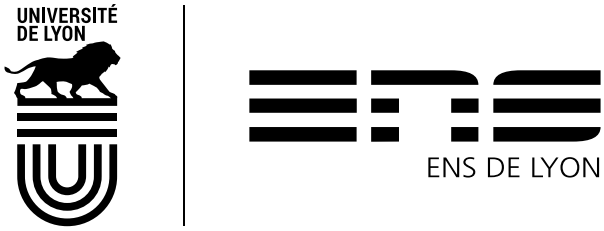
Kaluza-Klein Compactification, Exceptional Geometry and Holography

Bastien Duboeuf



THÈSE

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CHAPTER

1

SUMMARY OF THE THESIS

A Résumé en Français

Les Théories des Cordes sont nos meilleurs candidats dans la persepective d'unification de toutes les interactions de la Nature : électromagnétisme, interaction faible, interaction forte et gravité. Cette dernière est naturellement encodée dans toutes les Théories des Cordes, qui sont exemptes de divergences dans l'ultraviolet. C'est cette dernière caractéristique qui font des Théories des Cordes de potentiels candidats pour une théorie de gravitation quantique, et constitue une des raisons pour laquelle les Théories des Cordes sont largement étudiées.

Une caractéristique générale des Théories des Cordes est le nombre de dimensions de l'espace-temps recquis. Ce nombre est soit 26, soit 10. Cela nécessite un approfondissement de l'étude des Théories des Cordes, puisque l'Univers dans lequel nous vivons est a priori un espace-temps à 4 dimensions. L'une des façons d'aborder ce problème est de recourir à ce que l'on appelle les compactifications de Kaluza-Klein. Il s'agit d'un processus par lequel certaines dimensions de l'espace total sont rendues compactes et petites. L'objectif de ce procédé est qu'à notre échelle macroscopique et aux niveaux d'énergie accessibles au quotidien, les effets de ces dimensions "supplémentaires" soient suffisamment faibles pour que nous puissions les ignorer. Les Théories des Cordes étant des théories très complexes, avec de nombreux degrés de liberté, une façon d'étudier les procédures de compactification consiste à se restreindre à l'analyse des Supergravités. Ces dernières sont obtenues, entre autre, comme une limite à basse énergie des Théories des Cordes et constitutent l'objet principal d'étude de cette thèse. Dans ces théories, des symétries supplémentaires apparaissent lors des compactifications, appelées dualités, ce qui nous permet de reformuler les Supergravités en des "Théories des Champs Exceptionnelles", rendant ces dualités manifestes.

Les Supergravités sont d'autant plus intéressantes qu'elles jouent un rôle clef dans le contexte

de la correspondance AdS/CFT, qui est un autre type de dualité établissant l'équivalence entre une théorie de la gravité, ici une Supergravité définie sur un espace d'Anti-de-Sitter (AdS), et une Théorie Quantique des Champs, ici une Théorie Conforme des Champs vivant à la frontière de l'espace-temps AdS.

Dans la première partie de la thèse, nous ferons des commentaires généraux et les utiliserons pour introduire divers concepts nécessaires tout au long du manuscrit. Cela va d'une brève introduction aux Théories des Cordes et aux Supergravités à des éléments plus spécifiques tels qu'une introduction aux concepts de supersymétrie et aux réductions de Kaluza-Klein. Nous profiterons également de l'introduction pour faire un rappel sur les Théories des Champs Exceptionnelles. Enfin, nous introduirons la notion d'Holographie, trouvant une réalisation concrète dans la correspondance AdS/CFT.

Dans la deuxième partie de la thèse, nous montrerons comment utiliser les techniques de la Théorie des Champs Exceptionnelle pour calculer les spectres des solutions de la Supergravité à 11 dimensions sur $AdS_4 \times \Sigma^7$, avec Σ^7 un espace interne compact à 7 dimensions dont la topologie est celle de la sept-sphère. Nous commencerons par introduire les techniques pour la spectroscopie de Kaluza-Klein par un calcul direct, qui utilisent la théorie des groupes pour la diagonalisation des opérateurs internes. Dans un second temps nous introduiront les techniques de la Théorie des Champs Exceptionnelle pour les espaces dit parallélisables de Leibniz. Nous montrerons par la suite comment ces dernières techniques peuvent être étendues aux espaces avec la condition moins contraignante dite de parallélisabilité Généralisée. Nous illustrerons cette nouvelle approche dans le cas de la solution $AdS_4 \times S^7_{\text{squashed}}$ de la Supergravité à 11 dimension, pour laquelle nous donnons le spectre de masse dans sa totalité. Nous montrerons ensuite une réalisation concrète de la correspondance AdS_4/CFT_3, en calculant la solution solitonique des équations de Supergravité interpolant entre la sphère ronde et la sphère déformée. Cette solution est duale à un flux de renormalisation du côté de la Théorie des Champs Conformes de la correspondance.

Enfin, dans la dernière partie de la thèse, nous nous intéresserons aux couplages à n-points dans les Supergravités. Dans une première partie, nous montrerons comment calculer les couplages cubiques pour des champs spécifiques sur un espace $AdS_5 \times S^5$ de la Supergravité IIB à 10 dimensions via un calcul direct. Après avoir discuté des résultats obtenus grâce à ces techniques, nous montrerons comment nous pouvons utiliser les techniques de la Théorie des Champs Exceptionnelle pour calculer efficacement ces couplages. Ces calculs nous permettrons aussi de révéler des structures de la Théorie des Champs Exceptionnels dans les couplages à n-points en Supergravité.

B English Summary

STRING Theories are our most promising candidates in the goal of unifying all interactions in nature: electromagnetism, weak interaction, strong interaction, and gravity. The latter is naturally encapsulated in all String Theories, which are ultraviolet-free from divergences. A general feature of String Theories is the number of spacetime dimensions required. This number is either 26 or 10. In any case, this requires some further modifications of the theories since the world we are living in is a priori a 4-dimensional spacetime space. One way to tackle this problem is via so-called Kaluza-Klein compactifications. The latter is a process in which some of the dimensions of the total space are made compact and small. The idea is that at our macroscopic scale and daily energy level, the effects of those "extra" dimensions are sufficiently small so we can ignore them.

Because String Theories are very complex theories, with a lot of degrees of freedom, one way to study compactification procedures is by restricting the analysis to Supergravities, which can be obtained in a certain regime of String Theories.

Supergravities are the main focus of this thesis. In those theories, additional symmetries show up in dimensional compactifications, either called T- or U-duality, which allow us to reformulate Supergravities into so-called Exceptional Field Theories, making those dualities explicit. Let us finally remark that Supergravities are particularly important in the context of the AdS/CFT correspondence, which is another type of duality stating the equivalence between a theory of gravity, here a Supergravity defined on an Anti-de-Sitter (AdS) background, and a Quantum Field Theory, here a Conformal Field Theory living on the boundary of the AdS previous spacetime.

In the first part of the thesis, we will make general comments and use it to introduce various concepts needed throughout the thesis. This goes from a brief introduction of String Theories and Supergravities to more specific features such as an introduction to the concepts of supersymmetry and Kaluza-Klein reductions. We will also use the introduction to make a reminder about Exceptional Field Theories. Finally, we will introduce the notion of Holography with its concrete realization via the AdS/CFT correspondence.

In the second part of the thesis, we will show how to use Exceptional Field Theory techniques to compute spectra of solutions of 11-dimensional Supergravity on $AdS_4 \times \Sigma^7$, with Σ^7 a 7-dimensional compact internal space whose topology is that of the seven-sphere. After reviewing the state-of-the-art techniques for Kaluza-Klein spectroscopy, including direct calculations supplemented by diagonalization of internal operators via group theory techniques, as well as Exceptional Field Theory techniques for Leibniz parallelizable spaces, we will show how the latter can be extended to spaces with the less constraining condition of Generalized parallelizability. We will illustrate how this technology works in the case of $AdS_4 \times S^7_{\text{squashed}}$ solution of 11-dimensional Supergravity, for which we give a complete answer for the spectrum. We will then show a concrete realization of the AdS_4/CFT_3 correspondence, by computing the domain-wall solution of the Supergravity equations interpolating between the round and the squashed seven-sphere, which is dual to a Renormalization Group flow on the Conformal Field Theory side. This demonstrates that not only Exceptional Field Theory techniques can be used to compute spectra around Supergravities with AdS background, but it also allows us to compute quadratic couplings of Kaluza-Klein fluctuations around a domain-wall solution of Supergravity.

Finally, in the last section of the thesis, we will be interested in n-point couplings in Supergravities. In a first part, we will show how to compute cubic couplings for specific fields on AdS_5 background of IIB 10-dimensional Supergravity via a brute force calculation. After discussing the achievements made using these techniques, we will show how we can use Exceptional Field Theory techniques to efficiently compute these couplings. Not only will this prove more efficient, but it will also allow us to write the same couplings in a more compact form, to derive formulas that apply to any vacua that are Leibniz parallelizable, and also to reveal Exceptional Field Theory structures that prove long-standing conjectures. We will illustrate the power of these techniques on the example of AdS_5 background of IIB 10-dimensional Supergravity.

Foreword

This thesis represents the final accomplishment of my doctoral research, completed under the supervision of Henning Samtleben at the Laboratoire de physique at ENS de Lyon. It is based on the articles that I co-authored with Henning and our colleagues Emanuel Malek (Humboldt University Berlin), and Michele Galli (Humboldt University Berlin) which are detailed below.

[DB1] B. Duboeuf, E. Malek, and H. Samtleben, "Kaluza-Klein spectrometry beyond consistent truncations: The squashed S^7," JHEP, vol. 04, p. 062, 2023. DOI: 10.1007/JHEP04(2023)062. arXiv: 2212.01135 [hep-th].

[DB2] B. Duboeuf, M. Galli, E. Malek, and H. Samtleben, "Holographic RG flow from the squashed to the round S^7," Phys. Rev. D, vol. 108, no. 8, p. 086 002, 2023. DOI: 10.1103/PhysRevD.108.086002. arXiv: 2306.11789 [hep-th].

[DB3] B. Duboeuf, E. Malek, and H. Samtleben, "Cubic and higher-order supergravity couplings for ads vacua using exceptional field theory," Nov. 2023. arXiv: 2311.00742 [hep-th].

CHAPTER

2

INTRODUCTION

Contents

A General Introduction

GENERAL Relativity is currently the best theory of gravity we possess. It was formulated in 1915 by Albert Einstein [1]. In contrast to Isaac Newton's theory of gravity [2], which describes the gravitational force, General Relativity is a relativistic theory of space and time, effectively amalgamating both into what is known as spacetime. General Relativity provides a framework in which spacetime becomes a dynamic entity that can bend, curve, and oscillate, interacting with the matter it contains. This implies, for example, that Earth does not revolve around the Sun due to a force binding them together, but rather because the spacetime in which Earth evolves is curved due to the presence of the Sun. Consequently, Earth follows a straight path, albeit within curved geometry as illustrated in Figure 2.1 . Since its inception, General Relativity has led to several observations that Newton's theory of gravity could not predict: the bending of light around massive objects, first observed by Arthur Eddington in 1919 [3]; the advance in the perihelion of Mercury [4, 5]; the prediction of gravitational waves [6], small perturbations of spacetime itself that were successfully observed for the first time in 2016 [7]; and its extensive application in cosmology, where it predicts the expansion of the Universe [8, 9], as first observed by Edwin Hubble [10] and nowadays in [11]. Among all the predictions of General Relativity, one of peculiar importance for theoretical physicists is the existence of Black Holes (BH) [12]. These objects, predicted by General Relativity, are very dense objects, that bend so much spacetime that even light cannot escape their gravitational field, hence the name black hole. This region of spacetime is delimited by a so-called event horizon, behind which is hidden a "singularity", where General Relativity breaks down. BH are typically very massive objects, ranging in mass from the order of solar masses to billions of solar masses for supermassive BHs in our Universe. Note however that the mass of BHs can a priori be abitrarly small, and some theoretical models proposes primordial "light" BHs [13] as a candidate for dark matter. These objects fascinate physicists because they are predicted by General Relativity, yet the theory fails to properly explain them: physical quantities tend to infinity when approaching the inner core of the BHs, the singularity. Therefore, BH serve as a laboratory for fundamental physicists who test how new theories behave around BHs. Despite the difficulty in observing these objects due to their nature, many observations support their existence: the detection of gravitational waves [7] illustrated in Figure 2.3 and thought to be produced by the merger of two BH, observations of stars at the center of our galaxy orbiting a small but very massive object thought to be a supermassive BH [14], and more recently, the imaging of the accretion disk of the same BH at the center of our galaxy [15], as well as another supermassive BH called M87* [16], see Figure 2.2 .

Unlike gravity, which is important at large scales, Quantum Mechanics (QM) plays a crucial role at the microscopic scale. It was developed in the 1920s by several physicists, including Erwin Schrödinger, Wolfgang Pauli, Werner Heisenberg, Paul Dirac, Louis de Broglie, Niels Bohr, Max Planck, Marie Curie, Albert Einstein, Paul Langevin, and many others. Quantum mechanics offers a peculiar paradigm. Firstly, the results of experiments are inherently probabilistic: repeating the same procedure with the same initial conditions in the same environment will yield different outcomes. Quantum mechanics is intrinsically probabilistic. Measurements conducted by an observer cannot yield determined outcomes; instead, the realization of the measurement follows probabilistic laws. QM also asserts the equivalence between wave and particle behavior, known as the wave-particle duality [17]. This is illustrated in Figure 2.4 . Moreover, QM measurements have intrinsic uncertainity, stemming from the probabilistic nature of QM, which states, for example,

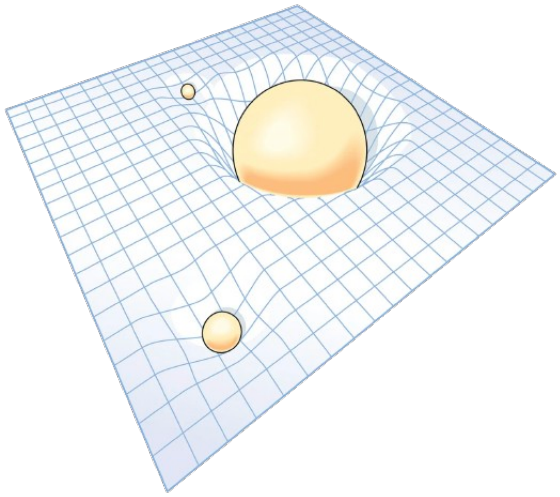


Figure 2.1: Schematics representation of General Relativity, with massive objects deforming space-time.

that both the speed and the position of a particle cannot be simultaneously measured with arbitrary precision. In other words, attempting to measure the position of a particle governed by QM with very high precision will introduce significant indetermination in its velocity. This principle is encapsulated in the Heisenberg Indeterminacy Principle [18]:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{2.1}$$

with σ_x representing the uncertainty in position and σ_p denoting the uncertainty in momentum. Another peculiar aspect of quantum mechanics is its intimate relationship with "observers." In quantum mechanics, an observer can be a person, a machine, or a measurement instrument coupled to the quantum system. It looks like laws of Nature kind of became subjective to this observer. Finally, QM leads to the quantization of physical quantities, hence its name. For instance, it predicts the quantization of the energy levels of hydrogen atoms, which matches the observations to a high degree [19, 20]. Despite its strange nature, quantum mechanics has led to numerous achievements and predictions, verified to a very high degree of accuracy.

In a sense, quantum physics is not merely a theory; it is more akin to a set of principles that one must apply to extend a given theory to the microscopic world. Using this "recipe," Quantum Field Theory (QFT) has been developed. QFT describes the behavior of relativistic particles: their motion, how they interact with one another, how they can be created or annihilated, and so forth. Its development led to the construction of the Standard Model (SM), which unifies three out of the four fundamental interactions of nature: electromagnetism, the weak force, and the strong force [21–27]. The success of the Standard Model relies heavily on the successful quantization of classical field theory.

Can we then apply the same methods to quantize General Relativity? First of all, let us clarify why we want to do so. We could have two different theories, General Relativity and the Standard Model (SM), both operating at vastly different scales, and live with it. It would be perfectly fine. However, there are at least two scenarios where a so-called theory of Quantum Gravity theory

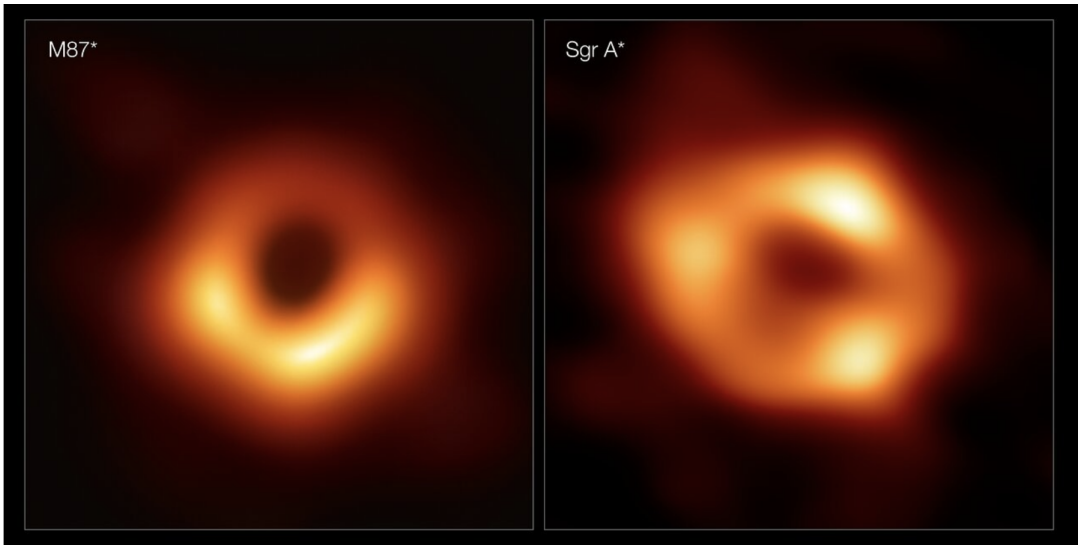


Figure 2.2: Pictures of M87* and Sagittarius A* taken by the Event Horizon Telescope collaboration [15, 16].

is needed. Firstly, to describe BH. They are characterized by the so-called singularity in General Relativity, which is hidden behind the event horizon. Therefore, we have an object operating at a microscopic scale involving strong gravitational interactions, hence the necessity of Quantum Gravity. The second scenario is in cosmology. The Universe is currently expanding, and if we rewind the expansion, it turns into contraction. According to the best-admitted model for the Universe, known as the Friedmann-Lemaître-Robertson-Walker (FLRW) model [8, 9, 28, 29], the Universe at its origin was small and compact, thus again highlighting the need for Quantum Gravity.

The quest for a quantum theory of gravity has spanned almost a century. Attempting to apply conventional quantum mechanics methods to General Relativity, such as canonical quantization, leads to unphysical results: physical quantities become infinite as they run with energy scale. This indicates that General Relativity is not renormalizable, or in other words that it diverges in the ultraviolet (high-energy) regime. Despite extensive efforts, the question of finding candidates for a theory of quantum gravity remains an open problem. Consequently, there are several promising candidates, with String Theory being one of the most prominent, and the focus of this thesis. Originally developed to explain strong interactions [30], String Theory postulates the existence of extended objects, strings. These strings can vibrate, oscillate, and interact, and String Theory provides a framework for studying their dynamics [31–34]. It is worth noting that when we refer to String Theory, it is more accurate to say String Theories, as there are five inequivalent theories. It was later realized that String Theorie could potentially serve as viable theories of quantum gravity and as theories to describe elementary particles and their interactions. This is known as the first String Theory revolution [35]. The second string revolution occurred latter when it has been realized that all String Theories could be realized as different limits of a single $11d$ theory, known as M-theory. We will come back to this last point later.

General Relativity naturally emerges from String Theories [36], whose ultraviolet behavior is free from divergences [31, 32], which is one of the reasons why String Theories are widely studied.

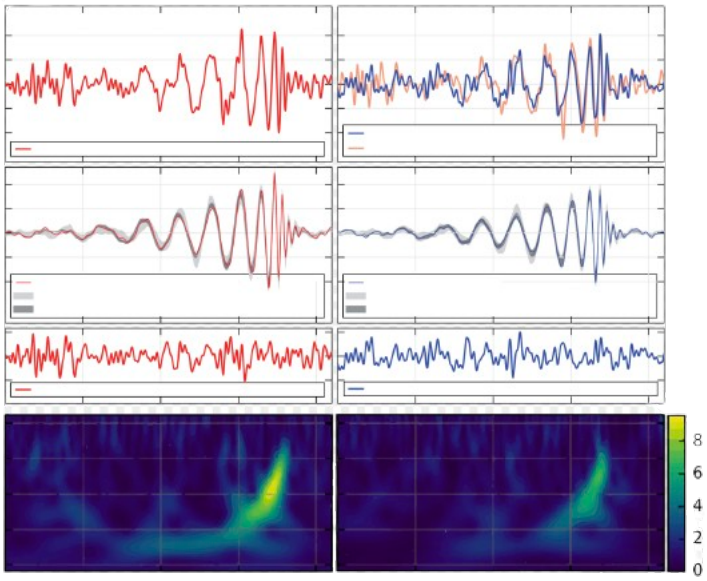


Figure 2.3: LIGO measurement of the gravitational waves of GW150914 at the Livingston (right) and Hanford (left) detectors [7], compared with the theoretical predicted values.

The focus of this thesis is on the low-energy limit of String Theories, called Supergravities. These theories of gravity incorporate an additional ingredient compared to General Relativity: supersymmetry. Supersymmetry relates bosonic degrees of freedom (such as photons) with fermionic degrees of freedom (such as electrons). While Supergravities may not directly serve as candidates for unifying all fundamental interactions [37], they play a crucial role in the advancement of fundamental physics. They can be utilized to study String Theories within certain regimes and are particularly significant in compactification procedures. Since String Theories operates in 10-dimensional spaces, it is necessary to eliminate the extra dimensions. This is achieved through a process known as compactification, firstly proposed by Kaluza and Klein [38, 39]. [1]. Super-

[1]Other possibilities such as brane world scenario, where the Universe is a brane inside a larger space, are possible to explain the existence of extra dimensions [40]. Brane are extended objects, which for example appear naturally in the

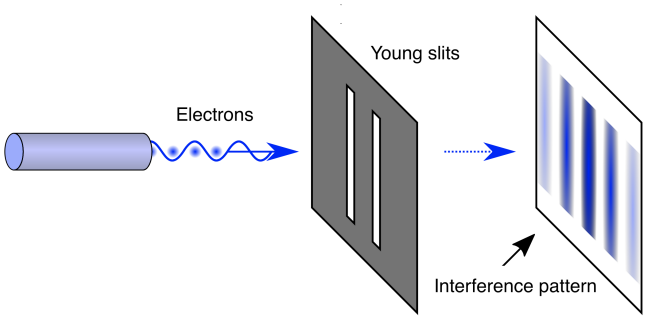


Figure 2.4: Experiment emphasizing the duality wave/particle of electrons, where an interference pattern is obtained in a two slits experiments with repeated electrons going through the setup (source : Wikipedia).

gravities are instrumental in the study of String Theories compactifications because much can be inferred about the compactifications by examining the Supergravity regime.

The final point we wish to discuss is a remarkably peculiar observation that has emerged in the quest for Quantum Gravity: the holographic principle. It appears that Quantum Gravity theories possess a holographic nature, wherein there exists a correspondence between a Quantum Gravity theory and a Quantum Field Theory (QFT) with no gravity. A concrete realization of this holographic correspondence was first conjectured in [41], taking the form of the so-called AdS/CFT correspondence: a theory of gravity defined on an Anti-de-Sitter (AdS) spacetime is equivalent to a Conformal Field Theory (CFT) existing on the boundary of the gravitational theory. Anti-de-Sitter spacetimes are spaces with negative constant curvature, while a CFT is a scale-invariant Quantum Field Theory. These conjectures have been extensively studied in the literature to date, leading to significant advancements in the development of fundamental physics.

The main focus of this thesis are Kaluza-Klein compactifications and the possible low-dimensional theories that can be obtained, not from the full String Theories, but from reductions of their low-energy limit, Supergravities. In the remainder of the introduction, we will review some of the main concepts that we will need throughout the text, such as the basics of String Theories, Supergravities, Kaluza-Klein reduction, and also the main tool of this work, Exceptional Field Theory.

B String theories and Supergravities

B.1 String Theories

THE aim of this section is to provide an overview of String Theories (ST) and to derive their massless sector. This gives a formulation of the so-called Supergravities, viewed as low energy limits of String Theories. String Theories are currently our most promising candidates for a consistent theory of quantum gravity. They postulate the existence of extended elementary constituents for matter, strings, which replace the point particles of Quantum Field Theory (QFT). The free action of a point particle in flat spacetime is simply described by the line element.

$$S = -mc \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}, \tag{2.2}$$

where c denotes the speed of light, X^μ the position of the particle in the targe-space, i.e. in spacetime, and d represents the number of dimensions. Henceforth, we set $c = 1$. $\eta_{\mu\nu}$ is the Minkowski metric with mostly plus sign. Therefore, in the absence of any interaction, a point particle that minimizes this action will trace a straight line while propagating, as this path is the shortest distance between two points. This principle can be generalized to extended objects, leading to what is called the Nambu-Goto action.

$$S_{\mathrm{NG}} = -T_p \int_{\Sigma_{p+1}} \mathrm{d}\sigma^{p+1} \sqrt{-\det(\gamma_{\alpha\beta})} \tag{2.3}$$

where $\gamma_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$, with $\alpha, \beta = 0, 1, \ldots, p$ and $\mu, \nu = 0, \ldots, d-1$, represents the induced metric of what is called the world-volume, into the target space. What is called the world-volume here, is the volume that is swept by the extended object in the target space, i.e. spacetime. See Figure 2.5 for an illustration in the case of a string. For the case $p = 0$, the Nambu-Gotto action

context of String Theory or Supergravity.

reduces to the action of a point-like particle. For $p = 1$, this action describes the world-sheet swept by the extended object in an d dimensional space: a line for a point-like particle, a surface for a string, and so forth. In the case of a string, T_p in (2.3) serves as the analogue of mass for the point-like particle and is called the string tension. It is related to the string length ℓ_s by $T_p = \frac{1}{2\pi\ell_s^2}$ and to the quantity known as the Regge slope α' by $T_p = \frac{1}{2\pi\alpha'}$. The latter parameter plays the role of the string energy scale. It encodes the energy level at which "stringy" effects begin to manifest. In the Nambu-Gotto action, X^μ are the embedding coordinates of the extended object in the target space which is d dimensional. The Nambu-Goto action is not entirely satisfactory due to its inclusion of

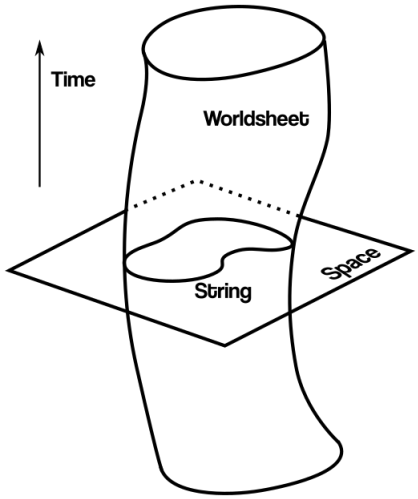


Figure 2.5: World sheet swipped by a string.

a square root, which complicates quantization. It can be replaced by another action, known as the Polyakov action

$$S_{\text{Pol}} = -\frac{T_p}{2}\int \mathrm{d}\sigma^{p+1}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu(\sigma^\alpha)\partial_\beta X^\nu(\sigma^\alpha)\eta_{\mu\nu} + \Lambda_p\int \mathrm{d}\sigma^{p+1}\sqrt{-h}\,, \tag{2.4}$$

where the auxiliary field $h_{\alpha\beta}$ has been introduced and is symmetric. It has the interpretation of a metric on the brane world-sheet, and on-shell, it is equal to the induced metric. The factor Λ_p is choosen such that the equations of motion from the the Nambu-Goto and the Polyakov actions are equivalent on-shell. The Polyakov action is therefore more commonly used for quantizing the theory.

We now specialize to the case of $p = 1$, i.e., for a string. Objects with p larger than one, known as p-branes, enter String Theories as non-perturbative objects. We will denote the two remaining coordinates of the string world-sheet as σ and τ. The Polyakov action describes a bosonic string, and we will restrict our analysis to the so-called bosonic String Theory as most of the important features arise from it. We will briefly mention fermionic string theories, which we will refer to as Superstring Theories.

The symmetries of the bosonic string action are:

- "Global" Poincaré invariance:

$$\begin{cases} X'^{\mu}(\sigma) = \Lambda^{\mu}{}_{\nu} X^{\nu} + c^{\mu}\,, \quad \Lambda \in \mathrm{O}(1, d-1), \quad c^{\mu} \in \mathbb{R}^{d} \\ \sigma'^{\alpha} = \sigma^{\alpha}\,. \end{cases} \tag{2.5}$$

- Local diffeomorphism on the string world-sheet:

$$\begin{cases} \sigma^{\alpha} \to \sigma'^{\alpha}\,, \\ h_{\alpha\beta} = h'_{\gamma\delta} \dfrac{\partial \sigma'^{\gamma}}{\partial \sigma'^{\alpha}} \dfrac{\partial \sigma^{\delta}}{\partial \sigma^{\beta}}\,, \\ X'^{\mu}(\sigma') = X^{\mu}(\sigma)\,. \end{cases} \tag{2.6}$$

- Weyl invariance:

$$\begin{cases} h'_{\alpha\beta} = e^{\phi(\sigma)} h_{\alpha\beta}\,, \\ \sigma'^{\alpha} = \sigma^{\alpha}\,. \\ X'^{\mu}(\sigma) = X^{\mu}(\sigma)\,. \end{cases} \tag{2.7}$$

By making use of those symmetries, and noting that h has three independent components, we can fix the gauge to be, at least locally $h_{\alpha\beta} = \eta_{\alpha\beta}$. The equations of motion for the embedding coordinates X^{μ} in this gauge are

$$\Box X^{\mu} \equiv \partial_{+}\partial_{-} X^{\mu} = 0\,, \tag{2.8}$$

where we introduced light-cone coordinates $\sigma_{+} = \sigma + \tau$ and $\sigma_{-} = \sigma - \tau$, in which the induced metric $h = \eta$ takes the form

$$\eta = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\,. \tag{2.9}$$

Those equations of motion are supplemented with two Virasoro constraints

$$\begin{cases} T_{++} \equiv \partial_{+} X^{\mu} \partial_{+} X_{\mu} = 0\,, \\ T_{--} \equiv \partial_{-} X^{\mu} \partial_{-} X_{\mu} = 0\,, \end{cases} \tag{2.10}$$

with T the energy-momentum tensor, which can be obtained by varying the action with respect to h. Those equations of motions are easily solved by what are called left and right movers

$$X^{\mu}(\sigma) = X^{\mu}_{L}(\sigma_{+}) + X^{\mu}_{R}(\sigma_{-})\,. \tag{2.11}$$

If we now ask to have a closed string (but this can be straightforwardly generalized to an opened

string), i.e. a string with periodical boundary conditions $X^\mu(\sigma + 2\pi) = X^\mu(\sigma)$ we can write

$$\begin{cases} X_R^\mu(\sigma_-) = \frac{1}{2}x_R^\mu + \frac{1}{2}\ell_s^2 p_R^\mu \sigma_- + \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2i\pi\sigma^-}\,, \\ X_L^\mu(\sigma_+) = \frac{1}{2}x_L^\mu + \frac{1}{2}\ell_s^2 p_L^\mu \sigma_+ + \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2i\pi\sigma^+}\,. \end{cases} \tag{2.12}$$

In these formulas, α_n^μ and $\tilde{\alpha}_n^\mu$ are the Fourier modes of the left and right movers, and will act as ladder operators in the quantum theory, similar to those used in the quantum harmonic oscillator. In (2.12), $x_{L/R}^\mu$ designated the position of the string center of mass in the d-dimensional space, $p_{L/R}^\mu$ the momentum of the string, and the rests describes the oscillations that propagate along the string, similarly to the fundamental and harmonic modes on a string musical instrument.

Canonical quatization can be used here, and a physical string state $|\varphi\rangle$ will be described in the quantum theory by a set of creation operators for right and left movers, a zero mode, as well as a momentum k characterizing the motion and velocity in the world volume

$$|\varphi\rangle = (a_{n_1}^\dagger)^{\mu_1} \dots (a_{n_N}^\dagger)^{\mu_N} (\tilde{a}_{m_1}^\dagger)^{\nu_1} \dots (\tilde{a}_{m_{\tilde{N}}}^\dagger)^{\nu_{\tilde{N}}} |0,k\rangle \tag{2.13}$$

such that

$$P^2|\varphi\rangle = k^2|\varphi\rangle \tag{2.14}$$

with P the momentum operator, N the number of right movers, $\tilde{N}$ the number of left movers, and $((a_n^\dagger)^\mu, (a_n)^\mu) = (\frac{1}{\sqrt{n}}\alpha_n^\mu, \frac{1}{\sqrt{n}}\alpha_{-n}^\mu)$ for $n \in \mathbb{N}^*$ (similarly for $\bar{a}_n^\mu$). One could follow a similar reasoning for open strings, where fixing the boundary conditions becomes necessary, such as Neumann (fixed derivative at the endpoints) or Dirichlet (fixed string endpoints). However, we will not delve into the details of these calculations here. We will now highlight the main features that arise from the quantization procedure.

The first and perhaps most important constraint is that, to preserve Weyl invariance at the quantum level, the quantized bosonic string must exist in $d = 26$ dimensions. If one extends the string to be fermionic, additional constraints come into play, and the quantized string must live in $d = 10$. This feature prompted physicists to seek ways to eliminate the extra dimensions. One of the main ideas is to compactify six out of ten dimensions: factorize the total manifold into a $4d$ external space that describes the Universe and a compact $6d$ manifold with a characteristic length much smaller than that of the external space. This approach effectively eliminates six dimensions. $6d$ compact spaces that fulfill the appropriate constraints are called Calabi-Yau manifolds [42], see Figure 2.6.

Another constraint that comes into play is what is called the level-matching formula, imposed at the quantum level by the Virasoro constraints (2.10), which states that the number N of right-moving modes and left-moving modes $\tilde{N}$ must be equal. This condition essentially ensures the existence of stationary waves propagating along the string. The on-shell mass formula is then given for a physical state $|\varphi\rangle$ by

$$M^2|\varphi\rangle = \frac{4}{\alpha'}(N-1)|\varphi\rangle\,. \tag{2.15}$$

We can have a look a the first few levels of the spectrum and look for massless modes. We intro-

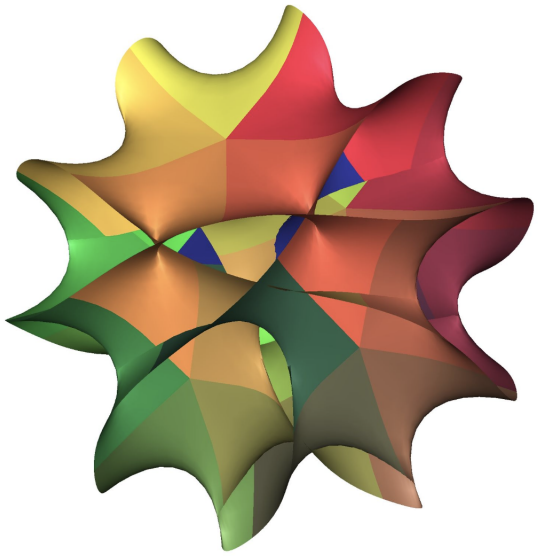


Figure 2.6: Example of a 2d slice of a Calabi-Yau manifold (source: Wikipedia).

duce light-cone coordinates in the target space

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1}), \qquad X^i = X^i, \quad i \in [\![1, \dots, d-2]\!]. \tag{2.16}$$

By utilizing the Virasoro constraint (2.10), one can show that the physical oscillations of the string are the transverse oscillations, i.e. the oscillations that are excited by the $(\alpha_n^i, \tilde{\alpha}_n^i)$. The spectrum takes the following form for the first few levels:

- $N = 0$: this leads to a negative mass. This is a tachyonic mode, which seems to imply that String Theories are not good physical theories. However, when delving into calculations of Superstring Theories, tachyonic modes disappear from the spectum.
- $N = 1$: this will constitute part of the "massless" sector of String Theories and we have

  $$\chi_{ij}(a^\dagger)^i(\tilde{a}^\dagger)^j|0,k\rangle. \tag{2.17}$$

  Fields described by χ_{ij} decompose into irreducible representations of $\mathrm{SO}(d-2)$. Therefore the χ_{ij} tensor can be decomposed into the irreducible representations

  - ➜ $\chi_{ii} \to \phi$ a singlet, called the dilaton scalar.
  - ➜ $\chi_{[ij]} \to B_{ij}$ which is a two-form, called Kalb-Ramond form.
  - ➜ $\chi_{((ij))} \to g_{ij}$ which is a symmetric traceless tensor. This will play the role of the graviton field.

These fields, even though derived in the context of the bosonic String Theories, persist in the case of Superstring Theories. Our aim with this analysis is to demonstrate that, as these are the massless modes of String Theories, these fields are part of the field content of all low-energy limits of String Theories, which are Supergravities.

Before including fermionic strings, let us do a few general remarks on String Theories. Strings can interact with one another. Those processes are similar to what happens in a quantum field theory, and the strength of the coupling in the case of String Theories is governed by a coupling constant g_s. If the strings are weakly coupled, i.e. when g_s is small, then we can use similar techniques as the one used in QFT to compute scattering amplitudes. The main difference here, is that the "lines" within Feynman diagram will be replaced by tubes as in Figure 2.7. Within a String

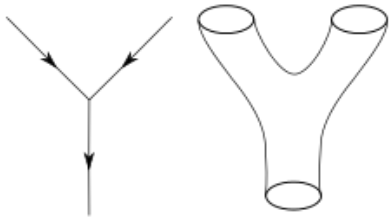


Figure 2.7: Lines in Feynman diagrams (left) are replaced by tubes on the String Theories scattering amplitude diagrams.

Theory diagram, holes may be present in the surface swipped by the string as shown in Figure 2.8 . For example a sphere does not have any hole, but the donut, the torus, has one hole. This is what controls the pertubative expansion. Therefore, the loop expansion is replaced by a genus (the number of hole in the surface) expansion, introducing g_s factors for each hole. Note that knowing the number of holes is enough to characterize the surface, because of the $2d$ conformal invariance of the string world-sheet. So the equivalent of the tree level is the diagram of the sphere in String

Figure 2.8: Examples of String Theory diagrams, each has a different genus [43].

Theories, the 1-loop level is the diagram of the torus and so on. Note also that in String Theories, one diagram usually encodes a number of Feynman diagrams, because there is only on surface with a hole, but several loop diagrams, see Figure 2.9.

To proceed in our exploration of the massless bosonic modes of String Theories, we would need to delve into Superstring Theory. However, we will not perform any calculations and proceed directly to the results. In Superstring Theories, there exists two different sectors that can combine to constitute fields: the Ramond (R) and the Neveu-Schwarz (NS) sector. The field content of the NS-NS sector aligns precisely with what we just described, featuring a dilaton, a Kalb-Ramond two-form, and a graviton. It is worth noting that because the graviton is present in every String Theory, each of them serves as a candidate for a theory of quantum gravity. Furthermore, the field equations for this graviton at low energy correspond exactly to the Einstein field equations. Hence,

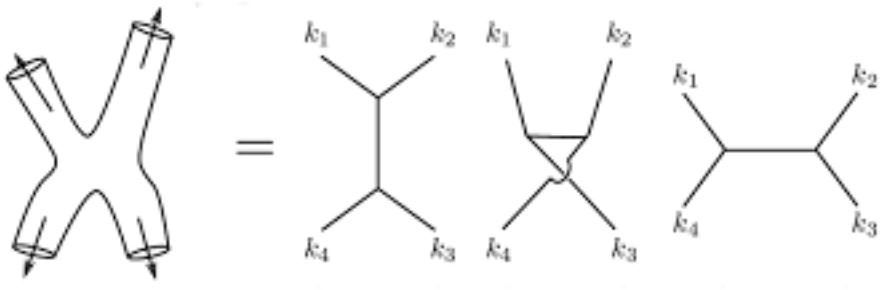


Figure 2.9: One String Theories diagram encodes the three channels (s,t,u) of the Feynman diagrams, at tree level [43]

String Theories encapsulate General Relativity and therefore represent promising candidates to extend it, as claimed before.

The R-NS sector provides us with fermionic fields, which we will not discuss in details. Finally, the last sector is the R-R sector, which is a sector of bispinors : one from the left sector and the other from the right sector. In this case, we have several options, primarily whether we desire a chiral theory or not. By introducing a chirality projector $\Gamma_{11} = \Gamma_0 \dots \Gamma_9$, where Γ's are the 10-dimensional gamma matrices. The chirality projection for a bispinor H, constituted from a Ramond left spinor and a Ramond right spinor takes the form

$$\Gamma_{11} H = \pm H \,. \tag{2.18}$$

The chiral theory is denoted as IIB String Theory for which one would take the minus sign in (2.18), while the non-chiral version is known as IIA String Theory for which one would take the plus sign in (2.18). The "II" designation signifies that for these theories, there are two supersymmetries. We will delve deeper into supersymmetries in the next section. There are other String Theories that exist with only one supersymmetry. The two so-called Heterotic String Theories, and Type I String Theory. The latter corresponds in fact to truncation to either IIA or IIB theory.

Depending on whether we are within IIA or IIB Superstring Theory, the R-R field content changes. For IIA Superstring Theory, there are additional potential one-form and potential three-form fields. For IIB Superstring Theory, there are additional 0-form potential, 2-form potential, and 4 potential fields which field strength is self-dual. Note that all those p-forms come with their dual partners. The massless bosonic sector of type II Superstring Theory is summarized in Table 2.1.

Before closing this section, note that all String Theories are nowadays thought to be connected via dimensional reduction or duality to a theory that unifies all String Theories: M-Theory. The latter is a theory living in $11d$, for which we do not yet have an explicit formulation, but its low-energy limit is $11d$ Supergravity. The connections between String Theories and M-theory are summarized in Figure 2.10.

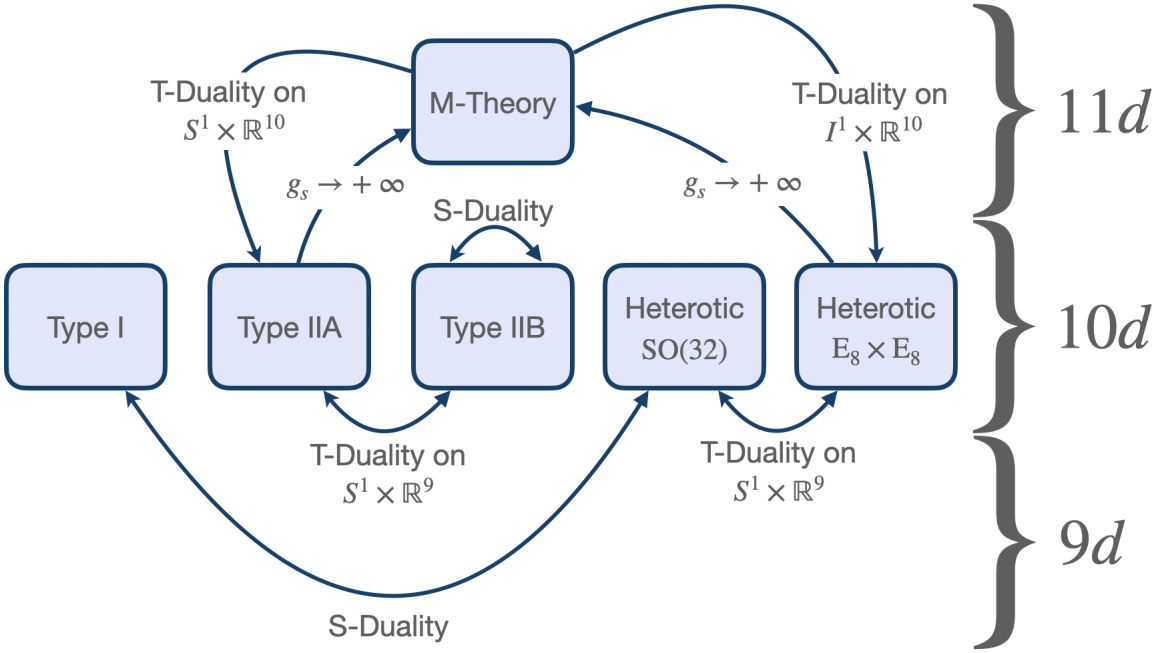


Figure 2.10: Duality transformations relating string theories all toghether and M-Theory [44]. T-duality transforms the radius $R \to \ell_s^2/R$ with R the radius of the compactified circle the length of the compactified interval I^1. S-duality, on the other hand, inverts the dimensionless string coupling constant $g_s \to 1/g_s$ (analog to electric-magnetic duality, or strong-weak coupling duality in four-dimensional gauge theories).

Bosonic sectors		
Sector	Type II A	Type II B
NS-NS	ϕ , $B_{(2)}$, g_{ij}	ϕ , $B_{(2)}$, g_{ij}
R-R	$C_{(1)}$, $C_{(3)}$	$C_{(0)}$, $C_{(2)}$, $C_{(4)}$

Table 2.1: Sum up of the bosonic field content of Type II A and B Supersting Theories, B_{ij} is the Kalb-Ramond 2-form, g_{ij} the graviton, et $C_{(n)}$ are n-forms.

B.2 Supersymmetry

We continue this introduction by providing a brief overview of a very important concept in theoretical physics: Supersymmetry. When the Standard Model of particle physics was established in its final form, theoretical physicists began exploring ways to extend it. In particular, Coleman and Mandula [45] investigated the extent to which the Poincaré group could be enlarged before the scattering amplitudes became trivial. Their result takes the form of a theorem that is summarized in the Table below.

Coleman and Mandula therorem (1967)

Let $\mathcal{G}$ be a connected symmetry group of the S-matrix, that encodes the scattering amplitudes, such that

- $\mathcal{G}$ contains a subgroup that is locally isomorphic to P(3, 1), the Poincaré group.
- For given $m > 0$ there exists only a finite number of one-particle states with mass less than m.
- Elastic scattering amplitudes are analytic functions of the Mandelstam variables in some neighbourhood of the physical region.
- The S-matrix is non-trivial, i.e. generically two one-particle momentum eigenstates scatter (except at isolated values of the Mandelstam variables).
- The generators of $\mathcal{G}$ in momentum space have distributions for their kernels.

Then $\mathcal{G}$ is locally isomorphic to the direct product of a compact Lie group Int and the Poincaré group

$$\mathcal{G} \simeq \mathrm{P}(3,1) \otimes \mathrm{Int} \tag{2.19}$$

Similarly, the Lie algebra g of $\mathcal{G}$ is the direct sum of the compact Lie algera int of Int and the Poincaré algebra

$$\mathfrak{g} \simeq \mathfrak{p}(3,1) \oplus \mathfrak{int} \tag{2.20}$$

Stated in this manner, the key-content of the theorem is that from a Poincaré algebra point of view, all 'internal' generators are forced to be scalars. There are however a number of loopholes, which we note in passing that String Theories realize all of them.

Loopholes in Coleman-Mandula theorem

1. We could work in $d = 1 + 1$ where the theorem does not hold.
2. Non local theories with extended objects do not enter the scope of the theorem.
3. Conformal Symmetry can be added in the case of a massless theory.
4. The generators are assumed to obey a Lie bracket, i.e. form a bosonic algebra. If one considers fermionic generators, then the Coleman-Mandula theorem does not hold.

This latter point is our focus in this section. It supposes the existence of generators of a symmetry, which are not commuting but anti-commuting. This is characteristic of fermionic symmetry, that exchanges bosonic and femionic degrees of freedom. If we call them Q, then we schematically

have in the theory

$$|F\rangle = Q|B\rangle \tag{2.21}$$

where $|F\rangle$ is a fermionic degree of freedom and $|B\rangle$ is a bosonic degree of freedom. Therefore, in a supersymmetric theory, couplings among fermions and bosons are not independent anymore, and the generators Q increase or decrease the helicity by $1/2$. As a consequence, the matter will be organized into so-called supermultiplets of fields with different helicities.

Depending on the number of supersymmetric generators and the dimension of the theory, the content of supermultiplets will vary. For a comprehensive review of supersymmetry and the structure of multiplets in various dimensions, see [46, 47].

Let us give a very simple example of a supersymmetric theory. Consider the Lagrangian

$$\mathscr{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + i\bar{\psi}\dot{\psi} - \frac{1}{2}(\bar{\psi}\psi - \psi\bar{\psi})\,, \tag{2.22}$$

which describes the Supersymmetric Harmonic Oscillator, where x describes the usual bosonic harmonic oscillator and ψ is a fermionic oscillators, i.e. with anti-commutation relations. Let ϵ and $\bar{\epsilon}$ be two fermionic variation parameters. Then this Lagrangian is invariant up to a total derivative according to

$$\begin{array}{ll} \delta_\epsilon x = \epsilon\bar{\psi}\,, & \delta_{\bar{\epsilon}} x = \bar{\epsilon}\psi\,, \\ \delta_\epsilon \psi = i\epsilon\dot{x} + \epsilon x\,, & \delta_{\bar{\epsilon}}\psi = 0\,, \\ \delta_\epsilon \bar{\psi} = 0\,, & \delta_{\bar{\epsilon}}\bar{\psi} = -i\bar{\epsilon}\dot{x} + \bar{\epsilon}x\,. \end{array} \tag{2.23}$$

Using Noether's theorem, we can now compute the conserved charges which will be the supersymmetric generators, and we obtain

$$\begin{cases} Q_\epsilon = (\dot{x} - ix)\bar{\psi}\,, \\ Q_{\bar{\epsilon}} = -(\dot{x} - ix)\psi\,. \end{cases} \tag{2.24}$$

Using this supersymmetry, we can now describe the full spectrum in term only of the bosonic degrees of freedom. Let us define the momentum associated to x

$$p = \frac{\partial\mathscr{L}}{\partial\dot{x}} = \dot{x}\,. \tag{2.25}$$

We can now introduce ladder operators for bosons and fermions

$$a = \frac{x + ip}{\sqrt{2}}\,, \quad a^\dagger = \frac{x - ip}{\sqrt{2}}\,, \quad f = \psi\,, \quad f^\dagger = \bar{\psi} \tag{2.26}$$

which verify the (anti)-commutation relations

$$[a, a^\dagger] = \mathbb{1}\,, \quad \{f, f^\dagger\} = \mathbb{1}\,, \quad \{f, f\} = \{f^\dagger, f^\dagger\} = 0 \tag{2.27}$$

Using those variables we can show that the Hamiltonian of the system can be written as

$$\mathscr{H} = a^\dagger a + f^\dagger f\,. \tag{2.28}$$

We can also re-express the supercharges as

$$Q_\epsilon \propto Q = a^\dagger f\,, \quad Q_{\bar\epsilon} \propto Q^\dagger = a f^\dagger\,. \tag{2.29}$$

The algebra satisfied by the Qs is

$$\{Q, Q^\dagger\} = \mathscr{H}\,, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0\,. \tag{2.30}$$

The Hilbert space is now described by the product of two Fock spaces $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_F$ whose states are $|n_B, n_F\rangle = \frac{1}{\sqrt{n_B!}}(a^\dagger)^{n_B}(f^\dagger)^{n_F}|0,0\rangle$, with here $n_F = 0$ or 1 because of the anticommutation relations of $f^\dagger$ (2.27). In this description $|n_B, 0\rangle$ is a boson and $|n_B, 1\rangle$ is a fermion. By acting with Q and $Q^\dagger$ on an arbitrary state $|n_B, n_F\rangle$ we can relate bosonic and fermionic states

$$Q|n_B, n_F\rangle = a^\dagger f|n_B, n_F\rangle = \sqrt{n_B + 1}\delta_{n_F,1}|n_B + 1, n_F - 1\rangle\,, \tag{2.31}$$

$$Q^\dagger|n_B, n_F\rangle = a f^\dagger|n_B, n_F\rangle = \sqrt{n_B}\delta_{n_F,0}|n_B - 1, n_F + 1\rangle\,. \tag{2.32}$$

Hence we have the relation between fermionic and bosonic states

$$|n_B, 0\rangle \underset{Q^\dagger}{\overset{Q}{\longleftrightarrow}} |n_B - 1, 1\rangle\,. \tag{2.33}$$

We can summarise the most important points of this model in a few comments. Firstly, the system has a non-degenerate (bosonic) ground state $|0,0\rangle$. Secondly, all the excited states come in pairs, one of which is a boson and the other is a fermion. We can see this from (2.33) by applying the Hamiltonian operator on both state

$$\mathscr{H}|n_B, n_F\rangle = (n_B + n_F)|n_B, n_F\rangle \tag{2.34}$$

$$\mathscr{H}|n_B - 1, n_F + 1\rangle = (n_B + n_F)|n_B - 1, n_F + 1\rangle \tag{2.35}$$

which therefore have the same energy. Finally, the supercharges Q and $Q^\dagger$ map into each other bosonic and fermionic degrees of freedom which have the same energy. They form a supermultiplet

$$\Big\{|n_B, n_F\rangle, |n_B - 1, n_F + 1\rangle\Big\}_{(n_B + n_F)}. \tag{2.36}$$

chacterized by a single quantum number : the energy. From the Lagrangian perspective, those supercharges stem supersymmetry (2.21), which also mixes fermionic on bosonic degrees of freedom and vice versa.

In the case of a QFT, the story is similar. Supercharges will come in pairs Q^I_α and $\bar{Q}^I_\alpha$, $I = 1, \ldots, \mathcal{N}$, with $\mathcal{N}$ the number of supersymmetries, and α being indices of the spinor representation of the Poincaré group. One supercharge increasing helicity by $1/2$ and the other lowering it by $1/2$. $\mathcal{N}$ is the number of supersymmetry. The Poincaré algebra is going to be supplemented by a graduated algebra similar to (2.30), i.e. with commuting and anticommuting relations, and the fields are be organized into supermultiplets.

The number of supersymmetries in a theory cannot be an arbitrary large number. If we restrict ourselves to theory with helicity lesser or equal to 2 in $4d$, then the maximum number of supersymmetries is $\mathcal{N} = 8$ (it takes 8 iterations to go from 2 to -2 by steps of $1/2$). Then we

have $8 \times 2 \times 2 = 32$ maximum number of components of supercharges. The first 2 is coming from $\alpha = 1, 2$ and the second because there are Q and $\bar{Q}$. If one starts from a theory in d dimensions and count the number of supercharges, one finds that $11d$ is the maximal dimension with $\mathcal{N} = 1$ in order to have at most 32 supercharges, and with the minimal number of supersymmetry $\mathcal{N} = 1$. After dimensional reduction to $4d$ this would give $\mathcal{N} = 8$. This explains why Supergravities have been constructed up to $11d$.

Note that in $4d$ with $\mathcal{N} = 8$ supersymmetries, we are forced to introduce a spin $3/2$ field. We can wonder what is the appropriate theory to describe this new field. The story is similar to what happens with spin 1 fields, which can be consistently described by gauge theories. The latter amounts to promote a rigid symmetry to a local symmetry, and the spin 1 field plays the role of the connection, and make sure that the different quantities in the theory transform convariantly under the local symmetry. For example, the derivative is promoted to a covariant derivative

$$\partial_\mu \longrightarrow \mathscr{D}_\mu = \partial_\mu - g A_\mu \tag{2.37}$$

with A_μ the spin 1 field and g will play the role of the coupling constant in the gauge theory. A simlilar process happens for the spin $3/2$ field. The appropriate theories to describe it are Supergravities, in which global supersymmetries are promoted to local supersymmetries, and the spin $3/2$ fields, also called gravitini fields will play the role of the connection for this local symmetry. The study of Supergravities in the subject of the next section.

B.3 Supergravities

As explained in the previous paragraph, Supergravities are the appropriate theories to study spin $3/2$ fields, by promoting the rigid supersymmtry into a local one. It turns out that when doing so, we are forced to introduce a spin 2 field, which corresponds to introducing gravity in the theory. We will not delve into the construction of Supergravities, for which the reader can refer to [48, 49]. We will rather highlight the main features of Supergravities that we will need through the thesis.

In the section on String Theories 2.B.1 , we claimed that the low-energy limits of String Theories give rise to Supergravities. It can indeed be shown that in $d = 10$, one can construct two inequivalent Supergravities with $\mathcal{N} = 2$, which correspond to the massless sectors of IIA and IIB String Theories. Moreover, it can be demonstrated that there is a unique way to construct a Supergravity in $d = 11$ with $\mathcal{N} = 1$ [50].

Supergravities can be constructed in dimensions with $2 \geq d \geq 11$, but all ungauged maximal Supergravities, can be obtained from $11d$ and $10d$ IIB Supergravities, except for some 6-dimensional Supergravities. Note that $10d$ IIA Supergravity can be obtained from $11d$ Supergravity by compactifying on S^1 [51–53]. However IIB Supergravity is peculiar in that sense : it cannot be obtained from dimensional reduction of $11d$ Supergravity.

We will use the remainder of this section to recap the field content and dynamics of $11d$ and type IIB Supergravities. We will not provide a detailed construction of Supergravities, for which we refer to [48, 49]. We give further details on basics of Supergravity compactification and gauging procedure in section 2.C.2 .

a $11d$ $\mathcal{N}=1$ Supergravity

$11d$ Supergravity is an $\mathcal{N}=1$ theory, in which all fields are regrouped in a single supermultiplet [54, 55]. This theory is thought to be the low energy limit of M-theory [56]. The $11d$ Supergravity theory consists of

$$\left\{G_{MN}, \Psi^M, C_{MNP}\right\}, \tag{2.38}$$

with G_{MN} the $11d$ metric, Ψ^M the gravitino field and C_{MNP} a three-form potential. Its Lagrangian is

$$\begin{aligned}\mathscr{L}_{11\mathrm{d}} = ER &- \frac{1}{2}E\bar{\psi}_K\Gamma^{KMN}D(\omega)_M\Psi_N - \frac{E}{48}F^{M_1M_2M_3M_4}F_{M_1M_2M_3M_4} \\ &+ \frac{1}{144^2}\epsilon^{M_1\dots M_{11}}C_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} \\ &+ \frac{1}{192}E\left(\bar{\Psi}_P\Gamma^{KLMNPQ}\Psi_Q + 12\bar{\Psi}^K\Gamma^{LM}\Psi^N\right)F_{KLMN},\end{aligned} \tag{2.39}$$

with $M, N, \dots = 1, \dots, 11$, E the determinant of the vielbein, R the $11d$ Ricci scalar, $F_{KLMN} = 4\partial_{[K}C_{LMN]}$, Γ with multiple indices products of antisymmetric Γ matrices of $\mathfrak{so}(1,10)$, $D(\omega)$ the covariant derivative involving the spin connection ω, and ϵ the Levi-Civita symbol. The symmetries of this Lagrangian can be found in [55].

This theory can be compactified to maximal Supergravities in lower dimensions via a so-called Freund-Rubin compactification procedure (2.C.4) on $\mathrm{AdS}_7 \times \mathrm{S}^4$ ans $\mathrm{AdS}_4 \times \mathrm{S}^7$. This process gives an ansatz to find a solution of 11 dimensional equations of motion descending from (2.39), which describes a background around which one can study the compactification of $11d$ to $4d$ or $7d$. For the $4d$ case, this process leads to a theory whose massless gravity supermultiplet aligns with maximal $\mathcal{N}=8$ $4d$ Supergravity [57], on top of which Kaluza-Klein towers are added up. The whole spectrum is organised into representations of the $\mathrm{Osp}(4|8)$ group. We will further develop this compactification scenario in section 3.B.2 where we give the full spectrum. This extension of $d=4$ Einstein gravity stands out notably for its exceptional degree of symmetry and the finiteness exhibited by its higher loop amplitudes [58–62].

b $10d$ $\mathcal{N}=2$ IIB Supergravity

IIB Supergravity is a $10d$ theory with $\mathcal{N}=2$, first constructed in [63–65] and whose field content is

$$\left\{G_{MN}, \phi^\alpha, C_{MN}{}^\alpha, C_{MNPQ}\right\}, \tag{2.40}$$

with G_{MN} the $10d$ metric, ϕ^α a doublet of $\mathrm{SL}(2,\mathbb{R})$ of scalar fields with $\alpha = 1,2$, $C_{MN}{}^\alpha$ a doublet of 2-forms and C_{MNPQ} a self-dual 4-form potential. The $\mathrm{SL}(2,\mathbb{R})$ symmetry under which scalars and 2-forms are doublets is the same. There are several formulations of type IIB Supergravity. In this formulation, the self-duality equation of the 5-form field strength must be imposed separetely

$$F_{M_1\dots M_5} = \frac{1}{5!}\epsilon_{M_1M_2M_3M_4M_5N_1N_2N_3N_4N_5}F^{N_1N_2N_3N_4N_5} \tag{2.41}$$

$$\begin{aligned}\mathscr{L}_{\text{IIB}} = & ER + \frac{1}{4}E\partial_M m_{\alpha\beta}\partial^M m^{\alpha\beta} - \frac{1}{12}F_{M_1M_2M_3}{}^{\alpha}F^{M_1M_2M_3\,\beta}m_{\alpha\beta} \\ & - \frac{1}{30}EF_{M_1M_2M_3M_4M_5}F^{M_1M_2M_3M_4M_5} \\ & - \frac{1}{864}\epsilon_{\alpha\beta}\epsilon^{M_1\dots M_{10}}C_{M_1M_2M_3M_4}F_{M_5M_6M_7}{}^{\alpha}F_{M_8M_9M_{10}}{}^{\beta} \\ & + \text{fermions}\,,\end{aligned} \tag{2.42}$$

with E the square root of the determinant of the metric; R the $10d$ Ricci scalar, $m_{\alpha\beta}$ is a 2×2 matrix in $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ that is parametrized by the two scalars, and the F are the various field strengths of the gauge potentials. Again we refer to [55] for further details on the symmetries.

This IIB Supergravity can be reduced to maximal Supergravity in $5d$ via a compactification on $\mathrm{AdS}_5\times\mathrm{S}^5$. The massless graviton supermultiplet aligns with maximal $5d$ Supergravity with gauge group $\mathrm{SO}(6)$. On top of this supermultiplet, there is an infinite number of massive $1/2$ BPS multiplets. The spectrum of the full compactified theory is given in Table 2.5 .

C Kaluza-Klein reductions

SOON after Einstein formulated his theory of gravity, it was proposed by the mathematician Theodor Kaluza [39] and later the physicist Oscar Klein [38] to add extra dimensions in an attempt to unify gravity with the electromagnetism force. After all, unifying theories is a fundamental principle for physicists : from Maxwell that unified electric and magnetic effects [66], to Einstein that unified space and time [1], unification has always been a guiding principle in theoretical physics. Latter with the emergence of String Theories in $10d$ and Supergravities in up to $11d$ as candidate for a theory of Quantum Gravity, it became a necessity to find a way to reduce this number of dimensions. There has been since a novel interest in ideas of Kaluza and Klein.

The basic idea is quite straightforward : take the total manifold $\mathcal{M}$ on which the higher-dimensional theory is defined and factorize it into an external and internal compact part. Then make the radius or characteristic length of the internal part much smaller than the one of the external part. The extra dimensions will effectively disappear from the theory, but with still effects on the lower dimensional theory. Let us illustrate this with the toy model of a scalar field ϕ living in $d+1$ dimensions. We will call $\{x^i\}$ the d first dimensions, time included, and y the last coordinate. This scalar field obey the free Klein-Gordon equation, and if it is massless we have

$$\Box\phi(\vec{x},y) = 0\,. \tag{2.43}$$

We now want to compactify one dimension, and therefore we are not left with many choices for the internal manifold and will compactify on S^1. Therefore the y coordinate becomes periodic, and we can Fourier expand the y dependance of the field ϕ

$$\phi(\vec{x},y) = \sum_{n\in\mathbb{N}}\phi_n(\vec{x})\exp(2i\pi ny/L)\,, \tag{2.44}$$

with L the radius of the circle. The d'Alembertian operator will split into two parts $\Box\to\Box_x+\partial_y^2$

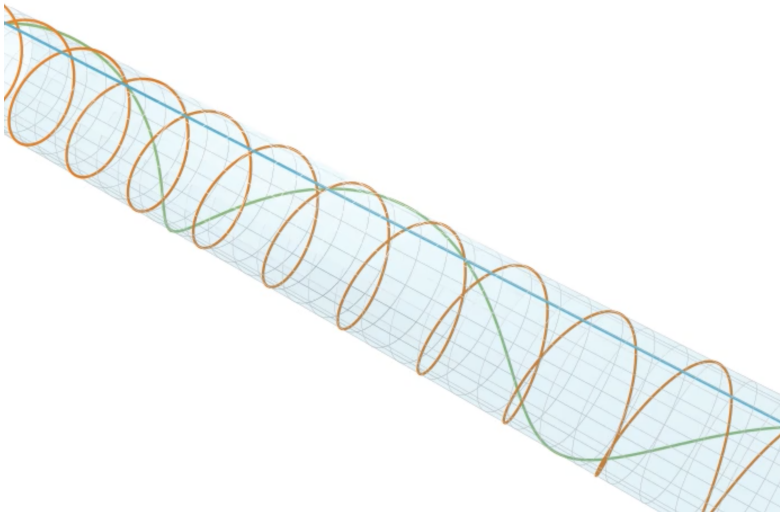


Figure 2.11: Schematics representation of the Kaluza-Klein reduction of a scalar field on S^1 with different modes which trajectories wrap the cylinder. The blue curve would represent the path of the 0 mode ; the orange and green curve would represent the path of two non zero modes of the Kaluza-Klein tower with increasing wrapping number. The higher in the tower the mode is, the mode it wraps around the cylinder.

and the Klein-Gordon equation (2.43) reduces to

$$\sum_{n\in\mathbb{N}} \left(\Box_x \phi_n(\vec{x}) - \frac{(2\pi n)^2}{L^2} \phi_n(\vec{x}) \right) e^{2i\pi n y/L} = 0\,. \tag{2.45}$$

There are already a number of things we can say from this very simple model, that will be resurgent of all compactifications. First of all, we see the emergence of a infinite amount of modes from what used to be a single field. All those modes constitute what we will call a Kaluza-Klein tower and is a distinguished feature that arises in all Kaluza-Klein reductions. In this example, they describe to what extent the trajectories of free massless fields wrap around the circle while propagating in the $d+1$ spacetime, as illustrated in Figure 2.11 . If a mode with a large n is excited, this means from the $d+1$-dimensional theory that the trajectory of scalar fields winds a lot around the circle. What we also see is that those modes, from the d-dimensional point of view, acquire a non zero mass, and this mass becomes bigger as the circle gets smaller. Therefore, if the circle shrinks sufficiently enough compared to a characteristic size of the external space, than we have an effective theory in d dimensions that is just again a free scalar field, corresponding to the zero mode in the expansion (2.44). All the masses of the other modes are too big and the corresponding fields cannot be excited at low energy.

C.1 Pure higher dimensional gravity

We already saw quite interesting features in the toy model of the $d+1$ dimensional scalar field. Let us now study another theory in $d+1$ dimensions and compactify it : Einstein gravity.

The only field we have is the $d+1$-dimensional metric G_{MN}, which depends on $\vec{x}$, the external coordinates, and y, the only internal coordinate. As before, we represent the total manifold as

the product of an external manifold and a circle S^1. However, this time we have a field with spin $s > 0$, and therefore we have to consider its index structure. How do we handle it? The answer is as follow: we will decompose the metric into different components, which will give rise to fields with different spins in the lower-dimensional theory

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu + 2G_{\mu y}dx^\mu dy + G_{yy}dy^2 \equiv e^{2\alpha\phi}g_{\mu\nu}dx^\mu dx^\nu + e^{2\beta\phi}(dy + A_\mu dx^\mu)^2 . \tag{2.46}$$

with α and β a priori free parameters. However, we will fix $\beta = -(d-2)\alpha$ in order to have a standard Einstein-Hilbert term in the d-dimensional theory, and $\alpha^{-2} = 2(d-1)(d-2)$ so the kinetic term we will obtain for ϕ is canonically normalized. Here μ, ν indices are the d-dimensional spacetime indices. We observe that the $(d+1)$-dimensional metric appears to give rise to a scalar field, a vector, and a d-dimensional metric. We use the term "appears" because we do not yet know how these objects transform under diffeomorphisms, nor if they satisfy the appropriate equations of motion. So, we substitute (2.46) into the $(d+1)$-dimensional Einstein-Hilbert Lagrangian. Then, we conduct an analysis similar to the one we conducted for the scalar fields in the previous section: we expand fields into harmonics of the internal space and split internal and external coordinates. Here, we take a step further: in the previous example, we observed that non-zero modes, i.e., modes that retained a dependence on the internal coordinate, would acquire a mass. In this example, since we are interested in the low-energy limit, we discard all massive modes and only retain the zero modes of the Kaluza-Klein towers. This is equivalent to retaining only the $\vec{x}$ dependence of the fields in (2.46), i.e.

$$\partial_y \Phi = 0 , \tag{2.47}$$

for any field Φ in the theory. By doing this, we can integrate out the y part of the Lagrangian and determine the d-dimensional effective theory we obtain. The result is as follows

$$S_{d+1} = \int dx^d \int dy R_{d+1} \quad \longrightarrow \quad S_d = \int dx^d \left(R_d - \frac{1}{4} e^{2\alpha(d-1)\phi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \right) . \tag{2.48}$$

where we have set $L = 1$. The effective theory we obtain consists of d-dimensional Einstein gravity, a Maxwell theory with field strength $F_{\mu\nu}$, and a scalar field. This is a remarkable result: starting from a $(d+1)$-dimensional theory, we can (almost) unify gravitational and electromagnetic interactions in such a simple manner. From a d-dimensional perspective, the Maxwell theory describes a photon. However, from the $(d+1)$-dimensional viewpoint, the vector field represents the geometry of the internal circle. Therefore, a photon can be seen as a sort of gravitational waves propagating along the internal space. Additionally, the scalar field, known as a dilaton, describes the overall volume of the internal circle

$$\text{vol}(S^1) = e^{\beta\phi} . \tag{2.49}$$

The larger ϕ is, the larger the radius of the circle becomes.

From this simple example, we can make several comments. Firstly, we can implement gauge interactions in the lower-dimensional theory from a geometrical standpoint. Depending on the internal space, if we start with more internal coordinates, providing more room, we could implement different gauge theories in the lower-dimensional theory. The isometries of the internal space correspond to the gauge group of the lower-dimensional theory. In our example, the isometries of the circle are given by $U(1)$, leading to $U(1)$ gauge theory, which corresponds to Maxwell's

theory of electromagnetism.

Secondly, we can see that ϕ controls the volume of the internal space. It can be seen from (2.46). Because the coordinate y on S^1 is periodical, $y + L \simeq y$, then the prefactor of dy^2 is the radius of the circle, i.e. $e^{2\beta\phi}$. As ϕ is a dynamical field, then we cannot simply make the internal space as small as we desire by hand, as we would like to achieve for a satisfactory compactification. Is there a dynamical way to make the internal space small and realize what is called scale separation? This is actually a non-trivial question, one of the many addressed by the so-called "Swampland program" [67–69].

Despite the simplicity and the elegance of the previous result, this model is unsatisfactory for a number of reasons. First of all, we see the emergence of a scalar field. This is a little unsatisfactory in the sense that in the SM there exists only one massive scalar field : the Higgs boson. Here there is also a single scalar fields, but when if instead of compactifying on a circle S^1, we compactified on an n torus T^n, we would have obtained more scalars fields describing internal geometry. The emergence of a number of scalar fields in dimensional compactifications is a feature that shows up all the time, and one has to deal with it. We cannot just throw away them and set them to zero, because this leads to inconsistencies as we will see later. A second point is the fact that here, none of the massless fields, i.e. the fields that consitute the d-dimensioncal Lagrangian in (2.48), are charged under the $U(1)$ gauge theory. This again is a common feature of toroidal compactifications. This can however be overcome using a gauging procedure. Note that if we decided to keep extra massive modes, then the latter would have been charged under the $U(1)$ gauge symmetry. However, as we saw before, ϕ not only controls the internal volume, but is also the coupling constant of Electromagnetism as we can see from the Lagrangian. Therefore in that case, we would need to find a way to stabilize it. One way to do it would be to add a potential for the scalar fields so they are stuck in a minimum of the potential. This can be done using only classical ingredients, such as fluxes, as we will see in the example of the Freund-Rubin compactification in section 2.C.4 .

The last thing we want to emphasise in this model is about symmetry. From the $(d+1)$-dimensional perspective, the symmetries are the diffeomorphisms with parameter $\xi^M(\vec{x}, y)$. These diffeomorphisms will split in the low dimensional theory to $\xi^M(\vec{x}, y) \to \{\xi^\mu(\vec{x}, y), \xi^5(\vec{x}, y)\}$. By dropping the y-dependence here as we did for all the fields, we end up with d-dimensional diffeomorphisms generated by $\xi^\mu(\vec{x})$ and $U(1)$ gauge symmetry as expected generated by $\xi^5(\vec{x})$. Note however that there is an additional global symmetry, because in order to preserve the Kaluza-Klein ansatz (2.47), we can still keep a y linear dependence in ξ^5 such that

$$\xi^5(\vec{x}, y) = \xi^5(\vec{x}) + cy\,, \tag{2.50}$$

with c a constant parameter. One can show that this symmetry leaves the equations of motion invariant. Therefore, there is an additional $\mathbb{R} = GL(1)$ global symmetry. If instead of compactifying from $d+1$ to d dimensions, we would have gone from $d+n$ to d with a compactification on a torus T^n, the previous equation becomes

$$\xi^m(x^\mu, y^m) = \xi^m(x^\mu) + \Lambda^m{}_n y^n\,, \tag{2.51}$$

whose last part induces an apparent $GL(n)$ symmetry. Note however that the trace $\Lambda^m{}_m$ induces a non-trivial transformation on the graviton field, which is not a symmetry of the Lagrangian. Therefore (2.51) only induces an $SL(n)$ symmetry, for $n \geq 2$. Note that there the story is slightly different in $3d$ and $2d$ where this $SL(n)$ symmetry is enhanced [70, 71]. We will come back on this

point in the next paragraph. Finally, that there is an additional symmetry entering the analysis: the trombone symmetry. This symmetry for a theory in D dimensions rescales the Lagrangian as

$$\delta\mathscr{L} = (D-2)\vartheta\mathscr{L}\,, \tag{2.52}$$

with ϑ the parameter inducing the trombone symmetry. Combining the trombone symmetry of the higher-dimensional gravity theory and the trace part of (2.51), we can construct an additional $\mathrm{GL}(1)$ symmetry, and with the previous $\mathrm{SL}(n)$, this makes a $\mathrm{GL}(n)$ symmetry.

C.2 Supergravity toroidal compactifications

In this section, we highlight the results of toroidal compactifications of type IIB $10d$ and $11d$ Supergravities, respectively to $5d$ and $4d$. When compactifying Supergravities on tori, the resulting theory contains a number of scalar fields and gauge fields, as illustrated in the previous section with the compactification of pure gravity. Here, we will encounter a richer structure and more fields due to the presence of extra form fields in both theories.

Starting from 10 or 11-dimensional Supergravities, we proceed similarly to what we did in the case of Einstein gravity, and compactify on a torus T^n. We then use the Kaluza-Klein ansatz (2.47) and discard all internal dependance. The resulting low-dimensional theories will be called ungauged Supergravities because, similar to the case of Einstein gravity compactification, none of the fields will be charged under the gauge fields. Moreover, the gauge fields we obtain in the low-dimensional theories will all be gauge fields of a $\mathrm{U}(1)^n$ theory. This setup closely resembles the case of pure gravity compactification. The difference lies mainly in the global symmetry and matter content. In the case of the pure gravity, the compactification on T^n led to a global $\mathrm{GL}(n)$ symmetry. However, in the case Supergravities, this symmetry is enhanced. If we are dealing with maximal Supergravities, this gives a group of the exceptional family. Let us illustrate how this goes.

Because of the presence of matter fields, extra symmetries, known has hidden symmetries, enter the lower-dimensional theory [57, 72, 73]. For example, we saw the presence of p-forms in $10d$ and $11d$ Supergravities. The gauge symmetries of those p-form will add global symmetries to the lower-dimensional symmetry in a similar fashion than (2.51). For example, if the theory of interest comprises a 2-form potential $B_{\mu\nu}$, the latter transform under diffeomorphisms generated the vector ξ^μ, plus gauge transformation generated by the vector λ^μ

$$\delta B_{\mu\nu} = \xi^\rho\partial_\rho B_{\mu\nu} + \partial_\mu\xi^\rho B_{\rho\nu} + \partial_\nu\xi^\rho B_{\mu\rho} + 2\partial_{(\mu}\lambda_{\nu)}\,. \tag{2.53}$$

Then one truncates the y dependance of λ which at the end enlarges the total symmetry group. One can show that the $\mathrm{GL}(n)$ group is enhanced, in general, to an $\mathrm{O}(n,n)$ group upon duallizing low dimensional fields. The latter is the universal symmetry arising from compactifications of Supergravities on tori. In some cases, this symmetry can be enhanced. For example, half-maximal Supergravities coupled to m vector multiplets, gives rise to $\mathrm{O}(n,n+m)$. In the case of maximal Supergravity, this $\mathrm{O}(n,n)$ group is further enhanced to an $\mathrm{E}_{n(n)}$ group for $11d$ Supergravity and $\mathrm{E}_{n+1(n+1)}$ for Type IIB Supergravity (2.2). Double Field Theory [74–82] and Exceptional Field Theory [83–88] are the framework that make those duality manifest in the higher dimensional theories, before compactification.

As seen in the previous section, the $\mathrm{GL}(n)$ global symmetry arises from higher-dimensional

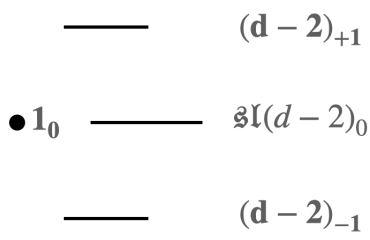


Figure 2.12: Decomposition of $\mathfrak{sl}(d-1)$ under $\mathfrak{sl}(d-2) \times \mathfrak{gl}(1)$. The subscripts denote the R^+ charges which establish the vertical grading. At level 0 one finds the algebra of $\mathfrak{sl}(d-2) \times \mathfrak{gl}(1)$ while level 1 and -1 contain an $\mathfrak{sl}(d-2)$ vector and a dual vector, respectively [70].

diffeomorphisms. Therefore, only a part of the exceptional symmetry is explained by those diffeomorphisms. In particular, part of the "extra" symmetries that enhance the $\mathrm{GL}(n)$ family to E_n become manifest only after dualization of some of the lower-dimensional fields. Hence, these additional symmetries are often referred to as "hidden" symmetries. We illustrate this mechanism with the example of Einstein gravity compactified from $d+1$ dimensions to 3 dimensions on a torus. From the previous discussion, we expect to have an $\mathrm{GL}(d+1-3) = \mathrm{GL}(d-2)$ global symmetry group. However, in $3d$ on can dualize the vector fields to scalars by

$$\rho^{(2+2/(d-2))} e M_{mn} F^{m\mu\nu} = \epsilon^{\mu\nu\rho} \partial_\rho \varphi_n \,, \tag{2.54}$$

with $m, n, \ldots = 1, \ldots, d-2$, M an unimodular matrix which encodes the scalars of the lower-dimensional theory, which parametrizes the coset $\mathbb{R} \times \mathrm{SL}(d-2)/\mathrm{SO}(d-2)$, ρ the dilaton field, e the determinant of the vielbein, $F^m_{\mu\nu} = 2\partial_{[\mu} A^m_{\nu]}$ and ϵ the Levi-Civita tensor. The equations of motions for the vector fields dual to the scalars φ_n are

$$\partial_\mu (\rho^{(1+2/(d-2))} e M_{mn} F^{m\mu\nu}) = 0 \,. \tag{2.55}$$

We see from (2.54) that the scalars φ^n are defined up to a global shifts $\varphi_n \to \varphi_n + \theta_n$. Therefore we see that when formulating the theory with scalars and metric only, upon dualizing the vectors, that there are additional symmetries entering the theory and the global symmetry group we obtain is $\mathrm{GL}(d-2) \ltimes \mathbb{R}^{d-2}$. However, this symmetry is actually enhanced and the total symmetry group is $\mathrm{SL}(d-1)$. The branching of $\mathrm{SL}(d-1)$ into $\mathrm{SL}(d-2) \times \mathrm{GL}(1)$ is represented in Figure 2.12. The generators denoted as $(\mathbf{d-2})_{-1}$ are the "hidden" generators enhancing the symmetry, often referred as generating "hidden" symmetries. They act on the dilaton scalar ϕ, the φ_n scalars and the scalars parametrizing the coset $\mathrm{SL}(d-2)/\mathrm{SO}(d-2)$ non-linearly. At the end, all the scalars can be arranged into the coset $\mathrm{SL}(d-1)/\mathrm{SO}(d-1)$. The reader can refer to [70, 89] for further details.

In the case of Supergravities, a similar mechanism is at work, with additional matter fields, higher rank forms, to dualize. These enhanced symmetries, the exceptional symmetries, organize the field content. In the case of type IIB and $11d$ Supergravities compactified to $5d$, the 42 scalar fields of the theory are organized into a scalar manifold given by the coset $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$. Similarly, for compactification to $4d$, where the 70 scalar fields reside on the scalar manifold $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ [57]. The vector fields reside in the fundamental representation of the exceptional group. The decompostion of the exceptional algebra according to the appropriate $\mathfrak{sl}$ algebra (before symmetry

Exceptional group serie		
d	G	K
9	$\mathrm{GL}(2)$	$\mathrm{SO}(2)$
8	$\mathrm{SL}(2) \times \mathrm{SL}(3)$	$\mathrm{SO}(2) \times \mathrm{SO}(3)$
7	$\mathrm{SL}(5)$	$\mathrm{SO}(5)$
6	$\mathrm{SO}(5,5)$	$\mathrm{SO}(5) \times \mathrm{SO}(5)$
5	$\mathrm{E}_{6(6)}$	$\mathrm{USp}(8)$
4	$\mathrm{E}_{7(7)}$	$\mathrm{SU}(8)$
3	$\mathrm{E}_{8(8)}$	$\mathrm{SO}(16)$
2	$\mathrm{E}_{9(9)}$	$K(\mathrm{E}_{9(9)})$

Table 2.2: Exceptional family showing up as global symmetry of the equations of motion in toroidal compactifications of Supergravities [90].

enhancement) is shown in Figure 2.13 [55].

C.3 Gauging procedure

The resulting theories obtained via toroidal compactificaitons in the previous sections, are ungauged low-dimensional theories, which can be gauged. This procedure is akin to what is done in usual gauge theories: select some of the vectors available and couple the matter fields to the gauge fields. However, the number of possibilities for gauging is not infinite; the maximal number of gauge fields is fixed, and the gauge group G_0 must be a subgroup of the global symmetry group. Furthermore, the consistency of the procedure imposes several constraints.

We briefly summarize the important features of the gauging procedure via the embedding tensor formalism. For more details on the embedding tensor formalism in Supergravity, we refer to [90]. This formalism is used to construct gauged Supergravities starting from ungauged ones, where the gauge couplings are controlled by an object called the embedding tensor. In a standard gauge theory, we introduce the covariant derivatives

$$\partial_\mu \mathbb{1} \to \mathscr{D}_\mu \mathbb{1} = \partial_\mu \mathbb{1} - g A_\mu{}^I \mathbb{T}_I^R \,, \tag{2.56}$$

where $\mathbb{T}_I^R$ are the generators of the gauge group in the appropriate representation. In the case of Supergravities, the vector fields are organized by some representations of the global symmetry group. Hence the number of vector fields is fixed. When gauging the theory, the total number of vector fields available may be higher than the number of gauge fields required to couple to a given gauge group G_0. To determine which fields among the available ones are needed for the coupling, one can introduce the embedding tensor, which selects the vector fields as needed

$$\partial_\mu \mathbb{1} \to \mathscr{D}_\mu \mathbb{1} = \partial_\mu \mathbb{1} - g A_\mu{}^M X_M \,, \tag{2.57}$$

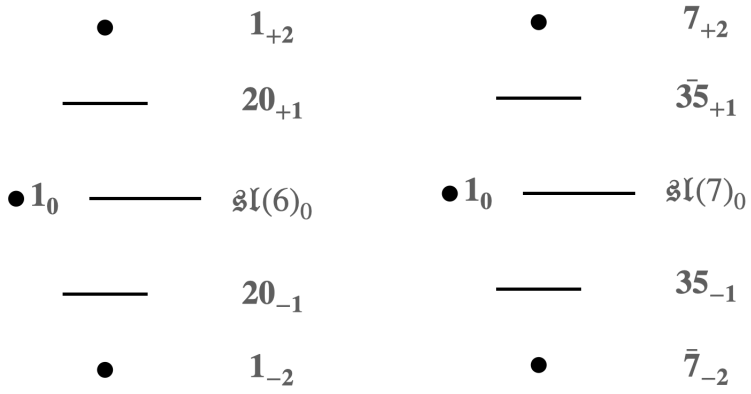


Figure 2.13: On the left, the branching of $\mathfrak{e}_6$ into $\mathfrak{sl}(6)$. On the right, the branching of $\mathfrak{e}_7$ into $\mathfrak{sl}(7)$. The subscript again denote the charge under $\mathfrak{gl}(1)$. The $\mathbf{1}_{+2}$ and $\mathbf{7}_{+2}$ generators are shift symmetries coming from dualization of higher rank forms. The $\mathbf{1}_{+2}$ and $\mathbf{7}_{+2}$ are the shift symmetries of the scalars as explained in the text. The generators with negative charges are the "accidental" symmetries responsible for the enhancement to the exceptional group, which are dual to the shift symmetries [70].

with

$$X_M = \Theta_M{}^{\alpha} \mathbb{T}_{\alpha} \,. \tag{2.58}$$

with g the coupling constant, $\mathbb{T}_{\alpha} \in \mathrm{G}$, where G is the total rigid symmetry group of the Supergravity, and $X_M \in \mathrm{G}_0$ the generators of the gauge group $\mathrm{G}_0 \subset \mathrm{G}$. Strictly speaking the embedding tensor is the Θ tensor, but by abuse of language we will also denote $X_{MN}{}^{P} = \Theta_M{}^{\alpha} \mathbb{T}_{\alpha,N}{}^{P}$ as the embedding tensor. The $\Theta_M{}^{\alpha} \mathbb{T}_{\alpha,N}{}^{P}$ in the previous formula are the generators of the generators of the group G expressed in the fundamental representation. This embedding tensor is subject to two constraints : one quadratic constraint that ensures the closure of the algebra

$$\left[X_{\underline{M}}, X_{\underline{N}}\right] = -X_{\underline{MN}}{}^{\underline{P}} X_{\underline{P}} \,, \tag{2.59}$$

which can be derived by demanding that the embedding tensor itself is invariant under the action of the gauge group. Explicitely, this leads to a quadratic constraints in Θ

$$0 = \delta_P \Theta_M{}^{\alpha} = \Theta_P{}^{\beta} \mathbb{T}_{\beta M}{}^{N} \Theta_N{}^{\alpha} + \Theta_P{}^{\beta} f_{\beta\gamma}{}^{\alpha} \Theta_M{}^{\gamma} \,, \tag{2.60}$$

where $f_{\beta\gamma}{}^{\alpha}$ are structure constants of the global symmetry group. By contracting the previous condition with generators $\mathbb{T}$ one recovers (2.59). There is an additional constraint, linear in Θ, that is recquired by supersymmetry. The embedding tensor Θ lives in the product of a fundamental representation of G and its adjoint. In general, this leads to a reducible representation, and the linear constraint project the latter to a particular representation in the decomposition of the product, which schematically gives

$$\textbf{Fundamental} \otimes \textbf{Adjoint} = \mathcal{R}_1 \oplus \ldots \mathcal{R}_n \tag{2.61}$$

The linear constraints select a subset of $\{\mathcal{R}_i\}$ on which the embedding tensor lives.

Let us now come back to the compactification of IIB and $11d$ Supergravities. We can compactify type IIB Supergravity on a torus to five dimensions. The resulting theory has total geometry

$\mathcal{M} = \mathbb{R}^{1,4} \times T^5$, and it contains 42 scalar fields described by the coset space $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ and 27 vector fields describing $\mathrm{U}(1)$ gauge theories. This is a maximal theory, in the sense that there are $\mathcal{N} = 8$ supersymmetries. For $11d$ Supergravity, a similar procedure but compactifying on T^7, leads to a theory living on $\mathbb{R}^{1,3} \times T^7$ with 70 scalar fields living on $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ and 56 vector fields describing again an abelian gauge theory. Here again the theory is maximal with $\mathcal{N} = 8$. Via the gauging procedure, we can enhance $G_0 \subset \mathrm{E}_{n(n)}$ to gauge group of the theory. By doing so, we can obtain the maximal $\mathcal{N} = 8$ $d = 5$ Supergravity with gauge group $\mathrm{SO}(6)$ from IIB Supergravity, and the maximal $\mathcal{N} = 8$ $d = 4$ Supergravity with gauge group $\mathrm{SO}(8)$ from $11d$ Supergravity [91, 92]. The results for ungauged compactification and gauging procedures are schematically represented in Figure 2.14. From the higher dimensional theory, this corresponds to deforming the geometry of the internal space. However, we cannot deform freely the internal space. As soon as we deform it, or as soon as we gauge the theory, a potential appears for the scalar fields. So can we wonder what are the allowed internal spaces, or at least can we find some compactifications which internal space are not tori? If so, are there other dimensional reductions leading to maximally symmetric theories? This is the object of the next section.

C.4 Spontaneous compactifications and Freud-Rubin ansatz

Suppose we have a $(d+n)$-dimensional theory living on $\mathcal{M}_{d+n}$ that we would like to compactify to a d-dimensional one. We split the set of full coordinates into two sets: internal coordinates y^m with $m \in \{1, \dots, n\}$, and external coordinates x^μ with $\mu \in \{1, \dots, d\}$. We are interested in finding a solution of our theory where $\mathcal{M}_{d+n}$ factors as a direct product of an internal $\mathcal{M}_n$ and an external $\mathcal{M}_d$

$$\mathcal{M}_{d+n} = \mathcal{M}_d \times \mathcal{M}_n \,. \tag{2.62}$$

We will also require that $\mathcal{M}_d$ is a maximally symmetric space with Lorentz signature $(-+\cdots+)$. This means that this space should be invariant under $SO(1, d-1)$ if it is flat Minkowski, or $SO(1, d)$ if it has constant positive curvature (de Sitter), or $SO(2, d-1)$ if it has negative constant curvate (Anti de Sitter), see Figure 2.15. We will further assume that our theory contains a "magnetic" $(n-1)$-form potential $C_{(n-1)}$ whose field strength we will denote $F_{(n)}$. This procedure we are about to demonstrate, naturally shows up in the context of Supergravity compactifications where numerous p-form are presents. Suppose that the Lagrangian takes the form, where we ignore the fermionic fields:

$$\int \sqrt{g_{d+n}} \left(R_{d+n} - \frac{1}{2} \frac{1}{n!} F_{M_1 \dots M_n} F^{M_1 \dots M_n} \right) \tag{2.63}$$

where here $M, N, \dots$ are indices running from 1 to $d+n$. The Einstein equations of motion read

$$R_{MN} = -\frac{n-1}{2(d+n+2)} \frac{1}{n!} F^2 g_{MN} + \frac{1}{2(n-1)!} F_{M M_1 \dots M_{n-1}} F_N{}^{M_1 \dots M_{n-1}} \tag{2.64}$$

Now because of our assumptions, the total metric factorizes as

$$\mathrm{d}s^2_{d+n} = \mathrm{d}s^2_d + \mathrm{d}s^2_n = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \rho \bar{g}_{mn} \mathrm{d}x^m \mathrm{d}x^n \,, \tag{2.65}$$

where $g_{\mu\nu}$ is a metric on $\mathcal{M}_d$ and $\bar{g}_{mn}$ a metric on $\mathcal{M}_n$. The ρ factor is a scalar field that controls the volume on $\mathcal{M}_n$ such that $\int_{\mathcal{M}_n} \sqrt{\bar{g}} = 1$. In 1980, Freund and Rubin proposed an ansatz for

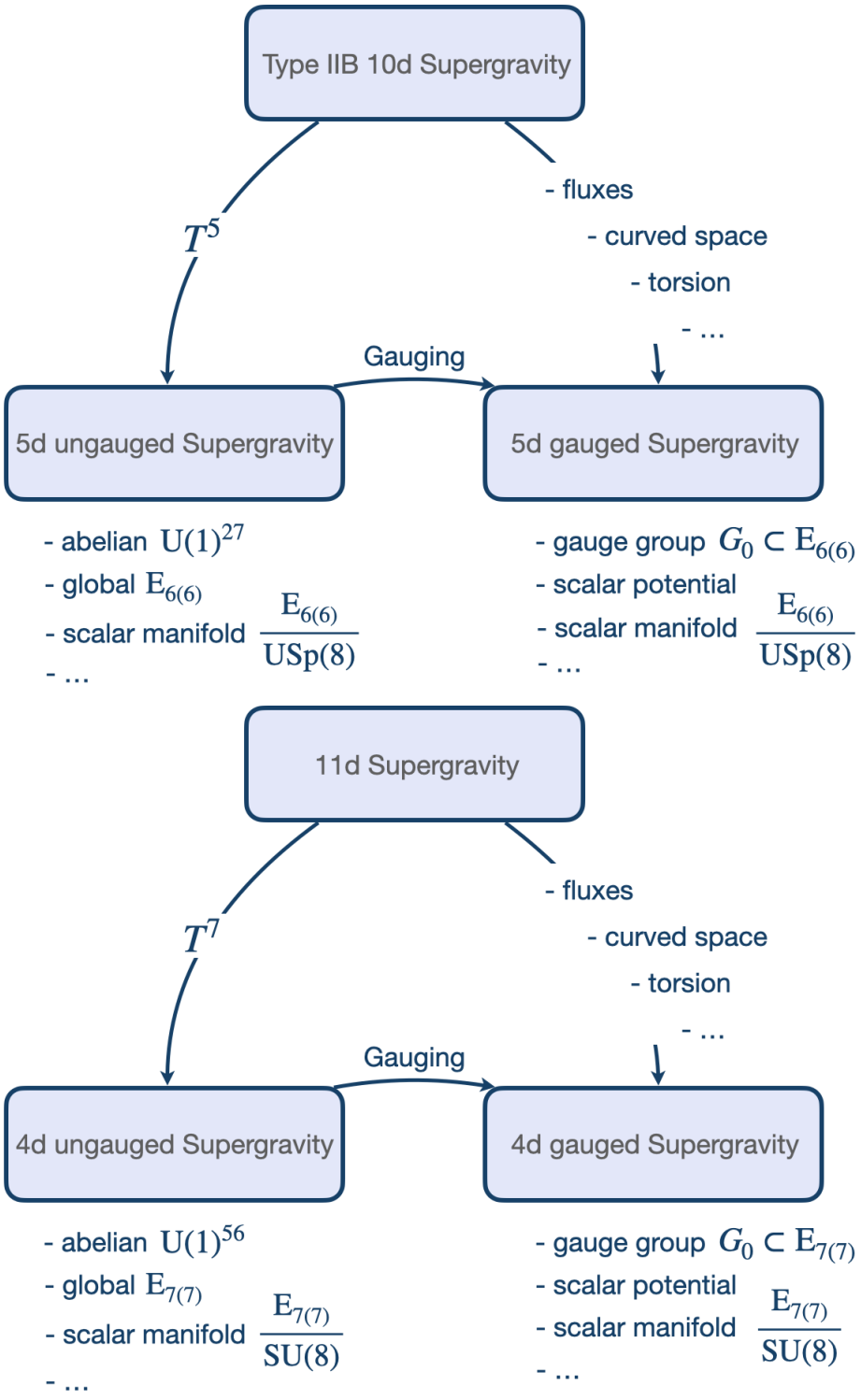


Figure 2.14: Schematics illustration of compactifications of Supergravities and gauging procedure.

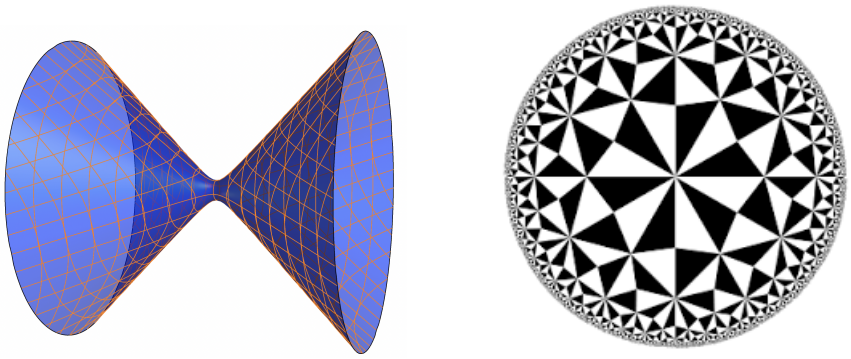


Figure 2.15: Example of $2d$ Anti de Sitter space, embbeded in $\mathbb{R}^3$ (left). Its topology is $\mathbb{R}\times S^1$. $2d$ AdS spaces can also be represented by Poincaré disks (right).

the form $F_{(n)}$, that leads to a solution of the equations of motion, around which we can study the compactification of the theory. This is the following

$$F_{(n)} = Q\epsilon_{(n)}\,, \tag{2.66}$$

with Q an a priori free parameter and ϵ the volume form of the external space. The Q parameter represents the amount of flux that is filling the internal manifold and one can show that it is in fact quantized. Injecting this ansatz in the Einstein equations leads to the following result for the Ricci tensor

$$\begin{cases} R_{\mu\nu} = -\dfrac{n-1}{2(d+n-2)}\dfrac{Q^2}{\rho^n}g_{\mu\nu}\,, \\ R_{mn} = \dfrac{d-1}{2(d+n-2)}\dfrac{Q^2}{\rho^n}\bar{g}_{mn}\,, \\ R_{m\mu} = 0\,, \end{cases} \tag{2.67}$$

and for the energy-momentum tensor

$$\begin{cases} F^2_{\mu\nu} = Q^2(n-1)!g_{\mu\nu}\,, \\ F^2_{mn} = 0\,, \\ F^2_{m\mu} = 0\,, \end{cases} \tag{2.68}$$

with $F^2_{MN} = F_{MM_1\ldots M_{n-1}}F_N^{M_1\ldots M_{n-1}}$. Several comments should be made at this point. Using the Freund-Rubin (FR) ansatz, we observe that there is no flux pointing toward the external dimensions, but only internal fluxes. These internal fluxes counteract the gravitational force, which tends to shrink the internal space due to gravitational collapse. The internal fluxes resist contraction, akin to how one experiences a repulsive force when attempting to bring magnetic field lines closer together. It is this competition between these two forces that leads to equilibrium.

From (2.67) we see that both the internal and external spaces are Einstein spaces. For the external space, due to the negative sign, we observe negative curvature. Since we have already required this space to be maximally symmetric, we can conclude that the external space is AdS.

Finally, the internal space has positive curvature. Therefore, the total space is

$$\mathcal{M}_{d+n} = \mathrm{AdS}_d \times \Sigma_n \,, \tag{2.69}$$

where Σ_n is a positively curved space, for example a sphere. With the example of 11d Supergravity, this leads to the background solution $\mathrm{AdS}_4 \times \mathrm{S}^7$ if we require the internal space to also be maximally symmetric.

Finally here again, similarly to what we observed for the compactification of pure gravity, we see the emergence of characteristic features that are resurgents of dimensional reductions. Parameters of the low dimensional theory are not free, but rather fixed by equations of motion or are linked to other parameters. This is the case for example for internal and external curvatures which must have opposite signs, but also the volume of the internal space ρ. If we take a step back, and promote ρ to a scalar field, we can build an effective Lagrangian for ρ in d dimensions. The Lagrangian takes the form

$$\mathscr{L}_d = \int \sqrt{g_d}\Big(R_d + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - V(\rho)\Big)\,, \tag{2.70}$$

with R_d the scalar curvature of the d dimensional space. The potential $V(\rho)$ controls the dynamics of the field ρ. To find a solution, one must minimize it such that $\partial V = 0$ at $\rho = \rho^*$. Then $V(\rho^*)$ acts as an effective cosmological constant. The higher dimensional origin of this potential is two fold: one part is coming from the higher dimensional scalar curvature

$$R_{d+n} = R_d + \rho^{-1}R_n + \dots\,, \tag{2.71}$$

with ellipses containing derivatives of ρ. The second part of the potential is coming from the F^2 term. The total potential is

$$V(\rho) = -\rho^a R_n + \rho^b \frac{Q^2}{2}\,, \tag{2.72}$$

with

$$a = -1 + \frac{d\gamma + n}{2}\,, \quad b = -n + \frac{d\gamma + n}{2}\,,$$

and $\gamma = -n/(d-2)$. The different coefficients are fixed when going from the higher dimensional Lagrangian to the lower-dimensional Lagragian, by doing a Weyl rescaling to get the Einstein frame for the effective action, as well as canonical normalization for the kinetic term of the ρ field. Around the vacuum, the effective theory we get is

$$\mathcal{S}_d = \int \mathrm{d}x^d \sqrt{-g_d}\left(R_d - \rho^{*-\frac{dn}{2(d-2)}} V(\rho^*)\right)\,. \tag{2.73}$$

This means that the volume of the internal space, which also controls the couplings of the scalar fields to the geometry, is not free, but rather fixed by the equations of motion. This raises questions like, what are the effective theories we can build from such a process? This again is related to the Swampland program.

Before closing this section, we note that it is possible to redo the same analysis, in the case of an "electric" flux. In that case, the form is taken to be $F_{(d)} = Q\epsilon_{(d)}$. This leads to a solution on an $\mathrm{AdS}_d \times \mathrm{S}^n$ vacuum, with this time an "electric" flux pointing in the non-compact directions. One can go from one picture to another by taking $F_{(n)} = \star F_{(d)}$.

Finally we note that in the case of Type IIB Supergravity compactified to $5d$, that one needs to include both a magnetic flux filling the internal space and an external electric flux pointing in the non-compact directions. This is because the five-form is self dual

$$F_{(5)} = \star F_{(5)} \,. \tag{2.74}$$

In particular this leads to the $AdS_5 \times S^5$ compactification of type IIB Supergravity.

D Exceptional Field Theory : a Review

In the preceding sections, we have reviewed Supergravities in $d = 11$ and $d = 10$, as well as their compactifications to maximal Supergravities in $d = 4$ and $d = 5$. We highlighted there was a global symmetry arising from compactification. Based on this observation, we can rephrase the $11d$ and type IIB Supergravities into what is called Exceptional Field Theories. The latter renders manifest the $E_{n(n)}$ global symmetry group before compactification. This reformulation of higher-dimensional Supergravities in terms of low dimensional objects will prove very useful in compactifications.

In this section, we will not review how to construct ExFT for which we refer to [83–88, 93–96]. Those are based on the idea of a generalized geometry that was first developped in [97, 98]. In this section, we will rather recap what are the main features of $E_{6(6)}$ and $E_{7(7)}$-ExFT which are suited for $5d$ and $4d$ reductions respectively [99].

$E_{6(6)}$-ExFT and $E_{7(7)}$-ExFT are the duality covariant formulations of the $11d$ Supergravity and IIB Supergravity. The bosonic field content of $E_{6(6)}$-ExFT is

$$\left\{ g_{\mu\nu}, \mathscr{M}_{MN}, \mathscr{A}_\mu{}^M, \mathscr{B}_{\mu\nu\, M} \right\} , \quad \mu,\nu = 0,\dots,4 \,, \quad M = 1,\dots 27 \,, \tag{2.75}$$

whereas the bosonic field content of $E_{7(7)}$-ExFT is

$$\left\{ g_{\mu\nu}, \mathscr{M}_{MN}, \mathscr{A}_\mu{}^M, \mathscr{B}_{\mu\nu\,\alpha}, \mathscr{B}_{\mu\nu\, M} \right\} , \qquad \mu,\nu = 0,\dots,3 \,, \quad M = 1,\dots,56 \,, \tag{2.76}$$

with $g_{\mu\nu}$ the $5d$ ($E_{6(6)}$) or $4d$ ($E_{7(7)}$) external metric, $\mathscr{M}_{MN}$ the so-called generalized metric parametrizing the $E_{6(6)}/USp(8)$ or $E_{7(7)}/SU(8)$ coset spaces for scalar fields, $\mathscr{A}_\mu{}^M$ are vectors labelleb by a fundamental index of the appropriate exceptional group. In both cases, $\mathscr{B}_{\mu\nu\, M}$ are 2-forms which are labeled by the index M living in the anti-fundamental representation of $E_{n(n)}$. Note the presence of the extra $\mathscr{B}_{\mu\nu\,\alpha}$ in the case of $E_{7(7)}$ living in the adjoint representation of $E_{7(7)}$. The $E_{6(6)}$ and for $E_{7(7)}$ bosonic Lagragian looks like

$$\mathscr{L}_{E_{6(6)}} = \sqrt{|g|}\left(\widehat{R} + \frac{1}{4\alpha_n}\, g^{\mu\nu} \mathscr{D}_\mu \mathscr{M}^{MN}\, \mathscr{D}_\nu \mathscr{M}_{MN} - \frac{1}{4}\, \mathscr{M}_{MN} \mathscr{F}^{\mu\nu M} \mathscr{F}_{\mu\nu}{}^N + \sqrt{|g|}^{-1} \mathscr{L}_{\text{top}} - V(g,\mathscr{M}) \right) . \tag{2.77}$$

with α_n depending on the theory. The $E_{7(7)}$ Lagrangian is actually a pseudo-Lagrangian and has to be supplemented by the twisted self-duality equation

$$\mathscr{F}_{\mu\nu}{}^M = -\frac{1}{2}\sqrt{|g|}\, \varepsilon_{\mu\nu\rho\sigma}\, \Omega^{MN} \mathscr{M}_{NK}\, \mathscr{F}^{\rho\sigma\, K} \,, \tag{2.78}$$

for the non-abelian vector field strengths, which are defined by

$$\begin{aligned}\mathscr{F}_{\mu\nu}{}^{M} \equiv{}& 2\,\partial_{[\mu}\mathscr{A}_{\nu]}{}^{M} - 2\,\mathcal{A}_{[\mu}{}^{K}\partial_{K}\mathcal{A}_{\nu]}{}^{M} - \frac{1}{2}\left(24\,(\mathbb{T}_{\alpha})^{MK}(\mathbb{T}^{\alpha})_{NL} - \Omega^{MK}\Omega_{NL}\right)\,\mathcal{A}_{[\mu}{}^{N}\,\partial_{K}\mathscr{A}_{\nu]}{}^{L} \\ &- 12\,(\mathbb{T}^{\alpha})^{MN}\partial_{N}\mathscr{B}_{\mu\nu\,\alpha} - \frac{1}{2}\,\Omega^{MN}\mathscr{B}_{\mu\nu\,N}\,.\end{aligned} \tag{2.79}$$

where $\mathbb{T}_{\alpha}$ are the $\mathrm{E}_{7(7)}$ generators and Ω^{MK} denotes the symplectic invariant tensor. It statisfies

$$\Omega_{MK}\Omega^{NK} = \delta^{N}_{M}\,. \tag{2.80}$$

For $\mathrm{E}_{6(6)}$ the Lagragian leads to the full set of equations of motion and the non-abelian field-strengths in (2.77) are defined

$$\mathscr{F}_{\mu\nu}{}^{N} = 2\,\partial_{[\mu}\mathscr{A}_{\nu]}{}^{N} - 2\,\mathscr{A}_{[\mu}{}^{K}\partial_{K}\mathscr{A}_{\nu]}{}^{N} + 10\,d^{NKR}d_{PLR}\,\mathscr{A}_{[\mu}{}^{P}\,\partial_{K}\mathscr{A}_{\nu]}{}^{L} + 10\,d^{NKL}\,\partial_{K}\mathscr{B}_{\mu\nu\,L}\,, \tag{2.81}$$

In both theories, extra coordinates are added in order to make the theory covariant under the $\mathrm{E}_{n(n)}$ group, and the full set of coordinates is now

$$\{x^{\mu}, \mathcal{Y}^{M}\}. \tag{2.82}$$

The x^{μ} coordinates are the coordinates on the external space, while $\mathcal{Y}^{M}$ are the internal coordiantes. In the case of $\mathrm{E}_{6(6)}$ the total number of coordinates is $5+27$ whereas for $\mathrm{E}_{7(7)}$ it is $4+56$. The internal y^{m} coordinates are embedded into the larger set of coordinates $\mathcal{Y}^{M}$. The derivatives with respect to this enlarge set of coordinates is denoted by ∂_{M}. Not that not all those internal coordinates have a good physical interpretation, and one needs to project out the non-physical ones. To do so, one uses the so-called section constraints. For the case of $\mathrm{E}_{6(6)}$, it takes the form

$$d^{KMN}\,\partial_{M}\Phi_{1}\partial_{N}\Phi_{2} = 0\,, \tag{2.83}$$

in term of the two fully symmetric invariant d-symbols of $\mathrm{E}_{6(6)}$ and for any couple of fields $\{\Phi_{1}, \Phi_{2}\}$. For $\mathrm{E}_{7(7)}$, the section constraints take the form

$$\Omega^{MK}\,(\mathbb{T}_{\alpha})_{K}{}^{N}\,\partial_{M}\Phi_{1}\partial_{N}\Phi_{2} = 0 = \Omega^{MN}\partial_{M}\Phi_{1}\partial_{N}\Phi_{2}\,, \qquad \alpha = 1,\dots 133\,, \tag{2.84}$$

where $(\mathbb{T}_{\alpha})_{M}{}^{N}$ are the 133 generators of $\mathrm{E}_{7(7)}$. Note that in the remaining of the thesis, we will use the same notation for the generators of $\mathrm{E}_{6(6)}$. On top of this constraint, the two-forms $\mathscr{B}_{\mu\nu\,M}$ are further constraints in the case of $\mathrm{E}_{7(7)}$

$$0 = \Omega^{MK}\,(\mathbb{T}_{\alpha})_{K}{}^{N}\,\mathscr{B}_{\mu\nu\,M}\,\partial_{N}\Phi = \Omega^{MK}\,(\mathbb{T}_{\alpha})_{K}{}^{N}\,\mathscr{B}_{\mu\nu\,M}\mathscr{B}_{\rho\sigma\,N}\,, \qquad \alpha = 1,\dots 133\,. \tag{2.85}$$

Gauge invariance of the ExFT actions is ensured by (2.83), (2.84) and (2.85).

Those Lagrangians are invariants under internal generalized diffeomorphisms whose action takes the form for a vector V^{M}

$$\delta V^{M} = \mathscr{L}_{\Lambda}V^{M} \equiv \Lambda^{K}\partial_{K}V^{M} - \alpha_{n}\mathbb{P}^{M}{}_{N}{}^{K}{}_{L}\partial_{K}\Lambda^{L}V^{N} + \lambda\partial_{P}\Lambda^{P}V^{M}\,, \tag{2.86}$$

with λ a density weight depending on the field and $\mathbb{P}^{M}{}_{N}{}^{K}{}_{L}$ the projector on the adjoint repre-

sentations, $\alpha_6 = 6$ and $\alpha_7 = 12$. The projectors can be expressed for $E_{6(6)}$ in term of the two fully symmetric invariant d-symbols

$$\mathbb{P}^{M}{}_{N}{}^{K}{}_{L} \equiv \mathbb{T}_{\hat{\alpha}N}{}^{M}\mathbb{T}^{\hat{\alpha}}{}_{L}{}^{K} = \frac{1}{18}\delta^{M}_{N}\delta^{K}_{L} + \frac{1}{6}\delta^{K}_{N}\delta^{M}_{L} - \frac{5}{3}d_{NLR}d^{MKR}\,, \tag{2.87}$$

with again $\mathbb{T}_{\alpha N}{}^{M}$ the generators of $E_{6(6)}$ in the fundamental 27 representation and $\alpha = 1,\ldots,78$ an adjoint index. For $E_{7(7)}$ this projector can be expressed as

$$\mathbb{P}^{K}{}_{M}{}^{L}{}_{N} = \frac{1}{24}\,\delta^{K}_{M}\delta^{L}_{N} + \frac{1}{12}\,\delta^{L}_{M}\delta^{K}_{N} + (\mathbb{T}_{\alpha})_{MN}(\mathbb{T}^{\alpha})^{KL} - \frac{1}{24}\,\Omega_{MN}\Omega^{KL}\,. \tag{2.88}$$

For the scalar matrix, the action of generalized diffeomorphisms becomes to

$$\mathscr{L}_{\Lambda}\mathscr{M}_{MN} = \Lambda^{K}\partial_{K}\mathscr{M}_{MN} + 2\alpha_{n}\partial_{L}\Lambda^{K}\mathbb{P}^{K}{}_{L}{}^{P}{}_{(M}\mathscr{M}_{N)P}\,, \tag{2.89}$$

The covariant derivative is accordingly defined as

$$\mathscr{D}_{\mu}\mathscr{M}_{MN} = (\partial_{\mu} - \mathscr{L}_{\mathscr{A}_{\mu}})\mathscr{M}_{MN}\,. \tag{2.90}$$

We note in passing that all the factors of the lagrangians are fixed by those internal generalized diffeomorphims. There is no room left for anything else.

The Einstein-Hilbert term is constructed from the modified Ricci scalar $\widehat{R}$, constructed from the external metric $g_{\mu\nu}$ in the standard way upon covariantising derivatives under internal diffeomorphisms

$$\partial_{\mu}g_{\nu\rho} \to \partial_{\mu}g_{\nu\rho} - \mathscr{A}_{\mu}{}^{M}\partial_{M}g_{\nu\rho} - g_{\nu\rho}\partial_{M}\mathscr{A}_{\mu}{}^{M}\,. \tag{2.91}$$

with a Stückelberg-type coupling to the two-form tensors $\mathscr{B}_{\mu\nu\,N}$. Finally the topological term $\mathscr{L}_{\text{top}}$ is defined via its derivative for $E_{6(6)}$

$$d\mathscr{L}_{\text{top}} \propto d_{MNK}\,\mathscr{F}^{M}\wedge\mathscr{F}^{N}\wedge\mathscr{F}^{K} - 40\,d^{MNK}\mathscr{H}_{M}\wedge\partial_{N}\mathscr{H}_{K}\,, \tag{2.92}$$

in terms of the field strengths $\mathscr{F}_{\mu\nu}{}^{M}$ and $\mathscr{H}_{\mu\nu\rho\,M} = 3\,\mathscr{D}_{[\mu}\mathscr{B}_{\nu\rho]\,M} + \ldots$, with the ellipses denoting Chern-Simons type couplings whose explicit form will not be relevant here. Similarly for $E_{7(7)}$

$$d\mathscr{L}_{\text{top}} \propto 24\,(\mathbb{T}^{\alpha})_{M}{}^{N}\mathscr{F}^{M}\wedge\partial_{N}\mathscr{H}_{\alpha} + \mathscr{F}^{M}\wedge\mathscr{H}_{M}\,, \tag{2.93}$$

with similar definition for $\mathscr{H}_{\alpha}$. Finally, the potential term $V(g,\mathscr{M})$ in (2.77) is built from bilinears in internal derivatives. It takes the universal form for both theories

$$\begin{aligned} V(g,\mathscr{M}) \;=\; & -\frac{1}{4\alpha_{n}}\mathscr{M}^{MN}\partial_{M}\mathscr{M}^{KL}\,\partial_{N}\mathscr{M}_{KL} + \frac{1}{2}\mathscr{M}^{MN}\partial_{M}\mathscr{M}^{KL}\partial_{L}\mathscr{M}_{NK} \\ & - \frac{1}{2}\,g^{-1}\partial_{M}g\,\partial_{N}\mathscr{M}^{MN} - \frac{1}{4}\mathscr{M}^{MN}g^{-1}\partial_{M}g\,g^{-1}\partial_{N}g - \frac{1}{4}\mathscr{M}^{MN}\partial_{M}g^{\mu\nu}\partial_{N}g_{\mu\nu}\,. \end{aligned} \tag{2.94}$$

Gauge invariance of the action (2.77) requires the so-called section constraint, expressed as a condition bilinear in internal derivatives

Let us further discuss the section constraints (2.83) and (2.84). Both can be solved in two different ways, allowing to recover 11d Supergravity and IIB Supergravity. In order to find the former,

one has to break

$$\begin{aligned} &\mathrm{E}_{6(6)} \supset \mathrm{SL}(6) \times \mathrm{SL}(2) \supset \mathrm{SL}(6) \times \mathrm{GL}(1)_{11}\,, \\ &\mathbf{27} \longrightarrow (6,2) + (15,1) \longrightarrow \mathbf{6}_{+1} + \mathbf{15}'_{\mathbf{0}} + \mathbf{6}_{-\mathbf{1}}\,, \end{aligned} \tag{2.95}$$

$$\begin{aligned} &\mathrm{E}_{7(7)} \supset \mathrm{SL}(7) \times \mathbb{R}\,, \\ &\mathbf{56} \longrightarrow \mathbf{7}_{+3} + \mathbf{21}'_{+1} + \mathbf{21}_{-1} + \mathbf{7}'_{-3}\,, \end{aligned} \tag{2.96}$$

and restrict the coordinate dependence of all fields to $\mathbf{6}_{+1}$ coordinates in the case of $\mathrm{E}_{6(6)}$ and to $\mathbf{7}_{+3}$ in the case of $\mathrm{E}_{7(7)}$. Upon this choice, the Lagrangians of ExFT become equivalent to full eleven-dimensional Supergravity. In a similar way, type IIB Supergravity is recovered by breaking

$$\begin{aligned} &\mathrm{E}_{6(6)} \supset \mathrm{SL}(5) \times \mathrm{SL}(2) \times \mathrm{GL}(1)_{\mathrm{IIB}}\,, \\ &\mathbf{27} \longrightarrow (5,1)_{+4} + (5',2)_{+1} + (10,1)_{-2} + (1,2)_{-5}\,, \end{aligned} \tag{2.97}$$

$$\begin{aligned} &\mathrm{E}_{7(7)} \supset \mathrm{SL}(6) \times \mathrm{SL}(2) \times \mathbb{R}\,, \\ &\mathbf{56} \longrightarrow (6,1)_{+2} + (6',2)_{+1} + (20,1)_{0} + (6,2)_{-1} + (6',1)_{-2}\,, \end{aligned} \tag{2.98}$$

and restrict internal coordinate dependence to $(5,1)_{+4}$ for $\mathrm{E}_{6(6)}$ and to $(6,1)_{+2}$ for $\mathrm{E}_{7(7)}$.

The explicit map of the ExFT fields (2.75) into the fields of ten- and eleven-dimensional Supergravity has been worked out in [83, 100]. Here, we just note that the internal part g_{mn} of the higher-dimensional metric can be straightforwardly identified within the components of the matrix $\mathscr{M}^{MN}$ according to

$$\mathscr{M}^{MN}\,\partial_M \otimes \partial_N = \begin{cases} (\det g)^{-1/3}\, g^{mn}\,\partial_m \otimes \partial_n\,, & \mathrm{E}_{6(6)}\,, \\ (\det g)^{-1/2}\, g^{mn}\,\partial_m \otimes \partial_n\,, & \mathrm{E}_{7(7)}\,, \end{cases} \tag{2.99}$$

where indices m, n label the derivatives along the physical coordinates embedded into the ∂_M according to (2.95) (2.96) (2.97) and (2.98).

E Holography and AdS/CFT correspondence

As we have seen through various examples throughout the introduction, unifying interactions or concepts has proven to be a powerful tool in theoretical physics. We observed how Kaluza-Klein reductions unified geometry and gauge theories, and how dualities related seemingly distinct theories, such as String Theories in $10d$ and their low-dimensional reductions. Now, we introduce another powerful unification concept that has emerged in the last 30 years: holography. Holography asserts the equivalence between a gravitational theory and a Quantum Field Theory without gravity. This duality is best understood in the context of what is now known as the AdS/CFT correspondence: a String Theory living on an AdS background is entirely equivalent to a Conformal Field Theory (CFT) residing on the flat Minkowski boundary of the AdS space, see Figure 2.16. This is profoundly intriguing and counterintuitive at first glance, as String Theories

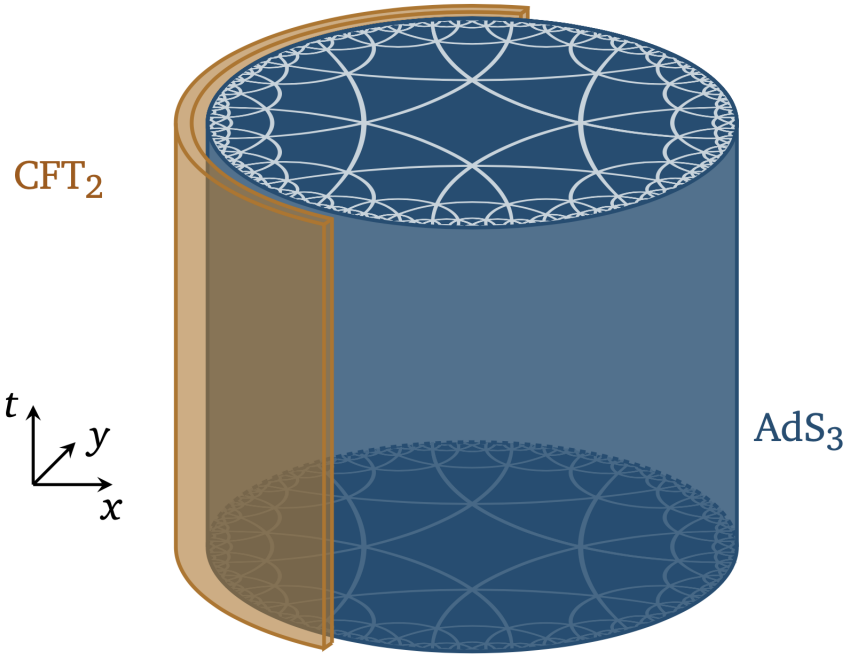


Figure 2.16: Schematic representation of the AdS_3/CFT_2 correspondence [101]. A gravity theory on the AdS (blue) background described by a cylinder and a Poincaré disk, is equivalent to a Conformal Field Theory defined on the boundary (orange) of the AdS spacetime.

are promising candidates for Quantum Gravity, yet this correspondence links them to a theory devoid of gravity altogether. This in particular means that the number of degrees of freedom for those gravity theories do not scale with the dimension of the bulk space $\mathcal{M}$, but rather with the dimension of its boundary $\partial\mathcal{M}$.

A CFT is a QFT which Lorentz invariance is supplemented with conformal invariance. It can be formulated with the metric, for which we have

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \rightarrow ds'^2 = \Lambda(x)g_{\mu\nu}dx^\mu dx^\nu \tag{2.100}$$

with Λ conformal factor. For $\Lambda(x) = 1$, the group generated by the transformation (2.100) is nothing but the Poincaré group. Geometrically, conformal symmetries are the symmetries that preserve not the norm of a vector x^μ but the angles between two vectors, see Figure 2.17.

The first appearance of the AdS/CFT correspondence [41] was in the context of type IIB String Theory. It states the equivalence of the latter with $\mathcal{N} = 4$ Super-Yang Mills (SYM) theory in $(3+1)$-dimensions, and gauge group $SU(N)$ and is formulated in Table 2.3. In this strongest form, the AdS/CFT correspondence states the complete equivalence between IIB String Theory on $AdS_5 \times S^5$ background with $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$, which is indeed conformaly invariant.

This conjecture is very interesting and brings some new insights on Quantum Gravity; however, this version is usually very difficult to study in practice. It can indeed be very dauting and difficult to conduct explicit calculations in String Theories or Supersymmetric Yang-Mills theories with arbitrary parameters. With this noted, we can refine the conjecture and specialize it. If we introduce the 't Hooft coupling $\lambda = g_{YM}^2 N$, we can explore various limits of the AdS/CFT corre-

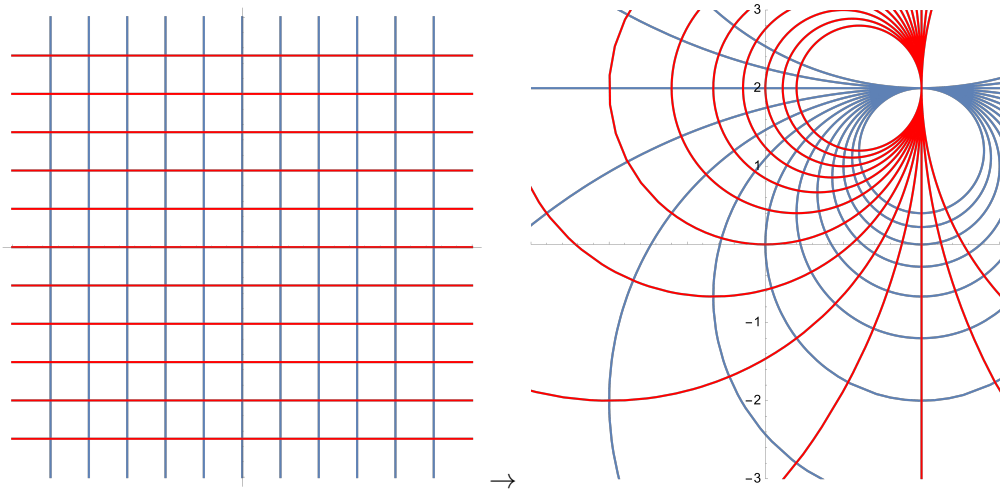


Figure 2.17: Example of a conformal dimensions in $2d$ Euclidian space.

AdS_5/CFT_4 correspondence

II B Superstring Theory with string length $\ell_s = \sqrt{\alpha'}$ on $AdS_5 \times S^5$ with radius L and N units of $F_{(5)}$ on S^5

$$\Leftrightarrow$$

$\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $SU(N)$ and Yang-Mills coupling g_{YM}

where we have the following map between the different free parameters

$$g_{YM}^2 = 2\pi g_s\,, \quad 2g_{YM}^2 N = \frac{L^4}{\alpha'^2} \tag{2.101}$$

Table 2.3: Statement of the AdS/CFT correspondence [102], and how the different parameters of the two theories are related to one another.

spondence. If we consider the limit in which the String Theory is weakly coupled, i.e., $g_s \ll 1$, then we recover the classical limit of String Theory. Additionally, if we keep $L/\sqrt{\alpha'}$ constant, or equivalently keep the 't Hooft coupling constant, this implies taking a large N limit. This limit is known as the 't Hooft limit and corresponds to the planar limit of the gauge theory, where we can realize a $1/N$ expansion. Therefore, we conclude that the $1/N$ expansion on the gauge theory side corresponds to a genus expansion in g_s on the String Theory side, with $1/N \propto g_s$ and fixed λ.

If we further take the limit where $\lambda \to +\infty$ on the field theory side, this corresponds on the String Theory side to $\sqrt{\alpha'}/L \to 0$. In this regime, stringy effects disappear, and one recovers the rigid limit of String Theories: Supergravities. In that sense, the AdS/CFT correspondence is sometimes referred to as a strong/weak duality: the gauge theory is strongly coupled with $\lambda \to +\infty$, whereas the Supergravity is weakly coupled on an $AdS_5 \times S^5$ background. The different regimes of the duality are summarized in Table 2.4 .

The different forms of the AdS_5/CFT_4 correspondence

	$\mathcal{N}=4$ SYM Theory	IIB ST on $AdS_5 \times S^5$
Strongest form	any N, λ	Quantum ST with $g_s \neq 0, \alpha'/L^2 \neq 0$
Intermediate form	$N \to \infty$, any fixed λ	Classical ST with $g_s \to 0, \alpha'/L^2 \neq 0$
Weak form	$N \to \infty, \lambda \to \infty$	Classical SUGRA with $g_s \to 0, \alpha'/L^2 \to 0$

Table 2.4: The different regimes of the AdS/CFT conjecture with the different limits for the various parameters [102].

Despite being originally a conjecture formulated between a $5d$ gravity theory and a $4d$ conformal theory, it is thought to hold for any dimensions $1 \geq d \geq 6$, and takes the form AdS_{d+1}/CFT_d. There are examples in the litterature of realization of this conjecture, or at least tests of the conjecture in other dimensions [103–105], and also in $5d$ with another internal space than S^5 [106].

The first step to check the AdS/CFT correspondence, is to make sure the symmetries agree on both sides. The symmetry of $\mathcal{N}=4$ SYM with $SU(N)$ gauge group is made of several parts. First the field theory is conformal which in $4d$ gives an $SO(4,2)$ group. Furthermore the theory preserves $\mathcal{N}=4$ supersymmetries, which means that there are 4 supercharges Q^I_α with $I = 1, \ldots, 4$ and $\alpha = 1, \ldots, 4$ the spinorial index in $4d$. Those supercharges are supplemented by superconformal generators S^I_α due to the presence of the conformal symmetry. Altogether this form an $SU(2,2|4)$ symmetry group. On the AdS side, type IIB Superstring Theory on $AdS_5 \times S^5$ is made of the following symmetries. First the theory is invariant under $SO(4,2)$, the isometries of AdS_5, and $SO(6)$, the isometries of S^5. The theory preserves $\mathcal{N}=8$ supersymmetries. Combining this with the isometries, it can be shown that the total theory in invariant under $SU(2,2|4)$

Let us further check the ideas of the AdS/CFT correspondence for the weakest form of the conjecture. On the AdS side, we have IIB Supergravity on $AdS_5 \times S^5$. The field content consist of supermutiplets organized by a $\mathcal{N}=8$ supersymmetry. The fields sit in representation $SU(2,2|4)$, and in particular are organized according to the isometry group of S^5, $SO(6)$. The full field content of the theory is summarized in Table 2.5 . On the CFT side, the elementary bosonic fields are a vector field A_μ and scalars ϕ^i, with $i = 1, \ldots, 6$ labeling the representation of the $SU(4)_R$ R-symmetry. The latter corresponds to the rotation of the supercharges Q^I, $I = 1, \ldots, 4$. For the AdS/CFT correspondence, the operators which are involved in the dictionary are composite operators. In the case of the weakest form of the conjecture, those are the so-called 1/2-BPS or chiral primary operators $\mathcal{O}_\Delta(x)$. For $SU(N)$ $\mathcal{N}=4$ SYM in $4d$, those operators belong to the representation $[0, \Delta, 0]$ of $SU(4)$. They can be expressed in terms of the elementary fields

$$\mathcal{O}_\Delta(x) = C^\Delta_{i_1 \ldots i_\Delta} \mathrm{Tr}(\phi^{i_1}(x) \ldots \phi^{i_\Delta}(x)) \tag{2.102}$$

with $C^\ell_{i_1 \ldots i_\ell}$ a totally symmetric tensor. Upon identifying $\Delta = \ell$, the fields dual to those operators are the fluctuations s^ℓ on the Supergravity side that sit in the $[\ell, 0, 0]$, as they belong to the same representation of $SO(6)$ upon identification of $SO(6) \simeq SU(4)$.

Let us now have a look at the dynamics of those fields. It can be shown that the low dimen-

Δ	
$\ell+2$	$[\ell+2,00](0\,0)$
$\ell+\frac{5}{2}$	$[\ell+1,10](0\,\frac{1}{2})+[\ell+1,01](\frac{1}{2}\,0)$
$\ell+3$	$[\ell,02](0\,0)+[\ell,20](0\,0)+[\ell+1,00](0\,1)+[\ell+1,00](1\,0)+[\ell,11](\frac{1}{2}\,\frac{1}{2})$
$\ell+\frac{7}{2}$	$[\ell,10](0\,\frac{1}{2})+[\ell-1,12](0\,\frac{1}{2})+[\ell,01](\frac{1}{2}\,0)+[\ell-1,21](\frac{1}{2}\,0)+[\ell,01](\frac{1}{2}\,1)+[\ell,10](1\,\frac{1}{2})$
$\ell+4$	$2\cdot[\ell,00](0\,0)+[\ell-2,22](0\,0)+[\ell-1,02](0\,1)+[\ell-1,20](1\,0)+2\cdot[\ell-1,11](\frac{1}{2}\,\frac{1}{2})+[\ell,00](1\,1)$
$\ell+\frac{9}{2}$	$[\ell-1,10](0\,\frac{1}{2})+[\ell-2,12](0\,\frac{1}{2})+[\ell-1,01](\frac{1}{2}\,0)+[\ell-2,21](\frac{1}{2}\,0)+[\ell-1,01](\frac{1}{2}\,1)+[\ell-1,10](1\,\frac{1}{2})$
$\ell+5$	$[\ell-2,02](0\,0)+[\ell-2,20](0\,0)+[\ell-1,00](0\,1)+[\ell-1,00](1\,0)+[\ell-2,11](\frac{1}{2}\,\frac{1}{2})$
$\ell+\frac{11}{2}$	$[\ell-2,10](0\,\frac{1}{2})+[\ell-2,01](\frac{1}{2}\,0)$
$\ell+6$	$[\ell-2,00](0\,0)$

Table 2.5: $\frac{1}{2}$-BPS supermultiplets $\mathcal{B}_{[\ell,0,0]}$ of SU(2,2|4) in SO(6) $\times$ SO(4) notation $[k_1,k_2,k_3](j_1,j_2)$ with Dynkin labels k_i, and (j_1,j_2) denoting the spins of SO(4) $\sim$ SU(2) $\times$ SU(2). Δ is the conformal dimension associated to the field on the CFT side of the correspondence.

sional Lagrangian for the s^ℓ fields is [107–109]

$$S(s)=\frac{4N^2}{(2\pi)^5}\int \mathrm{d}^5x\sqrt{g_{\text{ext}}}\sum_I \frac{32\ell(\ell-1)(\ell+2)}{\ell+1}\Big(-\frac{1}{2}\nabla_\mu s^I\nabla^\mu s^I-\frac{1}{2}\ell(\ell-4)(s^I)^2\Big) \tag{2.103}$$

from which we can read off the mass by reintroducing the radius of AdS

$$m^2L^2=\ell(\ell-4)=\Delta(\Delta-4). \tag{2.104}$$

The sum on the I harmonic index labelling the $[\ell,0,0]$ representation here means

$$\sum_I s^I s^I=\sum_{i_1,\ldots,i_\ell} s^{((i_1\ldots i_\ell))}\,. \tag{2.105}$$

This gives a one-to-one correspondence between the conformal dimension Δ of the composite operator $\mathcal{O}^\Delta$ and the mass of scalar fields s^ℓ. In general, this relation, and the identification $\Delta=\ell$ we have made above, follows from a boudary analysis on AdS [110]. Here the boundary analysis matches the results from IIB Supergravity and $\mathcal{N}=4$ SYM.

This kind of relation between the conformal dimension on the CFT side and the masses on the AdS side, can be extended to fields with any spin, and to any dimension. For example, the massless graviton field is mapped to energy-momentum tensor, and massless vectors to currents of the R-symmetry. The relations between conformal dimensions and masses is summed up in Table 2.6.

The correspondence between the operators and the fields can be made more precise. The correspondence states that the asymptotic value of the bulk field plays the role of a source for the

Masses to conformal dimensions relations

Type of fields	Relation between m and Δ
scalars and massive spin 2	$m^2L^2 = \Delta(\Delta - d)$
massless spin 2	$m^2L^2 = 0\,, \Delta = d$
p-form	$m^2L^2 = (\Delta - p)(\Delta + p - d)$
spin 1/2 and 3/2	$\lvert m\rvert L = \Delta - d/2$
rankssymmetric traceless tensor	$m^2L^2 = (\Delta + s - 2)(\Delta - s + 2 - d)$

Table 2.6: Dictionary between conformal dimensions and masses for various fields in the context of $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence [102].

operator on the boundary CFT. This can be phrased in term of the generating functionals of both side. Let $\phi_{(0)}$ be the boundary value of an AdS field ϕ and $\mathcal{S}$ the action on the CFT. Then, the modified action with a source term for the dual operator $\mathcal{O}$ of ϕ can be used to compute the connected Green's functions of the operator $\mathcal{O}$ through

$$Z[\phi_{(0)}] = \left\langle \exp\Big(\int \mathrm{d}x^d \phi_{(0)}(x)\mathcal{O}(x) \Big) \right\rangle_{\mathrm{CFT}}\,, \tag{2.106}$$

with x the coordinates on the CFT. On the AdS side, the action for $\phi(z,x)$, $\mathcal{S}_{\mathrm{sugra}}[\phi]$, can be derived via Kaluza-Klein procedure. For example, for s^ℓ on $\mathrm{AdS}_5 \times \mathrm{S}^5$ for IIB Supergravity, at the quadratic order it take the form (2.103).

The AdS/CFT conjecture states that the $\mathcal{S}_{\mathrm{sugra}}[\phi]$ is precisly the generating functional for connected Green's functions of the composite operator $\mathcal{O}$

$$W[\phi_{(0)}] = \mathcal{S}_{\mathrm{sugra}}[\phi]\Big|_{\lim\limits_{z\to 0}\phi(z,x)z^{\Delta-d}=\phi_{(0)}(x)}\,, \tag{2.107}$$

with z denoting the extra AdS coordinate and where we took the metric for the AdS space

$$\mathrm{d}s^2 = \frac{L^2}{z^2}(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu)\,, \tag{2.108}$$

hence the boundary limit is indeed $z \to 0$. The $z^{\Delta-d}$ in (2.107) takes care of the asymptotical behavior of the field ϕ. Using (2.106) and (2.107), we can compute the n-point functions of composite operators $\mathcal{O}_i$ by taking functional derivatives of the generating functional W with respect to the sources $\phi^i_{(0)}$

$$\Big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\Big\rangle_{\mathrm{CFT}} = -\frac{\delta^n W}{\delta\phi^1_{(0)}(x_1)\dots\delta\phi^n_{(0)}(x_n)}\bigg|_{\phi^i_{(0)}=0}\,. \tag{2.109}$$

In principle, this allows us to compute any n-point functions on the CFT side of operators, if we can identify the dual field. However, there are divergences due to the asymptotic behavior. In order to take care of them, we need to regularize the theory through a procedure called holographic renormalization [111–113].

In the example of the AdS/CFT correspondence between IIB Supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM, the number of symmetries on both side is high. We can ask if the correspondence holds in less symmetrical setups. For example, $\mathcal{N} = 4$ SYM can be deformed by relevant or marginal operators, that trigger an Renormalisation Group (RG) flow. In general those deformations break part of the symmetry, and the relevant deformations break the conformal symmetry. These RG flows interpolate between an ultraviolet (UV) fixed point, and an infrared (IR) fixed point. A question that the AdS/CFT correspondence tries to address, is what is the equivalent of these RG flows on the gravity side. In general, the RG flow is dual to a domain-wall solution of the gravity equations, and the RG flow is said to be holographic. The domain-wall interpolates between two stationary points of the gravity potential. In that case, the correspondence between n-point functions and couplings (2.109) is thought to hold at every step along the flow. In (3.F) we compute the domain wall dual to an RG flow in the case of AdS_4/CFT_3.

Let us for the moment forget about Superstring Theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM, to focus on the simpler example of a scalar field ϕ living in $d+1$ dimensions. We will use it to show how the asymptotic analysis works. We assume that the asymptotic value of the field ϕ sources an operator $\mathcal{O}$ on the CFT living on the boundary of the AdS space. Consider the action

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}z\mathrm{d}^d x\sqrt{-g}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2)\,, \tag{2.110}$$

with $m, n, \ldots = 0, \ldots, d$. The equation of motion for the action (2.110) are reduced to the Klein-Gordon equation

$$(\Box_{\mathrm{AdS}} - m^2)\phi = 0\,. \tag{2.111}$$

We enforce the metric to be AdS and express it in the so-called Poincaré coordinates

$$\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right)\,, \tag{2.112}$$

with $\mu, \nu, \ldots = 0, \ldots, d-1$, for which the AdS boundary is located at $z \longrightarrow 0$. Within this set of coordinates, the Klein-Gordon equation explicitely becomes

$$\frac{1}{L^2}\left(z^2\Box_{\mathrm{Mink}}\phi(z,\vec{x}) + z^2\partial_z^2\phi(z,\vec{x}) + (1-d)z\partial_z\phi(z,\vec{x})\right) = 0\,. \tag{2.113}$$

We can look for solutions of this equations which are plane-waves in the x^μ coordinates such that $\phi(z,\vec{x}) = e^{ip^\mu x_\mu}\phi_p(z)$. By looking at the asymptotic value of the field around $z \longrightarrow 0$, we see that there are two inequivalent solutions to (2.113)

$$\phi_p(z) \simeq z^{\delta_+}\,, \quad \phi_p(z) \simeq z^{\delta_-} \tag{2.114}$$

with $\delta_\pm$ the two roots of $\delta(\delta - d) = m^2L^2$. It remains to identify δ with the conformal dimension Δ of the operator $\mathcal{O}$. To do so, one could compute the two-points function of the latter. From the conformal symmetry, we know that the two-points function should be proportional to

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle \propto |x-y|^{-2\Delta}\,. \tag{2.115}$$

After some computations, for which we refer to [110], it can be shown that the two point functions

for $\mathcal{O}$ comes with the form (2.115), with $\Delta = \delta$. This explicitely demonstrates that

$$\Delta(\Delta - d) = m^2 L^2 \,. \tag{2.116}$$

F Content of the Thesis

We are now done with the general introduction. The rest of this thesis is organized as follows. In the next Chapter, we will study how masses emerge in a compactification process. The first section will serve as a warm-up where we will explore more ideas of Kaluza-Klein compactifications and in particular how we can obtain spectra in the low-dimensional theory through brute force calculation and making use of the power of group theory. In the second section, we will introduce recently developed techniques that make use of Exceptional Field Theories. Those techniques allow to access entire Kaluza-Klein mass spectra around backgrounds with low symmetry remaining. We will then extend these techniques to a broader range of vacua, which we will refer to as beyond consistent truncation. The two following sections of the third Chapter will be used to illustrate these techniques: we will compute the spectra of the squashed seven-sphere S^7 for 11-dimensional Supergravity and then find a domain-wall solution interpolating between the round and squashed S^7 of 11-dimensional Supergravity.

Chapter 4 will extend the study to n-point functions. Once again, the first section of this chapter will review how cubic and quartic couplings have been obtained in the literature so far through brute force calculations. Despite their considerable success, these techniques suffer from severe limitations. In the subsequent section, we will explore how ExFT techniques can be employed to access these couplings, similar to our approach in Chapter 3 for spectra. We will demonstrate the capabilities of ExFT technology in revealing underlying structures and how it provides a natural framework for computing couplings. Subsequently, we will focus on $E_{6(6)}$-ExFT, which is specialized for reductions to 5 dimensions. We will compute 3-point couplings in this theory and utilize the following section to apply these results to the $AdS_5 \times S^5$ background of IIB 10-dimensional Supergravity.

CHAPTER

3

KALUZA-KLEIN SPECTROMETRY USING EXFT

Contents

A Introduction

COMPACTIFICATIONS are ubiquitous in String Theories and their low-energy limit, 10-dimensional Supergravities, as well as in 11-dimensional Supergravity. Compactification is a process wherein the total manifold is factored into an external and an internal manifold, the latter of which is required to be compact. The original motivation is, by sufficiently shrinking the characteristic radius of the internal manifold, to make the internal space unobservable in daily observations. One of the hallmarks of the compactification is the appearance of infinitely many massive fields in the lower-dimensional theory. These fields are organised into so-called Kaluza-Klein (KK) towers: zero modes sit at the bottom of a tower and other fields are organized on top of them. From the lower-dimensional perspective, these KK towers encode the information about the geometry and fluxes of the compactification. The Kaluza-Klein modes, i.e. the fields constituting the towers, also play an important role in applications. For example, for non-supersymmetric backgrounds, their masses, denoted Kaluza-Klein masses, determine their perturbative stability. Moreover, in AdS vacua, the Kaluza-Klein masses encode the anomalous conformal dimensions of single-trace operators in the holographically dual CFTs, as in Table 2.6.

Despite their importance, computing the Kaluza-Klein masses is a formidable challenge for most String/Supergravity compactifications. Until recently, the full tower of Kaluza-Klein masses could only be computed in few examples, for coset spaces with large isometry group [114]. Beyond this, there were some techniques to access subsets of the Kaluza-Klein towers. For example, if the compactification admits a consistent truncation to a lower-dimensional Supergravity theory, then the Kaluza-Klein masses of the (finitely many) fields kept in the truncation can be computed in the lower dimensional theory. However, for generic compactifications, a systematic method, althought not always easy to implement and solve, was accessible only for spin-2 towers. [115].

A new method for computing all the Kaluza-Klein masses for a large class of String compactifications was presented in [99, 116]. This method uses the formalism of Exceptional Field Theory (ExFT), a reformulation of 10 and 11-dimensional Supergravities that unifies the gravitational and flux degrees of freedom, and as a result makes manifest an exceptional symmetry group [117]. Using ExFT, [99, 116] showed how to compute the full Kaluza-Klein spectrum of any vacuum that can be uplifted from a consistent truncation to maximal gauged Supergravity. Unlike the traditional Kaluza-Klein technology, which requires solutions of the eigenvalue spectrum of various internal Laplace operators acting on tensorial harmonics, together with a diagonalisation of the coupled system of higher-dimensional fluctuations, the ExFT method is exclusively based on the tower of scalar harmonics on the internal manifold. The relevant internal vector and tensor harmonics are implicitly taken care of by combining the scalar harmonics with the twist matrix encoding the underlying consistent truncation. Furthermore, to a large extent, the diagonalisation problem has already been resolved by the covariant formulation of ExFT. As a result, the lower dimensional mass spectrum can be computed separately KK level by KK level.

These techniques, for the first time, have given access fo the full Kaluza-Klein spectrum for compactifications with few or no remaining (super-)symmetries [99, 116, 118–130] and led to many interesting insights. For supersymmetric AdS vacua, the protected part of the Kaluza-Klein spectrum can be matched with the superconformal index of the CFT, as was done for the Pilch-Warner AdS_5 vacuum in [122] and for the $SU(3) \times U(1)$-invariant AdS_4 vacuum in $\mathcal{N} = 8$ SUGRA in [99, 131]. Moreover, the Kaluza-Klein spectrum can be used to determine compactness of the conformal manifold, which may not be visible in the consistent truncation [126, 127], see also [132]. For

non-supersymmetric vacua that are perturbatively stable within the consistent truncation, [118] showed that instabilities can be nonetheless triggered from higher Kaluza-Klein modes. Finally, the Kaluza-Klein spectroscopy can also be used to prove the existence of perturbatively stable non-supersymmetric AdS vacua in 10-dimensional Supergravity [120, 126].

Despite these successes, one drawback of the method of [99, 116] is that it only applies to vacua that can be uplifted from $\mathcal{N} = 8$ Supergravity via a consistent truncation. This means that these vacua arise by deforming round spheres/hyperboloids by the scalar fields that live in the lower-dimensional $\mathcal{N} = 8$ supermultiplet. Yet there are many interesting vacua that arise from deformations by scalar fields that are not part of the $\mathcal{N} = 8$ truncation, such as AdS vacua obtained by TsT deformations [133–135] or the $AdS_4 \times S^7_{\text{squashed}}$ vacua with $\mathcal{N} = 1$ and $\mathcal{N} = 0$ supersymmetry [136, 137]. The difficulties in computing spectra for the latter have been overcome in [DB1].

In this chapter, we will go throught different techniques to compute KK spetra. After a short warm up with a toy model, we will compute the entire spectrum in the old fashion on $AdS_4 \times S^7_{\text{round}}$. This can be done by a brute force calculation within $d = 11$ Supergravity, using the power of group theory. We will then introduce the ExFT technology for spectroscopy and give a flavor on how it works. Then we compute spectra beyond consistent truncation of the maximal theory. To this end, we will present the original results from [DB1], which will start in 3.D. As a illustration of those techniques, we will compute the full Kaluza-Klein spectra of $11d$ Supergravity on $AdS_4 \times S^7_{\text{squashed}}$. In the latter theory, only $\mathcal{N} = 1$ supersymmetry is preserved and the isometry group of the internal manifold is reduced to $USp(4) \times SU(2) \subset SO(8)$. This small amount of symmetry explains why the answer for the spectrum was incomplete so far. We will also give the spectrum of the cousin theory with $\mathcal{N} = 0$ related by changing the sign of the flux, a procedure often referred as skew-whiffing. Following this analysis, we will establish the domain-wall solution interpolating between the $AdS_4 \times S^7_{\text{round}}$ and $AdS_4 \times S^7_{\text{squashed}}$ solutions of $11d$ Supergravity in 3.F, firstly presented in [DB2]. According to the AdS/CFT correpondence, this domain wall solution is dual to an holographic Renormalization Group (RG) flow between two CFTs. This will allow us to extend the previous results and compute quadratic couplings around the domain wall solution dual to information on 2-point correlation functions along the flow. Stated differently, this example illustrates that not only can we compute spectra around solutions with few symmetries, but it allows us to calculate quadratic couplings for spaces which are no longer AdS solutions of the equations of motion.

B Warm up

B.1 Toy model

As previously mentioned, the exploration of the mass spectrum is an almost unavoidable task when including extra dimensions that are compactified. In higher-dimensional theories, each field is responsible for generating an infinite tower of massive states in the lower-dimensional theory. We illustrated this with the simple example of a scalar field on $\mathbb{R}^{1,d} \times S^1$. From this example, obtaining the masses seems to be a simple problem. We just had to Fourier expand the field and obtaining the modes and masses was almost given for free as n^2/L^2. However, as soon as the set up becomes more complicated, the calculations get more involved and obtaining the full Kaluza-Klein spectrum becomes a very tedious task. Consider, for example a more complicated

scenario where, instead of compactifying only one dimension from $d+1$ to d dimensions, we begin with a $d+n$ dimensional theory and compactify n dimensions. In such a scenario for a scalar field ϕ with mass m, the equation of motion reverts to the usual Klein-Gordon equation in $d+n$ dimensions

$$\Box\phi - m^2\phi = 0\,. \tag{3.1}$$

We now decompose the Laplacian into an internal operator and an external one. This decomposition is feasible when the total manifold can be expressed as the direct product of an internal part and an external part, denoted as $\mathcal{M}_{d+n} = \mathcal{M}_d \times \mathcal{M}_n$. We then have

$$\Box_x\phi + \Box_{\text{int}}\phi - m^2\phi = 0\,, \tag{3.2}$$

where $\Box_x$ denotes the Laplacian on $\mathcal{M}_d$ and $\Box_{\text{int}}$ the Laplacian on $\mathcal{M}_n$. If we want to find the d-dimensional mass of the field, we see that we must find eigenvalues and eigenvectors of the operator $\Box_{\text{int}}$. If we have a basis of functions diagonalising the internal Laplacian, with states that can be labelled by Σ, then we can expand the scalar field on such a basis $\phi(x,y) = \sum \phi^\Sigma(x)\mathcal{Y}^\Sigma(y)$ and obtain

$$\sum_\Sigma \left(\Box_x\phi^\Sigma(x) - m_\Sigma^2\phi^\Sigma(x) - m^2\phi^\Sigma(x)\right)\mathcal{Y}^\Sigma(y) = 0\,, \tag{3.3}$$

where m_Σ^2 are nothing but eigenvalues of associated the the eigenstate $\mathcal{Y}^\Sigma$ of $\Box_{\text{int}}$. We can now read off the new masses of the sets of fields in the lower dimensional theory. $\mathcal{Y}^\Sigma$ is here playing the role of the $(\exp(2i\pi ny/L))$ for S^1. Here again we see the emergence of an infinite number of states, as in our simple S^1 example. What further complicates the analysis is the necessity to be able to effectively diagonalize the Laplacian operator on the internal manifold. In the case of S^1, this was feasible because we know a basis of functions on the circle : Fourier modes. If, instead of S^1, we consider the round S^2, we could employ spherical harmonics for this purpose, just like we do when analyzing the hydrogen atom energy state. In fact, for any round sphere, it is possible to identify a set of basis functions and diagonalize the internal Laplacian. This is due to the large isometry group of the spheres, which can be expressed as the coset space $\frac{SO(n+1)}{SO(n)}$. Then [114] prescribes a method to derive the harmonics. However, if the number of isometries is low, obtaining a basis of functions that diagonalizes the internal Laplacian can prove exceedingly challenging or even unknown. This example demonstrates the limitations of the brute force calculation when the symmetry of the theory diminishes. Additional complexity may arise in the presence of coupled equations of motion. In such cases, identifying the appropriate basis of fields to diagonalize the set of equations of motion may require field redefinitions. Even with a small number of fields, this process can be intricate.

B.2 Example of spectrum : the round S^7

In the introduction, we presented the idea of a Freund-Rubin compactification that can be used in the presence of extra form fields. Using an appropriate ansatz, a solution to the equations of motion can be found, where gravitational and matter effects compete. This led to a solution, where the total manifold was expressed as the product of an external AdS manifold and an internal manifold with positive curvature. In this section, we will use this technique on $11d$ Supergravity and study it on $AdS_4 \times S^7$ background to illustrate the state of the art until recently for computing KK masses around a given background. This spectrum was first derived in [138–140]

We can use the Freund-Rubin ansatz to compactify 11-dimensional Supergravity down to $d = 4$, as it incorporates a three-form potential. As it is imposed by (2.67), the internal space must have positive curvature. We will further demand it is maximally symmetric hence we end up with the round S^7, which possesses the highest number of isometries among all possible 7-dimensional candidates [141]. We will therefore study $11d$ Supergravity on $AdS_4 \times S^7$ background. Equations of motion for $11d$ Supergravity are

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3} F_{MPQR} F_N{}^{PQR} - \frac{1}{24} g_{MN} F_{PQRS} F^{PQRS} , \tag{3.4}$$

$$\nabla_M F^{MNPQ} = -\frac{1}{576} \epsilon^{M_1 \dots M_8 NPQ} F_{M_1 \dots M_4} F_{M_4 \dots M_8} . \tag{3.5}$$

The solution in this case is similar to (2.66) and (2.67), except that here we are working in what we called the "electric" framework in section 2.C.4, where the form fills in the external space. The 4-form is then given by

$$\begin{cases} F_{\mu\nu\rho\sigma} = Q \epsilon_{\mu\nu\rho\sigma} , \\ F_{mnpq} = 0 , \end{cases} \tag{3.6}$$

where $\mu, \nu, \dots = 0, 1, 2, 3$ are external indices, $m, n, \dots = 1, \dots, 7$ are internal indices, Q is an integer number controlling the flux, and

$$\begin{cases} R_{\mu\nu} = -\frac{Q^2}{3} \mathring{g}_{\mu\nu} \equiv -\frac{6}{L_{\text{AdS}}^2} \mathring{g}_{\mu\nu} , \\ R_{ab} = \frac{Q^2}{6} \mathring{g}_{mn} \equiv \frac{3}{L_{S^7}^2} \mathring{g}_{mn} , \\ R_{m\mu} = 0 , \end{cases} \tag{3.7}$$

for the Ricci tensor. Here we defined the AdS radius L_{AdS} and the sphere radius L_{S^7}. Upon using the equations of motion, we can show that L_{AdS} and L_{S^7} are the same, and we define a mass scale $m = 1/L_{\text{AdS}}$ which is related to Q via $Q = 3m$.

The full compactified theory comprises a massless $\mathcal{N} = 8$ supermultiplet coupled with an infinite tower of massive $\mathcal{N} = 8$ supermultiplets. This is not obvious from the starting point, but at the end of the journey to obtain the spectrum, one can notice that fields are organised into $\mathcal{N} = 8$ supermultiplets. We note in passing that the theory described by only massless $\mathcal{N} = 8$ supermutiplet corresponds to the maximal $d = 4$ Supergravity with $SO(8)$ gauge group, described in [91]. By combining the internal $SO(8)$ gauge symmetry, $\mathcal{N} = 8$ supersymmetry, and the $SO(3, 2)$ isometries of the AdS background, we can conclude that the total symmetry of the theory is $OSp(4|8)$. Consequently, the massless and massive sectors of the theory will fit into irreducible representations of $OSp(4|8)$. The algebra and representation theory of $OSp(4|8)$ can be found, for example, in [141].

We can now start to study the mass spectrum of the theory. To process in the calculations, we add fluctuations on top of the background solution given by the Freund-Rubin compactification (3.6) and (3.7). The latter comprises a background metric $\mathring{G}_{MN}$ describing $AdS_4 \times S^7$ and a non vanishing three-form potential $\mathring{C}_{MNP}$ for which only the external part is non-vanshing. The analysis we will now present is similar to the S^1 example. We will be able in our case to extract the full spectrum thanks to powerful group theorical arguments. The fluctuation Ansatz is

Spin	SO(8) rep	Dynkin Label
2	**1**	[0,0,0,0]
3/2	$\mathbf{8}_s$	[0,0,0,1]
1	**28**	[0,1,0,0]
1/2	$\mathbf{56}_s$	[1,0,1,0]
0^+	$\mathbf{35}_v$	[2,0,0,0]
0^-	$\mathbf{35}_c$	[0,0,2,0]

Table 3.1: Massless gravity supermultiplet of 11-dimensional Supergravity compactifed on $AdS_4 \times S^7$, which aligns with the field content of $\mathcal{N} = 8$ $4d$ Supergravity. The scalar fields living in the $\mathbf{35}_v$ will be denoted 0^+ and the ones living in $\mathbf{35}_c$ will be denoted 0^-.

$$\begin{cases} G_{MN}(x,y) = \mathring{G}_{MN}(x,y) + h_{MN}(x,y)\,, \\ C_{MNP}(x,y) = \mathring{C}_{MNP}(x,y) + c_{MNP}(x,y)\,, \\ \Psi_M(x,y) = 0 + \psi_M(x,y)\,. \end{cases} \tag{3.8}$$

where $\Psi_M(x,y)$ is the gravitini of $11d$ Supergravity, and we splitted the full $11d$ coordinates into $\{x^\mu, y^m\}$ with x^μ external coordinates of the AdS space and y^m internal ones of the compact space S^7. In order to get the lower dimensional fields from the $11d$ ones, we need to split the fields as we did for $d+1$ Einstein gravity in the introduction. This gives for the metric

$$\mathrm{d}s^2 = (g_{\mu\nu}(x,y) + \dots)\mathrm{d}x^\mu \mathrm{d}x^\nu + (A_\mu{}^m \Phi_{mn}(x) + \dots)\mathrm{d}x^\mu \mathrm{d}x^n + \Phi_{mn}(x,y)\mathrm{d}x^m \mathrm{d}x^n\,. \tag{3.9}$$

for the 11d metric, giving rise to a $4d$ external metric $g_{\mu\nu}$, a number of vector fields $A_\mu{}^m$ and scalar fields encoded in Φ_{mn}, which is the internal part of the $11d$ metric. The split (3.9) recquires a little bit more explanation. The ellipses denote terms that are non linear, and which would not contribute the linearized equations of motion. The metric in $\mathrm{d}x^\mu \mathrm{d}x^\nu$ term comprises the external AdS background, as well as fluctuations, whereas the internal part of the metric $\Phi_{mn}(x)$ in the $\mathrm{d}x^\mu \mathrm{d}x^n$ term is only the background metric, as fluctuations of the metric would contribute to quadratic order because of the vector fields. Finally in the $\mathrm{d}x^m \mathrm{d}x^n$ term, the metric $\Phi_{mn}(x,y)$ appearing comprises the background and the fluctuations. We can process similarly for the 3-form that will give rises to vector fields and scalar fields. Note that the lower dimensional forms can be dualised to vectors or scalars.

Once the split to obtain low dimensional fields is done, we can continue to find the masses for KK fluctuations in (3.8). We expand the fields in harmonics of the internal space. For the scalar fields we will denote those harmoics $\mathcal{Y}^{I_1}$, for vector fields $\mathcal{Y}_m{}^{I_8}$, for two-forms $\mathcal{Y}_{mn}{}^{I_{28}}$ and so forth ... All of the harmonics are eigenfunctions of the appropriate internal operator, which eigenvalues will give the mass of the fields. Those eigenfunctions are functions of the internal coordinates. As a example, the harmonic of the scalar fields on the round S^7 will be polynomials of the embedding internal coordinates Y^i

$$(Y^1, Y^2, Y^3, Y^4, Y^5, Y^7, Y^8) = (y^1, y^2, y^3, y^4, y^5, y^6, y^7, \sqrt{1 - y^i y^i})\,, \tag{3.10}$$

Spin	Dynkin Label
2	[n,0,0,0]
3/2	[n,0,0,1]+[n-1,0,1,0]
1	[n,1,0,0]+[n-1,0,1,1]+[n-2,1,0,0]
1/2	[n+1,0,1,0]+[n-1,1,1,0]+[n-2,1,0,1]+[n-2,0,0,1]
0^+	[n+2,0,0,0]+[n-2,2,0,0]+[n-2,0,0,0]
0^-	[n,0,2,0]+[n-2,0,0,2]

Table 3.2: Field content of 11-dimensional Supergravity compactified on $AdS_4 \times S^7$. Here the 0^+ scalars are the Kaluza-Klein towers on top of the 0^+ scalars in Table 3.1, and similarly for 0^-.

with y^i the coordinates on the sphere. The harmonics transform in irreducible representations of the isometry group $SO(8)$ of the round S^7. More precisely, as the round sphere S^7 can be written as $\frac{SO(8)}{SO(7)}$, [114] tells us that I_1 runs over all representation $\mathcal{R}$ of $SO(8)$ that decomposes as $\mathcal{R} \to \mathbf{1} + \dots$ under branching to $SO(7)$. For the scalar fields, this exactly give the $[n,0,0,0]$ representations, $n \in \mathbb{N}$. Therefore, scalar harmonics live in the $[n,0,0,0]$ representations of $SO(8)$. In terms of embedding coordinates Y^i, this $[n,0,0,0]$ denotes the polynomial $\mathcal{Y}^{i_1 \dots i_n} = Y^{((i_1} \dots Y^{i_n))}$. The $((\dots))$ stands for symmetric and traceless. In other words, the order n harmonic functions for the scalar are polynomials in the embedding coordinates. Similar arguments can be used in order to find tensorial harmonics on the round sphere. For example, the vector harmonics are representations $\mathcal{R}_v$ of $SO(8)$ that decomposes as $\mathcal{R}_v \to \mathbf{vector} + \dots$.

Using these harmonics, we can now factor the dependance in the internal coordinates for the various fields

$$\begin{cases} h_{\mu\nu}(x,y) = h_{\mu\nu,I_1}(x)\mathcal{Y}^{I_1}\,, \\ h_{\mu m}(x,y) = h_{\mu,I_8}(x)\mathcal{Y}_m{}^{I_8}\,, \\ h_{mn}(x,y) = h_{I_{35}}(x)\mathcal{Y}_{mn}{}^{I_{35}}\,, \\ c_{\mu\nu\rho}(x,y) = c_{\mu\nu\rho,I_1}(x)\mathcal{Y}^{I_1}\,, \\ c_{\mu\nu m}(x,y) = c_{\mu\nu,I_8}(x)\mathcal{Y}_m{}^{I_8}\,, \\ c_{\mu mn}(x,y) = c_{\mu,I_{28}}(x)\mathcal{Y}_{mn}{}^{I_{28}}\,, \\ c_{mnp}(x,y) = c_{I_{56}}(x)\mathcal{Y}_{mnp}{}^{I_{56}}\,. \end{cases} \tag{3.11}$$

where the sum over harmonics has been made implicit. One then injects those expansions into the equations of motion (3.4) and (3.5), linearize them, and obtain a number of equations, for which the internal operators need to be diagonalized. Schematically, the result for a field Φ looks like

$$\mathcal{D}_{ext}\Phi + \mathcal{D}_{int}\Phi = 0\,, \tag{3.12}$$

where $\mathcal{D}_{ext}$ is the appropriate external Laplacian operator for the given field ($\Box$ if it is a scalar field for example as we had in (3.2)) and $\mathcal{D}_{int}$ in the operator to diagonalise on the internal space. Let us illustrate this with the example of $h^{TT}_{\mu\nu}$, being the traceless and transverse graviton. We define

the mass matrix M on an AdS background for spin 2 fieds as follow

$$\Delta_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^{\rho}h_{\nu)\rho} - \nabla_{(\mu}\nabla_{\nu)}h^{\rho}{}_{\rho} + 24m^2 h_{\mu\nu} + M^2 h_{\mu\nu} = 0\,. \tag{3.13}$$

The equation of motion for the fluctuations of the external part of the metric is descending from (3.4) upon splitting internal and external indices. This yields to the equation for the graviton

$$\begin{aligned}\Delta_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^{\rho}h_{\nu)\rho} - \nabla_{\mu}\nabla_{\nu}h^{\rho}{}_{\rho} + 2\nabla_{(\mu}\nabla^{m}h_{\nu)m} \\ = \frac{2}{3}m\mathring{g}_{\mu\nu}\epsilon^{\rho\sigma\tau\lambda}f_{\rho\sigma\tau\lambda} + 24m^2\mathring{g}_{\mu\nu}h^{\rho}{}_{\rho} - 24m^2 h_{\mu\nu}\,,\end{aligned} \tag{3.14}$$

with $f_{MNPQ} = 4\partial_{[M}c_{NPQ]}$, and Δ_L the Lichnerowicz operator defined by

$$\Delta_L h_{MN} = -\Box h_{MN} + R_{MNPQ}h^{PQ} + 2R_{(M}{}^{P}h_{N)P}\,, \tag{3.15}$$

with R_{MNPQ} the $11d$ Riemann tensor. The equation (3.14) gives an explicit expression for the mass operator M. Note that at the linear level in the equations of motion, we can compute this operator with the background metric $\mathring{G}$. We now use the following decomposition

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_{(\mu}V_{\nu)} + (\nabla_{\mu}\nabla_{\nu} + \frac{1}{4}\mathring{g}_{\mu\nu}\Delta_{4d})\phi + \frac{1}{4}\mathring{g}_{\mu\nu}h^{\mu}{}_{\mu}\,, \tag{3.16}$$

with $\mathring{g}_{\mu\nu}$ the AdS_4 background metric, V_μ a transverse vector and ϕ a scalar field. Δ is the Hodge-de-Rham operator for a scalar fields, which reduces to the $4d$ d'Alembartian operator

$$\Delta_{4d} = -\Box. \tag{3.17}$$

After injecting (3.8) and (3.11) in (3.4), the field equations for $h^{TT,I_I}_{\mu\nu}$ are

$$(\Delta_{4d}h^{TT,I_I}_{\mu\nu})\mathcal{Y}^{I_1} + h^{TT,I_I}_{\mu\nu}(\Delta_{7d}\mathcal{Y}^{I_1}) + 24m^2 h^{TT,I_I}_{\mu\nu}\mathcal{Y}^{I_1} = 0\,, \tag{3.18}$$

with Δ_k is this time the d'Alembertian operator for scalar fields in k dimensions. In order to obtain the mass for the $h^{TT,I_I}_{\mu\nu}$ field, it remains to simplify $(\Delta_{7d}\mathcal{Y}^{I_1})$. We follow the prescription of [114], and write for $S^7 = SO(8)/SO(7)$

$$\nabla_m \mathcal{Y}^{I_k}_{\text{tensor}} \propto e_m{}^{a}T_a\mathcal{Y}^{I_k}_{\text{tensor}}\,, \tag{3.19}$$

with $\mathcal{Y}^{\Sigma}_{\text{tensor}}$ a tensorial harmonic, ∇_m the covariant derivative on S^7, T_a generators of $SO(8)$ which are not in $SO(7)$ and $e_m{}^{a}$ the vielbein on S^7. We can write the previous expression because S^7 is a symmetric space on which $SO(8)$ acts transitively, and because $\mathcal{Y}^{I_k}_{\text{tensor}}$ are in irreducible representation of $SO(8)$. By squaring the expression and fixing the normalization we obtain

$$-\Box\mathcal{Y}^{I_k} = m^2(\mathcal{C}_{SO(8)} - \mathcal{C}_{SO(7)})\mathcal{Y}^{I_k}\,, \tag{3.20}$$

with $\mathcal{C}_{SO(n)}$ the quadratic Casimir operator of $SO(n)$, and the normalisation factor m^2 can be fixed through an example. It is now straighforward to extract the mass for the fields $h^{TT,I_I}_{\mu\nu}$. The Casimir

operators for SO(8) and SO(7) for the representation $[n,0,0,0]$ are given by

$$\mathcal{C}_{\mathrm{SO}(8)} = n(n+6)+1\,, \tag{3.21}$$

$$\mathcal{C}_{\mathrm{SO}(7)} = 1\,. \tag{3.22}$$

We can now read off the mass for $h_{\mu\nu}^{TT,I_I}$ from (3.18). According to the definition of the mass matrix for spin 2 field (3.13)

$$M(h_{\mu\nu}^{TT,I_I}\mathcal{Y}^{I_1}) = \Big((n+3)^2-9\Big)\,. \tag{3.23}$$

Very similar calculations would give the masses for vectors and scalars, as well as fermionic fields, involving Hodge-de Rham operators for higher rank form, but also the Lichnerowicz operator Δ_L and fermionic operators [141].

By this direct calculation, it is a priori not clear whether or not the fields will rearrange into supermultiplets. However, once the derivation of the mass spectrum is done, it can be realized that fields indeed fit into $\mathcal{N}=8$ supermultiplets. The summary of the mass spectrum for the round S^7 is given in 3.3 and one can refer to [138–141] for further details.

Δ		Mass^2
$\frac{n+2}{2}$	$[n+2,0,0,0](\Delta,0)$	$(n-1)^2-3$
$\frac{n+3}{2}$	$[n+1,0,1,0](\Delta,\frac{1}{2})$	n^2
$\frac{n+4}{2}$	$[n,0,2,0](\Delta,0)+[n,1,0,0](\Delta,1)$	$(n+1)^2-3\,\vert\,(n+1)^2-1$
$\frac{n+5}{2}$	$[n-1,1,1,0](\Delta,\frac{1}{2})+[n,0,0,1](\Delta,\frac{3}{2})$	$(n+2)^2\,\vert\,n^2$
$\frac{n+6}{2}$	$[n,0,0,0](\Delta,2)+[n-1,0,1,1](\Delta,1)+[n-2,2,0,0](\Delta,0)$	$(n+3)^2-9\,\vert\,(n+3)^2-1\,\vert\,(n+3)^2-3$
$\frac{n+7}{2}$	$[n-2,1,0,1](\Delta,\frac{1}{2})+[n-1,0,1,0](\Delta,\frac{3}{2})$	$(n+6)^2\,\vert\,(n+4)^2$
$\frac{n+8}{2}$	$[n-2,2,0,0](\Delta,0)+[n-2,1,0,0](\Delta,1)$	$(n+5)^2-3\,\vert\,(n+5)^2-1$
$\frac{n+9}{2}$	$[n-2,0,0,1](\Delta,\frac{1}{2})$	$(n+6)^2$
$\frac{n+10}{2}$	$[n-2,0,0,0](\Delta,0)$	$(n+7)^2-3$

Table 3.3: $\frac{1}{2}$-BPS supermultiplets $\mathcal{B}_{[n,0,0,0]}$ of OSp(4|8) in a notation $[k_1,k_2,k_3,k_4](\Delta,j)$ where $[k_1,k_2,k_3,k_4]$ denotes the Dinkyn label of SO(8) and (Δ,j) the representations of isometry group of AdS_4 of SO(2,3) for which Δ is the energy and j the total angular momentum.

C Kaluza-Klein spectroscopy using Exceptional Field Theory

In the previous section, we presented the traditional way to extract spectrum in KK reductions, with the example of $AdS_4\times S^7_{\mathrm{round}}$ in $11d$ Supergravity, and how we can extract the whole spectrum by direct calculation and diagonalization of internal operators. However, this method

fails at computing spectrum if the lower dimensional theories with lower symmetry. The problem of finding eigenvalues of internal operators is still well-defined, but extracting eigenvalues and identifying the origin of low dimensional fields can be more involved. For example, there exist another solution to Einstein equation in $11d$ Supergravity Freund-Rubin compactifications to $4d$ leading to another vacuum, where the internal space is now a squashed sphere. This will be the object of study of section 3.D . Despite a number of discoveries, the standard method did not give a complete answer since the 1980's at computing the whole spectrum of the theory, because the number of isometries and supersymmetries is lower on this background.

In order to overcome this problem, one can use Exceptional Field Theory. Before presenting those techniques, we want to first present a tool that will be very useful in the next sections : consistent truncations.

C.1 Consistent truncations

Let us consider the following simple Lagrangian of two real scalar fields

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2 + \lambda_{112}\phi_1^2\phi_2\,. \tag{3.24}$$

We can easily derive the equations of motion for the fields

$$\begin{cases} \Box\phi_1 + m_1^2\,\phi_1 = 2\lambda_{112}\,\phi_1\,\phi_2\,, \\ \Box\phi_2 + m_2^2\,\phi_2 = \lambda_{112}\,\phi_1^2\,. \end{cases} \tag{3.25}$$

We observe the following. If we put ϕ_1 to zero, then the first equation is trivially satisfied and the second equation reduces to the free massive Klein-Gordon equation for the field ϕ_2. Therefore it is consistent to put ϕ_1 to zero, i.e. we can consistently solve the system of equations when doing so. Let us now put $\phi_2 = 0$. If we do this, then the first equation is the free massive Klein-Gordon equation for ϕ_1, but the second equation becomes algebraic in ϕ_1 and implies that ϕ_1 should be zero. In that sense, it is not consistent to put ϕ_2 to zero and then solve the dynamical system for ϕ_1. Therefore we say that the conditions $\phi_2 \to 0$ is not a consistent truncation of the system (3.25), but $\phi_1 \to 0$ is.

Consistent truncations are really a matter of couplings : certains couplings in the theory must vanish to have a consistent truncation, otherwise, the non truncated fields will excite the truncated ones. In the previous example, because of the λ_{112} couplings, the field ϕ_2 plays the role of a source for the field ϕ_1, and therefore ϕ_2 cannot be set to zero. More precisly, the couplings that make a truncation failing at being consistent, are couplings which are linear in the truncated fields.

Let us do a final remark. Consistent truncations are deeply related to symmetries. In the above example, we have a discrete symmetry of the Lagrangian

$$\phi_1 \to -\phi_1\,, \quad \phi_2 \to \phi_2\,. \tag{3.26}$$

The presence of this symmetry ensures the consistency of the truncation $\phi_1 \to 0$, i.e. to the truncation that keep fields which are invariant under (3.26). This is a general argument: if one keeps only the fields that are invariant under a given symmetry, then this will give a consistent truncation. This comes from the fact that singlets under a given symmetry, cannont source non singlets. We

will encounter a lot of similar arguments in the rest of the thesis, for which symmetry arguments are formulated in terms of G-structures [142].

A more sophisticated example is $d+1$-dimensional Einstein gravity on $\mathbb{R}^{1,d-1}\times S^1$ that we already encountered in section (2.C.1). We saw before that the metric gives rise to a vector field and a scalar field. However, back in 1919 at the time the theory was constructed, the usual way of proceeding was to simply discard the scalar field from the theory. However the equation of motion descending from the Lagrangian (2.48) for the scalar field is

$$\Box\phi \propto F_{\mu\nu}F^{\mu\nu}. \tag{3.27}$$

Therefore, discarding the scalar field is not consistent is this example because it leads to an algebraic equation for the vector field.

This is the first example we give of a truncation in dimensional reduction, but will encounter many more, because dimensional reductions and consistent truncations are intimately related. This is true because when truncating fields, the remaining set of fields effectively describe a lower dimensional theory. This is why we will often talk about consistent truncations to a lower dimensional theory. For example, in the example of Einstein gravity in $d+1$ dimensions, reducing on S^1 and keeping only massless modes is consistent. This amounts to discard any dependence on the internal coordinates, as in (2.47). In that case, the remaining fields after truncation describre a d-dimensional theory, hence the name consistent truncation to a lower-dimensional theory.

Note that we could remove fields and then ignore some of the equations of motion in dimensional reductions. The main issue is that one loses the higher dimensional interpretation of the equations if it is done in a non consistent way, as described above with the example of Einstein gravity. A solution found in a non-consistent truncation, is not a solution anymore of the higher dimensional equations. This is the key point about consistent truncations and what make them of tremendous importance when doing Kaluza-Klein reduction. By turning around the argument, we can use consistent truncations to find solutions of the higher dimensional theory. Directly in higher dimensional theory, this can be a very hard task because of the number of fields, of the non-linearities, ... However, by restricting to a subsector of fields, it may be easier to find a solution of the set of coupled, non-linear equations of motion. Therefore, consistent truncations can be used as a toolmaker to build solutions of higher dimensional theories. This is very useful in the context of Supergravities, where we often start from $10d$ or $11d$ Supergravities.

We actually already saw a more sophisticated example of a consistent truncation in this thesis. The truncation to massless spin 2 multiplet of 11d Supergravity on $AdS_4\times S^7$ is a consistent truncation of the full theory, which corresponds to $d=4$ $\mathcal{N}=8$ Supergravity with $SO(8)$ gauge symmetry. The consistency of the truncation can be proven using G-structures arguments. This is a systematic way to construct consistent truncation in a given theory, that I will use later in this work. For now the reader can refer to [142]. Another way of proving the consistency of the truncation would be to show that a number of couplings are equal to zero, namely the couplings involving at least a massless mode and a single massive mode, as we saw in (3.25). We will provide such a proof latter in section 4.E.1 .

Let us do a final remark on consistent truncations. They are not to be confused with effective theories. In the latter, we integrate out massive fields, because the energy scale at which we work is lower than the energy scale at which the mode is excited. But this is very different to consistent truncation, where we rigorously set fields to zero. The discarded modes may have a mass scale

comparable to some modes we keep. Consistent truncations are really a tool, at least in this thesis, in order to build solutions, and compute couplings in the compactified theory.

C.2 Exceptional field theory techniques

The goal of this section is now to present the tools of Exceptional Field Theory for Kaluza-Klein spectroscopy. This was first developed in [99, 116].

In the introduction, we presented the basics of ExFT, and in the previous section, we introduced the concept of consistent truncations. We will now combine these two tools to demonstrate how they can be used for spectroscopy. Schematically, this will proceed as follows: employing ExFT and what is referred to as Generalized Scherk-Schwarz reduction, we will identify a consistent truncation of the higher-dimensional Supergravity, which will define the background solution. This process is akin to what was undertaken to discover the $AdS_4 \times S^7$ solution of $11d$ Supergravity, where the consistent truncation coincides with the massless sector of the theory. Subsequently, we will introduce fluctuations around this background and calculate the linearized equations of motion. This will yield a set of operators that will need to be diagonalized, from which we can extract the spectrum of the entire theory. While this may appear similar to previous approaches, it is now framed within the language of ExFT. What sets this setup apart? What are the advantages of employing ExFT for this purpose? Instead of providing an immediate answer, we aim to elucidate this throughout this section, with clarity emerging by its conclusion. For the remainder of this work, the focus will be on $E_{6(6)}$-ExFT and $E_{7(7)}$-ExFT that have been reviewed in (2.D).

The first task is to address how to efficiently implement consistent truncations within the framework of ExFT. This is accomplished through Generalized Scherk-Schwarz (GSS) reduction. The ansatz for ExFT fields is as follows:

$$\begin{cases} g_{\mu\nu}(x,y) = \rho^{-2}(y)\, g_{\mu\nu}(x)\,, \\ \mathscr{A}_\mu{}^M(x,y) = \rho^{-1}(y)\, (U^{-1})_{\underline{M}}{}^M(y)\, A_\mu{}^{\underline{M}}(x)\,, \\ \mathscr{M}_{MN}(x,y) = U_M{}^{\underline{M}}(y)\, U_N{}^{\underline{N}}(y)\, M_{\underline{MN}}(x)\,. \end{cases} \tag{3.28}$$

Here $U \in E_{d(d)}$ is the twist matrix and ρ is a scale factor, which factorises the dependance on internal and external coordinates. This ansatz bears similarity to Scherk-Schwarz (SS) reduction on a group manifold [143]. One of the main differences is that for standard SS reduction, what is called the twist matrix U takes its value in $GL(n)$, whereas here it takes its value in $E_{n(n)}$. Otherwise, the two ansätze work in a similar way and lead to similar consistency conditions, as we will see. We also introduce $\mathscr{U}_{\underline{M}}{}^M \equiv \rho^{-1}(U^{-1})_{\underline{M}}{}^M \in (E_{d(d)} \times \mathbb{R}^+)$ and also often refer to it as the twist matrix.

Let us try make a comment on the different indices here. The fields with curly letters on the left hand side and the $M, N, \ldots$ indices are fields of the full theory. The fields on the right hand side, with straigh letters and $\underline{M}, \underline{N}, \ldots$ indices do not carry any y-dependance. Those are the fields of the lower dimensional theory. For example for $M_{\underline{MN}}(x)$, $A_\mu{}^{\underline{M}}(x)$, and $g_{\mu\nu}(x)$ describe the fields of the maxinal $\mathcal{N} = 8$ theory in $4d$. Furthermore, objects with underlined indices designate objects of the low-dimensional theory. For example, writing $X_{MN}{}^P$ does not have any meaning, since X is intrepreted as the embedding tensor of the low dimensional theory, therefor we will only deal with $X_{\underline{MN}}{}^{\underline{P}}$.

The consistency of the ansatz (3.28) is now encoded in a set of differential equations on the

twist matrix $\mathscr{U}$ that takes the form

$$\left[\Gamma_{\underline{MN}}{}^{\underline{P}}\right]_{\mathcal{R}_n} = -\gamma_n X_{\underline{MN}}{}^{\underline{P}}, \quad \Gamma_{\underline{MN}}{}^{\underline{M}} = (1-d)\,\rho^{-1}\partial_{\underline{N}}\rho \tag{3.29}$$

where Γ is the current algebra

$$\Gamma_{\underline{MN}}{}^{\underline{P}} \equiv (U^{-1})_{\underline{N}}{}^{\underline{L}}\partial_{\underline{M}}U_{\underline{L}}{}^{\underline{P}}, \quad \partial_{\underline{M}} \equiv \rho^{-1}(U^{-1})_{\underline{M}}{}^{N}\partial_N. \tag{3.30}$$

Here, γ_n and n depend on the theory. If d is the dimension of the resulting lower-dimensional theory, then $n = 11-d$. Hence we have for $d = 5$ the $\mathrm{E}_{6(6)}$ group and for $d = 4$ the $\mathrm{E}_{7(7)}$ group. We also have $\gamma_6 = \frac{1}{5}$ for $\mathrm{E}_{6(6)}$ and $\gamma_7 = \frac{1}{7}$ for $\mathrm{E}_{7(7)}$ ExFT. The notation $[\dots]_{\mathcal{R}_n}$ denotes the projection onto the $\mathcal{R}_n$ irreducible representation within the tensor product of a fundamental and an adjoint representation (2.61), as we explained in section (2.C.2)

$$\mathrm{E}_{6(6)}: \quad \Theta_{\underline{M}}{}^{\alpha}; \quad \mathbf{27}\otimes\mathbf{78} = \mathbf{27}\oplus\mathbf{351}\oplus\mathbf{1728}. \tag{3.31}$$

$$\mathrm{E}_{7(7)}: \quad \Theta_{\underline{M}}{}^{\alpha}; \quad \mathbf{56}\otimes\mathbf{133} = \mathbf{56}\oplus\mathbf{912}\oplus\mathbf{6480}. \tag{3.32}$$

For $\mathrm{E}_{6(6)}$, we have $\mathcal{R}_6 = 351$, and for $\mathrm{E}_{7(7)}$, we have $\mathcal{R}_7 = 912$. The X tensor is of tremendous importance in this setup. This corresponds to the intrinsic torsion of the lower-dimensional theory and therefore will play the role of the embedding tensor of the theory. This means that it controls the gauge couplings of the lower-dimensional theory as we already saw in (2.C.2). The most crucial aspect, which may seem trivial, is that the embedding tensor is constant with respect to internal coordinates. Naturally, this is the case because in lower-dimensional Supergravity, there is no notion of internal coordinates, so it must be constant. However, when we compute the embedding tensor in ExFT via (3.29), it is not guaranteed a priori that the obtained embedding tensor will be constant with respect to y's. Being constant ensures the consistency of the truncation and allows us to identify the X tensor from ExFT as the embedding tensor of the lower-dimensional Supergravity.

The twist matrix U plays a central role in ExFT spectroscopy. As it defines a generalized vielbein for the generalized metric on the background

$$\mathscr{M}_{MN} = \Delta_{MN} = U_M{}^{\underline{M}}U_N{}^{\underline{N}}\delta_{\underline{MN}}. \tag{3.33}$$

U encodes both internal geometry and internal p-form field strengths, as it can be seen from (2.99)

$$\mathscr{M}^{MN}\,\partial_M\otimes\partial_N = (\det g)^{-1/2}\,g^{mn}\,\partial_m\otimes\partial_n\,. \tag{3.34}$$

where we see that the generalized metric, hence U, encodes the internal geometry. Moreover, the $\mathscr{U}$ twist matrix is globally well defined in a consistent truncation, therefore defining an everywhere non vanishing set of vector fields on the lower dimensional theory, and thus the background is called generalised parallelisable. The condition of generelised parallelisabilty and consistency

of the truncation (3.29) and (3.30) can be rephrased as

$$\mathscr{L}_{\mathscr{U}_{\underline{M}}}\mathscr{U}_{\underline{N}} = X_{\underline{MN}}{}^{\underline{P}}\mathscr{U}_{\underline{P}} \tag{3.35}$$

where $\mathscr{L}$ are generalised diffeomorphims that have been defined previously in (2.86), and $X_{\underline{MN}}{}^{\underline{P}}$ now play the role of structure constants of what is now called generalized Leibniz parallelisable space. The condition of generalized parallelisability, where X is not necessarly constant, but there still exist globally well defined twist matrix U, is weaker than the condition of generalized Leibniz parallelisability, where X must define a Leibniz algebra (and therefore must be constant). We will see later how the algebra must be deformed in the case of generalised parallelisability only.

So far, our discussion has centered on the twist matrix $U_M{}^{\underline{M}}$, i.e. on the background geometry and fluxes around which we define our consistent truncation. However, the consistent truncation ansatz (3.28) shows that any internal space contained by the truncation is generalized Leibniz parallelizable. To demonstrate this, let us introduce a vielbein for the lower-dimensional Supergravity scalar matrix $M_{\underline{MN}}$, namely

$$M_{\underline{MN}}(x) = \mathscr{V}_{\underline{M}}{}^{\underline{A}}(x)\mathscr{V}_{\underline{N}}{}^{\underline{B}}(x)\delta_{\underline{AB}}\,. \tag{3.36}$$

Now we can define a generalised frame field for every internal space obtained by the consistent truncation by dressing the twist matrix U with a vielbein $\mathscr{V}$ as

$$U_M{}^{\underline{A}}(x,y) = U_M{}^{\underline{M}}(y)\mathscr{V}_{\underline{M}}{}^{\underline{A}}(x)\,. \tag{3.37}$$

Therefore the generalised metric takes the form

$$\mathscr{M}_{MN}(x,y) = U_M{}^{\underline{A}}(x,y)U_N{}^{\underline{B}}(x,y)\delta_{\underline{AB}} = U_M{}^{\underline{M}}(y)U_N{}^{\underline{N}}(y)M_{\underline{MN}}(x)\,. \tag{3.38}$$

Futhermore, thank to the internal vielbein $\mathscr{V}$, we can now define the embedding tensor at every point of the space, by dressing the one defined in (3.35). We obtain

$$\mathscr{L}_{\mathscr{U}_{\underline{A}}}\mathscr{U}_{\underline{B}} = X_{\underline{AB}}{}^{\underline{C}}\mathscr{U}_{\underline{C}}\,, \tag{3.39}$$

with

$$X_{\underline{AB}}{}^{\underline{C}} = (\mathscr{V}^{-1})_{\underline{A}}{}^{\underline{M}}(\mathscr{V}^{-1})_{\underline{B}}{}^{\underline{N}}\mathscr{V}_{\underline{P}}{}^{\underline{C}}X_{\underline{MN}}{}^{\underline{P}}\,. \tag{3.40}$$

Doing this allows us to deform the maximally symmetric background to any background that can be described by a consistent truncation. To illustrate this more clearly, let us provide an example. Let us consider the compactification of 11-dimensional Supergravity. The maximally symmetric point gives rise to $d = 4$ $\mathcal{N} = 8$ Supergravity with SO(8) gauge symmetry. This corresponds to 11-dimensional Supergravity on $AdS_4 \times S^7$, as encountered before, and the appropriate ExFT to study this background is $E_{7(7)}$-ExFT. The background solution on $AdS_4 \times S^7$ is described by a consistent truncation of the full theory. We can however find a "further" consistent truncation living in the maximally symmetric theory, which would be on its own right a consistent truncation of the full theory. To do so, we choose a subgroup of $E_{7(7)}$, determine the singlet scalars of this subgroup within the $\mathcal{N} = 8$ theory, and use those singlets to construct a generalized vielbein U. These singlets will create a consistent truncation because of the singlets argument we have made before.

Let us illustrate this with an example. Let us consider an SO(7) subgroup of the SO(8) isometry

of the round sphere on $AdS_4 \times S^7$, which is itself a subgroup of $E_{7(7)}$. Because of the triality property of SO(8) (there are three inequivalent 8-dimensional representations), there are three ways to embbed SO(7) within SO(8). If we consider the following branching

$$8_v \underset{SO(7)}{\longrightarrow} 1+7\,, \tag{3.41}$$

where 8_v is the fundamental representation of SO(8), $[1,0,0,0]$ in terms of the Dynkin labels, then there is going to be in total one singlet within $\mathcal{N}=8$ Supergravity and therefore keeping this single field is a consistent truncation of $\mathcal{N}=8$. The resulting theory contains $\mathcal{N}=0$ supersymmety. This is known is the litterature as the $SO(7)_+$ solution [144]. There is another inequivalent solution which is known as $SO(7)_-$ in which

$$8_c \underset{SO(7)}{\longrightarrow} 1+7\,, \tag{3.42}$$

where the c stand for cospinor. In terms of the Dynkin labels this is the $[0,0,1,0]$ representation. Here again there are no residual supersymmetries. In order to describe this solution whitin the ExFT language, we need to deform the generalized vielbein as

$$U(x,y)_M{}^{\underline{A}} = \mathring{U}(x)_M{}^{\underline{B}} \exp\left(\phi(x)\mathbb{T}_{SO(7)_{+/-}}\right)_{\underline{B}}{}^{\underline{A}}\,, \tag{3.43}$$

with $\mathring{U}$ the twist matrix on the maximally symmetric point and $\mathbb{T}_{SO(7)_{+/-}}$ the generator which is a singlet either under $SO(7)_-$ or $SO(7)_+$ within SO(8). Using this vielbein, we can now describe both vacua and study the spectrum of the different fields around those deformed background.

Now we can describe background solutions within consistent truncations, we can add fluctuations on top of these background solutions. The generalized metric for this background is:

$$\mathscr{M}_{MN}(x,y) = U_M{}^{\underline{M}}(y)U_N{}^{\underline{N}}(y)M_{\underline{MN}}(x) = U_M{}^{\underline{A}}(x,y)U_N{}^{\underline{B}}(x,y)\delta_{\underline{AB}}\,, \tag{3.44}$$

and the metric $\mathring{g}_{\mu\nu}$ describes the external geometry. The fluctuations for the latter will simply be parametrised by

$$g_{\mu\nu}(x,y) = \rho^{-2}(y)(\mathring{g}_{\mu\nu}(x) + h_{\mu\nu}(x,y)). \tag{3.45}$$

For the vector fields we can parametrize the fluctuations as

$$\mathscr{A}_\mu{}^M(x,y) = \rho^{-1}(U^{-1})_{\underline{A}}{}^{\underline{M}}(y)\, A_\mu{}^{\underline{A}}(x,y). \tag{3.46}$$

In this equation, the background fields are hidden in the U twist matrix, and $A_\mu{}^{\underline{A}}(x,y)$ designates the fluctuation of the vector fields. Finally, it remains to parametrize the fluctuations for the scalar fields. Since the vielbein $\mathscr{V}$ parametrizes the coset space $E_{d(d)}/K_{d(d)}$, with $K_{d(d)}$ the maximal compact subgroup of $E_{d(d)}$, it can be expressed as

$$\mathscr{V}_{\underline{M}}{}^{\underline{A}}(x,y) = \exp(\phi^\alpha(x,y)\mathbb{T}_\alpha)_{\underline{M}}{}^{\underline{A}} \approx \delta_{\underline{M}}{}^{\underline{A}} + \phi^\alpha(x,y)\mathbb{T}_{\alpha,M}{}^{\underline{A}} + \dots \tag{3.47}$$

where ϕ^α are the fluctuations and $\mathbb{T}$ are the generators of $\mathfrak{e}_{d(d)} \ominus \mathfrak{k}_{d(d)}$, then α is a non-compact index. This leads for the scalar metric to

$$M_{\underline{MN}}(x,y) = \delta_{\underline{MN}} + 2\phi^\alpha(x,y)\mathbb{T}_{\alpha\underline{M}}{}^{\underline{P}}\delta_{\underline{PN}} + \dots \tag{3.48}$$

Notice that since α is a non compact index, we have

$$\mathbb{T}_{\alpha\underline{M}}{}^{\underline{P}}\delta_{\underline{PN}} = \mathbb{T}_{\alpha\underline{N}}{}^{\underline{P}}\delta_{\underline{PM}}\,. \tag{3.49}$$

Let us now explain how to expand fields into harmonics and how to deal with it within the language of ExFT. We expand each fluctuation $h_{\mu\nu}(x,y)$, $A_\mu{}^{\underline{A}}(x,y)$, and $\phi^\alpha(x,y)$ in terms of a complete basis of functions of the internal manifold. At this stage, we notice a very important feature of this machinery: we only need a basis of functions for the scalar fields on the internal manifold because all the tensorial internal structure has already taken care of by the ExFT. Another crucial feature of this technology, is that the topology of the internal space is the same for any deformation. This is true because we are dealing with continuous deformations of the internal space. Since the topology of the compactification to the lower-dimensional Supergravity is the same for any solution we can reach, we can choose a harmonic basis $\mathcal{Y}^\Sigma$ of any internal space. In particular, we can choose this basis of functions to form representations of the largest possible group G_{max}, which corresponds the maximally symmetric lower dimensional Supergravity. For $E_{7(7)}$ ExFT, this maximally symmetric theory is $d = 4$ $\mathcal{N} = 8$ gauged Supergravity with SO(8) gauge group, leading to $G_{\text{max}} = \text{SO}(8)$ and the compactification on $\text{AdS}_4 \times \text{S}^7$; for $\text{E}_{6(6)}$ ExFT, this maximally symmetric point is $d = 5$ $\mathcal{N} = 8$ with gauge group SO(6), leading to $G_{\text{max}} = \text{SO}(6)$ and the compactification on $\text{AdS}_5 \times \text{S}^5$. Note however that this basis of functions do not need to be defined on a point which satisfies the equations of motion. We can choose the internal space on which we define $\mathcal{Y}^\Sigma$ to be any configuration of the scalar fields, even if the latters do not correspond to an extrema of the scalar potential.

We can then choose $\mathcal{Y}^\Sigma$ to be representations of the maximally symmetric group. This is also around the maximally symmetric point that we typically define the consistent truncation (3.28), that is constructed using $\mathscr{U}$. Therefore, we have

$$\mathscr{L}_{\mathscr{U}_{\underline{M}}}\mathcal{Y}^\Sigma = \mathscr{U}_{\underline{M}}{}^{M}\partial_M\mathcal{Y}^\Sigma \equiv -\mathscr{T}_{\underline{M}}{}^{\Sigma}{}_{\Omega}\mathcal{Y}^\Omega\,. \tag{3.50}$$

Because the twist matrices generate the Lie algebra of G_{max}, the $\mathscr{T}$ matrices are generators of G_{max} in the representation Σ. It can be shown that [99]

$$\left[\mathscr{T}_{\underline{M}}, \mathscr{T}_{\underline{N}}\right] = X_{[\underline{MN}]}{}^{\underline{P}}\mathscr{T}_{\underline{P}}\,. \tag{3.51}$$

Those generators of G_{max} acting on harmonics $\mathcal{Y}^\Sigma$, are defined around the maximally symmetric where the consistent truncation is defined. However we saw before how to deform ExFT objects to take them from the maximally symmetric point, to a deformation of it. This is done by dressing the objects with the vielbein $\mathscr{V}_{\underline{M}}{}^{\underline{A}}$. In particular, with the set of globally well defined twist matrix (3.37), we can obtain the generators of the deformed background by the action of a generalized diffeomorphism on the harmonics of the maximally symmetric point

$$\mathscr{L}_{\mathscr{U}_{\underline{A}}}\mathcal{Y}^\Sigma = -\mathscr{T}_{\underline{A}}{}^{\Sigma}{}_{\Omega}\mathcal{Y}^\Omega\,, \tag{3.52}$$

where

$$\mathscr{T}_{\underline{A}}{}^{\Sigma}{}_{\Omega} = \mathscr{V}_{\underline{A}}{}^{\underline{M}}\mathscr{T}_{\underline{M}}{}^{\Sigma}{}_{\Omega}\,, \tag{3.53}$$

are the dressed generators of G_{max}. Their commutator is now given by

$$\left[\mathscr{T}_{\underline{A}}, \mathscr{T}_{\underline{B}}\right] = X_{[\underline{AB}]}{}^{\underline{C}}\mathscr{T}_{\underline{C}}\,. \tag{3.54}$$

Using all the ingredients we have introduced so far, we can write the fluctuation ansatz

$$\begin{cases} g_{\mu\nu}(x,y) = \rho^{-2}(y)\Big(\mathring{g}_{\mu\nu}(x) + \sum_{\Sigma} \mathcal{Y}_{\Sigma}\, h_{\mu\nu}{}^{\Sigma}(x)\Big)\,, \\ \mathscr{A}_{\mu}{}^{M}(x,y) = \rho^{-1}(y)\,(U^{-1})_{\underline{A}}{}^{M}(y) \sum_{\Sigma} \mathcal{Y}_{\Sigma}(y)\, A_{\mu}{}^{\underline{A}\Sigma}(x)\,, \\ \mathscr{M}_{MN}(x,y) = U_{M}{}^{\underline{A}}(y)\, U_{N}{}^{\underline{B}}(y)\Big(\delta_{\underline{AB}} + \delta_{\underline{AC}}\, \mathbb{T}_{\alpha,\underline{B}}{}^{\underline{C}} \sum_{\Sigma} \mathcal{Y}_{\Sigma}(y)\, \phi^{\alpha,\Sigma}(x)\Big)\,. \end{cases} \tag{3.55}$$

We now have everything in hand to compute masses. What we remains to be done is to inject this ansatz into the field equations derived from (2.77) for $E_{6(6)}$-ExFT and $E_{7(7)}$-ExFT, then to linearize these equations of motion and finally extract the mass operators.

Before closing this section, let us introduce a piece of notation. Say we have a generalized metric constructed as described above, which provides us a background solution of the equations of motion

$$\mathscr{M}_{MN} = U_{M}{}^{\underline{M}} U_{N}{}^{\underline{N}} \mathscr{V}_{\underline{M}}{}^{\underline{A}} \mathscr{V}_{\underline{N}}{}^{\underline{A}} \tag{3.56}$$

where $\mathscr{V}_{\underline{M}}{}^{\underline{A}} \mathscr{V}_{\underline{N}}{}^{\underline{A}} = \mathscr{V}_{\underline{M}}{}^{\underline{A}} \mathscr{V}_{\underline{N}}{}^{\underline{B}} \delta_{\underline{AB}}$. This is the convention we will consistently employ from now on. Whenever two indices are identical and in the same position, this means that a δ is contracting the two indices.

C.3 Mass matrices

We illustrate the previous method by applying it to $E_{6(6)}$-ExFT in this section. $E_{6(6)}$-ExFT can apply to vacua of 5-dimensional maximal gauged Supergravity as we discussed in section 2.D , however, the structure for other dimensions is very similar as we already explained in the previous section. Further details can be found in [99].

By plugging in the Kaluza-Klein fluctuation Ansatz (3.55) into the linearised equations of motion of $E_{6(6)}$ ExFT, we can easily compute the mass matrices for any vacuum of the 5-dimensional consistent truncation. These mass matrices will be expressed in terms of the embedding tensor dressed by the scalar coset representative (3.40) as well as the action of the dressed generalised frame on the scalar harmonics (3.52)

$$X_{\underline{AB}}{}^{\underline{C}} = \Big(\mathscr{V}^{-1}\Big)_{\underline{A}}{}^{\underline{M}} \Big(\mathscr{V}^{-1}\Big)_{\underline{B}}{}^{\underline{N}} \mathscr{V}_{\underline{P}}{}^{\underline{C}}\, X_{\underline{MN}}{}^{\underline{P}}\,, \qquad \mathcal{L}_{\mathscr{U}_{\underline{A}}} \mathcal{Y}_{\Sigma} = -\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Omega}\, \mathcal{Y}_{\Omega}\,. \tag{3.57}$$

a Level diagonalisation

Before computing the mass matrices explicitly, we can already observe a dramatic simplification in this approach compared to the standard one. With the ExFT fluctuation Ansatz (3.55), the mass matrices will be expressed in terms of the embedding tensor and the matrices $\mathscr{T}_{\underline{A},\Sigma}{}^{\Omega}$. Crucially, the only object acting on harmonics are $\mathscr{T}_{\underline{M}\,\Sigma}{}^{\Omega}$ matrices, which appear in their dressed form as

$$\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Omega} = \Big(\mathscr{V}^{-1}\Big)_{\underline{A}}{}^{\underline{M}} \mathscr{T}_{\underline{M}\,\Sigma}{}^{\Omega}\,. \tag{3.58}$$

The $\mathscr{T}_{\underline{M}\,\Sigma}{}^{\Omega}$ form representations of the round S^5 isometries, and therefore do not mix different levels of scalar harmonics. Similarly, the dressed $\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Omega}$ matrices will also not mix different S^5 KK levels. This means that the KK levels of the round S^5 are preserved for any other vacuum of the

consistent truncation. As a result, it becomes efficient to organise the KK spectrum of any vacuum of the consistent truncation in terms of the KK levels defined by the round S^5, something that is miraculous from the perspective of the remnant symmetry group of the vacuum. This fact drastically simplifies the computation of the KK spectrum compared to the traditional Supergravity approach, and is just one of the reasons why the ExFT approach is so powerful when studying vacua with small remnant symmetry groups. However, even more intriguingly, the same mechanism will give rise to a strong identities among cubic and higher couplings, as we will see in section 4.C.2.

b Spin-2 mass matrix

The equations of motion for the spin-2 fields read

$$\mathcal{Y}^{\Sigma}\Box_x h_{\mu\nu}{}^{\Sigma} = h_{\mu\nu}{}^{\Omega}\Big(\mathbb{M}_{\text{spin}-2}\Big)_{\Omega}{}^{\Sigma}\mathcal{Y}^{\Sigma} + \dots \tag{3.59}$$

The spin-2 mass matrix is simply given by

$$\Big(\mathbb{M}_{\text{spin}-2}\Big)_{\Sigma}{}^{\Omega} = -\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Lambda}\,\mathscr{T}_{\underline{A}\,\Lambda}{}^{\Omega}\,, \tag{3.60}$$

the repeated downstairs $\underline{A}$ index convention refers to summation by δ, e.g. for two tensors in the $\overline{\mathbf{27}}$,

$$V_{\underline{A}}\,W_{\underline{A}} = V_{\underline{A}}\,W_{\underline{B}}\,\delta^{\underline{A}\underline{B}}\,. \tag{3.61}$$

The ellipses in (3.59) denotes couplings between spin-1 and spin-2 fields, and is responsible for the Higgs mechanism. However those term do not affect the spin-2 masses. They will play a role when we would like to identify what are the physical modes in the theory, in other words to gauge fix the unphysical fluctuations. The minimal couplings (2.91) gives rise to couplings of the form

$$\pi_{\underline{A}}\partial_{(\mu}A_{\nu)}{}^{\underline{A}}\,, \quad \pi_{\underline{A}} = \mathscr{T}_{\underline{A}} \tag{3.62}$$

hence $\pi_{\underline{A}}$ controls the spin-2 Higgs mechanism.

c Vector & tensor mass matrix

The fields equations of motion obtained by varying the Lagrangian w.r.t. vector fields are of Yang-Mills type

$$\nabla_\nu(\mathscr{M}_{MN}\mathscr{F}^{\nu\mu N}) = (\mathscr{J}^{\mu}_{\text{EH}})_M + (\mathscr{J}^{\mu}_{\text{scal}})_M + (\mathscr{J}^{\mu}_{\text{top}})_M \tag{3.63}$$

where the currents on the r.h.s. denote the contributions from the Einstein-Hilbert (EH) term, the scalar kinetic term, and the topological term, respectively. The contributions from the topological term are in gereral of higher order in fluctuations and therefore dropped, and the EH contribution only gives rise to the Higgs mechanism. Therefore it does not contribute to the physical result. Upon linearizing and using the truncation ansatz (3.28), the equations of motion for the vector fields reduces to

$$\Box_x A^{\mu\,\underline{A}\Sigma} = \Big(\mathbb{M}_{\text{spin}-1}\Big)^{\underline{A}\,\Sigma}{}_{\underline{B}\,\Omega}A^{\mu\,\underline{B}\Omega} \tag{3.64}$$

The vector mass matrix is given by

$$\left(\mathbb{M}_{\text{spin}-1}\right)^{\underline{A}\,\Sigma}{}_{\underline{B}\,\Omega} = \frac{1}{12}\,\Pi_{\underline{A}\,\alpha}{}^{\Sigma}{}_{\Lambda}\,\Pi^{\alpha}{}_{\underline{B}}{}^{\Lambda}{}_{\Omega}\,, \tag{3.65}$$

with

$$\Pi^{\alpha}{}_{\underline{A}}{}^{\Sigma}{}_{\Omega} = -2\left(X_{\underline{A}}{}^{\alpha}\,\delta^{\Sigma}_{\Omega} - 6\,\mathbb{T}^{\alpha}{}_{\underline{A}}{}^{\underline{B}}\,\mathscr{T}_{\underline{B}}{}^{\Sigma}{}_{\Omega}\right)\,, \tag{3.66}$$

and its adjoint

$$\Pi_{\underline{A}\,\alpha}{}^{\Sigma}{}_{\Omega} = \Pi_{\alpha\,\underline{A}\,\Omega}{}^{\Sigma} = -2\left(X_{\underline{A}\alpha}\,\delta^{\Sigma}_{\Omega} + 6\,\mathbb{T}_{\alpha\underline{A}}{}^{\underline{B}}\,\mathscr{T}_{\underline{B}}{}^{\Sigma}{}_{\Omega}\right)\,. \tag{3.67}$$

Here, we defined the non-compact projection of the embedding tensor in its adjoint indices

$$X_{\underline{A}}{}^{\alpha} = X_{\underline{AB}}{}^{\underline{C}}\,\mathbb{T}^{\alpha}{}_{\underline{C}}{}^{\underline{B}}. \tag{3.68}$$

On the other hand, the tensor mass matrix is

$$\left(\mathbb{M}_{\text{tensor}}\right)_{\underline{A}\,\Sigma}{}^{\underline{B}\,\Omega} = \frac{1}{\sqrt{10}}\left(-Z^{\underline{AB}}\delta^{\Omega}_{\Sigma} + 10\,d^{\underline{ABC}}\,\mathscr{T}_{\underline{C}\,\Sigma}{}^{\Omega}\right)\,, \tag{3.69}$$

where we used the antisymmetric combination of the intrinsic torsion given by

$$Z^{\underline{AB}} = 2\,d^{\underline{CDA}}\,X_{\underline{CD}}{}^{\underline{B}} = -Z^{\underline{BA}}\,. \tag{3.70}$$

These two expressions can be combined to find the following interesting structure arising.

$$\begin{aligned}\left(\mathbb{M}_{\text{spin}-1}\right)^{\underline{A}\,\Sigma}{}_{\underline{B}\,\Omega} + \left(\mathbb{M}^2_{\text{tensor}}\right)_{\underline{A}\,\Sigma}{}^{\underline{B}\,\Omega} = \mathbb{M}_{0\,\underline{A}}{}^{\underline{B}}\,\delta^{\Omega}_{\Sigma} + 2\,\mathbb{N}_{\underline{CA}}{}^{\underline{B}}\mathscr{T}_{\underline{C}\,\Sigma}{}^{\Omega} + \delta_{\underline{AB}}\left(\mathbb{M}_{\text{spin}-2}\right)_{\Sigma}{}^{\Omega} \\ - \frac{5}{3}\pi_{\underline{A}\,\Sigma}{}^{\Lambda}\pi_{\underline{B}\,\Lambda}{}^{\Omega}\,,\end{aligned} \tag{3.71}$$

where

$$\begin{aligned}\mathbb{M}_{0\,\underline{A}}{}^{\underline{B}} &= \frac{1}{3}X_{\underline{A}}{}^{\alpha}\,X_{\underline{B}\alpha} + \frac{1}{10}Z^{\underline{AC}}\,Z^{\underline{BC}}\,, \\ \mathbb{N}_{\underline{CA}}{}^{\underline{B}} &= \mathbb{N}_{\underline{C}}{}^{\hat{\alpha}}\,\mathbb{T}_{\hat{\alpha}\,\underline{A}}{}^{\underline{B}} = \left(X_{\underline{C}}{}^{\hat{\alpha}} + 3X_{\underline{DE}}{}^{\underline{C}}\,\mathbb{T}^{\hat{\alpha}}{}_{\underline{D}}{}^{\underline{E}}\right)\mathbb{T}_{\hat{\alpha}\,\underline{A}}{}^{\underline{B}}\,,\end{aligned} \tag{3.72}$$

are the 5-dimensional gauged Supergravity contribution and the tensor $\mathbb{N}_{\underline{C}}{}^{\hat{\alpha}}$ which mixes the level-0 and higher-level contributions. The $\mathbb{T}_{\hat{\alpha}\,\underline{A}}{}^{\underline{B}}$ are the compact generators with $\hat{\alpha}$ labelling the **36** of USp(8) and raised/lowered with the Cartan-Killing metric $\kappa_{\hat{\alpha}\hat{\beta}}$. Moreover, $\pi_{\underline{A}\,\Sigma}{}^{\Omega} = \mathscr{T}_{\underline{A}\,\Sigma}{}^{\Omega}$ corresponds to the coupling between vectors and spin-2 and thus $\pi_{\underline{A}\,\Sigma}{}^{\Lambda}\pi_{\underline{B}\,\Lambda}{}^{\Omega}$ only acts on the spin-1 Goldstones eaten by the massive spin-2 states. We see that the $\mathscr{T}^2$ terms act on physical modes exactly like the spin-2 mass operator (3.60)

$$\mathbb{M}_{\text{spin}-2} = -\mathscr{T}_{\underline{A}}\mathscr{T}_{\underline{A}}\,. \tag{3.73}$$

d Scalar mass matrix

The variation of the potential (2.94) with respect to the scalar fluctuations $\phi^\alpha = \phi^{\alpha,\Sigma}\mathcal{Y}_\Sigma$ gives us the scalar equations of motion and obtain

$$\Box_x \phi^\alpha = \left(\mathbb{M}_{\text{spin}-0}\right)^\alpha{}_\beta \phi^\beta + \dots\,, \tag{3.74}$$

The scalar mass matrix is given by

$$\begin{aligned}\left(\mathbb{M}_{\text{spin}-0}\right)^{\alpha\,\Sigma}{}_{\beta\,\Omega} = \Big[& X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BF}}{}^{\underline{E}}\, \mathbb{T}^\alpha{}_{\beta\underline{A}}{}^{\underline{B}} \\ & + \frac{1}{5}\left(X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BE}}{}^{\underline{F}} + X_{\underline{EA}}{}^{\underline{F}} X_{\underline{EB}}{}^{\underline{F}} + X_{\underline{EF}}{}^{\underline{A}} X_{\underline{EF}}{}^{\underline{B}}\right) \mathbb{T}^\alpha{}_{\beta\underline{A}}{}^{\underline{B}} \\ & + \frac{2}{5}\left(X_{\underline{AC}}{}^{\underline{E}} X_{\underline{BD}}{}^{\underline{E}} - X_{\underline{AE}}{}^{\underline{C}} X_{\underline{BE}}{}^{\underline{D}} - X_{\underline{EA}}{}^{\underline{C}} X_{\underline{EB}}{}^{\underline{D}}\right) \mathbb{T}^\alpha{}_{\underline{A}}{}^{\underline{B}}\, \mathbb{T}_{\beta\underline{C}}{}^{\underline{D}}\Big]\delta^\Sigma_\Omega \\ & + 2\left(\mathbb{T}^\alpha{}_{\underline{A}}{}^{\underline{B}} X_{\underline{A}\beta} - \mathbb{T}_{\beta\underline{A}}{}^{\underline{B}} X_{\underline{A}}{}^\alpha\right) \mathscr{T}_{\underline{B}}{}^\Sigma{}_\Omega - 2\left[\mathbb{T}^\alpha, \mathbb{T}_\beta\right]_{\underline{A}}{}^{\underline{B}} X_{\underline{CB}}{}^{\underline{A}}\, \mathscr{T}_{\underline{C}}{}^\Sigma{}_\Omega \\ & - \delta^\alpha_\beta\, \mathscr{T}_{\underline{A}}{}^\Sigma{}_\Lambda \mathscr{T}_{\underline{A}}{}^\Lambda{}_\Omega + 12\, \mathbb{T}^\alpha{}_{\beta\underline{A}}{}^{\underline{B}}\, \mathscr{T}_{\underline{A}}{}^\Sigma{}_\Lambda \mathscr{T}_{\underline{B}}{}^\Lambda{}_\Omega\,,\end{aligned} \tag{3.75}$$

where we introduced the notation for products of generators

$$\mathbb{T}_{\alpha\beta\underline{A}}{}^{\underline{B}} = \mathbb{T}_{\alpha\underline{A}}{}^{\underline{C}}\, \mathbb{T}_{\beta\underline{C}}{}^{\underline{B}}\,. \tag{3.76}$$

We can simplify this by adding terms which only act on Goldstone modes. We can easily implement this via the Π tensor of (3.66). The mass for spin-1 boson in KK reduction indeed arise by a standard Higgs mechanism. If we consider the covariant derivative of the scalar fields, we have

$$\mathscr{D}_\mu(\phi^{\alpha\Sigma} t_{\alpha M}{}^N \mathcal{Y}^\Sigma) = \partial_\mu \phi^{\alpha\Sigma} \mathcal{Y}^\Sigma t_{\alpha M}{}^N - \mathscr{L}_{\mathscr{A}_\mu}(\phi^{\alpha\Sigma} t_{\alpha M}{}^N \mathcal{Y}^\Sigma) - t_{\alpha M}{}^N \Pi^\alpha_{\underline{A}}\, \mathscr{A}_\mu{}^{\underline{A}} \tag{3.77}$$

Therefore we see that $\Pi^\alpha{}_{\underline{A}}$ controls what are the spin-0 that couple to spin-1 fields via the Higgs mechanism. Therefore, adding a term like $\Pi^\alpha{}_{\underline{A}}{}^\Lambda{}_\Sigma \Pi^{\underline{A}}{}_\beta{}^\Lambda{}_\Omega$ in the scalar mass matrix will only act on Goldstone modes, which are not physical. We find

$$\left(\mathbb{M}_{\text{spin}-0}\right)^{\alpha\,\Sigma}{}_{\beta\,\Omega} + \frac{1}{12}\Pi^\alpha{}_{\underline{A}}{}^\Lambda{}_\Sigma \Pi^{\underline{A}}{}_\beta{}^\Lambda{}_\Omega = \mathbb{M}_0{}^\alpha{}_\beta\, \delta^\Sigma{}_\Omega + 2\,\mathbb{N}_{\underline{A}}{}^\alpha{}_\beta \mathscr{T}_{\underline{A}}{}^\Sigma{}_\Omega + \delta^\alpha{}_\beta \left(\mathbb{M}_{\text{spin}-2}\right)^\Sigma{}_\Omega\,. \tag{3.78}$$

Here

$$\begin{aligned}\mathbb{M}_0{}^\alpha{}_\beta &= X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BF}}{}^{\underline{E}}\, \mathbb{T}^\alpha{}_{\beta\underline{A}}{}^{\underline{B}} + \frac{1}{2} X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BE}}{}^{\underline{F}}\, \mathbb{T}^\alpha{}_{\beta\underline{A}}{}^{\underline{B}} \\ &\quad + X_{\underline{AC}}{}^{\underline{E}} X_{\underline{BD}}{}^{\underline{E}} (\mathbb{T}^\alpha)_{\underline{A}}{}^{\underline{B}} (\mathbb{T}_\beta)_{\underline{C}}{}^{\underline{D}} + \frac{1}{3} X_{\underline{A}}{}^\alpha\, X_{\underline{A}\beta}\,, \\ \mathbb{N}_{\underline{A}}{}^\alpha{}_\beta &= \mathbb{N}_{\underline{A}}{}^{\hat{\alpha}}\, \mathbb{T}_{\hat{\alpha}}{}^\alpha{}_\beta\,,\end{aligned} \tag{3.79}$$

where $\mathbb{N}_{\underline{A}}{}^{\hat{\alpha}}$ is the object from (3.72) that controls the mixing between levels 0 and higher levels.

C.4 $AdS_5 \times S^5$ spectrum

To conclude this section, we apply the ExFT-techniques to $AdS_5 \times S^5$ spectrum and give the results in terms of quadratic Casimir operators of SO(6) and U(1). This can be done straightforwardly

using (3.60), (3.71) and (3.79). The field content of IIB Supergravity on $AdS_5 \times S^5$ in recalled in (2.5).

Let us note the explicit form of the embedding tensor at this vacuum

$$X_{\underline{AB}}{}^{\underline{C}} \to \begin{cases} X_{ab,cd}{}^{ef} = 2\sqrt{2}\delta^{[e}_{[a}\delta_{b][c}\delta^{f]}_{d]} \\ X_{ab}{}^{cu}{}_{dv} = -\sqrt{2}\delta^{c}_{[a}\delta_{b]d}\delta^{u}_{v} \end{cases} , \tag{3.80}$$

where we decompose $E_{6(6)} \to SL(6) \times SL(2)$, such that the $\mathbf{27} \to (\mathbf{15},\mathbf{1}) \oplus (\overline{\mathbf{6}},\mathbf{2})$, with $a, b, \ldots = 1, \ldots, 6$ and we use $u, v = 1, 2$ for the $SL(2)$ doublet. The action of the $\mathscr{T}$ matrix on the harmonics is given by

$$\mathscr{T}_{ab,c}{}^{d} = \sqrt{2}\delta_{c[a}\delta^{d}_{b]} , \tag{3.81}$$

on fundamental harmonics $\mathcal{Y}^a$, and naturally extended on the higher harmonics, with $\mathscr{T}^{au}{}_{\Sigma}{}^{\Lambda} = 0$.

We can now immediately compute the spin-2 mass spectrum. From (3.81), we see that

$$\mathscr{T}_{\underline{A}\Sigma}{}^{\Omega} = \sqrt{2}\, t_{\underline{A}\Sigma}{}^{\Omega} , \tag{3.82}$$

where $t_{\underline{A}\Sigma}{}^{\Omega}$ are the canonically normalised $SO(6)$ generators acting on the scalar harmonics. Thus,

$$\mathbb{M}_{\text{spin}-2} = -\mathscr{T}_{\underline{A}}\mathscr{T}_{\underline{A}} = 2\,\mathcal{C}_{SO(6)} = \ell(\ell+4) , \tag{3.83}$$

where $\mathcal{C}_{SO(6)}$ is the $SO(6)$ Casimir

$$\mathcal{C}_{SO(6)} = -t_{\underline{A}}\, t_{\underline{A}} \tag{3.84}$$

and we expressed the spectrum in terms of the level, ℓ, of the spin-2 fluctuation. On the other hand, the level-0 mass operators for vector/tensors, $\mathbb{M}_{0\,\underline{A}}{}^{\underline{B}}$, and for the scalars, $\mathbb{M}_0{}^{\alpha}{}_{\beta}$, can be evaluated explicitly and is, in both cases, given by

$$\mathbb{M}_0 = 8 - 2\,\mathcal{C}_{SO(6)} - 8\,\mathcal{C}_{U(1)} , \tag{3.85}$$

where $\mathcal{C}_{U(1)}$ denotes the Casimir of the compact $U(1) \subset SL(2)_S$ that remains of the IIB S-duality.

Finally, the tensor that controls the mixing between level-0 and higher-levels, $\mathbb{N}_{\underline{AB}}{}^{\underline{C}}$ in (3.79), is given by

$$\mathbb{N}_{\underline{AB}}{}^{\underline{C}} = -X_{\underline{AB}}{}^{\underline{C}} . \tag{3.86}$$

In fact, this relation holds for any embedding tensor that lives in the **36** of $USp(8)$, such as for the $AdS_5 \times S^5$ vacuum. From the algebra of the $\mathscr{T}$-matrices (3.54) and using (3.86), we see that the $\mathbb{N}$-tensor is given in terms of the canonically normalised $SO(6)$ generators by

$$\mathbb{N}_{\underline{AB}}{}^{\underline{C}} = \sqrt{2}\, t_{\underline{AB}}{}^{\underline{C}} . \tag{3.87}$$

Putting the level-0 result for (3.85) together with the relation (3.87), this now allows us to evaluate the vector-tensor (3.71) and scalar mass matrices (3.78). For this, we need to disinguish the action of $SO(6)$ generators on level-0 fields and the scalar harmonics, which we denote by t^0 and t^h, respectively. Thus, we obtain for the mass matrix $\mathbb{M}$, corresponding to $\mathbb{M}_{\text{vec}-\text{tensor}}$ or $\mathbb{M}_{\text{spin}-0}$,

the following

$$\begin{aligned}\mathbb{M} &= \mathbb{M}_0 + 2\,\mathbb{N}_{\underline{A}}\mathscr{T}_{\underline{A}} - \mathscr{T}_{\underline{A}}\mathscr{T}_{\underline{A}} \\ &= 8 + 2\,t^0_{\underline{A}}\,t^0_{\underline{A}} - 8\,\mathcal{C}_{\mathrm{U}(1)} + 4\,t^0_{\underline{A}}\,t^h_{\underline{A}} - \mathscr{T}_{\underline{A}}\mathscr{T}_{\underline{A}} \\ &= 8 + 2\left(t^0_{\underline{A}} + t^h_{\underline{A}}\right)\left(t^0_{\underline{A}} + t^h_{\underline{A}}\right) - 8\,\mathcal{C}_{\mathrm{U}(1)} - 2\,\mathscr{T}_{\underline{A}}\mathscr{T}_{\underline{A}} \\ &= 8 - 2\,\mathcal{C}_{\mathrm{SO}(6)} - 8\,\mathcal{C}_{\mathrm{U}(1)} + 2\,\ell\,(\ell+4)\ ,\end{aligned} \tag{3.88}$$

where in going to the second line we replaced $\mathcal{C}_{\mathrm{SO}(6)}$ in $\mathbb{M}_0$ by (3.84), in going to the third line, we completed the square, and, in going to the final line, we recognised $\left(t^0_{\underline{A}} + t^h_{\underline{A}}\right)$ as the SO(6) generator acting on the tensor product of the level-0 fields and scalar harmonics, corresponding to our field basis.

Let us thus summarise our results for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spectrum. We have shown that the mass matrices at S^5 level ℓ are given by

$$\begin{aligned}\mathbb{M}_{\mathrm{spin}-2} &= -2\,\mathcal{C}_{\mathrm{SO}(6)} + 2\,\ell(\ell+4)\,, \\ \mathbb{M}_{\mathrm{vec}-\mathrm{tensor}} &= 8 - 2\,\mathcal{C}_{\mathrm{SO}(6)} - 8\,\mathcal{C}_{\mathrm{U}(1)} + 2\,\ell(\ell+4)\,, \\ \mathbb{M}_{\mathrm{spin}-0} &= 8 - 2\,\mathcal{C}_{\mathrm{SO}(6)} - 8\,\mathcal{C}_{\mathrm{U}(1)} + 2\,\ell(\ell+4)\,,\end{aligned} \tag{3.89}$$

for a field in a representation $[n,p,q]_{\mathrm{SO}(6)} \otimes [j]_{\mathrm{U}(1)}$. The Casimir operators are given by

$$\begin{aligned}\mathcal{C}_{\mathrm{SO}(6)}[n,p,q] &= \frac{1}{8}\left(4n^2 + 3p^2 + 3q(4+q) + 2p(6+q) + 4n(4+p+q)\right)\ , \\ \mathcal{C}_{\mathrm{U}(1)}[j] &= j^2\,,\end{aligned} \tag{3.90}$$

are the SO(6) and U(1) Casimirs. Note that our expression (3.89) in terms of the SO(6) and U(1) Casimirs only appears once we write the fields as a tensor product of the 5-dimensional gauged Supergravity fields with scalar harmonics. The compact expressions (3.89) are already a signal that this new field basis we are using is very efficient in computing KK spectra. As we will see in 4.C, this field basis also simplifies the computation and the results of n-point couplings.

D Kaluza-Klein spectroscopy beyond consistent truncation

We presented how we can use ExFT in order to compute efficiently the full spectrum of a Kaluza-Klein reduction to a lower dimensional theory in the previous section. This proved very powerful in getting the spectrum of backgrounds with few symmetry. We will now present how we managed to extent those techniques, to what we claim is beyond standard consistent truncations. Let us explain this naming. All background described in the previous section are deformations of the maximally symmetric point, defined in a consistent truncation. What we mean by consistent truncation in that case, is to select a subset of the field content of the maximally symmetric point, which is itself a consistent truncation of the full theory with all the fields from KK towers, and restrict the lower dimensional theory field content to the selected modes. Another way of saying it, is that the consistent truncations are restricted to truncations of the zero modes of the KK towers of the full theory. In the case of $11d$ Supergravity, this means for example that consistent truncations are living inside $\mathcal{N} = 8$ $d = 4$ maximal Supergravity. What about consistent

truncations including modes from higher KK levels? For example, what if I can make a consistent truncation with a mode from level 1 of a KK tower. Can I describe this in the previous language? Will this describe a valid lower dimensional theory? A priori, this is hill defined as it leads for example to a non constant embedding tensor. This can be seen straightforwardly because the vielbein $\mathscr{V}$ will schematically look like

$$\mathscr{V}(x,y) = \exp\left(\phi^i_{KK}\mathbb{T}_{KK}(x)\mathcal{Y}^i\right)\mathring{\mathscr{V}}(x). \tag{3.91}$$

with ϕ^i_{KK} a mode coming from a non zero KK level, $\mathcal{Y}^i$ and $\mathbb{T}_{KK}$ the corresponding harmonics and generators, and $\mathring{\mathscr{V}}$ the vielbein defined by the zero modes. We see that this new internal vielbein $\mathscr{V}(x,y)$ is now y dependant, hence does the embedding tensor on the lower dimensional theory (3.40)

$$X_{\underline{AB}}{}^{\underline{C}}(x,y) = (\mathscr{V}^{-1})_{\underline{A}}{}^{\underline{M}}(x,y)(\mathscr{V}^{-1})_{\underline{B}}{}^{\underline{N}}(x,y)\mathscr{V}_{\underline{P}}{}^{\underline{C}}(x,y)X_{\underline{MN}}{}^{\underline{P}}(x) \tag{3.92}$$

However as we will show in this section, there is a way to make sense of it, and to include modes from higher level of the KK towers. This is in that sense we mean beyond consistent truncation: this is not a standard consistent truncation of the maximally symmetric theory, but a consistent truncation of the full theory to a lower dimensional one. Extending the techniques of [99, 116] would lead to interesting insights, because many interesting vacua arise from deformations by scalar fields that are not part of the $\mathcal{N}=8$ truncation, such as the $AdS_4\times$ squashed S^7 vacua with $\mathcal{N}=1$ and $\mathcal{N}=0$ supersymmetry [136, 137], or AdS vacua obtained by TsT deformations [133–135].

This section is organized as follow : first we will present what changes compared to the standard method of ExFT, namely in the fluctuation ansatz. Then we will talk about quadratic constraints. We need to make sense of the non constant embedding tensor we found. Finally we will compute the new mass operators in the ExFT language. From now on in this section, we will focus only on $E_{7(7)}$-ExFT.

D.1 Fluctuation Ansatz

While [99, 116] showed how to obtain the full Kaluza-Klein spectrum around any vacuum of maximal gauged Supergravity that arises from a consistent truncation, here we will go further and treat more general deformations of vacua of $\mathcal{N}=8$ Supergravity that take us outside the consistent truncation as explained in the previous paragraph. Any background within the consistent truncation to $\mathcal{N}=8$ gauged Supergravity is generalised Leibniz parallelisable, which consists of the following two conditions:

- *Generalised parallelisable*: The generalised tangent bundle is trivial, i.e. there is a globally well-defined generalised frame. Put differently, we can define the globally well-defined $E_{7(7)}$ objects

$$\left\{\mathscr{U}_M{}^{\underline{A}} = U_M{}^{\underline{M}}\mathscr{V}_{\underline{M}}{}^{\underline{A}}, \rho\right\}, \tag{3.93}$$

  where $U_M{}^{\underline{A}}$ is the $E_{7(7)}$-valued twist matrix, and ρ is the nowhere-vanishing scalar density of weight $-1/2$. Crucially, generalised parallelisability is a topological condition.

- *Leibniz*: For a generalised parallelisable space to be generalised Leibniz parallelisable re-

quires $U_M{}^{\underline{A}}$ and ρ to additionally satisfy the differential condition

$$\mathscr{L}_{\mathscr{U}_{\underline{A}}}\mathscr{U}_{\underline{B}}{}^{M} = X_{\underline{AB}}{}^{\underline{C}}\,\mathscr{U}_{\underline{C}}{}^{M}\,, \qquad \text{for } \mathscr{U}_{\underline{A}}{}^{M} = \rho^{-1}\,(U^{-1})_{\underline{A}}{}^{M}\,, \tag{3.94}$$

where the so-called intrinsic torsion $X_{\underline{AB}}{}^{\underline{C}}$ must be constant. In this case, there exists a consistent truncation to maximal $d = 4$ Supergravity, with the reduction ansatz for the higher-dimensional fields encoded by the twist matrix U and scalar density ρ [145, 146]. The constant intrinsic torsion defines the embedding tensor of the maximal gauged Supergravity arising from the consistent truncation.

Let us now consider a general deformation of a background of $\mathcal{N} = 8$ Supergravity, which is not triggered by the 70 scalar fields of the $\mathcal{N} = 8$ Supergravity. Thus, the deformed background will no longer be part of the consistent truncation. Nonetheless, because we are considering a continuous deformation and generalised parallelisability is a topological condition, our deformed background is still generalised parallelisable. That is, our background admits a globally well-defined twist matrix $U_M{}^{\underline{A}}$ and scalar density ρ. However, if we compute the analogue of (2.59), we find

$$\mathcal{L}_{\mathscr{U}_{\underline{A}}}\mathscr{U}_{\underline{B}}{}^{M} = X_{\underline{AB}}{}^{\underline{C}}(y)\,\mathscr{U}_{\underline{C}}{}^{M}\,, \tag{3.95}$$

where the intrinsic torsion, $X_{\underline{AB}}{}^{\underline{C}}(y)$, is no longer constant. Rather, the dependence of the intrinsic torsion, $X_{\underline{AB}}{}^{\underline{C}}(y)$, on the internal coordinates is the obstruction to the deformation being part of the consistent truncation. Nonetheless, we can still employ the same ideas as in [99, 116] to compute the full KK spectrum. In the previous case with constant intrinsic torsion, the $d = 4$ mass matrices can be given as algebraic expressions in terms of the intrinsic torsion tensor. In the more general case (3.95) the $d = 4$ mass operators are most conveniently expressed in terms of differential operators

$$\partial_{\underline{A}} = \mathscr{U}_{\underline{A}}{}^{M}\partial_{M}\,, \tag{3.96}$$

in the internal space. While the expression of the lower dimensional mass operator in terms of differential operators on the internal manifold is of course at the heart of any Kaluza-Klein analysis, the $\mathrm{E}_{7(7)}$ covariant formulation (2.77) together with an appropriate fluctuation ansatz reduces the analysis to a differential problem involving exclusively the tower of scalar harmonics. As a result, the computation of the full Kaluza-Klein spectrum can be done in analogy to the spin-2 sector and leads to universal covariant expressions for all $d = 4$ mass operators, which can be straightforwardly diagonalized in the concrete examples.

The non-constant intrinsic torsion (3.95) now generically gives rise to level-mixing, i.e. mass eigenstates of the deformed background will come from mixing states amongst different KK levels of the undeformed background. By contrast, when the deformation is caused by the 70 scalars of the consistent truncation, the deformed KK mass eigenstates will be linear combinations of states of the *same* KK level of the undeformed background [99, 116].

Just as in previous sections, because our vacuum is generalised parallelisable, we can expand any tensor fluctuations in terms of the generalised frame $U_M{}^{\underline{A}}$ and ρ. Thus, we can write any deformation of a generalised parallelisable space as in (3.45)(3.46)(3.48)

$$\begin{cases} g_{\mu\nu}(x,y) = \rho^{-2}(y)\left(\mathring{g}_{\mu\nu}(x) + h_{\mu\nu}(x,y)\right), \\ \mathscr{A}_\mu{}^M(x,y) = \rho^{-1}(y)\,(U^{-1})_{\underline{A}}{}^M(y)\,A_\mu{}^{\underline{A}}(x,y), \\ \mathscr{M}_{MN}(x,y) = U_M{}^{\underline{A}}(y)\,U_N{}^{\underline{B}}(y)\,\mathscr{M}_{\underline{AB}}(x,y), \end{cases} \tag{3.97}$$

where $\mathring{g}_{\mu\nu}$ is a given $d = 4$ background metric and $h_{\mu\nu}(x,y)$, $A_\mu{}^{\underline{A}}(x,y)$ and $\mathscr{M}_{\underline{AB}}(x,y)$ are now scalars on the internal space, whose y-dependence we can further expand in a complete basis of scalar functions, $\mathcal{Y}_\Sigma(y)$. Moreover, if we are interested in linearised deformations, we can rewrite (3.97) to linear order in fluctuation similarly to (3.55)

$$\begin{cases} g_{\mu\nu}(x,y) = \rho^{-2}(y)\Big(\mathring{g}_{\mu\nu}(x) + \sum_\Sigma \mathcal{Y}_\Sigma\, h_{\mu\nu}{}^\Sigma(x)\Big), \\ \mathscr{A}_\mu{}^M(x,y) = \rho^{-1}(y)\,(U^{-1})_{\underline{A}}{}^M(y)\sum_\Sigma \mathcal{Y}_\Sigma(y)\,A_\mu{}^{\underline{A},\Sigma}(x), \\ \mathscr{M}_{MN}(x,y) = U_M{}^{\underline{A}}(y)\,U_N{}^{\underline{B}}(y)\Big(\delta_{\underline{AB}} + \delta_{\underline{AC}}\,\mathbb{T}_{\alpha,\underline{B}}{}^{\underline{C}}\sum_\Sigma \mathcal{Y}_\Sigma(y)\,\phi^{\alpha\Sigma}(x)\Big) \end{cases} \tag{3.98}$$

where the $\mathbb{T}_{\alpha,\underline{A}}{}^{\underline{B}}$ are the non-compact generators of $\mathfrak{e}_{7(7)}$, such that $\delta_{\underline{AC}}\,\mathbb{T}_{\alpha,\underline{B}}{}^{\underline{C}} = \delta_{\underline{BC}}\,\mathbb{T}_{\alpha,\underline{A}}{}^{\underline{C}}$ as we previously encountered in (3.47). Finally, we are ignoring the two-forms since these do not contribute to the linearised equations of motion. In the following, we will omit the explicit summation symbol over the harmonics Σ.

Since $\underline{A}$, $\underline{B}$ are really SU(8) indices, we have two invariants with which we can contract: the $\mathrm{E}_{7(7)}$ symplectic invariant $\Omega_{\underline{AB}}$ and the SU(8) invariant $\delta_{\underline{AB}}$. In order to keep track which tensor is involved in an index contraction in SU(8) language, we will use the following conventions. All $\mathrm{E}_{7(7)}$ indices, including the flattened $\underline{A}$, $\underline{B}$, are raised and lowered using the $\mathrm{E}_{7(7)}$ symplectic invariant $\Omega_{\underline{AB}}$ in the conventions of (2.80). Thus,

$$V^{\underline{A}}\,W_{\underline{A}} = V^{\underline{A}}\,W^{\underline{B}}\,\Omega_{\underline{BA}}\,. \tag{3.99}$$

On the other hand, we will suppress the effect of raising/lowering indices with $\delta_{\underline{AB}}$, i.e. they will simply be written in the same position. For example, repeated flattened $\underline{A}$, $\underline{B}$ indices at the same position are contracted with $\delta_{\underline{AB}}$, as we already encountered in 3.61, i.e.

$$V^{\underline{A}}\,V^{\underline{A}} = V^{\underline{A}}\,V^{\underline{B}}\,\delta_{\underline{AB}}\,. \tag{3.100}$$

In particular, with these conventions, the non-compact generators $\mathbb{T}_{\alpha,\underline{A}}{}^{\underline{B}}$ satisfy

$$\mathbb{T}_{\alpha,\underline{A}}{}^{\underline{B}} = \mathbb{T}_{\alpha,\underline{B}}{}^{\underline{A}}. \tag{3.101}$$

The benefit of the Kaluza-Klein Ansatz (3.55) is that the linearised equations of motion of ExFT obtained from the pseudo-action (2.77), (2.78) drastically simplify. As a result, we can read off the Kaluza-Klein mass operators, which are determined by the intrinsic torsion $X_{\underline{AB}}{}^{\underline{C}}(y)$ from (3.95), and flattened derivative (3.96).

Before we compute the mass operators, let us comment on the applicability of the technique developed here. Allowing for a y-dependent X-matrix in (3.95) looks like it may apply to any background of 10-/11-dimensional Supergravity, since any such background can be described by

a generalised vielbein, akin to (3.93). However, for a generic background, the generalised vielbein will not be globally well-defined, but only up to $\mathrm{SU}(8)$ transformations. This is a problem, since the intrinsic torsion defined in (3.95) is not invariant under such $\mathrm{SU}(8)$ transformations.[1] Moreover, the fluctuations in (3.55) would not be globally well-defined and instead require not just scalar harmonics but also tensor harmonics in various $\mathrm{E}_{7(7)}$ representations. As a result, the setup here does not readily apply for general backgrounds, but only for those which are generalised parallelisable, i.e. with a globally well-defined twist matrix. Such generalised parallelisable backgrounds include deformations of backgrounds that can be uplifted from maximal gauged Supergravity, even when the deformation does not correspond to one of the 70 scalar fields of the $\mathcal{N}=8$ gauged Supergravity. We will discuss concrete examples below.

D.2 Quadratic constraints

By plugging the fluctuation Ansatz (3.55) into the linearised equations of motion, we will obtain the mass operators for the full Kaluza-Klein tower. Since the equations of motion are quadratic and involve the generalised Lie derivative, our fluctuation Ansatz (3.55) implies that the mass operators will involve X^2, ∂X, $X\partial$ and ∂^2 terms. This is just like in [99, 116] (where the action of ∂ on harmonics was explicitly parametrised by a representation matrix $\mathscr{T}$), with the important difference that we now also obtain ∂X terms, since the intrinsic torsion is no longer constant.

However, the quadratic terms in X and ∂X are not all independent. Rather, some of them are linked by quadratic constraints, generalising the quadratic constraints of gauged Supergravity, governing constant X. Just as in that case, the quadratic constraints are a consequence of the section condition (2.85) and thus of closure of the algebra of generalised Lie derivatives, i.e.

$$\left[\mathscr{L}_{\mathscr{U}_{\underline{A}}}, \mathscr{L}_{\mathscr{U}_{\underline{B}}}\right] = \mathscr{L}_{[\![\mathscr{U}_{\underline{A}}, \mathscr{U}_{\underline{B}}]\!]}\,, \tag{3.102}$$

where we defined the shorthand

$$[\![\mathscr{U}_{\underline{A}}, \mathscr{U}_{\underline{B}}]\!] = \mathscr{L}_{\mathscr{U}_{\underline{A}}}\mathscr{U}_{\underline{B}}\,. \tag{3.103}$$

Applying (3.102) to ρ^{-2} give the following relation

$$\partial_{\underline{C}} X_{\underline{AB}}{}^{\underline{C}} = \partial_{[\underline{A}}\vartheta_{\underline{B}]} + X_{\underline{AB}}{}^{\underline{C}}\vartheta_{\underline{C}}\,, \tag{3.104}$$

where

$$\vartheta_{\underline{A}} = \rho^2 \mathscr{L}_{\mathscr{U}_{\underline{A}}}\rho^{-2} = \frac{1}{28} X_{\underline{AB}}{}^{\underline{B}}, \tag{3.105}$$

is the analogue of the trombone tensor of $\mathcal{N}=8$ gauged Supergravity. Most interesting vacua have $\vartheta_{\underline{A}}=0$, which will be what we restrict ourselves to in the following. Thus, $X_{\underline{AB}}{}^{\underline{C}}$ satisfies the quadratic constraint

$$\partial_{\underline{C}} X_{\underline{AB}}{}^{\underline{C}} = 0\,. \tag{3.106}$$

On the other hand, applying (3.102) to $\mathscr{U}_{\underline{A}}{}^{M}$ gives the quadratic constraint

$$X_{\underline{AC}}{}^{\underline{E}} X_{\underline{BE}}{}^{\underline{D}} - X_{\underline{BC}}{}^{\underline{E}} X_{\underline{AE}}{}^{\underline{D}} + X_{\underline{AB}}{}^{\underline{E}} X_{\underline{EC}}{}^{\underline{D}} = -2\,\partial_{[\underline{A}} X_{\underline{B}]\underline{C}}{}^{\underline{D}} + 12\,\mathbb{P}_{\underline{F}}{}^{\underline{E}}{}_{\underline{C}}{}^{\underline{D}}\partial_{\underline{E}} X_{\underline{AB}}{}^{\underline{F}}\,. \tag{3.107}$$

[1] Put differently, the intrinsic torsion (3.95) is defined with respect the generalised identity structure given by the generalised parallelisation of the well-defined $U_{\underline{A}}{}^{M}$. For more general backgrounds, we can define an intrinsic torsion with respect to the G-structure of the background, but this will have a different algebraic structure than $X_{\underline{AB}}{}^{\underline{C}}$.

For constant X, with the r.h.s. vanishing, this reproduces the quadratic constraints of gauged Supergravity (2.59) [147]. By tracing (3.107), we recover (3.106) as well as the relation

$$X_{\underline{AC}}{}^{\underline{D}}\, X_{\underline{BD}}{}^{\underline{C}} - 2\, X_{\underline{CA}}{}^{\underline{D}}\, X_{\underline{DB}}{}^{\underline{C}} = 0\,. \tag{3.108}$$

Finally, the section condition (2.85) also implies a quadratic relation linking the flat derivatives $\partial_{\underline{A}}$ and $X_{\underline{AB}}{}^{\underline{C}}$. This can be deduced from the closure of the generalised Lie derivative (3.102) when acting on scalar functions $f(y)$. Then, we have

$$2\,\partial_{[\underline{A}}\partial_{\underline{B}]} - X_{\underline{AB}}{}^{\underline{C}}\,\partial_{\underline{C}} = 0\,. \tag{3.109}$$

We will use these relations throughout the following sections in deriving the mass operators.

D.3 Mass operators

We are now ready to plug the fluctuation Ansatz (3.55) into the linearised equations of motions resulting from (2.77) and (2.78) to obtain the mass operators. The computation closely follows the one presented in [99, 116] with additional contributions arising from internal derivatives acting on the intrinsic torsion which is no longer constant.

a Spin-2

We begin with the linearised equations of motion for the spin-2 fields. The mass terms come from the final two terms of the potential (2.94), i.e.

$$\mathcal{L}_{\text{mass},g} = \frac{1}{4}\sqrt{|g|}\left(\mathscr{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu} + \mathscr{M}^{MN} g^{-2}\partial_M g\partial_N g\right)\,. \tag{3.110}$$

Upon computing the resulting linearised equations of motion and inserting the spin-2 fluctuation Ansatz (3.55), we obtain (upon gauge fixing in the external space)

$$\mathcal{Y}^{\Sigma}\,\Box_x h_{\mu\nu\,\Sigma} = h_{\mu\nu\,\Sigma}\left(\mathbb{M}_{\text{spin}-2}\right)\mathcal{Y}^{\Sigma} + \dots\,, \tag{3.111}$$

with $\Box_x$ the 4-dimensional Laplace operator and the mass operator $\left(\mathbb{M}_{\text{spin}-2}\right)$ given by

$$\left(\mathbb{M}_{\text{spin}-2}\right) = -\partial_{\underline{A}}\partial_{\underline{A}}\,, \tag{3.112}$$

acting on scalar harmonics $\mathcal{Y}_{\Sigma}$. The ellipses $\dots$ in (3.111) refer again to couplings to vector and scalar modes, which accounts for the spin-2 Higgs mechanism and do not affect the spin-2 masses. As we explained above in (3.60), the $\pi_{\underline{A}}$ (3.62) matrix will control the spin-2 Higgs mechanism. Note that because we came back to operators, and not matrices anymore, we have here

$$\pi_{\underline{A}} = \partial_{\underline{A}}\,. \tag{3.113}$$

b Spin-1

The vector masses arise from the standard Higgs mechanism and thus from the couplings between vectors and scalar fields in the scalar kinetic of (2.77). Thus, let us consider the linearised covariant

derivatives of the scalar fields, similarly to what we did in (3.77)

$$\mathscr{D}_\mu \mathscr{M}_{MN} = U_M{}^{\underline{A}}\, U_N{}^{\underline{B}} \left[\mathbb{T}_{\alpha,\underline{A}}{}^{\underline{B}}\, \partial_\mu \phi^{\alpha,\Sigma}\, \mathcal{Y}_\Sigma - \left(\mathscr{A}_\mu \bullet \phi\right)_{\underline{A}}{}^{\underline{B}} - \mathbb{T}_{\alpha,\underline{A}}{}^{\underline{B}}\, \Pi^\alpha{}_{\underline{C}} \left(\mathscr{A}_\mu{}^{\underline{C}}\right) \right] , \tag{3.114}$$

where we defined

$$\begin{aligned} \left(\mathscr{A}_\mu \bullet \phi\right)_{\underline{A}}{}^{\underline{B}} &\equiv \mathbb{T}_{\alpha,\underline{C}}{}^{\underline{D}}\, \phi^{\alpha,\Sigma}\, U_{\underline{A}}{}^{(M}\, U_{\underline{B}}{}^{N)}\, \mathscr{L}_{\mathscr{A}_\mu} \left(U_M{}^{\underline{C}}\, U_N{}^{\underline{D}}\, \mathcal{Y}_\Sigma \right) , \\ \Pi^I{}_{\underline{A}} \left(\mathscr{A}_\mu{}^{\underline{A}}\right) &\equiv 2\, \mathbb{T}^\alpha{}_{\underline{C}}{}^{\underline{B}}\, U_{\underline{C}}{}^M\, \mathscr{L}_{\mathscr{A}_\mu} U_M{}^{\underline{B}} \\ &= -2 \left(X_{\underline{A}}{}^\alpha - 12\, \mathbb{T}^\alpha{}_{\underline{A}}{}^{\underline{B}}\, \partial_{\underline{B}} \right) A_\mu{}^{\underline{A}} , \\ X_{\underline{A}}{}^\alpha &\equiv X_{\underline{A}\underline{B}}{}^{\underline{C}}\, \mathbb{T}^{\alpha,\underline{B}}{}_{\underline{C}} , \end{aligned} \tag{3.115}$$

and $\mathscr{A}_\mu{}^{\underline{A}} = \mathscr{A}_\mu{}^{\underline{A}\Sigma}\, \mathcal{Y}_\Sigma$. Coset indices α, β are raised and lowered with the non-compact part of the Cartan-Killing form. From (3.114), (3.115), we obtain the operator

$$\Pi^I{}_{\underline{A}} = -2 \left(X_{\underline{A}}{}^I - 12\, \mathbb{T}^\alpha{}_{\underline{A}}{}^{\underline{B}}\, \partial_{\underline{B}} \right) , \tag{3.116}$$

which will be responsible for the vector masses. To express the mass operator, we will also need the adjoint operator, $\Pi_{\underline{A}}{}^I$, of (3.116), defined as

$$\int dY \phi_\alpha\, \Pi^\alpha{}_{\underline{A}} \left(A_\mu{}^{\underline{A}} \right) = \int dY\, \Pi_{\underline{A}}{}^\alpha\, (\phi_\alpha)\, A_\mu{}^{\underline{A}} , \tag{3.117}$$

where $\phi_\alpha = j_{\alpha\,\Sigma} \mathcal{Y}^\Sigma$. Evaluating (3.117) explicitly, we get

$$\Pi_{\underline{A},\alpha} = -2 \left(X_{\underline{A}\alpha} + 12\, \mathcal{P}_{\alpha,\underline{A}}{}^{\underline{B}}\, \partial_{\underline{B}} \right) . \tag{3.118}$$

From the linearised equation of motion following from (2.77), upon usual gauge fixing in the external space

$$\Box_x A_\mu{}^{\underline{A}} = \left(\mathbb{M}_{\text{spin}-1} \right)^{\underline{A}}{}_{\underline{B}}\, \mathscr{A}_\mu{}^{\underline{B}} + \dots , \tag{3.119}$$

we find the mass operator given by the self-adjoint combination

$$\left(\mathbb{M}_{\text{spin}-1} \right)^{\underline{A}}{}_{\underline{B}} = \frac{1}{24}\, \Pi_{\underline{A},\alpha}\, \Pi^\alpha{}_{\underline{B}} . \tag{3.120}$$

Using (3.116) and (3.118), the mass operator takes the explicit form

$$\left(\mathbb{M}_{\text{spin}-1} \right)^{\underline{A}}{}_{\underline{B}} = \frac{1}{6} X_{\underline{A}}{}^\alpha\, X_{\underline{B}\alpha} + 2\, \mathbb{T}_{\alpha,\underline{A}}{}^{\underline{C}}\, \partial_{\underline{C}} X_{\underline{B}}{}^\alpha + 4\, \mathbb{T}_{\alpha,[\underline{A}}{}^{\underline{C}} X_{\underline{B}]}{}^\alpha\, \partial_{\underline{C}} - 24\, \mathbb{T}_{\alpha,\underline{A}}{}^{\underline{C}}\, \mathbb{T}^\alpha{}_{\underline{B}}{}^{\underline{D}}\, \partial_{\underline{C}}\, \partial_{\underline{D}} . \tag{3.121}$$

The mass operator (3.121) contains not only the physical spin-1 fields, but also the Goldstone vectors for the massive gravitons, as well as massless magnetic duals to all these. Thus, in evaluating the mass spectrum from (3.121), care must be taken to remove all these unphysical modes, e.g. by proper gauge fixing in the internal space.

However, we can further simplify the structure of the mass operator (3.121) by shifting the masses assigned to these unphysical modes. In particular, consider the magnetic dual of the mass

operator (3.121) given by

$$\begin{aligned}\left(\hat{\mathbb{M}}_{\text{spin}-1}\right)^{\underline{A}}{}_{\underline{B}} &\equiv \Omega_{\underline{AC}}\,\Omega^{\underline{BD}}\left(\mathbb{M}_{\text{spin}-1}\right)^{\underline{C}}{}_{\underline{D}} \\ &= \frac{1}{6}X^{\underline{B}}{}_{\alpha}\,X^{\underline{A}\alpha} + 2\,\mathbb{T}_{\alpha}{}^{\underline{AC}}\,\partial_{\underline{C}}X^{\underline{B}\alpha} + 4\,\mathbb{T}_{\alpha}{}^{\underline{C}[\underline{A}}X^{\underline{B}]\alpha}\,\partial_{\underline{C}} - 24\,\mathbb{T}_{\alpha}{}^{\underline{AC}}\,\mathbb{T}^{\alpha,\underline{BD}}\,\partial_{\underline{C}}\,\partial_{\underline{D}}\,,\end{aligned} \tag{3.122}$$

which carries the same eigenvalues as (3.121) while satisfying the orthogonality condition

$$\left(\mathbb{M}_{\text{spin}-1}\right)^{\underline{A}}{}_{\underline{B}}\left(\hat{\mathbb{M}}_{\text{spin}-1}\right)^{\underline{B}}{}_{\underline{C}} = 0\,. \tag{3.123}$$

Using (3.122) and (3.113), we can then deduce the relation

$$\begin{aligned}\left(\mathbb{M}_{\text{spin}-1}\right)^{\underline{A}}{}_{\underline{B}} + \left(\hat{\mathbb{M}}_{\text{spin}-1}\right)^{\underline{A}}{}_{\underline{B}} = \left(\mathbb{M}^{(0)}_{\text{spin}-1}\right)^{\underline{A}}{}_{\underline{B}} + \left(\mathbb{N}^{\underline{A}}{}_{\underline{B}}{}^{\underline{C}} - \mathbb{N}^{\underline{B}}{}_{\underline{A}}{}^{\underline{C}}\right)\partial_{\underline{C}} + \partial_{\underline{C}}\mathbb{N}^{\underline{A}}{}_{\underline{B}}{}^{\underline{C}} + \delta^{\underline{A}}_{\underline{B}}\left(\mathbb{M}_{\text{spin}-2}\right) \\ + 2\left(\Pi_{\underline{A}}\Pi_{\underline{B}} + \Pi^{\underline{A}}\Pi^{\underline{B}}\right)\,,\end{aligned} \tag{3.124}$$

with

$$\begin{aligned}\left(\mathbb{M}^{(0)}_{\text{spin}-1}\right)^{\underline{A}}{}_{\underline{B}} &= \frac{1}{6}\left(X_{\underline{A}\alpha}\,X_{\underline{B}}{}^{\alpha} + X^{\underline{B}}{}_{\alpha}\,X^{\underline{A}\alpha}\right)\,, \\ \mathbb{N}^{\underline{A}}{}_{\underline{B}}{}^{\underline{C}} &= 2\left(\mathbb{T}_{\alpha,\underline{A}}{}^{\underline{C}}\,X_{\underline{B}}{}^{\alpha} + \mathbb{T}_{\alpha}{}^{\underline{AC}}\,X^{\underline{B},\alpha}\right)\,.\end{aligned} \tag{3.125}$$

The second line in (3.124) acts only on the unphysical Goldstone fields, and can thus be ignored when computing the masses of the propagating degrees of freedom. Moreover, relations (3.122) and (3.123) imply that the first line of (3.124) carries as eigenvalues all the masses of the physical vector fields with an (unphysical) multiplicity of two, which has to be divided out. Equation (3.124) turns out to be very useful for the concrete computations as in particular the quadratic action on the scalar harmonics is exactly given by the spin-2 mass operator $\left(\mathbb{M}_{\text{spin}-2}\right)$ (3.112).

c Spin-0

The scalar masses arise from the scalar potential (2.94) in the action (2.77). Let us first rewrite the equations of motion coming from the scalar potential (2.94) in terms of the intrinsic torsion. We find that a vacuum must satisfy the equation

$$0 = \left[2\,\partial_{\underline{C}}X_{\underline{CA}}{}^{\underline{B}} - X_{\underline{AC}}{}^{\underline{D}}X_{\underline{BD}}{}^{\underline{C}} - \frac{1}{7}\left(X_{\underline{AC}}{}^{\underline{D}}X_{\underline{BC}}{}^{\underline{D}} + X_{\underline{CA}}{}^{\underline{D}}X_{\underline{CB}}{}^{\underline{D}} - X_{\underline{CD}}{}^{\underline{A}}X_{\underline{CD}}{}^{\underline{B}}\right)\right]\mathbb{T}_{\alpha,\underline{A}}{}^{\underline{B}}\,. \tag{3.126}$$

When the intrinsic torsion $X_{\underline{AB}}{}^{\underline{C}}$ is constant, this precisely matches the variation of the $\mathcal{N} = 8$ gauged Supergravity potential.

Similar to the computation presented in [99, 116], we can plug in the fluctuation Ansatz (3.55) and compute the variation of the potential (2.94) with respect to the scalar fluctuations $\phi^{\alpha} = \phi^{\alpha,\Sigma}\mathcal{Y}_{\Sigma}$ to obtain the equation of motions

$$\Box_x\phi^{\alpha} = \left(\mathbb{M}_{\text{spin}-0}\right)^{\alpha}{}_{\beta}\phi^{\beta} + \dots\,, \tag{3.127}$$

where we can obtain the explicit form of the mass operator as

$$\begin{aligned}\left(\mathbb{M}_{\text{spin}-0}\right)^{\alpha}{}_{\beta} &= X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BF}}{}^{\underline{E}} \left(\mathbb{T}^{\alpha}\mathbb{T}_{\beta}\right)_{\underline{A}}{}^{\underline{B}} \\ &+ \frac{1}{7}\left(X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BE}}{}^{\underline{F}} + X_{\underline{EA}}{}^{\underline{F}} X_{\underline{EB}}{}^{\underline{F}} + X_{\underline{EF}}{}^{\underline{A}} X_{\underline{EF}}{}^{\underline{B}}\right)\left(\mathbb{T}^{\alpha}\mathbb{T}_{\beta}\right)_{\underline{A}}{}^{\underline{B}} \\ &+ \frac{2}{7}\left(X_{\underline{AC}}{}^{\underline{E}} X_{\underline{BD}}{}^{\underline{E}} - X_{\underline{AE}}{}^{\underline{C}} X_{\underline{BE}}{}^{\underline{D}} - X_{\underline{EA}}{}^{\underline{C}} X_{\underline{EB}}{}^{\underline{D}}\right)\left(\mathbb{T}^{\alpha}\right)_{\underline{A}}{}^{\underline{B}}\left(\mathbb{T}_{\beta}\right)_{\underline{C}}{}^{\underline{D}} \\ &- 2\left(\mathbb{T}_{\beta}\right)_{C}{}^{D}\,\partial_{C} X_{\underline{D}}{}^{\alpha} - \left[\mathbb{T}^{\alpha}, \mathbb{T}_{\beta}\right]_{\underline{A}}{}^{\underline{B}}\partial_{\underline{C}} X_{\underline{CB}}{}^{\underline{A}} \\ &+ 2\left(\left(\mathbb{T}^{\alpha}\right)_{\underline{A}}{}^{\underline{B}} X_{\underline{A}\beta} - \left(\mathbb{T}_{\beta}\right)_{\underline{A}}{}^{\underline{B}} X_{\underline{A}}{}^{\alpha}\right)\partial_{B} - 2\left[\mathbb{T}^{\alpha}, \mathbb{T}_{\beta}\right]_{\underline{A}}{}^{\underline{B}} X_{\underline{CB}}{}^{\underline{A}}\partial_{\underline{C}} \\ &- \delta^{\alpha}_{\beta}\,\partial_{\underline{A}}\partial_{\underline{A}} + 24\left(\mathbb{T}^{\alpha}\mathbb{T}_{\beta}\right)_{\underline{A}}{}^{\underline{B}}\partial_{\underline{B}}\partial_{\underline{A}}\,.\end{aligned} \tag{3.128}$$

Note the similarities of this mass operator with (3.75). This is expected: by back setting X to a y-constant tensor and substituting $\partial_{\underline{A}}$ with $\mathscr{T}_{\underline{A}}$, one can recover (3.75). It is straightforward to verify that (3.128) is self-adjoint and thus has real eigenvalues. Once again, this operator yields mass eigenvalues not just for the physical scalars but also for the Goldstone scalars that are eaten by the massive spin-1 and spin-2 fields. Since we are not interested in these unphysical fields, and can gauge fix them away, we are free to shift their mass eigenvalues in a way that simplifies the structure of (3.128). Thanks to the Higgs mechanism, the operators (3.116), (3.118) provide us with projection matrices onto the Goldstone scalars, which we can therefore use to add to (3.121) terms of the form $\Pi^{\alpha}{}_{\underline{A}}\,\Pi^{\underline{A}}{}_{\beta}$ which only affect the eigenvalues of the non-physical Goldstone modes.

This allows us to rewrite (3.128) as

$$\begin{aligned}\left(\mathbb{M}_{\text{spin}-0}\right)^{\alpha}{}_{\beta} = \left(\mathbb{M}^{(0)}_{\text{spin}-0}\right)^{\alpha}{}_{\beta} + \left(\mathbb{N}^{\alpha}{}_{\beta}{}^{\underline{C}} - \mathbb{N}_{\beta}{}^{\alpha\underline{C}}\right)\partial_{\underline{C}} + \partial_{\underline{C}}\mathbb{N}^{\alpha}{}_{\beta}{}^{\underline{C}} + \delta^{\alpha}_{\beta}\left(\mathbb{M}_{\text{spin}-2}\right) \\ - \frac{1}{24}\Pi^{\alpha}{}_{\underline{A}}\,\Pi^{\underline{A}}{}_{\beta}\,,\end{aligned} \tag{3.129}$$

which is easily verified to be self-adjoint. The operators in (3.129) consist of

$$\begin{aligned}\left(\mathbb{M}^{(0)}_{\text{spin}-0}\right)^{\alpha}{}_{\beta} &= X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BF}}{}^{\underline{E}} \left(\mathbb{T}^{\alpha}\mathbb{T}_{\beta}\right)_{\underline{A}}{}^{\underline{B}} \\ &+ \frac{1}{7}\left(X_{\underline{AE}}{}^{\underline{F}} X_{\underline{BE}}{}^{\underline{F}} + X_{\underline{EA}}{}^{\underline{F}} X_{\underline{EB}}{}^{\underline{F}} + X_{\underline{EF}}{}^{\underline{A}} X_{\underline{EF}}{}^{\underline{B}}\right)\left(\mathbb{T}^{\alpha}\mathbb{T}_{\beta}\right)_{\underline{A}}{}^{\underline{B}} \\ &+ \frac{2}{7}\left(X_{\underline{AC}}{}^{\underline{E}} X_{\underline{BD}}{}^{\underline{E}} - X_{\underline{AE}}{}^{\underline{C}} X_{\underline{BE}}{}^{\underline{D}} - X_{\underline{EA}}{}^{\underline{C}} X_{\underline{EB}}{}^{\underline{D}}\right)\left(\mathbb{T}^{\alpha}\right)_{\underline{A}}{}^{\underline{B}}\left(\mathbb{T}_{\beta}\right)_{\underline{C}}{}^{\underline{D}} \\ &+ \frac{1}{6}\,X_{\underline{A}}{}^{\alpha}\,X_{\underline{A},\beta}\,,\end{aligned} \tag{3.130}$$

which is quadratic in the intrinsic torsion $X_{\underline{AB}}{}^{\underline{C}}$ and does not act on the scalar harmonics, and the combination

$$\mathbb{N}^{\alpha}{}_{\beta}{}^{\underline{C}} = -2X_{\underline{A}}{}^{\alpha}\mathbb{T}_{\beta,\underline{C}}{}^{\underline{A}} - 2X_{\underline{A}\beta}\mathbb{T}^{\alpha}{}_{\underline{C}}{}^{\underline{A}} - \left[\mathbb{T}^{\alpha},\,\mathbb{T}_{\beta}\right]_{\underline{A}}{}^{\underline{B}}\left(X_{\underline{CB}}{}^{\underline{A}} + \frac{7}{2}X_{\underline{AB}}{}^{\underline{C}}\right)\,, \tag{3.131}$$

which multiplies a linear differential operator on the scalar harmonics. Just as in the vector mass matrix (3.124), the quadratic differential operator on the scalar harmonics in (3.129) is simply given by the graviton mass operator (3.112). For the case of constant intrinsic torsion, the formula (3.129) consistently reduces to the expression derived in [120].

E The squashed S^7 : the spectrum

As an application of the presented methods, we will now apply the mass formulas to compute the Kaluza-Klein spectrum of the squashed S^7 in 11-dimensional Supergravity. The sphere S^7 admits two Einstein metrics: the round metric with $\mathrm{SO}(8)$ isometry, and the "squashed metric" which only preserves $\mathrm{USp}(4) \times \mathrm{SU}(2) \subset \mathrm{SO}(8)$ isometry. These give rise to two supersymmetric Freund-Rubin $\mathrm{AdS}_4 \times S^7$ vacua of 11-dimensional Supergravity: the $\mathcal{N} = 8$ vacuum, when the S^7 is round and an $\mathcal{N} = 1$ vacuum for the squashed S^7 [136]. For the squashed S^7, the isometry group $\mathrm{USp}(4) \times \mathrm{SU}(2)$ is embedded into $\mathrm{SO}(8)$ such that

$$\mathbf{8}_v \to (\mathbf{4},\mathbf{2})\,, \qquad \mathbf{8}_s \to (\mathbf{4},\mathbf{2})\,, \qquad \mathbf{8}_c \to (\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})\,, \tag{3.132}$$

often also referred to as the left-squashed S^7.[2] Note that there are two other embeddings of $\mathrm{USp}(4) \times \mathrm{SU}(2) \subset \mathrm{SO}(8)$, related to (3.132) by triality. The embedding

$$\mathbf{8}_s \to (\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})\,, \tag{3.133}$$

gives rise to the right-squashed S^7, with the same metric as (3.132), but different sign of flux, yielding a non-supersymmetric AdS_4 vacuum [137]. Finally, the embedding

$$\mathbf{8}_v \to (\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})\,, \tag{3.134}$$

does not give rise to an Einstein space, and hence no AdS_4 vacuum. Here we will mostly focus on the supersymmetric, left-squashed $\mathrm{AdS}_4 \times S^7$ vacuum (3.132), but the results also allow to fully determine the non-supersymmetric spectrum on the right-squashed S^7.

E.1 The squashed S^7 in ExFT

The round S^7 has already been extensively studied in the ExFT framework. It is a generalised Leibniz parallelisable background, whose twist matrix U consists of an $\mathrm{SL}(8) \subset \mathrm{E}_{7(7)}$ matrix [145, 146]. As a Freund-Rubin solution, the generalised vielbein of the S^7 solution lives on the coset space

$$\frac{\mathrm{SL}(8)}{\mathrm{SO}(8)} \subset \frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)}\,, \tag{3.135}$$

which contains precisely the right degrees of freedom to capture a 7-dimensional internal metric and 6-form potential. Explicitly, a general Freund-Rubin solution is described a generalised vielbein of the form

$$U_{\mathrm{FR}} = \exp[-6\,\alpha\,\mathring{\omega}\,\zeta^n T_n] \begin{pmatrix} \mathring{\omega}^{3/4} & 0 \\ 0 & \mathring{\omega}^{-1/4}\,\mathring{e}_m{}^i \end{pmatrix} \in \mathrm{SL}(8)\,, \tag{3.136}$$

where $\mathring{e}_m{}^i$ is the vielbein on the internal seven-dimensional space, $\mathring{\omega} \equiv \det \mathring{e}_m{}^i$, and ζ^n is a vector field with $\mathring{\nabla}_n \zeta^n = 1$. The T_n are the generators which extend $\mathfrak{gl}(7)$ to $\mathfrak{sl}(8)$, and α is a constant related to the seven-form flux of the solution. In our conventions, the round S^7 solution corresponds to a sphere of radius 1, and $\alpha = 1$. Upon embedding $\mathrm{SL}(8) \hookrightarrow \mathrm{E}_{7(7)}$, the generalised vielbein is

[2] Here, we use standard triality conventions, in which the $\mathcal{N} = 8$ gravitinos transform in the $\mathbf{8}_s$ and the round S^7 sphere harmonics in symmetric tensor products of the $\mathbf{8}_v$.

related to the generalised metric of (2.77) as $\mathscr{M} = UU^T$. The twist matrix describing the consistent truncation around the S^7 background is explicitly given by [145, 146]

$$\mathring{U}_{\underline{m}}{}^{a}(\mathcal{Y}) = \begin{pmatrix} \mathring{\omega}^{3/4}\,(\mathcal{Y}^a - 6\,\alpha\,\zeta^n \partial_n \mathcal{Y}^a) \\ \mathring{\omega}^{-1/4}\,\partial_m \mathcal{Y}^a \end{pmatrix} \Bigg|_{\alpha=1} \in \mathrm{SL}(8)\,, \qquad \underline{m} = \{0, m\}\,, \quad a = \{1, \ldots, 8\}\,, \tag{3.137}$$

in terms of the fundamental sphere harmonics $\mathcal{Y}^a\mathcal{Y}^a = 1$. It differs from (3.136) by an $\mathrm{SO}(8)$ rotation from the right, such that consistently $\mathring{U}\mathring{U}^T = \mathscr{M} = UU^T$. The scale factor ρ is given by $\rho = \mathring{\omega}^{-1/2}$. This twist matrix satisfies (3.94) with constant intrinsic torsion.

We will now give a similar ExFT description of the squashed S^7. First of all, since the topology is the same as the round S^7, the squashed S^7 is also a generalised parallelisable background, i.e. it can be described by a globally defined $\mathrm{E}_{7(7)}$ twist matrix. Moreover, since the squashed S^7 is a Freund-Rubin vacuum, by the argument above, the twist matrix should again be an $\mathrm{SL}(8) \subset \mathrm{E}_{7(7)}$ element. Finally, the twist matrix must be a continuous deformation of the one corresponding to the round S^7, and the deformation must preserve $\mathrm{USp}(4) \times \mathrm{SU}(2)$, see (3.132).

Let us thus consider the decomposition $\mathrm{E}_{7(7)} \to \mathrm{SL}(5) \times \mathrm{SL}(3) \times \mathbb{R}^+$, such that the isometry of the squashed S^7 is embedded as the compact subgroup $\mathrm{USp}(4) \times \mathrm{SU}(2) \subset \mathrm{SL}(5) \times \mathrm{SL}(3)$. Under this decomposition, the $\mathrm{E}_{7(7)}$ adjoint representation branches as follows

$+3$	$(\mathbf{5}, \mathbf{1})$	$[0,1,0]$
$+2$	$(\overline{\mathbf{5}}, \overline{\mathbf{3}})$	$[0,1,2]$
$+1$	$(\mathbf{10}, \mathbf{3})$	$[2,0,2]$
0	$(\mathbf{1}, \mathbf{1}) \oplus \mathfrak{sl}(5) \oplus \mathfrak{sl}(3)$	$[0,0,0] \oplus [2,0,0] \oplus [0,2,0] \oplus [0,0,2] \oplus [0,0,4]$
-1	$(\overline{\mathbf{10}}, \overline{\mathbf{3}})$	$[2,0,2]$
-2	$(\mathbf{5}, \mathbf{3})$	$[0,1,2]$
-3	$(\overline{\mathbf{5}}, \mathbf{1})$	$[0,1,0]$

(3.138)

where the vertical axis labels the $\mathbb{R}^+$ charge of the representations. The last column collects the $\mathrm{USp}(4) \times \mathrm{SU}(2)$ representation content described by their Dynkin labels. To construct $\mathrm{USp}(4) \times \mathrm{SU}(2)$-invariant deformations, we consider linear combinations of the $\mathrm{E}_{7(7)}$ generators which depend on the S^7 coordinates

$$c(y)_\alpha\, \mathbb{T}^\alpha = \sum_\Sigma c_{\alpha,\Sigma}\, \mathcal{Y}^\Sigma\, \mathbb{T}^\alpha\,, \tag{3.139}$$

with the scalar harmonics $\mathcal{Y}^\Sigma$ on the round S^7. These harmonics combine into the tower of representations

$$\sum_n [n,0,0,0]_{\mathrm{SO}(8)} \to \sum_{n,q} [n-2q, q, n-2q]_{\mathrm{USp}(4)\times\mathrm{SU}(2)}\,, \tag{3.140}$$

under $\mathrm{SO}(8)$ and $\mathrm{USp}(4) \times \mathrm{SU}(2)$, respectively. Combining this expansion with the decomposition (3.138) shows four $\mathrm{USp}(4) \times \mathrm{SU}(2)$ invariant combinations in (3.139): one at KK level $n = 0$, coming from the $[0,0,0]$ generator, two at KK level $n = 2$, coming from the noncompact generators

in the $[2,0,2]$ and $[0,1,0]$, and one at KK level four from the generator in the $[0,2,0]$. Closer inspection shows that only the generators in the $[2,0,2]$ and $[0,1,0]$ belong to the $\mathfrak{sl}(8)$ subalgebra of (3.135) corresponding to Freund-Rubin configurations.[3] We can thus construct a two-parameter family of SL(8) twist matrices, interpolating between the round S^7 and the squashed S^7. We choose to parametrise them as

$$U(\alpha,\eta) = \mathring{U}(\alpha)\, e^{\eta\, T_{(5,1)}(\mathcal{Y})}\ , \tag{3.141}$$

with $\mathring{U}(\alpha)$ from (3.137), however with free flux parameter α, and η an extra parameter associated with the generator $T_{(5,1)}(\mathcal{Y})$, which denotes the $\mathrm{USp}(4)\times\mathrm{SU}(2)$ invariant contraction of non-compact generators in the $[0,1,0]$ in (3.138) with the round S^7 harmonics. Apart from the twist matrix for the round sphere $U(1,0)$, the intrinsic torsion (3.95) associated to (3.141) depends on the S^7 coordinates. The field equations of $D=11$ Supergravity for this background can be expressed in terms of the intrinsic torsion as (3.126) and turn out to be identically satisfied for the values

$$\{\eta=0\,,\ \alpha=\pm 1\}\ , \qquad \left\{\eta=-\frac{1}{2}\log 5\,,\ \alpha=\pm\frac{3}{5}\right\}\ . \tag{3.142}$$

It is straightforward to verify that the internal seven-dimensional metric obtained from (3.141) is an Einstein metric precisely for these values of the parameters. The first solution in (3.142) describes the round sphere and its skew-whiffed counterpart, obtained by flipping the sign of the seven-form flux. The second solution in (3.142) describes the left- and right-squashed spheres. With the Killing vector fields on the round sphere given by $K_{ab}{}^{m} = 2\,\mathcal{Y}_{[a}\partial^{m}\mathcal{Y}_{b]}$ (where the vector index on the r.h.s. is raised with the round S^7 metric), the above generalised vielbein induces the following explicit metric

$$g^{mn}_{(\eta)} = \frac{1}{2}\,\mathcal{K}_{ab}{}^{m}\mathcal{K}_{ab}{}^{n} + \frac{1}{8}\,(e^{-2\eta}-1)\,\Gamma^{ab}_{ij}\Gamma^{cd}_{ij}\,\mathcal{K}_{ab}{}^{m}\mathcal{K}_{cd}{}^{n}\ , \tag{3.143}$$

on the squashed sphere. Here, the Γ^{ab} are the SO(8) Γ matrices, with spinor indices i,j running over the range $\{1,2,3\}$, in accordance with the breaking (3.132).

E.2 The Kaluza-Klein spectrum on the left-squashed sphere

We can now use the methods outlined in section 3.D to compute the full Kaluza-Klein spectrum of the squashed S^7. Since the squashed S^7 can also be described as the coset space

$$\frac{\mathrm{USp}(4)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{SU}(2)}\ , \tag{3.144}$$

the KK spectrum can also, in principle, be computed using group theory techniques [114]. However, because the squashed S^7 is not a symmetric space, the resulting procedure is still rather intricate, although many partial results have been collected over the years [141, 148–152]. In particular, while the set of potential mass eigenvalues of all different bosonic KK towers has been analysed to some extent (and is complete as we shall show), the traditional Kaluza-Klein computational scheme struggles to assign the eigenvalues with possible multiplicities to the correct eigenstates.

Here, we will determine the full KK spectrum on both, left- and right-squashed S^7 by evaluat-

[3] It is important to note that this Freund-Rubin $\mathfrak{sl}(8)$ does not fully contain the $\mathfrak{sl}(5)\oplus\mathfrak{sl}(3)$ subgroup appearing at zero charge in (3.138).

ing the ExFT mass formulas. This straightforwardly provides not only the mass eigenvalues but also the corresponding eigenstates and multiplicities. In order to diagonalise the differential operators (3.112), (3.121), (3.128), we evaluate them on general polynomials in the fundamental round harmonics $\mathcal{Y}^q$. In contrast to previous ExFT computations based on [99, 116], the y-dependence of the intrinsic torsion computed from (3.141), thus of the coefficients in the mass operators, induces a level mixing for the mass eigenstates. I.e. the action of the mass operators on a given polynomial of harmonics does not preserve the order of the polynomial. However, the symmetry group $\mathrm{USp}(4) \times \mathrm{SU}(2)$ is still large enough to keep the problem manageable. In particular, the fact that the tower of harmonics (3.140) does not carry any non-trivial multiplicities implies that any given representation appears only a finite number of times in the full KK spectrum. In order to determine its mass eigenvalues, it is thus sufficient to evaluate the corresponding mass operator on a sufficiently large polynomial of harmonics after projection onto the relevant representation. The computation can be further reduced by further projecting the polynomial onto highest weight states of $\mathrm{USp}(4) \times \mathrm{SU}(2)$. Concretely, we have pushed the computation up to Kaluza-Klein level $n = 8$ which together with the underlying supersymmetry is sufficient to extract the generic structure.

In the following, we summarise our results as well as details allowing us to match the results of [141, 148–152]. We start with the left-squashed sphere, for which the states organise into long $\mathcal{N} = 1$ supermultiplets. A generic long multiplet $L[J, \Delta]$ is identified by the (space-time) spin J and the conformal dimension Δ of its superconformal primary. The conformal dimension of the primary is bounded by $\Delta > J + 1$. The long multiplets with fields of spin no higher than 2 consist of the following Supergravity fields:

$$\begin{aligned}
L[\tfrac{3}{2}, \Delta] &: \psi_\mu \xrightarrow{Q} g_{\mu\nu} \oplus A_\mu \xrightarrow{Q} \psi_\mu \,, \\
L[1, \Delta] &: A_\mu \xrightarrow{Q} \psi_\mu \oplus \lambda \xrightarrow{Q} A_\mu \,, \\
L[\tfrac{1}{2}, \Delta] &: \lambda \xrightarrow{Q} A_\mu \oplus \phi \xrightarrow{Q} \lambda \,, \\
L[0, \Delta] &: \phi \xrightarrow{Q} \lambda \xrightarrow{Q} \phi \,,
\end{aligned} \tag{3.145}$$

where $g_{\mu\nu}$ denotes a spin-2 field, ψ_μ denotes a spin-3/2 field, A_μ denotes a spin-1 field, λ denotes a spin-1/2 field and ϕ a scalar. In addition, the spectrum contains short multiplets $A_1[J]$ with $\Delta = 1 + J$, which carry the gauge fields. These are

$$\begin{aligned}
A_1[\tfrac{3}{2}] &: \psi_\mu \xrightarrow{Q} g_{\mu\nu} \,, \\
A_1[\tfrac{1}{2}] &: \lambda \xrightarrow{Q} A_\mu \,.
\end{aligned} \tag{3.146}$$

The entire masse spectrum organises into a sum of long multiplets

$$\bigoplus L[J, \Delta] \otimes [p, q, r] \,, \tag{3.147}$$

in the different $\mathrm{USp}(4) \times \mathrm{SU}(2)$ representations $[p,q,r]$. Each such multiplet comes with a certain multiplicity. Remarkably, we find that the conformal dimensions of all these multiplets are captured by the universal formula

$$\Delta_{J,s} = 1 + \frac{5}{3}s + \frac{1}{3}\sqrt{(3J+2s^2)^2 + 5\,\mathcal{C}_3}\,, \tag{3.148}$$

in terms of the spin J and the combination

$$\mathcal{C}_3 = \mathcal{C}_{p,q} + 3\,\mathcal{C}_r\,, \tag{3.149}$$

of the $\mathrm{USp}(4)$ and the $\mathrm{SU}(2)$ Casimir operators

$$\mathcal{C}_{p,q} = \frac{1}{2}\left(p^2 + 2\,q^2 + 4\,p + 6\,q + 2\,p\,q\right)\,, \qquad \mathcal{C}_r = \frac{1}{4}r(r+2)\,. \tag{3.150}$$

The parameter $s \in \frac{1}{2}\mathbb{Z}$ in (3.148) is an additional label that organises the spectrum and counts the multiplicities. To present the spectrum in compact form, we use the following notation

$$L[J] \otimes \{s_1, s_2, \ldots, s_p\} \equiv \bigoplus_{i=1}^{p} L[J, \Delta_{J,s_i}]\,, \tag{3.151}$$

with conformal dimensions $\Delta_{J,s}$ given by (3.148). Remarkably, for all but a handful of small representations, s in fact appears like the $\mathbb{R}^+ \subset \mathrm{SL}(2)$ charge of a full $\mathrm{SL}(2)$ representation. Accordingly, we use the notation

$$L[J] \otimes [S] \equiv L[J] \otimes \{-S, -S+1, \ldots, S\}\,. \tag{3.152}$$

Let us take as an example the states in a $[k,q,k]$ of $\mathrm{USp}(4) \times \mathrm{SU}(2)$ for generic values of k,q (i.e. $k > 1$, $q > 1$). The KK spectrum exhibits one spin-2 state, 9 vectors and 16 scalar fields in this representation. They turn out to fall into 13 $\mathcal{N} = 1$ supermultiplets which in the notation (3.152) take the form

$$[k,q,k]_{k>1,q>1}: \quad L[\tfrac{3}{2}] \otimes [0] \;\oplus\; L[1] \otimes [\tfrac{1}{2}] \;\oplus\; L[\tfrac{1}{2}] \otimes [\tfrac{1}{2} \otimes \tfrac{1}{2}] \;\oplus\; L[0] \otimes [\tfrac{1}{2} \otimes 1]\,. \tag{3.153}$$

Similarly, the supermultiplets in the other generic towers of $\mathrm{USp}(4) \times \mathrm{SU}(2)$ representations can be summarised as

$$\begin{aligned} [k,q,k+2]_{k>0,q>1}\,\&\,[k+2,q,k]_{k>0,q>0}: &\quad L[0] \otimes [\tfrac{1}{2}] \;\oplus\; L[1] \otimes [\tfrac{1}{2}] \;\oplus\; L[\tfrac{1}{2}] \otimes [\tfrac{1}{2} \otimes \tfrac{1}{2}]\,, \\ [k,q,k+4]_{q>1}\,\&\,[k+4,q,k]: &\quad L[\tfrac{1}{2}] \otimes [0] \;\oplus\; L[0] \otimes [\tfrac{1}{2}]\,. \end{aligned} \tag{3.154}$$

Translating the conformal dimensions into Supergravity masses, we find that all mass eigenvalues fit into the list of potential eigenvalues identified in [141, 148–152]. Seemingly missing eigenvalues on that list are explained by non-trivial multiplicities arising in the expansion of (3.153), (3.154). For vector fields, and sufficiently large values of k and q, the masses found in (3.153), (3.154) span the entire list of [149, 151]. For scalar fields, the masses realised in the KK spectrum (3.153), (3.154) fix all potential sign ambiguities in the general analysis. For small values of k and q, the general structure of the spectrum degenerates, as spelled out in (3.155), such that only a subset of the potential mass eigenvalues are realised.

Before comparing the above method to the traditional one, we summarise the $\mathcal{N} = 1$ supermultiplets in the Kaluza-Klein spectrum on the left-squashed S^7 which appear in $\mathrm{USp}(4) \times \mathrm{SU}(2)$ representations with small Dynkin labels, such that some of the generic structures (3.153), (3.154) degenerate. In particular, in some of these representation, the values of the label s do no longer combine into full $\mathrm{SL}(2)$ representations, such that in these cases we revert to the notation of (3.151). The full list of these supermultiplets is given by

$$
\begin{aligned}
[1,q,1]_{q>1}: &\quad L[\tfrac{3}{2}]\otimes[0] \,\oplus\, L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes[1] \,\oplus\, L[0]\otimes[\tfrac{3}{2}]\,,\\
[0,q,0]_{q>1}: &\quad L[\tfrac{3}{2}]\otimes[0] \,\oplus\, L[0]\otimes[\tfrac{3}{2}]\,,\\
[k,1,k]_{k>1}: &\quad L[\tfrac{3}{2}]\otimes[0] \,\oplus\, L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes[\tfrac{1}{2}\otimes\tfrac{1}{2}] \,\oplus\, L[0]\otimes[\tfrac{3}{2}] \,\oplus\, L[0]\otimes\{+\tfrac{1}{2}\}\,,\\
[k,0,k]_{k>1}: &\quad L[\tfrac{3}{2}]\otimes[0] \,\oplus\, L[1]\otimes\{-\tfrac{1}{2}\} \,\oplus\, L[\tfrac{1}{2}]\otimes\{0,+1\} \,\oplus\, L[0]\otimes[\tfrac{3}{2}]\,,\\
[1,1,1]: &\quad L[\tfrac{3}{2}]\otimes[0] \,\oplus\, L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes[1] \,\oplus\, L[0]\otimes\{-\tfrac{3}{2},+\tfrac{1}{2},+\tfrac{3}{2}\}\,,\\
[1,0,1]: &\quad L[\tfrac{3}{2}]\otimes[0] \,\oplus\, L[1]\otimes\{-\tfrac{1}{2}\} \,\oplus\, L[\tfrac{1}{2}]\otimes\{+1\} \,\oplus\, L[0]\otimes\{-\tfrac{3}{2},+\tfrac{3}{2}\}\,,\\
[0,1,0]: &\quad L[\tfrac{3}{2}]\otimes[0] \,\oplus\, L[0]\otimes\{-\tfrac{3}{2},+\tfrac{1}{2},+\tfrac{3}{2}\}\,,\\
[0,0,0]: &\quad L[0]\otimes\{-\tfrac{1}{2},\tfrac{3}{2}\} \,\oplus\, A_1[\tfrac{3}{2}]\,,\\
[k,1,k+2]_{k>0}: &\quad L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes[1] \,\oplus\, L[0]\otimes[\tfrac{1}{2}]\,,\\
[k,0,k+2]_{k>0}: &\quad L[1]\otimes\{+\tfrac{1}{2}\} \,\oplus\, L[\tfrac{1}{2}]\otimes\{-1,0\}\,,\\
[0,q,2]_{q>1}: &\quad L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes[1]\,,\\
[0,1,2]: &\quad L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes\{-1,+1\}\,,\\
[0,0,2]: &\quad L[1]\otimes\{+\tfrac{1}{2}\} \,\oplus\, A_1[\tfrac{1}{2}]\,,\\
[k+2,0,k]_{k>0}: &\quad L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes[1] \,\oplus\, L[0]\otimes\{+\tfrac{1}{2}\}\,,\\
[2,q,0]_{q>0}: &\quad L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes[1]\,,\\
[2,0,0]: &\quad L[1]\otimes[\tfrac{1}{2}] \,\oplus\, L[\tfrac{1}{2}]\otimes\{+1\} \,\oplus\, A_1[\tfrac{1}{2}]\,,\\
[k,1,k+4]: &\quad L[\tfrac{1}{2}]\otimes[0] \,\oplus\, L[0]\otimes\{-\tfrac{1}{2}\}\,,\\
[k,0,k+4]: &\quad L[0]\otimes\{-\tfrac{1}{2}\}\,.
\end{aligned}
\tag{3.155}
$$

Note that for the $L[0]\otimes\{-\frac{1}{2}\}\otimes[0,0,0]$ multiplet, our formula (3.148) yields the value $\Delta = \frac{1}{3}$, which lies below the unitary bound. This arises because KK spectroscopy strictly computes the mass eigenvalues in the AdS bulk, whereas their translation into conformal dimensions via $\Delta(\Delta-3) = m^2L^2$ allows for two solutions. For this multiplet, the other choice of solution of Δ gives the correct conformal dimension.[4]

This completes the full Kaluza-Klein spectrum.

It is instructive to compare the above results for the KK spectrum on the squashed spheres to the results obtained in the traditional computational scheme [141, 148–152]. In that approach, the eigenvalue spectra of the different internal Laplacian operators on the squashed sphere are determined by the coset space techniques based on the representation (3.144) of the internal space. These spectra are then combined with the universal formulas for mass operators appearing in Freund-Rubin compactifications [141]. While extensive knowledge of the Laplacian eigenvalue spectra has been accumulated in [141, 148–152], the assignment of these eigenvalues and their multiplicities to specific mass eigenstates appears less straightforward in that approach.

[4]We thank Joel Karlsson for drawing our attention to this.

Let us consider as an example the KK states in the $[k,q,k]$ representation (for generic values $k>1, q>1$) for the $\mathcal{N}=1$ left-squashed sphere. From (3.153), we find the multiplet structure

$$[k,q,k]_{k>1,q>1} \;:\; L[\tfrac{3}{2}]\otimes[0] \;\oplus\; L[\tfrac{3}{2}]\otimes[\tfrac{1}{2}] \;\oplus\; L[\tfrac{1}{2}]\otimes[\tfrac{1}{2}\otimes\tfrac{1}{2}] \;\oplus\; L[0]\otimes[\tfrac{1}{2}\otimes 1]\,. \tag{3.156}$$

Formula (3.148) yields the conformal dimensions of all fields. We may translate them into Supergravity masses by the standard $D=4$ formulas

$$\begin{aligned} \text{spin-0, 2}: &\quad \Delta(\Delta-3) = m^2\,\ell^2\,, \\ \text{spin-1}: &\quad (\Delta-1)(\Delta-2) = m^2\,\ell^2\,. \end{aligned} \tag{3.157}$$

In our conventions, and with the twist matrix from (3.141) the AdS length ℓ for the squashed S^7 is given by $\ell^2=\frac{5}{72}$. Evaluating the field content of the various supermultiplets in (3.156), we obtain the following masses for the different spin-2 and spin-1 modes

	Δ	m^2	$L[J]$	#
$g_{\mu\nu}$	$\frac{3}{2}+\frac{1}{6}\sqrt{81+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3$	$L[\frac{3}{2}]$	1
A_μ	$\frac{3}{2}+\frac{1}{6}\sqrt{81+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{144}{5}$	$L[\frac{3}{2}]$	1
	$\frac{1}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{208}{5}-\frac{32}{5}\sqrt{49+20\,\mathcal{C}_3}$	$L[1]\otimes\{-\frac{1}{2}\}$	1
	$\frac{7}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{88}{5}-\frac{8}{5}\sqrt{49+20\,\mathcal{C}_3}$	$L[1]\otimes\{-\frac{1}{2}\}$	1
	$\frac{11}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{88}{5}+\frac{8}{5}\sqrt{49+20\,\mathcal{C}_3}$	$L[1]\otimes\{+\frac{1}{2}\}$	1
	$\frac{17}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{208}{5}+\frac{32}{5}\sqrt{49+20\,\mathcal{C}_3}$	$L[1]\otimes\{+\frac{1}{2}\}$	1
	$-\frac{1}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{280}{5}-\frac{40}{5}\sqrt{49+20\,\mathcal{C}_3}$	$L[\frac{1}{2}]\otimes\{-1\}$	1
	$\frac{3}{2}+\frac{1}{6}\sqrt{9+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3$	$L[\frac{1}{2}]\otimes\{0\}$	2
	$\frac{19}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{280}{5}+\frac{40}{5}\sqrt{49+20\,\mathcal{C}_3}$	$L[\frac{1}{2}]\otimes\{+1\}$	1

(3.158)

where we also list their multiplicities and the supermultiplets to which they belong. Comparing to the previous results, we find that all the mass eigenvalues exhibited in (3.158) fit into and fully span the list identified in [149, 151, 152] (up to an overall normalisation factor in the definition of mass). On the other hand, the seemingly missing eigenvalue (of the operator Δ_2 in the notation of [151]) is precisely taken care of by the non-trivial multiplicity in the penultimate line of (3.158).

Similarly, we may extract the scalar masses from (3.156) as

	Δ	m^2	$L[J]$	#
ϕ	$-\frac{1}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{136}{5}-8\sqrt{49+20\,\mathcal{C}_3}$	$L[\frac{1}{2}]\otimes\{-1\}$	1
	$\frac{3}{2}+\frac{1}{6}\sqrt{9+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3-\frac{144}{5}$	$L[\frac{1}{2}]\otimes\{0\}$	2
	$\frac{19}{6}+\frac{1}{6}\sqrt{49+20\,\mathcal{C}_3}$	$8\,\mathcal{C}_3+\frac{136}{5}+8\sqrt{49+20\,\mathcal{C}_3}$	$L[\frac{1}{2}]\otimes\{+1\}$	1
	$-\frac{3}{2}+\frac{1}{6}\sqrt{81+20\,\mathcal{C}}$	$8\,\mathcal{C}_3+\frac{648}{5}-\frac{72}{5}\sqrt{81+20\mathcal{C}_3}$	$L[0]\otimes\{-\frac{3}{2}\}$	1
	$-\frac{1}{2}+\frac{1}{6}\sqrt{81+20\,\mathcal{C}}$	$8\,\mathcal{C}_3+\frac{288}{5}-\frac{48}{5}\sqrt{81+20\,\mathcal{C}_3}$	$L[0]\otimes\{-\frac{3}{2}\}$	1
	$\frac{1}{6}+\frac{1}{6}\sqrt{1+20\,\mathcal{C}}$	$8\,\mathcal{C}_3-\frac{32}{5}-\frac{32}{5}\sqrt{1+20\,\mathcal{C}_3}$	$L[0]\otimes\{-\frac{1}{2}\}$	2
	$\frac{7}{6}+\frac{1}{6}\sqrt{1+20\,\mathcal{C}}$	$8\,\mathcal{C}_3-\frac{152}{5}-\frac{8}{5}\sqrt{1+20\mathcal{C}_3}$	$L[0]\otimes\{-\frac{1}{2}\}$	2
	$\frac{11}{6}+\frac{1}{6}\sqrt{1+20\,\mathcal{C}}$	$8\,\mathcal{C}_3-\frac{152}{5}+\frac{8}{5}\sqrt{1+20\mathcal{C}_3}$	$L[0]\otimes\{+\frac{1}{2}\}$	2
	$\frac{17}{6}+\frac{1}{6}\sqrt{1+20\,\mathcal{C}}$	$8\,\mathcal{C}_3-\frac{32}{5}+\frac{32}{5}\sqrt{1+20\,\mathcal{C}_3}$	$L[0]\otimes\{+\frac{1}{2}\}$	2
	$\frac{7}{2}+\frac{1}{6}\sqrt{81+20\,\mathcal{C}}$	$8\,\mathcal{C}_3+\frac{288}{5}+\frac{48}{5}\sqrt{81+20\,\mathcal{C}_3}$	$L[0]\otimes\{+\frac{3}{2}\}$	1
	$\frac{9}{2}+\frac{1}{6}\sqrt{81+20\,\mathcal{C}}$	$8\,\mathcal{C}_3+\frac{648}{5}+\frac{72}{5}\sqrt{81+20\mathcal{C}_3}$	$L[0]\otimes\{+\frac{3}{2}\}$	1

(3.159)

Again, all these eigenvalues fit into the list of eigenvalues identified in [141, 148, 151] (up to an overall normalisation factor and shift in the definition of scalar mass). Just as before, the seemingly missing eigenvalues (of the operators Δ_3, Δ_L in the notation of [151]) are precisely taken care of by the non-trivial multiplicities in the last column of (3.159). In that same notation of [151], the eigenvalues of Δ_3 pick a definite sign, fixing all the potential ambiguities.

Similar, one can extract the masses of all fields in the other representation towers (3.154), as well as in the lower representations (3.155). In the latter, the general pattern of (3.158), (3.159), degenerates and only some of the potential eigenvalues are realised, the explicit values follow from the multiplet structure together with (3.148).

E.3 The Kaluza-Klein spectrum on the right-squashed sphere

The right-squashed sphere is obtained by flipping the sign of the seven-form flux of the solution. The precise relation between the spectra on the left-squashed sphere and its "skew-whiffed" right-squashed counterpart can be inferred from the general results of [137, 141]. Combining this with the explicit form of the left-squashed spectrum (3.153), (3.154), we may describe the passage from the left-squashed to the right-squashed spectrum multiplet by multiplet (of course the right-squashed sphere breaks all supersymmetries, such that the resulting structure is no longer a supermultiplet). In the following, we give the bosonic part of the spectrum for the right-squashed sphere.

For all multiplets we find the following picture. First of all, the bosonic masses in all spin-2

multiplets as well as in vector multiplets, i.e. $L[\frac{3}{2}]$ and $L[1]$ multiplets, remain unchanged. For the bosonic states of the $L[\frac{1}{2}]$ multiplets, the transition works as follows

$$\begin{aligned} L[\tfrac{1}{2}]\otimes\{-1\}: &\quad \begin{cases} \text{vector}: \Delta_{\text{RS}} = \Delta_{\text{LS}}\,, \\ \text{scalar}: \Delta_{\text{RS}} = \Delta_{\text{LS}} + 2|s| + 1\,, \end{cases} \\ L[\tfrac{1}{2}]\otimes\{1\}: &\quad \begin{cases} \text{vector}: \Delta_{\text{RS}} = \Delta_{\text{LS}}\,, \\ \text{scalar}: \Delta_{\text{RS}} = \Delta_{\text{LS}} - 2|s| - 1\,, \end{cases} \end{aligned} \tag{3.160}$$

while the bosonic states of $L[\frac{1}{2}]\otimes\{0\}$ remain unchanged. This is illustrated in Fig. 3.1.

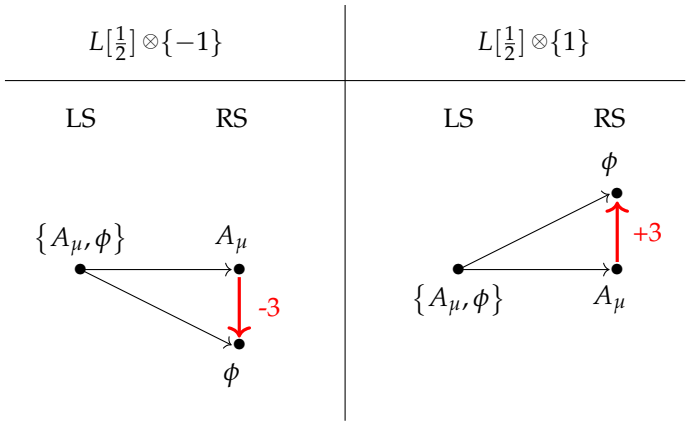


Figure 3.1: Shift patterns of the conformal dimensions of the bosonic states within $L[\frac{1}{2}]$, as we go from the left- to the right-squashed S^7.

Between the left and right squashing, the scalar is shifted by ± 3 in the direction of $\text{sign}(-s)$. As a result, in the case of the right squashing, the scalar and vector states which used to have the same conformal dimension are now separated by ± 3 in the right-squashed sphere, again depending on the sign of s.

The $L[0]$ multiplets behave similarly. First, there is always a scalar state, whose conformal dimension is unchanged between the left and the right squashing. The other scalar state gets shifted by ± 3 depending on the sign of s. In order to identify which of the states gets shifted, one notes that for the right squashing, the difference in the conformal dimension between bosonic states is no longer 1 but changes to $2|s| + 1$ as for the $L[\frac{1}{2}]$ multiplets. This can be summed up as in Fig. 3.2.

Again, the conformal dimensions of half of the states are shifted by ± 3 between the right and left squashing, such that the conformal dimensions of the bosonic states from the same multiplet now differ by an s dependent shift.

The shift patterns for all the multiplets and fields are summed up in the Figures 3.3, 3.4, 3.5 and 3.6.

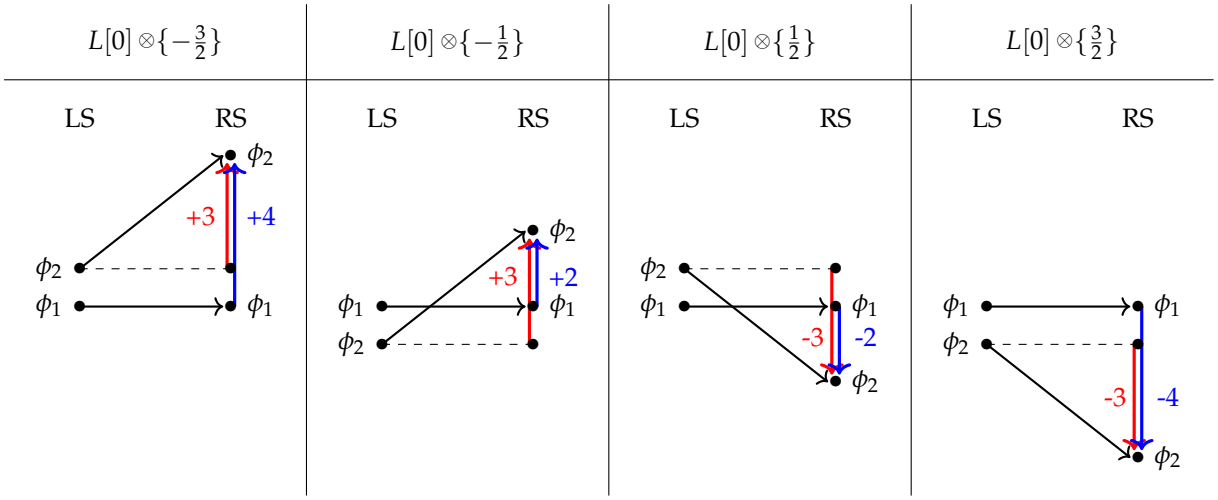


Figure 3.2: Shift patterns of the conformal dimensions of the bosonic states within $L[0]$, as we go from the left- to the right-squashed S^7.

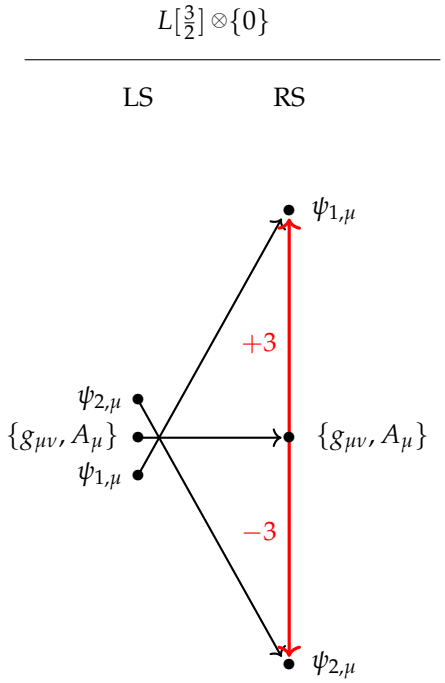


Figure 3.3: Shift patterns of the conformal dimensions within $L[\frac{3}{2}]$, as we go from the left- to the right-squashed S^7. Here, $g_{\mu\nu}$ is a graviton, A_μ a vector and ψ_μ a gravitino.

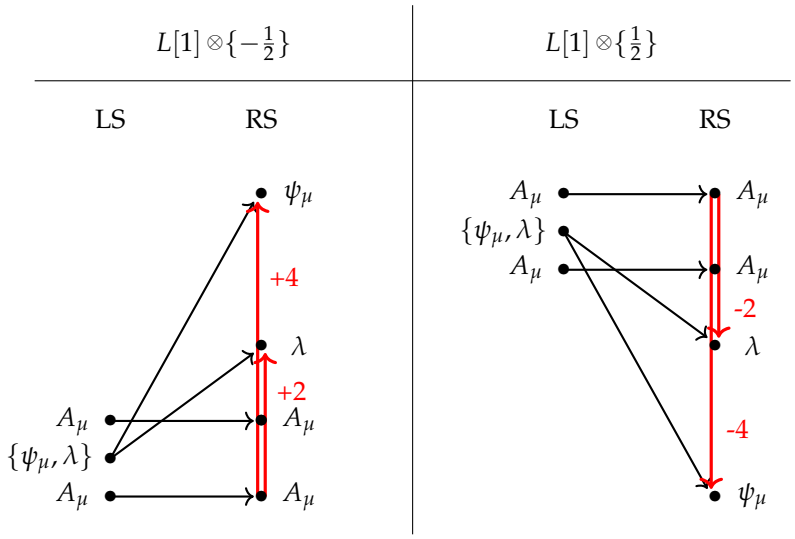


Figure 3.4: Shift patterns of the conformal dimensions of the states within $L[1]$, as we go from the left- to the right-squashed S^7. Here, A_μ are vectors, ψ_μ is a gravitino, and λ a fermion with spin $\frac{1}{2}$.

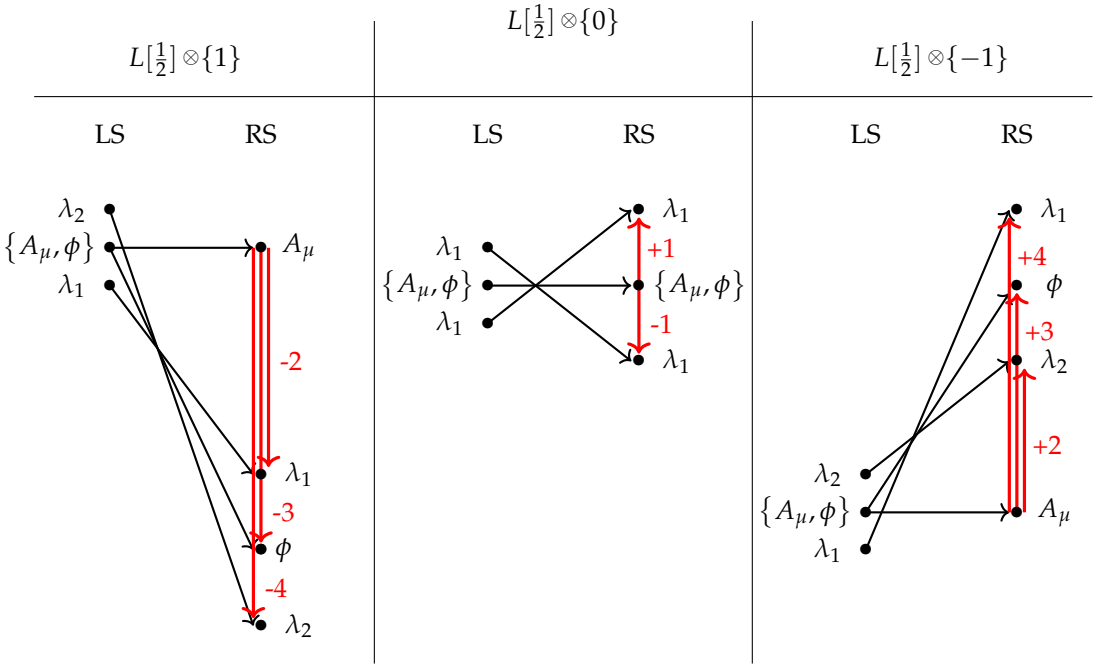


Figure 3.5: Shift patterns of the conformal dimensions of the states within $L[\frac{1}{2}]$, as we go from the left- to the right-squashed S^7. Here, A_μ is a vector, λ are fermions with spin $\frac{1}{2}$ and ϕ is a scalar field.

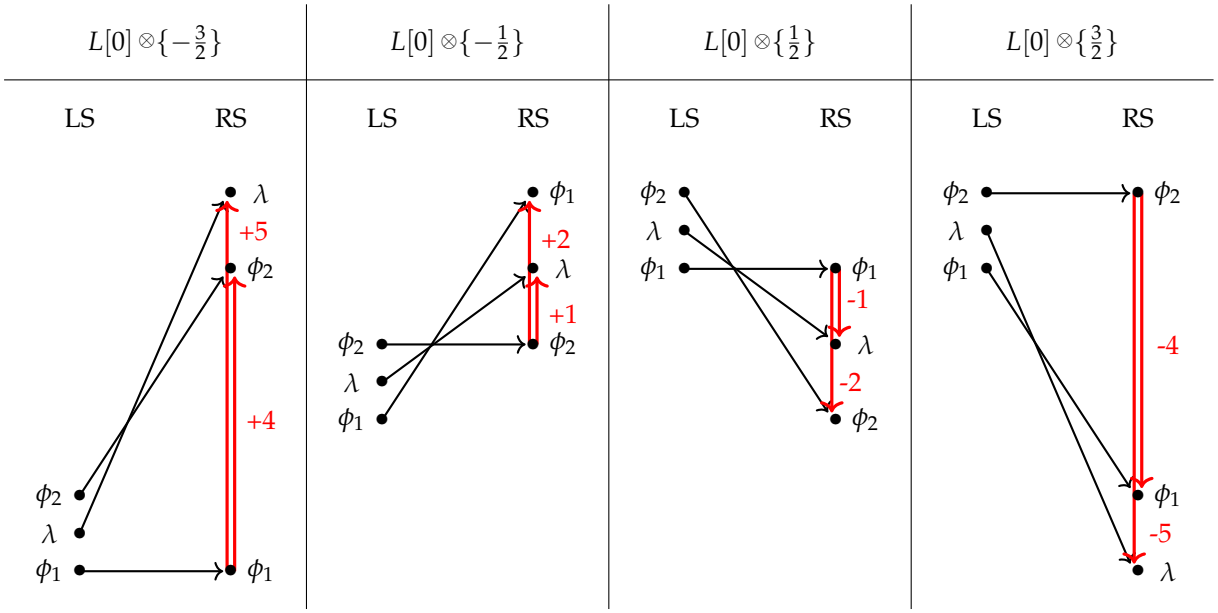


Figure 3.6: Shift patterns of the conformal dimensions of the states within $L[0]$, as we go from the left- to the right-squashed S^7. Here, λ is a fermion with spin $\frac{1}{2}$ and ϕ are scalar fields.

E.4 Rational conformal dimensions and marginal deformations

Since the AdS_4 vacuum only preserves $\mathcal{N}=1$ supersymmetry, all multiplets are unprotected, i.e. they are either long or sit at the unitarity bound where they can recombine into long multiplets. Still, we observe infinitely many rational conformal dimensions in the KK spectrum. In particular, these arise from the following towers (in the notation of (3.151))

$$\begin{aligned}
L[0]\otimes\{-\tfrac{1}{2},\tfrac{1}{2}\}\otimes[k,1,k]_{k>1}: &\quad \begin{cases}\Delta=\frac{10+5k}{6}\,,\\ \Delta=\frac{20+5k}{6}\,,\end{cases}\\
L[1]\otimes\{-\tfrac{1}{2}\}\otimes[k,0,k]_{k>1}: &\quad \Delta=\frac{8+5k}{6}\,,\\
L[\tfrac{1}{2}]\otimes\{1\}\otimes[k,0,k]_{k>1}: &\quad \Delta=\frac{23+5k}{6}\,,\\
L[1]\otimes\{\tfrac{1}{2}\}\otimes[k,0,k+2]: &\quad \Delta=\frac{24+5k}{6}\,,\\
L[\tfrac{1}{2}]\otimes\{-1\}\otimes[k,0,k+2]: &\quad \Delta=\frac{9+5k}{6}\,,\\
L[0]\otimes\{\tfrac{1}{2}\}\otimes[k+2,0,k]: &\quad \Delta=\frac{22+5k}{6}\,,\\
L[0]\otimes\{-\tfrac{1}{2}\}\otimes[k,1,k+4]: &\quad \Delta=\frac{22+5k}{6}\,,\\
L[0]\otimes\{-\tfrac{1}{2}\}\otimes[k,0,k+4]: &\quad \Delta=\frac{20+5k}{6}\,,
\end{aligned} \tag{3.161}$$

whose conformal dimensions are manifestly rational. More generally, we can study rational solutions of (3.148). In order for the conformal dimension to be rational, this requires

$$\sqrt{A+70k+25k^2+60q+20kq+20q^2}\;\in\;\mathbb{N}\,, \tag{3.162}$$

with $A\in\{1,9,49,81\}$. In other words, we need to solve the following order two diophantine equation[5]

$$A+70k+25k^2+60q+20kq+20q^2-N^2=0\,. \tag{3.163}$$

In order to get a feeling on general rational solutions of (3.162), we numerically plot integer solutions of (3.162) in the (k,q) plane. The results are shown in Fig. 3.7. We can see for the values $A=1$ and $A=49$ lines emerging from the graph, whereas there are no such lines on the graph $A=9$ (and similarly for $A=81$). As a consequence, we look for solutions of the form $q=ak+b, (a,b)\in\mathbb{Q}^2$. Plugging this into (3.162) we find an order two polynomial in k, whose discriminant Δ must vanish. As Δ is a function of a and b, we can solve the equation $\Delta=0$ for a. In order to have $a\in\mathbb{Q}$ we must find b such that $\sqrt{-(A-65)(A+20b(3+b))}\in\mathbb{Q}$. For $A=81$, the number in the square root is always a negative number, which explains why we do not see any line in the $A=81$ plot. For $A=1$ and $A=49$, the first factor $(A-65)$ gives an exact square number and we must find b such that $\sqrt{A+20b(3+b)}\in\mathbb{Q}$. The problem finally reduces to finding b

[5]Similar structures have been revealed by Gubser [153] in the KK spectrum on type IIB Supergravity on $AdS_5\times T^{1,1}$, see also [154].

such that $A + 60b + 20b^2 = y^2$, $y \in \mathbb{Q}$, and substituting $x = b - \frac{3}{2}$, we must solve

$$A - 45 + 20x^2 - y^2 = 0\,. \tag{3.164}$$

This is a Pell equation, whose integers solutions can be found using Mathematica. However, we are not only interested in integer solutions, but also in rational solutions of this equation. We must solve $y^2 - 20\,x^2 = A - 45$. In order to find solutions, we first solve what we will call the homogeneous Pell equation $y^2 - D\,x^2 = 1$. It can be shown that rational solutions of the homogeneous Pell equation can be written as $(x, y) = (\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1})$, $t \in \mathbb{Q}$, $t^2 \neq D$. Solutions to the original Pell equation can eventually be found using a particular solution, and multiplying it by the homogeneous solutions. Indeed, let (x_0, y_0) be a particular solution and (x, y) a solution to the homogeneous Pell equation, then $A - 45 = (y_0^2 - Dx_0^2)(y^2 - Dx^2) = (x_0 x \pm y_0 y)^2 - D(x_0 y \pm y_0 x)^2$, allowing us to generate families of solutions of the Pell equation. This method works as long as D is not a square number. We also want to emphasise that this may not be all solutions of the Pell equation, as a different particular solution may lead to a different family of solutions. We illustrate our findings with the orange lines in Fig. 3.7 for the case $A = 1$ and $A = 49$.

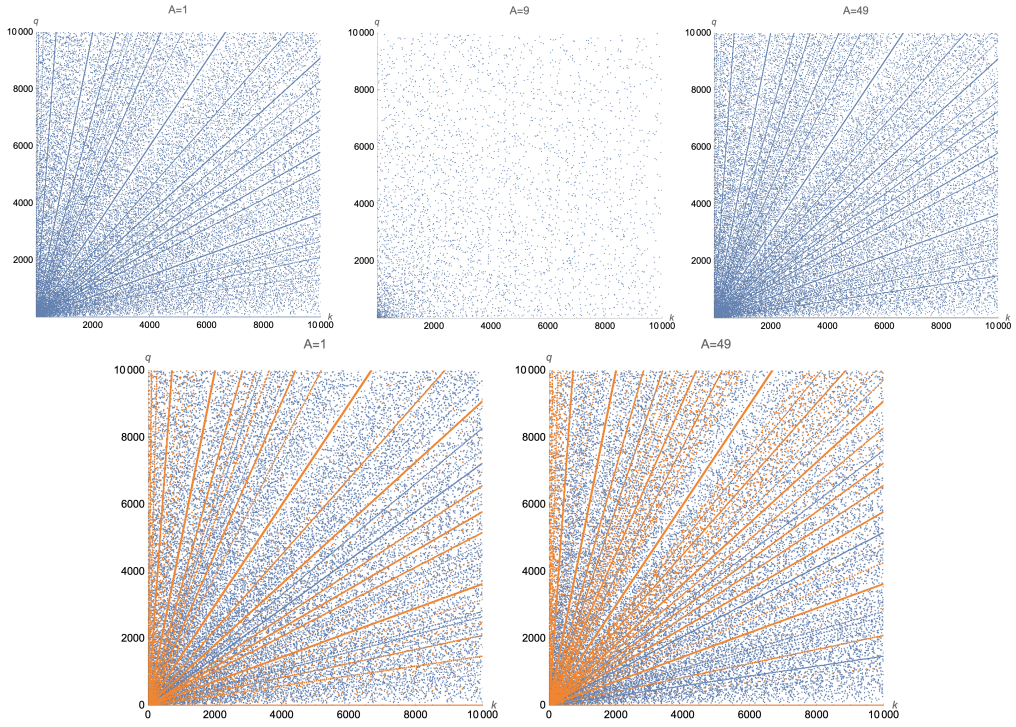


Figure 3.7: In the first row of figures, the blue points give integer solutions to (3.162) for the special values $A = 1, 9, 49$. In the second row, we superpose these plots with orange lines corresponding to analytical solutions to the Pell equation (3.164) as discussed in the text.

Particularly interesting are the multiplets with marginal deformations. For the $\mathcal{N} = 1$ left-squashed sphere these are

$$L[0] \otimes \{-\tfrac{3}{2}\} \otimes [0,3,0]\,, \quad L[0] \otimes \{-\tfrac{3}{2}\} \otimes [2,1,2]\,. \tag{3.165}$$

These are D-terms and preserve the $\mathcal{N} = 1$ supersymmetry. We note that one of the massless scalars preserves the $\mathrm{USp}(4) \times \mathrm{SU}(2)$ symmetry, while the second one breaks $\mathrm{USp}(4)$ to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and preserves $\mathrm{SU}(2)$. For the right-squashed sphere, all massless scalars in (3.165) turn massive. However, massive scalars from the following multiplets of the left-squashed sphere

$$L[\tfrac{1}{2}] \otimes \{1\} \otimes [2,1,0]\,, \quad L[\tfrac{1}{2}] \otimes \{1\} \otimes [2,0,2]\,, \tag{3.166}$$

become massless for the right-squashed sphere by the pattern displayed in Figure 3.5 above. It would be very interesting to study whether any of these massless scalars for the left-/right-squashed sphere can be integrated up to finite moduli.

F The squashed S^7 : holographic RG flow

EXFTspectroscopy techniques have been applied in the previous section in order to get the full mass spectrum on the left and right squashed spheres. We will now construct and analyze the domain wall solution in $11d$ Supergravity connecting the $\mathcal{N} = 1$, $\mathrm{AdS}_4 \times \mathrm{S}^7_{\text{squashed}}$ vacuum to the $\mathcal{N} = 8$, $\mathrm{AdS}_4 \times \mathrm{S}^7_{\text{round}}$ vacuum. This domain wall describes the holographic renormalization group flow from an $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ symmetric UV fixed point to the $\mathrm{SO}(8)$ symmetric IR fixed point. It breaks all supersymmetries which are (partially) restored at its endpoints.

F.1 Squashed sphere, S^7 and domain wall in D = 11 Supergravity

We first briefly review the round and the squashed S^7 vacua in $d = 11$ Supergravity language and construct the interpolating domain wall solution. Both S^7 backgrounds are Freund-Rubin solutions of $d = 11$ Supergravity, preserving $\mathcal{N} = 8$ and $\mathcal{N} = 1$ supersymmetry, respectively [136]. The general ansatz for the $d = 11$ metric with an internal space S^7 preserving $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ isometries can be put into the form [155]

$$ds^2 = e^{-7u}\, ds^2_{(4)} + \frac{1}{4}\, e^{2u} \left(e^{3v} \left(d\mu^2 + \frac{1}{4} \sin^2\mu \sum_i \omega_i^2 \right) + \frac{1}{4}\, e^{4v} \sum_i (\nu_i + \cos\mu\, \omega_i)^2 \right)\,, \tag{3.167}$$

with S^7 size parameter u and squashing parameter v that are taken as scalar functions over the four-dimensional space-time, and $i = 1, 2, 3$. The one forms $\omega_i = \sigma_i - \Sigma_i$, $\nu_i = \sigma_i + \Sigma_i$, satisfy

$$d\sigma_i = -\frac{1}{2}\, \varepsilon_{ijk}\, \sigma_j \wedge \sigma_k\,, \quad d\Sigma_i = -\frac{1}{2}\, \varepsilon_{ijk}\, \Sigma_j \wedge \Sigma_k\,. \tag{3.168}$$

The 4-form flux for these solutions is of the form

$$F_{\mu\nu\rho\sigma} = Q\, e^{-21u}\, \varepsilon_{\mu\nu\rho\sigma}\,, \tag{3.169}$$

with conserved charge Q. The ansatz (3.167), (3.169) is, in fact, a consistent truncation of $d = 11$ Supergravity as shown in [155]. More precisely, plugging (3.167), (3.169), into the $d = 11$ field

equations leads to field equations that are obtained from the $d = 4$ Lagrangian

$$\begin{aligned} |g|^{-1/2}\,\mathcal{L}^0_{(4)} &= R_{(4)} - \frac{63}{2}\,\partial_\mu u\,\partial^\mu u - 21\,\partial_\mu v\,\partial^\mu v - V_{\text{pot}}\,, \\ V_{\text{pot}} &= -6\,e^{-9u+4v} - 48\,e^{-9u-3v} + 12\,e^{-9u-10v} + 2\,Q^2\,e^{-21u}\,. \end{aligned} \tag{3.170}$$

The parameter Q may be absorbed into a shift of u together with a rescaling of the $d = 4$ metric $g_{\mu\nu}$. In the following, we set $Q = 3$.

Extremisation of the potential V_{pot} from (3.170) yields two critical points corresponding to $\text{AdS}_4 \times S^7$ solutions of $d = 11$ Supergravity, with the round and the squashed sphere located at

$$\begin{aligned} \text{round } S^7: &\quad u = 0\,,\ \ v = 0\,,\quad \ell_{\text{round}} = \tfrac{1}{2}\,, V_{\text{pot}} = -24\,, \\ \text{squashed } S^7: &\quad u = u_0 \equiv \tfrac{5}{42}\ln 5 - \tfrac{1}{6}\ln 3\,,\ \ v = v_0 \equiv \tfrac{1}{7}\ln 5\,,\quad \ell_{\text{squashed}} = \tfrac{5^{5/4}}{2\cdot 3^{7/4}}\,, V_{\text{pot}} = -20.0775\,, \end{aligned} \tag{3.171}$$

and the respective AdS lengths given by $\ell = \sqrt{-\frac{6}{V_{\text{pot}}}}$. With the standard ansatz for an interpolating domain wall solution

$$u = u(r)\,,\ \ v = v(r)\,,\quad ds^2_{(4)} = dr^2 + e^{2A(r)}\,\eta_{ij}\,dx^i dx^j\,,\quad i,j = 1,2,3\,, \tag{3.172}$$

variation of (3.170) yields the equations

$$\begin{aligned} u'' + 3\,A'u' &= -6\,e^{-21u} - \frac{12}{7}\,e^{-9u-10v} + \frac{48}{7}\,e^{-9u-3v} + \frac{6}{7}\,e^{-9u+4v}\,, \\ v'' + 3\,A'v' &= -\frac{20}{7}\,e^{-9u-10v} + \frac{24}{7}\,e^{-9u-3v} - \frac{4}{7}\,e^{-9u+4v}\,, \\ 3\,(A')^2 - \frac{63}{4}\,(u')^2 - \frac{21}{2}\,(v')^2 &= -9\,e^{-21u} - 6\,e^{-9u-10v} + 24\,e^{-9u-3v} + 3\,e^{-9u+4v}\,. \end{aligned} \tag{3.173}$$

The existence of a domain wall solution to these equations, interpolating between the two vacua (3.171), was discussed in [156], and later questioned in [157]. Its holographic interpretation was further elaborated in [158]. The domain wall represents a holographic renormalisation group flow from an $\mathcal{N} = 1$ superconformal UV fix point (dual to the squashed S^7, located a $r \to +\infty$) to an $\mathcal{N} = 8$, SO(8) symmetric IR fix point (dual to the round S^7, located a $r \to 0$). Note that the flow equations (3.173) are invariant under a shift of r, which allows us to set the origin at $r = 0$. Moreover, the discussion on which point is the UV and which is the IR can be decided by looking at the values of the potential in (3.171). As the round sphere is below the squashed one, this makes the former the UV point of the flow. Futhermore, it can be shown [156] it is stable under squashing, hence the name of UV fix point.

Unlike most explicitly known domain wall solutions, there is no description of the flow equations connecting (3.171) in terms of first order differential equations and a superpotential. More

precisely, [157] noted that the potential (3.170) can be written in terms of a superpotential W as

$$\begin{aligned} V_{\text{pot}} &= \frac{16}{63}\,(\partial_u W)^2 + \frac{8}{21}\,(\partial_v W)^2 - 12\,W^2\,, \\ W &= -\frac{3}{\sqrt{8}}\,e^{-9u/2}\left(e^{2v} + 2\,e^{-5v} - e^{-6u}\right)\,. \end{aligned} \tag{3.174}$$

However only the squashed S^7 represents a critical point of this superpotential. With hindsight, this is a manifestation of the fact that the Lagrangian (3.170) lives within an $\mathcal{N} = 1$ four-scalar truncation of $d = 11$ Supergravity [159], in which the round sphere appears as an $\mathcal{N} = 0$ vacuum, since all the $\mathcal{N} = 8$ massless gravitinos around this vacuum are truncated out.[6] Accordingly, the round S^7 does not correspond to a critical point of the associated superpotential (3.174).

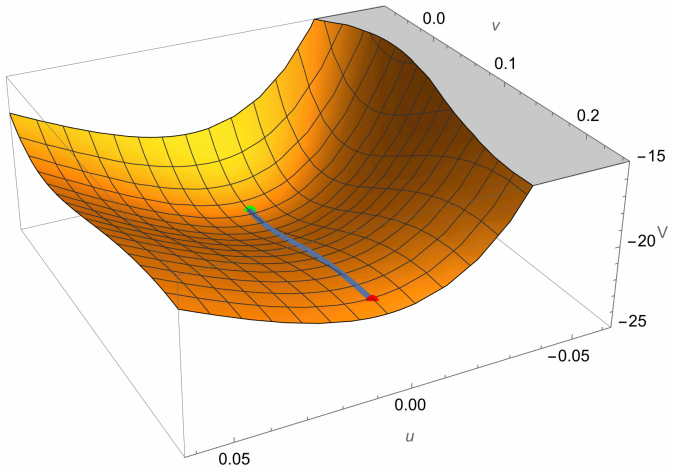


Figure 3.8: The domain wall (blue line) in the scalar potential of (3.170). The green and the red dots represent the round and the squashed sphere (3.171), respectively.

The interpolating domain wall solution can be found numerically, by solving equations (3.173) and fine-tuning the initial conditions. To this end, it is instructive to first study the general asymptotic behaviour of the scalar fields. As usual, this is correlated with the conformal dimensions of the associated dual operators, given by

$$\begin{aligned} \mathcal{O}_u: &\quad \Delta_{\text{UV}} = 6 = \Delta_{\text{IR}}\,, \\ \mathcal{O}_v: &\quad \Delta_{\text{UV}} = \tfrac{5}{3}\,,\ \ \Delta_{\text{IR}} = 4\,, \end{aligned} \tag{3.175}$$

for the scalar fields. Introducing the radial coordinate

$$\rho = e^{-2r/\ell_{\text{squashed}}}\,, \tag{3.176}$$

and expanding the flow equations (3.173) near the UV boundary $\rho = 0$, we find the following

[6] In the notation of [159], all this is on the $k < 0$ branch.

asymptotic expansions of its general regular solution

$$\begin{aligned} u(\rho) &= \tfrac{1}{6}\ln\tfrac{5^{5/7}}{3} + \rho^{\frac{8}{6}}\,\Omega_{1,8} + \rho^{\frac{9}{6}}\,\Omega_{1,9} + \rho^{\frac{10}{6}}\,\Omega_{1,10} + \cdots + \rho^{18/3}\,\underline{\Omega}_{1,18} + \dots\,, \\ v(\rho) &= \tfrac{1}{7}\ln 5 + \rho^{\frac{4}{6}}\,\underline{\Omega}_{2,4} + \rho^{\frac{5}{6}}\,\underline{\Omega}_{2,5} + \rho^{\frac{8}{6}}\,\Omega_{2,8} + \rho^{\frac{9}{6}}\,\Omega_{2,9} + \rho^{\frac{10}{6}}\,\Omega_{2,10} + \dots\,, \\ 2A(\rho) &= -\log\rho + A_8\,\rho^{\frac{8}{6}} + A_9\,\rho^{\frac{9}{6}} + A_{10}\,\rho^{\frac{10}{6}} + \dots\,. \end{aligned} \tag{3.177}$$

Equations (3.173) fix all coefficients $\Omega_{i,j}$ in the above expressions, except $\underline{\Omega}_{2,4}$, $\underline{\Omega}_{2,5}$, and $\underline{\Omega}_{1,18}$ which source the others. Explicitly, we find that the coefficients of the lowest powers are given by

$$\begin{aligned} \Omega_{1,8} &= -\tfrac{6}{17}\,\underline{\Omega}_{2,4}^2\,, & \Omega_{1,9} &= -\tfrac{20}{27}\,\underline{\Omega}_{2,4}\,\underline{\Omega}_{2,5}\,, \\ \Omega_{2,8} &= -\tfrac{15}{4}\underline{\Omega}_{2,4}^2\,, & \Omega_{2,9} &= -\tfrac{9}{2}\,\underline{\Omega}_{2,4}\,\underline{\Omega}_{2,5}\,, \\ A_8 &= -\tfrac{21}{4}\,\underline{\Omega}_{2,4}^2\,, & A_9 &= -\tfrac{280}{27}\,\underline{\Omega}_{2,4}\,\underline{\Omega}_{2,5}\,. \end{aligned} \tag{3.178}$$

In (3.177), we have imposed regularity at $\rho = 0$, which sets to zero a potential $\rho^{-\frac{3}{2}}$ term in the u expansion, allowed by equations (3.173). The general solution of (3.173), regular at $\rho = 0$, thus carries three independent constants. It furthermore admits the scaling symmetry $\rho \to \lambda\rho$, which can be used to set $\underline{\Omega}_{2,4} = -1$. For the interpolating solution, the remaining two coefficients are then fixed by further demanding regularity at the other end of the flow $\rho \to \infty$.

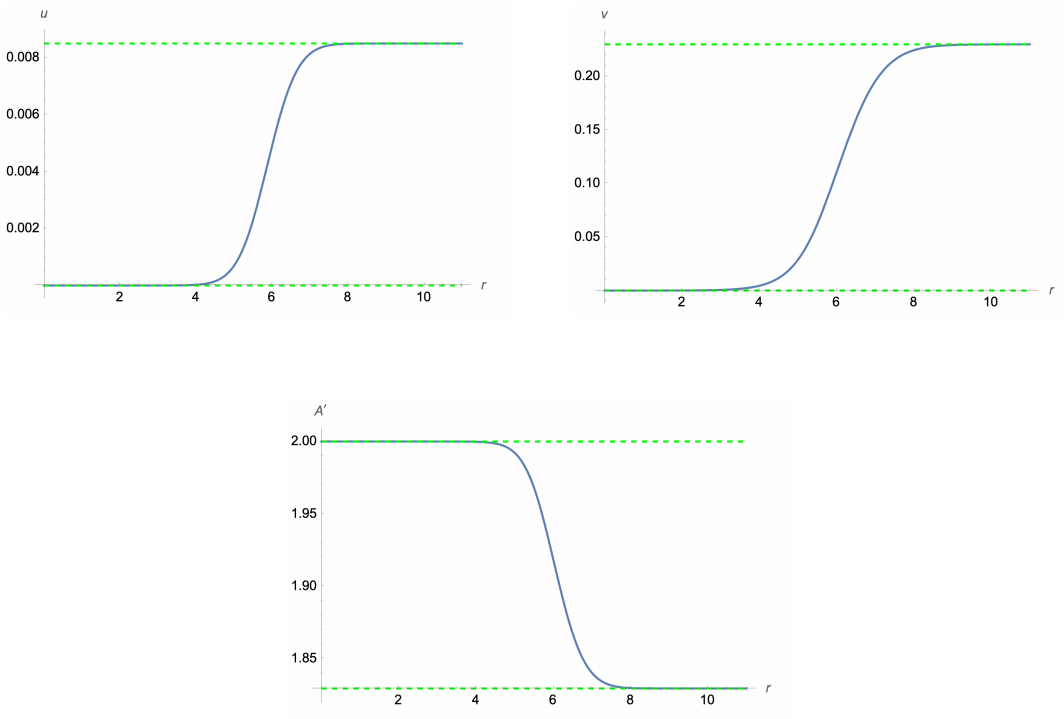


Figure 3.9: Numerical solutions for the two scalar fields, as well as the derivative A'. The horizontal green dashed line represent the asymptotic values of the various fields. The UV boundary is located at $r \to \infty$, the IR boundary is at $r \to -\infty$.

Imposing regularity at both ends of the flow, we have solved equations (3.173) numerically, and plot the result in Figures 3.8, 3.9 . As expected, the solution is of kink type for the scalar fields u, v, as well as for the derivative $\partial_r A$. In particular, with $\underline{\Omega}_{2,4} = -1$, we find for the coefficient $\underline{\Omega}_{2,5}$ the approximate numerical value

$$\underline{\Omega}_{2,5} \approx -1.4 \,. \tag{3.179}$$

In Figure 3.10, we plot the asymptotics of the scalar fields, which confirms the UV expansion (3.177), and in particular the fact that the leading coefficient $\underline{\Omega}_{2,4}$ is non-vanishing. This is the expected power-law behaviour

$$v(\rho) - v_0 \propto \rho^{\frac{3-\Delta}{2}} \,, \tag{3.180}$$

consistent with the interpretation that the holographic dual of this domain wall solution is an operator deformation (rather than a vev) of the UV CFT [113, 160], by a relevant operator of conformal dimension $\Delta_{\rm UV} = \frac{5}{3}$.

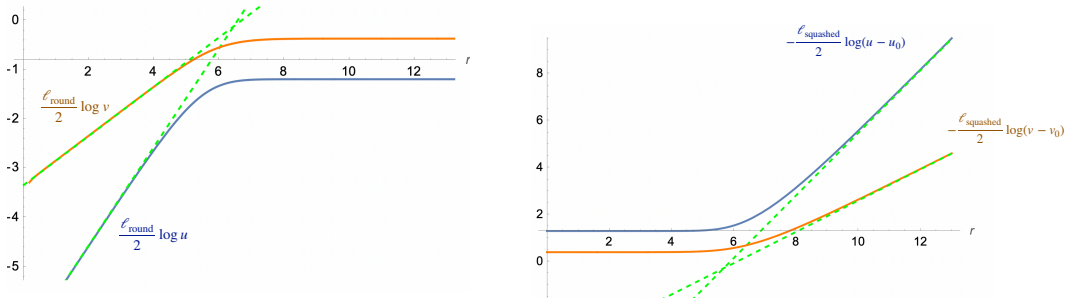


Figure 3.10: Asymptotic behavior of the numerical solution. In the UV ($r \to \infty$), the green dashed line have respectively a slope of $\frac{4}{3}$ for u and $\frac{2}{3}$ for v. This matches the asymptotic expansion (3.177). In the IR ($r \to -\infty$), the green dashed line have respectively a slope of 1 for u and $\frac{1}{2}$ for v. This matches the asymptotic expansion (3.181).

Similarly, one can work out the asymptotic behaviour in the IR. With the new radial variable $\tilde{\rho} = e^{2r/\ell_{\rm round}}$, the expansion of a general solution of (3.173), regular at the IR boundary $\tilde{\rho} = 0$, is given by

$$\begin{aligned} u(\tilde{\rho}) &= \omega_{1,2}\,\tilde{\rho} + \underline{\omega}_{1,3}\,\tilde{\rho}^{3/2} + \psi_{1,3}\,\tilde{\rho}^{3/2}\log\rho + \omega_{1,4}\,\tilde{\rho}^2 + \dots \,, \\ v(\tilde{\rho}) &= \underline{\omega}_{2,1}\,\tilde{\rho}^{1/2} + \omega_{2,2}\,\tilde{\rho} + \omega_{2,3}\,\tilde{\rho}^{3/2} + \omega_{2,4}\,\tilde{\rho}^2 + \psi_{2,4}\,\tilde{\rho}^2\log\rho + \dots \,, \\ 2A(\tilde{\rho}) &= \log\tilde{\rho} + \tilde{A}_2\,\tilde{\rho} + \tilde{A}_3\,\tilde{\rho}^{3/2} + \tilde{A}_4\,\tilde{\rho}^2 + \dots \,, \end{aligned} \tag{3.181}$$

in accordance with the IR conformal dimensions (3.175). Here, imposing regularity at $\tilde{\rho} = 0$ has set to zero two of the free coefficients of the general solution of (3.173). All other coefficients in the expansion (3.181) are then determined in terms of the remaining two free coefficients $\underline{\omega}_{1,3}$, $\underline{\omega}_{2,1}$,

e.g.

$$\begin{aligned}
&\omega_{1,2} = \tfrac{3}{2}\,\underline{\omega}_{2,1}^2\,, \qquad \psi_{1,3} = 11\,\underline{\omega}_{2,1}^3\,, \qquad \omega_{1,4} = -\tfrac{15525}{56}\,\underline{\omega}_{2,1}^4\,,\\
&\omega_{2,2} = -\tfrac{11}{2}\,\underline{\omega}_{2,1}^2\,, \quad \omega_{2,3} = \tfrac{5261}{168}\,\underline{\omega}_{2,1}^3\,, \quad \omega_{2,4} = -\tfrac{3}{2}\,\underline{\omega}_{2,1}\underline{\omega}_{1,3} - \tfrac{10583}{56}\,\underline{\omega}_{2,1}^4\,, \quad \psi_{2,4} = -\tfrac{33}{2}\,\underline{\omega}_{2,1}^4\,,\\
&\tilde{A}_2 = -\tfrac{21}{4}\,\underline{\omega}_{2,1}^2\,, \quad \tilde{A}_3 = \tfrac{154}{3}\,\underline{\omega}_{2,1}^3\,, \quad \tilde{A}_4 = -\tfrac{27081}{64}\,\underline{\omega}_{2,1}^4\,.
\end{aligned} \tag{3.182}$$

Again, the leading terms are confirmed by the plots of the numerical solution in Figure 3.10 . Regularity at the UV end of the flow finally fixes $\underline{\omega}_{1,3}$ as a function of $\underline{\omega}_{2,1}$. From the above numerical domain wall solution, we find the approximate value

$$\frac{\underline{\omega}_{1,3}}{(\underline{\omega}_{2,1})^3} \;\approx\; -239\,, \tag{3.183}$$

for the combination invariant under the scaling symmetry $\tilde{\rho} \to \lambda\tilde{\rho}$.

F.2 Generalised parallelisation of the domain wall in ExFT

a Background and domain wall in ExFT

In this section, we will identify the $\mathrm{AdS}_4 \times S^7_{\text{squashed}}$ background and the domain wall (3.167) within ExFT, i.e. within the duality-covariant formulation of $d = 11$ Supergravity. This allows us to construct consistent truncations around this vacuum, just as we did in section 3.E.1, as well as to compute the quadratic couplings of Kaluza-Klein fluctuations around the domain wall background in section 3.F.3 . All a lot of things that are going to be said in this section are similar to what is in section 3.E.1 .

The generalized vielbein is defined here as

$$U \equiv \exp\left[A_{klmnpq}\, t^{klmnpq}_{(+4)}\right] \exp\left[A_{kmn}\, t^{kmn}_{(+2)}\right]\; U_{\mathrm{GL}(7)}\,, \tag{3.184}$$

i.e. as a coset representative of $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ in a particular triangular gauge. Here, $U_{\mathrm{GL}(7)} \in \mathrm{GL}(7) \subset \mathrm{E}_{7(7)}$ is the internal block of the 11D vielbein (up to some power of its determinant), while A_{kmn} and A_{klmnpq} denote the internal components of the $d = 11$ three-form and six-form, respectively, with $k, l, m = 1, \ldots, 7$. The $t_{(+n)}$ refer to the $\mathrm{E}_{7(7)}$ generators of positive grading in the algebra decomposition

$$\mathfrak{e}_{7(7)} \quad \longrightarrow \quad 7'_{-4} \oplus 35_{-2} \oplus \mathfrak{gl}(7)_0 \oplus 35'_{+2} \oplus 7_{+4}\,. \tag{3.185}$$

All generators in (3.184) are evaluated in the fundamental **56** representation of $\mathrm{E}_{7(7)}$. For a Freund-Rubin background (i.e. a solution with $A_{kmn} = 0$), the parametrization (3.184) only involves generators within the $\mathfrak{sl}(8)$ subalgebra of $\mathfrak{e}_{7(7)}$ and the associated generalized vielbein can be represented as an 8×8 matrix. This twist matrix is the one given in (3.136)

$$U_{\mathrm{FR}} \in \mathrm{SL}(8)/\mathrm{SO}(8)\,. \tag{3.186}$$

For the round S^7 with 7-form flux, we can rewrite (3.136) as

$$U_{S^7} = \exp\left[-6\, t^{lmnpqr}_{(+4)}\, \zeta^k \mathring{\omega}_{klmnpqr}\right] \begin{pmatrix} \mathring{\omega}^{3/4} & 0 \\ 0 & \mathring{\omega}^{-1/4}\, \mathring{e}_m{}^i \end{pmatrix} \in \mathrm{SL}(8)\,, \tag{3.187}$$

where $\mathring{e}_m{}^i$ is the S^7 vielbein, $\mathring{\omega}_{klmnpqr}$ is the associated volume form, $\mathring{\omega} \equiv \det \mathring{e}_m{}^i$, ζ^k is a vector field with $\mathring{\nabla}_k \zeta^k = 1$ and $t_{(+4)}$ is evaluated in (3.187) in the **8** representation of SL(8).

b The squashed S^7 family in ExFT

The consistent truncation

$$U(x,y) = \mathring{U}(y)\, \mathscr{V}(x)\,, \tag{3.188}$$

to the lowest KK multiplet contains the round S^7 vacuum (corresponding to $\mathscr{V}(x) = \mathbb{I}$) but no other solution of the family of squashed backgrounds (3.167). Rather, these squashings correspond to excitations of scalars from higher KK levels. Specifically, the squashing (3.167) preserves an $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ subgroup of isometries, which is embedded into the SO(8) isometry group of the round S^7 such that the fundamental representations decompose as

$$\begin{gathered} \mathrm{SO}(8) \longrightarrow \mathrm{Sp}(2) \times \mathrm{Sp}(1)\,, \\ [1,0,0,0] \to [1,0,1]\,, \quad [0,0,1,0] \to [0,1,0] \oplus [0,0,2]\,, \quad [0,0,0,1] \to [1,0,1]\,. \end{gathered} \tag{3.189}$$

A scan of the scalar spectrum in (3.155) shows that the spectrum contains four scalar fields that are singlet under $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$, sitting at KK levels 0, 2, and 4, respectively, descending from the SO(8) representations

$$\begin{aligned} \ell = 0: &\quad [0,0,2,0]_2\,, \\ \ell = 2: &\quad [0,0,0,0]_6 \oplus [0,2,0,0]_4\,, \\ \ell = 4: &\quad [2,0,0,2]_6\,. \end{aligned} \tag{3.190}$$

The squashed background (3.167) requires a non-vanishing contribution from the $[0,2,0,0]$ at level $\ell = 2$ [141] and, thus, lives beyond the consistent truncation (3.188). This has been referred as the space invader scenario in [141], which is a reflexion of the level mixing we already mentionned. Another manifestation of this effect can be seen within the gravitini sector of the theory. Looking at singlets gravitini in the spectrum of $\mathrm{S}^7_{\text{squashed}}$ gives only one singlet at level $\ell = 1$, responsible for the $\mathcal{N} = 1$ supersymmetry on the left-squashed sphere, and no singlets at level $\ell = 0$

$$\begin{gathered} [0,0,0,1] \underset{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\longrightarrow} 1 \times [1,0,1]\,, \\ [0,0,0,1] \otimes [1,0,0,0] = 1 \times [1,0,0,1] + 1 \times [0,0,1,0] \underset{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\longrightarrow} 1 \times [0,0,0] + \dots\,. \end{gathered} \tag{3.191}$$

This is the reason for the non-supersymmetric flow : none of the original supersymmetries are preserved and all original massless gravitini become massive on the squashed sphere and along the flow ; one gravitini from a higher level becomes massless. This is illustrated in Figure 3.11.

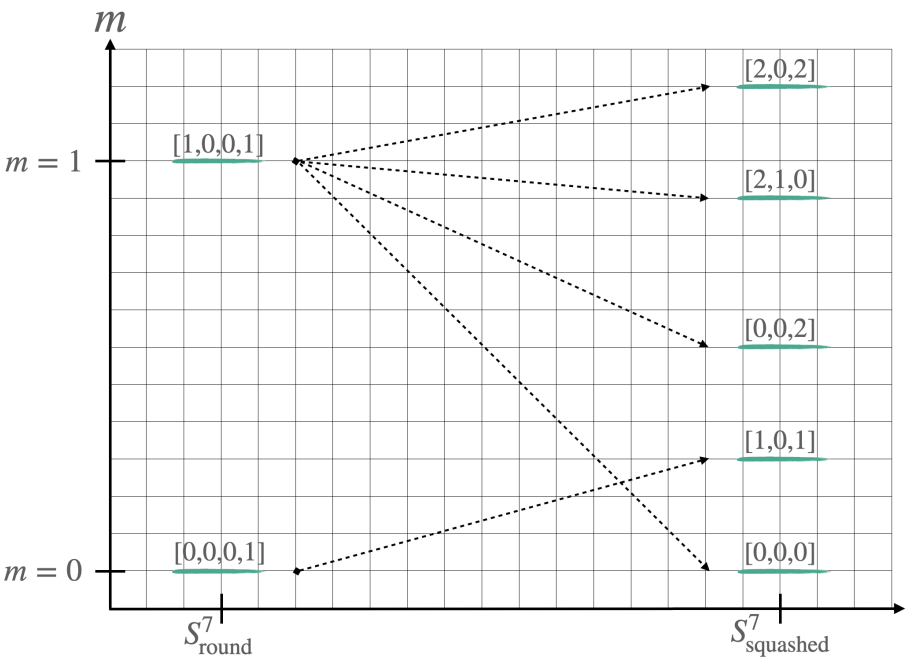


Figure 3.11: Schematic representation of the space invader scenario for the first two levels of the gravitini. All massless fields in the $[0,0,0,1]$ representation on S^7_{round} become massive on S^7_{squashed}, and one massive field in $[1,0,0,1]$ at $\ell = 1$ on S^7_{round} becomes massless on S^7_{squashed}.

The embedding of the scalar KK fluctuations according

$$U(x,y) = \mathring{U}(y)(\mathbb{1} + 2\mathbb{T}_\alpha \Phi^\alpha(x,y)) = \mathring{U}(y)(\mathbb{1} + 2\mathbb{T}_\alpha \sum_\Sigma \phi^{\alpha\Sigma}(x)\mathcal{Y}^\Sigma(y)) \tag{3.192}$$

allows us to explicitly construct different alternative truncations to other subsets of fields which in particular allow to embed the solution (3.167). E.g. the truncation of the full KK spectrum keeping all singlets under $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ defines a consistent truncation by the standard symmetry argument: by simple representation reasoning, singlet fields can never define non-vanishing sources for non-singlet fields, it is thus consistent to truncate out all the non-singlet fields. This truncation retains one field from the KK tower of gravitino fields, and thus corresponds to a $d = 4$, $\mathcal{N} = 1$ theory [159], further discussed in [161]. In the scalar sector, the truncation to $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ singlets only keeps the four scalar fields (3.190) and can be described in closed form by integrating the corresponding fluctuations (3.192) to

$$U(x,y) = \mathring{U}(y)\, \exp\Big[\sum_{\text{singlets}} \phi^i(x)\, s_i^{\alpha,\Sigma}\, \mathbb{T}_\alpha \mathcal{Y}_\Sigma(y) \Big] . \tag{3.193}$$

The index i here labels the four $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ singlets found in the tensor product of the 70 non-compact generators of $\mathrm{E}_{7(7)}$ and the scalar harmonics, thereby defining the constant tensor $s_i^{\alpha,\Sigma}$.

The group theoretical structure underlying this truncation can be made more transparent by representing the seven sphere as a coset space [162]

$$S^7 = \frac{\mathrm{G}}{\mathrm{H}} = \frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)_0}{\mathrm{Sp}(1)_\mathrm{L} \times \mathrm{Sp}(1)_\mathrm{D}} , \tag{3.194}$$

where the different $\mathrm{Sp}(1)$ factors are embedded as

$$\mathrm{Sp}(2) \supset \mathrm{Sp}(1)_\mathrm{L} \times \mathrm{Sp}(1)_\mathrm{R} , \qquad \mathrm{Sp}(1)_\mathrm{D} = \big(\mathrm{Sp}(1)_0 \times \mathrm{Sp}(1)_\mathrm{R}\big)_{\mathrm{diag}} , \tag{3.195}$$

and for clarity we have added a '$_0$' subscript to the $\mathrm{Sp}(1)$ subgroup of (3.189). The seven sphere (3.194) can be represented by an $(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$-valued coset representative $S(y)$, such that the infinitesimal action of the $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ isometry group on the coordinates is realized as

$$\delta_\Lambda S(y) = \Lambda S(y) - S(y) h_\Lambda(y) , \qquad \Lambda \in \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)_0 , \quad h_\Lambda(y) \in \mathfrak{sp}(1)_\mathrm{L} \oplus \mathfrak{sp}(1)_\mathrm{D} . \tag{3.196}$$

The consistent truncation (3.193) can then be given in more compact form as

$$U(x,y) = \mathring{U}(y)\, S(y)\, W(x)\, S^{-1}(y) , \tag{3.197}$$

with the $(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$ coset representative $S(y)$ and an $\mathrm{E}_{7(7)}$ matrix $W(x)$ defined to live in the commutant of the denominator group $\mathrm{H} = \mathrm{Sp}(1)_\mathrm{L} \times \mathrm{Sp}(1)_\mathrm{D}$ in $\mathrm{E}_{7(7)}$, i.e.

$$W(x)\, h = h\, W(x) \quad \forall\, h \in \mathfrak{sp}(1)_\mathrm{L} \oplus \mathfrak{sp}(1)_\mathrm{D} . \tag{3.198}$$

Indeed, (3.196) and (3.198) imply that the factor $S(y)\, W(x)\, S^{-1}(y)$ in (3.197) is invariant under the action of $(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$ up to a compact gauge transformation acting from the right. The representation (3.197) of the consistent truncation immediately reveals the geometry of its scalar target space, given by the commutant of $\mathrm{H} = \mathrm{Sp}(1)_\mathrm{L} \times \mathrm{Sp}(1)_\mathrm{D}$ in $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ as

$$\mathcal{M}_{\mathrm{scalar}} = \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \times \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} , \tag{3.199}$$

a Kähler manifold parametrized by the four scalar fields from (3.190). Let me show what this implies for us in this example. [7]

The $d = 4$ theory can be obtained by plugging the ansatz (3.197) into the ExFT Lagrangian of [84]. For the bosonic sector, parametrizing the matrix W by four scalar fields $\{u, v, c, \chi\}$, this gives

[7] The same reasoning shows that the analogous construction based on the coset representation $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)_\mathrm{L}$ of the seven sphere, yields an $\mathcal{N} = 4$ consistent truncation to the $\mathrm{Sp}(2)$ singlets with scalar target space given by the commutant of the denominator group $\mathrm{Sp}(1)_\mathrm{L}$ in $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$:

$$\mathcal{M}_{\mathrm{scalar}} = \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(6,3)}{\mathrm{SO}(6) \times \mathrm{SO}(3)} , \tag{3.200}$$

as explicitly found in [159].

rise to the four-scalar Lagrangian

$$
\begin{aligned}
|g|^{-1/2}\,\mathcal{L}_{(4)} &= R_{(4)} - \frac{63}{2}\,\partial_\mu u\,\partial^\mu u - 21\,\partial_\mu v\,\partial^\mu v - 3\,e^{-6u-2v}\partial_\mu c\,\partial^\mu c - \frac{1}{2}\,e^{-6u+12v}\,\partial_\mu\chi\,\partial^\mu\chi - V_{\rm pot}\,,\\
V_{\rm pot} &= -6\,e^{-9u+4v} - 48\,e^{-9u-3v} + 12\,e^{-9u-10v} - 72\,e^{-15u-12v}\,c^2\\
&\quad - 12\,e^{-15u+2v}\,(c+\chi)^2 - 18\,e^{-21u}\,\big(1+c^2+2\,c\,\chi\big)^2\,.
\end{aligned}
\tag{3.201}
$$

It is straightforward to verify that the intersection of the scalar target space (3.199) with the SL(8) of (3.186) capturing the Freund-Rubin solutions corresponds to a further consistent truncation to two scalar fields which precisely reproduces the result (3.170) of [155]. The $\mathcal{N}=1$ consistent truncation to four scalars (3.201) has already been found in [159]. What is new in the present construction is its explicit embedding (3.197) via a twist matrix in ExFT which allows us to apply the universal mass formulas such as (3.129) to any background within this truncation. With the frame given by

$$
U = \mathring{U}\,S\,W\,S^{-1}\,,
\tag{3.202}
$$

a general background will satisfy (3.94), but, in general, with y-dependent intrinsic torsion $X_{\underline{AB}}{}^{\underline{C}}(y)$.[8] Thus, it is no longer generalised Leibniz parallelisable, but still generalised parallelisable. As a consequence the mass formulae (3.129) still apply. This has been used in [DB1] in order to derive the full Kaluza-Klein spectrum around the $\mathcal{N}=1$, $\mathrm{AdS}_4\times S^7_{\rm squashed}$ vacuum. In the following, we will extend this analysis to also derive quadratic couplings around the domain wall background (3.167), or any other background of the consistent truncation.

F.3 Couplings around the domain wall background

We can efficiently describe the linearised fluctuations around the round S^7 using its generalised Leibniz parallelisation in ExFT. Similarly, using the generalised parallelisation of the family of squashed S^7's of section 3.F.2.b, we obtain a simple expression of the linearised fluctuations around the family of squashed S^7 that describe the flow of section 3.F.1 . While this was used in section 3.E ([DB1]) to obtain the full Kaluza-Klein spectrum of the squashed S^7, here we will further extend the computation to obtain the quadratic couplings of the higher Kaluza-Klein modes around the domain wall solution. These couplings encode all the information about the holographic 2-point functions along the dual RG flow.

In computing the quadratic couplings, we have to choose a field basis for the Kaluza-Klein fluctuations, or choice of "frame". Two natural choices are to multiply the generalised vielbein of the 2-scalar truncation, $U(x,y)$ of (3.197), from the right or left, leading to different kinetic terms for the scalars u, v and the Kaluza-Klein fluctuations. The kinetic term of ExFT is given by $D_\mu M^{\underline{MN}} D^\mu M_{\underline{MN}}$ with $M = \mathscr{V}\mathscr{V}^T$, which can be expressed in terms of the current $J_\mu(\mathscr{V})_{\underline{M}}{}^{\underline{N}} = (\mathscr{V}^{-1}D_\mu\mathscr{V})_{\underline{M}}{}^{\underline{N}}$ as

$$
D_\mu M^{\underline{MN}} D^\mu M_{\underline{MN}} = -2\left(\mathrm{Tr}\,J^\mu J_\mu^T + \mathrm{Tr}\,J^\mu J_\mu\right)\,.
\tag{3.203}
$$

Let us denote the Kaluza-Klein fluctuations by the matrix $\delta = \exp[\mathbb{T}_\alpha \sum_\Sigma \phi^{\alpha,\Sigma}(x)\,\mathcal{Y}_\Sigma(y)]$. Then, it

[8] Because of the scalar fields in $W(x)$, the intrinsic torsion will also depend on x, but this is standard already in gauged Supergravity. In particular, it does not interfere with the generalised (Leibniz) parallelisability.

is easy to see that J_μ becomes

$$\begin{aligned} J_\mu(\mathscr{V}\delta) &= \delta^{-1} J_\mu(\mathscr{V})\delta + J_\mu(\delta)\,, \\ J_\mu(\delta\mathscr{V}) &= \mathscr{V}^{-1} J_\mu(\delta)\mathscr{V} + J_\mu(\mathscr{V})\,, \end{aligned} \tag{3.204}$$

depending on whether we multiply the generalised $\mathscr{V}$ with the Kaluza-Klein fluctuations from the left or the right. We clearly see that multiplying $\mathscr{V}$ from the left leaves the kinetic term for the background, i.e. the kinetic term of the scalars u, v, invariant, but introduces u, v factors into the kinetic term of the Kaluza-Klein fluctuations. On the other hand, using $\mathscr{V}\delta$, we find that the Kaluza-Klein fluctuations enter the kinetic terms of u, v and hence introduce corrections to the kinetic terms of (3.201). In order to simply retain the normalisation (3.201), we will choose to define our Kaluza-Klein fluctuations as multiplying the truncation $\mathscr{V}$ from the left. Note that this differs from the conventions of the Kaluza-Klein spectrometry [99, 116, DB1], used in (3.192). The quadratic couplings around the domain wall solutions are then straightforwardly obtained by evaluating the action of the mass operator (3.129) in this basis.

Since the consistent truncation (and thus domain wall) preserves $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$, different $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ representations will not mix along the flow. Therefore, we can restrict our attention to any fixed $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ representation to compute their quadratic couplings. As an illustration of our method, we will evaluate the mass operator (3.129) on the scalars fields in the $[0,0,0]$, the $[0,1,2]$ and $[0,2,4]$ representations, which at the squashed S^7 sit in the long multiplets (3.155) as follows

$$\begin{aligned} L[\tfrac{1}{2},\Delta] : \lambda \quad &\xrightarrow{Q}\quad A_\mu \oplus \phi \quad \xrightarrow{Q}\quad \lambda\,, \\ L[0,\Delta] : \phi \quad &\xrightarrow{Q}\quad \lambda \quad\;\; \xrightarrow{Q}\quad \phi\,. \end{aligned} \tag{3.205}$$

a $[0,0,0]$ sector

As a warm up, let us compute the quadratic couplings in the sector of scalar fields singlet under $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$, forming the bosonic sector of the $\mathcal{N}=1$ truncation of [159, 161]. This sector carries four scalar field fluctuations whose basis we label as φ_i. The computation is straightforward: we first compute the intrinsic torsion from (3.202), where W is evaluated on the domain wall solution. Next we build the associated mass operator (3.129) and evaluate its action on the scalar fluctuations in this sector. The final result for the quadratic fluctuations in this sector takes the form

$$\begin{aligned} \mathcal{L}_{[0,0,0]} = &-\frac{1}{2}\left(\partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2\right) - \frac{1}{2}\,e^{-6u-2v}\,\partial_\mu\varphi_3\partial^\mu\varphi_3 - \frac{1}{2}\,e^{-6u+12v}\,\partial_\mu\varphi_4\partial^\mu\varphi_4 \\ &- \left(54\,e^{-21u} + 16\,e^{-9u-10v} - 36\,e^{-9u-3v} - 2\,e^{-9u+4v}\right)\varphi_1^2 - 12\,e^{-15u+2v}\,\varphi_4^2 \\ &+ 2\sqrt{6}\left(9\,e^{-21u} - 4\,e^{-9u-10v} - e^{-9u+4v}\right)\varphi_1\varphi_2 - 3\left(3\,e^{-21u} + 2\,e^{-9u-10v} - e^{-9u+4v}\right)\varphi_2^2 \\ &- 2\left(3\,e^{-21u} + 6\,e^{-15u-12v} + e^{-15u+2v}\right)\varphi_3^2 - 4\sqrt{6}\left(3\,e^{-21u} + e^{2v-15u}\right)\varphi_3\varphi_4\,, \end{aligned} \tag{3.206}$$

where u and v denote the domain wall solution constructed in section 3.F.1 above. One verifies that at the endpoints of the flow, the Lagrangian (3.206) reproduces the masses of the scalar fields for

the squashed and the round S^7. In fact, since all four scalar singlets lie within the $\mathcal{N}=1$ truncation (3.199), we could have arrived at this result directly by linearising the potential (3.201). It is a good consistency check of our method, that the results indeed agree. In contrast, the scalar fluctuations at higher KK levels do not lie within a consistent truncation and the respective couplings can only be obtained by the mass operator (3.129), as we shall discuss in the following. We finally note that in order to study the complete set of fluctuation equations in this singlet sector, one also has to take into account the fluctuations of the metric around the AdS_4 background.

b $[0,1,2]$ sector

Let us now extend the computation to scalar modes that do not lie within the consistent truncation. As an example, we choose the sector in the $[0,1,2]$ representation of $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$. This is the representation of the vector fields that become massless on the round S^7. As a consequence, the associated scalar Goldstone modes on the squashed sphere become physical scalars on the round sphere. There are in total eight scalar fluctuations ϕ_i transforming in the $[0,1,2]$ representation. Repeating the above computation in this sector leads to the Lagrangian

$$\begin{aligned}\mathcal{L}_{[0,1,2]} = &-\frac{1}{2}e^{-7v}\left(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2\right)-\frac{1}{2}e^{-6u-9v}\,\partial_\mu\phi_3\partial^\mu\phi_3-\frac{1}{2}e^{-6u-2v}\,\partial_\mu\phi_4\partial^\mu\phi_4\\&-\frac{1}{2}e^{-6u+5v}\left(\partial_\mu\phi_5\partial^\mu\phi_5+\partial_\mu\phi_6\partial^\mu\phi_6\right)-\frac{1}{2}e^{-12u-4v}\,\partial_\mu\phi_7\partial^\mu\phi_7-\frac{1}{2}e^{-12u+3v}\,\partial_\mu\phi_8\partial^\mu\phi_8\\&-\left(10\,e^{-9u-10v}-2\,e^{-9u-3v}\right)\phi_1^2-4\,e^{-15u-5v}\,\phi_{345}^2-4\sqrt{2}\left(e^{-15u-5v}+3e^{-21u}\right)\phi_6\phi_{345}\\&-\left(6\,e^{-15u-5v}+4\,e^{-15u+2v}+4\,e^{-15u+9v}+6\,e^{-21u}\right)\phi_6^2\,,\end{aligned}\tag{3.207}$$

with $\phi_{345}=\phi_3+\phi_4+\phi_5$. The result shows that the potential only depends on three out of the eight ϕ_i's. Therefore, the remaining five fluctuations are Goldstone modes along the flow. It is interesting to compare this with the squashed sphere endpoint. From equations (3.154) and (3.205) we see that at the squashed S^7 vacuum there are only two physical scalars in the $[0,1,2]$ in the spectrum. Indeed, plugging in the values of u and v for the squashed S^7 (3.171), we see that ϕ_1 additionally drops out of the potential at the squashed S^7 endpoint, thus reproducing the expected number of Goldstone scalars. Moreover, the potential (3.207) reproduces the masses of the scalar fields at the squashed and round S^7 points.

c $[0,2,4]$ sector

As a last example, we study the six scalars ψ_i in the $[0,2,4]$ representation. Their quadratic couplings are obtained by the analogous computation and read

$$\begin{aligned}\mathcal{L}_{[0,2,4]} = &-\frac{1}{2}e^{-7v}\,\partial_\mu\psi_1\partial^\mu\psi_1-\frac{1}{2}e^{-6u-2v}\left(\partial_\mu\psi_3\partial^\mu\psi_3+\partial_\mu\psi_4\partial^\mu\psi_4\right)-\frac{1}{2}\left(\partial_\mu\psi_5\partial^\mu\psi_5+\partial_\mu\psi_6\partial^\mu\psi_6\right)\\&-\frac{1}{2}e^{-6u+5v}\,\partial_\mu\psi_2\partial^\mu\psi_2-6\,e^{-9u}\left(3\,e^{-10v}+e^{-3v}\right)\psi_1^2-8\,e^{-9u-10v}\left(\sqrt{5}\,\psi_5-\sqrt{3}\,\psi_6\right)\psi_1\\&-4\,e^{-9u}\left(e^{-10v}+5\,e^{-3v}+e^{4v}\right)\psi_5^2-4\,e^{-9u}\left(e^{-10v}-e^{-3v}+3\,e^{4v}\right)\psi_6^2\\&-\left(6\,e^{-15u-5v}+8\,e^{-15u+2v}+12\,e^{-21u}\right)\tilde{\psi}_3^2-\left(10\,e^{-15u-5v}+8\,e^{-15u+2v}-12\,e^{-21u}\right)\tilde{\psi}_4^2\\&+4\sqrt{15}\,e^{-15u-5v}\,\tilde{\psi}_3\tilde{\psi}_4\,,\end{aligned}\tag{3.208}$$

where $\tilde{\psi}_3 = \psi_3 - \frac{1}{2}\sqrt{3}\,\psi_2$, $\tilde{\psi}_4 = \psi_4 - \frac{1}{2}\sqrt{5}\,\psi_2$. The potential only depends on five out of the six scalars. Therefore, one scalar field will be a Goldstone mode along the entire flow. Finally, as a consistency check, the potential (3.208) again reproduces the masses at both of the endpoints.

d Comments

We have shown how the mass operator (3.129) allows to determine the quadratic couplings of scalar fluctuations around the domain wall background constructed in section 3.F.1. We have restricted to spelling out three examples, but the method of course extends to all higher KK levels. The resulting couplings (3.206), (3.207), (3.208), carry the full information for the computation of the holographic 2-point functions along the dual RG flow. The corresponding computation requires a careful setup of the holographic renormalisation procedure along the lines of [113, 163, 164], and will be interesting to take on in the future.

G Conclusion

THROUGHOUT this chapter, we have reviewed the state-of-the-art techniques for computing spectra in a Kaluza-Klein scenario. We began by conducting direct calculations of spectra, leveraging the power of group theory. This technique have been to a large extend used in the passed to access vacua with a lot of symmetry. We illustrated this method by computing the Kaluza-Klein spectrum around $AdS_4 \times S^7_{\text{round}}$. Despite the success of this method, it encountered challenges in computing spectra around vacua with lower symmetry. For instance, the complete spectrum around $AdS_4 \times S^7_{\text{squashed}}$, both for $\mathcal{N} = 1$ and $\mathcal{N} = 0$, remained incomplete.

In recent years, new techniques utilizing ExFT have been developed to compute spectra in scenarios with lower symmetry. This approach has proven to be highly effective and has successfully been applied to a wide range of vacua across various dimensions as we showed in section (3.C). We then used those techniques to give the full spectrum of IIB Supergravity on $AdS_5 \times S^5$ background in the language of ExFT. However, these techniques heavily rely on having a consistent truncation living in the maximally symmetric theory. This has posed a limiting factor for ExFT techniques despite significant advancements.

In section (3.D), we showed how to compute the full Kaluza-Klein spectrum of Supergravity compactifications which are not part of a consistent truncation, but are still generalised parallelisable. Examples of such vacua are deformations of compactifications within $\mathcal{N} = 8$ Supergravity by scalar fields which are not part of the $\mathcal{N} = 8$ truncations. This includes the Supergravity duals of RG flows of $\mathcal{N} = 4$ SYM or $\mathcal{N} = 8$ ABJM triggered by single-trace operators. Thus, our formalism can be used to compute the Kaluza-Klein spectrum for the end-point of such flows or even along such flows.

As an application of the method, we computed the full spectrum of the $AdS_4 \times S^7_{\text{squashed}}$ solution of 11-dimensional Supergravity. This preserves only $\mathcal{N} = 1$ supersymmetry (or $\mathcal{N} = 0$ in the case of the right-squashed S^7), and thus has no protected operators. This background is a coset space, so that traditional techniques [114] can in principle be used. However, because the squashed S^7 is not a symmetric space, this is still rather difficult and had not been completed until now, despite however many explicit results [141, 148–151]. Using our technology, we were able to compute the full Kaluza-Klein spectrum, which is captured by the remarkably simple formula (3.148), which depends only on the $\mathrm{USp}(4) \times \mathrm{SU}(2)$ representation of the multiplet, the spin of the

superconformal primary and the charge under an additional $\mathbb{R}^+$ factor. Intriguingly, for all but the smallest representations, this $\mathbb{R}^+$ charge appears to descend from representations of a bonus $\mathrm{SL}(2)$, whose origin is mysterious.

The method we described here opens up the possibility of computing the Kaluza-Klein spectrum of many more interesting string compactifications which do not reside in a $\mathcal{N} = 8$ consistent truncation. This includes TsT transformations of vacua in $\mathcal{N} = 8$ Supergravity, such as the marginal deformations of $\mathrm{AdS}_5 \times S^5$, $\mathrm{AdS}_4 \times S^7$ or $\mathcal{N} = 1$ AdS_4 vacua of IIB string theory [135]. Our method might also apply to the cubic deformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$, which has recently been described implicitly in generalised geometry [165]. It would be interesting to see if this implicit description may still be sufficient to apply our method here.

Finally, it would be interesting to better understand our results from a CFT perspective. It is remarkable that the spectrum of the squashed S^7, which only preserves $\mathcal{N} = 1$ supersymmetry and thus has no protected multiplets, displays such a simple structure. A particularly interesting questions is the origin of the additional s charge (and its enhancement to a bonus $\mathrm{SL}(2)$ group) which appears to organise the spectrum. Perhaps computing the spectrum along the RG flow from the round S^7 to the squashed one may shed some light into this.

Then in section (3.F) we revisited the problem of constructing a domain wall solution interpolating between the $\mathcal{N} = 1$ $\mathrm{AdS}_4 \times S^7_{\text{squashed}}$ and $\mathcal{N} = 8$ $\mathrm{AdS}_4 \times S^7_{\text{round}}$ vacua of 11-dimensional Supergravity. There is no supersymmetric domain wall preserving the $\mathrm{Sp}(2) \times \mathrm{Sp}(1) \subset \mathrm{SO}(8)$ isometry of the squashed S^7, as can, for example, be seen by noting that the $\mathcal{N} = 8$ gravitini of the round S^7 are not $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ singlets. Instead, we construct an explicit non-supersymmetric flow by using a consistent truncation to 4-dimensional $\mathcal{N} = 1$ Supergravity [159, 166] and solving the second-order flow equations numerically. Interestingly, within this $\mathcal{N} = 1$ truncation, the $\mathcal{N} = 8$ round S^7 appears non-supersymmetric, since its massless gravitini reside amongst the higher KK modes.

Using the techniques developed in previous sections and [99, 116, DB1], we were able to compute the quadratic couplings of KK fluctuations around the domain-wall solution (or any other solution of the 2-dimensional truncation). This relied on the fact that the family of squashed S^7 still admits a trivialisation of the generalized tangent bundle [DB1], allowing us to construct a globally well-defined generalised frame $\mathcal{V}(x, y)$. However, the intrinsic torsion $X_{\underline{MN}}{}^{\underline{P}}(y)$ of this generalised frame is not constant, reflecting the fact that the squashed S^7 only admits consistent truncations to $\mathcal{N} = 4$ or $\mathcal{N} = 1$ but not $\mathcal{N} = 8$. We presented the quadratic couplings of some low-lying KK scalars, but the same method can be applied to any other KK tower. This encodes the information needed to extract all the holographic two-point functions along the flow.

This work opens up several natural directions for future investigations. One would be to understand the analytic structure underlying the quadratic couplings of section 3.F.3, similar to the group-theoretic formula that appear in the KK spectra of vacua with sufficient isometries, e.g. [99, 120]. For this it would be useful to extend our computation of the quadratic couplings to other KK towers. Another interesting question would be to push the calculations of section 3.F.3 to cubic order, giving access to 3-point functions along the RG flow on the field theory side. These results yield the relevant input for the holographic renormalisation procedure [113, 163, 164], that would allow us to precisely obtain the field theory correlators along the flow.

Finally, our results here, together with [DB1], provide the first extension of the ExFT formalism of Kaluza-Klein spectroscopy [99, 116] beyond generalised parallelisable spaces, i.e. vacua of maximally supersymmetric consistent truncations. While the family of squashed S^7's still admits a

generalised parallelisation, there is no longer an $\mathcal{N} = 8$ consistent truncation that the solutions belong to. Hence our method here provides a first window into how to generalise the Kaluza-Klein spectroscopy to more general truncations that break some supersymmetry.

All of these questions will allow us to address the ultimate question: how does having a consistent truncation affect the structure of correlation functions in the holographic dual? We leave these exciting research endeavours for the future.

CHAPTER

4

HIGHER-ORDER SUPERGRAVITY COUPLINGS USING EXFT

Contents

A Introduction

So far, our discussions have only focused on Kaluza-Klein spectroscopy. This has strong relevance in the AdS/CFT correspondence because masses obtained through KK spectroscopy are related to conformal dimensions of operators in the dual CFT, see Table 2.6. However, the AdS/CFT correspondence states the equivalence between n-point couplings on the Supergravity side and correlators on the dual CFT not only for $n = 2$ but for all values of n (2.109). For example, in the maximally symmetric $AdS_4 \times S^7$ and $AdS_5 \times S^5$ vacua of M-theory and String Theory, couplings between Kaluza-Klein modes naturally arise in the compactification procedure. These couplings are conjectured to encode the information about correlators of the ABJM $3d$ theory and $4d$ $\mathcal{N} = 4$ SYM [104], respectively. Therefore, using the AdS/CFT correspondence, we can probe the dual CFT by computing objects on the Supergravity side. This guiding principle hase been used and some cubic and quartic couplings have been computed on the Supergravity side to access 3-point and 4-point correlators of local operators on the CFT side [108, 109, 111, 167–169]. However, these results are only for specific fields, and no general results are known for n-point couplings, even for highly symmetric vacua. There are indeed 20-year-old conjectures [170, 171] about the vanishing of extremal and near-extremal n-point couplings for the maximally supersymmetric $AdS_5 \times S^5$, $AdS_4 \times S^7$, and $AdS_7 \times S^4$ vacua of string and M-theory, whose proof appears inaccessible from the Supergravity side. Furthermore, as we have already seen in (3.B.2), computing masses for Kaluza-Klein modes was already a daunting task and often impossible for backgrounds with few symmetries. Consequently, obtaining higher-order couplings by direct calculation is even more complex. Even if difficult, this is doable and we will first review the techniques used in calculations that have been done so far to compute the couplings in Supergravity.

The use of different variables has often led to new insights in physics, for example elucidating new structures or simplifying computations. Similarly here, the reorganisation of fields to make exceptional symmetries manifest in Supergravity section 2.D, has proven powerful in computing Kaluza-Klein spectra around any vacuum that belongs to a maximally supersymmetric consistent truncation of 10-/11-dimensional Supergravity and beyond, as shown in section 3.C and in section 3.D. With this progress in computing the Kaluza-Klein mass spectrum, a natural question is whether the ExFT formalism can help in computing n-point couplings of the Kaluza-Klein modes, where even less is known.

In this chapter, we show how to extend the ExFT techniques to compute higher-order couplings. This is done follwing [DB3]. We will show that the ExFT field basis (2.77) leads to a streamlined computation for n-point couplings, simplifying the laborious computations arising using traditional Supergravity techniques. These traditional approaches typically involve gauge fixing, and field redefinitions that invoke new higher-derivative terms and generally lack much structure. By contrast, in ExFT, we expand the two-derivative action (2.77) using a fluctuation ansatz which can immediately be used for any vacuum of a consistent truncation of a maximally supersymmetric consistent truncation. Not only is the computation simplified, ExFT also reveals a universal structure underlying the n-point couplings, which for any vacuum of a maximally supersymmetric consistent truncations are controlled by the same n-point invariant of scalar harmonics of the maximally symmetric point. In particular, this implies that infinitely many couplings vanish for any vacuum of the consistent truncation, even when allowed by group theory. For the maximally supersymmetric $AdS_5 \times S^5$, $AdS_4 \times S^7$ and $AdS_7 \times S^4$ vacua, this allows us to prove the conjectured vanishing of extremal and near-extremal n-point bulk couplings [170, 171]. Moreover,

focusing on 3-point functions in $E_{6(6)}$ ExFT, we will obtain universal formulae that can be used to easily compute the cubic couplings all Kaluza-Klein modes for any vacuum of a 5-dimensional maximally supersymmetric consistent truncation.

This chapter is organized as follows. We start by first reviewing the old way of computing cubic and quartic couplings in Supergravity. In section 4.C, we will start presenting the original results of [DB3]. We extend the ExFT analysis to n-point couplings, where we show that they are controlled by certain invariants of n scalar harmonics of the maximally symmetric points of the truncations. We show how this causes infinitely many n-point couplings to vanish, despite being allowed by the symmetry group of the vacua. In section 4.D, we specialise to cubic couplings and work out the explicit universal formulae which are valid for any vacuum that belongs to a 5-dimensional $\mathcal{N} = 8$ consistent truncation for several fields of the $E_{6(6)}$ ExFT. Finally in section 4.E, we apply our results to the $\mathcal{N} = 8$ $AdS_5 \times S^5$ vacuum of IIB string theory and prove the conjecture that extremal and near-extremal n-point couplings vanish. Moreover, we use our universal formulae to compute cubic couplings for $AdS_5 \times S^5$, comparing those that were already known with the literature and presenting some new results.

B Cubic couplings in Supergravity : brute force

THE AdS/CFT correspondence postulates the equivalence between String/M theory on AdS spacetime multiplied by a compact manifold and a conformal field theory known as the dual CFT. An example of this conjecture is the proposed equivalence between Type IIB string theory on $AdS_5 \times S^5$ and $d = 4$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, or between M-theory on $AdS_4 \times S^7$ and the $3d$ ABJM theory. This correspondence is holographic, meaning the CFT lives on the boundary of the AdS space in String Theory. The operators of the CFT are mapped to on-shell bulk fields, whose asymptotic values interact with the CFT's operators, effectively acting as sources for the local operators of the CFT. Consequently, the partition function of String Theory with fixed asymptotic values is identified with the partition function on the CFT side. This equivalence enables us to establish a link between n-point couplings on the gravity side and correlators on the dual CFT, where the sources are the boundary values of bulk fields. In the context of the large N limit on the SYM side, where N is the N appearing in the gauge group $SU(N)$ of the gauge theory, the conjecture is refined: as the 't Hooft coupling $\lambda = g_{YM}^2 N \gg 1$, String Theory is replaced by its low-energy limit, Supergravity.

In the rest of this chapter, we will mainly focus on Type IIB Supergravity on $AdS_5 \times S^5$ for which the spectrum and field content is given in section 3.C.4. We now work out some cubic couplings for this theory by brute force calculation. This is done following [108, 109].

The approach to obtaining couplings shares similarities with what we have done in the calculation of mass spectra in section 3.B.2. Initially, one seeks a solution to the equations of motion, which establishes the background, here $AdS_5 \times S^5$. Subsequently, fluctuations are introduced on top of these background fields, and these fluctuations are expanded into harmonics of the internal space. The computation involves to expand equations of motion to quadratic order. The effectiveness of this procedure also relies on gauge fixing. The process starts with the equations of motion which read for type IIB

$$R_{MN} = \frac{1}{3!} F_{MM_1M_2M_3M_4} F_N{}^{M_1M_2M_3M_4}\,, \tag{4.1}$$

$$F_{M_1\dots M_5} = \frac{1}{5!}\epsilon_{M_1M_2M_3M_4M_5N_1N_2N_3N_4N_5}F^{N_1N_2N_3N_4N_5}\,, \tag{4.2}$$

with $M, N, \dots \in [\![0,9]\!]$, R_{MN} the $10d$ Ricci tensor, and F the five-form field strength, defined as

$$F_{M_1M_2M_3M_4M_5} = 5\partial_{[M_1}C_{M_2M_3M_4M_5]}, \tag{4.3}$$

The equation of motion for the 4-form potential is a self-duality equation (4.2) for the field strength. The background solution leading to $AdS_5 \times S^5$ geometry is

$$\begin{cases} \mathring{g}_{MN}dx^M dx^N = \frac{1}{x_0^2}(dx_0^2 + \eta_{\bar{\mu}\bar{\nu}}dx^{\bar{\mu}}dx^{\bar{\nu}}) + d\Omega_5^2\,, \\ R_{\mu\nu\rho\sigma} = m^2(-\mathring{g}_{\mu\rho}\mathring{g}_{\nu\sigma} + \mathring{g}_{\mu\sigma}\mathring{g}_{\nu\rho})\,; \quad R_{\mu\nu} = -4m^2\mathring{g}_{\mu\nu}\,, \\ R_{mnpq} = m^2(\mathring{g}_{mp}\mathring{g}_{nq} - \mathring{g}_{mq}\mathring{g}_{nq})\,; \quad R_{mn} = m^2 4\mathring{g}_{mn}\,, \\ \mathring{F}_{\mu\nu\rho\sigma\lambda} = m\epsilon_{\mu\nu\rho\sigma\lambda}\,; \quad \mathring{F}_{mnpqr} = m\epsilon_{mnpqr} \end{cases} \tag{4.4}$$

with $\eta_{\bar{\mu}\bar{\nu}}$ the $4d$ Minkowski metric and $d\Omega_5^2$ the metric of S^5. The convention for the indices in the same as in section 3.B.2 with $m, n, \dots = 1, \dots, 5$ for internal indices and $\mu, \nu, \dots = 1, \dots 5$ for external indices. m is here an overall factor which fixes the mass scale for the compactification, and its inverse is the radius of both S^5 and AdS_5. In the following we set $m = 1$. Note that this ansatz is the Freund-Rubin ansatz we already described earlier and used to find spontaneous compactification of $11d$ Supergravity, and that we announced at the end of section 2.C.4. The existence of such solution was first discovered in [64] and further developped in [172].

The next step is to add fluctuations on top of this background. This gives

$$g_{MN}(x,y) = \mathring{g}_{MN}(x,y) + h_{MN}(x,y) \tag{4.5}$$

which we further decompose as

$$\begin{cases} h_{mn} = h_{((mn))} + \frac{h_2}{5}\mathring{g}_{mn}\,; \quad \mathring{g}^{mn}h_{((mn))} = 0\,, \\ h_{\mu\nu} = h'_{\mu\nu} - \frac{h_2}{3}\mathring{g}_{\mu\nu}\,; \quad h'_{\mu\nu} = h'_{((\mu\nu))} + \frac{h'}{5}\mathring{g}_{\mu\nu}\,; \quad \mathring{g}^{\mu\nu}h_{((\mu\nu))} = 0\,. \end{cases} \tag{4.6}$$

and non trivial mixed factors $h_{m\mu}$. Finally, for the field strength, the fluctuations can be parameterized

$$F = \mathring{F} + \delta F\,; \quad \delta F = 5\nabla_{[M}c_{NPQR]}\,. \tag{4.7}$$

At this stage we could expand the fluctuations in harmonics. Let us first impose the de Donder gauge conditions

$$\nabla^m h_{mn} = 0\,; \quad \nabla^m h_{m\mu} = 0\,; \quad \nabla^m c_{mabc} = 0 \tag{4.8}$$

The most general expansions in harmonics is now given by

$$\begin{cases} h'_{\mu\nu}(x,y) = \sum_{I_1} h'^{I_1}_{\mu\nu}(x)\mathcal{Y}^{I_1}\,, \quad h_{mn}(x,y) = \sum_{I_{14}} h^{I_{14}}_{\mu}(x)\mathcal{Y}^{I_{14}}_{mn}\,, \\ h_{\mu m}(x,y) = \sum_{I_5} h^{I_5}_{\mu}(x)\mathcal{Y}^{I_5}_{m}\,, \quad h_2(x,y) = \sum_{I_1} h_2^{I_1}(x,y)\mathcal{Y}^{I_1}\,, \end{cases} \tag{4.9}$$

for the metric part, and for the 4-form potential

$$\begin{cases} c_{\mu\nu\rho\sigma}(x,y) = \sum_{I_1} b^{I_1}_{\mu\nu\rho\sigma}(x)\mathcal{Y}^{I_1}\,, \\ c_{\mu\nu\rho m}(x,y) = \sum_{I_5} b^{I_5}_{\mu\nu\rho}(x)\mathcal{Y}^{I_5}_{m}\,, \\ c_{\mu\nu mn}(x,y) = \sum_{I_{10}} b^{I_{10}}_{\mu\nu}(x)\mathcal{Y}^{I_{10}}_{[mn]}\,, \\ c_{\mu mnp}(x,y) = \sum_{I_5} b^{I_5}_{\mu}(x)\epsilon_{mnp}{}^{qr}\nabla_q\mathcal{Y}^{I_5}_{r}\,, \\ c_{\mu mnp}(x,y) = \sum_{I_5} b^{I_5}(x)\epsilon_{mnpq}{}^{q}\nabla_r\mathcal{Y}^{I_5}\,. \end{cases} \tag{4.10}$$

Here, we once again observe the emergence of tensorial harmonics, which are eigenfunctions of the Hodge-de Rham operators for p-forms. Given that the internal space is S^5, which can be represented by $\frac{SO(6)}{SO(5)}$, we can employ similar reasoning as before to identify harmonics. Specifically, the scalar harmonics $\mathcal{Y}^{I_1}$ will be completely traceless representations $[k,0,0]$ of $SO(6)$, akin to the representations $[k,0,0,0]$ of $SO(8)$ for S^7. Decomposing the Einstein equations (4.1) and the self-duality equations (4.2) will yield the 5-dimensional equations of motion. Linearizing the latter equations will result in mass operators that need to be diagonalized, similar to the process described in (3.B.2). This methodology was initially undertaken in [107], where the spectrum of IIB Supergravity on $AdS_5 \times S^5$ was derived. We provide the ExFT answer for the spectrum in section 3.C.4 . We now extend this techniques to compute the cubic couplings by going one order higher in the fluctuations. While this procedure can be applied to scalars, vectors, and gravitons, our focus here will be solely on scalars. This will be enough to sketch the main ideas. The field equations of interest for us that we obtain from (4.1) and (4.2), by injecting (4.9) and (4.10) and after linearization are

$$\begin{cases} \left(\frac{1}{2}h'^{I_1} - \frac{8}{15}h_2^{I_1}\right)\nabla_{(m}\nabla_{n)}\mathcal{Y}^{I_1} = 0 \\ \left(\nabla_\mu h'^{\mu\nu I_1} - \nabla^\nu\left(h'^{I_1} - \frac{8}{15}h_2^{I_1} + 8b^I\right) - \frac{8}{4!}\epsilon^{\nu\mu_1\mu_2\mu_3\mu_4}b^{I_1}_{\mu_1\mu_2\mu_3\mu_4}\right)\nabla_m\mathcal{Y}^{I_1} = 0 \\ \left(b^{I_1}_{\mu_1\mu_2\mu_3\mu_4} + \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5}b^{I_1}\right)\nabla_m\mathcal{Y}^{I_1} = 0\,. \end{cases} \tag{4.11}$$

We now do the following field redefenition

$$\begin{cases} s^{I_1} = \frac{1}{20(k+2)}(h_2^{I_1} - 10(k+4)b^{I_1}) \\ t^{I_1} = \frac{1}{20(k+2)}(h_2^{I_1} + 10kb^{I_1}). \end{cases} \tag{4.12}$$

With the definitions (4.12) and fields equations (4.11), s^{I_1} and t^{I_1}, which we will denote s^I and t^I from now on, obey at linear level

$$\begin{cases} \nabla_\mu\nabla^\mu s^I = k(k-4)s^I \\ \nabla_\mu\nabla^\mu t^I = (k+4)(k+8)t^I. \end{cases} \tag{4.13}$$

where $\nabla_m \nabla^m \mathcal{Y}^I = -k(k+4)\mathcal{Y}^I$ as been used. This allows to identify s^I as being the chiral primary of the gravity multiplet and t^I to be a descendent, see Table 2.5 , where s^I are the scalars in $[k+2,0,0](0,0)$ representations and t^I the scalars in the $[k-2,0,0](0,0)$ representations. It is now straighforward to find an action that leads to those equations, which reads

$$\begin{aligned} S(s) &= \frac{4N^2}{(2\pi)^5}\int \mathrm{d}^5\mathrm{x}\sqrt{g_{\text{ext}}}\sum_I \frac{32k(k-1)(k+2)}{k+1}\Big(-\frac{1}{2}\nabla_\mu s^I\nabla^\mu s^I - \frac{1}{2}k(k-4)(s^I)^2\Big)\,, \\ S(t) &= \frac{4N^2}{(2\pi)^5}\int \mathrm{d}^5\mathrm{x}\sqrt{g_{\text{ext}}}\sum_I \frac{32(k+2)(k+4)(k+5)}{k+3}\Big(-\frac{1}{2}\nabla_\mu t^I\nabla^\mu t^I - \frac{1}{2}(k+4)(k+8)(t^I)^2\Big)\,. \end{aligned} \tag{4.14}$$

The normalization of this action can be fixed by comparing it with the full action of IIB Supergravity. We will skip this as it is only a technical details and the reader can refer to [108, 109].

To get the 3-point functions of the chiral primary s^I we need to go one step further in the expansion. In particular, we must find the quadratic corrections of (4.11) which implies to compute R_{MN} and $F_{MPQRS}F_N{}^{PQRS}$ to next order in s. One can carry out this calculations, and after some calculations one finds [173]

$$\begin{aligned} R_{mn} =& \frac{1}{2}(Y + \frac{1}{10}Z_\ell{}^\ell)g_{mn} + \frac{1}{4}Z_{(mn)}, \\ Y \equiv& V_1V_2\nabla^\ell(s_1\nabla_\ell s_2) + U_1V_2\nabla^\mu(s_1\nabla_\mu s_2) + W_1V_2\nabla_\mu(\nabla^{(\mu}\nabla^{\nu)}s_1\nabla_\nu s_2), \\ Z_{mn} \equiv& (3V_1V_2 + 5U_1U_2)(\nabla_m s_1\nabla_n s_2 + 2s_1\nabla_m\nabla_n s_2), \\ &+ W_1W_2(\nabla_m\nabla^{(\mu}\nabla^{\nu)}s_1\nabla_n\nabla_{(\mu}\nabla_{\nu)}s_2 + 2\nabla^{(\mu}\nabla^{\nu)}s_1\nabla_m\nabla_n\nabla_{(\mu}\nabla_{\nu)}s_2) \\ \frac{4}{4!}F_{mPQRS}F_n{}^{PQRS} =& 4g_{mn}\{X_1X_2(\nabla^\ell\nabla_\ell s_1\nabla^k\nabla_k s_2 + \nabla^\mu\nabla^\ell s_1\nabla_\mu\nabla_\ell s_2) \\ &- 8V_1X_2 s_1\nabla^\ell\nabla_\ell s_2 + 10V_1V_2 s_1 s_2\} - 8X_1X_2\nabla_m\nabla_\mu s_1\nabla_\mu\nabla^\mu s_2. \end{aligned} \tag{4.15}$$

with

$$V_k = \frac{5}{3}U_k = 2k\,, \quad W_k = \frac{4}{k+1}\,, \quad X_k = 1\,, \tag{4.16}$$

and s_i is a shortand for $s^{I_i}Y^{i_1}$. Note that the i here is not the same anymore than the one in (4.9) for example. Here it is just a label for the s field, whereas in (4.9) it denoted a particular representation in which the harmonic tensor lies in. This leads for s^I to the following corrected equations of motion

$$(\nabla_\mu\nabla^\mu - m^2_{I_1})s^{I_1} = \sum_{I_2,I_3}\Big\{D_{I_1I_2I_3}s^{I_2}s^{I_3} + E_{I_1I_2I_3}\nabla_\mu s^{I_2}\nabla^\mu s^{I_3} + F_{I_1I_2I_3}\nabla^{(\mu}\nabla^{\nu)}s^{I_2}\nabla_{(\mu}\nabla_{\nu)}s^{I_3}\Big\} \tag{4.17}$$

with the functions D, E and F that can be found in [108] and are functions of U and V. At this stage, we observe an important characteristic: through this direct calculation, we find that the couplings involve derivatives of the fields. They can be eliminated through nonlinear field redefinition

$$s^{I_1} = s'^{I_1} + \sum_{I_2,I_3}\Big\{J_{I_1I_2I_3}s'^{I_2}s'^{I_3} + L_{I_1I_2I_3}\nabla^\mu s'^{I_2}\nabla_\mu s'^{I_3}\Big\} \tag{4.18}$$

with s'^I the "old" fields and s^I the new ones, J and L are given by

$$L_{I_1I_2I_3} = \frac{1}{2}F_{I_1I_2I_3}\,, \quad J_{I_1I_2I_3} = \frac{1}{2}E_{I_1I_2I_3} + \frac{1}{4}F_{I_1I_2I_3}(m_{I_1}^2 - m_{I_2}^2 - m_{I_3}^2 + 8). \tag{4.19}$$

With this, (4.17) now becomes

$$(\nabla_\mu\nabla^\mu - k(k-4))s^{I_1} = \sum_{I_2,I_3}\lambda_{I_1I_2I_3}s^{I_2}s^{I_3}, \tag{4.20}$$

$$\begin{aligned}\mathcal{G}_{I_1I_2I_3} &= \text{Normalisation} \times \lambda_{I_1I_2I_3} \\ &= a(k_1,k_2,k_3)\frac{128\tilde{\sigma}\{(\frac{1}{2}\tilde{\sigma})^2-1\}\{(\frac{1}{2}\tilde{\sigma})^2-4\}\alpha_1\alpha_2\alpha_3}{(k_1+1)(k_2+1)(k_3+1)}\mathcal{C}^{I_1I_2I_3}\end{aligned} \tag{4.21}$$

where $\tilde{\sigma} = k_1+k_2+k_3$, $\alpha_i = \frac{1}{2}\tilde{\sigma} - k_i$; $\mathcal{C}^{I_1I_2I_3}$ and $a(k_1,k_2,k_3)$ are the unique $SO(6)$ invariant that can be formed with those three representations and a normalisation, defined in (4.83). We introduced $\mathcal{G}_{I_1I_2I_3}$ because this is the quantity appearing in the Lagrangian. The normalisation in (4.21) is the normalisation coming from the kinetic term of (4.14), so we can write

$$\begin{aligned}S(s) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_{\text{ext}}}\Bigg[&\sum_I \frac{32k(k-1)(k+2)}{k+1}\Big(-\frac{1}{2}(\nabla_\mu s^I)^2 - \frac{1}{2}k(k-4)(s^I)^2\Big) \\ &+ \sum_{I_1,I_2,I_3}\frac{1}{3}\mathcal{G}_{I_1I_2I_3}s^{I_1}s^{I_2}s^{I_3}\Bigg]\end{aligned} \tag{4.22}$$

This provides the final answer for cubic couplings of s's.

Let us note a number of important elements :

- This method has been successfully used to compute a number a cubic couplings in IIB Supergravity on $AdS_5 \times S^5$. On top of cubic couplings for s^I, cubic couplings involving other scalars, descendant of chiral primaries, but also vectors and graviton have been computed.
- Those results have been of tremendous importance in the AdS/CFT correspondance. Bootstrapping methods have been developped [174] and one-loop corrections have been calculated [175–177] as well as two-loop [178]. Furthermore, gluon scattering in AdS backgrounds has become feasible. A comprehensive examination of holographic correlators across a wide range of SCFTs with non-maximal superconformal symmetry has been presented in [179].
- In a great effort, similar techniques have been used to carry out 4-point functions of scalars in IIB $AdS_5 \times S^5$ [168].
- This procedure has been conducted for other theories, such as $d = 6$, $\mathcal{N} = (2,0)$ Supergravity on $AdS_3 \times S^3$ [180] but also for $11d$ Supergravity on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ [181–183].

Despite the immense success of these techniques, it is important to note some limitations. Firstly, the procedures involved are laborious. One must utilize various types of tensorial harmonics, numerous field redefinitions are performed in order to diagonalize the mass operators ...The final field redefinition performed is nonlinear and aims at eliminating derivatives acting

on scalar fields. These field redefinitions suggest that from the outset, the setup for studying the couplings may not be appropriate, necessitating several changes of basis to ultimately arrive at the correct set of fields. Additionally, aside from the $a(k_1, k_2, k_3)$ numerical factor and the $\mathcal{C}^{I_1 I_2 I_3}$ invariant tensor in (4.21), the factors appear random, and no organizing principle seems to emerge.

In the remainder of this chapter, we aim to demonstrate how ExFT techniques can address these questions. They provide a framework in which it appears more natural to study couplings without the need for field redefinitions and offer robust and enriching structures that enable the identification of important organizing principles.

C Exceptional Field Theory n-point couplings

C.1 Fluctuation Ansatz

CUBIC and higher-order, couplings of the lower dimensional Supergravity can be accessed using similar techniques as the ones used for Kaluza-Klein spectroscopy, reviewed above. To obtain these n-point couplings, we need to extend the fluctuation ansatz of (3.55) by expanding it to higher order. The key change comes from the scalar fluctuation Ansatz, which around any vacuum of the consistent truncation, can be written as an exponential

$$\begin{aligned} &\mathscr{M}_{MN}(x,y) = \mathcal{V}_M{}^{\underline{A}}(x,y)\mathcal{V}_N{}^{\underline{B}}(x,y)\,\delta_{\underline{AB}}\,, \\ \text{with}\quad &\mathcal{V}_M{}^{\underline{A}}(x,y) = U_M{}^{\underline{B}}(y)\left(\exp\left(\tfrac{1}{2}\,\mathbb{T}_\alpha \sum_\Sigma \mathcal{Y}_\Sigma(y)\phi^{\alpha\,\Sigma}(x)\right)\right)_{\underline{B}}{}^{\underline{A}}\,, \end{aligned} \tag{4.23}$$

since the generalised metric can be written in terms of the coset representative of $\frac{\mathrm{E}_{6(6)}}{\mathrm{USp}(8)}$ as discussed above (4.23). For example, in order to identify the cubic couplings from the equations of motion, we need its expansion to quadratic order in the fluctuations, given by

$$\begin{aligned} \mathscr{M}_{MN}(x,y) = U_M{}^{\underline{A}}(y)\,U_N{}^{\underline{B}}(y)\,\Big(&\delta_{\underline{AB}} + \mathbb{T}_{\alpha\underline{A}}{}^{\underline{B}}\sum_\Sigma \mathcal{Y}_\Sigma(y)\,\phi^{\alpha\,\Sigma}(x) \\ &+ \frac{1}{2}\mathbb{T}_{\alpha\beta\underline{A}}{}^{\underline{B}}\sum_{\Sigma,\Omega}\mathcal{Y}_\Sigma(y)\,\mathcal{Y}_\Omega(y)\,\phi^{\alpha\,\Sigma}(x)\,\phi^{\beta\,\Omega}(x) + \ldots\Big)\,, \end{aligned} \tag{4.24}$$

where we used the notation for products of generators

$$\mathbb{T}_{\alpha\beta\underline{A}}{}^{\underline{B}} = \mathbb{T}_{\alpha\underline{A}}{}^{\underline{C}}\mathbb{T}_{\beta\underline{C}}{}^{\underline{B}}\,, \tag{4.25}$$

that we will similarly use for higher products in the following. We recall from (3.55) that we have chosen a gauge such that the sum over generators in the exponential (4.23) and the expansion (4.24) only runs over the non-compact generators $\mathbb{T}_\alpha$, $\alpha = 1, \ldots, 42$, of $\mathrm{E}_{6(6)}$.

Plugging the Ansatz (3.55), (4.23) into the equations of motion or the Lagrangian then gives access to the n-point couplings.

C.2 Structure of n-point couplings

In section 3.C.3.a, we observed that the only action on harmoncis in the mass matrices comes from the $\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Lambda}$ matrices. Since these do not mix different KK levels, we immediately concluded that

the mass eigenstates do not mix different KK levels (defined with respect to the round S^5) either, even though the remnant symmetry group of the vacuum might allow mixing.

We can apply the same reasoning to n-point couplings using any ExFT, not just that based on $E_{6(6)}$, to find a powerful structure which implies that many couplings vanish, even though they might be allowed by the symmetry group of the vacuum being studied. This is because also in the n-point couplings, the fluctuation Ansatz (3.55), (4.23) implies that the only derivatives of the scalar harmonics will come from the terms

$$\mathcal{L}_{\mathscr{U}_{\underline{A}}}\mathcal{Y}_{\Sigma} = -\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Omega}\,\mathcal{Y}_{\Omega}\,, \tag{4.26}$$

and thus will simply give rise to the dressed $\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Omega}$ matrices. All other derivatives will necessarily act on a twist matrix and give rise to the (dressed) embedding tensor

$$X_{\underline{AB}}{}^{\underline{C}} = (\mathscr{V}^{-1})_{\underline{A}}{}^{\underline{M}}\,(\mathscr{V}^{-1})_{\underline{B}}{}^{\underline{N}}\,\mathscr{V}_{\underline{P}}{}^{\underline{C}}\,X_{\underline{MN}}{}^{\underline{P}}\,. \tag{4.27}$$

Therefore, we see that the only action on the scalar harmonics comes from the dressed $\mathscr{T}_{\underline{A}\,\Sigma}{}^{\Omega}$ matrices, which do not mix different KK levels, with the level notion defined with respect to the $SO(6)$ isometries of the round S^5.

We can, therefore, see that the n-point couplings will schematically appear in the action as

$$\mathcal{G}(\Phi^{\mathcal{A}_1\Sigma_1},\dots,\Phi^{\mathcal{A}_n\Sigma_n}) \sim \int d\omega\,\Phi^{\mathcal{A}_1\Sigma_1}\,\Phi^{\mathcal{A}_2\Sigma_2}\,\dots\Phi^{\mathcal{A}_n\Sigma_n}\left(\lambda_{\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n,\Sigma_1\Sigma_2\dots\Sigma_n}{}^{\Delta_1\Delta_2\dots\Delta_n}\right)\mathcal{Y}_{\Delta_1}\mathcal{Y}_{\Delta_2}\dots\mathcal{Y}_{\Delta_n}\,, \tag{4.28}$$

where $\Phi^{\mathcal{A}_i\Sigma_i}$, $i=1,\dots,n$ denotes some generic KK excitation in a representation $\mathcal{A}_i$ of $USp(8)$ and some representation of the 5-dimensional Lorentz group that we supressed and $d\omega$ represents the volume form of S^5, while the n-point coupling is encoded in $\lambda_{\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n\,\Sigma_1\Sigma_2\dots\Sigma_n}{}^{\Delta_1\Delta_2\dots\Delta_n}$. In (4.28), the only internal dependence is now carried by the KK harmonics $\mathcal{Y}_{\Delta_1}\mathcal{Y}_{\Delta_2}\dots\mathcal{Y}_{\Delta_n}$, whose integral gives the n-point invariant

$$c_{\Delta_1\Delta_2\dots\Delta_n} = \int d\omega\,\mathcal{Y}_{\Delta_1}\mathcal{Y}_{\Delta_2}\dots\mathcal{Y}_{\Delta_n}\,. \tag{4.29}$$

Since we can use the same scalar harmonics for any vacuum of the consistent truncation, $c_{\Sigma_1\dots\Sigma_n}$ is the same object for any vacuum. For example, for all vacua within the $SO(6)$ gauged Supergravity obtained by a consistent truncation on S^5, the cubic $c_{\Sigma\Delta\Gamma}$ is the unique cubic $SO(6)$-invariant, up to normalisation.

Thus, the n-point couplings take the form

$$\mathcal{G}(\Phi^{\mathcal{A}_1\Sigma_1},\dots,\Phi^{\mathcal{A}_n\Sigma_n}) \sim \Phi^{\mathcal{A}_1\Sigma_1}\,\Phi^{\mathcal{A}_2\Sigma_2}\,\dots\Phi^{\mathcal{A}_n\Sigma_n}c_{\Delta_1\Delta_2\dots\Delta_n}\left(\lambda_{\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n,\Sigma_1\Sigma_2\dots\Sigma_n}{}^{\Delta_1\Delta_2\dots\Delta_n}\right)\,. \tag{4.30}$$

Crucially, because $\lambda_{\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n,\Sigma_1\Sigma_2\dots\Sigma_n}{}^{\Delta_1\Delta_2\dots\Delta_n}$ does not mix different KK levels, i.e. the Σ_i are the same KK levels as Δ_i, $i=1,\dots,n$, this implies that $c_{\Delta_1\Delta_2\dots\Delta_n}$ will vanish when $c_{\Sigma_1\Sigma_2\dots\Sigma_n}$ vanishes. We immediately conclude that if the scalar harmonics of the n fields $\Phi^{\mathcal{A}_i\Sigma_i}$, $i=1,\dots n$, do not yield a non-vanishing n-point invariant $c_{\Sigma_1\Sigma_2\dots\Sigma_n}$, then their n-point couplings will vanish. Importantly, this vanishing of n-point couplings is controlled by scalar harmonics of the round sphere but holds for all vacua of the consistent truncation. This, therefore, implies that infinitely many n-point couplings vanish for vacua whose remnant symmetry group would have allowed such couplings. In fact, as we will show in section 4.E.1, even for the round spheres, e.g. the maximally supersymmetric $AdS_5\times S^5$, $AdS_4\times S^7$ and $AdS_7\times S^4$ vacua, our results imply that more couplings vanish

than would be expected from the isometries of the round sphere. Instead, our results show that extremal and non-extremal n-point couplings vanish, which was conjectured in [170, 171].

We see that having a consistent truncation not only implies that all the n-point couplings vanish between any $n-1$ modes of the truncation and a mode that is not part of the truncation. In addition, the consistent truncation leaves a remnant for all higher KK levels, whose n-point couplings vanish if the KK levels, inherited from the round sphere, do not give a non-vanishing n-fold invariant. This result hold for all consistent truncations to maximal gauged Supergravity, not just those in five dimensions.

D Cubic couplings from $E_{6(6)}$ ExFT

D.1 Scalar cubic couplings

We are now ready to explore the cubic couplings in detail using $E_{6(6)}$ ExFT and focusing on vacua that can be uplifted from 5-dimensional maximal gauged Supergravity. The cubic couplings of three scalar fields are obtained from the potential, expanding the scalar matrix $\mathscr{M}$ (4.23) in the potential. As we discussed in section 4.C.2, the couplings will be quadratic in the dressed embedding tensor $X_{\underline{AB}}{}^{\underline{C}}$ and the dressed $\mathscr{T}_{\underline{A}\Sigma}{}^{\Omega}$ matrix. Thus, schematically, the cubic result is given by

$$\mathcal{G}(\phi^{\alpha\Sigma},\phi^{\beta\Delta},\phi^{\gamma\Gamma}) \propto \phi^{\alpha\Sigma}\phi^{\beta\Delta}\phi^{\gamma\Gamma}\left(XX_{\alpha\beta\gamma\Sigma\Delta\Gamma}+X\mathscr{T}_{\alpha\beta\gamma\Sigma\Delta\Gamma}+\mathscr{T}\mathscr{T}_{\alpha\beta\gamma\Sigma\Delta\Gamma}\right), \tag{4.31}$$

where XX, $X\mathscr{T}$ and $\mathscr{T}\mathscr{T}$ refer to terms that are quadratic in X, linear in X and $\mathscr{T}$ and quadratic in $\mathscr{T}$, respectively. The XX part of the result is simply the coupling between the scalars from level 0 of the KK towers, and therefore can be obtained using 5-dimensional gauged Supergravity without any ExFT analysis. However, for the couplings involving higher KK levels, we need the $X\mathscr{T}$ and $\mathscr{T}\mathscr{T}$ terms and thus the ExFT couplings.

While the naive cubic couplings come from plugging the fluctuation Ansatz (4.23) into the ExFT potential, our fluctuation Ansatz still includes unphysical Goldstone modes. Since we are not interested in these unphysical couplings, we can change the couplings involving Goldstone fields in an effort to simplify the cubic coupling formulae. The Goldstones modes are encoded in the Π matrix that was introduced earlier in (3.66)

$$\mathcal{D}_\mu\phi = \partial_\mu\phi + \Pi A_\mu + \text{quadratic}\ldots, \tag{4.32}$$

and describes how the Goldstone scalars couple to massive vector fields. Adding corrections projected with the Π matrix to the cubic couplings will not change the physical couplings, so that we can harmlessly simplify the expressions using such terms.

We choose to add terms involving the Π matrices in such a way as to simplify the $\mathscr{T}\mathscr{T}$ terms as much as possible, analogously to what we did when simplifying the mass matrices in 3.C.3 . The resulting $\mathscr{T}\mathscr{T}$ term in (4.31) is then given by

$$\mathscr{T}\mathscr{T}_{\alpha\beta\gamma\Sigma\Delta\Gamma} = \mathscr{T}_{\underline{B}\Sigma}{}^{\Lambda}\mathscr{T}_{\underline{A}\Lambda}{}^{\Omega}c_{\Delta\Omega\Gamma}\left(6\mathbb{T}_{\gamma\alpha\beta\underline{B}}{}^{\underline{A}}-\frac{1}{2}\mathbb{T}_{\gamma\underline{B}}{}^{\underline{A}}\kappa_{\alpha\beta}\right), \tag{4.33}$$

and the $X\mathscr{T}$ part given by

$$X\mathscr{T}_{\alpha\beta\gamma\Sigma\Delta\Gamma} = \frac{1}{6}\mathcal{T}_{\underline{B}\Sigma}{}^{\Lambda} c_{\Gamma\Delta\Lambda}\Big[X_{\underline{AC}}{}^{\underline{D}}\left(-6\,\mathbb{T}_{[\alpha\gamma]\underline{D}}{}^{\underline{C}}\mathbb{T}_{\beta\underline{B}}{}^{\underline{A}} - 6\,\mathbb{T}_{\alpha\underline{D}}{}^{\underline{C}}\mathbb{T}_{\beta\gamma\underline{B}}{}^{\underline{A}}\right) - X_{\underline{BA}}{}^{\underline{C}}\left(6\,\mathbb{T}_{\beta\alpha\gamma\underline{C}}{}^{\underline{A}} + \frac{1}{8}\kappa_{\gamma\beta}\mathbb{T}_{\alpha\underline{C}}{}^{\underline{A}}\right)\Big]\,. \tag{4.34}$$

Note that because (4.33) and (4.34) are contracted with three scalars in (4.31), the r.h.s. of (4.33) and (4.34) should be viewed as symmetrised in the exchange of the pairs $\alpha\Sigma$, $\beta\Delta$, and $\gamma\Gamma$.

The XX part of the couplings in (4.31), which of the three parts is the longest and corresponds to the 5-dimensional gauged Supergravity result, reads

$$\begin{aligned} XX_{\alpha\beta\gamma\Sigma\Delta\Gamma} = c_{\Sigma\Delta\Gamma}\Big[&X_{\underline{AB}}{}^{\underline{C}}\,X_{\underline{DC}}{}^{\underline{B}}\left(\frac{1}{6}\,\mathbb{T}_{\beta\gamma\alpha\underline{A}}{}^{\underline{D}} + \frac{1}{6}\kappa_{\gamma\alpha}\,\mathbb{T}_{\beta\underline{A}}{}^{\underline{D}}\right) \\ &+ X_{\underline{AB}}{}^{\underline{C}}\,X_{\underline{DE}}{}^{\underline{F}}\left(\frac{3}{8}\delta^{\underline{BE}}\,\mathbb{T}_{\alpha\underline{A}}{}^{\underline{D}}\,\mathbb{T}_{\beta\gamma\underline{F}}{}^{\underline{C}} + \frac{37}{72}\delta^{\underline{AD}}\,\mathbb{T}_{\alpha\underline{B}}{}^{\underline{E}}\,\mathbb{T}_{\beta\gamma\underline{F}}{}^{\underline{C}}\right. \\ &\left. + \frac{19}{24}\,\delta_{\underline{CF}}\,\mathbb{T}_{\beta\underline{B}}{}^{\underline{E}}\,\mathbb{T}_{\alpha\gamma\underline{A}}{}^{\underline{D}} + \frac{97}{72}\,\delta^{\underline{AD}}\,\mathbb{T}_{\alpha\underline{F}}{}^{\underline{C}}\,\mathbb{T}_{\beta\gamma\underline{E}}{}^{\underline{B}} - \frac{23}{18}\,\delta^{\underline{AD}}\,\delta^{\underline{BE}}\,\mathbb{T}_{\beta\gamma\alpha\underline{C}}{}^{\underline{F}}\right)\Big]\,. \end{aligned} \tag{4.35}$$

Once again, because this term is multiplied by three scalars in (4.31), the r.h.s. of (4.35) should be symmetrised in the exchange of $\alpha\Sigma$, $\beta\Delta$, and $\gamma\Gamma$.

Further simplification for embedding tensors in the 36 of USp(8)

We can significantly simplify the previous results by introducing additional assumptions. Recall that the embedding tensor of 5-dimensional maximal gauged Supergravity transforms in the **351** representation of $E_{6(6)}$. When breaking $E_{6(6)}$ to its maximal compact subgroup USp(8), the **351** decomposes into **315** $\oplus$ **36**. We are interested in exploring the implications of the embedding tensor being solely in the **36**. This arises, for instance, for the maximally supersymmetric $AdS_5 \times S^5$ vacuum.

Let us start by examining the $X\mathscr{T}$ terms in (4.34). As we emphasised above, the multiplication of (4.34) by three scalars in (4.31) implies that the r.h.s. of (4.34) should be symmetrised when exchanging the index pairs $\alpha\Sigma$, $\beta\Delta$, and $\gamma\Gamma$. However, the contraction of $\mathscr{T}$ with a c-symbol in (4.34) imposes a hook symmetry on the harmonic indices, since the c-symbol is invariant under the action of $\mathscr{T}$. As a result, the adjoint indices α, β, γ must also appear in the hook symmetric fashion. Therefore, stripping the $\mathscr{T}$ term from (4.34), the remaining tensor $\Xi_{\underline{B}\alpha\beta\gamma}$ must exhibit a hook symmetry on the adjoint indices and, since it must be some USp(8)-invariant combination of the dressed embedding tensor X, it must reside in the **36**. Given these requirements, we find that only one term remains in all of 4.34. By conducting a similar analysis on the XX-like terms in (4.35), we arrive at a single remaining term. After calculating the relative coefficients, we obtain the general formula for the cubic couplings of scalars for vacua where the embedding tensor is in

the **36**

$$\begin{aligned}\mathcal{G}(\phi^{\alpha\Sigma},\phi^{\beta\Delta},\phi^{\gamma\Gamma}) \propto \phi^{\alpha\Sigma}\phi^{\beta\Delta}\phi^{\gamma\Gamma}\Big[&-\frac{1}{6}c_{\Sigma\Delta\Gamma}X_{\underline{AB}}{}^{\underline{C}}X_{\underline{DC}}{}^{\underline{B}}\mathbb{T}_{\beta\gamma\alpha\underline{A}}{}^{\underline{D}} \\ &-\mathcal{T}_{\underline{B}\Sigma}{}^{\Lambda}c_{\Lambda\Delta\Gamma}X_{\underline{AC}}{}^{\underline{D}}\mathbb{T}_{[\alpha\gamma]\underline{D}}{}^{\underline{C}}\mathbb{T}_{\beta\underline{B}}{}^{\underline{A}} \\ &+\mathcal{T}_{\underline{B}\Sigma}{}^{\Lambda}\mathcal{T}_{\underline{A}\Lambda}{}^{\Omega}c_{\Omega\Delta\Gamma}\left(6\,\mathbb{T}_{\gamma\alpha\beta\underline{B}}{}^{\underline{A}}-\frac{1}{2}\,\mathbb{T}_{\gamma\underline{B}}{}^{\underline{A}}\kappa_{\alpha\beta}\right)\Big]\,.\end{aligned} \tag{4.36}$$

D.2 Couplings involving spin-1

In this section, we will give the cubic couplings involving at least one vector field.

We begin with the cubic couplings between one vector and two scalar fields. These come from the scalar kinetic term in the Lagrangian, which is given by

$$\mathcal{L}_{\text{kin, scalar}} = \frac{1}{24}\sqrt{-g}g^{\mu\nu}\mathcal{D}_\mu\mathscr{M}_{MN}\mathcal{D}_\nu\mathscr{M}^{MN}\,, \tag{4.37}$$

where $D_\mu = \partial_\mu - \mathcal{L}_{A_\mu}$ is the 5-dimensional derivative covariantised with respect to the ExFT generalised diffeomorphisms. Since we are interested in vector-scalar-scalar couplings, we will take $g^{\mu\nu} = \mathring{g}^{\mu\nu}$ to be the background metric of AdS_5 spacetime. By plugging in the fluctuation Ansatz (3.55), (4.23) into (4.37), we find

$$\begin{aligned}\mathcal{G}(A,\phi,\partial\phi) \propto A_\mu{}^{\underline{A}\Sigma}\Big[&2\,c_{\Omega\Lambda\Sigma}X_{\underline{AB}}{}^{\underline{C}}\,\partial^\mu\phi^{\alpha\Omega}\,\phi^{\beta\Lambda}\,\mathbb{T}^{\hat{\gamma}}{}_{\alpha\beta}\,\mathbb{T}_{\hat{\gamma}\underline{C}}{}^{\underline{B}}+2\,c_{\Delta\Lambda\Sigma}\,\phi^{\beta\Omega}\,\partial^\mu\phi^{\alpha\Delta}\,\mathcal{T}_{\underline{A}\,\Omega}{}^{\Lambda}\,\kappa_{\alpha\beta} \\ &-12\,c_{\Lambda\Omega\Delta}\,\partial^\mu\phi^{\alpha\Lambda}\,\phi^{\beta\Delta}\,\mathcal{T}_{\underline{B}\Sigma}{}^{\Omega}\,\mathbb{T}^{\hat{\gamma}}{}_{\beta\alpha}\,\mathbb{T}_{\hat{\gamma}\underline{A}}{}^{\underline{B}}\Big]\,,\end{aligned} \tag{4.38}$$

where $\mathbb{T}$ are the generators of E_6, with $\alpha,\beta,\ldots$ labelling the non-compact and $\hat{\alpha},\hat{\beta},\hat{\gamma}$ labelling the compact indices of $E_{6(6)}$, i.e. respectively the **42** and **36** of $USp(8)$ and we suppressed the indices on A, ϕ and $\partial\phi$ on the l.h.s. for simplicity.

We now turn to couplings involving a vector field as well as 2-forms. They can be obtained by inspecting the field equations for the field strength $\mathcal{F}$, which is defined as

$$\mathcal{F}_{\mu\nu}{}^{M} = 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^{M} - 2\mathcal{A}_{[\mu}{}^{K}\partial_K\mathcal{A}_{\nu]}{}^{M} + 10\,d_{PLK}\,d^{MNK}\mathcal{A}_{[\mu}{}^{P}\partial_N\mathcal{A}_{\nu]}{}^{L} + 10\,d^{MNK}\partial_N\mathcal{B}_{\mu\nu K}\,. \tag{4.39}$$

The corresponding field equations are

$$\mathcal{D}\star\mathcal{F}_M \propto d_{MNK}\,\mathcal{F}^N\wedge\mathcal{F}^K\,, \tag{4.40}$$

given in differential form notation. Here $\star$ is the 5-dimensional Hodge dual, $\wedge$ is the 5-dimensional wedge operator and $\mathcal{D}$ is the ExFT covariantised 5-dimensional derivative operator, which in (4.40) acts as an exterior derivative on $\mathcal{F}_M$ on the l.h.s.

Plugging in the fluctuation Ansatz (3.55) in the action, we find

$$\begin{aligned}\mathcal{G}(A,B,B) \propto A_\mu{}^{\underline{A\Sigma}}\, B_{\nu\rho\underline{B\Delta}}\, B_{\sigma\tau\underline{C\Gamma}}\, \epsilon^{\mu\nu\rho\sigma\tau} \Big(& d_{\underline{ADE}}\, d^{\underline{CFE}}\, d^{\underline{DBQ}}\, \mathscr{T}_{\underline{F}}{}^{\Gamma}{}_{\Lambda}\, \mathscr{T}_{\underline{Q}}{}^{\Delta}{}_{\Omega}\, c_\Sigma{}^{\Omega\Lambda} \\ & -\frac{1}{10} d_{\underline{ADE}}\, d^{\underline{CQE}}\, d^{\underline{GHD}}\, \mathscr{T}_{\underline{Q}}{}^{\Gamma}{}_{\Lambda} X_{\underline{GH}}{}^{\underline{B}}\, c_\Sigma{}^{\Delta\Lambda} \\ & -\frac{1}{10} d_{\underline{AED}}\, d^{\underline{BFD}}\, d^{\underline{EGH}}\, \mathscr{T}_{\underline{F}}{}^{\Delta}{}_{\Lambda} X_{\underline{GH}}{}^{\underline{C}}\, c_\Sigma{}^{\Lambda\Gamma} \\ & +\frac{1}{100} d_{\underline{ADE}}\, d^{\underline{FQD}}\, d^{\underline{GHE}}\, X_{\underline{FQ}}{}^{\underline{B}} X_{\underline{GH}}{}^{\underline{C}}\, c_\Sigma{}^{\Gamma\Delta} \Big)\,. \end{aligned} \tag{4.41}$$

Finally we work out the couplings of three vectors. There are couplings of type $AA\partial A$, which come from the kinetic term

$$\mathcal{L}_{\text{kin, vec}} = -\frac{1}{4}\sqrt{-g}\mathscr{M}_{MN}\mathcal{F}^{\mu\nu\,M}\mathcal{F}_{\mu\nu}{}^{N}\,, \tag{4.42}$$

and are given by

$$\mathcal{G}(A,A,\partial A) \propto A^{\mu\underline{A\Sigma}} A^{\nu\underline{B\Omega}} \partial_{[\mu} A_{\nu]}{}^{\underline{C\Lambda}} \Big(2\, c_{\Sigma\Lambda\Delta}\, \mathscr{T}_{\underline{A}}{}_{\Omega}{}^{\Delta} \delta_{\underline{BC}} + 10\, c_{\Sigma\Delta\Lambda} d_{\underline{ABD}} d^{\underline{FCD}}\, \mathscr{T}_{\underline{F}}{}_{\Omega}{}^{\Delta} + c_{\Sigma\Omega\Lambda} X_{[\underline{AB}]}{}^{\underline{C}} \Big)\,. \tag{4.43}$$

On the other hand, there are $A\partial A\partial A$ couplings that come from the topological term. Because both derivatives act along the external 5-dimensional space, these couplings have the very simple structure

$$\mathcal{G}(A,\partial A,\partial A) \propto \epsilon^{\mu\nu\rho\sigma\lambda} A_\mu{}^{\underline{A\Sigma}}\, \partial_\nu A_\rho{}^{\underline{B\Lambda}}\, \partial_\sigma A_\lambda{}^{\underline{C\Delta}}\, d_{\underline{ABC}}\, c_{\Sigma\Lambda\Delta}\,. \tag{4.44}$$

D.3 Couplings between spin-2 and scalars

Here we give the couplings between the fluctuations of the metric, which we will denote by h, and two scalars. From the structure of the indices, there is only one term that one can write for these couplings

$$\mathcal{G}(h,\partial\phi,\partial\phi) \propto \frac{1}{6}\kappa_{\alpha\beta}\, \partial_\mu\phi^{\alpha\Sigma}\, \partial_\nu\phi^{\beta\Delta}\, h^{\mu\nu\Lambda}\, c_{\Sigma\Delta\Lambda}\,. \tag{4.45}$$

It can be checked that this is what is indeed obtained by expanding the scalar kinetic term from the Lagrangian (4.37).

E Example : AdS$_5\times$S^5

In this section, we use the previously introduced formalism in order to compute couplings on the background AdS$_5\times$S^5. This background preserves maximal supersymmetry, i.e. states fall into supermultiplets of SU$(2,2|4)$ and transform in representations $[n,p,q]$ of the SO(6) R-symmetry group. Table 2.5 recapitulates the structure of the $\frac{1}{2}$-BPS supermultiplets $\mathcal{B}_{[\ell,0,0]}$ into which the Supergravity spectrum decomposes.

As has been discussed above, our fluctuation ansatz (3.55) introduces a different way of labelling the Kaluza-Klein states by a couple of SO(6) indices, $\Phi^{\mathcal{A}\Sigma}$, of which the first index refers to the field content of the $\mathcal{N}=8$ Supergravity multiplet while the second index runs over the scalar harmonics on S^5. The latter are defined as polynomials in the fundamental harmonics $\mathcal{Y}^a$,

$a = 1, \dots, 6$, (satisfying $\mathcal{Y}^a \mathcal{Y}^a = 1$) as

$$\mathcal{Y}^I = \mathcal{Y}^{a_1 \dots a_\ell} = \mathcal{Y}^{((a_1} \dots \mathcal{Y}^{a_\ell))} \equiv \mathcal{Y}^{(a_1} \dots \mathcal{Y}^{a_\ell)} - \text{traces}\,, \tag{4.46}$$

and transform in the symmetric vector representations $[\ell, 0, 0]$.

All fields $\Phi^{\mathcal{A}\Sigma}$ with the second index in a given SO(6) representation $[\ell, 0, 0]$ combine into the $\frac{1}{2}$-BPS supermultiplet $\mathcal{B}_{[\ell,0,0]}$. For example, for the scalar fluctuations, the index α on $\phi^{\alpha\Sigma}$ counts the 42 scalars of $\mathcal{N} = 8$ Supergravity. Under SO(6) these decompose according to

$$\mathbf{42} \to \mathbf{1}_{+2} \oplus \mathbf{1}_{-2} \oplus \mathbf{10}_{+1} \oplus \overline{\mathbf{10}}_{-1} \oplus \mathbf{20}_{0}\,, \tag{4.47}$$

where subscripts refer to the different SO(2) charges. Accordingly, the fluctuations $\phi^{\alpha\Sigma}$ carry representations

$$\begin{aligned}
\mathbf{20} \otimes [\ell, 0, 0] &= [\ell{+}2, 0, 0] \oplus [\ell, 0, 0] \oplus [\ell{-}2, 0, 0] \oplus [\ell, 1, 1] \oplus [\ell{-}2, 1, 1] \oplus [\ell{-}2, 2, 2]\,, \\
(\mathbf{10} \oplus \overline{\mathbf{10}}) \otimes [\ell, 0, 0] &= [\ell, 0, 2] \oplus [\ell, 2, 0] \oplus 2 \cdot [\ell{-}1, 1, 1] \oplus [\ell{-}2, 2, 0] \oplus [\ell{-}2, 0, 2]\,, \\
(\mathbf{1} \oplus \mathbf{1}) \otimes [\ell, 0, 0] &= 2 \cdot [\ell, 0, 0]\,,
\end{aligned} \tag{4.48}$$

where SO(2) charges are easily restored. Comparing to Table 2.5, not all of these representations correspond to physical scalar fields in the supermultiplet $\mathcal{B}_{[\ell,0,0]}$, rather the representations given in blue and green in (4.48) appear as Goldstone modes for the massive spin-1 and spin-2 fields, respectively.

E.1 Near extremal scalar n-point couplings

Let us consider the n-point couplings of scalar fields transforming in fully symmetric traceless vector representations of type $[k, 0, 0]$. Explicitly, we label such representations by an index I

$$[k, 0, 0]: \quad \mathcal{R}^I = \mathcal{R}^{((i_1 \dots i_k))}\,, \quad \text{and define } |I| \equiv k\,, \tag{4.49}$$

where $i_1, \dots i_k$ label the fundamental vector representation of SO(6), and $((\dots))$ denotes traceless symmetrisation. We denote the corresponding fields in (4.48) as[1]

$$s^I: [\ell{+}2, 0, 0]\,, \qquad t^I: [\ell{-}2, 0, 0]\,, \qquad \phi^I_\pm: 2 \cdot [\ell, 0, 0]\,. \tag{4.50}$$

In particular, the s^I are the chiral primaries of the supermultiplet, while the $\phi^I_\pm$ carry non-trivial SO(2) charge.

We have seen in section 4.C.2 that the parametrisation of scalar fluctuations as $\phi^{\alpha\Sigma}$ together with the structure of the ExFT action allows to deduce strong constraints on the existence of possible couplings. Let us first illustrate this for the chiral primaries s^I. Consider an n-point coupling between this fields, that we shall denote as

$$\mathcal{G}(s^{I_1}, s^{I_2}, \dots, s^{I_n})\,. \tag{4.51}$$

[1] For s^I and t^I this corresponds to the notation introduced in [108, 109]. In contrast, the ϕ^I of [109] correspond to the (green) spin-2 Goldstone modes in (4.48), whereas the $\phi^I_\pm$ in (4.50) denote physical scalars.

SO(6) representation theory immediately poses the condition

$$|I_i| \leq \sum_{j \neq i} |I_j| \quad \forall i \,, \tag{4.52}$$

necessary for a non-vanishing coupling (4.51), more precisely for the appearance of a singlet in the tensor product $\bigotimes_j \mathcal{R}^{I_j}$. However, due to the fact that the fields s^I appear as

$$s^I \in \phi^{ab,\Sigma} \,, \quad |I| = |\Sigma| + 2 \,, \tag{4.53}$$

in the ExFT action, the general structure of couplings (4.30) shows that a non-vanishing coupling (4.51) in fact requires the stronger constraint

$$|\Sigma_i| \leq \sum_{j \neq i} |\Sigma_j| \quad \forall i \qquad \Longleftrightarrow \qquad |I_i| + 2\,(n-2) \leq \sum_{j \neq i} |I_j| \quad \forall i \,. \tag{4.54}$$

Put differently, we can conclude that

$$\Big(\sum_{j \neq i} |I_j| \Big) - |I_i| \;\leq\; 2n - 5 \qquad \Longrightarrow \qquad \mathcal{G}(s^{I_1}, s^{I_2}, \ldots, s^{I_n}) = 0 \,, \tag{4.55}$$

thus the vanishing of extremal and near-extremal couplings. This precisely corresponds to the conjecture first stated in [170], based on the lowest order explicit results [108, 109, 168], and found necessary to match the factored structure of the near-extremal correlation functions in weakly-coupled $\mathcal{N} = 4$ SYM via AdS/CFT. On the SYM side the analogous statement for single particle operators was shown in [175]. The above reasoning gives a proof of this conjecture for arbitrary n-point couplings.

The conjecture (4.55) has been reviewed in [171] and been put into the context of consistent truncations. As discussed above, the truncation of the full Kaluza-Klein spectrum on $AdS_5 \times S^5$ to the $\mathcal{N} = 8$ Supergravity multiplet $\mathcal{B}_{[2,0,0]}$ is consistent. Another way of stating this property is that the Supergravity fields do not source the (truncated) higher Kaluza-Klein states, or equivalently, the absence of couplings linear in higher Kaluza-Klein fields, i.e.

$$I_1 > 2 \text{ and } |I_{i \geq 2}| = 2 \qquad \Longrightarrow \qquad \mathcal{G}(s^{I_1}, s^{I_2}, \ldots, s^{I_n}) = 0 \,, \tag{4.56}$$

given that s^I, $|I| = 2$, is the chiral primary of the Supergravity multiplet. With hindsight, this is nothing but a very particular case of the general structure (4.55). In turn, the ExFT structure of the action together with the above construction shows how the existence of a consistent truncation in fact implies the absence of numerous potential couplings, far beyond the lowest Supergravity multiplet.

In [171], the conjecture on vanishing near-extremal couplings has further been generalised to the compactifications of eleven-dimensional Supergravity on $AdS_4 \times S^7$ and $AdS_7 \times S^4$. In the ExFT framework, similar to the computations presented here, such couplings can be derived within $E_{7(7)}$ and SL(5) ExFT [84, 95], respectively. While the technical details differ, the structure of the chiral primaries is still of the form (4.53), now with respect to the R-symmetry SO(8) and SO(5), respectively. The same general pattern thus applies and the conjecture (4.55) is proven also in these cases for arbitrary n-point couplings.

In order to demonstrate the power of the framework, we may straightforwardly generalise the

argument to other scalar fields, proving the vanishing of several other near extremal couplings. Let us consider couplings of the form

$$\mathcal{G}(s^{I_1},\dots,s^{I_m},t^{J_1},\dots,t^{J_n})\,, \tag{4.57}$$

between scalars of type s^I and t^J, both appearing in fully symmetric traceless vector representations. As above, SO(6) group theory requires that

$$\begin{aligned} |I_i| &\leq \sum_{j\neq i}|I_j| + \sum_{\ell}|J_\ell| \quad \forall i\,,\\ |J_k| &\leq \sum_{j}|I_j| + \sum_{\ell\neq k}|J_\ell| \quad \forall k\,, \end{aligned} \tag{4.58}$$

in order to allow for a non-vanishing coupling (4.58). Again, from the general structure of couplings (4.30), together with the embedding of fields as

$$s^I \in \phi^{ab,\Sigma}\,, \quad |I| = |\Sigma| + 2\,, \qquad t^J \in \phi^{ab,\Sigma}\,, \quad |J| = |\Sigma| - 2\,, \tag{4.59}$$

into the fluctuation ansatz, we may by reasoning similar to the above obtain the far stronger statements

$$\begin{aligned} \Big(\sum_{j\neq i}|I_j| + \sum_{\ell}|J_\ell|\Big) - |I_i| \;\leq\; 2\,(m-n-3) \quad &\Longrightarrow \quad \mathcal{G}(s^{I_1},\dots,s^{I_m},t^{J_1},\dots,t^{J_n}) = 0\,,\\ \Big(\sum_{j}|I_j| + \sum_{\ell\neq k}|J_\ell|\Big) - |J_k| \;\leq\; 2\,(m-n+1) \quad &\Longrightarrow \quad \mathcal{G}(s^{I_1},\dots,s^{I_m},t^{J_1},\dots,t^{J_n}) = 0\,, \end{aligned} \tag{4.60}$$

which imply the vanishing of many near-extremal couplings whose presence would be compatible with SO(6) symmetry (4.58). It is straightforward to generalise this pattern to n-point couplings which further include the fields $\phi^I_\pm$ from (4.50). Similarly, one may derive conditions for vanishing couplings involving the fields in other SO(6) representations within the multiplet $\mathcal{B}_{[\ell,0,0]}$ of Table 2.5.

E.2 Harmonics

Let us do an intermediate section to introduce notations that we will need later in the thesis. We collect our conventions and some formulae for the harmonics. We will explain our conventions for S^5, but it can be generalised to any coset space G/H [114].

Starting from the fundamental sphere harmonics $\mathcal{Y}^a$, $a = 1,\dots,6$ with $\mathcal{Y}^a\mathcal{Y}^a = 1$, we define the higher scalar harmonics by

$$\mathcal{Y}^I \equiv \mathcal{Y}^{a_1\dots a_n} \equiv \mathcal{Y}^{((a_1}\dots\mathcal{Y}^{a_n))} = \mathcal{Y}^{(a_1}\dots\mathcal{Y}^{a_n)} - \text{traces}\,, \quad |I| \equiv n\,, \tag{4.61}$$

with total weight 1 on the r.h.s.. These harmonics transform in the symmetric vector representation $[n,0,0]$ of SO(6). We denote the integral over products of harmonics as

$$\int_{S^5} d\omega\, \mathcal{Y}^{i_1}\dots\mathcal{Y}^{i_n}\mathcal{Y}^{j_1}\dots\mathcal{Y}^{j_n} = A_n\, m_n\, \delta^{(i_1 i_2}\dots\delta^{j_{n-1}j_n)}\,, \tag{4.62}$$

where

$$A_n = \frac{\pi^3}{2^{n-1}(n+2)!}\,, \tag{4.63}$$

and

$$m_n = \frac{1}{n!}\binom{2n}{2}\cdots\binom{2}{2} = \frac{(2n)!}{2^n n!}\,, \tag{4.64}$$

counts the number of possibilities to distribute the indices over the δ's. For the integral over a product of harmonics $\mathcal{Y}^I\mathcal{Y}^J$ with $|I| = |J| = n$, we then find

$$\begin{aligned}\int_{S^5} d\omega\, \mathcal{Y}^I\mathcal{Y}^J &= A_n \left(\delta_{b_1}^{(a_1}\cdots\delta_{b_n}^{a_n)}\right) n! - \text{traces}\,,\\ &= \frac{\pi^3}{2^{n-1}(n+1)(n+2)}\,\delta_{b_1}^{((a_1}\cdots\delta_{b_n}^{a_n))} \equiv z(n)\,\delta^{IJ}\,,\end{aligned} \tag{4.65}$$

with

$$z(n) = \frac{\pi^3\, n!}{2^{n-1}(n+2)!}\,. \tag{4.66}$$

Similarly, the triple product of harmonics $\mathcal{Y}^{I_i}$ with $I_i = n_i$, $i = 1,2,3$, is found to be

$$\int_{S^5} d\omega\, \mathcal{Y}^{I_1}\mathcal{Y}^{I_2}\mathcal{Y}^{I_3} = a(n_1,n_2,n_3)\,\mathcal{C}^{I_1I_2I_3}\,, \tag{4.67}$$

with

$$a(n_1,n_2,n_3) = \frac{n_1!n_2!n_3!}{(\frac{1}{2}\sigma+2)!\,2^{\frac{1}{2}\sigma-1}}\,\frac{\pi^3}{\alpha_1!\alpha_2!\alpha_3!}\,, \tag{4.68}$$

where $\sigma = n_1+n_2+n_3$, $\alpha_i = \frac{1}{2}\sigma - n_i$, and $\mathcal{C}^{I_1I_2I_3}$ is the (up to normalisation) unique SO(6) invariant structure in the tensor product of the three representations, explicitly given by

$$\mathcal{C}^{a_1\ldots ia_{n_1},b_1\ldots b_{n_2},c_1\ldots c_{n_3}} = \delta^{a_1b_1}\ldots\delta^{a_{\alpha_3}b_{\alpha_3}}\delta^{a_{\alpha_3+1}c_1}\ldots\delta^{a_{n_1}c_{\alpha_2}}\delta^{b_{\alpha_3+1}c_{\alpha_2+1}}\ldots\delta^{b_{n_2}c_{n_3}}\,, \tag{4.69}$$

where the indices in the same colour on the r.h.s. are understood to be projected onto the totally symmetric traceless part, such that the total weight is one. SO(6) group theory implies that a non-vanishing $\mathcal{C}^{I_1I_2I_3}$ requires the triangle inequality

$$\alpha_i \geq 0\,, \tag{4.70}$$

which has been stated in equivalent form in (4.52). This is illustrated in Figure 4.1 .

We now also define the vector harmonics $\mathcal{Y}^I_m$. We never explicitely used them in our calculations, but this will allow us to introduce $t(k_1,k_2,k_3)$ and T_{123} that appeared in (4.107) and (4.106). Vector harmonics are defined in terms of the embedding coordinates $\{\mathcal{Y}^a\}$ such that

$$\mathcal{Y}^I_m = C^I_{m;a_1\ldots a_k}\mathcal{Y}^{a_1}\ldots\mathcal{Y}^{a_k} \tag{4.71}$$

where $C^I_{m;a_1\ldots a_k}$ is a completely symmetric and traceless tensor in $a_1,\ldots,a_k$, and its symmetric part vanishes. The vector harmonics satisfy the orthogonality condition

$$\int_{S^5}\mathcal{Y}^I_m\mathcal{Y}^J_n = z(n)\delta^{IJ}\delta_{mn} \tag{4.72}$$

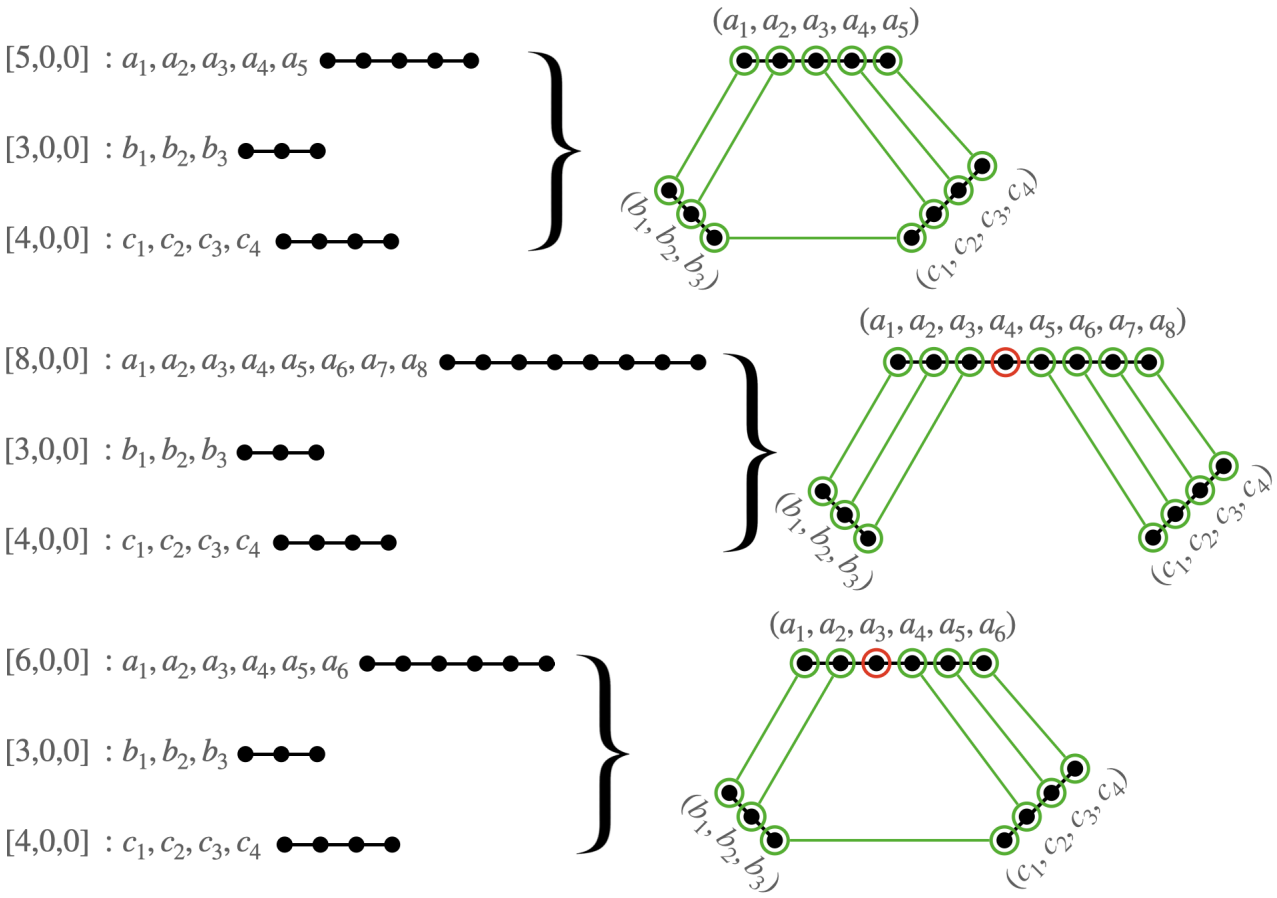


Figure 4.1: Illustration of the triangle inequality. A green line means that the two indices are on the same δ. For example the first scheme corresponds to $\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3c_2}\delta_{a_4c_3}\delta_{a_5c_4}\delta_{a_3c_1}$. In the first line, the $\mathcal{C}$ tensor exist because the α_is are in $\mathbb{N}$. Note that the α_is count the number of connections between the different sets of indices. For example, in the first line $\alpha_1 = 1$ is the number of connections between the second and third sets, i.e. between the b_is and the c_is, $\alpha_2 = 3$ is the number of connections between the first and third sets, and $\alpha_3 = 2$ is the number of connections between the first and second sets. In the second line, the $\mathcal{C}$ tensor does not exist because at least one of the α_is is negative (in that case α_1). We see here that one leg is too long and there is always going to be at least one index left alone is that case. In the last line, we see that all α_is are positives, but they are not integers. In that case, one index is left alone, and the $\mathcal{C}$ tensor does not exist.

with z defined in (4.66). On top of this orthogonality condition, the vector harmonics satisfy

$$\int_{S^5} \nabla^m \mathcal{Y}^{I_1} \mathcal{Y}^{I_2} \mathcal{Y}^{I_3}_m = t(k_1, k_2, k_3) \tag{4.73}$$

with

$$t(k_1, k_2, k_3) = \frac{\pi^3}{k_3 + 1} \frac{1}{(\frac{1}{2}(\sigma+3))! 2^{\frac{1}{2}(\sigma-3)}} \frac{k_1! k_2! k_3!}{(\alpha_1 - \frac{1}{2})!(\alpha_2 - \frac{1}{2})!(\alpha_3 - \frac{1}{2})!} T_{123} \tag{4.74}$$

The T_{123} tensor carries the index structure of the previous integral. Explicitely we have

$$\begin{aligned} T_{123} = \quad & C^{I_1}_{m a_1 \dots a_{\alpha_2 - 1/2} b_1 \dots b_{\alpha_3 - 1/2}} C^{I_2}_{b_1 \dots b_{\alpha_3 - 1/2} c_1 \dots \alpha_{1+1/2}} C^{I_3}_{m; c_1 \dots c_{\alpha_1 + 1/2} a_1 \dots a_{\alpha_2 - 1/2}} \\ & C^{I_1}_{a_1 \dots a_{\alpha_2 + 1/2} b_1 \dots b_{\alpha_3 - 1/2}} C^{I_2}_{b_1 \dots b_{\alpha_3 - 1/2} c_1 \dots \alpha_{1-1/2} m} C^{I_3}_{m; c_1 \dots c_{\alpha_1 - 1/2} a_1 \dots a_{\alpha_2 + 1/2}} . \end{aligned} \tag{4.75}$$

Upon contraction with fields, we will recover the definition on $\mathcal{C}^{I_1 I_2 I_3}$ that we will introduce latter in (4.106).

E.3 Cubic scalar couplings

Having shown that the general structure of our ansatz and the ExFT action impose the vanishing of numerous couplings, we will now use the explicit formulas (4.36) obtained for the cubic couplings in order to compute explicit expressions for some of the non-vanishing cubic couplings. Before evaluating these equations, let us consider the expansion of the kinetic term for the scalar fields.

$$\frac{1}{24} \mathcal{M}^{KL} \partial_\mu \mathcal{M}_{LM} \mathcal{M}^{MN} \partial^\mu \mathcal{M}_{NK} \longrightarrow \frac{1}{24} \partial_\mu \phi^{\alpha \Sigma} \partial^\mu \phi^{\beta \Lambda} \delta_{\alpha\beta} \mathcal{Y}^\Sigma \mathcal{Y}^\Lambda + \mathcal{O}(\phi^4) . \tag{4.76}$$

We note that in our basis of fluctuations, the expansion of this term does not give rise to cubic terms of type $\phi \partial_\mu \phi \partial^\mu \phi$. This is because in our parametrisation of fluctuations, the kinetic term respects USp(8) invariance in the first index of the $\phi^{\alpha\Sigma}$, and there is no USp(8) invariant cubic tensor $d^{\alpha\beta\gamma}$ that could define such a term.

Let us further note that for the fields s^I from (4.50), the expansion of the kinetic term (4.76) simply yields a normalisation constant proportional to $z(n)$ defined in (4.65), (4.66), with $\ell = |I| - 2$. Comparing this to the normalisation used in [108, 109] for the same fields (c.f. equations (3.15), (3.23) in [108], with $k = |I|$)

$$\mathcal{A}_I \propto \frac{k(k-1)(k+2)}{k+1} z(k) \propto \frac{k^2 (k-1)^2}{(k+1)^2} z(\ell) , \tag{4.77}$$

where we are ignoring overall constants that are independent of k and where we have used the relation

$$z(\ell) = z(k-2) = 4 z(k) \frac{(k+1)(k+2)}{k(k-1)} . \tag{4.78}$$

From (4.77) we see that our fields s^I are related to the fields of [108, 109] by rescaling

$$s^I \longrightarrow \tilde{s}^I \equiv \frac{k+1}{k(k-1)} s^I , \tag{4.79}$$

where for clarity we denote by $\tilde{s}^I$ for the fields of [108, 109]. Such rescaling factors will become relevant when comparing the results for the cubic couplings.

Let us now turn to evaluating the general formula (4.36) for $AdS_5 \times S^5$. Recall that (4.36) is the reduction of the general formulae (4.31), (4.33), (4.34), (4.35), to the class of vacua whose associated embedding tensor lives in the **36** of USp(8), which is the case for the round S^5. While this formula reduces the contributions of general cubic scalar couplings to four terms, it turns out that depending on the type of scalar fields not even all terms may contribute to the result. The towers of scalar fields have been listed in (4.48), organised according to the representation of

the first index α on $\phi^{\alpha,\Sigma}$. In turn, for a given Kaluza-Klein fluctuation the representation of the associated α the can be inferred from its SO(2) charge. Cubic couplings only exist among states whose SO(2) charges q_i add up to zero, and we find that for the different possible combinations only the following terms from (4.36) give non-vanishing contributions

$\langle q_1, q_2, q_3\rangle$	XX	$X\mathscr{T}$	$(\mathscr{T}\mathscr{T})_1$	$(\mathscr{T}\mathscr{T})_2$
$\langle 0,0,0\rangle$	✓	✓	✓	✓
$\langle 0,+1,-1\rangle$	–	✓	✓	✓
$\langle 0,+2,-2\rangle$	–	–	–	✓
$\langle \pm 1,\pm 1,\mp 2\rangle$	–	–	✓	–

(4.80)

Here, XX, $X\mathscr{T}$, $(\mathscr{T}\mathscr{T})_1$, and $(\mathscr{T}\mathscr{T})_2$ denote the four terms in (4.36). For example, if the charges of scalars in a cubic coupling

$$\mathcal{G}\left(\phi_1^{\alpha\Sigma}, \phi_2^{\beta\Delta}, \phi_3^{\gamma\Gamma}\right), \tag{4.81}$$

are 0, $+1$, and -1, the associated representations in α, β, γ are **20**, **10** and $\overline{\mathbf{10}}$, respectively. The existence of a non-vanishing first term in XX in (4.36) would require the existence of an SO(6) invariant tensor in the tensor product of these three representations, which does not exist. Similarly, one deduces the absence of several terms for other charges.

As a concrete example, let us compute the cubic coupling among three chiral primary fields

$$\mathcal{G}(s^{I_1}, s^{I_2}, s^{I_3}), \tag{4.82}$$

with s^I from (4.50). According to (4.80), all terms in (4.36) are present. First of all, the structure of (4.36) shows that the final result will be proportional to the structure constants defined in (4.29)

$$c_{\Sigma_1\Sigma_2\Sigma_3} = \int_{S^5} d\omega\, \mathcal{Y}_{\Sigma_1}\mathcal{Y}_{\Sigma_2}\mathcal{Y}_{\Sigma_3} = a(\ell_1,\ell_2,\ell_3)\, \mathcal{C}_{\Sigma_1\Sigma_2\Sigma_3}, \quad \ell_i \equiv |\Sigma_i| = |I_i| - 2, \tag{4.83}$$

with the objects on the r.h.s. explicitly defined in (4.68), (4.69). Let us develop the explicit calculations here for (4.36). The XX-like terms are straightforward to compute. They are independent of the length of the chains n_i and we can directly compute them with the explicit form of the X-tensor

$$X_{\underline{AB}}{}^{\underline{C}} \longrightarrow \begin{cases} X_{ab,cd}{}^{ef} = 2\sqrt{2}\delta^{[e}_{[a}\delta_{b][c}\delta^{f]}_{d]}, \\ X_{ab}{}^{c\alpha}{}_{d\beta} = -2\sqrt{2}\delta^{c}_{[a}\delta_{b]d}\delta^{\beta}_{\alpha}, \end{cases} \tag{4.84}$$

with $a, b, \ldots$ SO(6) indices and $\alpha, \beta, \ldots$ SO(2) indices. The more involved terms in this calculation, are the terms with at least one $\mathscr{T}$. They require to compute the action of generators of length ℓ_i's acting on the $[\ell, 0, 0]$ representations. Let us explain how we deal with it. First of all, the underlined capital indices $\underline{A}, \underline{B}, \ldots$ on the $\mathscr{T}$s are reduced to SO(6) $\times$ SO(2)

$$\mathscr{T}_{\underline{A}\Sigma}{}^{\Omega} \longrightarrow \begin{cases} \mathscr{T}_{[ab]\Sigma}{}^{\Omega}, \\ \mathscr{T}^{a\alpha}{}_{\Sigma}{}^{\Omega}. \end{cases} \tag{4.85}$$

The generators $\mathscr{T}_{\underline{A}}$ in the $[n,0,0]$ representation of SO(6) can be written

$$\mathscr{T}_{\underline{A},\Sigma}{}^{\Omega} = \ell\mathscr{T}_{[ab],a_1\dots a_\ell}{}^{b_1\dots b_\ell} = \ell\mathscr{T}_{[ab],((a_1}{}^{((b_1}\delta_{a_2}{}^{b_2}\dots\delta_{a_\ell))}{}^{b_\ell))} \equiv \ell\mathscr{T}_{[ab],A}{}^{B} \equiv \ell\mathscr{T}_{[ab],a_1}{}^{b_1}\delta_{A_1}{}^{B_1}\,, \quad (4.86)$$

$$\text{with}\quad \mathscr{T}_{\underline{A},a}{}^{b} = \begin{cases} \mathscr{T}_{[ab],c}{}^{d} = \sqrt{2}\delta_{c[a}\delta_{b]}{}^{d}\,, \\ \mathscr{T}^{a\alpha}{}_{c}{}^{d} = 0\,. \end{cases} \quad (4.87)$$

Let us explain the notations in (4.86). The $A,B,\dots$ are now used for the symmetric traceless representations of the same lower cases letters, i.e.

$$\begin{aligned} A &\longrightarrow ((a_1\dots a_\ell))\,, \\ B &\longrightarrow ((b_1\dots b_\ell))\,. \end{aligned} \quad (4.88)$$

hence, $A,B,\dots$ have replaced $\Sigma,\Omega,\dots$ indices. Those new $A,B,\dots$ indices are *completely* different to $\underline{A},\underline{B},\dots$ indices. Finally when those indices are dressed with the subscripts $i,j,\dots$ it means that that lower case index with subscripts $i,j,\dots$ has been extracted from the whole set

$$\begin{aligned} A_i &\longrightarrow ((a_1\dots a_{i-1}a_{i+1}\dots a_\ell))\,, \\ B_i &\longrightarrow ((b_1\dots b_{i-1}b_{i+1}\dots b_\ell))\,. \end{aligned} \quad (4.89)$$

Then the δ with a capital $A,B,\dots$ index represent all the δs with according lower case letters

$$\delta_A{}^B \equiv \delta_{((a_1}{}^{((b_1}\dots\delta_{a_\ell))}{}^{b_\ell))}\,. \quad (4.90)$$

The final point we have to explain is how to express the $\mathcal{C}$-symbol in these notations. This tensor reduces to a serie of δ as explained and illustrated in (4.1). We will therefore note it

$$\mathcal{C}^{\Sigma\Omega\Delta} \longrightarrow \mathcal{C}^{ABC} = \delta^{b_1}_{a_1}\dots\delta^{b_{\alpha_3}}_{a_{\alpha_3}}\delta^{c_1}_{b_{\alpha_3+1}}\dots\delta^{c_{\alpha_1}}_{b_{\ell_2}}\delta^{c_{\alpha_1+1}}_{a_{\alpha_3+1}}\dots\delta^{c_{\ell_3}}_{a_{\ell_1}} \equiv \delta^{\bar{A}\bar{B}}\delta^{\bar{B}\bar{C}}\delta^{\bar{C}\bar{A}}\,, \quad (4.91)$$

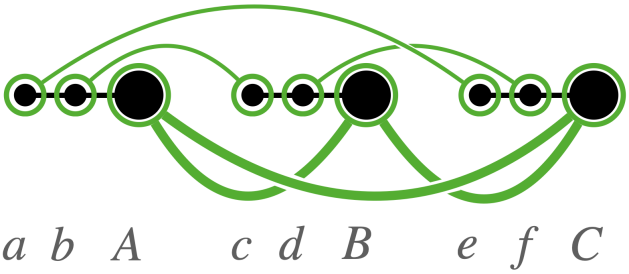
with $\ell_1=|A|$, $\ell_2=|B|$, $\ell_3=|C|$, $\alpha_1=\frac{1}{2}(\ell_2+\ell_3-\ell_1)$, similarly for α_2 and α_3 and with indices in the same color are understood to be symmetric and traceless. We are now ready to explain how we can compute the action of a $\mathscr{T}$ on the $\mathcal{C}$-symbol. The action of a $\mathscr{T}$ on the $\mathcal{C}$-symbol with those notations can be written

$$\begin{aligned} \mathscr{T}_{\underline{A}\Sigma}{}^{\Omega}\mathcal{C}^{\Omega\Delta\Gamma} \longrightarrow \mathscr{T}_{[ab]A}{}^{B}\mathcal{C}^{BCD} &= \ell_1\mathscr{T}_{[ab]a_j}{}^{b_j}\delta_{A_j}{}^{B_j}\mathcal{C}^{BCD} \\ &= \delta_{[a|a_j}\left(\alpha_3\delta_{|b|b_j}\delta_{\bar{D}_j\bar{B}_j}\delta_{\bar{D}\bar{C}}\delta_{\bar{B}\bar{C}} + \alpha_2\delta_{|b|c_j}\delta_{\bar{D}\bar{B}}\delta_{\bar{D}\bar{C}}\delta_{\bar{B}_j\bar{C}_j}\right)\,, \end{aligned} \quad (4.92)$$

with this time $\ell_1=|A|=|B|$, $\ell_2=|C|$, $\ell_3=|D|$. Note that there is still an antisymmetry in a,b indices in the previous identity. Because we are computing the couplings involving three s^I scalars, we know that the result should be proportional to $\mathcal{C}^{I_1I_2I_3}$. Therefore the result is automatically projected onto the symmetric and traceless representations $[\ell_i+2,0,0]$, which is the reason why we do not need to compute any trace term in (4.92) contracted with the scalar fields. On top of this, even if the index structure for the two terms in (4.92) looks different, there is only one index

structure that one can obtain for the coupling $\mathcal{G}(s^{I_1}, s^{I_2}, s^{I_3})$, which schematically looks like

$$\phi^{abA}\phi^{cdB}\phi^{efC}\delta_{ac}\delta_{be}\delta_{df}\mathcal{C}^{ABC} \longrightarrow$$



(4.93)

This picture is just a way of expressing how the indices are contracted together with the $\mathcal{C}$-symbol. Explicitely here this means that ϕ's are contracted with $\delta_{ae}\delta_{bc}\delta_{df}$ as well as properly symmetrized with the indices of $\mathcal{C}$. (4.93) is the only way we can contract the indices, up to symmetrization. We will see later examples where this is not the case anymore. We can now reduce the $(\mathscr{T}\mathscr{T})$-like term in (4.36) on the S^5 and find

$$\begin{gathered}
\mathscr{T}_{\underline{B}\Sigma}{}^{\Lambda}\mathscr{T}_{\underline{A}\Lambda}{}^{\Omega}c_{\Omega\Delta\Gamma}\left(6\,\mathbb{T}_{\gamma\alpha\beta\underline{B}}{}^{\underline{A}}-\tfrac{1}{2}\,\mathbb{T}_{\gamma\underline{B}}{}^{\underline{A}}\kappa_{\alpha\beta}\right)\\
\downarrow\\
\begin{aligned}
&-12\mathscr{T}_{ab\Sigma}{}^{\Omega}\mathscr{T}_{bc\Lambda}{}^{\Sigma}c^{\Omega\Delta\Gamma}\phi^{adA}\phi^{deA}\phi^{ecI}-12\mathscr{T}_{ab\Sigma}{}^{\Omega}\mathscr{T}_{cd\Lambda}{}^{\Sigma}c^{\Omega\Delta\Gamma}\phi^{aeA}\phi^{bcI}\phi^{edA}\\
&-12\mathscr{T}_{ab\Sigma}{}^{\Omega}\mathscr{T}_{cd\Lambda}{}^{\Sigma}c^{\Omega\Delta\Gamma}\phi^{aeA}\phi^{bcA}\phi^{edI}+12\mathscr{T}_{ab\Sigma}{}^{\Omega}\mathscr{T}_{cd\Lambda}{}^{\Sigma}c^{\Omega\Delta\Gamma}\phi^{acA}\phi^{beA}\phi^{edI}\\
&+6\mathscr{T}_{ab\Sigma}{}^{\Omega}\mathscr{T}_{bc\Lambda}{}^{\Sigma}c^{\Omega\Delta\Gamma}\phi^{acA}\phi_{de}{}^{\Lambda}\phi^{deI}\,.
\end{aligned}
\end{gathered}
\tag{4.94}$$

We can now illustrate how to explicitely calculate this for the couplings of three chiral primaries. We will show how this works with the first term in (4.94)

$$\begin{aligned}
-12\mathscr{T}_{ab,\Sigma}{}^{\Omega}\mathscr{T}_{bc,\Lambda}{}^{\Sigma}c^{\Omega\Delta\Gamma}\phi^{adA}\phi^{deA}\phi^{ecI} &\longrightarrow -12a(\ell_1,\ell_2,\ell_3)\mathscr{T}_{abD}{}^{A}\mathscr{T}_{bcE}{}^{D}\mathcal{C}^{ABC}\phi^{adB}\phi^{deE}\phi^{ceC}\\
&= -12a(\ell_1,\ell_2,\ell_3)\mathscr{T}_{ab,E}{}^{D}\mathscr{T}_{ac,D}{}^{A}\mathcal{C}^{ABC}\phi^{bdB}\phi^{ceE}\phi^{deC}\,,
\end{aligned}
\tag{4.95}$$

with $|E| = |D| = |A| = \ell_1$, $|B| = \ell_2$ and $|C| = \ell_3$. By using (4.92) twice and making the antisymmetry in a, b and a, c explicit we find

$$\begin{aligned}
&-12\mathscr{T}_{ab,E}{}^{D}\mathscr{T}_{ac,D}{}^{A}\mathcal{C}^{ABC}\phi^{bdB}\phi^{ceE}\phi^{deC} =\\
&-3\phi^{bdB}\phi^{ceE}\phi^{deC}\Big((-\delta_{be_i}\delta_{ab_i}\delta_{ae_j}\delta_{cb_j}+\delta_{be_i}\delta_{ab_i}\delta_{ab_j}\delta_{cc_j})\alpha_3(\alpha_3-1)\delta_{\bar{E}_{ij}\bar{B}_{ij}}\delta_{\bar{D}\bar{C}}\delta_{\bar{B}\bar{C}}\\
&\qquad(\delta_{ae_i}\delta_{bc_i}\delta_{aej}\delta_{cb_j}-\delta_{be_i}\delta_{ac_i}\delta_{ae_j}\delta_{cb_j})\alpha_3\alpha_2\delta_{\bar{E}_j\bar{B}_j}\delta_{\bar{E}_i\bar{C}_i}\delta_{\bar{B}\bar{C}}\\
&\qquad(-\delta_{be_i}\delta_{ab_i}\delta_{ae_j}\delta_{cc_j}+\delta_{be_i}\delta_{ab_i}\delta_{ce_j}\delta_{ac_j})\alpha_2\alpha_3\delta_{\bar{E}_i\bar{B}_i}\delta_{\bar{E}_j\bar{C}_j}\delta_{\bar{B}\bar{C}}\\
&\qquad(\delta_{ae_i}\delta_{bc_i}\delta_{ae_j}\delta_{cb_j}-\delta_{be_i}\delta_{ac_i}\delta_{cc_j}\delta_{ae_j})\alpha_2(\alpha_2-1)\delta_{\bar{E}_{ij}\bar{C}_{ij}}\delta_{\bar{E}\bar{B}}\delta_{\bar{B}\bar{C}}\Big).
\end{aligned}
\tag{4.96}$$

All terms in color are equal to zero, because of the tracelessness property of the $[\ell_i+2,0,0]$ representations. Explicitely, the terms in blue, upon supressing the a index, give either $\delta_{b_ib_j}$ or $\delta_{e_ie_j}$ which are traces and therefore projected out by the contractions with $\phi^{bdB}\phi^{ceE}\phi^{deC}$. The red term

is also equal to zero, because of δ_{ce_j}. Since it is contracted with $\phi^{ceE} = s^{ceE} = s^I$, this term gives a trace in a $[\ell+2,0,0]$ representation and can be discarded. By collecting all the remaining terms, and symmetrizing in α_i's, this term gives

$$-12\mathscr{T}_{ab,E}{}^{D}\mathscr{T}_{ac,D}{}^{A}\mathcal{C}^{ABC}\phi^{bdB}\phi^{ceE}\phi^{deC} = 3\sigma(\sigma-1)\delta_{bd}\delta_{dc}\delta_{ef}\mathcal{C}^{ABC}\phi^{bdB}\phi^{ceE}\phi^{dfC}\,, \tag{4.97}$$

with $\sigma = \ell_1+\ell_2+\ell_3$. By calculating all terms in (4.94) this way as well as $(X\mathscr{T})$ and (XX)-like terms in (4.36), we find the final result for $\mathcal{G}(s^{I_1},s^{I_2},s^{I_3})$

$$\begin{aligned}\mathcal{G}(s^{I_1},s^{I_2},s^{I_3}) &= a(\ell_1,\ell_2,\ell_3)\Big(\frac{\sigma}{2}+2\Big)\Big(\frac{\sigma}{2}+1\Big)\,\mathcal{C}^{\Sigma_1\Sigma_2\Sigma_3}\,\delta_{ac}\delta_{be}\delta_{df}\phi^{((ab,\Sigma_1))}\phi^{((cd,\Sigma_2))}\phi^{((ef,\Sigma_3))}\,,\\ &= a(\ell_1,\ell_2,\ell_3)\Big(\frac{\sigma}{2}+2\Big)\Big(\frac{\sigma}{2}+1\Big)\,\mathcal{C}^{I_1I_2I_3}\,s^{I_1}s^{I_2}s^{I_3}\,,\end{aligned} \tag{4.98}$$

with $\sigma = \ell_1+\ell_2+\ell_3$, and the symbols $\mathcal{C}^{\Sigma_1\Sigma_2\Sigma_3}$, $\mathcal{C}^{I_1I_2I_3}$ defined in (4.83) for different representations, $[\ell,0,0]$ and $[\ell+2,0,0]$, respectively.

In order to compare this rather compact result to the explicit expressions obtained in [108, 109], we first note the relation

$$a(\ell_1,\ell_2,\ell_3) = a(k_1,k_2,k_3)\,\frac{\alpha_1\,\alpha_2\,\alpha_3}{8\,k_1k_2k_3(k_1-1)(k_2-1)(k_3-1)}\,\frac{\tilde\sigma}{2}\Big(\frac{\tilde\sigma}{2}+1\Big)\Big(\frac{\tilde\sigma}{2}+2\Big)\,, \tag{4.99}$$

for $k_i = |I_i| = \ell_i+2$, with moreover $\tilde\sigma \equiv k_1+k_2+k_3 = \sigma+6$, and $\alpha_i \equiv \frac{1}{2}\tilde\sigma - k_i$. Further taking into account the rescaling (4.79), the cubic coupling (4.98) takes the form

$$\mathcal{G}(s^{I_1},s^{I_2},s^{I_3}) = \frac{\tilde\sigma\,\alpha_1\,\alpha_2\,\alpha_3\,a(k_1,k_2,k_3)}{16\,(k_1+1)(k_2+1)(k_3+1)}\Big(\Big(\tfrac{1}{2}\tilde\sigma\Big)^2-1\Big)\Big(\Big(\tfrac{1}{2}\tilde\sigma\Big)^2-4\Big)\,\mathcal{C}^{I_1I_2I_3}\,\tilde s^{I_1}\tilde s^{I_2}\tilde s^{I_3}\,, \tag{4.100}$$

and this indeed is the result obtained in [108, 109]. The non-trivial zeros in this explicit expression (for any $\alpha_i = 0$) precisely illustrate the theorem (4.55) which in the ExFT formulation follows from purely structural arguments.

E.4 Couplings involving spin-1 fields

Let us work out a few other examples involving the spin-1 fields. Similar to (4.47), (4.48), the spin-1 fields are parametrised as $A_\mu{}^{A,\Sigma}$ with the USp(8) index A decomposing as

$$\mathbf{27} \to \mathbf{6_{+1}} \oplus \mathbf{15_0} \oplus \mathbf{6_{-1}}\,, \tag{4.101}$$

under SO(6) $\times$ SO(2). Accordingly, the fluctuations $A_\mu{}^{A\Sigma}$ carry representations

$$\begin{aligned}\mathbf{15}\otimes[\ell,0,0] &= [\ell,1,1]\oplus[\ell,0,0]\oplus[\ell-1,0,2]\oplus[\ell-1,2,0]\oplus[\ell-2,1,1]\,,\\ (\mathbf{6}\oplus\mathbf{6})\otimes[\ell,0,0] &= 2\cdot[\ell+1,0,0]\oplus 2\cdot[\ell-1,0,0]\oplus 2\cdot[\ell-1,1,1]\,,\end{aligned} \tag{4.102}$$

where similarly to (4.48) we have denoted in blue and green the Goldstone modes for massive tensors and spin-2 modes, respectively. Following [109] we denote the vector fields in the $[\ell,1,1]$ and the $[\ell-2,1,1]$, as $a_\mu{}^{\underline{I}}$ and $c_\mu{}^{\underline{I}}$, respectively. We use the index $\underline{I}$ to label the 'hook' representations

$[k,1,1]$ with $|\underline{I}| \equiv k+1$. Explicitly, focusing on the ${a_\mu}^{\underline{I}}$, this corresponds to an embedding

$$ {A_\mu}^{ab,\Sigma} = \frac{1}{2}\left({a_\mu}^{a,((b\Sigma))} - {a_\mu}^{b,((a\Sigma))} \right) \equiv {a_\mu}^{\underline{I}}, \quad |\underline{I}| = |\Sigma| + 1\,. \tag{4.103} $$

Here, ${A_\mu}^{ab,\Sigma}$ is part of the general vector fluctuations in the basis of (3.55), with the index A decomposed according to (4.103) and the $[\ell,1,1]$ fluctuation living in the **15** part, as can be seen from (4.102). ${a_\mu}^{a,a_1\ldots a_{\ell+1}}$ denotes the part of the fluctuation in the $[\ell,1,1]$, i.e. it is symmetric traceless in $((a_1\ldots a_\ell))$ with $a^{(a,a_1\ldots a_{\ell+1})} = 0$.

We can now evaluate the above results for cubic vector couplings on the $\mathrm{AdS}_5 \times S^5$ background. Let us develop the explicit calculations for couplings of the form $\mathcal{G}(\partial s^{I_1}, s^{I_2}, a^{I_3})$ for s^I in (4.53) and a^I in (4.103). Reducing (4.38) to $\mathrm{AdS}_5 \times \mathrm{S}^5$ gives

$$ \begin{aligned} & A_\mu{}^{\underline{A}\Sigma}\Big[2\, c_{\Omega\Lambda\Sigma} X_{\underline{AB}}{}^{\underline{C}}\, \partial^\mu \phi^{\alpha\Omega}\, \phi^{\beta\Lambda}\, \mathbb{T}^{\hat{\gamma}}{}_{\alpha\beta}\, \mathbb{T}_{\hat{\gamma}\underline{C}}{}^{\underline{B}} + 2\, c_{\Delta\Lambda\Sigma}\, \phi^{\beta\Omega}\, \partial^\mu \phi^{\alpha\Delta}\, \mathscr{T}_{\underline{A}}{}^{\Omega\Lambda}\, \kappa_{\alpha\beta} \\ & \quad - 12\, c_{\Lambda\Omega\Delta}\, \partial^\mu \phi^{\alpha\Lambda}\, \phi^{\beta\Delta}\, \mathscr{T}_{\underline{B}}{}^{\Sigma\Omega}\, \mathbb{T}^{\hat{\gamma}}{}_{\beta\alpha}\, \mathbb{T}_{\hat{\gamma}\underline{A}}{}^{\underline{B}} \Big] \end{aligned} $$

$$ \downarrow \tag{4.104} $$

$$ \begin{aligned} & -24\sqrt{2} A_\mu{}^{ab\Sigma} \partial^\mu \phi^{ca\Omega} \phi^{bc\Lambda} c^{\Sigma\Omega\Lambda} + 12 A_\mu{}^{ab\Sigma}\, \mathscr{T}_{ab}{}^{\Omega\Lambda} \partial^\mu \phi^{cd\Delta} \phi^{cd\Omega} c^{\Sigma\Lambda\Delta} \\ & + 12 A_\mu{}^{ab\Sigma}\, \mathscr{T}_{cb}{}^{\Sigma\Omega} \partial^\mu \phi^{cd\Lambda} \phi^{ad\Delta} c^{\Omega\Lambda\Delta} - 12 A_\mu{}^{ab\Sigma}\, \mathscr{T}_{ca}{}^{\Sigma\Omega} \partial^\mu \phi^{cd\Lambda} \phi^{bd\Delta} c^{\Omega\Lambda\Delta} \\ & - 12 A_\mu{}^{ab\Sigma}\, \mathscr{T}_{cb}{}^{\Sigma\Omega} \partial^\mu \phi^{da\Lambda} \phi^{cd\Delta} c^{\Omega\Lambda\Delta} + 12 A_\mu{}^{ab\Sigma}\, \mathscr{T}_{ca}{}^{\Sigma\Omega} \partial^\mu \phi^{db\Lambda} \phi^{cd\Delta} c^{\Omega\Lambda\Delta}\,. \end{aligned} $$

We can now inject (4.103) and use similar techniques that we used for $\mathcal{G}(s^{I_1}, s^{I_2}, s^{I_3})$ to compute the action of $\mathscr{T}$ on $\mathcal{C}$. For the second term in the previous expression, we therefore obtain

$$ \begin{aligned} & 12 A_\mu{}^{ab\Sigma}\, \mathscr{T}_{ab}{}^{\Omega\Lambda} \partial^\mu \phi^{cd\Delta} \phi^{cd\Omega} c^{\Sigma\Lambda\Delta} \longrightarrow 12 A_\mu{}^{abD}\, \mathscr{T}_{ab}{}^{AC} \partial^\mu \phi^{cdB} \phi^{cdC} c^{ABD} = \\ & 3 a_\mu{}^{a,bD} \partial^\mu \phi^{cdB} \phi^{cdC} \Big(\alpha_2 (\delta_{ac_i}\delta_{bd_i} - \delta_{bc_i}\delta_{ad_i}) \delta_{D_iC_i}\delta_{BD}\delta_{BC} + \alpha_3 (\delta_{ac_i}\delta_{bb_i} - \delta_{bc_i}\delta_{ab_i}) \delta_{CD}\delta_{BD}\delta_{B_iC_i} \Big) \\ & -3 a_\mu{}^{b,aD} \partial^\mu \phi^{cdB} \phi^{cdC} \Big(\alpha_2 \delta_{bd_i}\delta_{ac_i} - \delta_{bd_i}\delta_{ac_i}) \delta_{C_iD_i}\delta_{BD}\delta_{BC} + \alpha_3 (\delta_{ac_i}\delta_{bb_i} - \delta_{bc_i}\delta_{ab_i}) \delta_{CD}\delta_{BD}\delta_{B_iC_i} \Big)\,, \end{aligned} \tag{4.105} $$

with the terms in red equal to zero again because of the tracelessness property of the $[\ell+2,0,0]$ representation. Note however that this time, the index structures arising from the action of $\mathscr{T}$ on $\mathcal{C}$ is not unique anymore. The different possibilites are listed in Figure 4.2 where we used same schemes as in (4.93), but this time we had to make a disctinction between the three fields. Those are the only three structure we obtain. By carefully working out the calculations, one finds that all the terms coming with the first structure cancel with one another, and the last two structures come with exactly the same numerical factor, up to a sign. This is exactly what is needed to reproduce the strucure of T_{123} in (4.75). The final result we found for this coupling is

$$ \begin{aligned} \mathcal{G}(\partial s^{I_1}, s^{I_2}, a^{\underline{I}_3}) & \propto a(\ell_1,\ell_2,\ell_3)\,(\sigma+4) \left(\delta_{ea}\delta_{fb}\delta_{gc} - \delta_{eb}\delta_{fa}\delta_{gc} \right) \mathcal{C}^{\Sigma_1\Sigma_2\Sigma_3} \partial^\mu \phi^{((ag\Sigma_1))} \phi^{((bc\Sigma_2))}\, {A_\mu}^{e,((f\Sigma_3))} \\ & \equiv a(\ell_1,\ell_2,\ell_3)\,(\sigma+4)\, \mathcal{C}^{I_1I_2\underline{I}_3}\, \partial^\mu s^{I_1} s^{I_2}\, {a_\mu}^{\underline{I}_3}\,, \\ & \ell_i \equiv |\Sigma_i|\,, \quad \sigma = \ell_1+\ell_2+\ell_3\,, \end{aligned} \tag{4.106} $$

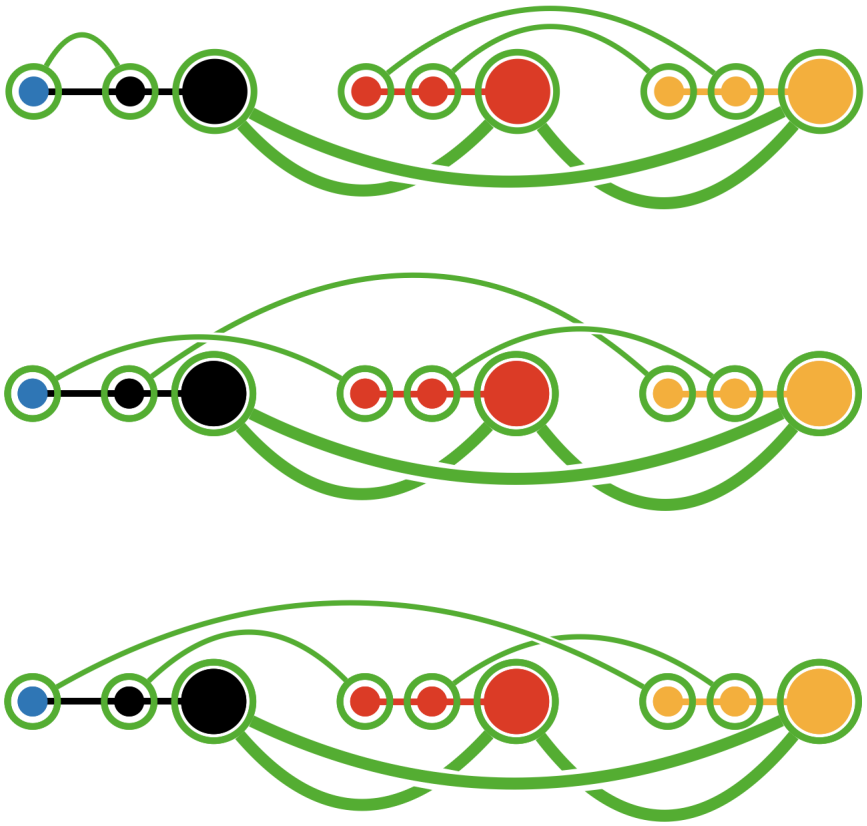


Figure 4.2: Schematic representation of the different index structure arising in the calculation of $\mathcal{G}(\partial s^{I_1}, s^{I_2}, a^{I_3}) = \mathcal{G}(\partial s^{ab\Sigma}, s^{cd\Omega}, a^{e,f\Delta})$. The set of black points represent the indices of the vector $a^{e,f\Delta}$, with the blue dot the extra index e which is not symmetrized and traceless, the red points stand for the indices of $s^{cd\Omega}$ and the orange points stand for the indices of $\partial_\mu s^{ab\Sigma}$.

where $|\Sigma_i| = |I_i| - 2$, for $i = 1,2$, and $|\Sigma_3| = |\underline{I_3}| - 1$. Here, the tensor $\mathcal{C}^{I_1 I_2 \underline{I_3}}$ is defined by the above equation as the (up to normalisation) unique invariant structure in this tensor product. It corresponds to what has been called T_{123} in (4.75), and the combination of the last two schemes in (4.2) with a minus sign.

In order to compare the full coupling (4.106) to the results of [109], we need the relation

$$
\begin{aligned}
a(\ell_1,\ell_2,\ell_3) = \frac{(k_3+1)(\alpha_3-\frac{1}{2})(\tilde{\sigma}+3)(\tilde{\sigma}+1)}{k_1k_2k_3(k_1-1)(k_2-1)} t(k_1,k_2,k_3)\,, \\
k_1 \equiv |I_1| = \ell_1+2\,, \quad k_2 \equiv |I_2| = \ell_2+2\,, \quad k_3 \equiv |\underline{I_3}| = \ell_3+1\,, \\
\tilde{\sigma} \equiv k_1+k_2+k_3 = \sigma+5\,, \quad \alpha_3 \equiv \tfrac{1}{2}\tilde{\sigma}-k_3\,,
\end{aligned}
\tag{4.107}
$$

to the $t(k_1,k_2,k_3)$ defined in (4.73). Furthermore, in order to match the normalisation of vector fields, we note that our kinetic term

$$
\mathcal{M}_{MN}\,\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}^{\mu\nu N} \longrightarrow \mathcal{F}_{\mu\nu}{}^{A\Sigma}\mathcal{F}^{\mu\nu A\Lambda}\,\mathcal{Y}^{\Sigma}\mathcal{Y}^{\Lambda} \propto \frac{\ell+2}{\ell+1}\,f_{\mu\nu}{}^{a,((b\Sigma))}\,f^{\mu\nu a,((b\Lambda))}\,\mathcal{Y}^{\Sigma}\mathcal{Y}^{\Lambda}\,, \tag{4.108}
$$

where the field strengths $f_{\mu\nu}$ are defined in analogy to (4.103). Comparing this to the normalisation of the kinetic term of [109] (whose fields we denote by $\tilde{a}_\mu{}^{\underline{I}}$, $\tilde{f}_{\mu\nu}{}^{\underline{I}}$), we find

$$\frac{\ell+2}{\ell+1}\, z(n)\, f_{\mu\nu}{}^{\underline{I}} f^{\mu\nu\,\underline{I}} = \frac{k+1}{k+2}\, z(k)\, \tilde{f}_{\mu\nu}{}^{\underline{I}} \tilde{f}^{\mu\nu\,\underline{I}}, \qquad k = \ell+1\,, \tag{4.109}$$

with z from (4.66). This yields the relation

$$\tilde{a}_\mu{}^{\underline{I}} = \frac{\ell+3}{\ell+1}\, a_\mu{}^{\underline{I}}\,. \tag{4.110}$$

Putting everything together, we can rewrite the result (4.106) as

$$\mathcal{G}(\partial s^{I_1}, s^{I_2}, a^{I_3}) \propto \frac{(k_3+1)(\alpha_3-\frac{1}{2})(\tilde{\sigma}+3)(\tilde{\sigma}+1)(\tilde{\sigma}-1)}{(k_1+1)(k_2+1)(k_3+2)}\, t(k_1,k_2,k_3)\, \mathcal{C}^{I_1 I_2 I_3}\, \partial^\mu \tilde{s}^{I_1} \tilde{s}^{I_2}\, \tilde{a}_\mu{}^{I_3}\,. \tag{4.111}$$

Indeed, this is the form in which this coupling has been found in [109].

Finally, we can turn to couplings involving three vectors from (4.43) and restrict our analysis again to the vector fields (4.103). Reducing to $\mathrm{SO}(6)\times\mathrm{SO}(2)$ gives

$$\begin{gathered} A^{\mu\,\underline{A}\Sigma} A^{\nu\,\underline{B}\Omega} \partial_{[\mu} A_{\nu]}{}^{\underline{C}\Lambda} \Big(2\, c_{\Sigma\Lambda\Delta} \mathscr{T}_{\underline{A}}{}_{\Omega}{}^{\Delta} \delta_{\underline{BC}} + 10\, c_{\Sigma\Delta\Lambda} d_{\underline{ABD}} d^{\underline{FCD}} \mathscr{T}_{\underline{F}}{}_{\Omega}{}^{\Delta} + c_{\Sigma\Omega\Lambda} X_{[\underline{AB}]}{}^{\underline{C}} \Big) \\ \downarrow \\ \begin{aligned} & A^{\mu ab\Sigma} A^{\nu cd\Omega}\, \mathscr{T}_{ab}{}^{\Omega\Delta} \partial_\mu A_\nu{}^{cd\Lambda} c^{\Sigma\Delta\Lambda} \\ & - A^{\mu ab\Sigma} A^{\nu cd\Omega}\, \mathscr{T}_{cd}{}^{\Sigma\Delta} \partial_\mu A_\nu{}^{ab\Lambda} c^{\Omega\Delta\Lambda} \\ & + 2\sqrt{2} A^{\mu ab\Sigma} A^{\nu bc\Omega} \partial_\mu A_\nu{}^{ac\Lambda} c^{\Sigma\Omega\Lambda}\,. \end{aligned} \end{gathered} \tag{4.112}$$

We can now inject (4.103) in this expression and conduct the calculations. Using the same tricks as before, we can compute those term and the action of $\mathscr{T}$ on $\mathcal{C}$. However, by a direct calculation, the index structure for the term with a $\mathscr{T}$ and the term without $\mathscr{T}$ are not the same. For the first we have something that is proportional to

$$\mathscr{T}\text{-like terms} \quad \longrightarrow \quad a_\mu{}^{d_i,b_i A} a_\nu{}^{c,dB} \partial^\mu a^{\nu c,dC} \mathcal{C}^{AB_i D_i}\,, \tag{4.113}$$

whereas for the second we have something that is proportional to

$$X\text{-like terms} \quad \longrightarrow \quad a_\mu{}^{a,bA} a_\nu{}^{b,cD} \partial^\mu a^{\nu a,cC} \mathcal{C}^{ADC}\,. \tag{4.114}$$

This is schematically illustrated in Figure 4.3. In order to match the two structures, we have to do the following transformations : first use the antisymmetry in the external $\mu \leftrightarrow \nu$ indices, and then notice that because of the representation the a_μ's are sitting in $[\ell,1,1]$, we have the following identity

$$a_\mu{}^{([ab],a_1\ldots a_n)} = 0 \Rightarrow a_\mu^{a,(ba_1\ldots a_n)} + a_\mu^{b,(aa_1\ldots a_n)} = 0\,. \tag{4.115}$$

Using both symmetries, we can show that the two appaerently distinguished structures are actually the same. This is illustrated in Figure 4.4 eii. Collecting the different factors, we find that the

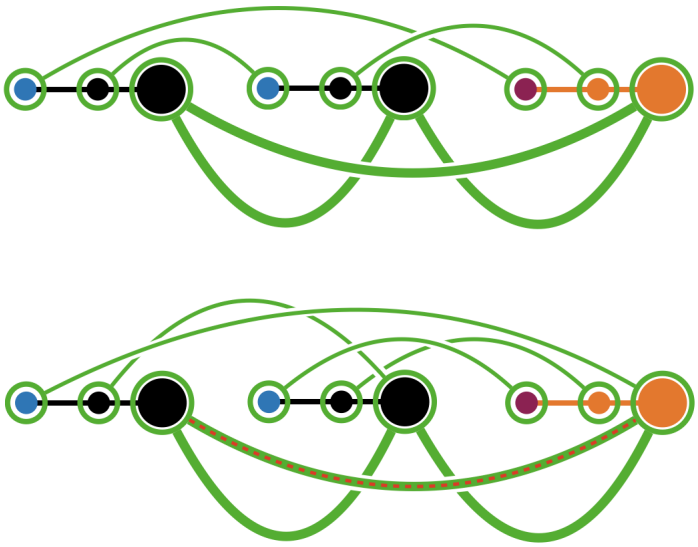


Figure 4.3: Schemes of the two different index structures arising for couplings $\mathcal{G}(a^{I_1}, a^{I_2}, \partial a^{I_3})$. The first one is for the X-like terms and the second for the $\mathscr{T}$-like term. The dotted red line in the second picture means that a δ from $\mathcal{C}$ has been cut in two and the indices have been redistributed elsewhere.

final result is

$$\mathcal{G}(a^{I_1}, a^{I_2}, \partial a^{I_3}) \propto a(\ell_1, \ell_2, \ell_3)(\sigma+6)\, \delta_{ae}\delta_{bc}\delta_{df}\, \mathcal{C}^{\Sigma_1\Sigma_2\Sigma_3}\, a_\mu{}^{a,((b\Sigma_1))} a_\nu{}^{c,((d\Sigma_2))} \partial_{[\mu} a_{\nu]}{}^{e,((f\Sigma_3))}\,,$$
$$\ell_i \equiv |\Sigma_i| = |I_i| - 1\,, \quad \sigma = \ell_1 + \ell_2 + \ell_3\,. \tag{4.116}$$

The index contraction on the r.h.s. of this equation defines an SO(6) invariant tensor $\mathcal{C}^{I_1 I_2 I_3}$, however unlike (4.83) there is not a unique invariant structure appearing in this tensor product, i.e. the particular contraction in the coupling (4.116) by itself carries non-trivial information.

E.5 Couplings involving spin-2 fields

As a final example, let us compute the coupling between a spin-2 field and two scalar fields, which again we take to be the chiral primaries s^I. It is straightforward to evaluate (4.45) for this case as

$$\begin{aligned}\mathcal{G}(\partial s^{I_1}, \partial s^{I_2}, h^{I_3}) &\propto a(\ell_1, \ell_2, \ell_3)\, \partial_\mu \phi^{((ab,\Sigma_1))} \partial_\nu \phi^{((ab,\Sigma_2))} h^{\mu\nu\Sigma_3}\, \mathcal{C}^{\Sigma_1\Sigma_2\Sigma_3} \\ &= a(\ell_1, \ell_2, \ell_3)\, \partial_\mu s^{I_1} \partial_\nu s^{I_2}\, h^{\mu\nu I_3}\, \mathcal{C}^{I_1 I_2 I_3}\,, \\ \ell_i &= |\Sigma_i|\,,\end{aligned} \tag{4.117}$$

where $|\Sigma_i| = |I_i| - 2$ for $i = 1, 2$, and $|\Sigma_3| = |I_3|$. In order to compare this result to the literature, we apply the rescaling (4.79) as well as the relation

$$a(\ell_1, \ell_2, \ell_3) = a(k_1 - 2, k_2 - 2, k_3) = a(k_1, k_2, k_3) \frac{(\tilde{\sigma}+4)(\tilde{\sigma}+2)}{k_1(k_1-1)k_2(k_2-1)}\,, \tag{4.118}$$

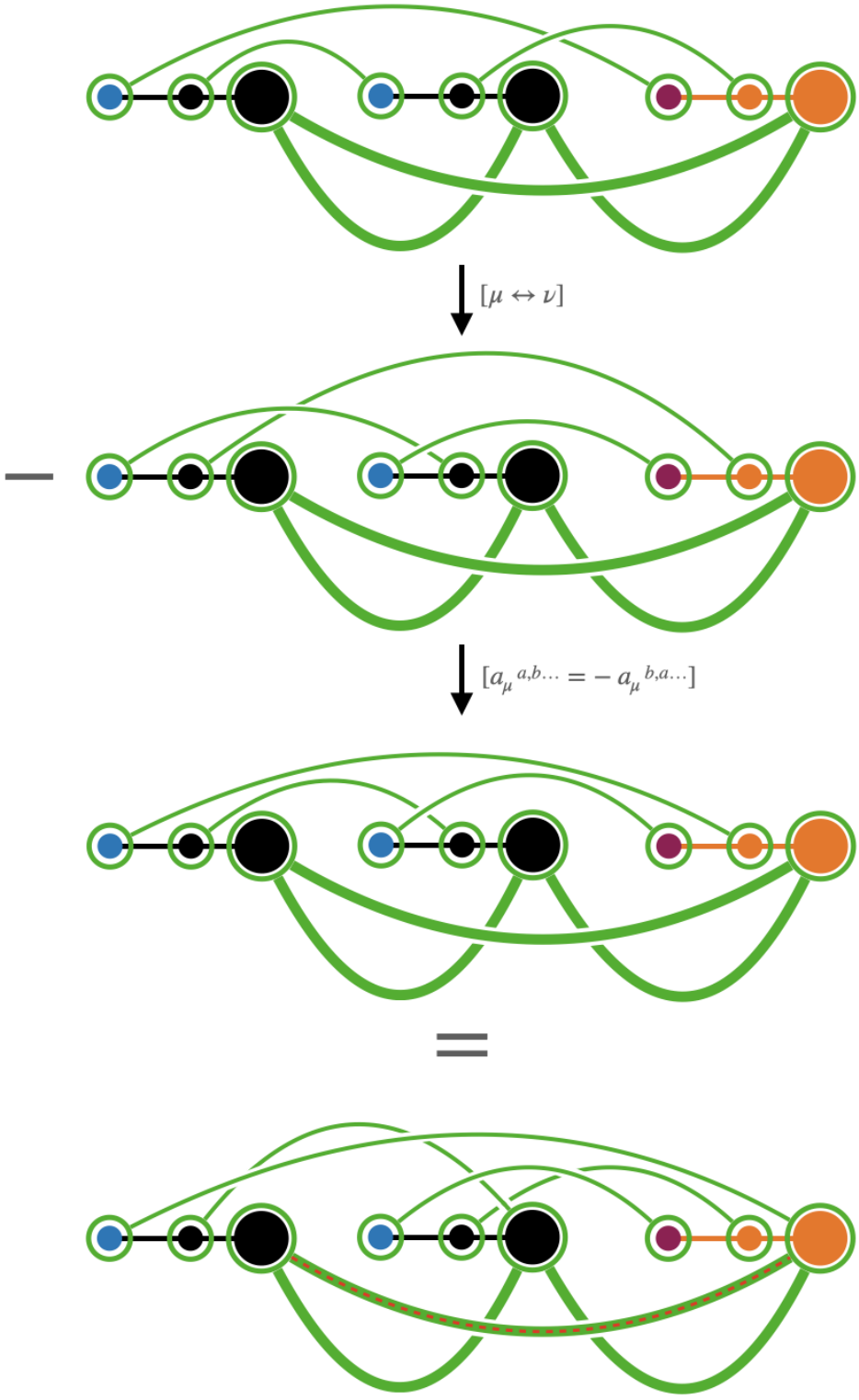


Figure 4.4: Schematic procedure to identify the two apparently disctincts index structures of Figure 4.3 arising in the calculation of $\mathcal{G}(a^{I_1}, a^{I_2}, \partial a^{I_3})$ couplings. The last equality holds because the small dots represent a single index that symmetric with all the indices represented in the big dots of the same color.

for $k_1 = \ell_1 + 2$, $k_2 = \ell_2 + 2$ and $k_3 = \ell_3$. With this, the result (4.117) turns into

$$\mathcal{G}(\partial s^{I_1}, \partial s^{I_2}, h^{I_3}) = \frac{(\tilde{\sigma}+4)(\tilde{\sigma}+2)(\alpha_3-1)\alpha_3}{(k_1+1)(k_2+1)}\, a(k_1,k_2,k_3)\, \partial_\nu \tilde{s}^{I_1} \partial_\mu \tilde{s}^{I_2} h^{\mu\nu I_3}\, \mathcal{C}^{I_1 I_2 I_3}\,,$$

$$k_i = |I_i|\,, \quad \tilde{\sigma} = k_1 + k_2 + k_3\,, \quad \alpha_3 = \tfrac{1}{2}\tilde{\sigma} - k_3\,, \tag{4.119}$$

which reproduces the result in the form found in [109].

Let us make a final remark about the calculations of couplings we showed in this thesis. For all the couplings, we had at some point to compute the action of a $\mathscr{T}$ on the $\mathcal{C}$ symbol. This was possible and not too lengthy because all the fields we considered were sitting in traceless representations. However, as soon as there is a trace part, as for example if we were to compute couplings with a $t^I \in [n-2,0,0]$ from (4.59), this would imply to compute extra terms involving at least a trace and the complexity of the calculations increases, because the number of terms to compute does. Note however that there is a way out, that we did not use in the thesis. All couplings in the ExFT language are of the form

$$(XX + X\mathscr{T} + \mathscr{T}\mathscr{T}) \times \text{"ExFT} - \text{Structure"} \times \Phi_1\Phi_2\Phi_3 \tag{4.120}$$

or

$$(X + \mathscr{T}) \times \text{"ExFT} - \text{Structure"} \times \Phi_1\Phi_2\Phi_3 \tag{4.121}$$

with Φ_i any fields. The "ExFT-Structure" is common for *all* couplings in the same class of couplings. For example all cubic scalars couplings of the form $\mathcal{G}(\phi,\phi,\phi)$ are of the form (4.33) (4.34) (4.35). The difference between the couplings involving diffferent scalar fields is made by the exact form the $(XX + X\mathscr{T} + \mathscr{T}\mathscr{T})$ takes for each individual couplings. However, this $(XX + X\mathscr{T} + \mathscr{T}\mathscr{T})$ is nothing more than a quadratic polynomials in the n_i, the length of the chain in the ExFT side

$$(XX + X\mathscr{T} + \mathscr{T}\mathscr{T}) \longrightarrow \sum_{k=0}^{2}\sum_{i=1}^{3} a_i n_i^k \tag{4.122}$$

The maximal order is fixed by the maximum number of $\mathscr{T}$ involved. Therefore, if one wants to know all the couplings $\mathcal{G}(s^{I_1}, s^{I_2}, t^{I_3})$, on would need to explicitely evaluate (4.36) on a finite number of non trivial examples to fix the coefficients in the polynomial. By non trivial, we mean on examples that are allowed by the SO(6) group (for example this amounts to have $\alpha_i \in \mathbb{N}$ for three scalar fields). Using this argument, we can access a much larger set of couplings on $AdS_5 \times S^5$, as long as we found the ExFT structure.

F Conclusion

Within this chapter, we revisited and extended the main approaches for computing cubic and higher order couplings within compactified Supergravities. In section (4.B), we showed the basic ideas and first derivations of cubic couplings by brut force calculation conducted in the 2000's. This approach has been applied to a few vacua, and we illustrated the calculation with the derivation of a cubic coupling for IIB Supergravity on $AdS_5 \times S^5$. Despite many successes of those techniques, they suffer an number of limitations, from which we can cite the lack of structure in results and the lengthyness of the calculations.

In sections (4.C) and (4.D), we showed how ExFT allows us to efficiently compute n-point couplings of KK modes around vacua of maximal gauged Supergravities that arise as consistent truncations of 10/11-dimensional Supergravity. These couplings are particularly interesting for AdS vacua, where they allow us to holographically compute correlators of single-trace operators of dual strongly-coupled CFTs. We showed that these n-point couplings inherit an ExFT structure, being controlled by an invariant $c_{\Sigma_1\dots\Sigma_n} = \int d\omega\, \mathcal{Y}_{\Sigma_1}\dots\mathcal{Y}_{\Sigma_n}$ that comes from integrating n scalar harmonics and which is the same for any vacuum of the consistent truncation. This structure implies that infinitely many couplings vanish in Supergravity, even though may be allowed by the symmetry group of the vacuum. This can be seen as a generalisation of the property of consistent truncation to higher KK levels. As we showed in section 4.E.1 , this structure even provides new results for the $\mathcal{N} = 8$ $AdS_5 \times S^5$ vacuum, where it allows us to prove the previously conjectured vanishing of near-extremal couplings [170, 171], as well as other vanishing n-point functions. For vacua that preserve fewer isometries, such as the $\mathcal{N} = 2$ $SU(2) \times U(1)$-invariant AdS_5 vacuum of 5-d maximal $SO(6)$ Supergravity [106], this structure is likely to lead to even more couplings vanishing.

In addition to investigating the structure of n-point couplings, we used ExFT to derive universal formulae that can be used to compute the cubic couplings of any vacuum of a consistent truncation to 5-dimensional maximal gauged Supergravity. These couplings are controlled by at most specific quadratic combinations of the 5-dimensional embedding tensor and the action of Killing vectors on scalar harmonics, $\mathscr{T}$, and can be evaluated for any vacuum of the consistent truncation by appropriately dressing the embedding tensor and $\mathscr{T}$-matrices by the coset representative of the vacuum. We used this to compute cubic scalar, scalar-scalar-vector, scalar-scalar-spin-2 and cubic vector couplings, the first three of which match against the known literature while the last is a new coupling. Here another benefit of our approach becomes apparent: not only can the expressions be evaluated for any vacuum of the consistent truncation, but even for the round S^5 our field basis, consisting of tensor products of the 5-dimensional Supergravity fields with the scalar harmonics, yields much more compact expressions for the couplings than the known results in the literature.

Our work opens up several avenues for further investigation. Since our method does not make any assumption about (super-)symmetries, our results can be used for vacua which preserve no isometries or are even non-supersymmetric. It would thus be interesting to investigate the consequence of the universal structure of n-point couplings and to explicitly compute the cubic couplings for less supersymmetric vacua, such as the $\mathcal{N} = 2$ $SU(2) \times U(1)$ AdS_5 vacuum [106, 184] dual to the Leigh-Strassler CFT [185]. Together with the recently-computed KK spectrum [122], this would effectively solve the single-trace sector of the Leigh-Strassler CFT at large N and coupling.

Another direction would be to compute the universal cubic formulae using ExFT for other dimensions. For example, there are consistent truncations to several 4-dimensional gauged Supergravities with interesting AdS vacua [145, 146, 186–191], whose cubic couplings could be explored using $E_{7(7)}$ ExFT. Yet another possibilty to extend this work, would be to push the techniques presented here to work beyond vacua of consistent truncations to maximal Supergravity, for example as was achieved for KK spectroscopy in the second chapter and in [DB1, DB2]. It would be interesting to see how the vanishing of couplings, as in (4.55), translates into those vacua which possess level mixing.

More generally, our work has uncovered a lot of structure in the n-point couplings for vacua that live in consistent truncations to maximal gauged Supergravity. What does this surprising

structure imply for dual CFTs? One would expect that the vanishing cubic couplings imply special OPE relations for the dual CFT. Can our results serves as useful input for the conformal bootstrap program or does the simplified basis and new structure arising from ExFT help with computing 4-point AdS correlators like in [192, 193]? And, how much, if any, of this structure survives beyond the large-'t Hooft coupling and large-N limits?

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