the associated superpotential and conformal Kähler potential

$$\mathcal{W} = \int_M W \quad \mathcal{N} = \int_M N. \quad (\text{IV.319})$$

Consider now a chiral field  $X$  on which the superpotential and the conformal Kähler potential depend. Under a holomorphic variation  $\delta X$  of  $X$  we define

$$(\delta\mathcal{W})_X := \delta_X \mathcal{W} - 3(\delta_X \log \mathcal{N}) \mathcal{W}. \quad (\text{IV.320})$$

We evaluate the variations associated to the two chiral fields at our disposal,  $\mathcal{Z}$  and  $\mathcal{T}$ . First for  $\mathcal{Z}$ , we distinguish two contributions coming from two distinct holomorphic deformations of  $\mathcal{Z}$

$$\delta\mathcal{Z} \in (V_1 \oplus V_3), \quad (\text{IV.321})$$

which we denote

$$(\delta\mathcal{W})_{\mathcal{Z}_{(1)}} \quad \text{and} \quad (\delta\mathcal{W})_{\mathcal{Z}_{(3)}} \quad (\text{IV.322})$$

respectively. They yield

$$(\delta\mathcal{W})_{\mathcal{Z}_{(1)}} = (-1)^{|\mathcal{Z}|+1} \pi \int_M \langle \delta\mathcal{Z}_{(1)}, G|_{V_{-1}} \rangle \quad (\text{IV.323})$$

$$(\delta\mathcal{W})_{\mathcal{Z}_{(3)}} = -\pi \int_M \left\langle \delta\mathcal{Z}_{(3)}, (-1)^{|\mathcal{Z}|} G|_{V_{-3}} + \frac{3i}{2} \frac{\mathcal{W}}{\mathcal{N}} e^{-4A} \bar{\mathcal{Z}} \right\rangle. \quad (\text{IV.324})$$

For  $\mathcal{T}$ , we find

$$(\delta\mathcal{W})_{\mathcal{T}} = i\pi \int_M \left\langle \delta\mathcal{T}, (-1)^{|\mathcal{Z}|} d_H \mathcal{Z} + 3i \frac{\mathcal{W}}{\mathcal{N}} e^{2A} \text{Im}t \right\rangle. \quad (\text{IV.325})$$

Imposing the vanishing of the variations (IV.323)-(IV.325) reproduces the Anti de Sitter version of the supersymmetric pure spinor equations (III.71) and (IV.308). If we also impose that the superpotential itself vanishes, they reduce to the corresponding equations in flat space

$$d_H \mathcal{Z} = 0 \quad G|_{V_{-1}} = 0 \quad G|_{V_{-3}} = 0. \quad (\text{IV.326})$$

One can therefore interpret the two  $\mathcal{N} = 1$  supersymmetry conditions (III.71) and

(IV.308) as F-term conditions and the vanishing of the superpotential

$$\mathcal{W} = 0 \quad (\delta\mathcal{W})_{\mathcal{Z}_{(1)}} = 0 \quad (\delta\mathcal{W})_{\mathcal{Z}_{(3)}} = 0 \quad (\delta\mathcal{W})_{\mathcal{T}} = 0. \quad (\text{IV.327})$$

On another note, the parametrisation in terms of the chiral fields  $\mathcal{T}$  and  $\mathcal{Z}$  has some redundancy, due to the RR gauge transformations  $\delta_\lambda C = d_H \lambda$  resulting in  $\delta_\lambda \mathcal{T} = -id_H \lambda$ . These symmetries are gauged in the effective theory, and their associated D-terms have been worked out in [7], yielding

$$\mathcal{D}(\lambda) = 2\pi \int_M \langle \lambda, \mathcal{D} \rangle \quad (\text{IV.328})$$

with

$$\mathcal{D} = d_H(e^{2A} \text{Im}t), \quad (\text{IV.329})$$

the D-term density. The last  $\mathcal{N} = 1$  supersymmetry condition (III.72) can thus be interpreted as the vanishing of these D-terms

$$\mathcal{D} = 0, \quad (\text{IV.330})$$

completing the four-dimensional picture.

One can also write down the corresponding covariant derivatives of the superpotential density. For a chiral field  $X$

$$\mathcal{D}_X W \equiv \partial_X W - 3(\partial_X \log N)W. \quad (\text{IV.331})$$

In the case of  $\mathcal{N} = 1$  Minkowski solutions, they are<sup>23</sup>

$$\mathcal{D}_{\mathcal{Z}_{(1)}} W = \pi(-1)^{|\mathcal{Z}|+1} G|_{V_{-1}} \quad (\text{IV.332})$$

$$\mathcal{D}_{\mathcal{Z}_{(3)}} W = \pi(-1)^{|\mathcal{Z}|+1} G|_{V_{-3}} \quad (\text{IV.333})$$

$$\mathcal{D}_{\mathcal{T}} W = i\pi(-1)^{|\mathcal{Z}|} d_H \mathcal{Z}. \quad (\text{IV.334})$$

## b) The type II effective potential

In this subsection, we recall the four-dimensional effective action for backgrounds with ten-dimensional space-time of the form  $X_4 \times M$  with the metric

$$ds_{10}^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{IV.335})$$

---

<sup>23</sup>We deliberately give the expressions with vanishing superpotential in order to make contact with the on-shell  $\mathcal{N} = 1$  Minkowski effective potential discussed in the next subsection.

where  $g_{\mu\nu}$  is for now a general four-dimensional metric depending only on the external coordinates, and all the other fields depend only on the internal coordinates.

The effective four-dimensional action is<sup>24</sup>

$$S_{\text{eff}} = \int_{X_4} d^4x \sqrt{-g_4} \left( \frac{1}{2} \mathcal{N} R_4 - 2\pi \mathcal{V}_{\text{eff}} \right), \quad (\text{IV.336})$$

where  $R_4$  is the four-dimensional scalar curvature, and

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \int_M \text{vol}_6 e^{4A} \left\{ e^{-2\phi} \left[ -\mathcal{R} + \frac{1}{2} H^2 - 4(d\phi)^2 + 8\nabla^2 A + 20(dA)^2 \right] - \frac{1}{2} \tilde{F}^2 \right\} \\ & + \sum_{i \in \text{loc. sources}} \tau_i \left( \int_{\Sigma_i} e^{4A-\phi} \sqrt{\det(g|_{\Sigma_i} + \mathcal{F}_i)} - \int_{\Sigma_i} C^{\text{el}}|_{\Sigma_i} \wedge e^{\mathcal{F}_i} \right) \end{aligned} \quad (\text{IV.337})$$

is the type II effective potential density, with  $\mathcal{R}$  the six-dimensional scalar curvature. Its first line corresponds to the closed string sector, while the second line is the localised sources contributions. For the O-planes, we set  $\mathcal{F} = 0$ , and we have  $\tau_{D_p} = 1$ ,  $\tau_{O_q} = -2^{q-5}$ . The sources couple to the RR potentials defined by  $d_H C^{\text{el}} = e^{4A} \tilde{F}$ .

The variations of the four dimensional action (IV.336) exactly reproduce the ten-dimensional equations of motion, as argued in [44]. They are given in Appendix A. Moreover, from the variation with respect to  $g_{\mu\nu}$ , one gets that the external space is Einstein, with

$$R_4 = 8\pi \mathcal{V}_{\text{eff}} / \mathcal{N}, \quad (\text{IV.338})$$

as expected. From the ten-dimensional perspective this is equivalent to the internal space integral of the external ten-dimensional Einstein equation's trace. As we consider Minkowski backgrounds, we focus on the cases where the effective potential vanishes at the solutions.

Finally, the RR equations of motion reproduce the usual Bianchi identities

$$d_H F = -j_{\text{tot}} = - \sum_i \tau_i j_i, \quad (\text{IV.339})$$

where, as described in [44] for instance, the  $j_i$  are the generalised currents for the localised sources.

From now on we will use the rewriting of the effective potential in terms of the pure spinors, derived in [44]:

$$\mathcal{V}_{\text{eff}} = \frac{1}{2} \int_M \text{vol}_6 \left[ |d_H(e^{2A-\phi} \text{Im} \Psi_1)|^2 + e^{-2A} |d_H(e^{3A-\phi} \Psi_2)|^2 \right]$$

---

<sup>24</sup>We use the convention  $2\pi\sqrt{\alpha'} = 1$ , so that all D-brane tensions are equal. We are also neglecting anomalous curvature-like corrections to the sources contribution: they can be easily added without affecting the results of the discussion. Finally we are also omitting the internal field kinetic terms, since they are taken to be constant along the external directions.

$$\begin{aligned}
 & + \frac{1}{2} \int_M \text{vol}_6 e^{4A} |\tilde{*}F - e^{-4A} d_H(e^{4A-\phi} \text{Re}\Psi_1)|^2 \\
 & - \frac{1}{4} \int_M e^{-2A} \left( \frac{|\langle \Psi_1, d_H(e^{3A-\phi} \Psi_2) \rangle|^2}{\text{vol}_6} + \frac{|\langle \bar{\Psi}_1, d_H(e^{3A-\phi} \Psi_2) \rangle|^2}{\text{vol}_6} \right) \\
 & - 4 \int_M \text{vol}_6 e^{4A-2\phi} [(u_R^1)^2 + (u_R^2)^2] \\
 & + \sum_{i \subset \text{D-branes}} \tau_i \int_M e^{4A-\phi} (\text{vol}_6 \rho_i^{\text{loc}} - \langle \text{Re}\Psi_1, j_i \rangle) \\
 & + \int_M \langle e^{4A-\phi} \text{Re}\Psi_1 - C^{\text{el}}, d_H F + j_{\text{tot}} \rangle. \tag{IV.340}
 \end{aligned}$$

The square of a polyform is defined in Appendix A, and we have

$$u_R^{1,2} = u_{Rm}^{1,2} dy^m \equiv (u_m^{1,2} + u_m^{*1,2}) dy^m, \tag{IV.341}$$

with

$$u_m^1 = \frac{i \langle \gamma_m \bar{\Psi}_1, d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle}{e^{2A-\phi} \langle \Psi_1, \bar{\Psi}_1 \rangle} + \frac{\langle \gamma_m \bar{\Psi}_2, d_H(e^{3A-\phi} \Psi_2) \rangle}{2e^{3A-\phi} \langle \Psi_2, \bar{\Psi}_2 \rangle} \tag{IV.342}$$

$$u_m^2 = \frac{i(-1)^{|\Psi_2|} \langle \Psi_1 \gamma_m, d_H(e^{2A-\phi} \text{Im}\Psi_1) \rangle}{e^{2A-\phi} \langle \Psi_1, \bar{\Psi}_1 \rangle} + \frac{(-1)^{|\Psi_1|} \langle \bar{\Psi}_2 \gamma_m, d_H(e^{3A-\phi} \Psi_2) \rangle}{2e^{3A-\phi} \langle \Psi_2, \bar{\Psi}_2 \rangle}. \tag{IV.343}$$

The gamma matrix conventions are given in Appendix A. We also introduced here the Born-Infeld density  $\rho_i^{\text{loc}}$  associated with a source wrapping a generalised submanifold  $(\Sigma_i, \mathcal{F}_i)$

$$\rho_i^{\text{loc}} = \frac{\sqrt{\det(g|_{\Sigma_i} + \mathcal{F}_i)}}{\sqrt{\det g}} \delta(\Sigma_i). \tag{IV.344}$$

It is useful to rewrite the algebraic inequality (IV.6) in terms of  $\rho^{\text{loc}}$ :

$$\rho_i^{\text{loc}} \geq \frac{\langle \text{Re}\Psi_1, j_i \rangle}{\text{vol}_6}, \tag{IV.345}$$

where the division by the volume form means that we remove the  $\text{vol}_6$  factor in the numerator.

Therefore, if one considers the effective potential of a (non-)supersymmetric background with calibrated sources, the above inequality is saturated and the fifth line of (IV.340) vanishes. Similarly, the last line of (IV.340) will vanish for solutions of ten-dimensional supergravity satisfying the Bianchi identities (IV.339).

In the case of an  $\mathcal{N} = 1$  Minkowski background, the closed-string sector of the on-shell effective potential can be rewritten as

$$\mathcal{V}_{\text{eff}} = \frac{1}{2} \int_M e^{4A} (|G_{-1}|^2 + |G_{-3}|^2) + e^{-2A} |d_H(\mathcal{Z})|^2 - \frac{e^{-2A+2\phi}}{8\text{vol}_6} |\langle \mathcal{Z}, G \rangle|^2$$

$$+ \frac{1}{2} \int_M |d_H(e^{2A-\phi} \text{Im}\Psi_1)|^2. \quad (\text{IV.346})$$

Though this is simply the  $\mathcal{N} = 1$  on-shell effective potential, this is the natural formulation to interpret each term as the vanishing of some D-terms, F-terms and superpotential.

The first three terms in the first line of (IV.346) can be identified with the covariant derivatives of the superpotential density (IV.332), (IV.333) and (IV.334) respectively, while the last term in the first line of (IV.346) can be identified with the superpotential density (IV.318). Finally, the last line of (IV.346) can be identified with the D-terms (IV.329).

We do not intend to make a rigorous identification with the usual four-dimensional  $\mathcal{N} = 1$  scalar potential here, we simply stress that each term in the closed-string sector of the effective potential of ten-dimensional  $\mathcal{N} = 1$  type II supergravity Minkowski solutions fits into the four-dimensional  $\mathcal{N} = 1$  supergravity description.

### IV.3.3 D-terms in generalised complex geometry

In this subsection we investigate the on-shell effective potential of non-supersymmetric solutions of type II supergravity with external Minkowski space. We identify some conditions that the modified pure spinor equations must satisfy in order for the non-supersymmetric solutions to have a clear interpretation in terms of four-dimensional  $\mathcal{N} = 1$  supergravity.

We also derive the equations of motion, in the language of pure spinors, associated to these backgrounds, with and without D-terms.

#### a) Effective potential and F-term conditions

Let us recall that we focus on non-supersymmetric solutions having only space-filling sources, in order to preserve the Poincaré symmetry of the external space, and that we consider only BPS sources. We also consider that our backgrounds satisfy the Bianchi identities (IV.61).

In this case, the type II effective potential is

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \frac{1}{2} \int_M \text{vol}_6 \left[ |d_H(e^{2A-\phi} \text{Im}\Psi_1)|^2 + e^{-2A} |d_H(e^{3A-\phi} \Psi_2)|^2 \right] \\ & - \frac{1}{4} \int_M e^{-2A} \left( \frac{|\langle \Psi_1, d_H(e^{3A-\phi} \Psi_2) \rangle|^2}{\text{vol}_6} + \frac{|\langle \bar{\Psi}_1, d_H(e^{3A-\phi} \Psi_2) \rangle|^2}{\text{vol}_6} \right) \\ & + \frac{1}{2} \int_M \text{vol}_6 e^{4A} |\tilde{*}F - e^{-4A} d_H(e^{4A-\phi} \text{Re}\Psi_1)|^2 \\ & - 4 \int_M \text{vol}_6 e^{4A-2\phi} [(u_R^1)^2 + (u_R^2)^2]. \end{aligned} \quad (\text{IV.347})$$

For non-supersymmetric Minkowski solutions, this potential still vanishes, but each term does not necessarily vanish identically.

We now want to restrict ourselves to the study of a subclass of backgrounds dimensionally reducing to a solution of  $\mathcal{N} = 1$  four-dimensional supergravity, in the sense that their scalar potential can be written as F-terms, D-terms, and superpotential contributions. From the ten-dimensional perspective, this means that we focus on backgrounds where each terms in the on-shell scalar potential (IV.347) has an interpretation in terms of the aforementioned contributions.

At this point the most pressing question to address is therefore the superpotential expression for such backgrounds.

As mentioned above, we consider backgrounds where supersymmetry is broken in a controlled way, as a perturbation around a certain supersymmetric backgrounds. For instance, we could think of the right-hand side contributions of the pure spinor equations (IV.301) and (IV.302) as controlled by some supersymmetry breaking parameters, whose vanishing would restore supersymmetry. The question now becomes: how is the superpotential in (IV.319) affected by switching on right-hand side contributions in (IV.301) and (IV.302).

In the supersymmetric case, the complexification of the RR potentials entering the superpotential is suggested by the coupling of a BPS space-filling D-brane to magnetic background fields [7, 8]. Indeed, the action of a space-filling D-brane wrapping a generalised cycle  $(\Sigma, \mathcal{F})$  can be written as<sup>25</sup>

$$S_{\text{D-brane}} = \int_{\Sigma} e^{4A-\phi} \sqrt{\det(g|_{\Sigma} + \mathcal{F})} \text{vol}_{\Sigma} - ie^{4A} C|_{\Sigma} \wedge e^{\mathcal{F}}, \quad (\text{IV.348})$$

with  $\text{vol}_{\Sigma}$  the volume form on the cycle  $\Sigma$ . The calibration of the space-filling D-brane imposes that

$$e^{4A-\phi} \sqrt{\det(g|_{\Sigma} + \mathcal{F})} \text{vol}_{\Sigma} = e^{4A-\phi} \text{Re}\Psi_1 \wedge e^{\mathcal{F}}, \quad (\text{IV.349})$$

so the resulting action is

$$S_{\text{D-brane}} = \int_{\Sigma} e^{4A} (e^{-\phi} \text{Re}\Psi_1 - iC) \wedge e^{\mathcal{F}} = \int_{\Sigma} e^{4A} \mathcal{T} \wedge e^{\mathcal{F}}, \quad (\text{IV.350})$$

putting in evidence  $\mathcal{T}$  as the natural complexification of the RR potentials.

We now consider the breaking of supersymmetry through switching on right-hand side contributions in (IV.301) and (IV.302). Crucially, the complexification of the RR flux entering the superpotential is unaltered. Indeed, this is because the supersymmetry breaking perturbations are such that the calibration of the space-filling D-branes (IV.349) is preserved, and thus the expression of the BPS space-filling D-brane action (IV.350)

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<sup>25</sup>The standard space-filling D-brane action is here Wick rotated to Euclidean space.



holds. The superpotential expression thus remains as in (IV.319), although it must be non-vanishing on-shell for non-supersymmetric Minkowski backgrounds, as it is the only negative contribution to the vanishing effective potential. The associated F-terms might not be vanishing either.

Non-supersymmetric backgrounds with BPS space-filling sources have been studied in the GCG literature, and the contributions to the scalar potential of these vacua have indeed been interpreted as F-terms and a superpotential contribution with the superpotential (IV.319) [44]. It is also interesting to mention that backgrounds respecting the algebraic calibration condition (IV.349) but not the differential one (IV.303) are successfully described using the superpotential (IV.319) in [110].

Generically we have no way to control the superpotential expression for non-supersymmetric backgrounds with non-BPS sources dimensionally reducing to solutions of four-dimensional  $\mathcal{N} = 1$  supergravity. We thus don't further address this case.

Coming back to the scalar potential (IV.347), the BPSness of the space-filling sources (IV.303) implies that its third line must vanish on-shell. In the supersymmetric case, this corresponds to the F-term conditions resulting from the vanishing of the variations  $(\delta\mathcal{W})_{\mathcal{Z}_{(1)}}$  and  $(\delta\mathcal{W})_{\mathcal{Z}_{(-3)}}$ . We investigate these conditions when the supersymmetry breaking contributions are switched on. Given that the BPSness of the space-filling sources is preserved on-shell, the third line of the effective potential (IV.347) vanishes and thus the F-term conditions from  $(\delta\mathcal{W})_{\mathcal{Z}_{(1)}}$  and  $(\delta\mathcal{W})_{\mathcal{Z}_{(-3)}}$  should still vanish.

Recall from the previous subsection that the variation of the superpotential with respect to  $\mathcal{Z}_{(1)}$  reads

$$(\delta\mathcal{W})_{\mathcal{Z}_{(1)}} = (-1)^{|\mathcal{Z}|+1} \pi \int_M \langle \delta Z_{(1)}, G|_{V_{-1}} \rangle. \quad (\text{IV.351})$$

The resulting F-term condition is therefore

$$G|_{V_{-1}} = 0. \quad (\text{IV.352})$$

However, in the case of generic domain-wall and D-string BPSness violation, (IV.303) is equivalent to

$$G|_{V_{-1}} = ie^{-2A} \mathcal{J}_1 \cdot d_H(e^{2A-\phi} \text{Im}\Psi_1)|_{V_{-1}} \quad (\text{IV.353})$$

$$G|_{V_3} = 0. \quad (\text{IV.354})$$

The F-term condition (IV.352) thus results in

$$\mathcal{J}_1 \cdot d_H(e^{2A-\phi} \text{Im}\Psi_1)|_{V_{-1}} = 0. \quad (\text{IV.355})$$

This is equivalent to

$$d_H(e^{2A-\phi}\text{Im}\Psi_1)|_{S_{2,-1}} = d_H(e^{2A-\phi}\text{Im}\Psi_1)|_{S_{-2,-1}} = 0, \quad (\text{IV.356})$$

where the sub-spaces  $S_{p,q}$  are defined in (III.61). Given that the polyform  $d_H(e^{2A-\phi}\text{Im}\Psi_1)$  is real, it implies

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) \subset U_0, \quad (\text{IV.357})$$

where  $U_n$  is the  $in$ -eigenspace of  $\mathcal{J}_1$ . Backgrounds which don't respect (IV.357) fall off of the class of solutions dimensionally reducing to solutions of four-dimensional  $\mathcal{N} = 1$  supergravity with the superpotential (IV.319). The effective theories associated to such solutions could be described as solutions of four-dimensional  $\mathcal{N} = 1$  supergravity with a different superpotential<sup>26</sup>, or could be described with a fake superpotential. Alternatively, it might not even be sensible to talk about effective theories associated to these ten-dimensional backgrounds, or it could be that their effective theories are non-supersymmetric with the field content of  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  supergravity with additional (massive) multiplets, or non-supersymmetric solutions of four-dimensional supergravities with higher supersymmetry. We don't address these possibilities further and for the rest of this section we focus on the backgrounds respecting (IV.357).

Interestingly, imposing the condition (IV.357) makes the last line of the potential (IV.347) vanish<sup>27</sup>

$$u_R^1 = u_R^2 = 0. \quad (\text{IV.358})$$

This last condition turns out to be crucial for the associated four-dimensional  $\mathcal{N} = 1$  supergravity, given that the last line of the potential (IV.347) is negative semi-definite and is not a superpotential contribution, so it must indeed vanish on-shell, which is guaranteed by (IV.358).

The modes set to zero by (IV.357) belong to vector representations under the  $SU(3) \times SU(3)$  structure, just like the massive spin  $\frac{3}{2}$ -multiplet degrees of freedom that appear when reducing the ten-dimensional theory to four-dimensions [38, 40], which are seen as non-physical degrees of freedom in the four-dimensional  $\mathcal{N} = 1$  supergravity and should be gauged away. It is then reasonable to interpret our condition (IV.357) as the requirement to keep only the four-dimensional  $\mathcal{N} = 1$  multiplets in the low-energy effective theories.

On another note, the violation of the domain-wall BPSness condition (IV.301) results in a non vanishing variation of the superpotential (IV.325). The corresponding contribution to the superpotential (the second term of the first line of (IV.347)) is therefore naturally interpreted as an F-term.

<sup>26</sup>And therefore not admitting BPS space-filling D-branes.

<sup>27</sup>This can be seen by imposing (IV.357) on the  $SU(3) \times SU(3)$  decomposition of the pure spinor equations given in Appendix C.2.

The condition (IV.357) ensures the vanishing of the F-term corresponding to the calibration of space-filling D-branes, but of course it does not guarantee that non-supersymmetric ten-dimensional solutions respecting this condition will have a four-dimensional  $\mathcal{N} = 1$  low-energy effective theory.

The type II supergravity solutions actually reducing to four-dimensional  $\mathcal{N} = 1$  models with vanishing D-terms automatically obey (IV.357), since then  $d_H(e^{2A-\phi}\text{Im}\Psi_1) = 0$ . This is the case for the GKP-like solutions of [3, 44] for example, but more generally it is true for backgrounds which reduce to no-scale models with non-vanishing F-terms and superpotential. In the GCG literature, the non-supersymmetric type II supergravity solutions with a supersymmetry breaking term violating the D-string BPSness (IV.302) do not respect the condition (IV.357), and therefore do not reduce to solutions of four-dimensional  $\mathcal{N} = 1$  supergravity [45, 92]. In the next section, we will investigate the constraint (IV.357) in a more concrete setting.

## b) Equations of motion

In this subsection we derive the equations of motion in the language pure spinors for the class of backgrounds discussed above, admitting BPS space-filling sources and respecting the condition (IV.357) as well as the Bianchi identities. We do so by requiring the vanishing of the variations of the effective potential with respect to the internal fields, which is equivalent to the ten-dimensional equations of motion given in Appendix A, (see [44]).

We consider the following potential<sup>28</sup>

$$\begin{aligned} \mathcal{V}_{\text{eff}} = & \frac{1}{2} \int_M e^{-2A} \langle \tilde{*}_6 [d_H(e^{3A-\phi}\Psi_2)], d_H(e^{3A-\phi}\bar{\Psi}_2) \rangle \\ & + \frac{1}{2} \int_M \langle \tilde{*}_6 [d_H(e^{2A-\phi}\text{Im}\Psi_1)], d_H(e^{2A-\phi}\text{Im}\Psi_1) \rangle \\ & - \frac{1}{4} \int_M e^{-2A} \left( \frac{|\langle \Psi_1, d_H(e^{3A-\phi}\Psi_2) \rangle|^2}{\text{vol}_6} + \frac{|\langle \bar{\Psi}_1, d_H(e^{3A-\phi}\Psi_2) \rangle|^2}{\text{vol}_6} \right). \end{aligned} \quad (\text{IV.359})$$

We introduce the polyforms

$$\Theta = e^{-2A} \tilde{*}_6 d_H(e^{3A-\phi}\Psi_2) + 2i(-1)^{|\Psi_1|} e^{A-\phi} (\bar{t}_1 \Psi_1 + \bar{t}_2 \bar{\Psi}_1) \quad (\text{IV.360})$$

$$\Xi = \tilde{*}_6 d_H(e^{2A-\phi}\text{Im}\Psi_1), \quad (\text{IV.361})$$

with

$$t_1 = 2(-1)^{|\Psi_1|} \frac{\langle d_H(e^{3A-\phi}\Psi_2), \Psi_1 \rangle}{\langle \Psi_1, \bar{\Psi}_1 \rangle} \quad t_2 = 2(-1)^{|\Psi_1|} \frac{\langle d_H(e^{3A-\phi}\Psi_2), \bar{\Psi}_1 \rangle}{\langle \Psi_1, \bar{\Psi}_1 \rangle}. \quad (\text{IV.362})$$

<sup>28</sup>The other terms in the effective potential give trivial contributions to the equations of motion since they are quadratic in quantities vanishing for the considered backgrounds.

The decompositions of  $\Theta$  and  $\Xi$  on the  $SU(3) \times SU(3)$  structure are given in Appendix C.2.

Then, varying the potential (IV.359) with respect to the dilaton, we find the following dilaton equation of motion

$$\text{Re}\{\langle e^{3A-\phi}\Psi_2, d_H\Theta \rangle\} + \langle e^{2A-\phi}\text{Im}\Psi_1, d_H\Xi \rangle = 0. \quad (\text{IV.363})$$

Note that, in our case, the solutions of the dilaton equation have an identically vanishing effective potential. The B-field equation of motion is

$$d\left[\text{Re}\{\langle e^{3A-\phi}\Psi_2, \Theta \rangle_3\} + \langle e^{2A-\phi}\text{Im}\Psi_1, \Xi \rangle_3\right] = 0. \quad (\text{IV.364})$$

We derive the internal component of the Einstein equation by varying the effective potential with respect to the internal metric. Given that the Hodge operator and the pure spinors depend implicitly on the metric, we use the following rules

$$\delta\sqrt{\det g} = -\frac{1}{2}\delta g^{mn}g_{mn}\sqrt{\det g} \quad (\text{IV.365})$$

$$\delta\langle \tilde{*}_6\chi_1, \chi_2 \rangle = \delta g^{mn}\left[\langle \tilde{*}_6\iota_m\chi_1, \iota_n\chi_2 \rangle - \frac{1}{2}g_{mn}\langle \tilde{*}_6\chi_1, \chi_2 \rangle\right] \quad (\text{IV.366})$$

$$\delta\Psi_i = -\frac{1}{2}\delta g^{mn}g_{k(m}dy^k \wedge \iota_n)\Psi_i \quad i = 1, 2. \quad (\text{IV.367})$$

We find the following internal Einstein equations

$$0 = \text{Re}\left\{\langle g_{k(m}dy^k \wedge \iota_n)(e^{3A-\phi}\Psi_2), d_H\Theta \rangle - \langle g_{k(m}dy^k \wedge \iota_n)d_H(e^{3A-\phi}\Psi_2), \Theta \rangle\right\} \\ + \langle g_{k(m}dy^k \wedge \iota_n)(e^{2A-\phi}\text{Im}\Psi_1), d_H\Xi \rangle - \langle g_{k(m}dy^k \wedge \iota_n)d_H(e^{2A-\phi}\text{Im}\Psi_1), \Xi \rangle. \quad (\text{IV.368})$$

These are equations of motion of backgrounds compatible with a four-dimensional supergravity solutions with non vanishing F-terms and D-terms.

A subclass of these backgrounds are those compatible with a four-dimensional supergravity solutions with non vanishing F-terms and vanishing D-terms. In this class we have

$$d_H(e^{2A-\phi}\text{Im}\Psi_1) = 0, \quad (\text{IV.369})$$

so obviously  $\Xi = 0$ .

However, the last equation of motion, the external component of the modified Einstein equation, must be discussed separately for the cases with and without D-terms.

The external component of the modified Einstein equation is equivalent to the van-

ishing of the following variation of the effective potential

$$\frac{\delta\mathcal{V}_{\text{eff}}}{\delta A} + 2\frac{\delta\mathcal{V}_{\text{eff}}}{\delta\phi} = 0, \quad (\text{IV.370})$$

and it is identically satisfied for backgrounds with calibrated sources, preserving the D-string BPSness (i.e. without D-terms) and satisfying the Bianchi identities, as shown in [44].

For backgrounds with non-vanishing D-terms, thus violating the D-string BPSness, we simply reduce the ten-dimensional equation (A.29) on our warped configurations

$$\nabla^m(e^{-2\phi}\nabla_m e^{4A}) = e^{4A}\tilde{F} \cdot \tilde{F} + e^{4A-\phi} \sum_{i \in \text{loc. sources}} \tau_i \rho_i^{\text{loc}}, \quad (\text{IV.371})$$

and rewrite it in terms of pure spinors as

$$\begin{aligned} -d(e^{-2\phi} *_6 de^{4A}) &= \langle \tilde{*}_6 \tilde{F}, e^{4A} \tilde{F} \rangle - \langle d_H \tilde{*}_6 \tilde{F}, e^{4A-\phi} \text{Re}\Psi_1 \rangle \\ &+ e^{4A-\phi} \sum_{i \in \text{loc. sources}} \tau_i [\rho_i^{\text{loc}} \text{vol}_6 - \langle \text{Re}\Psi_1, j_i \rangle] \end{aligned} \quad (\text{IV.372})$$

by using the Bianchi identity (IV.339) together with the RR-field-strength self-duality (III.64). For backgrounds admitting only calibrated sources, the second line in (IV.372) vanishes, and we are left with the external components of the modified Einstein equation

$$-d(e^{-2\phi} *_6 de^{4A}) = \langle \tilde{*}_6 \tilde{F}, e^{4A} \tilde{F} \rangle - \langle d_H \tilde{*}_6 \tilde{F}, e^{4A-\phi} \text{Re}\Psi_1 \rangle. \quad (\text{IV.373})$$

These equations of motion are drastically simpler than the ones one would obtain by varying the effective potential (IV.347). Considering the case where  $\Xi = 0$ , the complete set of equations of motion

$$\text{Re}\{\langle e^{3A-\phi}\Psi_2, d_H\Theta \rangle\} = 0 \quad (\text{IV.374})$$

$$d[\text{Re}\{\langle e^{3A-\phi}\Psi_2, \Theta \rangle_3\}] = 0 \quad (\text{IV.375})$$

$$\text{Re}\left\{ \langle g_{k(m} dy^k \wedge \iota_n)(e^{3A-\phi}\Psi_2), d_H\Theta \rangle - \langle g_{k(m} dy^k \wedge \iota_n) d_H(e^{3A-\phi}\Psi_2), \Theta \rangle \right\} = 0 \quad (\text{IV.376})$$

is simple enough to hope to solve them for new ten-dimensional type II flux vacua with BPS space-filling sources, which would dimensionally reduce to solutions of four-dimensional  $\mathcal{N} = 1$  supergravity with non-vanishing F-terms, like no-scale models for example.

### IV.3.4 The example of type IIB SU(3)-backgrounds with BPS O5-planes

In this section, we investigate again the condition (IV.357) we have to impose in order to interpret our solutions as dimensionally reducing to  $\mathcal{N} = 1$  four-dimensional supergravity. In particular, we want to determine how restrictive the condition (IV.357) can be. Concretely, we would like to consider the possibility of having non-zero D-terms  $d_H(e^{2A-\phi}\text{Im}\Psi_1) \neq 0$ , and see how constrained their expression is from requiring  $d_H(e^{2A-\phi}\text{Im}\Psi_1) \in U_0$ .

We focus on type IIB warped backgrounds with a four-dimensional Minkowski external space, admitting calibrated parallel space-filling O5-planes, and possibly D5-branes, and we restrict to internal manifolds with SU(3) structure. Introducing a local unwarped vielbein  $\{\tilde{e}^a\}$ , we choose the directions  $\tilde{e}^1$  and  $\tilde{e}^4$  to be tangent to the unique two-cycle wrapped by the sources. Our metric ansatz is thus

$$ds^2 = e^{2A} ds_{\mathbb{R}_{1,3}}^2 + ds_M^2 \quad (\text{IV.377})$$

$$ds_M^2 = e^{2A}[(\tilde{e}^1)^2 + (\tilde{e}^4)^2] + e^{-2A} \sum_{j=2,3,5,6} (\tilde{e}^j)^2. \quad (\text{IV.378})$$

For SU(3)-structure manifolds, the pure spinors (III.66) and (III.67) reduce to

$$\Psi_1 = e^{i\theta} e^{iJ} \quad \Psi_2 = e^{-i\theta} \Omega, \quad (\text{IV.379})$$

where  $\theta$  is the relative phase between the two parallel internal spinors  $\eta_1 = ie^{i\theta}\eta_2$ , the Kähler form  $J$  and the (3,0) form  $\Omega$  take the form

$$J = -(e^{2A}\tilde{e}^1 \wedge \tilde{e}^4 + e^{-2A}\tilde{e}^2 \wedge \tilde{e}^5 + e^{-2A}\tilde{e}^3 \wedge \tilde{e}^6) \quad (\text{IV.380})$$

$$\Omega = e^{-A}(\tilde{e}^1 + i\tilde{e}^4) \wedge (\tilde{e}^2 + i\tilde{e}^5) \wedge (\tilde{e}^3 + i\tilde{e}^6). \quad (\text{IV.381})$$

As discussed in [111], the orientifold projection sets

$$\theta = -\frac{\pi}{2}. \quad (\text{IV.382})$$

Notice that, combining the dilaton equation of motion with the appropriately traced external components of the Einstein equations, as done in [112], one can show that<sup>29</sup>  $\nabla^2(2A - \phi) = 0$ . Harmonic functions being constant on compact spaces, we set

$$e^{2A-\phi} \equiv g_s. \quad (\text{IV.383})$$

As shown in Appendix C.2, imposing that the gauge-BPSness condition still holds

<sup>29</sup>Note that both the Bianchi identities and the BPSness of the space-filling sources (IV.303) are crucial in this derivation.

already constrains the allowed non-supersymmetric deformations of the pure spinor equations to

$$d_H(e^{3A-\phi}\Omega) = -i e^{3A-\phi}K \quad (\text{IV.384})$$

$$d_H(e^{2A-\phi}\text{Re } e^{iJ}) = i e^{2A-\phi}\Upsilon \quad (\text{IV.385})$$

$$d_H(e^{4A-\phi}\text{Im } e^{iJ}) = e^{4A}\tilde{*}_6 F \quad (\text{IV.386})$$

with

$$K = \frac{i}{2} \left[ -t_2 e^{iJ} - t_1 e^{-iJ} + (u_m^1 + p_m^2)\hat{\gamma}^m \Omega - (u_m^2 + p_m^1)\Omega \hat{\gamma}^m + q_{mn}^1 \hat{\gamma}^n e^{-iJ} \hat{\gamma}^m + q_{mn}^1 \hat{\gamma}^n e^{iJ} \hat{\gamma}^m \right] \quad (\text{IV.387})$$

and

$$\Upsilon = \frac{i}{2} \left[ (r_1^* + t_2^*)\Omega - (u_m^1 + (p_m^2)^*)\hat{\gamma}^m e^{iJ} - ((u_m^2)^* + p_m^1)e^{iJ}\hat{\gamma}^m + (q_{nm}^1)^*\hat{\gamma}^m \Omega \hat{\gamma}^n \right] - c.c. \quad (\text{IV.388})$$

The gamma matrices  $\{\hat{\gamma}^m\}$  are defined in the local vielbein, and their action on polyforms is given in Appendix A. Imposing our condition (IV.357) further constrains the supersymmetry breaking terms  $\Upsilon$ :

$$\Upsilon = \frac{i}{2} \left[ (r_1^* + t_2^*)\Omega + (q_{nm}^1)^*\hat{\gamma}^m \Omega \hat{\gamma}^n \right] - c.c. \quad (\text{IV.389})$$

Expanding  $\Upsilon$  on the local vielbein yields

$$\Upsilon = \frac{1}{2} \left[ e^A(x_{32} - x_{23}) \left[ \tilde{e}^1 + \frac{1}{2}\tilde{e}^1 \wedge J \wedge J \right] + e^{-A}(x_{13} - x_{31}) \left[ \tilde{e}^2 + \frac{1}{2}\tilde{e}^2 \wedge J \wedge J \right] \right] \quad (\text{IV.390})$$

$$+ e^{-A}(x_{21} - x_{12}) \left[ \tilde{e}^3 + \frac{1}{2}\tilde{e}^3 \wedge J \wedge J \right] + e^A(y_{23} - y_{32}) \left[ \tilde{e}^4 + \frac{1}{2}\tilde{e}^4 \wedge J \wedge J \right] + e^{-A}(y_{31} - y_{13}) \left[ \tilde{e}^5 + \frac{1}{2}\tilde{e}^5 \wedge J \wedge J \right] + e^{-A}(y_{12} - y_{21}) \left[ \tilde{e}^6 + \frac{1}{2}\tilde{e}^6 \wedge J \wedge J \right] + X \quad (\text{IV.391})$$

with  $x_{ij}$ ,  $y_{ij}$  some real functions on the compact manifold, whose expressions are given in terms of the supersymmetry breaking parameters in Appendix C.2, and  $X$  a three-form specified in (C.46).

Moreover, the one-form components of  $\Upsilon$  is set to zero by (IV.383). The supersymmetry breaking parameters must therefore respect

$$x_{ij} = x_{ji} \quad y_{ij} = y_{ji} \quad i, j = 1, 2, 3. \quad (\text{IV.392})$$

It is then interesting to note that this requirement makes the five-form components of  $\Upsilon$  vanish, which imposes

$$d(J^2) = 0. \quad (\text{IV.393})$$

We already see at this stage that the condition (IV.357) highly constrains the possible D-terms for vacua with space-filling BPS-sources.

The only remaining possibility in order to have non-vanishing D-terms is through the NS flux. However, the most general NS field-strength compatible with (IV.357) and the orientifold projection<sup>30</sup> is also highly constrained

$$\begin{aligned} H = & y_{12}(e^A \tilde{e}^1 \wedge \tilde{e}^3 \wedge \tilde{e}^4 - e^{-3A} \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^5) \\ & + x_{13}(e^A \tilde{e}^1 \wedge \tilde{e}^4 \wedge \tilde{e}^5 + e^{-3A} \tilde{e}^3 \wedge \tilde{e}^5 \wedge \tilde{e}^6) \\ & - x_{12}(e^A \tilde{e}^1 \wedge \tilde{e}^4 \wedge \tilde{e}^6 - e^{-3A} \tilde{e}^2 \wedge \tilde{e}^5 \wedge \tilde{e}^6) \\ & - y_{13}(e^A \tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^4 + e^{-3A} \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^6). \end{aligned} \quad (\text{IV.394})$$

For these backgrounds, the Bianchi identity for the NS flux and the B-field equation of motion (IV.364) read

$$dH = 0 \quad d(\tilde{*}_6 H) = 0, \quad (\text{IV.395})$$

and together with the NS flux quantisation condition they would further constrain the possible NS flux, upon specifying some internal geometry.

Finally, it is also important to note that the orientifold projection sets

$$H \wedge \Omega = 0. \quad (\text{IV.396})$$

Since the F-terms from (IV.384) cannot be vanishing<sup>31</sup>, this means that the breaking of supersymmetry cannot originate purely from NS flux components.

Constructing ten-dimensional supergravity solutions with D-terms is difficult, and in this illustrative example we see that the mere requirement of consistency with the four-dimensional  $\mathcal{N} = 1$  description highly constrains the possible D-terms expression, potentially ruling out the possibility for non-vanishing D-terms for the whole class considered.

### IV.3.5 Discussion

In this section, we studied non-supersymmetric solutions of type II supergravity within the framework of generalised complex geometry. The interpretation of the supersymmetry conditions in terms of calibration conditions for different types of probe D-branes led us to

<sup>30</sup>The NS field-strength must be odd under the orientifold projection.

<sup>31</sup>There cannot be pure D-term breaking of supergravity in Minkowski space.



consider a subclass of non-supersymmetric solutions partially preserving supersymmetry, in the sense that the calibration condition for space-filling D-branes remains satisfied and such backgrounds admit only space-filling BPS sources. This calibration condition has been dubbed the gauge BPSness condition in [44]. On the other hand, the calibration conditions for string-like and domain-wall probe D-branes are allowed to be violated, which is encoded through the introduction of some supersymmetry breaking terms in these conditions.

The gauge BPSness condition has been interpreted in [7] as an F-term condition, making the connection with the four-dimensional  $\mathcal{N} = 1$  description.

We derived a generalisation of the gauge BPSness for our class of non-supersymmetric vacua. We then investigated under which conditions the gauge BPSness can still be interpreted as an F-term condition. Interestingly, this is the case when (IV.357) is respected, i.e. when some terms violating the string-like calibration condition are set to zero. These terms belong to vector representations of the  $SU(3) \times SU(3)$  structure, just like the modes identified with four-dimensional massive spin  $\frac{3}{2}$  multiplets degrees of freedom in [38]. Given that the violation of the string-like calibration condition has been interpreted as D-terms of the associated effective theory in [7], our condition (IV.357) restricts the possible D-terms for our class of backgrounds.

On another note, the vanishing of these vector-like modes results in some negative semi-definite contributions to the effective potential being set to zero on-shell (the last line of the effective potential (IV.347)). This is in agreement with the four-dimensional  $\mathcal{N} = 1$  picture, given that these contributions do not originate from the superpotential.

We derived the equations of motion for this class of backgrounds, and they are significantly simpler than the ones one would derive without imposing the constraint (IV.357), and an obvious extension of this work would be to search for such non-supersymmetric solutions.

A subclass of these backgrounds is the one containing vacua which would dimensionally reduce to four-dimensional  $\mathcal{N} = 1$  supergravity solutions with non-vanishing F-terms, and vanishing D-terms, like the abundantly discussed no-scale vacua. We also presented the remarkably simple general equations of motion for such backgrounds, and one could again look for new solutions of this type.

Finally, to illustrate this discussion we analysed how constraining it is to require (IV.357) for the class of  $SU(3)$  backgrounds with space-filling BPS O5-planes. We showed that non-vanishing D-terms could only arise through NS flux components, while the NS flux expression is itself highly constrained by (IV.357). It would then be interesting to investigate further the consequences of imposing our condition (IV.357) on different source configurations, and possibly rule out completely the possibility for D-terms in these cases, or find some new supergravity solutions with non-vanishing D-terms.

# Chapter V

## Non-supersymmetric flux vacua in Exceptional Generalised Geometry

In this chapter we discuss the  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry of non-supersymmetric flux backgrounds. We focus on type II supergravity backgrounds with a four-dimensional Minkowski external space.

As we have discussed at length in chapter III, an  $\mathcal{N} = 1$  type II supergravity background which has a four-dimensional Minkowski external space is in one-to-one correspondence with an integrable generalised  $SU(7)$  structure, and the integrability of the structure is a set of differential conditions equivalent to preserving  $\mathcal{N} = 1$  supersymmetry. The integrability of the generalised  $SU(7)$  structure amounts to the vanishing of the generalised intrinsic torsion associated to connections compatible with the  $SU(7)$  structure, the  $SU(7)$  intrinsic torsion. For a type II supergravity background compactified to four-dimensional Minkowski, if supersymmetry is broken, its corresponding  $SU(7)$  intrinsic torsion will thus be non-vanishing.

In this chapter we investigate the  $SU(7)$  intrinsic torsion of non-supersymmetric flux backgrounds. As we have discussed in chapter III, we can organise the  $SU(7)$  intrinsic torsion in terms of different irreducible representations of  $SU(7)$ .

This is useful because it allows one to rewrite the supergravity equations of motion in terms of the components of the associated intrinsic  $SU(7)$  torsion, as is done in section V.2. The hope is then to be able to identify no-go theorems for supersymmetry breakings associated to certain components of the  $SU(7)$  torsion.

We focus on the class of non-supersymmetric type II backgrounds which still admits two globally defined internal spinors. While not completely general<sup>1</sup>, this is still a broad class of backgrounds, which can be classified in  $O(6,6)$  generalised geometry in terms of

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<sup>1</sup>This class of solutions doesn't contain the case of an NS5-brane wrapping a Calabi-Yau for instance.

the different supersymmetry breaking parameters entering in the modified pure spinor equations, as introduced in chapter IV.

For this class of backgrounds, we provide a direct dictionary between the supersymmetry breaking parameters and the non vanishing irreducible  $SU(7)$  representations of the intrinsic torsion in section V.1. As such, this provides a bridge between the  $O(6,6) \times \mathbb{R}^+$  and  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry frameworks, and this thus allows to describe the known  $\mathcal{N} = 0$  backgrounds [3, 44, 45, 91] within  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry.

Another goal, which is still work in progress, is to associate non-supersymmetric type II solutions with the integrability of a smaller generalised geometric structure.

For a given non-supersymmetric type II solution, this could be done by defining the common stabiliser  $G \subset SU(7)$  of its  $SU(7)$  structure and its intrinsic torsion. This would also allow one to define deformations of said background and calculate its corresponding classical moduli space. Such deformations would be cast in terms of irreducible representations of  $G$ . We briefly initiate this discussion in section V.3.

Throughout this chapter, we use the conventions of section III.2, notably the pure spinors (III.156) and (III.157).

## V.1 The $SU(7)$ torsion

In this section, we study non-supersymmetric type II backgrounds with a four-dimensional Minkowski external space, and with the following general domain-wall and D-string supersymmetry breaking

$$d_H \Phi_2 = \omega_2 + \omega_0 \tag{V.1}$$

$$d_H(e^{-A} \text{Re} \Phi_1) = \omega_1 + \omega_{-1} + \omega_3 + \omega_{-3}, \tag{V.2}$$

with  $\omega_i$  polyforms of charge  $i$  under  $\mathcal{J}_2$ , and with  $\omega_1 = \bar{\omega}_{-1}$  and  $\omega_3 = \bar{\omega}_{-3}$ .

We focus on the subclass of backgrounds which satisfies the gauge BPSness, in order for the physical space-filling sources of the considered backgrounds to be BPS<sup>2</sup>. The non-supersymmetric reformulation of the gauge BPSness derived in section IV.3 now reads

$$F = -8d_H^{\mathcal{J}_2}(e^{-3A} \text{Im} \Phi_1) + 8ie^{-2A} \mathcal{J}_1 \cdot \omega_{-1}. \tag{V.3}$$

Let us rearrange this equation into

$$(F + d_H \Sigma)_{-1} = 8ie^{-2A} \mathcal{J}_1 \cdot \omega_{-1} \tag{V.4}$$

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<sup>2</sup>Recall that Minkowski flux backgrounds must have space-filling Orientifold sources to evade the no-go theorem from [66, 67], further discussed in [113].

$$(F + d_H \Sigma)_{-3} = 2d_H \Sigma|_{-3}, \quad (\text{V.5})$$

where we introduced  $\Sigma = 8ie^{-3A}\text{Im}\Phi_1$ . It is important to note that  $d_H \Sigma|_{-3}$  is non-vanishing only if  $\omega_0 \neq 0$ , it should hence be thought of as depending on  $\omega_0$ . We denote it  $d_H \Sigma|_{-3} \equiv Y^{\omega_0}$  in the following.

This reformulation of the gauge BPSness seems quite arbitrary for now but this will be the most natural formulation in order to connect with the exceptional generalised geometry framework, as we will discuss below.

The goal of this section is to derive expressions for the different irreducible  $SU(7)$  representations of the intrinsic torsion in terms of the supersymmetry breaking forms  $\{\omega_i\}$ .

### V.1.1 Connection, torsion and Dorfman derivative

In this subsection, we start by evaluating the  $SU(7)$  intrinsic torsion of non-supersymmetric type II flux backgrounds in terms of irreducible representation of the  $SU(3, 3)$  group associated to the almost complex structure  $\mathcal{J}_2$ .

We will reorganise this intrinsic torsion in terms of the irreducible representations of the  $SU(7)$  group associated with the generalised  $SU(7)$  structure in the following subsection.

Recall that the exceptional complex structure  $L_3 \subset E_{\mathbb{C}}$  and generalised  $SU(7)$  structure  $\psi \in \Gamma(\tilde{K}_{\mathbb{C}})$  for type II backgrounds with a four-dimensional Minkowski external space read

$$L_3 = e^{\Sigma} \cdot (L_1 \oplus \mathcal{U}_{\mathcal{J}_2}) \quad (\text{V.6})$$

$$\psi = e^{\Sigma} \cdot \Phi_2. \quad (\text{V.7})$$

We introduce the following seven-dimensional complex bundle

$$L_3^0 := L_1 \oplus \mathcal{U}_{\mathcal{J}_2}, \quad (\text{V.8})$$

which we refer to as the untwisted exceptional complex structure.

We consider a generalised connection  $D$ . Recall from section III.2 that we can define the corresponding generalised torsion  $T : \Gamma(E) \rightarrow \Gamma(\text{ad}\tilde{F})$  in the untwisted picture via

$$L_V^{H+F} \alpha = L_V^{D+H+F} \alpha - T(V) \cdot \alpha, \quad (\text{V.9})$$

for  $\alpha$  any tensors. We consider  $D$  to be compatible with the generalised  $SU(7)$  structure

$$(D + H + F)\psi = 0. \quad (\text{V.10})$$

The compatibility of the generalised connection with the  $SU(7)$  structure ensures

$$L_V^{H+F} \psi = -T(V) \cdot \psi \quad \forall V \in \Gamma(L_3). \quad (\text{V.11})$$

Introducing  $V = e^\Sigma \cdot V^0$  with  $V^0 \in \Gamma(L_3^0)$ , the above relation is equivalent to

$$L_{V^0}^{H+F+d_H\Sigma} \Phi_2 = -T^0(V^0) \cdot \Phi_2 \quad \forall V^0 \in \Gamma(L_3^0), \quad (\text{V.12})$$

with

$$T^0 = e^{-\Sigma} \cdot T, \quad (\text{V.13})$$

the untwisted  $SU(7)$  torsion.

Writing  $V^0 = W + \alpha \Phi_2 \in \Gamma(L_3^0)$  with  $\alpha \in \mathcal{C}^\infty(M)$ , the left-hand side of (V.12) reads

$$\begin{aligned} L_{V^0}^{H+F+d_H\Sigma} \Phi_2 &= \mathcal{W} d_H \Phi_2 - \alpha d_H \Phi_2 \cdot \Phi_2 \\ &\quad + (\mathcal{W}(F + d_H \Sigma) + \alpha \langle F + d_H \Sigma, \Phi_2 \rangle) \cdot \Phi_2. \end{aligned} \quad (\text{V.14})$$

The terms in the first line of this Dorfman derivative can be read off from the modified pure spinor equation (V.1), while the ones in the second line only depend on the modified pure spinor equations (V.4), and (V.5). We can therefore access  $T^0(V^0) \in \text{ad} \tilde{F}$  through (V.12). However, given that this section of the adjoint bundle acts on the untwisted generalised  $SU(7)$  structure  $\Phi_2 \in \Gamma(\tilde{K}_\mathbb{C})$  in (V.12), its expression will be up to the kernel of the adjoint action on the untwisted generalised  $SU(7)$  structure.

In order to do so, we introduce  $Z^{\omega_2} \in \Gamma(L_{-1})$ ,  $Z^{\omega_0} \in \Gamma(\Lambda^3(L_{-1}))$ , and  $SU(3,3)$  gamma matrices  $\Gamma^i$  with indices  $i = 1, \dots, 6$  such that

$$\omega_2 = Z_i^{\omega_2} \Gamma^i \Phi_2 \quad \omega_0 = Z_{ijk}^{\omega_0} \Gamma^{ijk} \Phi_2. \quad (\text{V.15})$$

We then evaluate the above Dorfman derivative, using (V.1), (V.4), and (V.5):

$$\begin{aligned} L_{V^0}^{H+F+d_H\Sigma} \Phi_2 &= \left( \eta(Z^{\omega_2}, W) + Z^{\omega_0} \odot W - \alpha \omega_0 \right. \\ &\quad \left. + 8ie^{-2A} \mathcal{W}(\mathcal{J}_1 \cdot \omega_{-1}) + 2\mathcal{W}Y^{\omega_0} + 2\alpha \langle Y^{\omega_0}, \Phi_2 \rangle \right) \cdot \Phi_2, \end{aligned} \quad (\text{V.16})$$

with  $(Z \odot W)_{ij} = W^k Z_{kij} \in \Gamma(\Lambda^2(L_{-1}))$ .

We therefore have

$$\begin{aligned} T^0(V^0) &= -[\eta(Z^{\omega_2}, W) + Z^{\omega_0} \odot W - \alpha \omega_0 \\ &\quad + 8ie^{-2A} \mathcal{W}(\mathcal{J}_1 \cdot \omega_{-1}) + 2\mathcal{W}Y^{\omega_0} + 2\alpha \langle Y^{\omega_0}, \Phi_2 \rangle] + \ker, \end{aligned} \quad (\text{V.17})$$

with  $\ker$  a section of the adjoint bundle sitting inside the kernel of the adjoint action on

the untwisted generalised  $SU(7)$  structure.

The terms entering  $T^0(V^0)$  depend explicitly on the supersymmetry breaking forms  $\{\omega_i\}$ , but they are irreducible representations of the  $SU(3,3)$  group associated with the pure spinor  $\Phi_2$ . We make the connection with the different irreducible  $SU(7)$  representations in the following subsection.

### V.1.2 The $SU(7)$ torsion from the adjoint bundle

We calculated the section of the adjoint bundle corresponding to the action of the untwisted  $SU(7)$  torsion on the  $L_3^0$  bundle  $T^0(V^0)$ . We therefore start by analysing further the adjoint bundle. In particular, we will parametrise the subspace of the adjoint bundle which acts non-trivially on the untwisted  $SU(7)$  structure in terms of irreducible  $SU(7)$  representations. Given that these are eigenstates of the exceptional complex structure, this will allow us to match the  $SU(3,3)$  representations entering in (V.17) with  $SU(7)$  representations by rearranging them into the corresponding eigenstates.

To do so, we consider the complex embedding  $SL(7, \mathbb{C}) \subset SL(8, \mathbb{C}) \subset E_{7\mathbb{C}}$  and its corresponding complex decomposition of the generalised tangent and adjoint  $E_{7\mathbb{C}}$  bundles

$$\mathbf{56}_{\mathbb{C}} \simeq \mathbf{7} \oplus \mathbf{21} \oplus \overline{\mathbf{21}} \oplus \overline{\mathbf{7}} \quad (\text{V.18})$$

$$E_{\mathbb{C}} \simeq L_3^0 \oplus \Lambda^5(L_3^0)^* \oplus \Lambda^2(L_3^0)^* \oplus [\Lambda^7(L_3^0)^* \otimes (L_3^0)^*] \quad (\text{V.19})$$

$$\mathbf{133}_{\mathbb{C}} \simeq \mathbf{1} \oplus \mathbf{48} \oplus (\mathbf{35} \oplus \overline{\mathbf{35}}) \oplus (\mathbf{7} \oplus \overline{\mathbf{7}}) \quad (\text{V.20})$$

$$\text{ad}\tilde{F}_{\mathbb{C}} \simeq \mathbb{C} \oplus [L_3^0 \otimes (L_3^0)^*]_0 \oplus \Lambda^3 L_3^0 \oplus \Lambda^3(L_3^0)^* \oplus \Lambda^6 L_3^0 \oplus \Lambda^6(L_3^0)^*, \quad (\text{V.21})$$

where  $[L_3^0 \otimes (L_3^0)^*]_0$  is a traceless  $7 \times 7$  complex matrix, corresponding to the  $SL(7, \mathbb{C})$  algebra. The  $SL(7, \mathbb{C}) \subset E_{7\mathbb{C}}$  adjoint action on the generalised tangent and the adjoint bundle are given in appendix B.2.1. From this adjoint action, we have that the following subspace  $\mathfrak{g}$  of the adjoint bundle

$$\mathbf{91} \simeq \mathbf{1} \oplus \mathbf{48} \oplus \overline{\mathbf{35}} \oplus \overline{\mathbf{7}} \quad (\text{V.22})$$

$$\mathfrak{g} \simeq \mathbb{C} \oplus [L_3^0 \otimes (L_3^0)^*]_0 \oplus \Lambda^3 L_3^0 \oplus \Lambda^6 L_3^0 \quad (\text{V.23})$$

stabilises the untwisted almost exceptional complex structure  $L_3^0$ . The stabiliser group  $G$  of the untwisted almost exceptional complex structure  $L_3^0$  is therefore  $G = ((SL(7, \mathbb{C}) \times \mathbb{C}^{35}) \times \mathbb{C}^7) \times \mathbb{C}$ . This group can be thought of as the complex analogue of the  $\mathbb{R}^+ \times U(7)$  group defining the exceptional complex structure.

The  $SL(7, \mathbb{C})$  subgroup corresponds to the complexification of the compact subgroup  $SU(7) \subset SU(8)/\mathbb{Z}_2 \subset E_{7(7)} \times \mathbb{R}^+$ , while the groups  $\mathbb{C}^{35}$ ,  $\mathbb{C}^7$ , and  $\mathbb{C}$  complexify part of the non-compact part of  $E_{7(7)} \times \mathbb{R}^+$ . In the following we will thus parametrise the

adjoint bundle in terms of irreducible representations of  $\mathrm{SL}(7, \mathbb{C}) \subset E_{7\mathbb{C}}$ , and these should be thought of as the complex analogues of the irreducible representations of the  $\mathrm{SU}(7)$  subgroup of the real  $E_{7(7)} \times \mathbb{R}^+$  algebra.

The group stabilising the untwisted generalised  $\mathrm{SU}(7)$  structure  $\Phi_2$  is the same as the one stabilising the untwisted exceptional complex structure, with the exception of the complex scalar part which rescales  $\Phi_2$ . We denote it  $H$  and we have  $H = (\mathrm{SL}(7, \mathbb{C}) \times \mathbb{C}^{35}) \times \mathbb{C}^7$ . Its algebra, as a subspace  $\mathfrak{h}$  of the adjoint bundle is

$$\mathbf{90} \simeq \mathbf{48} \oplus \overline{\mathbf{35}} \oplus \mathbf{7} \quad (\text{V.24})$$

$$\mathfrak{h} \simeq [L_3^0 \otimes (L_3^0)^*]_0 \oplus \Lambda^3 L_3^0 \oplus \Lambda^6 L_3^0. \quad (\text{V.25})$$

This leaves

$$\mathbf{133}/\mathbf{90} \simeq \mathbf{1} \oplus \mathbf{35} \oplus \mathbf{7} \quad (\text{V.26})$$

$$\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h} \simeq \mathbb{C} \oplus \Lambda^3 (L_3^0)^* \oplus \Lambda^6 (L_3^0)^* \quad (\text{V.27})$$

as the subspace of the adjoint bundle acting non-trivially on the untwisted generalised  $\mathrm{SU}(7)$  structure. We denote its sections as

$$R = R^1 + R^{35} + R^7. \quad (\text{V.28})$$

In terms of irreducible  $\mathrm{SU}(3, 3)$  representations, it reads

$$\mathbf{133}/\mathbf{90} \simeq \mathbf{1} \oplus \mathbf{15} \oplus \mathbf{6} \oplus \mathbf{20} \oplus \mathbf{1} \quad (\text{V.29})$$

$$\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h} \simeq \mathbb{C} \oplus \Lambda^2(L_{-1}) \oplus S_{-2} \oplus S_0 \oplus \Lambda^6 T_{\mathbb{C}}^*. \quad (\text{V.30})$$

The parametrisation of the generalised tangent and adjoint  $E_{7\mathbb{C}}$  bundles in terms of  $\mathrm{SU}(3, 3)$  representations is given in appendix B.2.2. As we have the expression (V.17) for  $T^0(V^0)$  as a section of (V.30), the goal is now to connect it with the  $\mathrm{SL}(7, \mathbb{C})$  representations in (V.27).

To do so, we make use of the fact that each  $\mathrm{SL}(7, \mathbb{C})$  representation entering (V.27) is an eigenspace of the almost exceptional complex structure. Indeed, as an  $\mathrm{SL}(8, \mathbb{C})$  representation, the exceptional complex structure  $J$  reads [61]

$$J = \mathrm{diag}(-1/2, -1/2, \dots, 7/2) \in \mathfrak{sl}(8, \mathbb{C}) \subset E_{7\mathbb{C}}. \quad (\text{V.31})$$

One can explicitly decompose  $J$  into irreducible  $\mathrm{SL}(7, \mathbb{C})$  representations, following appendix B.2.1. Acting on the different irreducible  $\mathrm{SL}(7, \mathbb{C})$  representations of  $\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h}$ , we

have

$$J \cdot R^1 = 0 \tag{V.32}$$

$$J \cdot R^7 = -4iR^7 \tag{V.33}$$

$$J \cdot R^{35} = -2iR^{35}. \tag{V.34}$$

The 1-, 7-, and 35-dimensional  $SL(7, \mathbb{C})$  representations living in  $\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h}$  are therefore eigenspaces of the exceptional complex structure  $J$  with eigenvalues 0,  $-4i$ , and  $-2i$  respectively.

We can now turn to evaluating the adjoint action of the exceptional complex structure  $J$  on irreducible  $SU(3, 3)$  representations of  $\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h}$ , and then construct the eigenstates corresponding to  $R^1$ ,  $R^7$ , and  $R^{35}$ .

First of all, in terms of  $SU(3, 3)$  representations, we have<sup>3</sup>

$$J = \mathcal{J}_2 + i(\Sigma - \Sigma^\sharp), \tag{V.35}$$

with  $\Sigma^\sharp = -\frac{1}{2}e^{3A}(\langle \Phi_2, \bar{\Phi}_2 \rangle)^{-1} \text{Re} \Phi_1$ . The corresponding untwisted exceptional complex structure  $J^0$ , equivalent to  $L_3^0$ , is

$$J^0 := e^{-\Sigma} J e^{\Sigma} = \mathcal{J}_2 - 2i - i\Sigma^\sharp, \tag{V.36}$$

where the scalar piece corresponds to a section  $l$  of the adjoint space in (B.89). We now write a section of  $\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h}$  in terms of irreducible  $SU(3, 3)$  representations as

$$X = p + \nu_{-2} + s_{-2} + s_0 + a, \tag{V.37}$$

where each term matches with the ones in (V.30) in the obvious way. We can evaluate the adjoint action of the untwisted exceptional complex structure  $J^0$  on  $X$  from appendix B.2.2.

From

$$J^0 \cdot X = -2i(\nu_{-2} + s_0) - 4i(s_{-2} + a) - i\Sigma^\sharp \cdot (s_0 + s_{-2} + a), \tag{V.38}$$

we can construct eigenstates of  $J^0$  as<sup>4</sup>

$$R^1 = p \tag{V.39}$$

$$R^7 = a + \frac{1}{2}\Sigma^\sharp \cdot a + s_{-2} + \frac{1}{2}\Sigma^\sharp \cdot s_{-2} \tag{V.40}$$

---

<sup>3</sup>This expression for  $J$  can be calculated from requiring  $J \cdot L_3 = 3iL_3$  for instance.

<sup>4</sup>We didn't expand some adjoint actions for the sake of simplicity, but it can be done following appendix B.2.2.



$$R^{35} = \nu_{-2} - \frac{1}{2}\Sigma^\sharp \cdot s_{-2} + s_0 - \frac{1}{2}\Sigma^\sharp \cdot a, \quad (\text{V.41})$$

where  $R^1$ ,  $R^7$ , and  $R^{35}$  have eigenvalues 0,  $-4i$ , and  $-2i$  respectively, under the action of  $J^0$ . We now have a concrete link between the  $\text{SU}(3, 3)$  and  $\text{SL}(7, \mathbb{C})$  representations entering in  $\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h}$ . The final step to complete the dictionary between the irreducible  $\text{SU}(7)$  representations of the  $\text{SU}(7)$  torsion and the non-vanishing  $\text{SU}(3, 3)$  representations involved in the modified pure spinor equations of a given non-supersymmetric background, is to connect  $T^0$  with the **1**, **7**, and **35**  $\text{SU}(7)$  representations in  $\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h}$ .

We denote as  $T_1^0$ ,  $T_7^0$ ,  $T_{21}^0$ , and  $T_{35}^0$ , the sections of the torsion bundle  $\tilde{K}_{\mathbb{C}}$  corresponding to the **1**,  $\bar{\mathbf{7}}$ , **21** and **35** representations of the  $\text{SU}(7)$  intrinsic torsion (III.135).

From  $\text{SL}(7, \mathbb{C})$  group theory, we have<sup>5</sup>

$$T_1^0(V^0) \equiv T_1^0 \odot V^0 \in \Gamma(\Lambda^6(L_3^0)^*) \quad (\text{V.42})$$

$$T_7^0(V^0) \equiv T_7^0 \odot V^0 \in \Gamma(\mathbb{C}) \quad (\text{V.43})$$

$$T_{35}^0(V^0) \equiv T_{35}^0 \odot V^0 \in \Gamma(\Lambda^3(L_3^0)^*) \quad (\text{V.44})$$

$$T_{21}^0(V^0) \equiv T_{21}^0 \odot V^0 \in \Gamma(\Lambda^3 L_3^0), \quad (\text{V.45})$$

for all  $V^0 \in \Gamma(L_3^0)$ , and with

$$\odot : \tilde{K}_{\mathbb{C}} \odot E_{\mathbb{C}} \rightarrow \text{ad}\tilde{F}_{\mathbb{C}} \quad (\text{V.46})$$

given in appendix B.2.1.

We conclude that  $T_1^0(V^0)$ ,  $T_7^0(V^0)$ , and  $T_{35}^0(V^0)$  correspond to sections  $R_7$ ,  $R_1$ , and  $R_{35}$  of  $\mathfrak{e}_{7\mathbb{C}}/\mathfrak{h}$ , respectively, while  $T_{21}^0(V^0)$  is a section of  $\mathfrak{h}$ , and as such it belongs to the kernel of the adjoint action on the untwisted generalised  $\text{SU}(7)$  structure. Its contribution to (V.12) hence vanishes.

### V.1.3 Backgrounds with non-vanishing $\text{SU}(7)$ intrinsic torsion

We can now come back to discussing non-supersymmetric type II backgrounds with the following general domain wall and D-string BPSness violations:

$$d_H \Phi_2 = \omega_2 + \omega_0 \quad (\text{V.47})$$

$$d_H(e^{-\Delta} \text{Re}\Phi_1) = \omega_1 + \omega_{-1} + \omega_3 + \omega_{-3}, \quad (\text{V.48})$$

---

<sup>5</sup>Alternatively, the sections  $T_1^0$ ,  $T_7^0$ ,  $T_{21}^0$ , and  $T_{35}^0$  of the torsion bundle  $\tilde{K}_{\mathbb{C}}$  can be written in terms of  $\text{SL}(7, \mathbb{C})$  indices following [42]. One can then evaluate these expressions explicitly, with  $\odot$  given in appendix B.2.1.

with  $\omega_1 = \bar{\omega}_{-1}$  and  $\omega_3 = \bar{\omega}_{-3}$ . These background satisfy the following gauge BPSness

$$(F + d_H \Sigma)_{-1} = 8ie^{-2A} \mathcal{J}_1 \cdot \omega_{-1} \quad (\text{V.49})$$

$$(F + d_H \Sigma)_{-3} = 2Y^{\omega_0}. \quad (\text{V.50})$$

Recall that for such backgrounds, we have

$$\begin{aligned} T^0(V^0) = & -[\eta(Z^{\omega_2}, W) + Z^{\omega_0} \odot W - \alpha\omega_0 \\ & + 8ie^{-2A} \mathcal{W}(\mathcal{J}_1 \cdot \omega_{-1}) + 2\mathcal{W}Y^{\omega_0} + 2\alpha \langle Y^{\omega_0}, \Phi_2 \rangle] + \ker. \end{aligned} \quad (\text{V.51})$$

We introduce the following notation:

$$T^0(V^0) = p + \nu_{-2} + s_0 + s_{-2} + a + \ker \quad (\text{V.52})$$

with

$$p = -\eta(Z^{\omega_2}, W) \quad (\text{V.53})$$

$$\nu_{-2} = -Z^{\omega_0} \odot W \quad (\text{V.54})$$

$$s_0 = \alpha\omega_0 - 8ie^{-2A} \mathcal{W}(\mathcal{J}_1 \cdot \omega_{-1}) \quad (\text{V.55})$$

$$s_{-2} = -2\mathcal{W}Y^{\omega_0} \quad (\text{V.56})$$

$$a = -2\alpha \langle Y^{\omega_0}, \Phi_2 \rangle, \quad (\text{V.57})$$

which are irreducible representations of the  $SU(3, 3)$  group associated with the pure spinor  $\Phi_2$ .

The different irreducible  $SU(7)$  representations of the  $SU(7)$  intrinsic torsion, acting on  $V^0 = W + \alpha\Phi_2 \in \Gamma(L_3^0)$ , are then

$$T_7^0(V^0) = p \quad (\text{V.58})$$

$$T_1^0(V^0) = a + \frac{1}{2} \Sigma^\sharp \cdot a + s_{-2} + \frac{1}{2} \Sigma^\sharp \cdot s_{-2} \quad (\text{V.59})$$

$$T_{35}^0(V^0) = \nu_{-2} - \frac{1}{2} \Sigma^\sharp \cdot s_{-2} + s_0 - \frac{1}{2} \Sigma^\sharp \cdot a. \quad (\text{V.60})$$

There are several lessons to draw from these expressions for the  $SU(7)$  intrinsic torsion of non-supersymmetric type II flux backgrounds with calibrated space-filling sources.

i) As we discussed in chapter III, only the  $\bar{\mathbf{7}}$  and  $\mathbf{21}$  components of the  $SU(7)$  intrinsic torsion enter the moment map (III.150).

We see here that the  $\bar{\mathbf{7}}$  component of the  $SU(7)$  torsion is solely dictated by  $\omega_2$ . From (III.183) we can thus conclude that a non-vanishing  $\mathbf{21}$  component of the torsion purely

results from

$$d_H(e^{-A}\text{Re}\Phi_1) \neq 0. \quad (\text{V.61})$$

ii) As we just discussed, the supersymmetry breaking form  $\omega_{-1}$  violating the D-string BPSness (V.48) generates some non-vanishing **21** torsion. We now learn from (V.60) that it also results in a non-vanishing **35** component of the SU(7) intrinsic torsion.

This ultimately comes from the non-supersymmetric expression of the gauge BPSness derived in chapter IV. In the O(6,6) generalised geometry context, it highlighted the interplay between the D-string BPSness violation and the gauge BPSness, while here in the  $E_{7(7)}$  generalised geometry framework, it shows the relationship between the **21** and **35** components of the SU(7) intrinsic torsion.

iii) The supersymmetry breaking forms  $\omega_{-3}$  and  $\omega_3$  don't appear in the **1**,  $\bar{\mathbf{7}}$ , and **35** components of the torsion, they hence correspond to pure **21** torsion. They are precisely the modes that are allowed for type II flux backgrounds with D-term supersymmetry breaking and calibrated space-filling sources, as discussed in chapter IV.

iv) Let us consider backgrounds with BPS space-filling sources, non-vanishing F-terms and superpotential. As discussed in chapter IV, they require a non-vanishing  $\omega_0 \neq 0$ , as it generates the F-terms, as well as a non-vanishing  $Y^{\omega_0}$ , which is proportional to the superpotential. We learn from (V.59) and (V.60) that such backgrounds will hence always have non vanishing **1** and **35** components of the SU(7) torsion.

As already mentioned, backgrounds with non-vanishing D-terms must have non-vanishing supersymmetry breaking forms  $\omega_{-3}$  and  $\omega_3$ , but they also need a non-vanishing  $Y^{\omega_0}$ , since it generates the required non-vanishing on-shell superpotential.  $Y^{\omega_0}$  can be non-vanishing only if  $\mathcal{I}_2$  is non-integrable, i.e if  $\omega_0 \neq 0$ . These backgrounds will hence also always have non vanishing **1** and **35** components of the SU(7) torsion.

v) Finally, let us mention that there can be backgrounds where supersymmetry is broken via only a  $\omega_{-1}$  supersymmetry breaking form, as in [45, 91]. As was discussed in the previous chapter, these do not dimensionally reduce to solutions of four-dimensional  $\mathcal{N} = 1$  supergravity with F-terms, D-terms and superpotential.

Before giving the explicit construction of the intrinsic SU(7) torsion for a few concrete examples, let us briefly discuss relaxing the gauge BPSness condition.

We hence discuss non-supersymmetric type II flux background which now have non-BPS space-filling sources. We consider the following completely general<sup>6</sup> breaking of supersymmetry

$$d_H\Phi_2 = \omega_2 + \omega_0 \quad (\text{V.62})$$

---

<sup>6</sup>With the usual caveat that only geometries admitting two globally defined internal spinors are considered.

$$d_H(e^{-\Delta} \text{Re}\Phi_1) = \omega_1 + \omega_{-1} + \omega_3 + \omega_{-3} \quad (\text{V.63})$$

$$(F + d_H \Sigma)_{-1} = \omega'_{-1} \quad (\text{V.64})$$

$$(F + d_H \Sigma)_{-3} = \omega'_{-3}, \quad (\text{V.65})$$

with again  $\omega_1 = \bar{\omega}_{-1}$  and  $\omega_3 = \bar{\omega}_{-3}$ . Evaluating (V.14) now yield

$$\begin{aligned} T^0(V^0) = & -[\eta(Z^{\omega_2}, W) + Z^{\omega_0} \odot W - \alpha\omega_0 \\ & + \mathcal{W}\omega'_{-1} + \mathcal{W}\omega'_{-3} + \alpha \langle \omega'_{-3}, \Phi_2 \rangle] + \ker. \end{aligned} \quad (\text{V.66})$$

Introducing

$$T^0(V^0) = p + \nu_{-2} + s_0 + s_{-2} + a + \ker \quad (\text{V.67})$$

with

$$p = -\eta(Z^{\omega_2}, W) \quad (\text{V.68})$$

$$\nu_{-2} = -Z^{\omega_0} \odot W \quad (\text{V.69})$$

$$s_0 = \alpha\omega_0 - \mathcal{W}\omega'_{-1} \quad (\text{V.70})$$

$$s_{-2} = -\mathcal{W}\omega'_{-3} \quad (\text{V.71})$$

$$a = -\alpha \langle \omega'_{-3}, \Phi_2 \rangle, \quad (\text{V.72})$$

we have the following irreducible  $SU(7)$  representations of the  $SU(7)$  intrinsic torsion acting on  $V^0 = W + \alpha\Phi_2 \in \Gamma(L_3^0)$

$$T_7^0(V^0) = p \quad (\text{V.73})$$

$$T_1^0(V^0) = a + \frac{1}{2}\Sigma^\sharp \cdot a + s_{-2} + \frac{1}{2}\Sigma^\sharp \cdot s_{-2} \quad (\text{V.74})$$

$$T_{35}^0(V^0) = \nu_{-2} - \frac{1}{2}\Sigma^\sharp \cdot s_{-2} + s_0 - \frac{1}{2}\Sigma^\sharp \cdot a. \quad (\text{V.75})$$

There are fewer lessons to draw from these expressions for the  $SU(7)$  intrinsic torsion of non-supersymmetric type II flux backgrounds with non-BPS space-filling sources. However, we note that:

i) The  $\bar{\mathbf{7}}$  component of the  $SU(7)$  torsion is again solely dictated by  $\omega_2$ . From (III.183) we can thus still conclude that a non-vanishing  $\mathbf{21}$  component of the torsion purely results from

$$d_H(e^{-A} \text{Re}\Phi_1) \neq 0. \quad (\text{V.76})$$

ii) The supersymmetry breaking forms  $\omega_{-3}$  and  $\omega_3$  don't appear in the  $\mathbf{1}$ ,  $\bar{\mathbf{7}}$ , and  $\mathbf{35}$  components of the torsion, they hence still correspond to pure  $\mathbf{21}$  torsion. However, as opposed to the case where the sources are BPS, it is now also the case for the  $\omega_{-1}$  and

$\omega_1$  supersymmetry breaking forms. The **21** and **35** components of the torsion can then be independent in principle.

iii) The supersymmetry breaking form  $\omega'_{-3}$  can be non-vanishing only if  $\omega_0 \neq 0$ . We thus find again that the **1** and **35** components of the SU(7) torsion are not independent.

We now illustrate the results found above to examples of non-supersymmetric type II flux backgrounds from the literature.

### a) GKP backgrounds

The type IIB GKP backgrounds [3] are non-supersymmetric flux backgrounds with an SU(3) structure and BPS D3-branes and O3-planes, where supersymmetry is broken by a (0,3)-component of the NS flux  $H_{-3}$ , with respect to the underlying complex structure of the background. They obey the following pure spinor equations

$$d_H \Phi_- = H_{-3} \wedge \Phi_- \quad (\text{V.77})$$

$$d_H(e^{-A} \text{Re} \Phi_+) = 0 \quad (\text{V.78})$$

$$(F + d_H \Sigma)_{-1} = 0 \quad (\text{V.79})$$

$$(F + d_H \Sigma)_{-3} = 2ie^{-\phi} H_{-3}. \quad (\text{V.80})$$

Using these, we evaluate the Dorfman derivative of the untwisted generalised SU(7) structure along  $V^0 = W + \alpha \Phi_- \in L_3^0$

$$\begin{aligned} L_{V^0}^{H+F+d_H \Sigma} \Phi_- &= \mathcal{W} d_H \Phi_- - d_H(\alpha \Phi_-) \cdot \Phi_- \\ &\quad + (\mathcal{W}(F + d_H \Sigma) + \alpha \langle F + d_H \Sigma, \Phi_- \rangle) \cdot \Phi_- \end{aligned} \quad (\text{V.81})$$

$$\begin{aligned} &= (v \lrcorner H_{-3} - \alpha H_{-3} \wedge \Phi_- \\ &\quad + 2ie^{-\phi} \mathcal{W} H_{-3} + 2ie^{-\phi} \alpha \langle H_{-3}, \Phi_- \rangle) \cdot \Phi_-. \end{aligned} \quad (\text{V.82})$$

From (V.12), we have

$$T^0(V^0) = -v \lrcorner H_{-3} + \alpha H_{-3} \wedge \Phi_- - 2ie^{-\phi} \mathcal{W} H_{-3} - 2ie^{-\phi} \alpha \langle H_{-3}, \Phi_- \rangle, \quad (\text{V.83})$$

up to the pieces acting trivially on  $\Phi_-$ . We decompose this expression in terms of the different irreducible SU(7) representations of the SU(7) intrinsic torsion

$$T_7^0(V^0) = 0 \quad (\text{V.84})$$

$$\begin{aligned} T_{35}^0(V^0) &= \alpha H_{-3} \wedge \Phi_- + i\alpha e^{-\phi} \Sigma^\sharp \cdot \langle H_{-3}, \Phi_- \rangle \\ &\quad - v \lrcorner H_{-3} + ie^{-\phi} \Sigma^\sharp \cdot (\mathcal{W} H_{-3}) \end{aligned} \quad (\text{V.85})$$

$$T_1^0(V^0) = -2ie^{-\phi} \alpha \langle H_{-3}, \Phi_- \rangle$$

$$\begin{aligned}
& -\alpha i e^{-\phi} \Sigma^\sharp \cdot \langle H_{-3}, \Phi_- \rangle \\
& -2i e^{-\phi} \mathcal{W} H_{-3} \\
& -i e^{-\phi} \Sigma^\sharp \cdot (\mathcal{W} H_{-3}),
\end{aligned} \tag{V.86}$$

where each line corresponds to a given SU(3, 3) representation. These backgrounds therefore have non-vanishing **1** and **35** contributions to the intrinsic torsion, while the **7** and **21** contributions vanish, from (V.84) and (V.78). This is a first example of a non-supersymmetric background dimensionally reducing to a solution of four-dimensional  $\mathcal{N} = 1$  supergravity with non-vanishing F-term and superpotential.

### b) LMMT backgrounds

A family of backgrounds T-duals to the original GKP construction has been introduced in [44], the LMMT backgrounds. They have been discussed at length in chapter IV. In type IIB, they obey the following pure spinor equations

$$d_H \Phi_- = i r j \tag{V.87}$$

$$d_H(e^{-A} \text{Re} \Phi_+) = 0 \tag{V.88}$$

$$(F + d_H \Sigma)_{-1} = 0 \tag{V.89}$$

$$(F + d_H \Sigma)_{-3} = 2 d_H \Sigma|_{-3} \tag{V.90}$$

with

$$j = (-1)^{|\Phi_-|} \text{Re} \Phi_+ + \frac{1}{2} \Lambda_{mn} \gamma^m \text{Re} \Phi_+ \gamma^n, \tag{V.91}$$

and  $r$  a complex supersymmetry breaking parameters.  $\Lambda_{mn}$  and the gamma matrices conventions are discussed in chapter IV.

Using these, we evaluate the Dorfman derivative of the untwisted generalised SU(7) structure along  $V^0 = W + \alpha \Phi_- \in L_3^0$

$$\begin{aligned}
L_{V^0}^{H+F+d_H \Sigma} \Phi_- &= \mathcal{W} d_H \Phi_- - d_H(\alpha \Phi_-) \cdot \Phi_- + (\mathcal{W}(F + d_H \Sigma) + \alpha \langle F + d_H \Sigma, \Phi_- \rangle) \cdot \Phi_- \\
&= \left( Z \odot W - i \alpha r j + 2 \mathcal{W}(d_H \Sigma|_{-3}) + 2 \alpha \langle d_H \Sigma|_{-3}, \Phi_- \rangle \right) \cdot \Phi_-,
\end{aligned} \tag{V.92}$$

with, in SU(3, 3) indices:  $(Z \odot W)_{ij} = W^k Z_{kij} \in \Gamma(\nu_{-2})$ , and  $Z_{ijk} \Gamma^{ijk} \Phi_- = i r j$ .

From (V.12), we have

$$T^0(V^0) = -Z \odot W + i \alpha r j - 2 \mathcal{W}(d_H \Sigma|_{-3}) - 2 \alpha \langle d_H \Sigma|_{-3}, \Phi_- \rangle, \tag{V.93}$$

up to pieces acting trivially on  $\Phi_-$ .

We decompose this expression in terms of the different irreducible SU(7) representa-

tions of the  $SU(7)$  intrinsic torsion

$$T_7^0(V^0) = 0 \quad (\text{V.94})$$

$$\begin{aligned} T_{35}^0(V^0) &= -Z \odot W + \Sigma^\sharp \cdot [\mathcal{W}(d_H \Sigma|_{-3})] \\ &\quad + i\alpha r j + \alpha \Sigma^\sharp \cdot \langle d_H \Sigma|_{-3}, \Phi_- \rangle \end{aligned} \quad (\text{V.95})$$

$$\begin{aligned} T_1^0(V^0) &= -2\alpha \langle d_H \Sigma|_{-3}, \Phi_- \rangle \\ &\quad - \alpha \Sigma^\sharp \cdot \langle d_H \Sigma|_{-3}, \Phi_- \rangle \\ &\quad - 2\mathcal{W}(d_H \Sigma|_{-3}) \\ &\quad - \Sigma^\sharp \cdot [\mathcal{W}(d_H \Sigma|_{-3})], \end{aligned} \quad (\text{V.96})$$

where each line corresponds to a given  $SU(3, 3)$  representation.

These backgrounds therefore have non-vanishing **1** and **35** contributions to the intrinsic torsion, while the **7** and **21** contributions vanish, from (V.94) and (V.88).

### c) Legramandi-Tomasiello backgrounds

Legramandi-Tomasiello backgrounds [45] are massive type IIA backgrounds with a static  $SU(2)$  structure and have been discussed in chapter IV. Their supersymmetry breaking comes from a violation of the D-string BPSness

$$d_H \Phi_+ = 0 \quad (\text{V.97})$$

$$d_H(e^{-\Delta} \text{Re} \Phi_-) = \frac{c}{8} e^{6\Delta - 2\phi} \text{vol}_6 \quad (\text{V.98})$$

$$(F + d_H \Sigma)_{-1} = ice^{4\Delta - 2\phi} \mathcal{J}_- \cdot (\text{vol}_6|_{-1}), \quad (\text{V.99})$$

with the subscript now denoting the charge under  $\mathcal{J}_+$ , and  $c$  a supersymmetry breaking parameter. As such, they are an example of string-like supersymmetry breaking backgrounds (SSB). These backgrounds also respect  $F_{-3} = d_H \Sigma|_{-3} = 0$ , which is imposed by (V.97).

Using these, we evaluate the Dorfman derivative of the untwisted generalised  $SU(7)$  structure along  $V^0 = W + \alpha_- \in L_3^0$

$$\begin{aligned} L_{V^0}^{H+F+d_H \Sigma} \Phi_- &= \mathcal{W} d_H \Phi_- - d_H(\alpha \Phi_-) \cdot \Phi_- + (\mathcal{W}(F + d_H \Sigma) + \alpha \langle F + d_H \Sigma, \Phi_- \rangle) \cdot \Phi_- \\ &= ice^{4\Delta - 2\phi} \mathcal{W}[\mathcal{J}_- \cdot (\text{vol}_6|_{-1})] \cdot \Phi_- . \end{aligned} \quad (\text{V.100})$$

From (V.12), we have

$$T^0(V^0) = -ice^{4\Delta - 2\phi} \mathcal{W}[\mathcal{J}_- \cdot (\text{vol}_6|_{-1})]. \quad (\text{V.101})$$

We decompose this expression in terms of the different irreducible SU(7) representations of the SU(7) intrinsic torsion

$$T_7^0(V^0) = 0 \quad (\text{V.102})$$

$$T_{35}^0(V^0) = -ice^{4\Delta-2\phi}\mathcal{W}[\mathcal{J}_- \cdot (\text{vol}_6|_{-1})] \quad (\text{V.103})$$

$$T_1^0(V^0) = 0. \quad (\text{V.104})$$

The **1** and  $\bar{\mathbf{7}}$  representation of the intrinsic torsion therefore vanish for these backgrounds, whereas the **35** representation is non-vanishing, as well as the **21** representation, from (V.98).

#### d) Other SSB Backgrounds

Focusing on the type IIB case, the family of backgrounds from [92], discussed at length in IV.2, obeys the following pure spinor equations

$$d_H\Phi_- = irj \quad (\text{V.105})$$

$$d_H(e^{-\Delta}\text{Re}\Phi_+) = \alpha_m[\gamma^m j + (-1)^{|\Phi_+|}j\gamma^m] \quad (\text{V.106})$$

$$(F + d_H\Sigma)_{-1} = 8ie^{-2\Delta}\alpha_m\mathcal{J}_1 \cdot [\gamma^m j + (-1)^{|\Phi_+|}j\gamma^m]_{-1} \quad (\text{V.107})$$

$$(F + d_H\Sigma)_{-3} = 2d_H\Sigma|_{-3} \quad (\text{V.108})$$

where  $\alpha_m$  are real supersymmetry breaking parameters that are responsible for the D-string BPSness violation, while  $r$  is a complex supersymmetry breaking parameter generating the domain-wall supersymmetry breaking contribution.

Using these modified pure spinor equations, we evaluate the Dorfman derivative of the untwisted generalised SU(7) structure along  $V^0 = W + \alpha\Phi_- \in L_3^0$

$$\begin{aligned} L_{V^0}^{H+F+d_H\Sigma}\Phi_- &= \mathcal{W}d_H\Phi_- - d_H(\alpha\Phi_-) \cdot \Phi_- + (\mathcal{W}(F + d_H\Sigma) + \alpha \langle F + d_H\Sigma, \Phi_- \rangle) \cdot \Phi_- \\ &= \left( Z \odot W - i\alpha rj + 8ie^{-2\Delta}\mathcal{W}(\alpha_m\mathcal{J}_1 \cdot [\gamma^m j + (-1)^{|\Phi_+|}j\gamma^m]_{-1}) \right. \\ &\quad \left. + 2\mathcal{W}(d_H\Sigma|_{-3}) + 2\alpha \langle d_H\Sigma|_{-3}, \Phi_- \rangle \right) \cdot \Phi_-, \end{aligned} \quad (\text{V.109})$$

with, in SU(3,3) indices:  $(Z \odot W)_{ij} = W^k Z_{kij} \in \Gamma(\nu_{-2})$ , and  $Z_{ijk}\Gamma^{ijk}\Phi_- = irj$ .

From (V.12), we have

$$\begin{aligned} T^0(V^0) &= -Z \odot W + i\alpha rj - 8ie^{-2\Delta}\mathcal{W}(\alpha_m\mathcal{J}_1 \cdot [\gamma^m j + (-1)^{|\Phi_+|}j\gamma^m]_{-1}) \\ &\quad - 2\mathcal{W}(d_H\Sigma|_{-3}) - 2\alpha \langle d_H\Sigma|_{-3}, \Phi_- \rangle, \end{aligned} \quad (\text{V.110})$$

up to pieces acting trivially on  $\Phi_-$ .



We decompose this expression in terms of the different irreducible  $SU(7)$  representations of the  $SU(7)$  intrinsic torsion

$$T_7^0(V^0) = 0 \tag{V.111}$$

$$\begin{aligned} T_{35}^0(V^0) &= -Z \odot W + \Sigma^\sharp \cdot [\mathcal{W}(d_H \Sigma|_{-3})] \\ &\quad + i\alpha r j - 8ie^{-2\Delta} \mathcal{W}(\alpha_m \mathcal{J}_1 \cdot [\gamma^m j + (-1)^{|\Phi_+|} j \gamma^m]_{-1}) + \alpha \Sigma^\sharp \cdot \langle d_H \Sigma|_{-3}, \Phi_- \rangle \end{aligned} \tag{V.112}$$

$$\begin{aligned} T_1^0(V^0) &= -2\alpha \langle d_H \Sigma|_{-3}, \Phi_- \rangle \\ &\quad - \alpha \Sigma^\sharp \cdot \langle d_H \Sigma|_{-3}, \Phi_- \rangle \\ &\quad - 2\mathcal{W}(d_H \Sigma|_{-3}) \\ &\quad - \Sigma^\sharp \cdot [\mathcal{W}(d_H \Sigma|_{-3})], \end{aligned} \tag{V.113}$$

where each line corresponds to a given  $SU(3, 3)$  representation.

These backgrounds therefore have non-vanishing **1** and **35** contributions to the intrinsic torsion, and the supersymmetry breaking term in (V.106) also results in a non-vanishing **21** representation of the intrinsic torsion.

Note that the subfamily of backgrounds with vanishing supersymmetry breaking parameter  $r$  obey the following set of pure spinor equations

$$d_H \Phi_- = 0 \tag{V.114}$$

$$d_H(e^{-\Delta} \text{Re} \Phi_+) = \alpha_m [\gamma^m j + (-1)^{|\Phi_+|} j \gamma^m] \tag{V.115}$$

$$(F + d_H \Sigma)_{-1} = 8ie^{-2\Delta} \alpha_m \mathcal{J}_1 \cdot [\gamma^m j + (-1)^{|\Phi_+|} j \gamma^m]_{-1} \tag{V.116}$$

with  $F_{-3} = d_H \Sigma|_{-3} = 0$ , which is imposed by (V.114). In this case, the intrinsic torsion is decomposed as

$$T_7^0(V^0) = 0 \tag{V.117}$$

$$T_{35}^0(V^0) = -8ie^{-2\Delta} \mathcal{W}(\alpha_m \mathcal{J}_1 \cdot [\gamma^m j + (-1)^{|\Phi_+|} j \gamma^m]_{-1}) \tag{V.118}$$

$$T_1^0(V^0) = 0. \tag{V.119}$$

Therefore only the **35** and **21** representations of the intrinsic torsion are non-vanishing for this subfamily of backgrounds.

## V.2 The equations of motion

Throughout this chapter, we discuss non-supersymmetric type II Minkowski flux backgrounds described through modified pure spinor equations. In V.1, we kept the super-

symmetry breaking forms entering these modified pure spinor equations as general as possible. However, a crucial question remains to be addressed: given that the supersymmetry conditions are violated, what is the subset of supersymmetry breaking forms such that the equations of motion are satisfied?

This is a very complicated question to answer in the general case, and it has been partially addressed in the  $O(6,6)$  generalised geometry formalism [44, 45, 91, 92] (see also chapter IV). We now investigate this problem from the  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry perspective.

We will derive here the equations of motion in the  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry framework, and we will see that they can be spelled out in terms of the non-vanishing  $SU(7)$  representations of the  $SU(7)$  intrinsic torsion.

Given that, in the above section, we established a dictionary between the supersymmetry breaking forms entering the modified pure spinor equations and the corresponding non-vanishing  $SU(7)$  representations of the  $SU(7)$  intrinsic torsion, these  $E_{7(7)} \times \mathbb{R}^+$  equations of motion will translate the complicated conditions to impose on the supersymmetry breaking forms into conditions on the  $SU(7)$  representations of the  $SU(7)$  intrinsic torsion.

To do so, we start by recalling the two following real  $SU(8)$  bundles  $S$  and  $J$ , the spinor and gravitino bundles respectively. These are

$$S = \mathbf{8} + \bar{\mathbf{8}} \equiv S^+ + S^- \quad J = \mathbf{56} + \bar{\mathbf{56}} \equiv J^+ + J^-. \quad (\text{V.120})$$

We then recall the equations of motion, discussed in section III.2:

$$D \times_J (D \times_J \eta) + 2D \times_J (D \times_S \eta) = 0 \quad (\text{V.121})$$

$$D \times_S (D \times_J \eta) + D \times_S (D \times_S \eta) = 0, \quad (\text{V.122})$$

with  $\eta$  the  $SU(8)$  spinor defined from the two internal supersymmetry parameters (III.155), and  $D$  a generalised connection.

We consider  $D$  to be compatible with the  $SU(8)/\mathbb{Z}_2$  structure, that is we have  $DG = 0$ , with  $G$  the generalised metric introduced in section III.2. If one considers another  $SU(8)/\mathbb{Z}_2$  compatible generalised derivative,  $D'$ , by definition it can be written as  $D' = D + \Sigma$ , and

$$\Sigma = D' - D \in \Gamma(K_{SU(8)}), \quad \text{with} \quad K_{SU(8)} = E^* \otimes \text{ad}P_{SU(8)}. \quad (\text{V.123})$$

In SU(8) indices, sections  $\Sigma$  of  $K_{\text{SU}(8)}$  are given by

$$\Sigma = (\Sigma_{\alpha\beta}{}^\gamma{}_\delta, \Sigma^{\alpha\beta\gamma}{}_\delta) \in (\mathbf{28} + \bar{\mathbf{28}}) \times \mathbf{63} = K_{\text{SU}(8)}. \quad (\text{V.124})$$

We can write down the  $\tau$  map discussed in section III.2,  $\tau : K_{\text{SU}(8)} \rightarrow W$  explicitly using SU(8) indices, where  $W$  is the space of generalised torsions  $W \simeq K \oplus E^*$ :

$$\tau(\Sigma) = T_{D'} - T_D \in \Gamma(W), \quad (\text{V.125})$$

with the usual torsion definition

$$T_D(V) \cdot V' = L_V^D V' - L_{V'} V \quad \forall V, V' \in \Gamma(E). \quad (\text{V.126})$$

It gives

$$\tau(\Sigma)_{\alpha\beta} = \Sigma_{\alpha\gamma}{}^\gamma{}_\beta, \quad \in \mathbf{28} + \mathbf{36} \quad (\text{V.127})$$

$$\tau(\Sigma)_{\alpha\beta\gamma}{}^\delta = \Sigma_{[\alpha\beta}{}^0{}_{\gamma]}{}^\delta, \quad \in \mathbf{420}, \quad (\text{V.128})$$

where the 0 superscript means that it is completely traceless. The **36** and **28** SU(8) representations correspond to the symmetrised and antisymmetrised two-forms of SU(8) respectively.

We now consider a specific torsion-free SU(8)-compatible generalised covariant derivative such that

$$D = \hat{D} - \Sigma, \quad (\text{V.129})$$

with  $\hat{D}$  a SU(7)-compatible connection. Since it is SU(7)-compatible, it's also SU(8)-compatible and therefore  $\Sigma \in \Gamma(K_{\text{SU}(8)})$ . The compatibility of  $\hat{D}$  with the generalised SU(7) structure is equivalent to

$$\hat{D} \times_S \eta = 0 \quad \hat{D} \times_J \eta = 0. \quad (\text{V.130})$$

We now decompose all our SU(8) objects into irreducible SU(7) representations, introducing SU(7) indices  $a, b, \dots = 1, \dots, 7$ . Recall from section III.2 that the SU(8) spinor  $\eta$  defines the generalised SU(7) structure. As such, it is a singlet representation of SU(7), and as a section of the  $S^+$  bundle, it reads

$$\eta \equiv \eta^8, \quad \eta^a = 0, \quad (\text{V.131})$$

where we used again a dead SU(8) index to highlight the SU(7)  $\subset$  SU(8) embedding. We choose  $\hat{D}$  such that  $\Sigma \in W_{\text{int}}^{\text{SU}(7)}$ . Following [42], we decompose  $\Sigma \in \Gamma(K_{\text{SU}(8)})$  into

irreducible SU(7) representations. In SU(7) indices, this gives:

$$\Sigma = (\Sigma_{8\gamma}{}^\gamma{}_8, \Sigma_{[ab}{}^8{}_{c]}, \Sigma_{a\gamma}{}^\gamma{}_8, \Sigma_{[ab}{}^8{}_{8]}, c.c.) \quad (\text{V.132})$$

$$\mathbf{1} \oplus \mathbf{35} \oplus \mathbf{7} \oplus \mathbf{21} \oplus c.c.. \quad (\text{V.133})$$

Similarly, we decompose the SU(7) compatible connection  $\hat{D}$  into SU(7) representations. It reads, in SU(8) indices,  $\hat{D} \equiv (\hat{D}^{[\alpha\beta]}, \hat{D}_{[\alpha\beta]})$ , which results in

$$\hat{D} = (\hat{D}^{[ab]}, \hat{D}^{a8}, \hat{D}_{[ab]}, \hat{D}_{a8}), \quad (\text{V.134})$$

in SU(7) indices. Finally, we decompose the  $S$  and  $J$  SU(8) bundles into SU(7) representations:

$$S^+ \rightarrow S_7^+ \oplus S_1^+ \quad J^+ \rightarrow J_{35}^+ \oplus J_{21}^+ \quad (\text{V.135})$$

$$\mathbf{8} \rightarrow \mathbf{7} \oplus \mathbf{1} \quad \mathbf{56} \rightarrow \mathbf{35} \oplus \mathbf{21} \quad (\text{V.136})$$

$$S^- \rightarrow S_7^- \oplus S_1^- \quad J^- \rightarrow J_{35}^- \oplus J_{21}^- \quad (\text{V.137})$$

$$\bar{\mathbf{8}} \rightarrow \bar{\mathbf{7}} \oplus \bar{\mathbf{1}} \quad \bar{\mathbf{56}} \rightarrow \bar{\mathbf{35}} \oplus \bar{\mathbf{21}}. \quad (\text{V.138})$$

We calculate the equations of motion (V.121) and (V.122), considering the connection  $D$  in (V.129), and the SU(7) projections given in appendix B.3, (B.136)-(B.152). This yields

$$-\frac{1}{2}\hat{D}^{[ab]}(\Sigma_{[ab}{}^8{}_{8]}) - \hat{D}^{[8a]}(\Sigma_{a\gamma}{}^\gamma{}_8) + \frac{1}{2}\Sigma_{[ab}{}^8{}_{8]}\Sigma^{[ab8]}{}_8 \quad (\text{V.139})$$

$$+ \frac{1}{2}\Sigma_{[ef}{}^8{}_{g]}\Sigma^{[efg]}{}_8 + \Sigma_{8\gamma}{}^\gamma{}_8\Sigma^{8\gamma}{}_\gamma{}^8 + \Sigma_{a\gamma}{}^\gamma{}_8\Sigma^{a\gamma}{}_\gamma{}^8 = 0 \quad (\text{V.140})$$

for the projection onto the bundle  $S_1^+$ , and

$$-\hat{D}^{[ab]}(\Sigma_{b\gamma}{}^\gamma{}_8) - \hat{D}^{[a8]}(\Sigma_{8\gamma}{}^\gamma{}_8) - \frac{1}{2}\hat{D}_{[bc]}(\Sigma^{[abc]}{}_8) \quad (\text{V.141})$$

$$- \frac{1}{2}\hat{D}_{[b8]}(\Sigma^{[ab8]}{}_8) + \Sigma^{a\gamma}{}_\gamma{}^8\Sigma_{8\gamma}{}^\gamma{}_8 + \frac{1}{2}\Sigma_{b\gamma}{}^\gamma{}_8\Sigma^{[ab8]}{}_8 = 0 \quad (\text{V.142})$$

for the component of the bundle  $S_7^+$ .

Then we find

$$\frac{1}{12}\epsilon_{ab8efghi}\hat{D}^{[ef]}(\Sigma^{[gh}{}^i]{}_8) + 2\hat{D}_{[8a]}(\Sigma_{b]\gamma}{}^\gamma{}_8) \quad (\text{V.143})$$

$$+ 2\hat{D}_{[ab]}(\Sigma_{8\gamma}{}^\gamma{}_8) - 2\Sigma_{[a|\gamma}{}^\gamma{}_8\Sigma_{|b]\gamma}{}^\gamma{}_8 - 2\Sigma_{[ab}{}^8{}_{8]}\Sigma_{8\gamma}{}^\gamma{}_8 = 0 \quad (\text{V.144})$$

for the projection onto the bundle  $J_{21}^-$ , and

$$\frac{1}{12}\epsilon_{abcd8efg}\hat{D}^{[d8]}(\Sigma^{[ef}{}_8{}^{g]}) + 2\hat{D}_{[ab}(\Sigma_{c]{}^\gamma}{}^\gamma{}_8) - 2\Sigma_{[ab}{}^8{}_{c]}\Sigma_{8\gamma}{}^\gamma{}_8 = 0 \quad (\text{V.145})$$

for the component of the bundle  $J_{35}^-$ .

This formulation of the equations of motion has the advantage that these are fairly simple first order partial differential equations. However, a few difficulties remains to be addressed in order to be able to try and solve these equations explicitly:

i) We only have the explicit expression for  $T^0(V^0)$  (V.66) in terms of  $SU(3, 3)$  representations, and extracting the exact expressions of the  $SU(7)$  torsion components  $T^0 \in \Gamma(\tilde{K}_{\mathbb{C}})$  would require the use of the cumbersome decomposition of the action

$$\odot : \tilde{K}_{\mathbb{C}} \odot E_{\mathbb{C}} \rightarrow \text{ad}\tilde{F}_{\mathbb{C}} \quad (\text{V.146})$$

in terms of irreducible  $SU(3, 3)$  representations.

ii) Establishing the expression of the  $SU(7)$  compatible generalised connection  $\hat{D}$  in terms of irreducible  $SU(3, 3)$  representations is a non-trivial task.

Investigating further these equations of motion is still a work in progress.

### V.3 Deformations of non-integrable $SU(7)$ structures

We conclude this chapter by initiating the investigation of the classical moduli space of non-supersymmetric Minkowski type II flux backgrounds within the framework of  $E_{7(7)} \times \mathbb{R}^+$  generalised geometry.

This discussion is intended as pointing toward an example of applications of the results derived in section V.1, and doesn't contain any definitive results.

In the supersymmetric case, the classical moduli space of  $\mathcal{N} = 1$  Minkowski type II flux backgrounds has been calculated in [61].

To do so, the authors introduced the following fibre bundle  $\mathcal{Q}_{\mathbb{R}^+ \times U(7)}$

$$\mathcal{Q}_{\mathbb{R}^+ \times U(7)} \rightarrow \mathcal{Q}_{\mathbb{R}^+ \times U(7)} \rightarrow M \quad (\text{V.147})$$

with

$$\mathcal{Q}_{\mathbb{R}^+ \times U(7)} = \frac{E_{7(7)} \times \mathbb{R}^+}{\mathbb{R}^+ \times U(7)}. \quad (\text{V.148})$$

One can then consider the space of deformations of the  $\mathbb{R}^+ \times U(7)$  structure as the principle adjoint bundle  $\text{ad}(\mathcal{Q}_{\mathbb{R}^+ \times U(7)})$ .

The integrable deformations of the  $\mathbb{R}^+ \times U(7)$  structure are then the ones preserving the integrability of said structure.

As we have discussed in section III.2, the integrability of the untwisted almost exceptional complex structure  $L_3^0$  is equivalent to

$$L_V^{H+F} V' \subset L_3^0 \quad \forall V, V' \in \Gamma(L_3^0). \quad (\text{V.149})$$

We therefore define a general deformation of the  $L_3^0$  bundle as

$$\hat{L}_3^0 := e^R \cdot L_3^0 \quad \text{with} \quad R \in \Gamma(\text{ad}(\mathcal{Q}_{\mathbb{R}^+ \times U(7)})), \quad (\text{V.150})$$

and the subset of integrable deformations correspond to the section  $R \in \Gamma(\text{ad}(\mathcal{Q}_{\mathbb{R}^+ \times U(7)}))$  such that

$$L_V^{H+F} V' \subset \hat{L}_3^0 \quad \forall V, V' \in \Gamma(\hat{L}_3^0). \quad (\text{V.151})$$

In order to calculate the moduli space of the integrable  $SU(7)$  structure, one should impose the vanishing of the moment map (III.151) and mod out by the trivial deformations between  $SU(7)$  structures, which correspond to the generalised diffeomorphisms  $\text{GDiff}$ , as discussed in section III.2.

However, given that

$$\{\psi \in \hat{\mathcal{Z}} \mid \mu = 0\} / \text{GDiff} \simeq \hat{\mathcal{Z}} / \text{GDiff}_{\mathbb{C}}, \quad (\text{V.152})$$

where  $\hat{\mathcal{Z}}$  is the space of  $SU(7)$  structures with an integrable associated exceptional complex structure, this is equivalent to simply modding out by the complexified generalised diffeomorphisms  $\text{GDiff}_{\mathbb{C}}$ . This allows to spell out the moduli space of  $\mathcal{N} = 1$  backgrounds in terms of generalised cohomologies, as discussed at length in [61].

In the non-supersymmetric case, we can't select the deformations preserving the integrability of the generalised structures, since these aren't integrable anymore. Instead, to define deformations, we can select the deformations preserving the value of the non-vanishing  $SU(7)$  intrinsic torsion.

To do so, we need to consider deformations that are irreducible representations of a group  $P \subset SU(7)$  such that both the  $SU(7)$  structure  $\psi$  and the  $SU(7)$  intrinsic torsion live in trivial representations of  $P$ . In that case, the deformations will preserve the value of the  $SU(7)$  intrinsic torsion.

Moving to the complex picture presented in section V.1, we must have

$$P_{\mathbb{C}} \subset H_{\text{comp}} \equiv \text{SL}(7, \mathbb{C}) \subset H \equiv (\text{SL}(7, \mathbb{C}) \ltimes \mathbb{C}^{35}) \ltimes \mathbb{C}^7, \quad (\text{V.153})$$

such that the complexification of  $P$  sits within the compact subgroup of  $H$ , denoted  $H_{\text{comp}}$ ,

with  $H$  the group stabilising the untwisted generalised  $SU(7)$  structure, as discussed in section V.1.

A convenient way to define  $P_{\mathbb{C}}$  is through its corresponding complex subalgebra  $\mathfrak{p}$

$$\mathfrak{p} \subset \mathfrak{h} \quad \text{such that} \quad (\mathfrak{p} \cdot [T^0(V^0)])/\mathfrak{h} \in T^0(V^0) \quad \forall V^0 \in \Gamma(L_3^0). \quad (\text{V.154})$$

This ensures that  $\mathfrak{p}$  stabilises  $T^0(V^0)$  and hence  $T^0$  since  $\mathfrak{p}$  is a subgroup of  $\mathfrak{h}$  which stabilises the untwisted generalised  $SU(7)$  structure  $\psi$  and thus  $V^0 \in \Gamma(L_3^0)$ .

This approach circumvents the complications which would result from directly searching for the group which stabilises  $T^0$  as a section of the torsion bundle  $\tilde{K}_{\mathbb{C}}$ .

We can define the following complex bundle

$$Q_H \rightarrow \mathcal{Q}_H \rightarrow M \quad (\text{V.155})$$

with

$$Q_H = \frac{E_{7\mathbb{C}}}{H}. \quad (\text{V.156})$$

Recall that for a given non-supersymmetric background, we have

$$L_{V^0}^{H+F+d_H\Sigma} \Phi_2 = -T^0(V^0) \cdot \Phi_2 \quad \forall V^0 \in \Gamma(L_3^0). \quad (\text{V.157})$$

We define the following deformations

$$\hat{L}_3^0 := e^R \cdot L_3^0 \quad \hat{\Phi}_2 := e^R \cdot \Phi_2 \quad (\text{V.158})$$

$$\hat{\Sigma} := e^R \cdot \Sigma \quad \hat{T}^0 := e^R \cdot T^0, \quad (\text{V.159})$$

with

$$R \in \Gamma(\text{ad}(\mathcal{Q}_H)). \quad (\text{V.160})$$

Crucially, we decompose  $R$  into irreducible representations of  $P$ , such that

$$T^0 := \alpha T'^0, \quad (\text{V.161})$$

with  $T'^0$  the section of  $\tilde{K}_{\mathbb{C}}$  deformed by  $R$ , and  $\alpha$  a complex constant which remains constant under the deformations.

We thus define the ‘‘integrable’’ deformations associated to a given non-supersymmetric background as the section  $R \in \Gamma(\text{ad}(\mathcal{Q}_H))$  such that

$$L_{\hat{V}^0}^{H+F+d_H\hat{\Sigma}} \hat{\Phi}_2 = -\hat{T}^0(\hat{V}^0) \cdot \hat{\Phi}_2 \quad \forall \hat{V}^0 \in \Gamma(\hat{L}_3^0). \quad (\text{V.162})$$

Although this approach allows to define the deformations preserving a given non-integrable exceptional complex structure, defining the method to calculate the moduli space of  $SU(7)$  structures is still work in progress. The missing piece is a definition of the trivial deformations of our backgrounds -the non-supersymmetric analogue of modding out by  $\mathbf{GDiff}_{\mathbb{C}}$ -. We conclude this section with a speculative comment about how to construct such deformations.

For a non-supersymmetric background and its associated non-integrable  $SU(7)$  structure  $\psi$ , one could define a deformed twisted Dorfman derivative

$$\mathbb{L}_V^{H+F} := L_V^{H+F} + T(V) \cdot \quad \forall V \in \Gamma(E), \quad (\text{V.163})$$

with  $T \in \tilde{K}_{\mathbb{C}}$  the  $SU(7)$  intrinsic torsion of  $\psi$ .

This would allow for a notion of integrability for the exceptional complex structure, spelled out in terms of the deformed Dorfman derivative:

$$\mathbb{L}_V^{H+F} V' \subset L_3^0 \quad \forall V, V' \in \Gamma(L_3^0). \quad (\text{V.164})$$

The associated deformed moment map would read<sup>7</sup>

$$\tilde{\mathfrak{m}}(\tilde{V}) := \frac{1}{3} \int_M s(\mathbb{L}_V^{H+F} \tilde{\psi}, \tilde{\psi}) (i s(\tilde{\psi}, \tilde{\psi}))^{-2/3}. \quad (\text{V.165})$$

Using this deformed moment map, we could define the trivial deformations of non-supersymmetric backgrounds in complete analogy with the supersymmetric case. Indeed, crucially, the integrability of the exceptional complex structure is necessary for the equivalence (V.152) to hold (see for instance [114]). It could therefore still hold through the introduction of the deformed Dorfman derivative. In this case, it reads

$$\{\psi \in \hat{\mathcal{Z}} \mid \mathfrak{m} = 0\} / \mathbf{GDiff} \simeq \hat{\mathcal{Z}} / \mathbf{GDiff}_{\mathbb{C}}. \quad (\text{V.166})$$

The modified complexified generalised diffeomorphisms  $\mathbf{GDiff}_{\mathbb{C}}$  are then generated by the deformed Dorfman derivative  $\mathbb{L}$ , and encode the trivial deformations of the corresponding non-supersymmetric background.

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<sup>7</sup>Recall the moment map (III.151), untwisted by the fluxes.





# Chapter VI

## Discussion

In this thesis, we set out to deepen the understanding of non-supersymmetric type II supergravity flux vacua. We focused on solutions compactified to four-dimensional Minkowski space, which we explored within the framework of generalised geometries, both complex and exceptional.

The results derived in the first part of the thesis, presented in chapter IV, all relied on a central feature: the generalised complex geometric description of the background sources - the D-branes and orientifolds-. For a given source, this description mainly revolves around the introduction of two objects: the generalised calibration and the generalised current associated to the source.

Generalised calibrations are natural extensions of ordinary calibrations: they are polyforms of given degree associated to submanifolds. If such a calibration obeys the calibrations conditions -a differential and an algebraic condition-, the corresponding submanifold is said to be calibrated. A D-brane wrapping a calibrated submanifold is then supersymmetric, it hence minimises its energy<sup>1</sup>. Crucially, the differential calibration conditions associated to sources extended in two, three, and four non-compact dimensions are exactly equivalent to preserving  $\mathcal{N} = 1$  supersymmetry for the corresponding ten-dimensional background.

Then, the generalised current associated to a given source is simply a generalisation of its ordinary current, the Poincaré dual to the calibrated submanifold, now calibrated by the generalised calibration form.

In this thesis, describing the sources in terms of their generalised calibrations and currents has allowed to establish the following results:

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<sup>1</sup>The difference between ordinary and generalised calibrations stems from the definition of the D-brane energy: in the ordinary case it is proportional to its volume, while the generalised case takes into account non-vanishing RR potential contributions.

i) We derived new classes of non-supersymmetric type II supergravity flux vacua compactified to four-dimensional Minkowski space. A key aspect to these constructions has been that supersymmetry was broken via the introduction of supersymmetry breaking terms into the supersymmetry conditions, and the generalised current associated to the background sources served as a building block for these supersymmetry breaking terms. This proved out to be a simple and natural ansatz, drastically simplifying the study of the corresponding equations of motion, that are very hard to solve in this formalism for general supersymmetry breaking terms.

ii) We partially addressed the question of perturbative stability for our newly introduced classes of non-supersymmetric flux backgrounds. Indeed, within the framework of generalised complex geometry, the generalised current associated to the background sources can enter the effective potential associated to a given ten-dimensional background, which is particularly useful as it allows to use powerful positivity arguments for the effective potential from the calibration bound, one of the calibration condition. We were able to prove that for a given truncation suggested by the geometry, the effective potential associated to our classes of solutions is positive semi-definite, and vanishes at the solutions. However, this is not quite equivalent to proving that our classes of solutions are perturbatively stable, given that we have little control over the aforementioned truncation.

iii) For a given non-supersymmetric background with BPS space-filling sources, we derived conditions on the supersymmetry breaking terms entering the corresponding deformed supersymmetry conditions such that this background dimensionally reduces to a non-supersymmetric solution of four-dimensional  $\mathcal{N} = 1$  supergravity.

The motivation to study such backgrounds comes from the fact that type II flux backgrounds compactified to four-dimensional Minkowski space must admit some space-filling orientifold sources in order to evade the no-go theorem of [66, 67]. We therefore focus on the case of well-behaved backgrounds with stable space-filling sources. In that regard, the formalism of generalised complex geometry is very helpful, as it allows to identify the supersymmetry condition associated to preserving the BPSness of such sources, which must be satisfied.

In order to derive the aforementioned constraints, we generalised to the non-supersymmetric setting the reformulation of the BPSness condition associated to space-filling sources derived in [6]. We then required that this condition is an F-term condition, coming from one of the variations of the superpotential, even when supersymmetry breaking terms are switched on. Interestingly, this procedure constrained the possible supersymmetry breaking terms generating D-terms in the corresponding effective theory.

We then derived the equations of motion for such backgrounds, and for the subclass of backgrounds having vanishing D-terms on shell. Solutions of this kind correspond to vacua

with non-vanishing superpotential and F-terms, and include the amply discussed class of no-scale vacua. These equations of motion were significantly simpler than the equations of motion associated to the most general supersymmetry breaking ansatz, sparking the hope to solve them for new non-supersymmetric solutions.

The above results further extend the generalised complex geometry literature on non-supersymmetric type II flux backgrounds, initiated in [44]. However, non-supersymmetric type II flux backgrounds were never studied within the exceptional generalised geometry framework. We set out to initiate this line of work in chapter V. Supersymmetric type II flux backgrounds were described within exceptional generalised geometry in [40, 42] and in [61], where a given  $\mathcal{N} = 1$  background was proven to be equivalent to a torsion-free generalised SU(7) structure. We argued in this thesis that a non-supersymmetric background is entirely defined by a generalised SU(7) structure, and its non-vanishing SU(7) compatible intrinsic torsion. We arranged the SU(7) torsion of a given  $\mathcal{N} = 0$  background into irreducible representations of SU(7), the generalised structure group, and established a dictionary between these intrinsic torsion components and the supersymmetry breaking terms entering the modified supersymmetry conditions in the generalised complex geometry formalism, bridging the two frameworks. Such a dictionary was then spelled out for different concrete examples of non-supersymmetric flux backgrounds.

Another line of work developed in chapter V concerns the reformulation of the supergravity equations of motion within the exceptional generalised geometry framework. These were written as first order differential conditions on the different irreducible SU(7) representations of the intrinsic torsion.

## VI.1 Future directions

There are many directions of work extending or completing the results derived in this thesis. First within the generalised complex geometry framework, the new classes of non-supersymmetric type II flux backgrounds presented in chapter IV must have heterotic analogues which would be interesting to study within  $O(6, 6 + n)$  generalised geometry, with  $n$  the rank of the heterotic gauge group. It would also be interesting to look for other  $\mathcal{N} = 0$  backgrounds with different patterns of supersymmetry breaking, still using the current associated to the space-filling sources of the background as a building block.

Another direct extension of the results presented in chapter IV would be to look for concrete ansatz for internal geometries within which one could solve the relatively simple equations of motion for the class of backgrounds with BPS space-filling sources

and dimensionally reducing to non-supersymmetric solutions of four-dimensional  $\mathcal{N} = 1$  supergravity.

From a broader perspective, there are other applications where the formalism of generalised calibrations must be insightful.

For instance, it would be interesting to develop the generalised calibration theory for backgrounds with a three-dimensional Minkowski and  $\text{AdS}_3$  external space, where the  $\mathcal{N} = 1$  supersymmetry conditions consists in two differential conditions and an algebraic one, the "pairing condition" [115, 116]. Supersymmetry thus wouldn't be quite equivalent to the calibrations of the different possible D-branes, given that the pairing condition has nothing to do with a calibration condition.

Additionally, this would prove to be very useful to study the non-supersymmetric solutions of [117], the massive type IIA  $\text{AdS}_3$  analogues of the DGKT solutions [118], which are poorly understood. These solutions are particularly interesting given that they provide one of the very few example of scale separated AdS vacua [119]. Reformulating such backgrounds within the  $O(7, 7)$  generalised geometry framework -as it has been done in [120], but only for the supersymmetric solutions- would spell the breaking of supersymmetry in terms of the violation of a calibration condition, providing insights into the stability of the sources present in these backgrounds.

Finally, there are many directions in which to extend the results derived within the exceptional generalised geometry framework, given that we have merely initiated the construction of the formalism suited to describe non-supersymmetric type II flux backgrounds.

The most obvious one is to exploit the equations of motion derived in section V.2. Solving them upfront will be a difficult task, but they can surely teach us interesting things about the different possible supersymmetry breaking, by highlighting the interplay between the different irreducible  $SU(7)$  representations of the intrinsic torsion, and thus the corresponding supersymmetry breaking terms.

Casting the  $\mathcal{N} = 1$  supersymmetry conditions in terms of integrability conditions for a generalised  $SU(7)$  structure has allowed the authors of [61] to calculate the classical moduli space of general  $\mathcal{N} = 1$  type II flux backgrounds<sup>2</sup>.

An interesting extension of the work presented in chapter V would be to set a deformation problem for non-supersymmetric type II flux backgrounds, and eventually calculate (part of) the classical moduli space of some of these backgrounds, if they exist. To define the deformations of these backgrounds, we cannot rely on requiring their integrability,

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<sup>2</sup>With the caveat that the two type II internal spinors must be globally defined.

as in the supersymmetric case. Instead, one could consider deformations that are irreducible representations of a group  $G \subset \text{SU}(7)$  stabilising both the  $\text{SU}(7)$  structure and the non-vanishing intrinsic torsion. The well behaved deformations would then be the ones preserving the intrinsic torsion.

Finally, let us mention the recent reformulation of consistent truncations within the exceptional generalised geometry framework [80, 121–125]. The key to these constructions -see [80]- is that the intrinsic torsion is stable under a specific group  $G$  within a given consistent truncation. This suits particularly well within the formalism developed in chapter V, and could allow one to naturally construct explicit non-supersymmetric consistent truncations.



# Appendix A

## Supergravity Conventions

### A.1 Notations

Given a  $p$ -vector  $v \in \Lambda^p T$  and a  $q$ -form  $\lambda \in \Lambda^q T^*$ , we write them in components as

$$v = \frac{1}{p!} v^{m_1 \dots m_p} \partial_{m_1} \wedge \dots \wedge \partial_{m_p} \quad (\text{A.1})$$

$$\lambda = \frac{1}{q!} \lambda_{m_1 \dots m_q} dy^{m_1} \wedge \dots \wedge dy^{m_q}. \quad (\text{A.2})$$

For  $v' \in \Lambda^{p'} TM$  and  $\lambda' \in \Lambda^{q'} T^* M$ , the wedge products and contractions are given by

$$(v \wedge v')^{m_1 \dots m_{p+p'}} = \frac{(p+p')!}{p!p'} v^{[m_1 \dots a_p} v'^{m_{p+1} \dots m_{p+p'}]}, \quad (\text{A.3})$$

$$(\lambda \wedge \lambda')_{m_1 \dots m_{q+q'}} = \frac{(q+q')!}{q!q'} \lambda_{[m_1 \dots m_q} \lambda'_{m_{q+1} \dots m_{q+q'}}] \quad (\text{A.4})$$

$$(v \lrcorner \lambda)_{m_1 \dots m_{q-p}} := \frac{1}{p!} v^{n_1 \dots n_p} \lambda_{n_1 \dots n_p m_1 \dots m_{q-p}} \quad \text{if } p \leq q \quad (\text{A.5})$$

$$(v \lrcorner \lambda)^{m_1 \dots m_{p-q}} := \frac{1}{q!} v^{m_1 \dots p-q n_1 \dots n_q} \lambda_{n_1 \dots n_q} \quad \text{if } p \geq q. \quad (\text{A.6})$$

We also use the “ $j$ -notation” from [88]

$$(j\lambda \wedge \rho)_{m, m_1 \dots m_d} := \frac{d!}{(q-1)!(d+1-q)!} \lambda_{m[m_1 \dots m_{q-1}} \rho_{m_q \dots m_d]} \quad (\text{A.7})$$

$$(jv \lrcorner j\lambda)^m_n := \frac{1}{(p-1)!} v^{m r_1 \dots r_{p-1}} \lambda_{n r_1 \dots r_{p-1}}. \quad (\text{A.8})$$

Given a basis  $\{\hat{e}_a\}$  for  $T$  and a dual basis  $\{e^a\}$  for  $T^*$ , there is a natural  $\mathfrak{gl}_d$  action on tensors of any rank. For a vector and a three-form, for example, this action gives

$$(r \cdot v)^m = r^m_n v^n \quad (\text{A.9})$$



$$(r \cdot \lambda)_{m_1 m_2 m_3} = -r^{m_1} \lambda_{n m_2 m_3} - r^{m_2} \lambda_{m_1 n m_3} - r^{m_3} \lambda_{m_1 m_2 n}. \quad (\text{A.10})$$

We introduce the following ten- and six-dimensional Hodge operators

$$\tilde{*}_{10} = *_{10} \circ \sigma \quad (\text{A.11})$$

$$\tilde{*}_6 = *_6 \circ \sigma \quad (\text{A.12})$$

with  $\sigma$  the reversal of all form indices, and with, for a  $p$ -form  $\omega$

$$*_{10} \omega_p = -\frac{1}{p!(10-p)!} \sqrt{-g} \epsilon_{M_1 \dots M_{10}} \omega^{M_{11-p} \dots M_{10}} dx^{M_1} \wedge \dots \wedge dx^{M_{10-p}} \quad (\text{A.13})$$

$$*_6 \omega_p = \frac{1}{p!(6-p)!} \sqrt{-g} \epsilon_{m_1 \dots m_6} \omega^{m_{7-p} \dots m_6} dy^{m_1} \wedge \dots \wedge dy^{m_{6-p}}. \quad (\text{A.14})$$

## A.2 Bosonic sector

Our supergravity conventions are identical to the ones of [44].

The bosonic sector of type II supergravity is composed of the NS sector and the RR sector. The NS sector contains the metric  $g$ , the dilaton  $\phi$ , and the NS three-form flux  $H$ , which can locally be written

$$H = dB \quad (\text{A.15})$$

away from the NS sources, with  $B$  its two-form potential.

The RR sector contains the RR field-strength: we use the democratic formulation of [69], where

$$F^{10} = \sum_q F_q^{10} \quad (\text{A.16})$$

with  $q = 0, 2, \dots, 10$  for type IIA and  $q = 1, 3, \dots, 9$  for type IIB. These fields obey the following self-duality condition

$$F^{10} = \tilde{*}F^{10}. \quad (\text{A.17})$$

Away from the RR sources, we write the RR fluxes from the RR potentials as

$$F^{10} = dC + H \wedge C \quad (\text{A.18})$$

with  $C = \sum_q C_{q-1}$ .

The type II pseudo-action in democratic formalism is

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi} [R + 4(d\phi)^2 - \frac{1}{2}H^2] - \frac{1}{4}(F^{10})^2 \right\} + S^{(\text{loc})}, \quad (\text{A.19})$$

where  $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$  and for any real p-form  $\omega$  we define  $\omega^2 = \omega \cdot \omega$  with  $\cdot$  defined as

$$\omega \cdot \chi = \frac{1}{p!} \omega_{M_1 \dots M_p} \chi^{M_1 \dots M_p}. \quad (\text{A.20})$$

If  $\omega$  is complex, we consider

$$|\omega|^2 = \omega \cdot \bar{\omega}. \quad (\text{A.21})$$

$S^{(\text{loc})}$  corresponds to the contributions of the localised sources to the action. Varying this action and imposing the self-duality condition (A.17) we find the following equations of motion.

The dilaton equation

$$\nabla^2 \phi - (d\phi)^2 + \frac{1}{4} R - \frac{1}{8} H^2 - \frac{1}{4} \frac{\kappa_{10}^2 e^{2\phi}}{\sqrt{-g}} \frac{\delta S^{(\text{loc})}}{\delta \phi} = 0, \quad (\text{A.22})$$

the  $B$ -field equation

$$-d(e^{-2\phi} *_{10} H) + \frac{1}{2} [*_{10} F^{10} \wedge F^{10}]_8 + 2\kappa_{10}^2 \frac{\delta S^{(\text{loc})}}{\delta B} = 0, \quad (\text{A.23})$$

the Einstein equation

$$e^{-2\phi} [g_{MN} + 2g_{MN} d\phi \cdot d\phi - 2g_{MN} \nabla^2 \phi + 2\nabla_M \nabla_N \phi] \quad (\text{A.24})$$

$$- \frac{1}{2} \iota_M H \cdot \iota_N H + \frac{1}{4} g_{MN} H \cdot H] - \frac{1}{4} \iota_M F^{10} \cdot \iota_N F^{10} - \kappa_{10}^2 T_{MN}^{(\text{loc})} = 0, \quad (\text{A.25})$$

with

$$T_{MN}^{(\text{loc})} = -\frac{2}{\sqrt{-g}} \frac{\delta S^{(\text{loc})}}{g^{MN}}, \quad (\text{A.26})$$

and the RR-fluxes variation gives the Bianchi identities

$$d_H F^{10} = -j_{\text{source}}. \quad (\text{A.27})$$

Combining the dilaton equation of motion with the Einstein equations, one can write the modified Einstein equations

$$R_{MN} + 2\nabla_M \nabla_N \phi - \iota_M H \cdot \iota_N H - \frac{1}{4} e^{2\phi} \iota_M F^{10} \cdot \iota_N F^{10} \quad (\text{A.28})$$

$$- \kappa_{10}^2 e^{2\phi} \left( T_{MN}^{(\text{loc})} + \frac{g_{MN}}{2\sqrt{-g}} \frac{\delta S^{(\text{loc})}}{\delta \phi} \right) = 0. \quad (\text{A.29})$$

We now specify the compactification ansatz: we consider type II solutions that are the warped product of four-dimensional Minkowski space  $X_4$  and a six-dimensional compact

manifold  $M$ , with the following metric

$$ds_{10}^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad (\text{A.30})$$

where again  $x^\mu$ ,  $\mu = 0, \dots, 3$  are the external coordinates on  $X_4$ , and  $y^m$ ,  $m = 1, \dots, 6$  are the coordinates on  $M$ .

The Poincaré invariance of  $X_4$  constrains the NS and RR-fluxes: the NS-field-strength  $H$  can only have internal legs, and the ten-dimensional RR-field-strength must take the form

$$F^{10} = F + e^{4A} \text{vol}_4 \wedge \tilde{F}, \quad (\text{A.31})$$

with  $\text{vol}_4$  the volume form on  $X_4$  and where  $F$  and  $\tilde{F}$  are purely internal and are related by the self-duality of  $F^{10}$  (A.17) as

$$\tilde{F} = \tilde{*}_6 F. \quad (\text{A.32})$$

### A.3 Gamma matrices

We use a real representation of the ten-dimensional gamma matrices  $\Gamma_M$ . The ten-dimensional chiral operator is

$$\Gamma_{(10)} = \Gamma^{01\dots 9} \quad (\text{A.33})$$

with flat ten-dimensional indices. For any p-form  $\omega$ , we denote its image under the Clifford map  $\phi$  with

$$\omega \equiv \frac{1}{p!} \omega_{M_1\dots M_p} dx^{M_1\dots M_p} \quad \longleftrightarrow \quad \phi = \frac{1}{p!} \omega_{M_1\dots M_p} \Gamma^{M_1\dots M_p}. \quad (\text{A.34})$$

We define the splitting of the ten-dimensional gamma matrices into four- and six-dimensional gamma matrices  $\hat{\gamma}^\mu$  and  $\gamma^m$  as

$$\Gamma^\mu = e^{-A} \hat{\gamma}^\mu \otimes \mathbb{1} \quad \Gamma^m = \gamma_{(4)} \otimes \gamma^m. \quad (\text{A.35})$$

The  $\hat{\gamma}^\mu$  are associated to the unwarped four-dimensional metric, and  $\gamma_{(4)} = i\hat{\gamma}^{0123}$  is the usual four-dimensional chiral operator. The six-dimensional chiral operator is  $\gamma_{(6)} = -i\gamma^{123456}$  so we have  $\Gamma_{(10)} = \gamma_{(4)} \otimes \gamma_{(6)}$ .

The chirality of the internal spinors is then

$$\gamma_{(6)} \eta_1 = \eta_1 \quad \gamma_{(6)} \eta_2 = \mp \eta_2 \quad \text{in type IIA/IIB.} \quad (\text{A.36})$$

# Appendix B

## Algebra Conventions

### B.1 $O(6,6) \times \mathbb{R}^+$ Algebra

The  $O(6,6) \times \mathbb{R}^+$  generalised tangent and adjoint spaces are

$$E = T \oplus T^* \tag{B.1}$$

$$\text{ad } \tilde{F} = \mathbb{R} \oplus (T \otimes T^*) \oplus \Lambda^2 T \oplus \Lambda^2 T^*, \tag{B.2}$$

and we write their sections as

$$V = v + \xi \tag{B.3}$$

$$R = l + r + B + \beta, \tag{B.4}$$

where each term matches with the expressions above in the obvious way.

The adjoint action on a section of the generalised tangent space is  $R \cdot V = V'$  with

$$v' = lv + r \cdot v - \beta \lrcorner \xi \tag{B.5}$$

$$\xi' = l\xi + r \cdot \xi + v \lrcorner B \tag{B.6}$$

The adjoint action on a section of the generalised adjoint space is  $[R', R] = R''$  with

$$l'' = 0 \tag{B.7}$$

$$r'' = [r', r] - (j\beta' \lrcorner jB - j\beta \lrcorner jB') \tag{B.8}$$

$$B'' = r' \cdot B - r \cdot B' \tag{B.9}$$

$$\beta'' = r' \cdot \beta - r \cdot \beta'. \tag{B.10}$$

We define the Mukai pairing for a pair of polyforms  $\omega$  and  $\chi$

$$\langle \omega, \chi \rangle = \omega \wedge \sigma(\chi)|_6, \quad (\text{B.11})$$

and more generally, we use throughout Chapter IV

$$\langle \omega, \chi \rangle_k = \omega \wedge \sigma(\chi)|_k. \quad (\text{B.12})$$

In the case of a six-dimensional manifold  $M$ , the Mukai pairing satisfies the following property

$$\int_M \langle d_H \omega, \chi \rangle = \int_M \langle \omega, d_H \chi \rangle. \quad (\text{B.13})$$

## B.2 $E_{7(7)} \times \mathbb{R}^+$ Algebra in type IIB

In type IIB, the decomposition into  $O(6,6)$  generalised bundles of the  $E_{7(7)} \times \mathbb{R}^+$  generalised tangent and adjoint spaces takes the form

$$E \simeq E_{O(6,6)} \oplus S^- \oplus (\Lambda^6 T^* \otimes E_{O(6,6)}) \quad (\text{B.14})$$

$$\text{ad} \tilde{F} \simeq \mathbb{R} \oplus \mathbb{R} \oplus \text{ad} \tilde{F}_{O(6,6)} \oplus S^+ \oplus (\Lambda^6 T \otimes S^+) \oplus \Lambda^6 T^* \oplus \Lambda^6 T. \quad (\text{B.15})$$

We write sections of these bundles as

$$V = X + \omega + \tilde{X} \quad (\text{B.16})$$

$$R = q + l + \nu + s + \tilde{s} + a + \tilde{a} \quad (\text{B.17})$$

respectively, with  $X \in E_{O(6,6)}$ ,  $\omega \in S^-$ , etc.

We recall the  $GL(6)$  decompositions of the  $O(6,6)$  bundles  $E_{O(6,6)}$  and  $\Lambda^6 T^* \otimes E_{O(6,6)}$

$$E_{O(6,6)} \simeq T \oplus T^* \quad (\text{B.18})$$

$$\Lambda^6 T^* \otimes E_{O(6,6)} \simeq \Lambda^6 T^* \otimes (T \oplus T^*) \quad (\text{B.19})$$

with the following sections

$$X = v + \lambda \quad (\text{B.20})$$

$$\tilde{X} = (\tilde{v} + \tilde{\lambda}) \otimes \text{vol}, \quad (\text{B.21})$$

with  $v, \tilde{v} \in T$ ,  $\lambda, \tilde{\lambda} \in T^*$ , and  $\text{vol}$  the volume form in the string frame.

We introduce  $O(6, 6)$  indices  $A = 1, \dots, 12$  and  $O(6, 6)$  gamma matrices, such that

$$(v_{\lrcorner} + \lambda \wedge) \omega = X^A \Gamma_A \omega \equiv \not{X} \omega, \quad (\text{B.22})$$

with  $\omega$  a form.

The adjoint action on a section of the generalised tangent space is  $R \cdot V = V'$  with

$$X'^A = qX^A - lX^A + \nu^A_B X^B + \langle \tilde{s}, \Gamma^A \omega \rangle \quad (\text{B.23})$$

$$\tilde{X}'^A = q\tilde{X}^A + l\tilde{X}^A + \nu^A_B \tilde{X}^B + \langle s, \Gamma^A \omega \rangle \quad (\text{B.24})$$

$$\omega' = q\omega + \frac{1}{4} \nu_{AB} \Gamma^{AB} \omega + X^A \Gamma_A s - \tilde{X}^A \Gamma_A \tilde{s}. \quad (\text{B.25})$$

The adjoint action on a section of the generalised adjoint space is  $[R', R] = R''$  with

$$q'' = \frac{1}{2} (\nu'^A_B \nu'^B_A - \nu^A_B \nu'^B_A + \langle s', \Gamma^A \tilde{s} \rangle - \langle \tilde{s}', \Gamma^A s \rangle) \quad (\text{B.26})$$

$$l'' = \tilde{a}a' - \tilde{a}'a + \langle s', \tilde{s} \rangle - \langle s, \tilde{s}' \rangle \quad (\text{B.27})$$

$$\nu'^A_B = \nu'^A_C \nu'^C_B - \nu^A_C \nu'^C_B + \langle s', \Gamma^A_B \tilde{s} \rangle - \langle \tilde{s}', \Gamma^A_B s \rangle \quad (\text{B.28})$$

$$s'' = l's - ls' + \tilde{s}a' - \tilde{s}'a + \frac{1}{4} \nu'_{AB} \Gamma^{AB} s - \frac{1}{4} \nu_{AB} \Gamma^{AB} s' \quad (\text{B.29})$$

$$\tilde{s}'' = -l'\tilde{s} + l\tilde{s}' + \tilde{a}'s - \tilde{a}s' + \frac{1}{4} \nu'_{AB} \Gamma^{AB} \tilde{s} - \frac{1}{4} \nu_{AB} \Gamma^{AB} \tilde{s}' \quad (\text{B.30})$$

$$a'' = 2l'a - 2la' - \langle s', s \rangle + \langle s, s' \rangle \quad (\text{B.31})$$

$$\tilde{a}'' = 2l\tilde{a}' - 2l'\tilde{a} + \langle \tilde{s}', \tilde{s} \rangle - \langle \tilde{s}, \tilde{s}' \rangle. \quad (\text{B.32})$$

The torsion bundle  $K \subset E^* \otimes \text{ad } \tilde{F}$  can locally be written in terms of  $O(6, 6)$  irreducible representations as

$$\begin{aligned} K \simeq & E_{O(6,6)} \oplus (E_{O(6,6)} \otimes \Lambda^6 T^*) \oplus S^\pm \oplus (\Lambda^6 T \otimes S^\pm) \oplus (\Lambda^6 T^* \otimes S^\pm) \\ & \oplus [E_{O(6,6)} \otimes S^\pm]_0 \oplus \Lambda^3(E_{O(6,6)}) \oplus (\Lambda^6 T^* \otimes \Lambda^3(E_{O(6,6)})), \end{aligned} \quad (\text{B.33})$$

where  $\Phi^A \in \Gamma([E_{O(6,6)} \otimes S^\pm]_0)$  respects  $\Gamma_A \Phi^A = 0$ . The fibres of this bundle transforms in the  $\mathbf{912}_1$  representation of  $E_{7(7)} \times \mathbb{R}^+$ , where the subscript denotes the  $\mathbb{R}^+$  weight.

The bundle  $N$  is a subbundle of the symmetric product  $S^2 E$ . The fibres of  $N$  belongs to the  $\mathbf{133}_2$  representation of  $E_{7(7)} \times \mathbb{R}^+$ , and  $N$  is isomorphic to

$$N \simeq \mathbb{R} \oplus \Lambda^6 T^* \oplus (\Lambda^6 T^* \otimes \Lambda^6 T^*) \oplus S^\mp \oplus (\Lambda^6 T^* \otimes S^\mp) \oplus (\text{ad } \tilde{F}_{O(6,6)} \otimes \Lambda^6 T^*). \quad (\text{B.34})$$

The Dorfman derivative reads

$$L_V V' = L_X X' + (d(\not{X} \omega') + \not{X} d\omega' - d\omega \cdot X')$$

$$+ (L_X \tilde{X}' + (\partial, \tilde{X}) \cdot X' - d\omega \cdot \omega'), \quad (\text{B.35})$$

with  $L_X X'$  the standard  $O(6, 6)$  Dorfman derivative, and  $\cdot$  the adjoint action.

We also introduced

$$L_X \tilde{X}' = \mathcal{L}_v \tilde{X}' + j(\tilde{v}' \lrcorner \text{vol}) \wedge d\lambda \quad (\text{B.36})$$

$$(\partial, \tilde{X}) = d(\tilde{v} \lrcorner \text{vol}). \quad (\text{B.37})$$

We define a flux-twisted Dorfman derivative as

$$L_V^{H+F} V' := L_V V' + (H \odot X + H \wedge \omega + \mathcal{X}F + \langle F, \omega \rangle) \cdot V', \quad (\text{B.38})$$

with  $H \odot X = v \lrcorner H$ , and where the NS and RR fluxes are sections of the torsion bundle  $H \in \Gamma(\Lambda^3(E_{O(6,6)})) \in \Gamma(K)$  and  $F \in \Gamma(S^\pm) \in \Gamma(K)$ .

The symplectic invariant reads

$$s(V, V') = \eta_{AB} \tilde{X}^A X'^B - \eta_{AB} \tilde{X}'^A X^B + \langle \omega, \omega' \rangle, \quad (\text{B.39})$$

where  $\eta_{AB}$  is the natural  $O(6, 6)$  inner product in indices.

### B.2.1 $\text{SL}(7, \mathbb{C}) \subset \text{SL}(8, \mathbb{C}) \subset E_{7(7)\mathbb{C}}$ embedding for type IIB

The complex generalised tangent, adjoint and torsion  $E_{7(7)\mathbb{C}}$  bundles can be decomposed into irreducible representations of  $\text{SL}(8, \mathbb{C})$

$$E_{\mathbb{C}} \simeq \mathbf{28} \oplus \overline{\mathbf{28}} \quad (\text{B.40})$$

$$\text{ad} \tilde{F}_{\mathbb{C}} \simeq \mathbf{63} \oplus \mathbf{70} \quad (\text{B.41})$$

$$K_{\mathbb{C}} \simeq \mathbf{36} \oplus \mathbf{420} \oplus \overline{\mathbf{36}} \oplus \overline{\mathbf{420}}. \quad (\text{B.42})$$

In  $\text{SL}(8, \mathbb{C})$  indices  $\alpha, \beta = 1, \dots, 8$ , the decomposition of sections of the generalised tangent, adjoint and torsion spaces are written respectively

$$V = (V^{\alpha\beta}, \bar{V}_{\alpha\beta}) \quad (\text{B.43})$$

$$R = (R^\alpha{}_\beta, R_{\alpha\beta\gamma\delta}) \quad (\text{B.44})$$

$$\phi = (\phi^{\alpha\beta}, \phi^{\alpha\beta\gamma}{}_\delta, \bar{\phi}_{\alpha\beta}, \bar{\phi}_{\alpha\beta\gamma}{}^\delta), \quad (\text{B.45})$$

with  $V^{\alpha\beta} = -V^{\beta\alpha}$ ,  $R^\alpha{}_\alpha = 0$ ,  $R_{\alpha\beta\gamma\delta}$  fully antisymmetric,  $\phi^{\alpha\beta} = \phi^{\beta\alpha}$ ,  $\phi^{\alpha\beta\gamma}{}_\delta = \phi^{[\alpha\beta\gamma]}{}_\delta$ , and  $\phi^{\alpha\beta\gamma}{}_\gamma = 0$ , and similarly for the barred object.

The adjoint action on a section of the generalised tangent space is then  $R \cdot V = V'$

with

$$V'^{\alpha\beta} = R^\alpha{}_\gamma V^{\gamma\beta} + R^\beta{}_\gamma V^{\alpha\gamma} + \star R^{\alpha\beta\gamma\delta} \bar{V}_{\gamma\delta} \quad (\text{B.46})$$

$$\bar{V}'_{\alpha\beta} = -R^\gamma{}_\alpha \bar{V}_{\gamma\beta} - R^\gamma{}_\beta \bar{V}_{\alpha\gamma} - R_{\alpha\beta\gamma\delta} V^{\gamma\delta}, \quad (\text{B.47})$$

where  $\star$  is the eight-dimensional Hodge operator.

The adjoint action on a section of the generalised adjoint space is  $[R', R] = R''$  with

$$R''^\alpha{}_\beta = R'^\alpha{}_\gamma R^\gamma{}_\beta - R^\alpha{}_\gamma R'^\gamma{}_\beta - \frac{1}{3}(\star R'^{\alpha\gamma\delta\epsilon} R_{\beta\gamma\delta\epsilon} - \star R^{\alpha\gamma\delta\epsilon} R'_{\beta\gamma\delta\epsilon}) \quad (\text{B.48})$$

$$R''_{\alpha\beta\gamma\delta} = 4(R'^\epsilon{}_{[\alpha} R_{\beta\gamma\delta]\epsilon} - R^\epsilon{}_{[\alpha} R'_{\beta\gamma\delta]\epsilon}). \quad (\text{B.49})$$

We also need the action

$$\odot : \mathbf{912}_\mathbb{C} \odot \mathbf{56}_\mathbb{C} \rightarrow \mathbf{133}_\mathbb{C} \quad (\text{B.50})$$

$$\phi \odot V \rightarrow R, \quad (\text{B.51})$$

which reads

$$R^\alpha{}_\beta = (V^{\gamma\alpha} \bar{\phi}_{\gamma\beta} + \bar{V}_{\gamma\beta} \phi^{\gamma\alpha}) + (\bar{V}_{\gamma\delta} \phi^{\gamma\delta\alpha}{}_\beta - V^{\gamma\delta} \bar{\phi}_{\gamma\delta\beta}{}^\alpha) \quad (\text{B.52})$$

$$R_{\alpha\beta\gamma\delta} = -4(\bar{\phi}_{[\alpha\beta\gamma}{}^\epsilon \bar{V}_{\delta]\epsilon} - \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta\mu_1\mu_2\mu_3\mu_4} \phi^{\mu_1\mu_2\mu_3}{}_\epsilon V^{\mu_4\epsilon}). \quad (\text{B.53})$$

We now give the decomposition of the generalised tangent, adjoint and torsion bundles in terms of  $\text{SL}(7, \mathbb{C}) \in \text{SL}(8, \mathbb{C})$  irreducible representations

$$E_\mathbb{C} \simeq \mathbf{7} \oplus \mathbf{21} \oplus \overline{\mathbf{21}} \oplus \overline{\mathbf{7}} \quad (\text{B.54})$$

$$\text{ad}\tilde{F}_\mathbb{C} \simeq \mathbf{1} \oplus \mathbf{48} \oplus (\mathbf{7} \oplus \overline{\mathbf{7}}) \oplus (\mathbf{35} \oplus \overline{\mathbf{35}}) \quad (\text{B.55})$$

$$K_\mathbb{C} \simeq \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{21} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{140} \oplus \mathbf{224} \oplus c.c.. \quad (\text{B.56})$$

Introducing  $\text{SL}(7, \mathbb{C})$  indices:  $a, b = 1, \dots, 7$ , we write the corresponding sections as<sup>1</sup>

$$V = (V^{a8}, V^{ab}, \bar{V}_{ab}, \bar{V}_{a8}) \quad (\text{B.57})$$

$$R = (R^8{}_8 + \frac{1}{7} R^a{}_a \mathbb{1}, R^{0a}{}_b, R^a{}_8, R^8{}_a, R_{abcd}, R_{abc8}) \quad (\text{B.58})$$

$$\begin{aligned} \phi = & (\phi^{88}, \phi^{a8}, \phi^{ab8}, \phi^{ab}{}_8, \phi^{abc}{}_8, \phi^{ab8}{}_d, \phi^{abc}{}_d, \\ & \bar{\phi}_{88}, \bar{\phi}_{a8}, \bar{\phi}_{ab8}, \bar{\phi}_{ab}{}^8, \bar{\phi}_{abc}{}^8, \bar{\phi}_{ab8}{}^d, \bar{\phi}_{abc}{}^d), \end{aligned} \quad (\text{B.59})$$

where each term matches with the expressions above in the obvious way, with  $\mathbb{1}$  the

<sup>1</sup>We use the notation of a dead  $\text{SL}(8, \mathbb{C})$  index "8", in order to see the  $\text{SL}(7, \mathbb{C}) \subset \text{SL}(8, \mathbb{C})$  embedding explicitly.



seven-dimensional identity matrix, and with

$$V^{ab} = V^{ba} \quad (\text{B.60})$$

$$\phi^{ab} = \phi^{ba} \quad (\text{B.61})$$

$$R^{0a}{}_a = 0 \quad (\text{B.62})$$

$$R_{abcd} = R_{[abcd]}, \quad R_{abc8} = R_{[abc]8} \quad (\text{B.63})$$

$$\phi^{abc}{}_c = \phi^{a8b}{}_b = 0 \quad (\text{B.64})$$

$$\phi^{abc}{}_d = \phi^{[abc]}{}_d, \quad \phi^{abc}{}_8 = \phi^{[abc]}{}_8, \quad \phi^{ab8}{}_d = \phi^{[ab]8}{}_d, \quad \phi^{ab8}{}_8 = \phi^{[ab]8}{}_8. \quad (\text{B.65})$$

and similarly for the barred objects.

The adjoint action on a section of the generalised tangent space is then  $R \cdot V = V'$  with

$$\begin{aligned} V'^{ab} &= (R^{0a}{}_c + \frac{1}{7}R^e{}_e\delta^a{}_c)V^{cb} + \frac{1}{2}R^a{}_8V^{8b} + (R^{0b}{}_c + \frac{1}{7}R^e{}_e\delta^b{}_c)V^{ac} + \frac{1}{2}R^b{}_8V^{a8} \\ &\quad + \star R^{abcd}\bar{V}_{cd} + \frac{1}{8}\star R^{abc8}\bar{V}_{c8} \end{aligned} \quad (\text{B.66})$$

$$V'^{a8} = \frac{1}{2}(R^{0a}{}_c + \frac{1}{7}R^e{}_e\delta^a{}_c)V^{c8} + \frac{1}{2}R^8{}_8V^{a8} + R^8{}_cV^{ac} + \frac{1}{4}\star R^{a8cd}\bar{V}_{cd} \quad (\text{B.67})$$

$$\begin{aligned} \bar{V}'_{ab} &= -(R^{0c}{}_a + \frac{1}{7}R^e{}_e\delta^c{}_a)\bar{V}_{cb} - \frac{1}{2}R^8{}_a\bar{V}_{8b} - (R^{0c}{}_b + \frac{1}{7}R^e{}_e\delta^c{}_b)\bar{V}_{ac} \\ &\quad - \frac{1}{2}R^8{}_b\bar{V}_{a8} - \frac{1}{8}R_{abc8}V^{c8} - R_{abcd}V^{cd} \end{aligned} \quad (\text{B.68})$$

$$\bar{V}'_{a8} = -\frac{1}{2}(R^{0c}{}_a + \frac{1}{7}R^e{}_e\delta^c{}_a)\bar{V}_{c8} - R^c{}_8\bar{V}_{ac} - \frac{1}{2}R^8{}_8\bar{V}_{a8} - \frac{1}{4}R_{a8cd}V^{cd}. \quad (\text{B.69})$$

The adjoint action on a section of the generalised adjoint space is  $[R', R] = R''$  with

$$R^8{}_8 + \frac{1}{7}R^a{}_a\mathbb{1} = R'^8{}_aR^a{}_8 - R^8{}_aR'^a{}_8 - \frac{1}{16}(\star R'^8abcR_{8abc} - \star R^{8abc}R'_{8abc}) \quad (\text{B.70})$$

$$\begin{aligned} &+ \frac{1}{7}[(R'^{0f}{}_c + \frac{1}{7}R'^e{}_e\delta^f{}_c)(R^{0c}{}_f + \frac{1}{7}R^e{}_e\delta^c{}_f) + R'^a{}_8R^8{}_f \\ &- (R^{0a}{}_c + \frac{1}{7}R^e{}_e\delta^f{}_c)(R'^{0c}{}_b + \frac{1}{7}R'^e{}_e\delta^c{}_b) - R^f{}_8R'^8{}_b \\ &- \frac{1}{3}\star R'^fcdeR_{fcde} - \frac{1}{16}\star R'^fcd8R_{fcd8} \\ &+ \frac{1}{3}\star R'^fcdeR'_{fcde} + \frac{1}{16}\star R'^fcd8R'_{fcd8}]\mathbb{1} \end{aligned} \quad (\text{B.71})$$

$$\begin{aligned} R'^{0a}{}_b &= (R^{0a}{}_c + \frac{1}{7}R'^e{}_e\delta^a{}_c)(R'^{0c}{}_b + \frac{1}{7}R'^e{}_e\delta^c{}_b) + R'^a{}_8R^8{}_b \\ &- (R^{0a}{}_c + \frac{1}{7}R^e{}_e\delta^a{}_c)(R'^{0c}{}_b + \frac{1}{7}R'^e{}_e\delta^c{}_b) - R^a{}_8R'^8{}_b \\ &- \frac{1}{3}\star R'^acdeR_{bcde} - \frac{1}{16}\star R'^acd8R_{bcd8} + \frac{1}{3}\star R'^acdeR'_{bcde} + \frac{1}{16}\star R'^acd8R'_{bcd8} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{7}[(R'^{0f}{}_c + \frac{1}{7}R'^e{}_e\delta^f{}_c)(R'^{0c}{}_f + \frac{1}{7}R'^e{}_e\delta^c{}_f) + R'^a{}_8R'^8{}_f \\
& - (R'^{0a}{}_c + \frac{1}{7}R'^e{}_e\delta^a{}_c)(R'^{0c}{}_b + \frac{1}{7}R'^e{}_e\delta^c{}_b) - R'^f{}_8R'^8{}_b \\
& - \frac{1}{3} \star R'^{fcd}{}_e R'^{fcd}{}_e - \frac{1}{16} \star R'^{fcd}{}_8 R'^{fcd}{}_8 \\
& + \frac{1}{3} \star R'^{fcd}{}_e R'^{fcd}{}_e + \frac{1}{16} \star R'^{fcd}{}_8 R'^{fcd}{}_8] \delta^a{}_b
\end{aligned} \tag{B.72}$$

$$\begin{aligned}
R''^a{}_8 &= (R'^{0a}{}_c + \frac{1}{7}R'^e{}_e\delta^a{}_c)R'^c{}_8 + R'^a{}_8R'^8{}_8 - (R'^{0a}{}_c + \frac{1}{7}R'^e{}_e\delta^a{}_c)R'^c{}_8 - R'^a{}_8R'^8{}_8 \\
& - \frac{1}{4}(\star R'^{acde} R'^{acde} - \star R'^{acde} R'^{acde})
\end{aligned} \tag{B.73}$$

$$\begin{aligned}
R''^8{}_a &= R'^8{}_c(R'^{0c}{}_a + \frac{1}{7}R'^e{}_e\delta^c{}_a) + R'^8{}_8R'^8{}_a - R'^8{}_c(R'^{0c}{}_a + \frac{1}{7}R'^e{}_e\delta^c{}_a) - R'^8{}_8R'^8{}_a \\
& - \frac{1}{4}(\star R'^{8cde} R'^{8cde} - \star R'^{8cde} R'^{8cde})
\end{aligned} \tag{B.74}$$

$$\begin{aligned}
R''_{abcd} &= (4(R'^{0e}{}_{[a} + \frac{1}{7}R'^f{}_f\delta^e{}_{[a})R'_{bcd]e} + 3R'^8{}_{[a}R'_{bcd]8} \\
& - 4(R'^{0e}{}_{[a} + \frac{1}{7}R'^f{}_f\delta^e{}_{[a})R'_{bcd]e} - 3R'^8{}_{[a}R'_{bcd]8})
\end{aligned} \tag{B.75}$$

$$\begin{aligned}
R''_{abc8} &= (3(R'^{0e}{}_{[a} + \frac{1}{7}R'^f{}_f\delta^e{}_{[a})R'_{bc8]e} + R'^8{}_8R'_{abc8} \\
& - 3(R'^{0e}{}_{[a} + \frac{1}{7}R'^f{}_f\delta^e{}_{[a})R'_{bc8]e} - R'^8{}_8R'_{abc8}).
\end{aligned} \tag{B.76}$$

The action

$$\odot : \mathbf{912}_C \odot \mathbf{56}_C \rightarrow \mathbf{133}_C \tag{B.77}$$

$$\phi \odot V \rightarrow R \tag{B.78}$$

reads

$$\begin{aligned}
R^8{}_8 + \frac{1}{7}R^a{}_a\mathbb{1} &= \frac{1}{4}(V^{a8}\bar{\phi}_{a8} + \bar{V}_{a8}\phi^{a8}) + (\bar{V}_{ab}\phi^{ab8}{}_8 - V^{ab}\bar{\phi}_{ab8}{}_8) \\
& + \frac{1}{7}[(V^{ce}\bar{\phi}_{ce} + \frac{1}{4}V^{8e}\bar{\phi}_{8e} + \bar{V}_{ce}\phi^{ce} + \frac{1}{4}\bar{V}_{8e}\phi^{8e}) \\
& + (\bar{V}_{cd}\phi^{cde}{}_e + \frac{1}{6}\bar{V}_{c8}\phi^{c8e}{}_e - V^{cd}\bar{\phi}_{cde}{}_e - \frac{1}{6}V^{c8}\bar{\phi}_{c8e}{}_e)]\mathbb{1}
\end{aligned} \tag{B.79}$$

$$\begin{aligned}
R^{0a}{}_b &= (V^{ca}\bar{\phi}_{cb} + \frac{1}{4}V^{8a}\bar{\phi}_{8b} + \bar{V}_{cb}\phi^{ca} + \frac{1}{4}\bar{V}_{8b}\phi^{8a}) \\
& + (\bar{V}_{cd}\phi^{cda}{}_b + \frac{1}{6}\bar{V}_{c8}\phi^{c8a}{}_b - V^{cd}\bar{\phi}_{cdb}{}_a - \frac{1}{6}V^{c8}\bar{\phi}_{c8b}{}_a) \\
& - \frac{1}{7}[(V^{ce}\bar{\phi}_{ce} + \frac{1}{4}V^{8e}\bar{\phi}_{8e} + \bar{V}_{ce}\phi^{ce} + \frac{1}{4}\bar{V}_{8e}\phi^{8e}) \\
& + (\bar{V}_{cd}\phi^{cde}{}_e + \frac{1}{6}\bar{V}_{c8}\phi^{c8e}{}_e - V^{cd}\bar{\phi}_{cde}{}_e - \frac{1}{6}V^{c8}\bar{\phi}_{c8e}{}_e)]\delta^a{}_b
\end{aligned} \tag{B.80}$$

$$R^a{}_8 = \frac{1}{2}(V^{ca}\bar{\phi}_{c8} + \bar{V}_{c8}\phi^{ca}) + (\bar{V}_{cd}\phi^{cda}{}_8 + \frac{1}{6}\bar{V}_{c8}\phi^{c8a}{}_8 - \frac{1}{3}V^{cd}\bar{\phi}_{cd8}{}_a) \tag{B.81}$$

$$R^8{}_a = \frac{1}{2}(V^{c8}\bar{\phi}_{ca} + \bar{V}_{ca}\phi^{c8}) + \left(\frac{1}{3}\bar{V}_{cd}\phi^{cd8}{}_a - \frac{1}{3}V^{cd}\bar{\phi}_{cda}{}^8 - \frac{1}{6}V^{c8}\bar{\phi}_{c8a}{}^8\right) \quad (\text{B.82})$$

$$\begin{aligned} R_{abcd} = & -4(\bar{\phi}_{[abc}{}^e\bar{V}_{d]e} + \frac{1}{2}\bar{\phi}_{[abc}{}^8\bar{V}_{d]8}) - \frac{1}{2 \cdot 3!}\epsilon_{abcdm_1m_2m_3}8\phi^{m_1m_2m_3}{}_e V^{8e} \\ & - \frac{1}{4 \cdot 3!}\epsilon_{abcdm_1m_2}8m_3\phi^{m_1m_2}{}_8 V^{m_3e} \\ & - \frac{1}{8 \cdot 3!}\epsilon_{abcdm_1m_2}8m_3\phi^{m_1m_2}{}_8 V^{m_38} \end{aligned} \quad (\text{B.83})$$

$$\begin{aligned} R_{abc8} = & -4\left(\frac{1}{2}\bar{\phi}_{[abc}{}^e\bar{V}_{8]e} - \frac{1}{4!}\epsilon_{abc8m_1m_2m_3m_4}\phi^{m_1m_2m_3}{}_e V^{m_4e}\right. \\ & \left. - \frac{1}{2 \cdot 4!}\epsilon_{abc8m_1m_2m_3m_4}\phi^{m_1m_2m_3}{}_8 V^{m_48}\right). \end{aligned} \quad (\text{B.84})$$

### B.2.2 The SU(3, 3) parametrisation

We can parametrise the complexified generalised tangent and adjoint  $E_{7(7)\mathbb{C}}$  bundles in terms of representations of the SU(3, 3) structure associated to the generalised complex structure  $\mathcal{J}_2 \equiv \mathcal{J}_\pm$  in type IIA/IIB. We focus on type IIB in the following

$$E_{\mathbb{C}} \simeq L_1 \oplus L_{-1} \oplus S_3^- \oplus S_1^- \oplus S_{-1}^- \oplus S_{-3}^- \oplus (\Lambda^6 T^* \otimes (L_1 \oplus L_{-1})) \quad (\text{B.85})$$

$$\begin{aligned} \text{ad}\tilde{F}_{\mathbb{C}} \simeq & \mathbb{C} \oplus \mathbb{C} \oplus (L_1 \otimes L_{-1})_0 \oplus \Lambda^2(L_1) \oplus \Lambda^2(L_{-1}) \oplus S_2^+ \oplus S_0^+ \oplus S_{-2}^+ \\ & \oplus (\Lambda^6 T \otimes (S_2^+ \oplus S_0^+ \oplus S_{-2}^+)) \oplus \Lambda^6 T_{\mathbb{C}}^* \oplus \Lambda^6 T_{\mathbb{C}}, \end{aligned} \quad (\text{B.86})$$

where the subscript denotes the charge under  $\mathcal{J}_-$ , and with

$$L_1 \simeq \bar{L}_{-1} \quad S_i \simeq \bar{S}_{-i}. \quad (\text{B.87})$$

We write sections of these bundles as

$$V = X_1 + X_{-1} + \omega_3 + \omega_1 + \omega_{-1} + \omega_{-3} + \tilde{X}_1 + \tilde{X}_{-1} \quad (\text{B.88})$$

$$R = l + p + \nu_0^0 + \nu_2 + \nu_{-2} + s_2 + s_0 + s_{-2} + \tilde{s}_2 + \tilde{s}_0 + \tilde{s}_{-2} + a + \tilde{a} \quad (\text{B.89})$$

respectively, with  $X_1 \in \Gamma(L_1)$ ,  $X_{-1} \in \Gamma(L_{-1})$ , etc, and with  $l \in \mathbb{C}$ ,  $p \in \mathbb{C}$ ,  $\nu_0^0 \in \Gamma((L_1 \otimes L_{-1})_0)$ , etc. We introduce SU(3, 3) indices  $i, j = 1, \dots, 6$  and in index notation we omit the subscript denoting the charge under  $\mathcal{J}_-$ . We have  $p = \frac{1}{6}\nu^i{}_i$  and

$$\nu^{0i}{}_j = \nu^i{}_j - \frac{1}{6}\nu^k{}_k\eta^i{}_j \quad (\text{B.90})$$

$$\bar{\nu}^{0\bar{i}}{}_{\bar{j}} = \bar{\nu}^{\bar{i}}{}_{\bar{j}} - \frac{1}{6}\nu^{\bar{k}}{}_{\bar{k}}\eta^{\bar{i}}{}_{\bar{j}} \quad (\text{B.91})$$

where  $\eta$  is the O(6, 6) metric, with  $\eta_{i\bar{j}}X^i = X_{\bar{j}}$ .

The adjoint action on a section of the generalised tangent space is then  $R \cdot V = V'$

with

$$X^i = (-l + p)X^i + \nu^{0i}{}_j X^j + \nu^i{}_{\bar{j}} X^{\bar{j}} + \langle \tilde{s}_{-2}, \Gamma^i \omega_3 \rangle + \langle \tilde{s}_0, \Gamma^i \omega_1 \rangle + \langle \tilde{s}_2, \Gamma^i \omega_{-1} \rangle \quad (\text{B.92})$$

$$X^{\bar{i}} = (-\bar{l} + \bar{p})X^{\bar{i}} + (\bar{\nu}^0)_{\bar{j}} X^{\bar{j}} + \nu^{\bar{i}}{}_{\bar{j}} X^{\bar{j}} + \langle \tilde{s}_{-2}, \Gamma^{\bar{i}} \omega_1 \rangle + \langle \tilde{s}_0, \Gamma^{\bar{i}} \omega_{-1} \rangle + \langle \tilde{s}_2, \Gamma^{\bar{i}} \omega_{-3} \rangle \quad (\text{B.93})$$

$$\tilde{X}^i = (l + p)\tilde{X}^i + \nu^{0i}{}_j \tilde{X}^j + \nu^i{}_{\bar{j}} \tilde{X}^{\bar{j}} + \langle s_{-2}, \Gamma^i \omega_3 \rangle + \langle s_0, \Gamma^i \omega_1 \rangle + \langle s_2, \Gamma^i \omega_{-1} \rangle \quad (\text{B.94})$$

$$\tilde{X}^{\bar{i}} = (\bar{l} + \bar{p})\tilde{X}^{\bar{i}} + (\bar{\nu}^0)_{\bar{j}} \tilde{X}^{\bar{j}} + \nu^{\bar{i}}{}_{\bar{j}} \tilde{X}^{\bar{j}} + \langle s_{-2}, \Gamma^{\bar{i}} \omega_1 \rangle + \langle s_0, \Gamma^{\bar{i}} \omega_{-1} \rangle + \langle s_2, \Gamma^{\bar{i}} \omega_{-3} \rangle \quad (\text{B.95})$$

$$\omega'_3 = \frac{1}{4} \nu^i{}_{\bar{j}} \Gamma^{\bar{i}\bar{j}} \omega_1 + X^i \Gamma_i s_2 - \tilde{X}^i \Gamma_i \tilde{s}_2 + \frac{1}{4} p \Gamma^{\bar{i}} \omega_3 + \frac{1}{4} \bar{p} \Gamma^i \omega_3 \quad (\text{B.96})$$

$$\omega'_1 = \frac{1}{4} \nu_{ij} \Gamma^{ij} \omega_3 + \frac{1}{4} \nu^i{}_{\bar{j}} \Gamma^{\bar{i}\bar{j}} \omega_{-1} + \frac{1}{4} \nu^0_{ij} \Gamma^{ij} \omega_1 + \frac{1}{4} \bar{\nu}^0_{\bar{i}\bar{j}} \Gamma^{\bar{i}\bar{j}} \omega_1 + X^i \Gamma_i s_0 + X^{\bar{i}} \Gamma_{\bar{i}} s_2 \quad (\text{B.97})$$

$$- \tilde{X}^i \Gamma_i \tilde{s}_0 - \tilde{X}^{\bar{i}} \Gamma_{\bar{i}} \tilde{s}_2 + \frac{1}{4} p \Gamma^{\bar{i}} \omega_1 + \frac{1}{4} \bar{p} \Gamma^i \omega_1 \quad (\text{B.98})$$

$$\omega'_{-1} = \frac{1}{4} \nu_{ij} \Gamma^{ij} \omega_1 + \frac{1}{4} \nu^i{}_{\bar{j}} \Gamma^{\bar{i}\bar{j}} \omega_{-3} + \frac{1}{4} \nu^0_{ij} \Gamma^{ij} \omega_{-1} + \frac{1}{4} \bar{\nu}^0_{\bar{i}\bar{j}} \Gamma^{\bar{i}\bar{j}} \omega_{-1} + X^{\bar{i}} \Gamma_{\bar{i}} s_0 + X^i \Gamma_i s_{-2} \quad (\text{B.99})$$

$$- \tilde{X}^i \Gamma_i \tilde{s}_{-2} - \tilde{X}^{\bar{i}} \Gamma_{\bar{i}} \tilde{s}_0 + \frac{1}{4} p \Gamma^{\bar{i}} \omega_{-1} + \frac{1}{4} \bar{p} \Gamma^i \omega_{-1} \quad (\text{B.100})$$

$$\omega'_{-3} = \frac{1}{4} \nu_{ij} \Gamma^{ij} \omega_{-1} + X^{\bar{i}} \Gamma_{\bar{i}} s_{-2} - \tilde{X}^{\bar{i}} \Gamma_{\bar{i}} \tilde{s}_{-2} + \frac{1}{4} p \Gamma^{\bar{i}} \omega_{-3} + \frac{1}{4} \bar{p} \Gamma^i \omega_{-3}. \quad (\text{B.101})$$

The adjoint action on a section of the generalised adjoint space is  $[R', R] = R''$  with

$$l'' = \tilde{a}a' - \tilde{a}'a + \langle s'_2, \tilde{s}_{-2} \rangle - \langle s_2, \tilde{s}'_{-2} \rangle + \langle s'_0, \tilde{s}_0 \rangle - \langle s_0, \tilde{s}'_0 \rangle + \langle s'_{-2}, \tilde{s}_2 \rangle - \langle s_{-2}, \tilde{s}'_2 \rangle \quad (\text{B.102})$$

$$p'' = \frac{1}{6} (\nu'^{ij} \nu_{ji} - \nu^{ij} \nu'_{ji} + \langle s'_2, \Gamma^i \tilde{s}_{-2} \rangle - \langle \tilde{s}'_2, \Gamma^i s_{-2} \rangle + \langle s'_{-2}, \Gamma^i \tilde{s}_2 \rangle - \langle \tilde{s}'_{-2}, \Gamma^i s_2 \rangle) \quad (\text{B.103})$$

$$\begin{aligned} \nu''^{0i}{}_j &= \nu^{0i}{}_k \nu^{0k}{}_j + \nu'^i{}_{\bar{k}} \nu^{\bar{k}}{}_j - \nu^{0i}{}_k \nu'^{0k}{}_j - \nu^i{}_{\bar{k}} \nu'^{\bar{k}}{}_j \\ &+ \langle s'_2, \Gamma^i \tilde{s}_{-2} \rangle - \langle \tilde{s}'_2, \Gamma^i s_{-2} \rangle + \langle s'_0, \Gamma^i \tilde{s}_0 \rangle - \langle \tilde{s}'_0, \Gamma^i s_0 \rangle \\ &+ \langle s'_{-2}, \Gamma^i \tilde{s}_2 \rangle - \langle \tilde{s}'_{-2}, \Gamma^i s_2 \rangle - \frac{1}{6} (\nu'^{kl} \nu_{lk} - \nu^{kl} \nu'_{lk} + \langle s'_2, \Gamma^k \tilde{s}_{-2} \rangle - \langle \tilde{s}'_2, \Gamma^k s_{-2} \rangle \\ &+ \langle s'_{-2}, \Gamma^k \tilde{s}_2 \rangle - \langle \tilde{s}'_{-2}, \Gamma^k s_2 \rangle) \eta^i{}_j \end{aligned} \quad (\text{B.104})$$

$$\begin{aligned} \nu''^i{}_{\bar{j}} &= \nu^{0i}{}_k \nu^k{}_{\bar{j}} + p' \nu^i{}_{\bar{j}} + \nu'^i{}_{\bar{k}} (\bar{\nu}^0)_{\bar{k}}{}^{\bar{j}} + \bar{p} \nu^i{}_{\bar{j}} - \nu^{0i}{}_k \nu'^k{}_{\bar{j}} - p \nu^i{}_{\bar{j}} - \nu^i{}_{\bar{k}} (\bar{\nu}'^0)_{\bar{k}}{}^{\bar{j}} - \bar{p}' \nu^i{}_{\bar{j}} \\ &+ \langle s'_2, \Gamma^i \tilde{s}_0 \rangle - \langle \tilde{s}'_2, \Gamma^i s_0 \rangle + \langle s'_0, \Gamma^i \tilde{s}_2 \rangle - \langle \tilde{s}'_0, \Gamma^i s_2 \rangle \end{aligned} \quad (\text{B.105})$$

$$\begin{aligned} \nu''^{\bar{i}}{}_{\bar{j}} &= \nu^{\bar{i}}{}_{\bar{k}} \nu^{\bar{k}}{}_{\bar{j}} + p \nu^{\bar{i}}{}_{\bar{j}} + (\bar{\nu}^0)_{\bar{k}}{}^{\bar{j}} \nu^{\bar{k}}{}_j + \bar{p}' \nu^{\bar{i}}{}_{\bar{j}} - \nu^{\bar{i}}{}_{\bar{k}} \nu'^{\bar{k}}{}_{\bar{j}} - p' \nu^{\bar{i}}{}_{\bar{j}} - (\bar{\nu}'^0)_{\bar{k}}{}^{\bar{j}} \nu^{\bar{k}}{}_j - \bar{p} \nu^{\bar{i}}{}_{\bar{j}} \\ &+ \langle s'_0, \Gamma^{\bar{i}} \tilde{s}_{-2} \rangle - \langle \tilde{s}'_0, \Gamma^{\bar{i}} s_{-2} \rangle + \langle s'_{-2}, \Gamma^{\bar{i}} \tilde{s}_0 \rangle - \langle \tilde{s}'_{-2}, \Gamma^{\bar{i}} s_0 \rangle \end{aligned} \quad (\text{B.106})$$

$$\begin{aligned} s''_2 &= l' s_2 - l s'_2 + \tilde{s}_2 a' - \tilde{s}'_2 a + \frac{1}{4} \nu^0_{ij} \Gamma^{\bar{i}\bar{j}} s_2 + \frac{1}{4} \bar{\nu}'^0_{\bar{i}\bar{j}} \Gamma^{\bar{i}\bar{j}} s_2 - \frac{1}{4} \nu^0_{ij} \Gamma^{\bar{i}\bar{j}} s'_2 - \frac{1}{4} \bar{\nu}'^0_{\bar{i}\bar{j}} \Gamma^{\bar{i}\bar{j}} s'_2 \\ &+ \frac{1}{4} \nu'^i{}_{\bar{j}} \Gamma^{\bar{i}\bar{j}} s_0 - \frac{1}{4} \nu^i{}_{\bar{j}} \Gamma^{\bar{i}\bar{j}} s'_0 + \frac{1}{4} p' \Gamma^{\bar{i}} s_2 + \frac{1}{4} \bar{p}' \Gamma^{\bar{i}} s_2 - \frac{1}{4} p \Gamma^{\bar{i}} s'_2 - \frac{1}{4} \bar{p} \Gamma^{\bar{i}} s'_2 \end{aligned} \quad (\text{B.107})$$

$$s''_0 = l' s_0 - l s'_0 + \tilde{s}_0 a' - \tilde{s}'_0 a + \frac{1}{4} \nu^0_{ij} \Gamma^{\bar{i}\bar{j}} s_0 + \frac{1}{4} \bar{\nu}'^0_{\bar{i}\bar{j}} \Gamma^{\bar{i}\bar{j}} s_0 - \frac{1}{4} \nu^0_{ij} \Gamma^{\bar{i}\bar{j}} s'_0 - \frac{1}{4} \bar{\nu}'^0_{\bar{i}\bar{j}} \Gamma^{\bar{i}\bar{j}} s'_0$$

$$\begin{aligned}
& + \frac{1}{4}\nu'_{ij}\Gamma^{ij}s_2 + \frac{1}{4}\nu'_{\bar{i}\bar{j}}\Gamma^{\bar{i}\bar{j}}s_{-2} - \frac{1}{4}\nu_{ij}\Gamma^{ij}s'_2 - \frac{1}{4}\nu_{\bar{i}\bar{j}}\Gamma^{\bar{i}\bar{j}}s'_{-2} \\
& + \frac{1}{4}p'\Gamma^{\bar{i}}_i s_0 + \frac{1}{4}p'\Gamma^i_i s_0 - \frac{1}{4}p\Gamma^{\bar{i}}_i s'_0 - \frac{1}{4}p\Gamma^i_i s'_0
\end{aligned} \tag{B.108}$$

$$\begin{aligned}
s''_{-2} & = \bar{l}'s_{-2} - \bar{l}s'_{-2} + \bar{s}_{-2}a' - \bar{s}'_{-2}a + \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}s_{-2} + \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}s_{-2} \\
& - \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}s'_{-2} - \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}s'_{-2} + \frac{1}{4}\nu'_{ij}\Gamma^{ij}s_0 - \frac{1}{4}\nu_{ij}\Gamma^{ij}s'_0 \\
& + \frac{1}{4}p'\Gamma^{\bar{i}}_i s_{-2} + \frac{1}{4}p'\Gamma^i_i s_{-2} - \frac{1}{4}p\Gamma^{\bar{i}}_i s'_{-2} - \frac{1}{4}p\Gamma^i_i s'_{-2}
\end{aligned} \tag{B.109}$$

$$\begin{aligned}
\tilde{s}''_2 & = -l'\tilde{s}_2 + l\tilde{s}'_2 + \tilde{a}'s_2 - \tilde{a}s'_2 + \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}_2 + \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}\tilde{s}_2 - \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}'_2 - \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}\tilde{s}'_2 \\
& + \frac{1}{4}\nu'_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}_0 - \frac{1}{4}\nu_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}'_0 + \frac{1}{4}p'\Gamma^{\bar{i}}_i \tilde{s}_2 + \frac{1}{4}p'\Gamma^i_i \tilde{s}_2 - \frac{1}{4}p\Gamma^{\bar{i}}_i \tilde{s}'_2 - \frac{1}{4}p\Gamma^i_i \tilde{s}'_2
\end{aligned} \tag{B.110}$$

$$\begin{aligned}
\tilde{s}''_0 & = -l'\tilde{s}_0 + l\tilde{s}'_0 + \tilde{a}'s_0 - \tilde{a}s'_0 + \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}_0 + \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}\tilde{s}_0 - \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}'_0 - \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}\tilde{s}'_0 \\
& + \frac{1}{4}\nu'_{ij}\Gamma^{ij}\tilde{s}_2 + \frac{1}{4}\nu'_{\bar{i}\bar{j}}\Gamma^{\bar{i}\bar{j}}\tilde{s}_{-2} - \frac{1}{4}\nu_{ij}\Gamma^{ij}\tilde{s}'_2 - \frac{1}{4}\nu_{\bar{i}\bar{j}}\Gamma^{\bar{i}\bar{j}}\tilde{s}'_{-2} \\
& + \frac{1}{4}p'\Gamma^{\bar{i}}_i \tilde{s}_0 + \frac{1}{4}p'\Gamma^i_i \tilde{s}_0 - \frac{1}{4}p\Gamma^{\bar{i}}_i \tilde{s}'_0 - \frac{1}{4}p\Gamma^i_i \tilde{s}'_0
\end{aligned} \tag{B.111}$$

$$\begin{aligned}
\tilde{s}''_{-2} & = -\bar{l}'\tilde{s}_{-2} + \bar{l}\tilde{s}'_{-2} + \tilde{a}'s_{-2} - \tilde{a}s'_{-2} + \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}_{-2} + \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}\tilde{s}_{-2} \\
& - \frac{1}{4}\nu'^0_{ij}\Gamma^{\bar{i}\bar{j}}\tilde{s}'_{-2} - \frac{1}{4}\bar{\nu}'^0_{ij}\Gamma^{ij}\tilde{s}'_{-2} + \frac{1}{4}\nu'_{ij}\Gamma^{ij}\tilde{s}_0 - \frac{1}{4}\nu_{ij}\Gamma^{ij}\tilde{s}'_0 \\
& + \frac{1}{4}p'\Gamma^{\bar{i}}_i \tilde{s}_{-2} + \frac{1}{4}p'\Gamma^i_i \tilde{s}_{-2} - \frac{1}{4}p\Gamma^{\bar{i}}_i \tilde{s}'_{-2} - \frac{1}{4}p\Gamma^i_i \tilde{s}'_{-2}
\end{aligned} \tag{B.112}$$

$$\begin{aligned}
a'' & = 2l'a - 2la' - \langle s'_2, s_{-2} \rangle - \langle s'_0, s_0 \rangle - \langle s'_{-2}, s_2 \rangle + \langle s_2, s'_{-2} \rangle \\
& + \langle s_0, s'_0 \rangle + \langle s_{-2}, s'_2 \rangle
\end{aligned} \tag{B.113}$$

$$\begin{aligned}
\tilde{a}'' & = 2\bar{l}\tilde{a}' - 2\bar{l}'\tilde{a} + \langle \tilde{s}'_2, \tilde{s}_{-2} \rangle + \langle \tilde{s}'_0, \tilde{s}_0 \rangle + \langle \tilde{s}'_{-2}, \tilde{s}_2 \rangle - \langle \tilde{s}_2, \tilde{s}'_{-2} \rangle \\
& - \langle \tilde{s}_0, \tilde{s}'_0 \rangle - \langle \tilde{s}_{-2}, \tilde{s}'_2 \rangle .
\end{aligned} \tag{B.114}$$

### B.3 SU(7) and SU(8) bundles and projections

We introduce two real SU(8) bundles  $S$  and  $J$ , which we refer to as the ‘‘spinor’’ bundle and the ‘‘gravitino’’ bundle respectively. These are

$$S = \mathbf{8} + \bar{\mathbf{8}} \equiv S^+ + S^- \quad J = \mathbf{56} + \bar{\mathbf{56}} \equiv J^+ + J^-, \tag{B.115}$$

with sections

$$\eta^\alpha \in \Gamma(S^+) \quad \eta_\alpha \in \Gamma(S^-) \tag{B.116}$$

$$\psi^{\alpha\beta\gamma} \in \Gamma(J^+) \quad \psi_{\alpha\beta\gamma} \in \Gamma(J^-), \tag{B.117}$$

in terms of SU(8) indices  $\alpha, \beta, \dots = 1, \dots, 8$ . In terms of SU(8) representations, we introduce the following generalised connection

$$D \equiv (D^{[\alpha\beta]}, \bar{D}_{[\alpha\beta]}), \quad (\text{B.118})$$

and throughout the text we use the following projections onto the  $S$  and  $J$  bundles between the generalised connection and sections of the  $S$  and  $J$  bundles

$$(D \times_J \eta)^{\alpha\beta\gamma} = D^{[\alpha\beta}\eta^{\gamma]} \in \Gamma(J^+) \quad (\text{B.119})$$

$$(D \times_S \eta)_\alpha = \bar{D}_{[\alpha\beta]}\eta^\beta \in \Gamma(S^-) \quad (\text{B.120})$$

$$(D \times_J \psi)_{\alpha\beta\gamma} = -\frac{1}{12}\epsilon_{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3} D^{[\delta\delta']}\psi^{\theta_1\theta_2\theta_3} \in \Gamma(J^-) \quad (\text{B.121})$$

$$(D \times_S \psi)^\alpha = -\frac{1}{2}\bar{D}_{[\beta\gamma]}\psi^{\alpha\beta\gamma} \in \Gamma(S^+) \quad (\text{B.122})$$

$$(D \times_J \eta)_{\alpha\beta\gamma} = D_{[\alpha\beta}\eta_{\gamma]} \in \Gamma(J^-) \quad (\text{B.123})$$

$$(D \times_S \eta)^\alpha = \bar{D}^{[\alpha\beta]}\eta_\beta \in \Gamma(S^+) \quad (\text{B.124})$$

$$(D \times_J \psi)^{\alpha\beta\gamma} = -\frac{1}{12}\epsilon^{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3} D_{\delta\delta'}\psi_{\theta_1\theta_2\theta_3} \in \Gamma(J^+) \quad (\text{B.125})$$

$$(D \times_S \psi)_\alpha = -\frac{1}{2}\bar{D}^{[\beta\gamma]}\psi_{\alpha\beta\gamma} \in \Gamma(S^-). \quad (\text{B.126})$$

We now decompose the  $S$  and  $J$  SU(8) bundles into irreducible SU(7) representations:

$$S^+ \rightarrow S_7^+ \oplus S_1^+ \quad J^+ \rightarrow J_{35}^+ \oplus J_{21}^+ \quad (\text{B.127})$$

$$\mathbf{8} \rightarrow \mathbf{7} \oplus \mathbf{1} \quad \mathbf{56} \rightarrow \mathbf{35} \oplus \mathbf{21} \quad (\text{B.128})$$

$$S^- \rightarrow S_7^- \oplus S_1^- \quad J^- \rightarrow J_{35}^- \oplus J_{21}^- \quad (\text{B.129})$$

$$\bar{\mathbf{8}} \rightarrow \bar{\mathbf{7}} \oplus \bar{\mathbf{1}} \quad \bar{\mathbf{56}} \rightarrow \bar{\mathbf{35}} \oplus \bar{\mathbf{21}}, \quad (\text{B.130})$$

with sections

$$\eta^a \in \Gamma(S_7^+) \quad \eta_a \in \Gamma(S_7^-) \quad (\text{B.131})$$

$$\eta^8 \in \Gamma(S_1^+) \quad \eta_8 \in \Gamma(S_1^-) \quad (\text{B.132})$$

$$\psi^{abc} \in \Gamma(J_{35}^+) \quad \psi_{abc} \in \Gamma(J_{35}^-) \quad (\text{B.133})$$

$$\psi^{ab8} \in \Gamma(J_{21}^+) \quad \psi_{ab8} \in \Gamma(J_{21}^-), \quad (\text{B.134})$$

in terms of SU(7) indices  $a, b, \dots = 1, \dots, 7$ . Similarly, we decompose the SU(7) compatible connection  $\hat{D}$  into SU(7) representations. It reads

$$\hat{D} = (\hat{D}^{[ab]}, \hat{D}^{a8}, \bar{\hat{D}}_{[ab]}, \bar{\hat{D}}_{a8}), \quad (\text{B.135})$$

Throughout the text we use the following projections onto the  $S_1^\pm$ ,  $S_7^\pm$ ,  $J_{35}^\pm$ , and  $J_{21}^\pm$  bundles between the generalised connection and sections of the said bundles

$$(\hat{D} \times_{J_{35}^+} \eta)^{abc} = \hat{D}^{[ab}\eta^{c]} \in \Gamma(J_{35}^+) \quad (\text{B.136})$$

$$(\hat{D} \times_{J_{21}^+} \eta)^{ab8} = \hat{D}^{[ab]}\eta^8 + \hat{D}^{8[a}\eta^{b]} \in \Gamma(J_{21}^+) \quad (\text{B.137})$$

$$(\hat{D} \times_{S_7^-} \eta)_a = \bar{D}_{[ab]}\eta^b + \bar{D}_{a8}\eta^8 \in \Gamma(S_7^-) \quad (\text{B.138})$$

$$(\hat{D} \times_{S_1^-} \eta)_8 = \bar{D}_{8a}\eta^a \in \Gamma(S_1^-) \quad (\text{B.139})$$

$$\begin{aligned} (\hat{D} \times_{J_{35}^-} \psi)_{abc} &= -\frac{1}{12}\epsilon_{abcd}e_1e_2e_8\hat{D}^{[dd']}\psi^{e_1e_2e_8} \\ &\quad -\frac{1}{12}\epsilon_{abcd}e_1e_2e_3\hat{D}^{d8}\psi^{e_1e_2e_3} \in \Gamma(J_{35}^-) \end{aligned} \quad (\text{B.140})$$

$$(\hat{D} \times_{J_{21}^-} \psi)_{ab8} = -\frac{1}{12}\epsilon_{ab8dd'}e_1e_2e_3\hat{D}^{[dd']}\psi^{e_1e_2e_3} \in \Gamma(J_{21}^-) \quad (\text{B.141})$$

$$(\hat{D} \times_{S_7^+} \psi)^a = -\frac{1}{2}\bar{D}_{[bc]}\psi^{abc} - \frac{1}{2}\bar{D}_{b8}\psi^{ab8} \in \Gamma(S_7^+) \quad (\text{B.142})$$

$$(\hat{D} \times_{S_1^+} \psi)^8 = -\frac{1}{2}\bar{D}_{[ab]}\psi^{ab8} \in \Gamma(S_1^+) \quad (\text{B.143})$$

$$(\hat{D} \times_{J_{35}^+} \eta)^{abc} = \hat{D}^{[ab}\eta^{c]} \in \Gamma(J_{35}^+) \quad (\text{B.144})$$

$$(\hat{D} \times_{J_{35}^-} \eta)_{abc} = \hat{D}_{[ab}\eta_{c]} \in \Gamma(J_{35}^-) \quad (\text{B.145})$$

$$(\hat{D} \times_{J_{21}^-} \eta)_{ab8} = \hat{D}_{[ab]}\eta_8 + \hat{D}_{8[a}\eta_{b]} \in \Gamma(J_{21}^-) \quad (\text{B.146})$$

$$(\hat{D} \times_{S_7^+} \eta)^a = \bar{D}^{[ab]}\eta_b + \bar{D}^{a8}\eta_8 \in \Gamma(S_7^+) \quad (\text{B.147})$$

$$(\hat{D} \times_{S_1^+} \eta)^8 = \bar{D}^{8a}\eta_a \in \Gamma(S_1^+) \quad (\text{B.148})$$

$$\begin{aligned} (\hat{D} \times_{J_{35}^+} \psi)^{abc} &= -\frac{1}{12}\epsilon^{abcd}e_1e_2e_8\hat{D}_{[dd']}\psi_{e_1e_2e_8} \\ &\quad -\frac{1}{12}\epsilon^{abcd}e_1e_2e_3\hat{D}_{d8}\psi_{e_1e_2e_3} \in \Gamma(J_{35}^+) \end{aligned} \quad (\text{B.149})$$

$$(\hat{D} \times_{J_{21}^+} \psi)^{ab8} = -\frac{1}{12}\epsilon^{ab8dd'}e_1e_2e_3\hat{D}_{[dd']}\psi_{e_1e_2e_3} \in \Gamma(J_{21}^+) \quad (\text{B.150})$$

$$(\hat{D} \times_{S_7^-} \psi)_a = -\frac{1}{2}\bar{D}^{[bc]}\psi_{abc} - \frac{1}{2}\bar{D}^{b8}\psi_{ab8} \in \Gamma(S_7^-) \quad (\text{B.151})$$

$$(\hat{D} \times_{S_1^-} \psi)_8 = -\frac{1}{2}\bar{D}^{[ab]}\psi_{ab8} \in \Gamma(S_1^-). \quad (\text{B.152})$$

# Appendix C

## Supersymmetry breaking and pure spinors

We consider a ten-dimensional  $SU(3) \times SU(3)$  background with a Minkowski four-dimensional space and a ten-dimensional bispinor  $\epsilon = (\epsilon_1, \epsilon_2)^T$  as in (II.59). We give here the parametrisation of the most general supersymmetry breaking, and we start by writing down the non-vanishing supersymmetry variations

$$\delta\psi_\mu^{(1)} = \frac{1}{2}e^A\hat{\gamma}_\mu\zeta \otimes \mathcal{V}_1 + c.c. \quad \delta\psi_\mu^{(2)} = \frac{1}{2}e^A\hat{\gamma}_\mu\zeta \otimes \mathcal{V}_2 + c.c. \quad (C.1)$$

$$\delta\psi_m^{(1)} = \zeta \otimes \mathcal{U}_m^1 + c.c. \quad \delta\psi_m^{(2)} = \zeta \otimes \mathcal{U}_m^2 + c.c. \quad (C.2)$$

$$\Delta\epsilon_1 = \zeta \otimes \mathcal{S}_1 + c.c. \quad \Delta\epsilon_2 = \zeta \otimes \mathcal{S}_2 + c.c., \quad (C.3)$$

where  $\mathcal{V}_{1,2}$ ,  $\mathcal{U}_m^{1,2}$  and  $\mathcal{S}_{1,2}$  are internal spinors parametrising the supersymmetry breaking

$$\begin{aligned} \mathcal{V}_1 &= \not{\partial}A\eta_1 + \frac{1}{4}e^\phi\gamma_{(6)}\not{F}\eta_2 \\ \mathcal{V}_2 &= \not{\partial}A\eta_2 - \frac{1}{4}e^\phi\gamma_{(6)}\not{F}^\dagger\eta_1 \\ \mathcal{S}_1 &= (\not{\nabla} - \not{\partial}\phi + 2\not{\partial}A + \frac{1}{4}\not{H})\eta_1 \\ \mathcal{S}_2 &= (\not{\nabla} - \not{\partial}\phi + 2\not{\partial}A + \frac{1}{4}\not{H})\eta_2 \\ \mathcal{U}_m^1 &= (\nabla_m + \frac{1}{4}\iota_m\not{H})\eta_1 + \frac{1}{8}e^\phi\not{F}\gamma_m\gamma_{(6)}\eta_2 \\ \mathcal{U}_m^2 &= (\nabla_m - \frac{1}{4}\iota_m\not{H})\eta_2 - \frac{1}{8}e^\phi\not{F}\gamma_m\gamma_{(6)}\eta_1. \end{aligned} \quad (C.4)$$

Following [44], we expand them in terms of supersymmetry breaking parameters in the following way

$$\mathcal{V}_1 = r_1\eta_1^* + s_m^1\gamma^m\eta_1 \quad \mathcal{V}_2 = r_2\eta_2^* + s_m^2\gamma^m\eta_2 \quad (C.5)$$

$$\mathcal{S}_1 = t_1\eta_1^* + u_m^1\gamma^m\eta_1 \quad \mathcal{S}_2 = t_2\eta_2^* + u_m^2\gamma^m\eta_2 \quad (C.6)$$



$$\mathcal{U}_m^1 = p_m^1 \eta_1 + q_{mn}^1 \gamma^n \eta_1^* \quad \mathcal{U}_m^2 = p_m^2 \eta_2 + q_{mn}^2 \gamma^n \eta_2^*. \quad (\text{C.7})$$

It has been shown in [44] that these parameters do not mix under T-duality. One can now rewrite the most general non-supersymmetric pure spinor equations, expanded on the generalised Hodge diamond, in terms of these supersymmetry breaking parameters

$$e^{-2A+\phi} d_H(e^{2A-\phi} \Psi_1) + 2dA \wedge \text{Re} \Psi_1 - e^\phi \tilde{*}_6 F = \Upsilon \quad (\text{C.8})$$

$$e^{-3A+\phi} d_H(e^{3A-\phi} \Psi_2) = K \quad (\text{C.9})$$

with

$$\begin{aligned} \Upsilon &= \frac{1}{2}(-1)^{|\Psi_1|} (r_1^* + t_2^*) \Psi_2 + \frac{1}{2}(-1)^{|\Psi_1|} (r_2 + t_1) \bar{\Psi}_2 + \frac{1}{2}(s_m^1)^* \gamma^m \bar{\Psi}_1 + \frac{1}{2}(-1)^{|\Psi_1|} s_m^2 \bar{\Psi}_1 \gamma^m \\ &\quad + \frac{1}{2}[u_m^1 + (p_m^2)^*] \gamma^m \Psi_1 + \frac{1}{2}(-1)^{|\Psi_1|} [(u_m^2)^* + p_m^1] \Psi_1 \gamma^m \\ &\quad + \frac{1}{2}(q_{mn}^2)^* \gamma^m \Psi_2 \gamma^n - \frac{1}{2} q_{mn}^1 \gamma^n \bar{\Psi}_2 \gamma^m \\ K &= \frac{1}{2}(-1)^{|\Psi_1|} t_2 \Psi_2 - \frac{1}{2}(-1)^{|\Psi_1|} t_1 \bar{\Psi}_2 + \frac{1}{2}(u_m^1 + p_m^2) \gamma^m \Psi_2 + \frac{1}{2}(-1)^{|\Psi_2|} (u_m^2 + p_m^1) \Psi_2 \gamma^m \\ &\quad + \frac{1}{2} q_{mn}^1 \gamma^n \bar{\Psi}_1 \gamma^m - \frac{1}{2} q_{mn}^2 \gamma^m \Psi_1 \gamma^n. \end{aligned} \quad (\text{C.10})$$

## C.1 String-like supersymmetry breaking

Let us now specify this expansion to the most general case of pure string-like supersymmetry breaking. We first impose domain-wall BPSness, namely

$$K = 0, \quad (\text{C.11})$$

which gives the following constraints on the supersymmetry breaking parameters

$$t_1 = t_2 = 0 \quad (\text{C.12})$$

$$u_m^1 = -\frac{1}{2}(1 + iJ_1)^k{}_m p_k^2 \quad (\text{C.13})$$

$$u_m^2 = -\frac{1}{2}(1 + iJ_2)^k{}_m p_k^1 \quad (\text{C.14})$$

$$(1 + iJ_2)^k{}_m q_{kn}^1 = 0 \quad (\text{C.15})$$

$$(1 + iJ_1)^k{}_m q_{kn}^2 = 0, \quad (\text{C.16})$$

then we impose the gauge BPSness, which amounts to requiring  $\Upsilon$  to be purely imaginary, which yields

$$r_1 = -r_2 \equiv r \quad (\text{C.17})$$

$$q_{mn}^1 = q_{nm}^2 \equiv q_{mn} \quad (\text{C.18})$$

$$s_m^1 = \frac{1}{2}(1 + iJ_1)^k{}_m p_k^2 - (p_m^2)^* \quad (\text{C.19})$$

$$s_m^2 = \frac{1}{2}(1 + iJ_2)^k{}_m p_k^1 - (p_m^1)^*. \quad (\text{C.20})$$

Imposing these constraints, the internal spinors describing the pure D-string supersymmetry breaking read

$$\mathcal{V}_1 = r\eta_1^* + (p_m^2 - (p_m^2)^*)\gamma^m\eta_1 \quad \mathcal{V}_2 = -r\eta_2^* + (p_m^1 - (p_m^1)^*)\gamma^m\eta_2 \quad (\text{C.21})$$

$$\mathcal{S}_1 = -p_m^2\gamma^m\eta_1 \quad \mathcal{S}_2 = -p_m^1\gamma^m\eta_2 \quad (\text{C.22})$$

$$\mathcal{U}_m^1 = p_m^1\eta_1 + q_{mn}\gamma^n\eta_1^* \quad \mathcal{U}_m^2 = p_m^2\eta_2 + q_{nm}\gamma^n\eta_2^*. \quad (\text{C.23})$$

The most general D-string supersymmetry breaking term is therefore

$$\begin{aligned} \Upsilon = & \frac{1}{2}(-1)^{|\Psi_1|}(r^*\Psi_2 - r\bar{\Psi}_2) + \frac{1}{4}(1 - iJ_1)^k{}_m (p_k^2)^* - \frac{1}{2}p_m^2\gamma^m\bar{\Psi}_1 \\ & + (-1)^{|\Psi_1|}\left(\frac{1}{4}(1 + iJ_2)^k{}_m p_k^1 - \frac{1}{2}(p_m^1)^*\right)\bar{\Psi}_1\gamma^m \\ & - \frac{1}{4}(1 + iJ_1)^k{}_m p_k^2 - \frac{1}{2}(p_m^2)^*\gamma^m\Psi_1 \\ & - (-1)^{|\Psi_1|}\left(\frac{1}{4}(1 - iJ_2)^k{}_m p_k^1 - \frac{1}{2}p_m^1\right)\Psi_1\gamma^m \\ & + \frac{1}{2}(q_{mn})^*\gamma^m\Psi_2\gamma^n - \frac{1}{2}q_{mn}\gamma^m\bar{\Psi}_2\gamma^n. \end{aligned} \quad (\text{C.24})$$

The non-supersymmetric backgrounds presented in [IV.2.1b](#)) have the following supersymmetry breaking parameters

$$r = 0 \quad (\text{C.25})$$

$$p_m^1 = e^A(-1)^{|\Psi_1|+1} \left[ \frac{1}{2}J_2^k{}_m(-\delta_k^n + \frac{1}{2}\Lambda^n{}_k)\alpha_n + \frac{1}{4}\Lambda_{nm}\alpha_q J_1^{qn} \right] \quad (\text{C.26})$$

$$+ \frac{3i}{2}(-\delta_m^n + \frac{1}{2}\Lambda^n{}_m)\alpha_n + \frac{i}{4}J_2^k{}_m\Lambda_{nk}\alpha_q J_1^{qn} \quad (\text{C.27})$$

$$p_m^2 = e^A \left[ (-1)^{|\Psi_1|+1} J_1^k{}_m\Lambda_{kn}\alpha_q J_2^{nq} - \frac{1}{2}((-1)^{|\Psi_2|}\delta_m^n + (-1)^{|\Psi_1|}\frac{1}{2}\Lambda_m{}^n)\alpha_n \right] \quad (\text{C.28})$$

$$- 3i(-1)^{|\Psi_1|}\Lambda_{mn}\alpha_q J_2^{nq} - \frac{i}{2}J_1^k{}_m((-1)^{|\Psi_2|}\delta_k^n + (-1)^{|\Psi_1|}\frac{1}{2}\Lambda_k{}^n)\alpha_n \quad (\text{C.29})$$

$$q_{mn} = -\frac{e^A}{2}\alpha_p \left[ (-1)^{|\Psi_1|}\Lambda_{mq}(\Omega_2^{qp}{}_n)^* - \Lambda_{qn}\Omega_1^{pq}{}_m \right], \quad (\text{C.30})$$

where  $J_{1,2}$  are the (almost) complex structures defined by  $\eta_1$  and  $\eta_2$

$$J_{1,2}{}^m{}_n = \frac{i}{\|\eta_{1,2}\|} \eta_{1,2}^\dagger \gamma^m{}_n \eta_{1,2}, \quad (\text{C.31})$$

and  $\Omega_{1,2mnp}$  are the  $(3,0)$ -forms with respect to the (almost) complex structures defined by  $\eta_1$  and  $\eta_2$ :

$$\Omega_{1,2mnp} = \frac{1}{\|\eta_{1,2}\|} \eta_{1,2}^T \gamma_{mnp} \eta_{1,2}. \quad (\text{C.32})$$

In order to write down the supersymmetry breaking parameters of the non-supersymmetric backgrounds with both SSB and DWSB contributions presented in c), one just have to add the pure DWSB parameters given in Appendix B of [44] to the one above.

The above supersymmetry breaking parameters correspond to the following decomposition of the supersymmetry breaking term in (IV.45) on the  $SU(3) \times SU(3)$  structure defined by the two pure spinors

$$\begin{aligned} d_H(e^{2A-\phi} \text{Im} \Psi_1) = & e^{3A-\phi} \left( \gamma^m \Psi_1 \left[ (-1)^{|j|+1} \Lambda_{mn} \alpha_p J_2^{np} - \frac{i}{2} \alpha_n ((-1)^{|\Psi_2|} \delta_m^n + (-1)^{|j|} \frac{1}{2} \Lambda_m^n) \right] \right. \\ & + \gamma^m \bar{\Psi}_1 \left[ (-1)^{|j|+1} \Lambda_{mn} \alpha_p J_2^{np} + \frac{i}{2} \alpha_n ((-1)^{|\Psi_2|} \delta_m^n + (-1)^{|j|} \frac{1}{2} \Lambda_m^n) \right] \\ & + \Psi_1 \gamma^m \left[ -\frac{1}{4} \Lambda_{nm} \alpha_p J_1^{pn} - \frac{i}{2} \alpha_n (-\delta_m^n + \frac{1}{2} \Lambda_m^n) \right] \\ & + \bar{\Psi}_1 \gamma^m \left[ -\frac{1}{4} \Lambda_{nm} \alpha_p J_1^{pn} + \frac{i}{2} \alpha_n (-\delta_m^n + \frac{1}{2} \Lambda_m^n) \right] \\ & + \frac{i}{4} \gamma^m \Psi_2 \gamma^n \alpha_p \left[ (-1)^{|j|} \Lambda_{mq} (\Omega_2^{qp})^* - \Lambda_{qn} \Omega_1^{pq} \right] \\ & \left. - \frac{i}{4} \gamma^m \bar{\Psi}_2 \gamma^n \alpha_p \left[ (-1)^{|j|} \Lambda_{mq} \Omega_2^{qp} - \Lambda_{qn} (\Omega_1^{pq})^* \right] \right). \quad (\text{C.33}) \end{aligned}$$

## C.2 Pure spinors and D-term supersymmetry-breaking

In this Section we write down the most general pure spinor expansion of the modified pure spinor equations respecting (IV.357).

We first impose the gauge BPSness (IV.303), which gives

$$r_1 + t_2 = -(r_2 + t_1) \quad (\text{C.34})$$

$$s_m^1 = -[u_m^1 + (p_m^2)^*] \quad (\text{C.35})$$

$$s_m^2 = -[u_m^2 + (p_m^1)^*] \quad (\text{C.36})$$

$$q_{mn}^1 = q_{nm}^2. \quad (\text{C.37})$$

Then, imposing the condition (IV.357) is equivalent to

$$u_m^1 = -\frac{1}{2}(1 + iJ_1)^k{}_m (p_k^2)^* \quad s_m^1 = -\frac{1}{2}(1 - iJ_1)^k{}_m (p_k^2)^* \quad (\text{C.38})$$

$$u_m^2 = -\frac{1}{2}(1 + iJ_2)^k{}_m (p_k^1)^* \quad s_m^2 = -\frac{1}{2}(1 - iJ_2)^k{}_m (p_k^1)^*, \quad (\text{C.39})$$

where  $J_{1,2}$  are the (almost) complex structures defined by  $\eta_1$  and  $\eta_2$

$$J_{1,2}{}^m{}_n = \frac{i}{\|\eta_{1,2}\|} \eta_{1,2}^\dagger \gamma^m{}_n \eta_{1,2}. \quad (\text{C.40})$$

This results in

$$\begin{aligned} \Upsilon &= \frac{1}{2}(-1)^{|\Psi_1|} (r_1^* + t_2^*) \Psi_2 - \frac{1}{2}(-1)^{|\Psi_1|} (r_1 + t_2) \bar{\Psi}_2 \\ &\quad + \frac{1}{2}(q_{nm}^1)^* \gamma^m \Psi_2 \gamma^n - \frac{1}{2} q_{nm}^1 \gamma^m \bar{\Psi}_2 \gamma^n \end{aligned} \quad (\text{C.41})$$

$$\begin{aligned} K &= \frac{1}{2}(-1)^{|\Psi_1|} t_2 \Psi_2 - \frac{1}{2}(-1)^{|\Psi_1|} t_1 \bar{\Psi}_2 + i \text{Im}(p_m^2) \gamma^m \Psi_2 + (-1)^{|\Psi_2|} i \text{Im}(p_m^1) \Psi_2 \gamma^m \\ &\quad + \frac{1}{2} q_{mn}^1 \gamma^n \bar{\Psi}_1 \gamma^m - \frac{1}{2} q_{mn}^1 \gamma^n \Psi_1 \gamma^m. \end{aligned} \quad (\text{C.42})$$

The polyforms  $\Theta$  and  $\Xi$  in (IV.360) and (IV.361) then take the form

$$\begin{aligned} \Theta &= \frac{i}{2} e^{A-\phi} \left[ (-1)^{|\Psi_1|} (t_2 + 4t_1^*) \Psi_1 + (-1)^{|\Psi_1|} (t_1 + 4t_2^*) \bar{\Psi}_1 \right. \\ &\quad \left. - 2i \text{Im}(p_m^2) \gamma^m \Psi_2 + (-1)^{|\Psi_2|} 2i \text{Im}(p_m^1) \Psi_2 \gamma^m \right. \\ &\quad \left. + q_{mn}^1 \gamma^n \Psi_1 \gamma^m + q_{mn}^1 \gamma^n \bar{\Psi}_1 \gamma^m \right] \end{aligned} \quad (\text{C.43})$$

$$\begin{aligned} \Xi &= \frac{1}{2} e^{A-\phi} \left[ (-1)^{|\Psi_1|} (r_1^* + t_2^*) \Psi_2 + (-1)^{|\Psi_1|} (r_1 + t_2) \bar{\Psi}_2 \right. \\ &\quad \left. - q_{mn}^{1*} \gamma^n \Psi_2 \gamma^m - q_{mn}^1 \gamma^n \bar{\Psi}_2 \gamma^m \right]. \end{aligned} \quad (\text{C.44})$$

Plugging the supersymmetry breaking expansion (C.42) in (IV.341) and using (C.38) and (C.39) gives

$$u_R^1 = u_R^2 = 0. \quad (\text{C.45})$$

Specifying the supersymmetry breaking terms violating the D-string BPSness condition

(C.41) to the case of Subsection IV.3.4 gives

$$\begin{aligned} \Upsilon &= \frac{1}{2} \left[ e^A (-x_{23} + x_{32}) \tilde{e}^1 + e^{-A} (x_{13} - x_{31}) \tilde{e}^2 + e^{-A} (-x_{12} + x_{21}) \tilde{e}^3 \right. \\ &\quad \left. + e^A (y_{23} - y_{32}) \tilde{e}^4 + e^{-A} (-y_{13} + y_{31}) \tilde{e}^5 + e^{-A} (y_{12} - y_{21}) \tilde{e}^6 \right. \\ &\quad \left. + e^{-A} (a - x_{11} - x_{22} - x_{33}) \tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^3 - e^{-A} (y_{23} + y_{32}) \tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^5 \right] \end{aligned}$$

$$\begin{aligned}
& + e^{-A}(-b + y_{11} + y_{22} - y_{33})\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^6 + e^{-A}(b - y_{11} + y_{22} - y_{33})\tilde{e}^1 \wedge \tilde{e}^3 \wedge \tilde{e}^5 \\
& + e^{-A}(y_{23} + y_{32})\tilde{e}^1 \wedge \tilde{e}^3 \wedge \tilde{e}^6 - e^{-A}(a - x_{11} + x_{22} + x_{33})\tilde{e}^1 \wedge \tilde{e}^5 \wedge \tilde{e}^6 \\
& + e^{-A}(-b - y_{11} + y_{22} + y_{33})\tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^4 + (y_{12} + y_{21})(e^A\tilde{e}^1 \wedge \tilde{e}^3 \wedge \tilde{e}^4 - e^{-3A}\tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^5) \\
& - (y_{13} + y_{31})(e^A\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^4 + e^{-3A}\tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^6) + e^{-A}(a + x_{11} - x_{22} + x_{33})\tilde{e}^2 \wedge \tilde{e}^4 \wedge \tilde{e}^6 \\
& - (x_{12} + x_{21})(e^A\tilde{e}^1 \wedge \tilde{e}^4 \wedge \tilde{e}^6 - e^{-3A}\tilde{e}^2 \wedge \tilde{e}^5 \wedge \tilde{e}^6) - e^{-A}(a + x_{11} + x_{22} - x_{33})\tilde{e}^3 \wedge \tilde{e}^4 \wedge \tilde{e}^5 \\
& + (x_{23} + x_{32})(e^{-A}\tilde{e}^2 \wedge \tilde{e}^4 \wedge \tilde{e}^5 - e^{-A}\tilde{e}^3 \wedge \tilde{e}^4 \wedge \tilde{e}^6) \\
& + (x_{13} + x_{31})(e^A\tilde{e}^1 \wedge \tilde{e}^4 \wedge \tilde{e}^5 + e^{-3A}\tilde{e}^3 \wedge \tilde{e}^5 \wedge \tilde{e}^6) + e^{-A}(b + y_{11} + y_{22} + y_{33})\tilde{e}^4 \wedge \tilde{e}^5 \wedge \tilde{e}^6 \\
& + e^{-A}(x_{12} - x_{21})\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^4 \wedge \tilde{e}^5 + e^{-A}(x_{13} - x_{31})\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^4 \wedge \tilde{e}^6 \\
& + e^{-3A}(x_{23} - x_{32})\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^5 \wedge \tilde{e}^6 + e^{-A}(-y_{12} + y_{21})\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^4 \wedge \tilde{e}^5 \wedge \tilde{e}^6 \\
& + e^{-A}(-y_{13} + y_{31})\tilde{e}^1 \wedge \tilde{e}^3 \wedge \tilde{e}^4 \wedge \tilde{e}^5 \wedge \tilde{e}^6 \\
& + e^{-3A}(-y_{23} + y_{32})\tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^4 \wedge \tilde{e}^5 \wedge \tilde{e}^6], \tag{C.46}
\end{aligned}$$

with

$$x_{ij} = \frac{1}{2}e^{2A-\phi}\text{Re}\{iq_{i,j+3}^1 + iq_{i+3,j}^1 - q_{ij}^1 - q_{i+3,j+3}^1\} \tag{C.47}$$

$$y_{ij} = \frac{1}{2}e^{2A-\phi}\text{Im}\{+q_{ij}^1 + q_{i+3,j+3}^1 - iq_{i,j+3}^1 - iq_{i+3,j}^1\} \tag{C.48}$$

$$a = -2e^{2A-\phi}\text{Re}(r_1 + t_2) \tag{C.49}$$

$$b = 2e^{2A-\phi}\text{Im}(r_1 + t_2). \tag{C.50}$$

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**Sujet : Géométrie Généralisée (Exceptionnelle) des vides  
non-supersymétriques avec Flux**

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**Résumé :** La construction de la théorie des cordes repose sur une symétrie reliant les bosons et les fermions, appelée supersymétrie, et qui doit être brisée à basse énergie. Les solutions supersymétriques de la théorie des cordes sont nettement plus simples que leurs homologues non-supersymétriques et peuplent donc la majorité de la littérature. Cependant, la supersymétrie peut être spontanément brisée à une énergie arbitrairement élevée. Dans cette thèse, nous étudions donc des solutions de supergravité de type II, une limite classique à basse énergie de la théorie des cordes, qui sont non-supersymétriques. Cette étude se place dans le cadre de la géométrie complexe généralisée, une généralisation de la géométrie différentielle qui unifie les transformations des coordonnées spatio-temporelles et les transformations de jauge de l'un des potentiels de la théorie des cordes, appelé le champ  $B$ . Nous construisons d'abord de nouvelles solutions de la supergravité de type II, où le mécanisme de brisure de la supersymétrie est dicté par la notion généralisée de stabilité pour les objets étendus qui sourcent les flux de supergravité. Nous dérivons ensuite une expression généralisée pour les flux de supergravité non-supersymétriques, et l'utilisons pour dériver les contraintes que les solutions de supergravité de type II non-supersymétriques doivent respecter afin que leurs théories effectives à basse énergie appartiennent à la classe bien connue de la supergravité quadridimensionnelle  $\mathcal{N} = 1$ . Enfin, nous décrivons les solutions non-supersymétriques de la supergravité de type II dans le cadre de la géométrie généralisée exceptionnelle, un formalisme qui unifie les transformations des coordonnées spatio-temporelles et les transformations de jauge de tous les potentiels de la théorie des cordes.

**Mots clés :** Théorie des cordes, compactification avec flux, géométrie généralisée, brisure de supersymétrie, calibrations.