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# Structures périphériques et invariants topologiques de sous-variétés nouées

Adrien Rodau

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# Peripheral structures and topological invariants of knotted submanifolds

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# Structures périphériques et invariants topologiques de sous-variétés nouées

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# Estructuras periféricas e invariantes topológicos de subvariedades anudadas

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**Keywords:** topology in small dimension, knot theory, algebraic curves

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*À ma famille.*

*Moi, j'ai trahi la musique respectable pour la musique concrète, et le violoncelle de mon enfance pour le magnétophone et pour le potentiomètre. [...] Nous autres dans nos studios avec nos armes automatiques, multipliant d'un coup de pouce le nombre des exécutants, gonflant le volume des orchestres, nous trichons.*

Me, I have betrayed respectable music for practical music, and the cello of my childhood for the tape recorder and for the potentiometer. [...]  
The rest of us in our studios, with our automatic weapons, multiplying the number of performers with a flick, swelling up entire orchestras, we cheat.  
*(translated)*

Yo, he traicionado a la música respetable para la música concreta, y al violonchelo de mi infancia para el magnetófono y para el potenciómetro. [...]  
Nosotros en nuestros estudios, con nuestras armas automáticas, multiplicando los intérpretes con una toba, hinchando las orquestas, hacemos trampas.  
*(traducido)*

— Pierre Schaeffer

LA LEÇON DE MUSIQUE

DE LA MUSIQUE CONCRÈTE À LA MUSIQUE MÊME, 1979



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**PERIPHERAL STRUCTURES AND TOPOLOGICAL INVARIANTS OF  
KNOTTED SUBMANIFOLDS**

**Abstract**

We study knotted codimension-two objects in manifolds of dimension 3 and 4: complex line arrangements in  $\mathbb{C}\mathbb{P}^2$  and links in  $S^3$ . We introduce new topological invariants of their embedding, derived from the interaction between their complement and their peripheral structure.

The motivation for line arrangements is to identify *Zariski pairs* which have the same combinatorics but different embeddings. Building on ideas developed by B. Guerville-Ballé and W. Cadiegan-Schlieper, we consider the inclusion map of boundary manifold to the exterior and its effect on homology classes. A careful study of the graph Waldhausen structure of the boundary manifold allows to identify specific generators of the homology. Their potential images are encoded in a group, the *graph stabiliser*, with a nice combinatorial presentation. The invariant related to the inclusion map is an element of this group. Using a computer implementation in Sage and the braid monodromy, we compute the invariant for some examples and exhibit new ordered Zariski pairs.

The second part concerns knot theory and a generalisation of a slope invariant developed by A. Degtyarev, V. Florens and A. G. Lecuona. Similarly to the context of line arrangements, we consider the inclusion map of the boundary components of a neighbourhood of a link in its exterior. On twisted homology, the kernel of this map is a Lagrangian subspace – for the intersection form – and its slopes provide a topological invariant of the link. We present two applications of this idea. In the first, developed in collaboration with L. Bénard, we consider knots and  $\mathrm{SL}_2(\mathbb{C})$  representations. This new slope invariant appears to be closely related to a higher-level invariant called the *A-polynomial*. The second application uses a Lagrangian characterisation method due to V. Arnol'd. It provides a concordance invariant with several relations to Sato-Levine invariant and Milnor linking numbers.

**Keywords:** topology in small dimension, knot theory, algebraic curves

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**STRUCTURES PÉRIPHÉRIQUES ET INVARIANTS TOPOLOGIQUES  
DE SOUS-VARIÉTÉS NOUÉES**

**Résumé**

On étudie des objets noués de codimension 2 dans des variétés de dimension 3 et 4 : des arrangements de droites complexes dans  $\mathbb{C}P^2$  et des entrelacs dans  $S^3$ . On introduit de nouveaux invariants topologiques de leurs plongements, dérivés des interactions entre leur complémentaire et leur structure périphérique.

La motivation concernant les arrangements de droites est d'identifier des *paires de Zariski* qui ont la même combinatoire mais des plongements différents. En utilisant des idées développées par B. Guerville-Ballé et W. Cadiegan-Schlieper, on considère l'application inclusion de la variété bord dans l'extérieur et son effet sur les classes d'homologie. Une étude approfondie de la structure graphée de Waldhausen de la variété bord permet d'identifier des générateurs spécifiques de son homologie. L'information de leurs images potentielles est collectée dans un groupe, le *stabilisateur du graphe*, qui a une présentation combinatoire simple. On utilise une implémentation en Sage et la monodromie de tresses pour calculer l'invariant dans certains exemples et produire de nouvelles paires de Zariski ordonnées.

La seconde partie est consacrée à la théorie des nœuds et à une généralisation d'un invariant de pente (« *slope* ») développé par A. Degtyarev, V. Florens et A. G. Lecuona. Similairement aux arrangements de droites, on considère l'application inclusion des composantes de bord d'un voisinage de l'entrelacs dans l'extérieur. Au niveau de l'homologie tordue, le noyau de cette application est un sous-espace Lagrangien – pour la forme d'intersection – et sa pente est un invariant topologique de l'entrelacs. On présente deux applications de cette construction. Dans la première, en collaboration avec L. Bénard, on considère le cas des nœuds et des représentations dans  $SL_2(\mathbb{C})$ . L'invariant *slope* obtenu est étroitement relié à un invariant de haut niveau, le *A-polynôme*. La seconde application utilise une caractérisation des sous-espaces lagrangiens due à V. Arnol'd. On construit un invariant de concordance qui a de nombreux liens avec l'invariant de Sato-Levine et les enlacements de Milnor.

**Mots clés :** topologie en petite dimension, théorie des nœuds, courbes algébriques

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**ESTRUCTURAS PERIFÉRICAS E INVARIANTES TOPOLÓGICOS DE  
SUBVARIETADES ANUDADAS**

**Resumen**

Estudiamos objetos anudados de codimensión dos en variedades de dimensión 3 y 4: configuraciones de rectas complejas en  $\mathbb{C}P^2$  y enlaces en  $S^3$ . Introducimos nuevos invariantes topológicos de su encaje, que provienen de la interacción entre el complementario y su estructura periférica.

La motivación para las configuraciones de rectas es identificar *pares de Zariski* que tienen la misma combinatoria pero diferentes encajes. Basándonos en las ideas desarrolladas por B. Guerville-Ballé y W. Cadiegan-Schlieper, consideramos el mapa de inclusión de la variedad límite hacia el exterior y su efecto sobre las clases de homología. Un estudio cuidadoso de la estructura de grafo de Waldhausen de la variedad del borde permite identificar generadores específicos de la homología. Sus imágenes potenciales están codificadas en un grupo, el *estabilizador del grafo*, con una elegante presentación combinatoria. El invariante relacionado con la inclusión es un elemento de este grupo. Utilizando una implementación informática en *Sagemath* y la monodromía de trenzas, calculamos el invariante para algunos ejemplos y encontramos nuevos pares ordenados de Zariski.

La segunda parte se refiere a la teoría de nudos y una generalización de un invariante llamado *pendiente* («slope») desarrollado por A. Degtyarev, V. Florens y A. G. Lecuona. De manera similar al contexto de configuraciones de rectas, consideramos la inclusión de los componentes del borde de un entorno de un enlace en su exterior. En homología torcida, el núcleo de esta aplicación es un subespacio lagrangiano –para la forma de intersección– y sus pendientes proporcionan un invariante topológico del enlace. Presentamos dos aplicaciones de esta idea. En el primero, desarrollado en colaboración con L. Bénard, consideramos nudos y representaciones  $SL_2(\mathbb{C})$ . Este nuevo invariante de pendiente parece estar estrechamente relacionado con un invariante de nivel superior llamada *A*-polinomio. La segunda aplicación utiliza un método de caracterización lagrangiano debido a V. Arnol'd. Proporciona un invariante de concordancia con varias relaciones con el invariante de Sato-Levine y los números de enlace de Milnor.

**Palabras claves:** topología en baja dimensión, teoría de los nudos, curvas algebraicas

# CONTENTS

<b>Abstracts</b>	<b>viii</b>
<b>Contents</b>	<b>xi</b>
<b>Introductions</b>	<b>1</b>
Introduction (in English)	2
Introduction (en français)	9
Introducción (en castellano)	16
<b>I Line arrangements</b>	<b>23</b>
1 Graph stabiliser	24
2 Homology inclusion of line arrangements	51
3 Computations of the homology inclusion	71
<b>II Slope invariants of links</b>	<b>87</b>
4 Character slope of links	88
5 The $SL_2(\mathbb{C})$ -slope of knots	95
6 Generalised slope of links	111
A Twisted (co-)homology	119
<b>Conclusions</b>	<b>127</b>
Conclusion (in English)	128
Conclusion (en français)	130
Conclusiones (en castellano)	132

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**Bibliography**

**134**

**Contents**

**140**

# INTRODUCTIONS

*‘Exactly!’ said Deep Thought. ‘So once you do know what the question actually is, you’ll know what the answer means.’*

— Douglas Adams

THE HITCHHIKER’S GUIDE TO THE GALAXY, 1979

# INTRODUCTION

## Outline of the current chapter

<b>Peripheral structures in co-dimension 2</b>	<b>2</b>
<b>Topology of line arrangements</b>	<b>3</b>
Generalities . . . . .	3
Review of common invariants . . . . .	3
Homology inclusion invariants . . . . .	3
<b>Summary of Part I</b>	<b>4</b>
<b>Link slopes and concordance</b>	<b>6</b>
General context . . . . .	6
Homotopical and homological invariants . . . . .	6
The character slope . . . . .	6
<b>Summary of Part II</b>	<b>7</b>

## Peripheral structures in co-dimension 2

The subject of this thesis is to study certain families of curves of co-dimension 2. The works of S. Smale [Sma62] and J. Milnor [Mil65] made it possible to determine an algebraic classification of smooth manifolds in dimension  $\geq 5$ . However, the study of smooth manifolds in dimension 3 and 4 is still a very active domain which involves a wide variety of tools from algebraic topology and geometry. In our work we notably make use of the fundamental group, several types of (co-)homology, intersections and mapping class groups. Graphs and related combinatorial methods also make a significant contribution.

Consider a manifold  $M$  of dimension 3 or 4 with boundary. In the cases we study this manifold is the complement of a regular neighbourhood of certain curves of co-dimension 2. Our approach is to study the inclusion

$$i : \partial M \hookrightarrow M$$

In several situations the boundary has a much simpler structure than the manifold itself. Yet this inclusion and its induced morphisms still contain significant topological information. We consider the morphisms induced by  $i$  on several types of algebraic topology structures, namely the fundamental groups, the first homology groups over  $\mathbb{Z}$  and the first twisted homology groups with respect to a representation  $\rho : \pi_1(M) \rightarrow \text{GL}(V)$  of the fundamental group over a vector space  $V$ . This last structure, also called *homology with local coefficients*, computes homological groups which have the structure of vector spaces. General construction of twisted homology is detailed for example in [Hat02], and more specialised results used in our work are presented in [Appendix A](#).

We apply this approach to the complements of two families of curves: singular plane algebraic curves embedded in  $\mathbb{CP}^2$ , in dimension 4, and knots and links embedded in  $S^3$ , in dimension 3.

# Topology of line arrangements

## Generalities

A *plane algebraic curve* is the zero locus of a complex homogenous polynomial. The topological study of these curves was initiated by O. Zariski which showed that all such smooth curves of a given degree are isotopic [Zar29], which enticed to focus on curves with singularities. The *combinatorics* of the curve is the topological type of the pair formed by a tubular neighbourhood of the curve and the curve itself. These data are determined by the topological type of the singularities and the incidence relations between the components. Following the works of O. Zariski [Zar31; Zar37] and E. R. van Kampen [Kam33] it is known that there exist pairs of curves with the same combinatorics but different embeddings in  $\mathbb{C}P^2$ , which were dubbed *Zariski pairs* by E. Artal in [Art94].

*Line arrangements* are finite collections of complex lines in  $\mathbb{C}P^2$  that is, curves whose defining polynomial has irreducible factors of degree 1. They form a subclass of plane algebraic curves whose combinatorics depends only on the incidence relations, which can be encoded in a graph called the *incidence graph*. The components of a line arrangement are non-singular and the singularities all belong to a same resolvable class. For all these reasons the study of line arrangements provides a favourable setting to create topological methods and invariants that could then be extended to algebraic curves in general. It also offers some interesting questions in itself. The first potential Zariski pair of line arrangements was discovered by G. Rybnikov [Ryb11]. It was definitively confirmed as a Zariski pair by E. Artal, J. Carmona, J. I. Cogolludo and M. Á. Marco in [Art+06]. They considered the two complements of the arrangements and showed that they have non-homeomorphic fundamental groups. This showed that the combinatorics does not determine the topological type of a curve even in the simplest case of line arrangements. The same team also found a Zariski pair of arithmetic complexified real arrangements in [Art+05]. The search for more Zariski pairs and a finer comprehension of the relationship between combinatorics and topology of curves and line arrangements has been a very active topic since the 2000s. Notably, S. Nazir, M. Yoshinaga [NY12] and F. Ye [Ye13] have completely determined the isotopy classes of line arrangements up to 9 lines and in particular that no Zariski pairs exist for these values. We also mention the works of B. Guerville-Ballé [Gue16] and J. Viu Sos [GV19] who discovered many new Zariski pairs with more than 10 lines. Readers can refer to [ACT08; Gue22] for a more detailed review of the subject.

## Review of common invariants

A wide variety of common invariants from algebraic topology have been applied to the study of line arrangements and Zariski pairs. E. R. van Kampen [Kam33] gave a method to compute a presentation of the fundamental group of the exterior of an algebraic curve, now called the *Zariski-van Kampen method*. Direct comparison of fundamental groups can sometimes give a Zariski pair, as for the original example of G. Rybnikov. However, there are known examples of Zariski pairs where the fundamental groups of the complements are isomorphic (see [Shi09; Shi19; ACM19a; Gue20]) and it has since then be shown that not even the characteristic varieties determine the topology of line arrangements in general, including for the subcategory of arrangements with real equations.

The Zariski-van Kampen method makes use of a construction called *braid monodromy*, introduced by O. Chisini in [Chi33], which encodes as a braid the relative position of each component of the curve in the vicinity of each singularity. This approach was later refined by B. Moishezon [Moi81] and for the specific case of real line arrangements by M. Salvetti [Sal88]. A. Libgober [Lib86] has shown that the braid monodromy does determine the homotopy type of a line arrangement, and J. Carmona [Car03] has extended it to the topological type (see also [ACC03]). The braid monodromy has appeared to be a slightly more effective invariant than the fundamental group and more Zariski pairs have been obtained using it, see [ACT08] for a survey.

## Homology inclusion invariants

As we mentioned earlier, another approach to build invariants of line arrangements is to focus on the *boundary manifold* of the exterior of the arrangement, defined as the boundary of a



regular neighbourhood. T. Jiang, S. S.-T. Yau [JY93] and E. R. Westlund [Wes97] have shown that this boundary manifold has the structure of a *graph manifold* as defined by F. Waldhausen [Wal67a; Wal67b] and W. D. Neumann [Neu81]. The topology of the boundary manifold is directly determined by the combinatorics of the line arrangement, encoded in the form of a graph. This graph provides a ‘blueprint’ to reconstruct the boundary manifold by gluing together circle bundles (Seifert pieces) corresponding to the boundary of local neighbourhoods around each line component and singularity of the arrangement.

Our own interest lies in the study of the inclusion of the boundary manifold inside the exterior of the line arrangement. This morphism was first considered for his induced action on the fundamental groups. Indeed, a presentation of the fundamental group of the exterior can be obtained with either the Zariski-van Kampen method mentioned earlier, or using an alternative result due to W. Arvola [Arv92]. For the boundary manifold, the fundamental group was obtained by E. R. Westlund [Wes97], see also [CS08]. Starting from these results, E. Hironaka [Hir01] studied the morphism induced by the inclusion between the fundamental groups for the case of real line arrangements, and this study was continued and generalised to complex line arrangements by V. Florens, B. Guerville-Ballé and M. Á. Marco in [FGM15]. Their results exposed that the study of the fundamental group ‘inclusion’ is made difficult by the algebraic complexity of the morphisms involved. Another approach is therefore to consider the morphism  $i_*$  induced by the inclusion on the first homology groups instead. Unlike what one might think, this is not a trivial matter. The study of that morphism was first considered by E. Artal, V. Florens and B. Guerville-Ballé in [AFG17], with a first generalisation having been later developed by W. Cadegan-Schlieper [Cad18].

We propose a new invariant that extends these attempts and fully exploits the homology ‘inclusion’ of the boundary manifold of any line arrangement inside its exterior. The main principle of the construction is as follows. The first homology group of the boundary has two types of generators: the meridians of the line components, and generators arising from the cycles of the graph. Then first homology group of the exterior is freely generated by the meridians. Even if both groups are combinatorially determined, the induced morphism  $i_*$  still contains significant topological information. The main difficulty of computing  $i_*$  lies in the ambiguity of defining the set of cycle generators. This depends on the choice of an *embedding of the graph* inside the boundary manifold.

**Proposition A.** *Any ordered graphed embedding inside the boundary manifold induces a group isomorphism between its first homology group and the group freely generated by the meridians and graph cycles.*  $\triangleleft$

To remove that ambiguity, we define the *graph stabiliser* as a combinatorial group which computes the homological differences between all ordered graphed embeddings. The morphism  $i_*$  then induces a topological invariant of ordered line arrangements called *homology inclusion* which is an element of the graph stabiliser. This invariant can be computed on any pair of line arrangements with the same combinatorics to determine if they form a Zariski pair. Its construction generalises the previous  $\mathcal{I}$ -invariant of [AFG17] and the *loop-linking number* of [Cad18]. However, unlike the  $\mathcal{I}$ -invariant it does not require any particular condition on the combinatorics of the pair to be defined, and it captures more topological information than the loop-linking number.

## Summary of Part I

**Chapter 1** is dedicated to the construction of the graph stabiliser. At this stage we consider the boundary manifold of a line arrangement as a purely combinatorial object, so **Chapter 1** does not yet reference line arrangements specifically.

The most elementary level of the structure of a graph manifold  $M$  are planar surfaces with boundary, so **Section 1.1** recalls general results on their mapping class groups. In **Section 1.2** we present the *ordered stars* which are the ‘elementary bricks’ used to build the graphed embeddings on the base surfaces. In **Section 1.3** we recall the structure of a graph manifold as a union of circle bundles over the base surfaces, joined by a set of tori along a graph  $\Gamma$ . The fundamental property of graph manifolds is the unicity of the minimal graph structure. As a consequence we consider a class of homeomorphisms which respect the graph structure and the local orientation of each Seifert piece and their bases (see **Definitions 1.3.22** and **1.3.23**). **Definition 1.3.20** introduce the *graph ordering*  $\Omega$  as the data of *local orders* on the neighbour set of each vertex of  $\Gamma$ . In **Section 1.4**

we define *ordered graphed embeddings* as the reunion of a set of ordered stars, one for each Seifert piece, respecting each local order. The dependency of the graphed embeddings on the graph ordering  $\Omega$  is the crucial point of our construction, which was absent from the previous similar invariants in [AFG17; Cad18]. **Theorem 1.4.9** establishes that the homeomorphisms of the graph manifold have a transitive action on the graphed embeddings. In **Section 1.5** we explain how the choice of an ordered graphed embedding determines the cycle generators of  $\pi_1(M)$  and  $H_1(M)$ , and by extension gives a presentation of these two groups. The transitivity of the action of the homeomorphisms implies the following result:

**Theorem B.** *There exists a presentation of  $\pi_1(M)$  that only depends on the graph ordering  $\Omega$ .*  $\triangleleft$

In **Section 1.6** we can finally present the graph stabiliser. Unlike its fundamental group, the first homology group of the exterior  $E_{\mathcal{A}}$  of a line arrangement  $\mathcal{A}$  is combinatorial. Therefore the group

$$\mathcal{H} := \text{Hom}(H_1(\Gamma_{\mathcal{A}}), H_1(E_{\mathcal{A}}))$$

is combinatorially determined, see **Proposition 2.4.22**. The *graph stabiliser*  $\mathcal{G}_{\Gamma}$  is the quotient of  $\mathcal{H}$  by the differences on the cycle generators between every two ordered graphed embeddings, see **Definition 1.6.1**. The computation of this difference can be reduced down to the ordered stars, so we get:

**Theorem C.** *The graph stabiliser  $\mathcal{G}_{\Gamma}$  admits an explicit combinatorial presentation that does not depend on the graph ordering  $\Omega$ .*  $\triangleleft$

All purely combinatorial matters having been settled, **Chapter 2** gives a presentation of line arrangements and the essential associated concepts, and then proceeds to the construction of the *homology inclusion*  $\mathcal{J}_{\Omega}(\mathcal{A})$  as an invariant of ordered line arrangements.

We start by recalling the basic definitions of line arrangements in **Section 2.1**, as well as the definition of combinatorics based on incidence data in **Section 2.2**. In **Section 2.3** we give the description of the minimal graph structure of the boundary manifold  $B_{\mathcal{A}}$  and of the circle bundles that compose it (see **Theorems 2.3.15** and **2.3.17**), using the *blow-up* operation to resolve the singularities. **Section 2.4** presents the main tools which we use to determine the topological type of the exterior manifold and its fundamental group, namely the *wiring diagram* and the *braid monodromy*. It is this last tool that we use as the primary invariant to get the topological information of the arrangements.

**Section 2.5** presents the homology inclusion invariant. The inclusion of the boundary  $B_{\mathcal{A}}$  inside the exterior  $E_{\mathcal{A}}$  induces a map  $i_*$  on the first homology groups. The restriction of this map on the cycle generators depends on the choice of a graph ordering  $\Omega$  and of an ordered graphed embedding  $\gamma$ . The *homology inclusion*  $\mathcal{J}_{\Omega}(\mathcal{A})$  is the class of the map  $i_* \in \mathcal{H}$  inside the quotient  $\mathcal{G}_{\Gamma}$ . By construction of the graph stabiliser, this class no longer depends on the choice of the embedding  $\gamma$ . Moreover, we have the invariance theorem:

**Theorem D.** *Let  $\mathcal{A}, \mathcal{A}' \subset \mathbb{C}\mathbb{P}^2$  be two line arrangements with the same combinatorics. Endow  $\mathcal{A}$  and  $\mathcal{A}'$  with the same order  $\theta$  on their set of lines, which induces a graph ordering  $\Omega^{\theta}$  on the incidence graph  $\Gamma$ . Suppose that there exists an homeomorphism*

$$(\mathbb{C}\mathbb{P}^2, \mathcal{A}) \xrightarrow{\sim} (\mathbb{C}\mathbb{P}^2, \mathcal{A}')$$

*that preserves the order  $\theta$ . Then  $\mathcal{J}_{\Omega^{\theta}}(\mathcal{A}) = \mathcal{J}_{\Omega^{\theta}}(\mathcal{A}')$  inside  $\mathcal{G}_{\Gamma}$ .*  $\triangleleft$

**Chapter 3** is dedicated to the practical method for computing the homology inclusion. Its principle, presented in **Section 3.1**, is to use the braid monodromy of the arrangement to construct a geometrical *standard graphed embedding*. This embedding is built inside an affine map of  $\mathbb{C}\mathbb{P}^2$  and requires a direct description of the graph structure of the boundary manifold without using the blow-up resolution. The values of the map  $i_*$  on the standard graph embedding are then determined by an algorithmic computation on the braids of the monodromy which we call the *braid linking*, presented in **Section 3.2**. Once projected inside the presentation of  $\mathcal{G}_{\Gamma}$  given by **Theorem C**, we obtain the value of  $\mathcal{J}_{\Omega}(\mathcal{A})$ . We then apply this method on potential Zariski pairs. A crucial point is that the graph ordering  $\Omega$  must be the same for both arrangements. This requires to run the ‘raw’ braid monodromies through a series of algorithms detailed in **Section 3.3**. The full computation process was implemented using the **Sage** language [Sag23]. Finally, in **Section 3.3.3** we give several examples of known and new Zariski tuples which we successfully detected using the homology inclusion.

## Link slopes and concordance

### General context

A link is an embedding of disjoint closed polygonal curves inside  $S^3$ . The modern study of links and knots as topological objects has really started in the 1920s and 1930s using the fundamental group developed and early homology theory developed by H. Poincaré. It has been quickly established that links and knots have many connections to other domains of topology, and even to physics.

Amongst other results, the theorem of W. B. R. Lickorish [Lic63] and A. H. Wallace [Wal60] allows to connect the study of knots and links to the study of oriented closed manifolds of dimension 3. The *concordance relation* is a natural extension of link equivalence in dimension 4 which was designed by R. Fox and J. Milnor [Fox62]. Two links are concordant if they co-bound a set of properly embedded disjoint cylinders in  $S^4$ , and a link is *slice* if it is concordant to the unlink. This relation creates a natural connection between link theory and the study of surfaces embedded in  $S^4$ .

### Homotopical and homological invariants

A method to obtain a presentation of the fundamental group of the exterior of a knot or link has been established by W. Wirtinger. A fundamental result of F. Waldhausen [Wal68], C. McA. Gordon and J. E. Luecke [GL89] establishes that the fundamental group along with the data of a peripheral structure, i.e. a meridian and longitude generator for each component, is a complete knot invariant. Even if counter-examples exist for the similar proposition with links [Gor02], this cements the idea that powerful knot and link invariants can be derived from the fundamental group.

J. W. Alexander II [Ale28] defined the polynomial that now bears his name in 1923. One of the first knot invariants ever discovered, the Alexander polynomial and the ideas behind its constructions has been extended into many different knot and link invariants with polynomial forms. A possible definition of the Alexander polynomial is as the order of the first abelian twisted homology group of the exterior of the link, which in this case has a module structure called the *Alexander module*. This illustrates the potential of links invariants based on twisted homology. Notable representatives of this type of invariants are the *twisted Alexander polynomials* of X.-S. Lin [Lin01] and M. Wada [Wad94] which are closely related to the Reidemeister torsion [Kit96].

### The character slope

The slope invariant has been developed by A. Degtyarev, V. Florens and A. G. Lecuona in [DFL22b] and has later been refined in [DFL21; DFL22a]. It appeared as an invariant of its own during a study of the signature of coloured links, and has been applied in this context. Relating with the ideas presented in the general introduction, its construction is based on the fact that the boundary of the exterior of the link  $M_L$  is merely a union of disjoint tori.

$$\partial M_L = \bigsqcup_{i=1}^n T_{L_i}$$

Applying the strategy similar as what we presented for line arrangements, they considered the induced application of the inclusion  $i : \partial M_L \hookrightarrow M_L$  in *twisted homology*:

$$i_* : H_1(\partial M_L; \rho) \longrightarrow H_1(M_L; \rho)$$

where  $\rho$  is a representation of the fundamental group  $\pi_1(M_L)$ . It turns out that the first twisted homology group of the boundary  $\partial M_L$  is a vector space entirely determined by the image of the chosen representation on the peripheral structure of the link, namely its meridians and longitudes. With a one-dimensional character  $\omega : \pi_1(M_L) \rightarrow \mathbb{C}^*$ , the dimension of  $H_1(\partial M_L; \omega)$  is a multiple of the number of non-trivial values of  $\omega$ . Therefore by imposing  $\omega$  to be trivial on a single component  $K$  called *distinguished*, they obtained:

**Theorem E.** *Let  $\omega$  be an admissible character and suppose that  $\dim \ker i_* = 1$ . Then  $\ker i_*$  is generated by a vector of  $H_1(\partial M_{K \cup L}; \omega)$  of the form*

$$\ker i_* = \langle a \cdot \ell + b \cdot m \rangle$$

where  $[a : b] \in \mathbb{C}P^1$  and  $m, \ell$  are the meridian and longitude of  $K$ . The  $(K/L)$ -slope is defined by the formula

$$s_{(K/L)}(\omega) := -\frac{b}{a} \in \mathbb{C} \cup \{\infty\} \quad \triangleleft$$

The slope is a rational function in the values of the character and has several interesting properties, some of which are recalled in **Chapter 4**. In particular it does not depend on the Fitting ideals of the Alexander module and it is a concordance invariant.

In fact the slope generalises the single-variable  $\eta$ -function developed by S. Kojima and M. Yamasaki [KY79] and the invariant developed by N. Sato and J. Levine [Sat84]. It is known that the  $\eta$ -function can be seen as a generating function of the  $\beta$ -invariants of T. Cochran [Coc85]. These are in turn [Coc90] linked to certain lifts of the linking numbers of J. Milnor [Mil54].

Our objective is to generalise the main idea of the first slope, which we now call the *character slope*, to other contexts. We present two such extensions. The first, the  $SL_2(\mathbb{C})$ -slope, was developed in collaboration with L. Bénard. It allows to define a slope on any knot using adjoint representations over  $SL_2(\mathbb{C})$ . The second extension, the *generalised slope*, uses realified representations in  $SO_2(\mathbb{R})$  and a symplectic method due to V. I. Arnold [Arn67]. With similar conditions as the character slope, this slope allows more than one component of the link to be distinguished.

## Summary of **Part II**

**Chapter 4** gives a quick presentation of the character slope and its main properties to establish the reference for the generalisations. **Section 4.1** makes a quick overview of basic knot and link theory definitions. **Section 4.2** gives the main **Definition 4.2.4** of the character slope. Some of the results use new more general proofs on twisted homology detailed in **Appendix A**, notably **Theorem 4.2.10** which establishes the concordance invariance. **Section 4.2** explains how to compute the slope using Fox calculus [Fox54] in **Theorem 4.3.1**. We also present a new implementation in **GAP** [GAP22] of that method which allows to compute the slope on any compatible link diagram thanks to [CD08; DT83].

**Chapter 5** is adapted from [BFR21] and is dedicated to the  $SL_2(\mathbb{C})$ -slope on knots. The choice of  $SL_2(\mathbb{C})$  is motivated by the fact that the set of all representations of the group of knot  $K$  in  $SL_2(\mathbb{C})$  carries naturally the structure of an algebraic set. This holds also for the characters of these representations, whose set is called the  $SL_2(\mathbb{C})$ -character variety of the knot. Given a peripheral structure of the knot, the character variety is a plane curve in  $\mathbb{C}^* \times \mathbb{C}^*$ , whose coordinates  $M$  and  $L$  correspond to the eigenvalues of the meridian and the preferred longitude of  $K$ . The polynomial  $A_K(L, M)$  defining this curve is an invariant of the knot, called the *A-polynomial* and was defined in [Coo+94]. This invariant contains many interesting informations on the knot; in particular, S. Boyer and X. Zhang [BZ05] and N. Dunfield and S. Garoufalidis [DG04] showed that  $A_K = L - 1$  if and only if  $K$  is trivial. One of our main motivations, as explained in **Section 5.1**, is to harness topological information from the *A-polynomial* using the slope. Indeed, the *A-polynomial* is a high-level invariant which is difficult to compute in general, whereas the  $SL_2(\mathbb{C})$ -slope can be computed with Fox calculus with a similar process as the character slope from **Chapter 4**. The definition and general results about the *A-polynomial* are covered in **Section 5.2**.

The construction of the  $SL_2(\mathbb{C})$ -slope is presented in **Section 5.3**. It is based on the fact that a perfect analogue of **Theorem E** exists for which all adjoint *non-parabolic* representations of the knot group in  $SL_2(\mathbb{C})$  are admissible, see **Definition 5.3.2** and **Lemma 5.3.4**. The invariant is thus defined again as the slope of  $\ker i_*$  with respect to the meridian-longitude basis. It should be noted that unlike the character slope, no restrictions are made on the knot  $K$ . The  $SL_2(\mathbb{C})$ -slope definition relies on constructions made by J. Porti [Por97] which observed that  $H_1(\partial M_K; \text{Ad} \circ \rho)$  has the structure of a *symplectic space* for the induced intersection form. J. Porti also defined the *Reidemeister torsion* for the  $\text{Ad} \circ \rho$ -twisted homology, and we prove that it is linked to the slope by a simple formula given in **Proposition 5.3.15**. **Section 5.4** contains the main **Theorem 5.4.1** which connects the slope and the *A-polynomial*. This result is notably used in **Corollary 5.4.5**

which establishes that the  $\mathrm{SL}_2(\mathbb{C})$ -slope can detect the unknot.

**Chapter 6** presents our other generalisation of the character slope. To simplify, we consider links with zero linking numbers between all components. The fundamental proposition for the construction is the following:

**Proposition F.** *Let  $\rho_\omega : \pi_1(M_L) \rightarrow \mathrm{SO}_2(\mathbb{R})$  be the realification of a character  $\omega$  of the link. Then the  $\mathbb{R}$ -vector space  $H_1(\partial E_L, \mathbb{R}(\rho_\omega))$  endowed with the intersection form is a symplectic space freely generated by the meridians and longitudes of the components  $L_i$  where  $\omega$  is trivial. Moreover,  $\ker i_*$  is a Lagrangian subspace, i.e. its own orthogonal for the symplectic form.  $\triangleleft$*

V. I. Arnol'd [Arn67] has developed a method based on the complexification of a symplectic real vector space, which allows to characterise all Lagrangian subspaces with a class of complex unitary matrices. This method is briefly recalled in **Section 4.2**. The *generalised slope* is defined in **Section 6.2** as the argument of the determinant of the matrix characterising  $\ker i_*$ , see **Definition 6.2.7**. Unlike the character slope, the generalised slope can be defined as a function on the entire character torus of the link  $L$ . Finally in **Section 6.3**, we prove that it is again a concordance invariant.

# INTRODUCTION

## Sommaire du présent chapitre

<b>Structures périphériques en co-dimension 2</b>	<b>9</b>
<b>Topologie des arrangements de droites</b>	<b>10</b>
Généralités . . . . .	10
Revue des invariants connus . . . . .	10
Inclusion homologique . . . . .	11
<b>Résumé de la Partie I</b>	<b>11</b>
<b>Pentes d'entrelacs et concordance</b>	<b>13</b>
Contexte général . . . . .	13
Invariants homotopiques et homologiques . . . . .	13
L'invariant de pente pour les caractères . . . . .	13
<b>Résumé de la Partie II</b>	<b>14</b>

## Structures périphériques en co-dimension 2

Le sujet de cette thèse est l'étude de certaines familles de courbes de co-dimension 2. Les travaux de S. Smale [Sma62] et de J. Milnor [Mil65] ont permis de déterminer une classification algébrique des variétés lisses de dimension  $\geq 5$ . Cependant, l'étude de ces mêmes variétés lisses en dimension 3 et 4 reste un sujet très actif, qui fait appel à une large variété d'outils issus de la topologie et de la géométrie algébrique. Notre travail utilise notamment le groupe fondamental, différents types de (co-)homologie ainsi que les groupes de difféotopie. Nous utilisons également les graphes et diverses méthodes combinatoires qui leurs sont associées.

Soit  $M$  une variété à bord de dimension 3 ou 4. Nous nous intéressons au cas où cette variété est le complémentaire d'un voisinage régulier d'une courbe de co-dimension 2 d'un certain type. Notre approche globale est d'étudier l'inclusion :

$$i : \partial M \hookrightarrow M$$

Dans certaines situations le bord possède une structure beaucoup plus simple que celle de la variété elle-même. Malgré cela, l'inclusion ci-dessus et ses morphismes induits contiennent des informations topologiques significatives. Plus précisément, nous étudions les morphismes induits par  $i$  sur différents types d'objets issus de la topologie algébrique : les groupes fondamentaux, les premiers groupes d'homologie standard sur  $\mathbb{Z}$ , ainsi que les premiers groupes d'*homologie tordue* par une représentation  $\rho : \pi_1(M) \rightarrow \mathrm{GL}(V)$  du groupe fondamental sur un espace vectoriel  $V$ . Cette dernière structure est aussi appelée *homologie à coefficients locaux* et est constituée de groupes homologiques ayant une structure d'espace vectoriel. Sa construction peut être trouvée par exemple dans [Hat02], mais nous présentons des résultats la concernant plus spécifiques aux besoins de nos travaux dans l'[Annexe A](#).

Nous appliquons notre approche globale aux complémentaires de deux types de courbes : d'une part les courbes algébriques planes singulières plongées dans  $\mathbb{C}\mathbb{P}^2$ , donc en dimension 4, et d'autre part les nœuds et entrelacs plongés dans  $S^3$ , donc en dimension 3.

# Topologie des arrangements de droites

## Généralités

Une *courbe algébrique plane* est le lieu des zéros d'un polynôme homogène complexe. L'étude de la topologie de ces courbes a été initiée par O. Zariski qui a notamment montré que toutes les courbes algébriques planes lisses de même degré sont isotopes [Zar29]. Dès lors, le problème se recentre sur l'étude des courbes algébriques singulières. La *combinatoire* d'une telle courbe est défini comme le type topologique de la paire constituée d'un voisinage régulier et de la courbe elle-même. La combinatoire est également déterminée par la donnée du type topologique des singularités et des relations d'incidence entre les composantes. Les travaux de O. Zariski [Zar31 ; Zar37] et E. R. van Kampen [Kam33] démontrèrent l'existence de couples de courbes ayant la même combinatoire mais des plongements différents dans  $\mathbb{C}P^2$ . Ces couples furent baptisés *couples de Zariski* par E. Artal dans [Art94].

Un *arrangement de droites* est une collection finie de droites complexes dans  $\mathbb{C}P^2$ , ce qui correspond à une courbe dont le polynôme de définition possède uniquement des facteurs irréductibles de degré 1. Les arrangements de droites forment une sous-classe des courbes algébriques planes dont les combinatoires sont déterminées uniquement par les relations d'incidence, lesquelles peuvent être encodées sous la forme d'un *graphe d'incidence*. Les composantes d'un arrangement de droites n'ont pas de singularité propre et toutes les singularités de l'arrangement sont du même types et sont résolubles. Pour toutes ces raisons, l'étude des arrangements de droites offre un cadre favorable à la création d'invariants topologiques qui pourront ensuite être étendus à d'autres types de courbes algébriques. Cette étude offre également des questions intéressantes pour elle-même. Le premier couple de Zariski potentiel constitué d'arrangements a été découvert par G. Rybnikov [Ryb11]. Son statut de couple de Zariski a été définitivement confirmé par E. Artal, J. Carmona, J. I. Cogolludo et M. Á. Marco dans [Art+06]. Pour ce faire ils ont considéré les complémentaires des arrangements et ont démontré que leurs groupes fondamentaux ne coïncident pas. La même équipe a également découvert un couple de Zariski composé d'arrangements arithmétiques réels (et donc complexifiés) dans [Art+05]. La quête de nouveaux couples de Zariski et d'une meilleure compréhension de la relation entre combinatoire et topologie des courbes et des arrangements est un sujet de recherche très actif depuis le début des années 2000. Entre autres découvertes, S. Nazir, M. Yoshinaga [NY12] et F. Ye [Ye13] ont complètement déterminé les classes d'isotopie des arrangements jusqu'à 9 droites inclus et ont ainsi constaté qu'aucun couple de Zariski n'existe en dessous de cette valeur. Mentionnons également les travaux de B. Guerville-Ballé [Gue16] et J. Viu Sos [GV19] qui ont découvert de nombreux couples de Zariski au delà de 10 droites. Le lecteur pourra se référer à [ACT08 ; Gue22] pour un historique plus détaillé des découvertes dans ce domaine.

## Revue des invariants connus

Une grande variété d'invariants connus de la topologie algébrique ont été appliqués à l'étude des arrangements de droites et à la recherche des couples de Zariski. E. R. van Kampen [Kam33] a conçu une méthode pour obtenir une présentation de groupe fondamental de l'extérieur d'une courbe algébrique, que l'on appelle désormais la *méthode de Zariski-van Kampen*. Une comparaison directe des groupes fondamentaux peut parfois identifier un couple de Zariski, comme cela a été le cas pour l'exemple original de G. Rybnikov. Toutefois, il existe des exemples connus de couples de Zariski dont les complémentaires ont des groupes fondamentaux isomorphes (cf. [Shi09 ; Shi19 ; ACM19a ; Gue20]). Il a également été démontré que les variétés caractéristiques ne détectent pas non plus la topologie en général, y compris pour les arrangements réels.

La méthode de Zariski-van Kampen utilise une construction appelée *monodromie de tresse*, introduite par O. Chisini dans [Chi33], et qui encode sous la forme d'une tresse la position relative de chaque composante de la courbe au voisinage de chaque singularité. La définition originale a ensuite été améliorée par B. Moishezon [Moi81] et précisée pour le cas spécifique des arrangements de droites par M. Salvetti [Sal88]. A. Libgober [Lib86] a démontré que la monodromie de tresse en elle-même suffit à déterminer le type d'homotopie d'un arrangement de droites, et J. Carmona [Car03] a étendu ce résultat au type topologique (cf. également [ACC03]). Malgré cela, la monodromie de tresse s'avère être un invariant plus puissant que le groupe fondamental, et un plus grand nombre de couples de Zariski ont été obtenus grâce à elle, cf. [ACT08].

## Inclusion homologique

Ainsi que nous l'avons mentionné plus tôt, une autre approche pour construire des invariants d'arrangements de droites consiste à se concentrer sur la *variété-bord* de l'arrangement, définie comme le bord d'un voisinage régulier. T. Jiang, S. S.-T. Yau [JY93] et E. R. Westlund [Wes97] ont démontré que cette variété-bord possède une structure de *variété graphée* au sens de F. Waldhausen [Wal67a ; Wal67b] et W. D. Neumann [Neu81]. La topologie de la variété-bord est donc directement déterminée par la combinatoire de l'arrangement, encodée sous la forme d'un graphe. Ce graphe peut être vu comme un « plan de montage » pour reconstituer la variété-bord en recollant ensemble des fibrés en cercles (un type de variété de Seifert). Chacun de ces fibrés correspond au bord d'un voisinage régulier local autour de chaque composante et de chaque singularité de l'arrangement.

Le problème auquel nous nous intéressons est celui de l'étude de l'inclusion de la variété-bord dans l'extérieur de l'arrangement de droites. Ce morphisme d'inclusion a déjà été étudié *via* le morphisme qu'il induit sur les groupes fondamentaux. En effet, une présentation du groupe fondamental de l'extérieur peut-être obtenue soit avec la méthode de Zariski-van Kampen mentionnée plus haut, soit à l'aide d'une autre méthode due à W. Arvola [Arv92]. De l'autre côté, le groupe fondamental de la variété bord a été déterminé par E. R. Westlund [Wes97], cf. également [CS08]. À partir de ces deux résultats, E. Hironaka [Hir01] a étudié le morphisme induit par l'inclusion entre les deux groupes fondamentaux, dans le cas des arrangements réels d'abord, puis cette étude a été étendue aux arrangements complexes par V. Florens, B. Guerville-Ballé et M. Á. Marco [FGM15]. Leurs résultats mettent en évidence que l'analyse de l'« inclusion » des groupes fondamentaux se heurte à la complexité algébrique des morphismes sous-jacents. Une autre approche consiste donc à considérer le morphisme  $i_*$  induit sur les premiers groupes d'homologie standard dans  $\mathbb{Z}$ . Contrairement à ce que l'on pourrait penser, l'étude de cet objet n'est pas triviale. Elle a été initiée par E. Artal, V. Florens et B. Guerville-Ballé [AFG17], et une première généralisation de leurs résultats a été ensuite développée par W. Cadegan-Schlieper [Cad18].

Nous proposons ici un nouvel invariant qui étend ces propositions antérieures en exploitant pleinement l'« inclusion » homologique de la variété-bord d'un arrangement dans son extérieur. Nous présentons maintenant le principe général de sa construction. Le premier groupe d'homologie de la variété-bord possède deux types de générateurs : les méridiens des composantes, et des générateurs issus des cycles du graphe d'incidence. De son côté, le premier groupe d'homologie de l'extérieur est librement engendré par les seuls méridiens. Même si les deux groupes sont donc déterminés par la combinatoire, le morphisme induit  $i_*$  contient des informations topologiques significatives. La difficulté principale du calcul exact de  $i_*$  est l'ambiguïté de la définition des générateurs issus des cycles. Ils sont en effet déterminés par un *plongement du graphe* dans la variété-bord.

**Proposition A.** *Tout plongement ordonné du graphe dans la variété-bord induit un isomorphisme de groupes entre son premier groupe d'homologie standard et le groupe librement engendré par les méridiens des composantes et les cycles du graphe.*  $\triangleleft$

Pour lever cette ambiguïté, nous définissons le *stabilisateur du graphe*. Il s'agit d'un groupe déterminé par la combinatoire qui calcule toutes les différences homologiques entre tous les plongements ordonnés du graphe. Le morphisme  $i_*$  induit alors un invariant topologique dénommé *inclusion homologique* et qui est un élément du stabilisateur du graphe. Cet invariant peut être calculé sur n'importe quel arrangement, et en particulier sur un couple d'arrangements avec la même combinatoire pour déterminer s'ils forment un couple de Zariski. Sa construction généralise les invariants antérieurs dénommés  *$\mathcal{I}$ -invariant* dans [AFG17] et *loop-linking number* dans [Cad18]. Contrairement au premier, l'inclusion homologique ne nécessite aucune restriction sur la combinatoire pour être bien défini, et elle capture une information topologique plus fine que le second.

## Résumé de la **Partie I**

Le **Chapitre 1** est dédié à la construction du stabilisateur du graphe. Pour cela nous considérons la variété-bord d'un arrangement de droites comme un objet purement combinatoire, de sorte que le **Chapitre 1** ne mentionne pas spécifiquement les arrangements de droites.

Le niveau le plus élémentaire de la structure d'une variété graphée  $M$  est constitué de surfaces planaires à bord. La **Section 1.1** rappelle donc des théorèmes généraux sur les groupes de



difféotopie de ces surfaces. Puis dans la **Section 1.2** nous présentons le concept d'*étoile ordonnée* qui constituent les « briques élémentaires » assemblées entre elles pour construire les plongements du graphe. La **Section 1.3** est consacrée aux rappels sur la définition d'une variété graphée comme réunion de fibrés en cercles par des tores de jonctions le long d'un graphe  $\Gamma$ . La propriété fondamentale de la théorie des variétés graphées est l'unicité de la structure de graphe minimale. Nous introduisons donc une classe d'homéomorphismes qui respecte cette structure de graphe ainsi que l'orientation de chaque fibré en cercle et de leurs bases (cf. **Définitions 1.3.22** et **1.3.23**). La **Définition 1.3.20** introduit la notion d'*ordre graphé*  $\Omega$ , défini comme la donnée d'un *ordre local* sur chacun des ensembles de voisins de chaque sommet de  $\Gamma$ . Dans la **Section 1.4** nous définissons enfin les *plongements ordonnés du graphe* comme réunion d'étoiles ordonnées, une pour chaque fibré, correspondant chacune à l'ordre local attribué par  $\Omega$ . Le point central de notre construction est que les plongements ordonnés du graphe dépendent de l'ordre graphé. Cette précision est absente des constructions des invariants antérieurs dans [AFG17; Cad18]. Le **Théorème 1.4.9** établit que les homéomorphismes de la variété graphée ont une action transitive sur l'ensembles des plongements ordonnés du graphe. Dans la **Section 1.5** nous expliquons comment le choix d'un plongement ordonné du graphe permet de déterminer les générateurs-cycles de  $\pi_1(M)$  et  $H_1(M)$ , et par extension une présentation complète de ces deux groupes. La transitivité de l'action des homéomorphismes implique le résultat suivant :

**Théorème B.** *Il existe une présentation de  $\pi_1(M)$  qui ne dépend que de l'ordre graphé  $\Omega$ .*  $\triangleleft$

Nous pouvons ensuite enfin définir le stabilisateur du graphe dans la **Section 1.6**. Contrairement à son groupe fondamental, le premier groupe d'homologie de l'extérieur  $E_{\mathcal{A}}$  d'un arrangement  $\mathcal{A}$  est déterminé par la combinatoire. Ainsi il en est de même pour le groupe de morphismes :

$$\mathcal{H} := \text{Hom}(H_1(\Gamma_{\mathcal{A}}), H_1(E_{\mathcal{A}}))$$

cf. **Proposition 2.4.22**. Le *stabilisateur du graphe*  $\mathcal{G}_{\Gamma}$  est défini comme le quotient de  $\mathcal{H}$  par la différence sur les générateurs-cycles entre tous les plongements ordonnés du graphes deux à deux, cf. **Définition 1.6.1**. Le calcul explicite de cette différence peut être ramené à un calcul similaire sur les étoiles ordonnées en dimension 2, de sorte que nous obtenons le résultat suivant :

**Théorème C.** *Le stabilisateur du graphe  $\mathcal{G}_{\Gamma}$  possède une présentation explicite déterminée par la combinatoire  $\Gamma$  qui ne dépend pas de l'ordre graphé  $\Omega$ .*  $\triangleleft$

Tous les aspects purement combinatoires ayant été réglés, le **Chapitre 2** donne une présentation formelle des arrangements de droites et des concepts essentiels associés, puis définit l'*inclusion homologique*  $\mathcal{J}_{\Omega}(\mathcal{A})$  en tant qu'invariant d'arrangements de droites ordonnés.

Nous commençons par rappeler les définitions basiques des arrangements de droites dans la **Section 2.1**, ainsi que la définition de la notion de combinatoire basée sur l'incidence dans la **Section 2.2**. La **Section 2.3** donne la description de la structure de graphe minimale de la variété-bord  $B_{\mathcal{A}}$  ainsi que la structure des fibrés en cercles qui la composent (cf. **Théorèmes 2.3.15** et **2.3.17**), ceci grâce à l'opération de *blow-up* qui permet de résoudre les singularités. La **Section 2.4** présente les outils principaux pour déterminer le type topologique de l'extérieur ainsi que son groupe fondamental. Il s'agit du *diagramme de câblage* et de la *monodromie de tresse*. C'est cette dernière en particulier que nous utilisons comme invariant primaire pour extraire l'information topologique sur les arrangements.

Enfin, la **Section 2.5** présente la construction de l'inclusion homologique. Le morphisme d'inclusion  $i$  de la variété-bord  $B_{\mathcal{A}}$  dans l'extérieur  $E_{\mathcal{A}}$  induit un morphisme  $i_*$  entre leurs premiers groupes d'homologie standard. La restriction de ce morphisme aux générateurs-cycles dépend du choix d'un ordre graphé  $\Omega$  et d'un plongement ordonné du graphe  $\gamma$ . L'*inclusion homologique*  $\mathcal{J}_{\Omega}(\mathcal{A})$  est définie comme la classe du morphisme  $i_* \in \mathcal{H}$  dans le quotient  $\mathcal{G}_{\Gamma}$ . Par construction du stabilisateur du graphe, cette classe ne dépend plus du choix du plongement  $\gamma$ . Nous obtenons ainsi le théorème d'invariance :

**Théorème D.** *Soient  $\mathcal{A}, \mathcal{A}' \subset \mathbb{C}\mathbb{P}^2$  deux arrangements de droites avec la même combinatoire. La donnée d'un ordre identique  $\theta$  sur les droites de  $\mathcal{A}$  et  $\mathcal{A}'$  induit un ordre graphé  $\Omega^{\theta}$  sur le graphe d'incidence  $\Gamma$ . Supposons maintenant qu'il existe un homéomorphisme*

$$(\mathbb{C}\mathbb{P}^2, \mathcal{A}) \xrightarrow{\sim} (\mathbb{C}\mathbb{P}^2, \mathcal{A}')$$

*qui préserve l'ordre  $\theta$ . Alors  $\mathcal{J}_{\Omega^{\theta}}(\mathcal{A}) = \mathcal{J}_{\Omega^{\theta}}(\mathcal{A}')$  dans le groupe  $\mathcal{G}_{\Gamma}$ .*  $\triangleleft$

Le **Chapitre 3** est dédié à l'établissement d'une méthode pratique de calcul de l'inclusion homologique.

Le principe de base de la méthode, présenté dans la **Section 3.1**, est d'utiliser la monodromie de tresse de l'arrangement pour construire géométriquement un plongement ordonné *standard* du graphe. Ce plongement standard vit à l'intérieur d'une carte affine de  $\mathbb{CP}^2$  et sa construction nécessite donc une description directe de la structure graphée de la variété-bord sans utiliser de *blow-up*. Les valeurs du morphisme  $i_*$  sur le plongement standard du graphe sont alors déterminées par un calcul algorithmique sur les tresses de la monodromie, baptisé *enlacement de tresse*, et que nous présentons dans la **Section 3.2**. Une fois ces valeurs projetées dans la présentation de  $\mathcal{G}_\Gamma$  donnée par le **Théorème C**, nous obtenons la valeur exacte de  $\mathcal{J}_\Omega(\mathcal{A})$ . Nous pouvons ensuite appliquer cette méthode à des couples de Zariski potentiels. Toutefois, un point crucial est que l'ordre graphé  $\Omega$  doit impérativement être identique sur les deux arrangements. Cela nécessite d'ajuster les monodromies de tresses « brutes » grâce à une série d'algorithmes détaillés dans la **Section 3.3**. Le processus complet de calcul de l'invariant a été implémenté dans le langage **Sage** [Sag23]. Dans la **Section 3.3.3** nous donnons enfin plusieurs exemples de couples de Zariski déjà connus mais aussi inconnus jusqu'ici que nous pouvons identifier grâce à l'inclusion homologique.

## Pentes d'entrelacs et concordance

### Contexte général

Un entrelacs est un plongement de courbes polygonales fermées disjointes dans  $S^3$ . L'étude moderne des nœuds et entrelacs en tant qu'objets topologiques a réellement commencé dans les années 1920 et 1930 grâce au groupe fondamental et aux premières versions de l'homologie développées par H. Poincaré. Il a assez vite été établi que les nœuds et entrelacs ont de nombreuses connexions avec d'autres domaines de la topologie, et même de la physique.

Entre autres résultats, le théorème de W. B. R. Lickorish [Lic63] et A. H. Wallace [Wal60] permet de relier l'étude des nœuds et entrelacs à celle des variétés fermées orientées de dimension 3. Par ailleurs, la relation dite de *concordance* créée par R. Fox et J. Milnor [Fox62] est une extension naturelle de l'équivalence d'entrelacs en dimension 4. Deux entrelacs sont dits concordants s'ils co-bordent un ensemble de cylindres disjoints plongés dans  $S^4$ , et un entrelacs est dit *slice* s'il est concordant à l'entrelacs trivial. Cette relation crée une connexion naturelle entre la théorie des entrelacs et l'étude des surfaces plongées dans  $S^4$ .

### Invariants homotopiques et homologiques

Une méthode pour obtenir une présentation du groupe fondamental de l'extérieur d'un nœud ou d'un entrelacs a été mise au point par W. Wirtinger. Un résultat fondamental de F. Waldhausen [Wal68], C. McA. Gordon et J. E. Luecke [GL89] établit que la donnée du groupe fondamental et d'une structure périphérique, c'est-à-dire un méridien et une longitude, est un invariant de nœud complet. Même s'il existe des contre-exemples à la proposition similaire pour les entrelacs [Gor02], ce théorème renforce l'idée que des invariants de nœud et d'entrelacs puissants peuvent être obtenus à partir du groupe fondamental.

J. W. Alexander II [Ale28] a défini le polynôme qui porte aujourd'hui son nom en 1923. Il s'agissait alors d'un des tous premiers invariants de nœud découverts. Le polynôme d'Alexander et les concepts utilisés dans sa construction ont inspiré de nombreux autres invariants polynomiaux de nœuds et d'entrelacs. Entre autres définitions équivalentes, le polynôme d'Alexander peut être vu comme l'ordre du premier groupe d'homologie tordue abélienne de l'extérieur de l'entrelacs. Ce groupe possède une structure de module et est ainsi nommé le *module d'Alexander*. Cette définition illustre le potentiel des invariants d'entrelacs développés à l'aide de l'homologie tordue. Parmi les représentants notables des invariants de ce type, on peut citer les *polynômes d'Alexander tordus* définis par X.-S. Lin [Lin01] et M. Wada [Wad94], qui sont très directement liés à la torsion de Reidemeister [Kit96].

### L'invariant de pente pour les caractères

L'invariant de pente a été développé par A. Degtyarev, V. Florens and A. G. Lecuona dans [DFL22b] et plus tard dans [DFL21 ; DFL22a]. Il est apparu en tant qu'invariant propre durant

une étude sur la signature des entrelacs colorés. En accord avec les idées présentées dans notre introduction générale, sa construction est basée sur le fait que le bord de l'extérieur d'un entrelacs  $M_L$  est tout simplement une union de tores disjoints :

$$\partial M_L = \bigsqcup_{i=1}^n T_{L_i}$$

En appliquant une stratégie similaire à celle que nous avons présentée pour les arrangements de droites, ils considèrent l'application induite par l'inclusion  $i : \partial M_L \hookrightarrow M_L$  en *homologie tordue* :

$$i_* : H_1(\partial M_L; \rho) \longrightarrow H_1(M_L; \rho)$$

où  $\rho$  est une représentation du groupe fondamental  $\pi_1(M_L)$ . Il s'avère que le premier groupe d'homologie tordue du bord  $\partial M_L$  est entièrement déterminé par les images de la représentation sur la structure périphérique de l'entrelacs, c'est-à-dire sur les méridiens et longitudes. Pour le cas d'un caractère  $\omega : \pi_1(M_L) \rightarrow \mathbb{C}^*$ , la dimension de  $H_1(\partial M_L; \omega)$  est un multiple du nombre de valeurs non-triviales indépendantes de  $\omega$ . Dès lors, en imposant à  $\omega$  d'être trivial sur une seule composante  $K$  dite *distinguée*, on obtient :

**Théorème E.** *Soit  $\omega$  un caractère admissible. Supposons que  $\dim \ker i_* = 1$ . Alors  $\ker i_*$  est engendré par un vecteur de  $H_1(\partial M_{K \cup L}; \omega)$  de la forme*

$$\ker i_* = \langle a \cdot \ell + b \cdot m \rangle$$

où  $[a : b] \in \mathbb{C}P^1$  et  $m, \ell$  désignent le méridien et la longitude de  $K$ . La  $(K/L)$ -pente est alors définie par la formule :

$$s_{(K/L)}(\omega) := -\frac{b}{a} \in \mathbb{C} \cup \{\infty\} \quad \triangleleft$$

La pente est une fonction rationnelle en les valeurs du caractère, et elle possède d'autres propriétés intéressantes, dont certaines sont rappelées dans le **Chapitre 4**. En particulier la pente ne dépend pas des idéaux de Fitting du module d'Alexander, et elle est également un invariant de concordance.

En réalité la pente généralise la  $\eta$ -fonction à une seule variable développée par S. Kojima and M. Yamasaki [KY79] ainsi que l'invariant développé by N. Sato et J. Levine [Sat84]. La  $\eta$ -fonction peut elle-même être vue comme une fonction génératrice des  $\beta$ -invariants de T. Cochran [Coc85]. Les  $\beta$ -invariants sont à leur tour liés ([Coc90]) à certains relèvements des enlacements définis par J. Milnor [Mil54].

Notre objectif dans cette partie est de généraliser à de nouveaux contextes l'idée principale de cet invariant de pente, que nous appelons désormais « *pente des caractères* ». Nous présentons deux généralisations distinctes. La première, la *pente*  $SL_2(\mathbb{C})$ , a été développée en collaboration avec L. Bénard. Elle permet de définir une pente sur n'importe quel nœud en utilisant des représentations adjointes dans  $SL_2(\mathbb{C})$ . La seconde, la *pente généralisée*, utilise des représentations réalistes à valeurs dans  $SO_2(\mathbb{R})$  et une méthode symplectique due à V. I. Arnold [Arn67]. Sous des conditions similaires à celles de la pente des caractères, cette pente permet à plusieurs composantes de l'entrelacs d'être distinguées.

## Résumé de la **Partie II**

Le **Chapitre 4** donne une présentation rapide de la pente des caractères et de ses propriétés principales, afin d'établir une référence pour les généralisations. La **Section 4.1** fait un rappel très rapide des définitions élémentaires de la théorie des nœuds et entrelacs. La **Section 4.2** donne la **Définition 4.2.4** principale de la pente des caractères. Certains des résultats présentés utilisent de nouvelles démonstrations générales sur l'homologie tordue détaillée dans l'**Annexe A**, notamment le **Théorème 4.2.10** qui établit l'invariance par concordance. La **Section 4.2** explique comment calculer la pente en utilisant le calcul de Fox [Fox54], cf. **Théorème 4.3.1**. Nous présentons également une nouvelle implémentation en **GAP** [GAP22] de cette méthode qui permet de calculer la pente directement à partir du diagramme d'un nœud compatible grâce à [CD08 ; DT83].

Le **Chapitre 5** est adapté de [BFR21] et est dédié à la pente  $SL_2(\mathbb{C})$  sur les nœuds. Le choix du groupe  $SL_2(\mathbb{C})$  est motivé par le fait que l'ensemble des représentations du groupe du nœud  $K$

dans  $\mathrm{SL}_2(\mathbb{C})$  porte naturellement une structure algébrique. Cela est encore le cas pour l'ensemble des caractères de ces représentations, que l'on nomme la *variété des caractères*  $\mathrm{SL}_2(\mathbb{C})$  du nœud. Étant donnée une structure périphérique du nœud, la variété des caractères est une courbe plane dans  $\mathbb{C}^* \times \mathbb{C}^*$ , dont les coordonnées  $M$  et  $L$  correspondent aux valeurs propres du méridien et de la longitude préférée du nœud. Le polynôme  $A_K(L, M)$  définissant cette courbe est un invariant du nœud appelé le  $A$ -polynôme, défini dans [Coo+94]. Cet invariant contient beaucoup d'informations intéressantes sur le nœud ; en particulier, S. Boyer et X. Zhang [BZ05] et N. Dunfield et S. Garoufalidis [DG04] ont démontré que  $A_K = L - 1$  si et seulement si  $K$  est trivial. Une de nos motivations principales, comme expliqué dans la **Section 5.1**, est de capturer une partie de l'information topologique du  $A$ -polynôme grâce à la pente. En effet, le  $A$ -polynôme est un invariant de haut niveau qui est en général difficile à calculer, tandis que la valeur de la pente  $\mathrm{SL}_2(\mathbb{C})$  peut être obtenue grâce au calcul de Fox *via* un processus similaire à celui de la pente des caractères du **Chapitre 4**. La définition et les résultats principaux sur le  $A$ -polynôme sont détaillées dans la **Section 5.2**.

La construction de la pente  $\mathrm{SL}_2(\mathbb{C})$  est présentée dans la **Section 5.3**. Celle-ci est basée sur le fait qu'un analogue parfait du **Théorème E** existe pour lequel toutes les représentations adjointes *non-paraboliques* à valeurs dans  $\mathrm{SL}_2(\mathbb{C})$  du groupe du nœud sont admissibles, cf. **Définition 5.3.2** et **Lemme 5.3.4**. L'invariant est alors défini encore une fois comme la pente du sous-espace  $\ker i_*$  relativement à la base méridien-longitude. Il est important de noter que contrairement à la pente des caractères, aucune restriction n'est imposée sur le nœud  $K$ . La définition de la pente  $\mathrm{SL}_2(\mathbb{C})$  utilise des constructions due à J. Porti [Por97] qui le premier a observé que  $H_1(\partial M_K; \mathrm{Ad} \circ \rho)$  possède une *structure symplectique* pour la forme d'intersection induite. J. Porti a également défini la *torsion de Reidemeister* pour l'homologie tordue par  $\mathrm{Ad} \circ \rho$ , et nous démontrons que la torsion est liée à la pente par une formule simple dans la **Proposition 5.3.15**. La **Section 5.4** contient le **Théorème 5.4.1** principal qui connecte la pente au  $A$ -polynôme. Ce résultat est notamment utilisé pour obtenir le **Corollaire 5.4.5** qui établit que la pente  $\mathrm{SL}_2(\mathbb{C})$  peut détecter le nœud trivial.

Le **Chapitre 6** présente notre autre généralisation de la pente des caractères. Pour simplifier les constructions, nous considérons des entrelacs avec enlacements nuls entre toutes les composantes. La proposition fondamentale de la construction de la pente généralisée est la suivante :

**Proposition F.** *Notons  $\rho_\omega : \pi_1(M_L) \rightarrow \mathrm{SO}_2(\mathbb{R})$  la réalification d'un caractère  $\omega$  de l'entrelacs. Le  $\mathbb{R}$ -espace vectoriel  $H_1(\partial E_L, \mathbb{R}(\rho_\omega))$  équipé de la forme d'intersection est un espace symplectique librement engendré par les méridiens et les longitudes des composantes  $L_i$  sur lesquelles  $\omega$  est trivial. De plus,  $\ker i_*$  est un sous-espace lagrangien, autrement dit son propre orthogonal pour la forme symplectique.  $\triangleleft$*

V. I. Arnol'd [Arn67] a développé une méthode basée sur la complexification d'un espace symplectique réel qui permet de caractériser tout sous-espace lagrangien par une classe de matrices complexes unitaires. La pente généralisée est définie dans la **Section 6.2** comme l'argument du déterminant de la matrice caractérisant  $\ker i_*$ , cf. **Définition 6.2.7**. Contrairement à la pente des caractères, la pente généralisée peut être définie comme une fonction sur la totalité du tore des caractères de l'entrelacs  $L$ . Enfin, dans la **Section 6.3**, nous démontrons que la pente généralisée est encore un invariant de concordance.

# INTRODUCCIÓN

## Índice del capítulo actual

<b>Estructuras periféricas en la codimensión 2</b>	<b>16</b>
<b>Topología de las configuraciones de rectas</b>	<b>17</b>
General	17
Revisión de invariantes conocidos	17
Inclusión homológica	18
<b>Resumen de la Parte I</b>	<b>18</b>
<b>Pendientes de enlaces y concordancia</b>	<b>20</b>
Contexto general	20
Invariantes homotópicos y homológicos	20
El invariante de pendiente para caracteres	20
<b>Resumen de la Parte II</b>	<b>21</b>

## Estructuras periféricas en la codimensión 2

El tema de esta tesis es el estudio de ciertas familias de curvas de codimensión 2. Gracias al trabajo de S. Smale [Sma62] y J. Milnor [Mil65] hay una clasificación algebraica de variedades diferenciables de dimensión  $\geq 5$ . Sin embargo, el estudio de estas mismas variedades diferenciables en las dimensiones 3 y 4 sigue siendo un tema muy activo, que requiere una amplia variedad de herramientas de topología y geometría algebraica. Nuestro trabajo utiliza en particular el grupo fundamental, diferentes tipos de (co)homología, así como grupos de difeotopía (isotopía diferenciable). También utilizamos grafos y varios métodos combinatorios asociados a ellos.

Sea  $M$  una variedad con borde de dimensión 3 o 4. Nos interesa el caso en el que esta variedad es el complemento de un entorno regular de una curva de codimensión 2. Nuestro enfoque general es estudiar la inclusión:

$$i : \partial M \hookrightarrow M$$

En determinadas situaciones el borde tiene una estructura mucho más sencilla que la variedad. A pesar de ello, la inclusión anterior y sus morfismos inducidos contienen información topológica significativa. Más precisamente, estudiamos los morfismos inducidos por  $i$  sobre diferentes tipos de objetos de la topología algebraica: los grupos fundamentales, los primeros grupos de homología estándar en  $\mathbb{Z}$ , así como los primeros grupos de *homología torcida* asociados a una representación  $\rho : \pi_1(M) \rightarrow \text{GL}(V)$  del grupo fundamental en un espacio vectorial  $V$ . Esta última estructura también se llama *homología con coeficientes locales* y está formada por grupos de homología que tienen una estructura de espacio vectorial. Su construcción se puede encontrar por ejemplo en [Hat02], pero presentamos resultados específicos para las necesidades de nuestro trabajo en el [Apéndice A](#).

Aplicamos nuestro enfoque global a los complementos de dos tipos de curvas: por un lado las curvas algebraicas planas singulares del  $\mathbb{CP}^2$ , por lo tanto en dimensión 4, y por otro lado los nudos y enlaces en  $S^3$ , por lo tanto en dimensión 3.

## Topología de las configuraciones de rectas

### General

Una *curva algebraica plana* es el lugar geométrico de los ceros de un polinomio homogéneo complejo. El estudio de la topología de estas curvas fue iniciado por O. Zariski, quien demostró que todas las curvas algebraicas planas lisas del mismo grado son isótopas [Zar29]. A partir de entonces, el problema se centra en el estudio de curvas algebraicas singulares. La *combinatoria* de dicha curva se define como el tipo topológico del par que consta de un entorno regular y la curva misma. La combinatoria también está determinada por los datos del tipo topológico de las singularidades y las relaciones de incidencia entre los componentes. Los trabajos de O. Zariski [Zar31; Zar37] y E. R. van Kampen [Kam33] demostraron la existencia de pares de curvas que tienen la misma combinatoria pero diferentes encajes en  $\mathbb{C}P^2$ . Estos pares fueron bautizados *pares de Zariski* por E. Artal en [Art94].

Una *configuración de rectas* es una colección finita de rectas complejas en  $\mathbb{C}P^2$ , que corresponde a una curva cuyo polinomio de definición tiene sólo factores irreducibles de grado 1. Las configuraciones de rectas forman una subclase de plano curvas algebraicas cuya combinatoria está determinada únicamente por las relaciones de incidencia, que pueden codificarse en forma de *grafo de incidencia*. Los componentes de una configuración de rectas no tienen singularidades, todas las singularidades de la configuración son del mismo tipo. Por todas estas razones, el estudio de configuraciones de rectas ofrece un marco favorable para la creación de invariantes topológicas que luego pueden extenderse a otros tipos de curvas algebraicas, además de problemas intrínsecos interesantes. El primer par de Zariski potencial para configuraciones fue descubierto por G. Rybnikov [Ryb11]. Su condición de par de Zariski fue confirmada definitivamente por E. Artal, J. Carmona, J. I. Cogolludo y M. Á. Marco en [Art+06]. Para ello consideraron los complementos de las configuraciones y demostraron que sus grupos fundamentales no coinciden. El mismo equipo también descubrió un par de Zariski compuesto por configuraciones *aritméticas* reales (y por lo tanto complejas) en [Art+05]. La búsqueda de nuevos pares de Zariski y una mejor comprensión de la relación entre combinatoria y topología de curvas y configuraciones ha sido un tema de investigación muy activo desde principios de este milenio. Entre otros descubrimientos, S. Nazir, M. Yoshinaga [NY12] y F. Ye [Ye13] determinaron completamente las clases de isotopía de las configuraciones hasta 9 rectas inclusive y, por lo tanto, descubrieron que no existe ningún par de Zariski debajo de este valor. Mencionemos también el trabajo de B. Guerville-Ballé [Gue16] y J. Viu Sos [GV19] que descubrieron numerosos pares de Zariski más allá de 10 rectas. El lector puede consultar [ACT08; Gue22] con una historia más detallada de los descubrimientos en este campo.

### Revisión de invariantes conocidos

Se han aplicado una amplia variedad de invariantes conocidos de topología algebraica al estudio de configuraciones de rectas y a la búsqueda de pares de Zariski. E. R. van Kampen [Kam33] ideó un método para obtener una presentación de grupo fundamental del exterior de una curva algebraica, que ahora se llama *método de Zariski-van Kampen*. Una comparación directa de grupos fundamentales a veces puede identificar un par de Zariski, como fue el caso del ejemplo original de G. Rybnikov. Sin embargo, se conocen ejemplos de pares de Zariski cuyos complementos tienen grupos fundamentales isomorfos (cf. [Shi09; Shi19; ACM19a; Gue20]). También se ha demostrado que las variedades características tampoco detectan todas las diferencias de topología, incluso para las configuraciones reales.

El método Zariski-van Kampen utiliza una construcción llamada *monodromía de trenzas*, introducida por O. Chisini en [Chi33], y que codifica en forma de trenza la posición relativa de cada componente de la curva en la proximidades de cada singularidad. La definición original fue luego mejorada por B. Moishezon [Moi81] y aclarada para el caso específico de configuración de rectas por M. Salvetti [Sal88]. A. Libgober [Lib86] demostró que la monodromía de trenzas por sí sola es suficiente para determinar el tipo de homotopía de una configuración de rectas, y J. Carmona [Car03] extendió este resultado al tipo topológico (ver también [ACC03]). A pesar de esto, la monodromía de trenzas resulta ser un invariante más poderoso que el grupo fundamental, y gracias a ella se ha obtenido un mayor número de pares de Zariski, cf. [ACT08].

## Inclusión homológica

Como mencionamos anteriormente, otro enfoque para construir invariantes de configuraciones de rectas consiste en enfocarse en el la variedad borde de la configuración, definida como el borde de un entorno regular. T. Jiang, S. S.-T. Yau [JY93] y E. R. Westlund [Wes97] demostraron que esta variedad de borde tiene una estructura de *variedad de grafo* en el sentido de F. Waldhausen [Wal67a; Wal67b] y W. D. Neumann [Neu81]. Por lo tanto, la topología de la variedad de borde está determinada directamente por la combinatoria de la configuración, codificada en forma de grafo. Este grafo puede verse como un «plan de ensamblaje» para reconstituir la variedad de borde pegando fibrados de circunferencias (un tipo de variedad de Seifert). Cada uno de estos paquetes corresponde al borde de un entorno regular local alrededor de cada componente y cada singularidad de la configuración.

El problema que nos interesa es el del estudio de la inclusión de la variedad de borde en el exterior de la configuración de rectas. Este morfismo de inclusión ya ha sido estudiado *vía* el morfismo que induce en grupos fundamentales. De hecho, se puede obtener una presentación del grupo fundamental del exterior con el método de Zariski-van Kampen mencionado anteriormente o con la ayuda de otro método debido a W. Arvola [Arv92]. Por otro lado, el grupo fundamental de la variedad de grafo fue determinado por E. R. Westlund [Wes97], cf. también [CS08]. A partir de estos dos resultados, E. Hironaka [Hir01] estudió el morfismo inducido por la inclusión entre los dos grupos fundamentales, primero en el caso de configuraciones reales, luego este estudio se extendió a configuraciones complejas por V. Florens, B. Guerville-Ballé y M. Á. Marco [FGM15]. Sus resultados resaltan que el análisis de la «inclusión» de grupos fundamentales choca con la complejidad algebraica de los morfismos subyacentes. Por lo tanto, otro enfoque consiste en considerar el morfismo  $i_*$  inducido en los primeros grupos de homología estándar en  $\mathbb{Z}$ . Al contrario de lo que podría pensarse, el estudio de este objeto no es baladí. Fue iniciado por E. Artal, V. Florens y B. Guerville-Ballé [AFG17], y luego W. Cadegan-Schlieper [Cad18] desarrolló una primera generalización de sus resultados.

Proponemos aquí una nueva invariante que amplía estas propuestas anteriores explotando plenamente la inclusión homológica de la variedad de borde de una configuración en su exterior. Presentamos ahora el principio general de su construcción. El primer grupo de homología de la variedad de borde tiene dos tipos de generadores: los meridianos de las componentes y los resultantes de los ciclos del grafo de incidencia. Por su parte, el primer grupo de homología procedente del exterior lo generan libremente únicamente los meridianos. Incluso si los dos grupos están determinados por combinatoria, el morfismo inducido  $i_*$  contiene información topológica significativa. La principal dificultad del cálculo exacto de  $i_*$  es la ambigüedad de la definición de los generadores resultantes de los ciclos. De hecho, están determinados por un *encaje del grafo* en la variedad de borde.

**Proposición A.** *Cualquier encaje ordenado del grafo en la variedad de borde induce un isomorfismo de grupos entre su primer grupo de homología estándar y el grupo generado por los meridianos de las componentes y los ciclos del grafo.*  $\triangleleft$

Para eliminar esta ambigüedad, definimos el *estabilizador de grafo*. Este es un grupo determinado por combinatoria que calcula todas las diferencias homológicas entre todos los encajes ordenados del grafo. El morfismo  $i_*$  induce un invariante topológico llamado *inclusión homológica* y que es un elemento del estabilizador de grafo. Este invariante se puede calcular sobre cualquier configuración, y en particular sobre un par de configuraciones con la misma combinatoria para determinar si forman un par de Zariski. Su construcción generaliza los invariantes anteriores llamados  *$\mathcal{I}$ -invariant* en [AFG17] y *loop-linking number* en [Cad18]. A diferencia de la primera, la inclusión homológica no requiere ninguna restricción en la combinatoria para estar bien definida y captura información topológica más fina que la segunda.

## Resumen de la **Parte I**

El **Capítulo 1** está dedicado a la construcción del estabilizador de grafo. Para esto consideramos la variedad de borde de una configuración de rectas como un objeto puramente combinatorio, de modo que el **Capítulo 1** no menciona específicamente configuraciones de rectas.

El nivel más básico de la estructura de una variedad de grafo  $M$  consiste en superficies planas con borde. El **Apartado 1.1** recuerda por lo tanto teoremas generales sobre los grupos de difeotopía de estas superficies. Luego en el **Apartado 1.2** presentamos el concepto de *estrella*

*ordenada* que constituye los «ladrillos elementales» ensamblados para construir los encajes del grafo. El **Apartado 1.3** está dedicado a recordar la definición de una variedad de grafo como una unión de fibrados en circunferencias pegados a largo de toros, donde las instrucciones de pegado están codificadas en un grafo  $\Gamma$ . La propiedad fundamental de la teoría de la variedad de grafos es la unicidad de la estructura de grafo mínima. Por lo tanto, introducimos una clase de homeomorfismos que respeta esta estructura de grafo así como la orientación de los fibrados en circunferencias y sus bases (cf. **Definiciones 1.3.22** y **1.3.23**). La **Definición 1.3.20** introduce la noción de *orden de grafo*  $\Omega$ , definido como los datos de un *orden local* en cada uno de los conjuntos de vecinos de cada vértice de  $\Gamma$ . En el **Apartado 1.4** finalmente definimos los *encajes ordenados del grafo* como una unión de estrellas ordenadas, una para cada paquete, cada una correspondiente al orden local asignado por  $\Omega$ . El punto central de nuestra construcción es que los encajes ordenados del grafo dependen del orden de grafo. Esta precisión está ausente en las construcciones de los invariantes previos en [AFG17; Cad18]. El **Teorema 1.4.9** establece que los homeomorfismos de la variedad de grafo tienen una acción transitiva sobre todas los encajes ordenados del grafo. En **Apartado 1.5** explicamos cómo la elección de un encaje ordenado del grafo permite determinar los ciclos generadores de  $\pi_1(M)$  y  $H_1(M)$ , y por extensión una presentación completa de estos dos grupos. La transitividad de la acción de los homeomorfismos implica el siguiente resultado:

**Teorema B.** *Existe una presentación de  $\pi_1(M)$  que solo depende del orden de grafo  $\Omega$ .*  $\triangleleft$

Entonces finalmente podemos definir el estabilizador de grafo en el **Apartado 1.6**. A diferencia de su grupo fundamental, el primer grupo de homología del exterior  $E_{\mathcal{A}}$  de un configuración  $\mathcal{A}$  está determinado por la combinatoria. Así es lo mismo para el grupo de morfismos:

$$\mathcal{H} := \text{Hom}(H_1(\Gamma_{\mathcal{A}}), H_1(E_{\mathcal{A}}))$$

cf. **Proposición 2.4.22**. El *estabilizador del grafo*  $\mathcal{G}_{\Gamma}$  se define como el cociente de  $\mathcal{H}$  por la diferencia en los ciclos del generador entre todos los pares de encajes ordenados del grafo, cf. **Definición 1.6.1**. El cálculo explícito de esta diferencia se puede reducir a un cálculo similar sobre estrellas ordenadas en dimensión 2, de modo que obtenemos el siguiente resultado:

**Teorema C.** *El estabilizador del grafo  $\mathcal{G}_{\Gamma}$  tiene una representación explícita determinada por la combinatoria  $\Gamma$  que no depende del orden de grafo  $\Omega$ .*  $\triangleleft$

Resueltos todos los aspectos puramente combinatorios, el **Capítulo 2** da una presentación formal de la configuración de rectas y los conceptos esenciales asociados, por lo que define la *inclusión homológica*  $\mathcal{J}_{\Omega}(\mathcal{A})$  como invariante de configuraciones ordenadas de rectas.

Comenzamos recordando las definiciones básicas de la configuración de rectas en el **Apartado 2.1**, así como la definición de la noción de combinatoria basada en la incidencia en el **Apartado 2.2**. El **Apartado 2.3** da la descripción de la estructura de grado mínima de la variedad de borde  $B_{\mathcal{A}}$  así como la estructura de los fibrados en circunferencias que la componen (cf. **Teoremas 2.3.15** y **2.3.17**), esto gracias a la operación de *explosión* que permite resolver las singularidades. El **Apartado 2.4** presenta las principales herramientas para determinar el tipo topológico del exterior así como su grupo fundamental. Estos son el *diagrama de cableado* y la *monodromía de trenzas*. Es este último en particular el que utilizamos como invariante principal para extraer la información topológica sobre las configuraciones.

Finalmente, el **Apartado 2.5** presenta la construcción de la inclusión homológica. El morfismo de inclusión  $i$  de la variedad de borde  $B_{\mathcal{A}}$  en el exterior  $E_{\mathcal{A}}$  induce un morfismo  $i_*$  entre sus primeros grupos de homología estándar. La restricción de este morfismo a los generadores de tipo ciclo depende de la elección de un orden de grafo  $\Omega$  y un encaje ordenado del grafo  $\gamma$ . La *inclusión homológica*  $\mathcal{J}_{\Omega}(\mathcal{A})$  se define como la clase del morfismo  $i_* \in \mathcal{H}$  en el cociente  $\mathcal{G}_{\Gamma}$ . Por construcción del estabilizador de grafo, esta clase ya no depende de la elección del encaje  $\gamma$ . Obtenemos así el teorema de invariancia:

**Teorema D.** *Sean  $\mathcal{A}, \mathcal{A}' \subset \mathbb{C}\mathbb{P}^2$  dos configuraciones de rectas con la misma combinatoria. Consideremos órdenes idénticos  $\theta$  en las rectas de  $\mathcal{A}$  y  $\mathcal{A}'$  que inducen el mismo orden de grafo  $\Omega^{\theta}$  en el grafo de incidencia  $\Gamma$ . Supongamos ahora que existe un homeomorfismo*

$$(\mathbb{C}\mathbb{P}^2, \mathcal{A}) \xrightarrow{\sim} (\mathbb{C}\mathbb{P}^2, \mathcal{A}')$$

*que preserva el orden  $\theta$ . Entonces  $\mathcal{J}_{\Omega^{\theta}}(\mathcal{A}) = \mathcal{J}_{\Omega^{\theta}}(\mathcal{A}')$  en el grupo  $\mathcal{G}_{\Gamma}$ .*  $\triangleleft$



El **Capítulo 3** está dedicado a establecer un método práctico para calcular la inclusión homológica.

El principio básico del método, presentado en el **Apartado 3.1**, es utilizar la monodromía de trenzas de la configuración para construir geoméricamente un encaje *estándar* ordenado del grafo. Este encaje estándar vive dentro de una carta afín de  $\mathbb{CP}^2$  y, por lo tanto, su construcción requiere una descripción directa de la estructura de grafo de la variedad de borde sin usar directamente las explosiones. Los valores del morfismo  $i_*$  en el encaje estándar del grafo se determinan luego mediante un cálculo algorítmico en las trenzas de la monodromía, llamado *enlazado de trenza*, y que presentamos en el **Apartado 3.2**. Una vez proyectados estos valores en la presentación de  $\mathcal{G}_\Gamma$  dada por el **Teorema C**, obtenemos el valor exacto de  $\mathcal{J}_\Omega(\mathcal{A})$ . Luego podemos aplicar este método a posibles pares de Zariski. Sin embargo, un punto crucial es que el orden de grafo  $\Omega$  debe ser idéntico en ambas configuraciones. Esto requiere ajustar las monodromías de las trenzas «en bruto» usando una serie de algoritmos detallados en el **Apartado 3.3**. El proceso completo de cálculo del invariante se implementó en el lenguaje **Sage** [Sag23]. En el **Apartado 3.3.3** damos finalmente ejemplos de pares de Zariski ya conocidos pero también nuevos<sup>8</sup> que podemos identificar gracias a la inclusión homológica.

## Pendientes de enlaces y concordancia

### Contexto general

Un enlace es un encaje de una unión disjunta de curvas poligonales cerradas en  $S^3$ . El estudio moderno de nudos y enlaces como objetos topológicos realmente comenzó en los años 1920 y 1930 gracias al grupo fundamental y las primeras versiones de homología desarrolladas por H. Poincaré. Rápidamente se estableció que los nudos y enlaces tienen numerosas conexiones con otras áreas de la topología e incluso con la física.

Entre otros resultados, el teorema de W. B. R. Lickorish [Lic63] y A. H. Wallace [Wal60] permite vincular el estudio de nudos y enlaces al de las variedades cerradas y orientadas de dimensión 3. Además, la llamada relación *concordancia* creada por R. Fox y J. Milnor [Fox62] es una extensión natural de la equivalencia de los enlaces en dimensión 4. Se dice que dos enlaces son concordantes si son el borde de un conjunto de cilindros disjuntos propiamente encajados en la bola de dimensión 4, y un enlace se llama *slice* si es concordante con el enlace trivial. Esta relación crea una conexión natural entre la teoría de enlaces y el estudio de superficies encajadas en la bola de dimensión 4.

### Invariantes homotópicos y homológicos

W. Wirtinger desarrolló un método para obtener una presentación del grupo fundamental del exterior de un nudo o enlace. Un resultado fundamental de F. Waldhausen [Wal68], C. McA. Gordon y J. E. Luecke [GL89] establece que los datos del grupo fundamental y de una estructura periférica, es decir, un meridiano y una longitud canónica, es un invariante completo del nudo. Aunque hay contraejemplos a una proposición similar para los enlaces [Gor02], este teorema refuerza la idea de que se pueden obtener invariantes fuertes de nudos y enlaces a partir del grupo fundamental.

J. W. Alexander II [Ale28] definió el polinomio que hoy lleva su nombre en 1923. Este fue uno de los primeros invariantes de nudos. El polinomio de Alexander y los conceptos utilizados en su construcción han inspirado muchos otros polinomios invariantes de nudos y enlaces. Entre otras definiciones equivalentes, el polinomio de Alexander puede verse como el orden del primer grupo de homología torcido abeliano del exterior de un enlace. Este grupo tiene una estructura de módulo y por eso se llama *módulo* de Alexander. Esta definición ilustra el potencial de entrelazar invariantes desarrollados utilizando homología torcida. Entre los representantes notables de invariantes de este tipo, podemos citar los *polinomios torcidos de Alexander* definidos por X.-S. Lin [Lin01] y M. Wada [Wad94], que están directamente vinculados a la torsión de Reidemeister [Kit96].

### El invariante de pendiente para caracteres

El invariante de pendiente fue desarrollado por A. Degtyarev, V. Florens y A. G. Lecuona en [DFL22b] y posteriormente en [DFL21; DFL22a]. Apareció como invariante durante el estudio sobre la signatura de enlaces coloreados. De acuerdo con las ideas expuestas en nuestra introducción

general, su construcción se basa en que el borde exterior de una enlace  $M_L$  es simplemente una unión de toros dos a dos disjuntos:

$$\partial M_L = \bigsqcup_{i=1}^n T_{L_i}$$

Aplicando una estrategia similar a la que presentamos para configuraciones de rectas, consideramos la aplicación inducida por la inclusión  $i : \partial M_L \hookrightarrow M_L$  en *homología torcida*:

$$i_* : H_1(\partial M_L; \rho) \longrightarrow H_1(M_L; \rho)$$

donde  $\rho$  es una representación del grupo fundamental  $\pi_1(M_L)$ . Resulta que el primer grupo de homología torcido de la arista  $\partial M_L$  está completamente determinado por las imágenes de la representación sobre la estructura periférica del enlace, es decir sobre los meridianos y longitudes. Para el caso de un carácter  $\omega : \pi_1(M_L) \rightarrow \mathbb{C}^*$ , la dimensión de  $H_1(\partial M_L; \omega)$  es un múltiplo del número de valores no triviales independientes de  $\omega$ . Por lo tanto, al requerir que  $\omega$  sea trivial en una sola componente *distinguida*  $K$ , obtenemos:

**Teorema E.** *Sea  $\omega$  un carácter admisible. Supongamos  $\dim \ker i_* = 1$ . Entonces  $\ker i_*$  está generado por un vector de  $H_1(\partial M_{K \cup L}; \omega)$  de la forma*

$$\ker i_* = \langle a \cdot \ell + b \cdot m \rangle$$

donde  $[a : b] \in \mathbb{CP}^1$  y  $m, \ell$  denotan el meridiano y la longitud de  $K$ . La pendiente  $(K/L)$  se define entonces mediante la fórmula:

$$s_{(K/L)}(\omega) := -\frac{b}{a} \in \mathbb{C} \cup \{\infty\} \quad \triangleleft$$

La pendiente es una función racional de los valores de los caracteres y tiene otras propiedades interesantes, algunas de las cuales se recuerdan en el **Capítulo 4**. En particular, la pendiente no depende de los ideales de Fitting del módulo de Alexander y también es un invariante de concordancia.

En realidad, la pendiente generaliza la función  $\eta$  a una sola variable desarrollada por S. Kojima y M. Yamasaki [KY79] así como el invariante desarrollado por N. Sato y J. Levine [Sat84]. La función  $\eta$  puede verse en sí misma como una función generatriz de los  $\beta$ -invariantes de T. Cochran [Coc85]. Los  $\beta$ -invariantes están a su vez vinculados ([Coc90]) a ciertas elevaciones de enlaces definidos por J. Milnor [Mil54].

Nuestro objetivo en esta parte es generalizar a nuevos contextos la idea principal de este invariante de pendiente, que ahora llamamos «*pendiente de caracteres*». Presentamos dos generalizaciones distintas. La primera, la *pendiente*  $SL_2(\mathbb{C})$ , fue desarrollada en colaboración con L. Bénard. Permite definir una pendiente en cualquier nudo usando representaciones adjuntas en  $SL_2(\mathbb{C})$ . La segunda, la *pendiente generalizada*, utiliza representaciones realizadas con valores en  $SO_2(\mathbb{R})$  y un método simpléctico debido a V. I. Arnold [Arn67]. En condiciones similares a las de la pendiente de caracteres, esta pendiente permite distinguir enlaces de varias componentes.

## Resumen de la **Parte II**

El **Capítulo 4** ofrece una descripción general rápida de la pendiente de caracteres y sus principales propiedades, con el fin de establecer una referencia para generalizaciones. El **Apartado 4.1** proporciona un recordatorio muy rápido de las definiciones elementales de la teoría de nudos y enlaces. El **Apartado 4.2** da la **Definición 4.2.4** principal de la pendiente de caracteres. Algunos de los resultados presentados utilizan nuevas demostraciones generales de homología torcida detalladas en el **Apéndice A**, en particular el **Teorema 4.2.10** que establece la invariancia por concordancia. El **Apartado 4.2** explica cómo calcular la pendiente usando el cálculo de Fox [Fox54], cf. **Teorema 4.3.1**. También presentamos una nueva implementación en **GAP** [GAP22] de este método que permite calcular la pendiente directamente desde el diagrama de un nudo compatible gracias a [CD08; DT83].

El **Capítulo 5** está adaptado de [BFR21] y está dedicado a la pendiente  $SL_2(\mathbb{C})$  en los nudos. La elección del grupo  $SL_2(\mathbb{C})$  está motivada por el hecho de que el conjunto de representaciones del grupo del nudo  $K$  en  $SL_2(\mathbb{C})$  tiene naturalmente estructura algebraica. Este sigue siendo el

caso para el conjunto de los caracteres de esas representaciones, a las que llamamos *variedad de caracteres*  $SL_2(\mathbb{C})$  del nudo. Dada una estructura periférica del nudo, la variedad de caracteres es una curva plana en  $\mathbb{C}^* \times \mathbb{C}^*$ , cuyas coordenadas  $M$  y  $L$  corresponden a los valores propios del meridiano y del longitud preferida del nudo. El polinomio  $A_K(L, M)$  que define esta curva es un invariante del nudo llamado  $A$ -polinomio, definido en [Coo+94]. Este invariante contiene mucha información interesante sobre el nudo; en particular, S. Boyer y X. Zhang [BZ05] y N. Dunfield y S. Garoufalidis [DG04] han demostrado que  $A_K = L - 1$  si y solo si  $K$  es trivial. Una de nuestras motivaciones principales, como se explica en el **Apartado 5.1**, es capturar una parte de la información topológica del  $A$ -polinomio gracias a la pendiente. En efecto, el  $A$ -polinomio es un invariante generalmente difícil de calcular, mientras que la pendiente  $SL_2(\mathbb{C})$  se puede obtener mediante cálculo de Fox *vía* un proceso similar al de la pendiente de caracteres del **Capítulo 4**. La definición y los resultados principales sobre el  $A$ -polinomio se detallan en el **Apartado 5.2**.

La construcción de la pendiente  $SL_2(\mathbb{C})$  se presenta en el **Apartado 5.3**. Este se basa en el hecho de que existe un análogo perfecto del **Teorema E** para el cual todas las representaciones adjuntas *no parabólicas* con valores en  $SL_2(\mathbb{C})$  del grupo de nudos son admisibles, cf. **Definición 5.3.2** y **Lema 5.3.4**. Luego, el invariante se define una vez más como la pendiente del subespacio  $\ker i_*$  en relación con la base formada por la longitud y el meridiano. Es importante tener en cuenta que, a diferencia de la pendiente de carácter, no existen restricciones en el nudo  $K$ . La definición de la pendiente  $SL_2(\mathbb{C})$  utiliza construcciones debidas a J. Porti [Por97] quien fue el primero en observar que  $H_1(\partial M_K; \text{Ad} \circ \rho)$  tiene una *estructura simpléctica* para la forma de intersección inducida. J. Porti también definió la *torsión de Reidemeister* para la homología torcida por  $\text{Ad} \circ \rho$ , y demostramos que el giro está relacionado con la pendiente mediante una fórmula simple en la **Proposición 5.3.15**. El **Apartado 5.4** contiene el **Teorema 5.4.1** principal que conecta la pendiente con el polinomio  $A$ . Este resultado se utiliza en particular para obtener el **Corolario 5.4.5** que establece que la pendiente  $SL_2(\mathbb{C})$  puede detectar el nudo trivial.

El **Capítulo 6** presenta nuestra otra generalización de la pendiente de carácter. Para simplificar las construcciones, consideramos enlaces con número de enlace cero entre todos los componentes. La proposición fundamental de la construcción de la pendiente generalizada es la siguiente:

**Proposición F.** *Notemos  $\rho_\omega : \pi_1(M_L) \rightarrow SO_2(\mathbb{R})$  la realización de un carácter  $\omega$  del enlace. El  $\mathbb{R}$ -espacio vectorial  $H_1(\partial E_L, \mathbb{R}(\rho_\omega))$  equipado con la forma de intersección es un espacio simpléctico generado libremente por meridianos y longitudes de los componentes  $L_i$  en los cuales  $\omega$  es trivial. Además,  $\ker i_*$  es un subespacio lagrangiano, es decir, coincide con su ortogonal para la forma simpléctica.*  $\triangleleft$

V. I. Arnol'd [Arn67] desarrolló un método basado en la complejización de un espacio simpléctico real que permite caracterizar cualquier subespacio lagrangiano mediante una clase de matrices complejas unitarias. La pendiente generalizada se define en el **Apartado 6.2** como el argumento del determinante de la matriz que caracteriza a  $\ker i_*$ , cf. **Definición 6.2.7**. A diferencia de la pendiente de carácter, la pendiente generalizada se puede definir como una función sobre todo el toro de caracteres del enlace  $L$ . Finalmente, en el **Apartado 6.3**, demostramos que la pendiente generalizada sigue siendo un invariante de concordancia.

# PART I

## LINE ARRANGEMENTS

*J'ai horreur de ces objets incultes à qui on n'a jamais appris que le génie  
de l'homme triomphe obligatoirement de la matière imbécile !*

— Michel Greg

ACHILLE TALON – LA TRAVERSÉE DU DESERT, 1982

# CHAPTER

# 1

# GRAPH STABILISER

## Outline of the current chapter

<b>1.1 Preliminaries</b>	<b>25</b>
1.1.1 Finitely presented groups	25
1.1.2 Braid groups	25
1.1.3 Mapping class group of planar surfaces	26
<b>1.2 Ordered stars</b>	<b>27</b>
1.2.1 Definition and standard model	27
1.2.2 Action of the pure mapping class group	28
<b>1.3 Graph manifolds</b>	<b>30</b>
1.3.1 Circle bundles	30
1.3.2 Definition of the graph manifolds	32
1.3.3 Ordered graphs	32
1.3.4 Graphed homeomorphisms	34
<b>1.4 Ordered graphed embeddings</b>	<b>34</b>
1.4.1 Ordered model of a graph manifold	34
1.4.2 Definition of an ordered graphed embedding	35
1.4.3 Action of the homeomorphisms	37
<b>1.5 Fundamental group of a graph manifold</b>	<b>37</b>
1.5.1 Cycles of the graph	37
1.5.2 Presentation of the fundamental group	38
1.5.3 First homology group of a graph manifold	41
<b>1.6 Graph stabiliser</b>	<b>42</b>
1.6.1 Definition of the graph stabiliser	42
1.6.2 Difference maps	42
1.6.3 Presentation of the graph stabiliser	45
1.6.4 Plumbing moves	47

This chapter presents the construction of the graph stabiliser as a purely combinatorial object related to graph manifolds in dimension 3 as defined by F. Waldhausen [Wal67a; Wal67b].

**Section 1.1** recalls general results on braid groups and their interpretation as mapping class groups of a 2-dimensional disc  $D_m$  with  $m$  holes. In **Section 1.2** we present the *ordered stars* drawn on  $D_m$  and prove **Proposition 1.2.6** which establishes that there is a transitive action of the pure mapping class group  $\mathcal{P}(D_m)$  on the set of ordered stars  $\mathcal{O}_m$ .

**Section 1.3** is dedicated to graph manifolds. We recall the definition of a graph manifold  $M$  with the associated minimal graph  $\Gamma$ . We also define the graph ordering  $\Omega$  and we present the group of homeomorphisms which respect all these structures. In **Section 1.4** we give the definition of the set  $E_\Gamma(\Omega)$  of *ordered graphed embeddings*. They allow to draw the graph  $\Gamma$  on the manifold  $M$  itself as a reunion of ordered stars, one for each Seifert piece. **Theorem 1.4.9** uses **Proposition 1.2.6** to establish that the selected homeomorphisms of the graph manifold have a transitive action on  $E_\Gamma(\Omega)$ . Then in **Section 1.5** we move on to build the fundamental group and the first homology group of the graph manifold  $M$ . Their presentations given in **Theorems 1.5.9** and **1.5.16** depend on the choice of an ordered graphed embedding which determines the cycle generators.

**Section 1.6** gives the construction of the graph stabiliser  $\mathcal{G}_\Gamma$  as a quotient of all potential images of the cycles as sums of meridians in order to remove the dependency on the graphed embedding. We use **Theorem 1.4.9** to determine an explicit presentation of  $\mathcal{G}_\Gamma$  in **Theorem 1.6.11**. Finally, **Section 1.6.4** in particular shows that the graph stabiliser is stable by plumbing moves and can thus be computed on a larger graph than the minimal graph  $\Gamma$ .

## 1.1 Preliminaries

### 1.1.1 Finitely presented groups

**Definition 1.1.1.** Let  $G$  be a group. Let  $F$  be a finitely-generated free group and let  $R$  be a finite set of elements in  $F$ . Let  $\langle R \rangle$  be the smallest normal subgroup of  $F$  generated by  $R$ . The data  $\mathcal{P} = \langle F \mid R \rangle$  is called a *presentation* of  $G$  if there is an exact sequence

$$0 \longrightarrow \langle R \rangle \longrightarrow F \longrightarrow G \longrightarrow 0 \quad \diamond$$

For  $A$  and  $B$  two  $\mathbb{Z}$ -modules we often make use of the equivalence

$$\mathrm{Hom}(A, B) \simeq A^* \otimes B$$

### 1.1.2 Braid groups

We recall some basic results about braid group presentation, which are taken from [Art47; Bir75].

The braid group on  $m$  strands  $\mathbb{B}_m$  is generated by the elementary braids  $\sigma_{i,j}$  that permutes the strands  $i$  and  $j$ , with the relations

$$\begin{aligned} \sigma_{s,t}\sigma_{q,r} &= \sigma_{q,r}\sigma_{s,t} && \text{if } (t-r)(t-q)(s-r)(s-q) > 0 \\ \sigma_{s,t}\sigma_{r,s} &= \sigma_{r,t}\sigma_{s,t} = \sigma_{r,s}\sigma_{r,t} && \text{for } 1 \leq r < s < t \leq n \end{aligned}$$

Alternatively,  $\mathbb{B}_m$  can also be generated by the elementary braids  $\sigma_i$  for  $1 \leq i \leq m-1$  that permutes the strands  $i$  and  $i+1$ , with the relations

$$\begin{aligned} \sigma_i \cdot \sigma_k &= \sigma_k \cdot \sigma_i && \text{if } |i-k| \leq 2 \\ \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i &= \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} && \text{for } 1 \leq i \leq n-1 \end{aligned}$$

The two presentations are connected by the inverse group isomorphisms

$$\begin{aligned} \mathbb{B}_m &\longleftarrow \mathbb{B}_m \\ \sigma_{i,j} &\longleftarrow (\sigma_i \cdots \sigma_{j-2}) \cdot \sigma_{j-1} \cdot (\sigma_i \cdots \sigma_{j-2})^{-1} \\ \sigma_{i,i+1} &\longleftarrow \sigma_i \end{aligned}$$

Let  $\mathfrak{S}_m$  be the permutation group on  $m$  elements. By convention, a permutations  $\sigma \in \mathfrak{S}_m$  performs a right action on the elements which is denoted  $i \cdot \sigma$ . There is a natural epimorphism  $\sigma : \mathbb{B}_m \twoheadrightarrow \mathfrak{S}_m$  defined by

$$\sigma(\sigma_{i,j}) := (i, j)$$

The *pure braid group* on  $m$  strands  $\mathbb{P}_m$  is defined as the kernel of  $\sigma$ . It is generated by the braids

$a_{i,j} := \sigma_{i,j}^2$  that performs a full twist on the strands  $i$  and  $j$  with the relations

$$a_{r,s} \cdot a_{i,j} \cdot a_{r,s}^{-1} = \begin{cases} a_{i,j} & \text{if } r < s < i < j \\ & \text{or } i < r < s < j \\ a_{r,j} \cdot a_{i,j} \cdot a_{r,j}^{-1} & \text{if } r < s = i < j \\ (a_{i,j} \cdot a_{s,j}) \cdot a_{i,j} \cdot (a_{i,j} \cdot a_{s,j})^{-1} & \text{if } r = i < j < s \\ \left( (a_{r,j} \cdot a_{s,j} \cdot a_{r,j}^{-1} \cdot a_{s,j}^{-1}) \cdot a_{i,j} \cdot (a_{r,j} \cdot a_{s,j} \cdot a_{r,j}^{-1} \cdot a_{s,j}^{-1}) \right)^{-1} & \text{if } r < i < s < j \end{cases}$$

For every subset  $I = \{i_1, \dots, i_k\}$  of  $\{1, \dots, n\}$ , the *full twist over  $I$*  is the pure braid defined by

$$\Delta_I^2 := (\sigma_{i_1, i_2} \cdots \sigma_{i_{k-1}, i_k})^k \quad (1.1)$$

### 1.1.3 Mapping class group of planar surfaces

Let  $\Sigma_r^m$  be a planar surface with  $m$  boundary components denoted  $D^1, \dots, D^m$  and  $r$  punctures denoted  $x^1, \dots, x^r$ . When there are no punctures we only write  $\Sigma^m$ . A homeomorphism  $\varphi : \Sigma_r^m \rightarrow \Sigma_r^m$  that restricts to the identity on  $\partial \Sigma_r^m$  is called a *boundary-homeomorphism*. We denote by  $\text{Homeo}_\partial(\Sigma_r^m)$  the group boundary-homeomorphism of  $M$  up to isotopies that also restrict to the identity on the boundary.

The *mapping class group*  $\mathcal{M}(\Sigma_r^m)$  is the group of isotopy classes of orientation-preserving diffeomorphisms of  $\Sigma_r^m$  that respect the boundary set. The *pure mapping class group*  $\mathcal{P}(\Sigma_r^m)$  is the subgroup of  $\mathcal{M}(\Sigma_r^m)$  consisting of elements that fix each boundary component  $\partial^i D$  and each puncture  $x^i$  individually.

The results of this section are well-known, see for example [FM11] and [Bir75].

**Theorem 1.1.2.** *For  $m \geq 2$ , the group  $\mathcal{M}(\Sigma^{m+1})$  (resp.  $\mathcal{P}(\Sigma^{m+1})$ ) is isomorphic to  $\mathbb{B}_m \times \mathbb{Z}^m$  (resp.  $\mathbb{P}_m \times \mathbb{Z}^m$ ), where the generator  $\sigma_{j,l} \in \mathbb{B}_m$  (resp.  $a_{j,l} \in \mathbb{P}_m$ ) corresponds to a Dehn half-twist (resp. full twist) along the curve  $\delta_{j,l}$ , and the generator  $d_i \in \mathbb{Z}^m$  is the full Dehn twist around a curve  $\delta_i$  parallel to  $\partial^i D$ , as shown on Figure 1.1.1.  $\triangleleft$*

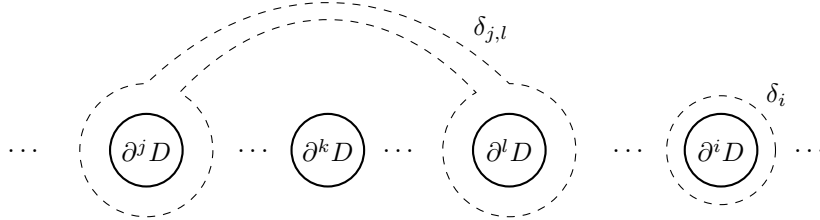


Figure 1.1.1: Dehn twist generators

**Theorem 1.1.3.** *Fill in a boundary component of  $\Sigma^{m+1}$  to obtain  $\Sigma^m$ . There is a surjective group homomorphism*

$$f_m : \mathcal{P}(\Sigma^{m+1}) \twoheadrightarrow \mathcal{P}(\Sigma^m)$$

*that respects the action of  $\mathcal{P}(\Sigma^{m+1})$  on the sub-surface  $\Sigma^{m+1} \subset \Sigma^m$ .  $\triangleleft$*

*Proof.* It is known from [Art47] that  $\mathcal{P}(\Sigma_r^1)$  is homeomorphic to  $\mathbb{P}_r$  for  $r \geq 3$ . The generator  $a_{j,l}$  corresponds to a Dehn full twist along a curve  $\delta_{j,l}$  that goes around  $x^j$  and  $x^l$  only. There is a relation that arises on mapping class groups by capping a boundary component with a punctured disc.

**Lemma 1.1.4.** *Fill in the boundary component  $D^m$  of  $\Sigma_r^m$  with a punctured disc  $B^2 \setminus \{x^{r+1}\}$  to get  $\Sigma_{r+1}^{m-1}$ . Then*

$$\mathcal{P}(\Sigma_r^m) = \mathcal{P}(\Sigma_{r+1}^{m-1}) \times \langle d_m \rangle$$

*where  $d_m$  is the full Dehn twist around a curve  $\delta_m$  parallel to  $D^m$ .  $\triangleleft$*

Starting from  $\Sigma^{m+1}$ , capping all boundary components but one using [Lemma 1.1.4](#) gives [Theorem 1.1.2](#).

A similar relation arises by filling in a puncture.

**Lemma 1.1.5** ([\[Bir75\]](#)). *Fill in the puncture  $x^{r+1}$  of  $\Sigma_{r+1}^m$  to obtain  $\Sigma_r^m$ . If  $m+r > 2$  then there is a short exact sequence*

$$1 \longrightarrow \pi_1(\Sigma_r^m, x^{r+1}) \longrightarrow \mathcal{P}(\Sigma_{r+1}^m) \longrightarrow \mathcal{P}(\Sigma_r^m) \longrightarrow 1 \quad \triangleleft$$

Fill in the boundary component  $D^{m+1}$  of  $\Sigma^{m+1}$  to obtain  $\Sigma^m$ . If  $m > 2$ , by combining [Lemma 1.1.4](#) and [Lemma 1.1.5](#), we get [Theorem 1.1.3](#).  $\square$

## 1.2 Ordered stars

An *ordered star* is a set of non-intersecting paths drawn on a disc  $D_m$  with  $m$  holes which are geometrically ordered. We denote by  $\mathcal{O}_m$  the set of all isotopy classes of ordered stars on  $D_m$ .

Ordered stars will be used in [Section 1.4.2](#) as elementary pieces to assemble the graphed embeddings on a graph manifold. Their role is to define a way to properly embed all half-edges of the graph with a common starting vertex  $v_i$  inside the corresponding circle bundle  $S_i$ . An important choice of our construction is that the half-edges, and thus the whole ordered star, are to be drawn on a *section*  $s_i : \Sigma_i \rightarrow S_i$ , where  $\Sigma_i$  is homeomorphic to a 2-sphere with  $m_i + 1$  holes.

We will also show in the beginning of [Section 1.4](#) that one can in fact always generically remove the interior of a disc from  $\Sigma_i$  disjoint from its boundary. Ordered stars are thus defined as objects on a 2-disc with a finite number  $m$  of holes.

### 1.2.1 Definition and standard model

Let  $R$  be a closed disc in the oriented plane  $\mathbb{C}$ . Let  $D^1, \dots, D^m$  be disjoint identical closed pairwise disjoint discs enclosed within  $R$  and ordered by their descending horizontal coordinates, and let  $x^j$  be the centre of  $D^j$ . By convention, we write  $\partial^\infty D$  for the boundary  $\partial R$  and  $\partial^j D$  for  $\partial(D^j)$ . These boundary circles are often seen as looping paths, denoted  $\partial_+^j D$  (resp.  $\partial_-^j D$ ) when travelled along in the positive (resp. negative) sense relatively to the orientation of  $\mathbb{C}$ . Finally, define

$$D_m := R \setminus \bigcup_{j=1}^m \overset{\circ}{D}^j \qquad \Delta_m := R \setminus \bigcup_{j=1}^m \{x^j\}$$

For any non-looping curve  $\alpha$  of the complex plane, we denote by  $\partial_- \alpha$  its starting point and  $\partial_+ \alpha$  its ending point.

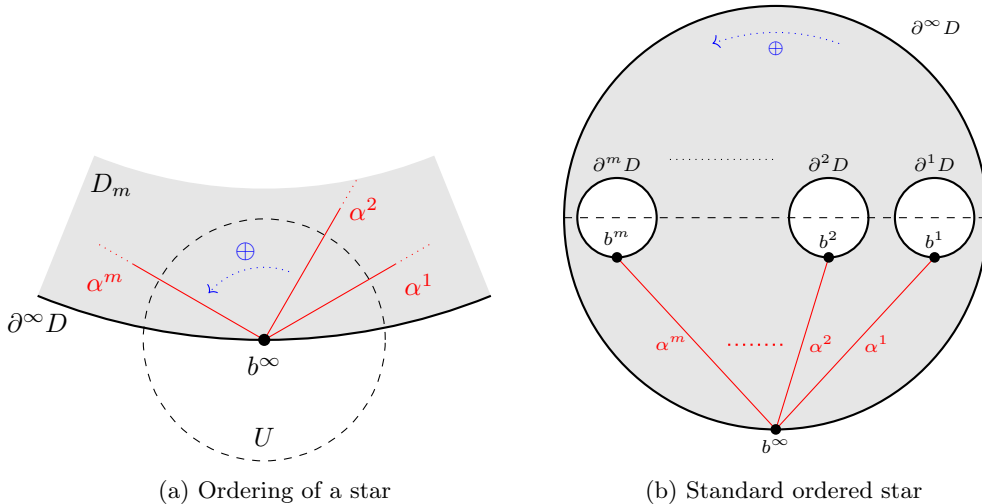


Figure 1.2.1: Ordered star



**Definition 1.2.1.** Let  $\omega \in \mathfrak{S}_m$  be a permutation. An  $\omega$ -star on  $D_m$  is a collection of properly embedded simple curves  $\alpha = (\alpha^j)_{1 \leq j \leq m}$  drawn on  $D_m$  such that:

- (i) for every curve  $\alpha^j$ ,  $\partial_+ \alpha^j := b^{j \cdot \omega} \in \partial^{j \cdot \omega} D$  and  $\partial_- \alpha^j$  is a common point  $b^\infty \in \partial^\infty D$ .
- (ii) for every pair of curves,  $\alpha^j \cap \alpha^k = \{b^\infty\}$ .
- (iii) there exists a disc  $U$  centred on  $b^\infty$  in the complex plane and an orientation-preserving homeomorphism  $\phi : U \rightarrow U$  that sends the pair  $(D_m \cap U, \alpha \cap U)$  to the pair shown in [Figure 1.2.1a](#).  $\diamond$

We denote by  $\mathcal{S}_m(\omega)$  the set of all isotopy classes of  $\omega$ -stars on  $D_m$  and by  $\mathcal{S}_m$  the reunion of all sets  $\mathcal{S}_m(\omega)$  for every  $\omega \in \mathfrak{S}_m$ .

**Definition 1.2.2.** An *ordered star* on  $D_m$  is a star of  $\mathcal{S}_m$  with respect to the permutation  $\omega = \text{Id} \in \mathfrak{S}_m$ . The set of all ordered stars is denoted  $\mathcal{O}_m$ .  $\diamond$

**Remark 1.2.3.** It is easy to construct a simple ordered star on  $D_m$  as shown in [Figure 1.2.1b](#), so the set  $\mathcal{O}_m$  is not empty.  $\square$

## 1.2.2 Action of the pure mapping class group

Let  $\omega \in \mathfrak{S}_m$  and write  $\xi := \omega^{-1}$ . Let  $\alpha \in \mathcal{S}_m(\omega)$  be a star on  $D_m$ . Up to isotopy, one can always move  $\alpha$  such that the loop  $\delta_{j,l}$  (resp.  $\delta_k$ ) of [Figure 1.1.1](#) does not intersect any path from  $\alpha$  except  $\alpha^{j \cdot \xi}$  and  $\alpha^{l \cdot \xi}$  (resp.  $\alpha^{k \cdot \xi}$ ), with only one transverse intersection for each. The respective actions of the Dehn twists on the curves of  $\alpha$  is then shown on [Figure 1.2.2](#). It is clear that the modified curves still respect the conditions from [Definition 1.2.1](#), and thus we obtain:

**Proposition 1.2.4.** *There is a well-defined action of  $\mathcal{M}(D_m)$  on  $\mathcal{S}_m$  induced by its natural action on  $D_m$ . Additionally, for every star  $\alpha \in \mathcal{S}_m(\omega)$  and every braid  $\beta \in \mathbb{B}_m$ , we have*

$$\beta \cdot \alpha \in \mathcal{S}_m(\omega \cdot \sigma(\beta)) \quad \triangleleft$$

**Corollary 1.2.5.** *There is a well-defined action of  $\mathcal{P}(D_m)$  on  $\mathcal{O}_m$  induced by its natural action on  $D_m$ .*  $\triangleleft$

The action of the pure braid generator  $a_{j,l} \in \mathbb{P}_m$  on an ordered star is shown on [Figure 1.2.3](#).

The rest of the section is dedicated to the proof of the following result, and can be skipped on first reading.

**Proposition 1.2.6.** *The group  $\mathcal{P}(D_m)$  acts transitively on the set  $\mathcal{O}_m$ .*  $\triangleleft$

*Proof.* We split the proof by considering separately the actions of the subgroups  $\mathbb{P}_m$  and  $\mathbb{Z}^m$  of  $\mathcal{P}(D_m)$ .

The fundamental group of  $\Delta_m$  is a free group  $\mathbb{F}_m$  with  $m$  generators. Let  $\alpha$  be an ordered star on  $D_m$ . We say that we travel positively along the curve  $\alpha^j$  when we go from  $b^0$  to  $b^j$ , and negatively otherwise, which we write as  $(\alpha^j)^{-1}$ . Then every curve  $\alpha^j$  of the ordered star can be associated to the closed curve of  $\Delta_m$  defined by:

$$\theta(\alpha)^j := \alpha^j \cdot \partial_+^j D \cdot (\alpha^j)^{-1} \quad (1.2)$$

where the closed path along  $\partial^j D$  is followed in the positive sense with respect to the orientation of  $\Delta_m$ . There is thus a map

$$\theta : \mathcal{O}_m \longrightarrow (\mathbb{F}_m)^m$$

which associates to any ordered star  $\alpha \in \mathcal{O}_m$  the homotopic classes of all the curves  $\theta(\alpha)^j$  inside  $\pi_1(\Delta_m) = \mathbb{F}_m$ . Note that the product  $\prod_{j=1}^m \theta(\alpha)^j$  is equal to the class of the loop  $\partial_+^\infty D$ .

**Lemma 1.2.7.** *The group  $\mathbb{P}_m$  acts transitively on the set  $\theta(\mathcal{O}_m)$ .*  $\triangleleft$

Go back to  $D_m$  by removing again the discs  $D^j$  from  $\Delta_m$ . We now consider the action on  $\mathcal{O}_m$  of the  $\mathbb{Z}^m$  subgroup of  $\mathcal{P}(D_m)$  generated by the Dehn twists along the  $\delta_k$  curves. We define the equivalence relation on  $\mathcal{O}_m$  by

$$\forall \alpha, \alpha' \in \mathcal{O}_m : \quad \alpha \simeq \alpha' \iff \theta(\alpha) = \theta(\alpha') \quad (1.3)$$

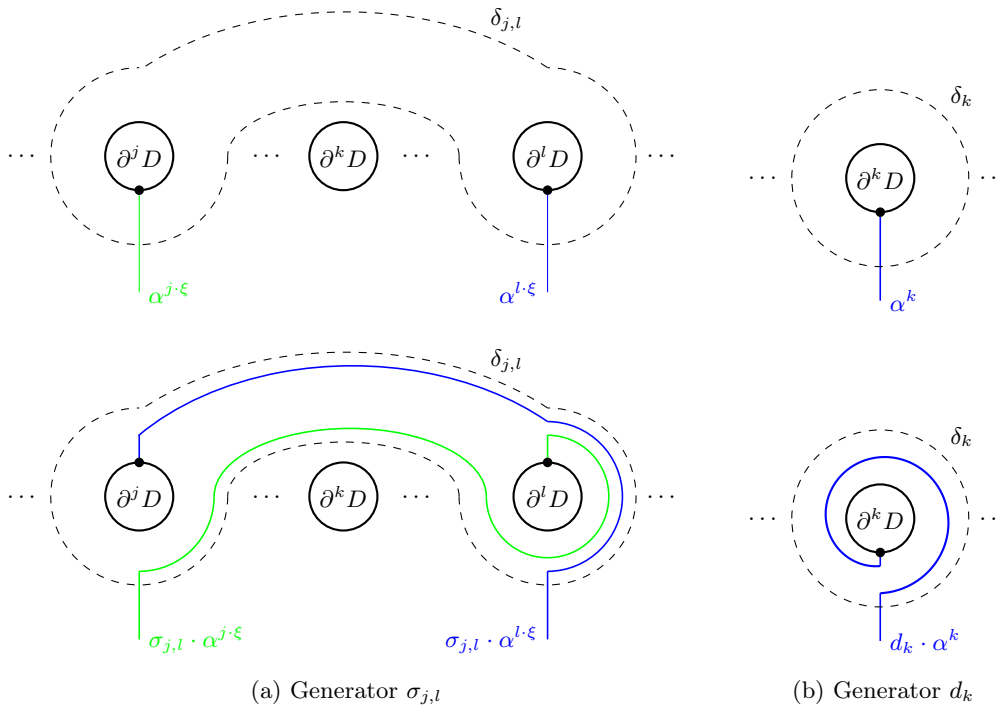


Figure 1.2.2: Action of  $\mathcal{M}(D_m)$  on a star

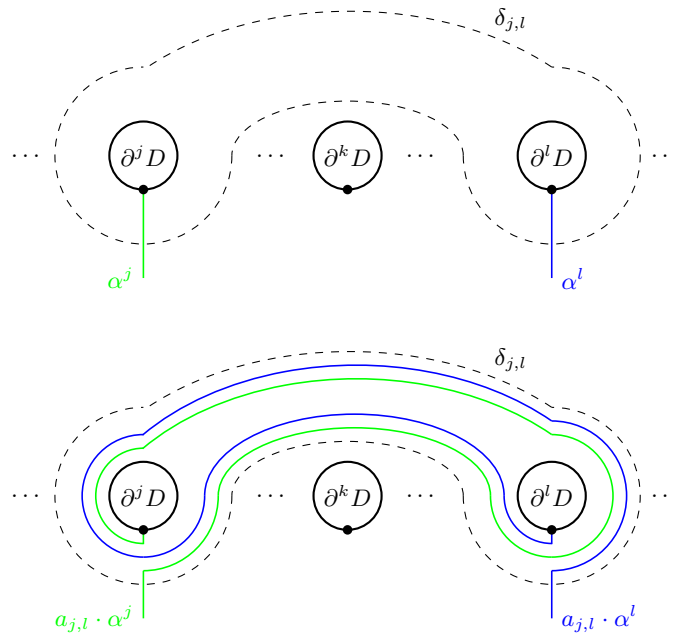


Figure 1.2.3: Action of  $a_{j,l} \in \mathcal{P}(D_m)$  on an ordered star

**Lemma 1.2.8.** *The group  $\mathbb{Z}^m$  acts transitively on every equivalence class of  $\mathcal{O}_m$ .*  $\triangleleft$

Let  $\alpha, \alpha' \in \mathcal{O}_m$  be two ordered stars on  $D_m$ . By [Lemma 1.2.7](#) there exists some pure braid  $\beta \in \mathbb{P}_m$  such that  $\beta \cdot \theta(\alpha) = \theta(\alpha')$  for the action of  $\mathcal{P}(D_m)$  on  $\pi_1(\Delta_m) \simeq \pi_1(D_m)$ . This action is induced by the natural action of  $\mathcal{P}(D_m)$  on  $D_m$  itself, which fixes all boundary components of  $D_m$ . So from [Eq. \(1.2\)](#) it is clear that  $\beta \cdot \theta(\alpha) = \theta(\beta \cdot \alpha)$ . This means that  $\beta \cdot \alpha$  and  $\alpha'$  are equivalent in the sense of [Eq. \(1.3\)](#), so by [Lemma 1.2.8](#) there exists some  $d \in \mathbb{Z}^m$  such that  $d \cdot (\beta \cdot \alpha) = (\beta, d) \cdot \alpha = \alpha'$ , which concludes the proof.  $\square$

*Proof of [Lemma 1.2.7](#).* Consider the standard ordered star  $\alpha^0$  on  $D_m$  shown on [Figure 1.2.1b](#). Set  $T = (t_1, \dots, t_n) := \theta(\alpha^0)$ . It is clear that  $T$  is a free generating family of  $\pi_1(\Delta_m) = \mathbb{F}_m$ . Now consider any other ordered star  $\alpha \in \mathcal{O}_m$  and set  $S = (s_1, \dots, s_m) := \theta(\alpha)$ . Every element  $s_j$  of  $S$  can be expressed in the basis  $T$ . It is also clear from [Eq. \(1.2\)](#) that each  $s_j$  is conjugated to the homotopical class of  $\partial_+^j D$ , i.e.  $t_j$ . There is thus a mapping  $f : (\mathbb{F}_m)^m \rightarrow (\mathbb{F}_m)^m$  such that  $f(t_j) = s_j = S_j t_j S_j^{-1}$  for every  $j$ , where  $S_j \in \mathbb{F}_m$ . By [[Art47](#), Theorem 14],  $f$  is canonically associated to the mapping-class-group action of a pure braid  $\beta \in \mathbb{P}_m$  on  $\pi_1(\Delta_m)$ .  $\square$

*Proof of [Lemma 1.2.8](#).* It is easy to see from [Figure 1.2.2b](#) that for every  $\alpha \in \mathcal{O}_m$  and every  $k$ ,  $\theta(d_k \cdot \alpha)$  is homotopic to  $\theta(\alpha)$  if  $D^k$  is filled in except for its centre, so the action is well-defined. Consider  $\alpha, \alpha' \in \mathcal{O}_m$  equivalent and define the collection of  $m$  closed curves  $\alpha \cdot \alpha'^{-1}$  by the formula

$$(\alpha \cdot \alpha'^{-1})^k := (\alpha'^k)^{-1} \cdot \alpha^k$$

Since  $(\alpha \cdot \alpha'^{-1})^k$  is a closed curve in  $D_m$  and thus in  $\Delta_m$ , we can see it as an element of  $\pi_1(\Delta_m)$ . Remember that  $\theta(\omega) = (t_1, \dots, t_m)$  is a free generating family of  $\pi_1(\Delta_m)$ . Using [Eq. \(1.2\)](#) inside the equation

$$1 = \theta(\alpha')^{-1} \cdot \theta(\alpha)$$

we obtain that  $(\alpha \cdot \alpha'^{-1})^k$  commutes with the homotopical class of  $\partial_+^k D$ , i.e.  $t_k$ . Since  $t_k$  is a free generator of  $\pi_1(\Delta_m) = \mathbb{F}_m$ , there must exist some integer  $r_k \in \mathbb{Z}$  such that  $(\alpha \cdot \alpha'^{-1})^k = t_k^{r_k}$ . This means exactly that up to homotopy  $\alpha^k = d_k^{r_k} \cdot \alpha'^k$ . Doing this for all  $k$ , we obtain an element  $d := \prod_k d_k^{r_k} \in \mathbb{Z}^m$  such that  $d \cdot \alpha = \alpha'$ .  $\square$

## 1.3 Graph manifolds

A graph manifold can be described as reunion of Seifert manifolds which are  $S^1$ -bundles over a disc with holes with a finite number of singular  $S^1$ -fibres. The Seifert pieces are glued together along their boundary tori with a map that permutes meridians and longitudes, and the graph indicates which gluings are made. The specific case of the boundary manifold of a line arrangement gives additional conditions on the subfamily of graph manifolds that we consider, in particular the Seifert pieces have no singular fibres and are thus *circle bundles*.

Graph manifolds were introduced by F. Waldhausen in [[Wal67a](#); [Wal67b](#)] and were further studied by D. Mumford [[Mum61](#)] and W. D. Neumann [[Neu81](#)]. The results of this section are well-known. See for example [[ST80](#)] and [[Hat99](#)] for circle bundles, and [[FM97](#)] or [[JS79](#)] for a modern approach on graph manifolds.

### 1.3.1 Circle bundles

**Definition 1.3.1.** A *circle bundle*  $S$  is a fibre bundle  $p : S \rightarrow \Sigma$ , where  $\Sigma$  is a compact 2-surface, such that for every  $s \in S$ , there exists an open neighbourhood  $U$  of  $p(s)$  and an isomorphism  $q : U \times S^1 \rightarrow p^{-1}(U)$  with  $(p \circ q)|_U$  corresponding to the projection on the first variable.  $\triangleleft$

The surface  $\Sigma$  is called the *basis* of the bundle. The pre-image  $p^{-1}(x)$  of a point  $x \in \Sigma$  is homeomorphic to a circle  $S^1$  which is called the *fibre over  $x$* . A *section* is an embedding  $s : \Sigma \hookrightarrow S$  such that  $p \circ s = \text{Id}_\Sigma$ .

**Definition 1.3.2.** A homeomorphism  $\Psi : S \rightarrow S'$  between two circle bundles is called *fibrewise* if it sends every fibre of  $S$  to a fibre of  $S'$ .  $\triangleleft$

**Definition 1.3.3.** An *oriented circle bundle* is an orientation-preserving fibre bundle  $p : S \rightarrow \Sigma$  along with the data of two of the following three:

- an orientation on  $S$ .
- an orientation on all fibres of  $p$ .
- an orientation of  $\Sigma$ .

◇

**Remark 1.3.4.** Fixing just the orientation of  $S$  does not fix the orientation on the basis and fibres. Indeed, there exists a homeomorphism  $\nu$  of  $S$  that changes the orientation of the basis and all fibres at the same time, without changing the global orientation of  $S$ .  $\square$

**Definition 1.3.5.** A fibrewise homeomorphism  $\Psi : S \rightarrow S$  is said to be *fibrewise-positive* if the induced homeomorphisms on the basis  $\Sigma$  and on every fibre  $S^1$  are all positive.  $\diamond$

We denote by  $\text{Homeo}_\partial^+(S)$  the group of fibrewise-positive boundary-homeomorphisms of  $S$ . We now assign a numbering in  $\{1, \dots, m\}$  to the boundary components of  $S$  and we denote by  $\partial_i S$  the  $i$ -th boundary component of  $S$ .

**Definition 1.3.6.** A *boundary section collection* on a circle bundle  $S$  with  $m$  components is the data of a collection of closed curves  $\bar{\mu} := (\mu_i)_{1 \leq i \leq m}$  such that  $\mu_i \subset \partial_i S$  and is transverse to the fibres of  $S$ .  $\diamond$

Circle bundles are a special case of Seifert manifolds, namely those without singular fibres. For our study we are only interested in a subclass of circle bundles that respect the following set of conditions:

**Conditions 1.3.7.** Let  $S$  be a circle bundle with basis  $\Sigma$ .

- (C1)  $S$  is orientable.
- (C2)  $\Sigma$  is planar.
- (C3)  $\Sigma$  has  $m \geq 3$  boundary components.

It must be noted that **Condition (C3)** can exceptionally be relaxed when we consider non-minimal graph structures.

We now describe the construction of the reference model that we use for all our circle bundles with fixed sections on the boundary.

**Definition 1.3.8.** Let  $\varepsilon \in \mathbb{Z}$ . Recall from **Section 1.2.1** the model  $D_m \subset \mathbb{C}$  homeomorphic to the surface  $\Sigma^{m+1}$ . Consider the direct product

$$T_m := D_m \times S^1$$

and a separate solid torus

$$T_\infty := D' \times S^1$$

Assign the same respective orientations on the sections and fibres of  $T_m$  and  $T_\infty$ . Let  $\check{s}$  be a fixed section of  $T_m$ . It is naturally associated with a collection  $\check{\mu}$  of sections on the boundary with  $\check{\mu}_i := \check{s}(\partial^i D_m)$ . Take a point  $x_\infty$  on  $\mu_\infty$  and let  $\check{\lambda}_\infty$  be the fibre over  $x_\infty$ . Let  $\check{s}'$  be a section of  $T_\infty$  and take  $\check{\mu}'_\infty$  the positive path  $\partial D'$  and  $\check{\lambda}'_\infty$  the fibre over a point  $x'_\infty$  on  $\check{\mu}'_\infty$ . Glue  $\partial T_\infty$  to the toric boundary component  $\partial^\infty D_m \times S^1$  of  $T_m$  using the gluing map:

$$g_\varepsilon : \begin{cases} \check{\mu}_\infty \mapsto -\check{\mu}'_\infty - \varepsilon \cdot \check{\lambda}'_\infty \\ \check{\lambda}_\infty \mapsto \check{\lambda}'_\infty \end{cases} \quad (1.4)$$

The manifold

$$S(m, \check{s}, \varepsilon) := T_m \cup_{g_\varepsilon} T_\infty$$

thus obtained is a circle bundle with  $m$  boundary components, and  $\varepsilon$  is called its *Euler number*.  $\diamond$

**Proposition 1.3.9.** *The manifold  $S(m, \check{s}, \varepsilon)$  does not depend on the choice of the section  $\check{s}'$  on the solid torus  $T_\infty$ .*  $\triangleleft$

*Proof.* This is a direct consequence from the fact that all fibre structures on the solid torus are isotopic.  $\square$

**Remark 1.3.10.** The notion of Euler number is usually defined on closed fibred spaces. The choice of the fixed section  $\check{s}$  allows to extend this definition and designate  $\varepsilon$  as the Euler number of  $S(m, \check{s}, \varepsilon)$ . Note that this could also have been achieved by fixing only a boundary section collection  $\check{\mu}$ .  $\square$

**Theorem 1.3.11** ([JS79]). *Let  $S$  be a circle bundle with  $m$  boundary components that respects [Conditions 1.3.7](#) and let  $\bar{\mu}$  be a collection of sections on the boundary of  $S$ . Then there exists an orientation-preserving fibrewise boundary-homeomorphism  $\chi : S(m, \check{s}, \varepsilon) \rightarrow S$  such that  $\chi(\check{\mu}_i) = \mu_i$  for some unique value of  $\varepsilon \in \mathbb{Z}$ .*  $\triangleleft$

The number  $\varepsilon \in \mathbb{Z}$  is called the *Euler number* of the oriented circle bundle  $S$ .

**Remark 1.3.12.** Orienting the manifold  $S(m, \check{s}, \varepsilon)$  induces an orientation of the bundle. Indeed, changing the orientation of just (say) the sections of  $T_m \subset S(m, \check{s}, \varepsilon)$  would also change the Euler number  $\varepsilon$  into its opposite  $-\varepsilon$ .  $\square$

**Remark 1.3.13.** For  $m = 2$ , the manifold  $S(2, \check{s}, \varepsilon)$  is homeomorphic to a thickened torus  $T \times I$  for any value of  $\varepsilon \in \mathbb{Z}$ . This means that the thickened torus has not a unique structure as a circle bundle, which justifies the requirement that  $m \geq 3$  in [Theorem 1.3.11](#).  $\square$

### 1.3.2 Definition of the graph manifolds

We shall begin by recalling some basics about the graph manifolds, in preparation of [Section 1.6](#). Let  $M$  be an oriented connected 3-manifold

**Definition 1.3.14.** A *graph structure* on  $M$  is a set  $\Theta$  of pairwise disjoint tori such that  $M \setminus \Theta$  is a disjoint union of Seifert manifolds. If  $M$  has a graph structure it is called a *graph manifold* and the elements of  $\Theta$  are called *joining tori*.  $\diamond$

**Theorem 1.3.15** ([Wal67b]). *Any graph manifold that does not belong to the exceptions listed in [Wal67b, Satz 8.1] admits a unique graph structure with a minimal number of tori.*  $\triangleleft$

In fact only a handful of exceptional classes arise in our applications to line arrangements. They are listed in [Section 2.2.3](#) and we do not intend to study them. From now on we only consider non-exceptional graph manifolds, for which we only use the unique minimal structure, which we will thus refer to as ‘the graph structure’.

According to [Theorem 1.3.11](#), for every circle bundle component  $S$  there exists an orientation-preserving fibrewise boundary-homeomorphism  $\chi : S(m, \check{s}, \varepsilon) \rightarrow S$  for some  $m \in \mathbb{N}$  and  $\varepsilon \in \mathbb{Z}$ . A joining torus  $T \in \Theta$  that joins two circle bundles  $S$  and  $S'$  corresponds to a component  $\partial^k D_m \times S^1$  in  $S(m, \check{s}, \varepsilon)$  and a component  $\partial^{k'} D_{m'} \times S^1$  in  $S(m', \check{s}', \varepsilon')$ .

We intend to apply the graph manifold theory to regular neighbourhoods of complex line arrangements. For this reason, we only consider a subclass of non-exceptional graph manifold with specific additional properties.

**Conditions 1.3.16.** Let  $M$  be a graph manifold with minimal graph structure  $\Theta$ .

- (M1) All irreducible components of  $M$  are oriented circle bundles (i.e. Seifert manifolds with no exceptional fibres).
- (M2) Every circle bundle component  $S$  of  $M$  verifies [Conditions 1.3.7](#).
- (M3) There is a choice of a section  $\bar{\mu} : D_m \rightarrow S$  on every circle bundle  $S$  of  $M$  oriented as the opposite of a base section.
- (M4) For every joining torus  $T \in \Theta$  between  $S$  and  $S'$ , let  $\mu$  be the chosen oriented section of  $T$  as a boundary component of  $S$ , and let  $\lambda$  be the oriented fibre over a point  $x_0$  on  $\mu$ . Define  $\mu'$  and  $\lambda'$  similarly inside  $S'$ . Then the gluing map from  $T \subset S$  to  $T \subset S'$  is given by

$$g_{\text{ext}} : \begin{cases} \mu \mapsto \lambda' \\ \lambda \mapsto \mu' \end{cases}$$

### 1.3.3 Ordered graphs

As the name obviously suggests, the graph structure of a given graph manifold can be represented as a graph.

**Definition 1.3.17.** The *graph*  $\Gamma$  of a graph manifold  $M$  with graph structure  $\Theta$  is given by the following description:

- every vertex  $v$  is associated to a circle bundle component  $S$  of  $M \setminus \Theta$ .
- every vertex  $v$  is decorated with the Euler number  $\varepsilon \in \mathbb{Z}$  of the corresponding circle bundle component  $S \simeq S(m, \check{s}, \varepsilon)$ .
- every edge  $e$  is associated to a unique joining torus in  $\Theta$  that co-bounds  $S$  and  $S'$ .  $\diamond$

For a non-exceptional graph manifold  $M$ , the graph is unique since it is entirely determined by the unique minimal graph structure  $\Theta$ .

The restrictions on the graph manifold given by [Conditions 1.3.16](#) induce new restrictions on the graph itself:

**Conditions 1.3.18.** The graph  $\Gamma$  of the graph manifold  $M$  is such that:

- (G1) no edge starts and ends at the same vertex.
- (G2) there is at most one edge between every two vertices.
- (G3) every vertex has at least *three* neighbours.

A graph that verifies the first two conditions is also called *simplicial*. The third condition is specific to the class of graph manifolds that we consider in this chapter.

**Remark 1.3.19.** In the most general case of graph manifolds as defined by [\[Wal67a; Neu81\]](#), vertices are also weighted by the genus of  $S$ , which in our case is set to zero for all vertices with [Condition \(C2\)](#). The gluing map is also characterised by a slope value for each boundary similar to [Eq. \(1.4\)](#), which is put as a decoration on the edges. Again, in our case this slope is set as  $\infty$  for all edges with [Condition \(M4\)](#).  $\square$

We denote by  $V$  the set of vertices and by  $E$  the set of edges of  $\Gamma$ . We fix an ordering  $\omega : V \rightarrow \{1, \dots, n\}$  which we denote by  $V = \{v_1, \dots, v_n\}$ . For every vertex  $v_i$ , the set of its neighbours is  $V_i$  with cardinal  $m_i \geq 3$ . By default edges are not oriented and can be seen as subsets  $\{v_i, v_j\} \subset V$ . The set of edges is denoted by  $E$ . We use the notation  $e_{i,j}$  (or  $e_{j,i}$ ) as a shortcut for the unique edge  $\{v_i, v_j\}$ . The corresponding joining torus in  $\Theta$  is denoted  $T_{i,j}$ . We also denote by  $\vec{e}_{i,j}$  and  $\vec{e}_{j,i}$  the two half-edges that compose  $e_{i,j}$  on the side of  $v_i$  and  $v_j$  respectively.

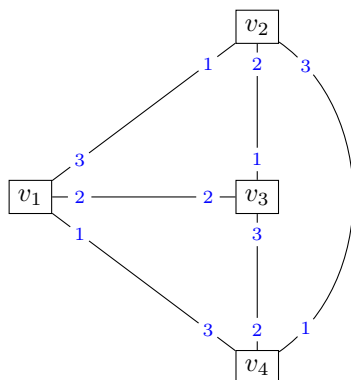


Figure 1.3.1: An ordered graph

**Definition 1.3.20.** A *graph ordering* over  $\Gamma$  is a collection of functions  $\Omega = (\omega_i)_{v_i \in V}$  where

$$\omega_i : V_i \rightarrow \{1, \dots, m_i\}$$

are bijections defined on the sets of neighbours. For every vertex  $v_i \in V$ , the function  $\omega_i$  is called the *local order* around  $v_i$ .  $\diamond$

In some contexts we make a greater use of the inverse of  $\omega_i$ , which we denote by

$$\xi_i : \{1, \dots, m_i\} \rightarrow V_i$$

A convenient way of representing a graph ordering is to put a decoration on every half-edge  $\vec{e}_{i,j}$  of the graph indicating the local order  $\omega_i(v_j)$  of the corresponding neighbour  $v_j$  around the starting vertex  $v_i$ , as illustrated on [Figure 1.3.1](#).

### 1.3.4 Graphed homeomorphisms

**Theorem 1.3.21** ([\[Wal67a\]](#)). *Let  $M$  and  $N$  be two graph manifolds with respective minimal graph structures  $\Theta_M$  and  $\Theta_N$ . Let  $\Phi : M \rightarrow N$  be a homeomorphism. Then, if  $M$  and  $N$  do not belong to one of the exceptional cases,  $\Phi$  is isotopic to a homeomorphism  $\Psi : M \rightarrow N$  such that  $\Psi(\Theta_M) = \Psi(\Theta_N)$ .*  $\triangleleft$

According to [Theorem 1.3.21](#), up to isotopy any homeomorphism of a graph manifold acts on the graph structure. Since joining tori are associated to edges of the graph, one can thus define a surjective morphism

$$G : \text{Homeo}(M) \longrightarrow \text{Aut}(\Gamma)$$

where  $\text{Homeo}(M)$  stands for the group of isotopy classes of homeomorphisms of  $M$ .

**Definition 1.3.22.** The kernel of  $G$  is called the group of *graphed homeomorphisms* of the graph manifold  $M$  and is denoted  $\text{Homeo}_\Gamma(M)$ .  $\diamond$

Graph manifolds are orientable as well as all of their circle bundle components. We thus take these orientations in consideration.

**Definition 1.3.23.** A homeomorphism  $\Psi : M \rightarrow M$  is said to be *positive* if it respects the global orientation of  $M$ . It is *strongly positive* if its restriction on every component  $S_i$  is fibre-positive in the sense of [Definition 1.3.5](#).  $\diamond$

We denote by  $\text{Homeo}^+(M)$  (resp.  $\text{Homeo}^{++}(M)$ ) the group of isotopy classes of positive (resp. strongly positive) homeomorphisms of  $M$ . The similar notations  $\text{Homeo}_\Gamma^+(M)$  (resp.  $\text{Homeo}_\Gamma^{++}(M)$ ) denotes the respective subgroups of graphed homeomorphisms of  $M$ . In fact  $\text{Homeo}_\Gamma^{++}(M)$  is a normal subgroup of index 2 of  $\text{Homeo}_\Gamma^+(M)$ . The quotient is generated by the positive homeomorphism of  $M$  defined as follows:

- On every circle bundle component  $S_i$  there is a homeomorphism  $\nu_i$  that changes the orientation of all sections and all fibres at the same time, without changing the global orientation of  $S_i$ .
- Since the gluing map given by [Condition \(M4\)](#) permutes fibres and sections, the action of  $\nu_i$  on one component  $S_i$  of  $M$  propagates to every other component. The ensuing global homeomorphism of  $M$  is denoted by  $\nu$  and does not change the global orientation of the manifold.

**Proposition 1.3.24.** *The group  $\text{Homeo}_\Gamma^{++}(M)$  is fixed by the action of  $\nu$  and we have:*

$$\text{Homeo}_\Gamma^+(M) = \text{Homeo}_\Gamma^{++}(M) \rtimes_\nu \mathbb{Z}/2\mathbb{Z} \quad \triangleleft$$

**Remark 1.3.25.** The definition of strongly positive homeomorphisms is motivated by the fact that the action of  $\nu$  on  $M$  does not change the global orientation of  $M$  but still affects the orientation of ordered stars drawn on the basis of every circle bundle component.  $\square$

## 1.4 Ordered graphed embeddings

In this section we explain how ordered stars can be assemble to draw the graph of a graph manifold on the manifold itself in a way that respects the graph structure. This requires to first select a collection of sections to draw the ordered stars on. To achieve this we use the explicit circle bundle model of [Definition 1.3.8](#) on each circle bundle, which we call an *ordered model* of the graph manifold.

### 1.4.1 Ordered model of a graph manifold

Let  $M$  be a graph manifold with associated (unique) graph structure  $\Theta$  and graph  $\Gamma$  and let  $\Omega$  be a graph ordering on  $\Gamma$ . For every vertex  $v_i \in V$ , we denote by  $S_i$  the associated circle bundle.

**Theorem 1.3.11** ensures that there exists a function

$$\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \longrightarrow S_i$$

where  $\varepsilon_i$  is the Euler number of  $S_i$ . The model circle bundle decomposes as:

$$S(m_i, \check{s}_i, \varepsilon_i) = T_{m_i} \cup_{g_{\varepsilon_i}} T_\infty$$

where  $T_{m_i} = D_{m_i} \times S^1$  for any  $m \geq 0$  and  $g_{\varepsilon_i}$  is the gluing map given in [Eq. \(1.4\)](#). The basis  $D_{m_i}$  is a disc with  $m_i$  holes described at the beginning of [Section 1.2.1](#). The boundary components of  $D_{m_i}$  have a *natural order* given by the horizontal coordinate of their basis circle in  $\mathbb{C}$ , which we extend to the boundary components of  $T_{m_i}$  and  $S(m_i, \check{s}_i, \varepsilon_i)$ .

**Definition 1.4.1.** Let  $M$  be a graph manifold with associated (unique) graph structure  $\Theta$  and graph  $\Gamma$ . Let  $\Omega$  be a graph ordering on  $\Gamma$ . An *ordered model* of  $M$  with respect to  $\Omega$  is a collection of functions  $X := (\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \rightarrow S_i)_{v_i \in V}$  such that:

- (i) for every  $v_i \in V$ ,  $\chi_i$  is a positive fibrewise boundary-homeomorphism.
- (ii) for every boundary torus  $T_{i,j} \in \Theta$ , the pre-image  $\chi_i^{-1}(T_{i,j})$  is the  $\omega_i(v_j)$ -th boundary component of  $S(m_i, \check{s}_i, \varepsilon_i)$  for the natural order.  $\diamond$

**Proposition 1.4.2.** *For any graph ordering  $\Omega$ , there always exists an ordered model  $X$  of  $M$  with respect to  $\Omega$ .*  $\triangleleft$

*Proof.* According to [Theorem 1.3.11](#), there exists an orientation-preserving fibrewise boundary-homeomorphism  $\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \rightarrow S_i$ . One then just needs to make the corresponding boundary tori coincide along the ordered graph by using homeomorphisms on the boundary that permutes the components.  $\square$

**Notation.** When

$$s : D_{m_i} \longrightarrow T_{m_i} \subset S(m_i, \check{s}_i, \varepsilon_i)$$

is a section of the trivial circle bundle and

$$\chi_i : S(m_i, \check{s}_i, \varepsilon_i) \longrightarrow S_i$$

is a local ordered model of a circle bundle component  $S_i$ , we write

$$\widehat{s}_i := \chi_i \circ s_i : D_{m_i} \longrightarrow S_i$$

The map  $\widehat{s}_i$  is not a full section of  $S_i$  but rather a section of the subset  $\chi_i(T_{m_i})$ . For simplicity, we call  $\widehat{s}_i$  a section nonetheless since we never consider objects in the  $\chi_i(T_\infty)$  part of  $S_i$ .

**Proposition 1.4.3.** *Let  $X = (\chi_i)_{v_i \in V}$  and  $X' = (\chi'_i)_{v_i \in V}$  be two ordered models of a graph manifold  $M$  with respect to the same graph ordering  $\Omega$ . Define the change of model  $\Psi$  as the unique homeomorphism of  $M$  which restricts to  $\Psi_i := \chi'_i \circ (\chi_i)^{-1}$  on every circle bundle component  $S_i$ . Then  $\Psi \in \text{Homeo}_\Gamma^{++}(M)$ .*  $\triangleleft$

*Proof.* We first prove that  $\Psi$  is well defined. By [Definition 1.4.1](#), for every  $v_i \in V$ ,  $\chi_i$  and  $\chi'_i$  are boundary-homeomorphisms on  $S_i$ , and thus so is  $\Psi_i$ . We can then extend  $\Psi_i$  by the identity on every gluing torus  $T_{i,j}$  to define  $\Psi \in \text{Homeo}(M)$ . It is clear that  $\Psi$  is graphed by construction. Besides,  $\chi_i$  and  $\chi'_i$  are fibrewise positive, and so is  $\Psi_i$ . This implies that  $\Psi$  respects the orientation of all fibres and sections of every  $S_i$ , and thus  $\Psi \in \text{Homeo}_\Gamma^{++}(M)$ .  $\square$

## 1.4.2 Definition of an ordered graphed embedding

Ordered graphed embeddings are defined piece by piece by tying together one ordered star on each circle bundle component of the graph manifold.

**Definition 1.4.4.** Let  $M$  be a graph manifold whose graph  $\Gamma$  is ordered by  $\Omega$ . Let  $\gamma : \Gamma \rightarrow M$  be a graphed embedding, i.e. a map that sends vertices of the graph to disjoint points and edges of the graph to disjoint simple curves. Suppose that there exists an ordered model  $X$  of  $M$  and, for every vertex  $v_i \in V$ , a section  $s_i : D_{m_i} \rightarrow T_{m_i}$  and an ordered star  $\alpha_i \in \mathcal{O}_{m_i}$  drawn on  $D_{m_i}$



such that  $\gamma \cap S_i = \widehat{s}_i(\alpha_i)$ . Then  $\gamma$  is called an *ordered graphed embedding* with respect to the graph ordering  $\Omega$ .

The set of isotopy classes of ordered graphed embeddings with respect to the graph ordering  $\Omega$  is denoted by  $E_\Gamma(\Omega)$ .  $\diamond$

**Remark 1.4.5.** We can always tie together an ordered star  $\alpha_i$  drawn on  $s_i$  and another ordered star  $\alpha_j$  drawn on  $s_j$  along an edge  $e_{i,j} \in E$ . Indeed, since the gluing map  $g$  given by **Condition (M4)** permutes longitudes and meridians, the circles  $g(\widehat{s}_i \cap T_{i,j})$  and  $\widehat{s}_j \cap T_{i,j}$  have exactly one point of intersection. One can thus always perform an isotopy on a small neighbourhood of  $\partial^{\omega_i(v_j)} D_{m_i}$  inside  $s_i$  so that the endpoint of  $\alpha_i^{\omega_i(v_j)}$  coincides with the intersection point, and likewise on the other side.  $\square$

**Example 1.4.6.** The schematic construction of a graphed embedding of the ordered graph of **Figure 1.3.1** is represented on **Figure 1.4.1**. The fibres of each circle bundle are assigned a specific colour. By the gluing map of **Condition (M4)**, each boundary curve of the basis of every circle bundle is identified with a fibre of the corresponding neighbour.  $\diamond$

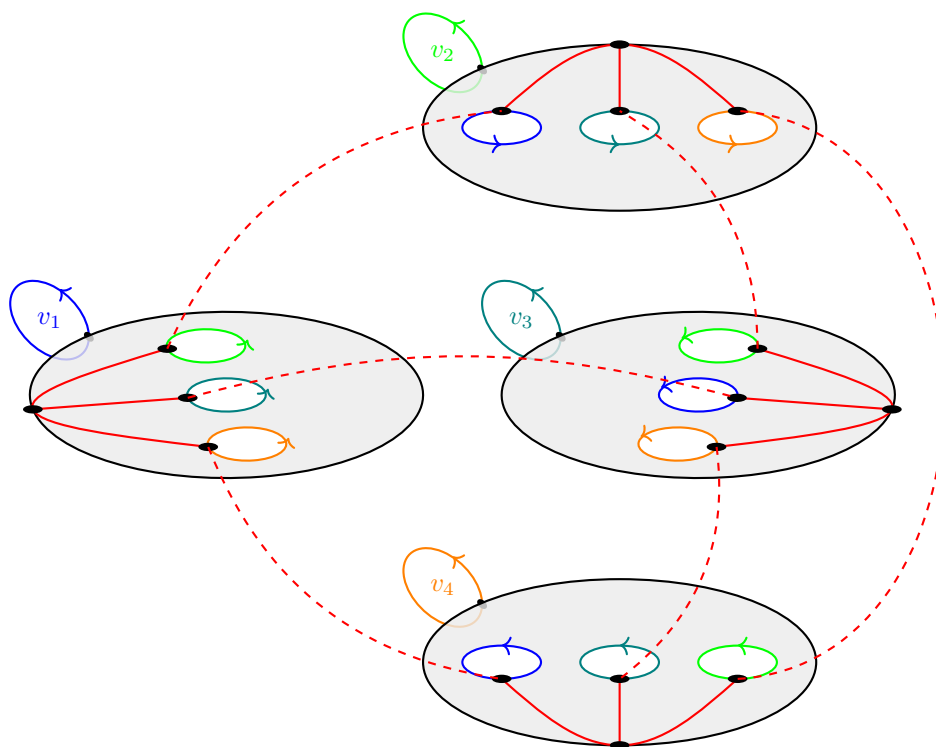


Figure 1.4.1: Construction of a graphed embedding

**Remark 1.4.7.** **Definition 1.4.4** is not the most general way of embedding the graph  $\Gamma$  inside  $M$  in a way that respects the graph structure. The specificity of our construction is that the image of every half-edge inside each circle bundle has to be drawn on a section. This allows to impose a circular order on the half-edges inside each circle bundle  $S_i$ , thus taking into account the local order  $\omega_i$ . This local order is indeed required to study the homotopy class of the half-edges, as evidenced by **Theorem 1.5.9**.  $\square$

**Proposition 1.4.8.** For any cycled graph manifold  $M$  with graph  $\Gamma$  and any choice of  $\Omega$ , the set  $E_\Gamma(\Omega)$  is non-empty.  $\triangleleft$

*Proof.* By **Proposition 1.4.2**, there always exists an ordered model  $X$  of  $M$ . By **Remark 1.2.3**, there always exists an ordered star  $\alpha_i \in \mathcal{O}_{m_i}$  for every possible value of  $m_i \geq 3$ . Choose a section  $\widehat{s}_i : D_{m_i} \rightarrow S_i$  for every  $v_i \in V$ . Now tie together the ordered stars  $\widehat{s}_i(\alpha_i)$  along every edge  $e_{i,j} \in E$  as described in **Remark 1.4.5**. The reunion of all the  $\alpha_i$  for  $1 \leq i \leq n$  is an ordered graphed embedding  $\gamma \in E_\Gamma(\Omega)$ .  $\square$

### 1.4.3 Action of the homeomorphisms

**Theorem 1.4.9.** *Let  $M$  be a graphed manifold ordered whose graph  $\Gamma$  is ordered by  $\Omega$ . There is a well-defined action of the group of strongly positive homeomorphisms  $\text{Homeo}_{\Gamma}^{++}(M)$  on  $E_{\Gamma}(\Omega)$ . This action is transitive.  $\triangleleft$*

*Proof.* We prove that the action is well defined. Consider  $\gamma \in E_{\Gamma}(\Omega)$ . There exists an ordered model  $X$  and for every  $v_i \in V$  a section  $s_i : D_{m_i} \rightarrow T_{m_i}$  and an ordered star  $\alpha_i \in \mathcal{O}_{m_i}$  such that  $\gamma \cap S_i = \widehat{s}_i(\alpha_i) = \chi_i \circ s_i(\alpha_i)$ . Let  $\Psi \in \text{Homeo}_{\Gamma}^{++}(M)$ . By [Definition 1.3.22](#), up to isotopy  $\Psi$  restricts to  $\Psi_i \in \text{Homeo}_{\partial}^+(S_i)$  for every  $v_i \in V$ . Define  $\Psi_i(\widehat{s}_i) := \Psi_i \circ \widehat{s}_i \circ \widetilde{\Psi}_i^{-1}$ . Then the morphism  $\Psi_i(\widehat{s}_i) : D_{m_i} \rightarrow S_i$  is a (partial) section, and we have

$$(\Psi_i \circ \widehat{s}_i)(\alpha_i) = \Psi_i(\widehat{s}_i) \circ \widetilde{\Psi}_i(\alpha_i) \quad (1.5)$$

The full basis of  $S_i$  is  $\Sigma_i \simeq \Sigma^{m_i}$  and the boundary-homeomorphism  $\widetilde{\Psi}_i$  is an element of  $\mathcal{P}(\Sigma^{m_i})$ . By [Theorem 1.1.3](#), there exists an element  $\psi_i \in \mathcal{P}(D_{m_i})$  such that  $\widetilde{\Psi}_i = f_{m_i} \circ \psi_i$ . By [Proposition 1.2.6](#),  $\psi_i(\alpha_i)$  is still an ordered star on  $D_{m_i}$ . Since  $f_{m_i}$  respects the inclusion  $D_{m_i} \subset \Sigma_i$  then  $\widetilde{\Psi}_i(\alpha_i)$  is still an ordered star on  $D_{m_i}$  for the same order  $\omega_i$ .

Thus, the action of  $\Psi$  sends every couple section/ordered star  $(\widehat{s}_i, \alpha_i)$  to another well-defined couple  $(\Psi_i(\widehat{s}_i), \widetilde{\Psi}_i(\alpha_i))$  on the same ordered model  $\chi_i$  of  $S_i$ . All the new ordered stars can then be tied together in the same points as  $\gamma$  since  $\Psi$  restricts to the identity on all joining tori. We therefore have formed a new ordered graphed embedding  $\Psi(\gamma) \in E_{\Gamma}(\Omega)$ .

We now prove that the action is transitive. Let  $\gamma, \gamma' \in E_{\Gamma}(\Omega)$  be two ordered graphed embeddings and let  $X, X'$  be the corresponding ordered models.

Consider  $v_i \in V$  and the two couples section/ordered stars  $(\widehat{s}_i, \alpha_i)$  and  $(\widehat{s}'_i, \alpha'_i)$  corresponding to  $\gamma$  and  $\gamma'$  respectively. There always exists a  $\Phi_i \in \text{Homeo}_{\partial}^+(T_{m_i})$  such that  $\Phi_i(s_i) = s'_i$  but also such that  $\Phi_i$  induces the identity on the basis  $D_{m_i}$ , and thus  $\Phi_i(\alpha_i) = \alpha_i$ .

By [Proposition 1.2.6](#), there also exists an element  $\pi_i \in \mathcal{P}(D_{m_i})$  such that  $\pi_i \cdot \alpha_i = \alpha'_i$ . Now extend  $\pi_i$  to  $T_{m_i}$  by taking the section  $s'_i$  as the basis of the fibration, to obtain  $\Pi_i \in \text{Homeo}_{\partial}^+(T_{m_i})$ . Now set  $\Pi_i \circ \Phi_i$  to  $S(m_i, \widehat{s}_i, \varepsilon_i)$  by the identity on  $T_{\infty}$  along the gluing map  $g_{\varepsilon_i}$  and define

$$\Psi_i := \chi'_i \circ \Pi_i \circ \Phi_i \circ (\chi_i)^{-1} \in \text{Homeo}_{\partial}^+(S_i)$$

By construction,  $\Psi_i(\widehat{s}_i) = \widehat{s}'_i$  and  $\widetilde{\Psi}_i(\alpha_i) = \alpha'_i$ .

Now define  $\Psi$  as the reunion of all the  $\Psi_i$ 's for every  $v_i \in V$ . Then  $\Psi \in \text{Homeo}_{\Gamma}^{++}(M)$  and  $\Psi(\gamma) = \gamma'$ .  $\square$

## 1.5 Fundamental group of a graph manifold

The data of a graph ordering  $\Omega$  and of an ordered graphed embedding  $\gamma \in E_{\Gamma}(\Omega)$  allows us to properly draw the cycles of the graph  $\Gamma$  on the graph manifold  $M$ . We can thus obtain a presentation of the fundamental group  $\pi_1(M)$  and the first homology group  $H_1(M)$  which are both combinatorially determined.

### 1.5.1 Cycles of the graph

Let  $M$  be a graph manifold with graph  $\Gamma$ . We reuse notations from [Section 1.3.3](#). Recall that  $V$  is the vertex set and  $E$  is the edge set.

There are two different ways to construct the cycles of the graph: either as generators of the fundamental group  $\pi_1(\Gamma)$  or as elements of the first homological group  $H_1(\Gamma)$ .

From now on it is convenient to orient the graph  $\Gamma$ .

**Definition 1.5.1.** An *orientation* of  $\Gamma$  is a function  $\delta : V^2 \rightarrow \{-1, 0, 1\}$  such that

- $\delta_{i,j} \in \{\pm 1\}$  if  $e_{i,j}$  is an edge of  $\Gamma$ .
- $\delta_{i,j} = 0$  if  $e_{i,j}$  is not an edge.
- $\delta_{i,j} = -\delta_{j,i}$ .  $\diamond$

Let  $r$  be a vertex. The generators of the fundamental group  $\pi_1(\Gamma, r)$  can be drawn directly on the graph  $\Gamma$  itself using a *spanning tree* based in a vertex  $r$  as shown on [Figure 1.5.1a](#).

**Definition 1.5.2.** A *spanning tree* of a graph  $\Gamma$  verifying [Conditions 1.3.18](#) is a connected acyclic subgraph  $\mathcal{T}$  of  $\Gamma$  containing all the vertices.  $\diamond$

**Proposition 1.5.3.** Let  $r \in V$ . For every vertex  $v \in V$ , there always exist a unique path from  $r$  to  $v$  inside  $\mathcal{T}$ . The vertex  $r$  is called a *root* of the tree  $\mathcal{T}$ .  $\triangleleft$

**Proposition 1.5.4.** Let  $\mathcal{T}$  be a spanning tree of a graph  $\Gamma$  and let  $r \in V$  be a root. Then the set of edges  $E \setminus \mathcal{T}$  is naturally associated with a basis of  $\pi_1(\Gamma, r)$  as follows: assign to every edge  $e_{i,j} \in E \setminus \mathcal{T}$  (with  $i, j$  ordered such that  $\delta_{i,j} = 1$ ) the unique cycle  $c_{i,j}$  on the graph  $\Gamma$  that goes along the unique path inside  $\mathcal{T}$  from  $r$  to  $v_i$ , then along  $e_{i,j}$ , and finally along the unique path inside  $\mathcal{T}$  from  $v_j$  back to  $r$  as illustrated on [Figure 1.5.1b](#).  $\triangleleft$

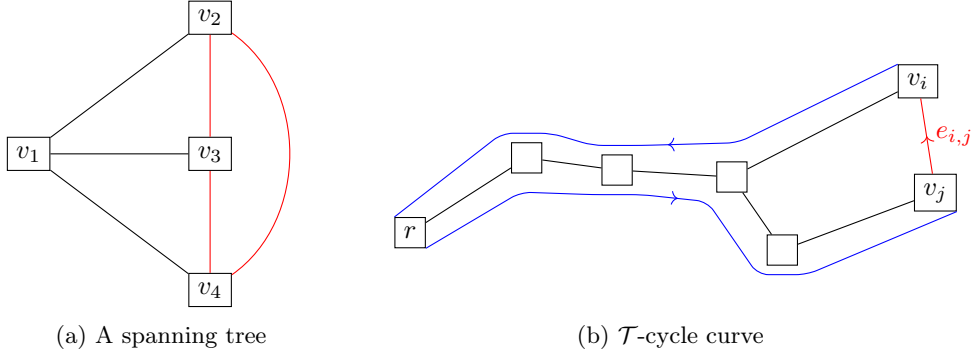


Figure 1.5.1: Spanning tree on a graph

The alternative way to describe the cycles of  $\Gamma$  is to use the fact that  $\Gamma$  has a natural structure of a CW-complex generated by the vertices and the edges. The boundary map  $\partial_1 : C_1(\Gamma) \rightarrow C_0(\Gamma)$  is defined on the basis  $E$  of  $C_1(\Gamma)$  as  $\partial_1(e_{i,j}) = \delta_{i,j}(v_j - v_i)$ .

The first homology group  $H_1(\Gamma)$  is defined as the kernel of  $\partial_1$  and does not depend on the choice of the orientation. There is a natural group homomorphism

$$\zeta_\delta : H_1(\Gamma) \longrightarrow C_1(\Gamma)$$

which decomposes every cycle into the sum of its oriented edges. Note that the choice of the orientation  $\delta$  only affects the signs of the decomposition.

The connection between the two definitions of the cycles of  $\Gamma$  is made by the natural abelianisation map

$$\text{Ab} : \pi_1(\Gamma, r) \longrightarrow H_1(\Gamma)$$

### 1.5.2 Presentation of the fundamental group

Fix a graph ordering  $\Omega$ , an orientation  $\delta$  on  $\Gamma$  and a graphed embedding  $\gamma \in E_\Gamma(\Omega)$ . Let  $\mathcal{T}$  be a spanning tree with root  $r$  and let  $b := \gamma(r)$ .

**Definition 1.5.5.** For every edge  $e_{i,j} \in E \setminus \mathcal{T}$ , let  $c_{i,j}$  be the unique cycle drawn on  $\Gamma$  going through  $e_{i,j}$  and the root  $r$  as per [Proposition 1.5.4](#). Then the simple closed curve  $\gamma_{i,j} := \gamma(c_{i,j})$  is called a  *$\mathcal{T}$ -cycle curve* of  $M$ .  $\diamond$

**Definition 1.5.6.** For every vertex  $v_i \in V$ , let  $f_i$  be the fibre curve of the circle bundle  $S_i$  over the point  $\gamma(v_i)$ , and let  $c_i$  be the unique path from the root  $r$  to  $v_i$  in  $\mathcal{T}$ . Then  $\mu_i := \gamma(c_i) \cdot f_i \cdot \gamma(c_i)^{-1}$  is called a  *$\mathcal{T}$ -meridian curve* of  $M$ .  $\diamond$

Recall from [Proposition 1.5.4](#) that the set  $E \setminus \mathcal{T}$  naturally determines a basis of the free group  $\pi_1(\Gamma, r)$ . We denote by  $\mathbb{F}_V$  the group freely generated by the vertex set  $V$ .

**Definition 1.5.7.** A graphed embedding  $\gamma \in E_\Gamma(\Omega)$  naturally induces a map

$$\gamma_{\mathcal{T},\delta} : \mathbb{F}_V * \pi_1(\Gamma, r) \longrightarrow \pi_1(M, b)$$

which sends  $v_i \in V$  to the class of the  $\mathcal{T}$ -meridian curve  $\mu_i$  of [Definition 1.5.6](#) and each cycle  $c_{i,j}$  for  $e_{i,j} \in E \setminus \mathcal{T}$  to the class of the  $\mathcal{T}$ -cycle curve  $\gamma_{i,j}$  of [Definition 1.5.5](#).  $\diamond$

**Remark 1.5.8.** If  $v_i, v_j \in V$  are linked by an edge  $e_{i,j}$  then by the gluing map of **Condition (M4)**, the meridian generator  $\mu_i$  can be retracted to the closed curve  $\widehat{s}_j(\partial^{\omega_j(v_i)} D_{m_j})$  inside the section  $\widehat{s}_j$  of  $S_j$ .  $\square$

Using **Remark 1.5.8** and a combination of Seifert-van Kampen's theorem and HNN-extensions along the graph  $\Gamma$  of the manifold, E. R. Westlund [Wes97] obtained a presentation of the fundamental group of  $M$ . Similar works were done in [Mum61; ACM19b; KN14]. We reformulate his result using our notations and the new concept of ordered graphed embeddings.

**Theorem 1.5.9.** *Let  $\delta$  be an orientation of  $\Gamma$ , let  $\mathcal{T}$  be a spanning tree of  $\Gamma$  and let  $r$  be the root of  $\mathcal{T}$ . Let  $\gamma \in E_\Gamma(\Omega)$  be an ordered graphed embedding and write  $b := \gamma(r)$ . Then there is an exact sequence*

$$0 \longrightarrow P_\Gamma(\mathcal{T}, \delta, \gamma) \longrightarrow \mathbb{F}_V * \pi_1(\Gamma, r) \xrightarrow{\gamma_{\mathcal{T}, \delta}} \pi_1(M, b) \longrightarrow 0$$

where  $P_\Gamma(\mathcal{T}, \delta, \gamma)$  is the subgroup of relations normally generated by

$$\bullet \forall v_i \in V : \prod_{k=1}^{m_i} \mu_{\xi_i(k)}^{u(i, \xi_i(k))} = \mu_i^{-\varepsilon_i}$$

$$\bullet \forall e_{i,j} \in E : [\mu_i, \mu_j^{u(i,j)}] = 1$$

$$\text{where } u(i, j) := \begin{cases} \gamma_{i,j} & \text{if } e_{i,j} \in E \setminus \mathcal{T} \text{ and } \delta_{i,j} = 1. \\ \gamma_{i,j}^{-1} & \text{if } e_{i,j} \in E \setminus \mathcal{T} \text{ and } \delta_{i,j} = -1. \\ 1 & \text{if } e_{i,j} \in \mathcal{T}. \end{cases} \quad \triangleleft$$

For  $x, y \in \pi_1(M, b)$  the notations are  $[x, y] = xyx^{-1}y^{-1}$  and  $x^y = y^{-1}xy$ .

**Remark 1.5.10.** This presentation can often be simplified, in particular when some Euler numbers  $\varepsilon_i$  are equal to 0, 1 or  $-1$ .  $\square$

In fact the presentation  $P_\Gamma(\mathcal{T}, \delta, \gamma)$  does not depend on the ordered graphed embedding.

**Theorem 1.5.11.** *With the spanning tree  $\mathcal{T}$  and orientation  $\delta$  of  $\Gamma$  being fixed, the presentation  $P_\Gamma(\mathcal{T}, \delta, \gamma)$  depends only on the graph ordering  $\Omega$ .*  $\triangleleft$

This result is crucial to study invariants directly derived from the fundamental group  $\pi_1(M)$ , as it ensures that the presentation is stable under the change of the graphed embedding induced by the action of  $\text{Homeo}_\Gamma^{++}(M)$ . This allows to compute representations of the fundamental group and other derived invariants such as twisted homology. Since these applications exceeds the scope of our presentation, we do not prove **Theorem 1.5.11** here. The remainder of this subsection is dedicated to the proof of **Theorem 1.5.9**.

**Lemma 1.5.12.** *Let  $S$  be a circle bundle with  $m$  boundary components and Euler number  $\varepsilon$  that verifies **Conditions 1.3.7**. Let  $b$  be a point in  $S$ . Then  $\pi_1(S, b)$  admits the presentation*

$$\text{Generators: } f, (x^1), \dots, (x^m)$$

$$\text{Relation: } \prod_{k=1}^m (x^k) = f^{-\varepsilon} \quad \triangleleft$$

*Proof.* This is obtained directly by using Seifert-van Kampen's theorem on the description of the model  $S(m, \delta, \varepsilon)$  given in **Section 1.3.1** and **Theorem 1.3.11**. The generator  $f$  is the positive fibre over  $b$ , and for every  $1 \leq r \leq m$ , the generator  $(x^r)$  is the positive curve  $\alpha^r \cdot \partial^r D_m \cdot (\alpha^r)^{-1}$  where  $\alpha \in \mathcal{O}_n$  is any ordered star on the basis  $D_m$  of  $T_m \subset S$ .  $\square$

*Proof of Theorem 1.5.9.* Let  $\mathcal{T}$  be a spanning tree of  $\Gamma$  and let  $\gamma \in E_\Gamma(\Omega)$ . Let  $V^*$  be the set of vertices of  $\Gamma$  that borders an edge of  $E \setminus \mathcal{T}$ . Consider  $v_i \in V$ . From **Lemma 1.5.12** we get the presentation  $P(m_i, \varepsilon_i)$  of  $\pi_1(S_i, \gamma(v_i))$ .

By **Remark 1.5.8**,  $(x_i^k)$  is homotopic inside  $M$  to  $f_j$  with  $k = \omega_i(v_j)$ , and  $\mu_j$  is homotopic to  $f_j$  inside  $S_j$ . Now consider  $e_{i,j} \in \mathcal{T}$  and glue together  $S_i$  and  $S_j$  along the gluing torus  $T_{i,j}$ . Using Seifert-van Kampen's theorem and the gluing map from **Conditions 1.3.16** along all edges of the tree  $\mathcal{T}$ , we obtain a sub-manifold  $M_{\mathcal{T}}$  of  $M$  whose fundamental group  $\pi_1(M_{\mathcal{T}}, \gamma(r))$  has the following presentation:

*Generators:*  $\mu_i$  for  $v_i \in V$  and  $x_i^{\omega_i(v_j)}$  for every  $e_{i,j} \in E \setminus \mathcal{T}$

*Relations:*

- $\forall e_{i,j} \in E \setminus \mathcal{T} : [\mu_i, \mu_j] = 1$
- $\forall v_i \in V \setminus V^* : \prod_{k=1}^{m_i} \mu_{\xi_i(k)} = \mu_i^{-\varepsilon_i}$
- $\forall v_i \in V^* : \prod_{k=1}^{m_i} (x_i^k) = \mu_i^{-\varepsilon_i}$  and  $(x_i^k) = \mu_j$  when  $\xi_i(k) \in V^*$  (which happens for only one value of  $k$ ).

Let  $e_{i,j} \in E \setminus \mathcal{T}$ , with  $v_i, v_j \in V^*$ . We glue back the tori  $T_{i,j}$  corresponding to the edge  $e_{i,j}$ . This corresponds to a HNN-extension of  $M_{\mathcal{T}}$  by gluing a handle from  $S^1 \times \partial^{\omega_i(v_j)} D_{m_i}$  to  $S^1 \times \partial^{\omega_j(v_i)} D_{m_j}$ . The fundamental group of the extension contains a new generator which corresponds to the homological class of the cycle curve  $\gamma_{i,j}$  from **Definition 1.5.5**. Suppose now that  $i < j$ . Since the gluing map once again permutes the meridian and longitude of  $T_{i,j}$ , the new relations are

$$\begin{aligned} (x_i^{\omega_i(v_j)}) &= \gamma_e^{-1} \cdot f_j \cdot \gamma_e & [f_i, (x_i^{\omega_i(v_j)})] &= 1 \\ f_i &= \gamma_e^{-1} \cdot (x_j^{\omega_j(v_i)}) \cdot \gamma_e & [f_j, (x_j^{\omega_j(v_i)})] &= 1 \end{aligned}$$

But  $f_i$  (resp.  $f_j$ ) is homotopic to  $\mu_i$  (resp.  $\mu_j$ ), which yields the simplified relations

$$(x_i^{\omega_i(v_j)}) = \mu_j^{\gamma_e} \quad (x_j^{\omega_j(v_i)}) = \mu_i^{(\gamma_e^{-1})} \quad [\mu_i, \mu_j^{\gamma_e}] = 1$$

Doing the HNN-extensions of  $\pi_1(M_{\mathcal{T}}, b)$  for all edges  $e_{i,j} \in E \setminus \mathcal{T}$ , and then replacing all  $(x_i^{\omega_i(v_j)})$  for every  $v_i \in V^*$  in the previous presentation gives the desired presentation of  $\pi_1(M, b)$ .  $\square$

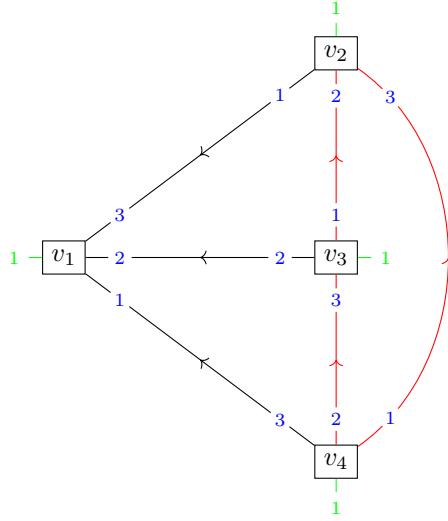


Figure 1.5.2: Minimal graph of the generic combinatorics with 4 vertices

**Example 1.5.13.** Consider the graph shown on **Figure 1.5.2**. Each vertex  $v_i$  is decorated with its Euler number in green and its local order  $\omega_i$  in blue. Each edge of  $E$  is oriented and the edges of  $E \setminus \mathcal{T}$  are drawn in red. The corresponding presentation  $P_{\Gamma}(\mathcal{T}, \delta, \Omega)$  of  $\pi_1(M, v_0)$  is given by:

*Generators:* meridian curves  $\mu_1, \mu_2, \mu_3, \mu_4$  ; cycle curves  $\gamma_{2,3}, \gamma_{3,4}, \gamma_{2,4}$ .

*Relations:*

$$\begin{aligned} \mu_4 \mu_3 \mu_2 &= \mu_1^{-1} & \mu_2 \mu_1 \mu_4 &= \mu_3^{-1} & \mu_1 \mu_3 \mu_4 &= \mu_2^{-1} & \mu_2 \mu_3 \mu_1 &= \mu_4^{-1} \\ [\mu_1, \mu_2] &= 1 & [\mu_1, \mu_3] &= 1 & [\mu_1, \mu_4] &= 1 & & \\ [\mu_2, \mu_3^{\gamma_{2,3}}] &= 1 & [\mu_3, \mu_4^{\gamma_{3,4}}] &= 1 & [\mu_2, \mu_4^{\gamma_{2,4}}] &= 1 & & \end{aligned}$$

$\square$

### 1.5.3 First homology group of a graph manifold

Recall that every vertex  $v_i \in V$  of the graph  $\Gamma$  is decorated with the Euler number  $\varepsilon_i \in \mathbb{Z}$  of the corresponding circle bundle component  $S_i$ .

**Definition 1.5.14.** The *meridian homology*  $V(\Gamma)$  of a graph  $\Gamma$  is the quotient of the free abelian vertex group  $C_0(\Gamma)$  by the normally generated subgroup

$$R(\Gamma) := \left\langle \varepsilon_i \cdot v_i + \sum_{v_j \in V_i} v_j, v_i \in V \right\rangle$$

with the exact sequence:

$$0 \longrightarrow R(\Gamma) \longrightarrow C_0(\Gamma) \xrightarrow{\eta} V(\Gamma) \longrightarrow 0$$

For any element  $v \in C_0(\Gamma)$ , we also note  $\bar{v}$  its class in  $V(\Gamma)$ .  $\diamond$

The terminology ‘meridian homology’ comes from the fact that  $V(\Gamma)$  corresponds precisely to the contribution of the meridians of the components to the first homology group of the graph manifold  $M$ .

**Example 1.5.15.** The meridian homology of the graph from [Figure 1.5.2](#) has four generators  $\mu_1, \mu_2, \mu_3, \mu_4$  with the relation:

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0 \quad \square$$

When a disambiguation is necessary, for a closed curve  $w$  inside a manifold  $M$  we use the notation  $[w]$  for its class inside  $H_1(M, \mathbb{Z})$ , which we shorten to  $H_1(M)$ .

**Theorem 1.5.16.** Let  $\gamma \in E_\Gamma(\Omega)$  be a graphed embedding. Then  $\gamma$  induces a group isomorphism

$$\gamma_* : V(\Gamma) \oplus H_1(\Gamma) \xrightarrow{\sim} H_1(M) \quad \triangleleft$$

*Proof.* Abelianising the exact sequence of [Theorem 1.5.9](#) yields a new exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_\Gamma(\mathcal{T}, \delta, \gamma) & \longrightarrow & \mathbb{F}_V * \pi_1(\Gamma, r) & \xrightarrow{\gamma_{\mathcal{T}, \delta}} & \pi_1(M, b) \longrightarrow 0 \\ & & \downarrow \text{Ab} & & \downarrow \text{Ab} & & \downarrow \text{Ab} \\ 0 & \longrightarrow & R(\Gamma) & \longrightarrow & C_0(\Gamma) \oplus H_1(\Gamma) & \xrightarrow{\text{Ab}(\gamma)} & H_1(M, \mathbb{Z}) \longrightarrow 0 \end{array}$$

which is exactly the defining exact sequence of  $V(\Gamma) \oplus H_1(\Gamma)$ . The morphism  $\gamma_*$  is defined as the morphism induced by  $\text{Ab}(\gamma)$  on the quotient, which does not depend on the choice of the presentation  $P_\Gamma(\mathcal{T}, \delta, \gamma)$ .  $\square$

**Remark 1.5.17.** Even if the morphism  $\gamma_*$  does not depend on the presentation  $P_\Gamma(\mathcal{T}, \delta, \gamma)$  and its underlying parameters, it is necessary to fix such a presentation to obtain a basis of  $C_0(\Gamma) \oplus H_1(\Gamma)$  and thus compute the exact values of  $\gamma_*$ .  $\square$

**Proposition 1.5.18.** For every pair of graphed embeddings  $\gamma, \gamma' \in E_\Gamma(\Omega)$ , we have

$$(\gamma_*^{-1} \circ \gamma'_*)|_{V(\Gamma)} = \text{Id}_{V(\Gamma)} \quad \triangleleft$$

*Proof.* Fix a spanning tree  $\mathcal{T}$  and a graph ordering  $\Omega$ . Let  $v_i \in V$ . By construction,  $\gamma'_*(\bar{v}_i) = [\mu_i]$ , which by [Definition 1.5.6](#) has the same class as the fibre  $f_i$  over  $\gamma(v_i)$  inside  $S_i$ . Similarly,  $\gamma_*(\bar{v}_i)$  is equal to the class of the fibre  $f'_i$  over  $\gamma'(v_i)$  inside  $S_i$ . But by [Lemma 1.5.12](#), all fibres of a circle bundle have the same homological class. Therefore all morphisms  $\gamma_*$  send  $\bar{v}_i$  to the same unique class  $[f_i] \in H_1(M)$  coming from  $V(\Gamma)$ . Thus  $\gamma'_*(\bar{v}_i) = [f_i]$  and  $\gamma_*^{-1}([f_i]) = \bar{v}_i$ .  $\square$

**Remark 1.5.19.** When  $M$  is the boundary of a regular neighbourhood of a complex algebraic curve embedded in the projective plane  $\mathbb{C}\mathbb{P}^2$ , the group  $V(\Gamma)$  is also isomorphic to the first homology group of the exterior of the curve. For the case of complex line arrangements this is proven in [Proposition 2.4.22](#).  $\square$

## 1.6 Graph stabiliser

The graph stabiliser  $\mathcal{G}_\Gamma$  is the quotient of  $\text{Hom}(H_1(M), V(\Gamma))$  by the differences between every two ordered graphed embeddings. The functions of the quotient are thus *stable* by all possible changes of the generators of  $H_1(M)$ . This definition makes it clear what the purpose of the graph stabiliser is but does not give a way to compute it in practice. The objective of this section is thus to use the different results on the homeomorphisms of the graph manifold  $M$ , in particular [Theorem 1.4.9](#), to give an explicit combinatorial presentation of  $\mathcal{G}_\Gamma$ .

### 1.6.1 Definition of the graph stabiliser

Let  $\Gamma$  be a graph verifying [Conditions 1.3.18](#) on page [33](#) and let  $M$  be the associated graph manifold. Let also  $\Omega$  be an ordering on  $\Gamma$ .

**Definition 1.6.1.** The *graph stabiliser*  $\mathcal{G}_\Gamma(\Omega)$  is defined as the quotient of

$$\text{Hom}(H_1(\Gamma), V(\Gamma))$$

by the subgroup

$$\left\langle \phi \circ (\gamma_* - \gamma'_*)|_{H_1(\Gamma)} \mid \begin{array}{l} \gamma, \gamma' \in E_\Gamma(\Omega), \phi \in \text{Hom}(H_1(M), V(\Gamma)) \\ (\phi \circ \gamma_*)|_{V(\Gamma)} = \text{Id}_{V(\Gamma)} \end{array} \right\rangle \quad \diamond$$

The remainder of this section is dedicated to establishing a combinatorial presentation of the graph stabiliser.

### 1.6.2 Difference maps

The difference maps are used to compute the homological difference between two ordered graphed embeddings. This difference lies exclusively on the cycle generators of  $H_1(M)$  since by [Proposition 1.5.18](#) two ordered graphed embeddings always coincide on the homological meridian generators.

**Definition 1.6.2.** Let  $v_i \in V$  be a vertex of the graph  $\Gamma$ . Let  $\langle V_i \rangle$  be the submodule of  $C_0(\Gamma)$  freely generated by  $V_i$ . The natural map

$$g_i : \langle V_i \rangle \longrightarrow H_1(D_{m_i})$$

which sends  $v_k$  to the class  $[\partial^{\omega_i(v_k)} D_{m_i}]$  for every neighbour vertex  $v_k \in V_i$  is called the *homological neighbour map*.  $\diamond$

Consider two ordered graphed embeddings  $\gamma, \gamma' \in E_\Gamma(\Omega)$  and let  $e_{i,j} \in E$  be an edge such that  $\delta_{i,j} = 1$ . Let  $\alpha_i$  (resp.  $\alpha'_i$ ) be the ordered star of  $\mathcal{O}_{m_i}$  associated to  $\gamma$  (resp.  $\gamma'$ ) for the vertex  $v_i$ . We can always suppose that  $\alpha_i$  and  $\alpha'_i$  have the same starting point in  $D_{m_i}$ . The branches  $\alpha_i^{\omega_i(v_j)}$  and  $\alpha'_i{}^{\omega_i(v_j)}$  both have their end points on the circular boundary component  $\partial_{\omega_i(v_j)} D_{m_i}$ . Let  $a_i^j$  be a simple circle arc that joins them, such that the curve

$$w_i^j(\alpha_i, \alpha'_i) := \alpha_i^{\omega_i(v_j)} \circ a_i^j \circ (\alpha'_i{}^{\omega_i(v_j)})^{-1} \subset D_{m_i}$$

is closed.

**Definition 1.6.3.** The *edge difference map* is the morphism

$$\Delta : E_\Gamma(\Omega)^2 \longrightarrow \text{Hom}(C_1(\Gamma), C_0(\Gamma))$$

defined by:

$$\Delta(\gamma, \gamma')(e_{i,j}) := g_i^{-1}(w_i^j(\alpha, \alpha')) - g_j^{-1}(w_j^i(\alpha, \alpha')) \in \langle V_i, V_j \rangle \subset C_0(\Gamma) \quad \diamond$$

**Proposition 1.6.4.** For every  $\gamma, \gamma' \in E_\Gamma(\Omega)$ , the edge difference map  $\Delta$  verifies the following properties:

- (i)  $\Delta(\gamma, \gamma)$  is the trivial group homomorphism.

(ii)  $\Delta(\gamma, \gamma')(e) = -\Delta(\gamma', \gamma)(e)$  for every  $e \in C_1(\Gamma)$ .  $\triangleleft$

**Definition 1.6.5.** Let  $\delta$  be an orientation of  $\Gamma$ . The *cycle difference map* is the map

$$\tilde{\Delta} : E_\Gamma(\Omega)^2 \longrightarrow \text{Hom}(H_1(\Gamma), V(\Gamma))$$

defined by

$$\tilde{\Delta}(\gamma, \gamma') := (\zeta_\delta^* \otimes \eta) \circ \Delta(\gamma, \gamma')$$

where  $\zeta_\delta : H_1(\Gamma) \rightarrow C_1(\Gamma)$  is the natural inclusion map and  $\eta : C_0(\Gamma) \rightarrow V(\Gamma)$  is the projection from [Definition 1.5.14](#).  $\triangleleft$

**Remark 1.6.6.** The cycle difference map  $\tilde{\Delta}$  verifies properties similar to [Proposition 1.6.4](#) which are induced by the properties of the edge difference map  $\Delta$ .  $\square$

As announced, the cycle difference map gives a first reformulation of the definition of the graph stabiliser.

**Proposition 1.6.7.** Let  $\gamma, \gamma' \in E_\Gamma(\Omega)$ . Then

$$\gamma_* \circ \tilde{\Delta}(\gamma, \gamma') = (\gamma_* - \gamma'_*)|_{H_1(\Gamma)} \in \text{Hom}(H_1(\Gamma), V(\Gamma)) \quad \triangleleft$$

**Theorem 1.6.8.** There is a natural identification

$$\mathcal{G}_\Gamma(\Omega) \simeq \text{Hom}(H_1(\Gamma), V(\Gamma)) / \langle \tilde{\Delta}(\gamma, \gamma') \mid \gamma, \gamma' \in E_\Gamma(\Omega) \rangle \quad \triangleleft$$

*Proof.* Let  $\gamma, \gamma' \in E_\Gamma(\Omega)$  and  $\phi \in \text{Hom}(H_1(\Gamma), V(\Gamma))$  such that  $(\phi \circ \gamma_*)|_{V(\Gamma)} = \text{Id}_{V(\Gamma)}$  as in [Definition 1.6.1](#) of  $\mathcal{G}_\Gamma(\Omega)$ . Then by [Proposition 1.6.7](#):

$$\phi \circ (\gamma_* - \gamma'_*)|_{H_1(\Gamma)} = \phi \circ \gamma_* \circ \tilde{\Delta}(\gamma, \gamma') = \tilde{\Delta}(\gamma, \gamma') \quad \square$$

The remainder of this subsection is dedicated to the proof of [Proposition 1.6.7](#). Let  $e_{i,j} \in E$  be an edge. By [Definition 1.4.4](#) on page 35 of  $\gamma \in E_\Gamma(\Omega)$ , there exist an ordered model  $X$ , sections  $s_i, s_j$  and ordered stars  $\alpha_i, \alpha_j$  such that

$$\gamma(\vec{e}_{i,j}) = \widehat{s}_i(\alpha_i^{\omega_i(v_j)}) \quad \gamma(\vec{e}_{j,i}) = \widehat{s}_j(\alpha_j^{\omega_j(v_i)})$$

where  $\vec{e}_{i,j}$  is the half-edge of  $\Gamma$  starting from  $v_i$  and going towards  $v_j$  (and reciprocally for  $\vec{e}_{j,i}$ ). The junction point of  $\gamma(\vec{e}_{i,j})$  and  $\gamma(\vec{e}_{j,i})$  is exactly the point  $\gamma \cap T_{i,j}$ . We therefore have

$$\gamma(e_{i,j}) = \gamma(\vec{e}_{i,j}) \cdot \gamma(\vec{e}_{j,i})^{-1} \subset \widehat{s}_i(D_{m_i}) \cup \widehat{s}_j(D_{m_j})$$

We use similar notations for  $\gamma'$ .

**Lemma 1.6.9.** The following diagram commutes

$$\begin{array}{ccc} H_1(D_{m_i}) & \xrightarrow{\widehat{s}_i^*} & H_1(M) \\ g_i \uparrow & & \uparrow \gamma_* \\ \langle V_i \rangle & \xrightarrow{\eta} & V(\Gamma) \end{array} \quad \triangleleft$$

*Proof.* As noted in [Remark 1.5.8](#) on page 39, for any  $v_k \in V_i$  the meridian curve  $\mu_k$  has the same homological class as the image by  $\widehat{s}_i$  of the corresponding boundary component  $\partial^{w_i(v_k)} D_{m_i}$ . This precisely means that  $\widehat{s}_i^*(v_k) = [\mu_k]$ . But by [Theorem 1.5.16](#), we also have  $\gamma_*(\overline{v_k}) = [\mu_k]$  by construction of  $\gamma_*$ .  $\square$

**Corollary 1.6.10.**  $(\widehat{s}_i^*)^{-1} \circ \widehat{s}_i'^* = \text{Id}_{H_1(D_{m_i})}$   $\triangleleft$

*Proof.* This is an immediate consequence of combining [Lemma 1.6.9](#) with [Proposition 1.5.18](#).  $\square$

*Proof of Proposition 1.6.7.* Up to isotopy, one can always suppose that the starting point  $b_i^0$  of  $\gamma_i(e_{i,j})$  and the starting point  $b_i^{0'}$  of  $\gamma_i'(e_{i,j})$  lie in the same fibre  $f_i$  of  $S_i$ , with a path  $w_i$



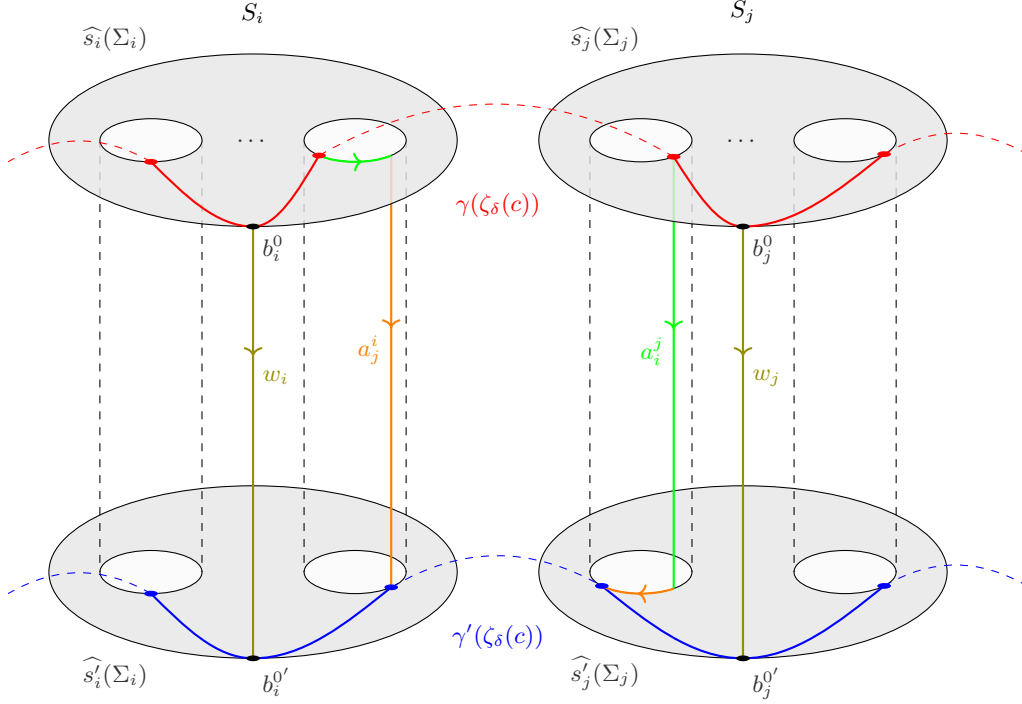


Figure 1.6.1: Difference between two graphed embeddings

joining them. Then the curves

$$w_L(\gamma, \gamma')(e_{i,j}) := \gamma(\vec{e}_{i,j}) \cdot a_i^j \cdot a_j^i \cdot \gamma'(\vec{e}_{i,j})^{-1} \cdot w_i^{-1} \subset S_i$$

$$w_R(\gamma, \gamma')(e_{i,j}) := \gamma(\vec{e}_{j,i}) \cdot a_i^j \cdot a_j^i \cdot \gamma'(\vec{e}_{j,i})^{-1} \cdot w_i^{-1} \subset S_j$$

are closed, as shown on **Figure 1.6.1**. By retracting the vertical fibre curves within  $S_i$  and  $S_j$ , we obtain the following equalities for the homological classes:

$$[w_L(\gamma, \gamma')(e_{i,j})]_M = \widehat{s}_i^* \left( [w_i^j(\alpha_i, \alpha'_i)]_{D_{m_i}} \right)$$

$$[w_R(\gamma, \gamma')(e_{i,j})]_M = \widehat{s}_j^* \left( [w_j^i(\alpha_j, \alpha'_j)]_{D_{m_j}} \right)$$

Let  $c \in H_1(\Gamma)$  and write  $\zeta_\delta(c) = \sum_k \delta_k \cdot e_{i_k, i_{k+1}}$  with  $\delta_k := \delta_{i_k, i_{k+1}}$ . By construction of the embedding  $\gamma \in E_\Gamma(\Omega)$  we have

$$\gamma(\zeta_\delta(c)) = \prod_k \gamma(e_{i_k, i_{k+1}})^{\delta_k}$$

Inside  $H_1(M)$ ,  $(\gamma_* - \gamma'_*)(c)$  can be seen as the class of the closed curve

$$w(\gamma, \gamma')(c) = \gamma(\zeta_\delta(c)) \cdot w_{i_0} \cdot \gamma'(\zeta_\delta(c))^{-1} \cdot w_{i_0}^{-1}$$

As shown on **Figure 1.6.1**, a 2-chain bordering that curve inside  $M$  can be decomposed into a sum of squared 2-chains bordering each of the closed curves  $w_L(\gamma, \gamma')(e_{i_k, i_{k+1}})$  and  $w_R(\gamma, \gamma')(e_{i_k, i_{k+1}})$  on each reunion  $S_{i_k} \cup S_{i_{k+1}}$ . Inside  $H_1(M)$ , this yields the equation:

$$(\gamma_* - \gamma'_*)(c) = \sum_k \delta_k \left( [w_L(\gamma, \gamma')(e_{i_k, i_{k+1}})]_M - [w_R(\gamma, \gamma')(e_{i_k, i_{k+1}})]_M \right)$$

Applying successively **Lemma 1.6.9** and **Definition 1.6.3** of the edge difference map  $\Delta(\gamma, \gamma')$ , we

get the following equality:

$$\begin{aligned} & [w_L(\gamma, \gamma')(e_{i_k, i_{k+1}})]_M - [w_R(\gamma, \gamma')(e_{i_k, i_{k+1}})]_M \\ &= \widehat{s}_i^* \circ g_i \circ g_i^{-1} \left( [w_i^j(\alpha_i, \alpha'_i)]_{D_{m_i}} \right) - \widehat{s}_j^* \circ g_j \circ g_j^{-1} \left( [w_j^i(\alpha_j, \alpha'_j)]_{D_{m_j}} \right) \\ &= \gamma_* \circ \eta \circ \Delta(\gamma, \gamma')(e_{i_k, i_{k+1}}) \end{aligned}$$

Replacing in the sum yields:

$$\begin{aligned} (\gamma_* - \gamma'_*)(c) &= \sum_k \delta_k \cdot \gamma_* \circ \eta \circ \Delta(\gamma, \gamma')(e_{i_k, i_{k+1}}) \\ &= \gamma_* \circ ((\zeta_\delta^* \otimes \eta) \circ \Delta(\gamma, \gamma'))(c) \\ &= \gamma_* \circ \widetilde{\Delta}(\gamma, \gamma')(c) \end{aligned} \quad \square$$

### 1.6.3 Presentation of the graph stabiliser

The results obtained in [Section 1.1.3](#) allows us to compute explicitly the image of the cycle difference map  $\widetilde{\Delta}$  and thus give a combinatorial presentation of  $\mathcal{G}_\Gamma(\Omega)$ .

**Theorem 1.6.11.** *The group  $\mathcal{G}_\Gamma(\Omega)$  is finitely presented and admits the presentation:*

Generators:  $c^* \otimes \bar{v}$  for every  $c \in H_1(\Gamma)$  and  $v \in V$ .

Relations: the images by  $\zeta_\delta^* \otimes \eta : \text{Hom}(C_1(\Gamma), C_0(\Gamma)) \rightarrow \text{Hom}(H_1(\Gamma), V(\Gamma))$  of

(GS1)  $e_{i,j}^* \otimes v_i = 0$  and  $e_{i,j}^* \otimes v_j = 0$  for every edge  $e_{i,j} \in E$ .

(GS2)  $e_{i,j}^* \otimes v_k - \delta_{i,j} \delta_{j,k} \cdot e_{j,k}^* \otimes v_i = 0$  for every pair of adjacent edges  $e_{i,j}$  and  $e_{j,k}$  in  $\Gamma$ .  $\triangleleft$

**Corollary 1.6.12.** *The graph stabiliser  $\mathcal{G}_\Gamma(\Omega)$  does not depend on the choice of the graph ordering  $\Omega$ .*  $\triangleleft$

*Proof of Theorem 1.6.11.* Applying [Definition 1.6.5](#), the subgroup

$$\langle \widetilde{\Delta}(\gamma, \gamma') \mid \gamma, \gamma' \in E_\Gamma(\Omega) \rangle \subset H^1(\Gamma) \otimes V(\Gamma)$$

is isomorphic to the image by  $\zeta_\delta^* \otimes \eta$  of the subgroup

$$\langle \Delta(\gamma, \gamma') \mid \gamma \in E_\Gamma(\Omega), \Psi \in \text{Homeo}_\Gamma^{++}(M) \rangle \subset C^1(\Gamma) \otimes C_0(\Gamma)$$

By [Theorem 1.4.9](#) on page 37, this subgroup is in turn isomorphic to

$$G_\Gamma := \langle \Delta(\Psi(\gamma), \gamma) \mid \gamma \in E_\Gamma(\Omega), \Psi \in \text{Homeo}_\Gamma^{++}(M) \rangle$$

Therefore, we just have to determine that  $G_\Gamma$  is freely generated by the elements of  $\text{Hom}(C_1(\Gamma), C_0(\Gamma))$  of type (GS1) and (GS2) listed above.

Let  $\Psi \in \text{Homeo}_\Gamma^{++}(M)$  and  $\gamma \in E_\Gamma(\Omega)$ . Recall from the proof of [Theorem 1.4.9](#) that the action of  $\Psi$  on  $\gamma$  can be seen as induced by the combined actions of elements  $\psi_i \in \mathcal{P}(D_{m_i})$  on each ordered star  $\alpha_i \in D_{m_i}$  that compose  $\gamma$ .

Let  $e_{i,j} \in E$ . We reuse the notations from [Sections 1.2](#) and [1.6.2](#). We want to compute  $\Delta(\psi_i \cdot \gamma, \gamma)(e_{i,j})$  where the action of  $\psi_i \in \mathcal{P}(D_{m_i})$  is extended by the identity everywhere outside  $S_i$ . In particular,  $\psi_i$  only acts on the  $\gamma(\vec{e}_{i,j})$ . Applying [Definition 1.6.3](#),  $\Delta(\psi_i \cdot \gamma, \gamma)(e_{i,j})$  can be seen as the image by  $g_i^{-1}$  of the homological class inside  $H_1(D_{m_i})$  of the loop

$$(\psi_i \cdot \alpha_i)^{\omega_i(v_j)} \cdot (\alpha_i^{\omega_i(v_j)})^{-1}$$

The generators of  $\mathcal{P}(D_{m_i})$  were given in [Theorem 1.1.2](#) on page 26: the Dehn twists  $d_k$  and  $d_{j,l}$  for  $1 \leq j, k, l \leq m_i$ . The actions of each of these generators on the path of ordered stars are shown on [Figure 1.2.3](#) on page 29. Denote by  $x_k$  the class of  $\partial_k D_{m_i}$  inside  $H_1(D_{m_i})$ . By superimposing

the upper part of [Figure 1.2.3](#) on the lower part, we see that for every  $v_j, v_k, v_l \in V_i$ :

$$\begin{aligned} [(d_r \cdot \gamma)(e_{i,k}) \cdot \gamma(e_{i,k})^{-1}]_{\Sigma_i} &= \begin{cases} \delta_{i,k} \cdot x_i^{\omega_i(v_k)} & \text{if } r = \omega_i(v_k). \\ 0 & \text{otherwise.} \end{cases} \\ [(d_{r,s} \cdot \gamma)(e_{i,j}) \cdot \gamma(e_{i,j})^{-1}]_{\Sigma_i} &= \begin{cases} \delta_{i,j} \cdot (x_i^{\omega_i(v_j)} + x_i^{\omega_i(v_l)}) & \text{if } \{r, s\} = \{\omega_i(v_j), \omega_i(v_l)\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Recall from [Definition 1.6.2](#) that  $g_i^{-1}(x_k) = v_k \in V_i$ . Hence we get that for every vertices  $v_j, v_k, v_l \in V_i$  the differences are:

$$\begin{aligned} \Delta(d_{\omega_i(v_k)} \cdot \gamma, \gamma) &= \delta_{i,k} \cdot e_{i,k}^* \otimes v_k \\ \Delta(d_{\omega_i(v_j), \omega_i(v_l)} \cdot \gamma, \gamma) &= \delta_{i,j} \cdot e_{i,j}^* \otimes (v_j + v_l) + \delta_{i,l} \cdot e_{i,l}^* \otimes (v_j + v_l) \end{aligned}$$

Repeating this for all vertices  $v_i \in V$  gives the generating set of the subgroup  $G_\Gamma$ . However, it can be simplified by removing the first generator from the second when  $k = j$  and  $k = l$ , which gives the presentation:

$$G_\Gamma = \langle e_{i,k}^* \otimes v_k, e_{i,j}^* \otimes v_l - \delta_{j,i} \delta_{i,l} \cdot e_{i,l}^* \otimes v_j \mid \forall v_i \in V, \forall v_j, v_k, v_l \in V_i \rangle$$

The group  $G_\Gamma$  is thus generated by the elements of types [\(GS1\)](#) and [\(GS2\)](#).  $\square$

**Example 1.6.13.** We apply [Theorem 1.6.11](#) to compute a simplified combinatorial presentation of the graph stabiliser  $\mathcal{G}_\Gamma$  for the data of the graph  $\Gamma$ , spanning tree  $\mathcal{T}$  and orientation  $\delta$  of [Figure 1.5.2](#) on page 40.

The basis of  $H_1(\Gamma)$  associated to  $E \setminus \mathcal{T}$  and  $\delta$  is given by:

$$c_{34} = -e_{31} + e_{34} + e_{41} \quad c_{42} = e_{21} + e_{42} - e_{41} \quad c_{32} = e_{21} + e_{32} - e_{31}$$

The dual map  $\zeta_\delta^* : C^1(\Gamma) \rightarrow H^1(\Gamma)$  is given by:

$$\begin{aligned} e_{21}^* &\mapsto c_{42}^* + c_{32}^* & e_{31}^* &\mapsto -c_{34}^* - c_{32}^* & e_{41}^* &\mapsto c_{34}^* - c_{42}^* \\ e_{32}^* &\mapsto c_{32}^* & e_{42}^* &\mapsto c_{42}^* & e_{34}^* &\mapsto c_{34}^* \end{aligned}$$

The generators of  $\mathcal{G}_\Gamma$  are thus:

$$\begin{aligned} c_{32}^* \otimes \overline{v_2} & \quad c_{34}^* \otimes \overline{v_2} & \quad c_{42}^* \otimes \overline{v_2} \\ c_{32}^* \otimes \overline{v_3} & \quad c_{34}^* \otimes \overline{v_3} & \quad c_{42}^* \otimes \overline{v_3} \\ c_{32}^* \otimes \overline{v_4} & \quad c_{34}^* \otimes \overline{v_4} & \quad c_{42}^* \otimes \overline{v_4} \end{aligned}$$

Now we compute the relations. This is done in two steps: first writing the relations of  $G_\Gamma$  inside  $C^1(\Gamma) \otimes C_0(\Gamma)$ , and then taking their image by the map  $\zeta_\delta^* \otimes \eta$ . We use the symbol  $\equiv$  to denote the equivalence of relations in the group  $\mathcal{G}_\Gamma$ . The first set of relations of type [\(GS1\)](#) is given by:

$$\begin{aligned} e_{32}^* \otimes v_3 &\mapsto c_{32}^* \otimes \overline{v_3} & e_{32}^* \otimes v_2 &\mapsto c_{32}^* \otimes \overline{v_2} \\ e_{34}^* \otimes v_3 &\mapsto c_{34}^* \otimes \overline{v_3} & e_{34}^* \otimes v_4 &\mapsto c_{34}^* \otimes \overline{v_4} \\ e_{42}^* \otimes v_4 &\mapsto c_{42}^* \otimes \overline{v_4} & e_{42}^* \otimes v_2 &\mapsto c_{42}^* \otimes \overline{v_2} \end{aligned}$$

The second set of relations of type [\(GS1\)](#) is given by:

$$\begin{aligned} e_{21}^* \otimes v_2 &\mapsto (c_{42}^* + c_{32}^*) \otimes \overline{v_2} &\equiv 0 \\ e_{21}^* \otimes v_1 &\mapsto -(c_{42}^* + c_{32}^*) \otimes (\overline{v_2} + \overline{v_3} + \overline{v_4}) &\equiv -c_{42}^* \otimes \overline{v_3} - c_{32}^* \otimes \overline{v_4} \\ e_{31}^* \otimes v_3 &\mapsto -(c_{32}^* + c_{34}^*) \otimes \overline{v_3} &\equiv 0 \\ e_{31}^* \otimes v_1 &\mapsto (c_{32}^* + c_{34}^*) \otimes (\overline{v_2} + \overline{v_3} + \overline{v_4}) &\equiv c_{32}^* \otimes \overline{v_4} + c_{34}^* \otimes \overline{v_2} \\ e_{41}^* \otimes v_4 &\mapsto (c_{34}^* - c_{42}^*) \otimes \overline{v_4} &\equiv 0 \\ e_{41}^* \otimes v_1 &\mapsto -(c_{34}^* - c_{42}^*) \otimes (\overline{v_2} + \overline{v_3} + \overline{v_4}) &\equiv -c_{34}^* \otimes \overline{v_2} + c_{42}^* \otimes \overline{v_3} \end{aligned}$$

Similar computations show that all relations of type (GS2) are redundant for this particular set of data.  $\square$

### 1.6.4 Plumbing moves

In [Neu81, Proposition 2.1], W. D. Neumann gives a series of *plumbing moves* that can be performed on any graph structure  $\Theta$  of a graph manifold  $M$  without changing the oriented diffeomorphism type of  $M$ . Using these moves one can reduce any decomposition of  $M$  in Seifert manifolds to the minimal graph structure of [Theorem 1.3.15](#).

The original topological [Definition 1.6.1](#) of the graph stabiliser is only valid as such on the minimal graph  $\Gamma$  of the manifold  $M$ , because it critically relies on [Theorem 1.3.21](#) that ensures that the minimal graph structure is preserved up to homeomorphism of  $M$ . However, Neumann's theorem still allows us to compute the purely combinatorial presentation of the graph stabiliser  $\mathcal{G}_\Gamma$  given in [Theorem 1.6.11](#) using non-minimal graphs.

W. D. Neumann's plumbing calculus is defined in a more general class of graphs than the ones verifying only [Conditions 1.3.18](#). In particular, he allows 'negative' edges which correspond to a gluing map that reverses the orientation of the meridian and longitude of the gluing torus. By this definition, our class of graphs contains only 'positive' edges. It turns out that only one flavour of the Neumann plumbing moves, the *blowing down*, can give graphs respecting [Conditions 1.3.18](#). The two applicable moves are denoted by (R1b) (*binary case*) and (R1u) (*unary case*) respectively. They are represented on [Figure 1.6.2](#), using the notation  $V_i = \widetilde{V}_i \sqcup \{v_a\}$ .

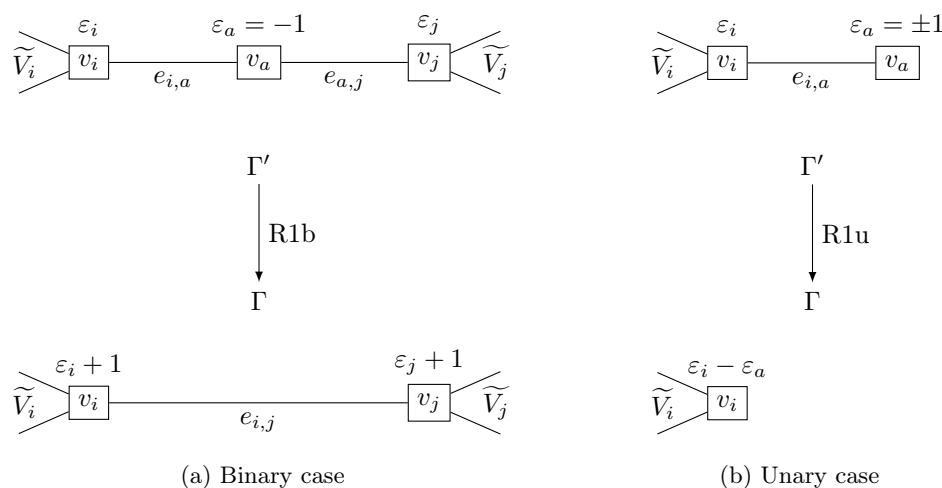


Figure 1.6.2: Blowing-down

For the subclass of graph manifolds that we consider, removing the minimality condition is therefore equivalent to alleviating [Condition \(G3\)](#). For the remainder of this section we thus consider graphs that verify the new set of conditions:

**Conditions 1.6.14.** The graph  $\Gamma'$  is such that:

(G1) no edge starts and ends at the same vertex.

(G2) there is at most one edge between every two vertices.

(G3') every vertex has at least three neighbours, *except*:

- vertices with  $\varepsilon = -1$  which can have *one or two* neighbours.
- vertices with  $\varepsilon = 1$  which can have *one* neighbour.

Thanks to [Theorem 1.6.11](#), the graph stabiliser can be seen as a combinatorial object which can thus be defined on graphs verifying only the new [Conditions 1.6.14](#). Then applying any of the blowing-down moves leave the graph stabiliser unchanged.

**Theorem 1.6.15.** *If  $\Gamma$  and  $\Gamma'$  are two graphs verifying [Conditions 1.6.14](#), such that  $\Gamma$  is obtained from  $\Gamma'$  by a series of blowing-down moves, then there is a natural isomorphism  $\mathcal{G}_{\Gamma'} \xrightarrow{\sim} \mathcal{G}_\Gamma$ .  $\triangleleft$*

**Remark 1.6.16.** In all generality, the (R1b) move with  $\varepsilon_a = +1$  could also be applied to the family of graphs that we consider. This case is nevertheless not considered for two reasons. Firstly because it is a combinatorial exception to [Theorem 1.6.15](#). Secondly because the non-minimal graphs which include this situation are precisely the graphs of the exceptional combinatorics of [Definition 2.2.18](#) on page 56, which correspond to graph manifolds that do not have a unique minimal graph structure, i.e. exceptions of [Theorem 1.3.15](#).  $\square$

The remainder of this section is dedicated to the purely combinatorial proof of [Theorem 1.6.15](#) and can be skipped on first reading.

**Lemma 1.6.17.** *If  $\Gamma'$  and  $\Gamma$  are related by a series of blowing-down moves then there exists a group isomorphism  $h$  and an epimorphism  $g$  such that the following diagram commutes:*

$$\begin{array}{ccc} C_0(\Gamma') & \xrightarrow{g} & C_0(\Gamma) \\ \eta' \downarrow & & \downarrow \eta \\ V(\Gamma) & \xrightarrow[h]{\sim} & V(\Gamma) \end{array} \quad \triangleleft$$

*Proof.* Suppose that  $\Gamma$  is obtained from  $\Gamma'$  by a (R1b) blowing-down move on the edges  $e_{i,a}$  and  $e_{a,j}$ . Define the morphism  $g$  by

$$v_a \mapsto v_i + v_j \qquad v \mapsto v \quad \text{if } v \neq v_a$$

By [Definition 1.5.14](#) inside  $V(\Gamma')$  we have the relations

$$\bar{v}_a = \bar{v}_i + \bar{v}_j \qquad -\varepsilon_i \bar{v}_i = \bar{v}_a + \sum_{v \in \tilde{V}_i} \bar{v} \qquad -\varepsilon_j \bar{v}_j = \bar{v}_a + \sum_{w \in \tilde{V}_j} \bar{w} \quad (1.6)$$

Replacing  $\bar{v}_a$  in the other two relations of [Eq. \(1.6\)](#) yields

$$-(\varepsilon_i + 1)\bar{v}_i = \sum_{v \in \tilde{V}_i} \bar{v} \qquad -(\varepsilon_j + 1)\bar{v}_j = \sum_{w \in \tilde{V}_j} \bar{w}$$

which are exactly the corresponding relations in  $V(\Gamma)$ . All other generators and relations of  $V(\Gamma')$  are identical in  $V(\Gamma)$  and are indeed preserved by the plumbing move. Therefore we can define  $h$  by  $h(\bar{v}) := \eta(g(v))$  for every  $v \in V'$ .

Suppose that  $\Gamma$  is obtained from  $\Gamma'$  by a (R1u) blowing-down move on the edge  $e_{i,a}$ . Define the morphism  $g$  by

$$v_a \mapsto -\varepsilon_a v_i \qquad v \mapsto v \quad \text{if } v \neq v_a$$

Inside  $V(\Gamma')$  we have the relations

$$-\varepsilon_a \bar{v}_a = \bar{v}_i \qquad -\varepsilon_i \bar{v}_i = \bar{v}_a + \sum_{v \in \tilde{V}_i} \bar{v} \quad (1.7)$$

Replacing  $\bar{v}_a$  in the other two relations of [Eq. \(1.7\)](#) yields

$$-(\varepsilon_i - \varepsilon_a)\bar{v}_i = \sum_{v \in \tilde{V}_i} \bar{v}$$

which is exactly the corresponding relation in  $V(\Gamma)$ . We can then define  $h$  by the same formula as in the (R1b) case.  $\square$

**Lemma 1.6.18.** *Let  $\Gamma$  and  $\Gamma'$  be two graphs verifying [Conditions 1.6.14](#) such that  $\Gamma$  is obtained from  $\Gamma'$  by a series of blowing-down moves. Let  $\delta'$  be an orientation of  $\Gamma'$ . Then there exists a monomorphism  $f : C_1(\Gamma) \hookrightarrow C_1(\Gamma')$ , an orientation  $\delta$  of  $\Gamma$ , and an isomorphism  $q : H_1(\Gamma) \xrightarrow{\sim} H_1(\Gamma')$  such that*

$$\zeta_\delta \circ q = f \circ \zeta_{\delta'} \quad \triangleleft$$

*Proof.* Suppose that  $\Gamma$  is obtained from  $\Gamma'$  by a (R1b) blowing-down move on the edges  $e_{i,a}$  and  $e_{a,j}$ .

Define the morphism  $f$  by

$$e_{i,j} \mapsto \delta'_{a,j} \cdot e_{i,a} + \delta'_{i,a} \cdot e_{a,j} \qquad e \mapsto e \quad \text{if } e \neq e_{i,j}$$

and define the orientation  $\delta$  by  $\delta_{i,j} := \delta'_{i,a} \delta'_{a,j}$  and  $\delta := \delta'$  everywhere else. The morphism  $q$  sends any cycle going through  $e_{i,j}$  inside  $\Gamma$  to the corresponding cycle going through  $e_{i,a}$  and  $e_{a,j}$  inside  $\Gamma'$ , and is the identity on all other cycles. One easily verifies that  $\zeta_{\delta'} \circ q = f \circ \zeta_{\delta}$ .

Suppose that  $\Gamma$  is obtained from  $\Gamma'$  by a (R1u) blowing-down move on the edge  $e_{i,a}$ . Define the morphism  $f$  by the natural inclusion of  $E$  inside  $E'$  and  $\delta'$  as equal to  $\delta$  on  $E$  and equal to any value in  $\{\pm 1\}$  on  $e_{i,a}$ . Any cycle  $c \in H_1(\Gamma)$  going through  $e_{i,a}$  does it twice in opposite directions. In other terms

$$\zeta_{\delta}(c) = \cdots + \delta_{j,i} \cdot e_{j,i} + \delta_{i,a} \cdot e_{i,a} - \delta_{i,a} \cdot e_{i,a} + \delta_{i,k} \cdot e_{i,k} + \cdots$$

Therefore  $q$  is the identity and we immediately have  $\zeta_{\delta'} \circ q = f \circ \zeta_{\delta}$ .  $\square$

*Proof of Theorem 1.6.15.* By Theorem 1.6.11, we have

$$\mathcal{G}_{\Gamma} = H^1(\Gamma) \otimes V(\Gamma) / (\zeta_{\delta}^* \otimes \text{Id})(G_{\Gamma})$$

where  $G_{\Gamma} \subset C^1(\Gamma) \otimes C_0(\Gamma)$  is the free group generated by the elements of type (GS1) and (GS2) listed in Theorem 1.6.11. Combining Lemmas 1.6.17 and 1.6.18, we have the following commutative diagram

$$\begin{array}{ccc} C^1(\Gamma) \otimes C_0(\Gamma) & \xleftarrow{f^* \otimes g} & C^1(\Gamma') \otimes C_0(\Gamma') \\ \zeta_{\delta}^* \otimes \eta \downarrow & & \downarrow \zeta_{\delta'}^* \otimes \eta' \\ H^1(\Gamma) \otimes V(\Gamma) & \xleftarrow{\sim_{q^* \otimes h}} & H^1(\Gamma') \otimes V(\Gamma') \end{array}$$

To prove the theorem it is enough to prove that  $(f^* \otimes g)(G_{\Gamma'}) = G_{\Gamma}$ . In both cases we only compute the image of the non trivial elements. It is understood that  $f^* \otimes g$  restricts to the identity on all other elements of  $G_{\Gamma'}$  not mentioned.

Suppose that  $\Gamma$  was obtained from  $\Gamma'$  by a (R1b) blowing-down move on the edges  $e_{i,a}$  and  $e_{a,j}$ . We have  $e_{i,j}^* = \delta'_{a,j} f^*(e_{i,a}^*) = \delta'_{i,a} f^*(e_{a,j}^*)$  and  $g(v_a) = v_i + v_j$ . On the elements of type (GS1) of  $G_{\Gamma'}$ , we have

$$\begin{aligned} f^*(e_{i,a}^*) \otimes g(v_i) &= \delta'_{a,j} \cdot e_{i,j}^* \otimes v_i & f^*(e_{i,a}^*) \otimes g(v_a) &= \delta'_{a,j} \cdot e_{i,j}^* \otimes (v_i + v_j) \\ f^*(e_{a,j}^*) \otimes g(v_j) &= \delta'_{i,a} \cdot e_{i,j}^* \otimes v_j & f^*(e_{a,j}^*) \otimes g(v_a) &= \delta'_{i,a} \cdot e_{i,j}^* \otimes (v_i + v_j) \end{aligned}$$

We obtain all the elements of type (GS1) of  $G_{\Gamma}$ . Now let  $v_l \in \widetilde{V}_i$  and  $v_k \in \widetilde{V}_j$ . On the elements of type (GS2) of  $G_{\Gamma'}$ , we have

$$\begin{aligned} f^*(e_{i,a}^*) \otimes g(v_j) - \delta'_{i,a} \delta'_{a,j} \cdot f^*(e_{a,j}^*) \otimes g(v_i) &= \delta'_{a,j} (e_{i,j}^* \otimes v_j + e_{i,j}^* \otimes v_i) \\ f^*(e_{i,i}^*) \otimes g(v_a) - \delta_{l,i} \delta'_{i,a} \cdot f^*(e_{i,a}^*) \otimes g(v_l) &= e_{l,i}^* \otimes (v_i + v_j) - \delta_{l,i} \delta'_{i,a} \delta'_{a,j} \cdot e_{i,j}^* \otimes v_l \\ &= e_{l,i}^* \otimes v_i + (e_{l,i}^* \otimes v_j - \delta_{l,i} \delta_{i,j} \cdot e_{i,j}^* \otimes v_l) \\ f^*(e_{a,j}^*) \otimes g(v_k) - \delta'_{a,j} \delta_{j,k} \cdot f^*(e_{j,k}^*) \otimes g(v_a) &= \delta'_{i,a} \cdot e_{i,j}^* \otimes v_k - \delta'_{a,j} \delta_{j,k} \cdot e_{j,k}^* \otimes (v_i + v_j) \\ &= -\delta'_{a,j} \delta_{j,k} \cdot e_{j,k}^* \otimes v_j + \delta'_{i,a} (e_{i,j}^* \otimes v_k - \delta_{i,j} \delta_{j,k} \cdot e_{j,k}^* \otimes v_i) \end{aligned}$$

We obtain all the elements of type (GS2) of  $G_{\Gamma}$ .

Suppose that  $\Gamma$  was obtained from  $\Gamma'$  by a (R1u) blowing-down move on the edge  $e_{i,a}$ . We have  $f^*(e_{i,a}^*) = 0$  and  $g(v_a) = -\varepsilon_a v_i$ . On the elements of type (GS1) of  $G_{\Gamma'}$ , we have

$$f^*(e_{i,a}^*) \otimes g(v_i) = 0 \qquad f^*(e_{i,a}^*) \otimes g(v_a) = 0$$

All other elements of type (GS1) of  $G_{\Gamma'}$  are preserved and we thus obtain all the corresponding elements of  $G_{\Gamma'}$ . Now let  $v_l \in \widetilde{V}_i$ . On the elements of type (GS2) of  $G_{\Gamma'}$ , we have

$$f^*(e_{l,i}^*) \otimes g(v_a) - \delta_{l,i} \delta'_{i,a} \cdot f^*(e_{i,a}^*) \otimes g(v_l) = -\varepsilon_i \cdot e_{l,i}^* \otimes v_i + 0$$

which is an element of type (GS1) of  $G_{\Gamma'}$ . Again all other elements of type (GS2) of  $G_{\Gamma'}$  are identically mapped to all of the corresponding elements in  $G_{\Gamma}$ .  $\square$

# CHAPTER

## 2

# HOMOLOGY INCLUSION OF LINE ARRANGEMENTS

### Outline of the current chapter

<b>2.1 Presentation of line arrangements</b>	<b>52</b>
<b>2.2 Combinatorics</b>	<b>53</b>
2.2.1 Definitions . . . . .	53
2.2.2 Graphs of a combinatorics . . . . .	55
2.2.3 Binary vertices . . . . .	56
<b>2.3 Boundary manifold of a line arrangement</b>	<b>58</b>
2.3.1 Blow-up . . . . .	59
2.3.2 Construction of the boundary manifold . . . . .	60
2.3.3 Graph structure . . . . .	61
<b>2.4 Exterior of a line arrangement</b>	<b>62</b>
2.4.1 Wiring diagram . . . . .	62
2.4.2 Braid monodromy . . . . .	63
2.4.3 Fundamental group of the exterior . . . . .	67
<b>2.5 Homology inclusion map</b>	<b>69</b>

In this chapter we present the construction of our ordered line arrangement invariant called the *homology inclusion*.

We start in [Section 2.1](#) by making a general presentation of line arrangements in  $\mathbb{CP}^2$  and giving the basic associated definitions. [Section 2.2](#) is dedicated to the concept of the combinatorics  $C_{\mathcal{A}}$  of a line arrangement  $\mathcal{A}$  and the associated *reduced incidence graph*  $\tilde{\Gamma}(\mathcal{A})$ .

In [Section 2.3](#) we give a construction of the boundary manifold  $B_{\mathcal{A}}$  of dimension 3 adapted from E. R. Westlund [[Wes97](#)]. Using the *blow-up* operation to resolve the singularities, we can describe the minimal graph structure of  $B_{\mathcal{A}}$  which is given by  $\tilde{\Gamma}(\mathcal{A})$  (see [Theorems 2.3.15](#) and [2.3.17](#)).

[Section 2.4](#) is dedicated to the *exterior manifold*  $E_{\mathcal{A}}$  of dimension 4 which is not combinatorial. We present the two main tools which are classically used to characterise the topological type of  $E_{\mathcal{A}}$  and to determine its fundamental group, namely the *wiring diagram* and the *braid monodromy*. This last object is the main concern of [Chapter 3](#) dedicated to the explicit computation of the invariant.



**Section 2.5** finally presents the homology inclusion invariant  $\mathcal{J}_\Omega(\mathcal{A})$  by applying the combinatorial constructions and the graph stabiliser of **Chapter 1** to the specific case of the manifolds  $B_{\mathcal{A}}$  and  $E_{\mathcal{A}}$ . The inclusion of the boundary  $B_{\mathcal{A}}$  inside the exterior  $E_{\mathcal{A}}$  induces a map  $i_*$  on the first homology groups. The restriction of this map on the cycle generators depends on the choice of a graph ordering  $\Omega$  and of an ordered graphed embedding  $\gamma \in E_\Gamma(\Omega)$ . The *homology inclusion*  $\mathcal{J}_\Omega(\mathcal{A})$  is the class of the map  $i_*$  inside the graph stabiliser  $\mathcal{G}_\Gamma$ . By construction of the graph stabiliser as a quotient, this class no longer depends on the choice of the embedding  $\gamma \in E_\Gamma(\Omega)$ . **Theorem 2.5.2** establishes the invariance of  $\mathcal{J}_\Omega$  for ordered line arrangements.

## 2.1 Presentation of line arrangements

**Definition 2.1.1.** A *line arrangement* is a union of complex lines

$$\mathcal{A} = \bigcup_{i=0}^n L_i$$

drawn on the projective complex plane  $\mathbb{C}\mathbb{P}^2$ .  $\diamond$

In particular, for given homogenous coordinates  $[x : y : z]$  on  $\mathbb{C}\mathbb{P}^2$ , there exist  $n$  homogenous polynomials  $f_i(x, y, z)$  of degree 1 such that

$$L_i = \{[x : y : z] \in \mathbb{C}\mathbb{P}^2 \mid f_i(x, y, z) = 0\}$$

We denote by  $\mathcal{L} = (L_i)_{1 \leq i \leq n}$  the set of line components of  $\mathcal{A}$ .

**Definition 2.1.2.** A *defining polynomial*  $f$  of  $\mathcal{A}$  is a square-free homogenous polynomial such that

$$\mathcal{A} = \{[x : y : z] \in \mathbb{C}\mathbb{P}^2 \mid f(x, y, z) = 0\} \quad \diamond$$

In particular, the product  $P_{\mathcal{A}} = \prod_{i=1}^n f_i$  is a defining polynomial of  $\mathcal{A}$ .

**Definition 2.1.3.** The restriction of a line arrangement  $\mathcal{A}$  to the *standard affine chart*

$$\mathbb{C}^2 \equiv \{[x : y : 1] \in \mathbb{C}\mathbb{P}^2\}$$

is called the *affine part*  $\mathcal{A}^{\text{aff}}$ .  $\diamond$

**Definition 2.1.4.** An *affine line arrangement* is a union of complex lines

$$\mathcal{A}^{\text{aff}} = \bigcup_{i=0}^n L_i^{\text{aff}}$$

drawn on the complex plane  $\mathbb{C}^2$ .  $\diamond$

If the line at infinity  $\{z = 0\}$  is not already in an arrangement  $\mathcal{A}$ , then by a linear change of coordinates one can always send any of the lines  $L_i$  at infinity. The resulting affine part  $\mathcal{A}^{\text{aff}}$  then coincides with the sub-arrangement  $(\mathcal{A} \setminus L_i)^{\text{aff}}$ . We often alternate between projective and affine points of view on arrangements. In general we will explicitly mention if the line at infinity is within the arrangement only when it is relevant.

The intersections between the lines are called *singular points*. We write

$$P_{i_1, i_2, \dots, i_m} = L_{i_1} \cap L_{i_2} \cap \dots \cap L_{i_m}$$

where  $m$  is called the *multiplicity* of the point. Alternatively, for a given singular point  $P$ , we write  $V_P = \{L_{i_1}, \dots, L_{i_m}\}$  the sub-arrangement composed only with the  $m$  lines that meet  $P$ . We denote by  $\mathcal{Q}$  the set of all singular points.

**Definition 2.1.5.** Two arrangements  $\mathcal{A}$  and  $\mathcal{A}'$  in  $\mathbb{C}\mathbb{P}^2$  are *topologically equivalent* if there exists a homeomorphism  $\Phi$  of the pair  $(\mathbb{C}\mathbb{P}^2, \mathcal{A})$  to the pair  $(\mathbb{C}\mathbb{P}^2, \mathcal{A}')$ .  $\diamond$

If two arrangements are topologically equivalent then there is a bijection between their sets of lines that naturally extends into a bijection between their sets of singular points.

An *ordering* of a line arrangement  $\mathcal{A}$  is a function  $\theta : \mathcal{L} \rightarrow \{1, \dots, n\}$ . An *ordered line arrangement* is the data of a line arrangement along with an ordering on its lines. In particu-

lar a topological equivalence homeomorphism  $\Phi$  between ordered line arrangements induces a permutation  $\theta_\Phi \in \mathfrak{S}_n$  on the set of lines  $\mathcal{L}$ .

**Definition 2.1.6.** Two *ordered* line arrangements  $(\mathcal{A}, \theta)$  and  $(\mathcal{A}', \theta')$  are topologically equivalent if  $\theta = \theta'$  and there exists a topological equivalence homeomorphism  $\Phi : (\mathbb{C}\mathbb{P}^2, \mathcal{A}) \rightarrow (\mathbb{C}\mathbb{P}^2, \mathcal{A}')$  such that  $\theta_\Phi = \text{Id}$ .  $\diamond$

An *orientation* of a line arrangement is the data of an orientation on every complex line component  $L_i \subset \mathbb{C}\mathbb{P}^2$  of  $\mathcal{A}$ . Two oriented line arrangements are equivalent if there exists an orientation-preserving topological equivalence between them.

## 2.2 Combinatorics

Combinatorics is a general concept that generalises incidence. It applies to all hyperplane arrangements but also to other types of algebraic curves in  $\mathbb{C}\mathbb{P}^2$ . However, since we are focused on line arrangements, we introduce it in a form specific to this context.

### 2.2.1 Definitions

**Definition 2.2.1.** A *line combinatorics* is a triple  $C = (\mathcal{L}, \mathcal{Q}, \in)$  where  $\mathcal{L}$  is a finite set,  $\mathcal{Q}$  is a finite subset of  $\mathcal{P}(\mathcal{L})$  and  $\in$  is a relation from  $\mathcal{Q}$  to  $\mathcal{L}$  such that:

- (i) For every element  $P \in \mathcal{Q}$ , there exist at least two distinct elements  $L, L' \in \mathcal{L}$  such that  $P \in L$  and  $P \in L'$ .
- (ii) For every pair  $L, L' \in \mathcal{L}$  with  $L \neq L'$  there exists a unique  $P \in \mathcal{Q}$  such that  $P \in L$  and  $P \in L'$ .  $\diamond$

For an element  $P \in \mathcal{Q}$  or  $L \in \mathcal{L}$ , we often make use of the *neighbour sets*:

$$V_P := \{L \in \mathcal{L} \mid P \in L\} \qquad V_L := \{P \in \mathcal{Q} \mid P \in L\} \qquad (2.1)$$

The *multiplicity* of an element  $P \in \mathcal{Q}$  or  $L \in \mathcal{L}$  are defined respectively as  $m(P) = \#V_P$  and  $m(L) = \#V_L$ . In particular, for  $P \in \mathcal{Q}$ , we have  $2 \leq m(P) \leq n$  where  $n = \#\mathcal{L}$ . Since every line has to meet all the others exactly once, we have the relation:

$$\forall L \in \mathcal{L} : \sum_{P \in L} m(P) = n - 1 + m(L)$$

The extreme values of the multiplicity correspond to two unique type of line combinatorics.

**Definition 2.2.2.** Let  $n \geq 2$ .

- There is a unique line combinatorics which contains a  $P \in \mathcal{Q}$  such that  $m(P) = n$ . In this case  $\mathcal{Q}$  is a singleton  $\{P\}$ . This is called the *trivial* combinatorics with  $n$  lines.
- There is a unique line combinatorics where  $m(P) = 2$  for every  $P \in \mathcal{Q}$ . This is called the *generic* combinatorics with  $n$  lines.  $\diamond$

**Definition 2.2.3.** An *ordered line combinatorics* is a line combinatorics  $C = (\mathcal{L}, \mathcal{Q}, \in)$  along with the data of an ordering  $\theta : \mathcal{L} \rightarrow \{1, \dots, n\}$ .  $\diamond$

Note that  $\mathcal{Q}$  is *not* ordered in general for the usual definition of an ordered combinatorics. There is however a canonical way to order  $\mathcal{Q}$  using the  $\theta$ -lexicographic ordering on  $\mathcal{P}(\mathcal{L})$ .

There is an alternative way to describe ordered line combinatorics. Replace every  $P \in \mathcal{Q}$  with the set  $\{\theta(L) \mid P \in L\}$ . Then the full ordered combinatorics can be retrieved from just the data of  $\mathcal{Q}$  as a set of sets. For example

$$(C, \theta) = [[1, 2, 3]; [1, 4]; [2, 4]; [3, 4]]$$

It is much easier to write down explicit ordered combinatorics than unordered ones. Therefore, we use the following notation

$$C = ([1, 2, 3]; [1, 4]; [2, 4]; [3, 4])$$

to denote an unordered combinatorics where the given order of both  $\mathcal{L}$  and  $\mathcal{Q}$  is purely indicative.

**Example 2.2.4.**

- $([1, 2, \dots, n])$  is the unique trivial combinatorics with  $n$  lines.
- $([1, 2]; [1, 3]; [2, 3])$  is the unique generic combinatorics with 3 lines.
- $([i, j] \mid 1 \leq i \neq j \leq n)$  is the unique generic combinatorics with  $n$  lines.  $\diamond$

**Definition 2.2.5.** Two combinatorics  $C = (\mathcal{L}, \mathcal{Q}, \in)$  and  $C' = (\mathcal{L}', \mathcal{Q}', \in')$  are *isomorphic* if there exists a bijection  $\lambda : \mathcal{L} \rightarrow \mathcal{L}'$  such that if  $\mu : \mathcal{Q} \rightarrow \mathcal{Q}'$  is the bijection naturally induced by  $\lambda$  on the points, we have:

$$\forall L \in \mathcal{P}, \forall P \in \mathcal{Q} : P \in L \iff \mu(P) \in' \lambda(L) \quad \diamond$$

The group of automorphism of a combinatorics  $C$  is denoted  $\text{Aut}_C$ .

**Example 2.2.6.** The previous combinatorics of [Example 2.2.4](#) have exceptionally large automorphism groups:

- The automorphism group of the trivial combinatorics with  $n$  lines is isomorphic to the permutation group  $\mathfrak{S}_n$ .
- The automorphism group of the generic combinatorics with  $n$  lines is also isomorphic to  $\mathfrak{S}_n$ .

For larger combinatorics, the automorphism group has in general a much simpler structure.  $\diamond$

For a line arrangement  $\mathcal{A}$ , there is a natural line combinatorics  $C_{\mathcal{A}}$  associated with  $\mathcal{A}$  and given by the incidence of the lines. If the arrangement is ordered, then  $C_{\mathcal{A}}$  inherits the ordering. However, not every combinatorics can be realised by an arrangement.

**Definition 2.2.7.** A *realisation* of a line combinatorics  $C$  is a line arrangement  $\mathcal{A} \subset \mathbb{C}\mathbb{P}^2$  such that  $C_{\mathcal{A}} = C$ . The combinatorics  $C$  is called *realisable*.  $\diamond$

**Example 2.2.8.** A realisation in  $\mathbb{R}\mathbb{P}^2$  of the ordered generic combinatorics with 4 lines is shown on [Figure 2.2.1](#).  $\diamond$

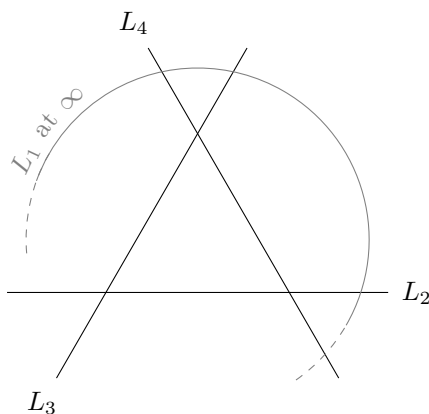


Figure 2.2.1: Realisation of the generic combinatorics with 4 lines

An ordered line arrangement with  $n$  lines can be seen as an element of  $(\check{\mathbb{C}\mathbb{P}^2})^n$  where  $\check{\mathbb{C}\mathbb{P}^2}$  denotes the dual space. The set of all ordered realisations of a realisable combinatorics is defined by

$$\Sigma(C) = \left\{ \mathcal{A} \in (\check{\mathbb{C}\mathbb{P}^2})^n \mid C_{\mathcal{A}} = C \right\}$$

There is a natural action of  $\text{PGL}_3(\mathbb{C})$  on  $\mathbb{C}\mathbb{P}^2$  which extends to its dual and then  $\Sigma(C)$ . The *moduli space* is defined as the quotient of  $\Sigma(C)$  by this natural action.

**Proposition 2.2.9.** *If two line arrangements  $\mathcal{A}$  and  $\mathcal{A}'$  are topologically equivalent then they have isomorphic combinatorics  $C_{\mathcal{A}}$  and  $C_{\mathcal{A}'}$ .*  $\triangleleft$

The converse is wrong. In fact, a large part of the theory of line arrangements revolves around measuring in which cases the combinatorics do not characterise the topological type.

**Definition 2.2.10.** A pair of line arrangements  $(\mathcal{A}, \mathcal{A}')$  is called a *combinatorial pair* if  $C_{\mathcal{A}}$  is isomorphic to  $C_{\mathcal{A}'}$ . A combinatorial pair where  $\mathcal{A}$  and  $\mathcal{A}'$  do not have the same topological type is called a *Zariski pair*.  $\diamond$

An usual way of building a combinatorial pair is to consider a defining polynomial  $f_{\omega}$  with a cyclotomic parameter  $\omega \in \mathbb{Q}(\sqrt{n}), n \in \mathbb{Z}$ . Then the conjugated arrangements have the same combinatorics.

### 2.2.2 Graphs of a combinatorics

Let  $C = (\mathcal{L}, \mathcal{Q}, \in)$  be a combinatorics. Then  $C$  can be represented by a graph. If  $C$  is realisable, this graph encodes the incidence data of every realisation of  $C$ .

**Definition 2.2.11.** The *full incidence graph*  $\Gamma(C)$  of  $C$  is defined by the following description:

*Vertices:* all elements of  $\mathcal{L} \cup \mathcal{Q}$ .

*Edges:*  $L \in \mathcal{L}$  and  $P \in \mathcal{Q}$  are linked by an edge  $e_{P,L}$  if and only if  $P \in L$ .  $\diamond$

The full incidence graph is a *bipartite graph*, which means that its set of vertices is separated in two subsets  $\mathcal{P}$  and  $\mathcal{Q}$ , and every edge only connects a vertex of  $\mathcal{P}$  to a vertex of  $\mathcal{Q}$ . The vertices corresponding to the subset  $\mathcal{P}$  are called *line-vertices* and the vertices corresponding to the subset  $\mathcal{Q}$  are called *point-vertices*.

If the combinatorics  $C$  is ordered then the graph  $\Gamma(C)$  inherits the ordering on the line-vertices. Every point-vertex can then be labelled with the set of its line-vertex neighbours.

Since two lines always intersect, the point-vertices with only two neighbours do not carry any characterising information. They can thus be removed without any loss of generality.

**Definition 2.2.12.** The *reduced incidence graph*  $\tilde{\Gamma}(C)$  of  $C$  is defined by the following description:

*Vertices:* all elements of  $\mathcal{L}$  and all elements  $P \in \mathcal{Q}$  such that  $m(P) \geq 3$ .

*Edges:*

- $L \in \mathcal{L}$  and  $P \in \mathcal{Q}$  such that  $m(P) \geq 3$  are connected by an edge  $e_{P,L}$  if and only if  $P \in L$ .
- two line-vertices  $L, L' \in \mathcal{L}$  are connected by an edge  $e_{L,L'}$  if and only if the element  $P \in \mathcal{Q}$  that verifies  $P \in L, P \in L'$  is such that  $m(P) = 2$ .  $\diamond$

Graphs may also bear additional combinatorial data, such as a graph ordering (see [Definition 1.3.20](#) on page 33), an orientation or a spanning tree (see [Definitions 1.5.1](#) and [1.5.2](#) on page 37 and on page 38).

**Definition 2.2.13.** The *standard orientation*  $\delta^0$  on  $\Gamma(C)$  is defined by

$$\delta_{L,P}^0 = -\delta_{P,L}^0 = -1 \quad \diamond$$

Now suppose that the combinatorics  $C$  is ordered by  $\theta : \mathcal{L} \rightarrow \{1, \dots, n\}$ .

**Definition 2.2.14.** The *induced spanning tree*  $\mathcal{T}^{\theta}$  on  $\Gamma(C)$  is defined for every edge  $e_{L_i,P}$  by:

- $e_{L,P} \in \mathcal{T}^{\theta}$  if  $P \in L_0$  or if  $\theta(L) = \min \theta(V_P)$ .
- $e_{L,P} \notin \mathcal{T}^{\theta}$  otherwise.  $\diamond$

The vertex  $v_{L_0}$  is a root of  $\mathcal{T}^{\theta}$  and every vertex  $v_P$  lying at distance greater than 2 of  $v_{L_0}$  has only one adjacent edge in  $\mathcal{T}^{\theta}$ .

A graph orientation of  $\Gamma(C)$  can be derived from the ordering  $\theta$  on  $\mathcal{L}$  and an additional ordering  $\nu$  on  $\mathcal{Q}$ . They induce two *graph sub-orderings*  $\Omega_{\mathcal{L}}^{\theta}$  and  $\Omega_{\mathcal{Q}}^{\nu}$  on the disjoint sets of vertices corresponding to  $\mathcal{L}$  and  $\mathcal{Q}$  respectively.

**Definition 2.2.15.** The induced graph sub-ordering  $\Omega_{\mathcal{Q}}^{\theta}$  on  $\Gamma(C)$  is defined on every vertex  $v_P$  by  $\theta|_{V_P}$  where  $V_P$  is seen as both the set of neighbour line-vertices of  $v_P$  and the subset of lines of  $\mathcal{L}$  that contain  $P$ .

The induced graph sub-ordering  $\Omega_{\mathcal{L}}^{\nu}$  is defined on every vertex  $v_L$  by  $\nu|_{V_L}$  where  $V_L$  is seen as both the set of neighbour point-vertices of  $v_L$  and the subset of points of  $\mathcal{Q}$  that are contained in  $L$ .  $\diamond$

Together,  $\Omega_Q^\theta$  and  $\Omega_{\mathcal{L}}^\nu$  form a full graph ordering  $\Omega^{\theta,\nu}$  of  $\Gamma(C)$ . Note that one could chose  $\nu$  as the  $\theta$ -lexicographic ordering and thus obtain a graph ordering  $\Omega^\theta$  of  $\Gamma(C)$  depending only on  $\theta$ .

**Proposition 2.2.16.** *Any graph ordering  $\Omega$  of the full incidence graph  $\Gamma(C)$  restricts to a graph ordering  $\tilde{\Omega}$  on the reduced incidence graph  $\tilde{\Gamma}(C)$  defined as follows:*

- $\tilde{\omega}_P = \omega_P$  for every  $P \in \mathcal{Q}$  with  $m(P) > 2$ .
- for every  $L \in \mathcal{L}$ :
  - $\tilde{\omega}_L(v_P) = \omega_L(v_P)$  if  $P \in V_L$  is such that  $m(P) > 2$ .
  - $\tilde{\omega}_L(v_{L'}) = \omega_L(v_P)$  if  $P \in V_L$  is such that  $m(P) = 2$  and  $V_P = \{L, L'\}$ . ◁

**Example 2.2.17.** Consider the ordered generic combinatorics with 4 lines given by

$$(G_4, \theta) := [[1, 2]; [1, 3]; [1, 4]; [2, 3]; [2, 4]; [3, 4]]$$

A realisation of  $G_4$  is shown on [Figure 2.2.1](#). The full incidence graph  $\Gamma(G_4)$  ordered with  $\Omega^\theta$  is shown on [Figure 2.2.2a](#). The edges of  $E \setminus \mathcal{T}^\theta$  are shown in red. The reduced incidence graph  $\tilde{\Gamma}(G_4)$  ordered with the reduced ordering  $\tilde{\Omega}^\theta$  is shown on [Figure 2.2.2b](#). ◻

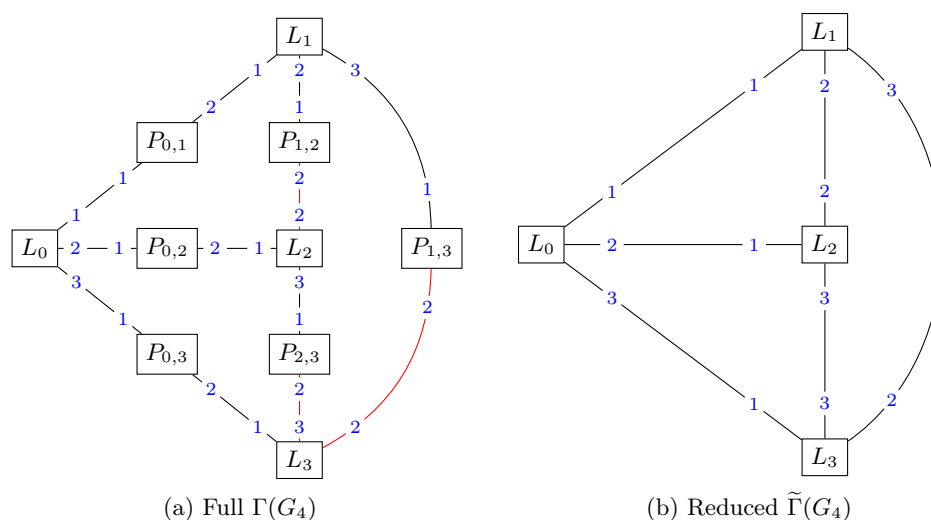


Figure 2.2.2: Ordered incidence graphs of the generic combinatorics  $G_4$

### 2.2.3 Binary vertices

In fact most of the time the reduced incidence graph coincides with the minimal graph of the graph manifold structure of the boundary manifold of a line arrangement. There are however some exceptions, which we establish in this section.

**Definition 2.2.18.** A combinatorics  $C$  is called *exceptional* if its reduced incidence graph  $\tilde{\Gamma}(C)$  contains reduced vertices of multiplicity 2, called *binary vertices*. A line arrangement whose combinatorics is exceptional is also called exceptional. ◊

Note that by [Definition 2.2.12](#) of the reduced incidence graph, binary vertices can only be line-vertices. The associated lines are also called binary. From the point of view of arrangements, a line  $L$  is binary if and only if  $m(L) = 2$ . We write  $\mathcal{L}^{>2}$  and  $\mathcal{Q}^{>2}$  the sets of non-binary lines and singular points of  $C$ .

In fact the combinatorics that contain binary vertices can be fully listed.

**Theorem 2.2.19.** *Let  $C$  be an exceptional combinatorics with  $n \geq 3$ . Then  $C$  is isomorphic to one of the following:*

- (B1) the generic combinatorics  $([1, 2]; [1, 3]; [2, 3])$  with three binary lines.

(B2) *the near-pencil combinatorics*

$$([1, 2]; [1, 3]; \dots; [1, n]; [2, \dots, n])$$

with exactly  $n - 1$  binary lines.

(B3) *one of the  $r$ -double-pencil combinatorics*

$$([1, 2, \dots, r]; [1, r + 1, \dots, n]) \cup ([a, b] \mid 2 \leq a \neq b \leq n), \quad 3 \leq r \leq \frac{n}{2}$$

with exactly one binary line. ◁

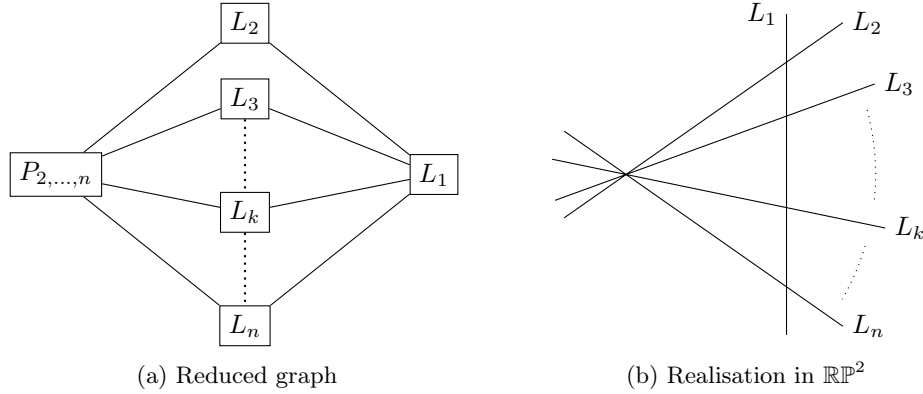


Figure 2.2.3: The near-pencil combinatorics

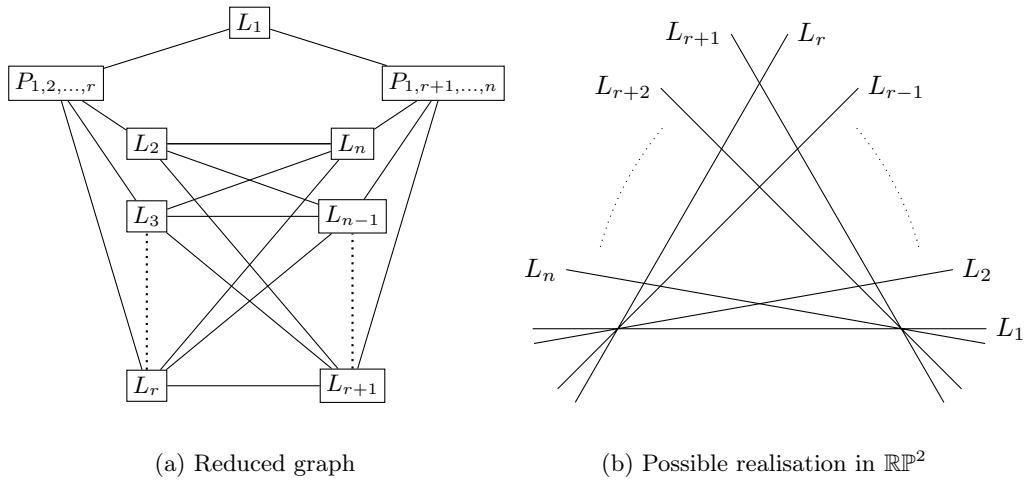


Figure 2.2.4: The  $r$ -double-pencil combinatorics

The proof of [Theorem 2.2.19](#) is decomposed in the following [Lemmas 2.2.20](#) to [2.2.23](#).

**Lemma 2.2.20.** *The generic combinatorics with 3 lines  $([1, 2]; [1, 3]; [2, 3])$  is the only combinatorics that contains two binary vertices connected by an edge. ◁*

*Proof.* Suppose that  $C$  has two connected binary lines  $L_1, L_2 \in \mathcal{L}$ . Then there exists a  $P_{1,2} \in \mathcal{Q}$  which meets no other line other than  $L_1$  and  $L_2$ . Consider a third line  $L_3$ . Then  $L_3$  has to meet both  $L_1$  and  $L_2$  in  $P_a$  and  $P_b$  respectively. If there is no other line in  $\mathcal{L}$ , then  $C$  is the generic combinatorics with three lines. Suppose that there exists a fourth line  $L_4 \in \mathcal{L}$ . Then  $L_4$  has to meet all of the previous lines. But  $L_1$  and  $L_2$  already meet two singular points, so  $L_4$  cannot create a third for either one of them. It cannot meet  $P_{1,2}$  either, since its multiplicity is fixed. Thus  $L_4$  has to meet  $L_1$  in  $P_a$  and  $L_2$  in  $P_b$ . But then  $L_4$  would meet  $L_3$  twice, which is impossible. ◻

**Lemma 2.2.21.** *Let  $C$  be a combinatorics with at least four lines that contains a binary line. Then any singular point that does not meet the binary line has multiplicity 2.*  $\triangleleft$

*Proof.* Let  $L_0$  be a binary line of  $C$ . It meets exactly two singular points  $P_a$  and  $P_b$ . Suppose that there exists another singular point  $P \in \mathcal{Q}^{>2}$  with  $m(P) \geq 3$ . Then there exist three lines  $L_1, L_2, L_3 \in \mathcal{L}$  that meet  $P$ . But at least two of these three lines also need to meet  $L_0$  in either  $P_a$  or  $P_b$ . They will thus meet twice, which is impossible.  $\square$

**Lemma 2.2.22.** *For  $n \geq 4$ , the near-pencil combinatorics is the only combinatorics with  $n$  lines that has more than one binary line.*  $\triangleleft$

*Proof.* We proceed by induction on the number of lines  $n$ .

Let  $C$  be a combinatorics with four lines that contains at least two binary lines  $L_1, L_2 \in \mathcal{L}$ . By [Lemma 2.2.20](#),  $L_1$  and  $L_2$  meet on a point  $P_0 \in \mathcal{Q}^{>2}$  with  $m(P_0) \geq 3$ . Then there exists at least another line  $L_0$  that meets  $P_0$ . Let  $P_a$  (resp.  $P_b$ ) be the other singular point of  $L_1$  (resp.  $L_2$ ). Then the fourth line  $L_3$  must meet  $L_1$  in  $P_a$  and  $L_2$  in  $P_b$ , and it cannot meet  $L_0$  in  $P_0$ . The result is that  $C$  is the near-pencil combinatorics with four lines.

Suppose that the proposition is true for  $n \geq 4$  lines. Consider a combinatorics  $C$  with  $n + 1$  lines and suppose that it contains at least two binary lines  $L_1, L_2 \in \mathcal{L}$ . By [Lemma 2.2.20](#),  $L_1$  and  $L_2$  meet on a point  $P_0 \in \mathcal{Q}^{>2}$  with  $m(P_0) \geq 3$ . Let  $P_a$  (resp.  $P_b$ ) be the other singular point of  $L_1$  (resp.  $L_2$ ). On  $L_1$  we have  $m(P_0) + m(P_a) = n + 2$  and on  $L_2$  we have  $m(P_0) + m(P_b) = n + 2$ . Therefore  $m(P_a) = m(P_b)$ . If  $m(P_0) < n$  then  $m(P_a), m(P_b)$  and  $m(P_0) > 2$ , which by [Lemma 2.2.21](#) is impossible. Thus  $m(P_a) = n > 3$ . Therefore there exists a line  $L_{n+1} \in \mathcal{L}$  such that  $P_a \in L_{n+1}$  but  $P_b, P_c \notin L_{n+1}$ . Consider the sub-combinatorics  $\widehat{C}$  obtained by removing  $L_{n+1}$  from  $C$ . Then  $\widehat{C}$  still has the two binary lines  $L_1$  and  $L_2$ , so by assumption  $\widehat{C}$  is the near-pencil combinatorics with  $n$  lines.

In  $\widehat{C}$  the lines  $L_1$  through  $L_{n-1}$  are binary and the line  $L_n$  meet all of them in double points. Now add back the line  $L_{n+1}$ . If  $L_{n+1}$  goes through the point  $P_{2,\dots,n}$ , then it meets  $L_1$  on a new singular point, and  $C$  is exactly the near-pencil combinatorics with  $n + 1$  lines. If  $L_{n+1}$  does not go through  $P_{2,\dots,n}$ , then it will create new singular points on  $n - 1$  or  $n - 2$  of the binary lines  $L_1, \dots, L_{n-1}$ , depending on whether he meets  $L_n$  at an existing singular point or not. In either case,  $C$  would have at most one binary line remaining, which is a contradiction.  $\square$

**Lemma 2.2.23.** *For  $n \geq 4$ , the  $r$ -double pencil combinatorics are the only combinatorics with exactly one binary line.*  $\triangleleft$

*Proof.* Let  $C$  be a combinatorics with  $n \geq 4$  lines and exactly one binary line  $L_1$ . Let  $P_a, P_b \in \mathcal{Q}$  be the two singular points that meet  $P_1$ . Denote by  $r = m(P_a)$ . Then  $m(P_b) = n + 1 - r$ . Since  $P_a$  and  $P_b$  have symmetric role, we can suppose that  $r \leq \frac{n}{2}$ . Suppose that  $r = 2$ , and let  $L_2$  be the other line that meets  $P_a$ . By [Lemma 2.2.21](#),  $L_2$  must meet all other lines in points of multiplicity 2. Then  $C$  is isomorphic to the near-pencil combinatorics with  $n$  lines, which contradicts  $C$  having exactly one binary line. Therefore  $r = m(P_a) \geq 3$  and  $m(P_b) \geq 3$ . By [Lemma 2.2.21](#), all other singular points of  $C$  have multiplicity 2. The lines that meet  $P_a$  form a pencil of  $r$  lines, and the lines that meet  $P_b$  form another pencil of  $n - r$  lines, with the binary line  $L_1$  being the only line shared by both pencils.  $\square$

## 2.3 Boundary manifold of a line arrangement

In this section, for a given arrangement  $\mathcal{A}$  in  $\mathbb{C}\mathbb{P}^2$  with  $n + 1$  lines, the line at infinity is part of the arrangement and is denoted by  $L_0 \in \mathcal{A}$ . In this case the affine part arrangement  $\mathcal{A}^{\text{aff}}$  coincides with the sub-arrangement  $(\mathcal{A} \setminus L_0)^{\text{aff}}$  with  $n$  lines in  $\mathbb{C}^2$ .

In [\[CS08\]](#), D. Cohen and A. Suciuc give a geometrical construction of a regular neighbourhood  $N_{\mathcal{A}}$  of a line arrangement  $\mathcal{A}$  as follows. Choose homogenous coordinates  $\mathbf{x} = [x : y : z]$  on  $\mathbb{C}\mathbb{P}^2$ . A closed, regular neighbourhood of  $\mathcal{A}$  may be constructed as follows. Define  $\phi : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{R}$  by

$$\phi(x) = |P_{\mathcal{A}}(\mathbf{x})|^2 / \|\mathbf{x}\|^{2(n+1)}$$

and define  $N_{\mathcal{A}} := \phi^{-1}([0, \delta])$ , for  $\delta > 0$  sufficiently small.

**Definition 2.3.1.** The *boundary manifold*  $B_{\mathcal{A}}$  of a line arrangement is defined as the boundary of the regular neighbourhood  $N_{\mathcal{A}}$ . The *exterior manifold*  $E_{\mathcal{A}}$  is defined as

$$E_{\mathcal{A}} := \mathbb{C}\mathbb{P}^2 \setminus \widehat{N_{\mathcal{A}}} \quad \diamond$$

**Remark 2.3.2.** The exterior manifold  $E_{\mathcal{A}}$  is a deformation retract of the *complementary* of the arrangement  $\mathbb{C}\mathbb{P}^2 \setminus \mathcal{A}$ .  $\square$

The boundary manifold of  $\mathcal{A}$  is a special case of a graph manifold, which means that it is constructed algorithmically from the incidence graph, by gluing together circle bundles as explained in [Section 1.3.2](#).

In this section we want to give a somewhat more detailed construction of  $B_{\mathcal{A}}$  due to [\[Wes97\]](#) which directly shows its graph manifold structure.

To do this we make use of the *blow-up* operation that helps to resolve the singularities of the line arrangements.

### 2.3.1 Blow-up

**Definition 2.3.3.** Let  $(x, y)$  be coordinates on  $\mathbb{C}^2$ . Consider  $\mathbb{C}\mathbb{P}^1$  with coordinates  $[x' : y']$ . The *blow-up* of the complex plane  $\mathbb{C}^2$  at the point  $(0, 0)$  is defined as the set

$$X_{(0,0)} := \{((x, y), [x' : y']) \in \mathbb{C}^2 \times \mathbb{C}\mathbb{P}^1 \mid xy' = x'y\} \quad \diamond$$

There exists a natural projection  $\sigma$  from  $X_{(0,0)}$  to  $\mathbb{C}^2$  defined by the diagram

$$\begin{array}{ccc} X_{(0,0)} & \hookrightarrow & \mathbb{C}^2 \times \mathbb{C}\mathbb{P}^1 \\ & \searrow \sigma & \downarrow \\ & & \mathbb{C}^2 \end{array}$$

By extension, the *blow-up* of an open neighbourhood  $U$  of  $(0, 0)$  in  $\mathbb{C}^2$  is defined as  $\widehat{U} := \sigma^{-1}(U)$ . Note that  $\sigma : \widehat{U} \rightarrow U$  is an isomorphism everywhere outside  $(0, 0)$ .

Using a change of coordinates, the blow-up can then be defined around any point  $P$  of  $\mathbb{C}^2$ .

**Definition 2.3.4.** Let  $S$  be a complex surface and let  $P \in S$ . Consider a local chart  $c : V \rightarrow U$  on a neighbourhood  $V$  of  $P$  inside  $S$  and with  $U \subset \mathbb{C}^2$ . The blow-up  $\widehat{S}_P$  of  $S$  in  $P$  is defined as

$$\widehat{S}_P := ((S \setminus \{P\}) \sqcup \widehat{U}) / f$$

where  $f = \sigma^{-1} \circ c$  is the isomorphism between  $V \setminus \{P\}$  and  $\widehat{U} \setminus \{\sigma^{-1}((0, 0))\}$ .  $\diamond$

We now apply this construction to the case of line arrangements, where all non-binary singular points will be blown-up separately.

**Definition 2.3.5.** The *full blow-up* of a line arrangement is the data of  $(X, \widehat{\mathcal{A}})$ , where  $X$  is the blow-up of all *non-binary* singular points  $P \in \mathcal{Q}^{>2}$  of  $\mathcal{A}$  (called the *blow-up space*), and  $\widehat{\mathcal{A}}$  is the pre-image of  $\mathcal{A}$  inside the blow-up space.

The *total blow-up* of  $\mathcal{A}$  is the data of  $(X', \widehat{\mathcal{A}}^{\max})$  where  $X'$  is the blow-up of *all* singular points  $P \in \mathcal{Q}$  of  $\mathcal{A}$ , and  $\widehat{\mathcal{A}}^{\max}$  is the pre-image of  $\mathcal{A}$ .  $\diamond$

We often call  $\widehat{\mathcal{A}}$  itself the ‘full blow-up of  $\mathcal{A}$ ’.

**Example 2.3.6.** The blow-up pre-image of a pencil with  $n$  lines is shown on [Figure 2.3.1](#).  $\square$

The irreducible components of  $\widehat{\mathcal{A}}$  are of two types:

- The components  $\widehat{L}_P$  arising from the blow-ups of the non-binary points  $P \in \mathcal{Q}^{>2}$  are called *exceptional lines*.
- The components  $\widehat{L}_i$  arising from the lines  $L_i$  of  $\mathcal{A}$  are called *regular lines*.

We denote by  $\widehat{\mathcal{L}}$  the set of irreducible components of  $\widehat{\mathcal{A}}$ . We have:

$$\widehat{\mathcal{L}} = (\widehat{L}_i)_{1 \leq i \leq n} \cup (\widehat{L}_P)_{P \in \mathcal{Q}^{>2}}$$



By construction, all irreducible components of  $\widehat{\mathcal{A}}$  intersect at binary singular points.

**Proposition 2.3.7.** *The pre-image  $\widehat{\mathcal{A}}$  inside the blow-up is a divisor with normal crossings of the line arrangement  $\mathcal{A}$ .  $\triangleleft$*

A divisor with normal crossings is a type of algebraic variety that contains only binary singular points and smooth irreducible components. It has a combinatorial structure encoded by a graph in a similar fashion to line arrangements but with simpler rules and no need to distinguish between two types of vertices.

**Definition 2.3.8.** Let  $\widehat{\mathcal{A}}$  be the full blow-up of a line arrangement. The *dual graph*  $\widehat{\Gamma}(\mathcal{A})$  of  $\widehat{\mathcal{A}}$  is defined by the following description:

*Vertices:*  $v_\ell$  for every irreducible component  $\ell$  of  $\widehat{\mathcal{A}}$ .

*Edges:*  $v_\ell$  and  $v_{\ell'}$  are linked by an edge  $e_{\ell,\ell'}$  if and only if  $\ell \cap \ell' \neq \emptyset$  in  $\widehat{\mathcal{A}}$ .  $\diamond$

**Proposition 2.3.9.** *The dual graph  $\widehat{\Gamma}(\mathcal{A})$  of the full blow-up  $\widehat{\mathcal{A}}$  is identical to the reduced incidence graph  $\widetilde{\Gamma}(C_{\mathcal{A}})$ , with every point-vertex of  $\widetilde{\Gamma}(C_{\mathcal{A}})$  replaced with the vertex of the corresponding exceptional-line component in  $\widehat{\Gamma}(\mathcal{A})$ .*

*Similarly, the dual graph  $\widehat{\Gamma}^{\max}(\mathcal{A})$  of the total blow-up  $\widehat{\mathcal{A}}^{\max}$  is identical to the full incidence graph  $\Gamma(C_{\mathcal{A}})$ .  $\triangleleft$*

### 2.3.2 Construction of the boundary manifold

Instead of constructing the boundary manifold on the original line arrangement  $\mathcal{A}$  inside  $\mathbb{C}\mathbb{P}^2$  as was done in [Definition 2.3.1](#), we build the boundary manifold of the full blow-up  $\widehat{\mathcal{A}}$  inside the blow-up space  $X$ . [Theorem 2.3.12](#) justifies that the two constructions lead to homeomorphic manifolds.

Just as with the line arrangement  $\mathcal{A}$  in [Definition 2.3.1](#), the boundary manifold of  $\widehat{\mathcal{A}}$  is obtained by taking the boundary of a regular neighbourhood of  $\widehat{\mathcal{A}}$ . This neighbourhood is obtained by gluing together local regular neighbourhoods of each irreducible component of  $\widehat{\mathcal{A}}$ .

For every line  $\ell \in \widehat{\mathcal{L}}$  of the full blow-up (exceptional or regular), let  $N_\ell$  be a regular neighbourhood of  $\ell$  inside the full blow-up space  $X$ . One can always take them small enough such that  $N_\ell \cap N_{\ell'} = \emptyset$  if  $\ell \cap \ell' = \emptyset$ .

**Definition 2.3.10.** Let

$$N_{\widehat{\mathcal{A}}} := \bigcup_{\ell \in \widehat{\mathcal{L}}} N_\ell$$

Then  $N_{\widehat{\mathcal{A}}}$  is a regular neighbourhood of  $\widehat{\mathcal{A}}$  inside the blow-up space  $X$ .  $\diamond$

**Definition 2.3.11.** The 3-manifold  $B_{\widehat{\mathcal{A}}} := \partial N_{\widehat{\mathcal{A}}}$  is called the *boundary manifold* of the full blow-up  $\widehat{\mathcal{A}}$ .  $\diamond$

The usefulness of the full blow-up construction is justified by the following

**Theorem 2.3.12.** *There is a homeomorphism between  $B_{\widehat{\mathcal{A}}}$  and  $B_{\mathcal{A}}$ .  $\triangleleft$*

*Proof.* Contracting each exceptional line  $\widehat{L}_P \in \widehat{\mathcal{L}}$  back into a point gives an isotopy from  $B_{\widehat{\mathcal{A}}}$  to  $B_{\mathcal{A}}$ . This can be seen by thickening the lines of [Figure 2.3.1](#): the ‘thick star’ around the pencil is homeomorphic to the ‘thick comb’ around its blow-up (recall that  $\sigma^{-1}(P)$  is compact).  $\square$

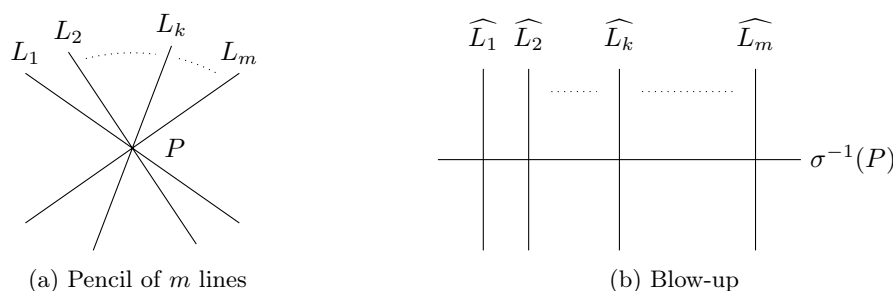


Figure 2.3.1: Blow-up of a line arrangement

### 2.3.3 Graph structure

To make the graph manifold structure of  $B_{\widehat{\mathcal{A}}}$  (and  $B_{\mathcal{A}}$ ) explicit we re-decompose it as a union of local boundary manifolds around each line of the blow-up.

**Definition 2.3.13.** For every line  $\ell \in \widehat{\mathcal{L}}$ , define

$$\begin{aligned}\Sigma_\ell &:= \ell \setminus \bigcup_{\ell \cap \ell' \neq \emptyset} \ell \cap N_{\ell'} \\ S_\ell &:= \partial N_\ell \setminus \bigcup_{\ell \cap \ell' \neq \emptyset} (\partial N_\ell \cap N_{\ell'})\end{aligned}\quad \diamond$$

For every line  $\ell \in \widehat{\mathcal{L}}$ , the surface  $\Sigma_\ell$  is homeomorphic to  $\Sigma^{m(\ell)}$ .

**Definition 2.3.14.** For a regular line  $L_i \in \mathcal{L}$ , define  $V_i^{>2} := \{P \in \mathcal{Q}^{>2} \mid P \in L_i\}$  and let  $b(L_i) := \#V_i^{>2}$  be the *blow-up number* of  $L_i$ .  $\diamond$

The Euler number of a circle bundle coincide with its self-intersection number. The regular neighbourhood of a regular complex line in  $\mathbb{C}\mathbb{P}^2$  has self-intersection 1. An exceptional line looks like a regular line in the neighbourhood  $\widehat{U}$  of the blow-up, but the change of charts  $f$  gives it a self-intersection  $-1$ . Since this also affects the line components that meet the exceptional line, their own self-intersection is reduced by 1 for each blow-up. One then obtains the following description of the structure of the local boundary manifolds.

**Theorem 2.3.15.** For every regular line  $L_i \in \mathcal{L}$ , the manifold  $S_{L_i} := \partial N_{\widehat{L}_i}$  is a circle bundle over  $\Sigma^{L_i} \simeq \Sigma^{m(L_i)}$  with Euler number  $\varepsilon_i = 1 - b(L_i)$ .

For every exceptional line  $\widehat{L}_P$  with  $P \in \mathcal{Q}^{>2}$ , the manifold  $S_P := \partial N_{\widehat{L}_P}$  is a circle bundle over  $\Sigma^{P_i} \simeq \Sigma^{m(P)}$  with Euler number  $\varepsilon_P = -1$ .

In particular,  $S_{L_i}$  and  $S_{P_i}$  verify **Conditions 1.3.7** on page 31.  $\triangleleft$

The local boundary neighbourhoods assemble along their boundary components to form a graph manifold which is exactly  $B_{\mathcal{A}}$ . It is however necessary to identify a collection of sections on the boundary (see **Definition 1.3.6** on page 31) on each of the circle bundles so that the graph manifold respects **Conditions 1.3.16** on page 32.

**Proposition 2.3.16.** For every line  $\ell \in \widehat{\mathcal{L}}$ ,  $(\ell \cap \partial N_{\ell'})_{\ell \cap \ell' \neq \emptyset}$  is a collection of section on the boundary of  $S_\ell$ .  $\triangleleft$

**Theorem 2.3.17** ([JY93]). Let  $\mathcal{A}$  be a non-exceptional line arrangement in the sense of **Definition 2.2.18**. The boundary manifold  $B_{\mathcal{A}}$  is a graph manifold whose minimal graph structure is given by the reduced incidence graph  $\widetilde{\Gamma}(C_{\mathcal{A}})$  decorated with the Euler numbers given in **Theorem 2.3.15**, and it verifies **Conditions 1.3.16**.  $\triangleleft$

**Corollary 2.3.18.** Let  $\mathcal{A}$  and  $\mathcal{A}'$  be two non-exceptional line arrangements with the same combinatorics  $C$ . Then there exists a homeomorphism  $\Phi : B_{\mathcal{A}} \rightarrow B_{\mathcal{A}'}$ .  $\triangleleft$

We can state a similar theorem when both arrangements are endowed with the same ordering  $\theta$ .

**Theorem 2.3.19.** Let  $(\mathcal{A}, \theta)$  and  $(\mathcal{A}', \theta)$  be two non-exceptional ordered line arrangements with the same combinatorics  $C$ . Then there exists a strongly positive graphed homeomorphism

$$\Phi : B_{\mathcal{A}} \longrightarrow B_{\mathcal{A}'} \quad \triangleleft$$

*Proof.* By **Corollary 2.3.18**, there exists a homeomorphism  $\Phi : B_{\mathcal{A}} \rightarrow B_{\mathcal{A}'}$ . We thus now write  $B := B_{\mathcal{A}} \simeq B_{\mathcal{A}'}$ . The graph structure of  $B$  is given by  $\Gamma = \widetilde{\Gamma}(C)$ . By **Theorem 1.3.21**,  $\Phi$  induces a permutation  $G(\Phi)$  on the graph  $\Gamma$ . However, since the ordered arrangements are equivalent, by **Definition 2.1.6** the homeomorphism  $\Phi$  induces the identity permutation on  $\mathcal{L}$ , and by extension on the whole combinatorics  $C$ . This means that  $G(\Phi) = \text{Id}_\Gamma$ , and thus  $\Phi$  is a graphed homeomorphism on  $B$ . Now take the image of  $\Phi$  in  $\text{Homeo}_\Gamma^{++}(B)$  by the quotient map of **Proposition 1.3.24** to obtain a strongly positive graphed homeomorphism from  $B_{\mathcal{A}}$  to  $B_{\mathcal{A}'}$ .  $\square$

One can make a similar construction of the boundary manifold of the total blow-up  $\widehat{\mathcal{A}}^{\max}$  of  $\mathcal{A}$ . **Theorem 2.3.12** extends to show that  $B_{\widehat{\mathcal{A}}^{\max}}$  is also homeomorphic to  $B_{\mathcal{A}}$ . However, the graph structure of  $B_{\widehat{\mathcal{A}}^{\max}}$ , which by **Proposition 2.3.9** is given by the full incidence graph  $\Gamma(C_{\mathcal{A}})$ , is not minimal.

As described in [Section 1.6.4](#), graphs for non-minimal structures can be obtained combinatorially from the minimal graph by performing *blowing-down* moves represented on [Figure 1.6.2](#) on page 47. It occurs that the binary blowing-down move corresponds exactly to the inverse of the action of the blow-up operation on the graph.

**Proposition 2.3.20.** *Let  $\mathcal{A}$  be a non-exceptional line arrangement. Then the reduced incidence graph  $\tilde{\Gamma}(C_{\mathcal{A}})$  is obtained from the full incidence graph  $\Gamma(C_{\mathcal{A}})$  by blowing down all vertices corresponding to binary singular points.*  $\triangleleft$

*Proof.* By [Theorem 2.3.15](#), the Euler numbers of the vertices of  $\Gamma(C_{\mathcal{A}})$  are:  $-1$  for every vertex  $v_P$  with  $P \in \mathcal{Q}$  and  $1 - \#N_i$  for every vertex  $v_i$  with  $L_i \in \mathcal{L}$ . Let  $P \in \mathcal{Q}$  be a binary point and write  $P = L_i \cap L_j$ . Blowing-down the vertex  $v_P$  as described on [Figure 1.6.2](#) will replace  $v_P$  by an edge exactly as in  $\tilde{\Gamma}(C_{\mathcal{A}})$ , and add 1 to the Euler numbers of  $v_i$  and  $v_j$ . Now consider a line component  $L_i$  and the associated vertex  $v_i$ . We have  $N_i = N_i^* \sqcup N_i'$  where  $N_i'$  is the set of all binary singular points  $P \in L_i$ . Blowing down all vertices  $v_P$  for  $P \in N_i'$  leaves  $1 - \#N_i^*$  as the new Euler number of  $v_i$ . This is exactly the corresponding Euler number of  $v_i$  in  $\tilde{\Gamma}(C_{\mathcal{A}})$ .  $\square$

## 2.4 Exterior of a line arrangement

Let  $\mathcal{A}$  be a line arrangement with  $n$  lines and let  $E_{\mathcal{A}}$  be the complement in  $\mathbb{C}\mathbb{P}^2$  of an open regular neighbourhood of  $\mathcal{A}$  as explained in [Definition 2.3.1](#). Suppose that  $L_0$  is the line at infinity in the standard affine chart. Throughout this section,  $\mathbb{C}^2$  is thus assumed to be  $\mathbb{C}\mathbb{P}^2 \setminus L_0$ . The complementary  $\mathbb{C}^2 \setminus \mathcal{A}^{\text{aff}}$  is naturally homeomorphic to  $\mathbb{C}\mathbb{P}^2 \setminus \mathcal{A}$ . We define

$$\mathcal{L}^* := \mathcal{L} \setminus \{L_0\} \qquad \mathcal{Q}^* := \mathcal{Q} \setminus V_{L_0}$$

### 2.4.1 Wiring diagram

The wiring diagram is a construction due to W. Arvola [[Arv92](#)] which allows to fully encode the topology of an ordered line arrangement on a planar diagram.

We consider linear projections  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  such that for every  $L \in \mathcal{L}^*$ , the restriction  $\pi|_{L^{\text{aff}}}$  is a homeomorphism of the complex plane.

**Definition 2.4.1.** A linear projection  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  is said to be *generic* if each multiple point  $P \in \mathcal{Q}^*$  lie in a different fibre of  $\pi$ .  $\diamond$

Let  $\nu : \mathcal{Q}^* \rightarrow \{1, \dots, r\}$  be an ordering on  $\mathcal{Q}^*$ . We denote by  $x_l := \pi(P_l)$  the ordered images of the singular points. Let  $R^0$  be an open rectangle in  $\mathbb{C}$  containing all the points  $x_1, \dots, x_r$  and let  $x_0$  and  $x_{r+1} \in \mathbb{C}$  be points such that

$$\operatorname{Re}(x_0) \leq \inf \operatorname{Re}(R_0) \qquad \operatorname{Re}(x_{r+1}) \geq \sup \operatorname{Re}(R_0)$$

**Definition 2.4.2.** A smooth path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  is said to be  $\nu$ -*admissible* if  $\gamma(0) = x_0$ ,  $\gamma(1) = x_{r+1}$ , and  $\gamma$  goes through  $x_1 \dots x_r$  in order.  $\diamond$

**Definition 2.4.3.** Let  $\pi$  be a generic linear projection and let  $\gamma$  be a  $\nu$ -admissible path. The *wiring diagram*  $W_{\mathcal{A}}(\pi, \gamma)$  associated to  $\pi$  and  $\gamma$  is defined as the graph inside  $[0, 1] \times \mathbb{C}$  of the multivalued function given by

$$\pi_{\mathcal{A}} : t \in [0, 1] \mapsto \pi^{-1}(\gamma(t)) \cap \mathcal{A}$$

Each continuous value of  $\pi_{\mathcal{A}}$  which corresponds to the trace of

$$w_L : t \in [0, 1] \mapsto \pi^{-1}(\gamma(t)) \cap L$$

is called the *wire* associated with the line  $L$ .  $\diamond$

Since  $W_{\mathcal{A}}$  is a real one-dimensional object inside a real three-dimensional space, it can then be projected back onto a real plane. By convention, we take the projection  $\pi_0 : [0, 1] \times \mathbb{C} \rightarrow \mathbb{R}^2$  given by

$$\pi_0(t, x + iy) := (t, y)$$

There are values of  $t \in [0, 1]$  for which several projected wires  $\pi_0(w_L(t))$  might cross. If  $\pi_{\mathcal{A}}(t) = P$  for some  $P \in \mathcal{Q}^*$ , then all the wires  $w_L(t)$  corresponding to the lines that contain  $P$  already

merged into a single point on  $W_{\mathcal{A}}$  before the projection  $\pi_0$ . The image  $\pi_0(P)$  on the real plane is thus called an *actual crossing*. However, if  $\pi_{\mathcal{A}}(t)$  do *not* correspond to a singular point of  $\mathcal{A}$ , then the crossing on the real plane was caused by  $\pi_0$  only. In this case, we replace it with a *virtual crossing* in a similar fashion as on a knot diagram.

The genericity of the projection  $\pi$  ensures that all crossings on the real plane (actual and virtual) correspond to different values of  $t$ . In practice a non-generic wiring diagram can always be slightly deformed to produce a generic one, with a new ordering on  $\mathcal{Q}^*$ . The admissible path  $\gamma$  can also always be adjusted so that only two projected wires meet on any virtual crossing.

The projection  $\pi_0(W_{\mathcal{A}})$  with the addition of virtual crossings as described above is used as the common representation of the wiring diagram  $W_{\mathcal{A}}$ .

## 2.4.2 Braid monodromy

The braid monodromy was first introduced for complex algebraic curves by O. Chisini [Chi33] and O. Zariski [Zar29] and was later redefined by B. Moishezon [Moi81]. For the case of line arrangements, it is a very similar construction to the wiring diagram and is in fact an equivalent way of presenting the same topological information. The construction of the braid monodromy we present was developed by E. Artal, J. Carmona and J. I. Cogolludo in [ACC03].

First we introduce another geometrical interpretation of the braid group. Let  $\mathbf{y} \subset \mathbb{C}$  be a subset of  $n$  points in the complex plane. The points of  $\mathbf{y}$  are naturally ordered by their ascending real parts. Fix a polygonal path  $p_{\mathbf{y}}$  in  $\mathbb{C}$  that joins the points of  $\mathbf{y}$  in order. Let  $\{\gamma_i : [0, 1] \rightarrow \mathbb{C}\}_{1 \leq i \leq n}$  be a set of  $n$  paths that start and end at  $\mathbf{y}$  such that for every  $t \in [0, 1]$ , the points  $\{\gamma_1(t), \dots, \gamma_n(t)\}$  are all distinct. Define the group  $\mathbb{B}_{\mathbf{y}}$  to be the set of homotopy classes of all such path sets  $\{\gamma_1, \dots, \gamma_n\}$  in  $\mathbb{C} \times [0, 1]$  relatively to  $p_{\mathbf{y}}^0$  and  $p_{\mathbf{y}}^1$ . Similarly, given two sets  $\mathbf{y}_1, \mathbf{y}_2$  one can define the groupoid  $\mathbb{B}_{\mathbf{y}_1, \mathbf{y}_2}$  as the set of homotopy classes of paths joining  $\mathbf{y}_1$  to  $\mathbf{y}_2$  relatively to the paths  $p_{\mathbf{y}_1}^0$  and  $p_{\mathbf{y}_2}^1$ .

**Proposition 2.4.4.** *There are natural isomorphisms*

$$\begin{aligned} I_{\mathbf{y}} : \quad \mathbb{B}_{\mathbf{y}} &\longrightarrow \mathbb{B}_n \\ I_{\mathbf{y}_1, \mathbf{y}_2} : \quad \mathbb{B}_{\mathbf{y}_1, \mathbf{y}_2} &\longrightarrow \mathbb{B}_n \end{aligned} \quad \triangleleft$$

The isomorphism  $I_{\mathbf{y}}$  restricts to the pure braid group  $\mathbb{P}_n$  for the subset of paths  $\{\gamma_1, \dots, \gamma_n\}$  that all start and end at the same respective point of  $\mathbf{y}$ . Consider a projection  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ , which is not necessarily generic in the sense of Definition 2.4.1. Several singular points can thus be sent to a same point in  $\mathbb{C}$ .

**Definition 2.4.5.** The projection  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  defines an *assignment function*

$$X_{\pi} : \pi(\mathcal{Q}^*) \longrightarrow \mathcal{P}(\mathcal{Q}^*)$$

that sends each point  $x \in \pi(\mathcal{Q}^*)$  to the subset  $\pi^{-1}(x) \cap \mathcal{Q}^*$ . ◇

The projection points  $x \in \pi(\mathcal{Q}^*)$  such that  $X_{\pi}(x)$  is a singleton are called *generic*. The others are called *non-generic*. By assumption the projection never sends a line component  $L \in \mathcal{L}^*$  to a single point. Therefore, two singular points sent to a common projection point  $x \in \pi(\mathcal{Q}^*)$  cannot be on the same component line of  $\mathcal{A}$ . In other words, if  $P, P' \in X_{\pi}(x)$  then  $V_P \cap V_{P'} = \emptyset$ .

We write  $r_{\pi} := \#\pi(\mathcal{Q}^*)$  and define  $\mathbb{C}_{\mathcal{A}} := \mathbb{C} \setminus \pi(\mathcal{Q}^*)$ . The set  $\mathbb{C}_{\mathcal{A}}$  is then a complex plane with  $r_{\pi}$  punctures.

The restriction  $\pi : \mathbb{C}^2 \setminus \mathcal{A}^{\text{aff}} \rightarrow \mathbb{C}_{\mathcal{A}}$  is a locally trivial bundle. For any point  $b \in \mathbb{C}_{\mathcal{A}}$ , the fibre  $\pi^{-1}(b)$  is isomorphic to  $\mathbb{C}$  with a set of  $n$  punctures  $\mathbf{y}(b)$  corresponding to the points  $w_L(b) := \pi^{-1}(b) \cap L$  for every  $L \in \mathcal{L}^*$ .

**Definition 2.4.6.** For every point  $b \in \mathbb{C}_{\mathcal{A}}$  there is a corresponding ordering

$$\theta^b : \mathcal{L}^* \longrightarrow \{1, \dots, n\}$$

which orders the set of points  $\{w_L(b) \mid L \in \mathcal{L}^*\}$  by their ascending horizontal coordinate in the complex plane  $\pi^{-1}(b)$ . ◇

**Definition 2.4.7.** Let  $b^-$  and  $b^+$  be two points in  $\mathbb{C}_{\mathcal{A}}$  and let  $\gamma : [0, 1] \rightarrow \mathbb{C}_{\mathcal{A}}$  be a path joining them. Consider the set of paths  $\{\gamma_L \mid L \in \mathcal{L}^*\}$  joining  $\mathbf{y}(b^-)$  to  $\mathbf{y}(b^+)$  defined by taking

the wires

$$\gamma_L : t \in [0, 1] \mapsto w_L(\gamma(t))$$

as shown on [Figure 2.4.1](#). Then the *lifted braid* of  $\gamma$  is defined by

$$\rho_{\mathcal{A},\pi}(\gamma) := I_{\mathbf{y}(b^-), \mathbf{y}(b^+)}(\{\gamma_L \mid L \in \mathcal{L}^*\}) \in \mathbb{B}_n \quad \diamond$$

By construction,  $\rho_{\mathcal{A},\pi}(\gamma)$  only depends on the homotopic class of  $\gamma$  relatively to  $b^-$  and  $b^+$ . Moreover,  $\rho_{\mathcal{A},\pi}$  restricts to  $\mathbb{P}_n$  when  $b^- = b^+$ .

Now recall that  $N_L$  (resp.  $N_P$ ) is a regular neighbourhood of  $L \in \mathcal{L}^*$  (resp.  $P \in \mathcal{Q}^*$ ) inside  $\mathbb{C}\mathbb{P}^2$ . We have  $N_P \simeq N_P^{\text{aff}}$  and the image  $\pi(N_P)$  is homeomorphic to a 2-disc. Let  $D$  be a closed disc in  $\mathbb{C}$  containing all the images  $\pi(N_P)$  for every  $P \in \mathcal{Q}^*$ . The basis

$$\Sigma := D \setminus \bigcup_{P \in \mathcal{Q}^*} \widehat{\pi(N_P)} \subset \mathbb{C}_{\mathcal{A}}$$

is homeomorphic to a 2-disc with  $r_\pi$  holes.

Let  $b^\infty \in \partial D$  be a base point. For any closed curve  $\gamma$  based in  $b^\infty$ , the lifted braid  $\rho_{\mathcal{A},\pi}$  is pure and only depends on the homotopic class of  $\gamma$  in  $\pi_1(\Sigma, b^\infty)$ . We therefore have a map

$$\rho_{\mathcal{A},\pi} : \pi_1(\Sigma, b^\infty) \longrightarrow \mathbb{P}_n$$

Changing the base point from  $b^\infty$  to  $b^{\infty'}$  is done by an isomorphism of  $\pi_1(\Sigma)$ . This corresponds to conjugating the map  $\rho_{\mathcal{A},\pi}$  with a braid  $\beta \in \mathbb{B}_n$  such that  $\theta^{\infty'} = \sigma(\beta) \circ \theta^\infty$ , where  $\sigma(\beta) \in \mathfrak{S}_n$  is the permutation associated with  $\beta$ .

**Definition 2.4.8.** The map

$$\rho_{\mathcal{A},\pi} : \pi_1(\Sigma) \longrightarrow \mathbb{P}_n$$

defined up to conjugation in  $\mathbb{B}_n$  is called the *braid monodromy* of the line arrangement  $\mathcal{A}$ .  $\diamond$

In general we do not use the morphism  $\rho_{\mathcal{A},\pi}$  itself, but rather its image on a set of generators of  $\pi_1(\Sigma, b^\infty)$ . This set is called a *representative* and depends on additional parameters which can be linked to the choice of a star on the basis  $\Sigma$ .

**Definition 2.4.9.** Let  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a linear projection. Up to a change of coordinates in  $\mathbb{C}^2$ , we suppose that all the points of  $\pi(\mathcal{Q}^*)$  have distinct real parts. The projection ordering  $\nu_\pi : \pi(\mathcal{Q}^*) \rightarrow \{1, \dots, r_\pi\}$  is defined such that the points  $x_1, \dots, x_{r_\pi}$  of  $\pi(\mathcal{Q}^*) \subset \mathbb{C}$  verify

$$\text{Re}(x_1) < \dots < \text{Re}(x_{r_\pi}) \quad \diamond$$

Let  $\theta$  be an ordering on  $\mathcal{L}^*$  and let  $b^\infty \in \partial^\infty \Sigma$  such that  $\theta^\infty = \theta$ . Let  $\nu$  be an ordering on  $\pi(\mathcal{Q}^*)$  and let  $\alpha \in \mathcal{S}_{r_\pi}$  be a star on  $\Sigma$  associated with the ordering  $\nu \circ \nu_\pi^{-1}$  (see [Definition 1.2.1](#) on page 28). Let also  $x_l \in \pi(\mathcal{Q}^*)$  be a projection point. The *local braid monodromy around  $x_l$*  is defined by  $\delta_l := \rho_{\mathcal{A},\pi}(\partial_l \Sigma)$ .

**Proposition 2.4.10.** For every  $P_l \in \mathcal{Q}^*$ , the local braid monodromy  $\delta_l$  is positive and

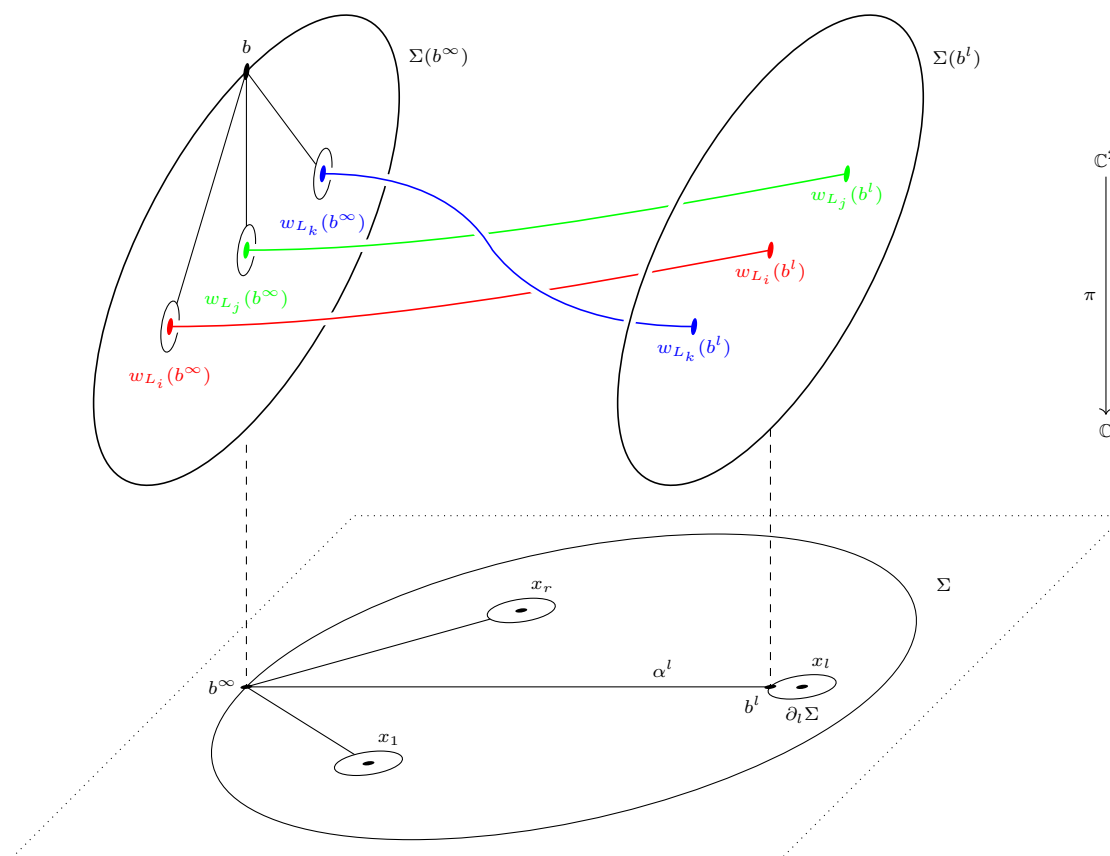
$$\rho_{\mathcal{A},\pi}(\partial_l \Sigma) = \prod_{P \in X_\pi(x_l)} \Delta_{\theta^l(V_P)}^2$$

where  $\theta^l$  is the order above the point  $b^l := \partial_+ \alpha^l$  at the extremity of the  $l$ -th branch of  $\alpha$  (as shown on [Figure 2.4.1](#)).  $\triangleleft$

Note that since the sets  $V_P$  are disjoint for all  $P \in X_\pi(x_l)$ , the full twists that compose  $\rho_{\mathcal{A},\pi}(\partial_l \Sigma)$  act on disjoint sets of strands and commute.

*Proof.* Write  $X_\pi(x_l) = \{P_1, \dots, P_k\}$ . Let  $(x, y)$  be a coordinate system of  $\mathbb{C}^2$  centred in  $P_1$ . Up to a change of coordinates, we can always suppose that  $\pi$  is the projection on the coordinate  $x$ . The points  $P_2, \dots, P_k$  have coordinates  $(0, y_2), \dots, (0, y_k)$ . Up to isotopy, the local equation of  $\mathcal{A}^{\text{aff}}$  around the singular point  $P_i$  with multiplicity  $m_i$  is of the form  $(y - y_i)^{m_i} - x^{m_i}$ . The lifted braid  $\rho_{\mathcal{A},\pi}(\partial_+^l \Sigma)$  is therefore a product of full twists  $\Delta^2$  over the  $m_i$  strands corresponding to the lines  $L \in V_{P_i}$  that contain  $P_i$ . The corresponding set of indices is  $\theta^l(V_{P_i}) \subset \{1, \dots, n\}$ .  $\square$

**Definition 2.4.11.** The *shift braid to  $x_l$*  is defined by  $\tau_l := \rho_{\mathcal{A},\pi}(\alpha^l) \in \mathbb{B}_n$ .  $\diamond$

Figure 2.4.1: Braid over the path  $\alpha^l \subset \Sigma$ 

**Definition 2.4.12.** The pure braid given by

$$\beta_l := \tau_l \cdot \delta_l \cdot (\tau_l)^{-1} \in \mathbb{P}_n$$

is called a *star braid monodromy around  $x_l$* . The set of braids

$$B_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \leq l \leq r_\pi} \in (\mathbb{P}_n)^{r_\pi}$$

is called a *representative* of the braid monodromy of  $\mathcal{A}$ .  $\diamond$

**Proposition 2.4.13.** The braid at infinity of the representative  $B_{\mathcal{A}}(\pi, \theta, \nu, \alpha)$  is defined by the equivalent formulas:

$$\beta_0 := \Delta_{\theta^\infty(\mathcal{L}^*)}^2 \cdot \left( \prod_{l=1}^{r_\pi} \beta_l \right)^{-1} = \prod_{P \in V_{L_0}} \Delta_{\theta^\infty(V_P)}^2 \quad \triangleleft$$

*Proof.* Choose a system of coordinates  $(x, y)$  of  $\mathbb{C}^2$  such that  $\pi$  is the projection on the horizontal plane  $\{y = 0\}$  and  $(0, 0)$  is the centre of  $D^0$ . Remember that the space  $\mathbb{C}^2$  is seen as the *standard* affine chart of  $\mathbb{C}\mathbb{P}^2$  given by  $\{[x : y : 1] \in \mathbb{C}\mathbb{P}^2\}$ . In the exterior of  $D \subset \mathbb{C}\mathbb{P}^2$  (where  $x \neq 0$ ) apply the change of coordinates

$$\begin{aligned} [x : y : 1] &\mapsto [Z^{-1} : YZ^{-1} : 1] \\ [1 : y/x : 1/x] &\longleftarrow [1 : Y : Z] \end{aligned}$$

The new chart  $\{[1 : Y : Z] \in \mathbb{C}\mathbb{P}^2\}$  is called the  $\infty$ -chart. Write  $D = \{|x| \leq r^0\}$ . The exterior  $D^\infty := \mathbb{C}\mathbb{P}^1 \setminus \overset{\circ}{D}$  becomes the 2-disc  $\{|Z| \leq 1/r^0\}$  in the  $\infty$ -chart. Similarly, the line  $L_0 = \{x = \infty\}$  becomes  $\{Z = 0\}$  and the projection  $\pi : (x, y) \rightarrow (x, 0)$  becomes  $\pi : (Y, Z) \rightarrow (0, Z)$ . The component  $L_0$  is now a vertical line for  $\pi$ , and all singular points of  $V_{L_0}$  are all sent to a same point

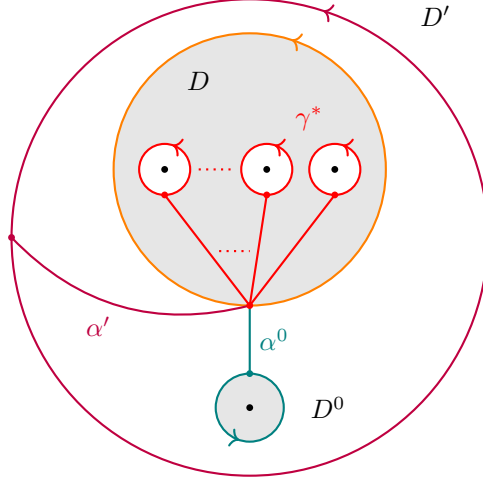


Figure 2.4.2: Construction of the braid at infinity in the standard prime chart

$x_0 = (Y = 0, Z = 0)$ . Let  $D^{\infty'}$  be a disc inside the interior of  $D^{\infty}$  that does not meet any of the  $\pi(N_P)$  for every  $P \in V_{L_0}$ . Note  $x_{\infty'}$  the centre of  $D^{\infty'}$ . By construction, the line  $L_{\infty} := \pi^{-1}(x_{\infty'})$  is generic to the arrangement  $\mathcal{A}$ , in the sense that all the intersection points  $L_{\infty} \cap L_i$  for  $L_i \in \mathcal{L}^*$  are double points. Now apply a new change of coordinates by making the translation

$$\begin{aligned} [1 : Y : Z] &\longmapsto [1 : Y' : Z'] := [1 : Y : Z] - x_{\infty'} \\ [x : y : 1] &\longmapsto [x' : y' : 1] := [x : y : 1] - x_{\infty'} \end{aligned}$$

This defines two new charts which we call the  $\infty'$ -chart and the standard prime chart respectively. The projection  $\pi$  is not affected. We also restrict the  $\infty'$ -chart to the interior of  $D^{\infty'}$ . Inside the standard prime chart, the line  $L_{\infty}$  is the new line at infinity, but  $L_0$  is still vertical. Now define  $D' := \mathbb{CP}^1 \setminus D^{\infty'}$ . Up to homeomorphism, we can suppose that all  $\pi(N_P)$  for  $P \in V_{L_0}$  are equal to a same disc  $D^0$ . We therefore have  $D^0 \sqcup D \subset D'$  as shown on [Figure 2.4.2](#). Now consider the path

$$\alpha^* := \prod_{l=1}^{r_{\pi}} \alpha^l \cdot \partial_+^l \Sigma \cdot (\alpha^l)^{-1}$$

which has the same homotopy type as  $\partial D$ . The lifted braid  $\rho_{\mathcal{A}, \pi}(\partial_+ D)$  is therefore equal to the product of all  $\beta_l$  for  $1 \leq j \leq m$ . Let  $\alpha^0$  be an extra branch to the star  $\alpha$  that joins  $b^{\infty}$  to  $\partial D^0$ . Up to homeomorphism, one can increase the diameter of  $D^0$  so that  $\rho_{\mathcal{A}, \pi}(\alpha^0) = 1$ . Let  $\gamma_0 := \alpha^0 \cdot \partial_+ D^0 \cdot (\alpha^0)^{-1}$ . By [Proposition 2.4.10](#), we have

$$\rho_{\mathcal{A}, \pi}(\gamma_0) = \rho_{\mathcal{A}, \pi}(\partial_+ D^0) = \prod_{P \in V_{L_0}} \Delta_{\theta^{\infty}(V_P)}^2$$

Similarly, let  $\alpha'$  be another extra branch that joins  $b^{\infty}$  to  $\partial D'$  and let  $\gamma' := \alpha' \cdot \partial_+ D' \cdot (\alpha')^{-1}$ . Since  $\pi^{-1}(D')$  contains all singular points of  $\mathcal{Q}$ , including those of  $V_{L_0}$ , then in an exterior neighbourhood of  $D'$  the local equation of the arrangement  $\mathcal{A} \setminus L_0$  is of the form  $(x')^n - (y')^n$ . This means that

$$\rho_{\mathcal{A}, \pi}(\partial_+ D') = \Delta_{\theta^{\infty}(\mathcal{L}^*)}^2$$

This full twist braid over all strands is central in the braid group  $\mathbb{B}^n$ . Therefore we have  $\rho_{\mathcal{A}, \pi}(\gamma') = \rho_{\mathcal{A}, \pi}(\partial_+ D')$ . To conclude the proof we only need to say that the loop  $\gamma'$  has the same homotopy type as the loop  $\gamma_0 \cdot \partial_+ D$ . By unicity of the lifting braid and using the previous results, we finally get

$$\rho_{\mathcal{A}, \pi}(\partial_+ D') \cdot \rho_{\mathcal{A}, \pi}(\gamma^*)^{-1} = \rho_{\mathcal{A}, \pi}(D^0) \quad \square$$

**Remark 2.4.14.** The path  $\gamma^*$  is also almost an admissible path in the sense of [Definition 2.4.2](#). The lifted braid over the branch parts of  $\gamma^*$ , seen as a geometrical object, corresponds exactly to the wiring diagram  $W_{\mathcal{A}}(\pi, \gamma)$  in  $\mathbb{C} \times [0, 1]$  with all crossings removed. This is the core argument to prove that the wiring diagram and the braid monodromy are topologically equivalent.  $\square$

It is possible to change the parameters  $\theta$ ,  $\nu$  and  $\alpha$  of a braid monodromy representative using two separate actions on the group  $(\mathbb{P}_n)^{r_\pi}$ . First consider the Hurwitz action of  $\mathbb{B}_{r_\pi}$  on the Cartesian product  $(\mathbb{B}_n)^{r_\pi}$ .

**Definition 2.4.15.** Let  $G$  be a group. The *right Hurwitz action* of  $\mathbb{B}_{r_\pi}$  on the Cartesian product  $G^{r_\pi}$  is defined on any element  $(a_1, \dots, a_{r_\pi}) \in G^{r_\pi}$  by

$$a_j \cdot \sigma_i^* := \begin{cases} a_j a_{j+1} a_j^{-1} & \text{if } j = i \\ a_{j-1} & \text{if } j = i + 1 \\ a_j & \text{otherwise} \end{cases}$$

The action of each elementary braid  $\sigma_i$  is called a *Hurwitz move*. ◇

Next consider the action of  $\mathbb{B}_n$  on  $(\mathbb{B}_n)^{r_\pi}$  by conjugation:

$$\forall \tau \in \mathbb{B}_n : (\beta_1, \dots, \beta_{r_\pi})^\tau := (\beta_1^\tau, \dots, \beta_{r_\pi}^\tau)$$

This action corresponds to the change of base point in  $\pi_1(\Sigma)$ .

**Proposition 2.4.16** ([Moi81; ACC03]). Let  $B_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \leq l \leq r_\pi}$  be a representative of the braid monodromy of  $\mathcal{A}$ . For every couple  $(\kappa, \tau) \in \mathbb{B}_{r_\pi} \times \mathbb{B}_n$ , define

- $\theta' := \sigma(\tau) \circ \theta$ .
- $\nu' := \sigma(\kappa) \circ \nu$ .
- $\alpha' := \kappa \cdot \alpha$  for the action of [Proposition 1.2.4](#) on page 28.
- $(\beta'_l)_{1 \leq l \leq r_\pi} = (\kappa, \tau) \cdot (\beta_l)_{1 \leq l \leq r_\pi}$  for the Hurwitz and conjugation actions.

Then  $B_{\mathcal{A}}(\pi, \theta', \nu', \alpha')$  coincides with  $(\beta'_l)_{1 \leq l \leq r_\pi}$  and is another representative of the braid monodromy of  $\mathcal{A}$ . ◇

### 2.4.3 Fundamental group of the exterior

Either the wiring diagram or the braid monodromy can be used to obtain a presentation of the fundamental group of the exterior of a line arrangement  $\mathcal{A}$ , and both presentations are equivalent [CS97]. The method using the wiring diagram was introduced by W. Arvola [Arv92]. The method using the braid monodromy was first introduced by O. Zariski, E. R. van Kampen and B. Moishezon, and was later readapted by A. Libgober [Lib86]. It is this method that we recall here. For the similar wiring diagram method, see for example [FGM15].

**Definition 2.4.17.** Let  $\mathbb{F}_r$  be the free group generated by  $r$  elements  $(f_1, \dots, f_r)$ . The *right Hurwitz action* of  $\mathbb{B}_r$  on  $\mathbb{F}_r$  is defined by

$$f_j^{\sigma_i} := \begin{cases} f_j f_{j+1} f_j^{-1} & \text{if } j = i \\ f_{j-1} & \text{if } j = i + 1 \\ f_j & \text{otherwise} \end{cases} \quad \diamond$$

We reuse notations from [Section 2.4.2](#). The arrangement  $\mathcal{A}$  is projected in  $\mathbb{C}^2$  with the line  $L_0$  at infinity. We fix  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  a projection. Let  $\omega$  be an ordering on  $\mathcal{L}^*$  and  $\nu$  be an ordering on  $\pi(\mathcal{Q}^*)$ .

For every  $x \in \mathbb{C}_{\mathcal{A}}$ , let  $D(x)$  be a closed disc in  $\pi^{-1}(x)$  containing all the intersections  $D_L(x) := \pi^{-1}(x) \cap N_L^{\text{aff}}$  for  $L \in \mathcal{L}^*$  and define

$$\begin{aligned} \Delta(x) &:= D(x) \setminus \{w_L(x) \mid L \in \mathcal{L}^*\} \\ \Sigma(x) &:= D(x) \setminus \bigcup_{L \in \mathcal{L}^*} \pi^{-1}(x) \cap \overset{\circ}{N}_L^{\text{aff}} \end{aligned}$$

as shown on [Figure 2.4.1](#).

**Definition 2.4.18.** Let  $\alpha_x \in \mathcal{O}_n$  be an ordered star drawn on  $\Sigma(x)$ . Then the curves

$$\mu_i := \alpha_x^i \cdot \partial^i \Sigma(x) \cdot (\alpha_x^i)^{-1} \quad \text{for } 1 \leq i \leq n$$



are called *exterior meridians* of the line arrangement  $\mathcal{A}$ . Each meridian  $\mu_i$  is associated with the line  $(\theta^x)^{-1}(i) \in \mathcal{L}^*$  for the ordering above  $x$ .  $\diamond$

**Theorem 2.4.19** ([Zar29; Moi81]). *Let  $B_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \leq l \leq r_\pi}$  be a representative of the braid monodromy of  $\mathcal{A}^{\text{aff}} = \mathcal{A} \setminus L_0$ . Let  $b^\infty$  be the base point of  $\alpha$  with  $\theta^\infty = \theta$ . Let  $\mu_1, \dots, \mu_n$  be exterior meridian curves of  $\mathcal{A}$  drawn on  $\Sigma(b^\infty)$ . For every  $x_l \in \pi(\mathcal{Q}^*)$ , let  $I_k \subset \{1, \dots, n\}$  be the reunion of the sets  $\theta(V_P)$  for every  $P \in X_\pi(x_l)$ . The fundamental group of  $\mathbb{C}^2 \setminus \mathcal{A}^{\text{aff}}$  admits the following presentation:*

Generators:  $\mu_1, \dots, \mu_n$ .

Relations:  $(\mu_i^{\delta_l} \cdot \mu_i^{-1})^{\tau_l^{-1}} = \mu_i$  for every  $1 \leq l \leq r_\pi$  and every  $i \in I_l$ .  $\triangleleft$

**Remark 2.4.20.** The product  $\mu_1 \cdots \mu_n$  has the same homotopy type as  $\partial_+ D(b^\infty)$  and corresponds to the inverse of a meridian  $\mu_0$  of the line at infinity.  $\square$

**Corollary 2.4.21.** *The first homology group of the exterior  $H_1(E_{\mathcal{A}}, \mathbb{Z})$  is a  $\mathbb{Z}$ -module of finite type given by the presentation*

$$H_1(E_{\mathcal{A}}, \mathbb{Z}) = \left\langle \mu_0, \mu_1, \dots, \mu_n \mid \sum_{i=0}^n \mu_i = 0 \right\rangle \triangleleft$$

Recall the *meridian homology*  $V(\Gamma)$  of a graph  $\Gamma$  from [Definition 1.5.14](#) on page 41. [Theorem 1.5.16](#) states that  $V(\Gamma)$  represents the contribution of the meridians to the first homology group of the graph manifold associated with  $\Gamma$ . It turns out that in the case of line arrangements that exact same construction also corresponds to the first homology group of the exterior in  $\mathbb{C}\mathbb{P}^2$ . This result is crucial for the definition of the *homology inclusion map* in [Section 2.5](#).

**Proposition 2.4.22.**  $H_1(E_{\mathcal{A}}, \mathbb{Z}) = V(\tilde{\Gamma}(C_{\mathcal{A}}))$ .  $\triangleleft$

*Proof.* By [Lemma 1.6.17](#) on page 48, one can compute  $V(\tilde{\Gamma}(C_{\mathcal{A}}))$  on a bigger graph which reduces to  $\tilde{\Gamma}(C_{\mathcal{A}})$  by blowing-down moves. By [Proposition 2.3.20](#) on page 62, the full incidence graph  $\Gamma(C_{\mathcal{A}})$  is such a graph. Therefore it is enough to prove that  $H_1(E_{\mathcal{A}}, \mathbb{Z}) = V(\Gamma(C_{\mathcal{A}}))$ . Recall that  $\Gamma(C_{\mathcal{A}})$  is the graph structure of the total blow-up boundary manifold  $B_{\widehat{\mathcal{A}}^{\text{max}}}$ , which by [Theorem 2.3.12](#) on page 60 is homeomorphic to  $B_{\mathcal{A}}$ . By [Definition 1.5.14](#),  $V(\tilde{\Gamma}(C_{\mathcal{A}}))$  is a free abelian module generated by one meridian curve for each vertex of the graph. In the case of  $B_{\mathcal{A}}$  these are the meridians  $\mu_0, \mu_1, \dots, \mu_n$  of [Definition 1.5.6](#) on page 38 corresponding to the line-vertices  $L_i \in \mathcal{L}$ , but also the meridians  $\mu_P$  for every  $P \in \mathcal{Q}$ , where  $\mu_P$  is a fibre curve inside the local boundary manifold  $S_P$  around  $P$ . The Euler number values of each vertex were given by [Theorem 2.3.15](#) on page 61. In the total blow-up  $\widehat{\mathcal{A}}^{\text{max}}$ , the blow-up number  $b(L_i)$  of a line  $L_i \in \mathcal{L}$  is equal to its multiplicity  $m(L_i)$ . Using the neighbour sets induced by the [Definition 2.2.11](#) on page 55 of  $\Gamma(C_{\mathcal{A}})$ , the relations of  $V(\Gamma(C_{\mathcal{A}}))$  become

$$\begin{aligned} \forall L_i \in \mathcal{L} : \quad & \sum_{P \in V_i} \mu_P = (m(L_i) - 1) \cdot \mu_i \\ \forall P \in \mathcal{Q} : \quad & \sum_{L_i \in V_P} \mu_i = \mu_P \end{aligned}$$

Fix  $L_i \in \mathcal{L}$ . Replacing  $\mu_P$  in the first relation for every  $P \in V_i$  yields:

$$(m(L_i) - 1) \cdot \mu_i = \sum_{P \in V_i} \sum_{L_k \in V_P} \mu_k = m(L_i) \cdot \mu_i + \sum_{P \in V_i} \sum_{\substack{L_k \in V_P \\ k \neq i}} \mu_k$$

since  $L_i \in V_P$  for every  $P \in V_i$  and  $m(L_i) = \#V_i$ . Thus we get:

$$0 = \mu_i + \sum_{P \in V_i} \sum_{\substack{L_k \in V_P \\ k \neq i}} \mu_k$$

But  $L_i$  meets every other line  $L_j \in \mathcal{L}$  exactly once. In other words:

$$\mathcal{L} = \{L_i\} \sqcup \bigsqcup_{P \in V_i} \{L_k \in \mathcal{L} \mid L_k \in V_P, k \neq i\}$$

For every  $L_i \in \mathcal{L}$ , the previous relation thus becomes

$$\sum_{L_j \in \mathcal{L}} \mu_j = 0$$

Since  $\mathcal{Q} = \bigcup_{L_i \in \mathcal{L}} V_i$ , we have replaced all the meridians  $\mu_P$  for  $P \in \mathcal{Q}$ , leaving only the  $\mu_i$  for  $L_i \in \mathcal{L}$  as generators. This simplification of the presentation of  $V(\Gamma(C_{\mathcal{A}}))$  thus gives exactly the presentation of  $H_1(E_{\mathcal{A}}, \mathbb{Z})$  given by [Corollary 2.4.21](#).  $\square$

## 2.5 Homology inclusion map

Let  $\mathcal{A}$  a non-exceptional line arrangement in  $\mathbb{C}\mathbb{P}^2$  and  $\Gamma := \tilde{\Gamma}(C_{\mathcal{A}})$  its reduced incidence graph. Let  $B_{\mathcal{A}}$  be the boundary manifold, as given in [Definition 2.3.1](#) on page 59. As per [Theorem 2.3.17](#) on page 61,  $B_{\mathcal{A}}$  is a graph manifold whose unique minimal graph structure is given by  $\Gamma$ . Order the arrangement  $\mathcal{A}$  with  $\omega$  and let  $\Omega^\theta$  be the reduced ordering on  $\Gamma$  induced by  $\theta$  as per [Definition 2.2.15](#) and [Proposition 2.2.16](#) on page 55 and on page 56.

Consider the inclusion map

$$i_{\mathcal{A}} : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$$

and the induced map on the first homology groups

$$i_{\mathcal{A}}^* : H_1(B_{\mathcal{A}}, \mathbb{Z}) \longrightarrow H_1(E_{\mathcal{A}}, \mathbb{Z})$$

By [Theorem 1.5.16](#) on page 41, every graphed embedding  $\gamma \in E_{\Gamma}(\Omega^\theta)$  induces an isomorphism

$$\gamma_* : V(\Gamma) \oplus H_1(\Gamma) \xrightarrow{\sim} H_1(B_{\mathcal{A}}, \mathbb{Z})$$

where the image of the subgroup  $V(\Gamma)$  is generated by the boundary meridian curves from [Definition 1.5.6](#) on page 38. Separately, by [Corollary 2.4.21](#) the homology of the exterior manifold  $E_{\mathcal{A}}$  is generated by the exterior meridian curves defined in [Section 2.4.3](#) and there is a natural group isomorphism

$$H_1(E_{\mathcal{A}}, \mathbb{Z}) \simeq V(\Gamma)$$

Therefore, we have a map

$$i_{\mathcal{A}}^* \circ \gamma_* : V(\Gamma) \oplus H_1(\Gamma) \longrightarrow V(\Gamma)$$

**Lemma 2.5.1.** *For every graphed embedding  $\gamma \in E_{\Gamma}(\Omega^\theta)$ , we have*

$$(i_{\mathcal{A}}^* \circ \gamma_*)|_{V(\Gamma)} = \text{Id}_{V(\Gamma)} \quad \triangleleft$$

*Proof.* By construction, for every line  $L_i$  of  $\mathcal{A}$  both the boundary meridian curve  $\mu_i$  and the exterior meridian curve  $\mu'_i$  have the same homology class as the boundary  $\partial_i D$  of a disc transverse to  $L_i$ .  $\square$

Now consider the restriction

$$(i_{\mathcal{A}}^* \circ \gamma_*)|_{H_1(\Gamma)} \in \text{Hom}(H_1(\Gamma), V(\Gamma))$$

which we simply write  $i_{\mathcal{A}}^* \circ \gamma_*$  for short. Remember the graph stabiliser  $\mathcal{G}_{\Gamma}$  from [Definition 1.6.1](#) on page 42, which by [Corollary 1.6.12](#) on page 45 only depends on the graph  $\Gamma$ . [Lemma 2.5.1](#) ensures that the class

$$\mathcal{J}_{\Omega^\theta}(\mathcal{A}) := |i_{\mathcal{A}}^* \circ \gamma_*| \in \mathcal{G}_{\Gamma}$$

is well-defined and by construction does not depend on the choice of  $\gamma \in E_{\Gamma}(\Omega^\theta)$ . Our main result states that this class element is a topological invariant of ordered line arrangements.

**Theorem 2.5.2.** *Let  $\mathcal{A}, \mathcal{A}' \subset \mathbb{C}\mathbb{P}^2$  be two non-exceptional line arrangements with the same combinatorics  $C$ . Endow  $\mathcal{A}$  and  $\mathcal{A}'$  with the same ordering  $\theta$  on their set of lines. If  $(\mathcal{A}, \theta)$  and  $(\mathcal{A}', \theta)$  are topologically equivalent then  $\mathcal{J}_{\Omega^\theta}(\mathcal{A}) = \mathcal{J}_{\Omega^\theta}(\mathcal{A}')$  inside the graph stabiliser  $\mathcal{G}_{\Gamma}$ .  $\triangleleft$*

*Proof.* The topological equivalence between  $(\mathcal{A}, \theta)$  and  $(\mathcal{A}', \theta)$  induces homeomorphisms  $\Psi$  and  $\Phi$  such that the following diagram commutes:

$$\begin{array}{ccc} B_{\mathcal{A}'} & \xrightarrow[\sim]{\Psi} & B_{\mathcal{A}} \\ i_{\mathcal{A}'} \downarrow & & \downarrow i_{\mathcal{A}} \\ E_{\mathcal{A}'} & \xrightarrow[\Phi]{\sim} & E_{\mathcal{A}} \end{array}$$

Write  $B = B_{\mathcal{A}'} \simeq B_{\mathcal{A}}$ . As described in the proof of [Theorem 2.3.19](#) on page 61, one can always suppose that  $\Psi \in \text{Homeo}_{\Gamma}^{++}(B)$ . Let  $\Omega$  be a graph ordering on  $\Gamma$  and let  $\gamma \in E_{\Gamma}(\Omega^{\theta})$  be an ordered graph embedding. By [Theorem 1.4.9](#) on page 37, the image  $\Psi(\gamma)$  is again an element of  $E_{\Gamma}(\Omega^{\theta})$ . The map  $\Psi$  induces a group automorphism  $\Psi^* : H_1(B, \mathbb{Z}) \rightarrow H_1(B, \mathbb{Z})$ . By construction we have  $\widetilde{\Psi(\gamma)} = \Psi^* \circ \gamma_*$  and  $(i_{\mathcal{A}} \circ \Psi)^* = i_{\mathcal{A}}^* \circ \Psi^*$ . Since the line arrangement equivalence is ordered, the map  $\Phi$  induces the identity map on  $H_1(E_{\mathcal{A}}, \mathbb{Z}) \simeq H_1(E_{\mathcal{A}'}, \mathbb{Z}) \simeq V(\Gamma)$  generated by the meridians. Then inside  $\text{Hom}(H_1(\Gamma), V(\Gamma))$  we have

$$i_{\mathcal{A}'}^* \circ \gamma_* = i_{\mathcal{A}}^* \circ \Psi^* \circ \gamma_* = i_{\mathcal{A}}^* \circ \widetilde{\Psi(\gamma)}$$

By [Definition 1.6.1](#) on page 42 of the graph stabiliser  $\mathcal{G}_{\Gamma}$ , this implies that inside the quotient

$$0 = \left| i_{\mathcal{A}}^* \circ \left( \widetilde{\Psi(\gamma)} - \gamma_* \right) \right| = |i_{\mathcal{A}'}^* \circ \gamma_*| - |i_{\mathcal{A}}^* \circ \gamma_*| = \mathcal{J}_{\Omega^{\theta}}(\mathcal{A}') - \mathcal{J}_{\Omega^{\theta}}(\mathcal{A}) \quad \square$$

**Remark 2.5.3.** By definition of the graph stabiliser, the class  $|i_{\mathcal{A}}^* \circ \gamma_*| \in \mathcal{G}_{\Gamma}$  does not depend on the graphed embedding  $\gamma_* \in E_{\Gamma}(\Omega^{\theta})$ . However, it does depend on the graph ordering  $\Omega^{\theta}$  and thus on the ordering on the combinatorics  $\theta$ . The homology inclusion is therefore an *ordered* line arrangement invariant. However, some combinatorics have trivial automorphism groups (see [Definition 2.2.5](#) on page 54). The restriction of the homology inclusion to this subclass of line arrangements becomes an unordered topological invariant.  $\square$

**Remark 2.5.4.** The graph stabiliser does not quotient the homology differences caused by the application of the complex conjugation inside  $\mathbb{C}\mathbb{P}^2$ , since it is not a positive homeomorphism of the boundary manifold. A line arrangement  $\mathcal{A}$  and its conjugate  $\overline{\mathcal{A}}$  might thus have different homology inclusion values. If one imposes the same orientation on both arrangements, then  $(\mathcal{A}, \overline{\mathcal{A}})$  becomes an *oriented* Zariski pair.  $\square$

# CHAPTER

# 3

## COMPUTATIONS OF THE HOMOLOGY INCLUSION

### Outline of the current chapter

<b>3.1 Standard ordered graphed embedding</b>	<b>71</b>
3.1.1 Combinatorial data	72
3.1.2 Boundary manifold and braid monodromy	72
3.1.3 Construction of the standard graphed embedding	75
<b>3.2 Values of the homology inclusion map</b>	<b>76</b>
3.2.1 Braid linking	76
3.2.2 Main computation theorem	80
<b>3.3 Comparison of a combinatorial pair</b>	<b>82</b>
3.3.1 Adjustments of the orderings	82
3.3.2 Comparison algorithm	84
3.3.3 Examples of Zariski pairs	85

Our objective is to compute the homology inclusion invariant map of a non-exceptional line arrangement  $\mathcal{A}$ . To achieve this, we explain in [Section 3.1](#) how to build a graph embedding  $\gamma_{\mathbf{B}} \in E_{\mathcal{T}}(\Omega)$  using a standard geometrical method on a representative  $\mathbf{B}$  of the braid monodromy of  $\mathcal{A}$ . We then use this graph embedding in [Section 3.2](#) to compute the value of the morphism  $i_{\mathcal{A}}^* \circ \gamma_{\mathbf{B}}^*$  of the embedded graph cycles using a simple tool called the braid linking. Finally in [Section 3.3](#) we explain how to modify the braid monodromies of two line arrangements  $\mathcal{A}, \mathcal{A}'$  with the same combinatorics  $C$  in order to associate them with a common graph ordering  $\Omega$  and spanning tree  $\mathcal{T}$ . The homology inclusion maps can then be computed in the same basis of the graph stabiliser module, and [Theorem 2.5.2](#) allows us to compare them to determine if the pair  $(\mathcal{A}, \mathcal{A}')$  is Zariski. We then describe the final algorithm written in [Sage \[Sag23\]](#) that sums up all these computations. Finally, we give our main examples of both known and new Zariski pairs, which were obtained in collaboration with B. Guerville-Ballé.

### 3.1 Standard ordered graphed embedding

Let  $\mathcal{A}$  be a non-exceptional line arrangement with combinatorics  $C_{\mathcal{A}}$ . As explained in [Section 2.3.3](#), the boundary manifold  $B_{\mathcal{A}}$  has several graph structures. The unique minimal one is given by the

reduced incidence graph  $\tilde{\Gamma}(C_{\mathcal{A}})$ , and another non-minimal one is given by the complete incidence graph  $\Gamma(C_{\mathcal{A}})$ .

Our objective is to build a *standard graphed embedding* of the full incidence graph  $\Gamma := \Gamma(C_{\mathcal{A}})$  using a representative of the braid monodromy of  $\mathcal{A}$  as defined in [Definition 2.4.8](#) on page 64. This graphed embedding is designed specifically to allow an algorithmic computation of the homology inclusion using only the braid monodromy representative as base data.

### 3.1.1 Combinatorial data

We reuse all notations from [Section 2.4](#) which we recall briefly. Most of them are also summed up on [Figure 2.4.1](#) on page 65. Fix a component  $L_0 \in \mathcal{A}$  as the line at infinity. We work with the affine part  $\mathcal{A}^{\text{aff}} \simeq \mathcal{A} \setminus L_0$  in the standard affine chart  $\mathbb{C}^2 \equiv \{[x : y : 1] \in \mathbb{C}\mathbb{P}^2\}$ . We use the sets  $\mathcal{L}^* = \mathcal{L} \setminus \{L_0\}$  and  $\mathcal{Q}^* = \mathcal{Q} \setminus V_{L_0}$ . The set  $\mathcal{L}^*$  is ordered by  $\theta$ .

Let  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a projection. We choose coordinates  $(x, y)$  on the standard affine chart such that  $\pi(x, y) = x$ . To simplify the descriptions, we also suppose that  $\pi$  is *generic* as in [Definition 2.4.1](#) on page 62 which means that  $\pi(\mathcal{Q}^*) \simeq \mathcal{Q}^*$ . In practice one can always adjust the projection  $\pi$  to make it generic. We write  $r := \#\mathcal{Q}^*$  and  $\mathbb{C}_{\mathcal{A}} := \mathbb{C} \setminus \pi(\mathcal{Q}^*)$ . For every point  $x \in \mathbb{C}_{\mathcal{A}}$ , the strand corresponding to  $L_i$  is  $w_{L_i}(x) = \pi^{-1}(x) \cap L_i$ . Let  $D(x)$  be a closed disc in  $\pi^{-1}(x)$  containing all the intersections  $D_{L_i}(x) := \pi^{-1}(x) \cap N_{L_i}$  for every  $L_i \in \mathcal{L}^*$ . Define

$$\Sigma(x) := D(x) \setminus \bigcup_{L_i \in \mathcal{L}^*} \widehat{D_{L_i}(x)}$$

Let  $D$  be a closed disc in  $\mathbb{C}$  containing all the images  $\pi(N_P)$  for  $P \in \mathcal{Q}^*$ . Up to isomorphism,  $D(x)$  can be chosen to have the same shape for all values of  $x \in D$ . We define

$$\Sigma := D \setminus \bigcup_{P \in \mathcal{Q}^*} \widehat{\pi(N_P)} \subset \mathbb{C}_{\mathcal{A}}$$

The projection  $\pi$  induces its own ordering  $\nu_{\pi} : \mathcal{Q}^* \rightarrow \{1, \dots, r\}$ . Let  $b^{\infty} \in \partial D$  be a base point and let  $\alpha \in \mathcal{S}_{r, \pi}$  be a star on  $\Sigma$  with respect to the permutation  $\nu \circ \nu_{\pi}^{-1}$ , where  $\nu$  is any ordering on  $\pi(\mathcal{Q}^*)$ .

These data allow to build a *representative of the braid monodromy*

$$B_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \leq l \leq r}$$

as explained in [Definition 2.4.12](#) on page 65.

### 3.1.2 Boundary manifold and braid monodromy

Before giving the definition of the standard graphed embedding we need to identify the graph structure of the boundary manifold  $B_{\mathcal{A}}$  associated with the complete incidence graph  $\Gamma(C_{\mathcal{A}})$  within the same construction of the braid monodromy representative we have fixed. To achieve this, we first need to identify the local regular neighbourhoods  $N_{L_i}$  and  $N_{P_l}$  for each  $L_i \in \mathcal{L}^*$  and  $P_l \in \mathcal{Q}^*$ , and we will then consider their boundaries. The cases of the line at infinity  $L_0$  and the singular points it meets must be treated separately.

Let  $V_{L_i}^* := V_{L_i} \setminus \{L_i \cap L_0\}$ . For every  $L_i \in \mathcal{L}^*$ , let

$$\Sigma^{L_i} := D \setminus \bigcup_{P_l \in V_{L_i}^*} \widehat{\pi(N_{P_l})}$$

Note that we have  $\Sigma \subset \Sigma^{L_i}$ . Separately, for every  $P_l \in \mathcal{Q}^*$  and every  $x \in \Sigma$ , let

$$\Sigma^{P_l}(x) := \Sigma(x) \cap N_{P_l}$$

By construction,  $\Sigma^{P_l}(x)$  is a subsurface of  $\Sigma(x)$  that contains only the boundary components

$\partial D_{L_i}(x)$  for  $L_i \in V_{P_l}$ . Recall that

$$S_{L_i}^{\text{aff}} = \partial N_{L_i}^{\text{aff}} \setminus \bigcup_{P_l \in V_{L_i}^*} \widehat{N}_{P_l} \quad S_{P_l} = \partial N_{P_l} \setminus \bigcup_{L_i \in V_{P_l}} \widehat{N}_{L_i}^{\text{aff}}$$

**Proposition 3.1.1.** *There are homeomorphisms*

$$\begin{aligned} S_{L_i}^{\text{aff}} &\simeq \bigcup_{x \in \Sigma^{L_i}} \partial D_{L_i}(x) \quad \cup \quad \bigcup_{x \in \partial D^0} D_{L_i}(x) \\ S_{P_l} &\simeq \bigcup_{x \in \partial \pi(N_{P_l})} \Sigma^{P_l}(x) \quad \cup \quad \bigcup_{x \in \pi(N_{P_l})} \partial^\infty \Sigma^{P_l}(x) \end{aligned}$$

for each  $L_i \in \mathcal{L}^*$  and  $P_l \in \mathcal{Q}^*$ .  $\triangleleft$

Both manifold types are divided between a trivial circle bundle which contains all boundary components, and a solid torus. The fibres of  $S_{P_l}$  are *horizontal* circles corresponding to the intersection of  $\{\Sigma^{P_l}(x) \mid x \in \partial \pi(N_{P_l})\}$  with an horizontal section of  $\pi$ . A section of  $S_{P_l}$  is simply  $\Sigma^{P_l}(x)$  for any choice of a  $x \in \partial \pi(N_{P_l})$ . Remember that we only consider sections of the sub-bundle that contains the boundary. This structure of  $S_{P_l}$  is represented schematically on **Figure 3.1.1b** on the next page. The situation is similar for  $S_{L_i}^{\text{aff}}$ . The fibres of  $S_{L_i}^{\text{aff}}$  are *vertical* circles  $\partial D_{L_i}(x)$  for  $x \in \Sigma^{L_i}$ . The intersection of a horizontal section of  $\pi$  with those vertical circles gives a partial section of  $S_{L_i}^{\text{aff}}$  in the sense that it lacks the part at infinity. Again we ignore the second solid torus. This structure of  $S_{L_i}^{\text{aff}}$  is represented schematically on **Figure 3.1.1a** on the following page.

*Proof of Proposition 3.1.1.* Let  $D^{P_l}(x)$  be the disc obtained by filling in the  $\partial D_{L_i}$  boundary components of  $\Sigma^{P_l}(x)$ . There are natural homeomorphisms

$$N_{L_i}^{\text{aff}} \simeq \bigcup_{x \in D} D_{L_i}(x) \quad N_{P_l} \simeq \bigcup_{x \in \pi(N_{P_l})} D^{P_l}(x)$$

Consider the boundaries

$$\begin{aligned} \partial N_{L_i}^{\text{aff}} &\simeq \bigcup_{x \in D} \partial D_{L_i}(x) \quad \cup \quad \bigcup_{x \in \partial D} D_{L_i}(x) \\ \partial N_{P_l} &\simeq \bigcup_{x \in \partial \pi(N_{P_l})} D^{P_l}(x) \quad \cup \quad \bigcup_{x \in \pi(N_{P_l})} \partial D^{P_l}(x) \end{aligned}$$

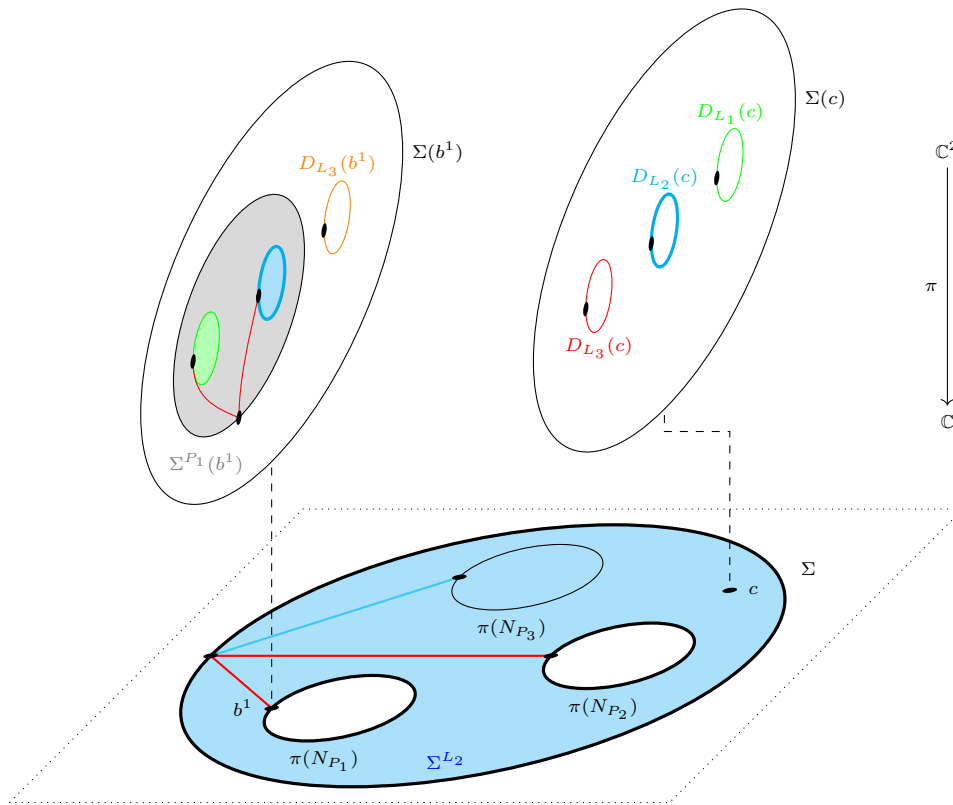
As expected, both boundaries are homeomorphic to  $S^3$  as the reunion of two solid tori. Suppose that  $P_l \in L_i$ . Then

$$T_{P_l, L_i} \simeq \partial N_{P_l} \cap \partial N_{L_i}^{\text{aff}} = \bigcup_{x \in \pi(N_{P_l})} \partial D_{L_i}(x)$$

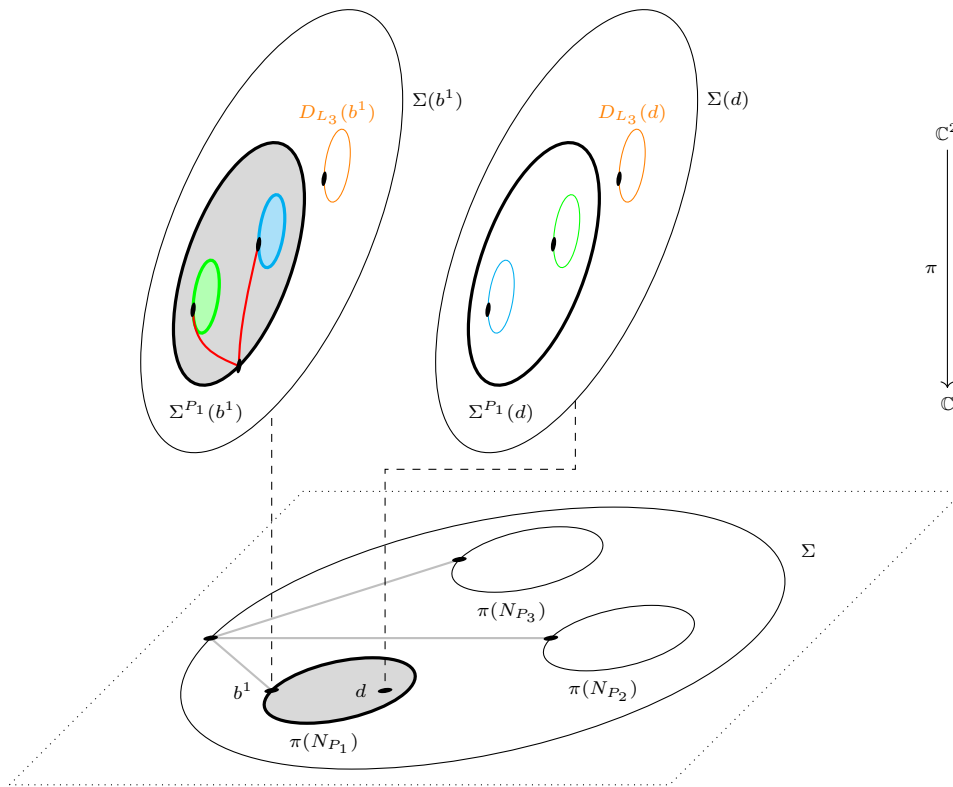
is the joining torus corresponding to the edge  $e_{L_i, P_l}$  for the  $\widetilde{\Gamma}(C_{\mathcal{A}})$  graph structure. Removing the joining tori from the boundaries  $\partial N_{L_i}^{\text{aff}}$  and  $\partial N_{P_l}$  give the expected description of  $S_{L_i}^{\text{aff}}$  and  $S_{P_l}$ .  $\square$

To complete  $S_{L_i}^{\text{aff}}$  into  $S_{L_i}$  we must now attend the situation at infinity. This is done by switching to a new chart called the  $\infty$ -chart, as described in the proof of **Proposition 2.4.13** on page 65. Denote by  $(Y, Z)$  the coordinates of the  $\infty$ -chart. The exterior  $D^\infty := \mathbb{C}\mathbb{P}^1 \setminus \mathring{D}$  is a 2-disc in the  $\infty$ -chart. The line  $L_0$  corresponds to  $\{Z = 0\}$  and the projection becomes  $\pi : (Y, Z) \rightarrow (0, Z)$ . This means that  $L_0$  is now a vertical line for  $\pi$ , and all singular points of  $V_{L_0}$  are all sent to a same point  $x_0 = (Y = 0, Z = 0)$ , and thus the extension of  $\pi$  to the  $\infty$ -chart is not generic.

**Remark 3.1.2.** The *braid at infinity* associated to the representative is the lifted braid over  $\partial D^\infty$  in the  $\infty$ -chart. But by **Proposition 2.4.13** it can also be computed from the product of all other braids of the representative. This means that in practice it is not necessary to consider direct transformations of the graph structure near  $L_0$  since they are automatically induced by the transformations in the standard chart.  $\square$



(a) Structure of  $S_{L_2}^{\text{aff}}$  with  $P_1, P_2 \in L_1$



(b) Structure of  $S_{P_1}$  with  $P_1 = L_1 \cap L_2$

Figure 3.1.1: Local boundary manifolds

We now describe the intersection of the boundary manifold with the  $\infty$ -chart. Let

$$N_{L_i}^\infty := N_{L_i} \cap \pi^{-1}(D^\infty)$$

Extending the notations of the standard chart, for every  $L_i \in \mathcal{L}^*$ ,  $P_l \in V_{L_0}$  and  $Y \in D^\infty$  we define  $D(Y)$ ,  $D_{L_i}(Y)$ ,  $\Sigma(Y)$  and  $\Sigma^{P_l}(Y)$ . The situation is different for the basis of the projection since  $\pi$  is no longer generic. By construction, for every  $P \in V_{L_0}$  we have  $N_P \setminus N_{L_0} \neq \emptyset$ . This means that up to homeomorphism we can suppose that  $\pi(N_{L_0})$  and all  $\pi(N_P)$  for all  $P \in V_{L_0}$  are a common disc  $D^0$ . We also define

$$\Sigma^\infty := D^\infty \setminus \widehat{D^0} \quad \Sigma^0 := D^0 \setminus \widehat{\pi(N_{L_0})}$$

**Proposition 3.1.3.** *There are homeomorphisms*

$$\begin{aligned} S_{L_i}^\infty &\simeq \bigcup_{x \in \Sigma^\infty} \partial D_{L_i}(Y) \cup \bigcup_{x \in \partial D^\infty} D_{L_i}(Y) \\ S_P &\simeq \bigcup_{x \in \partial D^0} D^P(Y) \end{aligned}$$

for every  $P \in V_{L_0}$  and every  $L_i \in \mathcal{L}^*$ .  $\triangleleft$

The structures of  $S_{L_i}^\infty$  and  $S_P$  are similar to their respective counterparts in the standard chart. However, for  $S_P$  the decomposition contains another horizontal torus  $\{\partial D^P(Y) \mid Y \in \partial D^0\}$  which corresponds to the joining torus with  $S_{L_0}$ . A section of  $S_P$  with  $P \in L_0$  is thus composed of the vertical surface  $\Sigma(Y)$  for  $Y \in \partial D^0$ , but unlike in the standard chart the additional boundary torus do not correspond to the Euler gluing map.

The two parts  $S_{L_i}^\infty$  and  $S_{L_i}^{\text{aff}}$  reconnect to form  $S_{L_i}$  along the torus

$$T_{L_i}^\infty := \bigcup_{x \in \partial D^\infty} \partial D_{L_i}(x)$$

A complete section of  $S_{L_i}$  is therefore determined by the choice of a point on  $D_{L_i}(x)$  and  $D_{L_i}(Y)$  for every  $x \in \Sigma^{L_i}$  and every  $Y \in \Sigma^\infty$ .

Finally, the circle bundle  $S_{L_0}$  is a special case since  $L_0^{\text{aff}}$  is vertical.

**Proposition 3.1.4.** *There is an homeomorphism*

$$S_{L_0}^{\text{aff}} \simeq \bigcup_{x \in \partial D^0} \left( \pi^{-1}(Y) \setminus \bigcup_{P \in V_{L_0}} D^P(Y) \right) \quad \triangleleft$$

A partial section of  $S_{L_0}^{\text{aff}}$  is thus the vertical surface

$$D_{L_0}(Y) := D(Y) \setminus \bigcup_{P \in V_{L_0}} D^P(Y)$$

for the choice of a point  $Y \in \partial \pi(N_{L_0})$ .

### 3.1.3 Construction of the standard graphed embedding

As explained in [Definition 1.4.4](#) on page 35, a graphed embedding is made of ordered stars drawn on sections of each of the circle bundles  $S_L$  and  $S_P$  for every  $L \in \mathcal{L}$  and every  $P \in \mathcal{Q}$ . Using the description of the graph structure of  $B_{\mathcal{A}}$  established in [Section 3.1.2](#), we can now explain how to build the standard graphed embedding associated with the representative  $\mathbf{B} = B_{\mathcal{A}}(\pi, \theta, \nu, \alpha)$  of the braid monodromy of the line arrangement  $\mathcal{A}$ .

A graphed embedding also depends on the choice of a graph ordering that must be fixed. Extend the ordering  $\theta$  defined on  $\mathcal{L}^*$  to  $\mathcal{L}$  by assigning 0 to  $L_0$ . Then [Definition 2.2.15](#) on page 55 allows to define two graph sub-orderings  $\Omega_{\mathcal{Q}}^\theta$  and  $\Omega_{\mathcal{L}^*}^\nu$  on  $\Gamma$ . The only neighbour set remaining to be ordered is  $V_{L_0}$ , which can be done using the (extended)  $\theta$ -lexicographic ordering. We thus get a full graph ordering  $\Omega^{\theta, \nu}$  of  $\Gamma(C_{\mathcal{A}})$ .



Recall that  $\alpha \in \mathcal{S}_r(\nu \circ \nu_\pi^{-1})$  is a star drawn on  $\Sigma \subset \mathbb{C}$  with  $b^\infty \in \partial^\infty \Sigma$  as its base point. Extend  $\alpha$  to  $\Sigma^\infty$  by adding a final branch  $\alpha^0$  joining  $b^\infty$  to  $\partial D^0$  to get the new star  $\tilde{\alpha} \in \mathcal{S}_{n+1}$ . We first need to adjust the local geometrical orderings  $\theta^l$  for each  $P_l \in \mathcal{Q}^*$  so that it coincides with the graph ordering  $\Omega^{\theta, \nu}$ .

**Lemma 3.1.5.** *For every  $P_l \in \mathcal{Q}^*$ , one can always make the sub-orderings*

$$\theta, \theta^l : V_{P_l} \longrightarrow \{1, \dots, m(P_l)\}$$

*coincide without changing the representative  $\mathbf{B}$ .*  $\triangleleft$

*Proof.* Let  $\sigma_l \in \mathfrak{S}_n$  be the permutation such that  $\sigma_l \circ \theta^l(V_{P_l}) = \theta(V_{P_l})$  and leave all other indices unchanged. Let  $\chi_l \in \mathbb{B}_n$  such that  $\sigma(\chi_l) = \sigma_l$ . Then the braid  $\chi_l$  commute with  $\Delta_{\theta^l(V_{P_l})}^2$ . We can thus write

$$\beta_l = \tau_l \cdot \Delta_{\theta^l(V_{P_l})}^2 \cdot (\tau_l)^{-1} = (\tau_l \cdot \chi_l) \cdot \Delta_{\theta(V_{P_l})}^2 \cdot (\tau_l \cdot \chi_l)^{-1}$$

Redefining the shift braid  $\tau_l$  as  $\tau_l \cdot \chi_l$  gives the intended result without changing the braid  $\beta_l$ .  $\square$

Now we can describe the pair section-ordered star that compose the standard graph ordering for each kind of vertices:

- For every  $L_i \in \mathcal{L}^*$ , consider the section  $s_{L_i}$  of  $S_{L_i}$  defined as such: for every  $x \in \Sigma_{L_i}$ ,  $s_{L_i}(x)$  is the point of  $\partial D_{L_i}(x)$  that minimises the value of  $\operatorname{Re}(y)$ . The section is completed by using the same definition on  $\Sigma^\infty$  in the  $\infty$ -chart. Now consider the ordered star  $\alpha_{L_i} \in \mathcal{O}_{m(L_i)}$  drawn on  $s_{L_i}$  and defined by  $\alpha_{L_i} := \pi^{-1}(\tilde{\alpha}) \cap s_{L_i}$ .
- For every  $P_l \in \mathcal{Q}^*$ , consider the section  $s_{P_l}$  of  $S_{P_l}$  defined as the vertical surface  $\Sigma^{P_l}(b^l)$ , where  $b^l := \partial_+ \alpha_l$ . Let  $\alpha_{P_l} \in \mathcal{O}_{m(P_l)}$  be an ordered star drawn on  $s_{P_l}$  that joins a base point to each of the points  $s_{P_l} \cap s_{L_i}$  for every  $L_i \in V_{P_l}$ .
- For  $L_0$ , consider the section  $s_{L_0}$  of  $S_{L_0}$  defined as the vertical surface  $\Sigma(b^0)$  where  $b^0 = \partial_+ \alpha^0$ . Let  $\alpha_{L_0} \in \mathcal{O}_{m(L_0)}$  be a star on  $s_{L_0}$ .
- For every  $P_l \in V_{L_0}$ , consider the section  $s_{P_l}$  of  $S_{P_l}$  defined as the surface  $\Sigma^{P_l}(b^0)$ , where  $b^0 := \partial_+ \alpha_0$ . Let  $\alpha_{P_l} \in \mathcal{O}_{m(P_l)}$  be an ordered star drawn on  $\Sigma^{P_l}(b^0)$  that joins a base point to each of the points  $s_{P_l} \cap s_{L_i}$  for every  $L_i \in V_{P_l}$ . Note that the centre of  $\alpha_{P_l}$  is *not* on  $\partial D^{P_l}$  as usual but must be placed in the interior of the section.

**Definition 3.1.6.** The *standard ordered graphed embedding*  $\gamma_{\mathbf{B}} \in \mathbf{E}_{\Gamma(C_{\mathcal{A}})}(\Omega^{\theta, \nu})$  is defined as the reunion of the ordered stars  $\alpha_{L_i} \in \mathcal{O}_{m_i}$  and  $\alpha_{P_l} \in \mathcal{O}_{m_l}$  described above for every  $L_i \in \mathcal{L}$  and  $P_l \in \mathcal{Q}$ .  $\diamond$

**Remark 3.1.7.** By construction, the image by  $\gamma_{\mathbf{B}}$  of the half-edge  $\vec{e}_{L_i, P_l}$  going from  $L_i \in \mathcal{L}^*$  to  $P_l \in V_{L_i}^*$  is homotopic to the  $i$ -th strand of the shift braid  $\tau_l$  of the representative  $\mathbf{B}$  (see [Definition 2.4.11](#) on page 64).  $\boxminus$

**Remark 3.1.8.** Since  $\pi$  is generic and therefore  $\pi(\mathcal{Q}^*) = \mathcal{Q}^*$ , the ordering  $\nu$  could have been fixed as the  $\theta$ -lexicographic ordering. We made this choice for the sake of simplicity in [Theorem 2.5.2](#) on page 69 which defines the homology inclusion of  $\mathcal{A}$ . The reasons to keep  $\nu$  as a separate value for the explicit computation are explained in [Section 3.3.1](#).  $\boxminus$

## 3.2 Values of the homology inclusion map

The specific geometrical construction of the standard graphed embedding  $\gamma_{\mathbf{B}}$  allows to compute the morphism  $i_{\mathcal{A}}^* \circ \gamma_{\mathbf{B}}^*$ . This reduces to the computation of linking numbers between strands of the shift braids extracted from the representative  $\mathbf{B}$  of the braid monodromy of the line arrangement  $\mathcal{A}$ .

### 3.2.1 Braid linking

The braid linking is a linking number computed between two strands of a braid according to specific rules.

Let  $\mathbf{y} \subset \mathbb{C}$  be a set of  $n$  disjoint points ordered by their descending horizontal coordinates. Recall from [Proposition 2.4.4](#) on page 63 that a braid  $\beta \in \mathbb{B}_n$  can be seen as the homotopy class of  $n$  non-crossing paths  $\gamma_1, \dots, \gamma_n$  inside  $\mathbb{C} \times [0, 1]$  that start and end at  $\mathbf{y}$ . We write  $\mathbb{C}(t) := \mathbb{C} \times \{t\}$ , where the parameter  $t$  is called the *height*. Fix a braid  $\beta := \{\gamma_1, \dots, \gamma_n\} \in \mathbb{B}_n$  and a braid decomposition

$$\beta = \prod_{s=1}^m \sigma_{i_s}$$

There is a set of heights  $H = \{t_1, \dots, t_s\}$  where two points  $\gamma_i(t)$  and  $\gamma_j(t)$  have the same horizontal coordinate in  $\mathbb{C}(t)$ . For every  $t \in [0, 1] \setminus H$ , the ordering

$$\omega_\beta^t : \{\gamma_1(t), \dots, \gamma_n(t)\} \longrightarrow \{1, \dots, n\}$$

corresponds to the decreasing horizontal coordinates of the points in  $\mathbb{C}(t)$ . The index  $i$  of a strand  $\gamma_i$  is therefore the order of its starting point inside  $\mathbb{C}(0)$ . By convention we suppose that every elementary braid  $\sigma_{i_s}$  that permutes two consecutive strands corresponds to a positive rotation of their projection points in the base plane.

Let  $1 \leq i < j \leq n$  be two fixed indices. For every  $t \in [0, 1]$ , let  $D_i(t) \subset \mathbb{C}(t)$  be a small disc such that  $\gamma_i(t)$  is the point of  $D_i(t)$  with the minimum horizontal coordinate. Finally, for every  $t \in [0, 1] \setminus H$ , let  $S_{i,j}(\beta, t)$  be the path drawn on  $\mathbb{C}(t)$  as shown on [Figure 3.2.1](#). Note that the *left support strand* and the *right support strand* of the path  $S_{i,j}(\beta, t)$  play non-symmetric roles. When the left and right support strands reverse their position, the modification of the path  $S_{i,j}(\beta, t)$  is shown on [Figure 3.2.3](#) on the next page. This is called a *reversal crossing*. Every other permutation of two strands involving only *one* of the support strands is called a *regular crossing*.

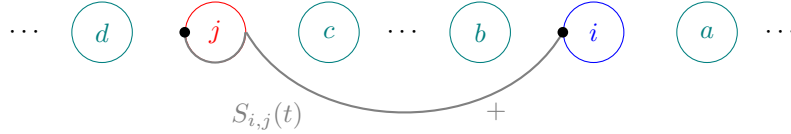


Figure 3.2.1: Section of the  $(i, j)$ -braid linking surface

**Definition 3.2.1.** The reunion

$$S_{i,j}(\beta) := \bigcup_{t \in [0,1]} S_{i,j}(\beta, t)$$

is an orientable surface within  $\mathbb{C} \times [0, 1]$  called the  $(i, j)$ -*braid linking surface* of the braid  $\beta$ .  $\diamond$

By convention,  $S_{i,j}(\beta)$  is oriented positively on the side that face the lowest vertical coordinates in the section  $\mathbb{C}(0)$ . At every reversal crossing, the positive and negative sides switch their positions, as illustrated on [Figure 3.2.3](#) on the following page.

For every height  $t \in [0, 1]$ , define

$$\mathbb{C}(\beta, t) := \mathbb{C}(t) \setminus \bigcup_{i=1}^n D_i(t)$$

and  $\mathbb{C}(\beta) := \bigcup_{t \in [0,1]} \mathbb{C}(\beta, t)$ . For any height  $t \in [0, 1]$ , there is a natural group isomorphism

$$\begin{aligned} \mathbb{Z}^n &\longrightarrow H_1(\mathbb{C}(\beta, t), \mathbb{Z}) \\ v_i &\longmapsto [D_i(t)] \end{aligned}$$

These isomorphisms extend into a global group isomorphism  $H_1(\mathbb{C}(\beta), \mathbb{Z}) \simeq \mathbb{Z}^n$  with the same generators given by the classes of the sections of the cylinders  $\{D_i(t) \mid t \in [0, 1]\}$ .

**Definition 3.2.2.** Let  $1 \leq i < j \leq n$  be two strand indices. The  $(i, j)$ -*braid linking function*

$$\lambda_{i,j} : \mathbb{B}_n \longrightarrow \mathbb{Z}^n$$

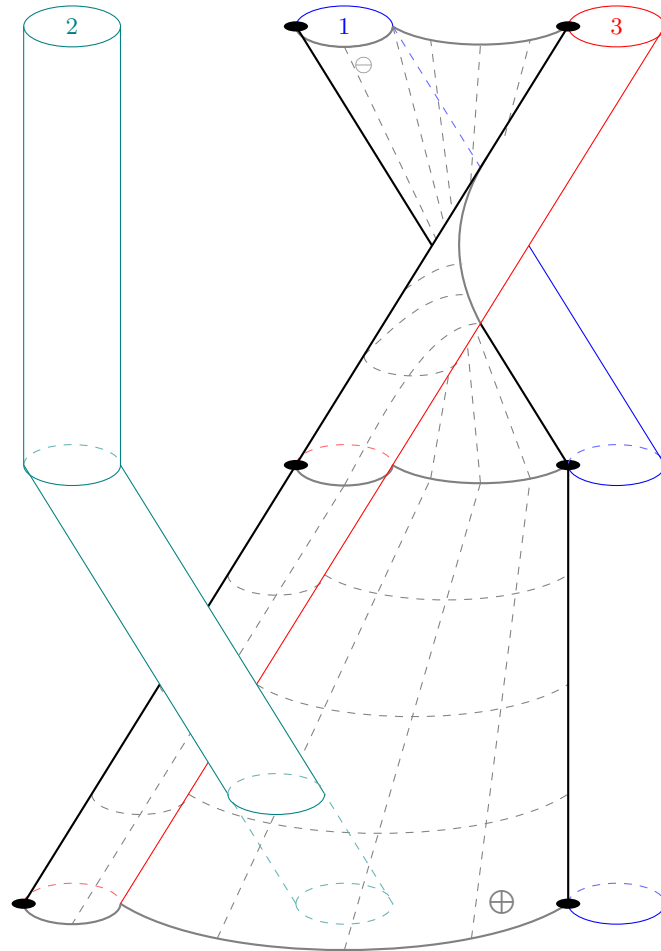


Figure 3.2.2: (1,3)-braid linking surface of  $\sigma_2^{-1}\sigma_1$

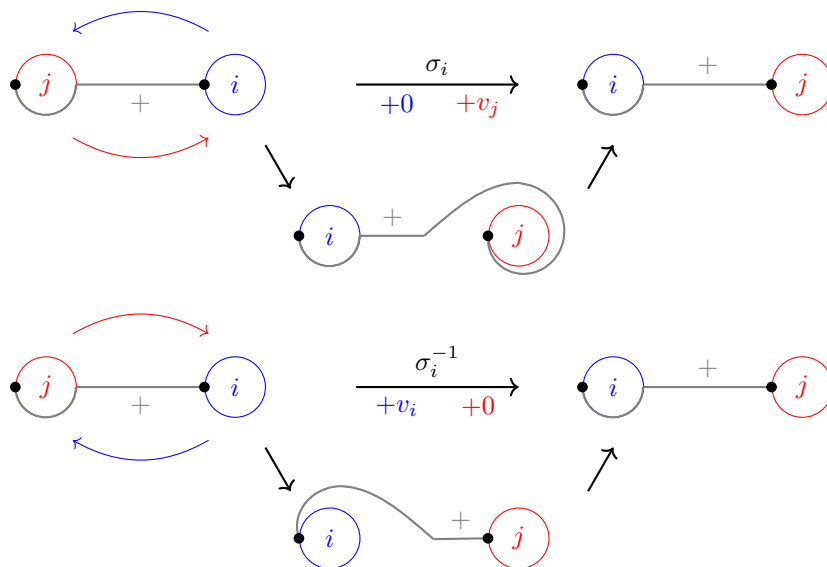
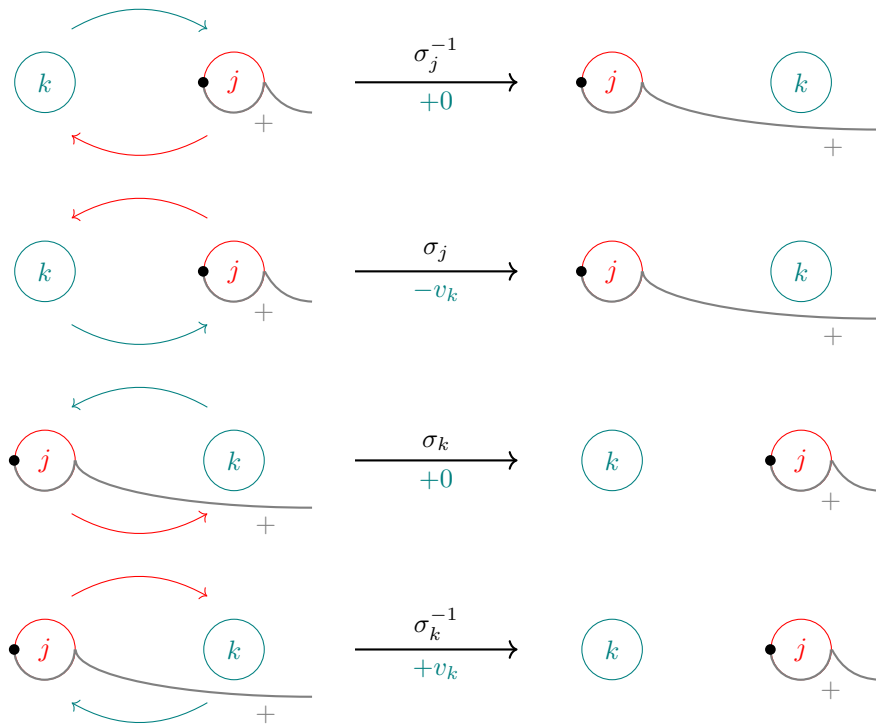
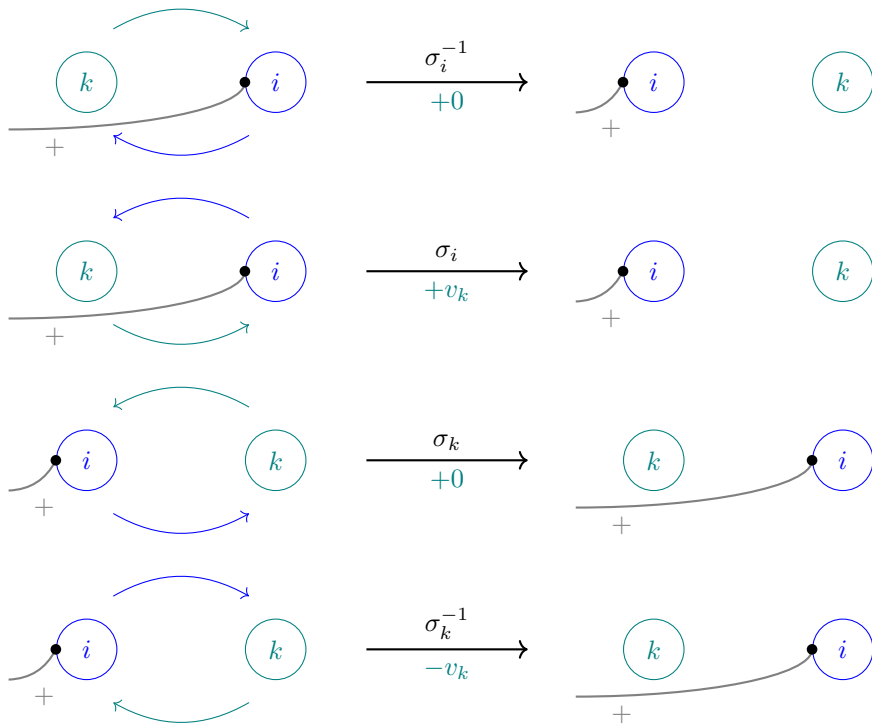


Figure 3.2.3: Braid linking surface at reversal crossings



(a) Regular crossings with left support strand



(b) Regular crossings with right support strand

Figure 3.2.4: Braid linking surface at regular crossings

associates to a braid  $\beta$  the homological class of the boundary  $\partial S_{i,j}(\beta)$  inside the free abelian group  $H_1(\mathbb{C}(\beta), \mathbb{Z}) \simeq \mathbb{Z}^n$ .  $\diamond$

**Example 3.2.3.** The  $(1, 3)$ -braid linking surface of the braid  $\sigma_2^{-1}\sigma_1 \in \mathbb{B}_3$  is shown on [Figure 3.2.2](#) on page [78](#). The corresponding  $(1, 3)$ -braid linking value is

$$\lambda_{1,3}(\sigma_2^{-1}\sigma_1) = v_2 + v_3 \quad \diamond$$

**Proposition 3.2.4.** *The  $(i, j)$ -braid linking of a braid  $\beta \in \mathbb{B}_n$  can be computed directly from any crossings decomposition*

$$\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r}, \quad 1 \leq i_s \leq n, \quad \varepsilon_{i_s} \in \{\pm 1\}$$

by adding up the values corresponding to each crossing  $\sigma_{i_s}^{\varepsilon_s}$  using the rules of [Figures 3.2.3](#) and [3.2.4](#). The indicated signs must be reversed after each reversal crossing.  $\triangleleft$

**Example 3.2.5.** The braid linking values of the braid  $\beta = \sigma_2^{-1}\sigma_1 \in \mathbb{B}_3$  are

$$\lambda_{1,2}(\beta) = v_3 \quad \lambda_{2,3}(\beta) = v_2 \quad \lambda_{1,3}(\beta) = v_2 + v_3 \quad \diamond$$

### 3.2.2 Main computation theorem

This section is dedicated to the statement and proof of [Theorem 3.2.6](#) which gives the main formula that links the braid linking function given in [Section 3.2.1](#) and the homology inclusion map defined in [Section 2.5](#). The statement of the theorem uses many objects and concepts introduced in the previous chapter, in addition of the specific construction of the standard graphed embedding detailed in [Section 3.1](#). We thus begin with a quick summary of these concepts before stating the theorem.

Recall from [Section 2.5](#) that the homology inclusion map is the morphism

$$i_{\mathcal{A}}^* \circ \gamma^* : H_1(\tilde{\Gamma}(C_{\mathcal{A}})) \longrightarrow V(\tilde{\Gamma}(C_{\mathcal{A}}))$$

inside the graph stabiliser  $\mathcal{G}_{\tilde{\Gamma}(C_{\mathcal{A}})}$ , where  $\gamma \in E_{\tilde{\Gamma}(C_{\mathcal{A}})}(\Omega^\theta)$  is any ordered graphed embedding. The class value depends on the graph ordering  $\Omega^\theta$  which itself depends on the ordering  $\theta$  of the line component set  $\mathcal{L}$ .

By [Proposition 2.3.20](#) on page [62](#), the reduced incidence graph  $\tilde{\Gamma}(C_{\mathcal{A}})$  is obtained from the full incidence graph  $\Gamma(C_{\mathcal{A}})$  by blowing-down moves. We can thus use [Theorem 1.6.15](#) on page [47](#) which gives a natural group isomorphism

$$\mathcal{G}_{\tilde{\Gamma}(C_{\mathcal{A}})} \xrightarrow{\sim} \mathcal{G}_{\Gamma(C_{\mathcal{A}})}$$

This means that both the graph stabiliser and the homology inclusion map can be computed using the full incidence graph  $\Gamma := \Gamma(C_{\mathcal{A}})$  rather than the reduced one  $\tilde{\Gamma}(C_{\mathcal{A}})$ . Following [Remark 3.1.8](#) on page [76](#), we also prefer to use the graph ordering  $\Omega^{\theta,\nu}$  from [Definition 2.2.15](#) on page [55](#) which depends on an additional ordering  $\nu$  on the set of singular points  $\mathcal{Q}$ .

Recall the induced spanning tree  $\mathcal{T}^\theta$  on  $\Gamma$  from [Definition 2.2.14](#) on page [55](#). By [Proposition 1.5.4](#) on page [38](#), the edges of  $E \setminus \mathcal{T}^\theta$  are naturally associated with a basis of the free abelian group  $H_1(\Gamma)$ . For every  $P_l \in \mathcal{Q}^*$ , define  $n_l := \min \theta(V_{P_l})$  and

$$V_{P_l}^* := V_{P_l} \setminus \{L_{n_l}\}$$

The set of edges outside  $\mathcal{T}^\theta$  is then given by

$$E \setminus \mathcal{T}^\theta := \{e_{L,P} \mid P \in \mathcal{Q}^*, L \in V_P^*\}$$

For any such edge  $e \in E \setminus \mathcal{T}^\theta$ , let  $c_e \in H_1(\Gamma)$  be the corresponding graph cycle. At the other end of the homology inclusion map, recall the quotient map

$$\eta : C_0(\Gamma) \longrightarrow V(\Gamma)$$

where  $C_0(\Gamma) \simeq \mathbb{Z}^{n+1}$  is the free abelian group generated by the vertices of  $\Gamma$ .

The standard graphed ordering  $\gamma_{\mathbf{B}} \in E_{\Gamma}(\Omega^{\theta,\nu})$  of the graph  $\Gamma$  is built with a geometrical

method from a representative of the braid monodromy

$$\mathbf{B} := B_{\mathcal{A}}(\pi, \theta, \nu, \alpha) = (\beta_l)_{1 \leq l \leq r}$$

The  $n$  strands of the braid  $\beta_l$  are naturally associated with the lines of  $\mathcal{L}^* := \mathcal{L} \setminus \{L_0\}$ . Moreover, each braid of the representative is a conjugate of the form

$$\beta_l = \tau_l \cdot \delta_l \cdot (\tau_l)^{-1} \in \mathbb{P}_n$$

where  $\tau_l \in \mathbb{B}_n$  is called the *shift braid* and  $\delta_l$  is a pure twist.

Finally, recall the braid linking function  $\lambda_{i,j} : \mathbb{B}_n \rightarrow \mathbb{Z}^n$  from [Section 3.2.1](#). The free abelian group  $\mathbb{Z}^n$  can be seen as the subgroup of  $C_0(\Gamma)$  generated by the set of vertices  $\{v_L \mid L \in \mathcal{L}^*\}$ .

All these constructions allow us to compute the values of the morphism  $i_{\mathcal{A}}^* \circ \gamma_{\mathbf{B}}^*$  on each element of the cycle basis given above.

**Theorem 3.2.6.** *The homology inclusion map  $i_{\mathcal{A}}^* \circ \gamma_{\mathbf{B}}^*$  admits the following value for every edge  $e_{L_i, P_l} \in E \setminus \mathcal{T}^\theta$  on the associated graph cycle:*

$$i_{\mathcal{A}}^* \circ \gamma_{\mathbf{B}}^*(c_e) = \eta \circ \lambda_{m(P_l), i}(\tau_l) + h_{0, \mathbf{B}}(e)$$

where  $h_{0, \mathbf{B}}(e) \in V(\Gamma)$  only depends on the braid at infinity  $\beta_0$  of the representative  $\mathbf{B}$  of the braid monodromy of  $\mathcal{A}$ . ◁

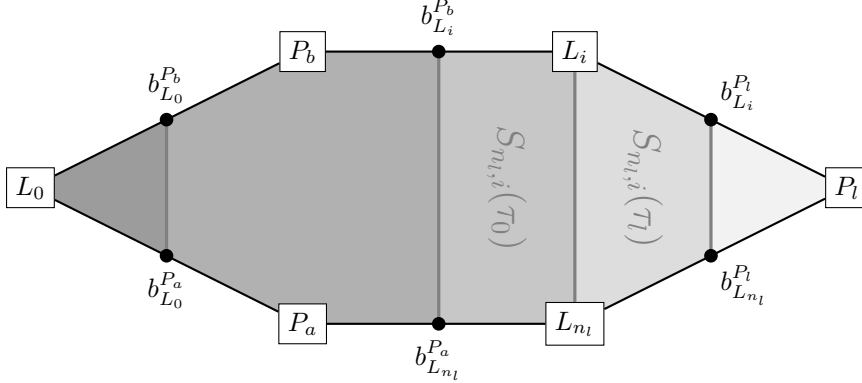


Figure 3.2.5: Decomposition of the exterior cycle surface

**Lemma 3.2.7.** *Let  $e_{L_i, P_l} \in E \setminus \mathcal{T}^\theta$ . Let  $P_a := L_{n_l} \cap L_0$  and  $P_b := L_i \cap L_0$ . Then the corresponding cycle  $c_e$  drawn on  $\Gamma$  is made of exactly six edges:*

$$e_{P_l, L_i} + e_{L_i, P_b} + e_{P_b, L_0} + e_{L_0, P_a} + e_{P_a, L_{n_l}} + e_{L_{n_l}, P_l} \quad \triangleleft$$

*Proof of Theorem 3.2.6.* Fix an edge  $e_{L_i, P_l} \in E \setminus \mathcal{T}^\theta$ . We reuse once more the notations from [Sections 2.3.3](#) and [3.1](#). Recall that  $\alpha$  is the star on the basis  $\Sigma$  of  $\pi$  (see [Figure 2.4.1](#) on page 65). One can add an extra branch  $\alpha^0$  joining the base point  $b^\infty$  to  $\partial D^0$  as explained in the proof of [Proposition 2.4.13](#) on page 65. The  $\mathcal{T}^\theta$ -cycle curve  $\gamma_e$ , which is the image of the cycle  $c_e \in H_1(\Gamma)$  by the ordered graphed embedding  $\gamma_{\mathbf{B}}$  borders a disc  $D_e$  in the exterior manifold  $E_{\mathcal{A}}$ . This disc can be decomposed into five parts as shown on [Figure 3.2.5](#). From left to right:

- The first part of the surface is contained within the vertical section  $s_{L_0}$ .
- The second part is a vertical surface contained within  $\pi^{-1}(b^0)$  that links the branch  $\alpha_{P_a}^i$  in the section  $s_{P_a}$  and the branch  $\alpha_{P_b}^{n_l}$  in the section  $s_{P_b}$ .
- The third part corresponds to the braid-linking surface  $S_{n_l, i}(\tau_0)$  of the shift braid  $\tau_0$  over the branch  $\alpha^0$ . As explained in the proof of [Proposition 2.4.13](#),  $\tau_0$  can always be chosen as the trivial braid.
- The fourth part corresponds to the braid-linking surface  $S_{n_l, i}(\tau_l)$  of the shift braid  $\tau_l$  over the branch  $\alpha^l$ . Indeed, the set  $\pi^{-1}(\alpha^l)$  can be identified as  $\mathbb{C} \times [0, 1]$ . The path  $S_{n_l, i}(\beta_l, 0)$

drawn on  $\Sigma(b^\infty)$  joins the two points  $\gamma(v_{L_i}) = \gamma_{\mathbf{B}} \cap T_{L_i}^\infty$  and  $\gamma(v_{L_{n_l}}) = \gamma_{\mathbf{B}} \cap T_{L_{n_l}}^\infty$ . On the other end, the path  $S_{n_l, i}(\beta_l, 1)$  drawn on  $\Sigma(b^l)$  joins the two points  $b_{L_i}^{P_l} := \gamma_{\mathbf{B}} \cap T_{L_i, P_l}^{\text{aff}}$  and  $b_{L_{n_l}}^{P_l} := \gamma_{\mathbf{B}} \cap T_{L_{n_l}, P_l}^{\text{aff}}$ .

- The fifth part is contained within the vertical section  $s_{P_l}$ .

Now we evaluate what are the contributions of each part to the homological value of the boundary.

- The first and fifth part are contained within sections of a line component circle bundle. This means that they do not meet any other line component and therefore do not contribute to the value.
- The second part's contribution corresponds to  $h_{0, \mathbf{B}}(e)$ . It is entirely determined by the local orderings around the line vertex  $v_{L_0}$  and the singular points vertices  $v_P$  for  $P \in V_{L_0}$ . This combinatorial information is contained within the braid at infinity by [Proposition 2.4.13](#) on page 65.
- The third part is a braid linking surface of a trivial braid, which does not meet any other line component by construction and therefore does not contribute to the value.

The morphism

$$\eta : C_0(\Gamma) \simeq H_1(\mathbb{C}(\tau_l)) \longrightarrow V(\Gamma)$$

corresponds to the induced morphism in homology of the inclusion

$$\pi^{-1}(\alpha^l) \simeq \mathbb{C}(\tau_l) \hookrightarrow E_{\mathcal{A}}$$

The fourth part's contribution can then be seen as the image by  $\eta$  of the homological value of the braid linking surface  $S_{n_l, i}(\tau_l)$  inside  $\mathbb{C} \times [0, 1] \simeq \pi^{-1}(\alpha^l)$ , which is precisely equal to  $\lambda_{m(P_l), i}(\tau_l)$  by [Definition 3.2.2](#).  $\square$

### 3.3 Comparison of a combinatorial pair

The following diagram sums up the combinatorial dependencies of all the objects involved in the computation of the homology inclusion.

$$\begin{array}{ccccccc}
 \theta, \nu & \longrightarrow & \Omega^{\theta, \nu} & & & & \\
 & \searrow & & \searrow & & & \\
 & & \mathbf{B} & \longrightarrow & \gamma_{\mathbf{B}} \in E_{\Gamma}(\Omega^{\theta, \nu}) & \longrightarrow & i_* \circ \gamma_{\mathbf{B}}^* \in \text{Hom}(H_1(\Gamma), V(\Gamma)) \\
 & \nearrow \text{topology} & & \nearrow & & & \downarrow \\
 \mathcal{A} & \longrightarrow & C \simeq \Gamma & \longrightarrow & \mathcal{G}_{\Gamma} & \longrightarrow & |i_* \circ \gamma_{\mathbf{B}}^*| \in \mathcal{G}_{\Gamma}
 \end{array}$$

From now on we consider  $(\mathcal{A}, \mathcal{A}')$  a combinatorial pair of line arrangements with the same combinatorics  $C$ .

#### 3.3.1 Adjustments of the orderings

As noted in [Remark 2.5.3](#) on page 70, the individual value of the homology inclusion of a line arrangement depends on the choice of  $\Omega^{\theta, \nu}$  and therefore  $\theta, \nu$ . However, the difference between the two values on the combinatorial pair does not as long as the parameters  $\theta, \nu$  are the same in both computations. In this situation the difference is only determined by the topological types of both line arrangements.

In practice the computational algorithms used to obtain the braid monodromy representative  $\mathbf{B}$  of a line arrangement  $\mathcal{A}$  take  $\nu = \nu_{\pi}$  to simplify computations, and the projection  $\pi$  itself might be chosen at random. Moreover, these algorithms do not decompose the individual star braid monodromies  $\beta_l$ , and do not explicitly give the orderings  $\theta, \nu$  associated with  $\mathbf{B}$ . We will thus use [Proposition 2.4.16](#) on page 67 to modify *a posteriori* the values of  $\theta$  and  $\nu$ , once they are determined.

E. Artal has written a function in Sage [Sag23] that provides a braid monodromy representative  $\mathbf{B}^{(0)}$  of a line arrangement  $\mathcal{A}$  as well as the associated initial ordering  $\theta^{(0)} : \mathcal{L}^* \rightarrow \{1, \dots, n\}$ .

**Algorithm 3.3.1:** Raw braid monodromy computation

**Data:** Defining polynomial of the affine arrangement  $\mathcal{A}^{\text{aff}}$   
**Result:** Braid monodromy representative  $\mathbf{B}^{(0)} = (\beta_l^{(0)})_{1 \leq l \leq r}$   
**Result:** Ordering  $\theta^{(0)}$

In the following we describe the algorithms to process the braid monodromy representative  $\mathbf{B}^{(0)}$  to prepare it for the final comparison. First we conjugate all star braid monodromies  $\beta_l^{(0)}$  to get the desired ordering  $\theta$ .

**Algorithm 3.3.2:** Adjustment of the ordering  $\theta$

**Data:** Braid monodromy representative  $\mathbf{B}^{(0)}$   
**Data:** Desired ordering  $\theta$   
**Result:** New braid monodromy representative  $\mathbf{B}^{(1)}$  with  $\theta^{(1)} = \theta$

The next algorithm uses a method due to N. Franco and J. González Meneses [FG03] to compute a decomposition of  $\beta_l^{(1)} \in \mathbb{P}_n$  of the form

$$\beta_l^{(1)} = \tau_l^{(1)} \cdot \Delta_{\theta^{(1)}(V_{P_l})}^2 \cdot (\tau_l^{(1)})^{-1}$$

**Algorithm 3.3.3:** Star braid assignation

**Data:** Star braid monodromy  $\beta_l^{(1)}$   
**Result:** Central full twist  $\Delta_{\theta^{(1)}(V_{P_l})}^2$   
**Result:** Shift braid  $\tau_l^{(1)}$

In fact this first decomposition is done only to associate each braid  $\beta_l^{(1)}$  with its corresponding singular point  $P \in \mathcal{Q}^*$ . Since the braids  $(\beta_l^{(1)})$  are already ordered, this also gives the initial ordering  $\nu_\pi$  associated with  $\mathbf{B}^0$ . The next algorithm applies Hurwitz moves on the representative  $\mathbf{B}^{(1)}$  to get the desired ordering  $\nu : \mathcal{Q}^* \rightarrow \{1, \dots, r\}$ .

**Algorithm 3.3.4:** Adjustment of the ordering  $\nu$

**Data:** Braid monodromy representative  $\mathbf{B}^{(1)}$   
**Data:** Desired ordering  $\nu$   
**Result:** New braid monodromy representative  $\mathbf{B}^{(2)}$  with  $\nu_\pi = \nu$

We now compute the new shift braids  $\tau_l^{(2)}$  by performing a second decomposition.

**Algorithm 3.3.5:** Star braid decomposition

**Data:** Star braid monodromy  $\beta_l^{(2)}$   
**Result:** Central full twist  $\Delta_{\theta^{(2)}(V_{P_l})}^2$   
**Result:** Shift braid  $\tau_l^{(2)}$

There is no guarantee that the ordering  $\theta^l$  coincides with the desired local ordering  $\theta|_{V_{P_l}}$ . The next step will multiply the shift braids on the right with a (non-full) twist braid to adjust the local ordering. This does not affect  $\beta_l^{(2)}$  since the central full twist can be conjugated by the inverse twist braid.



**Algorithm 3.3.6:** Adjustment of the local ordering  $\theta^l$ 

**Data:** Shift braid  $\tau_l^{(2)}$   
**Data:** Desired ordering  $\theta$   
**Result:** New shift braid  $\tau_l^{(3)}$  with  $\theta^l = \theta|_{V_{P_l}}$

**Algorithms 3.3.1** to **3.3.6** must be repeated for the other line arrangement  $\mathcal{A}'$ , thus giving a second set of shift braids  $\mathbf{T}^{(3)'}$ .

The braid at infinity (see **Proposition 2.4.13** on page 65) encodes all the local orderings for  $v_{L_0}$  and every  $v_P$  for  $P \in V_{L_0}$ . Individual adjustment of each of these local orderings is not required. Instead it is enough to make sure that the two braids at infinity  $b^0$  and  $b^{0'} \in \mathbb{P}_n$  of the combinatorial pair are equal. This is done by finding a conjugating braid and then multiplying all shift braids  $\mathbf{T}^{(3)'}$  of  $\mathcal{A}'$  on the left. The set of shift braids  $\mathbf{T}^{(3)}$  of  $\mathcal{A}$  is not modified.

**Algorithm 3.3.7:** Adjustment of the infinity braids

**Data:** Decomposed representative  $\mathbf{B}^{(3)}$   
**Data:** Decomposed representative  $\mathbf{B}^{(3)'}$   
**Result:** New decomposed representative  $\mathbf{B}^{(4)'}$  with  $\beta_0^{(4)' } = \beta_0^{(3)}$

**Remark 3.3.1.** The Hurwitz moves used on the braids of the representative to change  $\nu$  are computationally expensive because they require two separate decompositions, with the conjugation performed in-between by **Algorithm 3.3.4** significantly increasing the length of the braids ( $\beta_l^{(1)}$ ). This explains why we choose to define the standard graphed embedding  $\gamma_{\mathbf{B}}$  using  $\nu$  as an independent parameter, in order to minimise the number of such moves required to reach a common value.  $\square$

### 3.3.2 Comparison algorithm

The following procedure presents all the steps to compare the homology inclusion values of a combinatorial pair.

**Algorithm 3.3.8:** Comparison of a combinatorial pair

**Data:** Combinatorial pair  $(\mathcal{A}, \mathcal{A}')$   
**Data:** Orderings  $\theta, \nu$

- 1 Perform **Algorithms 3.3.1** to **3.3.6** on  $\mathcal{A}$  and  $\mathcal{A}'$  to get  $\mathbf{B}^{(3)}$  and  $\mathbf{B}^{(3)'}$ .
- 2 Perform **Algorithm 3.3.7** on  $\mathbf{B}^{(3)}$  and  $\mathbf{B}^{(3)'}$  to get  $\mathbf{B}^{(4)'}$ .  
*// Shift braids are now ready.*
- 3 Compute the full incidence graph  $\Gamma$ .
- 4 Compute the graph stabiliser  $\mathcal{G}_{\Gamma}$  and the projection map using **Theorem 1.6.11**.  
*// Graph stabiliser is now ready.*
- 5 **forall**  $e_{L_i, P_l} \in E \setminus \mathcal{T}^{\theta}$  **do**
- 6     Compute the values  $\lambda_{m(P_l), i}(\tau_l^{(3)})$  and  $\lambda_{m(P_l), i}(\tau_l^{(4)'})$  using **Proposition 3.2.4**.
- 7     Compute  $i_* \circ \gamma_{\mathbf{B}^{(3)}}^*$  and  $i_* \circ \gamma_{\mathbf{B}^{(4)'}}^*(c_e)$  using **Theorem 3.2.6**.
- 8     Take the image by the projection map.
- 9 **end**
- 10 Combine the images.

**Result:**  $|i_* \circ \gamma_{\mathbf{B}^{(3)}}^*| - |i_* \circ \gamma_{\mathbf{B}^{(4)'}}^*| \in \mathcal{G}_{\Gamma}$

Note that the graph stabiliser  $\mathcal{G}_{\Gamma}$  of a combinatorics is an abelian group since it is a quotient of the homomorphism group  $\text{Hom}(H_1(\Gamma), V(\Gamma))$  between two free abelian groups. As a  $\mathbb{Z}$ -module it admits a decomposition as a product of cyclic groups in an adequate basis, the so-called *Smith normal form*. It is this basis that we use to actually compare the two values of the homology inclusions.

### 3.3.3 Examples of Zariski pairs

We now give several examples of Zariski pairs which are identified by the homology inclusion.

**Example 3.3.2** (MacLane arrangements). The following lexicographically ordered combinatorics were discovered by S. MacLane [Mac36]:

$$[[0, 1, 2], [0, 3, 4], [0, 5, 6], [0, 7], [1, 3], [1, 5, 7], [1, 4, 6], [2, 3, 5], [2, 4, 7], [2, 6], [3, 6, 7], [4, 5]]$$

The automorphism group of the MacLane combinatorics is isomorphic to  $\mathrm{GL}_2(\mathbb{F}_3)$ . It is known (see [Bjö+99]) that this constitutes the smallest possible combinatorics that does not admit a realisation in  $\mathbb{R}\mathbb{P}^2$ . It admits two conjugated realisations  $M^+$  and  $M^-$  with 8 lines. The defining (ordered) equations in  $\mathbb{C}\mathbb{P}^2$  are

$$\begin{array}{lll} L_0 : 0 = z & L_3 : 0 = y & L_6 : 0 = -x - \omega^2 y + z \\ L_1 : 0 = -x + z & L_4 : 0 = \omega^2 x + \omega y + z & L_7 : 0 = \omega y + z \\ L_2 : 0 = x & L_5 : 0 = -x + y & \end{array}$$

where  $\omega = e^{\frac{2i\pi}{3}}$  for  $M$  and  $\omega = e^{-\frac{2i\pi}{3}}$  for  $\overline{M}$ . Each of the combinatorics automorphisms can be realised as projective automorphisms of  $\mathbb{C}\mathbb{P}^2$ . The ones in  $\mathrm{SL}_2(\mathbb{F}_3)$  preserve the order. We now give the results of the computations of the graph stabiliser and the homology inclusion of  $M$  and  $\overline{M}$  using [Algorithm 3.3.8](#).

The Smith normal form of the graph stabiliser is

$$\mathcal{G}_\Gamma \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}^{35}$$

The two values of the homology inclusion in the corresponding basis are given in [Figure 3.3.1](#).

$$\begin{array}{l} |i_* \circ \gamma_{\mathbf{B}}^*| : \bar{0} \ 0 \ 1 \ 2 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 2 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ -1 \ 1 \ -1 \ -1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ |i_* \circ \gamma_{\overline{\mathbf{B}}}^*| : \bar{2} \ 0 \ 1 \ 2 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 2 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ -1 \ 1 \ -1 \ -1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ \text{Difference: } \bar{2} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{array}$$

Figure 3.3.1: Homology inclusion values of the MacLane arrangements

The difference is non-zero in the torsion part of  $\mathcal{G}_\Gamma$ . The line arrangements  $(M, \overline{M})$  therefore form an ordered oriented Zariski pair (see [Remark 2.5.4](#) on page 70).

◻

**Example 3.3.3** (New Zariski quadruplet). Consider the polynomial

$$P = X^4 + 2X^3 + 4X^2 + 3X + 1$$

and the following equations given by

$$\begin{array}{ll} L_0 : 0 = z & L_6 : 0 = x \\ L_1 : 0 = \omega^2 x - y - \omega(\omega + 1)z & L_7 : 0 = x - z \\ L_2 : 0 = (3\omega^2 + 3\omega + 1)x + (\omega + 1)^2 y - (\omega^3 + 5\omega^2 + 5\omega + 2)z & \\ L_3 : 0 = \omega(\omega^2 + \omega + 1)x + y + \omega(\omega + 1)z & L_8 : 0 = y \\ L_4 : 0 = \omega x + y & L_9 : 0 = y - z \\ L_5 : 0 = \omega x + y - (\omega + 1)z & L_{10} : 0 = y + \omega(\omega^2 + 2\omega + 2)z \end{array}$$

where  $\omega = -\frac{1}{2} \pm \frac{1}{2}i\sqrt{5 \pm 2\sqrt{5}}$  take the values of the four roots of  $P$ . This defines four conjugated arrangements with 11 lines  $B_1, B_2, \overline{B}_1, \overline{B}_2$  whose common ordered combinatorics is given by

$$\begin{array}{l} [[0, 1, 2], [0, 3], [0, 4, 5], [0, 6, 7], [0, 8, 9, 10], [1, 3, 6], [1, 4, 7], [1, 5, 8], \\ [1, 9], [1, 10], [2, 3, 5], [2, 4], [2, 6, 10], [2, 7], [2, 8], [2, 9], [3, 4, 9], [3, 7, 10], \\ [3, 8], [4, 6, 8], [4, 10], [5, 6], [5, 7, 9], [5, 10], [6, 9], [7, 8]] \end{array}$$

The Smith normal form of the graph stabiliser is:

$$\mathcal{G}_\Gamma \simeq \mathbb{Z}/_5\mathbb{Z} \times \mathbb{Z}^{119}$$

The values of the homology inclusion of the four realisations are identical on the free part but differ on the torsion part:

$$B_1 : \bar{1} \qquad B_2 : \bar{4} \qquad \overline{B_1} : \bar{3} \qquad \overline{B_2} : \bar{0}$$

The automorphism group of the combinatorics is trivial, which means that the four realisations form an unordered oriented Zariski quadruplet. The two pairs  $(B_1, B_2)$  and  $(B_1, \overline{B_2})$  and their respective conjugates are unordered unoriented Zariski pairs.  $\square$

**Example 3.3.4** (Rybnikov quadruplet). This is the first Zariski pair of line arrangements identified by G. Rybnikov in [Ryb11]. Consider the equations:

$$\begin{array}{lll} L_0 : 0 = z & L_5 : 0 = -x + \omega y + z & L_9 : 0 = -2x + y + 3z \\ L_1 : 0 = x & L_6 : 0 = -(\omega + 1)x - y + z & L_{10} : 0 = (1 - 5\eta)x + 2\eta y + \eta z \\ L_2 : 0 = x - z & L_7 : 0 = -(\omega + 1)x + \omega x + z & L_{11} : 0 = (1 - 5\eta)x + 2\eta + 6\eta z \\ L_3 : 0 = y & L_8 : 0 = -4\eta x + 2\eta y + z & L_{12} : 0 = x + 2y + z \\ L_4 : 0 = y - z & & \end{array}$$

where

$$(\omega, \eta) = \left( e^{\pm \frac{2i\pi}{3}}, \frac{5\omega + 6}{31} \right) \text{ or } \left( e^{\pm \frac{2i\pi}{3}}, \frac{5\bar{\omega} + 6}{31} \right)$$

This defines four conjugated line arrangements with 13 lines  $R_1, R_2, \overline{R_1}, \overline{R_2}$ , whose common ordered combinatorics is given by

$$\begin{aligned} & [[0, 1, 2], [0, 3, 4], [0, 5, 6], [0, 7], [0, 8, 9], [0, 10, 11], [0, 12], [1, 3], [1, 4, 6], [1, 5, 7], [1, 8], \\ & [1, 9, 11], [1, 10, 12], [2, 3, 5], [2, 4, 7], [2, 6], [2, 8, 10], [2, 9, 12], [2, 11], [3, 6, 7], [3, 8], [3, 9], \\ & [3, 10], [3, 11], [3, 12], [4, 5], [4, 8], [4, 9], [4, 10], [4, 11], [4, 12], [5, 8], [5, 9], [5, 10], [5, 11], [5, 12], \\ & [6, 8], [6, 9], [6, 10], [6, 11], [6, 12], [7, 8], [7, 9], [7, 10], [7, 11], [7, 12], [8, 11, 12], [9, 10]] \end{aligned}$$

The Smith normal form of the graph stabiliser is:

$$\mathcal{G}_\Gamma \simeq \left( \mathbb{Z}/_3\mathbb{Z} \right)^2 \times \mathbb{Z}^{220}$$

The homology inclusion values differ not only on the torsion part but also on the free abelian part:

$$R_1 : (\bar{1}, \bar{1}, f_1) \qquad \overline{R_1} : (\bar{1}, \bar{0}, f_1) \qquad R_2 : (\bar{0}, \bar{0}, f_2) \qquad \overline{R_2} : (\bar{0}, \bar{1}, f_2)$$

where  $f_1, f_2 \in \mathbb{Z}^{220}$ . This means that the four realisations  $(R_1, R_2, \overline{R_1}, \overline{R_2})$  form an ordered oriented Zariski quadruplet. Moreover, the two pairs  $(R_1, R_2)$  and  $(R_1, \overline{R_2})$  and their respective conjugates are unoriented ordered Zariski pairs.  $\square$

**Remark 3.3.5.** For all the Zariski pairs we obtain, the graph stabiliser contains a torsion part (this is not always the case) and the value of the difference of the homology inclusions lies at least partly in the torsion part of the graph stabiliser. B. Guerville-Ballé has conjectured that only Zariski pairs whose graph stabiliser contains a torsion part can be distinguished by the homology inclusion itself. Even if this is true, a torsion-free graph stabiliser could nevertheless be used as an intermediary step to compute the *twisted* homology inclusion values. This new difference value might be able to detect the evading Zariski pairs.  $\square$

## PART II

# SLOPE INVARIANTS OF LINKS

*'It often seems to me that's all detective work is, wiping out your false starts and beginning again.'*

*'Yes, it is very true, that. And it is just what some people will not do. They conceive a certain theory, and everything has to fit into that theory. If one little fact will not fit it, they throw it aside. But it is always the facts that will not fit in that are significant.'*

— Agatha Christie

HERCULE POIROT – DEATH ON THE NILE, 1945

# CHAPTER

## 4

# CHARACTER SLOPE OF LINKS

### Outline of the current chapter

<b>4.1 Preliminaries on link theory</b>	<b>88</b>
4.1.1 Generalities . . . . .	88
4.1.2 Link group . . . . .	89
<b>4.2 Definition of the character slope</b>	<b>90</b>
4.2.1 Twisted homology of the boundary . . . . .	90
4.2.2 Definition of the character slope . . . . .	91
4.2.3 Properties of the slope . . . . .	92
<b>4.3 Slope computation</b>	<b>92</b>

This chapter makes a short presentation of the original link slope invariant developed in [DFL22b; DFL21; DFL22a], which we call ‘character slope’ to distinguish it from the new ‘slopes’ of Chapters 5 and 6.

We begin by making a quick recall of basic knot and link theory definitions in Section 4.1. Then Section 4.2 gives the main Definition 4.2.4 of the character slope and presents its properties, notably the concordance invariance. In Section 4.3 we present the method to compute the character slope using Fox calculus [Fox54] with a new implementation using GAP [GAP22] and the graphical interface PLink [CD08].

## 4.1 Preliminaries on link theory

### 4.1.1 Generalities

Let  $S^3$  be the oriented 3-sphere.

**Definition 4.1.1.** A *knot* is an embedding

$$K : S^1 \hookrightarrow S^3$$

such that  $K(S^1)$  is a simple polygonal curve.

A *link* is an embedding

$$L : \bigsqcup_{i=1}^n L_i \hookrightarrow S^3$$

of several components  $L_1, \dots, L_n$  whose images are disjoint simple polygonal curves.  $\diamond$

By extension, we identify the embeddings with their images and still call them ‘link’ and ‘knot’ respectively.

**Definition 4.1.2.** Let  $L$  and  $L'$  be two links. An *ambient isotopy* between  $L$  and  $L'$  is an application:

$$\begin{aligned} G : S^3 \times [0, 1] &\longrightarrow S^3 \\ (x, t) &\longmapsto g_t(x) \end{aligned}$$

such that  $G$  is a piecewise linear homeomorphism of  $S^3 \times [0, 1]$  into itself, with  $g_0 = \text{Id}_{S^3}$  and  $g_1(L) = L'$ .  $\diamond$

**Definition 4.1.3.** Two links  $L$  and  $L'$  are said to be *equivalent* if there exists an ambient isotopy of  $S^3$  that sends  $L$  to  $L'$ .  $\diamond$

The name ‘link’ now designate an equivalence class of links. For the remainder of this section, ‘link’ also include knots (i.e.  $n = 1$ ) unless otherwise specified.

**Definition 4.1.4.** A *link orientation* is a function

$$\delta_L : (L_1, \dots, L_n) \longrightarrow \{\pm 1\}^n$$

which assigns an orientation to every link component.  $\diamond$

**Definition 4.1.5.** A regular link projection is an application  $p : L \rightarrow \mathbb{R}^2$  such that for every point  $x$  of the *link diagram*  $p(\mathcal{E})$ , the fibre  $p^{-1}(x)$  has no more than two element, and has exactly two elements for a finite number of point in the diagram.  $\diamond$

The points of the plane which have exactly two antecedents are called *crossings*. Each crossing has an *upper* and *lower* antecedent in the link relatively to the projection  $p$ . The link  $L$  is therefore divided in *strands* which are continuous paths in  $L$  which joins two lower points.

**Proposition 4.1.6.** *Two links are equivalent if and only if their respective digrams obtained by the same regular projection on  $\mathbb{R}^2$  can be transformed into each other by a finite series of Reidemeister moves.*  $\triangleleft$

**Definition 4.1.7.** Let  $1 \leq \mu \leq n$ . A  $\mu$ -coloured link is an oriented link in  $S^3$  equipped with a surjective map

$$c : \{1, \dots, n\} \longrightarrow \{1, \dots, \mu\} \quad \diamond$$

Let  $\chi := c^{-1}$  such that  $\xi(j) \subset \{1, \dots, n\}$  for  $j \in \{1, \dots, \mu\}$ . We write  $L_{\chi(j)}$  for the union of individual components  $L_i$  of the link  $L$  that are  $j$ -coloured.

**Definition 4.1.8.** Two  $\mu$ -coloured links  $L^0$  and  $L^1$  are *concordant* if there exists a collection of properly embedded disjoint locally flat cylinders  $A := A_1 \sqcup \dots \sqcup A_\mu$  such that

$$\partial A_j \cap (S^3 \times 0) = -L_{\chi(j)}^0 \quad \text{and} \quad \partial A_j \cap (S^3 \times 1) = L_{\chi(j)}^1$$

for all  $1 \leq j \leq \mu$ . A  $n$ -coloured link concordant to the unlink is called *slice*.  $\diamond$

## 4.1.2 Link group

**Definition 4.1.9.** Let  $K \subset S^3$  be a knot. A *tubular neighbourhood*  $T(K)$  is a regular neighbourhood of  $K$  which is homeomorphic to a solid torus  $K \times D^2$ . The *exterior* of the knot  $M_K$  is defined as

$$M_K := S^3 \setminus \widehat{T(K)} \quad \diamond$$

This definition is naturally extended to links. Let  $b$  be a base point in  $\partial M_L$ . The *link group*  $\pi_L$  is the fundamental group  $\pi_1(M_L, b)$ .

**Definition 4.1.10.** A *meridian* of a knot  $K$  is a curve  $m$  whose class is null in  $H_1(T(K))$  but not in  $H_1(M_L)$ . A *longitude* of a knot  $K$  is a curve  $\ell$  is the intersection of  $T(K)$  with an orientable oriented surface  $S$  embedded in  $S^3$  with  $\partial S = K$ .  $\diamond$

Since we mostly work inside the fundamental group of either  $M_L$  or  $\partial T(K)$ , the terms ‘meridian’ and ‘longitude’ almost always designates the homotopy classes of these two types of curves.

**Definition 4.1.11.** The *peripheral system* of a knot  $K$  is the triple  $(\pi_L, \ell, m)$  where  $\ell$  and  $m$  are the homotopy classes of a longitude and meridian of  $K$  respectively such that  $m \cdot \ell = \ell \cdot m$ .

These are called the *preferred* longitude and meridians of  $K$ . The pair  $(m, \ell)$  is unique up to conjugation of a common element in  $\pi_K$ .  $\diamond$

**Theorem 4.1.12** (Waldhausen). *Two knots  $K, K'$  are equivalent if and only if there exists a group isomorphism  $\varphi : \pi_K \rightarrow \pi_{K'}$  such that  $(\varphi(m), \varphi(\ell)) = (m', \ell')$ .*  $\triangleleft$

**Definition 4.1.13.** The *Wirtinger presentation* of the oriented link group  $\pi_L$  is obtained from an link diagram of  $L$  in the following way: every strand is associated with a meridian generator as shown on [Figure 4.1.1a](#). Every crossing of the diagram is associated with the relation as shown on [Figure 4.1.1b](#). The *extended Wirtinger presentation* adds a peripheral couple  $(\ell_i, m_i)$  of preferred longitude and meridian for each knot component  $L_i$  of the link  $L$ .  $\diamond$

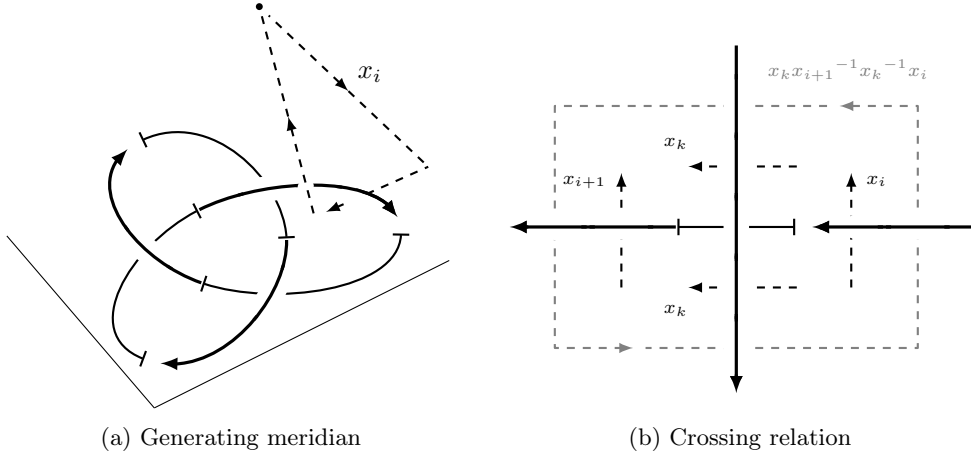


Figure 4.1.1: Wirtinger presentation of the link group

The abelianisation map

$$\text{Ab} : \pi_L \longrightarrow H_1(M_L)$$

induces a group isomorphism between  $H_1(M_L)$  and the free abelian group generated by a meridian for each link component. The preferred longitudes are all sent to the identity element 1.

**Definition 4.1.14.** Let  $K, K'$  be two knot components embedded in the same  $S^3$ . Then  $\text{lk}(K, K')$  is defined as the class of the curve  $\ell_K$  inside  $H_1(E_{K'}) \simeq \mathbb{Z}$ .  $\diamond$

## 4.2 Definition of the character slope

The construction presented in the section was developed by A. Degtyarev, V. Florens and A. G. Lecuona in [\[DFL22b; DFL21; DFL22a\]](#). It is the basis of the generalisations that we will present in the next chapters.

Let  $L$  be a  $\mu$ -coloured link. We denote by  $(m_i, \ell_i)$  the peripheral pairs of each component of  $L$ . The group  $H_1(E_L)$  is free abelian and generated by the classes of the meridians  $m_1, \dots, m_\mu$ . All meridian generators of a component  $L_i$  are conjugated to each other. A character of the link group  $\omega : \pi_1(E_L) \rightarrow \mathbb{C}^*$  is therefore determined by its values on the preferred meridian of each component and can then be seen as an element

$$(\omega(m_1), \dots, \omega(m_\mu)) \in (\mathbb{C}^*)^\mu$$

with  $\omega(m_i) = \omega(m_j)$  if  $L_i$  and  $L_j$  have the same colouring. We define

$$\omega^{-1} := (\omega_1^{-1}, \dots, \omega_\mu^{-1}) \quad \bar{\omega} := (\bar{\omega}_1, \dots, \bar{\omega}_\mu) \quad \omega^\dagger := (\bar{\omega})^{-1}$$

A character  $\omega$  is called *unitary* if  $\omega^\dagger = \bar{\omega}$ .

### 4.2.1 Twisted homology of the boundary

Let  $K \cup L$  be a  $(1, \mu)$ -coloured link where  $L = L_1 \cup \dots \cup L_\mu$  is a sublink and  $K$  is a special component called *distinguished*. We use the convention that  $L_0 = K$ .

**Definition 4.2.1.** A character  $\omega \in (\mathbb{C}^*)^{\mu+1}$  is called *admissible* if  $\omega(m_0) = 1$  and *non-vanishing* if  $\omega(m_i) \neq 1$  for every  $1 \leq i \leq \mu$ . The variety of admissible characters is denoted by  $\mathcal{A}(K/L)$ . The subvariety of non-vanishing admissible characters is denoted by  $\mathcal{A}^\circ(K/L)$ .  $\diamond$

We are interested in the restriction of the character  $\omega$  to  $H_1(\partial E_K)$  seen as a subgroup of  $H_1(E_L)$ . The subgroup  $H_1(\partial E_K)$  is free abelian generated by the classes of  $m_0$  and  $\ell_0$ . By [Definition 4.1.14](#), we have

$$\forall i \in \{1, \dots, \mu\} : \quad \omega(\ell_0) = \omega(m_i)^{\text{lk}(K, L_i)} \quad (4.1)$$

From now on, we suppose that the linking numbers  $\text{lk}(K, L_i)$  are zero for every  $1 \leq i \leq \mu$ . This means that  $\omega(\ell_0) = 1$  and

$$\mathcal{A}(K/L) = (\mathbb{C}^*)^\mu \quad \mathcal{A}^\circ(K/L) = (\mathbb{C}^* \setminus \{1\})^\mu$$

**Proposition 4.2.2.** *There is a natural vector space isomorphism*

$$H_1(\partial E_L; \omega) \simeq \bigoplus_{\omega(m_i)=1} \langle \ell_i, m_i \rangle \otimes \mathbb{C}(\omega) \quad \triangleleft$$

*Proof.* We have the decomposition

$$H_1(\partial E_L; \omega) \simeq \bigoplus_{\omega(m_i) \neq 1} H_1(\partial E_{L_i}; \omega) \oplus \bigoplus_{\omega(m_i)=1} H_1(\partial E_{L_i}; \omega)$$

For the components  $L_i$  where  $\omega(m_i) - 1 \neq 0$ , [Corollary A.2.2](#) gives  $H_1(\partial E_{L_i}; \omega) = \{0\}$ . For the components  $L_i$  where  $\omega(m_i) = 1$ , [Corollary A.2.3](#) gives a natural isomorphism

$$H_1(\partial E_{L_i}; \omega) \simeq \langle \ell_i, m_i \rangle \otimes \mathbb{C}(\omega) \quad \square$$

**Corollary 4.2.3.** *If  $\omega \in \mathcal{A}^\circ(K/L)$  then  $H_1(\partial E_L; \omega)$  is a  $\mathbb{C}$ -vector space of dimension 2 which admits the canonical basis  $(\ell_0, m_0)$ .*  $\triangleleft$

## 4.2.2 Definition of the character slope

Consider the inclusion map

$$i : \partial M_{KUL} \hookrightarrow M_{KUL}$$

and the induced map on the first twisted homology groups of  $(M_{KUL}, \omega)$ :

$$i_* : H_1(\partial M_{KUL}; \omega) \longrightarrow H_1(M_{KUL}; \omega)$$

We consider the space  $\mathcal{Z}(\omega) := \ker i_*$ .

**Definition 4.2.4.** Let  $\omega \in \mathcal{A}^\circ(K/L)$  and suppose that  $\dim \mathcal{Z}(\omega) = 1$ . The space  $\mathcal{Z}(\omega)$  is generated by a vector of  $H_1(\partial M_{KUL}; \omega)$  of the form

$$\mathcal{Z}(\omega) = \langle a \cdot \ell + b \cdot m \rangle$$

with  $[a : b] \in \mathbb{C}\mathbb{P}^1$ . The  $(K/L)$ -slope is defined by the formula

$$s_{(K/L)}(\omega) := -\frac{b}{a} \in \mathbb{C} \cup \{\infty\} \quad \diamond$$

**Proposition 4.2.5.** *If the slope  $s_{(K/L)}(\omega)$  is well defined for  $\omega \in \mathcal{A}(K/L)$ , then so are the slopes for  $\bar{\omega}$  and  $\omega^\dagger$ , and one has*

$$s_{(K/L)}(\omega^\dagger) = s_{(K/L)}(\omega) \quad s_{(K/L)}(\bar{\omega}) = \overline{s_{(K/L)}(\omega)} \quad \triangleleft$$

*Proof.* These results are immediate consequences of Poincaré duality as stated in [Lemma A.1.5](#) on page [120](#).  $\square$



**Proposition 4.2.6.** *If  $\omega$  is unitary then the slope is well-defined and*

$$s_{(K/L)}(\omega) \in \mathbb{R} \cup \{\infty\} \quad \triangleleft$$

*Proof.* The slope is well-defined as a direct application of [Theorems A.1.8](#) and [A.1.9](#) on page [121](#). The slope is real because of [Proposition 4.2.5](#).  $\square$

### 4.2.3 Properties of the slope

We recall some of the known properties of the slope. The proofs can be found in [[DFL22b](#); [DFL21](#); [DFL22a](#)].

**Definition 4.2.7.** The *characteristic varieties* of the  $(1, \mu)$ -coloured link  $L$  are the jump loci

$$\mathcal{V}_r(L) := \{\omega \in (\mathbb{C}^*)^\mu \mid \dim H_1(M_L; \omega) \geq r\}$$

They are nested algebraic sub-varieties of the character torus:

$$(\mathbb{C}^*)^\mu = \mathcal{V}_0(L) \supset \mathcal{V}_1(L) \supset \dots$$

A characteristic variety is called *proper* if  $\mathcal{V}_r(L) \neq (\mathbb{C}^*)^\mu$ . The first proper characteristic variety is denoted by  $\mathcal{V}_{\max}(L)$ .  $\diamond$

**Theorem 4.2.8** ([\[DFL22b](#), Theorems 3.19 and 3.21]). *If  $\text{lk}(K, L) = 0$ , the slope is a rational function, possibly identical to  $\{\infty\}$ , on  $\mathcal{A}^\circ(K/L) \setminus \mathcal{V}_{\max}(L)$ . If  $\mathcal{V}_{\max}(L) = \mathcal{V}_1(L)$ , one has*

$$s_{K/L}(\omega) = -\frac{\nabla'(1, \sqrt{\omega})}{2\nabla_L(\sqrt{\omega})} \in \mathbb{C} \cup \{\infty\}$$

where  $\nabla'$  is the derivative of the Conway polynomial  $\nabla_{K \cup L}(t, \cdot)$  with respect to  $t$ .  $\triangleleft$

**Definition 4.2.9.** Consider the subset of Laurent polynomials defined by

$$U := \{P \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}] \mid P(1, \dots, 1) = \pm 1\}$$

An element  $\omega \in \mathcal{A}(K/L)$  is called a *concordance root* if there is a polynomial  $P \in U$  such that  $P(\omega) = 0$ . The set of *non-concordance roots* is denoted by  $\mathcal{A}_c(K/L)$  and the subset of non-vanishing non-concordance roots by  $\mathcal{A}_c^\circ(K/L)$ .  $\diamond$

**Theorem 4.2.10** ([\[DFL22a](#), Theorem 3.2]). *Let  $K \cup L^0$  and  $K \cup L^1$  be two concordant  $(1, \mu)$ -coloured links. Then*

$$\mathcal{A}_c(K^0/L^0) = \mathcal{A}_c(K^1/L^1)$$

and the slope functions  $s_{(K^0/L^0)}(\omega)$  and  $s_{(K^1/L^1)}(\omega)$  are equal on  $\mathcal{A}(K/L)$ .  $\triangleleft$

**Corollary 4.2.11.** *If  $K \cup L$  is slice then  $s_{K/L}(\omega) = 0$  for every  $\omega \in \mathcal{A}_c(K/L)$ .*  $\triangleleft$

## 4.3 Slope computation

The slope computation is done using Fox calculus, whose definition is recalled in [Appendix A.3](#). Let  $K \cup L$  be a  $(1, \mu)$ -coloured link and let  $\omega \in \mathcal{A}^\circ(K/L)$  be a non-vanishing admissible character.

**Theorem 4.3.1.** *Let  $\mathcal{P}$  be a presentation of the link group  $\pi_1(E_{K \cup L})$  that contains  $m_K$  and  $\ell_K$ . Let  $A^\omega$  be the twisted Alexander matrix of  $M_{K \cup L}$  associated with  $\omega$  and  $\mathcal{P}$ . Let  $A_K^\omega$  be the sub-matrix of  $A^\omega$  containing only the columns associated with  $\ell_K$  and  $m_K$ , and let  $A_C^\omega$  be its complementary sub-matrix. Then*

$$A_K^\omega \cdot (\ker A_C^\omega)$$

is a generating matrix of  $\ker i_*$  and in particular has rank 1. If  $v$  is a generating vector of the row-space with

$$v = a \cdot d\ell_k + b \cdot dm_K$$

then  $s_{K/L}(\omega) = -\frac{b}{a}$ .  $\triangleleft$

*Proof.* This is a direct application of [Theorem A.4.1](#) on page [124](#) since  $M_{K \cup L}$  and  $\omega \in \mathcal{A}^\circ(K/L)$  verify [Eq. \(A.2\)](#).  $\square$

The algorithm used to compute the slope is written in `GAP` [GAP22]. [Theorem 4.3.1](#) reduces the computation of the slope to Fox calculus and elementary linear algebra. It is however necessary to obtain the extended Wirtinger presentation of the link  $K \cup L$  first.

To encode links, we use the oriented DT-code (which stands for C. H. Dowker and M. B. Thistlethwaite [DT83]). This code attributes a number to each strand of the link in-between two crossings while travelling along the link, starting from any crossing. The DT strands therefore do not coincide directly with the generators of the Wirtinger presentation, since over-strands are counted multiple times. The DT code also induces an ordering on the link components, dubbed the *DT order*.

**Algorithm 4.3.1:** Extended Wirtinger presentation

**Data:** Oriented DT code of  $K \cup L$

**Data:** DT order of the distinguished component  $K$

**Result:** Extended Wirtinger presentation of  $K \cup L$  with  $\ell_K$

**Remark 4.3.2.** It is crucial for all our studies of slope invariants to be able to compute the *extended* Wirtinger presentation, in such a way that the generators are *explicitly* associated with their corresponding strands in the link diagram. It turns out that amongst the many knot theory computer programs available, none meets these precise requirements. We therefore designed our own system to achieve this. It is based on the program `PLink` created by M. Culler and N. Dunfield [CD08], which is a lightweight graphical link editor which allows to draw a link by hand and can also give the oriented DT code of the link with an explicit display of the number assigned to each strand. Our own [Algorithm 4.3.1](#) will then compute the presentation. The initial DT numbers are kept inside the generators' names throughout all our computations, including after simplifications are performed.  $\square$

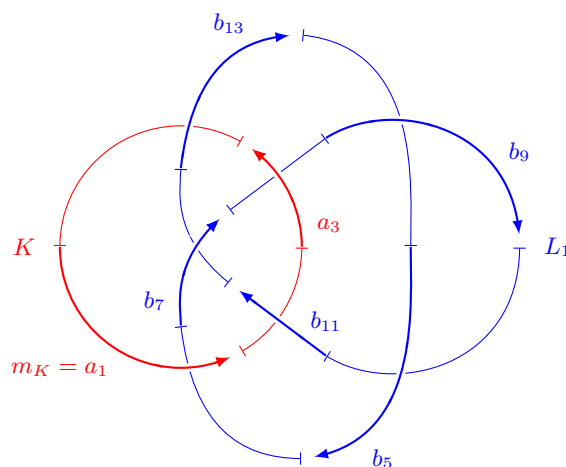


Figure 4.3.1: Link L7a1

**Example 4.3.3 (L7a1).** The link L7a1 represented on [Figure 4.3.1](#) has two components  $K, L$  with  $\text{lk}(K, L) = 0$ . A simplified extended Wirtinger presentation of the link group is given by

$$\pi_1(M_{K \cup L}) = \left\langle b_9, b_{13}, m_K, \ell_K \left| \begin{array}{l} r_0 : \ell_K^{-1} b_9^{-1} b_{13}^{-1} b_9 b_{13} b_9 b_{13}^{-1} \\ r_1 : \ell_K b_{13} m_K^{-1} b_9^{-1} b_{13}^{-1} b_9 m_K b_{13}^{-1} m_K^{-1} b_9^{-1} b_{13} b_9 m_K \\ r_2 : [\ell_K, m_K] \end{array} \right. \right\rangle$$

Let  $\omega$  be a unitary character with  $\omega|_K = 1$  and  $\omega|_L = b \in S^1$ . The twisted Alexander matrix  $A^\omega$

is given by

$$\begin{matrix} d\ell_K & dm_K & db_9 & db_{13} \\ \begin{matrix} dr_0 \\ dr_1 \\ dr_2 \end{matrix} \end{matrix} \begin{bmatrix} -1 & 0 & -b^{-1}(1-b^{-1}-b) & b^{-1}(1-b^{-1}-b) \\ 1 & -b+2-b^{-1} & b^{-1}(-b+2-b^{-1}) & -b^{-1}(-b+2-b^{-1}) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have

$$\ker A_C^\omega = \begin{bmatrix} dr_0 & dr_1 & dr_2 \\ 1 + (1-b^{-1}-b)^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$A_K^\omega(\ker A_C^\omega) = \begin{bmatrix} d\ell_K & dm_K \\ -(1-b^{-1}-b)^{-1} & -b+2-b^{-1} \\ 0 & 0 \end{bmatrix}$$

The value of the slope is then

$$s_{K/L}(\omega) = (1-b)(1-b^{-1})(1-(b+b^{-1}))$$

which is a real number as expected.  $\square$

The following result is a reformulation using the slope of a property of abelian Fox calculus already observed by R. Crowell [Cro71].

**Proposition 4.3.4.** *If  $L'$  is a boundary link then for any  $(1, \mu)$ -colouring  $L' = K \cup L$ , the slope function  $s_{(K/L)}$  is identically zero.  $\triangleleft$*

*Proof.* It is known that if  $L'$  is a boundary link, then all its longitudes are commutators of commutators, i.e. for every component  $K$  of  $L'$  there exist meridians  $a, b, c, d \in \pi_1(M_{L'})$  such that

$$\ell_K = [[a, b], [c, d]]$$

An elementary computation shows that inside  $\mathbb{Z}[H_1(M_{L'})]$  the derived vector  $d\ell_K$  is zero. In particular this is still the case after composition by a character  $\omega : \mathbb{Z}[H_1(L')] \rightarrow \mathbb{C}^*$ . As a consequence we have:

$$d\ell_K \in \text{im}(A_K^\omega \cdot (\ker A_C^\omega))$$

By [Theorem 4.3.1](#) this matrix has rank 1 and therefore  $d\ell_K$  generates its row-space. This necessarily means that  $s_{(K/L)}(\omega) = 0$ .  $\square$

CHAPTER

5

THE  $SL_2(\mathbb{C})$ -SLOPE OF KNOTS

Outline of the current chapter

<b>5.1 Introduction</b>	<b>95</b>
<b>5.2 Representation varieties and A-polynomial</b>	<b>97</b>
5.2.1 Representation and character varieties	97
5.2.2 The character variety of $\mathbb{Z}^2$	98
5.2.3 The A-polynomial	99
<b>5.3 The <math>SL_2(\mathbb{C})</math>-slope invariant</b>	<b>99</b>
5.3.1 Admissible representations	99
5.3.2 The slope of characters	100
5.3.3 Regularity and properties of the slope	101
5.3.4 Slope and Reidemeister torsion	103
5.3.5 Compute the slope	105
<b>5.4 The <math>SL_2(\mathbb{C})</math>-slope and the A-polynomial</b>	<b>107</b>
5.4.1 The derivation formula	107
5.4.2 Detecting the unknot	108
5.4.3 The slope at an ideal point	109

5.1 Introduction

The character set of all representations of the group of a knot  $K$  in  $SL_2(\mathbb{C})$  carries naturally the structure of an algebraic set. Given a peripheral structure of the knot, the character variety is a plane curve in  $\mathbb{C}^* \times \mathbb{C}^*$ , whose coordinates  $M$  and  $L$  correspond to the eigenvalues of the meridian  $m$  and the preferred longitude  $\ell$ . The polynomial  $A_K(L, M)$  defining this curve is an invariant of the knot, called the A-polynomial.

In this chapter, our motivations come, among others, from the following result of Boden:

**Theorem 5.1.1** ([Bod14]). *If the  $M$ -degree  $\deg_M A_K(L, M)$  of the A-polynomial is zero, then  $K$  is the trivial knot.* ◁

This result motivates the systematic study of the *logarithmic Gauss map* of the A-polynomial

$$\frac{M}{L} \cdot \frac{\partial_M A_K(L, M)}{\partial_L A_K(L, M)} \tag{5.1}$$

where  $\partial_M$  and  $\partial_L$  denote the partial derivatives. By [Theorem 5.1.1](#), this rational function vanishes identically on  $\{A_K = 0\}$  if and only if  $K$  is trivial.

The logarithmic Gauss map has introduced in [\[GKZ94\]](#) by I. M. Guelfand, M. M. Kapranov and A. V. Zelevinsky in order to study some determinantal varieties. Then it has been used for instance by G. Mikhalkin in [\[Mik00\]](#) for studying the topology of arrangements of real plane curves. In [\[GM21\]](#), A. Guilloux and J. Marché showed it is related with the volume function of the  $A$ -polynomial of knots, or more generally of exact polynomials.

Our proposal is to develop a homological point of view on this function, by extending the constructions of A. Degtyarev, V. Florens and A. G. Lecuona [\[DFL22b; DFL21\]](#) already presented in [Chapter 4](#) to the setting of non-abelian representations. Let  $K$  be an oriented knot in the 3-sphere  $S^3$  with exterior  $M_K$ . Denote by  $R(M_K)$  and  $X(M_K)$  the  $\mathrm{SL}_2(\mathbb{C})$ -representation and character varieties of the knot  $K$ . We consider representations  $\rho: \pi_1(M_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$  composed with the adjoint action of  $\mathrm{SL}_2(\mathbb{C})$  on the Lie algebra  $\mathrm{Ad}: \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))$ , and show that there is a non-empty Zariski open subset of  $X(M_K)$  such that for all  $\rho$  in this subset

- there is an element  $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$  such that  $(v_\rho \otimes \ell, v_\rho \otimes m)$  is a basis of the homology group  $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \simeq \mathbb{C}^2$  with coefficients twisted by  $\mathrm{Ad} \circ \rho$ , and
- the kernel of the homomorphism induced by the inclusion:

$$\mathcal{Z}(\partial M_K; \mathrm{Ad} \circ \rho) := \ker \left( H_1(\partial M_K; \mathrm{Ad} \circ \rho) \xrightarrow{i_*} H_1(M_K; \mathrm{Ad} \circ \rho) \right)$$

is generated by a single vector of the form  $a v_\rho \otimes \ell + b v_\rho \otimes m$  for some element  $[b : a] \in \mathbb{CP}^1$ .

The representations which verify these conditions are called *admissible*. We define the slope of  $K$  at the admissible representation  $\rho$  by

$$s_K(\rho) := -\frac{b}{a} \in \mathbb{CP}^1$$

We prove that representations which restrict to *non-parabolic* representations of the boundary  $\partial M_K$  of  $M_K$  are admissible, see [Lemma 5.3.4](#). If  $\rho$  is a boundary-parabolic representation, we define the slope  $s_K(\rho)$  as the modulus of the euclidean structure induced by the restricted representation on  $\pi_1(\partial M_K)$ , see [Section 5.3.3](#). It turns out that these two different definitions fit well and that the following holds.

**Proposition 5.1.2.** *The slope depends only on the conjugacy classes of the representations and induces a rational function*

$$s_K : X \subset X(M_K) \longrightarrow \mathbb{CP}^1$$

on each irreducible component  $X$  of the character variety. ◁

Note that if the representation is real or unitary, then  $s_K$  takes values in  $\mathbb{RP}^1$  (see [Proposition 5.3.13](#)). For any knot, the function  $s_K$  can be computed by Fox calculus, see [Section 5.3.5](#). We illustrate the method in the case of the trefoil knot, and further compute the slope of the figure-eight knot.

The following theorem relates  $s_K$  to the original motivation; a precise statement is given in [Theorem 5.4.1](#).

**Theorem 5.1.3.** *The slope function  $s_K$  equals minus the logarithmic Gauss map of the  $A$ -polynomial defined in [Eq. \(5.1\)](#). ◁*

We also relate  $s_K$  to the change of curve factor for the Reidemeister torsion. Let  $\mathbb{T}_{M_K, \ell}(\rho)$  and  $\mathbb{T}_{M_K, m}(\rho)$  be the Reidemeister torsions according to homology bases induced by the choices of the curves  $\ell$  and  $m$  in  $\partial M_K$ , see [Section 5.3.4](#).

**Proposition 5.1.4.** *The slope coincides with the quotient of Reidemeister torsion:*

$$s_K(\rho) = \frac{\mathbb{T}_{M_K, \ell}(\rho)}{\mathbb{T}_{M_K, m}(\rho)}$$

for all  $\rho$  such that this formula is well-defined. ◁

J. Porti had already observed ([\[Por97, Corollary 4.9\]](#)) that the logarithmic Gauss map of the  $A$ -polynomial could be expressed as a ratio of torsions -up to a sign-, and that this ratio of

torsions is equal to the modulus of  $\rho$  when it is a boundary-parabolic representation ([Por97, Proposition 4.7]). Our point of view permits to fix and compute the sign ambiguity. Moreover, our results [Proposition 5.1.2](#) and [Theorem 5.1.3](#) are more general, since they do not require the Reidemeister torsion to be well-defined, for instance they hold for high dimensional components of the character variety.

Finally, we consider *ideal* points of the  $A$ -polynomial, those are points added at infinity in a compactification of the curve  $\{A(L, M) = 0\}$  in  $\mathbb{C}^2$ . In [CS83], M. Culler and P. Shalen constructed incompressible surfaces in  $M_K$  associated to such points. Those surfaces have a non-empty boundary, whose slope is determined by a rational number  $p/q$ . We prove the following theorem:

**Theorem 5.1.5.** *Let  $y$  be an ideal point in a one-dimensional component  $Y$  of the  $A$ -polynomial. Then the value of the slope function at the ideal point  $y$  equals minus the boundary slope of an incompressible surface corresponding to  $y$  or minus the slope of the corresponding side of the Newton polygon of the  $A$ -polynomial.  $\triangleleft$*

This theorem sheds some light on the main theorem of [Coo+94], which states that the boundary slopes of the Culler–Shalen surfaces are boundary slopes of the Newton polygon of the  $A$ -polynomial. Indeed it is well-known that the logarithmic Gauss map converges at those ideal points to the value of the slope of the corresponding boundary of the Newton polygon.

To conclude this introduction, we mention that the slope invariant can be extended to orthogonal (real) representations of link groups. In this more general setting, the first twisted homology space  $H_1(\partial M_K, \rho)$  can have an arbitrary dimension higher than 2 and the kernel  $\mathcal{Z}(K, \rho)$  might not be a line any more. However, the space  $H_1(\partial M_K, \rho)$  carries a natural symplectic structure given by the (twisted) intersection form on  $\partial M_K$ , and  $\mathcal{Z}(K, \rho)$  is still a Lagrangian subspace. A construction of V. I. Arnol'd [Arn67] related to the Maslov index allows to construct a *generalised slope* for this context, lying in  $S^1 \subset \mathbb{C}^*$ . As it turns out, in the case of a representation  $\rho: \pi_1(M_K) \rightarrow \mathrm{SU}_2(\mathbb{C})$ , both theories coincide via the natural isomorphism  $\mathbb{R}\mathbb{P}^1 \simeq S^1$ .

In [Section 5.2](#) we collect basic definitions on character varieties and  $A$ -polynomials. In [Section 5.3](#) we define the slope invariant and we prove [Proposition 5.1.2](#) and [Proposition 5.1.4](#). In [Section 5.4](#) we prove [Theorem 5.1.3](#). Only the  $A$ -polynomial is concerned by [Section 5.4.2](#), where we prove [Theorem 5.1.1](#). Finally, in [Section 5.4.3](#) we prove [Theorem 5.1.5](#).

This chapter is adapted from work in collaboration with L. Bénard and V. Florens [BFR21].

## 5.2 Representation varieties and $A$ -polynomial

This section is devoted to definitions and properties of representations spaces and character varieties ([Section 5.2.1](#)). We compute the character variety of the group  $\mathbb{Z}^2$  in [Section 5.2.2](#) and define the  $A$ -polynomial of knots in [Section 5.2.3](#). References for character varieties are [Sha01; Sik12], the  $A$ -polynomial was first defined in [Coo+94], see also [CL98].

### 5.2.1 Representation and character varieties

Let  $\Gamma$  be a finitely generated group. The *representation variety* is the affine algebraic set

$$R(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$$

If  $\Gamma$  is generated by  $n$  elements, the representation variety is an algebraic subset of  $\mathrm{SL}_2(\mathbb{C})^n$  given by polynomial relations corresponding to the relations of the group  $\Gamma$ . Two different presentations yield naturally isomorphic algebraic sets.

A representation  $\rho: \Gamma \rightarrow \mathrm{SL}_2(\mathbb{C})$  is *abelian* if  $\rho(\Gamma)$  is an abelian subgroup of  $\mathrm{SL}_2(\mathbb{C})$ . A representation  $\rho$  is *reducible* if there exists a proper subspace of  $\mathbb{C}^2$  invariant under the action of  $\rho(\Gamma)$ . Equivalently,  $\rho(\Gamma)$  is conjugated to a subgroup of the group of upper-triangular matrices in  $\mathrm{SL}_2(\mathbb{C})$ . Abelian representations are reducible, but the converse does not hold. The non-reducible representations are called *irreducible*.

Two representations  $\rho$  and  $\rho'$  in  $R(\Gamma)$  are equivalent if they have the same trace:

$$\rho \sim \rho' \text{ if and only if } \mathrm{Tr} \rho(\gamma) = \mathrm{Tr} \rho'(\gamma), \text{ for any } \gamma \in \Gamma.$$

The set of equivalence classes of representations coincides with the algebro-geometric quotient of  $R(\Gamma)$  by the action of  $\mathrm{SL}_2(\mathbb{C})$  by conjugation. This quotient is usually constructed through invariant theory, and is denoted

$$X(\Gamma) = R(\Gamma) // \mathrm{SL}_2(\mathbb{C})$$

Points of the character variety are called *characters*. The equivalence class of a representation  $\rho$  (the character of  $\rho$ ) is denoted by  $\chi_\rho : \Gamma \rightarrow \mathbb{C}$  with  $\chi_\rho(\gamma) = \mathrm{Tr}(\rho(\gamma))$  for  $\gamma \in \Gamma$ . If  $\Gamma$  is the fundamental group of a manifold  $W$ , we simply write  $R(W)$  and  $X(W)$  for the representation and character varieties of the manifold  $W$ .

Despite being abelian is not a well-defined notion on the character variety, the notion of being reducible makes sense there, since a reducible representation  $\rho : \Gamma \rightarrow \mathrm{SL}_2(\mathbb{C})$  can be characterised by the fact that for any  $\gamma, \delta \in \Gamma$ , the following equality holds (see for instance [CS83, Lemma 1.2.1]):

$$\mathrm{Tr} \rho(\gamma \delta \gamma^{-1} \delta^{-1}) = 2. \quad (5.2)$$

The character variety  $X(\Gamma)$  can be decomposed as

$$X(\Gamma) = X^{\mathrm{irr}}(\Gamma) \cup X^{\mathrm{red}}(\Gamma)$$

where  $X^{\mathrm{red}}(\Gamma)$  is the set of reducible characters, and its complement  $X^{\mathrm{irr}}(\Gamma)$  is the set of irreducible characters. Eq. (5.2) implies that  $X^{\mathrm{red}}(\Gamma)$  is a Zariski closed subset of  $X(\Gamma)$ .

An algebraic set is *reducible* if it can be written as a union of two proper algebraic subset, else it is *irreducible*. An irreducible component of an algebraic set is a maximal irreducible algebraic subset.

**Remark 5.2.1.** Despite  $R(\Gamma)$  or  $X(\Gamma)$  are called *varieties*, they are not quite algebraic varieties in general: they are actually reducible, and might not be reduced as schemes (some points or subspaces might have multiplicity). On the other hand, any irreducible component is irreducible, and in particular reduced, by definition.  $\square$

Two representations  $\rho$  and  $\rho'$  are *conjugate* if there exists a matrix  $M \in \mathrm{SL}_2(\mathbb{C})$  such that  $\rho(\gamma) = M \rho'(\gamma) M^{-1}$  for every  $\gamma \in \Gamma$ . Two conjugate representations define the same character; the converse is false in general, but true for elements of  $X^{\mathrm{irr}}(\Gamma)$ . More precisely, the following holds:

**Theorem 5.2.2** ([CS83, Proposition 1.5.2]). *If  $\rho$  and  $\rho'$  are two representations  $\Gamma^A \rightarrow \mathrm{SL}_2(\mathbb{C})$  with  $\rho$  irreducible and  $\chi_\rho = \chi_{\rho'}$ , then  $\rho$  and  $\rho'$  are conjugate (and  $\rho'$  is irreducible as well).*  $\triangleleft$

Two non-conjugate representations having the same character in  $X(\Gamma)$  must be reducible. For the case where  $\Gamma$  is a knot group, G. Burde and G. de Rham [Bur67; Rha67] showed that the set of characters containing non-conjugate representations is finite.

### 5.2.2 The character variety of $\mathbb{Z}^2$

We describe explicitly the character variety of a 2-torus  $S^1 \times S^1$ . Pick a basis  $m, \ell$  of the fundamental group  $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$ . Any representation in  $\mathrm{SL}_2(\mathbb{C})$  is conjugate to a representation  $\rho$  given by two commuting matrices of the form

$$\rho(m) = \begin{bmatrix} M & * \\ 0 & M^{-1} \end{bmatrix} \quad \rho(\ell) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix}$$

for  $M, L \in \mathbb{C}^*$ . Each point of the character variety  $X(S^1 \times S^1)$  has a pre-image in  $R(S^1 \times S^1)$  of the form

$$\rho(m) = \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix} \quad \rho(\ell) = \begin{bmatrix} L & 0 \\ 0 & L^{-1} \end{bmatrix} \quad M, L \in \mathbb{C}^* \quad (5.3)$$

This pre-image is unique up to the involution  $\sigma$  of  $(\mathbb{C}^*)^2$  which sends  $(L, M)$  to  $(L^{-1}, M^{-1})$ , and  $X(S^1 \times S^1)$  can be identified with the singular affine complex surface  $(\mathbb{C}^*)^2 / \sigma$ . It embeds in  $\mathbb{C}^3$  as the zeros of the polynomial

$$\Delta = x^2 + y^2 + z^2 - xyz - 4$$

Indeed, the function algebra of  $X(S^1 \times S^1)$  naturally identifies with the  $\sigma$ -invariant sub-algebra  $\mathbb{C}[M + M^{-1}, L + L^{-1}]$  of  $\mathbb{C}[L^{\pm 1}, M^{\pm 1}]$ . This algebra of invariant functions is isomorphic with

$\mathbb{C}[x, y, z]/(\Delta)$  through

$$M + M^{-1} \mapsto x \qquad L + L^{-1} \mapsto y \qquad ML + (ML)^{-1} \mapsto z$$

From this description, one sees that the singular locus of  $X(S^1 \times S^1)$  consists on the four points  $\{(L, M) = (\pm 1, \pm 1)\}$ .

### 5.2.3 The $A$ -polynomial

Let  $K$  be an oriented knot in  $S^3$  with exterior  $M_K$ . The inclusion  $\partial M_K \subset M_K$  induces an injective group homomorphism  $\pi_1(\partial M_K) \hookrightarrow \pi_1(M_K)$ . Let  $r$  be the restriction map:

$$r: X(M_K) \longrightarrow X(\partial M_K) \simeq X(S^1 \times S^1)$$

For short we denote by  $\rho_\partial = r(\rho)$  the restriction of  $\rho$  to  $\pi_1(\partial M_K)$ . By [Section 5.2.2](#), the choice of the longitude  $\ell$  and the meridian  $m$  induces an identification of  $X(S^1 \times S^1)$  with a quotient of  $(\mathbb{C}^*)^2$ . The image of  $r$  is a union of points and curves, possibly with multiplicities, see for instance [\[DG04, Lemma 2.1\]](#). Discarding the 0-dimensional components, the  $A$ -polynomial of  $K$  is the unique polynomial  $A_K(L, M)$  in  $\mathbb{C}[L, M]$  whose zero locus in  $\mathbb{C}^2$  is exactly mapped onto the image of the algebraic map  $r$ . Note that  $A_K(L, M)$  is always divisible by  $L - 1$ . This factor corresponds to the curve of reducible characters. S. Boyer, X. Zhang, N. Dunfield and S. Garoufalidis have shown the following result.

**Theorem 5.2.3** ([\[BZ05; DG04\]](#)). *Let  $K$  be a knot in  $S^3$ . The  $A$ -polynomial  $A_K(L, M)$  is equal to  $(L - 1)^k$  for some  $k$ , if and only if  $K$  is the trivial knot (and in this case  $k = 1$ ).*  $\triangleleft$

## 5.3 The $\mathrm{SL}_2(\mathbb{C})$ -slope invariant

In this section we define the slope of an admissible representation ([Section 5.3.1](#)), and observe that generic  $\mathrm{SL}_2(\mathbb{C})$ -representations are admissible. In [Section 5.3.2](#) we show that the slope is invariant by conjugation of the representation. We prove in [Section 5.3.3](#) that it yields a rational function on irreducible components of the character variety and that the slope of a real representation is a real number. Then we prove in [Section 5.3.4](#) that the slope can be written as a quotient of Reidemeister torsions. Finally, in [Section 5.3.5](#) we describe a procedure to compute the slope with an Alexander matrix.

### 5.3.1 Admissible representations

Let  $V$  be a finite dimensional  $\mathbb{C}$ -vector space, and  $\rho: \pi_1(M_K) \rightarrow \mathrm{GL}(V)$  be a representation. The representation extends to a ring homomorphism and  $V$  can be viewed as a right  $\mathbb{Z}[\pi_1(M_K)]$ -module denoted by  $V(\rho)$ . Let  $H_*(M_K; \rho)$  be the  $\rho$ -twisted homology spaces of  $M_K$  as defined in [Appendix A](#).

**Definition 5.3.1.** A representation  $\rho: \pi_1(M_K) \rightarrow \mathrm{GL}(V)$  is *admissible* if it satisfies:

- there exists  $v_\rho \in V$  such that  $\{\ell \otimes v_\rho, m \otimes v_\rho\}$  is a basis of  $H_1(\partial M_K; \rho) \simeq \mathbb{C}^2$ ,
- the kernel of the homomorphism induced by the inclusion:

$$\mathcal{Z}(\partial M_K; \rho) := \ker \left( H_1(\partial M_K; \rho) \xrightarrow{i_*} H_1(M_K; \rho) \right)$$

has dimension one.  $\diamond$

We restrict to representations  $\rho: \pi_1(M_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$ . The composition of  $\rho$  with the adjoint action  $\mathrm{Ad}$  of  $\mathrm{SL}_2(\mathbb{C})$  on  $\mathfrak{sl}_2(\mathbb{C})$  induces the following representation:

$$\begin{aligned} \mathrm{Ad} \circ \rho: \pi_1(M_K) &\longrightarrow \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C})) \\ \gamma &\longmapsto (v \mapsto \rho(\gamma)v\rho(\gamma)^{-1}) \end{aligned}$$

**Definition 5.3.2.** Let  $\rho: \pi_1(M_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$  be such that  $\mathrm{Ad} \circ \rho$  is admissible. Let

$$a(\ell \otimes v_\rho) + b(m \otimes v_\rho)$$



be a generator of  $\mathcal{Z}(\partial M_K; \mathrm{Ad} \circ \rho)$  for some  $[a : b] \in \mathbb{C}\mathbb{P}^1$ . The *slope* of the knot  $K$  at the representation  $\rho$  is

$$s_K(\rho) := -\frac{b}{a} \in \mathbb{C} \cup \infty \quad \diamond$$

**Definition 5.3.3.** A representation  $\rho: \pi_1(M_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$  is *boundary-parabolic* if the restriction  $\rho_\partial: \pi_1(\partial M_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$  is parabolic, that is  $\mathrm{Tr} \rho(\gamma) = \pm 2$  for any  $\gamma \in \pi_1(\partial M_K)$ .

A boundary-parabolic character is the character of a boundary-parabolic representation.  $\diamond$

**Lemma 5.3.4.** *Let  $\rho: \pi_1(M_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$  be a non-parabolic representation. The vector space  $H_1(\partial M_K; \mathrm{Ad} \circ \rho)$  is isomorphic to  $\mathbb{C}^2$ , and the kernel  $\mathcal{Z}(M_K; \mathrm{Ad} \circ \rho)$  has dimension 1. Moreover, if  $\rho$  is not boundary-parabolic, then  $\mathrm{Ad} \circ \rho$  is admissible.*  $\triangleleft$

*Proof.* The group  $\pi_1(M_K)$  is generated by pairwise conjugate meridians. If  $\rho$  is non-parabolic, then the image of a meridian must differ from  $\pm I_2$ , otherwise we would have  $\rho(\pi_1(M_K)) \subset \{\pm I_2\}$ . Since the adjoint representation  $\mathrm{Ad} \circ \rho \in \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C})) \simeq \mathrm{SO}_3(\mathbb{C})$  is unitary, a direct application of [Lemma A.2.1](#) on page 122 gives the dimension of  $H_1(\partial M_K; \mathrm{Ad} \circ \rho)$ . Moreover, when  $\rho$  is not boundary-parabolic, for  $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$  invariant by  $\mathrm{Ad} \circ \rho_\partial$ , the pair of vectors  $(\ell \otimes v_\rho, m \otimes v_\rho)$  forms a basis of  $H_1(\partial M_K; \mathrm{Ad} \circ \rho)$ . The constructions of [Appendices A.1.3](#) and [A.1.4](#) can also be applied to  $\mathrm{Ad} \circ \rho$  using the  $\mathbb{C}$ -bilinear Killing form on  $\mathfrak{sl}_2(\mathbb{C})$  as the standard vector product, see [[Por97](#), Section 0.3]. [Theorem A.1.9](#) thus gives the dimension of the subspace  $\mathcal{Z}(M_K; \mathrm{Ad} \circ \rho)$ .  $\square$

As an example, we compute the slope for abelian non boundary-parabolic representations. Let  $\varphi: \pi_1(M_K) \rightarrow H_1(M_K) = \mathbb{Z}$  be the abelianisation. For any  $\lambda \in \mathbb{C}^*$ , there is an abelian representation defined by

$$\begin{aligned} \rho_\lambda: \pi_1(M_K) &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ \gamma &\longmapsto \begin{bmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{bmatrix} \end{aligned} \quad (5.4)$$

and any abelian, non boundary-parabolic representation is conjugate to a representation of this form.

**Lemma 5.3.5.** *For any  $\lambda \neq \pm 1$ , the slope at the abelian representation  $\rho_\lambda$  vanishes:*

$$s_K(\rho_\lambda) = 0 \quad \triangleleft$$

*Proof.* Up to conjugation, the representation  $\mathrm{Ad} \circ \rho$  has the form

$$\mathrm{Ad} \circ \rho(\gamma) = \begin{bmatrix} \lambda^{2\varphi(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2\varphi(\gamma)} \end{bmatrix}$$

and  $\mathfrak{sl}_2(\mathbb{C})$  splits as  $\mathbb{Z}[\pi_1(M_K)]$ -module as

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}_{\lambda^2} \oplus \mathbb{C} \oplus \mathbb{C}_{\lambda^{-2}}$$

This yields a splitting in twisted homology (with abelian coefficients), for  $U = \partial M_K$  or  $U = M_K$ :

$$H_1(U; \mathrm{Ad} \circ \rho) = H_1(U; \mathbb{C}_{\lambda^2}) \oplus H_1(U; \mathbb{C}) \oplus H_1(U; \mathbb{C}_{\lambda^{-2}})$$

Since  $\lambda \neq \pm 1$ , by [Corollaries A.2.2](#) and [A.2.3](#) on page 123 for  $U = \partial M_K$  the only non-trivial summand is  $H_1(\partial M_K, \mathbb{C})$ , and the map  $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \xrightarrow{i_*} H_1(M_K; \mathrm{Ad} \circ \rho)$  coincides with the map induced by the inclusion in homology with trivial coefficients  $H_1(\partial M_K, \mathbb{C}) \rightarrow H_1(M_K, \mathbb{C})$ , whose kernel is generated by  $\ell$ .  $\square$

### 5.3.2 The slope of characters

By the following lemma, the slope does not depend on the conjugacy class of an irreducible representation. Combined with [Theorem 5.2.2](#), it follows that the slope of an irreducible representation depends only on its character.

**Lemma 5.3.6.** *Let  $\rho$  and  $\rho' : \pi_1(M_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$  be two irreducible, non boundary-parabolic representations. If  $\rho$  and  $\rho'$  are conjugate, then  $s_K(\rho) = s_K(\rho')$ .  $\triangleleft$*

*Proof.* Let  $A$  be a matrix in  $\mathrm{GL}_2(\mathbb{C})$  such that  $\rho' = A\rho A^{-1}$ . Any  $\mathrm{Ad} \circ \rho$ -invariant vector  $v_\rho \alpha^\infty \in \mathfrak{sl}_2(\mathbb{C})$  yields an  $\mathrm{Ad} \circ \rho'$ -invariant vector  $v'_\rho = Av_\rho A^{-1}$ , and the conjugation by  $A$  induces an isomorphism

$$H_1(\partial M_K; \mathrm{Ad} \circ \rho) \longrightarrow H_1(\partial M_K, \mathrm{Ad} \circ \rho')$$

which sends the basis  $\{v_\rho \otimes \ell, v_\rho \otimes m\}$  to  $\{v'_\rho \otimes \ell, v'_\rho \otimes m\}$  and the subspace  $\mathcal{Z}(\partial M_K; \mathrm{Ad} \circ \rho)$  to the subspace  $\mathcal{Z}(K, \mathrm{Ad} \circ \rho')$ . Hence  $s_K(\rho) = s_K(\rho')$ .  $\square$

**Remark 5.3.7.** There exist pairs of reducible, non-conjugate representations with the same character. Indeed, let  $\chi$  be an arbitrary reducible character in  $X(M_K)$ . Consider a representation  $\rho$  of the form  $\begin{bmatrix} \lambda(\gamma) & \\ 0 & \lambda^{-1}(\gamma) \end{bmatrix}$ , where  $\lambda : \pi_1(M_K) \rightarrow \mathbb{C}^*$  is a group homomorphism, chosen such that  $\chi(\rho) = \chi$ . Note that  $\lambda$  can further be written  $\lambda(\gamma) = \lambda^{\varphi(\gamma)}$  for some  $\lambda \in \mathbb{C}^*$  and  $\varphi : \pi_1(M_K) \rightarrow \mathbb{Z}$ . Hence the abelian representation  $\rho_\lambda$  defined in Eq. (5.4) has also character  $\chi$ , but is not conjugated in general to  $\rho$ . It turns out that they can have different slope values.

For example, consider the right-handed trefoil knot  $T$  in  $S^3$ . The character variety  $X(M_T)$  is the union of a line  $X^{\mathrm{red}}$  and a conic  $X^{\mathrm{irr}}$  in the plane. The line contains only reducible characters, and any character in the conic is irreducible except the two intersection points  $X^{\mathrm{red}} \cap X^{\mathrm{irr}}$ . Let  $\chi$  be a point in  $X^{\mathrm{red}} \cap X^{\mathrm{irr}}$ . Since  $\chi$  is reducible, there exists a  $\lambda \in \mathbb{C}^*$  such that the abelian representation  $\rho_\lambda$  has character  $\chi$ . By Lemma 5.3.5, one has  $s_T(\rho_\lambda) = 0$ . However, we show in Example 5.3.19 that the slope defines a constant function on  $X^{\mathrm{irr}}$ , everywhere equal to  $-6$ .  $\square$

### 5.3.3 Regularity and properties of the slope

We extend the slope to a rational function –locally a quotient of polynomials– on the character variety  $X(M_K)$ .

There is a component  $X^{\mathrm{red}} \subset X(M_K)$  of reducible characters only. By Remark 5.3.7 any character in  $X^{\mathrm{red}}$  is the character of an abelian representation. Hence the slope is identically zero on  $X^{\mathrm{red}}$ , see Lemma 5.3.5. Suppose now that  $X \subset X(M)$  is an irreducible component containing an irreducible character.

**Proposition 5.3.8.** *Let  $X \subset X(M)$  be an irreducible component which contains an irreducible character. The slope extends to a rational function on  $X$ , still denoted  $s_K$ . Moreover, if  $\chi \in X$  is a boundary-parabolic character then*

$$s_K(\chi) = \tau(\chi), \tag{5.5}$$

where the modulus  $\tau(\chi) \in \mathbb{C}$  is defined by taking the representative  $\rho$  of  $\chi$  satisfying

$$\rho(m) = \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix} \qquad \rho(\ell) = \begin{bmatrix} \pm 1 & \tau(\chi) \\ 0 & \pm 1 \end{bmatrix} \qquad \triangleleft$$

**Remark 5.3.9.** If  $\chi$  is the character of an irreducible representation and lies at the intersection of several irreducible components, then the value of the slope at  $\chi$  is well-defined.  $\square$

The rest of the section is devoted to the proof of Proposition 5.3.8. Lemma 5.3.10 asserts that the slope is a rational function in the neighbourhood of any irreducible, non boundary-parabolic character. For boundary parabolic characters  $\chi$ , we define the slope by the relation in Eq. (5.5) and we show that the result is still a rational function on  $X$  in Lemma 5.3.12.

**Lemma 5.3.10.** *Let  $\chi_0$  an irreducible, non-boundary-parabolic character in  $X$ . The slope is a rational function in a neighbourhood of  $\chi_0$  in  $X$ .  $\triangleleft$*

*Proof.* Let  $\rho_0$  in  $R(M_K)$  be a representation with character  $\chi_0$ . By Lemma 5.3.4 one has  $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \simeq \mathbb{C}^2$ . The set of complex lines

$$\mathbb{P}(\rho) = \mathbb{P}(H_1(\partial M_K; \mathrm{Ad} \circ \rho))$$

is a complex algebraic variety isomorphic to  $\mathbb{CP}^1$ . If  $\rho$  and  $\rho'$  are conjugate, then there is a natural algebraic isomorphism  $\mathbb{P}(\rho) \simeq \mathbb{P}(\rho')$ . It defines an algebraic  $\mathbb{CP}^1$ -fibration on a neighbourhood of  $\chi_{\rho_0}$ , and for any  $\chi$ , the complex line  $Z(M_K; \mathrm{Ad} \circ \rho)$  is an algebraic section of this fibration, independent of the choice of representation  $\rho$  with character  $\chi$ .

It remains to show that the identification  $\mathbb{P}(\rho) \simeq \mathbb{CP}^1$  is algebraic, in other words, that the choice of the basis  $(v_\rho \otimes \ell, v_\rho \otimes m)$  depends algebraically on  $\rho$ . Since  $\rho_0$  is not boundary-parabolic, we can shrink the chosen neighbourhood so that no representation  $\rho$  near  $\rho_0$  is boundary-parabolic. Then, since  $\rho_\partial$  is conjugated to a diagonal representation, there is a unique  $\mathrm{Ad} \circ \rho_\partial$ -invariant vector  $v_\rho$  with norm 1 in  $\mathfrak{sl}_2(\mathbb{C})$ . This choice depends polynomially on the entries of the matrix  $\mathrm{Ad} \circ \rho(m)$ , and then the basis  $(v_\rho \otimes \ell, v_\rho \otimes m)$  depends algebraically on  $\rho$ .  $\square$

We now consider the case of boundary-parabolic characters.

**Lemma 5.3.11.** *Let  $\rho_0$  be a boundary-parabolic representation whose character  $\chi_{\rho_0}$  lies in  $X$ . Then  $\rho_0$  is irreducible, in particular  $\rho_0(m) \neq \pm I_2$ .*  $\triangleleft$

*Proof.* For  $\rho$  reducible in  $X$ , [Bur67; Rha67] implies that  $\rho(m)$  has eigenvalues  $\lambda, \lambda^{-1}$  in  $\mathbb{C}$ , whose square is a root of the Alexander polynomial  $\Delta_{M_K}(t)$ , in particular  $\lambda \neq \pm 1$ , and  $\rho$  is not boundary-parabolic. Now for irreducible  $\rho$ , the image of any meridian must be different of  $I_2$ , since meridians generate the group  $\pi_1(M_K)$ .  $\square$

**Lemma 5.3.12.** *Let  $X \subset X(M)$  be an irreducible component containing an irreducible character, and  $\chi_0 \in X$  a boundary-parabolic character. Then the slope function  $s_K$  is rational in a neighbourhood of  $\chi_0$ .*  $\triangleleft$

*Proof.* Suppose first that  $X$  contains only boundary-parabolic characters. Any  $\chi \in X$  is the character of a representation  $\rho$  such that  $\rho(m) = \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}$ . Hence  $\chi \mapsto \tau(\chi)$  is rational.

Now we assume that  $X$  contains a non boundary-parabolic character. By definition, boundary-parabolic characters form a Zariski closed subset of  $X$ . By Lemma 5.3.10, the slope function is rational on the open, non-empty subset of  $X$  consisting of non boundary-parabolic characters. By analytic continuation, it is enough to show that

$$\lim_{\chi \rightarrow \chi_0} \alpha^\infty s_K(\chi) = \tau(\chi_0)$$

By Lemma 5.3.11 any boundary-parabolic representation  $\rho_0$  with character  $\chi_0$  is irreducible. Moreover, since  $\rho_0(m)$  can not be trivial, we can chose such a  $\rho_0$  satisfying

$$\rho_0(m) = \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}$$

For any  $\chi$  close to  $\chi_0$ , we chose similarly a representation  $\rho$  with character  $\chi$  such that

$$\rho(m) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}$$

with  $M$  close to  $\pm 1$  in  $\mathbb{C}^*$ . For such  $\rho$ , let

$$v_\rho = \begin{bmatrix} M - M^{-1} & 2 \\ 0 & M^{-1} - M \end{bmatrix}$$

be an  $\mathrm{Ad} \circ \rho_\partial$ -invariant vector. The limit at  $\rho_0$  of  $v_\rho$  is the  $(\mathrm{Ad} \circ \rho_0)_\partial$ -invariant vector  $v_{\rho_0} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ . However, a direct computation shows that  $v_{\rho_0} \otimes \ell$  and  $v_{\rho_0} \otimes m$  are linearly dependent in  $H_1(\partial M_K, \mathrm{Ad} \circ \rho_0)$ , and we cannot compute the slope of the boundary parabolic representation  $\rho_0$  by means of Definition 5.3.2. Nevertheless, the subspace  $\mathcal{Z}(K, \mathrm{Ad} \circ \rho_0)$  is one-dimensional by Lemma 5.3.4.

It implies that the map  $i_*: H_1(\partial M_K; \mathrm{Ad} \circ \rho) \rightarrow H_1(M_K; \mathrm{Ad} \circ \rho)$  has rank one at any representation  $\rho$  near  $\rho_0$ , and at  $\rho_0$  as well. In particular, for any  $\rho$  near  $\rho_0$ , the slope can be computed as the ratio of  $i_*(v_\rho \otimes \ell)$  and  $i_*(v_\rho \otimes m)$  in  $i_*(H_1(\partial M_K; \mathrm{Ad} \circ \rho))$ . This actually makes sense for  $\rho = \rho_0$  as well. An explicit computation of the boundary operator  $\partial_1: C_2(\partial M_K, \mathrm{Ad} \circ \rho_0) \rightarrow C_1(\partial M_K, \mathrm{Ad} \circ \rho_0)$  shows that the vector  $v_{\rho_0} \otimes \ell - \tau(\chi_0) \cdot v_{\rho_0} \otimes m$  belongs to  $\mathrm{im} \partial_2$ , and the equality

$$v_{\rho_0} \otimes \ell = \tau(\chi_0) \cdot v_{\rho_0} \otimes m$$

holds in  $H_1(\partial M_K, \mathrm{Ad} \circ \rho_0)$ . This implies that the ratio of  $i_*(v_{\rho_0} \otimes \ell)$  and  $i_*(v_{\rho_0} \otimes m)$  coincides with the modulus  $\tau(\chi_0)$ . This proves the lemma, and achieves the proof of Proposition 5.3.8.  $\square$

We end this section with the following observation.

**Proposition 5.3.13.** *Let  $X \subset X(M)$  be an irreducible component which contains a non boundary-parabolic representation. If  $\rho \in X$  is a real representation  $\rho: \pi_1(M_K) \rightarrow \mathrm{SL}_2(\mathbb{R})$  or  $\mathrm{SU}_2(\mathbb{C})$ , then the slope is a real number in  $\mathbb{RP}^1$ .  $\triangleleft$*

*Proof.* First assume that  $\rho$  is non-boundary-parabolic. If  $\rho$  is real, denoting by  $\mathrm{Ad} \circ \rho_{\mathbb{R}}$  the action of  $\rho$  on the Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$  (resp.  $\mathfrak{su}(2)$ ) of  $\mathrm{SL}_2(\mathbb{R})$  (resp.  $\mathrm{SU}_2(\mathbb{C})$ ), then obviously the Lagrangian  $Z(M_K; \mathrm{Ad} \circ \rho) \subset H_1(\partial M_K; \mathrm{Ad} \circ \rho)$  is the complexification of the real Lagrangian  $\mathcal{Z}(M_K, \mathrm{Ad} \circ \rho_{\mathbb{R}})$  in the real symplectic vector space  $H_1(\partial M_K, \mathrm{Ad} \circ \rho_{\mathbb{R}})$  and the slope of this real Lagrangian is the slope of its complexification, a real number. If  $\rho$  is boundary-parabolic and real, then it takes value into  $\mathrm{SL}_2(\mathbb{R})$  and the proposition follows from the definition of the modulus  $\tau$ .  $\square$

### 5.3.4 Slope and Reidemeister torsion

In this section we show that the slope coincides with the ‘change of curve term’ for the Reidemeister torsion as stated in [Proposition 5.1.4](#).

If  $\rho$  is an irreducible representation in  $X(M_K)$ , we consider the torsion of the complex  $C_*(M_K; \mathrm{Ad} \circ \rho)$  defined in [Section 5.3.1](#). This complex is naturally based from a cell decomposition of  $M_K$  and a choice of a basis of  $\mathfrak{sl}_2(\mathbb{C})$ , but not acyclic. The Reidemeister torsion is usually defined for acyclic complexes. In the case we are considering, one needs to make some additional choices to define it, namely a basis of each homology group  $H_*(M_K; \mathrm{Ad} \circ \rho)$ .

According to [[Por97](#)], one can still define the Reidemeister torsion of the cellular complex  $C_*(M_K; \mathrm{Ad} \circ \rho)$  for representations  $\rho$  in  $R(M_K)$  such that  $H_1(M_K; \mathrm{Ad} \circ \rho)$  has dimension 1. For a given curve  $\gamma \in \pi_1(\partial M_K)$ , the representation  $\rho$  is  $\gamma$ -regular if there exists a vector  $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$  such that  $v_\rho \otimes \gamma$  spans  $H_1(M_K; \mathrm{Ad} \circ \rho)$ . In this case, since there is a natural choice of a basis of  $H_2(M_K; \mathrm{Ad} \circ \rho)$ , the curve  $\gamma$  determines a homology basis of the complex  $C_*(M_K; \mathrm{Ad} \circ \rho)$  and the torsion  $\mathbb{T}_{M_K, \gamma}(\mathrm{Ad} \circ \rho) \in \mathbb{C}^*$  is defined. Note that this torsion depends only on the conjugacy class of  $\rho$ , as well as the property of being  $\gamma$ -regular.

Let  $X \subset X(M_K)$  the component containing  $\chi$ , the torsion function is the rational function

$$\mathbb{T}_{M_K, \gamma}: X \longrightarrow \mathbb{C}$$

defined as the Reidemeister torsion of the complex  $C_*(M_K, \mathrm{Ad})$  if  $\chi$  is  $\gamma$ -regular, and by  $\mathbb{T}_{M_K, \gamma}(\chi) = 0$  otherwise.

We start with the following lemma, which provides the genuine setting to define the Reidemeister torsion.

**Lemma 5.3.14.** *If  $X$  has dimension one and contains the character of a scheme-smooth representation  $\rho$ , then  $\dim H_1(M_K; \mathrm{Ad} \circ \rho) = 1$ .  $\triangleleft$*

*Proof.* The proof of [Lemma 5.3.14](#) follows from the isomorphism between the Zariski tangent space of  $X(M_K)$  at  $\rho$  and the module  $H^1(M_K; \mathrm{Ad} \circ \rho)$ , see [[Sik12](#), Theorem 1]. Scheme-smoothness implies that the Zariski tangent space is the actual tangent space, which is one-dimensional because  $X$  is.  $\square$

Note that scheme-smoothness is a Zariski open condition.

It turns out that the character variety  $X(M_K)$  of a knot exterior is often one-dimensional. This is the case if the knot is *small* (if it does not contains a closed incompressible surface [[Coo+94](#), Proposition 2.4]). This is also the case for any component  $X \subset X(M_K)$  containing the character of a lift of the holonomy representation  $\bar{\rho}: \pi_1(M_K) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ , provided that the interior of  $M_K$  admits a hyperbolic structure.

The following proposition is the main result of this section.

**Proposition 5.3.15.** *Let  $X \subset X(M)$  be an irreducible one-dimensional component which contains a scheme-smooth, non-boundary parabolic character. For all  $\chi \in X$  the following holds*

$$s_K(\chi) = \frac{\mathbb{T}_{M_K, \ell}(\chi)}{\mathbb{T}_{M_K, m}(\chi)} \triangleleft$$

We provide two different proofs of this result: one uses the natural definition of the torsion while the other relies directly on some results on the *torsion form* proved by L. Bénard in [[Bén20](#)].

### Torsion and chain complexes

This section is devoted to the proof of [Proposition 5.3.15](#) by using the chain complex of  $M_K$ . The proof is very similar to [[DFL22b](#), Theorem 3.21] or [[DFL22b](#), Theorem 6.7]. We use the following technical lemma.

**Lemma 5.3.16.** *Let  $\gamma$  be a curve in  $\pi_1(\partial M_K)$ , and  $\chi$  be an irreducible  $\gamma$ -regular character in  $X(M_K)$ . There exists a Zariski open neighbourhood of  $\chi$  such that any character in this neighbourhood is irreducible and  $\gamma$ -regular.  $\triangleleft$*

*Proof.* Being irreducible is a Zariski open condition, see [Eq. \(5.2\)](#). The  $\gamma$ -regularity follows from lower semi-continuity of the rank of a linear map. Indeed the dimension of  $H_1(M_K; \mathrm{Ad} \circ \rho)$  is upper semi-continuous. It is at least one (the dimension of  $X$ ) again because it is isomorphic to the Zariski tangent space hence it is locally constant equal to one. On the other hand, the rank of the linear map  $H_1(\gamma, \mathrm{Ad} \circ \rho) \rightarrow H_1(M_K; \mathrm{Ad} \circ \rho)$  sending  $v_\rho \otimes \gamma$  to itself is lower semi-continuous. It is at most one (the dimension of  $H_1(\gamma, \mathrm{Ad} \circ \rho)$ ) and it cannot decrease on a neighbourhood of  $\chi$ . We deduce that  $H_1(\gamma, \mathrm{Ad} \circ \rho) \rightarrow H_1(M_K; \mathrm{Ad} \circ \rho)$  is an isomorphism on a Zariski open subset.  $\square$

*Proof of [Proposition 5.3.15](#).* Let  $\chi$  be an irreducible, scheme-smooth, non boundary-parabolic character and let  $\rho$  be a representation in  $R(M_K)$  with character  $\chi$ . We first assume that  $\rho$  is  $\ell$  and  $m$ -regular, that is for  $v \in \mathfrak{sl}_2(\mathbb{C})$  an  $\mathrm{Ad} \circ \rho_\partial$ -invariant vector, both  $v \otimes \ell$  and  $v \otimes m$  provide a basis of  $H_1(M_K; \mathrm{Ad} \circ \rho)$ .

The calculation of the torsions  $\mathbb{T}_{M_K, \ell}(\chi)$  and  $\mathbb{T}_{M_K, m}(\chi)$  involves different choices of homology basis of  $C_*(M_K; \mathrm{Ad} \circ \rho)$ . By [[Por97](#), Proposition 3.18], the bases of  $H_2(M_K; \mathrm{Ad} \circ \rho)$  are determined by the fundamental class of  $H_2(\partial M_K; \mathbb{C})$  and can be chosen to be the same. Hence, if  $b_1$  is a basis of  $\mathrm{im}(\partial_1)$ , the ratio of torsions corresponding to the choice of  $m$  or of  $\ell$  is reduced to

$$\frac{\mathbb{T}_{M_K, \ell}(\chi)}{\mathbb{T}_{M_K, m}(\chi)} = \frac{\det(b_1 \oplus (v \otimes \ell), c_1)}{\det(b_1 \oplus (v \otimes m), c_1)}.$$

In parallel, consider the affine equation in  $C_1(M_K; \mathrm{Ad} \circ \rho)$ :

$$y b_1 + x v \otimes m = v \otimes \ell$$

with at least a solution  $y = 0$  and  $x = s_K(\rho)$ . The Cramer determinants expressed in the common basis  $c_1$  show that  $s_K(\rho)$  coincides with the ratio of torsions.

If there exist a character in  $X$  which is  $\ell$ -regular and a character in  $X$  which is  $m$ -regular, then [Proposition 5.3.15](#) holds on the whole component  $X$  by [Lemma 5.3.16](#).

Assume that  $X$  contains only characters that are not (say)  $\ell$ -regular. Since the map  $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \rightarrow H_1(M_K; \mathrm{Ad} \circ \rho)$  is not trivial (by [Lemma 5.3.4](#)), it is onto on a Zariski open subset  $U \in X$ , again because  $H_1(M_K; \mathrm{Ad} \circ \rho)$  has dimension one generically. Thus all characters in  $U$  must be  $m$ -regular, and it follows from the definition that the slope and the quotient of torsions are identically zero on  $X$ . A similar argument works replacing  $\ell$  by  $m$  and zero by infinity.  $\square$

### The torsion form

In this paragraph, we present an alternative proof of [Proposition 5.3.15](#). We follow a slightly different point of view on the torsion, as a volume form on the character variety. The following lemma asserts that the cotangent space of the character variety [[Sik12](#), Section 8] is isomorphic to the first  $\mathrm{Ad} \circ \rho$ -twisted homology group.

**Lemma 5.3.17.** *Let  $\chi$  be an irreducible character in  $X(M_K)$ , and let  $\rho$  be a representation with character  $\chi$ . Let  $T_\chi^* X(M_K)$  be the Zariski tangent space of  $X(M_K)$  at  $\chi$ . There is a natural isomorphism*

$$H_1(M_K; \mathrm{Ad} \circ \rho) \simeq T_\chi^* X(M_K)$$

Moreover, if  $\chi$  is not boundary-parabolic, then

$$H_1(\partial M_K; \mathrm{Ad} \circ \rho) \simeq T_{r(\chi)}^* X(\partial M_K) \quad \triangleleft$$

The proof of [Lemma 5.3.17](#) follows from [[Sik12](#), Theorem 1]. Note that through the isomorphism, the space  $\mathcal{Z}(\partial M_K; \mathrm{Ad} \circ \rho)$  is the Zariski co-normal bundle of  $r(X(M_K))$  in  $X(\partial M_K)$ .

If  $X \subset X(M_K)$  is a one-dimensional component of the character variety which contains a scheme-smooth character, L. B enard proved in [[B en20](#), Proposition 5.1] that the *torsion form* can be written as

$$\mathrm{tor}(M_K) = \frac{1}{\mathbb{T}_{M_K, \ell}} r^* \left( \frac{dL}{L} \right) = \frac{1}{\mathbb{T}_{M_K, m}} r^* \left( \frac{dM}{M} \right) \quad (5.6)$$

where  $r^*$  is the cotangent map

$$r^*: T^*X(\partial M_K) \longrightarrow T^*X(M_K)$$

*Proof of [Proposition 5.3.15](#).* By [Eq. \(5.6\)](#), the ratio of torsions can be written as

$$\frac{\mathbb{T}_{M_K, \ell}}{\mathbb{T}_{M_K, m}} = \frac{r^*(dL/L)}{r^*(dM/M)}$$

If  $\chi$  is a non boundary-parabolic character, the character variety  $X(\partial M_K)$  is diffeomorphic to  $(\mathbb{C}^*)^2$  in a neighbourhood of  $r(\chi)$ . A local chart of  $X(\partial M_K)$  is given by taking  $\mathfrak{l}, \mathfrak{m} \in \mathbb{C}$  satisfying  $\exp \mathfrak{l} = L$  and  $\exp \mathfrak{m} = M$ . The latter ratio of torsions can be written

$$\frac{\mathbb{T}_{M_K, \ell}}{\mathbb{T}_{M_K, m}} = \frac{r^*(d\mathfrak{l})}{r^*(d\mathfrak{m})}$$

[Lemma 5.3.17](#) implies that the cotangent map  $r^*: T_{r(\chi)}^*X(\partial M_K) \rightarrow T_{\chi}^*X(M_K)$  coincides with the homomorphism in homology  $H_1(\partial M_K; \mathrm{Ad} \circ \rho) \rightarrow H_1(M_K; \mathrm{Ad} \circ \rho)$ , thus by [Lemma 5.3.4](#) the range of the map  $r^*$  is one-dimensional, and the images of the elements  $d\mathfrak{l}, d\mathfrak{m}$  are collinear. It turns out that the ratio  $\frac{r^*(d\mathfrak{l})}{r^*(d\mathfrak{m})}$  coincides with the slope by its very definition.

Finally, the formula extends to the whole  $X$  since irreducible, non boundary-parabolic character are Zariski dense in  $X$ .  $\square$

### 5.3.5 Compute the slope

In this section we compute the slope  $s_K(\rho)$  when  $\rho$  is an irreducible non-boundary parabolic representation, with Fox calculus, similarly to [Section 4.3](#) and [[DFL22b](#)]. Note that for the boundary-parabolic case, the slope can be computed directly from the representation using [Proposition 5.3.8](#).

The  $\mathrm{Ad} \circ \rho$ -twisted homology group  $H_1(M_K; \mathrm{Ad} \circ \rho)$  can be computed using the chain complex  $S_*(\mathrm{Ad} \circ \rho)$  and the Alexander matrix derived from an extended Wirtinger presentation of the knot group  $\pi_1(M_K)$ , as explained in [Appendix A.3](#).

The specific computation of the slope is achieved with the following result:

**Proposition 5.3.18.** *If  $\rho$  is irreducible and non-boundary parabolic, then there exist  $a, b \in \mathbb{C}$  and an  $\mathrm{Ad} \circ \rho$ -invariant vector  $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$  such that*

$$\mathrm{im}(\partial_1(\rho)) \cap \langle v_\rho \otimes d\ell, v_\rho \otimes dm \rangle = \langle a(v_\rho \otimes d\ell) + b(v_\rho \otimes dm) \rangle$$

and the slope is  $s_K(\rho) = -\frac{b}{a}$ .  $\triangleleft$

*Proof.* Set a base point  $p$  on  $\partial M_K$ . The subcomplex  $S_*(\rho_\partial)$  defined by considering only the generators  $x_1 = m, x_2 = \ell$  and the relation  $[m, \ell] = 1$  computes the space  $H_1(\partial M_K, p; \mathrm{Ad} \circ \rho)$ . There are natural identifications

$$\begin{aligned} H_1(\partial M_K, p; \mathrm{Ad} \circ \rho) &= H_1(\partial M_K; \mathrm{Ad} \circ \rho) \\ H_1(\partial M_K, p; \mathrm{Ad} \circ \rho) &\hookrightarrow H_1(\partial M_K; \mathrm{Ad} \circ \rho) \end{aligned}$$

and the following diagram commutes:

$$\begin{array}{ccc} S_1^\partial(\rho) & \xleftarrow{\quad} & S_1(\rho) \\ h_{\partial M_K} \downarrow & & \downarrow h \\ H_1(\partial M_K; \mathrm{Ad} \circ \rho) & \xrightarrow{i_*} & H_1(M_K; \mathrm{Ad} \circ \rho) \xleftarrow{\quad} H_1(M_K, p; \mathrm{Ad} \circ \rho) \end{array}$$

where  $h$  and  $h_{\partial M_K}$  are the quotient maps.

Let  $u \in \mathfrak{sl}_2(\mathbb{C})$  be an  $\mathrm{Ad} \circ \rho$ -invariant vector and  $\gamma \in \pi_1(M_K)$ . Any element  $u \otimes \gamma$  of  $H_1(\partial M_K; \mathrm{Ad} \circ \rho)$  can be lifted to  $u \otimes dw$  in  $S_1^\partial(\rho)$ . Since  $\rho$  is admissible, there exist  $a, b \in \mathbb{C}$  such that  $\mathcal{Z}(\partial M_K; \mathrm{Ad} \circ \rho) = \ker i_* = \langle a(v_\rho \otimes \ell) + b(v_\rho \otimes m) \rangle$ . Then

$$a(v_\rho \otimes d\ell) + b(v_\rho \otimes dm) \in \ker(h) = \mathrm{im}(\partial_1(\rho))$$

Reciprocally, suppose that there exist complex numbers  $a, b \in \mathbb{C}$  such that

$$dz := a(v_\rho \otimes d\ell) + b(v_\rho \otimes dm)$$

is a non-zero vector belonging to  $\mathrm{im}(\partial_1(\rho))$ . Then

$$h_{\partial M_K}(dz) = a(v_\rho \otimes \ell) + b(v_\rho \otimes m)$$

must be non-zero since  $(v_\rho \otimes \ell, v_\rho \otimes m)$  is a free basis of  $H_1(\partial M_K; \mathrm{Ad} \circ \rho)$ . However, we have  $h(dz) = i_*(h_{\partial M_K}(dz)) = 0$  hence  $h_{\partial M_K}(dz) \in \ker i_*$ . Since  $\ker i_*$  is one-dimensional, then  $\ker i_* = \langle h_{\partial M_K}(dz) \rangle$  and the slope is  $-\frac{b}{a}$ .  $\square$

**Example 5.3.19** (Trefoil knot). Let  $T$  be the exterior of the right-handed trefoil knot, with group  $\pi_1(M_K) = \langle u, v \mid uvu = vuv \rangle$ . Any irreducible representation is conjugate to  $\rho$  with

$$\rho(u) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \rho(v) = \begin{bmatrix} M^{-1} & 0 \\ -1 & M \end{bmatrix}$$

where  $M \in \mathbb{C}$ . If  $\ell = vuv^{-1}uvu^{-3}$  is the preferred longitude with corresponding meridian  $m = u$ , we obtain

$$\rho(\ell) = \begin{bmatrix} -M^{-6} & M^5 + M^3 + M + M^{-1} + M^{-3} + M^{-5} \\ 0 & -M^6 \end{bmatrix}$$

Whenever  $M \neq \pm 1$ , the vector  $v_\rho = [0, 1, \frac{1}{M-M^{-1}}]$  is right  $\mathrm{Ad} \circ \rho_\partial$ -invariant. The Alexander matrix (acting on the right on the coefficients) whose row-space is generating  $\mathrm{im}(\partial_1)$  is given by

$$\begin{array}{c} \mathfrak{sl}_2(\mathbb{C}) \otimes d\ell \quad \quad \quad \mathfrak{sl}_2(\mathbb{C}) \otimes dm \quad \quad \quad \mathfrak{sl}_2(\mathbb{C}) \otimes dv \\ \begin{array}{l} dr_1 \\ dr_2 \end{array} \left[ \begin{array}{ccccccc} 0 & 0 & 0 & \vdots & 1 & 0 & 0 & \vdots & -M^{-2} & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 & \vdots & -M^{-1} & -1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 & \vdots & 1 & 2M & -M^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \vdots & -2(2-M^{-2}) & 0 & 0 & \vdots & 1+M^{-2} & 0 & 0 \\ 0 & -1 & 0 & \vdots & 2M^{-1} & -2 & 0 & \vdots & M^{-1} & 2 & 0 \\ 0 & 0 & -1 & \vdots & -2 & -4M & 2M^2-4 & \vdots & -1 & -2M & M^2+1 \end{array} \right] \end{array}$$

where  $r_1$  is  $uvu = vuv$  and  $r_2$  is the longitude definition. By [Proposition 5.3.18](#), the space  $Z(M_K; \mathrm{Ad} \circ \rho)$  has generator

$$\left[0, 1, \frac{1}{M-M^{-1}}, 0, 6, \frac{6}{M-M^{-1}}, 0, 0, 0\right]$$

in the 2-dimensional subspace spanned by

$$\left\{ \left[0, 1, \frac{1}{M-M^{-1}}, 0, 0, 0, 0, 0\right], \left[0, 0, 0, 0, 1, \frac{1}{M-M^{-1}}, 0, 0, 0\right] \right\}$$

and the slope is  $s_T(\mathrm{Ad} \circ \rho) = -6$ . In particular it does not depend on  $\rho$ .  $\square$

**Example 5.3.20** (Figure-eight knot). Let  $K$  be the figure-eight knot. There is a unique component  $X \subset X(M_K)$  containing irreducible characters (see for instance [Bén20, Examples 1.6.2 and 5.5]). This component is a plane curve given by the equation

$$\{2x^2 + y^2 - x^2y - y - 1 = 0\} \subset \mathbb{C}^2$$

where  $x$  is the coordinate function given by  $\chi \mapsto \chi(m)$ . Note that the coordinate function of the longitude is  $\chi \mapsto \chi(\ell) = x^4 - 5x^2 + 2$ . Using [Por97, Théorème 4.1 (ii)] and Proposition 5.3.15 we compute

$$s_K(x, y)^2 = \frac{x^2 - 4}{(x^4 - 5x^2 + 2)^2 - 4} (4x^3 - 10x)^2 = \frac{4(2x^2 - 5)^2}{(x^2 - 5)(x^2 - 1)}$$

Expanding the denominator with the relation  $x^2 = \frac{y^2 - y - 1}{y - 2}$ , we obtain, up to sign

$$s_K(x, y) = \pm \frac{2(2x^2 - 5)(y - 2)}{(y - 1)(y - 3)} \quad \square$$

## 5.4 The $\mathrm{SL}_2(\mathbb{C})$ -slope and the $A$ -polynomial

In this section, we express the slope function in terms of the  $A$ -polynomial of the knot. As mentioned in Section 5.2.3,  $r(X)$  might have 0-dimensional components but they are omitted in the definition of the  $A$ -polynomial.

### 5.4.1 The derivation formula

**Theorem 5.4.1.** *Let  $X \subset X(M_K)$  be an irreducible component such that  $r(X)$  has dimension 1. For all  $\chi \in X$  with  $r(\chi) = (L, M)$ , the following holds*

$$s_K(\chi) = -\frac{M}{L} \cdot \frac{\partial_M A(L, M)}{\partial_L A(L, M)},$$

where  $A(L, M) = A_K(L, M)$  and  $\partial_L$  and  $\partial_M$  are the partial derivatives.  $\triangleleft$

**Remark 5.4.2.** Combining Proposition 5.3.15 with [Por97, Corollaire 4.9], the result of Theorem 5.4.1 follows directly, up to sign, in the case where  $X$  has itself dimension 1. We resolve those two issues. Moreover Theorem 5.4.1 does not require the characters in  $X$  to be scheme-reduced, and the factors of the  $A$ -polynomial might have multiplicities greater than 1.  $\square$

*Proof.* From Lemma 5.3.17 it follows that the Lagrangian  $Z(M_K; \mathrm{Ad} \circ \rho)$  generically identifies with the Zariski co-normal bundle of  $r(X(M_K))$  in  $X(\partial M_K)$ . Picking local coordinates  $l = \log L$ ,  $m = \log M$  around  $r(\chi)$ , the kernel of the cotangent map is generated by

$$dA(e^l, e^m) = \partial_l A(e^l, e^m) dl + \partial_m A(e^l, e^m) dm$$

in  $\mathbb{C}^2 = \langle dl, dm \rangle$ . Using the chain rule, we obtain that it is generated by the vector

$$\left( L \frac{\partial A(M, L)}{\partial L}, M \frac{\partial A(M, L)}{\partial M} \right)$$

and the proposition follows.  $\square$

**Remark 5.4.3.** Let  $T$  be the right-handed trefoil knot, with  $A_T(L, M) = 1 + LM^6$ . Applying Theorem 5.4.1 gives

$$s_T = -\frac{M}{L} \cdot \frac{6M^5 L}{M^6} = -6$$

Compare to Example 5.3.19.  $\square$



### 5.4.2 Detecting the unknot

Let  $K$  be an oriented knot in  $S^3$ , and  $A_K(L, M)$  be the  $A$ -polynomial of  $K$ . In this subsection we prove the following theorem, whose proof is a refinement of arguments due to S. Boyer and X. Zhang [BZ05]:

**Theorem 5.4.4.** *If  $\deg_M(A_K(L, M)) = 0$ , then  $K$  is the trivial knot.*  $\triangleleft$

The following corollary asserts that the slope detects the trivial knot:

**Corollary 5.4.5.** *Let  $K$  be a knot in  $S^3$  such that the slope  $s_K$  is identically zero. Then  $K$  is the trivial knot.*  $\triangleleft$

*Proof.* Since  $s_K(\mathrm{Ad} \circ \rho) \equiv 0$ , by [Theorem 5.4.1](#) the  $A$ -polynomial of  $K$  consists of a collection of lines  $L = \alpha_i$ . This is prohibited by [Theorem 5.4.4](#), unless  $K$  is the trivial knot.  $\square$

The rest of the section is devoted to the proof of [Theorem 5.4.4](#).

*Proof of [Theorem 5.4.4](#).* We assume that the  $M$ -degree of the  $A$ -polynomial is zero, and we will prove that  $K$  must be the trivial knot. The  $A$ -polynomial of  $K$  can be written as a finite product

$$A_K(M, L) = \prod_i (L - \alpha_i)$$

**Claim.** *The  $\alpha_i$  are roots of unity.*  $\triangleleft$

*Proof of the claim.* By [Coo+94, Proposition 3.1], compactifying the line  $L - \alpha_i$  in  $\mathbb{CP}^2$  yields an ideal point which produces an incompressible surface  $S$  in  $M$  whose boundary curves are parallel to the longitude  $\ell$ . Moreover, by the *root of unity phenomenon* ([Coo+94, Theorem 5.7]), there is an associated representation  $\rho: \pi_1(S) \rightarrow \mathrm{SL}_2(\mathbb{C})$  such that the eigenvalues  $\rho(\ell)$  are roots of unity. By construction, those eigenvalues are precisely  $\alpha_i^{\pm 1}$ , proving the claim.  $\square$

Assume now that  $K$  is not the trivial knot. Let  $M_K(r)$  the 3-manifold obtained by Dehn surgery on  $K$ , with coefficient  $r \in \mathbb{Q}$ . We use the following result due to P. Kronheimer and T. Mrowka:

**Theorem 5.4.6** ([KM04]). *For any non-zero integer  $n$ , there is an irreducible representation  $\rho_{1/n}: \pi_1(M_K(1/n)) \rightarrow \mathrm{SU}_2(\mathbb{C})$  with non-cyclic image.*  $\triangleleft$

Composing with the epimorphism  $\pi_1(M_K) \rightarrow \pi_1(M_K(1/n)) = \pi_1(M_K)/\langle\langle m\ell^n \rangle\rangle$ , the representations  $\rho_{1/n}$  yield a family of irreducible representations still denoted by

$$\rho_{1/n}: \pi_1(M_K) \longrightarrow \mathrm{SU}_2(\mathbb{C}) \subset \mathrm{SL}_2(\mathbb{C})$$

whose character is denoted  $\chi_{1/n} \in X(M_K)$ . For all  $i$ , the image  $r(\chi_{1/n})$  of the character  $\chi_{1/n}$  in  $X(\partial M_K)$  belongs to  $\{L - \alpha_i = 0\}$  if  $\rho_{1/n}$  is conjugated to

$$\rho(\ell) = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_i^{-1} \end{bmatrix} \quad \rho(m) = \begin{bmatrix} \alpha_i^{-n} & 0 \\ 0 & \alpha_i^n \end{bmatrix}$$

By the claim above,  $\alpha_i$  is a root of unity. Let  $d_i$  be its order.

**Claim.** *For any  $i$ , for any  $k \in d_i\mathbb{Z}$ , the image  $r(\chi_{1/k})$  does not belong to  $\{L - \alpha_i = 0\}$ .*  $\triangleleft$

*Proof of the claim.* Indeed  $d_i|k$  implies  $\rho_k(m) = \mathrm{Id}$ , contradicting the irreducibility of  $\rho_k$ .  $\square$

Finally, we have proved the intermediate result:

**Lemma 5.4.7.** *For  $d = \prod d_i$ , the characters  $\{r(\chi_{\rho_{1/d_n}})\}_{n \geq 1}$  do not belong to any of the lines  $\{L - \alpha_i = 0\}$  for all  $i$ . In particular, this whole family of characters collapse to a finite family of isolated points in  $r(X(M_K)) \subset X(\partial M_K)$ .*  $\triangleleft$

By similar arguments, the following holds:

**Claim.** *It there are  $k \neq k' \in d\mathbb{Z}$  such that  $r(\chi_{\rho_{1/k}}) = r(\chi_{\rho_{1/k'}})$  in  $X(\partial M_K)$ , then  $\rho_{1/k}(\ell)$  is conjugated to  $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$  where  $\alpha$  is a root of unity of order  $p_0 \mid (k - k')$ , but  $p_0$  does not divide neither  $k$  nor  $k'$ .*  $\triangleleft$

*Proof of the claim.* Since  $r(\chi_{\rho_{1/k}}) = r(\chi_{\rho_{1/k'}})$ , we have

$$\rho_{1/k}(m) = \rho_{1/k'}(m) \text{ and } \rho_{1/k}(\ell) = \rho_{1/k'}(\ell)$$

Moreover

$$\rho_{1/k}(m\ell^k) = \rho_{1/k'}(m\ell^{k'}) = \mathrm{Id}.$$

Hence  $\rho_{1/k}(\ell)^k = \rho_{1/k}(\ell)^{k'}$  and  $\rho_{1/k}(\ell)$  has order dividing  $k - k'$ . This implies that the order of the eigenvalue  $\alpha$  divides  $k - k'$  as well. As above, this order  $p_0$  cannot divide  $k$ , neither  $k'$ , otherwise the representation  $\rho_{1/k}$  would be trivial.  $\square$

Now, all isolated points  $x_i$  of  $X(\partial M_K)$  in  $r(\{\rho_{1/d_n}\}_{n \geq 1})$  yield an eigenvalue  $\alpha_i$  of finite order  $p_i$ . We showed that if  $\rho_{1/d_n}, \rho_{1/d'_n}$  are mapped to  $x_i$ , then  $d_n \equiv d'_n \not\equiv 0 \pmod{p_i}$ . In particular, with  $p = \prod p_i$ , the representation  $\rho_{np}$  is not mapped on any of the isolated points  $x_i$ . Since it is neither mapped into one of the lines  $\{L - \alpha_i = 0\}$ , this gives a contradiction and proves that  $K$  is the trivial knot.  $\square$

### 5.4.3 The slope at an ideal point

In this section we prove [Theorem 5.1.5](#). The context of this result is the work of M. Culler and P. Shalen (see for instance [\[Sha01\]](#)) which associates incompressible surfaces in  $M_K$  to ideal points of curves of  $X(M_K)$ .

Let  $X \subset X(M_K)$  be an irreducible component whose image  $r(X) = Y$  is a curve in  $X(\partial M_K)$ , defined as the zero locus of an irreducible factor  $P$  of  $A_K(L, M)$ . Its function ring is usually denoted by  $\mathbb{C}[Y] = \mathbb{C}[L, M]/(P)$ , and its function field is  $\mathbb{C}(Y) = \mathrm{Frac}(\mathbb{C}[Y])$ .

To any point  $y$  in  $Y$  one can associate a *discrete valuation*  $v$  on the multiplicative group  $\mathbb{C}(Y)^*$  in the field  $\mathbb{C}(Y)$  of rational functions on  $Y$ . A discrete valuation  $v: \mathbb{C}(Y)^* \rightarrow \mathbb{Z}$  is a group epimorphism satisfying  $v(f + g) \geq \min(v(f), v(g))$ . The valuation associated to a smooth point  $y$  is simply the map

$$f \longmapsto v_y(f) = \mathrm{ord}_y f$$

given the vanishing order of  $f$  at the point  $y$ . More generally, the smooth projective model  $\overline{Y}$  of  $Y$  is smooth compact curve bi-rational to  $Y$ , canonically defined up to isomorphism, and the points of  $\overline{Y}$  are bijectively associated to discrete valuations on the function field  $\mathbb{C}(Y) \simeq \mathbb{C}(\overline{Y})$ .

An *ideal point*  $y$  of  $Y$  is a point added ‘at infinity’ in the smooth projective model  $\overline{Y}$ , it corresponds to a valuation  $v_y$  on  $\mathbb{C}(Y)$  such that not every regular function  $f \in \mathbb{C}[Y]$  has non-negative valuation  $v_y(f)$ . In other words, some regular functions (at least one) should have poles at  $y$ .

In [\[CS83\]](#), M. Culler and P. Shalen gave a procedure to construct an *incompressible* surface  $\Sigma$  in  $M_K$  from the data of an ideal point  $x$  in a sub-curve  $C$  of  $X(M_K)$  together with the valuation  $v_x \alpha^\infty: \mathbb{C}(C)^* \rightarrow \mathbb{Z}$ . Not any ideal point  $x \in X(M_K)$  yields an ideal point  $y = r(x) \in X(\partial M_K)$ .

In this special case, the ideal point  $y$  in  $Y$  gives an incompressible surface in  $M_K$  of a particular kind: as observed in [\[Coo+94, Proposition 3.1\]](#), the incompressible surface  $\Sigma$  must have non-empty boundary  $\partial \Sigma \subset \partial M_K$ . The curve  $\partial \Sigma$  is a finite union of parallel circles in  $\partial M_K$  and uniquely determines a *boundary slope* in  $\mathbb{Q} \cup \{\infty\}$ : the slope of  $al + bm$  in  $H_1(\partial M_K; \mathbb{Z})$  is the rational number  $\frac{b}{a}$ .

On the other hand, the Newton polygon of  $A_K(L, M) = \sum_{i,j} a_{i,j} L^i M^j$  is the convex hull in  $\mathbb{C}^2$  of the points  $\{(i, j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$ . It is a convex polygon of  $\mathbb{C}^2$  with integral vertices, whose sides have a slope in  $\mathbb{Q} \cup \{\infty\}$ . In [\[Coo+94\]](#), D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen proved the following result:

**Theorem 5.4.8** ([\[Coo+94, Theorem 3.4\]](#)). *The slopes of the sides of the Newton polygon of the  $A$ -polynomial  $A_K(L, M)$  are boundary slopes of incompressible surfaces in  $M_K$  which correspond to ideal points of one-dimensional components of  $r^*(X(M_K))$  in  $X(\partial M_K)$ .*  $\triangleleft$

Our next statement states that the slope invariant studied in this chapter coincides with the slopes of [\[Coo+94\]](#) at ideal points.

**Theorem 5.1.5.** *Let  $y$  be an ideal point in a one-dimensional component  $Y$  of the  $A$ -polynomial. Then the value of the slope function at the ideal point  $y$  equals minus the boundary slope of an incompressible surface corresponding to  $y$  or minus the slope of the corresponding side of the Newton polygon of the  $A$ -polynomial.*  $\triangleleft$

*Proof.* The coordinate functions  $L, M$  define rational functions on  $Y$ , in particular their valuations  $v_y(L)$  and  $v_y(M)$  are well-defined. Since  $y$  is an ideal point and  $L, M$  generate the coordinate ring  $\mathbb{C}[Y]$  of the curve  $Y$ , at least one of this valuation must be negative, and at least one of these coordinate functions must have a pole at  $y$ .

**Lemma 5.4.9.** *The value of  $s_K$  at the ideal point  $y$  is  $\frac{v_y(L)}{v_y(M)}$ .*  $\triangleleft$

Now the theorem follows directly from **Lemma 5.4.9**, because it is proven in [Coo+94, Proposition 3.1] that the quantity  $-\frac{v_y(L)}{v_y(M)}$  is the boundary slope of an incompressible surface corresponding to  $y$ .  $\square$

*Proof of Lemma 5.4.9.* From the proof of **Proposition 5.3.15**, we deduce that the value of the slope at  $y$  is given by

$$s_K(y) = \lim_{(L,M) \rightarrow y} \frac{r^*(dL/L)}{r^*(dM/M)}$$

The following argument is an algebraic analogue of taking Taylor expansion of the functions  $L$  and  $M$  around the ideal point  $y$ . We pick  $t$  a local coordinate around  $y$ . It is characterised by  $v_y(t) = 1$ , and we can write

$$L = u_1 t^{v_y(L)}$$

for  $u_1 \in \mathbb{C}(Y)$ ,  $v_y(u_1) = 0$ , and similarly

$$M = u_2 t^{v_y(M)}$$

for  $u_2 \in \mathbb{C}(Y)$ ,  $v_y(u_2) = 0$ . Moreover, near  $y$  it follows that

$$\frac{r^*(dL/L)}{r^*(dM/M)} = \frac{v_y(L)/t}{v_y(M)/t} = \frac{v_y(L)}{v_y(M)}$$

and the claim follows.  $\square$

# CHAPTER

# 6

# GENERALISED SLOPE OF LINKS

## Outline of the current chapter

<b>6.1 Slope of a Lagrangian</b>	<b>111</b>
6.1.1 Complexification . . . . .	112
6.1.2 Characterisation of Lagrangians . . . . .	112
<b>6.2 Generalised slope construction</b>	<b>114</b>
6.2.1 Character torus . . . . .	114
6.2.2 Symplectic structure of the twisted homology . . . . .	114
6.2.3 Definition of the generalised slope . . . . .	115
6.2.4 Relation with the character slope . . . . .	116
<b>6.3 Concordance invariance</b>	<b>117</b>

This chapter is dedicated to a new generalisation of the character slope presented in [Chapter 4](#). The generalised slope function removes the restrictions on the character of the link group and can thus be defined on the entire character torus of the link.

The key construction used for the generalised slope is the classification of the Lagrangians of a symplectic space developed by V. I. Arnol'd [\[Arn67\]](#) and presented in [Section 6.1](#), which we slightly extend to define the *slope of a Lagrangian*. Using general results developed in [Appendix A](#), [Section 6.2](#) applies this construction to the homology of the link to define the generalised slope as a function on the character torus. We also give its relation with the character slope. Finally, [Section 6.3](#) establishes that the generalised slope is a concordance invariant.

## 6.1 Slope of a Lagrangian

A *symplectic space* is a couple  $(V, q_s)$  where  $V$  is a  $\mathbb{R}$ -vector space and  $q_s$  is a definite positive antisymmetric quadratic form on  $V$ .

**Definition 6.1.1.** A Lagrangian subspace  $U$  of  $(V, q_s)$  is a subspace such that  $q_s(x, y) = 0$  for all  $x, y \in U$ . ◇

Any symplectic space  $(V, q_s)$  is of even dimension, i.e.  $\dim V = 2r$  for some  $r \in \mathbb{N}$ , and any Lagrangian  $U$  of  $V$  has dimension  $r$ . The Lagrangian Grassmannian of  $(V, q_s)$  is denoted  $\Lambda(V)$ .

**Definition 6.1.2.** A basis  $\mathcal{B} = (x_1, \dots, x_r; y_1, \dots, y_r)$  of  $(V, q_s)$  is *symplectic* if :

$$\forall i, j \in \{1, \dots, r\} : \begin{cases} q_s(x_i, y_j) = \delta_{i,j} \\ q_s(x_i, x_j) = q_s(y_i, y_j) = 0 \end{cases} \quad \diamond$$

The characterisation of Lagrangian subspaces using complexification was introduced by V. I. Arnol'd [Arn67]. Readers can refer to [ACL03] for the proofs of the standard results relating to Lagrangians recalled in Sections 6.1.1 and 6.1.2. The objective of this section is to make the following new construction:

**Theorem 6.1.3.** *Let  $(V, q_s)$  be a real symplectic space and let  $\mathcal{B}$  be a symplectic basis. Then the triple  $(V, q_s, \mathcal{B})$  gives rise to a function*

$$s : \Lambda(V) \longrightarrow \left] -\frac{\pi}{2}; \frac{\pi}{2} \right]$$

called the slope of a Lagrangian. ◁

### 6.1.1 Complexification

Let  $(V, q_s)$  be a real symplectic vector space of dimension  $2r$ , and let  $\mathcal{B}$  be a given symplectic basis. The basis  $\mathcal{B}$  is naturally divided into two families  $\mathcal{B}^R = (b_1, \dots, b_r)$  and  $\mathcal{B}^I = (y_1, \dots, y_r)$ .

**Definition 6.1.4.** Let  $J$  be the automorphism of  $V$  defined by:

$$J(x_i) := y_i \qquad J(y_i) := -x_i$$

for every  $1 \leq i \leq r$ . Then  $J$  is called the *canonical complex structure* on  $(V, q_s)$  with respect to the basis  $\mathcal{B}$ . ◊

Note that  $J^2 = -\text{Id}_V$ . The space  $V$  is decomposed as  $V = V^I \oplus V^R$  where  $\mathcal{B}^R$  (resp.  $\mathcal{B}^I$ ) acts as a basis for  $V^R$  (resp.  $V^I$ ). Since  $J$  is an isomorphism from  $V^R$  to  $V^I$ , every vector  $v \in V$  can be uniquely be decomposed as  $v = J(v^I) + v^R$  where  $v^R$  and  $v^I$  are both vectors of rank  $r$  belonging to  $V^R$ . Now denote by  $j$  the multiplicative right action of  $J$  on  $V$ . Then  $V$  can be seen as a complex vector space  $V^{\mathbb{C}}$  of complex dimension  $r$ , called the *complexification of  $V$* . Moreover, there are two inverse  $\mathbb{R}$ -linear applications  $p_{\mathbb{R}}$  and  $p_{\mathbb{C}}$  defined by:

$$V \begin{array}{c} \xleftarrow{p_{\mathbb{R}}} \\ \xrightarrow{p_{\mathbb{C}}} \end{array} V^{\mathbb{C}}$$

$$v = (v^I) J + v^R \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} v^{\mathbb{C}} = v^I \cdot j + v^R$$

Note that  $p_{\mathbb{R}}$  is  $\mathbb{R}$ -linear but is *not*  $\mathbb{C}$ -linear, since the multiplication by a complex scalar is not defined directly on  $V$ . Subsequently, one can also define complexification and realification for endomorphisms:

$$\begin{array}{ccc} P_{\mathbb{C}} : E(V) & \longrightarrow & E(V^{\mathbb{C}}) \\ f & \longmapsto & p_{\mathbb{C}} \circ f \circ p_{\mathbb{R}} \end{array} \qquad \begin{array}{ccc} P_{\mathbb{R}} : E(V^{\mathbb{C}}) & \longrightarrow & E(V) \\ g & \longmapsto & p_{\mathbb{R}} \circ g \circ p_{\mathbb{C}} \end{array}$$

and the complexified basis  $\mathcal{B}^{\mathbb{C}}$  of  $V^{\mathbb{C}}$  by

$$\mathcal{B}^{\mathbb{C}} := (y_i \cdot j + x_i)_{1 \leq i \leq r}$$

### 6.1.2 Characterisation of Lagrangians

**Definition 6.1.5.** Let  $q_e$  be the Euclidean form on  $V$  defined by

$$q_e(x, y) := q_s((x)J, y) \tag{6.1}$$

Let  $q_h$  be the Hermitian form defined on  $V^{\mathbb{C}}$  by

$$q_h(z, w) := q_e(p_{\mathbb{R}}(z), p_{\mathbb{R}}(w)) + q_s(p_{\mathbb{R}}(z), p_{\mathbb{R}}(w)) \cdot j \tag{6.2}$$

These two forms are induced by the symplectic form  $q_s$ . ◊

**Definition 6.1.6.** Each of the three forms  $q_s, q_e, q_h$  has a corresponding subgroup of endomorphisms stabilising them:

- The subgroup of  $E(V)$  respecting  $q_s$  is called the *symplectic group*  $\text{Sp}(V)$ .

- The subgroup of  $E(V)$  respecting  $q_e$  is called the *orthogonal group*  $O(V)$ .
- The subgroup of  $E(V^{\mathbf{C}})$  respecting  $q_h$  is called the *unitary group*  $U(V^{\mathbf{C}})$ .  $\diamond$

These three groups are linked together by the following result:

**Theorem 6.1.7.**  $P_{\mathbf{C}}(\mathrm{Sp}(V) \cap O(V)) = U(V^{\mathbf{C}})$   $\triangleleft$

**Theorem 6.1.7** is the key to characterise every Lagrangian subspace. Indeed, any basis of a Lagrangian  $U$  can be extended into a symplectic basis  $\mathcal{B}_U$  of  $V$ . This basis is naturally associated to a symplectic endomorphism  $f_U$ , namely the change of basis from  $\mathcal{B}$ . One can then orthonormalise  $\mathcal{B}_U$  without affecting its symplectic property. According to **Theorem 6.1.7**, the endomorphism  $f_U$  can then be complexified into an Hermitian endomorphism of  $V^{\mathbf{C}}$ . Reciprocally, the realification of any Hermitian homeomorphism is symplectic and thus sends any Lagrangian to another Lagrangian. From these ideas one obtains the characterisation theorem:

**Theorem 6.1.8.** *The group  $U(V^{\mathbf{C}})$  acts transitively on  $\Lambda(V)$ . Moreover, the subgroup  $O(V^{\mathbf{C}})$  of  $U(V^{\mathbf{C}})$  is the stabiliser of the standard Lagrangian  $V^R$ .*  $\triangleleft$

**Corollary 6.1.9.** *There is a bijection between sets:*

$$G : \Lambda(V) \xrightarrow{\sim} U(V^{\mathbf{C}})/O(V^{\mathbf{C}}) \quad \triangleleft$$

$$U \longmapsto [g_U] \text{ such that } V^R \cdot g_U = U$$

**Remark 6.1.10.** The characterising class  $G(U)$  of a Lagrangian does not depend on the full symplectic basis  $\mathcal{B}$  but only on the decomposition  $\mathcal{B} = \mathcal{B}^R \oplus \mathcal{B}^I$ . In other words, if  $g \in O(V)$  is such that  $\langle g(\mathcal{B}^R) \rangle = \langle \mathcal{B}^R \rangle$  and  $\langle g(\mathcal{B}^I) \rangle = \langle \mathcal{B}^I \rangle$  then the map  $G$  built from the symplectic basis  $g(\mathcal{B})$  is identical to the one built from  $\mathcal{B}$ .  $\square$

Rather than using a class of Hermitian endomorphisms to fully characterise any Lagrangian, we define the slope as the determinant of this class. Since  $\det(g_U) \in S^1/\{\pm 1\}$ , by convention we only take the argument of its left representative in the unit circle.

**Definition 6.1.11.** The *slope of a Lagrangian* is the function

$$s_V : \Lambda(V) \longrightarrow ]-\frac{\pi}{2}; \frac{\pi}{2}] \quad \diamond$$

$$U \longmapsto \arg(\det(g_U))$$

Remark that the slope does not fully characterise a Lagrangian unlike  $G$ .

The following result explains how one can explicitly compute the slope of a Lagrangian using a generating matrix on  $\mathcal{B}$ .

**Proposition 6.1.12.** *Let  $U \in \Lambda(V)$  and  $M_U$  be a generating matrix of  $U$  in the basis  $\mathcal{B}$ , of size  $(r, 2r)$ . Suppose that the rows of  $M_U$  are  $q_e$ -orthonormal. Consider the two halves  $M_U^R$  and  $M_U^I$ , which are real square matrices of size  $r$ . Then the slope is given by*

$$s_V(U) = \arg(\det(M_U^R + M_U^I \cdot j)) \quad \triangleleft$$

*Proof.* Consider the real square matrix  $\overline{M_U}$  of size  $(2r, 2r)$  given by the block description:

$$\overline{M_U} := \begin{array}{c} V^I \quad V^R \\ V^I \left[ \begin{array}{c} (M_U) J \\ \cdots \\ M_U \end{array} \right] \\ V^R \end{array} = \begin{array}{c} V^I \quad V^R \\ V^I \left[ \begin{array}{c} M_U^R \quad \vdots \quad -M_U^I \\ \cdots \\ M_U^I \quad \vdots \quad M_U^R \end{array} \right] \\ V^R \end{array}$$

By assumption,  $\overline{M_U} \in O(V)$ . One easily checks that  $\overline{M_U} \in \mathrm{Sp}(V)$ . Then by **Theorem 6.1.7**, we have  $P_{\mathbf{R}}(\overline{M_U}) = M_U^R + M_U^I \cdot j \in U(V^{\mathbf{C}})$ . The matrix

$$S_r := \begin{array}{c} V^I \quad V^R \\ [0_r \quad : \quad I_r] \end{array}$$

generates the standard Lagrangian  $V^R$ . It is clear that  $S_r \overline{M_U} = M_U$ . Therefore  $P_{\mathbf{R}}(\overline{M_U}) = g_U$  as in the definition of  $s_V(U)$ .  $\square$

**Corollary 6.1.13.** *If  $r$  is even then  $s_V(V^R) = s_V(V^I) = 0$ .*  $\triangleleft$

*Proof.* The matrix  $S_r \cdot J = [I_r : 0_r]$  generates  $V^I$  in the basis  $\mathcal{B}$ . Applying [Proposition 6.1.12](#), one gets

$$s_V(V^I) = \arg(\det(I_r \cdot j)) = \arg(j^r \det I_r) = \arg(\det I_r) = s_V(V^r) = 0 \quad \square$$

## 6.2 Generalised slope construction

The principle of the generalised slope is to apply the constructions of [Section 6.1](#), and in particular [Definition 6.1.11](#), to the first twisted homology group of  $\partial M_L$  for a link  $L$ . To do this we must specify which types of representations we consider, and we must also establish that the homology group indeed has a symplectic structure. Then the kernel of the map  $i_*$  induced by the inclusion  $i : \partial M_L \hookrightarrow M_L$  is a Lagrangian subspace, so we can take its slope to define our new invariant.

### 6.2.1 Character torus

Let  $K$  be a knot with peripheral system  $(\pi_1(M_K), \ell, m)$ . Let  $\omega : \pi_1(M_K) \rightarrow S^1 \subset \mathbb{C}^*$  be a unitary character, which is determined by  $\omega(m) := e^{i\theta_m}$ . Denote  $\omega(\ell) := e^{i\theta_\ell}$ . Let  $\phi$  be the realification morphism of  $\mathbb{C}$  and define  $\rho_\omega := \phi(\omega)$ . The values of  $\rho_\omega$  are in  $\mathrm{SO}_2(\mathbb{R})$  and are given by

$$\rho_\omega(m) = \begin{bmatrix} \cos \theta_m & -\sin \theta_m \\ \sin \theta_m & \cos \theta_m \end{bmatrix} \quad \rho_\omega(\ell) = \begin{bmatrix} \cos \theta_\ell & -\sin \theta_\ell \\ \sin \theta_\ell & \cos \theta_\ell \end{bmatrix} \quad (6.3)$$

The following lemma describes the complete computation of the first twisted homology group of the torus for the representation  $\rho_\omega$ .

**Lemma 6.2.1.** *Let  $\mathbb{R}(\rho_\omega)$  be  $\mathbb{R}^2$  seen as a  $\mathbb{R}$ -module for the action of  $\rho_\omega$ . Then*

$$H_1(\partial M_K; \rho_\omega) = \begin{cases} \langle m, \ell \rangle \otimes \mathbb{R}^2 & \text{if } \omega(m) = \omega(\ell) = 1 \\ \{0\} & \text{otherwise} \end{cases} \quad \triangleleft$$

*Proof.* This is a direct application of [Lemma A.2.1](#) on page [122](#) with the additional remark that

$$V_{\rho_\omega} = \ker(\rho_\omega(\ell) - I_2) \cap \ker(\rho_\omega(m) - I_2) = \begin{cases} \mathbb{R}^2 & \text{if } \omega(m) = \omega(\ell) = 1 \\ \{0\} & \text{otherwise.} \end{cases} \quad \square$$

Let now  $L = L_1 \cup \dots \cup L_n$  be a link. We reuse some notations from [Section 4.2](#). We suppose that all linking numbers are zero, i.e.  $\mathrm{lk}(L_i, L_j) = 0$  for every pair  $i \neq j$ .

The group  $H_1(M_L)$  is free abelian, generated by the classes  $m_i$  of the meridians of the components  $L_i$ . Let  $\omega : \pi_1(M_L) \rightarrow S^1 \subset \mathbb{C}^*$  be a unitary character. Since  $\omega$  is abelian, it is fully determined by its value  $\omega_i$  on  $m_i$  for all  $1 \leq i \leq n$ . Let  $\mathbb{T}^n := \mathrm{Hom}(\pi_1(M_L), S^1)$  be the set of unitary characters. Then we have a natural identification

$$\mathbb{T}^n \simeq \{\omega = (\omega_1, \dots, \omega_n) \mid \omega_i \in S^1\} \subset (\mathbb{C}^*)^n$$

As the notation suggests, this set only depends on the number of components  $n$  of the link  $L$ .

### 6.2.2 Symplectic structure of the twisted homology

We now present a symplectic structure on the first homology group of  $\partial M_L$ .

For a given character  $\omega \in \mathbb{T}^n$ , we define

$$D(\omega) := \{i \mid \omega_i = 1\} \quad d(\omega) := \#D(\omega)$$

The components  $L_i$  where  $i \in D(\omega)$  are called *generating* for  $\omega$ .

**Proposition 6.2.2.** *The space  $H_1(\partial E_L, \mathbb{R}(\rho_\omega))$  endowed with the intersection form is a symplectic space of dimension  $4d(\omega)$ , and there is a canonical isomorphism*

$$H_1(\partial E_L; \rho_\omega) = \bigoplus_{i \in D(\omega)} H_1(\partial E_{L_i}; \rho_{\omega_i}) \quad \triangleleft$$

*Proof.* Each  $\mathbb{R}$ -vector space  $H_1(\partial E_{L_i}; \rho_{\omega_i})$  is symplectic as a direct application of [Theorem A.1.8](#) on page [121](#). The intersection form on  $H_1(\partial E_L; \rho_\omega)$  is given by the direct sum of all the forms since  $\omega(\ell_i) = 1$  and  $\rho_\omega(\ell_i) = I_2$  (and since  $\text{lk}(L_i, L_j) = 0$ , see [Eq. \(4.1\)](#) on page [91](#)). By [Lemma A.2.1](#) on page [122](#),  $H_1(\partial E_{L_i}; \rho_{\omega_i})$  has dimension 4 if  $\omega(m_i) = 1$ , i.e. if  $i \in D(\omega)$ , and dimension 0 otherwise.  $\square$

**Definition 6.2.3.** Let  $\langle c_1, c_2 \rangle$  be a fixed basis of  $\mathbb{R}^2$ . Let  $\mathcal{B}_L$  be the function:

$$\begin{aligned} \mathcal{B}_L : \quad \mathbb{T}^n &\longrightarrow \mathbb{R}^{4n} \\ \omega = (\omega_1, \dots, \omega_n) &\longmapsto (a_1, \dots, a_{2n}; b_1, \dots, b_{2n}) \end{aligned}$$

where

$$\begin{aligned} \forall i \in D(\rho), \forall j \in \{1, 2\} : &\begin{cases} a_{2(i-1)+j-1} = m_i \otimes c_j \\ b_{2(i-1)+j-1} = \ell_i \otimes c_j \end{cases} \\ \forall i \notin D(\rho), \forall j \in \{1, 2\} : &a_{2(i-1)+j-1} = b_{2(i-1)+j-1} = 0 \end{aligned} \quad \diamond$$

The rank of the vector family  $\mathcal{B}_L(\omega)$  is  $4d(\omega) \leq 4n$  and thus depends on  $\omega$ . Nevertheless, we slightly abuse notation and also call  $\mathcal{B}_L(\omega)$  the full-rank basis of the subspace of  $\mathbb{R}^{4n}$  it generates.

**Proposition 6.2.4.** *The family  $\mathcal{B}_L(\omega)$  is a symplectic basis of  $H_1(\partial M_L; \rho_\omega)$ .*  $\triangleleft$

*Proof.* The family is a basis thanks to [Lemma A.2.1](#). Using the notations of [Appendix A.1](#), on every component  $H_1(\partial M_{L_i}; \rho_{\omega_i})$  with  $i \in D(\omega)$  the intersection form particularises as

$$\begin{aligned} \langle \ell_i \otimes c_r \mid m_i \otimes c_s \rangle &= \langle c_r \mid c_s \rangle = \delta_{r,s} \\ \langle m_i \otimes c_r \mid \ell_i \otimes c_s \rangle &= \langle c_r \mid c_s \rangle = -\delta_{r,s} \\ \langle \ell_i \otimes c_r \mid \ell_i \otimes c_s \rangle &= 0 \\ \langle m_i \otimes c_r \mid m_i \otimes c_s \rangle &= 0 \end{aligned} \quad \square$$

**Remark 6.2.5.** It occurs that the coefficients of  $H_1(\partial M_L; \rho_\omega)$  come from the realification of  $\mathbb{C}(\omega)$  into  $\mathbb{R}(\rho_\omega) \simeq \mathbb{R}^2$  using  $\phi$ . However this symplectic structure on  $\mathbb{R}^2$  does not extend to the tensor product in general. As shown in [Theorem A.1.8](#), only the specific property of the intersection form on  $\partial M_L$  ensures that  $H_1(\partial M_L; \rho_\omega)$  is indeed symplectic, with the longitudes and meridians as the ‘real’ and ‘imaginary’ parts respectively. When a disambiguation is necessary, we call  $\phi$  the *coefficient* realification and  $p_{\mathbb{R}}$  the *intersection* realification.  $\boxtimes$

### 6.2.3 Definition of the generalised slope

Consider the inclusion map

$$i : \partial M_L \hookrightarrow M_L$$

and the induced map on the first twisted homology groups of  $(M_L, \rho_\omega)$ :

$$i_* : H_1(\partial M_L; \rho_\omega) \longrightarrow H_1(M_L; \rho_\omega)$$

We consider the space  $\mathcal{Z}(\rho_\omega) := \ker i_*$ .

**Lemma 6.2.6.** *The subspace  $\mathcal{Z}(\rho_\omega)$  is a Lagrangian subspace of the symplectic vector space  $H_1(\partial M_L; \rho_\omega)$  endowed with the intersection form.*  $\triangleleft$

*Proof.* This is a direct application of [Theorem A.1.9](#) on page [121](#).  $\square$

**Definition 6.2.7.** The *generalised slope* of the link  $L$  is the function

$$\begin{aligned} s_L : \quad \mathbb{T}^n &\longrightarrow \left] -\frac{\pi}{2}; \frac{\pi}{2} \right] \\ \omega &\longmapsto s_{H_\omega}(\mathcal{Z}(\rho_\omega)) \end{aligned}$$

where  $H_\omega := H_1(\partial M_L; \rho_\omega)$ . By convention, when we have  $\omega_i \neq 1$  for all  $i \in \{1, \dots, n\}$  and thus  $d(\omega) = 0$ , we set  $s_L(\omega) := 0$ .  $\diamond$



The generalised slope is computed in two steps: the first is to use [Algorithm 4.3.1](#) and [Theorem A.4.1](#) on page 93 and on page 124 to determine a generating matrix of  $\mathcal{Z}(\rho_\omega)$ , which is then orthonormalised. Then [Proposition 6.1.12](#) is used to compute the slope  $s_L(\omega)$  proper.

**Remark 6.2.8.** By [Corollary 6.1.13](#), the generalised slope  $s_L(\omega)$  cannot distinguish between the case where  $\ker i_*$  is generated only by the longitudes (i.e.  $\mathcal{Z}(\rho_\omega) = H_\omega^R$ ) and the case where  $\ker i_*$  is only generated by the meridians (i.e.  $\mathcal{Z}(\rho_\omega) = H_\omega^I$ ). However in practice the generating matrix of  $\mathcal{Z}(\rho_\omega)$  will contain only zeros in its  $L$ -side in the first case, and in its  $R$ -side in the second. Since computing this matrix is necessary to get  $s_L(\omega)$ , one can immediately identify these two cases before computing the Lagrangian slope.  $\square$

### 6.2.4 Relation with the character slope

We explain the relation between the generalised slope  $s_L$  of [Definition 6.2.7](#) and the ‘actual’ link character slope presented in [Chapter 4](#). Let  $L = L_1 \cup L'$  be a link with a single component  $L_1$  distinguished and such that  $\text{lk}(L_1, L_i) = 0$  for every  $2 \leq i \leq n$ . Reusing notations from [Section 4.2](#) we say that the character slope is defined on the subset of  $\mathbb{T}^n$  composed of non-vanishing unitary characters:

$$\mathcal{A}_u^\circ(L_1/L') := \{\omega = (1, \omega_2, \dots, \omega_n) \mid \omega_i \in S^1, \omega_i \neq 1\} \subset \mathbb{T}^n$$

The next proposition show how the generalised slope restricts to the character slope, for some specific characters.

**Proposition 6.2.9.** *Let  $\omega \in \mathcal{A}_u^\circ(L_1/L')$  be an admissible non-vanishing unitary character. Then  $s_{(L_1/L')}(\omega) \in \mathbb{R} \cup \{\infty\}$  and we have*

$$s_L(\omega) = \begin{cases} 2 \arctan s_{(L_1/L')}(\omega) - \pi & \text{if } s_{(L_1/L')}(\omega) > 1. \\ 2 \arctan s_{(L_1/L')}(\omega) & \text{if } -1 < s_{(L_1/L')}(\omega) \leq 1. \\ 2 \arctan s_{(L_1/L')}(\omega) + \pi & \text{if } s_{(L_1/L')}(\omega) \leq -1. \\ 0 & \text{if } s_{(L_1/L')}(\omega) = \infty. \end{cases} \quad \triangleleft$$

*Proof.* By construction, the *coefficient* realification morphism  $\phi$  makes the following diagram commute:

$$\begin{array}{ccc} H_1(\partial M_L; \omega) & \xrightarrow{i_*} & H_1(M_L; \omega) \\ \phi \downarrow & & \downarrow \phi \\ H_1(\partial M_L; \rho_\omega) & \xrightarrow{i_*} & H_1(M; \rho_\omega) \end{array}$$

One has therefore  $\phi(\mathcal{Z}(\omega)) = \mathcal{Z}(\rho_\omega)$ . Let  $\sigma := s_{(L_1/L')}(\omega)$  and suppose that  $\sigma \neq \infty$ . By [Definition 4.2.4](#) on page 91, one has

$$\mathcal{Z}(\omega) = \langle \sigma \cdot m_1 - \ell_1 \rangle$$

and thus

$$\mathcal{Z}(\rho_\omega) = \langle (\sigma \cdot m_1 - \ell_1) \otimes c_1, (\sigma \cdot m_1 - \ell_1) \otimes c_2 \rangle$$

since  $\sigma$  is a real number by [Proposition 4.2.6](#). An orthonormalised generating matrix of  $\mathcal{Z}(\rho_\omega)$  in the basis  $\mathcal{B}_L$  of  $H_1(\partial M_L; \rho_\omega)$  is given by

$$\frac{1}{\sqrt{1+\sigma^2}} \begin{bmatrix} m_1 \otimes c_1 & m_1 \otimes c_2 & \ell_1 \otimes c_1 & \ell_1 \otimes c_2 \\ \sigma & 0 & \vdots & -1 & 0 \\ 0 & \sigma & \vdots & 0 & -1 \end{bmatrix}$$

Applying [Proposition 6.1.12](#), we get that

$$s_{H_\omega}(\mathcal{Z}(\rho_\omega)) = \arg \frac{1}{1+\sigma^2} \begin{vmatrix} \sigma \cdot j + 1 & 0 \\ 0 & \sigma \cdot j + 1 \end{vmatrix} = \arg \frac{(1-\sigma^2) + 2\sigma \cdot j}{1+\sigma^2}$$

Using the tangent half-angle, one has  $\arctan\left(\frac{2\sigma}{1-\sigma^2}\right) = 2\sigma$ . Evaluating the argument of  $s_{H_\omega}(\mathcal{Z}(\rho_\omega))$  inside  $]-\frac{\pi}{2}; \frac{\pi}{2}]$  using  $\arctan$  gives the desired result thanks to this formula.

Finally, if  $\sigma = \infty$  then  $\mathcal{Z}(\omega) = \langle m_1 \rangle$  and  $\mathcal{Z}(\rho_\omega) = H_\omega^I$ . Using [Corollary 6.1.13](#), we obtain  $s_{H_\omega}(\mathcal{Z}(\rho_\omega)) = 0$ .  $\square$

### 6.3 Concordance invariance

In this section we show that the generalisation of the character slope from [Chapter 4](#) is still a concordance invariant.

Recall from [Definition 4.2.9](#) on page 92 that  $\omega \in \mathbb{T}^n$  is a concordance root if there exists no polynomial  $P \in U$  such that  $P(\omega) = 0$ . Let  $\mathbb{T}_1^n$  be the subset of  $\mathbb{T}^n$  composed of characters that are *not* concordance roots. We also exclude the characters  $\omega$  such that  $d(\omega) = 0$ . The set  $\mathbb{T}_1^n$  can be stratified to account for which values of the character are equal to 1. Let  $\mathcal{P}_n$  be the set of non-empty subsets of  $\{1, \dots, n\}$ . For every  $S \in \mathcal{P}_n$ , define

$$\mathbb{T}_1^S := \{\omega \in \mathbb{T}_1^n \mid D(\omega) = S\}$$

Then we have

$$\mathbb{T}_1^n = \{(1, \dots, 1)\} \sqcup \bigsqcup_{S \in \mathcal{P}_n} \mathbb{T}_1^S$$

**Definition 6.3.1.** Let  $S \in \mathcal{P}_n$  and write  $\nu = \#S$ . Let also  $\mu \in \{1, \dots, n - \nu\}$ . A  $(S, \mu)$ -colouring on a link  $L = L_1 \sqcup \dots \sqcup L_n$  is a surjective function

$$c : \{1, \dots, n\} \longrightarrow \{d_1, \dots, d_\nu\} \sqcup \{c_1, \dots, c_\mu\}$$

such that  $c(i) = d_i$  if and only if  $i \in S$ .  $\diamond$

For every  $j \in \{1, \dots, \mu\}$ , we write  $\chi(j) := c^{-1}(d_j)$  and  $L_{\chi(j)}$  the union of individual components of  $L$  that are  $c_j$ -coloured. As with [Definition 4.1.8](#),  $(S, \mu)$ -colouring induces a notion of concordance whose definition depends on the colouration.

**Theorem 6.3.2.** *Let  $L^0, L^1$  be links with  $n$  components such that  $\text{lk}(L_i^k, L_j^k) = 0$  for every  $1 \leq i \neq j \leq n$  and  $s = 0, 1$ . Fix  $S \in \mathcal{P}_n$  and  $\mu \in \{1, \dots, n - \nu\}$  and make the links  $(S, \mu)$ -coloured. If  $L^0$  and  $L^1$  are concordant then for every  $\omega \in \mathbb{T}_1^S$ , one has*

$$s_{L^0}(\omega) = s_{L^1}(\omega) \quad \triangleleft$$

The proof is mostly similar to the proof of the corresponding [Theorem 4.2.10](#) on page 92 for the character slope.

*Proof.* Up to reordering we suppose that  $S = \{1, \dots, \nu\}$ . For  $k = 0, 1$  we write  $L^k = L_d^k \sqcup L_c^k$  with

$$L_d^k = L_1^s \sqcup \dots \sqcup L_\nu^s \qquad L_c^k = L_{\nu+1}^s \sqcup \dots \sqcup L_\mu^s$$

Fix  $\omega \in \mathbb{T}_1^S$  and let  $D \cup C \subset S^3 \times [0, 1]$  be the concordance. We have  $D = D_1 \sqcup \dots \sqcup D_\nu$  and for every  $i \in \{1, \dots, \nu\}$ ,  $\partial D_i = -L_i^0 \sqcup L_i^1$ . Let  $T_{D \cup C}$  be a tubular neighbourhood of  $D \cup C$ . Define

$$E := S^3 \times [0, 1] \setminus T_{D \cup C} \qquad E_0 := S^3 \times [0, 1] \setminus T_C$$

Then  $E \cap (S^3 \times \{k\}) = M_{L^k}$  and  $E_0 \cap (S^3 \times \{k\}) = M_{L_c^k}$ . In particular the preferred meridian of each component of the links is sent to the preferred meridian of the corresponding cylinder inside  $D \cup C$ . Writing down the relative Mayer-Vietoris sequence in  $\mathbb{C}$ -homology of the pairs

$$(S^3 \times [0, 1], S^3 \times \{k\}) = (E, M_{L^k}) \cup (\bar{T}_{D \cup C}, \bar{T}_{L^k}) = (E_0, M_{L_c^k}) \cup (\bar{T}_C, \bar{T}_{L_c^k})$$

gives

$$H_*(E, M_{L^k}) = H_*(E_0, M_{L_c^k}) = 0 \quad (6.4)$$

Using this in the homology exact sequence of the pair  $(E, M_{L^k})$  gives that the inclusions  $M_{L^k} \hookrightarrow E$  induce an isomorphism  $H_1(M_{L^k}) \simeq H_1(E)$  that sends the meridians of the link  $L^k$  to the corresponding meridians of the cylinders that composes  $D \cup C$ .

Now write again a relative Mayer-Vietoris sequence in twisted homology for the decomposition

$$(E_0, M_{L_d^k}) = (E, M_L) \cup (\bar{T}_D, \bar{T}_{L_d^k})$$

This gives

$$\begin{array}{c} \cdots \longrightarrow H_1(D \times S^1, L_d^k \times S^1; \rho_\omega) \\ \qquad \qquad \qquad \downarrow \\ H_1(E, M_L; \rho_\omega) \oplus H_1(\bar{T}_D, \bar{T}_{L_d^k}; \rho_\omega) \longrightarrow H_1(E_0, M_{L_d^k}; \rho_\omega) \longrightarrow \cdots \end{array}$$

Since  $\omega$  is trivial on all the meridians of  $L_d^k$  and also on the meridians of  $D$  by the previous discussion, then  $H_1(D \times S^1, L_d^k \times S^1; \rho_\omega) = 0$ . Moreover, by [CNT20, Lemma 2.16] (see also [NP17]) the  $\mathbb{C}$ -homology of both pairs of Eq. (6.4) being null implies that their  $\rho_\omega$ -twisted homology is also null:

$$H_*(E, M_{L^k}; \rho_\omega) = H_*(E_0, M_{L_c^k}; \rho_\omega) = 0$$

Replacing this in the sequence above yields  $H_1(E, M_{L^k}; \rho_\omega) = 0$ . Once again using this in the  $\rho_\omega$ -twisted homology sequence of the pair gives an isomorphism

$$\psi : H_1(E; \rho_\omega) \xrightarrow{\simeq} H_1(M_{L^k}; \rho_\omega)$$

which preserves the meridians. Separately, since  $L^0$  and  $L^i$  are both  $(S, \mu)$ -coloured, there is a cylinder  $D_i$  joining each component  $L_i^0$  of  $L_c^0$  to the corresponding component  $L_i^1$  of  $L_c^1$ . Since  $\ell_i^0 = D_i \cap T_{L_i^0}$ , the cylinder  $D_i$  induces an homotopy between  $\ell_i^0$  and  $\ell_i^1$ . Both curves therefore have the same in  $H_1(E; \rho_\omega)$ , and therefore  $\psi$  sends  $\ell_i^0 \in H_1(M_{L^0}; \rho_\omega)$  to  $\ell_i^1 \in H_1(M_{L^1}; \rho_\omega)$ . Every element of the symplectic basis  $\mathcal{B}_{L^0}(\omega)$  is therefore sent to the corresponding element in  $\mathcal{B}_{L^1}(\omega)$ . Any vector of  $\mathcal{Z}^0(\rho_\omega)$  is also homologically null in  $\mathcal{Z}^1(\rho_\omega)$  and vice versa, so  $\psi$  restrains into an isomorphism  $\mathcal{Z}^0(\rho_\omega) \simeq \mathcal{Z}^1(\rho_\omega)$ . Therefore, the generalised slopes of the links  $L^0$  and  $L^1$  are equal.  $\square$

**Corollary 6.3.3.** *Let  $L^0$  and  $L^1$  be two  $n$ -coloured links with  $n$  components and all linking numbers equal to zero. If  $L^0$  and  $L^1$  are  $n$ -concordant then the restrictions of the generalised slope functions*

$$s_{L^k} : \mathbb{T}_!^n \longrightarrow \left] -\frac{\pi}{2}; \frac{\pi}{2} \right]$$

for  $k = 0, 1$  are equal.  $\triangleleft$

*Proof.* If  $L^0$  and  $L^1$  are  $n$ -concordant then for every  $S \in \mathcal{P}_n$  they are  $(S, n - \nu)$ -concordant. By Theorem 6.3.2 the generalised slope functions coincide on the entirety of  $\mathbb{T}_!^S$ . The slopes thus coincide on  $\mathbb{T}_!^n$ .  $\square$

## APPENDIX

# A

# TWISTED (CO-)HOMOLOGY

### Outline of the current chapter

<b>A.1 General results</b>	<b>119</b>
A.1.1 Definition . . . . .	119
A.1.2 Duality . . . . .	120
A.1.3 Intersection form . . . . .	120
A.1.4 Kernel of $i_*$ . . . . .	121
<b>A.2 Twisted homology of the torus</b>	<b>122</b>
<b>A.3 Fox calculus</b>	<b>123</b>
<b>A.4 Computation of <math>\ker i_*</math></b>	<b>124</b>

## A.1 General results

References for the twisted (co-)homology constructions of this section include [Por97; DFL22b].

### A.1.1 Definition

Let  $M$  be a smooth compact connected oriented  $n$ -dimensional manifold with boundary. Up to isotopy,  $M$  admits a unique p.l.-linear structure, which induces a triangulation and a CW-complex structure. Fix a base point  $b \in \partial M$ . Let  $\pi := \pi_1(M, b)$  be the fundamental group of  $M$  and denote by  $\widetilde{M}$  the universal covering of  $M$ . The group  $\pi$  has an action on the lifted cells from  $M$  inside  $\widetilde{M}$ . One can thus define the chain complex  $C_*(M; \mathbb{Z}[\pi])$  generated by the lifted cells over the group ring  $\mathbb{Z}[\pi]$ . If  $N$  is a submanifold of  $M$ , the relative chain complex  $C_*(M, N; \mathbb{Z}[\pi])$  is defined on the cells of  $\widetilde{M}$  with boundary operators relative to the cells of  $\widetilde{N}$ .

Let  $\mathbb{K}$  be  $\mathbb{R}$  or  $\mathbb{C}$ . Fix a basis  $\mathcal{B}$  of the  $\mathbb{K}$ -vector space  $\mathbb{K}^r$ . By convention we consider vectors as rows and the matrix action on the right. Consider  $\rho : \pi \rightarrow \mathrm{GL}_r(\mathbb{K})$  a representation of  $\pi$ . The group homomorphism  $\rho$  extends into a ring homomorphism on  $\mathbb{Z}[\pi]$  and induces a right action of  $\mathbb{Z}[\pi]$  on  $\mathbb{K}^r$ . The vector space  $\mathbb{K}^r$  can thus be seen as a right  $\mathbb{Z}[\pi]$ -module denoted  $\mathbb{K}(\rho)$ .

There is a canonical involutive anti-automorphism  $\dagger$  on  $\mathbb{Z}[\pi]$  given by

$$\dagger : \sum_{g \in \pi} n_g g \longmapsto \sum_{g \in \pi} n_g g^{-1}$$

Define  $\rho^\dagger := ({}^t\rho)^{-1}$  where  ${}^t$  is the matrix transposition. The image of  $\dagger$  is a left module for the action of  $\rho^\dagger$ , and up to transposition we have  $\mathbb{K}(\rho)^\dagger \simeq \mathbb{K}(\rho^\dagger)$ .

**Definition A.1.1.** Consider the (co-)chain complexes of  $\mathbb{K}$ -vector spaces:

$$\begin{aligned} C_*(M; \rho) &:= C_*(M; \mathbb{Z}[\pi]) \otimes_{\mathbb{Z}[\pi]} \mathbb{K}(\rho) \\ C^*(M; \rho) &:= \text{Hom}_{\mathbb{Z}[\pi]}(C_*(M; \mathbb{Z}[\pi]), \mathbb{K}(\rho^\dagger)) \end{aligned}$$

The  $\rho$ -twisted (co-)homology  $H_*(M; \rho)$  (resp.  $H^*(M; \rho^\dagger)$ ) is the (co-)homology of the (co-)chain complex  $C_*(M; \rho)$  (resp.  $C^*(M; \rho^\dagger)$ ), which have the structure of  $\mathbb{K}$ -vector spaces.  $\diamond$

The first module  $H_1(M; \rho)$  is also called the *twisted Alexander module of  $(M, \rho)$* .

**Definition A.1.2.** The *twisted Alexander polynomial of  $(M, \rho)$*  is the first order of the twisted Alexander module:

$$\Delta_M(\rho) := \text{ord } H_1(M; \rho) \in \mathbb{Z}[\text{GL}_r(\mathbb{K})] / \{\pm I_r\} \quad \diamond$$

### A.1.2 Duality

Let  $(\cdot | \cdot)$  be a vector product on the basis  $\mathcal{B}$ . The product can be bilinear if  $\mathbb{K} = \mathbb{R}$ , or bilinear or sesquilinear (i.e.  $(v | w) := v \cdot {}^t\bar{w}$ ) if  $\mathbb{K} = \mathbb{C}$ . The representation  $\rho$  is called *unitary* if

$$\begin{cases} \rho^\dagger = \bar{\rho} & \text{if } (\cdot | \cdot) \text{ is sesquilinear.} \\ \rho^\dagger = \rho & \text{if } (\cdot | \cdot) \text{ is bilinear.} \end{cases}$$

**Definition A.1.3.** The *Kronecker product* is defined by

$$\begin{aligned} [\cdot | \cdot] : C^k(M, \partial M; \rho) \times C_k(M, \partial M; \rho^\dagger) &\longrightarrow \mathbb{K} \\ (f, x \otimes v) &\longmapsto ({}^t f(x) | v) \end{aligned} \quad \diamond$$

The Kronecker product is invariant by the diagonal action of  $\rho \otimes \rho^\dagger$ . It is a perfect pairing and therefore induces a ‘universal coefficients’ formula:

**Lemma A.1.4.** *There is a natural vector-space isomorphism*

$$H^k(M, \partial M; \rho^\dagger) \simeq H_k(M, \partial M; \rho)^\vee$$

where  $\vee$  designates the dual vector space.  $\triangleleft$

The pair  $(M, \partial M)$  is a *simple Poincaré pair* as defined by C.T.C. Wall in [Wal99, Section 2]. This implies that Poincaré duality works as expected even for  $\rho$ -twisted coefficients.

**Lemma A.1.5.** *There are canonical Poincaré duality isomorphisms*

$$\begin{aligned} D_M : H^{n-k}(M, \partial M; \rho^\dagger) &\xrightarrow{\sim} H_k(M; \rho) \\ D_M : H^{n-k}(M; \rho^\dagger) &\xrightarrow{\sim} H_k(M, \partial M; \rho) \\ f &\longmapsto [M] \frown f \end{aligned}$$

where  $[M] \in H_n(M, \partial M; \rho)$  is the fundamental class of  $M$  and  $\frown$  is the usual cap product.  $\triangleleft$

### A.1.3 Intersection form

In this section we suppose that  $\rho$  is unitary.

The standard cup product on  $M$  is naturally defined on the twisted chain complex spaces

$$\smile : C^k(\partial M; \rho^\dagger) \times C^{m-k}(\partial M; \rho^\dagger) \longrightarrow C^m(M, \partial M; \rho^\dagger \otimes \rho^\dagger)$$

where  $\rho^\dagger \otimes \rho^\dagger$  designates  $\mathbb{K}(\rho^\dagger) \otimes \mathbb{K}(\rho^\dagger)$  seen as a  $\mathbb{Z}[\pi]$ -module for the diagonal action. When  $\rho$  is unitary, the vector product is invariant by this action and can thus be seen as a morphism

$$(\cdot | \cdot) : \mathbb{K}(\rho^\dagger) \otimes \mathbb{K}(\rho^\dagger) \longrightarrow \mathbb{K}$$

**Definition A.1.6.** The *cup product form* on  $\rho$ -twisted cohomology is defined by the formula

$$\begin{aligned} \langle \cdot | \cdot \rangle : H^k(M, \partial M; \rho^\dagger) \times H^{n-k}(M; \rho^\dagger) &\longrightarrow H^n(M, \partial M; \mathbb{K}) \simeq \mathbb{K} \\ (e, f) &\longmapsto (e \smile f)([M]) \end{aligned} \quad \diamond$$

One can also define the intersection form on  $\rho$ -twisted homology using the cap product.

**Definition A.1.7.** The *intersection form* on  $\rho$ -twisted homology is defined by the formula

$$\begin{aligned} \langle \cdot | \cdot \rangle : C_{n-k}(M; \rho) \times C_k(M, \partial M; \rho) &\longrightarrow \mathbb{K} \\ (x \otimes v, y \otimes w) &\longmapsto \sum_{\alpha \in \pi} (x\alpha \cdot y) \cdot (x\rho(\alpha) | y) \end{aligned}$$

where  $(\cdot)$  denotes the algebraic intersection number.  $\diamond$

The intersection form and the cup product are linked together by the Poincaré duality isomorphisms of [Lemma A.1.5](#), which make the following diagram commute:

$$\begin{array}{ccc} H^k(M, \partial M; \rho^\dagger) \times H^{n-k}(M; \rho^\dagger) &\longrightarrow & H^n(M, \partial M; \mathbb{K}) \\ \begin{array}{c} D_M \times D_M \downarrow \\ \downarrow \wr \end{array} & & \downarrow \wr \\ H_{n-k}(M; \rho) \times H_k(M, \partial M; \rho) &\longrightarrow & \mathbb{K} \end{array}$$

The properties of the cup product inherited by the intersection form give the fundamental result:

**Theorem A.1.8.** *Suppose that  $n = 2k + 1$  and  $\rho$  is unitary. Then  $H_k(\partial M; \rho)$  endowed with the intersection form is a symplectic  $\mathbb{K}$ -vector space.*  $\triangleleft$

#### A.1.4 Kernel of $i_*$

Consider the portion of the homology exact sequence of the pair  $(M, \partial M)$

$$\cdots \longrightarrow H_{k+1}(M, \partial M; \rho) \xrightarrow{\partial} H_k(\partial M; \rho) \xrightarrow{i_*} H_k(M; \rho) \longrightarrow \cdots$$

where  $i_*$  is induced by the inclusion  $i : \partial M \hookrightarrow M$ . Define

$$\mathcal{Z}_k(\partial M; \rho) := \ker i_* = \text{im } \partial$$

[Definition 6.1.1](#) gives the definition of a Lagrangian subspace of a symplectic space. The fundamental result of this section is the following:

**Theorem A.1.9.** *Suppose that  $n = 2k + 1$  and that  $\rho$  is unitary. Then  $\mathcal{Z}_k(\partial M; \rho)$  is a Lagrangian subspace of  $H_k(\partial M; \rho)$  for the intersection form. In particular,*

$$\dim \mathcal{Z}_k(\partial M; \rho) = \frac{1}{2} \dim H_k(\partial M; \rho) \quad \triangleleft$$

*Proof.* Using [Lemma A.1.5](#) between the homology and cohomology exact sequence of the pair  $(M, \partial M)$ , we get that the following diagram is commutative

$$\begin{array}{ccc} H^k(M; \rho^\dagger) &\xrightarrow{i^*}& H^k(\partial M; \rho^\dagger) \\ \begin{array}{c} D_M \downarrow \wr \\ \downarrow \wr \end{array} & & \downarrow \wr \\ H_{k+1}(M, \partial M; \rho) &\xrightarrow{\partial}& H_k(\partial M; \rho) \end{array}$$

In addition, [Lemma A.1.4](#) implies that  $i^* = (i_*)^\vee$  for the Kronecker product. On  $\partial M$ , the usual duality between the cup and cap products can be stated for  $f, e \in H^k(M, \partial M; \rho^\dagger)$  as

$$\langle f | e \rangle = (f \smile e)([\partial M]) = [f | [\partial M] \frown e] = [f | D_{\partial M}(e)]$$

Now consider two vectors  $a, b \in \mathcal{Z}_k(\partial M; \rho)$ . Since  $\ker i_* = \text{im } \partial$ , there exists  $A \in H_{k+1}(M, \partial M; \rho^\dagger)$

such that  $a = \partial A$ . Using the above diagram, we get

$$\begin{aligned}
 \langle a \mid b \rangle &= (D_{\partial M}^{-1}(a) \frown D_{\partial M}^{-1}(b))([\partial M]) \\
 &= [D_{\partial M}^{-1}(a) \mid b] \\
 &= [D_{\partial M}^{-1} \circ \partial(A) \mid b] \\
 &= [i^* \circ D_M^{-1}(A) \mid b] \\
 &= [D_M^{-1}(A) \mid i_*(b)] = 0
 \end{aligned}
 \quad \square$$

## A.2 Twisted homology of the torus

The value of the twisted homology of the 2-torus  $T$  is the fundamental case for the construction of all slope invariants we study, as the torus represents a boundary component of the exterior  $M_L$  of a link  $L$ . We make this computation in detail from the CW-complex of  $T$ .

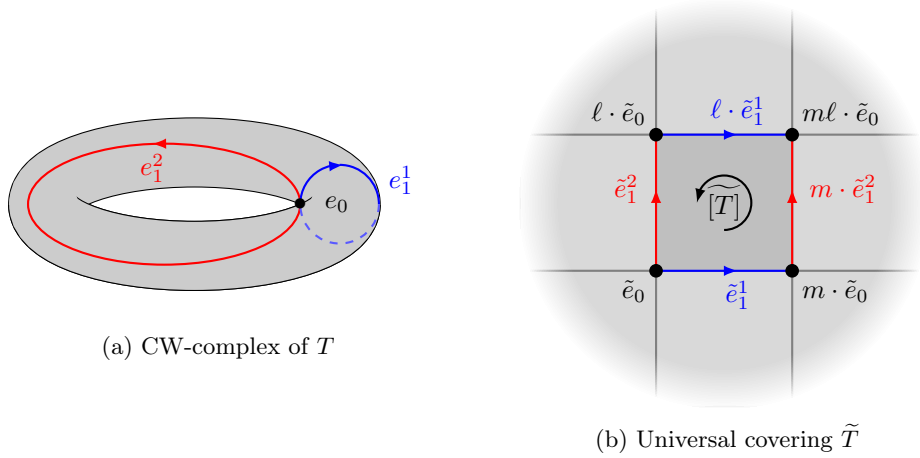


Figure A.2.1: Chain complex of the 2-torus

The generators of the CW-complex structure of  $T$  are shown on [Figure A.2.1a](#). The longitude curve is denoted  $e_1^1$  and the meridian curve is denoted  $e_1^2$ . The fundamental group  $\pi_1(T)$  is free abelian and generated by the classes  $m$  and  $\ell$  of the meridian and longitude of  $T$  respectively. The universal covering of the torus  $\tilde{T}$  is a  $\pi_1(T)$ -module homeomorphic to  $\mathbb{K}^2$ . The twisted chain complex of  $\tilde{T}$  as shown on [Figure A.2.1b](#) is given by:

$$\tilde{C}_2 = [\tilde{T}] \otimes \mathbb{K}(\rho) \xrightarrow{\partial_1} \tilde{C}_1 = \langle \tilde{e}_1^2, \tilde{e}_1^1 \rangle \otimes \mathbb{K}(\rho) \xrightarrow{\partial_0} \tilde{C}_0 = \{ \tilde{e}_0 \} \otimes \mathbb{K}(\rho) \longrightarrow 0$$

From [Figure A.2.1b](#) we immediately get that the matrices of the boundary operators  $\partial_1$  and  $\partial_0$  are given by

$$D_1 = \begin{bmatrix} \tilde{e}_1^1 \otimes \mathcal{B} & \tilde{e}_1^2 \otimes \mathcal{B} \\ I_r - \rho(\ell) & \vdots \\ \vdots & \rho(m) - I_r \end{bmatrix} \qquad D_0 = \begin{bmatrix} \rho(m) - I_r \\ \dots \\ \rho(\ell) - I_r \end{bmatrix} \begin{bmatrix} \tilde{e}_1^1 \otimes \mathcal{B} \\ \tilde{e}_1^2 \otimes \mathcal{B} \end{bmatrix}$$

**Lemma A.2.1.** *Suppose that  $\rho$  is unitary and let*

$$V_\rho := \ker(\rho(\ell) - I_r) \cap \ker(\rho(m) - I_r)$$

*Then there is a natural  $\mathbb{K}$ -vector space isomorphism*

$$H_1(T; \rho) \simeq \langle \tilde{e}_1^1, \tilde{e}_1^2 \rangle \otimes V_\rho \quad \triangleleft$$

*Proof.* Since  $[\ell, m] = 1$  inside  $H_1(T; \rho)$  and  $\rho(\ell)$  and  $\rho(m)$  are unitary then the two matrices are

co-diagonalisable (over  $\mathbb{C}$  if  $\mathbb{K} = \mathbb{R}$ ). In the adequate basis, the module  $\mathbb{K}(\rho)$  splits over  $\mathbb{C}$  into

$$\mathbb{K}(\rho) = \bigoplus_{i=1}^r \mathbb{C}(\omega_i) \tag{A.1}$$

where  $\omega_i : \pi_1(T) \rightarrow S^1$  for  $1 \leq i \leq r$  are characters corresponding to the (complex) eigenvalues of  $\rho$ . Let  $x \in \ker(\partial_0)$ . There exists  $a, b \in \mathbb{C}(\omega_i)$  such that  $x = \tilde{e}_1^1 \otimes a + \tilde{e}_1^2 \otimes b$  and :

$$\begin{aligned} \partial_0(x) &= (m-1) \cdot \tilde{e}_0 \otimes a + (\ell-1) \cdot \tilde{e}_0 \otimes b \\ &= \tilde{e}_0 \otimes (a(\omega_i(m)-1) + b(\omega_i(\ell)-1)) = 0 \end{aligned}$$

Then:

$$\begin{aligned} \partial_1(\widetilde{[T]} \otimes a) &= \tilde{e}_1^1 \otimes a(1 - \omega_i(\ell)) + \tilde{e}_1^2 \otimes a(\omega_i(m) - 1) \\ &= (\tilde{e}_1^1 \otimes a + \tilde{e}_1^2 \otimes b)(1 - \omega_i(\ell)) \\ &= x(1 - \omega_i(\ell)) \\ \partial_1(\widetilde{[T]} \otimes b) &= x(\omega_i(m) - 1) \end{aligned}$$

If (say)  $\omega_i(\ell) \neq 1$ , then every  $x \in \ker(\partial_0)$  can be written as

$$x = \partial_1(\widetilde{[T]} \otimes a(1 - \omega_i(\ell))^{-1})$$

Therefore  $\ker(\partial_0) = \text{im}(\partial_1)$  and  $H_1(T; \omega_i) = \{0\}$ . Conversely, if  $\omega_i(\ell) = \omega_i(m) = 1$ , then  $\partial_1$  and  $\partial_0$  are zero and there is a canonical isomorphism  $\langle \tilde{e}_1^1, \tilde{e}_1^2 \rangle \otimes \mathbb{C}(\omega_i)$ . If  $\mathbb{K} = \mathbb{R}$  then  $H_1(T; \omega_i)$  has the same dimension as  $H_1(T; \bar{\omega}_i)$ , so the recomposition of  $H_1(T; \rho)$  with **Eq. (A.1)** preserves the property.  $\square$

**Corollary A.2.2.** *If either one of the  $\mathbb{K}$ -endomorphisms  $(\rho(m) - I_r)$  or  $(\rho(\ell) - I_r)$  is bijective then  $H_1(T; \rho) = \{0\}$ .*  $\triangleleft$

**Corollary A.2.3.** *If both  $\rho(m) = I_r$  and  $\rho(\ell) = I_r$  then there is a natural  $\mathbb{K}$ -vector space isomorphism  $H_1(T; \rho) = \langle \tilde{e}_1^1, \tilde{e}_1^2 \rangle \otimes \mathbb{K}(\rho)$ .*  $\triangleleft$

### A.3 Fox calculus

In general the  $\rho$ -twisted Alexander module of a CW-complex is computed using Fox calculus, a combinatorial tool developed by R. Fox in [Fox54] which we present briefly.

Let  $\mathbb{F}_p$  be the free group generated by  $x_1, \dots, x_p$  and let  $\mathbb{Z}[\mathbb{F}_p]$  be its group module. Let the augmentation morphism be defined by

$$\begin{aligned} \text{aug} : \quad \mathbb{Z}[\mathbb{F}_p] &\longrightarrow \mathbb{Z} \\ \sum_{f \in \mathbb{F}_p} n_f \cdot f &\longmapsto \sum_{f \in \mathbb{F}_p} n_f \end{aligned}$$

**Definition A.3.1.** For every generator  $x_i \in \mathbb{F}_p$ , the  $i$ -th Fox derivative is the unique linear function

$$\frac{\partial}{\partial x_i} : \mathbb{Z}[\mathbb{F}_p] \longrightarrow \mathbb{Z}[\mathbb{F}_p]$$

such that

$$\begin{cases} \forall a, b \in \mathbb{Z}[\mathbb{F}_p] : \frac{\partial(ab)}{\partial x_i} = \text{aug}(b) \cdot \frac{\partial a}{\partial x_i} + a \frac{\partial b}{\partial x_i} \\ \frac{\partial x_i}{\partial x_i} = 1 \end{cases} \quad \diamond$$

Now consider the complex of  $\mathbb{Z}[\mathbb{F}_p]$ -modules

$$S_* := S_2 \xrightarrow{\partial_1} S_1 \xrightarrow{\partial_0} S_0$$



where

$$S_2 := \bigoplus_{j=1}^q \mathbb{Z}[\mathbb{F}_p] \otimes r_j \quad S_1 := \bigoplus_{j=1}^p \mathbb{Z}[\mathbb{F}_p] \otimes dx_i \quad S_0 := \mathbb{Z}[\mathbb{F}_p]$$

and  $dx_i$  is a formal generator corresponding to  $x_i$ . For every  $1 \leq i \leq p$  and  $1 \leq j \leq q$  the boundary operators are defined by

$$\partial_1 : r_j \mapsto dr_j \quad \partial_0 : dx_i \mapsto x_i,$$

where  $dw$  is the Fox differential of the word  $w \in \mathbb{Z}[\mathbb{F}_p]$  defined by

$$dw := \sum_{i=1}^p \frac{\partial w}{\partial x_i} dx_i \in S_1$$

We explain how to use Fox calculus to compute  $H_1(M; \rho)$ . Consider a presentation  $\mathcal{P}$  of the group  $\pi$  given by

$$0 \longrightarrow \langle r_1, \dots, r_q \rangle \longrightarrow \mathbb{F}_p \xrightarrow{\zeta} \pi \longrightarrow 0$$

where  $r_i \in \mathbb{F}_p$  for every  $1 \leq i \leq q$ . The morphism  $\zeta$  can naturally be extended over the group modules as  $\mathbb{Z}[\mathbb{F}_p] \rightarrow \mathbb{Z}[\pi]$ . Similarly, the representation  $\rho$  can be extended into  $\mathbb{Z}[\pi] \rightarrow \mathcal{M}_n(\mathbb{K})$ . Now consider the  $\rho$ -twisted chain complex over  $\mathbb{K}(\rho)$  defined by

$$S_*(\rho) := \mathbb{K}(\rho) \otimes_{\mathbb{Z}[\pi]} \zeta(S_*)$$

**Theorem A.3.2** ([Cro71]). *There is a natural vector space isomorphism between the  $\rho$ -twisted Alexander module  $H_1(M; \rho)$  and the first homology group  $H_1(S_*(\rho))$ .*  $\triangleleft$

**Definition A.3.3.** The  $\rho$ -twisted Alexander matrix of  $M$  associated with the presentation  $\mathcal{P}$  is the matrix of  $\partial_1(\rho)$  with coefficients in  $\mathbb{K}$  given by

$$A^\rho := \left[ (\rho \circ \zeta) \left( \frac{\partial r_i}{\partial x_j} \right) \right]_{1 \leq i \leq q, 1 \leq j \leq p} \quad \diamond$$

## A.4 Computation of $\ker i_*$

In this section we suppose that  $\partial M$  is a union of disjoint tori:

$$\partial M = \bigsqcup_{k=1}^{\mu} \partial_k M$$

We also suppose that  $\rho : \pi_1(M) \rightarrow \mathbb{K}(\rho)$  is such that

$$\begin{aligned} \forall i \in \{1, \dots, d\} : \quad & \rho(m_i) = \rho(\ell_i) = I_r \\ \forall j \in \{d+1, \dots, \mu\} : \quad & \begin{cases} \text{rank}(\rho(m_j) - I_r) < r \\ \text{rank}(\rho(\ell_j) - I_r) < r \end{cases} \end{aligned} \quad (\text{A.2})$$

We note  $\partial_D M = \sqcup_{i=1}^d \partial_i M$  the subset of the boundary  $\partial M$  where  $\rho$  is the identity.

Finally, let  $\mathcal{P}$  be a presentation of the fundamental group  $\pi_1(M)$ . Up to standard Tietze movements, one can always suppose that  $\mathcal{P}$  contains the generators  $m_i, \ell_i$  associated to  $\partial M_i$  for every  $1 \leq i \leq d$ .

**Theorem A.4.1.** *Let  $A^\rho$  be the  $\rho$ -twisted Alexander matrix of  $M$  associated with  $\mathcal{P}$ . Let  $A_D^\rho$  be the sub-matrix of  $A^\rho$  containing only the columns associated with the generators of  $\partial_D M$ , and let  $A_C^\rho$  be its complementary sub-matrix. Then*

$$\ker i_* \simeq \text{im}(A_D^\rho \cdot (\ker A_C^\rho)) \cap H(\partial_D M; \rho) \quad \triangleleft$$

Note that one can use the Zassenhaus algorithm to compute the intersection of the row-spaces

of two matrices written in the same basis.

*Proof.* By [Theorem A.3.2](#), all computation of twisted homology for  $M$  and the sub-sets of  $\partial M$  can be made with Fox calculus and the associated chain complex  $S_*(\rho)$ . The following diagram is commutative:

$$\begin{array}{ccc} S_1^\partial(\rho) & \xleftarrow{i_\#} & S_1(\rho) \\ h_\partial \downarrow & & \downarrow h \\ H_1(\partial M; \rho) & \xrightarrow{i_*} & H_1(M; \rho) \end{array}$$

By [Corollaries A.2.2](#) and [A.2.3](#) and [Eq. \(A.2\)](#), we have

$$H_1(\partial M; \rho) \simeq \bigoplus_{i=1}^d H_1(\partial_i M; \rho) \simeq \bigoplus_{i=1}^d \langle \ell_i, m_i \rangle \otimes \mathbb{K}(\rho)$$

Define

$$\begin{aligned} S_1^D(\rho) &:= \langle dl_i, dm_i \mid 1 \leq i \leq d \rangle \otimes \mathbb{K}(\rho) \\ S_1^C(\rho) &:= \langle dx \mid x \rangle \otimes \mathbb{K}(\rho) \end{aligned}$$

where  $x$  goes through every generator of  $\mathcal{P}$  that is not  $\ell_i, m_i$  for  $1 \leq i \leq d$ . We particularise the previous diagram to each component  $i \in \{1, \dots, d\}$ :

$$\begin{array}{ccc} S_1^D(\rho) & \xleftarrow{i_\#} & S_1(\rho) = S_1^D(\rho) \oplus S_1^C(\rho) \\ h_i \downarrow & & \downarrow h \\ H_1(\partial M_i; \rho) & \xrightarrow{i_*} & H_1(M; \rho) \end{array}$$

Let us note  $\mathcal{C}$  a complementary space of  $\ker(\partial_1)$  inside  $S_1^D(\rho)$ . We have the following decomposition:

$$S_1^D(\rho) = H \oplus \text{im}(\partial_2) \oplus \mathcal{C}$$

where  $H \simeq H_1(\partial M; \rho)$ . More precisely, we set an isomorphism

$$\theta : H_1(\partial M; \rho) \xrightarrow{\sim} H$$

such that for every  $W \in H_1(\partial M_i; \rho)$ , we have  $q^{-1}(W) = \theta(W) + \text{im}(\partial_2)$ .

**Lemma A.4.2.**  $(i_\# \circ \theta)(\ker i_*) = \ker h \cap \text{im}(i_\#) \cap i_\#(H)$  ◁

*Proof.* Consider  $a \in \ker i_*$ . We have

$$\begin{aligned} i_*(a) = 0 &\iff h \circ i_\# \circ h^{-1}(a) = \{0\} \\ &\iff i_\# \circ h^{-1}(a) \subset \ker h \cap \text{im}(i_\#) \\ &\iff i_\#(\theta(a) + \text{im}(\partial_2)) \subset \ker h \cap \text{im}(i_\#) \\ &\iff (i_\# \circ \theta)(a) + i_\#(\text{im}(\partial_2)) \subset \ker h \cap \text{im}(i_\#) \end{aligned}$$

Because  $i_\#$  is injective, it preserves the direct sum  $H \oplus \text{im}(\partial_2)$ . Then:

$$i_*(a) = 0 \iff (i_\# \circ \theta)(a) \in \ker h \cap \text{im}(i_\#) \cap i_\#(H) \quad \square$$

Since  $(i_\# \circ \theta_i)$  is a known isomorphism, to compute  $\ker i_*$  we only need to compute the space  $\text{im}(i_\#) \cap \ker h$ . First off,  $\text{im} A_\rho = \ker h$  by [Theorem A.3.2](#).

**Lemma A.4.3.**  $\text{im}(A_D^\rho(\ker A_C^\rho)) = \text{im}(i_\#) \cap \ker h$  ◁

*Proof.* For every vector  $a \in \ker A_C^\rho$  we have

$$aA^\rho = [aA_D^\rho \mid aA_C^\rho] = [aA_D^\rho \mid 0] \in \text{im}(i_\#) \cap \ker h$$

Reciprocally, let  $b \in \text{im}(i_{\#}) \cap \ker h$ , and  $a$  such that  $aA^{\rho} = b$ . We know that  $b$  is only supported by the columns of  $S_1^D(\rho)$ , and then necessarily  $a \in \ker A_C^{\rho}$ .  $\square$

This completes the proof of the theorem.  $\square$

# CONCLUSIONS

*Y así, del poco dormir y del mucho leer, se le secó el cerebro, de manera que vino a perder el juicio.*

— Miguel de Cervantes

EL INGENIOSO HIDALGO DON QUIXOTE DE LA MANCHA, 1605

# CONCLUSION AND FUTURE RESEARCHS

The work presented in the first part of this thesis is the continuation of a research program that started in the early 1990s to study Zariski pairs of curves and line arrangements. Our own contributions build on previous invariants that tried to properly describe the inclusion of the boundary manifold in the exterior. The induced map for the homology in  $\mathbb{Z}$  was not well understood, and we hope that the graph stabiliser has settled this part of the matter. However, our long-term objective is more ambitious as we seek to study twisted homology on line arrangements. Some research has already been made in the general case of graph manifolds [ACM19b] and we intend to apply it to study  $\ker i_*$  in twisted homology as we did with links and knots. For this one needs to describe characters or representations of the fundamental group of the exterior, and one must know what they induce to the boundary. This is precisely what we achieved with the homology inclusion for non-twisted integral coefficients. In addition of being an invariant of its own, it therefore also gives the framework to study the images of the cycle generators in twisted homology. The structure of  $\ker i_*$  will probably require more intermediate invariants which might also be able to detect new Zariski pairs on their own.

Another axis of research is based on our construction of ordered graphed embeddings, which are the base concept behind the definition of the graph stabiliser. [Theorem 1.5.11](#) allows to properly describe any representation of the group of the boundary induced by a representation on the exterior by removing the influence of the choice of the ordered graphed embedding. One could then use this presentation to study more general character varieties and other properties of the map  $i_*$  induced by the inclusion in twisted homology.

In addition, most of the standard tools and structures we used on line arrangements (and their computer implementations) also apply to more general type of complex algebraic curves containing non-ordinary singularities. The boundary manifold has again a graph structure with weaker restricting conditions. The Zariski-van Kampen method used to obtain the fundamental group of the exterior and the braid monodromy are defined as well. [Theorem 1.6.15](#) already established that the graph stabiliser is well defined on a wider class of graphs than the subclass corresponding to the minimal structure of line arrangements. We could therefore endeavour to extend the homology inclusion to algebraic curves whose boundary manifold has a graph structure of that type.

In the second part, we generalised the slope invariant of links. This invariant was again arising from the study of the inclusion of the boundary manifold inside the exterior, but directly in homology twisted by a character. We managed to extend the slope definition to every knot using  $\mathrm{SL}_2(\mathbb{C})$ -representations, and to remove some of the restrictions on the character on links using  $\mathrm{SO}_2(\mathbb{R})$ -representations and the characterisation of Lagrangians. This last generalisation is a concordance invariant for similar reasons as the original character slope. It is known that the character slope is related to with other known concordance invariants, mostly the Milnor linking numbers. Just like the slope, these invariants extract topological information on the links from a low-level analysis of the link group. We plan to investigate further the possible connection between the generalised slope and the Milnor linking numbers, but also with the Reidemeister torsion and twisted Alexander polynomials.

In a wider perspective, the general theorems of [Appendix A](#) and [Section 6.1](#) allow to define the slope using any real orthogonal representation of the link group with a non-trivial invariant

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space on the boundary. Finding new families of such representations could extend the scope of the generalised slope construction with the objective of detecting new non-slice links.

# CONCLUSION ET FUTURES RECHERCHES

Le travail présenté dans la première partie de cette thèse est la continuation d'un programme de recherche démarré au début des années 1990 et qui consiste à étudier les couples de Zariski de courbes et d'arrangements de droites. Notre contribution se base sur des invariants antérieurs dont l'objectif était de décrire of the first 3D shows I saw which was willing to stretch and break character models for a stylistic effecte correctement l'inclusion de la variété-bord dans l'extérieur. Le morphisme induit en homologie sur  $\mathbb{Z}$  n'était pas bien compris, et nous pensons que notre définition du stabilisateur du graphe permet de régler cette partie du problème. Toutefois, notre objectif à long terme est plus ambitieux et consiste à étudier l'homologie tordue des arrangements de droites. Des recherches ont déjà été menées sur le sujet [ACM19b] et nous avons l'intention de les appliquer pour étudier  $\ker i_*$  de la même manière que pour les nœuds et entrelacs. Pour cela il est nécessaire de décrire les représentations ou les caractères du groupe fondamental de l'extérieur, et de comprendre ce qu'ils induisent sur la variété-bord. C'est précisément ce que nous avons déterminé avec notre description de l'inclusion homologique pour des coefficients entiers non-tordus. En plus de représenter un invariant topologique en elle-même, l'inclusion homologique fournit donc le cadre permettant d'étudier les images dans l'extérieur des cycles de la variété-bord en homologie tordue. L'analyse complète de la structure de  $\ker i_*$  va cependant probablement nécessiter la création d'autres invariants intermédiaires, qui seront peut-être à même de détecter par eux-mêmes de nouveaux couples de Zariski.

Un autre axe de recherche est basé sur les plongements ordonnés du graphe qui sont au cœur de la définition du stabilisateur du graphe. En effet, le **Théorème 1.5.11** permet de maîtriser l'influence du choix du plongement du graphe lors de la description d'une représentation de la variété-bord induite par une représentation de l'extérieur. La présentation du groupe fondamental du bord ainsi obtenue peut être utilisée pour étudier les variétés caractéristiques ainsi que d'autres propriétés du morphisme  $i_*$  induit en homologie tordue par l'inclusion.

Par ailleurs, la plupart des outils et des structures standards que nous avons utilisés sur les arrangements de droites (ainsi que leurs implémentations informatiques) s'appliquent également à d'autres types de courbes algébriques complexes contenant des singularités non-transverses. Leur variété-bord a encore une structure de variété graphée mais avec des conditions plus faibles. La méthode de Zariski-van Kampen pour obtenir une présentation du groupe fondamental de l'extérieur fonctionne encore également. Le **Théorème 1.6.15** a déjà établi que le stabilisateur du graphe est bien défini sur une classe plus large de graphes que ceux qui constituent les graphes minimaux d'arrangements de droites. Nous envisageons ainsi d'étendre l'inclusion homologique à des courbes algébriques dont la combinatoire appartient à cette classe plus large.

Dans la seconde partie de la thèse, nous avons généralisé l'invariant de pente (« *slope* ») sur les entrelacs. Cet invariant est également issu de l'étude de l'inclusion de la variété-bord dans l'extérieur, mais en considérant directement le morphisme induit en homologie tordue par un caractère. Nous avons étendu la définition du *slope* à n'importe quel nœud en utilisant des représentations à valeurs dans  $SL_2(\mathbb{C})$ , et nous avons également levé une partie des restrictions du *slope* original sur les entrelacs en utilisant des représentations à valeurs dans  $SO_2(\mathbb{R})$  ainsi que la caractérisation des lagrangiens. Cette dernière généralisation est un invariant de concordance pour des raisons similaires au *slope* original. Il est déjà établi que celui-ci possède des liens avec d'autres invariants de concordance connus, en particulier les enlacements de Milnor. Tout comme le *slope*,

ces invariants extraient des informations topologiques des entrelacs en analysant directement le groupe fondamental à bas niveau. Nous prévoyons d'étudier plus loin la possible connexion entre le *slope* généralisé et les enlacements de Milnor, mais également la torsion de Reidemeister et les polynômes d'Alexander tordus.

Dans une perspective plus large, les théorèmes généraux de l'[Annexe A](#) et de la [Section 6.1](#) permettent de définir le *slope* avec n'importe quelle représentation réelle orthogonale du groupe de l'entrelacs ayant un espace invariant non-trivial sur la variété-bord. Trouver de telles familles de représentations permettrait d'étendre les champs d'application du *slope* généralisé dans l'objectif de découvrir de nouveaux entrelacs *non-slice*.



# CONCLUSIONES E INVESTIGACIONES FUTURAS

El trabajo presentado en la primera parte de esta tesis es la continuación de un programa de investigación que comenzó a principios de la década de 1990 para estudiar los pares de Zariski de curvas y configuraciones de rectas. Nuestras contribuciones se basan en invariantes anteriores que intentaron describir adecuadamente la inclusión de la variedad borde en el exterior. La aplicación inducida para la homología en  $\mathbb{Z}$  no estaba completamente desarrollada el invariante introducido, *estabilizador del grafo*, permite entenderla mejor. Sin embargo, nuestro objetivo a largo plazo es más ambicioso a medida que buscamos estudiar cómo se aplica para calcularlo en la homología torcida, para configuraciones de rectas y su variedad de borde. Ya se han realizado algunas investigaciones en el caso general de variedades de grafos [ACM19b] y pretendemos aplicarlas para estudiar  $\ker i_*$  en homología torcida como lo hicimos con enlaces y nudos. Para ello es necesario describir los caracteres o representaciones del grupo fundamental del exterior y saber qué inducen en el borde. Esto es precisamente lo que logramos para la inclusión en homología entera (con coeficientes no torcidos en  $\mathbb{Z}$ ) con el estabilizador del grafo. Además de ser un invariante en sí mismo, también proporciona el marco para estudiar las imágenes de los generadores de ciclos en homología torcida. La estructura de  $\ker i_*$  probablemente requerirá más invariantes intermedios que también podrían detectar nuevos pares de Zariski por sí solos.

Otro eje de investigación se basa en nuestra construcción de encajes ordenados del grafo, que son el concepto base detrás de la definición del estabilizador del grafo. El [Teorema 1.5.11](#) permite describir adecuadamente *cualquier* representación del grupo fundamental a la frontera inducida por una representación del exterior, eliminando la influencia de la elección del encaje ordenado del grafo. Luego se podría usar esta representación para estudiar variedades de caracteres más generales y otras propiedades de la aplicación  $i_*$  inducidas por la inclusión en homología torcida.

Además, la mayoría de las herramientas y estructuras estándar que utilizamos en configuraciones de rectas (y sus implementaciones informáticas) también se aplican a tipos más generales de curvas algebraicas complejas que contienen singularidades no ordinarias. La variedad límite nuevamente tiene una estructura de grafo con condiciones restrictivas más débiles. También se define el método de Zariski-van Kampen utilizado para obtener el grupo fundamental del exterior y la monodromía de trenzas. El [Teorema 1.6.15](#) ya estableció que el estabilizador del grafos está bien definido en una clase de grafos más amplia que la subclase correspondiente a la estructura más pequeña de configuraciones de rectas. Por lo tanto, podríamos intentar extender la inclusión de homología a curvas algebraicas cuya variedad borde tenga una estructura de grafo de ese tipo.

En la segunda parte, generalizamos el invariante de *pendiente* de los enlaces. Este invariante surgió nuevamente del estudio de la inclusión de la variedad borde en el exterior del enlace, pero directamente en homología torcida por un carácter. Logramos extender su definición a cada nudo usando  $SL_2(\mathbb{C})$ -representaciones, y eliminar algunas de las restricciones sobre el carácter en los enlaces usando  $SO_2(\mathbb{R})$ -representaciones y la caracterización de lagrangianos. Esta última generalización es un invariante de concordancia por razones similares a la pendiente del carácter original. Se sabe que la pendiente del carácter tiene vínculos con otros invariantes de concordancia conocidos, principalmente los números de enlace de Milnor. Al igual que la pendiente, estos invariantes extraen información topológica sobre los enlaces a partir de un análisis de bajo nivel de las relaciones del grupo del enlace. Planeamos investigar más a fondo la posible conexión entre la pendiente generalizada y los números de enlace de Milnor, pero también con la torsión de

Reidemeister y los polinomios torcidos de Alexander.

En una perspectiva más amplia, los teoremas generales del [Apéndice A](#) y del [Apartado 6.1](#) permiten definir la pendiente utilizando cualquier representación ortogonal real del grupo de enlaces con un espacio invariante no trivial en el borde. Encontrar nuevas familias de tales representaciones podría ampliar el alcance de la construcción de pendientes generalizadas con el objetivo de detectar nuevos enlaces no *slice*.

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# CONTENTS

<b>Abstracts</b>	<b>viii</b>
<b>Contents</b>	<b>xi</b>
<b>Introductions</b>	<b>1</b>
<b>Introduction (in English)</b>	<b>2</b>
<b>Introduction (en français)</b>	<b>9</b>
<b>Introducción (en castellano)</b>	<b>16</b>
<b>I Line arrangements</b>	<b>23</b>
<b>1 Graph stabiliser</b>	<b>24</b>
1.1 Preliminaries . . . . .	25
1.1.1 Finitely presented groups . . . . .	25
1.1.2 Braid groups . . . . .	25
1.1.3 Mapping class group of planar surfaces . . . . .	26
1.2 Ordered stars . . . . .	27
1.2.1 Definition and standard model . . . . .	27
1.2.2 Action of the pure mapping class group . . . . .	28
1.3 Graph manifolds . . . . .	30
1.3.1 Circle bundles . . . . .	30
1.3.2 Definition of the graph manifolds . . . . .	32
1.3.3 Ordered graphs . . . . .	32
1.3.4 Graphed homeomorphisms . . . . .	34
1.4 Ordered graphed embeddings . . . . .	34
1.4.1 Ordered model of a graph manifold . . . . .	34
1.4.2 Definition of an ordered graphed embedding . . . . .	35
1.4.3 Action of the homeomorphisms . . . . .	37
1.5 Fundamental group of a graph manifold . . . . .	37
1.5.1 Cycles of the graph . . . . .	37
1.5.2 Presentation of the fundamental group . . . . .	38
1.5.3 First homology group of a graph manifold . . . . .	41
1.6 Graph stabiliser . . . . .	42
1.6.1 Definition of the graph stabiliser . . . . .	42
1.6.2 Difference maps . . . . .	42

1.6.3	Presentation of the graph stabiliser . . . . .	45
1.6.4	Plumbing moves . . . . .	47
<b>2</b>	<b>Homology inclusion of line arrangements</b>	<b>51</b>
2.1	Presentation of line arrangements . . . . .	52
2.2	Combinatorics . . . . .	53
2.2.1	Definitions . . . . .	53
2.2.2	Graphs of a combinatorics . . . . .	55
2.2.3	Binary vertices . . . . .	56
2.3	Boundary manifold of a line arrangement . . . . .	58
2.3.1	Blow-up . . . . .	59
2.3.2	Construction of the boundary manifold . . . . .	60
2.3.3	Graph structure . . . . .	61
2.4	Exterior of a line arrangement . . . . .	62
2.4.1	Wiring diagram . . . . .	62
2.4.2	Braid monodromy . . . . .	63
2.4.3	Fundamental group of the exterior . . . . .	67
2.5	Homology inclusion map . . . . .	69
<b>3</b>	<b>Computations of the homology inclusion</b>	<b>71</b>
3.1	Standard ordered graphed embedding . . . . .	71
3.1.1	Combinatorial data . . . . .	72
3.1.2	Boundary manifold and braid monodromy . . . . .	72
3.1.3	Construction of the standard graphed embedding . . . . .	75
3.2	Values of the homology inclusion map . . . . .	76
3.2.1	Braid linking . . . . .	76
3.2.2	Main computation theorem . . . . .	80
3.3	Comparison of a combinatorial pair . . . . .	82
3.3.1	Adjustments of the orderings . . . . .	82
3.3.2	Comparison algorithm . . . . .	84
3.3.3	Examples of Zariski pairs . . . . .	85
<b>II</b>	<b>Slope invariants of links</b>	<b>87</b>
<b>4</b>	<b>Character slope of links</b>	<b>88</b>
4.1	Preliminaries on link theory . . . . .	88
4.1.1	Generalities . . . . .	88
4.1.2	Link group . . . . .	89
4.2	Definition of the character slope . . . . .	90
4.2.1	Twisted homology of the boundary . . . . .	90
4.2.2	Definition of the character slope . . . . .	91
4.2.3	Properties of the slope . . . . .	92
4.3	Slope computation . . . . .	92
<b>5</b>	<b>The <math>SL_2(\mathbb{C})</math>-slope of knots</b>	<b>95</b>
5.1	Introduction . . . . .	95
5.2	Representation varieties and $A$ -polynomial . . . . .	97
5.2.1	Representation and character varieties . . . . .	97
5.2.2	The character variety of $\mathbb{Z}^2$ . . . . .	98
5.2.3	The $A$ -polynomial . . . . .	99
5.3	The $SL_2(\mathbb{C})$ -slope invariant . . . . .	99
5.3.1	Admissible representations . . . . .	99
5.3.2	The slope of characters . . . . .	100
5.3.3	Regularity and properties of the slope . . . . .	101
5.3.4	Slope and Reidemeister torsion . . . . .	103
5.3.5	Compute the slope . . . . .	105
5.4	The $SL_2(\mathbb{C})$ -slope and the $A$ -polynomial . . . . .	107
5.4.1	The derivation formula . . . . .	107

5.4.2	Detecting the unknot . . . . .	108
5.4.3	The slope at an ideal point . . . . .	109
<b>6</b>	<b>Generalised slope of links</b>	<b>111</b>
6.1	Slope of a Lagrangian . . . . .	111
6.1.1	Complexification . . . . .	112
6.1.2	Characterisation of Lagrangians . . . . .	112
6.2	Generalised slope construction . . . . .	114
6.2.1	Character torus . . . . .	114
6.2.2	Symplectic structure of the twisted homology . . . . .	114
6.2.3	Definition of the generalised slope . . . . .	115
6.2.4	Relation with the character slope . . . . .	116
6.3	Concordance invariance . . . . .	117
<b>A</b>	<b>Twisted (co-)homology</b>	<b>119</b>
A.1	General results . . . . .	119
A.1.1	Definition . . . . .	119
A.1.2	Duality . . . . .	120
A.1.3	Intersection form . . . . .	120
A.1.4	Kernel of $i_*$ . . . . .	121
A.2	Twisted homology of the torus . . . . .	122
A.3	Fox calculus . . . . .	123
A.4	Computation of $\ker i_*$ . . . . .	124
	<b>Conclusions</b>	<b>127</b>
	<b>Conclusion (in English)</b>	<b>128</b>
	<b>Conclusion (en français)</b>	<b>130</b>
	<b>Conclusiones (en castellano)</b>	<b>132</b>
	<b>Bibliography</b>	<b>134</b>
	<b>Contents</b>	<b>140</b>

## Abstract

We study knotted codimension-two objects in manifolds of dimension 3 and 4: complex line arrangements in  $\mathbb{C}\mathbb{P}^2$  and links in  $S^3$ . We introduce new topological invariants of their embedding, derived from the interaction between their complement and their peripheral structure.

The motivation for line arrangements is to identify *Zariski pairs* which have the same combinatorics but different embeddings. Building on ideas developed by B. Guerville-Ballé and W. Cadiegan-Schlieper, we consider the inclusion map of boundary manifold to the exterior and its effect on homology classes. A careful study of the graph Waldhausen structure of the boundary manifold allows to identify specific generators of the homology. Their potential images are encoded in a group, the *graph stabiliser*, with a nice combinatorial presentation. The invariant related to the inclusion map is an element of this group. Using a computer implementation in *Sage* and the braid monodromy, we compute the invariant for some examples and exhibit new ordered Zariski pairs.

The second part concerns knot theory and a generalisation of a slope invariant developed by A. Degtyarev, V. Florens and A. G. Lecuona. Similarly to the context of line arrangements, we consider the inclusion map of the boundary components of a neighbourhood of a link in its exterior. On twisted homology, the kernel of this map is a Lagrangian subspace – for the intersection form – and its slopes provide a topological invariant of the link. We present two applications of this idea. In the first, developed in collaboration with L. Bénard, we consider knots and  $SL_2(\mathbb{C})$  representations. This new slope invariant appears to be closely related to a higher-level invariant called the *A*-polynomial. The second application uses a Lagrangian characterisation method due to V. Arnol'd. It provides a concordance invariant with several relations to Sato-Levine invariant and Milnor linking numbers.

**Keywords:** topology in small dimension, knot theory, algebraic curves

## STRUCTURES PÉRIPHÉRIQUES ET INVARIANTS TOPOLOGIQUES DE SOUS-VARIÉTÉS NOUÉES

## Résumé

On étudie des objets noués de codimension 2 dans des variétés de dimension 3 et 4 : des arrangements de droites complexes dans  $\mathbb{C}\mathbb{P}^2$  et des entrelacs dans  $S^3$ . On introduit de nouveaux invariants topologiques de leurs plongements, dérivés des interactions entre leur complémentaire et leur structure périphérique.

La motivation concernant les arrangements de droites est d'identifier des *paires de Zariski* qui ont la même combinatoire mais des plongements différents. En utilisant des idées développées par B. Guerville-Ballé et W. Cadiegan-Schlieper, on considère l'application inclusion de la variété bord dans l'extérieur et son effet sur les classes d'homologie. Une étude approfondie de la structure graphée de Waldhausen de la variété bord permet d'identifier des générateurs spécifiques de son homologie. L'information de leurs images potentielles est collectée dans un groupe, le *stabilisateur du graphe*, qui a une présentation combinatoire simple. On utilise une implémentation en *Sage* et la monodromie de tresses pour calculer l'invariant dans certains exemples et produire de nouvelles paires de Zariski ordonnées.

La seconde partie est consacrée à la théorie des nœuds et à une généralisation d'un invariant de pente (« *slope* ») développé par A. Degtyarev, V. Florens et A. G. Lecuona. Similairement aux arrangements de droites, on considère l'application inclusion des composantes de bord d'un voisinage de l'entrelacs dans l'extérieur. Au niveau de l'homologie tordue, le noyau de cette application est un sous-espace Lagrangien – pour la forme d'intersection – et sa pente est un invariant topologique de l'entrelacs. On présente deux applications de cette construction. Dans la première, en collaboration avec L. Bénard, on considère le cas des nœuds et des représentations dans  $SL_2(\mathbb{C})$ . L'invariant *slope* obtenu est étroitement relié à un invariant de haut niveau, le *A*-polynôme. La seconde application utilise une caractérisation des sous-espaces lagrangiens due à V. Arnol'd. On construit un invariant de concordance qui a de nombreux liens avec l'invariant de Sato-Levine et les enlacements de Milnor.

**Mots clés :** topologie en petite dimension, théorie des nœuds, courbes algébriques

## ESTRUCTURAS PERIFÉRICAS E INVARIANTES TOPOLÓGICOS DE SUBVARIEDADES ANUDADAS

## Resumen

Estudiamos objetos anudados de codimensión dos en variedades de dimensión 3 y 4: configuraciones de rectas complejas en  $\mathbb{C}\mathbb{P}^2$  y enlaces en  $S^3$ . Introducimos nuevos invariantes topológicos de su encaje, que provienen de la interacción entre el complementario y su estructura periférica.

La motivación para las configuraciones de rectas es identificar *pares de Zariski* que tienen la misma combinatoria pero diferentes encajes. Basándonos en las ideas desarrolladas por B. Guerville-Ballé y W. Cadiegan-Schlieper, consideramos el mapa de inclusión de la variedad límite hacia el exterior y su efecto sobre las clases de homología. Un estudio cuidadoso de la estructura de grafo de Waldhausen de la variedad del borde permite identificar generadores específicos de la homología. Sus imágenes potenciales están codificadas en un grupo, el *estabilizador del grafo*, con una elegante presentación combinatoria. El invariante relacionado con la inclusión es un elemento de este grupo. Utilizando una implementación informática en *Sagemath* y la monodromía de trenzas, calculamos el invariante para algunos ejemplos y encontramos nuevos pares ordenados de Zariski.

La segunda parte se refiere a la teoría de nudos y una generalización de un invariante llamado *pendiente* («*slope*») desarrollado por A. Degtyarev, V. Florens y A. G. Lecuona. De manera similar al contexto de configuraciones de rectas, consideramos la inclusión de los componentes del borde de un entorno de un enlace en su exterior. En homología torcida, el núcleo de esta aplicación es un subespacio lagrangiano –para la forma de intersección– y sus pendientes proporcionan un invariante topológico del enlace. Presentamos dos aplicaciones de esta idea. En el primero, desarrollado en colaboración con L. Bénard, consideramos nudos y representaciones  $SL_2(\mathbb{C})$ . Este nuevo invariante de pendiente parece estar estrechamente relacionado con un invariante de nivel superior llamada *A*-polinomio. La segunda aplicación utiliza un método de caracterización lagrangiano debido a V. Arnol'd. Proporciona un invariante de concordancia con varias relaciones con el invariante de Sato-Levine y los números de enlace de Milnor.

**Palabras claves:** topología en baja dimensión, teoría de los nudos, curvas algebraicas