# Exploring infinite distance limits in string moduli spaces

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*Explorer les limites à distance infinie dans les espaces des modules en théorie des cordes*

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**Titre :** Explorer les limites à distance infinie dans les espaces des modules en théorie des cordes

**Mots clés :** Théorie des cordes, Paysage des cordes, Dualités, Compactifications, Gravité quantique

**Résumé :** Le paysage de la théorie des cordes est vaste et, à bien des égards, encore inexploré. Néanmoins, après compactification, il a été observé que des caractéristiques communes apparaissent, qui sont indépendantes de la compactification elle-même, et qui sont supposées être intrinsèquement liées aux effets de gravité quantique. Dans cette optique, le « Swampland program » vise à comprendre les conditions auxquelles une théorie effective des champs doit obéir afin d'être complétée de manière cohérente par la gravité quantique, pour laquelle la théorie des cordes est, à l'heure actuelle, le cadre le plus satisfaisant. Ces contraintes sont actuellement exprimées sous la forme d'une série de conjectures qui, à une énergie donnée, divisent l'espace des théories effectives en « Landscape » (à savoir, celles qui proviennent de la gravité quantique) et « Swampland » (celles qui, bien que cohérentes en tant que théories quantiques des champs, par exemple sans anomalie, ne peuvent pas être couplées de manière cohérente à la gravité quantique). Une caractéristique typique des compactifications de cordes est la présence de modules, des champs scalaires sans masse avec un potentiel plat dont le terme cinétique dans l'action de la supergravité joue le rôle d'une métrique dans l'espace paramétré par ces modules. Dans ce contexte, l'une des conjectures les plus largement acceptées, la « Distance Conjecture », stipule qu'en se déplaçant à une distance infinie dans l'espace des modules d'une théorie de la gravité quantique, une tour infinie d'états deviennent légers de façon exponentielle en la distance géodésique. Dans cette thèse, nous nous sommes concentrés sur certains aspects de cette conjecture dans des contextes spécifiques de la théorie des cordes. Tout d'abord, nous avons étudié comment ces états infiniment nombreux étendent les algèbres de symétrie dans les limites de décompactification des cordes hétérotiques et CHL sur des tores à d dimensions, qui correspondent à des théories de jauge de rang 16+d et 8+d, respectivement. Dans les deux cas, en utilisant la théorie sur la surface de la corde, nous avons montré qu'en allant à l'infini dans $k \leq d$ directions, les algèbres qui émergent sont la version affine de celles de la théorie de dimension supérieure 10-d+k vers laquelle nous décompactifions. De plus, nous montrons que la décompactification de la théorie CHL en théorie hétérotique, qui s'accompagne d'une augmentation du rang de la symétrie de jauge, est associée à la présence d'une version tordue de l'algèbre affine. Enfin, nous montrons que dans le cadre simple des espaces de modules symétriques (tels que ceux des compactifications toroïdales de la théorie M et de la théorie des cordes), il existe un lien naturel entre la géométrie de l'espace de modules et le spectre des cordes. Nous paramétrons la frontière de ces espaces de modules et caractérisons le comportement des géodésiques à l'infini, et en supposant un réseau d'états ainsi qu'en utilisant la complétude du spectre, nous prouvons la « Distance Conjecture » dans ces configurations.

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**Title :** Exploring infinite distance limits in string moduli spaces

**Keywords :** String theory, String landscape, Dualities, Compactifications, Quantum gravity

**Abstract :** The string landscape is vast, and to many extents still unexplored. Nevertheless, in string compactifications it has been observed that common features appear, which are independent of the compactification itself, and that are believed to be deeply related to the quantum gravitational nature of the theory. Along these lines, the Swampland program aims to understand from a bottom-up perspective the conditions that an effective field theory should obey in order to be consistently UV completed to quantum gravity, for which string theory is, as of now, the most satisfactory framework. These constraints are currently expressed as a series of conjectures, which at a given energy divide the space of effective field theories into Landscape (namely, the ones that come from quantum gravity) and Swampland (the ones that despite being consistent as quantum field theories e.g. anomaly free, cannot be consistently coupled to quantum gravity). A typical feature of string compactifications is the presence of moduli, massless scalar fields with flat potential whose kinetic term in the supergravity action plays the role of a metric in a space parametrized by the moduli themselves, the so-called moduli space. Related to this, one of the most widely accepted conjectures, the Distance Conjecture, states that moving an infinite distance in the moduli space of a theory of quantum gravity, an infinite tower of states becomes exponentially light in the geodesic distance. In this thesis, we focused on some aspects of this conjecture in specific string theory settings. Firstly, we studied how these infinitely many states enhance the symmetry algebras in decompactification limits of Heterotic and CHL strings on d-dimensional tori, which correspond to gauge theories of rank 16+d and 8+d, respectively. In both cases, using the worldsheet theory we showed that taking $k \leq d$ radii to infinity, the algebras that emerge are the affine version of the ones in the higher 10-d+k dimensional theory we are decompactifying to. Moreover, we show that the decompactification of the CHL theory to the Heterotic one, which is accompanied by rank enhancement of the gauge symmetry, is associated with the presence of a twisted version of the affine algebra. Finally, we show that in the simple setting of symmetric moduli spaces (such as those of toroidal M- and string theory compactifications), there is a natural connection between the geometry of the moduli space and the string spectrum. We parametrize the boundary of these moduli spaces and characterize the behaviour of the geodesics at infinity, and assuming a lattice of states as well as using completeness of the spectrum we prove the Distance Conjecture in these setups.

This thesis is based on the following papers

[1] *$E_9$ symmetry in the heterotic string on $S^1$ and the weak gravity conjecture*,
**V. Collazuol**, Mariana Graña and Alvaro Herráez,
JHEP 06 (2022) 083, [arXiv:2203.01341]

[2] *Affine algebras at infinite distance limits in the Heterotic String*,
**V. Collazuol**, Mariana Graña, Alvaro Herráez and H. Parra De Freitas,
JHEP 07 (2023) 036, [arXiv:2210.13471]

[3] *A twist at infinite distance in the CHL string*,
**V. Collazuol** and I. V. Melnikov, [arXiv:2402.01606]

and on still unpublished work in collaboration with Stephanie Baines, Bernardo Fraiman, Mariana Graña and Daniel Waldram.

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# Summary

In this thesis, we focus on some aspects related to the Distance Conjecture and the presence of infinite massless towers in the context of the Swampland Program.
First, we analyse the boundaries of the moduli spaces of the heterotic string compactified on $T^d$. We compute the current algebras on the string worldsheet as we approach all the infinite distance limits that correspond to (possibly partial) decompactification limits in some dual frame. When decompactifying $k$ directions of the $T^d$, we find that the infinite towers of states becoming light enhance the algebra arising at a given point in the moduli space of the $T^{d-k}$ compactification to its $k$-loop version from the lower dimensional point of view. The central extensions are given by the $k$ KK vectors, and in the case of semi-simple gauge groups, they make all the factors affine at the same time. In the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic theory on $S^1$ we also prove that these towers of modes satisfy the Lattice Weak Gravity and Repulsive Force Conjectures. Moreover, we make particular emphasis on $d = 2$ and its F-theory duals. For $T^2$ compactifications, we reproduce all the affine algebras that arise in the F-theory dual, and show all the towers explicitly, including some that are not manifest in their F-theory counterparts. Furthermore, we construct the affine $SO(32)$ algebra arising in the full decompactification limit, both in the heterotic and in the F-theory sides, showing that not only affine algebras of exceptional type arise in the latter.
We then consider an orbifold constructed from the 9 dimensional $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic theory on $S^1$, namely the CHL string, and analyse a space-time algebra of BPS states that emerges in the infinite distance limit as the theory decompactifies, focusing especially on the full decompactification limit to 10 dimensions. In particular, we show that decompactifications of the cycle related to the orbifold lead to twisted versions of affine algebra. Moreover, through the kind of affine algebras that can be found at the boundary of the CHL moduli space, we can clearly see that the 9 dimensional theory can only decompactify to the 10 dimensional $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic theory, while the 8 dimensional one can decompactify also to the 10 dimensional $\mathrm{Spin}(32)/\mathbb{Z}_2$.
Finally, we find geodesics and characterize the infinite distance boundary in symmetric moduli spaces (of the coset form $G/K$), like the ones that appear in compactifications with maximal or half maximal supersymmetry, respectively of the kind $E_{d(d)}/K$ and $O(d, d')/(O(d) \times O(d'))$, quotiented out by the discrete U- or T-duality group. The infinite distance points are characterised by rational parabolic subgroups of $G$: due to the quotient by the discrete group, the infinite distance points are reached only by a (measure zero) set of very particular geodesics, while most of them have an ergodic motion, reaching only finite distance points. Assuming completeness of the spectrum, we show that there always is a tower of states becoming massless exponentially in the geodesic distance, thus proving the Swampland Distance Conjecture in those setups.

# Contents

# Part I

# Preliminaries

# Chapter 1

# Introduction

One of the most compelling aspects of physics, and a cornerstone of its development as a science, is the quest for unification – finding the most fundamental laws of nature to describe in a single framework the broadest possible range of phenomena over different scales. Over the past few decades, this has become a more and more advanced theoretical – and technical – effort aimed at formulating a "theory of everything" that could explain all we know about nature and more. Despite seeming then something far from what we can commonly experience about the world, and somehow mainly of philosophical interest, this is actually deeply rooted in observations of nature. Indeed, the first big step towards unification came from Newton's realisation that the force governing the fall of an apple and the motion of celestial bodies is described by the same gravitational law. Another such example is given by electromagnetism, introduced after electricity and magnetism were understood to be just two different manifestations of a single underlying electromagnetic field, as described by Maxwell's laws.
This synthesis paved the way for a fundamental step towards the unification process, culminated with the birth of the Standard Model (SM) of particle physics. In the second half of the last century, they discovered what are the constituents of nature at the energy scales that we can now probe more and more precisely at LHC, and they characterised the interactions among the various particles, which are electromagnetism, weak and strong forces, and gravity.
Leaving gravity aside for the moment, our ability to make very precise predictions that can be tested experimentally was due to the development of Quantum Field Theory (QFT) [4–6], which allows to describe Quantum Mechanical (QM) systems in a Special Relativistic fashion. It is in this language that it became evident that all the known forces could be described based on the concept of symmetries using the same mathematical framework, the one of gauge theories. With this insight, it was understood that electromagnetism and the weak force are just two different low-energy manifestations of the so-called "electroweak force", and despite not having experimental evidence for an additional unification with the strong force at higher energy scales, one can use the tools of QFT to make different proposals for this even more "fundamental" possible interaction.

Regardless of the huge success of QFT and the SM in accounting for collider physics, for which gravitational effects are irrelevant, it seems that this paradigm of unification fails at accounting also for gravity. The current formulation of gravitational interactions is Einstein's General Relativity (GR) [7] (see also [8]), a semiclassical description that is very accurate at

our cosmological scales, but that can only be treated as an effective field theory, as it is non-renormalisable in 4 dimensions. Despite this apparent challenge, it is natural to look for a unique theory that reduces to GR and the SM in the appropriate regimes, and which is able to describe all these forces at a quantum level. This would allow us to explain phenomena such as black holes and the Big Bang, where gravitational interactions are expected to be crucial, and strong at very short scales.

The main candidate for a framework that provides such a consistent formulation of Quantum Gravity (QG) in addition to all the other known interactions is String Theory (ST) [9, 10], which is based on the assumptions that, at the most fundamental level, the constituents of nature are not point-like, but one-dimensional, i.e. strings. The fact that, being extended objects, they are associated with a length scale, non-trivially provides gravity with a way to cut-off UV divergences in a way that evades the problem of non-renormalisability. The strings oscillations in space-time give rise to different states; these include particles with properties that are similar to the ones described by the SM, such as for instance gauge bosons and matter fields, but in addition to this, the spectrum of string theory always includes a massless spin-two particle that has the characteristics that one would expect from the graviton, the "gauge field" of gravity.
In its supersymmetric formulation, internal consistency of ST requires it to be formulated in ten space-time dimensions. Despite this rigidity, there are still five possible string theories that can be constructed in ten dimensions, which feature closed and sometimes open strings depending on the boundary conditions we imposed on them. They are the Type I, Type IIA and IIB, Heterotic $SO(32)$ and Heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ String Theories. They are believed to be deeply related to an eleven-dimensional theory, called M-theory, which at low energies reduces to eleven-dimensional supergravity, as shown in Figure 1.1.

Even though they are different and unrelated theories in 10 dimensions, it turns out that they are all connected (or more precisely, "dual" to one another) once we compactify them, namely when some of their ten spatial directions are taken to be compact. This is moreover required in order to connect ST with phenomenology, as one should account for the fact that we can perceive only a four-dimensional space-time. In order to achieve this, six out of the ten dimensions need to be compactified on an internal manifold whose characteristic length scale is below the current experimental sensitivity, so that they could not have been detected so far and such that effectively the theory has only 4 extended "external" directions. This introduces a huge number of possible compactification spaces which in turn give a huge number of possible lower-dimensional effective theories, since the particle content and the interactions specifically depend on the geometry of the compact manifold being chosen. In particular, one can consider a larger class of vacua without restricting necessarily to four dimensions by choosing a compact manifold with any dimension between 1 and 9.[1] Due to the need for compactification, a common feature of string EFTs is the presence of moduli, namely massless scalar fields without a potential that are not stabilised.[2] From the supergravity perspective, they can be thought of as parameters on a metric space of connected string vacua having the same energy, called

[1] The time-direction cannot be compactified.

[2] Since moduli would give rise to long-range interactions, they need to be stabilised, namely given a fixed vacuum expectation value. This can be done within ST, for example adding fluxes, see e.g. [11]. This is outside the scope of the thesis and will not be further considered.

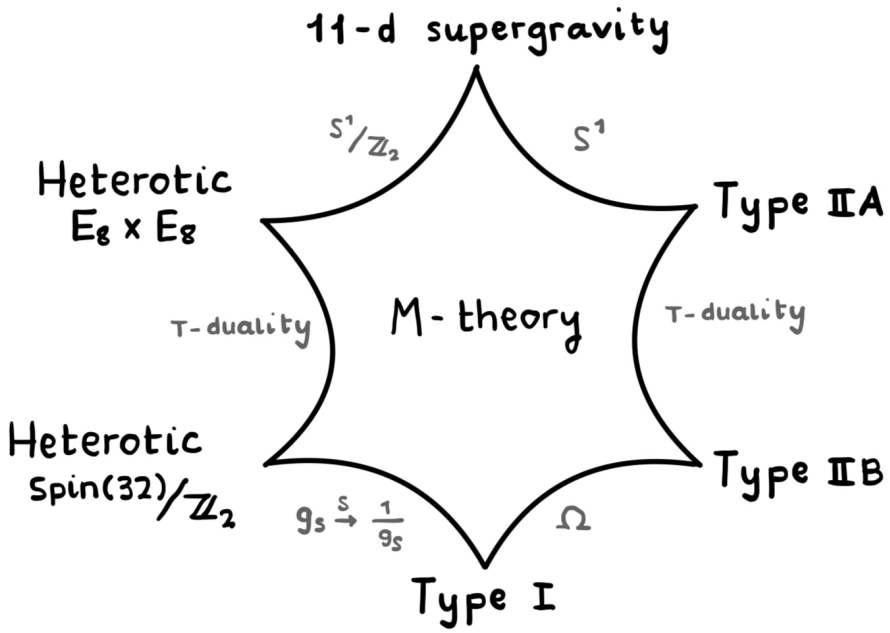



**Figure 1.1:** The duality web among the 10 dimensional string theories and M-theory, whose low-energy description is the one of 11 dimensional $\mathcal{N} = 1$ (maximal) supergravity.

*moduli space*.

The set of different Effective Field Theories (EFTs) that can be obtained through string compactifications is known as "Landscape" [12], and it is very vast, with early estimations of the order of $\sim 10^{500}$ theories. This may raise the question of whether ST has actually any predictive power, as it seems able to describe an very large amount of low energy phenomena. In this light, an important research direction to test the effectiveness of ST in describing phenomenology is to try to understand the generic properties of theories coming from it, which have recently proven to be actually quite constrained. Along these lines, the Swampland program [13] aims at understanding the features that an effective field theory should have in order to be consistently UV completed to QG, by separating the low energy theories that can (belonging to the Landscape) from those that cannot (the so-called Swampland) through a series of conjectures. These are proposed and tested – and sometimes proven/disproven – either from explicit string constructions (pointing out common patterns arising in string compactifications), from heuristic arguments, usually coming from black hole physics, or using holography in the context of e.g. AdS/CFT, with the goal of building a better understanding of the low-energy implications of QG. Despite the fact that all the consistent EFTs can apparently be coupled to GR through minimal coupling and describe gravitational theories, these must obey additional constraints in order to be lifted to full QG.

## Plan of the thesis

One of the most widely accepted claims of the Swampland Program is the universal presence of infinite towers of states that become light when approaching the boundaries of moduli spaces.

This feature of QG theories was originally suggested in [14] with the formulation of the *Distance Conjecture* (SDC), which will be presented in more detail in the following. This thesis mainly focuses on two aspects related to this conjecture, touching also upon some related ones, and is structured as follows.

- In Chapter 2, we will introduce and comment on the SDC, together with other conjectures that will be mentioned in the following and that are related as well to the presence of infinite towers of states with specific properties, namely the *Completeness Hypothesis* [15, 16], the *(Lattice) Weak Gravity Conjecture* (LWGC) [17–19] and the closely related *(Lattice) Replulsive Force Conjecture* (LRFC) [20, 21].

- Part II is related to the characterisation of those infinite distance limits that correspond to decompactification limits in the heterotic and CHL string theories compactified on tori, which display perturbative gauge sectors, based on the symmetries that can arise. In particular, the heterotic string in ten dimensions can have either $\mathrm{E}_8 \times \mathrm{E}_8$ or $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2}$ symmetry, while the CHL string is an intrinsically nine-dimensional construction that comes from an orbifold of the $\mathrm{E}_8 \times \mathrm{E}_8$ theory compactified on a circle. We present them and their toroidal compactifications in Chapter 3, in order to set the notation and describe the phenomenon of symmetry enhancements that can occur at different points in the interior moduli space, in order to connect it with the behaviour at its boundaries. This will be done in Chapters 4 for the heterotic case and 5 for the CHL case; there we focus on decompactification limits of such theories, arising in the infinite distance regions of moduli space where some or all the radii of the torus are taken to have infinite size (up to T-dualities), and we discuss the related symmetry enhancements that can arise. The mathematical background on this kind of algebras, known as affine Lie algebras, is summarised in Appendix A, while details on the computations can be found in Appendix B. Moreover, for the specific case of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string on $S^1$, we explicitly comment on how the tower of states fits within the LWGC, the LRFC and the SDC in Chapter 4.2, leaving the details of the computation for Appendix C. Finally, in Chapter 4.5 we show that, for $T^2$ compactifications, the Heterotic results match those obtained in the dual F-theory on K3 framework in the context of Kulikov models and K3 complex structure degenerations [22,23], and how all the different features appear. Some additional background and comment is given in Appendix D.

- In Part III, we focus on the SDC in the case of compactifications that have a symmetric moduli space, such as the ones of toroidal compactifications of M- and string theory before the action of U- or T- duality, but that actually cover a wider class. After presenting the mathematical tools to characterise and parametrise the boundary of such spaces in Section 6.1.2, we show how this can be used to explicitly prove the SDC in these settings.

# Chapter 2

# Swampland conjectures and infinite towers

In this Chapter, we introduce the Swampland Conjectures that are relevant in the following, without being able to present an exhaustive list[1]. Indeed there exist a large number of different conjectures characterising the theories that belong to the Swampland of theories that cannot be embedded in QG. Despite of this, actually it seems to be possible that the most general and fundamental QG principle are in fact limited in number, as many of these conjectures are connected among one another. This is shown for a subset of them in Figure 2.1, and sharpening the relations among them is one of the main goals of the Swampland Program.
One of the most used arenas to connect and test these conjectures is String Theory, as a realisation of QG, but it is important to stress that in principle they are meant to be defined in relation to a more generic framework of QG.
Here we focus on the Swampland Distance conjecture and its relation with the completeness conjecture [15,16], the Weak Gravity and Repulsive Force Conjectures in the asymptotic limits of moduli spaces.

## 2.1 The Distance Conjecture

This conjecture can be stated as follows.

**Swampland Distance Conjecture (SDC) [14].** In a gravitational theory with a moduli space (with metric given by the kinetic terms of the scalars), starting at a point $P$ in such moduli space and moving towards a point $Q$ an infinite geodesic distance away, an infinite tower of states becomes exponentially light (in Planck units) as

$$M_{\text{tower}}(Q) \sim M_{\text{tower}}(P)\, e^{-\alpha d(P,Q)}\,, \tag{2.1.1}$$

where $d(P,Q)$ is the geodesic distance between the points $P$ and $Q$, and $\alpha$ is some positive, order-one number.

This came originally with the related conjecture that at least one point at infinite distance from $P$ always exists [14].

[1] For a broad overview on these topics, see the recent reviews [24–29].

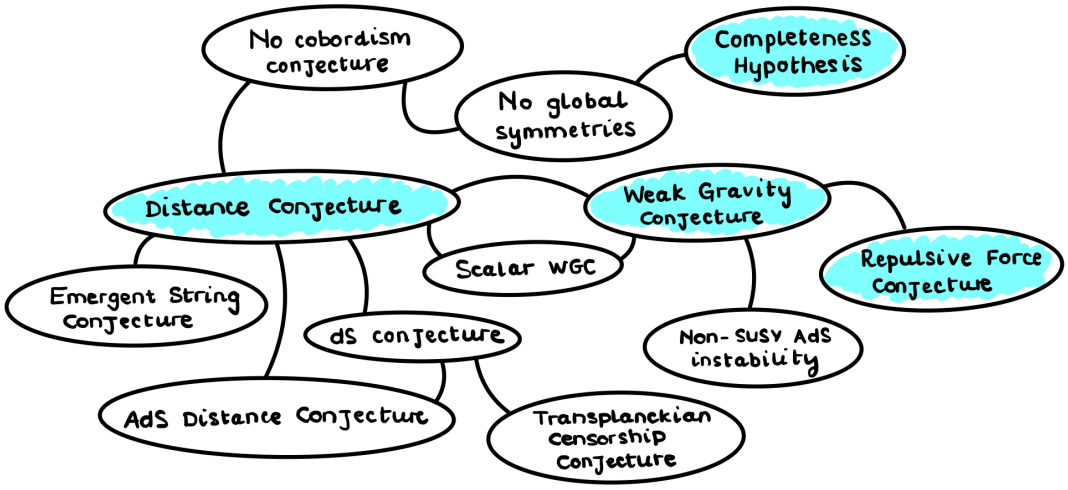



**Figure 2.1:** A non-complete list of Swampland conjectures and some of the connections among them.

The classical example for this conjecture is the bosonic string compactified on $S^1$ of radius $R$; in this case, without accounting for the dilaton, the moduli space is one-dimensional and is parametrized by the radius.
From the string spectrum, one can argue that the contribution to the mass coming from the compact momenta reads, in string frame

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2}\,, \tag{2.1.2}$$

where $n$, $w \in \mathbb{Z}$ are respectively the momentum and winding numbers of the state.
As stated above, the moduli space metric can be read from the Einstein frame 25-dimensional effective theory obtained from dimensional reduction of the 26-dimensional one

$$S_{\text{moduli}} \sim \int d^{25}x\sqrt{-g}\frac{1}{R^2}\partial_\mu R \partial^\mu R\,, \tag{2.1.3}$$

which we present up to $\mathcal{O}(1)$ factors that depend on the number of dimensions, and will not be relevant in the following. This means that distances in moduli space between two points, $R_1$ and $R_2$, are to be computed with respect to $g_{RR} \sim \frac{1}{R^2}$, namely

$$d(R_1, R_2) \sim \int_{R_1}^{R_2} \frac{1}{R} dR = \log\left(\frac{R_2}{R_1}\right)\,. \tag{2.1.4}$$

There are two points at infinite distance, given by $R \to 0$ and $R \to \infty$, and in both cases there is a tower which is exponentially massless. In the $R \to \infty$ limit, this is given by the momentum tower, also called Kaluza-Klein (KK) tower

$$m_n \sim \frac{n}{R} \sim n e^{-d(R_1,R)}\,, \tag{2.1.5}$$

while in the $R \to 0$ limit it is the winding tower that becomes exponentially massless $m_w \sim e^{-d}$. Let us stress that the presence of winding modes is a direct consequence of the stringy nature of the theory, as they would be absent in a pure QFT framework.
This conjecture has received a lot of interest in the community, and it has been extensively studied and checked in several string examples [18, 30–47], and refined in several ways. For example, the fact that $\alpha$ in (2.1.1) is a constant is needed in order not to spoil the exponential behaviour in the distance. A more precise characterisation of it has been done [40, 41, 48, 49], and it has been lower bounded precisely with the *Sharpened Distance Conjecture* [50] as

$$\alpha \geq \frac{1}{\sqrt{d-2}} \tag{2.1.6}$$

in a theory with $d$ external dimensions. Moreover, it turned out to be possible to relate the exponential behaviour in the moduli of the various light towers to a geometric, convex hull formulation and to characterise precisely the polytopes that arise in string settings [51–56], also in relation to the asymptotic behaviour of the species scale [55, 57–60], which is the natural cutoff of QG in the presence of light species [61–65]. Another interesting direction to study the SDC has been the holographic approach, with the help of CFT techniques [66–68].
The fact that infinite field excursions[2] lead to the presence of infinitely many massless states means that an EFT built around the vacuum characterised by $R_0$ has a finite range of validity, and in particular that in this specific case the boundary of the moduli space cannot be described with a 25-dimensional effective field theory. Indeed, approaching infinite distance points one needs a new physical description for the system, as clarified by the *Emergent String conjecture* [69, 70]. According to this, the nature of the towers that become exponentially massless can be

- (dual to) a KK tower. This corresponds to a decompactification limit, so that the new description that is needed is a higher-dimensional one. In this case $\alpha = \sqrt{\frac{d-1}{d-2}}$ in a $d$-dimensional theory.

- (dual to) an emergent tensionless critical and weakly coupled string, with a tower of oscillator modes. The new description must then be a critical string theory, namely in 26 (10) space time dimensions in the case of bosonic (supersymmetric) string. Such a tower is characterised by an exponential rate $\alpha = \frac{1}{\sqrt{d-2}}$, which corresponds to the lower bound of (2.1.6).

This has been tested in several examples [57, 71–75].
It then is clear that there is a strong connection between these conjectures and the notion of duality in String Theory. In this specific example, this can be seen from the T-duality transformation

$$n \leftrightarrow w\,, \quad R \leftrightarrow \frac{1}{R}\,, \tag{2.1.7}$$

that connects KK modes at $R \to \infty$ with winding modes at $R \to 0$.

[2] It has been argued that the exponential behaviour should be seen at most after a critical distance of the order of the Planck mass [46, 47].

## 2.2 The Completeness Hypothesis

**Completeness Hypothesis [15,16].** In a gravitational theory with a gauge symmetry, there must exist states with all the possible gauge charges allowed by Dirac quantization.

This is not necessary in a normal QFT, because charged particles with mass above the physical cutoff can be consistently integrated out. In the presence gravity, this statement can be understood in two different ways, namely from black hole physics and in connection with another Swampland principle, namely the absence of global symmetries [16, 76].
Starting with the first one, a particle cannot be decoupled from the theory just sending the mass to infinity, as this would be a black hole which would be present in the full UV complete theory. Since black holes can have any gauge charge, one expects to have a state - not necessarily a fundamental particle - for each gauge charge.
As for the second argument, the absence of global symmetry, one can show that despite the fact that gauge interactions are not inconsistent with QG, pure gauge interactions, for example the Einstein-Maxewll theory, lead to the presence of a global higher-form symmetry (whose charged operators are higher-dimensional objects) which is incompatible with QG. Indeed, consider the following action

$$S = \int d^4\sqrt{-g}\left(\frac{M_{\mathrm{p},4}^2}{2}R - \frac{1}{4g^2}F^2\right), \tag{2.2.1}$$

with $F = dA$ a 2-form, invariant under $A \to A + d\lambda$. This leads to a 2-form symmetry whose current is given by the field strength $F$ itself, $d * F = 0$, whose charged operators are Wilson lines

$$\mathcal{O} = \exp\left(iq \oint_\gamma A\right). \tag{2.2.2}$$

A way to break this 1-form symmetry is to add electrically charged matter[3], such that $d{*}F \neq 0$

$$d * F = *j_{\mathrm{elect}}\,. \tag{2.2.3}$$

Despite the fact that (2.2.3) would hold also in the presence of just one charged particle, it actually turns out that in the case of higher form symmetries, the completeness is required in order to break all the possible discrete symmetries that would otherwise be preserved [77].

## 2.3 The Weak Gravity Conjecture

The Weak Gravity Conjecture (WGC) [17][4] comes in two versions, the electric and the magnetic one, which are dual statements as follows. Consider a gravitational theory in $d$ dimensions with a gauge symmetry with coupling $g$. Then

**Electric Weak Gravity Conjecture.** There must be at least one charged state with charge $q$ and mass $m$ such that its charge to mass ratio is greater or equal to that of a large semiclassical extremal black hole. We will call such a particle "superextremal".

[3] Another way to make the theory consistent with the no global symmetry principle is to gauge this symmetry by coupling $F$ with a 2-form field.

[4] See [78, 79] for reviews, and references therein. See also [80] for a recent proof in the bosonic string theory from a worldsheet approach.

**Magnetic Weak Gravity Conjecture.** There must be a magnetically charged magnetic monopole with charge to mass ratio greater or equal to that of an extremal, magnetically charged black hole.

In the case of a $U(1)$ gauge theory, with $g$ defined in such a way to have particles with integer quantized charge

$$S = \int d^d x \sqrt{-g} \left( \frac{(M_{\mathrm{p},d})^{d-2}}{2} R_{(d)} - \frac{1}{4g^2} F^2 + ... \right) . \tag{2.3.1}$$

then these two statements can be interpreted as follows.

- Electric WGC: there must exist a particle in the spectrum of the theory such that

$$m \leq \sqrt{\frac{d-2}{d-3}} g q (M_{\mathrm{p},d})^{\frac{d-2}{2}} . \tag{2.3.2}$$

- Magnetic WGC: from the low energy perspective, the monopole scale is associated with the cutoff of validity of the EFT $\Lambda$, such that it is upper bounded by the gauge coupling

$$\Lambda \lesssim g (M_{\mathrm{p},d})^{\frac{d-2}{2}} .^{5} \tag{2.3.3}$$

In particular, the electric version of the WGC should hold because one should allow any black hole of mass $M$ and physical charge $Q$ under a gauge group to discharge by decaying into some products with masses $m_i$ and charges $q_i$ (everything expressed in Planck units), which by energy and charge conservation should satisfy

$$M \geq \sum_i m_i \quad Q = \sum_i q_i, \tag{2.3.4}$$

so that

$$\frac{M}{Q} \geq \frac{\sum_i m_i}{Q} = \frac{\sum_i \frac{m_i}{q_i} q_i}{Q} \geq \left( \frac{m}{q} \right)_{min} \cdot \frac{\sum_i q_i}{Q} = \left( \frac{m}{q} \right)_{min} , \tag{2.3.5}$$

which is exactly the statement of the electric WGC if we consider that the biggest charge-to-mass ratio for "physical" black holes is $\frac{Q}{M} = 1$ for an extremal black hole charged under a $U(1)$. The bound (2.3.2) with reversed sign of the inequality holds in the case of Reissner-Nordström (RN) black holes charged under a 1-form gauge field. Indeed, the RN black hole solution for $d \geq 4$ is characterized by

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2, \quad f(r) = 1 - \frac{2\mu}{r^{d-3}} + \frac{\tilde{q}}{r^{2(d-3)}} \tag{2.3.6}$$

in spherical coordinates, where $d\Omega_{d-2}^2$ is the line element of the $d-2$ dimensional unit sphere. If the black hole has mass $M$ and $U(1)$ charge $Q$, the parameters of the theory enter the black hole solution as

$$\mu = \frac{8\pi G_{(d)} M}{(d-2) V_{d-2}}, \ \tilde{q} = \frac{8\pi G_{(d)} Q^2}{(d-3)(d-2) V_{d-2}^2} , \tag{2.3.7}$$

[5] This can be deduced from known example, see for instance [25].

where $G_{(d)}$ is the $d$ dimensional Newton's constant

$$\frac{1}{8\pi G_{(d)}} = (M_p^{(d)})^{d-2}\,, \tag{2.3.8}$$

and $V_{d-2}$ is the volume of the $d-2$ dimensional unit sphere

$$V_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\,. \tag{2.3.9}$$

This solution has two coordinate singularities

$$r_\pm^{d-3} = \mu \pm \sqrt{\mu^2 - \tilde{q}}\,, \tag{2.3.10}$$

$r_+$ corresponding to the horizon, and in order to satisfy the cosmic censorship conjecture one needs $\mu^2 \geq \tilde{q}$, namely

$$M^2 \geq \frac{d-2}{d-3} Q^2 (M_{\mathrm{p},d})^{d-2}\,, \tag{2.3.11}$$

where the equality corresponds to extremal black holes. Then, the condition (2.3.2) is indeed describing a superextremal particle.

This expression can be further generalized to the case of black hole solutions charged under a $p$-form field with a massless dilaton field $\phi$ coupling to the gauge field as $\sim e^{\alpha\phi}F^2$ (see for instance [18]). The WGC in dimension $d$ then requires the existence of a $(p-1)$-brane of tension $T$ and integer charge $Q$ such that

$$8\pi G_{(d)} \gamma_{p,d}(\alpha) T^2 \leq g^2 Q^2, \tag{2.3.12}$$

where the convention is that the field $F_{\mu_1\ldots\mu_p}$ should have mass dimension $+p$ and be normalized in such a way that the charges are integers, and

$$\gamma_{p,d}(\alpha) \equiv \left[\frac{\alpha'}{2} + \frac{p(d-p-2)}{d-2}\right]^{-1}\,, \tag{2.3.13}$$

with the convention $\gamma \equiv \gamma_{1,d+1}(\alpha)$.

The meaning of this conjecture is precisely that gravity is not strong enough to build a stable structure; indeed, black holes (which are intrinsically characterized by a mass greater than their charge) are unstable.

In general, in dimensional reductions more than one $U(1)$ factor appear. In the case of multicharged particles, it is not enough to fulfil the WGC separately for each $U(1)$ [81], but a stronger statement is needed. This can again be argued in terms of black hole decays. Indeed, consider a black hole with mass $M$ charged under $U(1)^N$ gauge symmetry with charge vector

$$\vec{Q} = (Q_1, ..., Q_N)\,, \tag{2.3.14}$$

and suppose that it decays into $n_i$ particles of kind $i$, each having charge $\vec{q_i}$ and mass $m_i$. In Planck units it must hold

$$M \geq \sum_i n_i m_i \text{ and } \vec{Q} = \sum_i n_i \vec{q_i}\,. \tag{2.3.15}$$

Let

$$\sigma_i = \frac{n_i m_i}{M} \tag{2.3.16}$$

be the fraction of the initial mass that is converted in the species $i$ in the final state, which for energy conservation obey $\sum_i \sigma_i \leq 1$. Then the charge-to-mass vector $\vec{Z}$ of the black hole satisfies

$$\vec{Z} \equiv \frac{\vec{Q}}{M} = \frac{\sum_i n_i \vec{q_i}}{M} = \sum_i \frac{\vec{q_i}\sigma_i}{m_i} \equiv \sum_i \sigma_i \vec{z}_i, \text{ where } \vec{z}_i \equiv \frac{\vec{q_i}}{m_i}\,. \tag{2.3.17}$$

The second hand side is a vector in a region of the space of charge-to-mass ratios $\vec{z}$ which is delimited by the convex hull whose vertices are the $\vec{z}_i$ vectors of the particles in the spectrum of the theory. Equation (2.3.17) then tells that states that are able to decay are those whose charge-to-mass ratio vector $\vec{Z}$ lies inside such a convex hull, while the ones whose $\vec{Z}$ lies outside the convex hull are stable. This leads to the

**Convex Hull Condition (CHC).** In a theory with multiple $U(1)$ gauge symmetries, the convex hull defined by the $\vec{z}_i$ of all the fundamental particles in the spectrum must contain the black hole region.[6]

Semiclassically, the black hole region describing subextremal and extremal black holes is the unit ball $|\vec{Z}| \leq 1$, and this is the case presented in Figure 2.2. On the right, the WGC is satisfied marginally for each single $U(1)$, but the CHC is not, because the black hole region (shaded) is not fully contained in the convex hull, so that the black holes with $\vec{Z}$ in the outer region cannot decay. On the left, the CHC is satisfied, which makes it clear that, at least in a theory with a finite number of fundamental particles, in order for them to be final states of black hole decay their $z_i$ should be strictly larger than the extremal value. In particular, in the case of a $U(1)^N$ gauge symmetry and the black hole region given by the unit ball, the condition is $z_i > \sqrt{N}$ instead of $z_i \geq 1$, which would be valid valid for a single $U(1)$.

### 2.3.1 The Lattice WGC

The WGC as stated in the previous section is not preserved under dimensional reduction for an arbitrary value of the compactification radii [18]. Indeed, compactification on $S^1$ of a theory with metric $g_{\mu\nu}$ and $B$-field $B_{\mu\nu}$ leads to two additional $U(1)$ gauge symmetries in the lower dimensional theory. The gauge fields are $Z_\mu = g_{\mu R}$ (known as graviphoton) and $W_\mu = B_{\mu R}$[7] and the charged particles under them are respectively the momentum and winding states; from the lower dimensional point of view they are extremal states, which marginally satisfy the WGC if considered seprately. As explained above, the CHC is then not satisfied, since a black hole which is charged under both these $U(1)$'s cannot in general decay.
As an example, consider a $D$-dimensional $U(1)$ gauge theory of the kind (2.3.1) with gauge coupling $g_D$ with only one state $(m_0, q_{F,0})$ satisfying the WGC (2.3.12)

$$z_0 \equiv g_D (M_{\mathrm{p},D})^{\frac{D-2}{2}} \gamma^{\frac{1}{2}} \frac{q_{F,0}}{m_0} \geq 1\,, \tag{2.3.18}$$

[6] One speaks generically of "black hole region" because its shape depends on the couplings in the theory -possibly with dilaton-like fields.

[7] $R$ refers to the polarization along the compact direction.

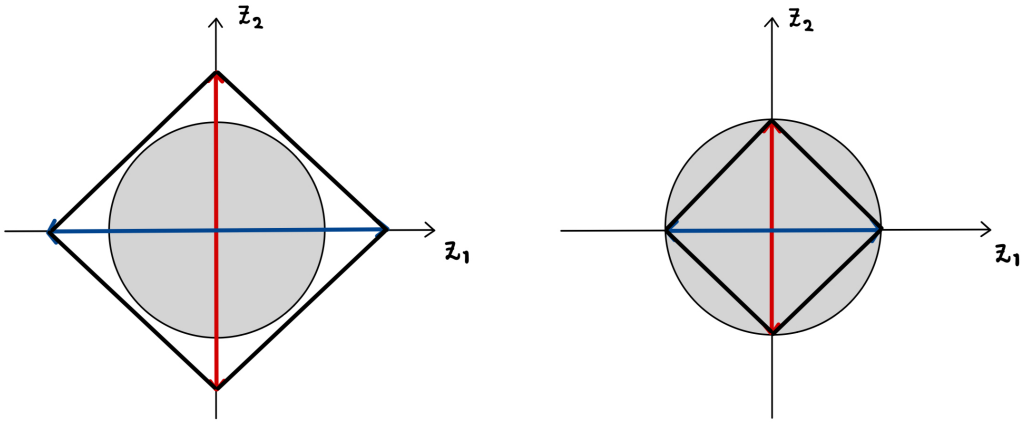



**Figure 2.2:** Convex hull condition in the case of $U(1)^2$ for two different sets of only two particles $\vec{z}_{1,2}$ (1 in blue, 2 in red) in the spectrum.

and compactify it on a $S^1$ of radius $R$, in such a way that the $d = (D-1)$ dimensional gauge theory has a graviphoton $U(1)$ symmetry, which marginally satisfies the WGC, in addition to the $U(1)$ coming from the dimensional reduction of the $D$-dimensional photon.
In the absence of axions, a generic state with mass $m$ in the $d$-dimensional theory is characterized by

$$\vec{z} = \frac{1}{m}\left(g_d (M_p^{(d)})^{\frac{d-2}{2}} \gamma^{\frac{1}{2}} q_F, \frac{q_{KK}}{R}\right) \equiv (z_F, z_{KK})\,, \tag{2.3.19}$$

where $q_F$ is the charge under the $D$-dimensional photon and $q_{KK} = n$ is the charge under the graviphoton, given by the quantized internal momentum, and in this notation the black hole region for such a theory is $z_F^2 + z_{KK}^2 \leq 1$.
In particular, the particle $(m_0, q_0)$ satisfying the WGC in $D$-dimensions has, in the compactified theory, a full tower of KK modes with mass

$$m^2 = m_0^2 + \frac{n^2}{R^2}\,, \tag{2.3.20}$$

and from (2.3.19) the charge-to-mass-ratio of the copy with internal momentum $n$ is

$$\vec{z}_{(n)} = \frac{(m_0 R z_0, n)}{\sqrt{(m_0 R)^2 + n^2}}\,, \tag{2.3.21}$$

which lie on $\frac{z_F^2}{z_0^2} + z_{KK}^2 = 1$, shown in Figure 2.3 (not shaded) together with the black hole region (shaded). The states $\vec{z}_{(n)}$ (in red) discretely populate the ellipsoid, becoming more and more dense towards $(0, \pm 1)$, so the convex hull they define does not in general entirely contain the black hole region. In particular, for each value of $z_0$ there exists one value of the radius below which the CHC is not satisfied [18].
In order for the WGC to be robust under toroidal compactification and valid in every point of moduli space, an infinite number of states must exist that satisfy the WGC in the higher dimensional theory, which leads to the following strong version

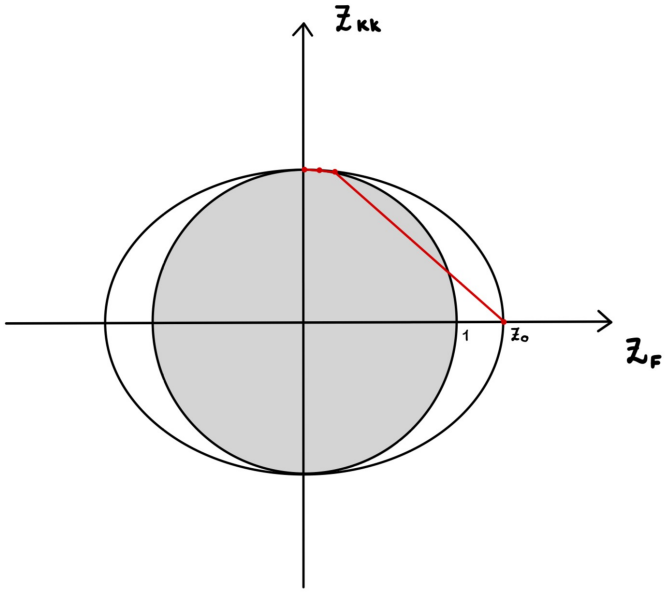



**Figure 2.3:** Example in which the CHC does not hold in the case of dimensional reduction from $D$ to $d = D-1$ dimensions with one $U(1)$ for which the WGC is satisfied in $D$ dimensions, and the second $U(1)$ associated to the graviphoton.

**Lattice Weak Gravity Conjecture (LWGC) [18, 19].** In the presence of multiple $U(1)$ gauge groups, for every point $\vec{q}$ in the charge lattice $\Gamma$ of a theory which can be consistently coupled to quantum gravity there must exist a superextremal particle.

For completeness, let us mention that under compactification on orbifolds or in the presence of Wilson lines, the LWGC proves not to hold on the whole charge lattice, but only on a finite index sublattice. The refined statement is then the

**Sublattice Weak Gravity Conjecture [19].** For a theory with charge lattice $\Gamma$ there exists a sublattice $\Gamma_{WGC}$ of finite coarseness such that for each $\vec{q} \in \Gamma_{WGC}$ there exists a superextremal particle of corresponding charge.

## 2.4 The Repulsive Force Conjecture

Related (but not equivalent) to the LWGG there is another statement, preserved under dimensional reduction, known as the

**Lattice Repulsive Force Conjecture (LRFC) [20].** In order for a multiple-$U(1)$ gauge theory with charge lattice $\Gamma$ to be coupled consistently to quantum gravity, in any site $\vec{q} \in \Gamma$ there exists a self-repulsive particle. A self-repulsive particle is characterised by a long range repulsive force due to the gauge bosons mediation with another identical particle which is at least as strong as the sum of the attractive ones.

Generically, in $D$ dimensions the long range force between two particles of masses $M_1$, $M_2$ with charges $Q_{F,i;1}$, $Q_{F,i,2}$ under the $U(1)^N$ gauge fields ($i = 1, ..., N$) and $\mu_{\alpha;1}$, $\mu_{\alpha,2}$ ($\alpha = 1, ..., n$)

under the interactions mediated by the $n$ neutral and massless scalars, takes the form[8]

$$F_{12} = \frac{k^{ij} Q_{F,i;1} Q_{F,j;2}}{r^{D-2}} - \frac{G M_1 M_2}{r^{D-2}} - \frac{g^{\alpha\beta} \mu_{\alpha;1} \mu_{\beta;2}}{r^{D-2}} \tag{2.4.1}$$

for $k^{ij}$ and $g^{\alpha\beta}$ given by the kinetic terms of the gauge fields and of the massless scalars respectively.
The LRFC amounts to ask that each point of the charge lattice is populated by a particle such that $F_{11} \geq 0$, in such a way that a bound state among such particles is not guaranteed.[9]
From (2.4.1) one can argue that the LRFC directly implies the LWGC only in the case of no neutral massless scalars in the theory, but in general this is not guaranteed. Nevertheless, even though in the general case the two theories do not imply each other, it is argued [20] that it is very unlikely that only one of them is satisfied for a given system.
Again, a refined version of the LRFC has been proposed, which is valid under any Ricci-flat compactification. It is the

**Sublattice Repulsive Force Conjecture (SRFC) [20].** In order for a multiple-$U(1)$ gauge theory with charge lattice $\Gamma$ to be coupled consistently with quantum gravity, there must exist an integer $n > 0$ such that for any site $\vec{q} \in \Gamma$ there exists a self-repulsive particle of charge $n\vec{q}$.

In particular, in the case of an asymptotic weak coupling point in moduli space, the SRFC and the SLWGC should become equivalent [82].

It is worth to mention that the connection that exists between the SDC and the WGC/RFC at the infinite distance points in moduli space at which a gauge couplings goes to zero has been studied in the literature [31,40,82], as it allows to connect the tower of light particles predicted by the former with the charged states becoming light as the gauge coupling decreases, predicted by the latter. In light of the magnetic version of the WGC, the fact that the cutoff of the EFT vanishes with the gauge coupling is then naturally associated with infinitely many massless states, predicted both by the SDC and by the LWGC/LRFC, as can be seen in explicit string theoretic examples.

[8] This holds strictly speaking only in the case of vanishing cosmological constant.
[9] A special case is given by BPS states, which satisfy the zero-force condition and thus are by definition self repulsive.

# Part II

# Affine algebras at the boundaries of moduli spaces

# Chapter 3

# The heterotic and CHL String Theories

The main goal of this Chapter is to introduce the heterotic and the CHL string theories, that will be considered in the following for what concerns infinite distance limits in moduli space in relation to symmetry enhancements. We will review their bosonic construction in order to set the notation, and explain the symmetry enhancement patterns in the bulk of moduli space to present the tools which will be used in the following discussions.

## 3.1 The heterotic string theory

The heterotic string theory [83–85][1] is a theory of closed strings in 10 dimensions with $(\mathcal{N}, \bar{\mathcal{N}}) = (1, 0)$ supersymmetry constructed from two independent sectors:

- the right-moving sector is the one of the superstring, which propagates in 10 space-time dimensions $X^M(\bar{\tau})$, where $M = 0, ..., 9$ and $\bar{\tau}$ is the anti-holomorphic coordinate on the string worldsheet. Due to supersymmetry, they come together with 10 fermionic superpartners, $\psi^M(\bar{\tau})$.

- the left-moving one of the bosonic string, consistently defined in a 26-dimensional target space. In the bosonic constructions, they are split like $\{X^M(\tau), X^I(\tau)\}$, where again $M = 0, ..., 9$ are the holomorphic counterpart of the bosonic sector of the superstring, and $I = 1, ..., 16$ are the 16 additional bosons that are compactified on a torus.

Being compact, the 16 $X^I(\tau)$ have discrete momenta $\pi^I$ which belong to an Euclidean lattice $\Gamma_{16}$, generated by $e_i^I \in \Gamma_{16}$, such that

$$\pi^I = \pi^i e_i^I, \quad \pi^i \in \mathbb{Z}\,, \tag{3.1.1}$$

and the metric of $\Gamma_{16}$ is

$$\kappa_{ij} = \sum_{I=1}^{16} e_i^I e_j^I\,, \tag{3.1.2}$$

[1] Here we follow the presentation of [86].

where the contraction with respect to the flat metric $\delta_{IJ}$ is understood.
From the requirement of modular invariance of the partition function, the lattice $\Gamma_{16}$ defining the heterotic torus must be even and self dual [87], namely

$$\pi \in \Gamma_{16} \implies |\pi|^2 \equiv \sum_{i,j} \pi^i \pi^j \kappa_{ij} \in 2\mathbb{Z} \quad \text{and} \quad \Gamma_{16} = \Gamma_{16}^* . \tag{3.1.3}$$

There are only two 16 dimensional Euclidean lattices with these properties (3.1.3), which are the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$, $\Gamma_{\mathrm{Sp}(32)}$, and the direct product of two copies of the root lattice $\Gamma_{\mathrm{E}_8}$ of $\mathrm{E}_8$, namely $\Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8}$, so there are two heterotic 10 dimensional strings with associated gauge groups respectively $\mathrm{Spin}(32)/\mathbb{Z}_2$ and $\mathrm{E}_8 \times \mathrm{E}_8$.[2] At the level of the algebra, this can be shown by determining the representation of the massless gauge bosons, as follows. Since they belong to the NS sector of the theory, we will focus on this.
In the bosonic sector, the spectrum reads

$$\alpha' m_L^2 = |\pi|^2 + 2(N-1) \tag{3.1.4}$$

where $N$ is the number of left-moving excitations, $N \in \mathbb{N}_+$, while in the right supersymmetric NS sector[3]

$$\alpha' m_{R;\mathrm{NS}}^2 = 2\left(\bar{N} - \frac{1}{2}\right) , \tag{3.1.5}$$

where $\bar{N}$ is the number of right-moving excitations, $\bar{N} \in \mathbb{N} + \frac{1}{2}$. Tensoring the two sectors together, the spectrum of the NS sector of the theory arranges with respect to the mass formula

$$\alpha' M_{\mathrm{NS}}^2 = \alpha'(m_L^2 + m_{R;\mathrm{NS}}^2) = |\pi|^2 + 2\left(N + \bar{N} - \frac{3}{2}\right) \tag{3.1.6}$$

and the Level Matching Condition (LMC) for physical NS states reads

$$|\pi|^2 + 2\left(N - \bar{N} - \frac{1}{2}\right) = 0 . \tag{3.1.7}$$

In 10 dimensions, the massless states in the NS sector, which are space-time bosons, are the following.

- $\pi = 0$, $N = 1$, $\bar{N} = \frac{1}{2}$:
  - The gravitational sector

$$\alpha_{-1}^M \bar{\psi}_{-\frac{1}{2}}^N |0\rangle_{NS} \longrightarrow g_{MN},\, B_{MN},\, \Phi . \tag{3.1.8}$$

  - The Cartan sector of the gauge group

$$\alpha_{-1}^I \bar{\psi}_{-\frac{1}{2}}^M |0\rangle_{NS} \longrightarrow A_M^I . \tag{3.1.9}$$

- $|\pi_\alpha|^2 = 2$, $N = 0$, $\bar{N} = \frac{1}{2}$: these are the gauge bosons associated with the ladder operators

[2] They are both 496-dimensional groups of rank 16.
[3] For completeness, the Ramond sector mass reads $\alpha' m_{R;\mathrm{R}}^2 = 2\bar{N}$.

with roots $\pi_\alpha$ of either $D_{16}$ or $\mathrm{E}_8 \oplus \mathrm{E}_8$, $\alpha = 1, ..., 480$

$$\bar{\psi}^M_{-\frac{1}{2}} |0, \pi_\alpha\rangle_{NS} \longrightarrow A^\alpha_M \,, \tag{3.1.10}$$

and it is not hard to see that, once the lattice $\Gamma_{16}$ is fixed to be either $\Gamma_{\mathrm{Sp}(32)}$ or $\Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8}$, the spectrum arranges in representations of the corresponding groups also at the massive level, such that indeed the theories have $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $\mathrm{E}_8 \times \mathrm{E}_8$ gauge symmetry, with massless gauge bosons (3.1.9) and (3.1.10). The bosonic part of the sigma model for the heterotic theory is

$$\begin{aligned} S_1 = &- \frac{1}{4\pi\alpha'} \int d^2\sigma (\eta^{\rho\sigma} G_{MN} + \epsilon^{\rho\sigma} B_{MN}) \partial_\rho X^M \partial_\sigma X^N \\ &- \frac{1}{8\pi} \int d^2\sigma (\eta^{\rho\sigma} \partial_\rho X^I \partial_\sigma X^J \delta_{IJ} + 2\epsilon^{\rho\sigma} A^I_M \partial_\rho X^M \partial_\sigma X^I) \,, \end{aligned} \tag{3.1.11}$$

where $\rho$, $\sigma = 1, 2$ label the worldsheet coordinates, which are taken to be dimensionless. The first line is the universal contribution to the action in closed string theories, while the second line the contribution of the 16 chiral bosons $X^I$ compactified on the maximal torus of the gauge group, and in particular their coupling to a background gauge field $A^I_M$. The $X^I$ are taken to be dimensionless, so that the gauge fields have mass dimension +1. In their coupling, one could equivalently choose to use $\eta^{\alpha\beta}$ instead of $\epsilon^{\alpha\beta}$, since the difference vanishes for chiral bosons $X^I$

$$(\partial_\tau - \partial_\sigma) X^I = 0 \,. \tag{3.1.12}$$

Given $\epsilon^{\tau\sigma} = -1$ and $\eta^{\tau\tau} = -1$

$$\begin{aligned} &\int d^2\sigma (\epsilon^{\alpha\beta} A^I_M \partial_\alpha X^M \partial_\beta X^I - \eta^{\alpha\beta} A^I_M \partial_\alpha X^M \partial_\beta X^I) \\ &= \int d^2\sigma A^I_M (\partial_\tau - \partial_\sigma) X^I (\partial_\tau + \partial_\sigma) X^M = 0 \,. \end{aligned} \tag{3.1.13}$$

Our normalization is such that

$$X^I(z) X^J(w) \sim -\delta^{IJ} \log(z - w) \,, \tag{3.1.14}$$

$$X^M(z, \bar{z}) X^N(w, \bar{w}) \sim -\frac{G^{MN} \alpha'}{2} \log |z - w|^2 \,. \tag{3.1.15}$$

### 3.1.1 $S^1$ compactification of the heterotic String

Let us start by considering the compactification of the heterotic string on the background $\mathbb{R}^{1,8} \times S^1$, and $M = \{\mu, 9\}$ with $\mu = 0, ..., 8$. The circle is parameterised by the (dimensionful) coordinate $x^9 \equiv y \sim y + 2\pi\sqrt{\alpha'}$, with a constant background metric $G_{99} = \frac{R^2}{\alpha'}$. From now on, unless specified otherwise, we will set $\alpha' = 1$.
The boundary condition for the string is

$$Y(\tau, \sigma + 2\pi) = Y(\tau, \sigma) + 2\pi \mathrm{w} \,, \tag{3.1.16}$$

where $\mathrm{w} \in \mathbb{Z}$ is the winding number, namely the number of times the string wraps the circle. It can be seen as a topological charge, since strings with different w's cannot be deformed one into the other without breaking them.
In heterotic string compactifications, there is the possibility of a non-trivial background for the gauge fields, with constant value in the compact directions and vanishing field strength

$$F^I_{MN} = \partial_{[M} A^I_{N]} + f^I_{JK} A^J_M A^K_N = 0\,, \tag{3.1.17}$$

where $f^I_{JK}$ are the structure constants of the gauge algebra. Setting $F^I_{MN} = 0$ impose a restriction to commuting gauge fields $A^I$, namely to the Cartan subgroup, so that generically the symmetry is broken to $U(1)^{16}$ and the background is specified by the 16 scalars $A^I_9$[4], which are moduli of the theory together with the circle radius $R$. The gauge background implies the presence of Wilson lines, defined on a closed path $\gamma$ as follows

$$W(\gamma) = \mathcal{P}\exp\left(i \oint_\gamma A\right)\,, \tag{3.1.18}$$

where $\mathcal{P}$ is the path ordering. If $\gamma$ is non-contractible, then a constant one-form $A$ gives a non-trivial Wilson line, and since then the Wilson line depends only on the homotopy class of the loop, there are as many Wilson lines (valued in $U(1)^{16}$) as non-contractible loops in the internal manifold, so one in the case at hand. This is why $A^I$ is referred to as "Wilson line".
The solution of the equations of motion for (3.1.11) compatible with the boundary condition (3.1.16) is

$$Y(\tau,\sigma) = \frac{1}{\sqrt{2}}(Y_L(\tau+\sigma) + Y_R(\tau-\sigma)) = y_0 + p^9\tau + \mathrm{w}\sigma + \text{ oscillators}\,, \tag{3.1.19}$$

$$X^I(\tau+\sigma) = x^I + p^I(\tau+\sigma) + \text{ oscillators}\,, \tag{3.1.20}$$

and by defining $|A|^2 \equiv \delta_{IJ} A^I A^J$, the quantized canonical momenta read

$$\pi_9 = p_9 + A^I \pi_I - \frac{1}{2}|A|^2 \equiv \mathrm{n} \in \mathbb{Z}\,, \tag{3.1.21}$$

$$\pi^I = p^I - A^I \mathrm{w} \in \Gamma_{16}\,, \tag{3.1.22}$$

with n the momentum number along the circle. In particular, from (3.1.19)

$$p_9 = p_L + p_R\,, \tag{3.1.23}$$

$$\frac{p_L - p_R}{\sqrt{2}} = \mathrm{w}R^2\,. \tag{3.1.24}$$

[4]When considering the $S^1$ compactification, for the sake of brevity we will drop the index 9, switching to the notation $A^I$.

The momentum along the maximal torus, and the left/right (L/R) momenta along the circle then are, respectively [88]

$$p_I = \pi_I + A_I \mathrm{w}\,, \tag{3.1.25}$$

$$p_{L,R} = \frac{1}{\sqrt{2}}\left(\mathrm{n} \pm \mathrm{w}R^2 - A^I\pi^I - \frac{\mathrm{w}|A|^2}{2}\right). \tag{3.1.26}$$

For any value of the moduli, the momentum vector $\boldsymbol{p} = (p_R, p_L, p^I) \equiv (p_R, \boldsymbol{p}_L)$ spans an even and self dual Lorentzian lattice with signature $(1, 17)$ endowed with the scalar product

$$\boldsymbol{p}\cdot\boldsymbol{p} \equiv \boldsymbol{p}_L^2 - p_R^2 = 2\mathrm{wn} + |\pi|^2 \in 2\mathbb{Z}\,. \tag{3.1.27}$$

This is the so called Narain lattice $\Gamma_{(1,17)}$, and it is unique up to $O(1, 17, \mathbb{R})$ transformations. The mass of the NS states, which depends on the value of the moduli ($R$, $A^I$), is

$$M^2 = \boldsymbol{p}_L^2 + p_R^2 + 2\left(N + \bar{N} - \frac{3}{2}\right), \tag{3.1.28}$$

where the left and right oscillator numbers, respectively $N$ and $\bar{N}$, should satisfy the LMC

$$\boldsymbol{p}_L^2 - p_R^2 + 2\left(N - \bar{N} - \frac{1}{2}\right) = 0\,, \tag{3.1.29}$$

which from (3.1.27) is moduli-independent $R$ and $A^I$. In particular, the states that are massless everywhere in moduli space are the $N = 1$, $\bar{N} = \frac{1}{2}$, $\boldsymbol{p}_L = 0$, $p_R = 0$ states, which split according to their nine-dimensional indices giving rise to the following spectrum.

- The gravitational sector:

$$\alpha^\mu_{-1}\bar{\psi}^\nu_{-\frac{1}{2}}\,|0\rangle_{NS} \longrightarrow g_{\mu\nu}, B_{\mu\nu}, \Phi\,. \tag{3.1.30}$$

- The vector bosons:

$$\alpha^\mu_{-1}\bar{\psi}^9_{-\frac{1}{2}}\,|0\rangle_{NS}\ ,\ \alpha^9_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}\,|0\rangle_{NS}\ ,\ \alpha^I_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}\,|0\rangle_{NS} \longrightarrow (g_{\mu 9} \mp B_{\mu 9}),\ A_\mu{}^I\,, \tag{3.1.31}$$

  giving in general a $U(1)_L^{17} \times U(1)_R$ symmetry.

- The scalars:

$$\alpha^I_{-1}\bar{\psi}^9_{-\frac{1}{2}}\,|0\rangle_{NS}\ ,\ \alpha^9_{-1}\bar{\psi}^9_{-\frac{1}{2}}\,|0\rangle_{NS} \longrightarrow A^I,\ g_{99}\,. \tag{3.1.32}$$

As we will point out in the following, by looking at the additional massless vectors, if any, at a given point in moduli space one can find the gauge symmetry of the theory at that point. Equivalently, the algebra can be characterised from the worldsheet analysis. Indeed, the algebra that is realised on the worldsheet in the left-moving sector at a given point in moduli space is determined by the set of holomorphic currents $(\mathrm{h}, \bar{\mathrm{h}}) = (1, 0)$, which is equivalent to the massless condition for spin $\mathrm{s} = \mathrm{h} - \bar{\mathrm{h}} = 1$ states.[5]

[5] Here we are considering only holomorphic currents because in heterotic toroidal compactifications only the $L$ sector of the gauge group can be enhanced.

### 3.1.2 Toroidal compactification of the heterotic String

The $\mathbb{R}^{1,9-d}\times T^d$ compactification of the heterotic String is just a straightforward generalization of the $S^1$ case presented in the previous Section, which we briefly discuss here in order to introduce the notation. Here, $M=\{\mu,i\}$ with $\mu=0,...9-d$ and $i=10-d,...,9$, and we write $X^M=(X^\mu,Y^i)$ with $Y^i$ the compact spacetime bosons.
Let $T^d$ be associated with a lattice $\Lambda$ generated by the vectors $e_i$, $i=1,...,d$, and dual lattice $\tilde{\Lambda}$ generated by the dual vectors $e^{*i}$. The background of the theory compactified on this space is specified by[6]

- the metric on the torus $G_{ij}=e^a_i\delta_{ab}e^b_j,\ a=1,...,d$,
- the internal B-field $B_{ij}$ and
- the Wilson lines $A^I_i,\quad I=1,...,16$.

The heterotic states are characterized by the winding and momentum numbers associated to the $T^d$ directions, respectively $\mathrm{w}^i,\ \mathrm{n}_i\in\mathbb{Z}$, and by the heterotic momenta $\pi^I$ along the 16 dimensional heterotic torus. These can be arranged in a charge vector

$$Z=(\pi^I,\ \mathrm{w}^i,\ \mathrm{n}_i)\ . \tag{3.1.33}$$

Again, the vector $\pi$ lies either in the root lattice of $\mathrm{E}_8\times\mathrm{E}_8$, or in the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$. Moreover, the states have the following right ($R$) and left ($L$) internal momenta along the $T^d$ directions,

$$\begin{aligned} p_{R,i}&=\frac{1}{\sqrt{2}}(\mathrm{n}_i-E_{ij}\mathrm{w}^j-\pi\cdot A_i)\,,\\ p_{L,i}&=\frac{1}{\sqrt{2}}(\mathrm{n}_i+(2G_{ij}-E_{ij})\mathrm{w}^j-\pi\cdot A_i)\,, \end{aligned} \tag{3.1.34}$$

where

$$E_{ij}=G_{ij}+\frac{1}{2}A_i\cdot A_j+B_{ij}\,, \tag{3.1.35}$$

and along the heterotic torus,

$$p^I=\pi^I+A^I_i\mathrm{w}^i\,. \tag{3.1.36}$$

The vector $\boldsymbol{p}=(p_{R,i}e^{*i}_a;p_{L,i}e^{*i}_a,p^I)=(p_{R,a};p_{L,a},p^I)\equiv(\boldsymbol{p}_R;\boldsymbol{p}_L)$ lies in an even and self-dual lattice $\Gamma_{d,d+16}$ with Lorentzian signature $(-_d,+_{d+16})$, as seen from the expression

$$\boldsymbol{p}^2=-\boldsymbol{p}_R^2+\boldsymbol{p}_L^2=2\mathrm{n}_i\mathrm{w}^i+|\pi|^2\in 2\,\mathbb{Z}\,. \tag{3.1.37}$$

The mass formula and the LMC for the NS sector of the spectrum then read

$$\begin{aligned} M^2&=\boldsymbol{p}_L^2+p_R^2+2\left(N+\bar{N}-\frac{3}{2}\right)\,,\\ 0&=\boldsymbol{p}_L^2-p_R^2+2\left(N-\bar{N}-\frac{1}{2}\right)\,, \end{aligned} \tag{3.1.38}$$

[6] One should also include the dilaton, but we are taking it to be fixed (and small).

where $N$ and $\bar{N}$ are respectively the left and right-moving oscillator numbers. These formulas determine the group of symmetries of the spectrum to be $O(d, d+16, \mathbb{Z})$, the T-duality group. Analogously to the $S^1$ case, for generic values of the moduli, the massless vectors generate a $U(1)_R^d \times U(1)_L^{16+d}$ gauge symmetry, and they are characterized by $N = 1, \bar{N} = \frac{1}{2}, \boldsymbol{p}_L = 0, \boldsymbol{p}_R = 0$:

$$\alpha^{\mu}_{-1}\bar{\psi}^{i}_{-\frac{1}{2}}\left|0\right\rangle_{NS} , \alpha^{i}_{-1}\bar{\psi}^{\mu}_{-\frac{1}{2}}\left|0\right\rangle_{NS} , \alpha^{I}_{-1}\bar{\psi}^{\mu}_{-\frac{1}{2}}\left|0\right\rangle_{NS} \longrightarrow (g_{\mu i} \mp B_{\mu i}),\ A_{\mu}{}^{I} . \tag{3.1.39}$$

The internal metric, B field and Wilson lines parametrise the moduli space

$$O(d, d+16, \mathbb{Z})\backslash\frac{O(d, d+16, \mathbb{R})}{O(d) \times O(d+16)} . \tag{3.1.40}$$

### 3.1.3 Moduli space and T-duality

As already mentioned, in dimension 16 the only two Euclidean integer even and self-dual lattices are $\Gamma_{\mathrm{Sp}(32)}$ or $\Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8}$ defined above. The charge lattice of the $S^1$ compactification of these two theories is the Narain lattice

$$\Gamma_{(1,17)} \simeq \Gamma_{(1,1)} \oplus \Gamma_{\mathrm{Sp}(32)} \simeq \Gamma_{(1,1)} \oplus \Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8} , \tag{3.1.41}$$

where $\Gamma_{(1,1)}$ is the even, two dimensional Lorentzian lattice.
One can show [89] that $(p, q)$ even integer self-dual Lorentzian lattices exist for $p - q = 8n$ with $n \in \mathbb{Z}$, and every such lattice can be obtained from a reference one $\Gamma^0$ through an $O(p, q, \mathbb{R})$ transformation, in this case $O(1, 17, \mathbb{R})$. Since the heterotic spectrum (3.1.28), (3.1.29) is invariant under separate rotations of the left and right momenta, $O(17)_L \times O(1)_R$, theories that differ by such a transformation are physically equivalent. Moreover, the theory is also invariant under the $O(1, 17, \mathbb{Z})$ T-duality, so that the global structure of the moduli space of the theory parametrising inequivalent backgrounds $(R,\ A^I)$ is

$$O(1, 17, \mathbb{Z})\backslash\frac{O(1, 17, \mathbb{R})}{O(17) \times O(1)} . \tag{3.1.42}$$

The uniqueness of the Narain lattice then implies that there exists an $O(1, 17)$ transformation interpolating between the $\mathrm{E}_8 \times \mathrm{E}_8$ theory on $S^1$ and the $\mathrm{Spin}(32)/\mathbb{Z}^2$ on $S^1$, as in (3.1.41) [90], namely the two theories belong to the same moduli space.
This structure, in the case of $T^d$ compactifications, allows for a $O(d, d+16, \mathbb{R})$-covariant formulation. The $O(d, d+16)$ invariant metric is

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_{d\times d} & 0 \\ \mathbb{1}_{d\times d} & 0 & 0 \\ 0 & 0 & \kappa_{IJ} \end{pmatrix} , \tag{3.1.43}$$

where $\kappa_{IJ}$ is the Killing metric of the Cartan subgroup of $D_{16}$ or $\mathrm{E}_8 \oplus \mathrm{E}_8$.
Defining

$$C_{ij} = B_{ij} + \frac{1}{2}A^I_i \kappa_{IJ} A^J_j , \tag{3.1.44}$$

and the generalised metric associated to this compactification

$$\mathcal{H} = \begin{pmatrix} G_{ij} + C_{li}G^{lk}C_{kj} + A^I_i A_{jI} & -G^{jk}C_{ki} & C_{ki}G^{kl}A_{lJ} + A_{iJ} \\ -G^{ik}C_{kj} & G^{ij} & -G^{ik}A_{kJ} \\ C_{kj}G^{kl}A_{lI} + A_{jI} & -G^{jk}A_{kI} & \kappa_{IJ} + A_{kI}G^{kl}A_{kJ} \end{pmatrix} , \tag{3.1.45}$$

the mass formula and LMC (3.1.38) can be recast in an $O(d, d+16)$ covariant form, respectively

$$M^2 = Z^t \mathcal{H} Z + 2\left(N + \bar{N} - \frac{3}{2}\right) , \tag{3.1.46}$$

$$0 = Z^t \eta Z + 2\left(N - \bar{N} - \frac{1}{2}\right) . \tag{3.1.47}$$

T-duality $O(1, 17; \mathbb{Z}) \ni T$ acts as

$$Z' = \eta^{-1} T \eta Z \, , \quad \mathcal{H}' = T \mathcal{H} T^T \tag{3.1.48}$$

and it has the following generators (see for instance [91]).

- Wilson line shifts by a vector $\Lambda \in \Gamma_{16}$

$$T_\Lambda = \begin{pmatrix} \mathbb{1}_{d\times d} & -\frac{1}{2}\Lambda\Lambda^t & \Lambda \\ 0 & \mathbb{1}_{d\times d} & 0 \\ 0 & -\Lambda^t & \mathbb{1}_{16\times 16} \end{pmatrix} . \tag{3.1.49}$$

- $B$-field shifts by an integer-valued antisymmetric matrix $\Theta$, $\Theta_{ij} \in \mathbb{Z}$

$$T_\Theta = \begin{pmatrix} \mathbb{1}_{d\times d} & \Theta & 0 \\ 0 & \mathbb{1}_{d\times d} & 0 \\ 0 & 0 & \mathbb{1}_{16\times 16} \end{pmatrix} . \tag{3.1.50}$$

- Lattice basis changes by $M \in GL(d, \mathbb{Z})$

$$T_M = \begin{pmatrix} M & 0 & 0 \\ 0 & (M^T)^{-1} & 0 \\ 0 & 0 & \mathbb{1}_{16\times 16} \end{pmatrix} . \tag{3.1.51}$$

- Rotation of the Wilson lines $R \in O(16, \mathbb{Z})$

$$T_M = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 & 0 \\ 0 & \mathbb{1}_{d\times d} & 0 \\ 0 & 0 & R \end{pmatrix} . \tag{3.1.52}$$

- The so-called "factorized dualities"

$$T_{D_i} = \begin{pmatrix} \mathbb{1}_{d\times d} - D_i & D_i & 0 \\ D_i & \mathbb{1}_{d\times d} - D_i & 0 \\ 0 & 0 & \mathbb{1}_{16\times 16} \end{pmatrix} , \tag{3.1.53}$$

where $D_i \in GL(d,\mathbb{Z})$ has all the components 0 except for a 1 in the $ii$ component.

- Transformations of the dual Wilson lines by $\Gamma \in \Gamma_{16}$

$$T_\Gamma = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 & 0 \\ -\frac{1}{2}\Gamma\Gamma^T & \mathbb{1}_{d\times d} & -\Gamma^T \\ \Gamma & 0 & \mathbb{1}_{16\times 16} \end{pmatrix}. \tag{3.1.54}$$

- Shifts by an integer-valued bivector $\beta^{ij} = -\beta^{ji} \in \mathbb{Z}$

$$T_\Gamma = \begin{pmatrix} \mathbb{1}_{d\times d} & 0 & 0 \\ \beta & \mathbb{1}_{d\times d} & 0 \\ 0 & 0 & \mathbb{1}_{16\times 16} \end{pmatrix}. \tag{3.1.55}$$

In particular, (3.1.53) is the generalisation of the radius inversion (2.1.7). Indeed, in the case of $S^1$ compactifications, its action on the generalized metric is

$$\mathcal{H}' = T_D \mathcal{H} T_D^T = \begin{pmatrix} \frac{1}{R^2} & -\frac{1}{2R^2}|A|^2 & -\frac{1}{R^2}A_J \\ -\frac{1}{2R^2}|A|^2 & R^2 + |A|^2 + \frac{1}{4R^2}|A|^4 & \frac{1}{2R^2}|A|^2 A_J + A_J \\ -\frac{1}{R^2}A_I & \frac{1}{2R^2}|A|^2 A_I + A_I & \kappa_{IJ} + \frac{1}{R^2}A_I A_J \end{pmatrix}, \tag{3.1.56}$$

namely it acts on the charges exchanging momentum and winding, and on the moduli as

$$R' = \frac{R}{R^2 + \frac{1}{2}|A|^2} \text{ and } A'^I = -\frac{A^I}{R^2 + \frac{1}{2}|A|^2}, \tag{3.1.57}$$

which reduces to $R \to \frac{1}{R}$ in the case vanishing Wilson line.

### 3.1.4 Symmetry enhancements in $O(d,d+16)/(O(d)\times O(d+16))$

In this Section, let us focus on the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic theory. The case of $\mathrm{Spin}(32)/\mathbb{Z}_2$ is analogous, by replacing $\Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8}$ with $\Gamma_{\mathrm{Sp}(32)}$.

As we said above, in order to characterise the gauge symmetry at a particular point of the moduli space one needs to consider the massless vectors, which correspond to the mediators of the gauge interaction. In particular, at a generic point of the moduli space this is $U(1)_R^d \times U(1)_L^{d+16}$ (3.1.39). Due to the moduli dependence of (3.1.38), it turns out that at fixed points of the T-duality group there can be additional massless vectors with $N = 0, \bar{N} = \frac{1}{2}, |\boldsymbol{p}_L|^2 = 2,\ \boldsymbol{p}_R = 0$, of the form

$$\bar{\psi}^\mu_{-\frac{1}{2}} |\pi_\alpha, \mathrm{w}^i, \mathrm{n}_i\rangle_{NS} \longrightarrow A^\alpha_\mu \tag{3.1.58}$$

for some particular charge vectors. These enhance the gauge group in the left sector, as classified in [91–93].

In particular, the internal momenta $\boldsymbol{p}$ of the vectors (3.1.58) are the roots of a group $G_r$ of rank $r \leq d+16$, which is a product of $ADE$ groups, whose weight lattice can be embedded in the Narain lattice. The gauge group of the theory is then $G_r \times U(1)_L^{d+16-r} \times U(1)_R$. For example, in the $S^1$ case for vanishing Wilson lines all the states with $N = 0, \bar{N} = \frac{1}{2}$ and

$\pi^I \in \Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8}$, $|\pi|^2 = 2$ are massless for any value of $R$. In particular, the massless vectors

$$\bar{\psi}^{\mu}_{-\frac{1}{2}} |0, \pi_\alpha\rangle_{NS} \longrightarrow A^{\alpha}_{\mu} \tag{3.1.59}$$

with $\alpha = 1, ..., 480$ the roots of $\mathrm{E}_8 \times \mathrm{E}_8$, together with the ones in (3.1.39) enhance the symmetry to $\mathrm{E}_8 \times \mathrm{E}_8 \times U(1)_L \times U(1)_R$.[7]
In the case of the circle compactification, all the enhancement algebras and the point in moduli space where they occur (up to T-dualities) are encoded in the Extended Dynkin Diagram (EDD) of $\Gamma_{(1,17)}$ [94] (as described for instance in [95]), displayed in Figure 3.1.

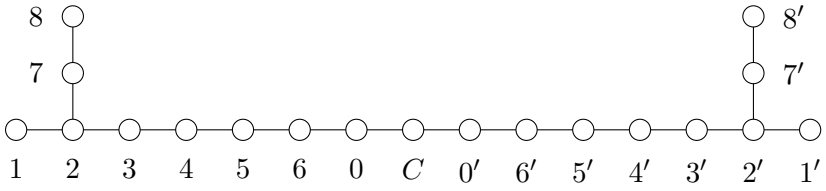



**Figure 3.1:** EDD of the $\Gamma_{(1,17)}$ lattice.

The labels to the nodes show how the $\mathrm{E}_8 \times \mathrm{E}_8$ lattice is embedded in $\Gamma_{(1,17)}$: the EDD is made of two extended $\mathrm{E}_8$ Dynkin diagrams (adding the highest root 0 to the Dynkin diagram of each $\mathrm{E}_8$) linked by a central node $C$. The primed indices are meant to distinguish the two copies of $\mathrm{E}_8$. The nodes are described by a charge vector

$$Z = (\mathrm{n}, \mathrm{w}, \pi^i, \pi^{i'}) \in \Gamma_{(1,17)}\,, \tag{3.1.60}$$

with $i = 1, ..., 8$, as follows

$$\begin{aligned} Z_i &= (0, 0, \alpha_i, 0_8)\,, \\ Z_0 &= (0, -1, \alpha_0, 0_8)\,, \\ Z_C &= (1, 1, 0_8, 0_8)\,, \\ Z_{0'} &= (0, -1, 0_8, \alpha_{0'})\,, \\ Z_{i'} &= (0, 0, 0_8, \alpha_{i'})\,. \end{aligned} \tag{3.1.61}$$

Here $\alpha_i$ are the simple roots of $\mathrm{E}_8$ and $\alpha_0$ is the highest root, as explicitly shown in the Table 3.1.
The primed quantities are taken to be equal to the unprimed ones, so they amount just to an identical copy of $\mathrm{E}_8$, with the same convention.
With the following definition for the imaginary root

$$\delta = (0, -1, 0_8, 0_8)\,, \tag{3.1.62}$$

the nodes 0 and 0' are embedded in $\Gamma_{(1,17)}$ respectively as

$$(0, 0, \alpha_0, 0_8) + \delta \;\text{ and }\; (0, 0, 0_8, \alpha_{0'}) + \delta\,. \tag{3.1.63}$$

[7] This is the case for all $R \neq 1$. If $R = 1$ the symmetry is enhanced to $(\mathrm{E}_8 \times \mathrm{E}_8 \times SU(2))_L \times U(1)_R$.

| $i$ | $\alpha_i$ |
|---|---|
| 1 | (1,-1,0,0,0,0,0,0) |
| 2 | (0,1,-1,0,0,0,0,0) |
| 3 | (0,0,1,-1,0,0,0,0) |
| 4 | (0,0,0,1,-1,0,0,0) |
| 5 | (0,0,0,0,1,-1,0,0) |
| 6 | (0,0,0,0,0,1,-1,0) |
| 7 | (-1,-1,0,0,0,0,0,0) |
| 8 | $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ |
| 0 | (0,0,0,0,0,0,1,-1) |

**Table 3.1:** Simple roots ($i = 1, ..., 8$) and highest root ($i = 0$) of $E_8$.

This means that the extension of the two copies of the $E_8$ Dynkin diagram that build the EDD is performed with the same imaginary root.
We could have equivalently defined the charge vector as

$$Z = (\mathrm{w}, \mathrm{n}, \pi^i, \pi^{i'}) \tag{3.1.64}$$

instead of (3.1.60), leaving (3.1.61) unchanged. In the first case, the nodes 0 and 0' have winding charge, in the second one momentum, and the relation (3.1.63) still holds. The two cases are related by the T-duality transformation (3.1.53).
All the possible enhancement groups, which are of the form $G_r \times U(1)^{17-r}$, are obtained by deleting $n = 19 - r$ nodes from the EDD in such a way to obtain the Dynkin diagram of the group $G_r$. The point or region in moduli space where this group arises is found by imposing that the remaining nodes satisfy the condition $\frac{p_R}{R} = 0$. Each node gives a constraint on the moduli, that we collect in Table 3.2. As explained in [95], the co-dimension one plane in moduli space corresponding to each node is a fixed plane under a given T-duality transformation. Points of symmetry enhancement are therefore fixed points of T-duality, and they arise at boundaries of a fundamental domain of moduli space.

| Node | Boundary of moduli space |
|---|---|
| $i$ | $\frac{A}{R} \cdot (\alpha_i, 0_8) = 0$ |
| 0 | $\frac{A}{R} \cdot (\alpha_0, 0_8) = R\left(1 + \frac{\lvert A\rvert^2}{2R^2}\right)$ |
| C | $R\left(\frac{1}{R^2} - \frac{\lvert A\rvert^2}{2R^2}\right) = R$ |
| 0' | $\frac{A}{R} \cdot (0_8, \alpha_{0'}) = R\left(1 + \frac{\lvert A\rvert^2}{2R^2}\right)$ |
| $i'$ | $\frac{A}{R} \cdot (0_8, \alpha_{i'}) = 0$ |

**Table 3.2:** Equations defining the co-dimension one boundary planes in moduli space corresponding to each node of the EDD, where the nodes 0 and 0', given in (3.1.61) have winding charge (i.e. $Z$ is that of Eq. (3.1.60)).

Note that the assignment of the nodes with momentum and/or winding to a boundary in moduli space changes if we interchange momentum and winding. For the conventions given by (3.1.64), the nodes given in (3.1.61) correspond instead to the boundaries given in Table 3.3. The original assignment is more convenient for analysing the region of small $R$, while the latter

is more suitable for large $R$. The point or region in moduli space where enhancements of the

| Node | Boundary of moduli space |
|---|---|
| $i$ | $\frac{A}{R}\cdot(\alpha_i, 0_8) = 0$ |
| 0 | $\frac{A}{R}\cdot(\alpha_0, 0_8) = -\frac{1}{R}$ |
| C | $R\left(\frac{1}{R^2} - \frac{\lvert A\rvert^2}{2R^2}\right) = R$ |
| 0' | $\frac{A}{R}\cdot(0_8, \alpha_{0'}) = -\frac{1}{R}$ |
| $i'$ | $\frac{A}{R}\cdot(0_8, \alpha_{i'}) = 0$ |

**Table 3.3:** Equations defining the co-dimension one boundary planes in moduli space corresponding to each node of the EDD, where the nodes 0 and 0', given in (3.1.61) have momentum charge (i.e. $Z$ is that of Eq. (3.1.64)).

form $G_r \times U(1)_L^{17-r} \times U(1)_R$ occur are found by imposing the $r$ equations given in Table 3.2 for the remaining nodes in the EDD (or analogously one can use the conventions in Table 3.3). If the node C is not deleted[8], these arise at a radius that is fixed in terms of the norm squared of the Wilson line, $R^2 = 1 - \frac{|A|^2}{2}$.

## 3.2 The CHL String

In this Section we recall the main features of the CHL string that we will use in the following to characterize the current algebra arising in the decompactification limit, this time focusing more on the worldsheet approach.

### 3.2.1 Orbifold construction of the CHL string

In the construction of [96], which is most suited to our analysis, the CHL theory [97] in nine dimensions is obtained as a $\mathbb{Z}_2$ orbifold of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory compactified on $S^1$. For our purposes it is sufficient to work with the Narain sector of the internal CFT, a theory with central charges $c = 17$ and $\bar{c} = 1$, which is described as follows in the convention $\alpha' = 1$.
Let $X^I(z)$ $(I = 1, ..., 16)$ be the holomorphic compact chiral bosons on the heterotic torus and $X^9(z, \bar{z})$ the non-chiral boson for the circle of radius $R$, normalized as

$$X^I(z)X^J(w) \sim -\delta^{IJ}\log(z-w)\,, \tag{3.2.1}$$

$$X^9(z,\bar{z})X^9(w,\bar{w}) \sim -\frac{1}{2R^2}\log|z-w|^2\,. \tag{3.2.2}$$

We decompose the non-chiral boson as

$$X^9(z,\bar{z}) = \frac{1}{\sqrt{2}}\left(X_L^9(z) + X_R^9(\bar{z})\right)\,. \tag{3.2.3}$$

We split the Cartan indices of the heterotic bosons between the two $\mathrm{E}_8$ factors as $I = (i, i+8)$, with $i = 1, .., 8$, and denote $X_1^i(z) = X^i(z)$ and $X_2^i(z) = X^{i+8}(z)$, so that the subscripts 1 and

[8] It is easy to see that for finite ADE groups of maximal rank ($r = 17$), node C cannot be deleted.

2 signify the copy of $E_8$. The orbifold action consists of the exchange $\mathcal{E}$ of the two $E_8$ factors, together with a half-shift $\mathcal{S}$ in the compact circle direction:

$$\mathcal{E}: \quad X_1^i(z) \leftrightarrow X_2^i(z)\,, \tag{3.2.4}$$

$$\mathcal{S}: \quad X^9(z,\bar{z}) \to X^9(z,\bar{z}) + \pi\,. \tag{3.2.5}$$

This is a symmetry of the heterotic $E_8 \times E_8$ theory on $S^1$ in the subspace of its moduli space

$$\mathcal{M}_{\rm het} = \frac{O(1,17;\mathbb{R})}{O(17;\mathbb{R})}\Big/ O(1,17;\mathbb{Z})\,, \tag{3.2.6}$$

characterized by a symmetric Wilson line $A = (a,a)$.[9] In 9 dimensions this is a freely acting orbifold, whose effect is to reduce the rank of the gauge group from 17 to 9, while not breaking any supersymmetry. The primary states in the Narain CFT are labelled by integer momentum n and winding w along the $S^1$ and by heterotic momenta $\pi = (\pi_1, \pi_2)$, and the orbifold acts on these as

$$\mathcal{E}\mathcal{S}\,|\mathrm{n},\mathrm{w},\pi_1,\pi_2\rangle = (-1)^{\mathrm{n}}\,|\mathrm{n},\mathrm{w},\pi_2,\pi_1\rangle\ . \tag{3.2.7}$$

The Hilbert space of the orbifold theory has the untwisted sector and a single twisted sector. In order to describe these spaces, it is useful to change the basis for the heterotic bosons from $\{X_1^i(z), X_2^i(z)\}$ to the symmetric and antisymmetric combinations

$$X_\pm^i(z) = \frac{1}{\sqrt{2}}\left(X_1^i(z) \pm X_2^i(z)\right) \xrightarrow{\mathcal{E}\mathcal{S}} \pm X_\pm^i(z)\,. \tag{3.2.8}$$

The states belonging to the untwisted sector are constructed from bosons with untwisted boundary conditions

$$X_\pm^i(e^{2\pi i}z) = X_\pm^i(z) + \frac{1}{\sqrt{2}}Q_\pm^i\,, \tag{3.2.9}$$

$$X^9(e^{2\pi i}z, e^{2\pi i}\bar{z}) = X^9(z,\bar{z}) + 2\pi\mathrm{w}\,, \tag{3.2.10}$$

where $Q_\pm \in \Gamma_8$ are fixed vectors in the $E_8$ root lattice, and $\mathrm{w} \in \mathbb{Z}$ is the winding number; in the Hilbert space of the Narain theory there are sectors labelled by each allowed choice of $\{Q_\pm, \mathrm{w}\}$. The untwisted states are the heterotic states that are invariant under the orbifold, and they have the following internal momenta $P = (p_+^i, p_-^i, p_L^9, p_R^9) \equiv (\mathbf{p}_L, p_R)$

$$\begin{aligned}
p_+^i &= \frac{1}{\sqrt{2}}(\rho^i + 2\mathrm{w}a^i)\,,\\
p_-^i &= \frac{1}{\sqrt{2}}(\pi_1^i - \pi_2^i)\,,\\
p_L^9 &= \frac{1}{\sqrt{2}R}(\mathrm{n} + R^2\mathrm{w} - a^2\mathrm{w} - \rho\cdot a)\,,\\
p_R^9 &= \frac{1}{\sqrt{2}R}(\mathrm{n} - R^2\mathrm{w} - a^2\mathrm{w} - \rho\cdot a)\,,
\end{aligned} \tag{3.2.11}$$

[9] Here $a$ is valued in the torus $\mathbb{R}^8/\Gamma_8$, where $\Gamma_8$ is the $E_8$ lattice.

where the momenta along the circle direction are taken in flat indices, and $\rho \in \Gamma_8$ can be written in terms of the heterotic momenta as $\rho^i = \pi_1^i + \pi_2^i$.
The untwisted spectrum with NS right movers, which is the one describing space-time bosons, is characterized by the following mass formula and level matching condition

$$\begin{aligned} M^2 &= \mathbf{p}_L^2 + p_R^2 + 2\left(N + \bar{N} - \frac{3}{2}\right) , \\ 0 &= \mathbf{p}_L^2 - p_R^2 + 2\left(N + \bar{N} - \frac{1}{2}\right) , \end{aligned} \tag{3.2.12}$$

where $N\,(\bar{N}) \in \mathbb{N}_+$ is the oscillator number in the left (right) moving sector.
In the sector twisted with respect to the $\mathbb{Z}_2$ orbifold action, closed strings have boundary conditions which are additionally twisted with respect to the $\mathcal{ES}$ transformation

$$X_\pm^i(e^{2\pi i} z) = \pm X_\pm^i(z) + \frac{1}{\sqrt{2}} Q'^i_\pm \,, \tag{3.2.13}$$

$$X^9(e^{2\pi i} z, e^{2\pi i}\bar{z}) = X^9(z,\bar{z}) + \pi + 2\pi \mathrm{m} = X^9(z,\bar{z}) + 2\pi \mathrm{w} \,, \tag{3.2.14}$$

where $Q'_\pm \in \Gamma_8$ and $\mathrm{m} \in \mathbb{Z}$, so that $\mathrm{w} \in \mathbb{Z} + \frac{1}{2}$. Being antisymmetric under the orbifold, in the twisted sector $X_-^i(z)$ has half-integer mode expansion

$$X_-^i(z) = \frac{1}{2\sqrt{2}} Q_-^i + \sqrt{\frac{1}{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{\alpha_{-;r}^i}{r z^r} \,. \tag{3.2.15}$$

The twisted states are characterized by the following internal momenta

$$\begin{aligned} p_+^i &= \frac{1}{\sqrt{2}} (\rho^i + 2\mathrm{w} a^i) \,, \\ p_-^i &= 0 \,, \\ p_L^9 &= \frac{1}{\sqrt{2}R} (\mathrm{n} + R^2 \mathrm{w} - a^2 \mathrm{w} - \rho \cdot a) \,, \\ p_R^9 &= \frac{1}{\sqrt{2}R} (\mathrm{n} - R^2 \mathrm{w} - a^2 \mathrm{w} - \rho \cdot a) \,, \end{aligned} \tag{3.2.16}$$

and the physical states have

$$\begin{aligned} M^2 &= \boldsymbol{p}_L^2 + p_R^2 + 2(N + \tilde{N} - 1) \,, \\ 0 &= \boldsymbol{p}_L^2 - p_R^2 + 2(N - \tilde{N}) \,, \end{aligned} \tag{3.2.17}$$

due to the different zero-point energy with respect to the untwisted sector.
Considering both sectors, the charges belong to the lattice $II_{1,9}$, which is associated to the Generalized Dynkin Diagram (GDD) $E_{10}$ [98]. By defining the charge vector of each state as

$$Z = |\mathrm{n}, 2\mathrm{w}, \rho\rangle \ , \tag{3.2.18}$$

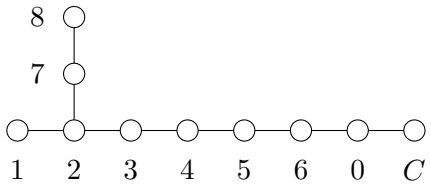



**Figure 3.2:** Generalized Dynkin Diagram of the $II_{(1,9)}$ lattice.

each node corresponds to, respectively

$$Z_i = |0, 0, \alpha_i\rangle \ , \tag{3.2.19}$$

$$Z_0 = |-1, 0, \alpha_0\rangle \ , \tag{3.2.20}$$

$$Z_C = |1, 1, 0\rangle \ , \tag{3.2.21}$$

where $i = 1, ..., 8$ and $\alpha^i$ is the $i$-th simple root of $\mathrm{E}_8$, as in Table 3.1.
The central node $C$ is associated to the twisted sector, as it has half-integer winding. The physical momenta of the corresponding states are determined by the polarization of this lattice, which depends on the point of the moduli space. For the CHL theory, it is given by

$$\mathcal{M}_{\mathrm{CHL}} = O(1, 9; \mathbb{Z}) \Big\backslash \frac{O(1, 9; \mathbb{R})}{O(9; \mathbb{R})} \, , \tag{3.2.22}$$

and it is parametrised by the radius $R$ and Wilson line $a$. $O(1, 9; \mathbb{Z})$ is the T-duality group, and it corresponds to the group of automorphisms of the lattice $II_{1,9}$ spanned by the charge vectors $Z$ of the physical states. A particularly interesting duality transformation is inherited from the radius inversion $R \to \frac{1}{R}$ duality of the Narain CFT, which in the CHL case interchanges states between the two sectors of the CHL orbifold. Its action on the moduli and charges is

$$\begin{aligned} R^2 + a^2 &\to \frac{4}{R^2 + a^2} \, , \\ a &\to \frac{2a}{R^2 + a^2} \, , \\ 2\mathrm{w} &\leftrightarrow \mathrm{n} \, , \\ \rho &\to -\rho \ , \end{aligned} \tag{3.2.23}$$

which implies that untwisted states with even momentum are mapped to untwisted states ($\mathrm{w} \in \mathbb{Z}$), while the ones with odd momentum are dual to twisted states ($\mathrm{w} \in \mathbb{Z} + \frac{1}{2}$).

### 3.2.2 Symmetry enhancements

In supersymmetric perturbative heterotic compactifications, space-time gauge bosons arise in both the right (supersymmetric) and left moving sectors of the string. The former are universal, namely they are present at each point in moduli space, and give a $\mathfrak{u}(1)^d$ anti-holomorphic current algebra in the case of $T^d$ compactifications. The latter are in one-to-one correspondence with holomorphic currents, which depend on the point of moduli space. In this analysis, we

will be concerned with the second set.
At a generic point $\{R, a\}$, they consist of

- the untwisted current
$$J^9(z) = i\partial X^9(z)\,, \tag{3.2.24}$$
corresponding to the state generated by the left oscillator $\alpha^9_{-1}\,|0\rangle$,

- the 8 untwisted currents
$$J^i_+(z) = i(\partial X^i_1(z) + \partial X^i_2(z))\,, \tag{3.2.25}$$
corresponding to the untwisted states $\frac{1}{\sqrt{2}}(\alpha^i_{-1} + \alpha^{i+8}_{-1})\,|0\rangle$ ,

which together build the algebra $\mathfrak{u}(1)^9$ in the holomorphic sector of the theory. At particular loci in the moduli space there appear additional conserved currents $J^\alpha(z)$ that can belong either to the untwisted or twisted sector, corresponding to primary states $|\mathrm{n}, \mathrm{w}, \rho\rangle$ with weights $\mathrm{h} = 1, \bar{\mathrm{h}} = 0$. In nine dimensions, the modes of the currents $\{J^\alpha(z)\}$ include the ladder operators associated to the roots $\{\alpha\}$ of a semisimple ADE algebra $\mathfrak{g}_r$ [10] of rank $r \leq 9$ and give the symmetry enhancement pattern

$$\mathfrak{u}(1)^9 \to (\mathfrak{g}_r)_2 \oplus \mathfrak{u}(1)^{9-r}\ , \tag{3.2.26}$$

where the subscript 2 denotes the level of the algebra. The allowed enhancements $\mathfrak{g}_r$ are those whose Dynkin diagram can be obtained from the GDD in Figure 3.2 by deleting one ($r = 9$) ore more ($r < 9$) nodes. The specific point in moduli space where each enhancement occurs are the ones for which the remaining simple roots are massless, which can be enforced by asking that $p_R = 0$ for their specific charge $Z$ assignment.

As an example, let us consider the locus in moduli space given by $a = 0$ and generic $R$. Here the theory displays the symmetry $(\mathrm{E}_8)_2 \oplus \mathfrak{u}(1)$.[11] This is obtained by removing the nodes 0 and $C$ from the GDD, which leaves us only with untwisted simple roots. Asking that the $\mathrm{E}_8$ roots be massless leads to the following constraints on the moduli

$$a^1 = a^2 = a^3 = a^4 = a^5 = a^6 = a^7 = 0, \quad a^1 + a^2 = 0, \quad \sum_{i=1}^{8} a^i = 0, \tag{3.2.27}$$

so that $a = 0$, with no constraint on $R$. The $\mathfrak{u}(1)$ current is in (3.2.24); the Cartan generators of $(\mathrm{E}_8)_2$ are given by (3.2.25), and the ladder operators are the symmetric combinations of the $(\mathrm{E}_8)_1 \oplus (\mathrm{E}_8)_1$ ones (which have positive eigenvalue under the $\mathcal{ES}$ action, and so are kept by the orbifold projection):

$$J^\alpha_+(z) = c_\alpha(:e^{i\alpha^i X^i_1(z)}: + :e^{i\alpha^i X^i_2(z)}:)\,,\ \alpha \in \Gamma_8\,, \tag{3.2.28}$$

where $c_\alpha$ is the cocycle factor ensuring the correct commutation properties of the currents. Denoting by $a$ the adjoint representation of $\mathrm{E}_8$, (3.2.25) and (3.2.28) together satisfy the level

[10] For a full classification of the possible $\mathfrak{g}_r$ in 9 dimensions (as well as the $r = 10$ case in 8 dimensions), see [99].

[11] At $R = \sqrt{2}$ the $\mathfrak{u}(1)$ factor is further enhanced to $\mathfrak{su}(2)$.

2 current algebra

$$J^a_+(z_1)J^b_+(z_2) \sim \frac{2\delta^{a,b}}{z_{12}^2} + \frac{1}{z_{12}} i f^{ab}{}_c J^c_+(z_2) \ , \tag{3.2.29}$$

where $z_{12} \equiv z_1 - z_2$, and $f^{ab}{}_c$ are the structure constants of $\mathrm{E}_8$ defined relative to the normalization where the roots have length squared 2. Alternatively, the current correlation functions are determined by

$$\langle J^a_+(z_1)J^b_+(z_2)\rangle = \frac{2\delta^{a,b}}{z_{12}^2} \ , \qquad \langle J^a_+(z_1)J^b_+(z_2)J^c_+(z_3)\rangle = \frac{1}{z_{12}z_{23}z_{13}} i\mathcal{F}^{abc} \ , \tag{3.2.30}$$

where $\mathcal{F}^{abc} = f^{ab}{}_d 2\delta^{c,d}$.

Turning this around, given the two- and three-point functions of any set of currents we can obtain the Cartan-Killing metric $\mathcal{G}^{ab}$ and the structure constants $f^{ab}{}_c$ of the algebra as

$$\mathcal{G}^{ab} \equiv z_{12}^2 \langle J^a_+(z_1)J^b_+(z_2)\rangle \ , \tag{3.2.31}$$

$$i f^{ab}{}_c \equiv i\mathcal{F}^{abd}\mathcal{G}_{dc} = z_{12}z_{23}z_{13}\langle J^a_+(z_1)J^b_+(z_2)J^c_+(z_3)\rangle \mathcal{G}_{dc} \ . \tag{3.2.32}$$

# Chapter 4

# Heterotic decompactifications and affine algebras

As we have already pointed out, one of the most interesting results on the Swampland Program is the presence of infinite towers of states that become light at the boundaries of moduli space, which is believed to be a universal feature of Quantum Gravity.
On the other hand, the set of massless vectors at a given point in moduli space is a signature of the gauge symmetry group, and all massless states arrange into representations of this group. Special points of moduli space of string theory compactifications, where symmetry enhancements can occur, are then of particular interest in this context, as they also feature additional massless states. Furthermore, an exhaustive scan of these points is also crucial for the question of string universality and whether the string lamppost can cover all viable theories of Quantum Gravity. This has recently been explored in the context of heterotic/heterotic orbifolds compactifications and lattice methods [91–93, 99–105], F-theory constructions [22, 23, 106, 107] and also from general supergravity and Swampland arguments [107–110]. A subset of special points lie at infinite distances, where the tower of massless modes predicted by the SDC fits very naturally with the idea of a symmetry enhancement. The following two Chapters are devoted to the characterisation of such algebras.
The goal of this Chapter is to understand the symmetry enhancements that arise at infinite distance points in the moduli space of the heterotic string on $T^d$. As in the bosonic string, one gets a tower of winding (momentum) modes that becomes massless in this limit, in the form proposed by the SDC. The intriguing aspect in the heterotic String is that, as we will show, a subset of this tower enhances the symmetries of the lower-dimensional theory to the affine version of the Lie algebra of the higher dimensional theory which is reached by decompactification. Affine Lie algebras are briefly reviewed in Appendix A.
The appearance of infinite-dimensional symmetries in the circle compactification of the heterotic string can be anticipated from the Generalised Dynkin diagram of the Narain lattice. This 19-node Dynkin diagram contains, for the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string, two copies of the affine algebra $\mathrm{E}_9$, where the affine nodes are charged with respect to the circle, connected by a central node. By deleting nodes in this diagram one can obtain all the enhancement groups that arise in the circle compactifications of the heterotic string, as well as the point or subspace in moduli space where they arise, as explained above. This diagram tells us that an enhancement

to $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ is possible in the decompactification limit $R \to \infty$, and that the affine algebra $\widehat{D}_{16}$ is obtained in the limit $R \to 0$ by a specific choice of Wilson line, as we will present in the following.
We will extend this intuition to compactifications to generic dimensions, where in general there is no Generalised Dynkin Diagram associated to the Narain lattice. For compactifications on $T^2$, this appearance of loop algebras in the dual F-theory on K3 was analysed first in [111] and recently studied in great detail in [22, 23]. Our results from the heterotic perspective match the expectations from these results obtained in the F-theory picture.

## 4.1 Decompactifcation limits of the heterotic String on $S^1$

In the limit $R \to \infty$, the following states become massless

$$(0, \mathrm{n}, \pi^\alpha), \quad \mathrm{n} \in \mathbb{Z},\ \pi^\alpha \in \Gamma_{\mathrm{E}_8} \times \Gamma_{\mathrm{E}_8} \text{ or } \Gamma_{\mathrm{Sp}(32)},\ |\pi|^2 = 2\ . \tag{4.1.1}$$

In the T dual case $R \to 0$, we get an equivalent tower of states becoming massless, labelled by the winding numbers

$$(0, \mathrm{w}, \pi^\alpha), \quad \mathrm{w} \in \mathbb{Z},\ \pi^\alpha \in \Gamma_{\mathrm{E}_8} \times \Gamma_{\mathrm{E}_8} \text{ or } \Gamma_{\mathrm{Sp}(32)},\ |\pi|^2 = 2 \tag{4.1.2}$$

Our goal now is to show that these states are actually the roots of an affine Lie algebra.
In the limit $R \to \infty$, and for any finite Wilson line $A$, the equations for all the nodes in Table 3.3, apart from the one corresponding to node C, are satisfied. The corresponding algebra from the EDD is naively $\mathrm{E}_9 \oplus \mathrm{E}_9$. Taking into account the fact that the two copies of $\mathrm{E}_9$ share the same imaginary root, we will denote the corresponding algebra as $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$. As expected, there is also an enhancement to $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ at $R \to 0$ for zero Wilson line if we use the T-dual convention of Table 3.2, where the extended nodes 0 and 0' (c.f. eq. (3.1.61)) have momentum instead of winding charge.
This pattern and the actual enhanced algebra can be understood by explicitly building the gauge vectors. For concreteness, we focus on the $R \to \infty$ limit here, keeping in mind that the opposite limit is related to this one by T-duality. Moreover, all finite values for the Wilson lines $A^I$ (including $A^I = 0$) are physically equivalent in the large $R$ limit, since all of them correspond to the situation $\frac{A}{R} \to 0$. For this reason, we approach the enhancement point along the path with vanishing Wilson line

$$(A^I, R) = (0, R)\,, \tag{4.1.3}$$

but we remark that our results, namely the structure of the algebra, are the same even turning on a non-trivial (finite) $A^I$, as discussed in detail in Appendix B. Note that along the path (4.1.3) the symmetry is $(\mathrm{E}_8 \times \mathrm{E}_8 \times U(1)_L) \times U(1)_R$ for all finite values of $R$. All these massless states are accompanied by a tower of momentum and winding states. We are interested in particular in the tower of momentum states of the $\mathrm{E}_8 \times \mathrm{E}_8$ vectors in (3.1.39) and (3.1.59), given by

$$\alpha^I_{-1} \bar{\psi}^\mu_{-\frac{1}{2}} |0, n\rangle_{NS}\ , \qquad \bar{\psi}^\mu_{-\frac{1}{2}} |0, n, \pi_\alpha\rangle_{NS}\ , \tag{4.1.4}$$

where we recall that $\alpha = 1, ..., 480$ and $\mu = 0, ..., 8$. These are associated to the following asymptotically conserved left currents

$$J^a_{\rm n}(z) \equiv J^a(z) e^{i{\rm n}Y(z)}\,, \tag{4.1.5}$$

with $a = \{I, \alpha\}$ and the finite algebra currents given by

$$J^I(z) = i\partial X^I(z)\,, \qquad J^\alpha(z) = c_\alpha e^{i\pi^I_\alpha X^I(z)}\,. \tag{4.1.6}$$

Here $c_\alpha$ are the cocycle factors, that satisfy

$$c_\alpha c_\beta \;=\; \epsilon(\alpha,\beta)\, c_{\alpha+\beta}\,, \qquad \epsilon(\alpha,\beta) = \pm 1 \tag{4.1.7}$$

when $\alpha + \beta$ is a root, and zero otherwise (and $c_0 \equiv 1$). Let us mention that in principle the currents (4.1.5) are not purely holomorphic, as the exponential dipendence would strictly speaking be of the form

$$e^{ipY(z,\bar z)}\,, \quad Y(z,\bar z) = Y(z) + \bar Y(\bar z)\,. \tag{4.1.8}$$

As we will explicitly see in 5.1, though, the antiholomorphic part does not contribute to the currents correlation functions in the limit; this can be naively explained by the fact that the $J^a_{\rm n}(z)$ have antiholorphic conformal dimension $\bar{\rm h} = \frac{\rm n}{2R^2}$ that vanishes as $R \to \infty$. On the other hand, the holomorphic conformal dimension ${\rm h} = 1 + \frac{\rm n}{2R^2}$ tends to 1 as $R \to \infty$, so indeed they correspond to asymptotically conserved currents.
By computing the OPEs of these currents we will prove that they define an $({\rm E}_9 \oplus {\rm E}_9)/\sim$ algebra.
First, recall the OPEs of a set of currents associated to an ADE algebra $\mathfrak{g}$ at level 1, with conformal dimension $({\rm h}, \bar{\rm h}) = (1, 0)$, is given by

$$J^a(z) J^b(w) \sim \frac{1}{(z-w)^2} K^{ab} + i \frac{f^{ab}{}_c}{z-w} J^c(w)\,, \tag{4.1.9}$$

where $K^{ab}$ is the Cartan-Killing metric and $f^{ab}{}_c$ are the structure constants. In the case of a finite algebra, eq. (4.1.9) is equivalent to the current algebra between the Laurent expansion coefficients defined by

$$J^a_k = \oint \frac{dz}{2\pi i} z^k J^a(z)\,, k \in \mathbb{Z}\,. \tag{4.1.10}$$

This current algebra is thus given by

$$[J^a_j, J^b_k] = j\delta^{ab}\delta_{j,-k} + i f^{ab}{}_c J^c_{j+k}\,, \tag{4.1.11}$$

and we can read the algebra $\mathfrak{g}$, formed by the currents $J^a(z)$, from the algebra of the set $\{J_0^a\}$. The structure constants of an $E_9$ algebra can be read off from the commutators

$$\begin{aligned}
&[H_{\rm n}^{\hat{I}}, H_{\rm m}^{\hat{J}}] = \hat{k}\,{\rm n}\,\delta^{\hat{I}\hat{J}}\delta_{\rm n+m,0}\,,\\
&[H_{\rm n}^{\hat{I}}, E_{\rm m}^{\alpha}] = \alpha^{\hat{I}} E^{\alpha}_{\rm n+m}\,,\\
&[E_{\rm n}^{\alpha}, E_{\rm m}^{\beta}] = \begin{cases} N_{\alpha,\beta}E^{\alpha+\beta}_{\rm n+m} & \text{for } \alpha+\beta \text{ root},\\ \alpha^I H^I_{\rm n+m} + \hat{k}\,{\rm n}\,\delta_{\rm n+m,0} & \text{for } \alpha=-\beta\,,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned} \tag{4.1.12}$$

where $\hat{I}=1,...8$, and $\hat{k}$ represents the central extension, which commutes with all the other generators. The $E_8$ roots $\alpha$ are normalized to $|\alpha|^2=2$, and $N_{\alpha,\beta}$ is a constant.
Let us now turn to the calculation of the OPEs of the currents in (4.1.6) (see Appendix B for the detailed computation). For the towers of two Cartan currents we find

$$J^I_{\rm n}(z)J^J_{\rm m}(w) \sim (z-w)^{\frac{\rm nm}{2R^2}}\left(\delta^{IJ}\frac{e^{i({\rm n+m})Y(w)}}{(z-w)^2} + i\delta^{IJ}{\rm n}\frac{:\partial Y(w)e^{i({\rm n+m})Y(w)}:}{z-w} + \mathcal{O}(1)\right), \tag{4.1.13}$$

whereas the OPE of the towers of two roots takes the form

$$J^{\alpha}_{\rm n}(z)J^{\beta}_{\rm m}(w) \sim (z-w)^{\frac{\rm nm}{2R^2}}\cdot\begin{cases}\frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\rm n+m}(w)}{z-w}+\mathcal{O}(1) & \alpha+\beta\,\text{root},\\ \frac{:e^{i({\rm n+m})Y(w)}:}{(z-w)^2}+\frac{\pi^I_\alpha J^I_{\rm n+m}(w)+in:\partial Y(w)e^{i({\rm n+m})Y(w)}:}{z-w}+\mathcal{O}(1) & \alpha=-\beta\,,\\ \mathcal{O}(1) & \text{otherwise.}\end{cases} \tag{4.1.14}$$

A Cartan and a root tower give

$$J^I_{\rm n}(z)J^{\alpha}_{\rm m}(w) \sim (z-w)^{\frac{\rm nm}{2R^2}}\left(\frac{\pi^I_\alpha J^{\alpha}_{\rm n+m}(w)}{z-w}+\mathcal{O}(1)\right). \tag{4.1.15}$$

Finally, the OPEs involving $\partial Y$ are

$$J^I_{\rm n}(z)i\partial Y(w) \sim \frac{i}{2R^2}{\rm n}\frac{:\partial X^I(w)e^{i{\rm n}Y(w)}:}{z-w}+\mathcal{O}(1)\,, \tag{4.1.16}$$

$$J^{\alpha}_{\rm n}(z)i\partial Y(w) \sim \frac{i}{2R^2}{\rm n}\frac{:e^{i\pi^I_\alpha X^I(w)}e^{i{\rm n}Y(w)}:}{z-w}+\mathcal{O}(1)\,, \tag{4.1.17}$$

$$i\partial Y(w)i\partial Y(w) \sim \frac{1}{2R^2}\frac{1}{(z-w)^2}+\mathcal{O}(1)\,. \tag{4.1.18}$$

The factors $(z-w)^{\frac{\rm nm}{2R^2}}$ in eqs. (4.1.13)-(4.1.15) tend to 1 in the limit $R\to\infty$, thus recovering the pole structure in (4.1.9), which describes an algebra of (asymptotically, in this case) conserved currents. In this limit, all the OPEs involving the circle direction, namely (4.1.16)-(4.1.18) vanish.

Defining the generators as the zero modes of the asymptotically conserved currents ($a = \{I, \alpha\}$)

$$(J^a_{\rm n})_0 \equiv \oint \frac{dz}{2\pi i} J^a_{\rm n}(z)\,, \tag{4.1.19}$$

and using the OPEs (4.1.13)-(4.1.15), we obtain that they satisfy the following algebra (details of the computation can be found in Appendix B)

$$\begin{aligned}
&[(J^I_{\rm n})_0, (J^J_{\rm m})_0] = i{\rm n}\delta^{IJ}\delta_{\rm n+m,0}(\partial Y)_0\,,\\
&[(J^I_{\rm n})_0, (J^\alpha_{\rm m})_0] = \pi^I_\alpha (J^\alpha_{\rm n+m})_0\,,\\
&[(J^\alpha_{\rm n})_0, (J^\beta_{\rm m})_0] = \begin{cases} \epsilon(\alpha,\beta)(J^{\alpha+\beta}_{\rm n+m})_0 & \alpha+\beta \,\text{root},\\ \pi^I_\alpha (J^I_{\rm n+m})_0 + i{\rm n}\delta_{\rm n+m,0}(\partial Y)_0 & \alpha = -\beta\,,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned} \tag{4.1.20}$$

where $(\partial Y)_0$ corresponds to the zero mode of the Laurent expansion of $\partial Y(z)$.
Recall that all the commutators involving the zero mode of the circle direction vanish in the limit $R \to \infty$, as shown in (4.1.16)-(4.1.18).
With a similar analysis, the non-vanishing components of the Cartan-Killing metric are found to be

$$K(J^I_{\rm n}(z), J^J_{\rm m}(w)) = \delta_{\rm n+m,0}\delta^{IJ}\,, \quad K(J^\alpha_{\rm n}(z), J^\beta_{\rm m}(w)) = \delta_{m+n,0}\delta^{\alpha+\beta,0}\,. \tag{4.1.21}$$

These are precisely the commutators and the Cartan-Killing metric for two copies of the algebra $\mathrm{E}_9$ with central extension given by

$$\hat{k} = (\partial Y)_0\,. \tag{4.1.22}$$

Since both $\mathrm{E}_9$ share the central extension, the total algebra is $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$. The fact that the two $\mathrm{E}_9$ share the central extension is consistent with the fact that the maximum rank of the left-moving algebra is 17.
Notice that since we did not need to specify the roots of the algebra, the same applies to the $SO(32)$ case, which at the same boundary of moduli space gets enhanced to $\widehat{D}_{16}$ (whose rank is 17 as well), in agreement with what one gets from the EDD with $SO(32)$ embedding (see [92] for details).
In fact, it is known [112] that the $\mathrm{E}_9 \oplus \mathrm{E}_9/\sim$ (or $\widehat{D}_{16}$) states, together with the other BPS states which are Lorentz scalars in the left-moving sector and in the right-movers ground state, give rise an BPS algebra of the generalized Kac-Moody type even at finite distances in moduli space, where generically all of them are massive. The $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ (or $\widehat{D}_{16}$) algebra is then realized as a subalgebra at the boundary of the moduli space by the BPS vectors that become massless.
Note that $\mathrm{E}_{10}$ (or $\mathrm{E}_{11}$) cannot be realized in this setting, as this should appear at infinite distance in moduli space where clearly the equation for the node C can never be satisfied. Another interesting point to make is that even if the EDD suggests that it is possible to find groups of maximal enhancement in the left sector with only one copy of $\mathrm{E}_9$, these cannot be realised in heterotic compactifications on the circle. This can be seen either from Table 3.3, which says that for $R \to \infty$ the equations for both $\mathrm{E}_9$ are satisfied at the same time, as well as from the explicit construction of the algebra, where the central extension extends both $\mathrm{E}_8$

factors at the same time.
Still focusing on the $\mathrm{E}_8 \times \mathrm{E}_8$ theory on $S^1$, inside the fundamental domain of the moduli space (that we chose to contain the point with $R \to \infty$), there is a second point at infinite distance, namley $R \to 0$ with Wilson line $A = (0_7, 1, 0_7, 1)$. This Wilson line breaks the $\mathrm{E}_8 \times \mathrm{E}_8 \times U(1)$ to $SO(16) \times SO(16) \times U(1)$. In the limit $R \to 0$, the massless left-moving vectors in the NS sector have charges

$$Z = (\pi_\alpha, \mathrm{w}, \mathrm{n}) \quad \text{with} \quad \begin{cases} \mathrm{w} = -2\mathrm{m} + 2\,\pi_\beta \cdot \Lambda\,, \\ \mathrm{n} = 2\mathrm{m} + \pi_\beta \cdot (A - 2\Lambda)\,, \\ \pi_\alpha = 2\mathrm{m}A + \pi_\beta - 2(\Lambda \cdot \pi_\beta)A\,, \end{cases} \qquad \forall \mathrm{m} \in \mathbb{Z}\,, \pi_\beta \in \Gamma_{\mathrm{Sp}(32)}\,, |\pi_\beta|^2 = 0, 2\,, \tag{4.1.23}$$

where $\pi_\alpha \in \Gamma_{\mathrm{Sp}(32)}$, $\Lambda = \left(\left(\frac{1}{2}\right)_8, 0_8\right)$ is the Wilson line that in the $SO(32)$ embedding breaks the gauge group to $SO(16) \times SO(16) \times U(1)$, and corresponding internal momenta

$$\begin{aligned} p_R &= \sqrt{2}R^2(\mathrm{m} - \pi_\alpha \cdot \Lambda)\,, \\ p_L &= -\sqrt{2}R^2(\mathrm{m} - \pi_\alpha \cdot \Lambda)\,, \\ p^I &= \pi^I_\alpha\,. \end{aligned} \tag{4.1.24}$$

The asymptotically holomorphic currents associated to these states are

$$\alpha^I_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}\,|Z\rangle_{NS} \to J^I_{\mathrm{m}}(z) = i\partial X^I(z) e^{-i\sqrt{2}\mathrm{m}R^2 Y(z)}\,, \tag{4.1.25}$$

$$\alpha^9_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}\,|Z\rangle_{NS} \to J^9_{\mathrm{m}}(z) = i\partial Y(z) e^{-i\sqrt{2}\mathrm{m}R^2 Y(z)}\,, \tag{4.1.26}$$

$$\bar{\psi}^\mu_{-\frac{1}{2}}\,|Z\rangle_{NS} \to J^\alpha_{\mathrm{m}}(z) = c_\alpha e^{i\pi^I_\alpha X^I(z)} e^{i\sqrt{2}R(-\mathrm{m}+\pi^I_\alpha \Lambda^I)Y(z)}\,, \tag{4.1.27}$$

and the currents of $\widehat{D}_{16}$ are obtained by introducing

$$Y^I(z) = X^I(z) + \sqrt{2}R^2\Lambda^I Y(z)\,, \tag{4.1.28}$$

which from (3.1.14) and (3.1.15) has OPE

$$Y^I(z)Y^J(w) = -(\delta^{IJ} + 2\Lambda^I\Lambda^J R^2) log(z-w) \to -\delta^{IJ} log(z-w) = X^I(z)X^J(w)\,. \tag{4.1.29}$$

Then the currents of $\widehat{D}_{16}$ are[1]

$$J^I_{\mathrm{m}}(z) = i\partial Y^I(z) e^{-i\sqrt{2}\mathrm{m}R^2 Y(z)}\,, \tag{4.1.30}$$

$$J^\alpha_{\mathrm{m}}(z) = c_\alpha e^{i\pi^I_\alpha Y^I(z)} e^{-i\sqrt{2}mR^2 Y(z)}\,. \tag{4.1.31}$$

They satisfy the commutation relations (4.1.20) with $\mathrm{n} \to \mathrm{m}$ and $\pi_\alpha$ being $D_{16}$ roots. This algebra is thus the $\widehat{D}_{16}$ affine algebra.
We can also decompactify the $SO(32)$ theory on $S^1$ to get the $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ taking the $R \to 0$ limit and Wilson line $\Lambda$.

[1] Again, we write explicitly only the holomorphic part.

## 4.2 Swampland conjectures approaching $(\mathbf{E}_9 \oplus \mathbf{E}_9)/\sim$

Let us now try to look at this symmetry enhancement from the point of view of the Swampland program. As a full stringy construction, it is expected that our setup fulfils all the Swampland Conjectures, so it is interesting to see how the relevant ones are realized in this particular example. Moreover, it is particularly illuminating to see how constraining these conjectures are, in the sense of understanding what aspects about the enhancement could have been anticipated had we not known the full string construction, but some 9d EFT data instead.

On the one hand, it seems very natural to try to connect our setup to the SDC (2.1.1). Since the points we have studied (i.e. $R \to 0,\ \infty$) are both infinite distance points, it is expected to find the corresponding towers when approaching them. In this case there is nothing unexpected, as these are just the winding and KK-states, respectively. Given the fact that these towers are also charged under the other gauge groups that are present in the 9d theory, it is interesting though to analyze how they fit in with the WGC and the (related) RFC, which we presented in their lattice versions respectively in Sections 2.3.1 and 2.4.

In the remaining of this section, we focus on the states charged under the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge bosons (more precisely under their Cartan generators), as well as those with winding and momentum charge. In the corresponding infinite distance limit, some subset of these gauge couplings go to zero and we show how the states predicted by the LWGC and the LRFC become massless, as required by the superextremality or self-repulsiveness conditions in the vanishing gauge coupling limit. These states in fact correspond to the $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ vectors, therefore connecting the conjectures to the presence of the affine enhancement.

The extremality bound for black holes charged under the heterotic group in toroidal compactifications was computed in [113] and takes the form

$$\frac{\alpha'}{4} M^2 \geq \frac{1}{2} \max(\boldsymbol{p}_L^2, p_R^2)\,, \tag{4.2.1}$$

so that states fulfilling the bound are subextremal. This is the relevant bound to study the superextremal particles predicted by the WGC.

On the other hand, we are going to compute the long range interaction between probe particles charged under these gauge groups in the Einstein frame, which are the relevant quantities for the RFC. We use the compactified heterotic low energy action as in [114], with the following metric ansatz

$$ds^2 = e^{\frac{4}{7}\tilde{\Phi}} e^{-\frac{2}{7}\tilde{\sigma}} g^E_{\mu\nu} dx^\mu dx^\nu + e^{2\tilde{\sigma}} (dy + Z_\mu dx^\mu)^2,\ \ y \sim y + 2\pi R\,, \tag{4.2.2}$$

where $\tilde{\Phi}$ and $\tilde{\sigma}$ denote the dynamical parts of the dilaton field, $\Phi = \Phi_0 + \tilde{\Phi}$, and the (dimensionless) radion, $\sigma = \sigma_0 + \tilde{\sigma}$, where we have also explicitly included a background piece for the scalars. We recall that the physical radius of the circle in the string frame and in string units is $R = e^{\sigma_0}\sqrt{\alpha'}$, and it is included in the definition of the coordinate $y$ instead of in the metric for convenience. The tensor $g^E_{\mu\nu}$ gives the 9 dimensional metric in the Einstein frame, and $Z_\mu$ represents the (dimensionless) graviphoton.

To compute the long range force we need the action up to quadratic order in the dynamical fields and including the minimal coupling to the physical states in the string spectrum, which act as semi-classical sources. We leave the details of the computation for Appendix C, but

outline the main points here. Using the metric ansatz in (4.2.2), one needs to perform the following field redefinition to work with conventionally normalized fields without mixed kinetic terms (as explained in detail around eq. (C.1.21))

$$\lambda = \frac{4}{\sqrt{7}}\Big(\frac{\Phi}{4} - \sigma\Big)\,. \tag{4.2.3}$$

Using this, the expression for the action up to quadratic order and including the minimally coupled sources that describe the heterotic states reads

$$\begin{aligned} S =& \frac{M_{\rm p,9}^7}{2}\int d^9x\sqrt{-g}\Big(R - \frac{1}{2}\partial_\mu\lambda\partial^\mu\lambda - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{\alpha'}{2}\partial_\mu A_9^I\partial^\mu A_9^I - \frac{R^2}{4}\tilde{Z}_{\mu\nu}\tilde{Z}^{\mu\nu} - \\ &- \frac{\alpha'}{4}\tilde{F}^I_{\mu\nu}\tilde{F}^{\mu\nu I} - \frac{\alpha'^2}{4R^2}\tilde{W}_{\mu\nu}\tilde{W}^{\mu\nu}\Big) - \int M(\Phi,\lambda,A_9^I)\,ds - w\int\tilde{W} - \pi^I\int A^I - n\int\tilde{Z}\,, \end{aligned} \tag{4.2.4}$$

where $ds$ is the line element along the particle world-line. The 9 dimensional Planck mass has the following expression

$$\frac{M_{\rm p,9}^7}{2} = \frac{e^{-2\Phi_0}\,R}{(2\pi)^6\alpha'^4}\,, \tag{4.2.5}$$

whereas $\tilde{Z}_\mu$ is the dimensionful graviphoton (see eq. (C.1.12)), and $A_\mu^I$ and $\tilde{W}_\mu \equiv \boldsymbol{B}_{\mu 9} + \frac{\alpha'}{2}A_9^I A_{\mu I}$ are the Cartan gauge fields, which are the only ones we are interested in to compute the long range force. All these gauge fields are normalized to have mass dimension +1 and in such a way that the charged states have integer charges, and $\tilde{Z}_{\mu\nu}$, $\tilde{W}_{\mu\nu}$ and $\tilde{F}^I_{\mu\nu}$ are their respective field strengths.

The repulsive force condition (for vanishing Wilson lines) is

$$\begin{aligned} M^2 \leq M_{\rm p,9}^2(32\pi^6)^{\frac{2}{7}}\max\Big\{&\Big(2|\pi|^2 e^{\frac{1}{2}\Phi_0}e^{\frac{\sqrt{7}}{14}\lambda_0} + \mathrm{n}^2 e^{\frac{4\sqrt{7}}{7}\lambda_0} + \mathrm{w}^2 e^{\Phi_0}e^{-\frac{3\sqrt{7}}{7}\sigma_0} + 2\mathrm{n}\ \mathrm{w}e^{\frac{1}{2}\Phi_0}e^{\frac{\sqrt{7}}{14}\lambda_0}\Big)\,, \\ &\Big(\mathrm{n}^2 e^{\frac{4\sqrt{7}}{7}\lambda_0} + \mathrm{w}^2 e^{\Phi_0}e^{-\frac{3\sqrt{7}}{7}\sigma_0} - 2\mathrm{n}\ \mathrm{w}e^{\frac{1}{2}\Phi_0}e^{\frac{\sqrt{7}}{14}\lambda_0}\Big)\Big\}\,. \end{aligned} \tag{4.2.6}$$

Let us remark that this expression is equivalent to the extremality bound given in eq. (4.2.1), so that all superextremal states are self-repulsive. This means that the WGC and the RFC are equivalent in this case (as expected, see [20]), but one must keep in mind that the long range force computation only makes sense in the perturbative regime.

The expressions of the three relevant gauge couplings as a function of the moduli and in Planck

units are

$$\frac{1}{g_Z^2} = \frac{M_{\rm p,9}^5}{2(32\pi^6)^{\frac{2}{7}}} e^{-\frac{4\sqrt{7}}{7}\lambda_0}\,, \tag{4.2.7}$$

$$\frac{1}{g_W^2} = \frac{M_{\rm p,9}^5}{2(32\pi^6)^{\frac{2}{7}}} e^{\frac{3\sqrt{7}}{7}\lambda_0} e^{-\Phi_0}\,, \tag{4.2.8}$$

$$\frac{1}{g_A^2} = \frac{M_{\rm p,9}^5}{2(32\pi^6)^{\frac{2}{7}}} e^{-\frac{\sqrt{7}}{14}\lambda_0} e^{-\frac{1}{2}\Phi_0}\,. \tag{4.2.9}$$

Thus, in the infinite distance decompactification limit $e^{\lambda_0} \to 0$ ($\frac{R}{\sqrt{\alpha'}} \to \infty$) we obtain in Planck units

$$e^{\lambda_0} \to 0: \quad g_Z \sim e^{\frac{2\sqrt{7}}{7}\lambda_0} \to 0, \quad g_W \sim e^{\frac{-3\sqrt{7}}{14}\lambda_0} \to \infty, \quad g_A \sim e^{\frac{\sqrt{7}}{28}\lambda_0} \to 0\,. \tag{4.2.10}$$

Since we are particularly interested in the states charged under the graviphoton and the heterotic gauge group (as these are the ones that form the $E_9 \oplus E_9/\sim$ algebra), it is clear that the large radion limit is the right one, as the relevant gauge couplings are small and the perturbative calculation under control. Let us remark also that even though the gauge theory associated to $W_\mu$ is non-perturbative in this corner of moduli space, we do not consider states charged under it here. We emphasize that the correspondence between the WGC and the RFC can be made precise in this limit, so that we can then interpret the presence of the $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ gauge fields in the light of these conjectures. Restricting to the aforementioned charged states, we obtain the following superextremality/self-repulsiveness condition

$$M \leq M_{\rm p,9}^{\frac{7}{2}} \sqrt{|\pi|^2 g_A^2 + \frac{\mathrm{n}^2 g_Z^2}{2}}\,. \tag{4.2.11}$$

For finite gauge couplings, this bound allows for massive states. Moreover, since the states that become the $\mathrm{E}_9 \oplus \mathrm{E}_9/\sim$ vectors in the infinite distance limit are BPS states, they saturate the inequality. In the decompactification limit, the fact that the two relevant gauge couplings tend to zero means that the LWGC/LRFC require an infinite tower of states (one for each point in the infinite charge lattice) to become massless. In particular, note that both the KK replicas of the Cartans and the roots of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic group studied in the previous section, which have arbitrary n and $|\pi|^2 = 0, 2$ are forced to become massless. This means that the states of the $\mathrm{E}_9 \oplus \mathrm{E}_9/\sim$ BPS algebra that we found in the limit can be understood to become massless as a consequence of the LWGC/LRFC. Note that in general there are many more (generically non-BPS) states that become massless in the limit according to (4.2.11), so that only a subset of all these asymptotically massless states, namely the BPS ones, form the $\mathrm{E}_9 \oplus \mathrm{E}_9/\sim$ algebra. Moreover, following the reasoning in [31, 40], we can argue for the appearance of this tower also as a way to prevent the restoration of a global symmetry, which is known to be forbidden in quantum gravity (see e.g. [25] and references therein). Since in the limit of vanishing coupling a gauge symmetry behaves as a global one, there must be a way to prevent this from happening at the infinite distance point, and this is indeed the case due to the presence of the infinite lattice of states becoming light. Note that this is also in complete agreement with the magnetic WGC if we identify the cutoff as the scale of the tower of states

becoming light, as it would predict a cutoff (in Planck units) $\Lambda_{\text{cutoff}} \lesssim g \to 0$. Let us finally mention that the limit of vanishing radius would give analogous results upon application of T-duality, which is not directly manifest in the computations above as we are working in the Einstein frame instead of the string frame.

## 4.3 Decompactifications of $T^d$

The decompactification pattern from nine to ten dimensions of the heterotic theory on $S^1$ can be generalized to the decompactification of several dimensions of the theory on $T^d$.
In the next section we show the possible decompactification patterns in the case $d = 2$, which is particularly interesting in light of its dual realisation in F-theory on K3, analyzed in [22,23]. In section 4.3.2, following the intuition in 8 dimensions, we extend to generic $d$.

### 4.3.1 Decompactification limits of the heterotic string on $T^2$

Consider a generic background given by

$$G_{ij} = \begin{pmatrix} R_8^2 & R_8 R_9 \cos\alpha \\ R_8 R_9 \cos\alpha & R_9^2 \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad A_i^I, \quad i = 8, 9. \tag{4.3.1}$$

where $R_8$ and $R_9$ are the lengths of the two dimensional vectors $e_{1,2}$ generating the $T^2$ lattice and $\alpha$ is the angle between them. In what follows we will also use the inverse metric, given by

$$G^{ij} = \frac{1}{\sin^2\alpha} \begin{pmatrix} \frac{1}{R_8^2} & -\frac{\cos\alpha}{R_8 R_9} \\ -\frac{\cos\alpha}{R_8 R_9} & \frac{1}{R_9^2} \end{pmatrix}. \tag{4.3.2}$$

One can distinguish two types of decompactification limits[2], according to the values of $R_8$, $R_9$ and $\cos\alpha$:

- only one radius diverges: decompactification from eight to nine dimensions;
- both $R_8$, $R_9 \to \infty$, independently of their ratio ($\frac{R_8}{R_9} \to 0$, const, $\infty$): decompactification from eight to ten dimensions.

The possible interesting profiles for $\cos\alpha$ along the path taken in moduli space depend on the type of decompactification, and they will be specified case by case.

#### $8d \to 9d$ decompactification

Let us focus on the first situation, where only one radius diverges, $R_8 \to \infty$.
The massless states can only come from states with no winding in the eighth direction, $\mathrm{w}^8 = 0$.

[2] T-dual versions are considered as equivalent. For example, in the case of diagonal torus metric $\alpha = \frac{\pi}{2}$ to which we will restrict in the following, T duality acts as a shift of the Wilson lines by a vector in $\Gamma_8 \times \Gamma_8$ or as $A_i' = -\frac{A_i}{R_i^2 + \frac{A_i^2}{2}}$, $R_i' = \frac{R_i}{R_i^2 + \frac{A_i^2}{2}}$, $\mathrm{w}^i \leftrightarrow \mathrm{n}_i$, $i = 8, 9$.

The massless and the level-matching conditions reduce then to

$$\begin{aligned} 0 &= \frac{1}{\sin^2\alpha}\left(\frac{1}{R_9^2}(p_{R,9}^2+p_{L,9}^2)-(\mathrm{w}^9)^2R_9^2\cos^2\alpha\right)+p_I^2+2\left(N+\tilde{N}-\frac{3}{2}\right)\,,\\ 0 &= 2\mathrm{n}_9\mathrm{w}^9+|\pi_\alpha|^2+2\left(N-\tilde{N}-\frac{1}{2}\right)\,, \end{aligned} \tag{4.3.3}$$

where we have used the inverse metric in (4.3.2), and

$$\begin{aligned} p_{R,9} &= \frac{1}{\sqrt{2}}\left(\mathrm{n}_9-\left(R_9^2+\frac{1}{2}A_9\cdot A_9\right)\mathrm{w}^9-\pi_\alpha\cdot A_9\right)\,,\\ p_{L,9} &= \frac{1}{\sqrt{2}}\left(\mathrm{n}_9+\left(R_9^2-\frac{1}{2}A_9\cdot A_9\right)\mathrm{w}^9-\pi_\alpha\cdot A_9\right)\,,\\ p^I &= \pi_\alpha^I+A_9^I\mathrm{w}^9\,. \end{aligned} \tag{4.3.4}$$

Note that (4.3.3) with the expressions (4.3.4) for the internal momenta correspond, in the limit $\cos\alpha\to 0$, to the massless and the level-matching conditions of the nine-dimensional theory. On the other hand, when $\cos\alpha\to c\neq 0$, which corresponds to the asymptotic behaviour $G_{89}\to\infty$, nine-dimensional Lorentz invariance is broken. We therefore restrict to $\cos\alpha=0$. Similarly, nontrivial values of the Kalb-Ramond field break Lorentz invariance, so we also restrict to $b=0$.

If we choose the moduli such that $R_9$ and $A_9^I$ correspond to a point with gauge algebra $\mathfrak{g}$ in the left sector of the nine-dimensional theory, with roots of the form $Z_\mathfrak{g}=(\pi_\alpha,\mathrm{w}^9,\mathrm{n}_9)$ satisfying $p_{L,9}=2$ and $p_{R,9}=0$, the massless spectrum of the eight-dimensional theory will contain the momentum tower of the nine-dimensional gauge vectors. This is given by the following left-moving vectors together with their associated currents

- Cartan sector:

$$\begin{aligned} \alpha_{-1}^I\bar{\psi}^\mu_{-\frac{1}{2}}\left|0,\mathrm{n}_8\right\rangle_{NS} &\to J^I_{\mathrm{n}_8}(z)=i\partial X^I(z)e^{i\mathrm{n}_8Y^8(z)}\,,\\ \alpha_{-1}^8\bar{\psi}^\mu_{-\frac{1}{2}}\left|0,\mathrm{n}_8\right\rangle_{NS} &\to J^8_{\mathrm{n}_8}(z)=i\sqrt{2}\partial Y^8(z)e^{i\mathrm{n}_8Y^8(z)}\,; \end{aligned} \tag{4.3.5}$$

- root sector:

$$\bar{\psi}^\mu_{-\frac{1}{2}}\left|Z_\mathfrak{g},\mathrm{n}_8\right\rangle_{NS}\to J^\alpha_{\mathrm{n}_8}(z)=c_\alpha e^{ip_{\alpha;I}X^I(z)}e^{i\mathrm{n}_8Y^8(z)}\,, \tag{4.3.6}$$

where the first number in the kets represents the nine dimensional quantum numbers, $\mathrm{n}_8\in\mathbb{Z}$, and $\mathrm{w}^8=0$ is implied. We have defined the index $\hat{I}=\{9,I\}$, the left moving $X^{\hat{I}}(z)=\left(\sqrt{2}e^9_{\ 9}Y^9(z),X^I(z)\right)$, and $p_{\alpha;\hat{I}}=(e_9^{*9}p_{L,9},p_I)$. We have also restricted to $A_8^I=0$, which together with the choices $\cos\alpha$, $b=0$ implies in particular that $E_{89}$, $E_{98}=0$; the case $A_8^I\neq 0$ is discussed in Appendix B.2, but in both cases the resulting BPS algebra is the same and so we present the former for simplicity.

As derived in Appendix B.2, for $R_8\to\infty$ the OPEs among the asymptotic currents of equations

(4.3.5) and (4.3.6) are

$$J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\hat{J}}_{\mathrm{m}_8}(w) \sim \delta^{\hat{I}\hat{J}}\frac{:e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{(z-w)^2} + i\delta^{\hat{I}\hat{J}}\mathrm{n}_8\frac{:\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{z-w} + \mathcal{O}(1)\,, \tag{4.3.7}$$

$$J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\alpha}_{\mathrm{m}_8}(w) \sim \frac{p^{\hat{I}}_{\alpha}J^{\alpha}_{\mathrm{n}_8+\mathrm{m}_8}(w)}{z-w} + \mathcal{O}(1)\,, \tag{4.3.8}$$

$$J^{\alpha}_{\mathrm{n}_8}(z)J^{\beta}_{\mathrm{m}_8}(w) \sim \begin{cases} \frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8}(w)}{z-w} + \mathcal{O}(1) & \alpha+\beta\,\text{root}, \\ \frac{:e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w):}}{(z-w)^2} + \frac{p_{\alpha;\hat{I}}J^{\hat{I}}_{\mathrm{n}_8+\mathrm{m}_8}(w)+i\mathrm{n}_8:\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{z-w} + \mathcal{O}(1) & \alpha=-\beta\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.9}$$

All the OPEs involving $J^8_{\mathrm{n}_8}(z)$ are trivial in the limit $R\to\infty$.
The asymptotic algebra is obtained from the commutation relations of the zero modes of the currents (4.3.5)-(4.3.6), defined as in (4.1.19). We find (see Appendix B.2 for the detailed computation)

$$\begin{aligned} &[(J^{\hat{I}}_{\mathrm{n}})_0,(J^{\hat{J}}_{\mathrm{m}})_0] = i\mathrm{n}\delta^{\hat{I}\hat{J}}\delta_{\mathrm{n}+\mathrm{m},0}(\partial Y^8)_0\,, \\ &[(J^{\hat{I}}_{\mathrm{n}})_0,(J^{\alpha}_{\mathrm{m}})_0] = p^{\hat{I}}_{\alpha}(J^{\alpha}_{\mathrm{n}+\mathrm{m}})_0\,, \\ &[(J^{\alpha}_{\mathrm{n}})_0,(J^{\beta}_{\mathrm{m}})_0] = \begin{cases} \epsilon(\alpha,\beta)(J^{\alpha+\beta}_{\mathrm{n}+\mathrm{m}})_0 & \alpha+\beta\,\text{root}, \\ p_{\alpha;\hat{I}}(J^{\hat{I}}_{\mathrm{n}+\mathrm{m}})_0 + i\mathrm{n}\delta_{\mathrm{n}+\mathrm{m},0}(\partial Y^8)_0 & \alpha=-\beta\,, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{4.3.10}$$

which is precisely the algebra of $\hat{\mathfrak{g}}$, with central extension $(\partial Y^8)_0$. Importantly, if the algebra $\mathfrak{g}$ is semisimple, $\mathfrak{g}=\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_{\mathrm{n}}$, (4.3.10) describes the algebra $\hat{\mathfrak{g}}=(\widehat{\mathfrak{g}}_1\oplus\ldots\oplus\widehat{\mathfrak{g}}_{\mathrm{n}})/\sim$, where $\sim$ means that the central extension of all the factors (including $\mathfrak{u}(1)$'s) is identified, namely as $(\partial Y^8)_0$. This signifies that all the factors are made affine by the universal presence of the momentum tower.
As an example, let us show explicitly the case of $(\widehat{\mathrm{E}}_8\oplus\widehat{\mathrm{E}}_8\oplus\widehat{A}_1)/\sim$ in the decompactification of the $\mathrm{E}_8\times\mathrm{E}_8$ theory from 8 to 9 dimensions. We start from the background

$$G_{ij} = \begin{pmatrix} R_8^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad b=0, \quad A^I_8, \quad A^I_9=0\,, \tag{4.3.11}$$

and take the limit $R_8\to\infty$. As we argued above, without loss of generality we can take $A^I_8$ to vanish. The choice $G_{99}=1$ and $A^I_9=0$ is instead dictated by the fact that we want to decompactify to a theory with gauge algebra $\mathrm{E}_8\oplus\mathrm{E}_8\oplus A_1$ in 9 dimensions. The massless states here are the momentum towers along $y^8$ of the $\mathrm{E}_8\oplus\mathrm{E}_8$ states,

$$\alpha^I_{-1}\bar{\psi}^{\mu}_{-\frac{1}{2}}\,|0,\mathrm{n}_8\rangle_{NS} \;\to J^I_{\mathrm{n}_8}(z) = i\partial X^I(z)e^{i\mathrm{n}_8Y^8(z)}\,, \tag{4.3.12}$$

$$\bar{\psi}^{\mu}_{-\frac{1}{2}}\,|Z_{\mathrm{E}_8\oplus\mathrm{E}_8},\mathrm{n}_8\rangle_{NS} \;\to J^{\alpha}_{\mathrm{n}_8}(z) = c_{\alpha}e^{i\pi_{\alpha;I}X^I(z)}e^{i\mathrm{n}_8Y^8(z)}\,, \tag{4.3.13}$$

where $Z_{\mathrm{E}_8\oplus\mathrm{E}_8} = (\pi_\alpha, 0, 0)$, $\pi_\alpha$ are the roots of $\mathrm{E}_8\oplus\mathrm{E}_8$, and in addition we have the corresponding momentum towers of the $A_1$ states

$$\alpha^9_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}\,|0,\mathrm{n}_8\rangle_{NS} \to J^9_{\mathrm{n}_8}(z) = i\sqrt{2}\partial Y^9(z)e^{i\mathrm{n}_8 Y^8(z)}, \tag{4.3.14}$$

$$\bar{\psi}^\mu_{-\frac{1}{2}}\,|Z_{A_1},\mathrm{n}_8\rangle_{NS} \to J^\pm_{\mathrm{n}_8}(z) = e^{\pm i2Y^9(z)}e^{i\mathrm{n}_8 Y^8(z)}\,, \tag{4.3.15}$$

with $Z_{A_1} \equiv (\pi, \mathrm{w}^9, \mathrm{n}_9) = (0, \pm 1, \pm 1)$. Here (4.3.12) and (4.3.14) are associated to the Cartan currents and (4.3.13) and (4.3.15) to the root generators. The affinization of the $\mathrm{E}_8 \oplus \mathrm{E}_8$ part of the algebra works exactly as in the decompactification from 9 to 10 dimensions presented in Section 4, with the replacements $\mathrm{n} \to \mathrm{n}_8$ and $Y \to Y^8$.
The affinisation of the $A_1$ factor is derived from the general structure of the OPEs (B.2.11), (B.2.15) and (B.2.16) among the affine currents (4.3.14) and (4.3.15). These give the subalgebra

$$\begin{aligned}
&[(J^9_{\mathrm{n}_8})_0, (J^9_{\mathrm{m}_8})_0] = i\mathrm{n}_8\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8)_0\,,\\
&[(J^9_{\mathrm{n}_8})_0, (J^\pm_{\mathrm{m}_8})_0] = \pm\sqrt{2}(J^\pm_{\mathrm{n}_8+\mathrm{m}_8})_0 = \pi_{A_1,\pm}(J^\pm_{\mathrm{n}_8+\mathrm{m}_8})_0\,,\\
&[(J^\pm_{\mathrm{n}_8})_0, (J^\pm_{\mathrm{m}_8})_0] = 0\,,\\
&[(J^\pm_{\mathrm{n}_8})_0, (J^\mp_{\mathrm{m}_8})_0] = \pi_{A_1,\pm}(J^\pm_{\mathrm{n}_8+\mathrm{m}_8})_0 + i\mathrm{n}_8\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8)_0
\end{aligned} \tag{4.3.16}$$

among the current zero modes, which corresponds to $\widehat{A}_1$ with central extension $(\partial Y^8)_0$, the commutators between $\widehat{\mathrm{E}}_8 \oplus \widehat{\mathrm{E}}_8$ modes and $\widehat{A}_1$ ones vanishing in the limit. This is indeed a $(\widehat{\mathrm{E}}_8 \oplus \widehat{\mathrm{E}}_8 \oplus \widehat{A}_1)/\sim$ algebra for the current zero modes.
We emphasize that the Cartans corresponding to the algebra of the decompactified theory get affinized independently of them being enhanced or not to nonabelian algebras. In fact, the same is true for those Cartans lying in the gravity multiplet, which do not admit enhancements. These get affinized by the antiholomorphic counterpart $(\bar{\partial}\bar{Y}^8)_0$ of the left-moving central extension (see Appendix B.2 for details).

### $8d \to 10d$ decompactification

In this limit both $R_8$ and $R_9$ diverge. They can go to infinity with different speeds $\frac{R_8}{R_9} \to 0, \infty$ or at the same rate, $\frac{R_8}{R_9} \to$ const. The first can effectively capture a decompactification by steps, $8 \to 9 \to 10$ dimensions, while the second is always a one-step process $8 \to 10$ dimensions. Again, in the limit $R_8, R_9 \to \infty$, the massless spectrum is characterised by $\mathrm{w}^8 = \mathrm{w}^9 = 0$, with no restriction on $\mathrm{n}_8$, $\mathrm{n}_9$.
The non-trivial contributions to the massless and level-matching conditions are then

$$\begin{aligned}
0 &= |\pi_\alpha|^2 + 2\left(N + \tilde{N} - \frac{3}{2}\right)\,,\\
0 &= |\pi_\alpha|^2 + 2\left(N - \tilde{N} - \frac{1}{2}\right)\,,
\end{aligned} \tag{4.3.17}$$

regardless of the values of $\cos\alpha$, $b$, $A^I_8$ and $A^I_9$. The massless left-moving vectors are then the momentum towers of the $\mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \mathfrak{u}(1)^2$ vectors or $D_{16} \oplus \mathfrak{u}(1)^2$ vectors, depending on which theory one starts with. From the algebra point of view turning on $B_{ij}$ or the Wilson lines is

equivalent to setting them to zero, as shown in Appendix B.2, so we take for simplicity $B_{ij}=0$, $A^I_8=0$, $A^I_9=0$ and $G_{89}=0$ in the following expressions for the currents. The two KK Cartans are associated to $(i=8,9)$

$$\alpha^i_{-1}\bar\psi^\mu_{-\frac12}\,|0,\mathrm{n}_8,\mathrm{n}_9\rangle_{NS}\to J^i_{\mathrm{n}_8,\mathrm{n}_9}(z)=i\sqrt{2}\partial Y^i(z)e^{i\mathrm{n}_8Y^8(z)}e^{i\mathrm{n}_9Y^9(z)}\,, \tag{4.3.18}$$

the $\mathrm{E}_8\oplus\mathrm{E}_8$ Cartans to

$$\alpha^I_{-1}\bar\psi^\mu_{-\frac12}\,|0,\mathrm{n}_8,\mathrm{n}_9\rangle_{NS}\to J^I_{\mathrm{n}_8,\mathrm{n}_9}(z)=i\partial X^I(z)e^{i\mathrm{n}_8Y^8(z)}e^{i\mathrm{n}_9Y^9(z)}\,, \tag{4.3.19}$$

and the $\mathrm{E}_8\oplus\mathrm{E}_8$ roots to

$$\bar\psi^\mu_{-\frac12}\,|Z_{\mathrm{E}_8\oplus\mathrm{E}_8},\mathrm{n}_8,\mathrm{n}_9\rangle_{NS}\to J^\alpha_{\mathrm{n}_8,\mathrm{n}_9}(z)=c_\alpha e^{i\pi_{\alpha;I}X^I(z)}e^{i\mathrm{n}_8Y^8(z)}e^{i\mathrm{n}_9Y^9(z)}\,. \tag{4.3.20}$$

As shown in detail in Appendix B.2, in the decompactification limit the OPEs among these currents are, regardless of $\frac{R_8}{R_9}$,

$$J^I_{\mathrm{n}_8,\mathrm{n}_9}(z)J^J_{\mathrm{m}_8,\mathrm{m}_9}(w)=\frac{\delta^{IJ}:e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}:}{(z-w)^2}+i\delta^{IJ}\mathrm{n}_i\frac{:\partial Y^i(w)e^{i(\mathrm{n}_j+\mathrm{m}_j)Y^j(w)}:}{z-w}+\mathcal{O}(1)\,, \tag{4.3.21}$$

$$J^I_{\mathrm{n}_8,\mathrm{n}_9}(z)J^\alpha_{\mathrm{m}_8,\mathrm{m}_9}(w)=\frac{\pi^I_\alpha J^\alpha_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)}{z-w}+\mathcal{O}(1)\,, \tag{4.3.22}$$

$$J^\alpha_{\mathrm{n}_8,\mathrm{n}_9}(z)J^\beta_{\mathrm{m}_8,\mathrm{m}_9}(w)=\begin{cases}\frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)}{z-w}+\mathcal{O}(1) & \alpha+\beta\,\text{root},\\ \frac{:e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}:}{(z-w)^2}+\frac{\pi_{\alpha;I}J^I_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)+i\mathrm{n}_i:\partial Y^i(w)e^{i(\mathrm{n}_j+\mathrm{m}_j)Y^j(w)}:}{z-w}+\mathcal{O}(1) & \alpha=-\beta\,,\\ 0 & \text{otherwise},\end{cases} \tag{4.3.23}$$

All the other OPEs are either finite or vanishing as the inverse metric components. These have the structure of the double loop versions of the ten dimensional algebra ($\mathfrak{g}=\mathrm{E}_8\oplus\mathrm{E}_8$, $D_{16}$) with the addition of two central extensions, which can be identified from the structure of the single pole in the OPEs and are the zero modes $(\partial Y^i)_0$. The algebra of the zero modes is

$$\begin{aligned}
&[(J^I_{\mathrm{n}_8,\mathrm{n}_9})_0,(J^J_{\mathrm{m}_8,\mathrm{m}_9})_0]=i\delta^{IJ}(\mathrm{n}_8\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8_{0,\mathrm{n}_9+\mathrm{m}_9})_0+\mathrm{n}_9\delta_{\mathrm{n}_9+\mathrm{m}_9,0}(\partial Y^9_{\mathrm{n}_8+\mathrm{m}_8,0})_0)\,,\\
&[(J^I_{\mathrm{n}_8,\mathrm{n}_9})_0,(J^\alpha_{\mathrm{m}_8,\mathrm{m}_9})_0]=\pi^I_\alpha(J^\alpha_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9})_0\,,\\
&[(J^\alpha_{\mathrm{n}_8,\mathrm{n}_9})_0,(J^\beta_{\mathrm{m}_8,\mathrm{m}_9})_0]=\begin{cases}\epsilon(\alpha,\beta)(J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9})_0 & \alpha+\beta\,\text{root},\\ \pi_{\alpha;I}(J^I_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9})_0+ & \\ \quad+i(\mathrm{n}_8\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8_{0,\mathrm{n}_9+\mathrm{m}_9})_0+\mathrm{n}_9\delta_{\mathrm{n}_9+\mathrm{m}_9,0}(\partial Y^9_{\mathrm{n}_8+\mathrm{m}_8,0})_0) & \alpha=-\beta\,,\\ 0 & \text{otherwise},\end{cases}
\end{aligned} \tag{4.3.24}$$

where

$$(\partial Y^i_{\mathrm{n}_j+\mathrm{m}_j,0})_0=\oint\frac{dz}{2\pi i}\partial Y^i(z)e^{i(\mathrm{n}_j+\mathrm{m}_j)Y^j(z)}\,,\quad i\neq j \tag{4.3.25}$$

are the momentum modes of the two central extensions of the algebra, $(\partial Y^8)_0$ and $(\partial Y^9)_0$, along the $y^9$ and $y^8$ directions, respectively. This gives the double loop version $(\widehat{\mathrm{E}}_9 \oplus \widehat{\mathrm{E}}_9)/\sim$ or $\widehat{\widehat{D}}_{16}$, and we see that indeed the operators associated to the two torus directions are good central extensions.
Let us notice that in the limit $R_{8,9} \to \infty$ the only possible decompactification is to the ten dimensional theory one started with, independently of their ratio. Since the toroidally compactified $\mathrm{E}_8 \times \mathrm{E}_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ theories are T-dual, it is also possible to reach the 10 dimensional $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic theory from the eight dimensional $\mathrm{Spin}(32)/\mathbb{Z}_2$ one and viceversa.[3]

### 4.3.2 Decompactification limits of the heterotic String on $T^d$

One can easily generalize the above discussion to the decompactification of an arbitrary number $k$ of dimensions starting from the heterotic theory compactified on $T^d$, $k \leq d$.
We start from a fixed point in the moduli space of $T^{d-k}$, specified by a given $G_{\hat{\imath}\hat{\jmath}}$, $B_{\hat{\imath}\hat{\jmath}}$ and $A^I_{\hat{\imath}}$ ($\hat{\imath} = 1, .., d-k$), giving an enhanced gauge algebra $\mathfrak{g}$. As explained in Section 4.3.1, in order to decompactify to a Lorentz invariant vacuum of a higher dimensional toroidally compactified heterotic theory, the $T^d$ background must be such that $T^d = T^k \times T^{d-k}$ as follows

$$G_{ij} = \begin{pmatrix} G_{\bar{\imath}\bar{\jmath}} & 0 \\ 0 & G_{\hat{\imath}\hat{\jmath}} \end{pmatrix} = e_i \cdot e_j, \quad B_{ij} = \begin{pmatrix} B_{\bar{\imath}\bar{\jmath}} & 0 \\ 0 & B_{\hat{\imath}\hat{\jmath}} \end{pmatrix}, \quad A^I_i = (A^I_{\bar{\imath}}, A^I_{\hat{\imath}}) \tag{4.3.26}$$

with $i = 1, ..., d$, $i = (\bar{\imath}, \hat{\imath})$. $G_{\bar{\imath}\bar{\jmath}}$ is parametrised by $k$ radii $R_{\bar{\imath}}$ and $\frac{k(k-1)}{2}$ angles. The decompactification limit is realized by taking $R_{\bar{\imath}} \to \infty$, and a generic finite value for $A^I_{\bar{\imath}}$. Again, T-dual backgrounds can be obtained via an $O(d, d+16, \mathbb{Z})$ transformation to the one described above.
The general pattern is that, under these assumptions, the $10-d$ dimensional theory displays the $k$-times loop version of the gauge algebra in $10-(d-k)$ dimensions at the point in moduli space given by $\{g_{\hat{\imath}\hat{\jmath}}, B_{\hat{\imath}\hat{\jmath}} A^I_{\hat{\imath}}\}$, with central extensions $(\partial Y^{\bar{\imath}})_0$. In the case of a semi-simple algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_{\mathrm{n}}$ all the factors are made affine.
For concreteness, let us introduce the index $\hat{I} = \{\hat{\imath}, I\}$, $X^{\hat{I}}(z) = (\sqrt{2} e^{\hat{a}}_{\hat{\imath}} Y^{\hat{\imath}}(z), X^I(z))$, and call and call $p_{\alpha;\hat{I}}$ the roots of the algebra $\mathfrak{g}$ in dimension $10-(d-k)$, corresponding to the $16+d-k$ momenta $p_{\alpha;\hat{I}} = (e^{*\hat{\imath}}_{\hat{a}} p_{L,\hat{\imath}}, p_I)$.
The left-moving massless states and their associated holomorphic conserved currents, related to the Cartan and ladder generators respectively, are then

$$\begin{aligned} \alpha^{\hat{I}}_{-1} \bar{\psi}^{\mu}_{-\frac{1}{2}} |0\rangle_{NS} &\longrightarrow J^{\hat{I}}(z) = i \partial X^{\hat{I}}(z) \,, \\ \bar{\psi}^{\mu}_{-\frac{1}{2}} |Z_{\mathfrak{g}}\rangle_{NS} &\longrightarrow J^{\alpha}(z) = c_\alpha e^{i p_{\alpha;\hat{I}} \cdot X^{\hat{I}}(z)} \,. \end{aligned} \tag{4.3.27}$$

and they satisfy the massless and level-matching conditions

$$\begin{cases} 0 = G^{\hat{\imath}\hat{\jmath}}(p_{L,\hat{\imath}} p_{L,\hat{\jmath}} + p_{R,\hat{\imath}} p_{R,\hat{\jmath}}) + p_I^2 + 2\Big(N + \tilde{N} - \dfrac{3}{2}\Big) \,, \\ 0 = 2 \mathrm{n}_{\hat{\imath}} \mathrm{w}^{\hat{\imath}} + 2\Big(N - \tilde{N} - \dfrac{1}{2}\Big) \,. \end{cases} \tag{4.3.28}$$

[3] For this we can take $R_9 \to 0$ and Wilson line $\Lambda = \big(\big(\frac{1}{2}\big)_8, 0_8\big)$ and then take $R_8 \to \infty$ with any Wilson line, or equivalently $8 \leftrightarrow 9$.

Additionally compactifying $k$ directions as in (4.3.26), the internal momenta $P$ of the $10-d$ dimensional theory in terms of the ones in the $10-(d-k)$ dimensional ones, $p$, are

$$\begin{aligned} P_{R,\hat{\imath}} &= p_{R,\hat{\imath}} - \frac{1}{\sqrt{2}} E_{\hat{\imath}\bar{\jmath}} \mathrm{w}^{\bar{\jmath}} \,, \\ P_{L,\hat{\imath}} &= p_{L,\hat{\imath}} - \frac{1}{\sqrt{2}} E_{\hat{\imath}\bar{\jmath}} \mathrm{w}^{\bar{\jmath}} \,, \\ P^I &= p^I + A^I_{\bar{\jmath}} \mathrm{w}^{\bar{\jmath}} \,, \end{aligned} \tag{4.3.29}$$

while $P_{R/L,\bar{\imath}}$ are independent from $p_{R/L,\hat{\imath}}$.
The inverse internal metric components satisfy $G^{\bar{\imath}\bar{\jmath}} \sim \frac{1}{R_{\bar{\imath}} R_{\bar{\jmath}}} \to 0$. In order to have massless states one should set to zero the possibly divergent term in the $10-d$ dimensional mass formula

$$G^{\bar{\imath}\bar{\jmath}} \left( (2g_{\bar{\imath}\bar{k}} - E_{\bar{\imath}\bar{k}})(2g_{\bar{\jmath}\bar{h}} - E_{\bar{\jmath}\bar{h}}) + E_{\bar{\imath}\bar{h}} E_{\bar{\jmath}\bar{h}} \right) \mathrm{w}^{\bar{k}} \mathrm{w}^{\bar{h}} \sim g_{\bar{k}\bar{h}} \mathrm{w}^{\bar{k}} \mathrm{w}^{\bar{h}} = 0 \tag{4.3.30}$$

which is achieved by requiring $\mathrm{w}^{\bar{\imath}} = 0$. The level-matching condition in $10-d$ dimensions reduces to

$$0 = 2\mathrm{n}_{\hat{\imath}} \mathrm{w}^{\hat{\imath}} + |\pi_\alpha|^2 + 2\left( N - \tilde{N} - \frac{1}{2} \right) \tag{4.3.31}$$

which is exactly the same expression in $10-d+k$ dimensions with the only difference that the oscillators can be turned on also in the additional $k$ directions. This will provide the states related to the central extensions.
Moreover, having vanishing winding along the directions we are decompactifying in (4.3.29) gives asymptotically in moduli space

$$P_{R,\hat{\imath}} = p_{R,\hat{\imath}} \,, \quad P_{L,\hat{\imath}} = p_{L,\hat{\imath}} \,, \quad P^I = p^I \,, \tag{4.3.32}$$

so that the massless condition is

$$\begin{aligned} G^{\hat{\imath}\hat{\jmath}} \left( p_{L,\hat{\imath}} p_{L,\hat{\jmath}} + p_{R,\hat{\imath}} p_{R,\hat{\jmath}} \right) + G^{\bar{\imath}\bar{\jmath}} \left( P_{L,\bar{\imath}} P_{L,\bar{\jmath}} + P_{R,\bar{\imath}} P_{R,\bar{\jmath}} \right) + \\ + p_I^2 + 2\left( N + \tilde{N} - \frac{3}{2} \right) = 0 \,. \end{aligned} \tag{4.3.33}$$

where for $\mathrm{w}^{\bar{\imath}} = 0$

$$\begin{aligned} P_{R,\bar{\imath}} &= \frac{1}{\sqrt{2}} \left( \mathrm{n}_{\bar{\imath}} - E_{\bar{\imath}\hat{\jmath}} \mathrm{w}^{\hat{\jmath}} - \pi_\alpha \cdot A_{\bar{\imath}} \right) \,, \\ P_{L,\bar{\imath}} &= \frac{1}{\sqrt{2}} \left( \mathrm{n}_{\bar{\imath}} - E_{\bar{\imath}\hat{\jmath}} \mathrm{w}^{\hat{\jmath}} - \pi_\alpha \cdot A_{\bar{\imath}} \right) \,. \end{aligned} \tag{4.3.34}$$

Since at leading order $G^{\bar{\imath}\bar{\jmath}} \sim \frac{1}{R_{\bar{\imath}} R_{\bar{\jmath}}} \to 0$ and $G^{\hat{\imath}\hat{\jmath}} \sim \mathcal{O}(1)$, (4.3.33) reduces to

$$G^{\hat{\imath}\hat{\jmath}} \left( p_{L,\hat{\imath}} p_{L,\hat{\jmath}} + p_{R,\hat{\imath}} p_{R,\hat{\jmath}} \right) + p_I^2 + 2\left( N + \tilde{N} - \frac{3}{2} \right) = 0 \,, \tag{4.3.35}$$

and together with (4.3.31) this implies that the massless spectrum coincides with the momentum towers of the massless spectrum of the $10-(d-k)$ dimensional theory (4.3.28), with

$k$ additional left KK vectors (with related momentum towers) to the Cartans in $10-d+k$ dimensions, so that the massless vectors are

$$\begin{aligned} \alpha^{\bar{I}}_{-1}\bar{\psi}^{\mu}_{-\frac{1}{2}}\left|0,\mathrm{n}_{\bar{j}}\right\rangle_{NS} &\longrightarrow J^{\bar{I}}(z)=i\partial X^{\bar{I}}(z)e^{i\mathrm{n}_{\bar{j}}Y^{\bar{j}}(z)}\,, \\ \bar{\psi}^{\mu}_{-\frac{1}{2}}\left|Z_{\mathfrak{g}},\mathrm{n}_{a}\right\rangle_{NS} &\longrightarrow J^{\alpha}(z)=c_{\alpha}e^{ip_{\alpha;I}X^{I}(z)}e^{i\mathrm{n}_{\bar{j}}Y^{\bar{j}}(z)}\,. \end{aligned} \tag{4.3.36}$$

where $\bar{I}=(i,I)=(\bar{\imath},\hat{I})$, $X^{\bar{I}}=(\sqrt{2}Y^{\bar{\imath}},X^{\hat{I}})$.
The OPEs among these states are found in Appendix B.3, where one can see that the algebra is the $k$-th loop version of $\mathfrak{g}$ (with the corresponding $k$ central extensions).

## 4.4 String junctions

In the previous Sections 4.1 and 4.3 we argued that infinite distance points in the moduli space of the perturbative heterotic theory on $T^d$ are characterised by the presence of affine BPS algebras. In particular, the predictions in 4.3.1 for the case $d=2$ can be matched to the ones in the dual framework of F-theory on K3, worked out in [23]. The presence of affine algebras in these limits is locally detected through the intersection patterns of string junctions supported by stacks of 7-branes as argued already in [115, 116].
In this section we give a brief overview of string junctions, their properties under monodromies and some key concepts for the following study of affine algebras. We follow mainly [116] (see also [117]).
The reader who is already familiar with these concepts can skip this section and go directly to Section 4.5.

### 4.4.1 Basic concepts on String Junctions

The spectrum of F-theory includes $(p,q)$-strings, bound states of $p$ F1 strings and $q$ D1 strings. 7-branes also carry $[p,q]$-indices, which indicate the kind of $(p,q)$-strings that end on them. A string junction is a web of $(p,q)$-strings that may join at a point[4] in a way that conserves their $(p,q)$ charge, as shown in Figure 4.1(a). A segment of $(p,q)$ string that starts or ends at a $[p,q]$ brane is called a prong, see Figure 4.1(b). Junctions may have prongs at different $[p,q]$-branes and/or include strings extending away from them that carry some asymptotic $(p,q)$ charge away from the branes. Upon encircling a 7-brane $X_{[p,q]}$ (i.e. when crossing the branch cut), an $(r,s)$-string gets transformed by the corresponding monodromy as (see Figure 4.2(a))

$$\begin{pmatrix} r \\ s \end{pmatrix} \to M_{[p,q]}\begin{pmatrix} r \\ s \end{pmatrix}, \qquad M_{[p,q]}=\begin{pmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{pmatrix}=\mathbb{I}_{2x2}+\begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix} \tag{4.4.1}$$

Moreover, this segment of the junction can be moved across the 7-brane to a position in which it does not cross the corresponding branch-cut anymore. For this crossing to be consistent with the monodromy transformation above, there should be an extra prong leaving $X_{[p,q]}$ and joining the original segment of the junction (in analogy with the original Hanany-Witten effect [118]), as shown in Figure 4.2(b). Note that the junction that is obtained after the monodromy

[4] This is a pictorial way of seeing the intersection of two-dimensional worldsheets

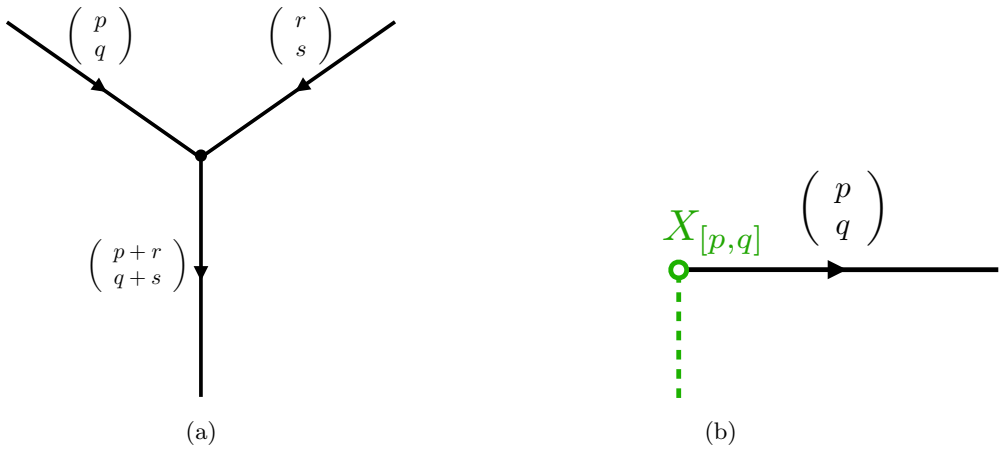



**Figure 4.1:** **(a)** String junction from joining two strings with $(p, q)$ and $(r, s)$ charges at a point. **(b)** A $(p, q)$ prong. The dotted vertical line represents the branch cut of the 7-brane, which is not intersected by the string.

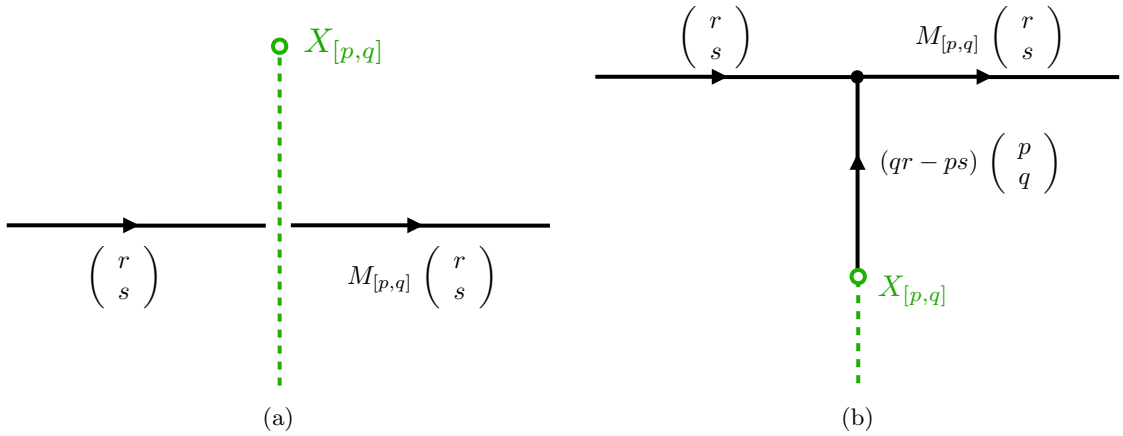



**Figure 4.2:** The Hanany-Witten effect in terms of string junctions

transformation acts on the $(r, s)$-string is precisely

$$M_{[p,q]} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix} + (rq - sp) \begin{pmatrix} p \\ q \end{pmatrix} , \tag{4.4.2}$$

where the second term on the right hand side means there are $rq - sp$ strings of $(p, q)$ type. This shows that both configurations are indeed equivalent. 7-brane configurations related by a global $SL(2, \mathbb{Z})$ transformation or by a relocation of branch-cuts (i.e. a reordering of the branes) are physically equivalent [111, 119]. Moving an $X_{[p,q]}$ brane across the branch-cut of $X_{[r,s]}$ is equivalent to (see Figure 4.3)

$$X_{[p,q]}\, X_{[r,s]} \to X_{[r,s]}\, X_{[p',q']} \qquad \text{with} \qquad p' = p - (rq - sp)r\,, \quad q' = q - (rq - sp)s \tag{4.4.3}$$

In a configuration with several 7-branes, one can always choose the branch-cuts to be vertical

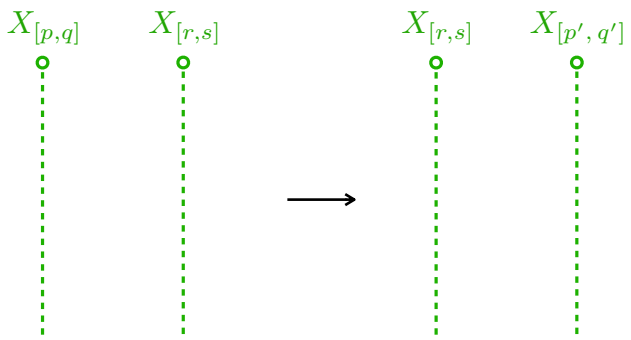



**Figure 4.3:** Reordering of the branch-cuts. After displacing the $X_{[p,q]}$ brane from left to right across the branch cut of the $X_{[r,s]}$ brane it becomes a $X_{[p',q']}$ brane with $p' = p + (sp - rq)r$ and $q' = q + (sp - rq)s$.

lines going downwards. For such a stack of branes ordered from left to right, the full monodromy can be calculated by the product of the individual monodromies from right to left, that is

$$M_{X_{[p_1,q_1]}X_{[p_2,q_2]}\ldots X_{[p_n,q_n]}} \;=\; M_{[p_n,q_n]} \,\ldots\, M_{[p_2,q_2]}\, M_{[p_1,q_1]}\,. \tag{4.4.4}$$

Moreover, in eight dimensions, the tadpole cancellation condition requires a total of 24 $[p,q]$ 7-branes with total monodromy equal to the identity.
Any configuration of 7 branes can be decomposed into the combination of three particular 7 branes called $A$, $B$ and $C$

$$\begin{aligned}
A \equiv X_{[1,0]}: \qquad & M_A \;=\; M_{[1,0]} \;=\; \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \\
B \equiv X_{[1,-1]}: \qquad & M_B \;=\; M_{[1,-1]} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \\
C \equiv X_{[1,1]}: \qquad & M_C \;=\; M_{[1,1]} \;=\; \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{4.4.5}$$

Finally, let us mention that the $(p,q)$ strings have a natural interpretation in M-theory as M2-branes wrapping the $pA + qB$ cycle in the elliptic fiber, and the 1-cycle that characterizes the string on the base. String junctions can then be understood as bound states of the corresponding M2-branes wrapping the relevant cycles in the fiber and the base. The intersection patterns of the relevant 2-cycles in homology, which are crucial for the identification of gauge groups, can be described effectively by the so called junction lattice [116], which we introduce now.

### 4.4.2 The Junction Lattice

Let us introduce a charge vector for a given string junction, whose entries are the number of prongs starting/ending on a given 7-brane (i.e. these vectors should have 24 entries in a global eight dimensional compactification). By convention the charge is positive (negative) for a prong with outwards (inwards) orientation with respect to a brane. That is, if we introduce a basis

for the charge space, by denoting $\vec{x}_{[p,q]}$ the unit vector associated to a particular $X_{[p,q]}$ brane in the configuration, the charge of $n$ $(p,q)$-strings leaving that brane would be $\vec{Q} = n\,\vec{x}_{[p,q]}$. In general, for a stack of branes ordered from left to right (with branch-cuts pointing downwards) $X_{[p_1,q_1]}X_{[p_2,q_2]}\dots X_{[p_n,q_n]}$ we can characterize a string junction by the charge vector

$$\vec{Q} = \sum_i Q^i \vec{x}_{[p_i,q_i]}\,, \qquad Q^i \in \mathbb{Z}\,. \tag{4.4.6}$$

The lattice of all possible charges is the so-called junction lattice, and one can define a basis of strings as the ones that correspond to the basis vectors in the charge lattice. A junction can also carry some asymptotic $(p,q)$ charge away from the 7-branes, and it is given by the expression

$$\begin{pmatrix} p \\ q \end{pmatrix}_{\text{asymptotic}} = \sum_i Q^i \begin{pmatrix} p_i \\ q_i \end{pmatrix}\,. \tag{4.4.7}$$

It is also possible to introduce a symmetric intersection bilinear pairing between two junctions, $(\cdot\,,\cdot)$ [116, 120]. This pairing captures the intersection properties of the corresponding 2-cycles in the M-theory uplift. By defining the self-intersection of any two basis strings to be

$$(\vec{x}_{[p_i,q_i]},\, \vec{x}_{[p_i,q_i]}) \;=\; -1\,, \tag{4.4.8}$$

exploiting bilinearity and the invariance under junction transformations (c.f. Figure 4.2) one finds that the intersection between different basis elements takes the form

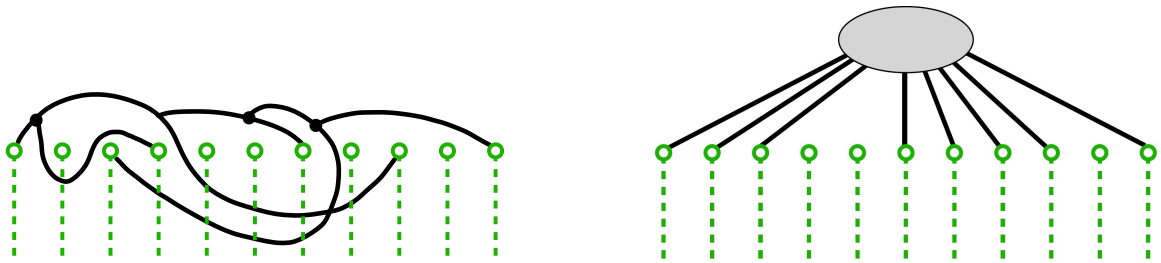
$$(\vec{x}_{[p_i,q_i]},\, \vec{x}_{[p_j,q_j]}) \;=\; \frac{1}{2}(p_i q_j - p_j q_i)\,, \tag{4.4.9}$$

when the position of the brane $X_{[p_i,q_i]}$ is on the left of $X_{[p_j,q_j]}$.[5] For a set of branes given by $A_1\dots A_{n_a}\,B_1\dots B_{n_b}\,C_1\dots C_{n_c}$, denoting the corresponding basis vectors by $\vec{a}_i$, $\vec{b}_i$ and $\vec{c}_i$, the relevant pairings are

$$\begin{array}{lll} (\vec{a}_i,\vec{a}_j) = -\delta_{ij}\,, & (\vec{b}_i,\vec{b}_j) = -\delta_{ij}\,, & (\vec{c}_i,\vec{c}_j) = -\delta_{ij}\,, \\ (\vec{a}_i,\vec{b}_j) = -1/2\,, & (\vec{a}_i,\vec{c}_j) = 1/2\,, & (\vec{b}_i,\vec{c}_j) = 1\,. \end{array} \tag{4.4.10}$$

Let us remark that, by construction, the charge and the pairing are invariant under the operations of moving a junction across a 7-brane and reordering of the branes, as displayed in figs. 4.2 and 4.3, respectively. The former operation can actually be used to define a *canonical* representation for junctions, which is the one in which the junction itself does not cross any branch cut. That is, given a general string junction, we can construct its canonical representation by just moving any segment crossing a branch cut across the corresponding brane by adding the extra prongs that appear due to the Hanany-Witten effect, until no segment of the junctions crosses any branch cut. This is displayed schematically in Figure 4.4. The latter operation, namely rearranging the branch-cuts and changing the order of the branes, corresponds to a base change that is taken into account by the transformation properties of

[5] Let us remark that the pairing is defined to be symmetric, but in order to compute it for a given configuration the ordering of the branes matters. That is, in this case one can define $(\vec{x}_{[p_j,q_j]},\, \vec{x}_{[p_i,q_i]}) := (\vec{x}_{[p_i,q_i]},\, \vec{x}_{[p_j,q_j]}) = \frac{1}{2}(p_i q_j - p_j q_i)$.



(a) (b)

**Figure 4.4:** **(a)** String junction in an arbityrary representation. **(b)** The same string junction in the canonical representation, including the existence of new prongs after moving the original junction across the branes in order for it not to cross any branch-cut.

the branes given in eq. (4.4.3) [117].

### 4.4.3 String Junctions and (Affine) Lie Algebras

So far, we have introduced the string junctions, together with their allowed invariant charges, which form the junction lattice. This plays a crucial role in the discussion of algebras, as certain string junctions turn out to represent the roots of the algebra (and their corresponding affine extensions), and their pairwise intersections (given by the pairing defined above) reproduce the negative of the Cartan matrix of the associated algebra [111, 115, 116, 121, 122].
In the following, we will restrict to junctions with no asymptotic charge, as they will suffice to describe the root sector of the groups (and their corresponding affine or loop versions) that we study here.[6]
Therefore, in order to characterize the algebra associated to a stack of 7-branes, one needs to specify the branes and find the particular string junctions that give rise to the (simple) roots.

#### Example: The $A_n$ Algebra

Let us introduce the simple example of the $A_n = \mathfrak{su}(n+1)$ algebra, following [116]. This algebra is realized by a stack of $(n+1)$ $A$ branes, with monodromy

$$M_{A^{n+1}} = \begin{pmatrix} 1 & -n-1 \\ 0 & 1 \end{pmatrix}, \tag{4.4.11}$$

The junctions that represent the $n$ simple roots have charges

$$\vec{\alpha}_i = \vec{a}_i - \vec{a}_{i+1} \tag{4.4.12}$$

that is, the $i$-th entry is 1, the $(i+1)$-th entry is -1 and the rest are zero. From this charges we can conclude that the simple roots are strings starting at one brane and ending at the next, as displayed in Figure 4.5. Using eq. (4.4.10) one can calculate the intersection form, which

[6] More general junctions including asymptotic charges are necessary to describe more general weight sectors of the corresponding groups [116].

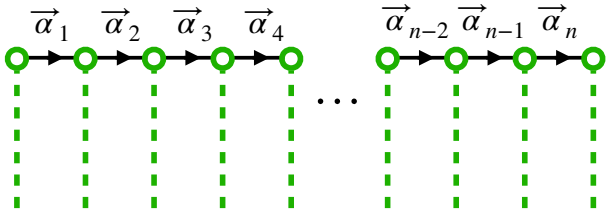



**Figure 4.5:** String junctions that realize the simple roots $\vec{\alpha}_i$ of the $A_n$ algebra in a stack of $(n+1)$ A-branes.

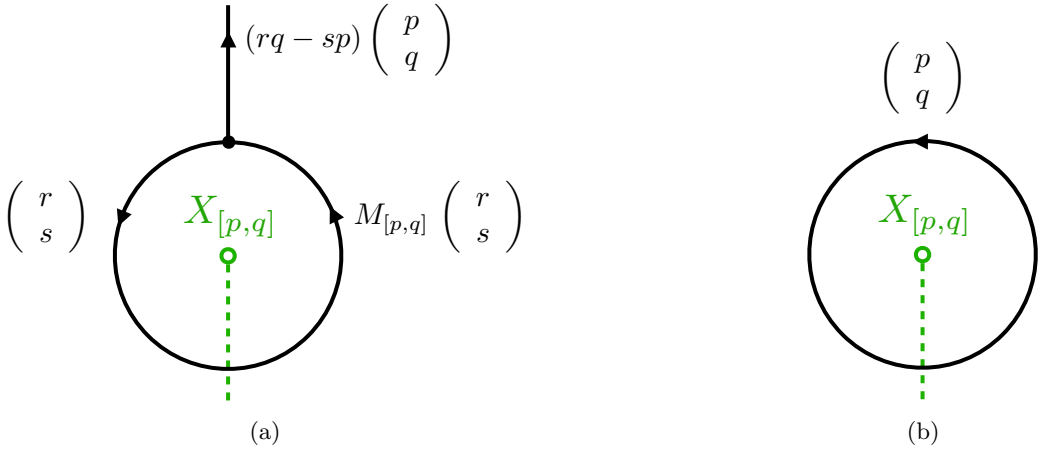



**Figure 4.6:** Two loop junctions (in their non-canonical form)

yields

$$(\vec{\alpha}_i, \vec{\alpha}_j) = \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 \end{pmatrix}, \tag{4.4.13}$$

and indeed reproduces the negative of the Cartan Matrix for $A_n$.

### Loop Junctions and Affine/Loop Algebras

Consider now a junction that takes the form of a loop around a (stack) of 7-branes, and possibly an extra piece attached to it that may carry some asymptotic charge, as displayed in Figure 4.6(a). The monodromy transformation associated to the (stack of) 7-brane(s) acts on the junction as it crosses the branch cut(s) such that the asymptotic charge is compatible with it. Furthermore, one can bring the junction into its canonical representation by moving it through the brane and taking into account the corresponding Hanany-Witten transition. In the case displayed in Figure 4.6(a) this would correspond to a charge vector given by $\vec{Q} = (rq-sp)\vec{x}_{[p,q]}$, so the charge in that case is equivalent to the asymptotic charge under the brane surrounded

by the loop junction. The situation becomes much richer when several different branes are encircled by the loop, as we will see in the following. Additionally, one can consider the simple case of a $(p,q)$ string surrounding a $X_{[p,q]}$ brane, as shown in Figure 4.6(b), where the junction carries no asymptotic charge since the monodromy acts trivially on it. More generally, loop junctions surrounding a stack of branes and carrying no asymptotic charge can be built whenever the full monodromy action of the stack is such that it leaves the junction invariant. This situation will be particularly relevant in the identification of imaginary roots for the affine algebras below.

Let us now remark a couple of particularly interesting properties of loop junctions. First, it can be checked that loop junctions with no asymptotic charge have vanishing self-intersection, making them the perfect candidates for a junction realization of the imaginary root associated to the affine or loop version of an algebra. Second, recall that the roots of an algebra on a stack of branes are also realized by string junctions with no asymptotic charge, so they always start and end on different branes. It is straightforward to see that a loop junction surrounding the whole configuration does not intersect any of roots, which is the other key condition that an imaginary root must fulfil. Note that these two properties can be properly checked by computing the intersection from the charges in the canonical representation.

We must stress here the importance of the loop junction not having an asymptotic charge, as it is the one that has vanishing self-intersection, and it is the one that indirectly indicates what must be added to a finite algebra in order to affinize it. As explained above, in order for a loop junction with no asymptotic charge to be supported around a brane configuration, it must be left invariant under the total monodromy. That is, for a given stack of branes, the only allowed loop junctions are the ones that correspond to the eigenspaces that are left invariant by such monodromy. In many general stacks of branes (as e.g. the ones that realize the finite exceptional groups) this space is just empty.[7] Thus, the standard procedure in this case is to add one extra brane to the stack such that the new total monodromy allows for a loop junction in order to realize the affine version of the previous algebra.

In general, one must also check that the imaginary root is non-trivial, meaning it has non-zero charge in the junction lattice. This might look like a meaningless check, but it turns out to be key in some cases. Consider for example the $\mathfrak{su}(n)$ algebra discussed above. Its total monodromy (4.4.11) leaves invariant the space spanned by $(1,0)$ strings. However, a $(1,0)$ loop junction surrounding the whole configuration does not make the algebra affine because it is actually contractible (one can just move the loop across all branes without generating any extra junction by the Hanany-Witten effect). This can be detected by the fact that its charge vector is $\vec{Q}=0$, as can be easily seen from its canonical representation. As we will see in section 4.5, a different way to affinize such an algebra is by having some extra branes far away from their stack such that the $(1,0)$ loop junction has some non-trivial charge under them in the junction lattice. This is in contrast with the usual procedure (typically used for exceptional algebras) where one adds some extra brane to change the total monodromy in the right way.

[7] To be precise, it cannot be realized by $(p,q)$-strings with integer charges.

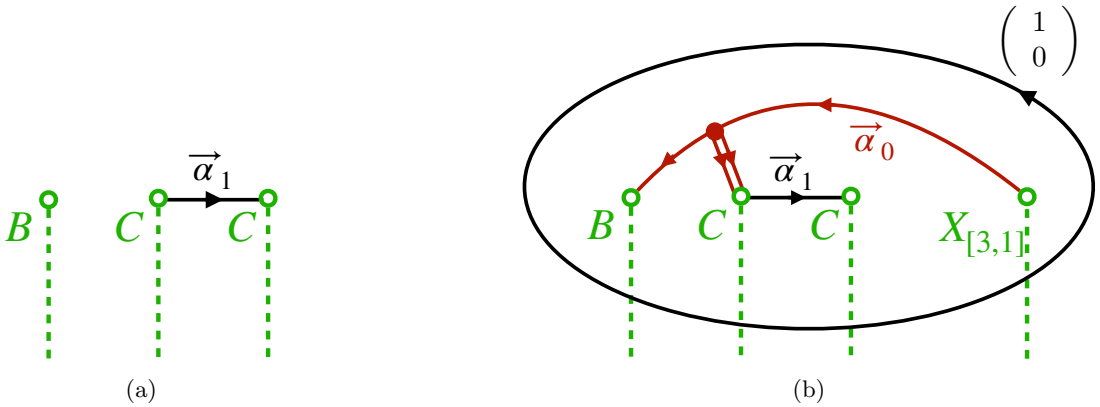



**Figure 4.7:** **(a)** Realization of the $A_1$ algebra on a BCC stack of branes, where $\vec{\alpha}_1$ gives the simple root. **(b)** Realization of the $\widehat{A}_1$ algebra by adding a $X_{[3,1]}$ brane to the BCC stack. The imaginary root, $\vec{\delta}_{(1,0)}$ is given by a $(1,0)$ string loop around the whole configuration, and the affine root (in red) is built as $\vec{\alpha}_0 = \vec{\delta}_{(1,0)} - \vec{\alpha}_1$ .

### Example: The $\widehat{A}_1$ Algebra

Let us consider the case of the affinization of the $A_1$ algebra. Apart from $A^2$, the $A_1 = E_1$ algebra can be realized by a configuration of $BCC$ branes. The single simple root, $\vec{\alpha}_1$, which is also the highest root, is given by a junction with charge $\vec{\alpha}_1 = (0, 1, -1)$ (under the original $BCC$ stack),[8] as shown in Figure 4.7(a). The associated monodromy matrix is given by

$$M_{BCC} = \begin{pmatrix} -2 & -7 \\ -1 & -4 \end{pmatrix} , \tag{4.4.14}$$

and it cannot support any physical loop junction around it. By adding a $X_{[3,1]}$ brane to the stack, the total monodromy yields

$$M_{BCCX_{[3,1]}} = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix} , \tag{4.4.15}$$

which can support a $(1,0)$ string loop surrounding the four branes. One can easily calculate its charge in the junction lattice by going to the canonical representation, and also build the affine root $\vec{\alpha}_0 = \vec{\delta}_{(1,0)} - \vec{\alpha}_1$, obtaining

$$\vec{\alpha}_1 = (0, 1, -1, 0) , \qquad \vec{\delta}_{(1,0)} = (-1, -1, -1, 1) , \qquad \vec{\alpha}_0 = (-1, -2, 0, 1) . \tag{4.4.16}$$

[8] In this simple case this is easy to see, since the requirement of absence of asymptotic charge straightforwardly restricts the possible junctions to those with as many prongs starting in one of the $C$ branes as ending in the remaining $C$ brane. For more general brane configurations these conditions give rise to a more cumbersome set of diophantine equations, studied more systematically in [115].

By using the pairings given in eqs. (4.4.8)-(4.4.10), the intersection matrix for the $BCCX_{[3,1]}$ configuration is

$$(\cdot,\cdot) = \begin{pmatrix} -1 & 1 & 1 & 2 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{pmatrix} . \tag{4.4.17}$$

With this pairing, it can be checked that the imaginary root is indeed orthogonal to the simple roots, namely

$$(\vec{\delta}_{(1,0)}, \vec{\alpha}_1) = (\vec{\delta}_{(1,0)}, \vec{\alpha}_0) = 0 , \tag{4.4.18}$$

and the Gramm matrix between the two simple roots $\vec{\alpha}_0$ and $\vec{\alpha}_1$ takes the form

$$(\vec{\alpha}_i, \vec{\alpha}_j) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \tag{4.4.19}$$

which is indeed the negative of the Cartan Matrix for $\widehat{A}_1$.

## 4.5 Heterotic/F-theory duality at infinite distance

The string junction picture can only give local information about the affinisation process. The types of affine algebras that can actually be realised in F-theory on an elliptically fibered K3 were analysed in detail in [22, 23] relying on the global information contained in the degenerations of elliptically fibered K3 lying at infinite distance in the complex structure moduli space. In turn, these can be described and classified in terms of Kulikov models and can be broadly divided into four classes, which we introduce briefly here in relation to the heterotic theory, referring the reader to the Appendix D for the detailed geometrical description

- Type II.a: the degenerate K3 surface has non-minimal singularities according to the Kodaira classification. This limit is dual to the full decompactification limit to the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic theory, and realises the algebra $(\widehat{\mathrm{E}}_9 \oplus \widehat{\mathrm{E}}_9)/\sim$ as derived in 4.5.1.

- Type II.b: this is an Emergent String limit, where the associated K3 surface has singular fibers already in codimension zero loci on the base. Since it has no counterpart in the weakly coupled heterotic theory, we will not discuss it further.

- Type III.a: the K3 has non-minimal fibers and it corresponds to a partial decompactification of the heterotic dual from eight to nine dimensions. One example is given in 4.5.2.

- Type III.b: the degenerate K3 surface has both non minimal singularites and codimension zero singular fibers, and it corresponds to a weak coupling limit accompanied by a full decompactification in the Type IIB string frame. It realises the decompactification to the $SO(32)$ dual heterotic theory in ten dimensions as shown in 4.5.3.

The states that are dual to the KK towers in the heterotic decompactification limits are partially captured in the M-theory picture as M2 branes wrapping vanishing calibrated 2-tori in the degenerate K3 geometry, as explained in the following paragraphs.

It is important to remark that while in the F-theory setting one can reach all the infinite distance points in moduli space, one can do it in a controlled way only along certain paths, namely the ones compatible with the algebraic nature of the associated Weierstrass models. We elaborate on this in Appendix D.3.
In this Section, we consider the brane configurations studied in [23] and examine the affine enhancements that arise in the language of string junctions. The geometric idea is that at the infinite distance limits, the junctions, and therefore all the states realizing the affine algebra, become asymptotically massless. This geometric picture is studied in detail in the references above, so we will not elaborate further here, but instead we will consider the BPS algebra that can arise at each configuration by using their geometric results.

### 4.5.1 $(\widehat{\mathbf{E}}_9 \oplus \widehat{\mathbf{E}}_9)/\sim$

The Type II.a Kulikov model is related to the presence of non minimal Kodaira singularities in the K3 degeneration. Following the general procedure, these can be resolved by a blowup of the base that zooms in their brane content. In the F-theory setup, the basis degenerates into two components, each of which has twelve branes. This corresponds to the so-called stable degeneration limit, at which the geometric picture consists of two del Pezzo surfaces intersecting at a 2-torus, which is known to be dual to the heterotic theory on the torus at the intersection in the limit in which the torus becomes large. We know from the heterotic computation in Section 4.3.1 that the algebra in the full decompactification limit to the ten dimensional heterotic should be $(\widehat{\mathrm{E}}_9 \oplus \widehat{\mathrm{E}}_9)/\sim$, where the $\sim$ indicates that the imaginary roots enhancing both factors are identified.
There is an $\mathrm{E}_8$ factor on each component, which is realized by the brane configuration $A^7BCC$ made of ten out of the twelve branes in each basis component. This algebra gets enhanced to $E_9$ by adding a $X_{[3,1]}$ brane, in analogy to the the example in section 4.4.3. The total monodromy of this configuration is then

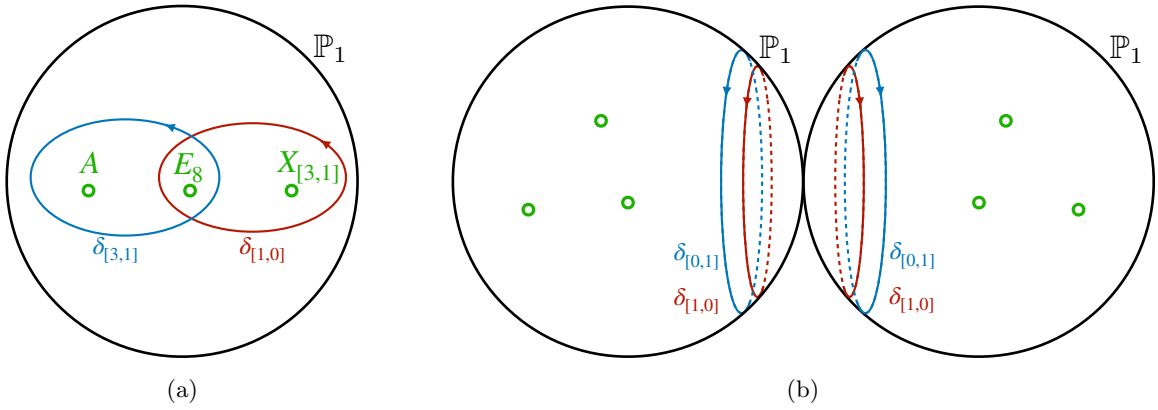
$$M_{A^7BCCX_{[3,1]}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{4.5.1}$$

which clearly allows for a $\delta_{[1,0]}$ loop junction surrounding the whole configuration yielding the imaginary root for the affine enhancement. Alternatively, by considering the $\mathrm{E}_8$ stack together with the extra $A$ brane which is present in the same component of the basis, one obtains the brane configuration $A^8BCC$. This brane configuration supports another imaginary root, given by the loop junction $\delta_{[3,1]}$, as can be seen from the invariant eigenspace of the corresponding monodromy matrix

$$M_{A^8BCC} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}. \tag{4.5.2}$$

By considering the whole configuration, one obtains the loop algebra $\widehat{\mathrm{E}}_9$, as can be seen from the fact that the full monodromy matrix is the identity

$$M_{\widehat{\mathrm{E}}_9} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.5.3}$$




(a) (b)

**Figure 4.8:** **(a)** Brane configuration realizing the loop algebra $\widehat{\mathrm{E}}_9$. The two imaginary roots, $\delta_{[1,0]}$ and $\delta_{[3,1]}$, are realized by loop junctions surrounding the $\mathrm{E}_8$ stack and one additional brane (note they can also be represented as two independent junctions surrounding the whole configuration). The branes are represented in one of the basis components given by a $\mathbb{P}_1$. **(b)** Configuration realizing the algebra $(\widehat{\mathrm{E}}_9 \oplus \widehat{\mathrm{E}}_9)/\sim$, with the loop junctions which are identified in the junction lattice represented in both base components. This corresponds to the stable degeneration limit which is dual to the full decompactification of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string.

so that both imaginary roots are present, as displayed in Figure 4.8(a). Note that both string junctions can also be brought to a form in which they surround the whole configuration, and one can equally choose the basis $\delta_{[1,0]}$ and $\delta_{[0,1]}$ for the string junctions. In the M-theory lift these come from wrapping an M2-brane in the $(1,0)$ or $(0,1)$ cycle in the elliptic fiber, and the one cycle wrapping the whole brane configuration in each component of the base.
Each component supports then a pair of the aforementioned imaginary roots (see Figure 4.8(b)), but these two pairs are identified in the junction lattice. They can indeed be seen to be homologous in the M-theory lift, as nothing special happens to any of the fibre cycles at the intersection points. This is the reason behind the identification of the imaginary roots producing the affinizations in both components.
In this setting the towers of states are given by the $(n,0)$ and $(0,m)$ strings, which become massless since they are allowed to shrink to the base intersection point in the blown up version, or to the non minimal singularity from the perspective of the degenerate K3.
One interesting remark is the following. On the heterotic side, we can decompactify from 8 to 10 dimensions $R_8$, $R_9 \to \infty$ either in one step (corresponding to the case $\frac{R_8}{R_9} = \text{const}$) or in two steps ($\frac{R_8}{R_9} \to 0, \infty$) as already noticed in 4.3.1. In the F-theory picture, this Kulikov model seems to capture only the first possibility.

### 4.5.2 $(\widehat{\mathbf{E}}_{8-n} \oplus \widehat{\mathbf{E}}_8 \oplus \widehat{A}_{n+1})/\sim$

Consider now the situation where in the F-theory picture both end components of the base are $dP_9$ surfaces (see section 3.4 in [23]), which corresponds to a Type III.a degeneration. In particular, this includes the case where the heterotic dual realizes $\mathrm{E}_8 \oplus \mathrm{E}_8 \oplus A_1$ as explicitly shown in Section 4.3.1, whose F-theory dual is schematically displayed in Figure 4.9. The affinization process is similar to the previous one, but with some crucial differences. First of all, the monodromy around the exceptional factors $\widehat{\mathrm{E}}_{8-n}$ in the end components (including the

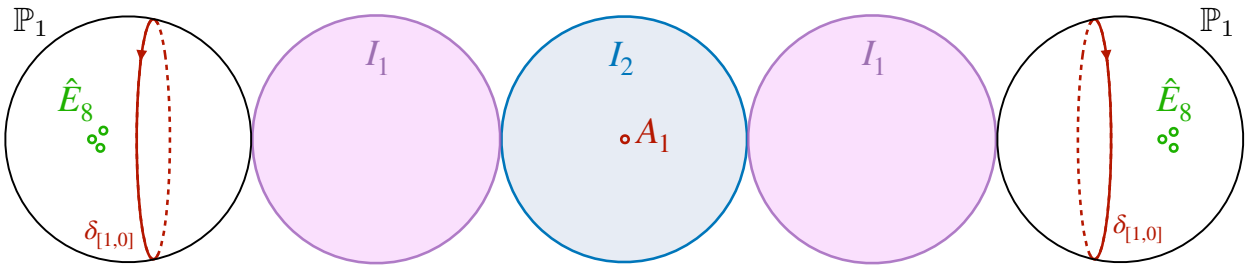



**Figure 4.9:** Schematic configuration of the branes and codimension zero singularities realizing the loop algebra $(\widehat{\mathrm{E}}_8 \oplus \widehat{\mathrm{E}}_8 \oplus \widehat{A}_1)/\sim$, with the loop junction $\vec{\delta}_{[1,0]}$ giving rise to the imaginary root. For the details on the intermediate components and their corresponding $I_k$ singularities see [23].

extra $X_{[3,1]}$ brane responsible for the affinization) takes the form

$$M_{\widehat{\mathrm{E}}_{8-n}} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix} = M_{I_{n+1}}^{-1} , \tag{4.5.4}$$

so that it only leaves invariant the space of loop junctions generated by $\vec{\delta}_{[1,0]}$, that is, there will be only one tower, as oppossed to the double loop enhancement associated to the two towers before. This can be understood in terms of a dual KK tower from decompactification from eight to nine dimensions, instead of full decompactification to ten dimensions.
Moreover, the internal components of the base, displayed in different colours for the $(\widehat{\mathrm{E}}_8 \oplus \widehat{\mathrm{E}}_8 \oplus \widehat{A}_1)/\sim$ example in Figure 4.9, have codimension zero singularities of the form $I_k$. Their existence is consistent with the full monodromy composition, which gets more involved in the presence of non-trivial codimension zero singularities. More precisely, from the general blow up procedure [22], given two components intersecting at a point with generic fibers $I_n$ and $I_m$, respectively, the contribution to the monodromy at the component where the generic $I_n$ is located is equivalent to the one from a codimension one singularity (located at the intersection) with monodromy

$$M_{I_{m-n}} = M_{I_n}^{-1} M_{I_m} . \tag{4.5.5}$$

Note that the contribution from the same intersection point, as seen from the component where the codimension zero $I_m$ singularity is located, is given by the inverse monodromy, which can indeed be consistently described by $m \leftrightarrow n$. For the $(\widehat{\mathrm{E}}_8 \oplus \widehat{\mathrm{E}}_8 \oplus \widehat{A}_1)/\sim$ example, this is consistent with the monodromy structure seen by each of the first three base components (starting from the left) in Figure 4.9, namely

$$M_{\widehat{E}_8} M_{I_1} = \mathbb{I}_{2x2} , \qquad M_{I_{-1}} M_{I_1} = \mathbb{I}_{2x2} , \qquad M_{I_{-1}} M_{A_1} M_{I_{-1}} = \mathbb{I}_{2x2} . \tag{4.5.6}$$

Note that the final two components get the same contributions as the first two.
Finally, the fact that the intermediate components are mutually local is key for the affinization of all the different components of the algebra by one unique imaginary root (given by the loop junction), so that the full algebra is $(\widehat{\mathrm{E}}_{8-n} \oplus \widehat{\mathrm{E}}_8 \oplus \widehat{A}_{n+1})/\sim$ (for $0 \leq n \leq 8$), with the three imaginary roots identified.

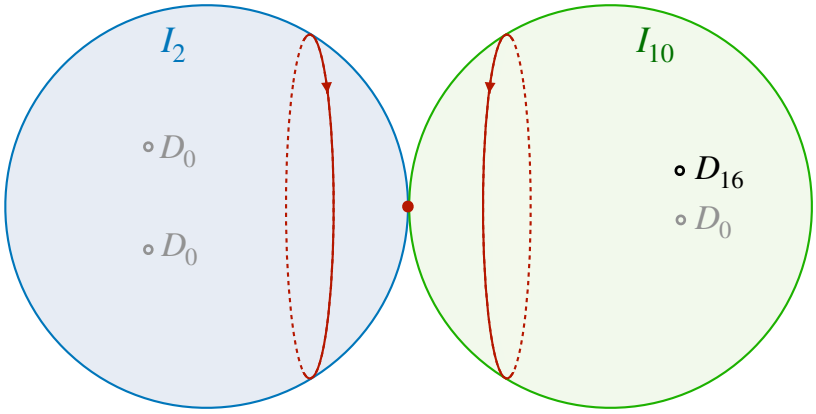



**Figure 4.10:** Configuration associated to the $\widehat{\widehat{D}}_{16}$, with the only loop junction one can detect in the Kulikov model framework in red. The KK tower giving the second imaginary root cannot be realised from this analysis.

Let us clarify this point, and the reason why one unique $\delta_{[1,0]}$ loop junction, which is invariant under crossing the intermediate component, is now enough for the affinization of the intermediate $A_{n+1}$ factor. Recall that the loop junction given by $\vec{\delta}_{[1,0]}$ is uncharged under the intermediate $A$ branes, and also under the $I_k$ singularities over generic points of the internal components of the base. This means the junction can just be transported from one end component to the other, crossing the $A^{n+2}$ branes forming the stack, as well as the components with codimension zero singularities, without any effect. Therefore such a junction could never give rise to an affinization of that algebra alone, as it would be trivial because it has no charge in the junction lattice. However, the existence of the exceptional factors in the end components gives this loop junction non-trivial charges in the junction lattice, and thus the $A_{n+1}$ factor can be affinized. In other words, it is the presence of the end components, and the fact that the loop junction has non-trivial charge under them in the charge lattice what allows for the affinization of all factors. In this sense, this affinization is a non-local effect as the imaginary root for all three factors, in the canonical representation, is such that the affine root for the $\widehat{A}_{n+1}$ factor includes string junctions supported in the exceptional factors at the end components. Note that this is precisely what one would expect for an affinization of the $A_{n+1}$ factor, since the imaginary root must not intersect with the rest of the roots, but at the same time it has to be non-trivial in the junction lattice.

### 4.5.3 $\widehat{\widehat{D}}_{16}$

The explicit realisation of this algebra, dual to the full decompactification to the $SO(32)$ heterotic theory in 4.3.1, is shown in Appendix D.2 together with the general features of the corresponding Type III.b Kulikov model. We summarise here the main points. As shown in Figure 4.10, the base splits into two $\mathbb{P}^1$ components, $B^0$ and $B^1$, both with non trivial codimension zero singular fibers, respectively of Kodaira type $I_{10}$ and $I_2$. Additionally, there are codimension one singular fibers: on $B^0$ there are one $D_{16}$ and one $D_0$ singularities, while on $B^1$ there are two $D_0$ ones. The affinisation of the algebra $D_{16}$ can be seen from the $BC$ system that is approaching the $A^{16}BC$ stack of branes, such that the monodromy of the final

configuration is

$$M_{A^{16}BCBC} = \begin{pmatrix} 1 & -8 \\ 0 & 1 \end{pmatrix} . \tag{4.5.7}$$

This is consistent with the monodromy composition in the case of non-trivial codimension zero singularities $I_n$ and $I_m$ discussed around eq. (4.5.5) (see also [22]). In this case the monodromy as seen from each of the two base components reads

$$M_{BCBC}\, M_{I_8} \;=\; \mathbb{I}_{2x2} \;=\; M_{A^{16}BCBC}\, M_{I_8}^{-1} . \tag{4.5.8}$$

The $A^{16}BCBC$ system then supports one loop junction, consisting of a $(1,0)$ string, that is related to the imaginary root making $D_{16}$ affine. From the K3 degeneration perspective which we present in detail in Appendix D.1, the imaginary root is given by the M2 brane wrapping the fiber of the vanishing transcendental torus of the Calabi-Yau, which in this case is only one. Thus, apparently the Type III.b degeneration is dual to a partial decompactification limit from eight to nine dimensions, and as such one can indeed realise the affine versions of nine dimensional algebras that are compatible with the weak coupling limit, namely those without E factors. Nevertheless, even though from the geometric perspective only the winding tower is manifest, one should remember the fact that the degeneration limit is taken at fixed Kähler moduli, so that in the type IIB string theory the volume of the $T^2$ is constant. This means that the shrinking of one cycle must be accompanied by the growing of the other one, giving a KK tower that provides the second imaginary root whose presence is geometrically obscure. So, we expect only the algebra $\widehat{\widehat{D}}_{16}$ to be consistently realised in this limit.

# Chapter 5

# CHL decompactifications and twisted algebras

In this Chapter, we focus on a slightly more complicated case: the nine-dimensional CHL string [97], which can be obtained as an orbifold of the $E_8 \times E_8$ heterotic theory on $S^1$, whose effect is to reduce the rank of the gauge symmetry from 17 to 9. As was argued in [96], in decompactifying the circle direction one recovers the Cartan generators that were projected out by the orbifold, obtaining back the ten dimensional $E_8 \times E_8$ heterotic theory. Thus, to this extent, the decompactification limit in the CHL string is the same as the one of the $E_8 \times E_8$ heterotic string on $S^1$. We show that from the nine-dimensional point of view there is an important difference. The algebra in the heterotic theory enhances to $(E_9 \oplus E_9)/\sim$ due to the presence of the KK tower, where the $\sim$ stands for the identification of the central extensions of the two $E_9$ factors. In the case at hand the momentum towers arrange in a twisted version of the affine $(E_9 \oplus E_9)/\sim$ , while the central extension is still identified between the two $E_9$ factors. This is related to the enhancement of the rank of the algebra in the decompactification limit.

## 5.1 Decompactification limit

In this Section, we focus on the decompactification limit of the CHL String in the duality frame with zero Wilson line and $R \to \infty$. Geometrically, the CHL string is obtained from the $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ string compactified on $S^1$ with a holonomy for the $\mathbb{Z}_2$ exchange gauge symmetry along the circle direction, and therefore it is expected that by taking the decompactification limit the effect of the holonomy should disappear, thus giving back the ten dimensional $E_8 \times E_8$ heterotic theory. In this limit, the $(E_8)_2 \oplus \mathfrak{u}(1)$ currents acquire a tower of approximately holomorphic light operators obtained by "dressing" the currents with vertex operators constructed from the

compact boson:[1]

$$\mathcal{J}^a_{+\mathrm{n}}(z,\bar{z}) = J^a_+(z)e^{i2\mathrm{n}X^9(z,\bar{z})}\,,\ \mathrm{n}\in\mathbb{Z}\,, \tag{5.1.1}$$

$$\mathcal{J}^9_{\mathrm{n}}(z,\bar{z}) = J^9(z)e^{i2\mathrm{n}X^9(z,\bar{z})}\,,\ \mathrm{n}\in\mathbb{Z}\,, \tag{5.1.2}$$

where the normal ordering of the operators is understood. These operators have weights $(\mathrm{h},\bar{\mathrm{h}}) = \left(1+\frac{\mathrm{n}^2}{R^2},\frac{\mathrm{n}^2}{R^2}\right)\to(1,0)$, and we refer to such operators as asymptotic currents. Note that the orbifold projection requires the momentum quantum number to be even.
Another set of asymptotic currents is constructed by dressing the antisymmetric combinations of the $(\mathrm{E}_8)_1\oplus(\mathrm{E}_8)_1$ currents:

$$\mathcal{J}^{\dot{a}}_{-\mathrm{r}}(z,\bar{z}) = J^{\dot{a}}_-(z)e^{i2\mathrm{r}X^9(z,\bar{z})}\,,\ \ \mathrm{r}\in\mathbb{Z}+\frac{1}{2} \tag{5.1.3}$$

where

$$J^a_-(z) = J^a_1(z) - J^a_2(z)\,, \tag{5.1.4}$$

and in (5.1.3) the momentum $\mathrm{n}=2\mathrm{r}$ surviving the projection is odd. Note that we assign half-integer labels to the $\mathcal{J}_{-\mathrm{r}}$ currents, and we introduced a dotted index $\dot{a}$, also valued in the adjoint of $\mathrm{E}_8$, to distinguish the $\mathcal{J}_\pm$ asymptotic currents in what follows.[2]
We can now use (3.2.31) to obtain the Cartan-Killing metric and structure constants of the algebra of the $\mathcal{J}_\pm$ and $\mathcal{J}^9$ currents in the limit $\epsilon=\frac{1}{2R^2}\to 0$. To do so, we first note that the two- and three-point functions of the even and odd currents are

$$\langle J^a_+(z_1)J^b_-(z_2)\rangle = 0\ , \qquad\qquad \langle J^a_\pm(z_1)J^b_\pm(z_2)\rangle = \frac{2\delta^{a,b}}{z_{12}^2}\ , \tag{5.1.5}$$

and

$$\langle J^a_+(z_1)J^b_+(z_2)J^c_+(z_3)\rangle = \langle J^a_-(z_1)J^b_-(z_2)J^c_+(z_3)\rangle = \frac{i\mathcal{F}^{abc}}{z_{12}z_{23}z_{13}}\ , \tag{5.1.6}$$

$$\langle J^a_-(z_1)J^b_+(z_2)J^c_+(z_3)\rangle = \langle J^a_-(z_1)J^b_-(z_2)J^c_-(z_3)\rangle = 0\ . \tag{5.1.7}$$

These results follow since the $J^a_-$ are found to be in the adjoint representation of the $(\mathrm{E}_8)_2$ and are odd under the exchange symmetry. We also have the three-point function for the compact boson (the normal ordering of the exponentials is understood)

$$\langle e^{i\mathrm{n}_1X^9(z_1,\bar{z}_1)}e^{i\mathrm{n}_2X^9(z_2,\bar{z}_2)}e^{i\mathrm{n}_3X^9(z_3,\bar{z}_3)}\rangle = \frac{\delta_{n_1+n_2+n_3,0}}{|z_{12}|^{2(h_1+h_2-h_3)}|z_{23}|^{2(h_2+h_3-h_1)}|z_{13}|^{2(h_1+h_3-h_2)}}\ , \tag{5.1.8}$$

[1] Note that the dressing we introduce has winding mode number $\mathrm{w}=0$, since all states with $\mathrm{w}\neq 0$ have large conformal dimensions in the $R\to\infty$ limit. For the same reason, the twisted sector does not contribute to light operators, since there $\mathrm{w}\in\mathbb{Z}+\frac{1}{2}$.

[2] The asymptotic currents are also charged with respect to the translation current $J^9$, which leads to the extension of the algebra generated by the $\mathcal{J}_\pm$.

where $h_i = \frac{\mathrm{n}_i}{4R^2}$.
As far as the asymptotic current algebra is concerned, only the zero-momentum copy of the $\mathfrak{u}(1)$ contributes [1]; more precisely, including $\mathcal{J}^9_{\mathrm{n}}$ with $\mathrm{n} \neq 0$ would lead to an additional pole in the three-point functions that is inconsistent with the form of a current algebra. Thus, we restrict to $\mathcal{J}^9_0 \equiv \mathcal{J}^9$ in what follows. Its correlation functions with insertions of $\mathcal{J}_\pm$ are fixed by the $\mathfrak{u}(1)$ Ward identity.
Putting together these explicit correlation functions, we find that the non-trivial two-point functions are

$$\begin{aligned}
\widetilde{\mathcal{G}}^{a\mathrm{m};b\mathrm{n}} &:= z_{12}^2 \langle \mathcal{J}^a_{+\mathrm{m}}(z_1,\bar{z}_1)\mathcal{J}^b_{+\mathrm{n}}(z_2,\bar{z}_2)\rangle = 2\delta^{a,b}\delta^{\mathrm{m}+\mathrm{n},0}\left(1+\mathrm{O}(\epsilon\log|z_{12}|)\right), \\
\widetilde{\mathcal{G}}^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}} &:= z_{12}^2 \langle \mathcal{J}^{\dot{a}}_{-\mathrm{r}}(z_1,\bar{z}_1)\mathcal{J}^{\dot{b}}_{-\mathrm{s}}(z_2,\bar{z}_2)\rangle = 2\delta^{\dot{a},\dot{b}}\delta^{\mathrm{r}+\mathrm{s},0}\left(1+\mathrm{O}(\epsilon\log|z_{12}|)\right), \\
\widetilde{\mathcal{G}}^{9,9} &:= z_{12}^2 \langle \mathcal{J}^9_n(z_1,\bar{z}_1)\mathcal{J}^9_n(z_2,\bar{z}_2)\rangle = \frac{1}{R^2} = 2\epsilon\ .
\end{aligned} \tag{5.1.9}$$

As expected, all anti-holomorphic dependence disappears in the $\epsilon \to 0$ limit.
Similarly, the non-zero three-point functions are

$$\begin{aligned}
\widetilde{\mathcal{F}}^{a\mathrm{m};b\mathrm{n};c\mathrm{p}} &:= z_{12}z_{23}z_{13}\langle \mathcal{J}^a_{+\mathrm{m}}(z_1,\bar{z}_1)\mathcal{J}^b_{+\mathrm{n}}(z_2,\bar{z}_2)\mathcal{J}^c_{+\mathrm{p}}(z_3,\bar{z}_3)\rangle = 2f^{abc}\delta^{\mathrm{m}+\mathrm{n}+\mathrm{p},0}\left(1+\mathrm{O}(\epsilon\log|z_{ij}|)\right)\ , \\
\widetilde{\mathcal{F}}^{\dot{a}\mathrm{r};\dot{b}\mathrm{s};c\mathrm{p}} &:= z_{12}z_{23}z_{13}\langle \mathcal{J}^{\dot{a}}_{-\mathrm{r}}(z_1,\bar{z}_1)\mathcal{J}^{\dot{b}}_{-\mathrm{s}}(z_2,\bar{z}_2)\mathcal{J}^c_{+\mathrm{p}}(z_3,\bar{z}_3)\rangle = 2f^{\dot{a}\dot{b}c}\delta^{\mathrm{r}+\mathrm{s}+\mathrm{p},0}\left(1+\mathrm{O}(\epsilon\log|z_{ij}|)\right)\ , \\
\widetilde{\mathcal{F}}^{a\mathrm{m};b\mathrm{n};9} &:= z_{12}z_{23}z_{13}\langle \mathcal{J}^a_{+\mathrm{m}}(z_1,\bar{z}_1)\mathcal{J}^b_{+\mathrm{n}}(z_2,\bar{z}_2)\mathcal{J}^9(z_3)\rangle = -2i\frac{2\mathrm{m}}{R^2\sqrt{2}}\delta^{a,b}\delta^{\mathrm{m}+\mathrm{n},0}\left(1+\mathrm{O}(\epsilon\log|z_{ij}|)\right)\ , \\
\widetilde{\mathcal{F}}^{\dot{a}\mathrm{r};\dot{b}\mathrm{s};9} &:= z_{12}z_{23}z_{13}\langle \mathcal{J}^{\dot{a}}_{-\mathrm{r}}(z_1,\bar{z}_1)\mathcal{J}^{\dot{b}}_{-\mathrm{s}}(z_2,\bar{z}_2)\mathcal{J}^9(z_3)\rangle = -2i\frac{2\mathrm{r}}{R^2\sqrt{2}}\delta^{a,b}\delta^{\mathrm{r}+\mathrm{s},0}\left(1+\mathrm{O}(\epsilon\log|z_{ij}|)\right)\ .
\end{aligned} \tag{5.1.10}$$

Thus, the asymptotic currents give rise to an algebra whose non-trivial components of the Cartan-Killing metric are

$$\widetilde{\mathcal{G}}^{a\mathrm{m};b\mathrm{n}} = 2\delta^{a,b}\delta^{\mathrm{m}+\mathrm{n},0}\ , \quad \widetilde{\mathcal{G}}^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}} = 2\delta^{\dot{a},\dot{b}}\delta^{\mathrm{r}+\mathrm{s},0}\ , \tag{5.1.11}$$

and whose structure constants are given by

$$\begin{gathered}
\widetilde{f}^{a\mathrm{m};b\mathrm{n}}{}_{c\mathrm{p}} = f^{ab}{}_c\delta^{\mathrm{m}+\mathrm{n}+\mathrm{p},0}\ , \quad \widetilde{f}^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}}{}_{c\mathrm{p}} = f^{\dot{a}\dot{b}}{}_c\delta^{\mathrm{r}+\mathrm{s}+\mathrm{p},0}\ , \quad \widetilde{f}^{a\mathrm{m};b\mathrm{n}}{}_9 = -2i\mathrm{m}\sqrt{2}\delta^{a,b}\delta^{\mathrm{m}+\mathrm{n},0}\ , \\
\widetilde{f}^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}}{}_9 = -2i\mathrm{r}\sqrt{2}\delta^{\dot{a},\dot{b}}\delta^{\mathrm{r}+\mathrm{s},0}\ .
\end{gathered} \tag{5.1.12}$$

Because $\mathrm{r} \in \mathbb{Z} + \frac{1}{2}$, they correspond to the twisted version of the algebra $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ (see for instance [123] and [124]), which we will denote as $(\mathrm{E}_9 \oplus \mathrm{E}_9)_{\mathrm{tw}}/\sim$. The $\sim$ stands for the identification of the central extensions for the two $\mathrm{E}_9$, corresponding to $\mathcal{J}^9$.

### Comparison with heterotic decompactification: the twist

We will now compare this with the asymptotic currents in the decompactification limit of the full heterotic theory from nine to ten dimensions. The generators of the $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ in that

case are given by

$$\mathcal{K}^a_{+\mathrm{n}}(z,\bar{z}) = J^a_+(z)e^{i\mathrm{n}X^9(z,\bar{z})}\,,\ \mathrm{n}\in\mathbb{Z}\,, \tag{5.1.13}$$

$$\mathcal{K}^{\dot{a}}_{-\mathrm{r}}(z,\bar{z}) = J^{\dot{a}}_-(z)e^{i\mathrm{r}X^9(z,\bar{z})}\,,\ \mathrm{r}\in\mathbb{Z}\,, \tag{5.1.14}$$

$$\mathcal{K}^9(z) = J^9(z)\,. \tag{5.1.15}$$

These currents have both even and odd momentum, and they are labelled by integers. In this case, following the same procedure, one obtains

$$\begin{gathered}\mathcal{G}^{a\mathrm{m};b\mathrm{n}} = 2\delta^{a,b}\delta^{\mathrm{m}+\mathrm{n},0}\,,\qquad \mathcal{G}^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}} = 2\delta^{\dot{a},\dot{b}}\delta^{\mathrm{r}+\mathrm{s},0}\,,\qquad f^{a\mathrm{m};b\mathrm{n}}{}_{c\mathrm{p}} = f^{ab}{}_c\delta^{\mathrm{m}+\mathrm{n}+\mathrm{p},0}\,,\\ f^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}}{}_{c\mathrm{p}} = f^{\dot{a}\dot{b}}{}_c\delta^{\mathrm{r}+\mathrm{s}+\mathrm{p},0}\,,\quad f^{a\mathrm{m};b\mathrm{n}}{}_{9} = -i\mathrm{m}\sqrt{2}\delta^{a,b}\delta^{\mathrm{m}+\mathrm{n},0}\,,\quad f^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}}{}_{9} = -i\mathrm{r}\sqrt{2}\delta^{\dot{a},\dot{b}}\delta^{\mathrm{r}+\mathrm{s},0}\,.\end{gathered} \tag{5.1.16}$$

In the heterotic case the structure constants involving the $\mathcal{K}^9(z)$ are smaller by a factor 2 than in the CHL case. This can be absorbed into a rescaling of $\mathcal{K}^9(z)$: we set $\mathcal{K}^{9'}(z) = 2\mathcal{K}^9(z)$, which leads to

$$\widetilde{\mathcal{G}}^{9',9'} = \frac{4}{R^2} = 8\epsilon\,, \tag{5.1.17}$$

so that

$$\widetilde{\mathcal{F}}^{a\mathrm{m};b\mathrm{n};9'} = 2\widetilde{\mathcal{F}}^{a\mathrm{m};b\mathrm{n};9}\,, \tag{5.1.18}$$

and

$$f^{a\mathrm{m};b\mathrm{n}}{}_{9'} = \frac{1}{2}f^{a\mathrm{m};b\mathrm{n}}{}_{9}\,. \tag{5.1.19}$$

Similarly $f^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}}{}_{9'} = \frac{1}{2}f^{\dot{a}\mathrm{r};\dot{b}\mathrm{s}}{}_{9}$.
The rescaling has a simple interpretation: the CHL orbifold naturally gives us a geometry with radius $R_{\mathrm{CHL}} = R/2$. On the other hand, the fact that in (5.1.3) $\mathrm{r}\in\mathbb{Z}+\frac{1}{2}$ while in (5.1.14) $\mathrm{r}\in\mathbb{Z}$ is a structural difference: the CHL algebra is obtained through a twisting of the heterotic one by an outer automorphism of $\mathrm{E}_8\times\mathrm{E}_8$, and the decompactification retains this feature as $R\to\infty$. This is our key result.
While we obtained our result by setting $a=0$, it can be easily generalized to the case of generic Wilson line because in the $R\to\infty$ limit its contribution to the conformal dimension of the $(\mathrm{E}_9\oplus\mathrm{E}_9)_{\mathrm{tw}}/\sim$ currents vanishes: $(\mathrm{h},\bar{\mathrm{h}}) = \left(1+\frac{(\mathrm{n}-\alpha\cdot a)^2}{4R^2},\frac{(\mathrm{n}-\alpha\cdot a)^2}{4R^2}\right)\to(1,0)$ for each root $\alpha\in\Gamma_8$ (see [2] for the equivalent in the full heterotic picture), so that the affine enhancement is the same.

## 5.2 Decompactification limit, dual frame

In this section, we consider the T-dual decompactification limit, which according to (3.2.23) is given by $R\to 0$ and $a=0$. We do this for several reasons. First, we want to directly check the presence of $(\mathrm{E}_9\oplus\mathrm{E}_9)_{\mathrm{tw}}/\sim$. This is not trivial because in this limit the asymptotic

currents arise in both sectors of the orbifold, leading to significant differences with respect to the heterotic decompactification and its T-dual. In addition, the explicit realization that we will obtain may be useful for understanding how twisted and untwisted currents concur to give level two enhancements at finite distance in moduli space.
To understand the asymptotic currents in the $R \to 0$ limit, we will need to describe the twisted Hilbert space in the orbifold CFT. This is most easily accomplished by first constructing the exchange orbifold by $\mathcal{E}$ of the ten dimensional heterotic theory. We know that in this case we recover an isomorphic theory [96, 125].

### 5.2.1 The exchange orbifold in ten dimensions

Following [125], one can decompose the $\mathrm{E}_8 \times \mathrm{E}_8$ internal holomorphic CFT with $c = 16$ of the heterotic string, with energy-momentum tensor $T_{\mathrm{E}_8\times\mathrm{E}_8}(z)$, in terms of the two commuting factors

$$T_{\mathrm{E}_8\times\mathrm{E}_8}(z) = T_{\mathrm{E}_{8,2}}(z) + T_{\mathrm{Ising}}(z)\,, \tag{5.2.1}$$

where the $\mathrm{E}_8$ Kac-Moody algebra is realized at level 2 on the worldsheet by the modes of the $\{J^a_+(z)\}$ and has central charge $c_2 = \frac{31}{2}$. There are three unitary, highest weight, integrable representations $\lambda$ of the $(\widehat{\mathrm{E}}_8)_2$ sector, labeled by the $\mathrm{E}_8$ irreducible representation [126, 127] :

$$\begin{aligned} &\lambda = \mathbf{1}\,, && \mathrm{h}_{\mathbf{1}} = 0\,,\\ &\lambda = \mathbf{248}\,, && \mathrm{h}_{\mathbf{248}} = \frac{15}{16}\,,\\ &\lambda = \mathbf{3875}\,, && \mathrm{h}_{\mathbf{3875}} = \frac{3}{2}\,. \end{aligned} \tag{5.2.2}$$

The familiar primary fields of the Ising CFT have conformal weights

$$\mathrm{h} = 0\,, \quad \mathrm{h} = \frac{1}{16}\,, \quad \mathrm{h} = \frac{1}{2}\,, \tag{5.2.3}$$

corresponding respectively to the vacuum, the (holomorphic) spin field and the (holomorphic) Majorana-Weyl fermion.
From (5.2.1), it is natural to express the partition function $Z_{\mathrm{E}_8\times\mathrm{E}_8}(\tau) = Z_{\mathrm{E}_8}(\tau)^2$ in terms of the characters of $(\widehat{\mathrm{E}}_8)_2$ [125], based on the primaries described in (5.2.2):

$$\begin{aligned} \mathcal{X}^{\mathrm{E}_{8,2}}_{\mathbf{1}} &= q^{-31/48}\left(1 + 248q + 31\,124q^2 + 871\,627q^3 + \mathrm{O}(q^4)\right)\,,\\ \mathcal{X}^{\mathrm{E}_{8,2}}_{\mathbf{248}} &= q^{14/48}\left(248 + 34\,504q + 1\,022\,752q^2 + \mathrm{O}(q^3)\right)\,,\\ \mathcal{X}^{\mathrm{E}_{8,2}}_{\mathbf{3875}} &= q^{41/48}31\left(125 + 5863q + 116\,899q^3 + \mathrm{O}(q^4)\right)\,, \end{aligned} \tag{5.2.4}$$

and of the Ising CFT ones, associated to the primaries in (5.2.3):

$$\begin{aligned} \mathcal{X}^{\mathrm{I}}_{0} &= q^{-1/48}\left(1 + q^2 + q^3 + 2q^4 + \mathrm{O}(q^5)\right)\,,\\ \mathcal{X}^{\mathrm{I}}_{1/2} &= q^{-1/48+1/2}\left(1 + q + q^2 + q^3 + \mathrm{O}(q^4)\right)\,,\\ \mathcal{X}^{\mathrm{I}}_{1/16} &= q^{2/48}\left(1 + q + q^2 + 2q^3 + \mathrm{O}(q^4)\right)\,. \end{aligned} \tag{5.2.5}$$

As usual, $q$ denotes the modular parameter $q = e^{2\pi i\tau}$. The result of [125] is that

$$Z_{\mathrm{E}_8}(\tau)^2 = \mathcal{X}_0^{\mathrm{I}}\mathcal{X}_{\mathbf{1}}^{\mathrm{E}_{8,2}} + \mathcal{X}_{1/2}^{\mathrm{I}}\mathcal{X}_{\mathbf{3875}}^{\mathrm{E}_{8,2}} + \mathcal{X}_{1/16}^{\mathrm{I}}\mathcal{X}_{\mathbf{248}}^{\mathrm{E}_{8,2}} \ , \tag{5.2.6}$$

where the first two terms are even under the permutation symmetry, while the last one is odd.[3] To connect this presentation with the previous discussion, the $(\widehat{\mathrm{E}}_8)_2$ current algebra is generated by the $J_+^a$, while the $J_-^a$ are in the adjoint representation and correspond to the last ($\mathcal{E}$-odd) term in (5.2.6).

To obtain the $\mathcal{E}$ exchange orbifold we can follow [129], which gave a general construction for the partition function of a cyclic permutation orbifold.[4] Rather than discuss the general situation, we focus on the case with a $\mathbb{Z}_2$ permutation. Let $\mathcal{C}$ be a CFT with a modular-invariant partition function $Z_{\mathcal{C}}$. Then the product theory has partition function

$$Z_0(\tau,\overline{\tau}) = Z_{\mathcal{C}}(\tau,\overline{\tau})^2 \ . \tag{5.2.7}$$

By taking modular orbits, one arrives at the $\mathcal{C}\otimes\mathcal{C}/\mathbb{Z}_2$ partition function:[5]

$$Z_{\mathrm{orb}} = \frac{1}{2}Z_0(\tau,\overline{\tau}) + \frac{1}{2}Z_{\mathcal{C}}(2\tau,2\overline{\tau}) + \frac{1}{2}Z_{\mathcal{C}}\left(\tfrac{\tau}{2},\tfrac{\overline{\tau}}{2}\right) + \frac{1}{2}Z_{\mathcal{C}}\left(\tfrac{\tau+1}{2},\tfrac{\overline{\tau}+1}{2}\right) \ . \tag{5.2.8}$$

We cannot apply their construction verbatim to the partition function for $\mathrm{E}_8$ because the latter is not modular–invariant, and instead it picks up a phase under the T transformation $\tau\to\tau+1$:

$$Z_{\mathrm{E}_8}(\tau+1) = e^{-2\pi i/3}Z_{\mathrm{E}_8}(\tau) \ , \qquad\qquad Z_{\mathrm{E}_8}(-1/\tau) = Z_{\mathrm{E}_8}(\tau) \ . \tag{5.2.9}$$

The phase factor just arises from the overall factor of $q^{-c/24} = q^{-1/3}$ in $Z_{\mathrm{E}_8}(\tau)$.

However, we can attempt to construct a partition function that would have the same covariance. To do this in the most obvious fashion, write the full modular-invariant partition function:

$$Z_0(\tau,\overline{\tau}) = Z_{\mathrm{E}_8}(\tau)^2\widetilde{Z}(\tau,\overline{\tau}) \ , \tag{5.2.10}$$

where the second factor satisfies

$$\widetilde{Z}(\tau+1,\overline{\tau}+1) = e^{4\pi i/3}\widetilde{Z}(\tau,\overline{\tau}) \ , \qquad\qquad \widetilde{Z}(-1/\tau,-1/\overline{\tau}) = \widetilde{Z}(\tau,\overline{\tau}) \ . \tag{5.2.11}$$

Taking invariant states, we obtain

$$Z_{\mathrm{inv}}(\tau,\overline{\tau}) = \frac{1}{2}\left(Z_{\mathrm{E}_8}(\tau)^2 + Z_{\mathrm{E}_8}(2\tau)\right)\widetilde{Z}(\tau,\overline{\tau}) \ . \tag{5.2.12}$$

This is invariant under the T transformation $\tau\to\tau+1$ because $Z_{\mathrm{E}_8}(\tau)^2$ and $Z_{\mathrm{E}_8}(2\tau)$ come with the same overall factor of $q^{-2\times 8/24}$. While the first term is also invariant under the S transformation $\tau\to-\frac{1}{\tau}$, the second one is not, but by taking the modular orbit and using S invariance and T covariance of $Z_{\mathrm{E}_8}(\tau)$ we find a candidate for the modular-invariant partition

[3] This decomposition has also been recently reviewed in the context of classification of holomorphic CFTs with $c\le 16$ [128].

[4] This was then generalized in, e.g. [130, 131].

[5] See also [132] for a modern presentation and discussion of possible anomalies.

function:

$$\begin{aligned} Z_{\rm orb} &= \frac{1}{2}\left(Z_{\rm E_8}(\tau)^2 + Z_{\rm E_8}(2\tau)\right)\widetilde{Z}(\tau,\overline{\tau}) \\ &\quad + \frac{1}{2}Z_{\rm E_8}(\tfrac{\tau}{2})\widetilde{Z}(\tau,\overline{\tau}) + \frac{1}{2}Z_{\rm E_8}(\tfrac{\tau+1}{2})e^{4\pi i/3}\widetilde{Z}(\tau,\overline{\tau}) \\ &= Z'_{\rm orb}(\tau)\widetilde{Z}(\tau,\overline{\tau}) \ , \end{aligned} \tag{5.2.13}$$

where

$$\begin{aligned} Z'_{\rm orb}(\tau) &= \frac{1}{2}\left(Z_{\rm E_8}(\tau)^2 + Z_{\rm E_8}(2\tau) + Z_{\rm E_8}(\tfrac{\tau}{2}) + e^{4\pi i/3}Z_{\rm E_8}(\tfrac{\tau+1}{2})\right) \\ &= Z_{\rm E_8}(\tau)^2 \ . \end{aligned} \tag{5.2.14}$$

The last equality is the non-trivial statement [125] that this orbifold is equivalent to the original theory. This can be seen explicitly by writing each term in $Z'_{\rm orb}(\tau)$ in terms of the $(\widehat{\rm E}_8)_2$ and Ising characters

$$\begin{aligned} Z_{\rm E_8}(\tau)^2 &= +\mathcal{X}^{\rm I}_0\mathcal{X}^{\rm E_{8,2}}_{\mathbf{1}} + \mathcal{X}^{\rm I}_{1/2}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{3875}} + \mathcal{X}^{\rm I}_{1/16}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{248}} \ , \\ Z_{\rm E_8}(2\tau) &= +\mathcal{X}^{\rm I}_0\mathcal{X}^{\rm E_{8,2}}_{\mathbf{1}} + \mathcal{X}^{\rm I}_{1/2}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{3875}} - \mathcal{X}^{\rm I}_{1/16}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{248}} \ , \\ Z_{\rm E_8}(\tfrac{\tau}{2}) &= +\mathcal{X}^{\rm I}_{1/2}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{1}} + \mathcal{X}^{\rm I}_0\mathcal{X}^{\rm E_{8,2}}_{\mathbf{3875}} + \mathcal{X}^{\rm I}_{1/16}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{248}} \ , \\ e^{4\pi i/3}Z_{\rm E_8}(\tfrac{\tau+1}{2}) &= -\mathcal{X}^{\rm I}_{1/2}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{1}} - \mathcal{X}^{\rm I}_0\mathcal{X}^{\rm E_{8,2}}_{\mathbf{3875}} + \mathcal{X}^{\rm I}_{1/16}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{248}} \ , \end{aligned} \tag{5.2.15}$$

so that indeed the sum correctly reproduces $2Z_{\rm E_8}(\tau)^2$ as expanded in (5.2.6).
The terms appearing in (5.2.14) can be interpreted in terms of a projection in the untwisted and twisted Hilbert spaces $\mathcal{H}_{\rm ut}$ and $\mathcal{H}_{\rm t}$ respectively as

$$\begin{aligned} Z_{\rm ut}(\tau) &= {\rm Tr}_{\mathcal{H}_{\rm ut}}\left[(1+\mathcal{E})q^{L_0-c/24}\right] = Z_{\rm E_8}(\tau)^2 + Z_{\rm E_8}(2\tau) \ , \\ Z_{\rm t}(\tau) &= {\rm Tr}_{\mathcal{H}_{\rm t}}\left[(1+\mathcal{E})q^{L_0-c/24}\right] = Z_{\rm E_8}(\tfrac{\tau}{2}) + e^{4\pi i/3}Z_{\rm E_8}(\tfrac{\tau+1}{2}) \ . \end{aligned} \tag{5.2.16}$$

In particular, focusing on (5.2.16), we can characterize the $\mathcal{E}$-even and -odd states by expanding the two terms separately. For our purposes, it suffices to examine the leading terms in the expansions:

$$\begin{aligned} Z_{\rm E_8}(\tfrac{\tau}{2}) &= q^{-1/6}\left(+1 + 248q^{1/2} + 4\,124q + 34\,752q^{3/2} + 213\,126q^2 + {\rm O}(q^{5/2})\right) \ , \\ e^{4\pi i/3}Z_{\rm E_8}(\tfrac{\tau+1}{2}) &= q^{-1/6}\left(-1 + 248q^{1/2} - 4\,124q + 34\,752q^{3/2} - 213\,126q^2 + {\rm O}(q^{5/2})\right) \ . \end{aligned} \tag{5.2.17}$$

Writing $q^{-1/6} = q^{-2/3}q^{1/2}$ , we learn that the twisted ground state is $\mathcal{E}$-odd and has ${\rm h} = \frac{1}{2}$. We denote the corresponding twist field as $\Lambda(z)$.
The additional currents can be read off directly from the $Z_{\rm t}(\tau)$ as coming from the ${\rm h} = 1$ primaries in $\mathcal{X}^{\rm I}_{1/16}\mathcal{X}^{\rm E_{8,2}}_{\mathbf{248}}$ of the last line in (5.2.15), [6] which are the twisted version of the $J^a_-(z)$

[6] This could have equivalently been argued from the $q^{1/3}$ term in (5.2.17).

in (5.1.4). While $J^a_-$ are $\mathcal{E}$-odd, we have the OPE

$$J^a_-(z_1)\Lambda(z_2) \sim \frac{K^a(z_2)}{z_{12}^{1/2}}\,, \tag{5.2.18}$$

where the $K^a(z)$ are the $\mathcal{E}$-even primary operators associated to the $\mathcal{X}^{\rm I}_{1/16}\mathcal{X}^{{\rm E}_{8,2}}_{\mathbf{248}}$ characters in the twisted Hilbert space. Since $\Lambda(z)$ is a singlet under $(\widehat{\rm E}_8)_2$, the $K^a(z)$ are in the adjoint representation of the algebra:

$$J^a_+(z_1)K^b(z_2) \sim \frac{if^{ab}{}_cK^c(z_2)}{z_{12}}\,, \tag{5.2.19}$$

with $f^{ab}{}_c$ the ${\rm E}_8$ structure constants. The conformal and $(\widehat{\rm E}_8)_2$ Ward identities imply that

$$\langle J^a_+(z_1)K^b(z_2)K^c(z_3)\rangle = if^{ab}{}_d\langle K^d(z_2)K^c(z_3)\rangle\frac{z_{23}}{z_{12}z_{13}}\,. \tag{5.2.20}$$

We normalize the $K^a(z)$ as

$$\langle K^a(z_1)K^b(z_2)\rangle = \frac{\mathcal{G}^{ab}}{z_{12}^2}\,, \tag{5.2.21}$$

with $\mathcal{G}^{ab} = 2\delta^{a,b}$ as the Cartan-Killing metric of $({\rm E}_8)_2$ (this just amounts to choosing accordingly the normalization of the twist operator) and obtain

$$\langle J^a_+(z_1)K^b(z_2)K^c(z_3)\rangle = \frac{i\mathcal{F}^{abc}}{z_{12}z_{23}z_{13}}\,, \tag{5.2.22}$$

with the $\mathcal{F}^{abc}$ as in (5.1.6).
Moreover, the quantum symmetry of the orbifold, which assigns charge $+1$ to the untwisted and $-1$ to the twisted sector, implies that

$$\langle K^a(z_1)J^b_+(z_2)J^c_+(z_3)\rangle = 0\,, \tag{5.2.23}$$

$$\langle K^a(z_1)K^b(z_2)K^c(z_3)\rangle = 0\,, \tag{5.2.24}$$

so that we recover the full description of the algebra $({\rm E}_8)_1 \oplus ({\rm E}_8)_1$ , but now we also have a detailed understanding of the orbifold action on the currents, as well as the structure of the twisted sector.

### 5.2.2 Decompactification limit in the dual $R \to 0$ frame

Having reviewed the pure exchange orbifold $\mathcal{E}$, we now combine it with the shift $\mathcal{S}$ on the circle, focusing on the $R \to 0$ limit in the absence of Wilson lines. In this case, the holomorphic currents are dressed with winding modes, and the orbifold pairs the $J^a_+$ with integer winding in the untwisted sector and $K^a$ with half-integer winding in the twisted sector. This is what we expect by T-duality (3.2.23) applied to the asymptotic currents found in Section 5.1, with

the correspondence being

$$\mathcal{J}^a_{+\mathrm{n}}(z,\bar{z}),\ \mathrm{n}\in\mathbb{Z} \leftrightarrow J^a_+(z)e^{i\frac{\mathrm{w}}{\sqrt{2}}R^2\left(X^9_L(z)-X^9_R(\bar{z})\right)},\ \mathrm{w}\in\mathbb{Z}\,, \tag{5.2.25}$$

$$\mathcal{J}^{\dot{a}}_{-\mathrm{r}}(z,\bar{z}),\ \mathrm{r}\in\mathbb{Z}+\frac{1}{2} \leftrightarrow K^{\dot{a}}(z)e^{i\frac{\mathrm{w}}{\sqrt{2}}R^2\left(X^9_L(z)-X^9_R(\bar{z})\right)},\ \mathrm{w}\in\mathbb{Z}+\frac{1}{2}\,. \tag{5.2.26}$$

In the $R\to\infty$ limit, the twisting in the asymptotic current algebra can be traced back to the different moding of the $J_+$ and $J_-$ towers. We now see that in the T-dual frame this is a consequence of the fact that in the two sectors the winding has either integer or half-integer values.

## 5.3 $8d \to 10d$ decompactification

As it was pointed out in the previous Sections, there is only one decompactification limit of the 9 dimensional CHL string, namely the one to the 10 dimensional $\mathrm{E}_8\times\mathrm{E}_8$ heterotic theory. On the other hand, starting from 8 dimensions it is possible to decompactify also to the 10 dimensional $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory. Indeed, the CHL theory further compactified on $S^1$ is dual to the $T^2$ compactification without vector structure of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic theory [133]. This decompactification can be seen by the affinisation of the corresponding $D_{16}$ algebra as follows.

Let us consider the $S^1$ compactification of the CHL string to 8 dimensions. Let us denote the $S^1$ coordinate by $y^8$ and the CHL cycle by $y^9$, and $a,b=8,9$ (while the indices $i,j=1,...,8$ label the heterotic torus directions). In this case, focusing on the locus of the heterotic moduli space given by $A=(a,a)$, (3.1.35) reads

$$E_{ab}=G_{ab}+B_{ab}+a_a\cdot a_b, \tag{5.3.1}$$

and the internal momenta (3.1.34) and (3.1.36) in the $\pm$ basis are

$$\begin{aligned}
p_{R,a}&=\frac{1}{\sqrt{2}}(\mathrm{n}_a-E_{ab}\mathrm{w}^b-\rho\cdot a_a)\,,\\
p_{L,a}&=\frac{1}{\sqrt{2}}(\mathrm{n}_a+(2G_{ab}-E_{ab})\mathrm{w}^b-\rho\cdot a_a)\,,\\
p_+&=\frac{1}{\sqrt{2}}(\rho+2a_b\mathrm{w}^b)\,,\\
p_-&=\begin{cases}\frac{1}{\sqrt{2}}(\pi_1-\pi_2) & \text{untwisted sector,}\\ 0 & \text{twisted sector}\end{cases}
\end{aligned} \tag{5.3.2}$$

$\rho=\pi_1+\pi_2\in\Gamma_{\mathrm{E}_8}$ and $a\in\mathbb{R}^8$. $\mathrm{n}_8,\mathrm{n}_9$ and $\mathrm{w}^8\in\mathbb{Z}$, while $\mathrm{w}^9\in\mathbb{Z}$ in the untwisted sector and $\mathrm{w}^9\in\mathbb{Z}+\frac{1}{2}$ for twisted states.

The desired decompactification limit is obtained for

$$G_{ab}=\begin{pmatrix}1/R^2 & 0\\ 0 & R^2\end{pmatrix},\quad B_{ab}=0,\quad a_8=(1,0_7),\quad a_9=0. \tag{5.3.3}$$

for $R \to \infty$[7].
The massless vectors turn out to belong to the NS untwisted sector, $\mathrm{w}^9 = 0$, and they are characterised by $\mathrm{n}_9 \equiv \mathrm{n} \in \mathbb{Z}$ and the $\rho$, $\mathrm{n}_8$, $\mathrm{w}^8$ that must be related to the ones in (4.1.23).
The holomorphic currents are the following.

- In the Cartan sector, the states with even momentum along the CHL cycle must be the symmetric combination of the corresponding heterotic ones, while the states with odd momentum must involve the antisymmetric combination to survive the projection. Defining $\vec{\mathrm{n}} = (\mathrm{n}, \mathrm{m})$, $\vec{\mathrm{r}} = (\mathrm{r}, \mathrm{m})$, and recalling $X^i_\pm$ from (3.2.8), the related vertex operators are

$$\begin{aligned} J^i_{+,\vec{\mathrm{n}}}(z,\bar{z}) &= i\partial X^i_+(z) e^{i\sqrt{2}\mathrm{m}/R^2(-Y^8(z)+\bar{Y}^8(\bar{z}))} e^{i2\mathrm{n}Y^9(z,\bar{z})}\,, \mathrm{n} \in \mathbb{Z} \\ J^i_{-,\vec{\mathrm{r}}}(z,\bar{z}) &= i\partial X^i_-(z) e^{i\sqrt{2}\mathrm{m}/R^2(-Y^8(z)+\bar{Y}^8(\bar{z}))} e^{i2\mathrm{r}Y^9(z,\bar{z})}\,, \mathrm{r} \in \mathbb{Z} + \frac{1}{2}\,, \\ C^8(z) &= \frac{1}{R^2}\partial Y^8(z)\,, \quad C^9(z) = \partial Y^9(z)\,, \end{aligned} \tag{5.3.4}$$

  not considering the towers of the central extensions $C^{8,9}$, which do not contribute to the affine algebra.

- In the ladder operator sector

$$\begin{aligned} J^\alpha_{+,\vec{\mathrm{n}}}(z,\bar{z}) &= c_\alpha e^{ip_+X_+(z)}(e^{ip_-X_-(z)} + e^{-ip_-X_-(z)}) e^{i\sqrt{2}/R^2(\pi_{\alpha,1}\cdot\lambda - \mathrm{m})(Y^8(z)-\bar{Y}^8(\bar{z}))} e^{i2\mathrm{n}Y^9(z,\bar{z})}\,, \\ J^\alpha_{-,\vec{\mathrm{r}}}(z,\bar{z}) &= c_\alpha e^{ip_+X_+(z)}(e^{ip_-X_-(z)} - e^{-ip_-X_-(z)}) e^{i\sqrt{2}/R^2(\pi_{\alpha,1}\cdot\lambda - \mathrm{m})(Y^8(z)-\tilde{Y}^8(\bar{z}))} e^{i2\mathrm{r}Y^9(z,\bar{z})}\,. \end{aligned} \tag{5.3.5}$$

Then, from the heterotic example, the currents that give the $\widehat{\widehat{D}}_{16}$ are a redefinition of (5.3.4) and (5.3.5). First of all, notice that (focusing on the $Y^1$ contribution)

$$e^{i\frac{\sqrt{2}}{R^2}(\pi_{\alpha,1}\cdot\lambda - m)(Y^1(z)-\bar{Y}^1(\bar{z}))} = e^{i\frac{1}{R^2}p_+\cdot\lambda(Y^1(z)-\bar{Y}^1(\bar{z}))} e^{i\frac{1}{R^2}p_-\cdot\lambda(Y^1(z)-\bar{Y}^1(\bar{z}))}\,. \tag{5.3.6}$$

Moreover, we can define

$$Y_+(z) = X_+(z) + \frac{1}{R^2}\lambda Y^8(z) \xrightarrow{\mathcal{ES}} Y_+(z)\,, \tag{5.3.7}$$

$$Y_-(z) = X_-(z) + \frac{1}{R^2}\lambda Y^8(z) \xrightarrow{\mathcal{ES}} -Y_-(z) + \frac{2}{R^2}\,, \tag{5.3.8}$$

shown together with their transformation under the $\mathbb{Z}_2$ orbifold action. Their OPEs are the same as the $X_\pm$ ones.

[7] Even though the volume of this torus is actually constant as one varies $R$, it still corresponds to a decompactification limit in a T-dual frame. Also, in principle one could consider the two $S^1$ radii to scale at a different rate in $R$.

The currents

$$K^i_{+,\vec{n}}(z,\bar{z}) = i\partial Y^i_+(z) e^{i\sqrt{2}\mathrm{m}/R^2(-Y^8(z)+\bar{Y}^8(\bar{z}))} e^{i2\mathrm{n}Y^9(z,\bar{z})}\,, \tag{5.3.9}$$

$$K^i_{-,\vec{r}}(z,\bar{z}) = i\partial Y^i_+(z) e^{i\sqrt{2}\mathrm{m}/R^2(-Y^8(z)+\bar{Y}^8(\bar{z}))} e^{i2\mathrm{r}Y^9(z,\bar{z})}\,, \tag{5.3.10}$$

$$J^\alpha_{+,\vec{n}}(z,\bar{z}) = c_\alpha e^{ip_+Y_+(z)}(e^{ip_-Y_-(z)} + e^{-ip_-Y_-(z)}) e^{-i\sqrt{2}\mathrm{m}/R^2(Y^8(z)-\bar{Y}^8(\bar{z}))} e^{i2\mathrm{n}Y^9(z,\bar{z})}\,,$$

$$J^\alpha_{-,\vec{r}}(z,\bar{z}) = c_\alpha e^{ip_+Y_+(z)}(e^{ip_-Y_-(z)} - e^{-ip_-Y_-(z)}) e^{-i\sqrt{2}\mathrm{m}/R^2(Y^8(z)-\bar{Y}^8(\bar{z}))} e^{i2\mathrm{r}Y^8(z,\bar{z})}\,, \tag{5.3.11}$$

can be shown to satisfy a $\widehat{\widehat{D}}_{16}$ algebra with central extensions $C^1$ and $C^2$ in (5.3.4), signalling a decompactification to the 10 dimensional $\mathrm{Spin}(32)/\mathbb{Z}_2$.
The reason why all this works in 8 dimensions but not in 9 is that the $SO(32) - \mathrm{E}_8 \times \mathrm{E}_8$ map only works for integer winding, so that in the case of the 9 dimensional CHL, where the affine algebra should be built by states with non-trivial winding as $R \to 0$, this does not happen.

# Part III

# The Distance Conjecture in symmetric moduli spaces

# Chapter 6

# The Distance Conjecture in symmetric moduli spaces

The aim of this Chapter is to show a proof for the SDC (2.1.1) in the case of locally symmetric spaces, which we will introduce in the following. Under mild conditions on their rank, spaces with finite volume and non-positive sectional curvature are locally symmetric. The most straightforward examples are the moduli spaces of M- and string theory toroidal compactifications,[1] of the form

$$\mathcal{M} = \Gamma \backslash G/K \ , \tag{6.0.1}$$

(where $G/K$ is a quotient of Lie groups of the non-compact, globally symmetric type and $\Gamma$ a discrete subgroup of $G$), but the proof includes a much wider class, as e.g. moduli spaces of Kähler deformations in certain Calabi-Yau compactifications [135].[2]

As examples of (6.0.1), compactifying the Heterotic string on $T^d$, the low energy theory in $10-d$ dimensions is half-maximal supergravity with $d^2$ scalars from deformations of the metric and B-field on the torus, and $16d$ components of the $d$ Wilson lines in the torus. These $d(d+16)$ degrees of freedom parameterise the coset $O(d+16,d)/(O(d+16)\times O(d))$, and can be combined in a generalized metric. The subsector of the moduli given by the $d^2$ deformations of metric and B-field, which is also a subsector of the moduli space of Type II toroidal compactifications and the full moduli space of toroidal compactifications of bosonic string, spans a coset $G/K = O(d,d)/(O(d)\times O(d))$. On the other hand, compactifying M-theory on $T^d$, or Type II theories on $T^{d-1}$ leads to maximal supergravities in $11-d$ dimensions with massless scalar fields coming from deformations of the metric and gauge fields. These scalar fields parameterise the cosets $G/K = E_{d(d)}/K_d$, where $K_d$ is the maximal compact subgroup of $E_{d(d)}$.

In string theory compactifications, there is a T-duality group (or more generally, a U-duality group for maximal supergravities), that maps the lattice of charges of the string spectrum to itself. For example, in the case of $O(d,d)$ the charges are given by the momenta and winding numbers on the torus, and the T-duality group is $O(d,d,\mathbb{Z})$. It acts on the moduli by conjugation of the generalized metric, thus identifying $B \simeq B+1$, and in the case of $B=0$ and a diagonal metric, $R_i \simeq 1/R_i$ for any radius $i$ (while for non-zero $B$ or non-diagonal metric it gives the standard Buscher rules [136]). Also in the case of $E_{d(d)}$ the U-duality group is

[1] Some of the arguments given here also appear in [134].

[2] In this thesis we will focus on the first case.

just the restriction to the rationals $E_{d(d)}(\mathbb{Z})$, though this statement is not true in general. For example, for the compactification of the non-supersymmetric Heterotic string, locally the classical[3] moduli space is also $O(d+16,d)/(O(d+16)\times O(d))$, but the T-duality group is not $O(d,d+16,\mathbb{Z})$ [104]. In mathematics terms, the T- (or U-) duality group corresponds to a discrete arithmetic subgroup $\Gamma$ of $G$ as above.
In the rest of the Chapter, we will first introduce the definition of symmetric spaces and how their boundaries can be characterised only through group theoretic information. With this knowledge, we can prove the Distance Conjecture just assuming the hypothesis of completeness of the spectrum.

## 6.1 Symmetric spaces: geodesics and boundaries

As just mentioned, in an attempt to prove the distance conjecture in a formal setting, we describe geodesic motion and the various infinite distance limits on symmetric spaces.
Following mainly [137–139], in this section we introduce the definition of symmetric spaces as smooth manifolds diffeomorphic to quotients of Lie groups. Geodesics and boundaries can be studied from the group theory perspective systematically, and the infinite distance limits are characterised by parabolic subgroups. We illustrate the concepts with the simplest example, namely $SL(2,\mathbb{R})/SO(2)$ (the moduli space of Type IIB supergravity in 10 dimensions, which is diffeomorphic to the upper half plane). In this Section, we will not yet consider the discrete quotient by $\Gamma$, which will be discussed in 6.2.

### 6.1.1 Globally symmetric spaces

There are several classes of globally symmetric spaces. We will be interested in particular in *symmetric spaces of non-compact type*[4] but with finite volume, which are the conjectured properties of the moduli spaces arising in string theory compactifications[5] [14]. These spaces are Riemannian and have negative sectional curvature.
Formally, a manifold $\mathcal{S}$ is a globally (locally) symmetric space if it has an inversion symmetry $\mathcal{S}$ that acts at every point $x\in\mathcal{S}$ as the identity and on the tangent space as $e^X\to e^{-X}$ for every $X\in T_x\mathcal{S}$, which is a global (local) isometry. This in turn implies that $\mathcal{S}$ is geodetically complete and homogeneous, namely its isometry group, which we will denote as $G$, acts transitively on $\mathcal{S}$, i.e. given two points $x,\,y\in\mathcal{S}$, there exists a group element $g\in G$ such that $g(x)=y$. Moreover, by denoting as $K\subseteq G$ the isotropy group that stabilizes any point $o\in\mathcal{S}$, namely[6]

$$K=\{k\in G\,|\,k\cdot o=o\}\,, \tag{6.1.1}$$

it can be shown that $\mathcal{S}\cong G\cdot o$, and that there exist a diffeomorphism[7]

$$\mathcal{S}\cong G/K\,. \tag{6.1.2}$$

[3] Without considering quantum corrections that arise due to the dilaton potential.
[4] For example, the spheres are also symmetric spaces, but of compact type: $S^n\cong SO(n+1)/SO(n)$
[5] $S^1$ compactifications, having infinite volume, are an exception.
[6] The $\cdot$ denotes the $G$ action on $\mathcal{S}$.
[7] More generally, a symmetric space is diffeomorphic to $\mathbb{R}^n\times G/K$. Here we will not consider the $\mathbb{R}^n$ factor, but everything can be straightforwardly generalised if one wants to incorporate it.

If $\mathcal{S}$ is non-compact, as is the case of interest here, then $K$ is the maximal compact subgroup of $G$. We denote the corresponding algebras as $\mathfrak{k}$ and $\mathfrak{g}$ respectively. Then the inversion symmetry $\mathcal{S} \in G$[8] guarantees that the algebra $\mathfrak{g}$ splits into

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}\,, \tag{6.1.3}$$

which are orthogonal with respect to the Killing form $\mathcal{K}$ of $\mathfrak{g}$. This is known as Cartan decomposition of the algebra $\mathfrak{g}$ of $G$ and is unique. Moreover, given this decomposition, it holds

$$[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}\,, \tag{6.1.4}$$

where $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$ tells us that $K$ rotates tangent vectors on $\mathcal{S}$.
In symmetric spaces of non-compact type $\mathfrak{g}$ is non-compact, and the Killing form on it is positive definite on $\mathfrak{p}$, while it is negative definite on $\mathfrak{k}$, and they have negative sectional curvature. The positive definite metric tells us $\mathcal{S}$ is Riemannian.
The Cartan decomposition of the algebra (6.1.3) induces the so-called *Cartan decomposition of the Lie group* $G$ [137, 138]

$$\forall g \in G: \ \exists\, k_1, k_2 \in K,\ h \in \mathfrak{h}: \ g = k_1\, e^h\, k_2 \quad \Leftrightarrow \quad G = K\, e^{\mathfrak{h}}\, K\ , \tag{6.1.5}$$

where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra. More specifically, it will be useful in the following to connect to a refinement of the above Cartan decomposition. Let $\Phi^+ = \{\alpha \in \Phi: \ \alpha(H) > 0\}$ be the set of positive roots, and $\Phi^-$ the set of negative roots. The positive Weyl chamber of $\mathfrak{g}$ is defined as

$$\mathfrak{h}^+ = \{h \in \mathfrak{h} : (h,\alpha) > 0\, \forall \alpha \in \Phi^+\} \subset \mathfrak{h}\,, \tag{6.1.6}$$

and let us denote by $\overline{\mathfrak{h}^+}$ its closure in $\mathfrak{h}$. Then the Weyl group $W \subset K$ maps one Weyl chamber to another, so that one can restrict only to the positive one

$$G = K\, e^{\overline{\mathfrak{h}^+}}\, K\,. \tag{6.1.7}$$

It is useful to also decompose the Cartan subalgebra into the subset $\mathfrak{a}$ of mutually commuting generators in $\mathfrak{p}$, and the rest

$$\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}\ . \tag{6.1.8}$$

One thus has

$$r \equiv \mathrm{rank}_{G/K} = \dim \mathfrak{a}\,. \tag{6.1.9}$$

We illustrate these definitions on the group $SL(2,\mathbb{R})/SO(2)$ (and when this is too simple, with other cosets), with $\mathcal{S}$ the upper half plane $\mathbb{H}^2$, parameterised by

$$\mathbb{H}^2 = \{\tau = \tau_1 + i\tau_2 : \tau_2 \geq 0\}\,, \tag{6.1.10}$$

[8] More precisely, its pull back $\Theta$ to $\mathfrak{g}$.

where $\tau$ can be interpreted as the complex structure of (possibly geometrically equivalent) tori, such that its metric reads

$$g(\tau) = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} . \tag{6.1.11}$$

Its isometry group is

$$G = SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \, a,b,c,d \in \mathbb{R}, \, ad - bc = 1 \right\}, \tag{6.1.12}$$

acting on $\tau \in \mathbb{H}^2$ as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{6.1.13}$$

while the isotropy group, fixing for example the point $o = i$, is $SO(2) \ni k(\theta)$

$$k(\theta)\cdot i = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot i = \frac{i\cos\theta - \sin\theta}{-i(-\sin\theta + i\cos\theta)} = i\,, \tag{6.1.14}$$

so that $\mathbb{H}^2 \cong SL(2,\mathbb{R})/SO(2)$. The algebra $\mathfrak{sl}(2,\mathbb{R})$ is generated by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{6.1.15}$$

$h$ generates the Cartan subalgebra, namely $\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\langle h\rangle$, and $e_2$ generates an $\mathfrak{so}(2) \subset \mathfrak{sl}(2)$. Then $\mathfrak{p} = \mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2) = \mathrm{span}_{\mathbb{R}}\langle h, e_1\rangle$ is orthogonal to $\mathfrak{so}(2)$ with respect to the Killing form $\mathcal{K} = \frac{1}{2}\mathrm{Tr}$, and moreover is isomorphic to $T_i\mathbb{H}^2$, the tangent space to $\mathbb{H}^2$ at $o = i$. The Cartan decomposition of the group $SL(2,\mathbb{R})$ is given by

$$\forall\, g \in SL(2,\mathbb{R}) \;\; \exists\, \theta,\phi \in [0,2\pi], \; t \in \mathbb{R} \;:\; \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} e^{t\,h} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} . \tag{6.1.16}$$

In this case, for which $SL(2)$ splits over the reals, it turns out that $\mathfrak{a} = \mathfrak{h}$, but obviously this is not the general situation. For instance, consider $SO(1,3)/SO(3)$, with $\mathfrak{g} = \mathfrak{so}(1,3)$ and $\mathfrak{k} = \mathfrak{so}(3)$. We choose a basis for $\mathfrak{so}(1,3)$ that stabilises $o = \mathrm{diag}(1,-1,-1,-1)$ and the following representation of the Cartan subalgebra $\mathfrak{h} = \langle h_1, h_2\rangle_{\mathbb{R}}$,

$$h_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \;\; h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{6.1.17}$$

where $h_2$ is also a generator in $\mathfrak{so}(3) \subset \mathfrak{so}(1,3)$. While $h_1 \in \mathfrak{p}$, $h_2$ is not in $\mathfrak{p}$. Hence, $\mathfrak{a} = \langle h_1 \rangle_{\mathbb{R}} \subset \mathfrak{h}$. In this representation, the rest of the algebra is given by

$$b_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.1.18}$$

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{6.1.19}$$

The algebra $\mathfrak{so}(3) = \langle h_2, r_1, r_2 \rangle_{\mathbb{R}}$ is generated by the antisymmetric generators, while $\mathfrak{p} = \langle h_1, b_1, b_2 \rangle_{\mathbb{R}} \cong T_o\mathcal{S}$ is generated by the symmetric ones, which physically can be interpreted as boosts.

### 6.1.2 Geodesics and boundaries on globally symmetric spaces

One of the reasons why we want to think of $\mathcal{S}$ as a quotient of Lie groups is that geodesics on Lie group manifolds have a simple expression. In fact, for any $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, $e^{tX}$ is a geodesic on $G$. More precisely, a geodesic on a symmetric space $\mathcal{S}$ of the form (6.1.2), passing through the point $x = g \cdot o$, is a curve

$$\gamma(t) = e^{tX}\, g \cdot o\,. \tag{6.1.20}$$

In particular, using the Cartan decomposition, to describe geodesics on the coset space one can restrict to $X \in \mathfrak{p}$. The physical meaning is the following: choosing as initial condition the point $x \in \mathcal{S}$, all the geodesics through it are specified by the choice of the initial velocity $X \in \mathfrak{p}$.
The geodesic distance between any two points $x_1 = g_1 \cdot o$ and $x_2 = g_2 \cdot o$ in $\mathcal{S}$ is given by

$$d(x_1, x_2) = |\mathcal{K}(X, X)|^{1/2}, \quad \text{where} \;\; \exp(X) \cdot o = g_2^{-1} g_1 \cdot o\,, \tag{6.1.21}$$

with $\mathcal{K}$ the Killing form of $\mathfrak{g}$.
Since these spaces are geodesically complete, we can use the geodesic flow on $\mathcal{S}$ to describe its boundary $\partial\mathcal{S}$, for the non-compact symmetric spaces we want to characterise. Intuitively, since geodesics are 1-parameter subgroups of the $G$-action on $\mathcal{S} \cong \exp(\mathfrak{p})$ [138], one expects the boundary $\partial\mathcal{S}$ to correspond to the limit $t \to \infty$ of 1-parameter subgroups $g(t)$ generated by $\mathfrak{p}$ acting on $o \in \mathcal{S}$, as is made clear by the Cartan decomposition. From this perspective,

$$\partial\mathcal{S} \cong \lim_{t\to\infty} e^{t\, X_1} \cdot o \cong S^{n-1}\,, \tag{6.1.22}$$

where the subscript 1 on $X \in \mathfrak{p}$ denotes a restriction to norm 1 vectors. Hence, the boundary is isomorphic to the unit sphere in $\mathcal{S}$, whose dimension we denote by $n$ ($n = \mathrm{dim}\mathrm{G} - \mathrm{dim}K - 1$). Additionally, restricting to norm 1 vectors makes $t$ an affine parameter for the geodesic. With this intuition, one can define the boundary as the set of *points at infinity*.

**Definition 6.1.1.** (Point at infinity) One point at the boundary of a symmetric space is defined as an equivalence class of asymptotic geodesics. Two geodesics $\gamma_1(t)$ and $\gamma_2(t)$ are asymptotic if and only if the distance $d(\gamma_1(t), \gamma_2(t))$ is finite for $t \to +\infty$.

In particular, given the Cartan decomposition (6.1.7), it can be seen that one point at infinity is characterised by the subalgebra $\overline{\mathfrak{a}_1^+} = \overline{\mathfrak{h}_1^+} \cap \mathfrak{p}$ and the compact subgroup $K$. For example, going back to the upper half plane, from the hyperbolic metric

$$ds^2 = \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2}\,, \tag{6.1.23}$$

one can see that two geodesics of the kind

$$\gamma_1(t) = b_1 + i\,e^t\,, \quad \gamma_2(t) = b_2 + i\,e^t\,, \tag{6.1.24}$$

with $b_1 \neq b_2$, have $\lim_{t\to+\infty} d(\gamma_1(t), \gamma_2(t)) \to 0$, so they define the same boundary point $\tau_\infty = i\infty$. On the other hand, $\lim_{t\to-\infty} d(\gamma_1(t), \gamma_2(t)) \to +\infty$, corresponding to two distinct boundary points $\tau_1 = b_1$ and $\tau_2 = b_2$. This is in agreement with the fact that

$$\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\} \cong S^1\,. \tag{6.1.25}$$

Let us show this equivalently from the coset perspective. It turns out that one element of each equivalence class (under the action of $G$) of asymptotic geodesics can be described by a geodesic from the point $i \in \mathbb{H}^2$ of the form

$$\begin{aligned} \gamma_0(t) &= e^{2t} i, \\ \gamma_1(t) &= e^{-2t} i + x\,, \end{aligned} \tag{6.1.26}$$

for any given $x$. All the other geodesics are obtained just acting with $g \in SL(2,\mathbb{R})$ on these, which does not modify the asymptotic behaviour. In the limit $t \to \infty$, $\gamma_0(t) \cdot i$ goes to a single point on the boundary, while $\gamma_1(t) \cdot i$ goes to a 1-parameter family of points parametrised by $x \in \mathbb{R}$, which correspond respectively to the $\tau = i\infty$ and the $\tau \in \mathbb{R}$ components of the boundary of the upper half plane. They correspond respectively to the following geodesics in $SL(2,\mathbb{R})/SO(2)$

$$c_0(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\,, \tag{6.1.27}$$

$$c_1(t) = \begin{pmatrix} -xe^t & e^{-t} \\ -e^t & 0 \end{pmatrix}\,. \tag{6.1.28}$$

In order to study later how the towers of states evolve along geodesic paths, we need to determine how the metric on the moduli space depends on the parameter $t$. This metric, usually called $\mathcal{H}$, can be written in terms of a veilbein as $\mathcal{H} = \mathcal{E}\mathcal{E}^T$, with $\mathcal{E} \in G$ with a right $K$ action.

For the $SL(2)/SO(2)$ example, take the metric given in (6.1.11), and evaluate it at $\tau = i$

$$\mathcal{H}\Big|_{\tau=i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1.29}$$

The vielbeine are given by $\mathcal{E} = \text{diag}(1,1)$ up to an $SO(2)$ right action. We now act on the vielbeine using the group elements $\{c_0(t), c_1(t)\}$ that generate the geodesics. We find

$$\begin{aligned} \mathcal{H}_0(t) &= c_0(t)\mathcal{E}\ (c_0(t)\mathcal{E})^T = e^{2ht} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}, \\ \mathcal{H}_1(t) &= c_1(t)\mathcal{E}\ (c_1(t)\mathcal{E})^T = \begin{pmatrix} e^{2t}x^2 + e^{-2t} & e^{2t}x \\ e^{2t}x & e^{2t} \end{pmatrix}, \end{aligned} \tag{6.1.30}$$

showing again that along the geodesic $c_0(t)$, $\tau$ evolves as

$$\tau(t) = \tau_1(t) + i\tau_2(t): \quad \tau_1(t) = 0,\ \tau_2(t) = e^{2t}, \tag{6.1.31}$$

and along geodesic $c_1(t)$, $\tau$ evolves as

$$\tau(t) = \tau_1(t) + i\tau_2(t): \quad \tau_1(t) = x \in \mathbb{R},\ \tau_2(t) = e^{-2t}. \tag{6.1.32}$$

In the $t \to \infty$ limit, we find $\tau$ parametrises $\mathbb{R} \cup \{\infty\}$ reproducing the geometric boundary at infinity.

### 6.1.3 Boundaries and parabolic subgroups

An alternative description of the boundary of symmetric spaces is through the notion of *parabolic subgroups* of the group $G$. Qualitatively, one can think of a parabolic transformation as fixing one point at infinity. For example, if we consider again the case of $SL(2,\mathbb{R})/SO(2)$, we can easily check that the elements of the form

$$P = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \tag{6.1.33}$$

for $a, b \in \mathbb{R}$, leave the point $\tau = i\infty$ fixed. These are indeed parabolic transformations for $SL(2,\mathbb{R})$, as we define in the following.
Given a semi-simple Lie algebra $\mathfrak{g}$ of a group $G$, let us introduce the the subalgebras associated to positive and negative roots, respectively

$$\mathfrak{n}^+ = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{a\in\Phi^+} \mathfrak{g}_{-\alpha} = \sum_{a\in\Phi^-} \mathfrak{g}_\alpha, \tag{6.1.34}$$

which are two nilpotent subalgebras, whose exponentiation gives the corresponding unipotent groups $N^+ = e^{\mathfrak{n}^+}$ and $N^-$. A *minimal parabolic subgroup* $P_{\min} \subset G$ is defined as the subgroup of $G$ whose algebra $\mathfrak{p}_{\min}$ is the complement of the nilpotent subalgebra $\mathfrak{n}^-$ of $\mathfrak{g}$, i.e. $\mathfrak{p}_{\min} =$

$\mathfrak{g} \ominus \mathfrak{n}^- = (\mathfrak{n}^-)^\perp$ [138]. It is a closed subgroup of $G$ which decomposes as [137]

$$P_{\min} = MAN^+ \,, \tag{6.1.35}$$

where $A = \exp(\mathfrak{h})$ is the maximal Abelian (Cartan) subgroup in $G$, and $M \subset K$ is the centraliser of $\mathfrak{h}$, namely

$$M = \{k \in K : k^{-1}hk = h \ \ \forall h \in \mathfrak{h}\} \,. \tag{6.1.36}$$

There are also bigger parabolic subgroups $P$, defined generically as related to a subalgebra $\mathcal{L}(P) \subset \mathfrak{g}$ whose perpendicular space with respect to the Killing metric $(\mathcal{L}(P))^\perp$ is a nilpotent subalgebra.
For example, in the case of $SL(2,\mathbb{R})$, we have that

$$\mathfrak{h} = \mathfrak{a} = \langle h \rangle_{\mathbb{R}}, \ \ \mathfrak{n}^+ = \langle \frac{1}{2}(e_1 - e_2) \rangle_{\mathbb{R}}, \ \ \mathfrak{n}^- = \langle \frac{1}{2}(e_1 + e_2) \rangle_{\mathbb{R}} \,, \tag{6.1.37}$$

where $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent, with roots (normalized as $|\alpha|^2 = 1$)

$$\alpha_+ = 1, \quad \alpha_- = -1. \tag{6.1.38}$$

The subgroup

$$N^+ = e^{x\mathfrak{n}^+} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \tag{6.1.39}$$

is unipotent. The centraliser $M$ of $\mathfrak{h}$ is trivial, so the parabolic subgroup (there is a single class here) is generated by $h$ and $n^+$, and given by

$$P = e^{\mathfrak{n}^+} e^{\mathfrak{a}} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} . \tag{6.1.40}$$

In this case[9] elements of $P$ are upper triangular matrices, which fix the point $\tau = i\infty$ as noticed around (6.1.33). In order to obtain the full boundary one should then account for the additional $SO(2)$ adjoint action on $P$, as in general parabolic subgroups come in $K$-conjugacy classes $[P]_K$. For example, each point in the $\mathbb{R}$ component of the boundary of $\mathbb{H}^2$ is associated to a parabolic subgroup which is obtained from the minimal one (6.1.40) by an $SO(2) \ni k(\theta)$ adjoint action. Indeed,

$$p_\theta = k(\theta) p k(\theta)^{-1} \in [p]_{SO(2)} \tag{6.1.41}$$

fixes the point $\tau = \cot\theta \in \mathbb{R}$. In the general case, for which one can have more than one nilpotent subalgebra than $\mathfrak{n}^-$ (that is, when it can have non-trivial subalgebras), there are more $K$-conjugacy classes of parabolic subgroups. It turns out that one possible representative $p_I$ of each $K$-conjugacy class $[p_I]_K$ is specified by the choice of a set of simple roots $I \subseteq \Sigma$, and the algebra of the parabolic subgroup, $\mathcal{L}(P_I)$ is Killing perpendicular to

$$\sum_{\alpha \in (\Sigma^+ - I)} \mathfrak{g}_{-\alpha} \,, \tag{6.1.42}$$

[9] As in in $SL(n,\mathbb{R})$ for any $n \in \mathbb{Z}$.

which is the subspace associated to the negative roots generated by $\Sigma - I$. In particular, the choice of a set of simple roots allows to define the set of Cartan generators in $\mathfrak{a}$ that are orthogonal to $\sum_{\alpha\in I}\mathfrak{g}_\alpha$ as

$$\mathfrak{a}_I = \{h \in \mathfrak{a} : \alpha(h) = 0\,, \forall \alpha \in I\}\,, \tag{6.1.43}$$

and from (6.1.22) we know that each class of asymptotic geodesics describing the boundary can be described by restricting to Cartan generators $h \in \mathfrak{a}$ with norm 1, and from (6.1.7) that one can restrict to the closure of the positive Weyl chamber, such that the minimal description of the boundary is contained in the set of

$$\mathfrak{a}^+_{I;1} = \{h \in \overline{\mathfrak{a}^+} : \alpha(h) = 0\,, \forall \alpha \in I\,, |h| = 1\}\,, \tag{6.1.44}$$

which corresponds to the intersection of the closure of the positive Weyl chamber with the unit sphere in $\mathfrak{a}$. It turns out that minimal parabolic subgroups are associated to $I_{\min} =$ and $h_{\min} \in \mathfrak{a}^+_1$, while the non-minimal case to a non-trivial set $I$ and to Cartan generators in $\partial(\mathfrak{a}^+_1)$.

This way of building parabolic subgroups then allows to associate one representative in the class of asymptotic geodesics for each point at infinty. Given such a parabolic subgroup $P_I$ and a Cartan generator $h_I \in \mathfrak{a}^+_{I;1}$, one point at infinity is described by the equivalence class of

$$\gamma_I(t) = e^{th_I} \cdot o\,, \quad h \in \mathfrak{a}^+_{I;1}\,. \tag{6.1.45}$$

To reconstruct the full boundary, as already explained, we then have to take the orbit under $K$ of every point at infinity described by each $(P_I, h_I)$, such that

$$\partial\mathcal{S} = \bigsqcup_I \lim_{t\to\infty} K e^{t\,h_I}(o)\,. \tag{6.1.46}$$

This construction makes the structure of the boundary $\partial\mathcal{S}$ clear: it is a $K$ fibration over the base simplex which is given by the union over parabolics of $\mathfrak{a}^+_{I;1}$. It can be then explicitly parametrised using the Cartan decomposition as follows. We (locally) choose coordinates on the base simplex to be $\{\varphi_i\}_{i\in[1,r-1]}$, so that $h(\{\varphi_i\}_{i\in[1,r-1]})$ is a generator in $\mathfrak{a}^+_{I;1}$. We also choose a set of angles $\{\theta_n\}_{n\in[1,\dim K]}$, representing the position in the fibre. We have that points $x \in \partial\mathcal{S}$ correspond to pairs $(k(\{\theta_n\}_{n\in[1,\dim K]}), h(\{\varphi_i\}_{i\in[1,r-1]}))$ representing generators $\mathrm{Ad}_{k(\{\theta_n\}_{n\in[1,\dim K]})}h(\{\varphi_i\}_{i\in[1,r-1]})$ of geodesics that reach from $o$ to the boundary.[10]

Consider the example of $\mathbb{H}^2 \cong SL(2;\mathbb{R})/SO(2)$. The boundary can be easily described as a fibration over $\overline{\mathfrak{a}^+_1}$, which in this case corresponds to $h$ as in (6.1.15), so the simplex corresponding to the base of the fibration is a point. The fibre is given by $\mathrm{Ad}_{SO(2)/M_H}(h)$ where $M_H$, the subgroup of $SO(2)$ stabilizing $h$, is trivial. Hence, $\partial\mathcal{S} \cong \mathrm{Ad}_{SO(2)}(h) \cong SO(2)$. Clearly, as in (6.1.22), the unit sphere in $T_o\mathbb{H}^2$ is just an $S^1$ which is diffeomorphic to $SO(2)$.

Now consider the boundary of $\mathcal{S} \cong SO(1,3)/SO(3)$. Here, the simplex forming the base of the fibration, given by $\overline{\mathfrak{a}^+_1} = h_1$, is once again a point. Hence, $\partial\mathcal{S} \cong \mathrm{Ad}_{SO(3)/M_H}(h_1) \cong SO(3)/M_H$. In this case, $M_H = E^{\theta h_2} \cong SO(2)$ is an $SO(2)$ subgroup of $SO(3)$. Hence, $\partial\mathcal{S} \cong SO(3)/SO(2) \cong S^2$ is a 2-sphere.

[10] It is understood that the use of a chart with more than one patch may be necessary.

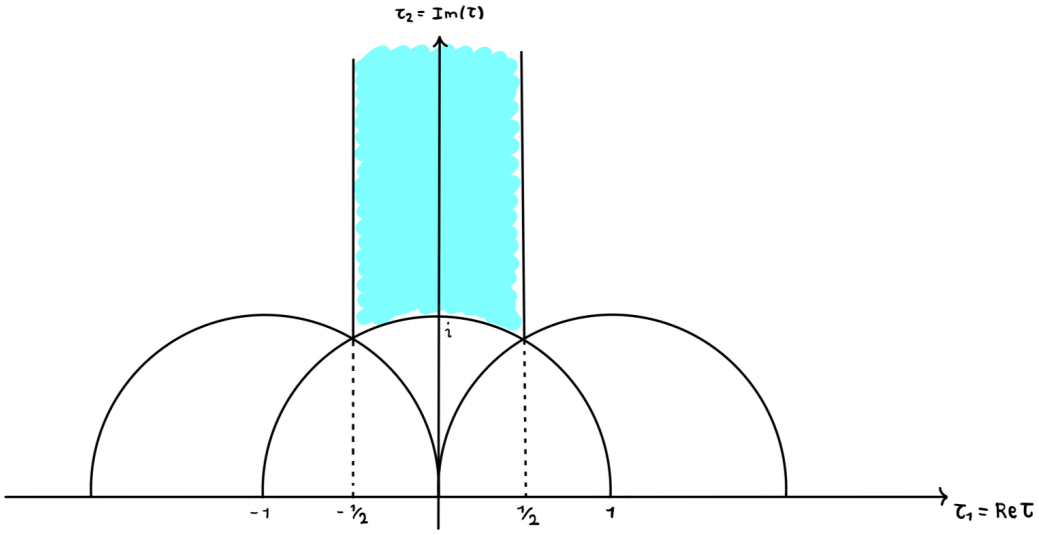



**Figure 6.1:** One possible choice of fundamental domain for $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$.

## 6.2 Quotient by discrete subgroups

In this section, motivated by the string theoretic U- and T- dualities that relate different points in the moduli spaces, we present the mathematical framework to account for the modifications to the structure of the boundary and to the geodesics on such a space quotiented out by a discrete group

$$\mathcal{M} = \Gamma\backslash G/K\,, \tag{6.2.1}$$

where as before $G$ and $K$ are defined over the reals and their quotient is a symmetric space. The fact that the structure of the boundary changes is clear for instance considering the case of the $SL(2,\mathbb{Z})$ quotient on $\mathbb{H}^2$, which consists in the restriction to one fundamental domain as for example in Figure 6.1, which clearly contains only one boundary point. Moreover, only geodesics that are associated to rational values of $\tau_1$ reach the point at infinty up to $SL(2,\mathbb{Z})$ transformations [140].

### 6.2.1 Discrete subgroups and how to choose them

Let $G(\mathbb{Q})$ be an algebraic group defined over the rationals. Then $G(\mathbb{Q}) \subset GL(n,\mathbb{Q})$. An arithmetic subgroup is a subgroup $\Gamma \subset G(\mathbb{Q})$ such that $\Gamma \subseteq GL(n,\mathbb{Z}) \cap G(\mathbb{Q})$. The discrete subgroup $\Gamma$ in (6.2.1) is an arithmetic group. These are defined by a set of polynomial conditions with rational coefficients (e.g. det $M$=1, for $SL(n,\mathbb{Z})$, or $M^T = M^{-1}$ for $O(n,\mathbb{Z})$ etc.). To be more concrete, one can always embed $G \in GL(n,\mathbb{C})$, and define the arithmetic group using

this embedding, namely

$$\begin{aligned} G &\subset GL(n,\mathbb{R}) && \text{real group}, \\ G(\mathbb{Q}) &= G \cap GL(n,\mathbb{Q}) && \text{rational group}, \\ \Gamma &= G \cap GL(n,\mathbb{Z}) && \text{arithmetic group}. \end{aligned} \tag{6.2.2}$$

This is one way to build an arithmetic subgroup, but there is an infinite number of different arithmetic groups in a given Lie group. By different, we mean that a pair $\Gamma_1 \subset G(\mathbb{Q})$ and $\Gamma_2 \subset G(\mathbb{Q})$ which are isomorphic as groups defined over the rationals may not be isomorphic as groups defined over the integers (that is, the homomorphism $U$ between them $\Gamma_1 = U\Gamma_2 U^{-1}$ is not in $GL(n,\mathbb{Z})$). In general, a discrete subgroup $\Gamma_1$ of $G$ is arithmetic if it is commensurate to $\Gamma$ as defined above (i.e. if $\Gamma/(\Gamma \cap \Gamma_1)$ and $\Gamma_1/(\Gamma \cap \Gamma_1)$ are finite). One intuitive way to see that there are actually infinitely many arithmetic subgroups is by noticing that they can be equivalently defined as the subgroups of $GL(n,\mathbb{Z})$ that preserve a given invariant $J$ in $\Gamma(\mathbb{Q})$. The invariant also defines a lattice $\Lambda$ generated by vectors in $\mathbb{Z}^n$ that are orthonormal with respect to $J$.

The choice of arithmetic group $\Gamma$ determines the fundamental domain, a representative region on the coset spaces $G/K$ that contains the largest number of points in $G/K$ such that no two points related by $\Gamma$.[11] Furthermore, it determines whether the fundamental domain is compact or has a boundary at infinity, how many points at infinity it has[12] and moreover if it has finite or infinite volume.

From what said above, there is no canonical or natural way to choose an arithmetic subgroup $\Gamma$, see for example Footnote 12. Nevertheless, the structure of the toroidally compactified string theory selects naturally is the so-called Chevalley arithmetic subgroup, which comes from the exponentiation of the algebra $\mathfrak{g}$ in the Chevalley basis $\mathcal{B}_{Ch}$, which allows to define the group purely over the integers[13] and gives a way of constructing semi-simple Lie groups $G$ and their arithmetic subgroups $\Gamma$, so that one can study how $\Gamma \subseteq G(\mathbb{Z})$ fits into $G$ and what $\Gamma\backslash G$ might look like.

In particular, *Chevalley groups* are defined as groups formed by taking the exponential map of the Chevalley basis. For example, the Chevalley group $G(\mathbb{C})$ defined over $\mathbb{C}$ is given by $e^{t\hat{A}}$ for $\hat{A} \in \mathcal{B}_{Ch}$ and $t \in \mathbb{C}$, the Chevalley group $G(\mathbb{Z})$ defined over $\mathbb{Z}$ is given by $e^{t\hat{A}}$ for $\hat{A} \in \mathcal{B}_{Ch}$ and $t \in \mathbb{Z}$ and so on.

As defined up to now, $\Gamma$ is the reflective part of the duality group. There can be other transformations one can quotient by (that often in the literature are given for granted and not explicitly written) given by the fact that in addition to the inner automorphisms (related to the arithmetic group $\Gamma$), in general a group $G$ also displays outer automorphisms, which are related to symmetries of the Dynkin diagram of the corresponding algebra $\mathfrak{g}$ but are not in

[11] From the string theory point of view, it determines the polygon on the coset space that represents the physical moduli space of inequivalent vacua.

[12] For example, the fundamental domain in $SL(2;\mathbb{Z})\backslash SL(2;\mathbb{R})/SO(2;\mathbb{R})$ can have a width of 1 (horizontal domain $[-1/2, 1/2]$ in analogy with IIB string theory's moduli space), or a width of 2 (horizontal domain $[-1, 1]$ in analogy with the heterotic theory's moduli space). In the former case, the only one point reaches the boundary at infinity, namely $\tau \to i\infty$. In the latter case, the boundary is the union of $tau \to i\infty$ and $\tau \to 1$.

[13] For a more formal definition of Chevalley groups, see Appendix E.

$G(\mathbb{Z})$,

$$\mathrm{Aut}(G) = \mathrm{In}(G) \ltimes \mathrm{Out}(G)\,. \tag{6.2.3}$$

For example, in the case of $O(2,2) \cong SL(2) \times SL(2)$ there is a $\mathbb{Z}_2$ outer automorphism that consists of the exchange of the two nodes of the Dynkin diagram in Figure 6.2.

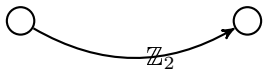



**Figure 6.2:** Dynkin diagram of $O(2,2)$, displaying an exchange symmetry between the two nodes.

We will provide an explicit example of this in Section 6.3.1.

### 6.2.2 Geodesics and boundaries in the fundamental domain

As said above, the action of $\Gamma$ divides the coset space into fundamental domains, delimited by hyperplanes of co-dimension one that are either fixed, or mapped to one another under the action of a given element in $\Gamma$. For example, for $SL(2,\mathbb{Z})$, the vertical walls that delimit the moduli space in Figure 6.1 at $\tau = \pm 1/2$ are mapped to one another by the $SL(2,\mathbb{Z})$ transformation $\tau \to \tau + 1$. On the other hand, outer automorphisms do not have an associated hyperplane.

We saw that, before the quotient by $\Gamma$, all geodesics that reach the boundary are of the form (6.1.45), and that for a fixed $h_I \in \mathfrak{a}^+_{I;1}$ all of them define the same boundary point irrespectively of $N_I^+$, which allowed to deduce that the boundary of a symmetric space is isomorphic to the unit sphere in $\mathfrak{p}$. This changes after the discrete quotient, and the way it changes depends on the specific arithmetic subgroup.

In particular, it can be shown that the information on the boundary of a space like (6.2.1) is fully contained in the restriction of the groups to their elements over the rationals, $G(\mathbb{Q})$ and $K(\mathbb{Q})$, as the discrete quotient is not sensitive to irrational boundary points. This is a generalisation of the known behaviour in the case of $SL(2,\mathbb{Z})$ (e.g. see [140]). For definiteness, let us choose as fundamental domain the one in Figure 6.1, containing the boundary point $\tau_\infty = i\infty$. It can be easily seen that the only points in the $\mathbb{R}$ component of the boundary, $\tau = \tau_1 + i\tau_2$ with $\tau_2 \to 0$, that can be mapped to $\tau_\infty$ via an $SL(2,\mathbb{R})$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{6.2.4}$$

are of the kind $\tau_1 = -\frac{d}{c} \in \mathbb{Q}$.

This result can again be obtained studying the boundary through asymptotic geodesics, accounting for the additional identifications given by the discrete quotient as follows. In order to study the geodesics in the fundamental domain $\mathcal{F}$, one can for instance start from a point $x = g \cdot o \in \mathcal{F}$ and evolve assigning a tangent vector $X \in T_x\mathcal{F}$, as in (6.1.20). Such a geodesic then starts in the fundamental domain but generically ends outside, namely on the real line, crossing the boundary of $\mathcal{F}$ at finite $t$. If we want to restrict to the quotient space, at that point it should be mapped back to $\mathcal{F}$ with a $G(\mathbb{Z})$ transformation, and so on. Two qualitatively

different behaviours arise, depending on the geodesics' properties in the covering space $G/K$. For simplicity, let us take the $SL(2)$ case specifically.

- The geodesic in the covering space crosses a finite number of fundamental regions, namely it can be mapped with a finite amount of $SL(2,\mathbb{Z})$ transformations to the straight geodesics going to $i\infty$. This happens when the endpoint of the geodesic has $\tau_1 \in \mathbb{Q}$.

- The geodesic in the covering space crosses infinitely many fundamental regions when its endpoint is irrational. It can not be mapped to the geodesic reaching $i\infty$, and by repeatedly applying $SL(2,\mathbb{Z})$ transformations one can see that it has either a periodic orbit ($\tau_{1,end}$ can be written as a periodic continued fraction) or an ergodic one (all the other cases).

From the group-theoretic point of view, this is reflected in the statement that points at infinity now are associated to equivalence classes under $K$ of *rational parabolic subgroups* $P_{\mathbb{Q}} \subset G$, namely $[P_{\mathbb{Q}}]_K$, together with an element of norm 1 in the corresponding $\mathfrak{a}^+_{P_{I;1}}$, as defined in (6.1.44).

$$x \in \partial\mathcal{M} \;\leftrightarrow\; (P_{I,\mathbb{Q}}, h \in \mathfrak{a}^+_{P_{I;1}}) + \; K \text{ action} \,. \tag{6.2.5}$$

The boundary is then of the form [139]

$$\partial\mathcal{M} \sim \Gamma \backslash \bigsqcup_I \mathfrak{a}^+_{P_{I,\mathbb{Q};1}} \,, \tag{6.2.6}$$

where again $I$ labels the $K$-conjugacy classes of parabolic subgroups, and the subscript 1 stands for the unit normalization of the Cartan generators. Let us be more concrete and focus on the example of $SL(2,\mathbb{Z})\backslash\mathbb{H}^2$, which as already mentioned has only one point at infinity. This is reflected in the fact that all the rational parabolic subgroups are in fact $SL(2,\mathbb{Z})$-equivalent for example to the minimal (upper-triangular) one (6.1.33) corresponding to $\tau_\infty = i\infty$. This can indeed be restricted to the rationals

$$P_{rat} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \; a,\, b \in \mathbb{Q} \,. \tag{6.2.7}$$

As for the other rational parabolics in the same $SO(2)$ conjugacy class, they are obtained by an $SO(2,\mathbb{Q})$ transformation

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \alpha/\gamma & -\beta/\gamma \\ \beta/\gamma & \alpha/\gamma \end{pmatrix} : \quad \alpha,\beta,\gamma \in \mathbb{Z},\, \alpha^2+\beta^2=\gamma^2 \,, \tag{6.2.8}$$

namely

$$P_{\theta;rat} = k(\theta) P_{rat} k(\theta)^{-1} \,, \tag{6.2.9}$$

and they correspond indeed to a rational boundary point $\tau = -\cot\theta = -\frac{\alpha}{\beta} \in \mathbb{Q}$. We can see that all these points are $SL(2,\mathbb{Z})$-equivalent to $\tau_\infty$ because $P_{rat}$ and all the $P_{\theta;rat}$ belong to

the same $SL(2,\mathbb{Z})$ equivalence class, namely there exists $A \in SL(2,\mathbb{Z})$

$$A = \begin{pmatrix} a_1 & a_3 \\ a_2 & \frac{1+a_2a_3}{a_1} \end{pmatrix} : \quad a_1, a_2, a_3 \in \mathbb{Z}\,, \tag{6.2.10}$$

such that

$$\mathrm{Ad}_A(p_1) = \mathrm{Ad}_{k(\theta)}(p_2) \;\Leftrightarrow\; \mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{k(\theta)}(P_{rat}) \subseteq P_{rat}, \tag{6.2.11}$$

for $p_1$, $p_2 \in P_{rat}$, and in particular

$$\frac{\beta}{\alpha} = \frac{\left(1 - a^2a_2a_3 - a^2 + aba_1a_3 + a_2a_3\right)}{a_1\left(aba_1 + a_2 - a^2a_2\right)} \text{ or } \; a_1\beta = \alpha a_3\,, \tag{6.2.12}$$

which admits a solution for $a_1$, $a_2$, $a_3 \in \mathbb{Z}$.
In the cases when the algebra splits over the reals (e.g. $G = SL(n)$, $O(d,d)$ and $E_{d(d)}$), it can be shown that it is always the case that there is only one $G(\mathbb{Z})$ equivalence class of parabolic subgroups [141]. Going back to the fibration representation of the boundary of $G/K$, this means that, for each element $kP_{rat;I}k^{-1}$, $k \in K(\mathbb{Q})$ of the fiber over the standard parabolic $P_{rat;I}$, there exist a $G(\mathbb{Z})$ transformation that trivialises the $k$ adjoint action, namely $[kP_{rat;I}k^{-1}]_{G(\mathbb{Q})} = [P_{rat;I}]_{G(\mathbb{Q})}$. This means that considering the discrete quotient, the boundary is isomorphic to the simplex given by the intersection of the positive Weyl chamber with the unit sphere, the fibre over each point being trivial under the arithmetic Chevalley subgroup.
This is not guaranteed for groups that do not split over the reals, such as $G = O(d,d')$ for $d \neq d'$,[14] but it has been shown in general that for any symmetric space there is a finite number of conjugacy classes of rational parabolics.[15]

**An example:** $SL(n)$

After having discussed explicitly $SL(2)$ in the previous discussion in order to exemplify the theoretical concepts, let us now present the case of $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/O(n)$, focusing first on $n = 3$ (the first non-trivial example after $SL(2)$) in order to then generalise to $n > 3$. These are the moduli spaces of $T^n$ at fixed volume, and as we will see in the next Section 6.3, the moduli spaces of M-theory on $T^d$ with $d \leq 4$ have this structure.
Since the positive Weyl chamber of $\mathfrak{sl}(3)$ is 2-dimensional, the boundary of the quotient space $\mathcal{S} = SL(3,\mathbb{R})/SO(3)$ is a fibration where the base is the simplex corresponding to a closed interval, whose interior is associated to Cartans which have non-degenerate eigenvalues, that is

$$h = \mathrm{diag}\,(a,b,c)\,,\; a+b+c = 0\,,\; a^2+b^2+c^2 = 1\,,\; a \neq b \neq c\,, \tag{6.2.13}$$

while the two extrema correspond to the two singular Cartans

$$h_1 = \mathrm{diag}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \text{ and } \; h_2 = \mathrm{diag}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right), \tag{6.2.14}$$

[14] But for example, in the case of $O(1,3)$ it is still the case that there is only one equivalence class.
[15] This is a generalization of the result of [140] for the case of $O(1,17)$, which we will recall in Section 6.3.3.

namely elements at the boundary of the positive Weyl chamber that have degenerate eigenvalues. The fibre is given by the non-trivial $SO(3)$ transformations on the base. Then, the extrema of the interval turn out to be stablised by an $SO(2,\mathbb{R}) \subset SO(3,\mathbb{R})$. This means that the fibre over the vertex mapped out by the $SO(3,\mathbb{R})$ is isomorphic to $SO(3,\mathbb{R})/SO(2,\mathbb{R}) \cong \mathbb{RP}^2$. The fibre over the open interval is instead $\mathbb{RP}^3$.
Moving then to $\mathcal{M} = SL(3,\mathbb{Z})\backslash\mathcal{S}$, where the information is contained in the group restricted to the rationals, we can use the lessons learnt from the $SL(2;\mathbb{R})/SO(2;\mathbb{R})$ case to see what $SO(3;\mathbb{Q})$ should look like. Any element of $SO(3;\mathbb{Q})$ can be written as a concatenation of rotations about the three axes. We have

$$k(\theta,\phi,\psi) = R(\theta)R(\phi)R(\psi)\,, \tag{6.2.15}$$

with

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \quad R(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}. \tag{6.2.16}$$

This implies we can think of the $SO(3,\mathbb{Q})$ action decomposed into 3 $SO(2,\mathbb{Q})$'s. In the same way as in the 2 dimensional case, we want to show that there exists a matrix $A \in SL(3,\mathbb{Z})$ such that

$$\mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{k(\theta,\phi,\psi)}P^0_{min,1,2} \subset P^0_{min,1,2}\,, \tag{6.2.17}$$

where $P^0_{min}$ is a reference minimal parabolic, $P^0_1$ is one of the non-minimal parabolics and $P^0_2$ is the other. Define $A \in SL(3,\mathbb{Z})$ as a concatenation of matrices $A = A_\theta A_\phi A_\psi$. Consider $\mathrm{Ad}_{A_\theta^{-1}}\mathrm{Ad}_{R(\theta)}P^0_{min}$. This reduces to the 2 dimensional case where the matrices $A_\theta$ and $R(\theta)$ act trivially on the 3rd row of any $p \in P^0_{min}$. Clearly, there always exists an $A_\theta$ such that

$$\mathrm{Ad}_{A_\theta^{-1}}\mathrm{Ad}_{k(\theta,\phi,\psi)}P^0_{min} \subset P^0_{min}\,. \tag{6.2.18}$$

as in the 2 dimensional case. The same logic applies for $R(\phi)$ and $R(\psi)$. Hence, there exists an $A = A_\theta A_\phi A_\psi \in SL(2,\mathbb{Z})$ that satisfies

$$\mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{k(\theta,\phi,\psi)}P^0_{min} \subset P^0_{min}\,. \tag{6.2.19}$$

Moving to the non-minimal case, we have

$$p_1 \in P^0_1: \quad p_1 = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix}, \tag{6.2.20}$$

which, under the action of $R(\theta)$, stays in $P^0_1$. Solving for

$$\mathrm{Ad}_{A_\phi^{-1}}\mathrm{Ad}_{R(\phi)}P^0_1 \subset P^0_1 \tag{6.2.21}$$

corresponds to requiring

$$\mathrm{Ad}_{A_\phi^{-1}}\mathrm{Ad}_{R(\phi)}\begin{pmatrix} a & c \\ 0 & g \end{pmatrix} \tag{6.2.22}$$

is upper triangular where $A_\phi$ and $R(\phi)$ have the 2nd row and 2nd column set to $(0,1,0)$ and $(0,1,0)^T$ and removed. This reduces to the 2 dimensional minimal case. Solving for

$$\mathrm{Ad}_{A_\psi^{-1}}\mathrm{Ad}_{R(\psi)}P_1^0 \subset P_1^0 \tag{6.2.23}$$

again reduces to the 2 dimensional minimal case where the 1st row and first columns of $A_\psi$ and $R(\psi)$ are set to $(1,0,0)$ and $(1,0,0)^T$ and removed. We require that

$$\mathrm{Ad}_{A_\psi^{-1}}\mathrm{Ad}_{R(\psi)}\begin{pmatrix} e & f \\ 0 & g \end{pmatrix} \tag{6.2.24}$$

is upper triangular. Of course, we know this to be possible. The exact same logic applies to $P_2^0$. Hence, we have shown that there exists and $A \in SO(3,\mathbb{Z})$ such that for any $(\theta,\phi,\psi)$ (6.2.17) holds, and hence there is only one $SL(3,\mathbb{Z})$ conjugacy class per point on the simplex of the boundary of $\mathcal{S}$. In conclusion, the boundary at infinity of $\mathcal{M}$ is just a closed interval.
The examples of $SL(2)$ and $SL(3)$ are not unique can be extended to the quotient space $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n,\mathbb{R})$ is given by simplex defined by the link over the closure of the positive Weyl chamber of $SL(n,\mathbb{R})$.

## 6.3 Lesson for symmetric moduli spaces from string theory

As already mentioned, the moduli spaces coming from toroidal compactifications of maximal and half-maximal supergravity are of symmetric type. In particular, let us recall their structure

- M-theory on $T^d$ (or Type II on $T^{d-1}$): $E_{d(d)}(\mathbb{Z})\backslash E_{d(d)}(\mathbb{R})/K_d$ as in Table 6.1.
- Heterotic theory on $T^d$: $O(d,d+16,\mathbb{Z})\backslash O(d,d+16,\mathbb{R})/(O(d)\times O(d+16))$.
- CHL string on $T^d$: $O(d,d+8,\mathbb{Z})\backslash O(d,d+8,\mathbb{R})/(O(d)\times O(d+8))$.
- Bosonic string on $T^d$: $O(d,d,\mathbb{Z})\backslash O(d,d,\mathbb{R})/(O(d)\times O(d))$.
- Classical moduli space of the non-supersymmetric $O(16)\times O(16)$ heterotic theory on $T^d$: $\Gamma_{(8,8)}\backslash O(d,d+16,\mathbb{R})/(O(d)\times O(d+16))$.
- Classical moduli space of the non-supersymmetric $E_8$ heterotic string: $\Gamma_8\backslash O(d,d+8,\mathbb{R})/(O(d)\times O(d+8))$.

| $d$ | $G$ | $K$ |
|---|---|---|
| 2 | $GL(2)$ | $SO(2)$ |
| 3 | $SL(2) \times SL(3)$ | $SO(2) \times SO(3)$ |
| 4 | $SL(5)$ | $SO(5)$ |
| 5 | $SO(5,5)$ | $SO(5) \times SO(5)$ |
| 6 | $E_{6(6)}$ | $USp(8)$ |
| 7 | $E_{7(7)}$ | $SU(8)$ |
| 8 | $E_{8(8)}$ | $SO(16)$ |

**Table 6.1:** Cosets $G/K$ of maximal supergravities in $11-d$ dimensions.

In all the cases involving orthogonal groups, there is an additional $\mathbb{R}$ factor accounting for the dilaton modulus.
Except in the non-supersymmetric cases, for which the duality group is not related to $G(\mathbb{Z})$ but has a more involved structure, all the discrete quotients involve the Chevalley groups over $\mathbb{Z}$, as already mentioned.
In the following, we will explicitly show the structure of the boundary for some examples in each class, in particular focusing on how the various supergravity degrees of freedom are embedded in the coset, in such a way to better understand how they evolve along geodesics.

### 6.3.1 Toroidal compactifications of the bosonic theory

In order to most easily understand the map between physical degrees of freedom and geometric data, let us first show the example of the bosonic theory, whose moduli space for $T^d$ compactifications has the form

$$\mathcal{M}_{bos} = O(d,d,\mathbb{Z})\backslash O(d,d,\mathbb{R})/(O(d)\times O(d))\,. \tag{6.3.1}$$

It turns out that given an element $g \in O(d,d,\mathbb{R})/(O(d)\times O(d))$, the metric moduli $G_{mn}$ and the $B$-field moduli $B_{mn}$, with $m,n = 1,...,d$ are embedded in the so-called generalised metric $\mathcal{H} = g^T g$ as follows [142]

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{6.3.2}$$

In particular, Cartan generators in $\mathfrak{a}$ are related to the radii $R_1$, ..., $R_d$ of the $T^d$, while the compact moduli (angles and $B$-field) are associated to the ladder operators. For simplicity, let us focus on the case $d=2$, for which

$$G_{mn} = \begin{pmatrix} R_1^2 & R_1 R_2 \cos\theta \\ R_1 R_2 \cos\theta & R_2^2 \end{pmatrix}, \quad B_{mn} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}. \tag{6.3.3}$$

In particular

$$\mathfrak{p} = \Big\langle H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{6.3.4}$$

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Big\rangle, \tag{6.3.5}$$

and

$$\mathfrak{k} = \Big\langle E_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \Big\rangle. \tag{6.3.6}$$

From (6.1.45), by exponentiation the norm 1 elements of the closure of the positive Weyl chamber, which are of the form

$$h = \mathrm{diag}(\lambda_1, \lambda_2, -\lambda_1, -\lambda_2), \quad \lambda_1 \geq \lambda_2,\ \lambda_1 \geq -\lambda_2, \tag{6.3.7}$$

which parametrise an interval,[16] and

$$\mathcal{H} = e^{th} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{-\lambda_1 t}, e^{-\lambda_2 t}). \tag{6.3.8}$$

With no $B$ field turned on, from (6.3.2) we see that the radii of the torus are $R_1 = e^{t\lambda_1}$ and $R_2 = e^{t\lambda_2}$ where $\lambda_1 \in [0, \sqrt{2}/2]$ and $\lambda_2 \in [-\sqrt{2}/2, \sqrt{2}/2]$. Hence, we find the following possible classes of limits

$$\lambda_1, \lambda_2 > 0, \ \lim_{t\to\infty} e^{ht}, \tag{6.3.9}$$

heads to boundary points corresponding to decompactification limits to 10D bosonic string theory. For

$$\lambda_1 > 0, \lambda_2 < 0, \ \lim_{t\to\infty} e^{ht}, \tag{6.3.10}$$

the trajectory heads to boundary points which describe the T-dual version of 10 dimensional bosonic string theory. Finally,

$$\lambda_1 > 0, \lambda_2 = 0, \ \lim_{t\to\infty} e^{ht}, \tag{6.3.11}$$

is a 9d partial decompactification limit.
Due to the fact that we have two T-dual descriptions (6.3.9) and (6.3.10), it seems that we did not properly account for the quotient by T-duality. This is indeed true, as we did not account

[16] Since the group $O(2,2)$ splits over $\mathbb{R}$, there is only one equivalence class of rational parabolics under the Chevalley group over the integers, so after the discrete quotient the boundary is an interval.

for the $\mathbb{Z}_2$ outer automorphism of $O(2,2)$, that acts as $\lambda_2 \to -\lambda_2$. This reduces the range of $\lambda_2 \in [0, \sqrt{2}/2]$, then make the case (6.3.10) redundant. Strictly speaking, then, the moduli space one is actually considering is

$$\mathcal{M} \cong (O(2,2,\mathbb{Z}) \times \mathbb{Z}_2) \backslash O(2,2;\mathbb{R}) / (O(2;\mathbb{R}) \times O(2;\mathbb{R}))\,. \tag{6.3.12}$$

### 6.3.2 Toroidal compactifications of M-theory

For the $E_{d(d)}$ case, let us focus on two explicative cases: the one of $E_{4(4)} \cong SL(5)$ and one of the actually exceptional cases, $E_{6(6)}$ for simplicity.

$E_{6(6)}$

The best way to understand the moduli space based on $E_6(\mathbb{R})$ is to once again, look at the level of the algebra and split it into representation spaces of $GL(6;\mathbb{R})$. The 78 of $\mathfrak{e}_6$ splits as follows

$$78 = 36 + 20 + 20 + 1 + 1\,, \tag{6.3.13}$$

$$\mathfrak{e}_6 = \mathfrak{gl}(6) \oplus (TM \otimes T^\star M) \oplus \Omega^3(TM) \oplus \Lambda^3(TM) \oplus \Omega^6(TM) \oplus \Lambda^6(TM)\,. \tag{6.3.14}$$

The Cartan appears in the $\mathfrak{sl}(6) \subset \mathfrak{gl}(6)$. We can write these as

$$H_1 = \mathrm{diag}(1,0,0,0,0,0),\ H_2 = \mathrm{diag}(0,1,0,0,0,0),\ ...,\ H_6 = \mathrm{diag}(0,0,0,0,0,1). \tag{6.3.15}$$

A general element of the maximal torus in $\mathfrak{e}_6$ looks like $H = \sum_{i\in 1}^{6} \lambda_i H_i$ and lives in the $TM \otimes T^\star M$ part of the decomposition. We choose 6 simple roots, 5 corresponding to generators in $TM \otimes T^\star M$ and one corresponding to a generator in $\Omega^3(TM)$.

$$e_{\alpha_1} = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},\quad e_{\alpha_2} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},\quad e_{\alpha_3} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \tag{6.3.16}$$

$$e_{\alpha_4} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},\quad e_{\alpha_5} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix},\quad (e_{\alpha_2})_{ijk} = \begin{cases} 1 & i=1, j=2, k=3 \\ 0 & otherwise \end{cases} \tag{6.3.17}$$

The root vectors for this choice look like

$$\alpha_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.3.18}$$

$$\alpha_2 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{6.3.19}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}, \tag{6.3.20}$$

$$\alpha_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \tag{6.3.21}$$

$$\alpha_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \tag{6.3.22}$$

$$\alpha_6 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.3.23}$$

A Weyl chamber can be isolated by fixing $\alpha_j(H) = \alpha_j(\sum_{i\in 1}^{6} \lambda_i H_i) \geq 0$ for all $j \in [1, 6]$. In the supergravity theory, $e^{\lambda_i}$ parameterise 6 geometric quantities, radii, on the internal space of a compactification of M-theory to $5D$. The resulting conditions of the geometric quantities are

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq \lambda_6,\ 0 \geq \lambda_1 + \lambda_2 + \lambda_3 \tag{6.3.24}$$

$$\Leftrightarrow \tag{6.3.25}$$

$$\lambda_6 \leq \lambda_5 \leq \lambda_4 \leq \lambda_3 \leq 0\ \wedge \tag{6.3.26}$$

$$\lambda_3 \leq \lambda_2 < -\frac{\lambda_3}{2} \wedge \lambda_3 \leq -\lambda_2 - \lambda_1 \leq -2\lambda_2 \vee \lambda_1 = \lambda_2 = -\frac{\lambda_3}{2} \wedge \lambda_6 \leq \lambda_5 \leq \lambda_4 \leq \lambda_3 \leq 0. \tag{6.3.27}$$

We define a trajectory in the moduli space to be parameterised by an affine parameter $t$. The constrains above tell us that at least 4 of the radii, $e^{-t\lambda_3}, ..., e^{-t\lambda_6}$ diverge at large distance $t$. Then

- If $\lambda_1, \lambda_2 < 0$, the trajectory corresponds to an 11D M-theory decompactification limit.
- If $\lambda_1 > 0, \lambda_2 < 0$, this corresponds to a 10D type IIA decompactification limit.
- If $\lambda_1 > 0, \lambda_2 = 0$, the boundary point is type IIA on an $S^1$.
- If $\lambda_1 = 0, \lambda_2 < 0$, this is M-theory on $S^1$, or type IIA.
- If $\lambda_1 = 0, \lambda_2 = 0$, this is M-theory on a $S^1$.
- If $\lambda_1, \lambda_2 > 0$, then this corresponds to type IIA on an $S^1$ dual to 10D type IIB.

$SL(5)$

The structure of the boundary in the case of $SL(5, \mathbb{Z})\backslash SL(5, \mathbb{R})/SO(5, \mathbb{R})$ was already described in Section 6.2.2. Here we want to show how the moduli of 11 dimensional supergravity compactified on $T^4$, that are in $GL(4, \mathbb{R})$ representations, are embedded into $SL(5)$. In this case, the generalised metric in terms of the internal metric $G_{ij}$ and 3-form $C_{ijk}$ $i$, $j$, $k = 1, ..., 4$

is [142]

$$\mathcal{H} = \begin{pmatrix} G + \frac{1}{2} C G^{-1} G^{-1} C & -\frac{1}{2} C G^{-1} G^{-1} \\ -\frac{1}{2} G^{-1} G^{-1} C & \frac{1}{2} G^{-1} G^{-1} \end{pmatrix} . \tag{6.3.28}$$

By denoting the $T^4$ radii as $R_i = e^{-\lambda_i}$, an element of the Cartan subalgebra of $\mathfrak{sl}(5)$ can be expressed as

$$h = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix} - \frac{2}{5} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \mathbb{1} . \tag{6.3.29}$$

The positive Weyl chamber is given by

$$\mathfrak{a}^+ = \{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \, : \, \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 < 0 \} , \tag{6.3.30}$$

and its closure $\overline{\mathfrak{a}}^+$ is given by replacing strict inequalities with non-strict ones.
In the case of the minimal parabolic (related to Cartan generators in the interior of the positive Weyl chamber), there are the following possibilities

- All $\lambda_i < 0$: decompactification to 11 dimensional M-theory.
- $\lambda_1 > 0$, $\lambda_{2,3,4} < 0$: decompactification to 10 dimensional Type IIA.
- $\lambda_{1,2} > 0$, $\lambda_{3,4} < 0$: decompactification to 10 dimensional Type IIB.

In the case of non-minimal parabolics, some inequalities in (6.3.30) are saturated, meaning that some radii stay constant along the geodesic. This corresponds to partial decompactification limits of M-theory to 8 or 9 dimensions.

### 6.3.3 Toroidal compactifications of the Heterotic theory

The moduli space from compactifications of the Heterotic theory on $T^d$ is built on $O(d, d+16)$ and, given $i, j = 1, ..., d$, it is parametrized by the internal metric $G_{ij}$, $B$-field $B_{ij}$ and $d$ 16-dimensional Wilson lines $A_i^I$, $I = 1, ..., 16$ which enter the generalised metric as

$$\mathcal{H} = \begin{pmatrix} G + C G^{-1} C + A A & -G^{-1} C & C G^{-1} A + A \\ -G^{-1} C & G^{-1} & -G^{-1} A \\ C G^{-1} A + A & -G^{-1} A & \kappa_{IJ} + A G^{-1} A \end{pmatrix} , \tag{6.3.31}$$

with

$$C_{ij} = B_{ij} + \frac{1}{2} A_{iI} \kappa^{IJ} A_{jJ} . \tag{6.3.32}$$

Depending on the embedding, $\kappa_{IJ}$ is either the $E_8 \times E_8$ or the $SO(32)$ Killing metric. This is related to the fact that there are two self-dual lattices in 16 dimensions, distinguished by the choice of metric with respect to which we get norm one lattice generators.
Since the moduli space is built on a group that does not split over the reals, the only contribution to the boundary comes from the $O(d, d)$ component, which is essentially related to the

radii of the $T^d$. Indeed, the $O(16)$ part, related to the internal Heterotic torus, is rigid and cannot be decompactified.
Moreover, in this case we can in principle have more than one equivalence classes of parabolic subgroups under $O(d, d+16, \mathbb{Z})$. This is indeed what happens, and is related to the fact that there exist two possible decompactification limits to 10 dimensions, to the $SO(32)$ and the $E_8 \times E_8$ Heterotic theories.
For the non-supersymmetric $O(16) \times O(16)$ heterotic theory compactified on a circle, the continuous part of the moduli space is the same, but as we pointed out before the action of $\Gamma$ is different, and there are seven points at the boundary, two corresponding to the two supersymmetric ten-dimensional heterotic theories, and the other five to the five non-supersymmetric theories [104].

## 6.4 The Swampland Distance Conjecture

What we have shown so far, is that, given any point $o$ on a symmetric space, it is possible to explicitly parametrise points at infinite geodesic distance from it. This allows us furthermore to make statements about how the masses of towers of states evolve along these geodesics, connecting to the Swampland Distance Conjecture [14] as stated in (2.1.1).
In order to do this explicitely, in addition to the coset structure of the moduli space we need to make some further assumptions regarding the spectrum of the theory, namely completeness. As we mentioned previously, related to the arithmetic duality group there is the notion of lattice. That is, let $V$ be the vector space over which $G \subset GL(n, \mathbb{R})$ acts, and $\Sigma \simeq \mathbb{Z}^n$ a lattice in $V$, that is to be physically identified with the lattice of charge under the $U(1)^n$ gauge symmetry of the perturbative states of the theory, which is left invariant by the action of $\Gamma$. For $G = O(d, d) \subset GL(2d)$ the lattice is given by the momenta and winding charges, corresponding respectively to the diffeomorphisms and gauge symmetries of the B-field. Similarly, for $G = E_{d(d)}$, $n$ is the the dimension of the vector representation of $E_d$. The gauge symmetries are (in M-theory compactifications on $d$ tori) the $d$ diffemorphisms, the M2 and M5 brane charges (and diffeomorphisms of the dual graviton for $d \geq 7$).
For each $[g] \in \mathcal{S} = G/K$, that is, for each point in moduli space, one can define a notion of metric on $V$ as

$$H = g^T g \,. \tag{6.4.1}$$

Being $K$ compact, $K \subset O(n)$ so all the elements $kg$, $k \in K$ in the same equivalence class define the same metric since $k^T k = \mathbb{1}$. Then, given two vectors in the lattice, $v, w \in \Sigma \subset V$, the distance between the two can be defined as

$$d(v, w) = v^T H w \,, \tag{6.4.2}$$

and in particular one can define the mass in the string frame of a state characterised by a charge vector $q \in \Sigma$ as

$$M_q^2 = q^T H q \,. \tag{6.4.3}$$

Notice that in principle we do not have to assume supersymmetry in order to make this statement, despite the fact that in the specific case of toroidal compactifications the sates in the

charge lattice are BPS. Conceptually, we are only asking for the mass of the states to be a function defined on the moduli space and covariant under the action of the discrete subgroup. This expression can be then connected to what we saw in Section 6.3, as it turns out that $H$ defined in (6.4.1) is the generalised metric $\mathcal{H}$ defined above for the various moduli spaces. Then, having computed explicitly its form along the geodesics flow, which is generically of the form

$$\mathcal{H}(t) \sim \text{diag}(e^{\lambda_1 t}, ..., e^{\lambda_n t}) \,, \tag{6.4.4}$$

with $t$ an affine parameter, we can follow the behavior of the mass associated to each charge as $t \to \infty$, which corresponds to an infinite distance limit in moduli spaces, provided that the geodesic is such to reach a rational value of the compact moduli. It turns out that generically the mass has the following form

$$M^2(t) \sim q_1^2 e^{\lambda_1 t} + q_2^2 e^{\lambda_2 t} + ... + q_n^2 e^{\lambda_n t} \,, \tag{6.4.5}$$

where depending on the $G-$representation of the charges $q$, the $\lambda$'s are not in general independent.
For all the moduli spaces, for each choice of infinite distance limit there is at least one $\lambda_i < 0$, such that the corresponding tower $q_i$ satisfies the distance conjecture.
Let us show this explicitly in two examples.

$SL(5)$

Focusing on M-theory compactified on a $T^4$. The charges are in the **10** representation of $SL(5)$, which under $GL(4)$ decomposes as

$$\mathbf{10} = \mathbf{4} + \mathbf{6} \,, \tag{6.4.6}$$

namely a vector (corresponding to the KK charges) and a 2-form (corresponding to the brane charges)

$$q = \left\{ \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} , \begin{pmatrix} 0 & q_5 & q_6 & q_7 \\ -q_5 & 0 & q_8 & q_9 \\ -q_6 & -q_8 & 0 & q_{10} \\ -q_7 & -q_9 & -q_{10} & 0 \end{pmatrix} \right\} . \tag{6.4.7}$$

Then the contribution to the mass coming from the compact directions reads

$$\begin{aligned} M_q^2 = e^{-8/5 t(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} (& e^{2t(2\lambda_1+\lambda_2+\lambda_3+\lambda_4)} q_1^2 + e^{2t(\lambda_1+2\lambda_2+\lambda_3+\lambda_4)} q_2^2 + e^{2t(\lambda_1+\lambda_2+2\lambda_3+\lambda_4)} q_3^2 \\ &+ e^{2t(\lambda_1+\lambda_2+\lambda_3+2\lambda_4)} q_4^2 + 2e^{2t(\lambda_1+\lambda_2)} q_5^2 + 2e^{2t(\lambda_1+\lambda_3)} q_6^2 + 2e^{2t(\lambda_1+\lambda_4)} q_7^2 + 2e^{2t(\lambda_2+\lambda_3)} q_8^2 \\ &+ 2e^{2t(\lambda_2+\lambda_4)} q_9^2 + 2e^{2t(\lambda_3+\lambda_4)} q_{10}^2) \,, \end{aligned} \tag{6.4.8}$$

where the $\lambda$'s are related to the $T^4$ radii as in (6.3.29). It can be shown that for each choice of $\{\lambda_1, ..., \lambda_4\} \neq \{0, ..., 0\}$ at least one of the exponents is negative, thus proving the Distance Conjecture.

$SL(2) \subset O(2,2)$

Let us now focus on bosonic string theory compactified on a $T^2$ at fixed volume with metric $g_{ij}$, $\det g = 1$ and with no $B$-field.[17] The moduli space is isomorphic to $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$. The mass spectrum is given by

$$M^2 = M_0^2 + \frac{2}{\alpha'}(N_R + \tilde{N}_L - 2): \quad M_0^2 = g_{ij}(\tilde{p}_L^i \tilde{p}_L^j + p_R^i p_R^j) = (W,N)\hat{\mathcal{H}}(W,N)^T\,, \tag{6.4.9}$$

where $M_0^2$ is the contribution from the internal directions to the mass, $\tilde{N}_L$ and $N_R$ are respectively the left and right moving oscillator numbers that give the 10 dimensional string spectrum, $\tilde{p}_L^i$ and $p_R^i$ are respectively the left and right moving internal momenta, and $(W,N)$ is a vector of winding and momentum charges, $W = (w_1, w_2)$ and $N = (n_1, n_2)$. $\hat{\mathcal{H}}$ is the embedding of the $SL(2)$ $\mathcal{H}$ into the $O(2,2,\mathbb{R})$ generalised metric. It was shown in [143] that

$$\hat{\mathcal{H}} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 & 0 & 0 \\ \tau_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\tau_1 \\ 0 & 0 & -\tau_1 & |\tau|^2 \end{pmatrix}. \tag{6.4.10}$$

Hence, we can deduce that the flow of $\hat{\mathcal{H}}$ along geodesics is given by

$$\hat{\mathcal{H}}(t) = \begin{pmatrix} g(t) & 0 \\ 0 & g^{-1}(t) \end{pmatrix} = \frac{1}{\tau_2(t)}\begin{pmatrix} |\tau(t)|^2 & \tau_1(t) & 0 & 0 \\ \tau_1(t) & 1 & 0 & 0 \\ 0 & 0 & 1 & -\tau_1(t) \\ 0 & 0 & -\tau_1(t) & |\tau(t)|^2 \end{pmatrix}. \tag{6.4.11}$$

with $\hat{\mathcal{H}}(0) = \mathrm{diag}(g(0), g^{-1}(0))$ and $g(0) = \mathrm{diag}(1,1)$ at $\tau(0) = i$. We find

$$M_0^2(t) = \frac{1}{\tau_2(t)}(|\tau(t)|^2 w_1^2 + 2\tau_1(t) w_1 w_2 + w_2^2 + |\tau(t)|^2 n_1^2 - 2\tau_1(t) n_1 n_2 + n_2^2). \tag{6.4.12}$$

Under $c_0(t)$ in (6.1.27), $M_0^2(t)$ flows as

$$M_0^2(t) = (e^{2t} w_1^2 + e^{-2t} w_2^2 + e^{2t} n_1^2 + e^{-2t} n_2^2)\,, \tag{6.4.13}$$

so that the towers $w_1 = n_1 = 0$ become massless as $t \to \infty$. Under $c_1(t)$ in (6.1.28), it flows as

$$M_0^2(t) = (e^{-2t} w_1^2 + 2x e^{2t} w_1 w_2 + e^{2t} w_2^2 + e^{-2t} n_1^2 - 2x e^{2t} n_1 n_2 + e^{2t} n_2^2)\,, \tag{6.4.14}$$

so that the tower $w_2 = n_2 = 0$ become massless as $t \to \infty$. In summary, we have shown that the distance conjecture (unsurprisingly) holds in this string compactification.

At this point, one should account for the discrete quotient by the duality group, that selects only rational geodesics as reaching the boundary of the physical moduli space. Nevertheless, we have shown that the Distance Conjecture in principle holds also before this quotient, thus the explicit form of the discrete quotient does not matter as long as the mass behaviour is concerned. It only gives the structure of the boundary, for example the number and the type

[17] The true moduli space has twice as many degrees of freedom and is isomorphic to $O(2,2,\mathbb{R})/(O(2)\times O(2))$.

of possible decompactification limits.
What was said up to now holds in the string frame and in string units, in which the effective action has the form

$$S = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g_s} e^{-2\Phi} (R_s + ...) \,, \tag{6.4.15}$$

while the Distance Conjecture is a statement about the behaviour of the mass in the Einstein frame and in Planck units

$$S = \frac{M_p^{d-2}}{2} \int d^d x \sqrt{-g_E} (R_E - G_{ij} \partial\phi^i \partial\phi^j) \,, \tag{6.4.16}$$

obtained through a Weyl rescaling of the metric.
If we want to express the mass in Planck units then we need to account for a multiplicative factor containing the 10 dimensional dilaton and the compactification volume

$$\frac{M^2}{M_P^2}(t) \sim \frac{1}{M_P(\alpha', \Phi, Vol_{T^d}(t))} (q_1^2 e^{\lambda_1 t} + q_2^2 e^{\lambda_2 t} + ... + q_n^2 e^{\lambda_n t}) \,. \tag{6.4.17}$$

Provided that the dilaton is at weak coupling, restricting to the positive Weyl chamber guarantees that the torus volume diverges,[18] so that also in Planck units the conjecture is satisfied.

[18] This is more subtle for e.g. the $R \to 0$ limit of the heterotic string on $S^1$. Here anyway it can be shown that the SDC is still satisfied, as expected.

# Part IV

# Conclusions

# Chapter 7

# Conclusions

In the first part of this work we studied the infinite distance points of the heterotic string compactified on $d$-dimensional tori corresponding to decompactification limits and their T-dual vanishing size limit. We showed that as one approaches these limits, an affine algebra arises.

- In the case $d = 1$, we find that starting from eithr the $\mathrm{E}_8 \times \mathrm{E}_8$ or $SO((32)$ heterotic theory, one of the two infinite distance points is characterised by the affine algebra $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ (the identification means that the two copies of $E_9$ share the same central extension), while the other one presents a $\widehat{D}_{16}$. These are the only two possibilities, in agreement with the claim that only the affine versions of the gauge groups realized in the decompactified theory (the 10 dimensional theory in our case) should be obtained in the decompactification limit of the lower dimensional one [23]. The limit in which they are realised depends on whether we consider the moduli space in the $\mathrm{E}_8 \times \mathrm{E}_8$ or $SO((32)$ parametrisation. Independently on this, though, the central extension that gives rise to the affinization of these algebras is the circle direction. In fact, these heterotic BPS states which become arbitrarily light as we approach the infinite distance points are known to be part of an even bigger BPS algebra realized at finite distances, where BPS states are generically massive [144].
  We motivated the appearance of these affine algebras by means of the Extended Dynkin Diagram of the Narain Lattice $\Gamma_{(1,17)}$, and also found them very explicitly by computing the relevant OPEs and taking the infinite distance limit (e.g. $R \to \infty$). To be precise, by computing the OPEs of the KK modes of the Cartan and root sectors of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic states, we were able to isolate the simple poles in the limit and read off the affine algebra, explicitly identifying its central extension.
  We have also analyzed these results in the light of some relevant Swampland Conjectures, namely the Weak Gravity Conjecture, the closely related Repulsive Force Conjecture, and the Distance Conjecture. We have explicitly performed the field theory computation of the force between two probe heterotic states in the 9 dimensional theory, and showed that it matches the corresponding extremality bound [113] (in the regime where perturbative computation can be trusted). The Lattice Weak Gravity/Repulsive Force Conjecture then requires one particle in each site of the charge lattice to be superextremal/self-repulsive, and in particular the BPS states saturate this bound. We showed explicitly

that when the relevant infinite distance point is approached, a subset of the gauge couplings becomes vanishing (this has been conjectured to be a general property of all infinite distance limits in QG [40]), forcing the relevant states on the lattice to become massless. In the decompactification limit, these vanishing gauge couplings are associated to the heterotic gauge bosons (or more precisely, their Cartan subsector) and the graviphoton, and a subset of the states charged under them (i.e. the BPS ones) are precisely the ones that form the $(\mathrm{E}_9 \oplus \mathrm{E}_9)/\sim$ or $\widehat{D}_{16}$ algebras.

- Affine algebras were also found in their dual F-theory reincarnation, at the boundaries of "open string moduli space" [22, 23, 111]. In compactifications of F-theory on K3 one finds other affine exceptional algebras, and thus we expect to find them in toroidal compactifications of the heterotic string. We thus studied the infinite distance limits in the moduli space of the heterotic string on $T^d$ correponding to decompactification of one or more dimensions, generalizing the $d = 1$ case. In these backgrounds, which have 16 supercharges, the heterotic string worldsheet neatly captures all the data relevant to gauge symmetry enhancements and in particular this makes it a suitable frame to study how these symmetries behave at infinite distance. We have seen that in every case, the gauge algebra is asymptotically enhanced to an affine BPS algebra with the same number of central extensions as decompactified dimensions. This algebra is always the affine version of an allowed gauge algebra in the (partially) decompactified theory, and the affinization works democratically on every one of its simple factors. This includes the nonabelian factors of type $A$, $D$ and $E$, as well as left- and right-moving Cartans.
  Restricting to the case of eight dimensions, we have studied the dual description given by type IIB string junctions on the sphere and the corresponding F-theory on an elliptic K3 surface. Infinite distance limits in the latter correspond to degenerations of the K3 surface and are exhaustively accounted for in Kulikov models [22]. In the case that only one dimension decompactifies in the heterotic dual, i.e. in a Type III.a degeneration, we have shown that the affine algebras emerge in the same way as for the heterotic string. This is a nontrivial statement insofar as only affine algebras of exceptional type had been considered in the string junction literature [111, 115] and constructed explicitly in F-theory [22, 23], but we emphasise how explicitly how this works for all other kinds. There are in turn two K3 degeneration limits corresponding to decompactification of the heterotic string to ten dimensions. One of them, the Type II.a degeneration, produces the double loop algebra $(\widehat{\mathrm{E}}_9 \oplus \widehat{\mathrm{E}}_9)/\sim$, in perfect agreement with the heterotic string. As we showed, there is also a Type III.b degeneration limit in which double loop algebra $\widehat{\widehat{D}}_{16}$ should appear. However, one of the two associated towers of light states is of KK type and is not realized by string junctions. In fact, the III.b degeneration type admits a priori the realization of various other algebras which we know do not correspond to ten dimensional theories, a symptom of the same underlying problem.
  Moreover, we have shown that in the heterotic setup we can explicitly describe some paths towards the infinite distance singularities that cannot be straightforwardly described by the one parameter Kulikov models in F-theory. These paths can be understood as a sequential two-step decompactification from the heterotic point of view. However, after the first step, once the effective physics becomes 9 dimensional, it may not be possible to

describe it as an F-theoretic vacuum any more (while it can be from the heterotic angle).

- Given the simplicity with which the heterotic string allows to study the appearance of affine algebras, it is interesting to consider other backgrounds with 16 supercharges constructed as asymmetric orbifolds. The simplest case is the CHL string, where $ADE$ algebras are realised at level 2, and in eight dimensions it exhibits algebras of type $C_n$. Furthermore, in lower dimensions also those of type $B_n$ and $F_4$ are realised. The eight dimensional CHL string and some of its decompactification to nine dimensions has been studied from the point of view of string junctions in [117]. We focused on the decompactification of the perturbative CHL string to ten dimensions, which is characterized by a twisted affine algebra. Our result can be generalized to the case of eight-dimensional compactifications with rank reduction of CHL type when one decompactifies the cycle carrying the holonomy. Thus, it is a non-trivial prediction to be reproduced in a dual M/F-theory description in the presence of frozen singularities on K3 in the appropriate infinite distance limit, though the explicit realisation is still an open question. In lower dimensions there are also other possibilities; for example, in the case of the CHL string on $S^1$, one could decompactify the $S^1$ without the holonomy recovering an affine, non-twisted algebra (from the Type II point of view, see [117]).
  Another possible avenue would be to understand the algebras arising in infinite distance limits in the case of Calabi-Yau compactifications of the heterotic theory (which in the K3 case is dual to F-theory on elliptic Calabi-Yau threefolds, whose degenerations at infinite distance have been recently considered in [145, 146]). However, even in the heterotic frame, this would require an understanding of both space-time and worldsheet non-perturbative effects. It may be possible and instructive to generalize our methods to special loci where the K3 is realized as an orbifold of $T^4$, and we hope that the lessons learned from those examples may be useful for understanding the more general situation.
  While we focused on decompactification limits that preserve the number of supercharges, we could also consider the case of nine-dimensional non-supersymmetric heterotic string, in which case we would expect to find both an asymptotic current algebra and supersymmetry enhancement, at least at string tree-level. It would then be interesting to reconsider these structures in light of the non-zero vacuum energy generated by string loops.

In the second part of the thesis, we focused on the case of symmetric moduli spaces. These include toroidal compactifications of M- and string theory, but the definition actually refers to a more general class. Indeed, starting with a "non-positively curved" space $\mathcal{M}$ (i.e. having non-positive sectional curvature), one can still define a notion of rank (with rank = 1, 2, 3, ...). There it can be shown that if $\mathcal{M}$ is finite volume and of rank $> 1$, then $\mathcal{M}$ has to be a locally symmetric space. In addition to this, string theoretic examples display also an action of a duality group, which mathematically can be accounted for with a quotient by a discrete subgroup.
We first presented the mathematical framework of global symmetric spaces, which are diffeomorphic to a quotient of Lie groups (locally symmetric spaces have globally symmetric ones as universal covers). In particular, we focused on the properties of the class of non-compact symmetric spaces, which clearly connects to the properties of string moduli spaces. This re-

quirement sets constraints on the type of group quotients, which makes it possible to give a unified description of their boundaries just in terms of group theoretic arguments. In particular, they can be studied in terms of geodesics that reach infinite distance; these are just a measure zero class of all the possible geodesics on such manifolds, and are characterised by rational values of the compact moduli. This can be directly connected with the SDC, which is a statement along geodesic flows at infinite distance. These are classified by rational parabolic subgroups and elements of the Cartan subalgebra of the structure group of the manifold, which allows for a systematic description of the geometry at infinity of such moduli spaces. In turn, this makes it possible to give a rigorous proof of the Distance Conjecture in these settings under the assumption of a fully populated charge lattice [134]. Despite having focused explicitly on cases with a high number of supercharges, in principle these methods are completely general and do not rely at any point in the amount of supersymmetry, as long as the theory displays a symmetric moduli space and a complete charge lattice.

# Part V

# Appendices

# Appendix A

# Affine Lie algebras

In this Appendix we briefly summarise the construction of loop and affine algebras, both in their unwisted and twisted versions. We will follow mainly [123, 124, 127].
Let us consider a compact simple[1] Lie group $G$ with algebra $\mathfrak{g}$ generated by an orthonormal set $\{J^a\}$ and characterized by structure constants $f_c^{ab}$. The notion of orthonormality comes from the Killing form, defined as folllows. The action of a generator $X \in \}$ in the adjoint representation $\mathrm{ad}(X)$ is

$$\mathrm{ad}(X)Y = [X, Y], \quad X, Y \in \mathfrak{g}\,, \tag{A.0.1}$$

and from this one can define the Killing form $K$ as

$$K(X, Y) = \frac{1}{2g}\mathrm{Tr}(\mathrm{ad}X\ \mathrm{ad}Y)\,, \tag{A.0.2}$$

where $g$ is the dual Coxeter number of $\mathfrak{g}$. Then by orthonormality of the basis $\{J^a\}$ one means

$$K(J^a, J^b) = \delta^{ab}\,. \tag{A.0.3}$$

Let us choose for $\mathfrak{g}$ the Cartan-Weyl basis, with the Hermitean and orthonormal $H^i$ (i=1,...,$r$=rank($\mathfrak{g}$)) in the maximal Abelian subalgebra of $\mathfrak{g}$, the Cartan subalgebra, and $E^\alpha$ such that

$$[H^i, H^j] = 0, \quad [H^i, E^\alpha] = \alpha^i E^\alpha\,. \tag{A.0.4}$$

$\alpha = (\alpha^1, ..., \alpha^r)$ is called the root associated to the step operator $E^\alpha$. Let us denote by $\Sigma$ the set of roots.
Moreover, from Jacobi identities it holds

$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha,\beta}E^{\alpha+\beta} & \text{for } \alpha+\beta \in \Sigma \\ \frac{2}{|\alpha|^2}\alpha^i H^i & \text{for } \alpha = -\beta\,, \\ 0 & \text{otherwise,} \end{cases} \quad , \tag{A.0.5}$$

[1] The algebra of a generic compact Lie group would just be the product of simple ones and Abelian ones.

where from the structure of the Killing form $|\alpha|^2 = \sum_{k=1}^{r} \alpha^k \alpha^k$ and the normalization of $E^\alpha$ is given from $[E^\alpha, E^{-\alpha}]$, and is

$$K(E^\alpha, E^\beta) = \frac{2}{|\alpha|^2} \delta^{\alpha+\beta,0} \,. \tag{A.0.6}$$

Moreover, let us define the pairing

$$\alpha \cdot H = \sum_{i=0}^{r} \alpha^i \, H^i \,. \tag{A.0.7}$$

One can always choose a basis of $r$ roots, the so-called simple roots $\{\alpha_{(1)}, ..., \alpha_{(r)}\} = \Phi$, such that every other root $\beta \in \Sigma$ can be written as

$$\beta = \sum_{i=1}^{r} b_i \alpha_{(i)} \,, \tag{A.0.8}$$

where either $b_i \geq 0$, $1 \leq i \leq r$ (and $\beta$ is called a positive root) or $b_i \leq 0$, $1 \leq i \leq r$ (and $\beta$ is called a negative root).
The Cartan matrix associated to $\mathfrak{g}$ is defined as

$$C_{ij} = \frac{2\alpha_{(i)} \cdot \alpha_{(j)}}{\alpha_{(j)}^2} \in \mathbb{Z} \,, \tag{A.0.9}$$

and it is determined by $\mathfrak{g}$, up to permutations of rows/columns. The information contained in (A.0.9) is the same as the one in the Dynkin diagram, which can be built from it as follows. To each simple root $\alpha_{(i)}$ corresponds one node (white for long roots and black for short roots), and the number of links between two nodes related to $\alpha_{(i)}$ and $\alpha_{(j)}$ is $C_{ij}C_{ji}$.
The roots are the eigenvalues of the Cartan generators are called roots in the adjoint representation. By considering an arbitrary representation, one can always find a basis $\{|\lambda\rangle\}$ for which

$$H^i \, |\lambda\rangle = \lambda^i \, |\lambda\rangle \ . \tag{A.0.10}$$

The vectors $\lambda = (\lambda^1, ..., \lambda^r)$ are called weights, and it turns out that they can be also expressed in the basis of simple roots, namely

$$\lambda = \sum_{i=1}^{r} l_i \alpha_{(i)} \,, \tag{A.0.11}$$

where in this expansion usually the coefficients $l_i$ are not integers.

### Untwisted affine Lie algebras

Associated to the algebra } there is a so-called loop algebra, denoted $\bar{\mathfrak{g}}$, which is built as the periodic map from the circle into the Lie algebra $\mathfrak{g}$

$$\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \,, \tag{A.0.12}$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in $t$. Namely, one is considering the elements $X \in \mathfrak{g}$ to be Laurent polynomials in $t$ as

$$X = \sum_{\mathrm{n}=-\infty}^{+\infty} X_{\mathrm{n}} t^{\mathrm{n}}\,, \quad \mathrm{n} \in \mathbb{Z}\,. \tag{A.0.13}$$

In particular, the generators of the loop algebra are

$$J^a_{\mathrm{n}} \equiv J^a \otimes t^{\mathrm{n}}\,, \tag{A.0.14}$$

and the multiplication rule is naturally inherited from the one of $\mathfrak{g}$

$$[J^a_{\mathrm{n}}, J^b_{\mathrm{m}}] = i f^{ab}{}_c J^c_{\mathrm{n+m}}\,. \tag{A.0.15}$$

An untwisted affine Lie algebra $\hat{\mathfrak{g}}$ can be realised as a central extension of $\bar{\mathfrak{g}}$ by a central element $\hat{k}$ commuting with all the $J^a_{\mathrm{n}}$'s

$$[J^a_{\mathrm{n}}, \hat{k}] = 0\,, \tag{A.0.16}$$

with the addition of a 'derivation operator' $d$ needed to remove the degeneracy in the affine roots, as will be explained in the following. Thus

$$\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}\hat{k} \otimes \mathbb{C}d\,. \tag{A.0.17}$$

with commutators

$$[J^a_{\mathrm{n}}, J^b_{\mathrm{m}}] = i f^{ab}_c J^c_{\mathrm{n+m}} + \hat{k}\mathrm{n} K(J^a, J^b)\delta_{\mathrm{n+m},0} \tag{A.0.18}$$

$$= i f^{ab}_c J^c_{\mathrm{n+m}} + \hat{k}\mathrm{n}\delta^{ab}\delta_{\mathrm{n+m},0}\,, \tag{A.0.19}$$

where the second equality holds in the convention (A.0.3). This central extension can be proved to be unique.
This means that in the Cartan-Weyl basis the brakets read

$$\begin{aligned}
[H^i_{\mathrm{n}}, H^j_{\mathrm{m}}] &= \hat{k}\mathrm{n}\delta^{ij}\delta_{\mathrm{n+m},0}\,,\\
[H^i_{\mathrm{n}}, E^\alpha_{\mathrm{m}}] &= \alpha^i E^\alpha_{\mathrm{n+m}}\,,\\
[E^\alpha_n, E^{-\alpha}_{\mathrm{m}}] &= \frac{2}{|\alpha|^2}\left(\alpha \cdot H_{\mathrm{n+m}} + \hat{k}n\delta_{\mathrm{n+m},0}\right)\,,
\end{aligned} \tag{A.0.20}$$

From (A.0.20), the set $\{H^i_0, ..., H^r_0, \hat{k}\}$ is Abelian, and also $\hat{k}$ commutes with all the generators. The roots built with this Abelian subalgebra, though, are infinitely degenerate: the eigenvalues of $\mathrm{ad}(H^i_0)$ and $\mathrm{ad}(\hat{k})$ on the generators $E^\alpha_{\mathrm{n}}$ are $\alpha^i$ and 0, in such a way that the associated root is $(\alpha^1, ..., \alpha^r, 0)$ for all n. Hence the need to have an operator that can discriminate among different n's, for instance

$$d = t\frac{d}{dt}\,, \tag{A.0.21}$$

such that from (A.0.14)

$$[d, J^a_{\mathrm{n}}] = \mathrm{n} J^a_{\mathrm{n}}\,, \tag{A.0.22}$$

and moreover

$$[d, \hat{k}] = 0\,, \tag{A.0.23}$$

such that the extension of the definition of the Killing form to the affine algebra is given by

$$\begin{aligned} K(J^a_{\rm n}, J^b_{\rm m}) = \delta^{ab}\delta_{\rm n+m,0}\,, \quad K(J^a_{\rm n}, \hat{k}) = 0\,, \quad K(\hat{k}, \hat{k}) = 0\,, \\ K(J^a_{\rm n}, d) = 0, \qquad K(d, \hat{k}) = 1\,, \quad K(d, d) = 0\,. \end{aligned} \tag{A.0.24}$$

The Cartan subalgebra is $\{H^i, \hat{k}, d\}$ and the ladder operators are $E^\alpha_{\rm n}$ for all ${\rm n} \in \mathbb{Z}$ and $H^i_{\rm n}$ for ${\rm n} \neq 0$. The root vectors are respectively $\hat{\alpha} = (\alpha, 0, {\rm n})$, each of which is non-degenerate, and $\hat{h} = (0_r, 0, {\rm n}) \equiv {\rm n}\delta$ (where $\delta = (0_r, 0, 1)$ is called imaginary root), each of which is $r$-fold degenerate.
From (A.0.24), and defining an affine weight vector $\hat{\lambda} = \lambda, \lambda_k, \lambda_d$ with $\lambda$ a weight vector for $\mathfrak{g}$, the scalar product $\hat{\cdot}$ is defined as

$$\hat{\lambda} \hat{\cdot} \hat{\mu} = \lambda \cdot \mu + \lambda_k \mu_d + \lambda_d \mu_k\,, \tag{A.0.25}$$

so that $|\hat{\alpha}|^2 = |\alpha|^2$, $|\delta|^2 = 0$ and $\hat{\alpha} \hat{\cdot} \delta = 0$, so that $\delta$ is a null vector, orthogonal to all the other roots.
The set of simple roots of the affine algebra can be defined from the simple roots of the finite algebra with ${\rm n} = 0$, namely $\hat{\alpha}_{(i)} = (\alpha, 0, 0)$ and the affine root

$$\hat{\alpha}_0 = (-\psi, 0, 1)\,, \tag{A.0.26}$$

where $\psi$ is the highest root of $\mathfrak{g}$, defined as the unique root such that, for any $\alpha \in \Sigma$, $\psi - \alpha$ is a sum of positive roots.
The Cartan matrix for the affine algebra is then

$$C_{ij} = \frac{2\alpha_{(i)} \cdot \alpha_{(j)}}{\alpha^2_{(j)}}\,, 0 \geq i,\, j \geq r\,, \tag{A.0.27}$$

and from it one can build the corresponding affine Dynkin diagram.

### Twisted affine Lie algebras

Let us now consider the case in which the algebra $\mathfrak{g}$ has an automorphism $\tau$

$$[\tau(J^a), \tau(J^b)] = i f^{ab}{}_c \tau(J^c) \tag{A.0.28}$$

of order $N$, $\tau^N = 1$. Then $\mathfrak{g}$ can be divided into eigenspaces $\mathfrak{g}_{({\rm m})}$ of $\tau$, such that

$$\tau(T) = e^{2\pi i {\rm m}/N} T \text{ for } T \in \mathfrak{g}_{({\rm m})} \tag{A.0.29}$$

by taking appropriate linear combinations $T$ of the generators. This defines a gradation of the algebra

$$\mathfrak{g} = \oplus_{{\rm m} \in \mathbb{Z}_N} \mathfrak{g}_{({\rm m})}\,, \quad [T_{({\rm m})}, T_{({\rm n})}] \in \mathfrak{g}_{({\rm m+n})} \text{ for } T_{({\rm m})} \in \mathfrak{g}_{({\rm m})}\,, T_{({\rm n})} \in \mathfrak{g}_{({\rm n})}\,. \tag{A.0.30}$$

The $\mathfrak{g}_{(0)} \subset \mathfrak{g}$ component is a Lie algebra, and it is the subalgebra of $\mathfrak{g}$ that commutes with the automorphism $\tau$.
Building then an affine algebra from the basis $\{T^a\}$ of eigenvectors of $\tau$ leads to the affine generators $\{T^a_{\mathrm{r}}\}$, $\mathrm{r} \in \mathbb{Z} + \frac{\mathrm{r}}{N}$ such that

$$[T^a_{\mathrm{m}}, T^b_{\mathrm{n}}] = i f^{ab}{}_c T^c_{\mathrm{m+n}} + \hat{k}\mathrm{m}\delta^{ab}\delta_{\mathrm{m+n},0} \tag{A.0.31}$$

gives what is referred to as twisted affine algebra $\hat{\mathfrak{g}}^\tau$. Again, one introduces a derivation $d$ such that

$$[d, T^a_{\mathrm{m}}] = \mathrm{m}T^a_{\mathrm{m}}\,, \quad [d, \hat{k}] = 0\,. \tag{A.0.32}$$

It turns out that using an inner automorphism $\tau_{inner}$ of $\mathfrak{g}$ yields a twisted algebra $\hat{\mathfrak{g}}^{\tau_{inner}}$ that is isomorphic to the untwisted $\hat{\mathfrak{g}}$. Indeed, in this case there exists a $g \in G$ (a Lie group with algebra $\mathfrak{g}$) such that

$$\tau(T) = gTg^{-1}\,, \tag{A.0.33}$$

and the twist coming from such a transformation can always be removed by a shift as follows. First of all, notice that twists generated by two conjugate elements $g_1$, $g_2 \in G$ lead to isomorphic twisted algebras. This means that by conjugation we can always choose $g$ to correspond to a Cartan generator, namely

$$g = e^{i\xi^i H^i}\,. \tag{A.0.34}$$

Then asking that $\tau$ has order $N$ is equivalent to $N\xi^i\alpha^i \in 2\pi\mathbb{Z}$ for all the roots $\alpha$ of $\mathfrak{g}$. Moreover, it holds

$$\tau(H^i) = H^i, \quad \tau(E^\alpha) = e^{i\xi^i\alpha^i}E^\alpha, \tag{A.0.35}$$

so that $\hat{\mathfrak{g}}^\tau$ has generators

$$H^i_{\mathrm{m}},\ \mathrm{m} \in \mathbb{Z}\,, \tag{A.0.36}$$

$$E^\alpha_{\mathrm{r}},\ \mathrm{r} \in \mathbb{Z} + \frac{\xi^i\alpha^i}{2\pi} \tag{A.0.37}$$

together with the central extension $\hat{k}$ and the derivation $d$ which satisfy an algebra which is formally the same as the one of the untwisted $\hat{\mathfrak{g}}$ with properly shifted indices as in (A.0.36). Then, it is easy to show that the operators

$$\begin{aligned} K^i_{\mathrm{m}} &= H^i_{\mathrm{m}} + \frac{1}{2\pi}\hat{k}\xi^i\delta_{\mathrm{n},0}\,, \\ F^\alpha_{\mathrm{m}} &= E^\alpha_{\mathrm{m}+\xi^i\alpha^i/2\pi}\,, \\ d' &= d - \frac{1}{2\pi}\xi^i H^i \end{aligned} \tag{A.0.38}$$

together with $\hat{k}$ satisfy the algebra $\hat{\mathfrak{g}}$.
It is important to stress though that in the two cases the physical interpretation is different, for instance states arrange differently. In particular, being the derivations $d$ and $d'$ different, the vacuum states in some representation of $\hat{\mathfrak{g}}^\tau$ will form a representation of $\mathfrak{g}_0$ instead of the full $\mathfrak{g}$. Also the $\mathfrak{g}_{\mathrm{m}}$, $\mathrm{m} > 0$, are representations of $\mathfrak{g}_0$.

Outer automorphisms exist when the Dynkin diagram of $\mathfrak{g}$ displays some symmetries, such as in the cases $A_n$ with $n \geq 2$, $D_n$ wiht $n \geq 4$ and $E_6$. In all these these cases $\tau$ has order 2, and $D_4$ has in addition an outer automorphism of order 3, related to the $S_3$ permutation symmetry. Let us define the action of the outer automorphism $\tau$ on the simple roots as

$$\tau\alpha_{(i)} = \alpha_{\tau(i)}\,, \quad i = 1, ..., r \tag{A.0.39}$$

which can be extended to $\mathfrak{g}$ as

$$\tau(\alpha) \cdot \tau(H) = \alpha \cdot H\,, \tag{A.0.40}$$

$$\tau(E^{\alpha_{(i)}}) = E^{\tau(\alpha_{(i)})}\,, \tag{A.0.41}$$

where (A.0.41) holds for simple roots, and in general by the requirement that $\tau$ leaves the Dynkin diagram unchanged, it holds

$$\tau(E^{\alpha}) = \epsilon_\alpha E^{\tau(\alpha)}\,, \tag{A.0.42}$$

with $\epsilon_\alpha = \epsilon_{-\alpha} + \pm 1$.
Let us now restrict for simplicity to the order 2 case. Then, $\tau$ has eigenvalue $+1$ on $\mathfrak{g}_{(0)}$ and $-1$ on $\mathfrak{g}_{(1)}$. Then, it can be easily seen that for $\alpha_{(i)}$ such that $\tau(i) = i$

$$\alpha_{(i)} \cdot H \in \mathfrak{g}_{(0)} \tag{A.0.43}$$

and if $\tau(i) \neq i$

$$\frac{1}{2}\left(\alpha_{(i)} + \alpha_{(\tau(i))}\right) \cdot H \in \mathfrak{g}_{(0)}\,, \quad \frac{1}{2}\left(\alpha_{(i)} - \alpha_{(\tau(i))}\right) \cdot H \in \mathfrak{g}_{(1)}\,, \tag{A.0.44}$$

and the Cartan subalgebra of $\mathfrak{g}_{(0)}$ is generated by the distinct operators of the form $\xi \cdot H$ with $\tau(\xi) = \xi$. Moreover

$$[\xi \cdot H, E^{\alpha} \pm E^{\tau(\alpha)}] = \xi\left(\alpha + \tau(\alpha)\right)\left(E^{\alpha} \pm E^{\tau(\alpha)}\right), \tag{A.0.45}$$

and since

$$\tau(E^{\alpha} \pm E^{\tau(\alpha)}) = \pm\epsilon_\alpha(E^{\alpha} \pm E^{\tau(\alpha)})\,, \tag{A.0.46}$$

it follows that

- if $\alpha = \tau(\alpha)$, then $E^{\alpha} \in \mathfrak{g}_{(0,1)}$ if $\epsilon_\alpha = \pm 1$,
- if $\alpha \neq \tau(\alpha)$, then $E^{\alpha} \pm E^{\tau(\alpha)} \in \mathfrak{g}_{(0,1)}$.

Then, the roots of $\mathfrak{g}_{(0)}$ are of the form

- $\alpha$ with $\alpha = \tau(\alpha)$ and $\epsilon_\alpha = 1$, or
- $\frac{1}{2}\left(\alpha + \tau(\alpha)\right)$ for $\alpha \neq \tau(\alpha)$ and $\epsilon_\alpha = 1$,

such that a set of simple roots of $g_{(0)}$ are $\beta_{(i)} = \frac{1}{2}\left(\alpha_{(i)} + \alpha_{(\tau(i))}\right)$. In order to get a basis of simple roots for $\hat{\mathfrak{g}}^{\tau}$, one has to add the vector

$$\beta_{(0)} = (-\phi, 0, \frac{1}{2}), \tag{A.0.47}$$

where $\phi$ is the highest vector of the form $\frac{1}{2}\alpha + \frac{1}{2}\tau(\alpha)$ for $\alpha \neq \tau(\alpha)$ or $\alpha$ for $\alpha = \tau(\alpha)$ and $\epsilon_{\alpha} = -1$. In particular, given the imaginary root $\delta$ as above, the roots of the twisted affine algebra are of the form

$$\{\Sigma(\mathfrak{g}_{(0)}) + \mathbb{Z}\delta\} \cup \{\Sigma(\mathfrak{g}_{(1)}) + (\mathbb{Z} + 1/2)\delta\} \cup \{\mathbb{Z}/2\delta\}, \tag{A.0.48}$$

where $\Sigma(\mathfrak{g}_{(0)})$ is the root system of $\mathfrak{g}_{(0)}$ and $\Sigma(\mathfrak{g}_{(1)})$ are the non-zero weights of $\mathfrak{g}_{(1)}$.

# Appendix B

# Worldsheet realization of the affine algebras

In the following, we present the affinisation of the current algebras from the worldsheet point of view. With the normalization in (3.1.11), the non trivial OPEs between the worldsheet bosons in the heterotic theory are

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \sim -\frac{1}{2}\eta^{\mu\nu}\log|z-w|^2\,, \tag{B.0.1}$$

$$Y^i(z,\bar{z})Y^j(w,\bar{w}) \sim -\frac{G^{ij}}{2}\log|z-w|^2\,, \tag{B.0.2}$$

$$X^I(z)X^J(w) \sim -\delta^{IJ}\log(z-w)\,, \tag{B.0.3}$$

where $Y^i(z,\bar{z}) = Y^i(z) + \bar{Y}^i(\bar{z})$ are the compact bosons of $T^d$, with $i = 1, ..., d$, and $X^I(z)$ are the compact chiral bosons of the heterotic torus.

## B.1 Affine algebras in the heterotic string on $S^1$

Let us show explicitly how the $(E_9 \oplus E_9)/\sim$ algebra is derived in the general case of finite constant Wilson line $A$. The vertex operators in the 0 picture associated to each $\mathrm{E}_8 \times \mathrm{E}_8 \times U(1)$ (or $SO(32) \times U(1)$) massless vector in the string spectrum are

$$\alpha^I_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}\,|0,\mathrm{n}\rangle_{NS} \to J^I(z)\left(i\sqrt{2}\bar{\partial}X^\mu(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^\mu(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i\mathrm{n}Y(z,\bar{z})}\,, \tag{B.1.1}$$

$$\bar{\psi}^\mu_{-\frac{1}{2}}\,|0,\mathrm{n},\pi_\alpha\rangle_{NS} \to J^\alpha(z)\left(i\sqrt{2}\bar{\partial}X^\mu(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^\mu(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i(\mathrm{n}-\pi^I_\alpha A^I)Y(z,\bar{z})}\,, \tag{B.1.2}$$

$$\alpha^9_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}\,|0\rangle_{NS} \to J^9(z)\left(i\sqrt{2}\bar{\partial}X^\mu(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^\mu(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i\mathrm{n}Y(z,\bar{z})}\,. \tag{B.1.3}$$

They depend on the right momentum $\kappa = (k_\mu, p_R)$, on the $\mathrm{E}_8 \times \mathrm{E}_8$ (or $SO(32)$) currents, $J^I$ and $J^\alpha$ as in (4.1.6) and on the $U(1)$ current

$$J^9(z) = i\sqrt{2}\partial Y(z)\,. \tag{B.1.4}$$

In the limit $R \to \infty$, in order to read the structure of the algebra in the presence of a non vanishing Wilson line we can generalise the definitions (4.1.5) of $\{J^a_{\rm n}\}$ $(a = I, \alpha)$ to

$$\begin{aligned} J^I_{\rm n}(z) &\equiv \left(J^I(z) - \frac{A^I}{\sqrt{2}} J^9(z)\right) e^{i{\rm n}Y(z)} = i(\partial X^I(z) - A^I \partial y(z)) e^{i{\rm n}Y(z)}\,, \\ J^\alpha_{\rm n}(z) &\equiv J^\alpha(z) e^{i({\rm n} - \pi^I_\alpha A^I)Y(z)} = c_\alpha e^{i\pi^I_\alpha (X^I(z) - A^I Y(z))} e^{i{\rm n}Y(z)}\,, \end{aligned} \tag{B.1.5}$$

which are still both associated to massless states. They can still be interpreted as asymptotically conserved currents, as their conformal dimension ($\mathrm{h}_I = 1 + \frac{\rm n}{2R^2}$ and $\mathrm{h}_\alpha = 1 + \frac{({\rm n} - \pi^I_\alpha A^I)}{2R^2}$ respectively) tends to 1 as $R \to \infty$. As one can see from (B.1.5), they can be rewritten using the field redefinition

$$X'^I(z) = X^I(z) - A^I Y(z) \tag{B.1.6}$$

(which makes it manifest that the effect of the Wilson line is to mix the $\mathrm{E}_8 \times \mathrm{E}_8$ Cartan states with the left KK one) yielding

$$J^I_{\rm n}(z) = i\partial X'^I(z) e^{i{\rm n}Y(z)}\,, \qquad J^\alpha_{\rm n}(z) = c_\alpha e^{i\pi^I_\alpha X'^I} e^{i{\rm n}Y(z)}\,, \tag{B.1.7}$$

which are in the same form as the ones in the zero Wilson line case. In terms of the redefined heterotic coordinate, the non trivial OPEs among the worldsheet fields are

$$X'^I(z)X'^J(w) \sim X^I(z)X^J(w) + A^I A^J Y(z)Y(w) \sim -\left(\delta^{IJ} + \frac{A^I A^J}{2R^2}\right) \log(z-w)\,, \tag{B.1.8}$$

$$X'^I(z)Y(w,\bar w) \sim - A^I Y(z)Y(w) \sim \frac{A^I}{2R^2} \log(z-w)\,, \tag{B.1.9}$$

as well as (B.0.1) and (B.0.2) which remain unchanged. It is clear that for $R \to \infty$ the fields $X'^I$ and $X^I$ satisfy the same OPEs, and so the cases $A^I = 0$ and constant $A^I \neq 0$ are equivalent in the limit when the structure of the algebra is concerned, namely the currents (4.1.5) and (B.1.5) satisfy the same relations. For simplicity we will restrict to the case $A^I = 0$, described by (4.1.5) and (4.1.6). One can compute the OPE between two of such affine currents as follows. Taking two states in the tower of the $\mathrm{E}_8 \times \mathrm{E}_8$ (or $SO(32)$) Cartan vectors

$$\begin{aligned} J^I_{\rm n}(z) J^J_{\rm m}(w) &= -:\partial X^I(z)\, e^{i{\rm n}Y(z)}:\,:\partial X^J(w)\, e^{i{\rm m}Y(w)}: \\ &= (z-w)^{\frac{\rm nm}{2R^2}} :e^{i{\rm n}Y(z)} e^{i{\rm m}Y(w)} \left(-\partial X^I(z)\partial X^J(w) + \frac{\delta^{IJ}}{(z-w)^2}\right):, \end{aligned}$$

and expanding around $z = w$

$$\begin{aligned} &J^I_{\rm n}(z) J^J_{\rm m}(w) = \\ &= (z-w)^{\frac{\rm nm}{2R^2}} :(1 + i{\rm n}\partial Y(w)(z-w) + \ldots)\, e^{i({\rm n}+{\rm m})Y(w)} \left(-\partial X^I(z)\partial X^J(w) + \frac{\delta^{IJ}}{(z-w)^2}\right): \\ &\sim (z-w)^{\frac{\rm nm}{2R^2}} \left(\delta^{IJ} \frac{:e^{i({\rm n}+{\rm m})Y(w)}:}{(z-w)^2} + i\delta^{IJ}{\rm n} \frac{:\partial Y(w) e^{i({\rm n}+{\rm m})Y(w)}:}{z-w} + \mathcal{O}(1)\right), \end{aligned} \tag{B.1.10}$$

which is indeed (4.3.7). As for the OPE between two affine root currents

$$\begin{aligned} J^\alpha_{\rm n}(z)J^\beta_{\rm m}(w) =& : c_\alpha e^{i\pi^I_\alpha X^I(z)} e^{i{\rm n}Y(z)} :: c_\beta e^{i\pi^J_\beta X^J(w)} e^{i{\rm m}Y(w)} : \\ =& c_\alpha c_\beta (z-w)^{\pi_\alpha\cdot\pi_\beta+\frac{\rm nm}{2R^2}} : e^{i\pi^I_\alpha X^I(z)} e^{i{\rm n}Y(z)} e^{i\pi^J_\beta X^J(w)} e^{i{\rm m}Y(w)} : \\ =& c_\alpha c_\beta (z-w)^{\pi_\alpha\cdot\pi_\beta+\frac{\rm nm}{2R^2}} : \big[1+i\pi^I_\alpha \partial X^I(w)(z-w)+...)\cdot \\ & \cdot (1+i{\rm n}\partial Y(w)(z-w)+\dots\big] e^{i(\pi^I_\alpha+\pi^I_\beta)X^I(w)} e^{i({\rm n}+{\rm m})Y(w)} : . \end{aligned} \tag{B.1.11}$$

The explicit result depend on the pair of roots we consider. If $\pi_\alpha+\pi_\beta=\pi_{\alpha+\beta}$ still belongs to the root system of $\mathrm{E}_8\times\mathrm{E}_8$ (or $SO(32)$), with the root normalization $|\pi|^2=2$ it holds $\pi_\alpha\cdot\pi_\beta=-1$ and (B.1.11) reads

$$\begin{aligned} J^\alpha_{\rm n}(z)J^\beta_{\rm m}(w) \sim& (z-w)^{\frac{\rm nm}{2R^2}} \left( \frac{\epsilon(\alpha,\beta) c_{\alpha+\beta} : e^{i\pi^I_{\alpha+\beta}X^I(w)} e^{i({\rm n}+{\rm m})Y(w)} :}{z-w} + \mathcal{O}(1) \right) \\ =& (z-w)^{\frac{\rm nm}{2R^2}} \left( \frac{\epsilon(\alpha,\beta) J^{\alpha+\beta}_{\rm n+m}(w)}{z-w} + \mathcal{O}(1) \right) . \end{aligned} \tag{B.1.12}$$

In the case $\pi_\alpha=-\pi_\beta$, $\pi_\alpha\cdot\pi_\beta=-|\pi_\alpha|^2=-2$, (B.1.11) reads

$$J^\alpha_{\rm n}(z)J^{-\alpha}_{\rm m}(w) \sim (z-w)^{\frac{\rm nm}{2R^2}} : e^{i({\rm n}+{\rm m})Y(w)} \left( \frac{1}{(z-w)^2} + \frac{i(\pi^I_\alpha \partial X^I(w) + {\rm n}\partial Y(w))}{z-w} + \mathcal{O}(1) \right) :, \tag{B.1.13}$$

which can be rewritten as

$$J^\alpha_{\rm n}(z)J^{-\alpha}_{\rm m}(w) \sim (z-w)^{\frac{\rm nm}{2R^2}} \left( \frac{: e^{i({\rm n}+{\rm m})Y(w)} :}{(z-w)^2} + \frac{\pi^I_\alpha J^I_{\rm n+m}(w) + i{\rm n} : \partial Y(w) e^{i({\rm n}+{\rm m})y(w)} :}{z-w} + \mathcal{O}(1) \right) . \tag{B.1.14}$$

In all the other cases we do not find any integer pole for $R\to\infty$. These results can be summarized as

$$J^\alpha_{\rm n}(z)J^\beta_{\rm m}(w) \sim (z-w)^{\frac{\rm nm}{2R^2}} \cdot \begin{cases} \frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\rm n+m}(w)}{z-w} + \mathcal{O}(1) & \alpha+\beta \,\text{root}, \\ \frac{:e^{i({\rm n}+{\rm m})Y(w)}:}{(z-w)^2} + \frac{\pi^I_\alpha J^I_{\rm n+m}(w) + i{\rm n}:\partial Y(w) e^{i({\rm n}+{\rm m})Y(w)}:}{z-w} + \mathcal{O}(1) & \alpha=-\beta\,, \\ \mathcal{O}(1) & \text{otherwise}, \end{cases} \tag{B.1.15}$$

For the OPE between one current in the Cartan tower and the other in the root tower

$$\begin{aligned} J^I_{\rm n}(z)J^\alpha_m(w) =& : i\partial X^I(z) e^{i{\rm n}Y(z)} :: c_\alpha e^{i\pi^J_\alpha X^J(w)} e^{i{\rm m}Y(w)} : \\ =& ic_\alpha (z-w)^{\frac{\rm nm}{2R^2}} \frac{-i\pi^I_\alpha : e^{i\pi^J_\alpha X^J(w)} e^{i{\rm n}Y(z)} e^{i{\rm m}Y(w)} :}{z-w} \\ \sim& c_\alpha (z-w)^{\frac{\rm nm}{2R^2}} \frac{\pi^I_\alpha : e^{i\pi^J_\alpha X^J(w)} e^{i({\rm n}+{\rm m})Y(w)} :}{z-w} = \frac{\pi^I_\alpha J^\alpha_{\rm n+m}(w)}{z-w} , \end{aligned} \tag{B.1.16}$$

as $R \to \infty$. Again, this is consistent with (4.3.8).
(4.1.16), (4.1.17) and (4.1.18) are straightforward from (B.0.2). From the OPEs between the currents one can compute the algebra between their zero modes (4.1.19) as follows. The commutator between two Cartan generators for $R \to \infty$ reads

$$[(J^I_{\rm n})_0, (J^J_{\rm m})_0] = \oint_{C'} \frac{dz}{2\pi i} \oint_C \frac{dw}{2\pi i} J^I_{\rm n}(z) J^J_{\rm m}(w) - \oint_{C'} \frac{dw}{2\pi i} \oint_C \frac{dz}{2\pi i} J^I_{\rm n}(z) J^J_{\rm m}(w) \tag{B.1.17}$$

where $C'$ is a contour external to $C$. Then one obtains

$$\begin{aligned}
[(J^I_{\rm n})_0, (J^J_{\rm m})_0] &= \oint \frac{dw}{2\pi i} \delta^{IJ} \mathrm{Res}_{z\to w} \Big[ \frac{: e^{i({\rm n}+{\rm m})Y(w)} :}{(z-w)^2} + i{\rm n} \frac{: \partial Y(w) e^{i({\rm n}+{\rm m})Y(w)} :}{z-w} \Big] \\
&= \oint \frac{dw}{2\pi i} i{\rm n} \delta^{IJ} \partial Y(w) e^{i({\rm n}+{\rm m})Y(w)} \\
&= \begin{cases} \oint \frac{dw}{2\pi i} i{\rm n}\delta^{IJ} \partial Y(w) = i{\rm n}\delta^{IJ} (\partial Y)_0 \text{ for } {\rm n}+{\rm m} = 0\,, \\ \oint \frac{dw}{2\pi i} \delta^{IJ} \frac{\rm n}{\rm n+m} \partial (e^{i({\rm n}+{\rm m})Y(w)}) = 0 \text{ for } {\rm n}+{\rm m} \neq 0 \end{cases} \\
&= i{\rm n}\delta^{IJ} \delta_{{\rm n}+{\rm m},0} (\partial Y)_0\,.
\end{aligned} \tag{B.1.18}$$

The commutator for a Cartan generator and a ladder operator is

$$\begin{aligned}
[(J^I_{\rm n})_0, (J^\alpha_m)_0] &= \oint_{C'} \frac{dz}{2\pi i} \oint_C \frac{dw}{2\pi i} J^I_{\rm n}(z) J^\alpha_m(w) - \oint_{C'} \frac{dw}{2\pi i} \oint_C \frac{dz}{2\pi i} J^I_{\rm n}(z) J^\alpha_m(w) \\
&= \oint \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} \Big[ \frac{\pi^I_\alpha c_\alpha : e^{i\pi^J_\alpha X^J(w)} e^{i({\rm n}+{\rm m})Y(w)} :}{z-w} \Big] \\
&= \oint \frac{dw}{2\pi i} \pi^I_\alpha c_\alpha : e^{i\pi^J_\alpha X^J(w)} e^{i({\rm n}+{\rm m})Y(w)} : \\
&= \pi^I_\alpha (J^\alpha_{\rm n+m})_0\,,
\end{aligned} \tag{B.1.19}$$

and finally the commutator between two ladder operators is

$$\begin{aligned}
[(J^\alpha_n)_0, (J^\beta_m)_0] &= \oint_{C'} \frac{dz}{2\pi i} \oint_C \frac{dw}{2\pi i} J^\alpha_{\rm n}(z) J^\beta_{\rm m}(w) - \oint_{C'} \frac{dw}{2\pi i} \oint_C \frac{dz}{2\pi i} J^\alpha_{\rm n}(z) J^\beta_{\rm m}(w) \\
&= \oint \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} \begin{cases} \frac{\epsilon(\alpha,\beta) c_{\alpha+\beta} e^{i(\pi^I_{\alpha+\beta}) X^I(w)} e^{i({\rm n}+{\rm m})Y(w)}}{z-w} + \mathcal{O}(1) & \alpha+\beta \,\text{root}, \\ \frac{: e^{i({\rm n}+{\rm m})Y(w)} :}{(z-w)^2} + \frac{: i\pi^I_\alpha \partial X^I(w) e^{i({\rm n}+{\rm m})Y(w)} + i{\rm n}\partial Y(w) e^{i({\rm n}+{\rm m})Y(w)} :}{z-w} + \mathcal{O}(1) & \alpha = -\beta, \\ \mathcal{O}(1) & \text{otherwise.} \end{cases} \\
&= \begin{cases} \epsilon(\alpha,\beta) (J^{\alpha+\beta}_{\rm n+m})_0 & \alpha+\beta \;\text{ root} \\ \pi^I_\alpha (J^I_{\rm n+m})_0 + i{\rm n}\delta_{{\rm n}+{\rm m},0} (\partial Y)_0 & \alpha = -\beta \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{B.1.20}$$

## B.2 Affine algebras in $T^2$ decompactifications

From (B.0.1)-(B.0.3), the only non-trivial OPEs among internal fields are

$$X^{\hat{I}}(z)X^{\hat{J}}(w) \sim -\delta^{\hat{I}\hat{J}} log(z-w)\,, \tag{B.2.1}$$

the others being either finite or vanishing as $\frac{1}{R_8^2}$.

### $8d \to 9d$ decompactification limit

The vertex operators in the 0 ghost picture associated to the massless vectors (4.3.5) and (4.3.6) in the string spectrum are

$$\begin{aligned}
&\alpha^{\hat{I}}_{-1}\bar{\psi}^{\mu}_{-\frac{1}{2}}\,|0,\mathrm{n}_8\rangle_{NS} \to\; i\partial X^{\hat{I}}(z)\left(i\sqrt{2}\bar{\partial}\bar{X}^{\mu}(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^{\mu}(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i\mathrm{n}_8 Y^8(z,\bar{z})}\,,\\
&\alpha^{8}_{-1}\bar{\psi}^{\mu}_{-\frac{1}{2}}\,|0,\mathrm{n}_8\rangle_{NS} \to\; i\sqrt{2}\partial Y^{8}(z)\left(i\sqrt{2}\bar{\partial}\bar{X}^{\mu}(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^{\mu}(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i\mathrm{n}_8 Y^8(z,\bar{z})}\,,\\
&\bar{\psi}^{\mu}_{-\frac{1}{2}}\,|Z_{\mathfrak{g}},\mathrm{n}_8\rangle_{NS} \to\; c_\alpha e^{ip_{\alpha;\hat{I}}X^{\hat{I}}(z)}\left(i\sqrt{2}\bar{\partial}\bar{X}^{\mu}(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^{\mu}(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i(\mathrm{n}_8 - E_{89}\mathrm{w}^9 - \pi_\alpha\cdot A_8)Y^8(z,\bar{z})}\,,
\end{aligned} \tag{B.2.2}$$

where $\kappa_{\mathrm{m}} = (k_\mu, p_{R,i})$ is the 10 dimensional right momentum.
In the decompactification limit $G_{88} = R_8^2 \to \infty$, the affine algebra in the presence of a non vanishing $E_{89} = \frac{1}{2}A_8\cdot A_9$ component is realised by currents generalising the definitions (4.3.5) and (4.3.6) of $\{J^a_{\mathrm{n}}\}$ $(a = \hat{I}, \alpha)$, as we will now show. Let us first redefine the internal fields as

$$\mathcal{X}^I(z) = X^I(z) - A_8^I Y^8(z) \tag{B.2.3}$$

and $\mathcal{Y}^9(z)$ through the relation

$$G_{99}\mathcal{Y}^9(z) = G_{99}Y^9(z) + \frac{1}{2}E_{98}Y^8(z)\,. \tag{B.2.4}$$

In particular, since $G^{99}G_{99} = 1$, (B.2.4) is equivalent to

$$\mathcal{Y}^9(z) = Y^9(z) + \frac{1}{2}G^{99}E_{98}Y^8(z)\,. \tag{B.2.5}$$

The leading terms in the OPEs among these redefined internal fields are equivalent to (B.0.2)-(B.0.3)

$$\mathcal{Y}^9(z)\mathcal{Y}^9(w) = Y^9(z)Y^9(w) + \mathcal{O}\left(\frac{1}{R_8^2}\right), \tag{B.2.6}$$

$$\mathcal{Y}^9(z)\mathcal{X}^I(w) = \mathcal{O}\left(\frac{1}{R_8^2}\right), \tag{B.2.7}$$

$$\mathcal{Y}^9(z)Y^8(w) = \mathcal{O}\left(\frac{1}{R_8^2}\right), \tag{B.2.8}$$

$$\mathcal{X}^I(z)\mathcal{X}^J(w) = X^I(z)X^J(w) + \mathcal{O}\left(\frac{1}{R_8^2}\right). \tag{B.2.9}$$

Finally, let us also define $\mathcal{X}^{\hat{I}}(z) = \left(\sqrt{2}R_9\mathcal{Y}^9(z), \mathcal{X}^I\right)$. The expression for the affine currents in the general case is then

$$\begin{aligned} J^{\hat{I}}_{\mathrm{n}_8}(z) &\equiv i\partial\mathcal{X}^I(z)e^{i\mathrm{n}_8 Y^8(z)}, \\ J^{\alpha}_{\mathrm{n}_8}(z) &\equiv c_\alpha e^{ip_{\alpha;\hat{I}}\mathcal{X}^{\hat{I}}(z)}e^{i\mathrm{n}_8 Y^8(z)}. \end{aligned} \tag{B.2.10}$$

They are in form equal to (4.3.5) and (4.3.6) holding in the case of trivial $A_8^I$, and due to (B.2.6)-(B.2.9) the two sets of currents have the same OPEs, so that they generate the same algebra. The OPEs between currents associated to Cartan vectors read, up to $\mathcal{O}\left(\frac{1}{R_8^2}\right)$ terms

$$\begin{aligned} J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\hat{J}}_{\mathrm{m}_8}(w) &= -:\partial\mathcal{X}^{\hat{I}}(z)\,e^{i\mathrm{n}_8 Y^8(z)}::\partial\mathcal{X}^{\hat{J}}(w)\,e^{i\mathrm{m}_8 Y^8(w)}: \\ &= (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}:e^{i\mathrm{n}_8 Y^8(z)}e^{i\mathrm{m}_8 Y^8(w)}\left(-\partial\mathcal{X}^{\hat{I}}(z)\partial\mathcal{X}^{\hat{J}}(w) + \frac{\delta^{\hat{I}\hat{J}}}{(z-w)^2}\right):. \end{aligned}$$

Performing an expansion around $z = w$

$$\begin{aligned} &J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\hat{J}}_{\mathrm{m}_8}(w) = \\ &= (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}:\left(1 + i\mathrm{n}_8\partial Y^8(w)(z-w) + \ldots\right)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}\left(-\partial\mathcal{X}^{\hat{I}}(z)\partial\mathcal{X}^{\hat{J}}(w) + \frac{\delta^{\hat{I}\hat{J}}}{(z-w)^2}\right): \\ &\sim (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}\left(\delta^{\hat{I}\hat{J}}\frac{:e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{(z-w)^2} + i\delta^{\hat{I}\hat{J}}\mathrm{n}_8\frac{:\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{z-w} + \mathcal{O}(1)\right), \end{aligned} \tag{B.2.11}$$

which is indeed (4.3.7). As for the OPE between two affine root currents, up to $\mathcal{O}\left(\frac{1}{R_8^2}\right)$ terms

$$\begin{aligned}
J^{\alpha}_{\mathrm{n}_8}(z)J^{\beta}_{\mathrm{m}_8}(w) =& : c_\alpha e^{ip_{\alpha,\hat{I}}\mathcal{X}^{\hat{I}}(z)}e^{i\mathrm{n}_8 Y^8(z)} :: c_\beta e^{ip_{\beta;\hat{J}}\mathcal{X}^{\hat{J}}(w)}e^{i\mathrm{m}_8 Y^8(w)} : \\
=& c_\alpha c_\beta (z-w)^{p_\alpha\cdot p_\beta+\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}} : e^{ip_{\alpha;\hat{I}}\mathcal{X}^{\hat{I}}(z)}e^{i\mathrm{n}_8 Y^8(z)}e^{ip_{\beta;\hat{J}}\mathcal{X}^{\hat{J}}(w)}e^{i\mathrm{m}_8 Y^8(w)} : \\
=& c_\alpha c_\beta (z-w)^{p_\alpha\cdot p_\beta+\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}} : \big[1+ip_{\alpha;\hat{I}}\partial\mathcal{X}^{\hat{I}}(w)(z-w)+\ldots)\cdot \\
& \cdot(1+i\mathrm{n}_8\partial Y^8(w)(z-w)+\ldots\big]e^{i(p_{\alpha;\hat{I}}+p_{\beta;\hat{I}})\mathcal{X}^I(w)}e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} : .
\end{aligned} \tag{B.2.12}$$

There are only two non trivial cases for the choice of $p_\alpha$ and $p_\beta$. Assume that $p_\alpha+p_\beta=p_{\alpha+\beta}$ is still a root of the algebra $\mathfrak{g}$ in 9 dimensions. Then, if the roots are normalised as $|p_\alpha|^2=2$, it holds $p_\alpha\cdot p_\beta=-1$ and (B.2.12) reads

$$\begin{aligned}
J^{\alpha}_{\mathrm{n}_8}(z)J^{\beta}_{\mathrm{m}_8}(w) \sim& (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}\left(\frac{\epsilon(\alpha,\beta)c_{\alpha+\beta} : e^{ip_{\alpha+\beta;\hat{I}}\mathcal{X}^{\hat{I}}(w)}e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} :}{z-w}+\mathcal{O}(1)\right) \\
=& (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}\left(\frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8}(w)}{z-w}+\mathcal{O}(1)\right).
\end{aligned} \tag{B.2.13}$$

If instead $p_\alpha=-p_\beta$, then $p_\alpha\cdot p_\beta=-|p_\alpha|^2=-2$, and so (B.2.12) gives

$$\begin{aligned}
J^{\alpha}_{\mathrm{n}_8}(z)J^{-\alpha}_{\mathrm{m}_8}(w) \sim& (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}} : e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}\left(\frac{1}{(z-w)^2}+\frac{i(p_{\alpha;\hat{I}}\partial\mathcal{X}^{\hat{I}}(w)+\mathrm{n}_8\partial Y^8(w))}{z-w}+\mathcal{O}(1)\right) :, \\
\sim& (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}\left(\frac{: e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} :}{(z-w)^2}+\frac{p_{\alpha;\hat{I}}J^{\hat{I}}_{\mathrm{n}_8+\mathrm{m}_8}(w)+i\mathrm{n}_8 : \partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} :}{z-w}+\mathcal{O}(1)\right).
\end{aligned} \tag{B.2.14}$$

All the OPEs for other choices of $p_\alpha$ and $p_\beta$ vanish since in these cases $c_\alpha c_\beta=0$

$$J^{\alpha}_{\mathrm{n}_8}(z)J^{\beta}_{\mathrm{m}_8}(w) \sim (z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}\cdot\begin{cases}\frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8}(w)}{z-w}+\mathcal{O}(1) & \alpha+\beta\,\text{root},\\ \frac{:e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{(z-w)^2}+\frac{p_{\alpha;\hat{I}}J^{\hat{I}}_{\mathrm{n}_8+\mathrm{m}_8}(w)+i\mathrm{n}_8:\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{z-w}+\mathcal{O}(1) & \alpha=-\beta\,,\\ 0 & \text{otherwise},\end{cases} \tag{B.2.15}$$

which taking the limit $R_8 \to \infty$ reduces indeed to (4.3.9).
Finally, considering one Cartan and one root current

$$\begin{aligned} J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\alpha}_{\mathrm{m}_8}(w) =& : i\partial\mathcal{X}^{\hat{I}}(z)e^{i\mathrm{n}_8 Y^8(z)} :: c_\alpha e^{ip_{\alpha;\hat{\jmath}}\mathcal{X}^{\hat{\jmath}}(w)}e^{i\mathrm{m}_8 Y^8(w)} : \\ =& ic_\alpha(z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}\frac{-ip^{\hat{I}}_\alpha : e^{ip_{\alpha;\hat{\jmath}}\mathcal{X}^{\hat{\jmath}}(w)}e^{i\mathrm{n}_8 Y^8(z)}e^{i\mathrm{m}_8 Y^8(w)} :}{z-w} + \mathcal{O}(1) \\ \sim& c_\alpha(z-w)^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}}\frac{p^{\hat{I}}_\alpha : e^{ip_{\alpha;\hat{\jmath}}\mathcal{X}^{\hat{\jmath}}(w)}e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} :}{z-w} + \mathcal{O}(1) \sim \frac{p^I_\alpha J^{\alpha}_{\mathrm{n}_8+\mathrm{m}_8}(w)}{z-w} + \mathcal{O}(1) \end{aligned} \tag{B.2.16}$$

as $R_8 \to \infty$, which is (4.3.8). The symmetry algebra can be defined as the set of commutation relations between the currents zero modes as follows. Let us start considering two Cartan generators, in the limit $G_{88} = R_8^2 \to \infty$

$$[(J^{\hat{I}}_{\mathrm{n}_8})_0, (J^{\hat{J}}_{\mathrm{m}_8})_0] = \oint_{C'}\frac{dz}{2\pi i}\oint_C \frac{dw}{2\pi i}J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\hat{J}}_{\mathrm{m}_8}(w) - \oint_{C'}\frac{dw}{2\pi i}\oint_C\frac{dz}{2\pi i}J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\hat{J}}_{\mathrm{m}_8}(w) \tag{B.2.17}$$

where $C'$ is a contour external to $C$.

$$\begin{aligned} [(J^{\hat{I}}_{\mathrm{n}_8})_0, (J^{\hat{J}}_{\mathrm{m}_8})_0] =& \oint\frac{dw}{2\pi i}\delta^{\hat{I}\hat{J}}\mathrm{Res}_{z\to w}\left[\frac{: e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} :}{(z-w)^2} + i\mathrm{n}_8\frac{:\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} :}{z-w}\right] \\ =& \oint\frac{dw}{2\pi i}i\mathrm{n}_8\delta^{\hat{I}\hat{J}}\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} \\ =& \begin{cases} \oint\frac{dw}{2\pi i}i\mathrm{n}_8\delta^{\hat{I}\hat{J}}\partial y^8(w) = i\mathrm{n}_8\delta^{\hat{I}\hat{J}}(\partial Y^8)_0 & \text{for } \mathrm{n}_8+\mathrm{m}_8 = 0\,, \\ \oint\frac{dw}{2\pi i}\delta^{\hat{I}\hat{J}}\frac{\mathrm{n}_8}{\mathrm{n}_8+\mathrm{m}_8}\partial(e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}) = 0 & \text{for } \mathrm{n}_8+\mathrm{m}_8 \neq 0 \end{cases} \\ =& i\mathrm{n}_8\delta^{\hat{I}\hat{J}}\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8)_0\,. \end{aligned} \tag{B.2.18}$$

The commutator between a Cartan and a root generator is

$$\begin{aligned} [(J^{\hat{I}}_{\mathrm{n}_8})_0, (J^{\alpha}_{\mathrm{m}_8})_0] =& \oint_{C'}\frac{dz}{2\pi i}\oint_C\frac{dw}{2\pi i}J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\alpha}_{\mathrm{m}_8}(w) - \oint_{C'}\frac{dw}{2\pi i}\oint_C\frac{dz}{2\pi i}J^{\hat{I}}_{\mathrm{n}_8}(z)J^{\alpha}_{\mathrm{m}_8}(w) \\ =& \oint\frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\left[\frac{p^{\hat{I}}_\alpha c_\alpha : e^{ip_{\alpha;\hat{\jmath}}\mathcal{X}^{\hat{\jmath}}(w)}e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} :}{z-w}\right] \\ =& \oint\frac{dw}{2\pi i}p^{\hat{I}}_\alpha c_\alpha : e^{ip_{\alpha;\hat{\jmath}}\mathcal{X}^{\hat{\jmath}}(w)}e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)} : \\ =& p^{\hat{I}}_\alpha(J^{\alpha}_{\mathrm{n}+\mathrm{m}})_0\,, \end{aligned} \tag{B.2.19}$$

and the commutator between two root generators is

$$
\begin{aligned}
[(J^{\alpha}_{\mathrm{n}_8})_0,(J^{\beta}_{\mathrm{m}_8})_0] &= \oint_{C'}\frac{dz}{2\pi i}\oint_{C}\frac{dw}{2\pi i}J^{\alpha}_{\mathrm{n}_8}(z)J^{\beta}_{\mathrm{m}_8}(w)-\oint_{C'}\frac{dw}{2\pi i}\oint_{C}\frac{dz}{2\pi i}J^{\alpha}_{\mathrm{n}_8}(z)J^{\beta}_{\mathrm{m}_8}(w)\\
&= \oint\frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\begin{cases}\frac{\epsilon(\alpha,\beta)c_{\alpha+\beta}e^{ip_{\alpha+\beta;\hat{I}}\mathcal{X}^{\hat{I}}(w)}e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}}{z-w}+\mathcal{O}(1) & \alpha+\beta\,\text{root},\\ \frac{:e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{(z-w)^2}+\frac{:ip_{\alpha;\hat{I}}\partial\mathcal{X}^{\hat{I}}(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}+i\mathrm{n}_8\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}:}{z-w}+\mathcal{O}(1) & \alpha=-\beta,\\ 0 & \text{otherwise.}\end{cases}\\
&= \begin{cases}\epsilon(\alpha,\beta)(J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8})_0 & \alpha+\beta\ \text{root}\\ p_{\alpha;\hat{I}}(J^{\hat{I}}_{\mathrm{n}_8+\mathrm{m}_8})_0+i\mathrm{n}_8\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8)_0 & \alpha=-\beta\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
\tag{B.2.20}
$$

### Affinisation of the right moving algebra

Let us now focus on the $\mathfrak{u}(1)^2_R$ right moving contribution to the algebra and its affinisation pattern. Restricting for simplicity to the case $A^I_8=0$, the associated massless states as $R_8\to\infty$ and their corresponding vertex operators in the 0 ghost picture are

$$
\alpha^{\mu}_{-1}\bar{\psi}^i_{-\frac{1}{2}}\left|0,\mathrm{n}_8\right\rangle\to i\sqrt{2}\partial X^{\mu}\left(i\sqrt{2}\bar{\partial}\bar{Y}^i(\bar{z})+\frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^i(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i\mathrm{n}_8Y^8(z,\bar{z})}\,,\tag{B.2.21}
$$

with $i=8,9$, and they also come in massless momentum towers as the eighth direction is decompactified. The associated antiholomorphic currents are

$$
\bar{\Upsilon}^i_{\mathrm{n}}(\bar{z})=\bar{\Upsilon}^i(\bar{z})e^{i\mathrm{n}_8Y^8(\bar{z})}\,,\tag{B.2.22}
$$

where

$$
\bar{\Upsilon}^i(\bar{z})=\left(i\sqrt{2}\bar{\partial}\bar{Y}^i(\bar{z})+\frac{1}{\sqrt{2}}k_{\mu}\bar{\psi}^{\mu}(\bar{z})\bar{\psi}^i(\bar{z})\right)\tag{B.2.23}
$$

are the currents of the finite $U(1)^2_R$. Indeed, using (B.0.2) and

$$
\bar{\psi}^M(\bar{z})\bar{\psi}^N(\bar{w})\sim\frac{\eta^{MN}}{\bar{z}-\bar{w}}\,,\tag{B.2.24}
$$

and the fact that in the decompactification limit the $T^2$ metric is diagonal, $G^{ij} = \frac{1}{R_i^2}\delta^{ij}$, one finds, up to $\mathcal{O}(1)$ factors

$$\begin{aligned}
\bar{\Upsilon}^i(\bar{z})\bar{\Upsilon}^j(\bar{w}) = \\
&= -2\bar{\partial}\bar{Y}^{\overbrace{i}}(\bar{z})\bar{\partial}\bar{Y}^{\overbrace{j}}(\bar{w}) - 2:\bar{\partial}\bar{Y}^i(\bar{z})\bar{\partial}\bar{Y}^j(\bar{w}): - \frac{1}{2}k_\mu k_\nu(\overbrace{\bar{\psi}^\mu(\bar{z})\bar{\psi}^\nu}(\bar{w}):\bar{\psi}^i(\bar{z})\bar{\psi}^j(\bar{w}): + \\
&+ :\bar{\psi}^\mu(\bar{z})\bar{\psi}^\nu(\bar{w}):\overbrace{\bar{\psi}^i(\bar{z})\bar{\psi}^j}(\bar{w}) + \overbrace{\bar{\psi}^\mu(\bar{z})\bar{\psi}^\nu}(\bar{w})\overbrace{\bar{\psi}^i(\bar{z})\bar{\psi}^j}(\bar{w}) + :\bar{\psi}^\mu(\bar{z})\bar{\psi}^i(\bar{z})\bar{\psi}^\nu(\bar{w})\bar{\psi}^j(\bar{w}):) \\
&= \frac{\delta^{ij}}{R_i^2}\frac{1}{(\bar{z}-\bar{w})^2} - \frac{1}{2}k_\mu k_\nu\left(\frac{\eta^{\mu\nu}:\bar{\psi}^i(\bar{z})\bar{\psi}^j(\bar{w}):}{\bar{z}-\bar{w}} + :\frac{\bar{\psi}^\mu(\bar{z})\bar{\psi}^\nu(\bar{w}):\delta^{ij}}{R_i^2(\bar{z}-\bar{w})} + \frac{\eta^{\mu\nu}\delta^{ij}}{R_i^2(\bar{z}-\bar{w})^3}\right) \\
&= \frac{\delta^{ij}}{R_i^2}\frac{1}{(\bar{z}-\bar{w})^2}\,.
\end{aligned} \tag{B.2.25}$$

The term with $k_\mu k_\nu \eta^{\mu\nu}$ vanish because $k^2 = 0$, while the term in $k_\mu k_\nu \bar{\psi}^\mu(\bar{z})\bar{\psi}^\nu(\bar{w}) = 0$ due to symmetry arguments. The absence of the single pole means that the zero modes $(\bar{\Upsilon}^i(\bar{z}))_0$ commute, so that they are indeed associated to a $U(1)_R^2$ symmetry. The full asymptotic currents (B.2.22) satisfy the following OPEs

$$\begin{aligned}
\bar{\Upsilon}^i_{\mathrm{n}_8}(\bar{z})\bar{\Upsilon}^j_{\mathrm{m}_8}(\bar{w}) = (\bar{z}-\bar{w})^{\frac{\mathrm{n}_8\mathrm{m}_8}{2R_8^2}} :e^{i\mathrm{n}_8\bar{Y}^8(\bar{z})}e^{i\mathrm{m}_8\bar{Y}^8(\bar{w})}\cdot \\
\cdot\left[\frac{\delta^{ij}}{R_i^2(\bar{z}-\bar{w})^2} - 2i\frac{\mathrm{n}_8\delta^{8j}\bar{\partial}\bar{Y}^i(\bar{z}) - \mathrm{m}_8\delta^{8i}\bar{\partial}\bar{Y}^j(\bar{w})}{R_8^2(\bar{z}-\bar{w})} - 2\frac{\mathrm{n}_8\mathrm{m}_8\delta^{8i}\delta^{8j}}{R_8^4(\bar{z}-\bar{w})^2} + \mathcal{O}(1)\right]:,
\end{aligned} \tag{B.2.26}$$

and as $R_8 \to \infty$ the only non trivial case is

$$\bar{\Upsilon}^9_{\mathrm{n}_8}(\bar{z})\bar{\Upsilon}^9_{\mathrm{m}_8}(\bar{w}) = \frac{e^{i(\mathrm{n}_8+\mathrm{m}_8)\bar{Y}^8(\bar{z})}}{R_8^2(\bar{z}-\bar{w})^2} + \frac{i\mathrm{n}_8\bar{\partial}\bar{Y}^8(\bar{w})e^{i(\mathrm{n}_8+\mathrm{m}_8)\bar{Y}^8(\bar{z})}}{\bar{z}-\bar{w}} + \mathcal{O}(1)\,, \tag{B.2.27}$$

all the other combinations giving only finite terms asymptotically in moduli space. In particular, eq. (B.2.27) means that the zero modes of the rescaled current $\bar{\Upsilon}^9_{\mathrm{n}_8}(\bar{z}) \to e^9{}_9\bar{\Upsilon}^9_{\mathrm{n}_8}(\bar{z})$ obey the OPEs of an $\hat{\mathfrak{u}}(1)$ algebra, where the central extension is given by $(\bar{\partial}\bar{Y}^8)_0$.

### $8d \to 10d$ decompactification limit

The vertex operators in the 0 picture associated to the massless vectors (4.3.18)-(4.3.20) for generic finite B field and Wilson lines components are

$$\begin{aligned}
\alpha^i_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}|0,\mathrm{n}_8,\mathrm{n}_9\rangle_{NS} &\to i\sqrt{2}\partial Y^i(z)\left(i\sqrt{2}\bar{\partial}\bar{X}^\mu(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^\mu(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i\mathrm{n}_jY^j(z,\bar{z})}\,, \\
\alpha^I_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}|0,\mathrm{n}_8,\mathrm{n}_9\rangle_{NS} &\to i\partial X^I(z)\left(i\sqrt{2}\bar{\partial}\bar{X}^\mu(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^\mu(\bar{z})\right)e^{ikX(z,\bar{z})}e^{i\mathrm{n}_jY^j(z,\bar{z})}\,, \\
\bar{\psi}^\mu_{-\frac{1}{2}}|Z_{\mathrm{E}_8\oplus\mathrm{E}_8},\mathrm{n}_8,\mathrm{n}_9\rangle_{NS} &\to c_\alpha e^{i\pi_{\alpha;I}X^I(z)}\left(i\sqrt{2}\bar{\partial}\bar{X}^\mu(\bar{z}) + \frac{1}{\sqrt{2}}\kappa\cdot\bar{\psi}(\bar{z})\bar{\psi}^\mu(\bar{z})\right)e^{i(\mathrm{n}_i-\pi_{\alpha;I}A_i^I)Y^i(z,\bar{z})}\,,
\end{aligned} \tag{B.2.28}$$

with the same conventions of B.2. Let us generalise the definition (B.2.3) to

$$\mathcal{X}^I(z) = X^I(z) - A^I_i Y^i(z)\,. \tag{B.2.29}$$

In the full decompactification limit, at first order the OPE between two such operators is

$$\mathcal{X}^I(z)\mathcal{X}^J(w) = X^I(z)X^J(w) \tag{B.2.30}$$

The relevant holomorphic currents giving the double loop algebra are

$$\begin{aligned} J^I_{\mathrm{n}_8,\mathrm{n}_9}(z) &\equiv i\partial\mathcal{X}^I(z)e^{i\mathrm{n}_i Y^i(z)}\,,\\ J^\alpha_{\mathrm{n}_8,\mathrm{n}_9}(z) &\equiv c_\alpha e^{i\pi_{\alpha;I}\mathcal{X}^I(z)}e^{i\mathrm{n}_i Y^i(z)}\,, \end{aligned} \tag{B.2.31}$$

related respectively to the Cartan and root sector. They are a generalisation of (4.3.19) and (4.3.20), the latters holding in the case of vanishing Wilson lines along the two torus directions. The non trivial OPEs among these currents are

$$\begin{aligned} J^I_{\mathrm{n}_8,\mathrm{n}_9}(z)J^J_{\mathrm{m}_8,\mathrm{m}_9}(w) &= -:\partial\mathcal{X}^I(z)e^{i\mathrm{n}_8Y^8(z)}e^{i\mathrm{n}_9Y^9(z)}::\partial\mathcal{X}^J(w)e^{i\mathrm{m}_8Y^8(w)}e^{i\mathrm{m}_9Y^9(w)}:\\ &= (z-w)^{\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}}:e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)}(1+i\mathrm{n}_8\partial Y^8(w)(z-w)+\ldots)\cdot\\ &\quad\cdot(1+i\mathrm{n}_9\partial Y^9(w)(z-w)+\ldots)\left(-\partial\mathcal{X}^I(z)\partial\mathcal{X}^J(w)+\frac{\delta^{IJ}}{(z-w)^2}\right):\\ &= (z-w)^{\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}}\left(\frac{\delta^{IJ}:e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}:}{(z-w)^2}+i\mathrm{n}_i\frac{:\partial Y^i(w)e^{i(\mathrm{n}_j+\mathrm{m}_j)Y^j(w)}:}{z-w}+\mathcal{O}(1)\right) \end{aligned} \tag{B.2.32}$$

which helds the following commutator of the zero modes

$$\begin{aligned} [(J^I_{\mathrm{n}_8,\mathrm{n}_9})_0,(J^J_{\mathrm{m}_8,\mathrm{m}_9})_0] &= \oint\frac{dw}{2\pi i}\delta^{IJ}\mathrm{Res}_{z\to w}\left[\frac{:e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}:}{(z-w)^2}+i\mathrm{n}_i\frac{:\partial Y^i(w)e^{i(\mathrm{n}_j+\mathrm{m}_j)Y^j(w)}:}{z-w}\right]\\ &= \oint\frac{dw}{2\pi i}i\delta^{IJ}(\mathrm{n}_8\partial Y^8(w)+\mathrm{n}_9\partial Y^9(w))e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)}\,. \end{aligned} \tag{B.2.33}$$

This integral can be solved as follows. For any choice of $\mathrm{n}_i+\mathrm{m}_i$, $i=8,9$ it must hold

$$\begin{aligned} 0 &= \frac{1}{i}\oint\frac{dw}{2\pi i}\partial\left(e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)}\right)\\ &= \oint\frac{dw}{2\pi i}(\mathrm{n}_8+\mathrm{m}_8)\partial Y^8(w)e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}+\oint\frac{dw}{2\pi i}(\mathrm{n}_9+\mathrm{m}_9)\partial Y^9(w)e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}\\ &= \oint\frac{dw}{2\pi i}(\mathrm{n}_8\partial Y^8(w)+\mathrm{n}_9\partial Y^9(w))e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}+\oint\frac{dw}{2\pi i}(\mathrm{m}_8\partial Y^8(w)+\mathrm{m}_9\partial Y^9(w))e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}\,, \end{aligned} \tag{B.2.34}$$

where in the last line one can recognize two integrals of the type (B.2.33). Focusing on the second line of (B.2.34) and taking into account the fact that $Y^8(z)$ and $Y^9(z)$ are independent, this means that the $\partial Y^8$ and $\partial Y^9$ contributions in the sum must vanish separately. Let us consider for concreteness

$$\oint \frac{dw}{2\pi i}(\mathrm{n}_8+\mathrm{m}_8)\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)} = 0\,. \tag{B.2.35}$$

- If $\mathrm{n}_8+\mathrm{m}_8=0$, (B.2.35) vanishes trivially, so that in this case one can have

$$\oint \frac{dw}{2\pi i}\partial Y^8(w)e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)} \neq 0\,, \tag{B.2.36}$$

  which is $(\partial Y^8_{0,\mathrm{n}_9+\mathrm{m}_9})_0$.

- If $\mathrm{n}_8+\mathrm{m}_8\neq 0$, then it must hold

$$\oint \frac{dw}{2\pi i}\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)} = 0\,. \tag{B.2.37}$$

  This statement is trivial for $\mathrm{n}_9+\mathrm{m}_9=0$, when the integrand in (B.2.37) reduces to a total derivative, but it must actually hold $\forall(\mathrm{n}_9+\mathrm{m}_9)$.

Summarising

$$\oint \frac{dw}{2\pi i}\partial Y^8(w)e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)} = \delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8_{0,\mathrm{n}_9+\mathrm{m}_9})_0\,, \tag{B.2.38}$$

and the same holds with $8\leftrightarrow 9$. Using these results in (B.2.33)

$$[(J^I_{\mathrm{n}_8,\mathrm{n}_9})_0,(J^J_{\mathrm{m}_8,\mathrm{m}_9})_0] = i\delta^{IJ}(\mathrm{n}_8\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8_{0,\mathrm{n}_9+\mathrm{m}_9})_0 + \mathrm{n}_9\delta_{\mathrm{n}_9+\mathrm{m}_9,0}(\partial Y^9_{\mathrm{n}_8+\mathrm{m}_8,0})_0)\,. \tag{B.2.39}$$

In the case of a Cartan and a root current

$$\begin{aligned} J^I_{\mathrm{n}_8,\mathrm{n}_9}(z)J^\alpha_{\mathrm{m}_8,\mathrm{m}_9}(w) &= ic_\alpha :\partial\mathcal{X}^I(z)e^{i\mathrm{n}_8Y^8(z)}e^{i\mathrm{n}_9Y^9(z)}:: e^{i\pi_{\alpha;I}\mathcal{X}^I(w)}e^{i\mathrm{m}_8Y^8(w)}e^{i\mathrm{m}_9Y^9(w)}: \\ &= (z-w)^{\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}}\left(c_\alpha\pi^I_\alpha\frac{:e^{i\pi_{\alpha;I}\mathcal{X}^I(w)}e^{i(\mathrm{n}_8+\mathrm{m}_8)Y^8(w)}e^{i(\mathrm{n}_9+\mathrm{m}_9)Y^9(w)}:}{z-w}+\mathcal{O}(1)\right) \\ &= (z-w)^{\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}}\left(\frac{\pi^I_\alpha J^\alpha_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)}{z-w}+\mathcal{O}(1)\right), \end{aligned} \tag{B.2.40}$$

which gives the following commutation relations

$$\begin{aligned} [(J^I_{\mathrm{n}_8,\mathrm{n}_9})_0,(J^\alpha_{\mathrm{m}_8,\mathrm{m}_9})_0] &= \oint\frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\left[\frac{\pi^I_\alpha J^\alpha_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)}{z-w}\right] \\ &= \oint\frac{dw}{2\pi i}\pi^I_\alpha J^\alpha_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w) = \pi^I_\alpha(J^\alpha_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9})_0\,. \end{aligned} \tag{B.2.41}$$

Finally, considering two root currents

$$\begin{aligned}
J^{\alpha}_{\mathrm{n}_8,\mathrm{n}_9}(z)J^{\beta}_{\mathrm{m}_8,\mathrm{m}_9}(w) &= c_\alpha c_\beta : e^{i\pi_{\alpha;I}\mathcal{X}^I(z)}e^{i\mathrm{n}_8 Y^8(z)}e^{i\mathrm{n}_9 Y^9(z)} :: e^{i\pi_{\beta;J}\mathcal{X}^J(w)}e^{i\mathrm{m}_8 Y^8(w)}e^{i\mathrm{m}_9 Y^9(w)} : \\
&= c_\alpha c_\beta (z-w)^{\pi_\alpha\cdot\pi_\beta+\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}} : e^{i(\pi_{\alpha;I}\mathcal{X}^I(z)+\pi_{\beta;J}\mathcal{X}^J(w))}e^{i(\mathrm{n}_i Y^i(z)+\mathrm{m}_i Y^i(w))} : \\
&= c_\alpha c_\beta (z-w)^{\pi_\alpha\cdot\pi_\beta+\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}} : (1+i\pi_{\alpha;J}\partial\mathcal{X}^J(w)(z-w)+\ldots)e^{i(\pi_{\alpha;I}+\pi_{\beta;I})\mathcal{X}^I(w)} \\
&\quad (1+i\mathrm{n}_j\partial Y^j(w)(z-w)+\ldots)e^{i(n_k+m_k)Y^k(w)} :,
\end{aligned} \tag{B.2.42}$$

which again splits in three different cases. If $\pi_\alpha+\pi_\beta=\pi_{\alpha+\beta}$, then

$$\begin{aligned}
J^{\alpha}_{\mathrm{n}_8,\mathrm{n}_9}(z)J^{\beta}_{\mathrm{m}_8,\mathrm{m}_9}(w) &= (z-w)^{\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}}\left(\frac{\epsilon(\alpha,\beta)c_{\alpha+\beta} : e^{i(\pi_{\alpha;I}+\pi_{\beta;I})\mathcal{X}^I(w)}e^{i(n_k+m_k)Y^k(w)} :}{z-w}+\mathcal{O}(1)\right) \\
&= (z-w)^{\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}}\left(\frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)}{z-w}+\mathcal{O}(1)\right).
\end{aligned} \tag{B.2.43}$$

If $\pi_\alpha=-\pi_\beta$

$$J^{\alpha}_{\mathrm{n}_8,\mathrm{n}_9}(z)J^{\beta}_{\mathrm{m}_8,\mathrm{m}_9}(w) = (z-w)^{\frac{G^{ij}\mathrm{n}_i\mathrm{m}_j}{2}} : e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}\left(\frac{1}{(z-w)^2}+\frac{i\pi_{\alpha;I}\partial\mathcal{X}^I+i\mathrm{n}_j\partial Y^j(w)}{z-w}+\mathcal{O}(1)\right) : . \tag{B.2.44}$$

For all the remaining $\pi_\alpha$, $\pi_\beta$ instead, $c_\alpha c_\beta=0$ and so to summarise

$$J^{\alpha}_{\mathrm{n}_8,\mathrm{n}_9}(z)J^{\beta}_{\mathrm{m}_8,\mathrm{m}_9}(w) = \begin{cases} \frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)}{z-w}+\mathcal{O}(1) & \alpha+\beta\,\text{root}, \\ \frac{:e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}:}{(z-w)^2}+\frac{\pi_{\alpha;I}J^I_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)+i\mathrm{n}_i:\partial Y^i(w)e^{i(\mathrm{n}_j+\mathrm{m}_j)Y^j(w)}:}{z-w}+\mathcal{O}(1) & \alpha=-\beta\,, \\ 0 & \text{otherwise}, \end{cases} \tag{B.2.45}$$

The algebra of the zero modes in this case is

$$
\begin{aligned}
[(J^{\alpha}_{\mathrm{n}_8,\mathrm{n}_9})_0,(J^{\beta}_{\mathrm{m}_8,\mathrm{m}_9})_0] &= \oint_{C'}\frac{dz}{2\pi i}\oint_{C}\frac{dw}{2\pi i}J^{\alpha}_{\mathrm{n}_8,\mathrm{n}_9}(z)J^{\beta}_{\mathrm{m}_8,\mathrm{m}_9}(w)-\oint_{C'}\frac{dw}{2\pi i}\oint_{C}\frac{dz}{2\pi i}J^{\alpha}_{\mathrm{n}_8,\mathrm{n}_9}(z)J^{\beta}_{\mathrm{m}_8,\mathrm{m}_9}(w)\\
&=\oint\frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\begin{cases}\frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}(w)}{z-w}+\mathcal{O}(1) & \alpha+\beta\,\text{root},\\ \frac{:e^{i(\mathrm{n}_i+\mathrm{m}_i)Y^i(w)}:}{(z-w)^2}+\frac{:\pi_{\alpha;I}J^I_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9}+i\mathrm{n}_i\partial Y^i(w)e^{i(\mathrm{n}_j+\mathrm{m}_j)Y^j(w)}:}{z-w}+\mathcal{O}(1) & \alpha=-\beta,\\ 0 & \text{otherwise.}\end{cases}\\
&=\begin{cases}\epsilon(\alpha,\beta)(J^{\alpha+\beta}_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9})_0 & \alpha+\beta\,\text{root},\\ \pi_{\alpha;I}(J^I_{\mathrm{n}_8+\mathrm{m}_8,\mathrm{n}_9+\mathrm{m}_9})_0+ & \\ \quad+i(\mathrm{n}_8\delta_{\mathrm{n}_8+\mathrm{m}_8,0}(\partial Y^8_{0,\mathrm{n}_9+\mathrm{m}_9})_0+\mathrm{n}_9\delta_{\mathrm{n}_9+\mathrm{m}_9,0}(\partial Y^9_{\mathrm{n}_8+\mathrm{m}_8,0})_0) & \alpha=-\beta\,,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}
\tag{B.2.46}
$$

## B.3 Affine algebras in $T^d$ decompactification

Let us explicitly derive the commutation relations for the algebra in a generic decompactification limit of the heterotic theory on $T^d$. Since from the $S^1$ and $T^2$ examples we clearly showed how the off-diagonal blocks of metric and B field and the Wilson lines along the directions we decompactify do not play any role in the limit, let us for simplicity set them to 0, the difference otherwise being just a field redefinition which does not affect the OPEs. The relevant OPEs in this case are

$$X^{\mu}(z,\bar z)X^{\nu}(w,\bar w)\sim-\frac{1}{2}\eta^{\mu\nu}\log|z-w|^2\,, \tag{B.3.1}$$

$$Y^{i}(z,\bar z)Y^{j}(w,\bar w)\sim-\frac{1}{2}G^{ij}\log|z-w|^2\,, \tag{B.3.2}$$

$$X^{I}(z)X^{J}(w)\sim-\delta^{IJ}\log(z-w)\,, \tag{B.3.3}$$

where (B.3.2) for $i=\hat\imath$ and (B.3.3) can be written in terms of the fields $X^{\hat I}(z)=(\sqrt{2}e^{\hat a}_{\hat\imath}Y^{\hat\imath}(z),X^I(z))$ defined in Section 4.3.2 as

$$X^{\hat I}(z)X^{\hat J}(w)\sim-\delta^{\hat I\hat J}\log(z-w)\,. \tag{B.3.4}$$

The vertex operators associated to the massless states (4.3.36) are

$$
\begin{aligned}
\alpha^{\hat I}_{-1}\bar\psi^{\mu}_{-\frac12}|0,\mathrm{n}_{\bar\jmath}\rangle_{NS}&\to i\partial X^{\hat I}(z)\left(i\sqrt{2}\bar\partial\bar X^{\mu}(\bar z)+\frac{1}{\sqrt2}\kappa\cdot\bar\psi(\bar z)\bar\psi^{\mu}(\bar z)\right)e^{ikX(z,\bar z)}e^{i\mathrm{n}_{\bar\jmath}Y^{\bar\jmath}(z,\bar z)}\,,\\
\alpha^{\bar\imath}_{-1}\bar\psi^{\mu}_{-\frac12}|0,\mathrm{n}_{\bar\jmath}\rangle_{NS}&\to i\sqrt{2}\partial Y^{\bar\imath}(z)\left(i\sqrt{2}\bar\partial\bar X^{\mu}(\bar z)+\frac{1}{\sqrt2}\kappa\cdot\bar\psi(\bar z)\bar\psi^{\mu}(\bar z)\right)e^{ikX(z,\bar z)}e^{i\mathrm{n}_{\bar\jmath}Y^{\bar\jmath}(z,\bar z)}\,,\\
\bar\psi^{\mu}_{-\frac12}|Z_{\mathfrak g},\mathrm{n}_{\bar\jmath}\rangle_{NS}&\to c_{\alpha}e^{ip_{\alpha;\hat I}X^{\hat I}(z)}\left(i\sqrt{2}\bar\partial\bar X^{\mu}(\bar z)+\frac{1}{\sqrt2}\kappa\cdot\bar\psi(\bar z)\bar\psi^{\mu}(\bar z)\right)e^{ikX(z,\bar z)}e^{i\mathrm{n}_{\bar\jmath}Y^{\bar\jmath}(z,\bar z)}\,.
\end{aligned}
\tag{B.3.5}
$$

The conserved currents related to the affine algebra are

$$J^{\hat{I}}_{\{\mathrm{n}_{\bar{\imath}}\}}(z) \equiv i\partial X^{\hat{I}}(z) e^{i\mathrm{n}_{\bar{\imath}} Y^{\bar{\imath}}(z)}\,, \tag{B.3.6}$$

$$J^{\alpha}_{\{\mathrm{n}_{\bar{\imath}}\}}(z) \equiv c_\alpha e^{ip_{\alpha;\bar{I}} X^{\bar{I}}(z)} e^{i\mathrm{n}_{\bar{\imath}} Y^{\bar{\imath}}(z)}\,. \tag{B.3.7}$$

$$\tag{B.3.8}$$

In the following we present their mutual OPEs, up to $\mathcal{O}\left(\frac{1}{R_{\bar{\imath}}^2}\right)$ terms and dropping the prefactor $(z-w)^{\frac{G^{\bar{\imath}\bar{\jmath}}\mathrm{n}_{\bar{\imath}}\mathrm{m}_{\bar{\jmath}}}{2}} \to 1$ asymptotically, which is common to all of them.

$$\begin{aligned}
J^{\hat{I}}_{\{\mathrm{n}_{\bar{\imath}}\}}(z) J^{\hat{J}}_{\{\mathrm{m}_{\bar{\imath}}\}}(w) &= -:\partial X^{\hat{I}}(z)\, e^{i\mathrm{n}_{\bar{\imath}} Y^{\bar{\imath}}(z)} :: \partial X^{\hat{J}}(w)\, e^{i\mathrm{m}_{\bar{\jmath}} Y^{\bar{\jmath}}(w)} : \\
&=: e^{i\mathrm{n}_{\bar{\imath}} Y^{\bar{\imath}}(z)} e^{i\mathrm{m}_{\bar{\jmath}} Y^{\bar{\jmath}}(w)} \left( -\partial X^{\hat{I}}(z)\partial X^{\hat{J}}(w) + \frac{\delta^{\hat{I}\hat{J}}}{(z-w)^2} \right) : \\
&=: \left(1 + i\mathrm{n}_{\bar{\imath}}\partial Y^{\bar{\imath}}(w)(z-w) + \ldots\right) e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} \left( \frac{\delta^{\hat{I}\hat{J}}}{(z-w)^2} + \mathcal{O}(1) \right) : \\
&\sim \delta^{\hat{I}\hat{J}} \frac{: e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} :}{(z-w)^2} + i\delta^{\hat{I}\hat{J}}\mathrm{n}_{\bar{\imath}} \frac{: \partial Y^{\bar{\imath}}(w) e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} :}{z-w} + \mathcal{O}(1)\,.
\end{aligned} \tag{B.3.9}$$

$$\begin{aligned}
J^{\alpha}_{\{\mathrm{n}_{\bar{\imath}}\}}(z) J^{\beta}_{\{\mathrm{m}_{\bar{\jmath}}\}}(w) &= :c_\alpha e^{ip_{\alpha,\hat{I}} X^{\hat{I}}(z)} e^{i\mathrm{n}_{\bar{\imath}} Y^{\bar{\imath}}(z)} :=: c_\beta e^{ip_{\beta;\hat{J}} X^{\hat{J}}(w)} e^{i\mathrm{m}_{\bar{\jmath}} Y^{\bar{\jmath}}(w)} : \\
&= c_\alpha c_\beta :\, e^{ip_{\alpha;\hat{I}} X^{\hat{I}}(z)} e^{i\mathrm{n}_{\bar{\imath}} Y^{\bar{\imath}}(z)} e^{ip_{\beta;\hat{J}} X^{\hat{J}}(w)} e^{i\mathrm{m}_{\bar{\jmath}} Y^{\bar{\jmath}}(w)} :\, \\
&= c_\alpha c_\beta :\, \big[1 + ip_{\alpha;\hat{I}}\partial X^{\hat{I}}(w)(z-w) + \ldots\big)\cdot \\
&\qquad \cdot \left(1 + i\mathrm{n}_{\bar{\imath}}\partial Y^{\bar{\imath}}(w)(z-w) + \ldots\right] e^{i(p_{\alpha;\hat{I}}+p_{\beta;\hat{I}})X^{\hat{I}}(w)} e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} :\,.
\end{aligned} \tag{B.3.10}$$

Again, one can choose $p_\alpha + p_\beta = p_{\alpha+\beta}$

$$\begin{aligned}
J^{\alpha}_{\{\mathrm{n}_{\bar{\imath}}\}}(z) J^{\beta}_{\{\mathrm{m}_{\bar{\jmath}}\}}(w) &\sim \frac{\epsilon(\alpha,\beta) c_{\alpha+\beta} :\, e^{ip_{\alpha+\beta;\hat{I}} X^{\hat{I}}(w)} e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} :}{z-w} + \mathcal{O}(1) \\
&= \frac{\epsilon(\alpha,\beta) J^{\alpha+\beta}_{\{\mathrm{n}_{\bar{\imath}}+\mathrm{m}_{\bar{\imath}}\}}(w)}{z-w} + \mathcal{O}(1)\,,
\end{aligned} \tag{B.3.11}$$

or $p_\alpha = -p_\beta$

$$\begin{aligned}
J^{\alpha}_{\{\mathrm{n}_{\bar{\imath}}\}}(z) J^{\beta}_{\{\mathrm{m}_{\bar{\jmath}}\}}(w) &\sim : e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} \left( \frac{1}{(z-w)^2} + \frac{i(p_{\alpha;\hat{I}}\partial X^{\hat{I}}(w) + \mathrm{n}_{\bar{\imath}}\partial Y^{\bar{\imath}}(w))}{z-w} + \mathcal{O}(1) \right) :\,, \\
&= \frac{: e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} :}{(z-w)^2} + \frac{p_{\alpha;\hat{I}} J^{\hat{I}}_{\{\mathrm{n}_{\bar{\imath}}+\mathrm{m}_{\bar{\imath}}\}}(w) + i\mathrm{n}_{\bar{\imath}} : \partial Y^{\bar{\imath}}(w) e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)} :}{z-w} + \mathcal{O}(1)\,.
\end{aligned} \tag{B.3.12}$$

All the other choices of $\pi$ give vanishing vanish OPEs.

$$J^{\alpha}_{\{\mathrm{n}_{\bar\imath}\}}(z)J^{\beta}_{\{\mathrm{m}_{\bar\jmath}\}}(w)\sim\begin{cases}\frac{\epsilon(\alpha,\beta)J^{\alpha+\beta}_{\{\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath}\}}}{z-w}+\mathcal{O}(1) & \alpha+\beta\,\text{root},\\ \frac{:e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:}{(z-w)^2}+\frac{p_{\alpha;\hat\imath}J^{\hat\imath}_{\{\mathrm{n}_{\bar\imath}+\mathrm{m}_{\bar\imath}\}}(w)+i\mathrm{n}_{\bar\imath}:\partial Y^{\bar\imath}(w)e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:}{z-w}+\mathcal{O}(1) & \alpha=-\beta\,,\\ 0 & \text{otherwise,}\end{cases}\tag{B.3.13}$$

$$\begin{aligned}J^{\hat I}_{\{\mathrm{n}_{\bar\imath}\}}(z)J^{\alpha}_{\{\mathrm{m}_{\bar\imath}\}}(w)&=:i\partial X^{\hat I}(z)e^{i\mathrm{n}_{\bar\imath}Y^{\bar\imath}(z)}::c_\alpha e^{ip_{\alpha;\hat\jmath}X^{\hat\jmath}(w)}e^{i\mathrm{m}_{\bar\jmath}Y^{\bar\jmath}(w)}:\\&=ic_\alpha\frac{-ip^{\hat I}_\alpha:e^{ip_{\alpha;J}X^J(w)}e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:}{z-w}+\mathcal{O}(1)\\&=c_\alpha\frac{p^{\hat I}_\alpha:e^{ip_{\alpha;\hat\jmath}X^{\hat\jmath}(w)}e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:}{z-w}+\mathcal{O}(1)=\frac{p^{\hat I}_\alpha J^{\alpha}_{\{\mathrm{n}_{\bar\imath}+\mathrm{m}_{\bar\imath}\}}(w)}{z-w}+\mathcal{O}(1)\,.\end{aligned}\tag{B.3.14}$$

The asymptotic algebra among the currents zero modes is

$$[(J^{\hat I}_{\{\mathrm{n}_{\bar\imath}\}})_0,J^{\hat\jmath}_{\{\mathrm{m}_{\bar\jmath}\}})_0]=\oint_{C'}\frac{dz}{2\pi i}\oint_C\frac{dw}{2\pi i}J^{\hat I}_{\{\mathrm{n}_{\bar\imath}\}}(z)J^{\hat\jmath}_{\{\mathrm{m}_{\bar\jmath}\}}(w)-\oint_{C'}\frac{dw}{2\pi i}\oint_C\frac{dz}{2\pi i}J^{\hat I}_{\{\mathrm{n}_{\bar\imath}\}}(z)J^{\hat\jmath}_{\{\mathrm{m}_{\bar\jmath}\}}(w)\tag{B.3.15}$$

where $C'$ is a contour external to $C$.

$$\begin{aligned}[(J^{\hat I}_{\{\mathrm{n}_{\bar\imath}\}})_0,J^{\hat\jmath}_{\{\mathrm{m}_{\bar\jmath}\}})_0]&=\oint\frac{dw}{2\pi i}\delta^{\hat I\hat\jmath}\mathrm{Res}_{z\to w}\left[\frac{:e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:}{(z-w)^2}+i\mathrm{n}_{\bar\imath}\frac{:\partial Y^{\bar\imath}(w)e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:}{z-w}\right]\\&=\oint\frac{dw}{2\pi i}i\delta^{\hat I\hat\jmath}\mathrm{n}_{\bar\imath}\partial Y^{\bar\imath}(w)e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}=i\delta^{\hat I\hat\jmath}\mathrm{n}_{\bar\imath}\delta_{\mathrm{n}_{\bar\imath}+\mathrm{m}_{\bar\imath},0}(\partial Y^{\bar\imath}_{\{\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath}\}})_0\,,\end{aligned}\tag{B.3.16}$$

where a sum over $\bar\imath$, the directions we are decompactifying, is understood, and between the second and third line we used the generalised version of (B.2.38)

$$\oint\frac{dw}{2\pi i}\partial Y^{\bar\imath}(w)e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}=\delta_{\mathrm{n}_{\bar\imath}+\mathrm{m}_{\bar\imath},0}(\partial Y^{\bar\imath}_{\{\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath}\}})_0\,.\tag{B.3.17}$$

Analogously

$$\begin{aligned}[(J^{\hat I}_{\{\mathrm{n}_{\bar\imath}\}})_0,(J^{\alpha}_{\{\mathrm{m}_{\bar\jmath}\}})_0]&=\oint\frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\left[\frac{c_\alpha p^{\hat I}_\alpha:e^{ip_{\alpha;\hat\jmath}X^{\hat\jmath}(w)}e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:}{z-w}\right]\\&=\oint\frac{dw}{2\pi i}p^{\hat I}_\alpha c_\alpha:e^{ip_{\alpha;\hat\jmath}X^{\hat\jmath}(w)}e^{i(\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath})Y^{\bar\jmath}(w)}:\\&=p^{\hat I}_\alpha(J^{\alpha}_{\{\mathrm{n}_{\bar\jmath}+\mathrm{m}_{\bar\jmath}\}})_0\,,\end{aligned}\tag{B.3.18}$$

and

$$
\begin{aligned}
&[(J^{\hat{I}}_{\{\mathrm{n}_{\bar{\imath}}\}})_0, (J^{\beta}_{\{\mathrm{m}_{\bar{\jmath}}\}})_0] = \\
&= \oint \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} \begin{cases} \frac{\epsilon(\alpha,\beta) J^{\alpha+\beta}_{\{\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}}\}}}{z-w} + \mathcal{O}(1) & \alpha+\beta \,\text{root}, \\ \frac{:e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)}:}{(z-w)^2} + \frac{p_{\alpha;\hat{I}} J^{\hat{I}}_{\{\mathrm{n}_{\bar{\imath}}+\mathrm{m}_{\bar{\imath}}\}}(w) + i\mathrm{n}_{\bar{\imath}} :\partial Y^{\bar{\imath}}(w) e^{i(\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}})Y^{\bar{\jmath}}(w)}:}{z-w} + \mathcal{O}(1) & \alpha = -\beta\,, \\ 0 & \text{otherwise}, \end{cases} \\
&\qquad = \begin{cases} \epsilon(\alpha,\beta)(J^{\alpha+\beta}_{\{\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}}\}})_0 & \alpha+\beta \ \text{root} \\ p_{\alpha;\hat{I}}(J^{\hat{I}}_{\{\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}}\}})_0 + i\mathrm{n}_{\bar{\imath}}\delta_{\mathrm{n}_{\bar{\imath}}+\mathrm{m}_{\bar{\imath}},0}(\partial Y^{\bar{\imath}}_{\{\mathrm{n}_{\bar{\jmath}}+\mathrm{m}_{\bar{\jmath}}\}})_0 & \alpha = -\beta \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
\tag{B.3.19}
$$

Only as a brief comment on the affinisation of the right moving $\mathfrak{u}(1)^{d-k}_R$ algebra, the general pattern is that the $\mathfrak{u}(1)$'s associated to the directions that remain compact are made affine with $k$ central extensions given by the $(\bar{\partial}\bar{Y}^{\bar{\imath}})$ that correspond to decompactified directions.

# Appendix C

# Test of the RFC by dimensional reduction

In this appendix we review the dimensional reduction of the 10d heterotic supergravity Lagrangian in detail, as well as the calculation of the long-range interactions between heterotic states in the field theory language.

## C.1 Dimensional reduction of the supergravity Lagrangian

We consider an $S^1$ compactification of the heterotic string from 10 to 9 dimensions, by identifying the 9th coordinate as $x^9 \equiv y \sim y + 2\pi R$. From the target spacetime point of view, the low energy heterotic supergravity action in 10 dimensions takes the form [114]

$$S = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-\mathbf{G}^s} e^{-2\mathbf{\Phi}} \Big( \mathbf{R} + 4\partial_M \mathbf{\Phi} \partial^M \mathbf{\Phi} - \frac{1}{12} \mathbf{H}^{MNR} \mathbf{H}_{MNR} - \\ - \frac{\alpha'}{4} \bar{\mathbf{F}}_I^{MN} \bar{\mathbf{F}}^I_{MN} \Big) , \tag{C.1.1}$$

$\mathbf{G}^s_{MN}$, $M, N = 0, ..., 9$ is the 10 dimensional metric in the string frame with Ricci scalar $\mathbf{R}$, $\mathbf{\Phi} = \bar{\mathbf{\Phi}} + \Phi_0$ is the 10 dimensional dilaton and the gauge invariant field strengths are defined as

$$\mathbf{H}_{MNR} = 3 \left( \partial_{[M} \mathbf{B}_{NR]} - \alpha' \mathbf{A}^I_{[M} \partial_{\mathrm{n}} \mathbf{A}_{R]I} - \frac{\alpha'}{3} f_{IJK} \mathbf{A}^I_M \mathbf{A}^J_N \mathbf{A}^K_R \right) , \tag{C.1.2}$$

$$\mathbf{F}^I_{MN} = \sqrt{\alpha'} (2\partial_{[M} \mathbf{A}^I_{N]} + f^I_{JK} \mathbf{A}^J_M \mathbf{A}^K_N) = \sqrt{\alpha'} \bar{\mathbf{F}}^I_{MN} , \tag{C.1.3}$$

where $\mathbf{B}_{MN}$ is the NSNS 2-form and $\mathbf{A}^I$ are the heterotic vector bosons, with $I = 1, ..., 496$ the gauge group index (taking into account both the Cartan and the root sectors). For the reduction, the spacetime indices are split as $M = (\mu, 9)$, with $\mu = 0, ..., 8$ and we take the following ansatz for the metric (in the string frame)

$$ds^2 = \mathbf{G^s}_{MN} dx^M dx^N = e^{\frac{4}{7}\bar{\Phi}} e^{-\frac{2}{7}\bar{\sigma}} g_{\mu\nu} dx^\mu dx^\nu + e^{2\bar{\sigma}} (dy + Z_\mu dx^\mu)^2 . \tag{C.1.4}$$

Here $g_{\mu\nu}$ is the 9 dimensional metric in the Einstein frame, $Z_\mu$ the (dimensionless) graviphoton, and $\bar{\sigma}$ is the dynamical part of the (dimensionless) radion field $\sigma = \bar{\sigma} + \sigma_0$, where $\sigma_0$ is the background value. The physical compactification radius of the circle is therefore $\mathcal{R} = e^{\bar{\sigma}} R$. We restrict to the massless sector upon compactification and therefore assume the 10 dimensional fields (in bold) to be independent of the coordinate $y$. The dimensionally reduced action in the Einstein frame in terms of the 9 dimensional fields (not in bold) reads

$$\begin{aligned} S =& \frac{e^{-2\Phi_0} R}{(2\pi)^6 \alpha'^4} \int d^9x \sqrt{-g} \Big( R - \frac{8}{7} \partial_\mu \bar{\sigma} \partial^\mu \bar{\sigma} - \frac{4}{7} \partial_\mu \bar{\Phi} \partial^\mu \bar{\Phi} + \frac{4}{7} \partial_\mu \bar{\Phi} \partial^\mu \bar{\sigma} - \\ & - \frac{\alpha'}{2} e^{-2\bar{\sigma}} \bar{F}^I_{\mu 9} \bar{F}^{\mu I}_9 - \frac{1}{4} e^{\frac{16}{7}\bar{\sigma}} e^{-\frac{4}{7}\bar{\Phi}} Z_{\mu\nu} Z^{\mu\nu} - \frac{\alpha'}{4} e^{-\frac{4}{7}\bar{\Phi}} e^{\frac{2}{7}\bar{\sigma}} \bar{F}^I_{\mu\nu} \bar{F}^{\mu\nu I} - \\ & - \frac{1}{12} e^{\frac{4}{7}\bar{\sigma}} e^{-\frac{8}{7}\bar{\Phi}} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} e^{-\frac{12}{7}\bar{\sigma}} e^{-\frac{4}{7}\bar{\Phi}} H_{\mu\nu 9} H_9^{\mu\nu} \Big), \end{aligned} \tag{C.1.5}$$

where

$$\bar{F}^I_{\mu 9} = \partial_\mu A^I_9 + f^I_{JK} A^J_\mu A^K_9 \,, \tag{C.1.6}$$

$$\bar{F}^I_{\mu\nu} = 2\partial_{[\mu} A^I_{\nu]} + f^I_{JK} A^J_\mu A^K_\nu + Z_{\mu\nu} A^I_9 \,, \tag{C.1.7}$$

and $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$. By defining also

$$W_\mu = \mathbf{B}_{\mu 9} + \frac{\alpha'}{2} A^I_9 A_{\mu I} \,, \qquad W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu \,, \tag{C.1.8}$$

we obtain

$$H_{\mu\nu 9} = W_{\mu\nu} - \alpha' (2\partial_{[\mu} A^I_{\nu]} + f^I_{JK} A^J_\mu A^K_\nu) A_{9I} - \frac{\alpha'}{2} Z_{\mu\nu} (A^I_9)^2 \,, \tag{C.1.9}$$

$$H_{\mu\nu\rho} = 3 \left( \partial_{[\mu} B_{\nu\rho]} - \alpha' A^I_{[\mu} \partial_\nu A_{\rho]I} - (\partial_{[\mu} Z_\nu) W_{\rho]} + Z_{[\mu} (\partial_\nu W_{\rho]}) - \frac{\alpha'}{3} f_{IJK} A^I_\mu A^J_\nu A^K_\rho \right) . \tag{C.1.10}$$

In the following, we choose to work with dimension 1 gauge fields, normalized so that string states have integer quantized charges. The momentum number is derived by looking at the diffeomorphism symmetry of the 10 dimensional action. If one wants the charge of the $n$-th KK state from the expansion

$$\phi\left(x^M\right) = \sum_{\mathrm{n}} \phi_{\mathrm{n}}(x^\mu) e^{i\frac{ny}{R}} \tag{C.1.11}$$

to be $n \in \mathbb{Z}$ under the redefined field $\bar{Z}_\mu$, the dimensionfull graviphoton must be taken as

$$\bar{Z}_\mu = \frac{Z_\mu}{R} \,. \tag{C.1.12}$$

The expressions for the 9 dimensional heterotic field strength (C.1.7) is thus

$$\bar{F}^I_{\mu\nu} = 2\partial_{[\mu} A^I_{\nu]} + 2R \partial_{[\mu} \bar{Z}_{\nu]} A^I_9 + f^I_{JK} A^J_\mu A^K_\nu \,. \tag{C.1.13}$$

Focusing now on the $W_\mu$ redefinition and the winding charge, its expression can be derived from the worldsheet action term [25]

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\Sigma\, W_\mu \partial_\tau X^\mu \partial_\Sigma Y\,, \tag{C.1.14}$$

where $(\tau,\, \Sigma)$ are the coordinates parameterizing the worldsheet, $\Sigma \in [0, 2\pi)$, and $X^M\,(\tau,\, \Sigma)$ are the embedding functions into target spacetime. Since the string winds around the $S^1$ direction $w$ times, one can write

$$Y = w\Sigma R\,. \tag{C.1.15}$$

The worldsheet action then yields

$$S = -\frac{wR}{\alpha'}\int d\tau W_\mu \partial_\tau X^\mu\,, \tag{C.1.16}$$

which is the worldline coupling of a particle with charge

$$Q = \frac{wR}{\alpha'} \tag{C.1.17}$$

with the dimensionless gauge field $W_\mu$. To get an integer charge, the dimensionful gauge field $\bar{W}_\mu$ must be defined as

$$\bar{W}_\mu = \frac{W_\mu R}{\alpha'}\,. \tag{C.1.18}$$

With the redefinitions (C.1.12) and (C.1.18), the 9 dimensional field strengths obtained by reducing $\mathbf{H}_{MNR}$ are

$$\begin{aligned} H_{\mu\nu 9} = \frac{\alpha'}{R}(\partial_\mu \bar{W}_\nu - \partial_\nu \bar{W}_\mu) - \alpha'(\partial_\mu A^I_\nu)A^I_9 + \alpha'(\partial_\nu A^I_\mu)A^I_9 - \frac{\alpha' R}{2}\partial_\mu \bar{Z}_\nu (A^I_9)^2 + \\ + \frac{\alpha' R}{2}\partial_\nu \bar{Z}_\mu (A^I_9)^2 - \alpha' f_{IJK} A^I_\mu A^J_\nu A^K_9 \end{aligned} \tag{C.1.19}$$

and

$$H_{\mu\nu\rho} = 3\left(\partial_{[\mu} B_{\nu\rho]} - \alpha' A^I_{[\mu}\partial_\nu A^I_{\rho]} - \alpha'((\partial_{[\mu}\bar{Z}_\nu)\bar{W}_{\rho]} + \bar{Z}_{[\mu}(\partial_\nu \bar{W}_{\rho]})) - \frac{\alpha'}{3} f_{IJK} A^I_\mu A^J_\nu A^K_\rho\right). \tag{C.1.20}$$

Furthermore, the Cartan heterotic gauge fields $A^I$ are correctly normalized in such a way for the charges to be $\pi^I \in \Gamma_8 \times \Gamma_8$. Finally, in order to work with conventionally normalized scalar fields, without cross terms in the kinetic part, we make the following field redefinition

$$\lambda = \frac{4}{\sqrt{7}}\left(\frac{\Phi}{4} - \sigma\right) \longrightarrow \lambda_0 = \frac{4}{\sqrt{7}}\left(\frac{\Phi_0}{4} - \sigma_0\right),\ \bar{\lambda} = \frac{4}{\sqrt{7}}\left(\frac{\bar{\Phi}}{4} - \bar{\sigma}\right). \tag{C.1.21}$$

Finally, the 9 dimensional action in the Einstein frame, with conventionally normalized scalars and gauge fields such that minimally coupled sources have integer charges takes the form

$$S = \frac{M_{\rm p,9}^7}{2}\int d^9x\sqrt{-g}\Big(R - \frac{1}{2}\partial_\mu\lambda\partial^\mu\lambda - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{\alpha'}{2}e^{-\frac{\bar\Phi}{2}}e^{\frac{\sqrt{7}}{2}\bar\lambda}\bar F^I_{\mu 9}\bar F_9^{\mu I} - \frac{1}{4}e^{-\frac{4\sqrt{7}}{7}\bar\lambda}R^2 Z_{\mu\nu}Z^{\mu\nu} - \\ - \frac{\alpha'}{4}e^{-\frac{1}{2}\bar\Phi}e^{-\frac{\sqrt{7}}{14}\bar\lambda}\bar F^I_{\mu\nu}\bar F^{\mu\nu I} - \frac{1}{12}e^{-\frac{\sqrt{7}}{7}\bar\lambda}e^{-\bar\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}e^{\frac{3\sqrt{7}}{7}\bar\lambda}e^{-\bar\Phi}H_{\mu\nu 9}H_9^{\mu\nu}\Big), \tag{C.1.22}$$

where the 9 dimensional Planck mass reads

$$\frac{M_{\rm p,9}^7}{2} = \frac{e^{-2\Phi_0}R}{(2\pi)^6\alpha'^4} = \frac{e^{-\frac{7}{4}(\Phi_0+\frac{1}{\sqrt{7}}\lambda_0)}}{(2\pi)^6\alpha'^4}\,. \tag{C.1.23}$$

To close this section, let us recast the mass formula for the heterotic string states in the 9 dimensional Einstein frame and in Planck units and in the absence of Wilson lines (with $R = e^{\sigma_0}\sqrt{\alpha'}$)

$$\frac{M^2}{M_{\rm p,9}^2} = (32\pi^6)^{\frac{2}{7}}\left\{e^{\frac{4\sqrt{7}}{7}\lambda}n^2 + e^{\Phi}e^{-\frac{3\sqrt{7}}{7}\lambda}w^2 + e^{\frac{1}{2}\Phi}e^{\frac{\sqrt{7}}{14}\lambda}\left[2\left(N+\bar N-\frac{3}{2}\right)+|\pi|^2\right]\right\}. \tag{C.1.24}$$

We also introduce the following notation for shortness

$$M^2 = M_{\rm n}^2 + M_w^2 + M_{\rm n}^2 + M_\pi^2\,, \tag{C.1.25}$$

where

$$M_{\rm n}^2 = (32\pi^6)^{\frac{2}{7}}e^{\frac{4\sqrt{7}}{7}\lambda}n^2M_{\rm p,9}^2\,, \tag{C.1.26}$$

$$M_w^2 = (32\pi^6)^{\frac{2}{7}}e^{\Phi}e^{-\frac{3\sqrt{7}}{7}\lambda}w^2M_{\rm p,9}^2\,, \tag{C.1.27}$$

$$M_{\rm n}^2 = 2(32\pi^6)^{\frac{2}{7}}e^{\frac{1}{2}\Phi}e^{\frac{\sqrt{7}}{14}\lambda}\left(N+\bar N-\frac{3}{2}\right)M_{\rm p,9}^2\,, \tag{C.1.28}$$

$$M_\pi^2 = (32\pi^6)^{\frac{2}{7}}e^{\frac{1}{2}\Phi}e^{\frac{\sqrt{7}}{14}\lambda}|\pi|^2M_{\rm p,9}^2\,. \tag{C.1.29}$$

## C.2 Computing the long range force

In order to test the RFC in this setting, we need to compute the long range force between two states in the heterotic string spectrum as prescribed in e.g. [147]. The part coming from the $U(1)$ interactions, in the system at hand and with the conventions introduced above, is mediated by $\bar Z_\mu$, $\bar W_\mu$ and the Cartan vectors of $\mathrm{E}_8\times\mathrm{E}_8$, $A^I$, $I = 1,...,16$. This perturbative computation makes sense only when the theory is weakly coupled, and we have just argued that in the two corners of interest in moduli space actually there is at least one divergent gauge coupling. This means that we cannot trust the full computation at all points in moduli space if the three interactions are turned on at the same time. Nevertheless, we will perform the computation including all gauge fields at the same time for completeness because it is valid as an abstract calculation, but we will then limit ourselves to its application to states charged

only under the groups whose gauge coupling is perturbative in the region of moduli space that we want to study. Consider then a generic state, with mass given by eq.(C.1.24), minimally coupled to gravity and to the gauge fields. The leading contribution to the force is obtained through the linearization of action (C.1.22), which yields

$$S = \frac{M_{\rm p,9}^7}{2}\int d^9x\sqrt{-g}\Big(R - \frac{1}{2}\partial_\mu\lambda\partial^\mu\lambda - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{\alpha'}{2}\partial_\mu A_9^I\partial^\mu A_9^I - \frac{R^2}{4}\bar{Z}_{\mu\nu}\bar{Z}^{\mu\nu} -$$

$$-\frac{\alpha'}{4}\bar{F}^I_{\mu\nu}\bar{F}^{\mu\nu I} - \frac{\alpha'^2}{4R^2}\bar{W}_{\mu\nu}\bar{W}^{\mu\nu}\Big) - \int M(\Phi,\lambda,A_9^I)\,ds - w\int\bar{W} - \pi^I\int A^I - n\int\bar{Z}, \quad \text{(C.2.1)}$$

where

$$\bar{F}^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]}\,. \quad \text{(C.2.2)}$$

We expand the Einstein frame metric around the Minkowski background, that is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \quad \text{(C.2.3)}$$

where $h_{\mu\nu} << 1$ in Planck units. We will use the trace-reversed metric perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \quad \text{(C.2.4)}$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$, and under the hypothesis of a static solution, so that $\partial_t$ is a Killing vector defining a conserved energy. Furthermore, we work in the Lorentz gauge for all the gauge fields and also for the trace-reversed metric perturbation: $\partial_\mu\bar{h}^{\mu\nu} = 0$, $\partial_\mu\bar{W}^{\mu\nu} = 0$, $\partial_\mu\bar{Z}^{\mu\nu} = 0$, $\partial_\mu\bar{F}^{\mu\nu I} = 0$. We now compute the perturbation on the background caused by a source particle in the heterotic string spectrum at rest, with worldline parameterized by its proper time as

$$x^\mu(\tau) = (\tau, \hat{x}^i)\,, \qquad \hat{x}^i = \text{const.} \quad \text{(C.2.5)}$$

The linerarized equations of motion and their respective solutions thus read (where $M \equiv M(\Phi_0, \lambda_0, 0)$ and also all the derivatives of the mass are evaluated at the background)

$$\Box \bar{h}_{\mu\nu}(x) = -\frac{2\,M}{M_{\mathrm{p},9}^7}\,\delta^0_\mu\,\delta^0_\nu\,\delta^{(8)}(\hat{x}^i - x^i) \implies \bar{h}_{\mu\nu}(r) = \frac{M}{3\,V_7\,M_{\mathrm{p},9}^7\,r^6}\,\delta^0_\mu\,\delta^0_\nu\,, \tag{C.2.6}$$

$$\Box \lambda(x) = \frac{2}{M_{\mathrm{p},9}^7}\frac{\partial M}{\partial \lambda}\,\delta^{(8)}(\hat{x}^i - x^i) \implies \bar{\lambda}(r) = -\frac{\partial_\lambda M}{3\,V_7\,M_{\mathrm{p},9}^7\,r^6}\,, \tag{C.2.7}$$

$$\Box A^I_9(x) = \frac{2}{M_{\mathrm{p},9}^7}\frac{1}{\alpha'}\frac{\partial M}{\partial A^9_I}\,\delta^{(8)}(\hat{x}^i - x^i) \implies A^I_9(r) = -\frac{\partial_{A^9_I} M}{3\,\alpha'\,V_7\,M_{\mathrm{p},9}^7\,r^6}\,, \tag{C.2.8}$$

$$\Box \Phi(x) = \frac{2}{M_{\mathrm{p},9}^7}\frac{\partial M}{\partial \Phi}\,\delta^{(8)}(\hat{x}^i - x^i) \implies \bar{\Phi}(r) = -\frac{\partial_\Phi M}{3\,V_7\,M_{\mathrm{p},9}^7\,r^6}\,, \tag{C.2.9}$$

$$\Box A^{\mu I}(x) = \frac{2}{M_{\mathrm{p},9}^7}\frac{1}{\alpha'}\,\pi^I\,\delta^\mu_0\,\delta^{(8)}(\hat{x}^i - x^i) \implies A^{\mu I}(r) = -\frac{\pi^I}{3\,\alpha'\,V_7\,M_{\mathrm{p},9}^7\,r^6}\,\delta^\mu_0\,, \tag{C.2.10}$$

$$\Box \bar{W}^\mu(x) = \frac{2}{M_{\mathrm{p},9}^7}\frac{R^2}{\alpha'^2}\,w\,\delta^\mu_0\delta^{(8)}(\hat{x}^i - x^i) \implies \bar{W}^\mu(r) = -\frac{w\,R^2}{3\,\alpha'^2\,V_7\,M_{\mathrm{p},9}^7\,r^6}\delta^\mu_0\,, \tag{C.2.11}$$

$$\Box \bar{Z}^\mu(x) = \frac{2}{M_{\mathrm{p},9}^7}\frac{1}{R^2}\,n\,\delta^\mu_0\,\delta^{(8)}(\hat{x}^i - x^i) \implies \bar{Z}^\mu(r) = -\frac{n}{3\,R^2\,V_7\,M_{\mathrm{p},9}^7\,r^6}\,\delta^\mu_0\,. \tag{C.2.12}$$

These solutions are taken to be static ($\Box = \vec{\nabla}^2$) and such that all the field perturbations vanish at infinity. $V_7$ is the volume of the 7 dimensional unit sphere, given by

$$V_7 = \frac{2\pi^4}{\Gamma(4)}\,, \tag{C.2.13}$$

and we defined the radial coordinate $r = |x^i - \hat{x}^i|$. The force felt by a probe particle with mass $M_2$ and charges ($n_2$, $w_2$, $\pi^I_2$) sitting a large distance $r$ away from another particle with mass $M_1$ and charges ($n_1$, $w_1$, $\pi^I_1$), can be derived from the potential felt by the probe

$$V_{12} = \left(1 - \frac{h_{00}(r)}{2}\right) M_2(\Phi(r), \lambda(r), A^I_9(r)) + w\,\bar{W}_0(r) + \pi^I A^I_0(r) + n\,\bar{Z}_0(r)\,, \tag{C.2.14}$$

where from the definition (C.2.4) we have

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{7} g_{\mu\nu}\bar{h} \implies h_{00} = \frac{6}{7}\bar{h}_{00}\,. \tag{C.2.15}$$

By expanding the mass around the background and keeping only the leading terms in $r$ one gets

$$\begin{aligned} V_{12} = &-\frac{h_{00}(r)}{2} M_2(\Phi_0, \sigma_0, 0) + \frac{\partial M_2}{\partial \Phi}(\Phi_0, \sigma_0, 0)\bar{\Phi}(r) + \frac{\partial M_2}{\partial \sigma}(\Phi_0, \sigma_0, 0)\bar{\sigma}(r) + \\ &+ \frac{\partial M_2}{\partial A^I_9}(\Phi_0, \sigma_0, 0) A^I_9(r) - w\,\bar{W}^0(r) - \pi^I A^{0I}(r) - n\,\bar{Z}^0(r) + \mathcal{O}(r^{-12})\,. \end{aligned} \tag{C.2.16}$$

By defining $M_i \equiv \mathrm{m}_i(\Phi_0, \lambda_0, 0)$ for $i = 1$ and 2, and evaluating all the derivatives of the mass at the background once again, the force takes the form

$$\begin{aligned} F_{12} = -\frac{\partial V_{12}}{\partial r} =& \frac{2}{V_7\, M_{\mathrm{p},9}^7\, r^7}\Bigg\{ -\frac{3}{7} M_1\, M_2 - \frac{\partial M_1}{\partial \Phi}\frac{\partial M_2}{\partial \Phi} - \frac{\partial M_1}{\partial \lambda}\frac{\partial M_2}{\partial \lambda} - \frac{1}{\alpha'}\frac{\partial M_1}{\partial A_I^9}\frac{\partial M_2}{\partial A_9^I} + \\ &+ \frac{w_1 w_2 e^{\frac{1}{2}\Phi_0} e^{\frac{\sqrt{7}}{2}\lambda_0}}{\alpha'} + \frac{\pi_1^I \pi_2^I}{\alpha'} + \frac{n_1 n_2 e^{-\frac{1}{2}\Phi_0} e^{-\frac{\sqrt{7}}{2}\lambda_0}}{\alpha'} \Bigg\}, \end{aligned} \tag{C.2.17}$$

and one can see that the scalar and gravitational forces give attractive contributions whereas the $U(1)$'s give repulsive ones, as expected for particles with equal charges. Keeping the 9 dimensional Planck mass fixed, the derivatives of the mass evaluated at the background read

$$\frac{\partial M}{\partial \Phi} = \frac{1}{2M}\left[M_w^2 + \frac{1}{2}M_\pi^2\right], \tag{C.2.18}$$

$$\frac{\partial M}{\partial \lambda} = \frac{1}{2M}\left[\frac{4\sqrt{7}}{7}M_{\mathrm{n}}^2 - \frac{3\sqrt{7}}{7}M_w^2 + \frac{\sqrt{7}}{14}(M_{\mathrm{n}}^2 + M_\pi^2)\right], \tag{C.2.19}$$

$$\frac{\partial M}{\partial A_9^I} = \sqrt{\alpha'}\,\frac{M_{\pi I}(M_w - M_{\mathrm{n}})}{M}, \tag{C.2.20}$$

so that, combining all the contributions

$$\begin{aligned} V_7\, M_{\mathrm{p},9}^7\, r^7\, F_{12}(r) =& - M_1 M_2 + 2M_{\pi,1}\cdot M_{\pi,2} + 2M_{n,1}\cdot M_{n,2} + 2M_{w,1}\cdot M_{w,2} - \\ &- \frac{(M_{n,1}^2 - M_{w,1}^2)(M_{n,2}^2 - M_{w,2}^2)}{M_1 M_2} - \\ &- \frac{2M_{\pi,1}M_{\pi,2}(M_{n,1} - M_{w,1})(M_{n,2} - M_{w,2})}{M_1 M_2}. \end{aligned} \tag{C.2.21}$$

There are several interesting cases in which this force vanishes, namely

(i) For states having only momentum mass $M_n \neq 0$, that is $M_i = M_{n,i}$ the force takes the form $F_{12} \propto -M_{n,1}M_{n,2} - M_{n,1}M_{n,2} + 2M_{n,1}M_{n,2} = 0$. The same holds for states having only winding charge, $\mathrm{m}_i = M_{w,i}$.

(ii) The force between one momentum state and one winding state vanishes as well: $F_{12} \propto -M_{1,n}M_{2,w} + M_{1,n}M_{2,w} = 0$.

(iii) Since the states we are most interested in (namely the ones that form the $(E_9 \oplus E_9)/\sim$ algebra) are BPS, let us consider the case of BPS heterotic states, which are characterized by the conditions [89]

$$M^2 = \frac{2}{\alpha'}p_R^2 \overset{\text{NS sector}}{\Longrightarrow} \bar{N} = \frac{1}{2}, \quad N = 1 - nw - \frac{|\pi|^2}{2}. \tag{C.2.22}$$

The force between two mutually BPS states vanishes, as can be easily seen by recasting

the expression (C.2.21) in the equivalent form (also found in [20])

$$V_7 r^7 F_{12} = -\frac{4}{\alpha'^2 M_1 M_2}\left(\frac{\alpha'}{2}M_1M_2 - \mathbf{p}_{L,1}\cdot\mathbf{p}_{L,2}\right)\left(\frac{\alpha'}{2}M_1M_2 - p_{R,1}p_{R,2}\right). \quad \text{(C.2.23)}$$

From the second factor, mutually BPS particles exert a vanishing force on each other.

The RFC states that for each point in the charge lattice there must exist a particle such that the force between two identical particles is non-negative, i.e. $F_{11} \geq 0$. From (C.2.21), the state having $N - \bar{N} - \frac{3}{2} = 0$ is self repulsive and it is among the ones predicted by the RFC. Indeed, in this case

$$V_7 M_{\mathrm{p},9}^7 r^7 M^2 F_{11}(r) = (M_\pi^2 + 2M_w M_{\mathrm{n}})^2 > 0. \quad \text{(C.2.24)}$$

Since the factor $V_7 M_{\mathrm{p},9}^7 r^7 M^4 > 0$, this means that $F_{11} > 0$. For completeness, let us mention that as we saw BPS particles satisfy also the repulsive condition (and by the BPS condition the second factor cannot be negative), as well as the ones having $M^2 \leq \frac{2}{\alpha'}\mathbf{p}_L^2$, which given the mass formula $\frac{\alpha'}{4}M^2 = \frac{1}{2}\mathbf{p}_L^2 + N - 1$ means that $N = 0, 1$ [20].

# Appendix D

# Decompactification limits in F-theory

## D.1 Review of Kulikov models

From the dual F-theory point of view, decompactification limits of the heterotic theory can be analysed geometrically by studying the complex structure degenerations of the elliptically fibered K3, in particular the ones occurring at infinite distance in the complex structure moduli space. In turn, these can be described and classified in terms of Kulikov models [22]. In order to define them, let us introduce a K3 degeneration as a one-parameter family $\{X_u\}$ of K3 surfaces fibered over a disk $u \in D = \{u \in \mathbb{C}, |u| < 1\}$, such that the fiber is smooth everywhere except at $u = 0$, the infinite distance limit, where it degenerates. The disk and the K3 surfaces form a threefold $\mathcal{X}$ with base $D$ and fiber $X_u$, and by construction it defines a Kulikov model, which is a degeneration with the following properties:

- semi-stability: $\mathcal{X}$ is smooth, and the fiber $X_0$ is a reduced variety where all components have multiplicity one and the singularities are of normal crossing type. This is guaranteed by the semi-stable reduction theorem.

- Ricci flatness, which can always be achieved for a semi-stable degeneration via base changes ($u \to u^k$) and/or birational transformations.

Since the case of interest is the one of elliptically fibered K3's, one can describe this family $\{X_u\}$ by blowing down each member $X_u$ and associating to the resulting surface a Weierstrass model,

$$Y_u : \quad y^2 = x^3 + f_u(s,t)xz^4 + g_u(s,t)z^6 \,, \tag{D.1.1}$$

and the corresponding family $\{Y_u\}$ is called Kulikov Weierstrass model. Here $[x : y : z]$ are homogeneous coordinates of $\mathbb{P}_{231}$ representing the fiber space and $[s : t]$ are the homogeneous coordinates of the base $\mathbb{P}^1$ of $X_u$. $f_u$ and $g_u$ are homogeneous polynomials in $[s : t]$ of degree 8 and 12 respectively, giving a singular surface $Y_0$ which can be obtained by blowing down $X_0$. It can be shown that $Y_0$ is an union of several components

$$Y_0 = \cup_{i=0}^{p} Y^i \,, \tag{D.1.2}$$

where each $Y^i$ is a fibration over a $\mathbb{P}^1$ base component $B_i = \{e_i = 0\}$ with $e_i$ coordinates on the base space, which arrange in a chain (each $B^i$ intersect two other components at most once), with possibly two kinds of degeneracies that can be read from the vanishing orders of the discriminant

$$\Delta_0 = 4f_0^3 + 27g_0^2 \,. \tag{D.1.3}$$

These can be

- non-minimal singularities in the Kodaira classification, namely points on $B^i$ with vanishing orders $ord(f_0, g_0, \Delta_0) = (\geq 4, \geq 6, \geq 12)$.
- codimension zero singularities, namely singular fibers over any point of the base component $B^i$ of the form $\Delta_0 = e_i^{\mathrm{n}_i}\Delta_0'$, where $\Delta_0'$ does not allow for further factorisation in $e_i$. These singularities must be of Kodaira Type $I_{\mathrm{n}_i}$ (namely, $f_0$ and $g_0$ do not vanish at these generic points) in order for them to be of normal crossing type. These singularities can get enhanced at co-dimension one loci to give either type A, or two type D singularities. This is in contrast to the usual smooth case where the codimension zero singularities are of Kodaira Type $I_0$, which can have all types of singular fibers over codimension one loci on the base.

We will not give the details of the classification of all the possible degenerate limits, for which we refer to [22], but just qualitatively introduce the structure of the degenerations which are dual to decompactifications in the heterotic framework. The towers in this approach are recovered - at least partially - from elliptic transcendental 2-cycles with asymptotically vanishing volume on which M2 branes can be wrapped arbitrarily many times. All the statements below hold up to base changes and birational transformations.

### Partial decompactification limits to 9 dimensions

This case, called 'Type III.a', is dual to 4.3.1. The base of the elliptically fibered K3 degenerates into a chain of two or more components. The intermediate components have codimension zero fiber of Kodaira Type $I_{\mathrm{n}_i}$, $\mathrm{n}_i > 0$ and at most codimension one singularities of type A. As for the end components, either they are both rational elliptic surfaces, $dP_9$, or one of them is a $dP_9$ surface and the other one has codimension zero fiber of Type $I_{n>0}$ and two D-type codimension one fibers. All these components intersect at a point, where they share a Type $I_m$ singularity. One example is given explicitly in 4.5.2. The vanishing transcendental torus is given by fibering the (1,0) vanishing cycle of the fiber at the intersection point over the vanishing 1-cycle of the base, which is denoted by $\delta_{[1,0]}$ in Figure 4.9.

### Full decompactification limits to 10 dimensions

In the F-theory setting there are two possible cases in which the dual heterotic theory fully decompactifies. The one that has been explicitly worked out in terms of the Kulikov Weierstrass model is the one corresponding to the algebra $(\widehat{E}_9 \oplus \widehat{E}_9)/\sim$, called 'Type II.a' and reviewed in Section 4.5.1, but we can also realise the qualitatively different 'Type III.b' limit the algebra $\widehat{\widehat{D}}_{16}$, as we will explicitly show in the following. In the Type III.b limit the base of the K3 surface degenerates into several components $B^i$, all of which have a codimension zero

singularity of Kodaira type $I_{\mathrm{n}_i}$, $\mathrm{n}_i > 0$. This corresponds to the weak coupling Sen's orientifold limit. In addition there is a non-minimal singular fiber, which is related to the presence of an affine algebra. Like in the Type III.a case, the Kulikov model captures only one vanishing transcendental elliptic curve, namely the fibration of the vanishing A-cycle in the fiber over the cycle surrounding the point of intersection between two base components. This account for only one of the towers, given in the dual M-theory by M2 branes wrapping this transcendental torus arbitrarily many times, or alternatively to the winding modes of the weakly coupled type IIB F1 wrapping a vanishing cycle on the $T^2$. Nevertheless, the Type III.b corresponds to a full decompactification limit if one accounts for the fact that the Kähler modulus is kept constant in the degeneration, which is possible if the size of the B-cycle goes to infinity, giving the second tower as KK modes along it. Let us now explicitly describe the limit corresponding to the algebra $\widehat{\widehat{D}}_{16}$, which is dual to the decompactification to the 10 dimensional $SO(32)$ heterotic theory.

## D.2 Realisation of double loop $D_{16}$ in F-theory

Let us consider the following Weierstrass model in the patch $z = 1$

$$y^2 = x^3 + f(s,t)x + g(s,t) \tag{D.2.1}$$

with

$$f(s,t) = -\frac{1}{3}s^2 h(s,t)^2 - ds^8\,, \tag{D.2.2}$$

$$g(s,t) = -\frac{2}{27}s^3 h(s,t)^3 - \frac{1}{3}ds^9 h(s,t)\,, \tag{D.2.3}$$

$$h(s,t) = bt^3 + at^2 s + s^3\,, \tag{D.2.4}$$

$a$, $b$, $d \in \mathbb{C}$. The discriminat is

$$\begin{aligned}\Delta(s,t) &= 4f(s,t)^3 + 27g(s,t)^2 \\ &= -d^2 s^{18}((1+4d)s^6 + 2as^4t^2 + 2bs^3t^3 + a^2s^2t^4 + 2abst^5 + b^2t^6)\,.\end{aligned} \tag{D.2.5}$$

There is a $D_{16}$ singularity at $s = 0$, with vanishing orders $ord(f, g, \Delta)|_{s=0} = (2, 3, 18)$, and 6 additional $I_1$ singularites, which restricting to the patch $s = 1$ are at

$$t_{1,k_\pm} = -\frac{a}{3b} + \frac{\sqrt[3]{2}a^2}{3bk_\pm} + \frac{k_\pm}{3\sqrt[3]{2}b}\,, \tag{D.2.6}$$

$$t_{2,k_\pm} = -\frac{a}{3b} - \frac{(1+i\sqrt{3})a^2}{3\sqrt[3]{4}bk_\pm} - \frac{(1-i\sqrt{3})k_\pm}{6\sqrt[3]{2}b}\,, \tag{D.2.7}$$

$$t_{3,k_\pm} = -\frac{a}{3b} - \frac{(1-i\sqrt{3})a^2}{3\sqrt[3]{4}bk_\pm} - \frac{(1+i\sqrt{3})k_\pm}{6\sqrt[3]{2}b} \tag{D.2.8}$$

where the two $\pm$ in $t_{2,k_\pm,\pm i}$ are not correlated, and we have defined the following quantites

$$k_\pm(a,b,d) \equiv k_\pm = \left(-2a^3 - 27b^2 + \sqrt{-4a^6 + (-2a^3 - 27b^2 \pm 54b^2\sqrt{-d})^2 \pm 54b^2\sqrt{-d}}\right)^{\frac{1}{3}} . \tag{D.2.9}$$

So, this Weierstrass model describes a compactification with algebra $D_{16}$. For $d = 0$ one has $k_+ = k_-$, but also $\Delta$ vanishes identically at order 2. This limit is consistent with a Kulikov model of Type IIb, with $d = d(u) = d_0 u$. We should then work with the redefined quantity

$$\Delta_0(s,t) = \left.\frac{\Delta(s,t)}{d^2}\right|_{d=0} , \tag{D.2.10}$$

(the subscript 0 stands for $d = 0$), which is regular as $d \to 0$, with

$$f(s,t)|_{d=0} = -\frac{1}{3}s^2 h(s,t)^2 , \tag{D.2.11}$$

$$g(s,t)|_{d=0} = -\frac{2}{27}s^3 h(s,t)^3 , \tag{D.2.12}$$

$$\Delta_0(s,t) = -s^{18} h(s,t)^2 . \tag{D.2.13}$$

$\Delta_0$ has zeros at $s = 0$, with vanishing orders $ord(f, g, \Delta_0)|_{s=0} = (2, 3, 18)$ and three additional zeros of vanishing orders $ord(f, g, \Delta_0) = (2, 3, 2)$, which in the patch $s = 1$ are at

$$t_1 = -\frac{a}{3b} + \frac{\sqrt[3]{2}a^2}{3bk_0} + \frac{k_0}{3\sqrt[3]{2}b} , \tag{D.2.14}$$

$$t_2 = -\frac{a}{3b} - \frac{(1+i\sqrt{3})a^2}{3\sqrt[3]{4}bk_0} - \frac{(1-i\sqrt{3})k_0}{6\sqrt[3]{2}b} , \tag{D.2.15}$$

$$t_3 = -\frac{a}{3b} - \frac{(1-i\sqrt{3})a^2}{3\sqrt[3]{4}bk_0} - \frac{(1+i\sqrt{3})k_0}{6\sqrt[3]{2}b} , \tag{D.2.16}$$

where $k_0 \equiv k_+(a,b,0) = k_-(a,b,0)$. Physically, the six 7-branes giving six $I_1$ singularities in (D.2.5) pair into three $D_0$ singularities.
Additionally, one can take $b \to 0$, so that, introducing $h_0(s,t) = at^2 s + s^3 = s(at^2 + s^2)$

$$f(s,t)_{b,d=0} = -\frac{1}{3}s^2 h_0(s,t)^2 , \tag{D.2.17}$$

$$g(s,t)|_{b,d=0} = -\frac{2}{27}s^3 h_0(s,t)^3 , \tag{D.2.18}$$

$$\Delta_0(s,t)|_{b=0} = -s^{18} h_0(s,t)^2 . \tag{D.2.19}$$

$\Delta_0(s,t)|_{b=0}$ has zeros at $s = 0$, with vanishing orders $ord(f|_{b,d=0}, g|_{b,d=0}, \Delta_0|_{b=0})\big|_{s=0} = (4, 6, 20)$, which is a non minimal singularity, and two zeros of vanishing orders $ord(f|_{b,d=0}, g|_{b,d=0}, \Delta_0|_{b=0}) = (2, 3, 2)$, which in the patch $s = 1$ are at

$$t'_{1,2} = \pm\frac{i}{\sqrt{a}} . \tag{D.2.20}$$

The singularity becoming non minimal can be interpreted as one of the $D_0$ stacks coming closer to the $D_{16}$ one, the other two having a fixed position far apart from each other and from the $D_{16} + D_0$ system.
The fact that in this case $\Delta$ vanishes at a generic point on the base at order two and there is a non minimal singularity makes this limit consistent with a Kulikov model of Type III.b, with $b = b(u) = b_0 u$. To resolve the non minimal singularity we should consider this Kulikov model explicitly, namely

$$y^2 = x^3 + f_u(s,t)x + g_u(s,t) \tag{D.2.21}$$

with

$$f_u(s,t) = -\frac{1}{3}s^2 h_u(s,t)^2 - d_0 u s^8\,, \tag{D.2.22}$$

$$g_u(s,t) = -\frac{2}{27}s^3 h_u(s,t)^3 - \frac{1}{3}d_0 u s^9 h_u(s,t)\,, \tag{D.2.23}$$

$$h_u(s,t) = b_0 u t^3 + a t^2 s + s^3\,, \tag{D.2.24}$$

with $b_0$, $d_0 \neq 0$. With this parametrisation of the vanishing coefficients it holds

$$\begin{aligned}\Delta_u(s,t) &= 4f_u(s,t)^3 + 27 g_u(s,t)^2 \\ &= -d_0^2 u^2 s^{18}((1+4d_0 u)s^6 + 2a s^4 t^2 + 2b_0 u s^3 t^3 + a^2 s^2 t^4 + 2ab_0 u s t^5 + b_0^2 u^2 t^6)\,.\end{aligned} \tag{D.2.25}$$

Indeed, one sees that at $u = 0$, $s = 0$ there is a non minimal singularity with

$$f_u(s,t)|_{s=u\to 0} = -\frac{1}{3}s^4(h'(s,t)^2 + 3d_0 s^5)\,, \tag{D.2.26}$$

$$g_u(s,t)|_{s=u\to 0} = -\frac{1}{27}s^6 h'(s,t)\left(2h'(s,t)^2 + 9d_0 s^5\right)\,, \tag{D.2.27}$$

$$\Delta(s,t)|_{s=u\to 0} = -d_0^2 s^{22}(h'(s,t)^2 + 4d_0 s^5)\,. \tag{D.2.28}$$

where

$$h_u(s,t)|_{s=u\to 0} = s(b_0 t^3 + a t^2 + s^2) \equiv s h'(s,t)\,, \tag{D.2.29}$$

so that the vanishing orders in the Kulikov model framework correspond to a non minimal singularity in the Kodaira classification, $ord(f|_{s=u\to 0}, g|_{s=u\to 0}, \Delta|_{s=u\to 0}) = (4,6,22)$. On the other hand, at $u = 0$, $t = 0$ the threefold has an $I_2$ minimal singularity

$$f_u(s,t)|_{t=u\to 0} = -\frac{1}{3}s^2\left(h_u(s,t)|_{t=u\to 0}\right)^2 - d_0 t s^8\,, \tag{D.2.30}$$

$$g_u(s,t)|_{t=u\to 0} = -\frac{2}{27}s^3\left(h_u(s,t)|_{t=u\to 0}\right)^3 - \frac{1}{3}d_0 t s^9\left(h_u(s,t)|_{t=u\to 0}\right)\,, \tag{D.2.31}$$

$$\Delta(s,t)|_{t=u\to 0} = -d_0^2 t^2 s^{18}(s^6 + 4d_0 s^6 t + 2a s^4 t^2 + a^2 s^2 t^4 + 2b_0 s^3 t^4 + 2ab_0 s t^6 + b_0^2 t^8)\,, \tag{D.2.32}$$

with

$$h_u(s,t)|_{t=u\to 0} = b_0 t^4 + a t^2 s + s^3\,, \tag{D.2.33}$$

which does not display any further factorisation of $t$ factors.
To get rid of the non minimal singularity one performs the base blowup

$$s \to s e_1 \,, \quad u \to e_0 e_1 \,, \tag{D.2.34}$$

accompanied by the rescalings (preserving the Calabi-Yau condition)

$$\begin{aligned} f_{e_0}(s,t,e_1) &\to \frac{f_{e_0}(s,t,e_1)}{e_1^4} \,, \\ g_{e_0}(s,t,e_1) &\to \frac{g_{e_0}(s,t,e_1)}{e_1^6} \,. \end{aligned} \tag{D.2.35}$$

After these rescalings, it holds

$$\Delta_{e_0}(s,t,e_1) = -d_0^2 e_0^2 e_1^{10} s^{18} \delta_{e_0}(s,t,e_1) \tag{D.2.36}$$

where $\delta_{e_0}(s,t,e_1)$ cannot be further factorised, and for completeness it reads

$$\delta_{e_0}(s,t,e_1) = e_1^4 s^6 + 4 d_0 e_0 e_1^5 s^6 + 2 a e_1^2 s^4 t^2 + 2 b_0 e_0 e_1^2 s^3 t^3 + a_0^2 s^2 t^4 + 2 a b_0 e_0 s t^5 + b_0^2 e_0^2 t^6 \,. \tag{D.2.37}$$

The blown up model has a minimal singularity for $e_1 = s = 0$, which from (D.2.35) and (D.2.36) can be seen to be of the type $ord(f_{e_0}(s,t,e_1)|_{s=e_1\to 0}, g_{e_0}(s,t,e_1)|_{s=e_1\to 0}, \Delta_{e_0}(s,t,e_1)|_{s=e_1\to 0}) = (2,3,28)$, so the base cannot be further blown up.
In order to read the physical singularities, namely the ones corresponding to 7-branes, one should define the $K3$ discriminant

$$\Delta'_{e_0}(s,t,e_1) = \frac{\Delta_{e_0}(s,t,e_1)}{e_0^2 e_1^{10}} \,. \tag{D.2.38}$$

(D.2.35) and (D.2.36) describe a degenerate $K3$ surface. The base degenerates in a chain of two intersecting surfaces

- $\{e_0 = 0\}$ with generic $I_2$ fiber. Restricting to this component we can set $s = 1$ since $s$ and $e_0$ cannot vanish simultaneously and

$$\begin{aligned} f_0(1,t,e_1) &= -\frac{1}{3}(e_1^2 + a t^2)^2 \,, \\ g_0(1,t,e_1) &= -\frac{2}{27}(e_1^2 + a t^2)^3 \,, \\ \Delta'_0(1,t,e_1) &= -d_0^2 (e_1^2 + a t^2)^2 \,. \end{aligned} \tag{D.2.39}$$

  In this component there are two minimal singularities with vanishing orders given by $ord(f_0(1,t,e_1), g_0(1,t,e_1), \Delta'_{e_0}(1,t,e_1)) = (2,3,2)$ corresponding to two $D_0$ singularities, separated one from the other.

- $\{e_1 = 0\}$ with generic $I_{10}$ fiber. Restricting to this component we can set $t = 1$ and

$$\begin{aligned} f_{e_0}(s,1,0) &= -\frac{1}{3}s^2(b_0e_0 + as)^2\,, \\ g_{e_0}(s,1,0) &= -\frac{2}{27}s^3(b_0e_0 + as)^3\,, \\ \Delta'_{e_0}(s,1,0) &= -d_0^2 s^{18}(b_0e_0 + as)^2\,, \end{aligned} \tag{D.2.40}$$

  so that in this component there is one $D_{16}$ singularity $ord(f_{e_0}(s,s,0), g_{e_0}(s,1,0), \Delta'_{e_0}(s,1,0)) = (2,3,18)$ and one $D_0$ singualrity as above.

The monodromy of the $D_{16} + D_0$ system is

$$M_{D_{16}/D_0} = \begin{pmatrix} 1 & -8 \\ 0 & 1 \end{pmatrix}\,, \tag{D.2.41}$$

so indeed it supports only one invariant $(1,0)$ string surrounding the brane configuration, corresponding to only one imaginary root extending the algebra to $\widehat{D}_{16}$. As already mentioned, the second one cannot be detected in this framework.

## D.3 Heterotic/F-theory duality map

Let us now examine how the F-theory and heterotic frames differ in the way they capture the decompactification limits. More concretely, as we will show, certain decompactification limits which can be described naturally from the point of view of the heterotic string cannot be encoded in the framework of Kulikov Weierstrass models, as these involve functions that are not rational. We work with an elliptic K3 surface with two generic $E_8$ singular fibers, described by the Weierstrass model

$$y = x^3 + as^4t^4x + t^5s^5(t^2 + bst + ds^2)\,, \tag{D.3.1}$$

where $a$, $b$, $d \in \mathbb{C}$. The dual $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string background is simply characterized by a vanishing Wilson line, with complex structure $\tau$ and complexified Kähler modulus $\rho$ arbitrary. The exact relation between the geometric and heterotic moduli is then [148]

$$j(\tau)j(\rho) = -1728^2\frac{a^3}{27d}\,, \qquad (j(\tau) - 1728)(j(\rho) - 1728) = 1728^2\frac{b^2}{4d}\,, \tag{D.3.2}$$

where $j(z)$ is the j-invariant modular function.
Consider a decompactification limit from eight to ten dimensions with square torus and vanishing B-field. We can parametrize

$$\tau = i\frac{R_8}{R_9} = \frac{i}{u^n}\,, \qquad\qquad \rho = iR_8R_9 = \frac{i}{u^m}\,, \tag{D.3.3}$$

with $u \in \mathbb{R}$ and $m > n > 0$, so that

$$-\tau\rho = R_8^2 = \frac{1}{u^{m+n}}\,, \qquad\qquad \frac{\rho}{\tau} = R_9^2 = \frac{1}{u^{m-n}}\,. \tag{D.3.4}$$

It follows that as $u$ goes to zero, $R_8$ diverges faster than $R_9$, describing a decompactification from eight to ten dimensions at which the two radii grow at different asymptotic rates. We consider other values for $m$ and $n$ below.

Now we ask how this decompactification with different rates is described in F-theory. From the map (D.3.2) we get the equations

$$\begin{aligned} \frac{a^3}{d} + \frac{b^2}{d} &\sim j(i/u^n) + j(i/u^m) \\ &\sim e^{\frac{2\pi}{u^n}} + e^{\frac{2\pi}{u^m}} \end{aligned} \tag{D.3.5}$$

and

$$\frac{b^2}{a^3} \sim 1 - 1728 \frac{e^{\frac{2\pi}{u^n}} + e^{\frac{2\pi}{u^m}}}{e^{\frac{2\pi}{u^n} + \frac{2\pi}{u^m}}}\,, \tag{D.3.6}$$

where we have used the expansion $q(z) = e^{-2\pi i z} + 744 + \cdots$ and kept only the leading orders as $u \to 0$. By redefining the degeneration parameter $u \mapsto u' = f(u)$, any of these equations can be made algebraic. However, this cannot be done for both equations at once, and so the parameters $a, b, c$ cannot be expressed as rational functions of any degeneration parameter; this makes it impossible to write down a Kulikov Weierstrass model of this kind describing the associated decompactification limit.

If we set $m = n$ instead, from eq. (D.3.4) we see that only $R_8$ goes to infinity, $R_9$ remaining constant. In turn, equations (D.3.5) and (D.3.6) can be written as

$$\frac{a^3}{d} + \frac{b^2}{d} \sim 2e^{\frac{2\pi}{u^n}}\,, \qquad \frac{b^2}{a^3} \sim 1 - 1728 \frac{2}{e^{\frac{2\pi}{u^n}}}\,, \tag{D.3.7}$$

hence reparametrizing $u \to u' = \exp(2\pi/u^n)$ makes both equations algebraic and so, as required for consistency, a Kulikov Weirstrass model description exists. A similar situation arises if we set $n = 0$, which corresponds to the two radii diverging at the same rate. The remaining choices for $m$ and $n$ come from exchanging $m \leftrightarrow n$ in the cases already studied and correspond to T-dual frames in which one of the radii goes to zero instead. The map (D.3.2) is T-duality invariant and so the conclusions are the same.

# Appendix E

# Chevalley groups

In this Appendix, we expand on concepts introduced in Section 6.2 and in particular summarize some concepts related to Chevalley groups, in connection with the string theory realizations of the T- and U-duality groups.

## E.1 Definition

Assume $G$ is semi-simple. Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$ be the complexified semi-simple Lie algebra of G, $r$ its rank, $\Phi$ the set of roots and $\Sigma$ the set of simple roots. In the Cartan-Weyl basis, it is generated by $\mathcal{B}_{CW} = \{H_i, X_\alpha\}_{i\in\Sigma,\alpha\in\Phi}$, where $\alpha$ are roots, $X_\alpha$ are nilpotent and $H_i$ generate the maximal torus.
The Chevalley basis $\mathcal{B}_{Ch} = \{\hat{H}_i, \hat{X}_\alpha\}_{i\in\Sigma,\alpha\in\Phi}$ corresponds to a rescaling of the Cartan-Weyl basis so that the eigenvalues of all elements with respect to the maximal torus are all integers. This is effectively a rescaling of the structure constants in the Cartan-Weyl basis and is possible due to a theorem by Chevalley [149, 150]. It is unique up to automorphisms of $\mathfrak{g}$ and signs of $\hat{X}_\alpha$. The algebra satisfies

$$\begin{array}{cc} [\hat{H}_\alpha, \hat{H}_\beta] = 0 & \alpha, \beta \in \Sigma\,, \\ [\hat{H}_\alpha, \hat{X}_\beta] = A_{\alpha\beta}\hat{X}_\beta & \alpha \in \Sigma, \beta \in \Phi\,, \\ [\hat{X}_\alpha, \hat{X}_{-\alpha}] = \hat{H}_\alpha & \alpha \in \Sigma\,, \\ [\hat{X}_\alpha, \hat{X}_\beta] = B_{\alpha\beta}\hat{X}_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi\,, \\ [\hat{X}_\alpha, \hat{X}_\beta] = 0 & \text{if } \alpha+\beta \notin \Phi\,. \end{array} \tag{E.1.1}$$

Notice that $A_{\alpha\beta} = 2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$, and by construction they are the components of the Cartan matrix. The structure constants $B_{\alpha\beta}$ are complex numbers in the Cartan-Weyl basis, but are integers in the Chevalley basis. The reason for doing this is that the algebra can now be fully defined over the integers. We call $\mathfrak{g}_{\mathbb{Z}}$ the $\mathbb{Z}$-module generated by $\mathcal{B}_{Ch}$, $\mathfrak{g}_{\mathbb{Z}} = \langle\mathcal{B}_{Ch}\rangle_{\mathbb{Z}}$, the $\mathbb{Z}$-form of the Lie algebra. We find $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$.
Clearly, $B_{\alpha\beta} \in \mathbb{Z}$ implies $\alpha \in \mathbb{Z}^r$ for all $\alpha \in \Phi$. This defines the $\mathbb{Z}$-form of the root space and generates an integer lattice.
We can now define *Chevalley groups*. First, we note that for all $\hat{X}_\alpha \in \mathcal{B}_{Ch}$, $(\mathrm{ad}_{\hat{X}_\alpha})^n = 0$ for

some $n \in \mathbb{N}$. We can define a unipotent automorphism of $\mathfrak{g}$,

$$x_\alpha^{\mathrm{ad}}(t) = \exp(tX_\alpha) = \sum_{n=0}^{\infty} (t\,\mathrm{ad}_{X_\alpha})^n/n!\,, \tag{E.1.2}$$

which terminates at finite $n = \bar{n}$. If $t \in \mathbb{Z}$, $x_\alpha^{\mathrm{ad}}(t)$ maps $\mathfrak{g}_\mathbb{Z}$ to itself, so that one can define the elementary adjoint Chevalley group of type $\Phi$ as the group of automorphisms of the Chevalley algebra $\mathfrak{g}_\mathbb{Z}$

$$E_{\mathrm{ad}}(\Phi, \mathbb{Z}) = \langle x_\alpha(t) : \alpha \in \Phi, t \in \mathbb{Z} \rangle \subset GL(\mathbb{Z}) = \mathrm{Aut}(\mathfrak{g}_\mathbb{Z})\,, \tag{E.1.3}$$

and is a linear algebraic group. Notice that it depends crucially on the existence of an admissible $\mathbb{Z}$-form of the adjoint representation[1]. The automorphism group of the associated group $G$ is a semidirect product of inner and outer automorphisms

$$\mathrm{Aut}(G) = \mathrm{In}(G) \ltimes \mathrm{Out}(G)\,, \tag{E.1.5}$$

where $\mathrm{In}(G) \cong \mathrm{ad}(G)$ and $\mathrm{Out}(G)$ is finite and contains the symmetries induced by the Dynkin diagram [149, 150].

## E.2 Lattices and Arithmetic Groups

Finite $\mathfrak{g}$-modules (modules with a $\mathfrak{g}$ action on them) contain $\mathbb{Z}$-lattices. Indeed, let $V$ be such a module and $\rho$ a representation of $\mathfrak{g}_\mathbb{Z}$ on $V$, there exists a $\mathbb{Z}$-lattice $\Lambda \subset V$ which is invariant under the group generated by

$$x_\alpha : \mathbb{Z} \to GL(V) \quad \forall\, \alpha \in \Phi\,. \tag{E.2.1}$$

It is the direct sum of the weight components, or eigenspaces of definite weight and isomorphic to $V_\mathbb{Z}$. The $\mathbb{Z}$-lattice is called an *admissible $\mathbb{Z}$-form* of the module $V$. Let $\mathcal{W}(\rho)$ be the set of weights of the $\mathfrak{g}$ action on $V$. A basis

$$(v^\lambda : \lambda \in \mathcal{W}(\rho))$$

is called the admissible basis of the lattice $V_\mathbb{Z}$ and contains all the weight vectors.
In the case $V = \mathbb{Q}$ and for $G(\mathbb{Q})$ algebraic group defined over the rationals, the group generated by (E.2.1) is an arithmetic subgroup of $G(\mathbb{Q})$. It is in fact the Chevalley arithmetic subgroup over $\mathbb{Z}$, $G(\mathbb{Z})$, which, in many cases, turns out to be precisely the discrete U- or T-duality groups defining string theory moduli spaces. In the case where $G$ is a simple Lie group, it also corresponds to the Coxeter groups of the root system of $G$.

[1] The construction of such a group is not unique to the adjoint representation. Indeed, if there exists an admissible $\mathbb{Z}$-form of a representation $\rho$ on a finite dimensional vector space $V$, then one can define the *Chevalley group* over a ring $\mathcal{R}$ as the group generated by the set of homomorphisms

$$x_\alpha : \mathcal{R} \to GL(V) : \quad \forall \alpha \in \Phi\,. \tag{E.1.4}$$

# Appendix F

# Introduction en français

## Synthèse

Dans cette thèse, dans le cadre du programme « Swampland », nous nous concentrons sur certains aspects liés à la « Distance Conjecture » et à la présence de tours infinies d'états légers à une distance infinie dans l'espace des modules.
Tout d'abord, nous analysons les frontières des espaces de modules de compactifications de la corde hétérotique sur des tores à $d$ dimensions. Nous calculons les algèbres de courant de symétrie sur la surface d'univers de la corde à mesure que nous approchons de toutes les limites de distance infinie qui correspondent à des limites de décompactification (éventuellement partielle) dans un certain cadre dual. Lors de la décompactification de $k$ directions du $T^d$, nous constatons que les tours infinies d'états devenant légers augmentent l'algèbre apparaissant en un point donné de l'espace de modules de la compactification $T^{d-k}$ à sa version à $k$-fois affine du point de vue de la dimension inférieure. Les extensions centrales sont données par les $k$ vecteurs de Kaluza-Klein, et dans le cas de groupes de jauge semi-simples, elles rendent tous les facteurs affines en même temps. Dans la théorie hétérotique $\mathrm{E}_8 \times \mathrm{E}_8$ sur $S^1$, nous prouvons également que ces tours de modes satisfont la « Weak Gravity Conjecture » et la « Repulsive Force Conjecture ». De plus, nous mettons un accent particulier sur $d = 2$ et ses duals en théorie F. Pour les compactifications sur $T^2$, nous reproduisons toutes les algèbres affines qui apparaissent dans le dual en théorie F et montrons explicitement toutes les tours, y compris certaines qui ne sont pas manifestes dans leurs homologues en théorie F. En outre, nous construisons l'algèbre affine $SO(32)$ apparaissant dans la limite de décompactification totale, à la fois dans le cadre hétérotique et dans le cadre de la théorie F, montrant que des algèbres affines de type exceptionnel apparaissent dans cette dernière.
Nous considérons ensuite un orbifold construit à partir de la théorie hétérotique $\mathrm{E}_8 \times \mathrm{E}_8$ en 9 dimensions sur $S^1$, à savoir la corde CHL, et analysons l'algèbre espace-temps des états BPS qui émerge dans la limite de distance infinie à mesure que la théorie se décompactifie, en nous concentrant particulièrement sur la limite de décompactification totale vers 10 dimensions. En particulier, nous montrons que les décompactifications du cycle lié à l'orbifold conduisent à des versions tordues de l'algèbre affine. De plus, par le type d'algèbres affines qui peuvent être trouvées à la frontière de l'espace des modules CHL, nous pouvons clairement voir que la théorie en 9 dimensions ne peut se décompactifier qu'en la théorie hétérotique $\mathrm{E}_8 \times \mathrm{E}_8$ en 10 dimensions, tandis que celle en 8 dimensions peut également se décompactifier en la théorie en

10 dimensions $\mathrm{Spin}(32)/\mathbb{Z}_2$.
Enfin, nous trouvons des géodésiques et caractérisons la frontière de distance infinie dans les espaces de modules symétriques (de type quotient $G/K$), comme ceux qui apparaissent dans les compactifications avec supersymétrie maximale ou semi-maximale, respectivement du type $E_{d(d)}/K$ et $O(d,d')/(O(d)\times O(d'))$, quotients par le groupe discret de U- ou T-dualité. Les points de distance infinie sont caractérisés par des sous-groupes paraboliques rationnels de $G$: en raison du quotient par le groupe discret, les points de distance infinie ne sont atteints que par un ensemble (de mesure nulle) de géodésiques très particulières, tandis que la plupart d'entre elles ont un mouvement ergodique, n'atteignant que des points de distance finie. En supposant la complétude du spectre, nous montrons qu'il y a toujours une tour d'états devenant légers de manière exponentielle en la distance géodésique, prouvant ainsi la ≪ Distance Conjecture ≫ dans ces configurations.

## Introduction

L'un des aspects les plus convaincants de la physique, et une pierre angulaire de son développement en tant que science, est la quête d'unification – trouver les lois les plus fondamentales de la nature pour décrire, dans un cadre unique, la gamme la plus large possible de phénomènes à différentes échelles. Au cours des dernières décennies, cela est devenu un effort théorique – et technique – de plus en plus avancé, visant à formuler une "théorie du tout" qui pourrait expliquer tout ce que nous savons sur la nature et plus encore. Bien que cela puisse sembler éloigné de ce que nous pouvons généralement expérimenter du monde, et en quelque sorte d'un intérêt principalement philosophique, c'est en réalité profondément ancré dans les observations de la nature. En effet, le premier grand pas vers l'unification est venu de la réalisation de Newton que la force gouvernant la chute d'une pomme et le mouvement des corps célestes est décrite par la même loi gravitationnelle. Un autre exemple similaire est donné par l'électromagnétisme, introduit après que l'électricité et le magnétisme aient été reconnus comme étant simplement deux manifestations différentes d'un seul champ électromagnétique sous-jacent, tel que décrit par les lois de Maxwell.
Cette synthèse a ouvert la voie à une étape fondamentale vers le processus d'unification, qui a culminé avec la naissance du Modèle Standard (MS) de la physique des particules. Dans la seconde moitié du siècle dernier, ils ont découvert quels sont les constituants de la nature aux échelles d'énergie que nous pouvons sonder au LHC, et ils ont caractérisé les interactions entre les différentes particules, qui sont l'électromagnétisme, les forces faible et forte, et la gravité.
En laissant de côté la gravité pour le moment, notre capacité à faire des prédictions très précises qui peuvent être testées expérimentalement est due au développement de la Théorie Quantique des Champs (TQC) [4–6], qui permet de décrire les systèmes de Mécanique Quantique (MQ) dans un cadre relativiste spécial. C'est dans ce langage qu'il est devenu évident que toutes les forces connues pouvaient être décrites en se basant sur le concept de symétries en utilisant le même cadre mathématique, celui des théories de jauge. Avec cette perspective, il a été compris que l'électromagnétisme et la force faible ne sont que deux manifestations à basse énergie de la soi-disant "force électrofaible", et bien qu'il n'y ait pas de preuve expérimentale d'une unification supplémentaire de la force forte à des échelles d'énergie plus élevées, on peut utiliser les outils de la TQC pour faire différentes propositions pour cette interaction encore

plus “fondamentale” possible.

En dépit du succès énorme de la TQC et du MS pour rendre compte de la physique des collisionneurs, pour laquelle les effets gravitationnels sont insignifiants, il semble que ce paradigme d'unification échoue à intégrer également la gravité. La formulation actuelle des interactions gravitationnelles est la Relativité Générale (RG) d'Einstein [7] (voir aussi [8]), une description semi-classique qui est très précise à nos échelles cosmologiques, mais qui ne peut pas être étendue au-delà du domaine de la théorie des champs classiques car elle est non-renormalisable en 4 dimensions, et donc elle ne peut pas être quantifiée de manière cohérente. Malgré ce défi apparent, il est naturel de chercher une théorie unique qui se réduise à la RG et au MS dans les régimes appropriés, et qui soit capable de décrire toutes ces forces au niveau quantique. Cela nous permettrait d'expliquer des phénomènes tels que les trous noirs et le Big Bang, où les interactions gravitationnelles sont censées être cruciales, et fortes à des échelles très courtes.

Le principal candidat pour un cadre qui offre une formulation cohérente de la Gravité Quantique (GQ) en plus de toutes les autres interactions connues est la Théorie des Cordes (TC) [9, 10], qui repose sur l'hypothèse qu'au niveau le plus fondamental, les constituants de la nature ne sont pas ponctuels, mais unidimensionnels, c'est-à-dire des cordes. Le fait que, étant des objets étendus, elles sont associées à une échelle de longueur non-triviale donne à la gravité un moyen de couper les divergences UV d'une manière qui évite le problème de la non-renormalisabilité. Les oscillations des cordes dans l'espace-temps donnent naissance à différents états ; ceux-ci incluent des particules aux propriétés similaires à celles décrites par le MS, telles que par exemple les bosons de jauge et les champs de matière, mais en plus de cela, le spectre de la théorie des cordes inclut toujours une particule de spin deux sans masse qui possède les caractéristiques que l'on attendrait du graviton, le “champ de jauge” de la gravité.
Dans sa formulation supersymétrique, la cohérence interne de la TC exige qu'elle soit formulée en dix dimensions spatio-temporelles. Malgré cette rigidité, il existe encore cinq théories de cordes possibles qui peuvent être construites en 10 dimensions, présentant des cordes fermées et parfois ouvertes selon les conditions aux limites que nous leur imposons. Il s'agit des théories des Cordes de Type I, Type IIA et IIB, Hétérotique $SO(32)$ et Hétérotique $E_8 \times E_8$. Elles sont censées être profondément reliées à une théorie à onze dimensions, appelée M-théorie, qui à basse énergie se réduit à la supergravité en onze dimensions, comme le montre la Figure F.1.

Bien qu'il s'agisse de théories différentes et non liées en 10 dimensions, il s'avère qu'elles sont toutes connectées (ou plus précisément, “duales” les unes aux autres) une fois que l'on compacte la théorie, c'est-à-dire lorsque certaines des dix directions spatiales sont compactifiées. Cela est par ailleurs nécessaire pour relier la TC à la phénoménologie, car il faut tenir compte du fait que nous ne percevons qu'un espace-temps à quatre dimensions. Pour y parvenir, six des dix dimensions doivent être compactifiées sur une variété interne dont l'échelle de longueur caractéristique est inférieure à la sensibilité expérimentale actuelle, de sorte qu'elles n'auraient pas pu être détectées jusqu'à présent, de telle sorte qu'effectivement la théorie n'a que 4 directions “externes” étendues. Cela introduit un grand nombre d'espaces de compactification possibles qui, à leur tour, donnent un grand nombre de théories effectives à basse énergie possibles, puisque le contenu en particules et les interactions dépendent spécifiquement de la géométrie de la variété compacte choisie. En particulier, on peut considérer une classe plus large de vides sans se limiter nécessairement à quatre dimensions en choisissant une variété

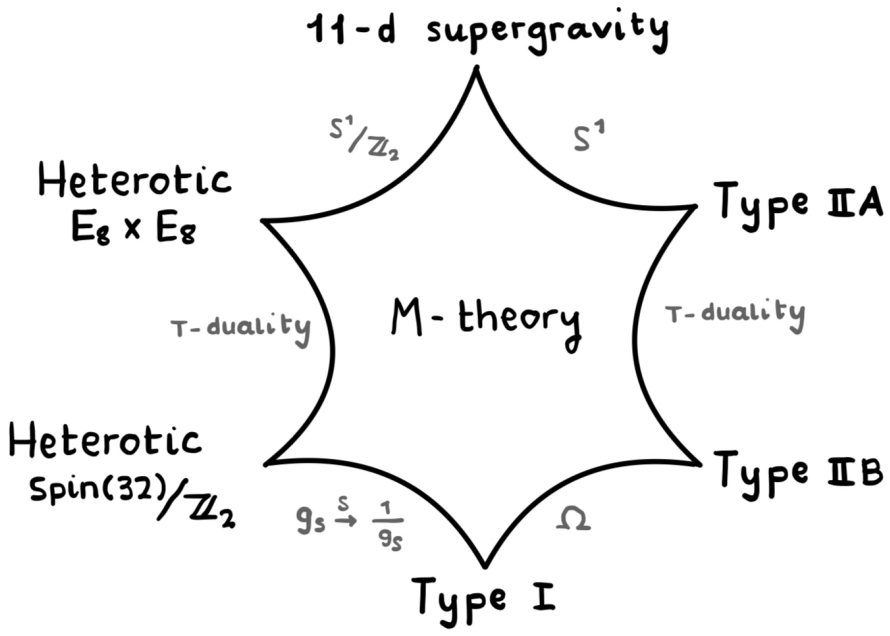



**Figure F.1:** La dualité entre les théories des cordes à 10 dimensions et la théorie M, dont la description à basse énergie est celle de la supergravité $\mathcal{N} = 1$ (maximale) à 11 dimensions.

compacte avec n'importe quelle dimension comprise entre 1 et 9.[1] En raison de la nécessité de compactification, une caractéristique commune des EFTs de cordes est la présence de moduli, c'est-à-dire des champs scalaires sans masse sans potentiel qui ne sont pas stabilisés.[2] Du point de vue de la supergravité, on peut les considérer comme des paramètres sur un espace métrique de vides de cordes connectés ayant la même énergie, appelé *espace des moduli*.

L'ensemble des différentes théories de champs effectifs (EFTs) qui peuvent être obtenues par compactifications de cordes est connu sous le nom de "Paysage" [12], et il est très vaste, avec des estimations initiales de l'ordre de $\sim 10^{500}$ théories. Cela peut soulever la question de savoir si la TC a réellement un pouvoir prédictif, car elle semble capable de décrire une très grande quantité de phénomènes à basse énergie. Dans cette optique, une direction de recherche importante pour tester l'efficacité de la TC dans la description de la phénoménologie est d'essayer de comprendre les propriétés génériques des théories qui en découlent, qui se sont récemment avérées être en réalité assez contraintes. Dans cette veine, le programme Swampland [13] vise à comprendre les caractéristiques qu'une théorie de champs effective doit posséder pour être cohérente avec une complétion UV en gravité quantique, en séparant les théories à basse énergie qui le peuvent (appartenant au Paysage) de celles qui ne le peuvent pas (le soi-disant Swampland) par une série de conjectures. Celles-ci sont proposées et testées – et parfois prouvées/réfutées – soit à partir de constructions explicites de cordes (en mettant en évidence des motifs communs apparaissant dans les compactifications de cordes), soit à partir

[1] La direction temporelle ne peut pas être compactifiée.

[2] Étant donné que les moduli donneraient lieu à des interactions à longue portée, ils doivent être stabilisés, c'est-à-dire qu'ils doivent avoir une valeur d'attente au vide fixe, ce qui peut être fait dans la TC, par exemple en ajoutant des flux, voir par exemple [11]. Cela dépasse le cadre de cette thèse et ne sera pas abordé davantage.

d'arguments heuristiques, venant généralement de la physique des trous noirs, ou en utilisant l'holographie dans le contexte, par exemple, de AdS/CFT, dans le but de mieux comprendre les implications à basse énergie de la gravité quantique. Malgré le fait que toutes les EFTs cohérentes puissent apparemment être couplées à la RG par couplage minimal et décrire des théories gravitationnelles, celles-ci doivent obéir à des contraintes supplémentaires pour être étendues à une gravité quantique complète.

### Plan de la thèse

L'une des affirmations les plus largement acceptées du programme Swampland est la présence universelle de tours infinies d'états qui deviennent légers en s'approchant des frontières des espaces de moduli. Cette caractéristique des théories de gravité quantique (QG) a été initialement suggérée dans [14] avec la formulation de la *Distance Conjecture* (SDC), qui sera présentée plus en détail par la suite. Cette thèse se concentre principalement sur deux aspects liés à cette conjecture, en abordant également certaines autres conjectures connexes, et est structurée comme suit.

- Dans le Chapitre 2, nous introduirons et commenterons la SDC, ainsi que d'autres conjectures qui seront mentionnées par la suite et qui sont également liées à la présence de tours infinies d'états avec des propriétés spécifiques, à savoir l'*Hypothèse de Complétude* [15,16], la *Conjecture de la Gravité Faible (sur un Réseau)* (LWGC) [17–19] et la *Conjecture de la Force Répulsive (sur un Réseau)* (LRFC) [20,21], qui y est étroitement liée.

- La Partie II est liée à la caractérisation de ces limites de distance infinie qui correspondent à des limites de décompactification dans les théories de cordes hétérotiques et CHL compactifiées sur des tores, qui présentent des secteurs de jauge perturbatifs, en se basant sur les symétries qui peuvent apparaître.
  En particulier, la corde hétérotique en dix dimensions peut avoir soit la symétrie $\mathrm{E}_8 \times E_8$, soit $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2}$, tandis que la corde CHL est une construction intrinsèquement neuf-dimensionnelle issue d'un orbifold de la théorie $\mathrm{E}_8 \times E_8$ compactifiée sur un cercle. Nous les présentons et leurs compactifications toroïdales dans le Chapitre 3, afin de définir les notations et de décrire le phénomène d'améliorations de symétrie qui peut se produire à différents points à l'intérieur de l'espace de moduli, afin de le relier au comportement à ses frontières. Cela sera fait dans le Chapitre 4 pour le cas hétérotique et dans le Chapitre 5 pour le cas CHL ; nous nous concentrons sur les limites de décompactification de ces théories, qui apparaissent dans les régions de distance infinie de l'espace de moduli où certains ou tous les rayons du tore sont pris pour avoir une taille infinie (jusqu'aux dualités T), et nous discutons des améliorations de symétrie liées qui peuvent survenir. Le contexte mathématique sur ce type d'algèbres, connues sous le nom d'algèbres de Lie affines, est résumé dans l'Annexe A, tandis que les détails des calculs peuvent être trouvés dans l'Annexe B. De plus, pour le cas spécifique de la corde hétérotique $\mathrm{E}_8 \times E_8$ sur $S^1$, nous commentons explicitement sur la façon dont la tour d'états s'inscrit dans le cadre de la LWGC, de la LRFC et de la SDC dans le Chapitre 4.2, en laissant les détails du calcul pour l'Annexe C. Enfin, dans le Chapitre 4.5, nous montrons que pour les compactifications sur $T^2$, les résultats hétérotiques correspondent à ceux obtenus dans le cadre dual de la théorie F sur K3 dans le contexte des modèles de Kulikov et des dégénérescences

de la structure complexe K3 [22, 23], et comment toutes les différentes caractéristiques apparaissent. Quelques éléments de contexte supplémentaires sont donnés dans l'Annexe D.

- Dans la Partie III, nous nous concentrons sur la SDC dans le cas des compactifications qui ont un espace de moduli globalement symétrique, comme ceux des compactifications toroïdales de la théorie M et de la théorie des cordes avant l'action de la dualité U ou T, mais qui couvrent en réalité une classe plus large. Après avoir présenté les outils mathématiques pour caractériser et paramétrer la frontière de tels espaces dans la Section 6.1.2, nous montrons comment cela peut être utilisé pour prouver explicitement la SDC dans ces contextes.

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