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Handling derivatives risks with XVAs in a one-period network model

Dorinel-Marian Bastide

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Handling derivatives risks with XVAs in a one-period network model

*Gestion des risques des produits dérivés avec XVAs dans un modèle
de réseau à une période*

École doctorale n° 574 mathématiques Hadamard (EDMH)
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Graduate School : Université Paris-Saclay GS Mathématiques
Réfèrent : Université d'Évry Val d'Essonne

Thèse préparée dans le **laboratoire de Mathématiques et Modélisation d'Évry** (Université Paris-Saclay, CNRS, Univ Évry), sous la direction de **Stéphane Crépey**, Professeur en Mathématiques appliquées à l'Université Paris Cité et la co-direction de **Emmanuel Gobet**, Professeur en Mathématiques appliquées à l'École polytechnique

Thèse soutenue à Paris-Saclay, le 25 septembre 2024, par

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Titre: Gestion des risques des produits dérivés avec XVAs dans un modèle de réseau à une période.....

Mots clés: Modélisation financière, Processus stochastiques en grande dimension, simulation d'évènements rares, Monte Carlo par Chaîne de Markov

Résumé: La réglementation requiert des établissements bancaires d'être en mesure de conduire des analyses de scénarios de tests de résistance (stress tests) réguliers de leurs expositions, en particulier face aux chambres de compensation (CCPs) auxquels ils sont largement exposés, en appliquant des chocs de marchés pour capturer le risque de marché et des chocs économiques pouvant conduire à l'état de faillite, dit aussi de défaut, divers acteurs financiers afin de refléter les risques de crédit et de contrepartie. Un des rôles principaux des CCPs est d'assurer par leur interposition entre acteurs financiers la réduction du risque de contrepartie associé aux pertes potentiels des engagements contractuels non respectés dus à la faillite d'une ou plusieurs des parties engagées. Elles facilitent également les divers flux financiers des activités de trading même en cas de défaut d'un ou plusieurs de leurs membres en re-basculant certaines des positions de ces membres et en allouant toute perte qui pourrait se matérialiser suite à ces défauts aux membres survivants. Pour développer une vision juste des risques et disposer d'outils performants de pilotage du capital, il apparaît essentiel d'être en mesure d'appréhender de manière exhaustive les pertes et besoins de liquidités occasionnés par ces divers chocs dans ces réseaux financiers ainsi que d'avoir une compréhension précise des mécanismes sous-jacents. Ce projet de thèse aborde différentes questions de modélisation permettant de refléter ces besoins, qui sont au cœur de la gestion des risques d'une banque dans les environnements actuels de trad-

ing centralisé. Nous commençons d'abord par définir un dispositif de modèle statique à une période reflétant les positions hétérogènes et possibilité de défauts joints de multiples acteurs financiers, qu'ils soient membres de CCPs ou autres participants financiers, pour identifier les différents coûts, dits de XVA, générés par les activités de clearing et bilatérales avec des formules explicites pour ces coûts. Divers cas d'usage de ce dispositif sont illustrés avec des exemples d'exercices de stress test sur des réseaux financiers depuis le point de vue d'un membre ou de novation de portefeuille de membres en défaut sur des CCPs avec les autres membres survivants. Des modèles de distributions à queues épaisses pour générer les pertes sur les portefeuilles et les défauts sont privilégiés avec l'application de techniques de Monte Carlo en très grande dimension accompagnée des quantifications d'incertitudes numériques. Nous développons aussi l'aspect novation de portefeuille de membres en défauts et les transferts de coûts XVA associés. Ces novations peuvent s'exécuter soit sur les places de marchés (exchanges), soit par les CCP elles-mêmes qui désignent les repreneurs optimaux ou qui mettent aux enchères les positions des membres défaillants avec des expressions d'équilibres économiques. Les défauts de membres sur plusieurs CCPs en commun amènent par ailleurs à la mise en équation et la résolution de problèmes d'optimisation multidimensionnelle du transfert des risques abordées dans ces travaux.

Title: Handling derivatives risks with XVAs in a one-period network model.....

Keywords: Rare events simulation, Markov chain Monte Carlo methods, Financial modelling, Stochastic processes with large dimension

Abstract: Finance regulators require banking institutions to be able to conduct regular scenario analyses to assess their resistance to various shocks (stress tests) of their exposures, in particular towards clearing houses (CCPs) to which they are largely exposed, by applying market shocks to capture market risk and economic shocks leading some financial players to bankruptcy, known as default state, to reflect both credit and counterparty risks. By interposing itself between financial actors, one of the main purposes of CCPs are to limit counterparty risk due to contractual payment failures due to one or several defaults among engaged parties. They also facilitate the various financial flows of the trading activities even in the event of default of one or more of their members by re-arranging certain positions and allocating any loss that could materialize following these defaults to the surviving members. To develop a relevant view of risks and ensure effective capital steering tools, it is essential for banks to have the capacity to comprehensively understand the losses and liquidity needs caused by these various shocks within these financial networks as well as to have an understanding of the underlying mechanisms. This thesis project aims at tackling modelling issues to answer those different needs that are at the heart of risk management practices for banks un-

der clearing environments. We begin by defining a one-period static model for reflecting the market heterogeneous positions and possible joint defaults of multiple financial players, being members of CCPs and other financial participants, to identify the different costs, known as XVAs, generated by both clearing and bilateral activities, with explicit formulas for these costs. Various use cases of this modelling framework are illustrated with stress test exercises examples on financial networks from a member's point of view or innovation of portfolio of CCP defaulted members with other surviving members. Fat-tailed distributions are favoured to generate portfolio losses and defaults with the application of very large-dimension Monte Carlo methods along with numerical uncertainty quantifications. We also expand on the novation aspects of portfolios of defaulted members and the associated XVA costs transfers. These innovations can be carried out either on the marketplaces (exchanges) or by the CCPs themselves by identifying the optimal buyers or by conducting auctions of defaulted positions with dedicated economic equilibrium problems. Failures of members on several CCPs in common also lead to the formulation and resolution of multidimensional optimization problems of risk transfer that are introduced in this thesis.

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Introduction (en français)

Suite à la crise des subprimes, dont les racines remontent à la bulle immobilière américaine des années 2000 qui entraîna la faillite de Lehman Brothers en septembre 2008, une crise de liquidité et de solvabilité d'établissements financiers, mais aussi de pays de la zone euro, a touché l'ensemble du secteur financier jusqu'à la fin de l'année 2010. Des répercussions se sont ensuivies durant les années 2011 et 2012 via la crise de la dette européenne avec les difficultés de paiement de la Grèce sur ses endettements, la nationalisation de plusieurs banques en Irlande (Anglo Irish Bank en 2009, Educational Building Society limited and Irish Nationwide Building Society en 2010, et Irish Life and Permanent en 2011, voir [Laeven and Valencia \(2012, Table A.3\)](#)) et des plans de sauvetage à grands renforts d'apports de liquidité via des prêts facilités de l'Union Européenne pour le Portugal, l'Espagne et Chypre (€77 mds, €41 mds et €7.3 mds respectivement, d'apport de liquidités provenant de fonds de l'UE et de l'IMF, voir [Gourinchas, Martin, and Messer \(2023, Table 1\)](#)). En réponse, les organismes internationaux de normalisation, comme le comité Bâlois (Basel Committee on Banking Supervision [BCBS]) et le comité de paiements et infrastructures des marchés financiers (Committee on Payments and Market Infrastructures [CPMI]) ont établi, dès 2008, des instructions à destination des établissements bancaires afin de garantir une meilleure robustesse du système financier. Ces instructions visent à limiter les risques de contagion de défauts d'acteurs, principalement bancaires, dans le réseau financier, tant concernant le manque de liquidité des établissements financiers que l'impact des niveaux en capital ou fonds propres requis à ces établissements. Les États-Unis adoptèrent en 2010 l'acte Dodd-Frank (Dodd-Frank Wall Street Reform and Consumer Protection Act, [The United States Congress \(2010\)](#)) alors que l'Union Européenne adopta en 2012 la régulation sur l'infrastructure du marché européen avec le texte EMIR (European Market Infrastructure Regulation, [The European Parliament and the Council of the European Union \(2012\)](#)). S'ensuivit la mise à jour des textes Bâlois, garants des accords internationaux régissant le ratio de solvabilité des établissements financiers. Une composante importante de cet arsenal réglementaire consiste à conduire annuellement des exercices de résistance (dit aussi de *stress test*) par les autorités des institutions bancaires telles que l'Autorité Bancaire Européenne [EBA] et la Réserve Fédérale des Etats-Unis [FED]. Le rôle prospectif de ces exercices permet de relever des premiers signaux de faiblesses des structures comptables des banques reconnues comme contributrices majeures du risque systémique financier.

Les exercices de stress test, dont les principes sont détaillés dans [Crouhy, Galai, and Mark \(2014\)](#) et [Bellini \(2017\)](#), ont pour base un ensemble de projections averses de facteurs économiques (PIB, inflation, taux de change, taux d'intérêts, indices boursiers, etc.) sous des hypothèses de conditions défavorables des marchés financiers, généralement sur un horizon de trois ans pour ce qui est des exercices purement financiers, et allant jusqu'à 50 ans lorsque ces exercices concernent l'impact des changements climatiques sur le système financier. Ces trajectoires prospectives de facteurs économiques sont alors injectées dans les probabilités de défaut (par exemple en exploitant des régressions linéaires des historiques

de ces probabilités, ou de leurs variables latentes, sur ces facteurs économiques) à divers horizons de temps futurs ainsi que dans les actifs composant les portefeuilles (on parle également d'expositions) des contreparties auxquelles sont exposées les banques. Les probabilités de défaut et montants d'expositions peuvent être ainsi projetés sur la base de cette ré-interprétation des variables macro-économiques pour en déduire l'impact sur leurs actifs pondérés à risque (en anglais *risk weighted assets* [RWA]) et sur les montants en capital à réserver que ce soit les fonds propres dans le cadre des instructions réglementaires Capital Requirements Regulation [CRR] ou les provisions de crédit dans le cadre des normes comptables International Financial Reporting Standards [IFRS]. Un établissement bancaire exposé à divers types de contreparties (entreprises, autres banques, fonds de pensions, etc.) aura une tâche d'autant plus complexe pour réaliser ces exercices de stress test que son portefeuille de clients sera diversifié. Par ailleurs, si l'objectif de la réglementation est bien de pouvoir limiter les risques systémiques et de contagion d'évènements économiques et financiers défavorables, ces exercices doivent pouvoir capturer et identifier les signaux de faiblesse de ces risques. Cela suppose en particulier, pour l'institution bancaire qui conduit l'exercice, d'intégrer les diverses interactions entre les acteurs financiers et économiques constituant le réseau du système financier. Ces demandes sont notamment exprimées dans l'article 302 du texte réglementaire [The European Parliament and the Council of the European Union \(2013\)](#) (mis à jour dans [The European Parliament and the Council of the European Union \(2019\)](#)):

Les établissements évaluent, par une analyse de scénario et des tests de résistance appropriés, si le niveau des fonds propres détenus au regard des expositions à une CCP, y compris les expositions de crédit potentielles futures ou éventuelles, les expositions découlant de contributions à un fonds de défaillance et, lorsque l'établissement agit en qualité de membre compensateur, les expositions découlant de dispositions contractuelles [...] et en proportion des risques inhérents à ces expositions.

Par ailleurs, les recommandations faites dans le paragraphe 105 du texte [European Bank Authority \(2018\)](#) en termes de conduite d'exercice de stress test pour l'activité de clearing stipulent:

Les établissements devraient veiller à ce que les tests de résistance relatifs au risque de crédit à l'échelle de l'établissement couvrent toutes leurs positions dans leur portefeuille bancaire et de négociation, y compris les positions de couverture et les expositions sur les chambres centrales de compensation [CCP].

Le réseau du système financier a profondément été modifié ces dix dernières années avec le rôle de plus en plus prépondérant attribué aux chambres centrales de compensation (dont l'abréviation est CCP et que l'on nomme plus communément chambre de compensation, *clearinghouse* en anglais), sous l'impulsion du G20. En septembre 2009 ([G20 Research Group, 2009](#)), le G20 impose la prise en charge systématique des contrats dérivés de gré-à-gré (en anglais *over-the-counter* [OTC]) lorsque ceux-ci peuvent être normalisés (dit aussi standardisés) par l'intermédiaire des CCPs entre deux acteurs financiers, alors reconnus comme membres "compensateurs" ou "clearers" (en anglais *clearing members*) des CCPs agissant comme leurs intermédiaires. Autrement dit, les flux financiers d'un contrat dérivé entre deux contreparties transitent de manière intermédiaire par une CCP, cette dernière devenant acheteuse de toute contrepartie vendeuse et vendeuse de toute contrepartie acheteuse de ce contrat. Les CCPs ont aussi pour rôle de garantir la continuité des versements de ces flux même en cas de défaut de l'une des parties. Ceux-ci sont garantis via la collection de collatéraux auprès des deux parties pour lesquelles les CCPs sont un intermédiaire, similaire aux principes appliqués par les bourses financières. Ainsi les contrats dérivés sont négociés entre deux parties sur les bourses sous-tendues aux CCPs sur lesquelles les parties sont ensuite re-dirigées pour mettre en œuvres les flux financiers de ces contrats. Les contrats

traités par les CCPs pour le compte des divers acteurs financiers sont dits *clearés*. Lorsqu'un établissement est exposé à diverses contreparties sur les marchés de produits dérivés, qui pour une grande partie des contrats ont pu être standardisés, cette exposition est indirecte puisque la CCP en est une intermédiaire. Tous les produits dérivés ne peuvent être standardisés puisque nombre d'entre eux sont spécifiquement définis aux seuls besoins des clients désireux de se prémunir contre certaines fluctuations du marché ou simplement à des fins spéculatives. Dès lors, la structure des marchés financiers s'est métamorphosée d'une structure d'accords bilatéraux opérés entre deux acteurs du réseau financier, en une structure plus complexe. Cette nouvelle structure se compose de dizaines de CCPs concentrant des montants gigantesques de collatéraux, quelques milliers d'acteurs financiers disposant pour la plupart d'un niveau conséquent de capital et faisant face à ces CCPs en leur fournissant ces collatéraux collectés auprès d'acteurs plus réduits financièrement et ne pouvant accéder directement aux services d'une CCP. Bien que simpliste et schématique avec une seule CCP représentée, la Figure 1 illustre le passage de relations, c'est-à-dire de positions financières, uniquement bilatérales entre six acteurs financiers (schéma de gauche) à des relations uniquement gérées par une seule CCP comme unique intermédiaire entre ces six acteurs (graphe du milieu) et une représentation plus en ligne avec la pratique industrielle consistant en un mélange de relations bilatérales et passant par la CCP (graphe de droite).

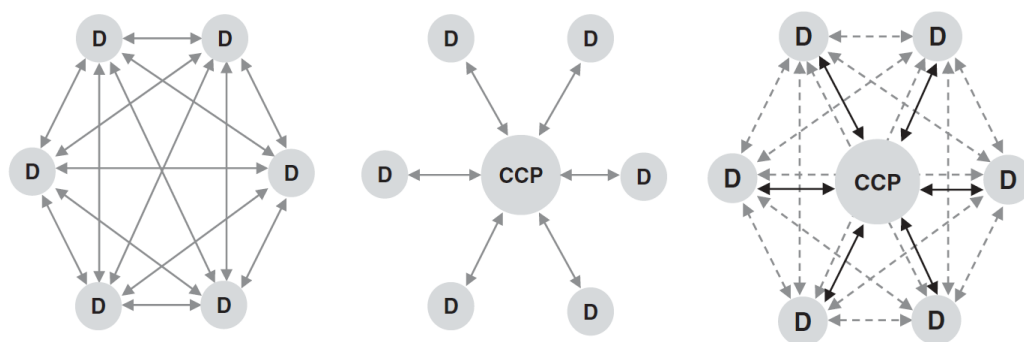


Figure 1: Illustration d'un marché de flux financiers, collatéraux compris, avec des liens uniquement bilatéraux (à gauche) comparé à un marché de flux financiers centralisés via une CCP (au centre). A droite, illustration d'un marché de flux financiers en partie centralisés (liens solides) et contenant des flux financiers bilatéraux entre les membres (liens en pointillés) de la CCP. Source: Gregory (2014, Figures 3.2 et 3.5, pages 28 et 32).

Ces montants de collatéraux sont d'ailleurs observables via des rapports trimestriels publiés par la majorité des CCPs et se chiffrent pour les CCPs les plus larges en centaines de milliards d'Euros (il s'agit des documents ou fichiers de *quantitative disclosure*). Par ailleurs, les positions provenant des produits dérivés encore OTC ne transitent pas par les CCPs qui ne peuvent les gérer et continuent de mailler le réseau financier de liens bilatéraux. Concernant la concentration de collatéraux, bien que ceux-ci permettent une certaine garantie de la continuité des opérations pour les CCPs et leurs membres suite à la défaillance potentielle d'un des leurs, ils soumettent les plus petits acteurs du réseau, non membres de ces CCPs, à des besoins de liquidité importants. Néanmoins le nombre d'événements de défaut d'un membre sur une CCP lors des quinze dernières années est de moins d'une dizaine. Il est alors reconnu que le risque de contrepartie, c'est-à-dire le risque qu'un acteur financier fasse défaut et ne puisse pas respecter ses engagements financiers contractuels envers sa contrepartie devant se tourner vers d'autres acteurs pour remplacer ces engagements avec des coûts additionnels, était un risque

majeur du réseau financier lorsque celui-ci était constitué principalement de liens financiers bilatéraux. Ce risque de contrepartie s'est transformé, avec l'importance croissante des CCPs, en un risque de liquidité, qui consiste pour un acteur financier au risque de ne pas pouvoir respecter un ou plusieurs de ses engagements financiers, y compris de fournir des collatéraux supplémentaires soit par manque de moyen pour les assurer ou par une accessibilité réduite et onéreuse pour certains de ces collatéraux. Un tel événement d'insuffisance de liquidité, et donc de défaillance, a une probabilité de développement bien plus faible étant donné les collatéraux postés par les membres des CCPs pour garantir jusqu'à un certain niveau de confiance les flux financiers. Cette probabilité est bien plus faible comparée à celle générée par le risque de contrepartie. Cependant, si les collatéraux venaient à être insuffisants — ce qui ne se produirait alors que dans un cas plus extrême qu'un ou plusieurs défaut(s) de contrepartie(s) plus isolé(s) dans le cadre plus simples d'échanges financiers bilatéraux —, les besoins de liquidité seraient bien plus significatifs pour de multiples membres, dont une partie ne sauraient pouvoir les combler. Un effet de contagion est alors possible, les défauts des membres, et potentiellement de leurs clients, peuvent par conséquent dans ce scénario plus rare mais plus extrême se propager rapidement et créer un événement systémique d'une intensité dévastatrice pour l'ensemble du réseau. Pour illustrer la complexité du réseau financier dans son ensemble, une analyse des liens entre les membres, leurs clients pour lesquels ils opèrent, et les CCPs, est souvent privilégiée car elle facilite une certaine lecture du réseau mais la simplifie en omettant les diverses connections entre les membres des CCPs ou entre les membres et d'autres acteurs financiers de ce réseau non-membre d'une CCP comme dans [Rosati and Vacirca \(2019, Figure 2\)](#), reprise dans la Figure 2.

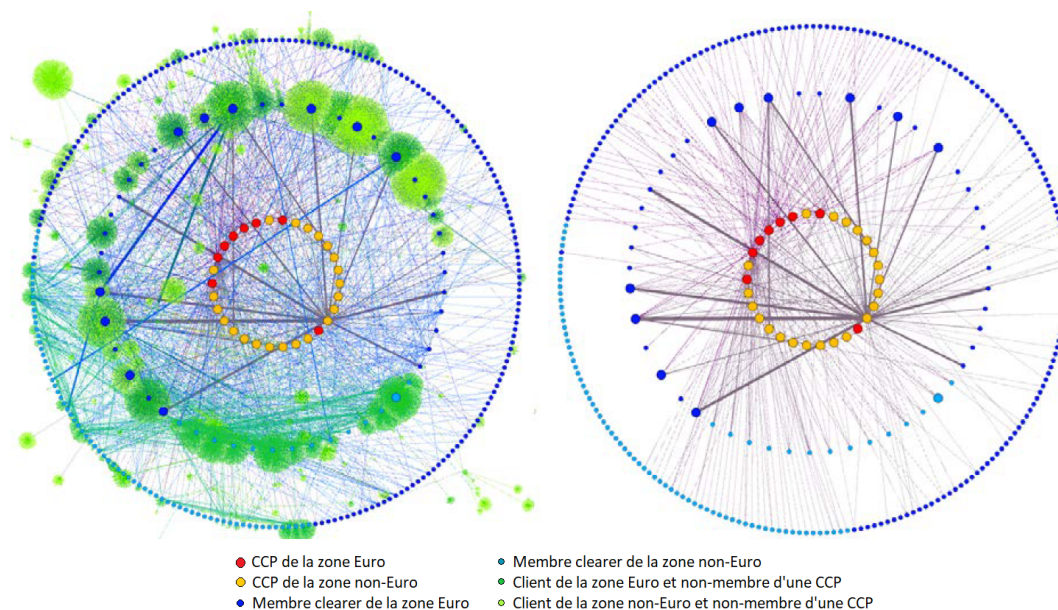


Figure 2: Réseaux financiers indiquant l'existence de positions Marked-to-Market (lignes solides) entre deux acteurs financiers (points foncés pour la zone Euro, clairs pour la zone non-Euro) comprenant 30 CCPs (points rouges et oranges), leurs membres (190 points bleus foncés et clairs) et les clients non-membres passant par ces membres (points verts foncés et clairs). Le réseau de droite indique uniquement les positions Marked-to-Market entre les CCPs et leurs membres sans les clients qui sont par contre inclus dans le graphe de gauche. Les cercles bleus intérieurs représentent les clearing members exposés aux plus grands nombres de CCPs. *Source: Rosati and Vacirca (2019, Figure 2).*

En pratique, il est également possible de générer de tels réseaux de manière régulière sur la base des analyses annuelles conduites concernant la qualité de crédit des CCPs où l'analyste de crédit, en charge de la notation interne de la qualité de crédit d'une telle institution collecte l'ensemble des noms de ses clearing members. En croisant ces informations sur diverses CCPs, il est par exemple possible de produire un réseau comme le montre la Figure 3. On peut y observer les relations qu'ont les membres avec une ou plusieurs CCPs mais les relations entre les membres eux-mêmes, dans le cadre d'une activité bilatérale de gré-à-gré, ne sont pas indiquées pour ne pas surcharger la figure.

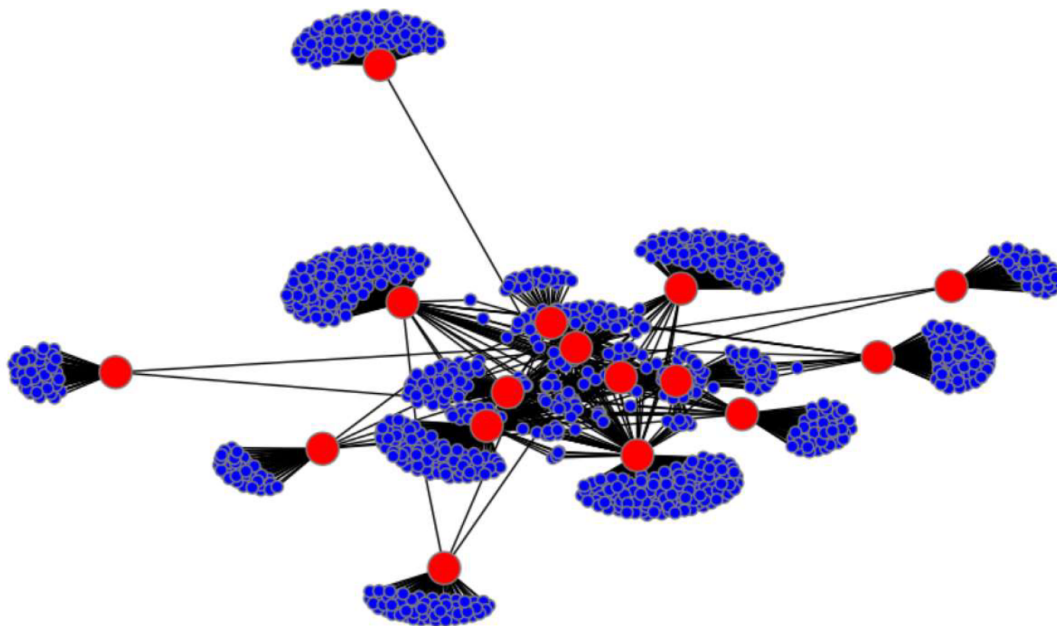


Figure 3: Réseau financier de l'activité de clearing en Europe avec 16 CCPs en rouge et leurs membres respectifs en bleu (1059 membres au total), dont 112 membres ont des accords sur plusieurs CCPs et concentrés au centre du réseau, sur la base de données à fin juin 2021.

Une représentation plus réaliste mais plus complexe à illustrer consiste à montrer les liens à la fois bilatéraux de portefeuilles pris sur les marchés de gré-à-gré et de l'activité de clearing comme [Fiedor, Lapschies, and Országhová \(2017, Figure 3\)](#), voir la Figure 4, mais qui se concentrent, étant donné la lourdeur des données à traiter, sur le marché des produits de dérivés de taux (qu'ils soient clearés ou de gré-à-gré).

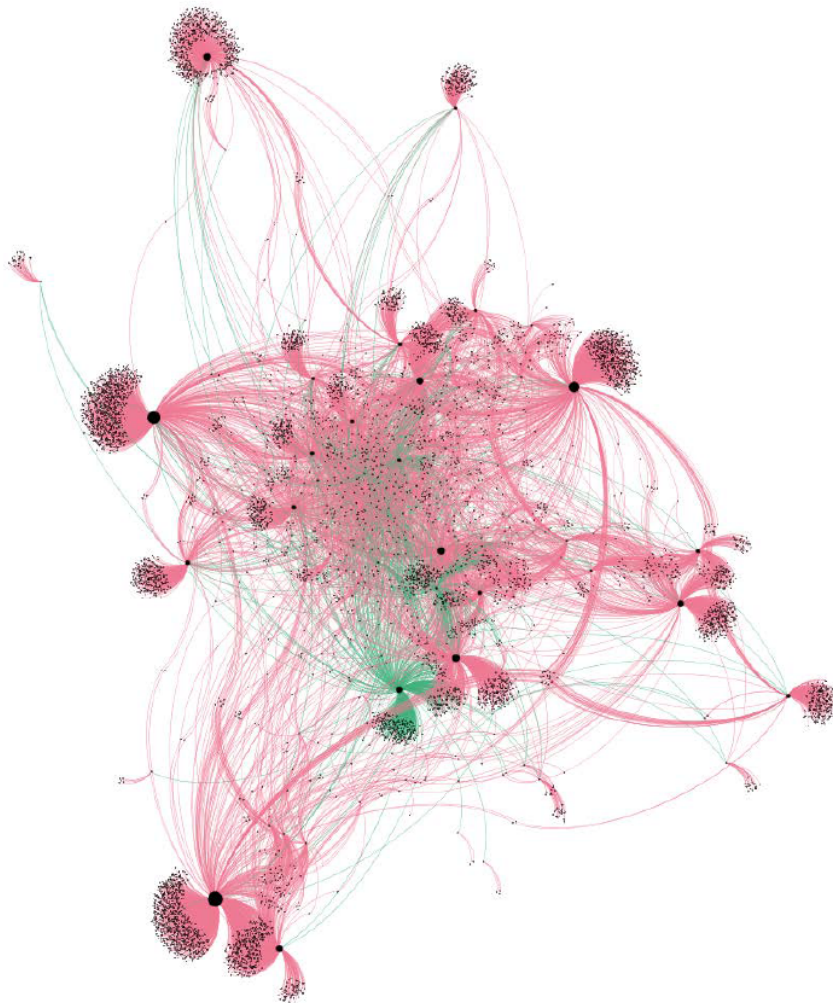


Figure 4: Relations entre les contreparties pour l'activité de clearing des produits dérivés de taux au 31 Déc. 2016. La taille des nœuds est relative au nombre de liens (degré). Les liens entre les CCPs et leurs membres sont en vert et ceux entre les membres et leurs clients pour lesquels ils opèrent sont en rouge. *Source: Fiedor, Lapschies, and Országhová (2017, Figure 3)*

Problématique générale : Les instructions réglementaires ont pris en compte ces récentes transformations du réseau financier et de ses risques en recommandant leur représentation et quantification adéquates pour les exercices de stress test. C'est également un point d'attention privilégié par le comité exécutif des institutions bancaires. Il demande des exercices introspectifs et prospectifs de la solidité de leur établissement financier via divers exercices réglementaires d'évaluations du bon fonctionnement de leurs opérations et de la garantie de leur continuité suite à des événements averses sur le système financier. Les cadres d'exercices réglementaires d'analyses Internal Capital Adequacy Assessment Process [ICAAP] et Internal Liquidity Adequacy Assessment Process [ILAAP], composantes du cadre réglementaire Supervisory Review and Evaluation Process [SREP], sont élaborés à ces fins. Ils requièrent l'analyse de la solidité financière des établissements financiers suite à diverses chocs économiques hy-

pothétiques (qu'ils soient de nature politique, économique et/ou climatique). Il convient alors de pouvoir élaborer des outils pouvant capturer ces mécaniques du réseau financier et de ses risques, quantification à l'appui, avec pour cible des approches et implémentation extensibles à large échelles du réseau financier mais en puissance et temps de calcul limités.

Cette thèse propose diverses contributions sur 4 axes de développement, liés les uns aux autres, pour apporter des réponses à cette problématique:

- une modélisation des diverses mécanismes et risques des marchés financiers sur les activités de clearing et bilatérales OTC avec la prise en compte de ses différents types d'acteurs et coûts qui en découlent, permettant divers exercices de gestion des risques;
- une analyse théorique et numérique de l'adéquation de cette modélisation reposant sur les propriétés de super-modularité de fonctions multi-dimensionnelles et de vecteurs aléatoires;
- l'exposition d'une approche basée sur le concept d'équilibre de Radner pour capturer le risque de liquidation au travers d'impact sur les prix de marché comprenant une mise en adéquation avec les divers types d'acteurs et d'activités de clearing et bilatérales OTC des deux axes précédents;
- la mise en application de résolution algorithmique approchée de l'allocation optimale des portefeuilles d'un membre en défaut sur plusieurs CCPs, abordé dans le premier axe de manière naïve et coûteuse en temps de calcul.

Nous détaillons ci-après ces axes faisant l'objet, pour chacun d'entre eux, d'un chapitre dans ce manuscrit.

§1 Risques des produits dérivés traduits en coûts dans un modèle de réseau à une période

Dans une première partie, nous proposons une description exhaustive des principaux mécanismes de fonctionnement du réseau financier sur l'activité des dérivés financiers, y compris les coûts qui en découlent, en prenant en compte à la fois les liens bilatéraux et ceux constitués par l'activité des CCPs. L'utilisation de modèles elliptiques permet de proposer diverses applications comme un ré-arrangement du réseau financier (que l'on peut également qualifier de déformation ou encore de restructuration de ce réseau) suite à un défaut d'un des acteurs financiers, membre d'une CCP, suivi de la reprise de son portefeuille à moindre coût par les autres membres de cette même CCP. Une autre application essentielle permettant de répondre aux exigences réglementaires consiste à la mise en place d'exercices de stress test prospectifs. Par ailleurs, des comportements, bien qu'intuitifs, de monotonie des métriques de coûts et de risques, sont observés numériquement avec ces modèles et amènent à valider théoriquement la pertinence de l'utilisation de tels modèles.

Contributions du Chapitre I : nous reprenons Bastide, Crépey, Drapeau, and Tadese (2023a) dans lequel

- nous décrivons les différents types d'acteurs et les flux financiers les concernant, permettant de maintenir leur positions dans le cas où des défauts de leurs contreparties se produisent, ainsi que les coûts de financement nécessaires prenant en compte leur propre risque de défaut (**Lemme I.1, Lemme I.2**);
- nous exprimons ces divers types de coûts en combinant les coûts de l'activité de clearing (Albanese, Armenti, and Crépey, 2020) et bilatérale OTC (Abbas-Turki, Crépey, and Saadeddine, 2023) dans un cadre statique et prenant en compte la multitude de CCPs auxquelles s'exposent un acteur financier majeur comme une large banque (**Théorème I.4 et Proposition I.5**);
- nous illustrons numériquement diverses applications possibles comme des exercices de stress test, exploratoires ou inverses (reverse stress test) et des reprises idéalisées de portefeuilles de membres en défaut.

Un acteur majeur du réseau financier tel qu'une banque systémique détient des expositions à la fois face à des CCPs et face à des contreparties bilatérales pour les produits ne pouvant être standardisés. Il subit par conséquent des coûts et des besoins en liquidité sur ses états financiers dus à leur détention. Ces coûts sont alors à transférer à ses clients, ces derniers étant à l'origine de la détention de ces produits financiers afin de répondre à leurs besoins économiques. Quant aux besoins de liquidité, tels que du collatéral en devise ou titres suffisamment liquides que ce soit dans des conditions de marchés averses ou non, ils se traduisent par la nécessité pour l'acteur financier les supportant de se tourner vers les marchés et autres acteurs financiers dédiés afin de disposer de cette liquidité, moyennant également des coûts correspondants. Par ailleurs, l'interconnexion des membres au travers des CCPs se retrouve par la pluralité des accès d'un membre à plusieurs CCPs, chacune couvrant un certain type de produits de son portefeuille.

L'analyse de ces mécanismes d'activités de clearing et bilatérales et de leurs diverses formulations remontent aux travaux de Brigo, Morini, and Pallavicini (2013) et Gregory (2015) pour la partie bilatérale OTC, Arnsdorf (2012), Gregory (2014), Ghamami (2015), Arnsdorf (2018), Arnsdorf (2019), Andersen and Dickinson (2019) et Andersen and Dickinson (2020) pour la partie clearing et CCP. Celle-ci est conduite au niveau de chaque contrat comprenant leurs coûts en temps continu dans Crépey (2022). Elle peut être mise en place numériquement à condition de disposer d'une grande capacité de calcul,

puisque les grands acteurs financiers comme les banques systémiques détiennent plusieurs dizaines de millions de ces transactions dans leurs portefeuilles.

Dans [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#), nous proposons de reprendre ces mécanismes au travers d'un modèle à une période inspiré de la version à temps continu de [Armenti and Crépey \(2017\)](#) pour la partie clearing et similaire à l'approche dans [Albanese, Crépey, Hoskinson, and Saadeddine \(2021, Section 3\)](#) dans le cas bilatérale OTC. Nous y proposons aussi une alternative aux variations aléatoires de valeur d'un portefeuille modélisées avec une seule variable aléatoire, au lieu de toutes les variables dont pourraient dépendre des milliers de contrats de ce portefeuille. Le réseau d'acteurs financiers et d'expositions financières, même s'il peut contenir plusieurs milliers de nœuds, est ainsi numériquement tractable (pour des approches similaires voir par exemple [Arnsdorf \(2012\)](#) où les variations aléatoires de valeur de portefeuilles sont supposées suivre des lois de Pareto, [Andersen and Dickinson \(2019\)](#) utilisant des lois elliptiques et [Barker, Dickinson, Lipton, and Virmani \(2017\)](#) pour une approche par contrat avec des applications numériques sur un réseau constitué de 4 CCPs et de leurs membres).

Cette adaptation permet aussi d'illustrer numériquement, via des modèles elliptiques pour les variable de variations de portefeuilles et latentes de défaut, des exercices réglementaires de stress test et le portage de position d'un membre en défaut d'une CCP vers l'un de ses membres survivants générant le moindre coût. cette dernière application s'apparente à une mise aux enchères idéale qui se résout avec un repreneur permettant d'obtenir ce coût moindre pour l'ensemble des membres survivants. Une approche stress test pour un réseau de CCPs et leur membres, limitée à quelques scénarios macro-économiques averses, mais sans prise en compte des coûts de contrepartie, de financement des collatéraux ou de capital, est présentée dans [Barker, Dickinson, Lipton, and Virmani \(2017\)](#) et repris par [Lipton \(2018\)](#). Un cadre plus complet est détaillé dans [Albanese, Crépey, and Iabichino \(2023\)](#) sans se focaliser sur un type d'activité particulier (clearing ou bilatérale OTC). Quant à l'analyse du portage des portefeuilles en défaut, il permet d'appréhender les restructurations possibles du réseau financier suite à un ou plusieurs défauts dans le système financier.

Notons J_i, J_c, J_b les indicateurs de survie au temps 1 respectivement des membres i des CCPs, des clients indexés par c du membre de référence 0 pour lesquels il opère un service de clearing et de ses contreparties bilatérales OTC indexées par b . Une CCP indexée ccp détient face à ses membres des expositions, comprenant les appels de marges journaliers, notées Y_i^{ccp} , $i \in I$ avec I l'ensemble des membres des CCPs. Les positions de couvertures face aux CCPs sont identifiées spécifiquement par \bar{Y}_i^{ccp} . Cette distinction sera particulièrement utile dans le **Chapitre III** pour établir les découvertes de prix sur les marchés et est illustré dans la Figure 5.

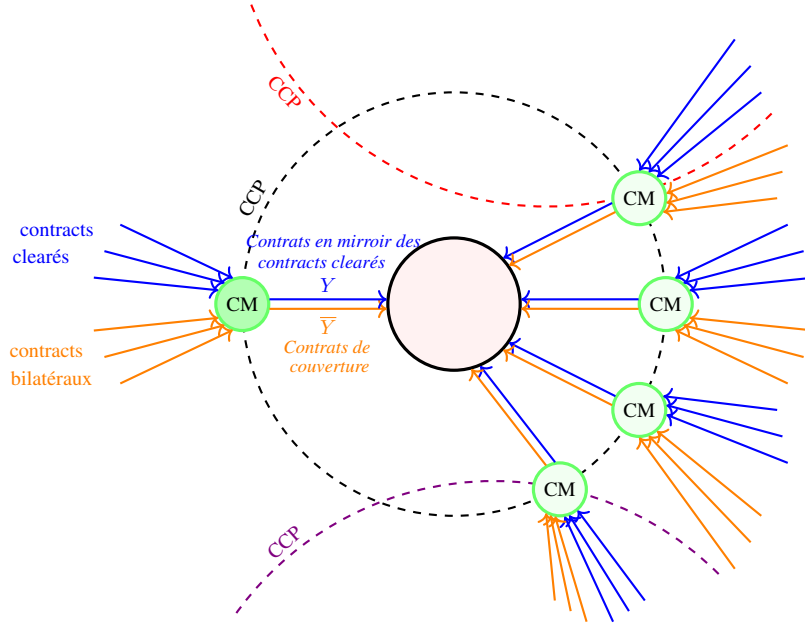


Figure 5: Flux contractuels entre les participants de marchés. Les flèches bleues représentent des engagements de positions clearées par les membres au nom de leurs clients, les flèches oranges sont des engagement bilatéraux et leur positions symétriques face à la CCP en tant que couverture (dits aussi de *proprietary trading*).

Une CCP applique une cascade (*waterfall*) de couches de protection $M_i^{ccp} = IM_i^{ccp} + \overline{IM}_i^{ccp} + DF_i^{ccp}$ pour chacun de ses membres $i \in I$ (voir le principe 6 dans l'ensemble des principes émis et recommandés par [Bank For International Settlements \(2012\)](#)). Les marges initiales (*initial margins*, [IM]) sont calculées séparément entre les positions clearées (dit de compte *client*) et les positions de couverture (dit de compte *house*), et notées respectivement IM_i^{ccp} et \overline{IM}_i^{ccp} . Le membre 0 reçoit de ses clients clearés et de ses contreparties bilatérales les flux Y_c^{ccp} et Y_b respectivement avec comme marges initiales correspondantes IM_c^{ccp} et IM_b . Pour les expositions bilatérales OTC, le membre 0 peut poster des marges initiales \overline{IM}_b , pour chacune de ses contreparties b . La plupart des CCPs requièrent également un fonds de garantie (ou fonds de défaut, en anglais default fund DF) noté DF_i^{ccp} pour chaque membre i sur la CCP ccp . Nous ignorons une couche additionnelle de capital propre aux CCPs (le *skin-in-the-game*) étant donné son faible montant par rapport aux autres couches de protections ([Gregory, 2014](#), Section 8.4.5).

La formulation de la perte \mathcal{C} , que le membre de référence indexé 0 supporte, constitue le point de départ des divers coûts de la détention de ses portefeuilles et s'écrit (**Proposition I.5** et équations I.(15) et I.(16))

$$\mathcal{C} = \sum_{ccp,c} (1 - J_c) \underbrace{(Y_c^{ccp} - IM_c^{ccp})^+}_{\text{perte face au client } c \text{ clearé par } 0} + \sum_{ccp} \underbrace{\mu^{ccp} \mathcal{L}^{ccp}}_{\text{perte sur la } ccp \text{ allouée à } 0} + \sum_b (1 - J_b) \underbrace{(Y_b - IM_b)^+}_{\text{perte face au client } b}, \quad (1)$$

avec

$$\mathcal{L}^{ccp} = \sum_i (1 - J_i) \left(\underbrace{(Y_i^{ccp} - \text{IM}_i^{ccp})^+}_{\text{position client sur la ccp}} + \underbrace{(\overline{Y}_i^{ccp} - \overline{\text{IM}}_i^{ccp})^+}_{\text{position house sur la ccp}} - \text{DF}_i^{\text{ccp}} \right)^+. \quad (2)$$

Les formulations (1) et (2) découlent des principes suivants:

1. Une contrepartie en défaut envers une autre et ayant des dettes au-delà des collatéraux postés, les rembourse à hauteur de ce qu'elle a posté en collatéral et non au-delà. Si cette dette est en-deçà du collatéral, l'ensemble des dettes remboursé sans qu'aucune perte ne se matérialise pour l'autre contrepartie.
2. Les flux entrants et sortants de toute CCP s'annulent, de même que pour tout membre faisant face à une CCP. Un membre est ainsi considéré comme étant parfaitement couvert sur ses risques de marchés émanant de ses propres positions que ce soit pour l'activité de clearing ou l'activité bilatérale OTC.

La plupart des membres étant des larges banques, celles-ci ont effectivement une activité de marché relativement bien couverte, comme en atteste leurs besoins en fonds propres analysés dans [European Bank Authority \(2022\)](#) (estimés comme une fraction des actifs pondérés reportés dans ce rapport).

Les coûts de détention des divers types de positions du membre 0, sont valorisés sous une probabilité notée \mathbb{P}^* , d'opérateur d'espérance \mathbb{E}^* , et avec pour numéraire un actif de taux sans risque permettant de simplifier les équations de valorisation de tous les flux financiers et des comptes de collatéraux. La probabilité \mathbb{P}^* est un mixte de mesures physique et risque-neutre, que l'on qualifiera à ce titre de mesure "finassurance", construit de la manière suivante: étant donnée une mesure de probabilité physique définissant complètement le modèle reflété par la σ -algèbre \mathcal{A} et équivalente à une mesure risque-neutre sur la σ -algèbre \mathcal{B} ne reflétant que l'incertitude de marché et contenue dans \mathcal{A} , \mathbb{P}^* est égale à la mesure risque-neutre sur \mathcal{B} et à la mesure physique conditionnellement à \mathcal{B} . Ces deux conditions caractérisent uniquement \mathbb{P}^* ([Artzner et al., 2023](#), Proposition 4.1). Cette construction remonte à [Plachky and Rüschendorf \(1984\)](#) et [Dybvig \(1992\)](#). La probabilité de défaut correspondante d'un membre i est notée $\gamma_i = \mathbb{P}^*(J_i = 0)$. Pour simplifier, J_0 est noté simplement J , et γ_0 simplement γ .

Concernant le financement des expositions pour le membre 0, et donc ses besoins de liquidité, les termes IM^{ccp} et $\sum_{b \in \mathcal{B}} \overline{\text{IM}}_b$, constituent des montants de collatéraux à emprunter et poster soit à la CCP, soit à ses contreparties bilatérales. Le membre 0 reçoit en échange un taux de rémunération que l'on considère en-deçà de son taux de financement pris comme sa probabilité de défaut γ si bien que le coût de financement se base sur un taux dilué $\tilde{\gamma} < \gamma$ (le membre 0 se finance au taux γ et se fait rémunérer au taux $\gamma - \tilde{\gamma}$). De plus, le membre 0 transfère ses coûts de risque de contrepartie et de financement de marges agrégés, résumé par le terme CA, à ses clients. CA est l'acronyme de *contra-asset* et joue le rôle d'un passif dans le bilan du membre 0 comptable venant contrebalancer la valeur des dérivés détenus comme actifs. Ses actionnaires doivent alors supporter la perte potentielle $\ell := \mathcal{C} + \mathcal{F} - \text{CA}$ pour laquelle un niveau de capital EC est calculé comme une mesure de risque de cette perte où la mesure est par exemple une expected shortfall ES_α^0 sous la probabilité \mathbb{P}^* conditionnée à la survie de 0, notée \mathbb{Q}^0 . Cette mesure de risque correspond, dans le cas où elle est appliquée à une variable aléatoire de distribution continue, à la moyenne des pertes sous \mathbb{Q}^0 au-delà d'un certain seuil $\mathcal{Q}_\alpha^0(\ell)$ pour un intervalle de confiance $\alpha \in (1/2, 1)$ également sous \mathbb{Q}^0 , soit $\text{EC} = \text{ES}^0(J\ell) = \mathbb{E}^0[\ell | \ell \geq \mathcal{Q}_\alpha^0(\ell)]$, avec $\mathbb{Q}^0(\ell \geq \mathcal{Q}_\alpha^0(\ell)) = 1 - \alpha$. Elle devient la mesure à considérer pour le standard réglementaire des mesures du risque de marché ([Bank for International Settlements, 2013](#)). Le comité exécutif de ce membre, doit assurer un dividende, à hauteur d'un taux dénoté h , sur le capital à risque de ses

actionnaires, soit $(EC - KVA)^+$ où KVA représente le coût en capital également chargé au client en plus de CA . On fait alors l'hypothèse qu'à temps 0, les montants de ces coûts collectés par le membre 0 sont déposés sur ses comptes de réserve en capital et de capital à risque et peuvent être utilisés pour ses besoins de financement de marges de variations. Ainsi les dépenses dues aux financements des diverses marges peuvent s'écrire comme le flux (**Proposition I.5** équation (17))

$$\mathcal{F} = \gamma \left(\sum_b V_b - CA - \max(EC, KVA) \right)^+ + \tilde{\gamma} \sum_{ccp} M^{ccp} + \tilde{\gamma} \sum_b \overline{IM}_b, \quad (3)$$

où V_b représente la différence entre le collatéral dû par le membre 0 à la contrepartie b et celui collecté auprès de cette même contrepartie. CA se décompose en divers termes reflétant les coûts de crédit et de financement de marge de la forme $CA = CCVA + BCVA + CMVA + BMVA + FVA$, où les différents termes sont définis dans la Table 1.

XVA	Expression	Nom complet et description
KVA	$\mathbb{E}^* [Jh(EC - KVA)^+ + (1 - J)KVA]$, where $EC = \mathbb{E}\mathbb{S}(J(\mathcal{C} + \mathcal{F} - CA))$	capital valuation adjustment
BCVA	$\mathbb{E}^* \left[J \sum_b (1 - J_b)(Y_b - IM_b)^+ + (1 - J)BCVA \right]$	credit valuation adjustment pour les expositions bilatérales
CCVA	$\mathbb{E}^* \left[J \sum_{ccp.c} (1 - J_c)(Y_c^{ccp} - IM_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} + (1 - J)CCVA \right]$	credit valuation adjustment pour les expositions d'activités de clearing
BMVA	$\mathbb{E}^* \left[J \tilde{\gamma} \sum_b \overline{IM}_b + (1 - J)BMVA \right]$	margin valuation adjustment pour les expositions bilatérales
CMVA	$\mathbb{E}^* \left[J \sum_{ccp} \tilde{\gamma} (IM^{ccp} + \overline{IM}^{ccp} + DF^{ccp}) + (1 - J)CMVA \right]$	margin valuation adjustment pour les expositions d'activités de clearing
FVA	$\mathbb{E}^* \left[J \gamma \left(\sum_b Y_b - CA - \max(EC, KVA) \right)^+ + (1 - J)FVA \right]$	funding valuation adjustment

Table 1: Définitions des coûts XVA (avec \mathcal{C} , \mathcal{F} et \mathcal{L}^{ccp} donnés respectivement par (1), (2) et (3)).

On remarque que ces coûts de risque de contrepartie, de financement de marges et de capital s'écrivent comme solution d'équation de type point fixe sous la probabilité "finassurance" et sont bien définis lorsqu'ils sont réécrits sous la probabilité de survie \mathbb{Q}^0 . Nous pouvons alors obtenir des formules explicites de ces différents coûts, que nous listons dans la Table 2 (**Théorème I.4** et **Proposition I.5**).

XVA	Formule explicite
CCVA	$\mathbb{E} \left[\sum_{ccp,c} (1 - J_c)(Y_c^{ccp} - \text{IM}_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} \right]$
CMVA	$\tilde{\gamma} \sum_{ccp} M^{ccp}$
BCVA	$\mathbb{E} \left[\sum_b (1 - J_b)(Y_b - \text{IM}_b)^+ \right]$
BMVA	$\tilde{\gamma} \sum_b \overline{\text{IM}}_b$
EC	$\mathbb{E}\mathcal{S}(J(\mathcal{C} - \text{CVA}))$
FVA	$\frac{\gamma}{1 + \gamma} \left(\sum_b Y_b - (\text{CCVA} + \text{CMVA} + \text{BCVA} + \text{BMVA}) - \text{EC} \right)^+$
KVA	$\frac{h}{1 + h} \text{EC}$

Table 2: Formules explicites des coûts XVA sous probabilité \mathbb{Q}^0 (avec \mathcal{C} et \mathcal{L}^{ccp} donnés par (1) et (2)).

Nous proposons plusieurs exemples numériques, sur la base de modèle elliptique avec des distributions de Student ayant un degré de liberté de 3 pour générer des distributions à queues épaisses concernant les variations aléatoires de portefeuilles (revenant à supposer l'existence des moments d'ordre 1 et 2 seulement, comme cela peut être observé en pratique sur certains portefeuilles), afin de souligner l'application possible du cadre des métriques XVA pour des exercices de stress test, exploratoires ou inverses (reverse stress test). L'un d'eux concerne la mise en défaut par deux CCPs du gestionnaire d'actifs Ronin Capital suite à la pandémie de Covid 19, en mars 2020. Un autre exemple met en avant l'identification d'un ré-agencement optimal qu'une CCP pourrait opérer suite au défaut de l'un de ses membres avec la reprise du portefeuille en défaut, et des positions symétriques correspondantes de ses clients, par un membre survivant, résultant aux moindres coûts pour l'ensemble de tous les membres survivants. Pour chaque métrique non-analytique XVA (comme les CVA et KVA) et quantiles (pris comme scénario d'exploration de stress test pour un intervalle de confiance donné), nous effectuons des simulations via la technique de Monte Carlo avec des intervalles de confiance non-paramétriques, basés pour ce qui est des quantiles estimés sur [Meeker et al. \(2017, Appendix G\)](#) qui utilise la distribution binomiale des statistiques d'ordres de variables aléatoires uniformes indépendantes.

§2 Monotonie des métriques de provisions de crédit et de capital économique

Etant donné le cadre elliptique considéré pour les variables aléatoires suivant des distributions de Student dans les tests numériques mentionnés dans le §1 et le Chapitre I, on observe que les mesures d'espérance et d'expected shortfall sont croissantes par rapport aux paramètres de corrélation. On peut alors se demander si c'est une propriété du cadre de modélisation consistant à l'application de mesures de risque convexes à des fonctions de variables aléatoires suivant des distributions elliptiques. Il s'avère que les distributions elliptiques appartiennent aux variables aléatoires obéissant à des propriétés dites de super-modularité et que les fonctions de pertes que nous utilisons pour les activités bilatérales et de clearing possèdent également la propriété fonctionnelle de même nom. En appliquant des mesures convexes à ces fonctions de vecteurs super-modulaires, on peut obtenir la monotonie de ces mesures en fonction des coefficients de covariance et donc de corrélation. Ces mesures sont par ailleurs appliquées du point de vue d'un acteur financier de référence, typiquement une banque systémique, également membre d'une CCP, sujette aux activités bilatérales OTC et de clearing, et que nous indexerons par 0.

Contributions du Chapitre II : Dans [Bastide and Crépey \(2024\)](#), qui constitue ce chapitre,

- nous démontrons et illustrons le comportement monotone de mesures de risque convexes pour des variables aléatoires multidimensionnelles ayant la propriété de super-modularité (**Théorème II.12**);
- nous appliquons ces résultats à des modèles elliptiques pour les facteurs régissant les mouvements des portefeuilles et les variables latentes de défaut, soulignant ainsi la monotonie des mesures de risque comme les opérateurs d'espérance et d'expected shortfall [ES] par rapport aux coefficients de corrélation entre ces facteurs (**Proposition II.13**, **Proposition II.14**, **Proposition II.16**, **Corollary II.17**), et que nous illustrons numériquement;
- nous étendons également les résultats de [Cousin and Laurent \(2008\)](#) concernant la monotonie des valorisations de tranches equity et senior des dérivés de titres garantis par des créances (en anglais *collateralized debt obligations* [CDO]) avec des modèles à variables latentes de défaut (**Proposition II.7** et **Corollary II.8**).

Une fonction super-modulaire est une fonction $f : \mathbb{R}^n \rightarrow \mathbb{R}$ qui, pour tout $x \in \mathbb{R}^n$, tout $\varepsilon > 0$, tout $\delta > 0$ et toute paire d'indices $i < j$ dans $1..n$, vérifie

$$\begin{aligned} f(x_1, \dots, x_i + \varepsilon, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i + \varepsilon, \dots, x_j, \dots, x_n) \geq \\ f(x_1, \dots, x_i, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_j, \dots, x_n). \end{aligned} \quad (4)$$

Les variables aléatoires que nous considérons par la suite dans cette section sont supposées appartenir à un sous-espace linéaire fermé \mathcal{X} de $L^1(\mathbb{Q})$, défini à partir d'un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{Q})$, avec \mathbb{E} l'opérateur d'espérance correspondant, et considéré sans atomes, c'est-à-dire que pour tout $\omega \in \Omega$, $\mathbb{Q}(\{\omega\}) = 0$. Ces hypothèses permettent que les résultats de [Müller \(1997\)](#); [Bäuerle and Müller \(1998\)](#); [Müller and Scarsini \(2000\)](#); [Müller and Stoyan \(2002\)](#); [Bäuerle and Müller \(2006\)](#) puissent s'appliquer sans particulièrement contraindre le cadre de modélisation.

Un vecteur aléatoire $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$ est plus petit que le vecteur aléatoire $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ pour l'ordre super-modulaire, que l'on écrit aussi $\mathbf{X} \leq_{sm} \mathbf{Y}$, si l'inégalité $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$ est vérifiée pour toutes les fonctions super-modulaires $f : \mathbb{R}^m \rightarrow \mathbb{R}$ telles que les espérances écrites existent.

On définit aussi l'ordre stop-loss de la manière suivante: \mathcal{X} précède \mathcal{Y} , tous deux dans \mathfrak{X} , pour l'ordre stop-loss, écrit $\mathcal{X} \leq_{sl} \mathcal{Y}$, si $\mathbb{E}[\mathcal{X} - A]^+ \leq \mathbb{E}[\mathcal{Y} - A]^+$ est vraie pour toute constante $A \geq 0$, ce qui équivaut (Bäuerle and Müller, 1998, Theorem 2.2 b)) à ce que $\mathbb{E}(f(\mathcal{X})) \leq \mathbb{E}(f(\mathcal{Y}))$ soit vraie pour toute fonction croissante convexe $f : \mathbb{R} \rightarrow \mathbb{R}$ et telle que les espérances écrites existent (c'est en particulier le cas pour $f = id$). Cet ordre permet d'obtenir le résultat énoncé dans Bäuerle and Müller (2006, Theorem 4.4) concernant la monotonie d'une mesure de risque ρ définie sur un espace probabilisé sans atomes, soit $[\mathcal{X} \leq_{sl} \mathcal{Y}] \Rightarrow \rho(\mathcal{X}) \leq \rho(\mathcal{Y})$.

La définition de mesure de risque que nous employons relâche l'hypothèse d'équivariance par translation couramment utilisée dans la théorie des mesures de risque (voir par exemple Shapiro (2013, Introduction (A3))). Elle comprend alors les 4 propriétés suivantes: (i) la mesure de risque ρ est dite propre ($\forall \mathcal{X} \in \mathfrak{X}, \rho(\mathcal{X}) > -\infty$, et $\text{dom } \rho := \{\mathcal{X} \in \mathfrak{X}; \rho(\mathcal{X}) < +\infty\} \neq \emptyset$); (ii) ρ est invariante pour les variables aléatoires de même loi ($\mathcal{X} \stackrel{d}{=} \mathcal{Y} \Rightarrow \rho(\mathcal{X}) = \rho(\mathcal{Y}), \mathcal{X}, \mathcal{Y} \in \mathfrak{X}$); (iii) ρ est monotone ($\mathcal{X} \leq \mathcal{Y} \Rightarrow \rho(\mathcal{X}) \leq \rho(\mathcal{Y}), \mathcal{X}, \mathcal{Y} \in \mathfrak{X}$) et (iv) ρ est convexe ($\rho(\lambda \mathcal{X} + (1 - \lambda)\mathcal{Y}) \leq \lambda \rho(\mathcal{X}) + (1 - \lambda)\rho(\mathcal{Y}), \forall \lambda \in (0, 1), \mathcal{X}, \mathcal{Y} \in \mathfrak{X}$). Les mesures de risque que nous employons dans Bastide and Crépey (2024) sont celles d'espérance et d'expected shortfall comme définie par Acerbi and Tasche (2002, Definition 2.6) à partir de la probabilité \mathbb{Q} . Ces deux mesures obéissent bien aux 4 propriétés précédemment citées sur l'espace $L^1(\mathbb{Q})$.

L'argument de départ pour obtenir nos résultats de monotonie des mesures de risque en fonction de l'ordre super-modulaire repose sur le **Lemme II.1** que nous rappelons ci-dessous:

Lemma .1. *Si $\mathbf{X} \leq_{sm} \mathbf{Y}$ dans \mathfrak{X} avec \mathbf{X}, \mathbf{Y} à valeurs dans \mathbb{R}^m , alors, pour toute fonction croissante super-modulaire $f : \mathbb{R}^m \rightarrow \mathbb{R}$, on a $f(\mathbf{X}) \leq_{sl} f(\mathbf{Y})$.*

Ce lemme découle directement de Müller and Stoyan (2002, Theorem 3.9.3 f), page 113) qui stipule que si $f : \mathbb{R}^m \rightarrow \mathbb{R}$ est croissante et super-modulaire et $\phi : \mathbb{R} \rightarrow \mathbb{R}$ est croissante et convexe, alors $\phi \circ f : \mathbb{R}^m \rightarrow \mathbb{R}$ est également super-modulaire. Par conséquent, d'après Bäuerle and Müller (2006, Theorem 4.4), si $\rho : \mathfrak{X} \rightarrow \mathbb{R}$ est propre, monotone, convexe, et invariante pour les variables aléatoires de même loi et si $f : \mathbb{R}^m \rightarrow \mathbb{R}$ est croissante super-modulaire, alors $\mathbf{X} \leq_{sm} \mathbf{Y} \Rightarrow \rho(f(\mathbf{X})) \leq \rho(f(\mathbf{Y}))$.

Par ailleurs le résultat Müller and Stoyan (2002, Theorem 3.9.3 f), page 113) permet de compléter les résultats de Cousin and Laurent (2008) concernant les propriétés de monotonie des valorisations de tranches equity et senior de CDO dans des modèles à variables latentes de défaut. Il suffit pour cela d'observer que les flux financiers correspondant de ces tranches s'écrivent comme des fonctions croissantes convexes composées avec des fonctions croissantes super-modulaires des variables latentes de défauts des obligataires sous-jacents. Nous illustrons numériquement ce résultat dans **II.§3.A**.

Concernant la monotonie des mesures de risque telles que nous les avons introduites auparavant appliquées à des fonctions de pertes des activités bilatérales OTC et de clearing, nous considérons la forme des fonctions de pertes sur une période de temps $[0, T]$ comme $\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)$ avec n un nombre donné de débiteurs (membres, contreparties OTC ou client clearé). Les temps de défaut τ_i de ces débiteurs sont indexés par $i \in 1 \dots n$ et sont modélisés par une variable latente \mathcal{X}_i dépassant un certain niveau B_i , typiquement \mathcal{X}_i représente le niveau des passifs qui peut excéder la somme des niveaux d'actifs et de capital du débiteur i , notée B_i . La variable $g_i(\mathcal{Y}_i)$ représente une perte positive étant donné le défaut du débiteur i . Les fonctions f_i sont positives et croissantes par rapport à chacun de ses arguments et possèdent la propriété de super-modularité (lorsque $n > 1$, le cas $n = 1$ se traitant directement sans difficulté particulière). Les fonctions g_i sont également positives et croissantes.

On établit alors (**Proposition II.11**) que $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i)$ est une fonction croissante super-modulaire sur \mathbb{R}^{2n} . Par exemple, on démontre dans la **Proposition II.16**

que les fonctions de la forme

$$\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \xrightarrow{f} \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}} \mathbb{1}_{\{x_i \geq B_i\}} \times (y_i - m_i)^+ \in \mathbb{R}_+ \quad (5)$$

sont croissantes super-modulaires avec $B_1, \dots, B_n \in \mathbb{R}$, $\beta_1, \dots, \beta_n \in \mathbb{R}_+$. Ces formes de fonction de pertes se retrouvent dans les activités de clearing face aux CCPs comme présentées dans les §1 et **Chapitre I**. Les formes de fonction de pertes pour les expositions bilatérales OTC, ou des clients clearés, sont plus simples et s'expriment comme $\sum_b \mathbb{1}_{\{x_b \geq B_b\}} (y_b - m_b)^+$ où la somme est prise sur toutes les contreparties et clients clearés b auxquelles la banque de référence est exposée, et auquel cas la forme de la fonction de perte est $f_b(x_b)g_b(y_b)$ pour une contrepartie b . Ainsi, d'après le Lemme .1, si $(\mathcal{X}_1, \dots, \mathcal{X}_n, \mathcal{Y}_1, \dots, \mathcal{Y}_n) \leq_{sm} (\mathcal{X}'_1, \dots, \mathcal{X}'_n, \mathcal{Y}'_1, \dots, \mathcal{Y}'_n)$, on obtient

$$\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i) \leq_{sl} \sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n)g_i(\mathcal{Y}'_i). \quad (6)$$

Bäuerle and Müller (2006, Théorème 4.4) permet de conclure que $\rho(\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)) \leq \rho(\sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n)g_i(\mathcal{Y}'_i))$.

Pour $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m) = (X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ et $\mathbf{X}' = (\mathcal{X}'_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m) = (X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ (avec $n > 0$) de telle sorte que $\mathbf{X} \leq_{sm} \mathbf{X}'$, nous définissons les fonctions de pertes comme

$$L = \sum_{i=1}^n f_i(X_1, \dots, X_n)g_i(Y_i) \text{ et } L' = \sum_{i=1}^n f_i(X'_1, \dots, X'_n)g_i(Y'_i), \quad (7)$$

avec les fonctions f_i croissantes super-modulaires et les fonctions g_i croissantes. Rappelons, comme cela est précisé dans les §1 et **Chapitre I**, que les mesures de risque sont prises sous la probabilité de survie de l'acteur financier de référence, notée ici \mathbb{Q}^0 , équivalente à une probabilité de départ, par exemple $\mathbb{Q} = \mathbb{P}^*$ la mesure de probabilité finassurance introduite au §1. Plus précisément, on peut écrire $\mathbb{Q}^0(\cdot) = \mathbb{Q}^*(\cdot, X_0 < B_0)/(1 - \gamma)$ où γ est la probabilité de défaut sous \mathbb{Q}^* sur la période du modèle statique des §1 et **Chapitre I** pour cet acteur financier de référence. Par conséquent, cela exige que $L, L' \in \mathcal{X}^0 = L^1(\mathbb{Q}^0)$ alors que l'ordre super-modulaire est initialement pris sous \mathbb{Q}^* . Néanmoins, le défaut de cet acteur 0 est généré par la variable latente X_0 si bien que l'on peut récrire $\mathbb{Q}^0(\cdot) = \mathbb{E}^*[h(X_0)\mathbb{1}_{\{\cdot\}}]$, avec $h(\cdot) = \mathbb{1}_{\cdot < B_0}/(1 - \gamma)$. Nous privilégions, afin de reproduire la pratique, les vecteurs aléatoires super-modulaires \mathbf{X}, \mathbf{X}' sous \mathbb{Q}^* dont les données sont observables sous cette probabilité. Ils sont par ailleurs supposés avoir la même composante \mathcal{X}_0 et vérifiés $[\mathbf{X}_{1:m} | \mathcal{X}_0] \leq_{sm} [\mathbf{X}'_{1:m} | \mathcal{X}_0]$ c'est-à-dire l'ordre super-modulaire sur les composantes autre que la composante 0 conditionnellement à celle-ci (que l'on écrit également $\mathbb{E}^*[f(\mathbf{X}_{1:m}) | \mathcal{X}_0] \leq \mathbb{E}^*[f(\mathbf{X}'_{1:m}) | \mathcal{X}_0]$ pour toute fonction super-modulaire $f : \mathbb{R}^m \rightarrow \mathbb{R}$). Cette hypothèse permet de démontrer alors que pour h définissant le changement de mesure de \mathbb{Q}^* à \mathbb{Q}^0 , on obtient $\mathbf{X}_{1:m} \leq_{sm^0} \mathbf{X}'_{1:m}$ où \leq_{sm^0} est l'ordre super-modulaire sous \mathbb{Q}^0 .

Pour illustrer numériquement ces résultats, on considère la famille des vecteurs aléatoires de distribution elliptique sous \mathbb{Q}^* de même générateur caractéristique, même moyenne, même première composante \mathcal{X}_0 , même variance pour toutes les composantes, mais dont les covariances $\Gamma_{\mathbf{X}}^{ij} = \text{Cov}^*(\mathcal{X}_i, \mathcal{X}_j)$, $\Gamma_{\mathbf{X}'}^{ij} = \text{Cov}^*(\mathcal{X}'_i, \mathcal{X}'_j)$ (avec Cov^* l'opérateur de covariance sous \mathbb{Q}^*) non-diagonales vérifient $\Gamma_{\mathbf{X}}^{ij} \leq \Gamma_{\mathbf{X}'}^{ij}$ pour $i \neq j \in 1..m$ (et $\Gamma_{\mathbf{X}}^{ii} = \Gamma_{\mathbf{X}'}^{ii}$ pour $i \in 1..m$ concernant les variances). Ces vecteurs sont alors super-modulaires conditionnellement à \mathcal{X}_0 . L'égalité des premières composantes

de \mathbf{X} et \mathbf{X}' impose par ailleurs que $\Gamma_{\mathbf{X}'}^{0j} = \Gamma_{\mathbf{X}}^{0j}$ pour tout $j \in 0..m$ et traduit le fait de garder le même point de vue de l'acteur de référence au travers duquel les mesures de risque sont considérées. Sous ces hypothèses, on a $\mathbf{X}_{1:m} \leq_{sm^*} \mathbf{X}'_{1:m}$ où \leq_{sm^*} est l'ordre super-modulaire sous \mathbb{Q}^* (Block and Sampson, 1988, Corollary 2.3). Il en résulte que $[\mathbf{X}_{1:m}|\mathcal{X}_0] \leq_{sm^*} [\mathbf{X}'_{1:m}|\mathcal{X}_0]$ (**Proposition II.5**) puis que $\mathbf{X}_{1:m} \leq_{sm^0} \mathbf{X}'_{1:m}$ (**Lemma II.6**). On peut ainsi appliquer notre **Lemme .1** avec f de la forme (5) (mais également pour toute fonction super-modulaire croissante de forme plus générale $\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)$) pour des vecteurs elliptiques super-modulaires sous \mathbb{Q}^* gardant cette propriété de super-modularité sous \mathbb{Q}^0 afin d'obtenir une relation d'ordre stop loss du type (6).

En se donnant sous \mathbb{Q}^0 une mesure de risque comme définie précédemment, on obtient la monotonie d'une telle mesure en fonction des coefficients de covariance des vecteurs super-modulaires (et donc des paramètres de corrélation pour ceux s'exprimant comme un facteur positif du coefficient de covariances correspondant). L'espérance et l'expected shortfall vérifient les hypothèses de mesures de risque sur $L^1(\mathbb{Q}^0)$. Par ailleurs l'espérance est associée à la métrique comptable de current expected credit loss utilisée pour les calculs de provisions de crédit (donc de CVA) et l'expected shortfall à une métrique de gestion des risques d'un niveau de capital économique (donc de KVA).

§3 Equilibre de Radner et application à la gestion de défauts d'une CCP

L'aspect d'impact sur les prix de la gestion d'un ou plusieurs défaut(s) par les CCPs est couvert dans un troisième chapitre basé sur [Bastide, Crépey, Drapeau, and Tadese \(2023b\)](#), exemples numériques à l'appui, en exploitant l'analyse convexe et les équilibres de Radner qui constituent un cas particulier des équilibres de Nash. Cet effet d'impact sur les prix concerne les positions propriétaires de membres en défaut sur des CCPs qui n'est pas couvert dans les applications du **Chapitre I** traitant de la résolution du défaut de membres ayant des positions clients. Ce troisième chapitre vise donc à traiter des positions propriétaires de ces membres non concernés par la ré-assignation de positions clients d'un membre en défaut vers un membre survivant. Ces résolutions de membres en défaut par les CCPs et de leurs positions propriétaires sont envisagées avec différentes stratégies que peut implémenter une CCP pour gérer un défaut d'un de ses membres, comme prescrit par les recommandations des institutions Bank for International Settlements [BIS] et International Organization of Securities Commissions [IOSCO] à savoir la couverture puis la mise aux enchères des portefeuilles du membre en défaut, ou sa liquidation sur un marché financier ([Bank For International Settlements, 2019a, 2020](#)). Ces organismes recommandent explicitement aux CCPs d'analyser les impacts de telles stratégies sur les prix des portefeuilles et marchés correspondants.

Contributions du Chapitre III :

- nous exprimons les pertes qu'un participant d'un ou plusieurs marchés peut subir sur ses différents types de portefeuilles qu'il détient suite aux fluctuations de leurs prix, modélisées sur une seule période de temps;
- nous formulons l'impact de défauts sur les prix des titres échangés sur les marchés en procédant à une résolution d'équilibres de Radner dans un réseau sans défaut et sur ce même réseau en considérant des reprises de portefeuilles suite aux défauts;
- nous démontrons l'existence (**Théorème III.2**) et l'unicité sous certaines conditions de tels équilibres (**Théorème III.3**), adaptant des résultats de [Cheridito, Horst, Kupper, and Pirvu \(2015\)](#) à un cadre statique à une période de temps mais pour des variables aléatoires de pertes non-bornées, que nous illustrons de plusieurs exemples couvrant diverses stratégies possibles qu'une CCP peut opérer dans le cas du défaut d'un ou plusieurs de ses membres;
- nous reprenons les formulations de ces impacts de prix dans les expressions de coûts de crédit, capital et financement abordés dans le **Chapitre I** et illustrons numériquement ces applications.

Les principaux risques auxquels sont exposés les acteurs du réseau financier, et pour lesquels les régulateurs requièrent auprès des institutions bancaires des montants de fonds propres suffisants avec un niveau de probabilité de 99.9% ([Basel Committee on Banking Supervision, 2005](#)), se résument en 4 types de risque ([European Bank Authority, 2022](#)): marché, contrepartie, crédit et opérationnel. Le risque de liquidité, lié à la nécessité pour un acteur bancaire d'honorer ses engagements de paiement en devises ou en titres, peut susciter indirectement des besoins supplémentaires en capital. Divers indicateurs permettent de tester la solvabilité d'un acteur bancaire concernant ses engagements contractuels de paiements et de remboursements. Par exemple, le ratio de couverture de liquidité, en anglais liquidity coverage ratio [LCR], est défini comme le ratio du montant d'actifs liquides de très haute qualité divisé par le montant des flux venant à être débités dans les 30 prochains jours ([Crouhy et al., 2014](#), Chapitres 3 et 8). De mauvaises performances de ces indicateurs pour une institution financière peuvent signaler

des perturbations de ses flux opérationnels et mener à des actions défavorables des régulateurs comme des surcharges en capital.

Lorsqu'un acteur financier exposé à une CCP vient à faire défaut, son portefeuille, négocié sur un marché financier doit être géré par la CCP, liée à ce marché. Par exemple, le Groupe EUREX dispose de ses services de marchés pour la négociation des termes des contrats et de ses propres services de clearing une fois négociés. La CCP peut soit totalement ou partiellement couvrir le portefeuille du membre en défaut puis le mettre aux enchères auprès des membres survivants, avec éventuellement d'autres acteurs invités, soit le liquider sur les marchés financiers en le vendant à perte ou en l'achetant au-delà de son prix selon que la CCP ait respectivement initialement acheté ou vendu ce portefeuille (Bank For International Settlements, 2019a, 2020). Ces choix de résolution peuvent générer des pertes sur ce portefeuille, qui seront supportées au-delà du collatéral du membre en défaut par les membres survivants: la CCP ne saurait ni avoir une position agrégée non nulle sur l'ensemble de ses membres survivants ni supporter un coût dû à des pertes du membre en défaut (hormis une partie infime de son capital dénommée le *skin-in-the-game*).

Afin de compléter la quantification du risque de crédit couvert dans le **Chapitre I** au travers des coûts XVA dans un cadre statique à une période, nous proposons de capturer l'impact de la couverture, de la liquidation totale, ou d'une combinaison des deux, d'un portefeuille d'un ou plusieurs membres en défaut d'une CCP. L'utilisation d'équilibre de Radner reposant sur les concepts d'équilibres économiques compétitifs Walrassiens avec incertitude apparaît naturelle dans ce contexte (Radner, 1968, 1972). Un équilibre de Radner est défini par une quantité prédéterminée de chaque titre du marché, négocié à un certain prix unitaire minimisant une métrique d'utilité ou de risque appliquée aux variables aléatoires des flux financiers futurs de chacun des participants à ce marché.

C'est un cas particulier des équilibres de Nash où une contrainte de clearing, en plus des titres communs traités sur le marché, relie l'ensemble des participants et leurs actions de minimisation de leur métrique de risque respective. On montre qu'un tel équilibre de Radner existe et peut se calculer en supposant des flux financiers de distributions elliptiques.

Pour un marché financier donné E , constitué de $m \geq 1$ titres, on considère que chacun de ses participants i supporte une perte de marché sur une période de temps $[0, T]$ où T représente la maturité de l'ensemble des contrats négociés. Chaque perte est exprimée comme

$$\underbrace{-R_i}_{\text{flux aléatoire exogène à } E} + \underbrace{q_i^\top (p - P)}_{\text{couverture}}. \quad (8)$$

Le flux $R_i \in \mathbb{R}$ est promis à maturité T au participant i , et est potentiellement corrélé aux titres de marché de valeur aléatoires $P \in \mathbb{R}^m$ à maturité pour une quantité $q_i \in \mathbb{R}^m$, de prix $p \in \mathbb{R}^m$ au temps 0. Les notations $\mathbf{q}_i \in \mathbb{R}^m$, $i \in E$ et $\mathbf{p} \in \mathbb{R}^m$ sont réservées pour les solutions de l'Equilibre de Radner sur E . Les variables aléatoires R_i et P définies sur un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$ appartiennent à un sous espace vectoriel \mathfrak{X} de $L^1(\Omega)$, contenant les constantes. On note \mathbb{E} , Var et Cov les opérateurs d'espérance, de variance et de covariance respectivement sous la probabilité \mathbb{P} .

Les portefeuilles d'un membre a représentés par les quantités à l'équilibre \mathbf{q}_a , face à une CCP traitant pour un marché E , peuvent contenir à la fois des positions de couverture (dites aussi de proprietary trading, enregistrées dans le compte propriétaire du membre, connu sous le terme *house* en anglais) et des positions de clearing du membre de la CCP (enregistrées dans le compte client du membre). Les couvertures sont vis-à-vis de l'ensemble des positions $R_a = \sum_{o \in O} R_a^o$ (où l'acteur financier o est en dehors du marché traité par la CCP) prises sur des marchés OTC et non traités par les CCPs. Les positions de clearing sont back-to-back, c'est-à-dire résulte en deux flux l'un étant l'opposé de l'autre, et

notées \mathbf{q}_b^a . Elles sont l'effet du rôle intermédiaire de la CCP entre un acteur financier b n'accédant pas à la CCP, par choix ou par contraintes dues à des ressources insuffisantes pour prétendre à un accès direct à la CCP, et le membre a , pour s'aligner aux demandes réglementaires (G20 Research Group, 2009).

Notons $E = A \cup B$, avec $A \cap B = \emptyset$, A l'ensemble des participants a ayant directement accès aux services de la CCP pour le marché représenté par ses participants E et B l'ensemble des participants b non membre de la CCP devant recourir aux participants de A pour y accéder indirectement. q_a définit une position de proprietary trading du membre a et q_b^a la position d'un de ses clients noté b pour lequel a exerce le rôle de membre de clearing. La position q_b^a ne fait alors que transiter de b vers a puis de a vers la CCP. Ainsi, même si la position totale de a envers la CCP est $q_a + \sum_{b \in B} q_b^a$, comme a détient $-\sum_{b \in B} q_b^a$ vis-à-vis de ses clients dont il est le membre de clearing, les positions de a et b impliquées dans la résolution de l'équilibre de Radner sur E sont respectivement q_a et $q_b = \sum_{a \in A} q_b^a$.

Pour ce qui est du cas pré-défaut, la perte de marché à l'équilibre supportée par un membre $a \in A$ sur le marché E s'exprime

$$-R_a + \mathbf{q}_a^\top (\mathbf{p}^E - P) = -\sum_{o \in O} R_a^o + \mathbf{q}_a^\top (\mathbf{p}^E - P)$$

et celle d'un participant b , non membre de la CCP traitant des positions prises sur E s'écrit

$$-R_b + \mathbf{q}_b^\top (\mathbf{p}^E - P) = -R_b + \left(\sum_{a \in A} \mathbf{q}_b^a \right)^\top (\mathbf{p}^E - P).$$

Il n'est pas nécessaire de connaître les termes $(\mathbf{q}_b^a)_{a \in A}$ mais seulement \mathbf{q}_b .

Lorsqu'une CCP choisit de couvrir le portefeuille du membre en défaut, elle devient un participant du marché sur lequel la couverture est prise. Alternativement elle peut choisir de liquider cette position sur le marché où elle opère ou sur un autre marché et ne fait pas partie des participants du marché où la position est liquidée. Une combinaison de ces deux stratégies de couverture et de liquidation sur un même marché ou des marchés différents est possible. Pour chaque marché E avant un évènement de défaut, dit pré-défaut, dont les titres ont pour prix initiaux à l'équilibre \mathbf{p} et considérés au temps 0-, on note E' et on qualifie de post-défaut ce même marché après l'avènement d'un ou plusieurs défauts instantanés d à la date 0. La composition en quantité et prix de titres de E' dépendent de la procédure de résolution considérée par la CCP avec des nouveaux prix des titres \mathbf{p}' considéré au temps 0.

Pour tout membre i sur un marché post-défaut E' dont l'ensemble des participants pré-défaut est E , sa part incrémentale de positions reprises $\Delta \mathbf{q}_i$ se décompose en une part $\Delta \mathbf{q}_i^l$ due à la liquidation des positions \mathbf{q}_d^l des membres en défaut, et une part $\Delta \mathbf{q}_i^h$ due à la couverture par la CCP des positions \mathbf{q}_d^h avec $\mathbf{q}_d = \mathbf{q}_d^l + \mathbf{q}_d^h$, $\mathbf{q}_d^l = \sum_{i \in E'} \Delta \mathbf{q}_i^l$ et $\mathbf{q}_d^h = \sum_{i \in E'} \Delta \mathbf{q}_i^h$. La quantité à l'équilibre post-défaut de i est notée $\mathbf{q}'_i = \mathbf{q}_i + \Delta \mathbf{q}_i$ avec $\mathbf{q}_i = 0$ si i est un nouveau participant, initialement non participant de E . La liquidation se fait sur la base des prix pré-défaut et contractuels \mathbf{p} alors que l'intervention de la CCP sur le marché pour couvrir une partie du portefeuille du membre en défaut génère des nouveaux prix \mathbf{p}' . La perte de marché post-défaut pour $i \in E'$ s'écrit alors (voir III.§3.B)

$$\underbrace{-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P)}_{\text{perte de marché au prix post-défaut}} + \underbrace{(\mathbf{q}_i + \Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - \mathbf{p}'^E)}_{=: LC_i \text{ (liquidation cost)}}, \quad (9)$$

où LC représente le coût de la liquidation des portefeuilles de membres en défaut. Il peut s'interpréter comme des paiements d'appel de marge, comme pour les marchés de contrats futures, pour transiter d'un marché pré-défaut basé sur des prix \mathbf{p} à un marché post-défaut basé sur des prix \mathbf{p}' . Ces appels

de marges, payés par tout participant survivant $i \in E$, ne s'appliquent que sur les contrats basés sur les anciens prix, y compris ceux qui sont liquidés par la CCP.

Afin de résoudre les équilibres de Radner, pré comme post-défaut, la minimisation du risque de tout participant $i \in E$ se traduit par une minimisation d'une mesure convexe et homogène positive de risque ρ_i appliquée à sa perte de marché. Cette minimisation se fait d'abord en considérant qu'aucun défaut ne s'est produit, puis suite à la réalisation d'un ou plusieurs défaut(s), pour établir les prix et positions à l'équilibre avant et après un évènement de défaut. Le coût du marché qui en résulte, noté MC_i , pour le participant $i \in E'$ est défini comme la différence de cette mesure de risque du participant i entre le marché ayant subi un évènement de défaut et le marché avant tout évènement de défaut, tous deux pris à l'équilibre:

$$MC_i = LC_i + \underbrace{\rho_i(-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P)) - \rho_i(-R_i + (\mathbf{q}_i)^\top (\mathbf{p}^E - P))}_{=:\Delta\rho_i}.$$

Le coût de liquidation global est $LC_E = \sum_{i \in E'} LC_i$ et son coût de marché global est $MC_E = \sum_{i \in E'} MC_i$, redistribué sur l'ensemble des membres de la CCP. Si la CCP choisit d'implémenter une telle stratégie de liquidation et couverture d'une partie du portefeuille d'un de ses membres en défaut sur plusieurs marchés, les coûts de marché et de liquidation sont agrégés sur l'ensemble de ces marchés: $MC = \sum_E MC_E$. Ce coût représente ainsi le prix des transferts de fonds (FTP pour funds transfer price) puisqu'il reflète le coût de transférer des positions en défaut à des participants survivants et/ou nouveaux du marché.

La condition de clearing, pour un marché E n'ayant pas subi de défaut, s'écrit $\sum_{i \in E} \mathbf{q}_i = 0$, $\mathbf{q}_i \in \mathbb{R}^m$, correspondant à une quantité de zéro pour tout titre du marché considéré. Pour chaque participant i , on considère une mesure de risque $\rho_i: \mathfrak{X} \rightarrow \mathbb{R}$, distribution invariante, monotone, convexe, équivariante par translation ($\rho_i(L + m) = \rho_i(L) + m$ pour tout $L \in \mathfrak{X}$ et $m \in \mathbb{R}$, dit aussi cash-équivariante) et normalisée ($\rho_i(0) = 0$), conduisant, pour \mathbf{p} donné, à une condition d'optimalité

$$\rho_i(-R_i + \mathbf{q}_i^\top (\mathbf{p} - P)) \leq \rho_i(-R_i + q_i^\top (\mathbf{p} - P)), q_i \in \mathbb{R}^m. \quad (10)$$

Un équilibre de Radner sur un marché représenté par son ensemble de participants E consiste en une paire $\{(\mathbf{q}_i)_{i \in E}, \mathbf{p}\}$ vérifiant la condition de clearing pour ce qui est des quantités et les conditions d'optimalité de tout participant $i \in E$. La résolution d'un équilibre de Radner repose sur les propriétés des mesures convexes ρ_i employées par chacun des participants i . Celles-ci doivent être convexes, monotones (pour $L_1, L_2 \in \mathfrak{X}$, $L_1 \leq L_2 \Rightarrow \rho_i(L_1) \leq \rho_i(L_2)$), équivalentes par translation, invariantes pour des variables aléatoires de même loi (pour $L_1 \stackrel{d}{=} L_2$, $\rho_i(L_1) = \rho_i(L_2)$), normalisées ($\rho_i(0) = 0$) et sensible aux grandes pertes ($\lim_{\lambda \rightarrow \infty} \rho_i(\lambda L) = \infty$ pour tout $L \in \mathfrak{X}$ tel que $\mathbb{P}[L > 0] > 0$). Par cash-équivariance, on réécrit la condition d'optimalité pour tout participant $i \in E$ comme $\rho_i(-R_i - \mathbf{q}_i^\top P) + \mathbf{q}_i^\top \mathbf{p} \leq \rho_i(-R_i - q_i^\top P) + q_i^\top \mathbf{p}$, $q_i \in \mathbb{R}^m$. Posant $r_i(q_i) = \rho_i(-R_i - q_i^\top P)$, on définit l'inf-convolution des mesures de risque des participants comme

$$r(q) = \inf \left\{ \sum_{i \in E} r_i(q_i); \sum_{i \in E} q_i = q \right\}, \quad q \in \mathbb{R}^m. \quad (11)$$

Un équilibre de Radner peut s'exprimer de manière équivalente avec les trois caractéristiques suivantes basées sur l'inf-convolution (11) (**Lemme III.1**): **(i)** $-\mathbf{p} \in \partial r(0)$, **(ii)** $r(0) = \sum_{i \in E} r_i(\mathbf{q}_i)$, et **(iii)** $\sum_{i \in E} \mathbf{q}_i = 0$. Le **Lemme III.1** combiné à l'hypothèse de sensibilité aux grandes pertes des mesures de risque ρ_i garantit l'existence d'un équilibre de Radner (**Théorème III.2**). L'unicité des prix peut

être obtenue si les mesures de risque ρ_i exprimées comme des fonctions des quantités détenues par tout participant i sont différentiables au vecteur de leur position d'équilibre respective \mathbf{q}_i (la sous-différentielle $\partial r_i(\mathbf{q}_i)$ au point \mathbf{q}_i est le singleton gradient $\nabla r_i(\mathbf{q}_i)$ évalué en ce même vecteur) alors que l'unicité des positions d'équilibre est elle garantie si ces mêmes fonctions sont différentiables et strictement convexes (**Théorème III.3**). Dans le cas de mesures de risque entropiques ou d'expected shortfall avec (P, R_i) pris comme des vecteurs de distributions elliptiques, il est possible d'obtenir une formule analytique unique pour les positions et les quantités à l'équilibre (voir **Proposition III.4**, **Proposition III.5** et **Remark III.5**).

Au coût de marché doit s'ajouter les coûts du crédit qui capture le risque de contrepartie au sens large, y compris ceux induits par les financements et besoins en capital pour constituer la totalité du FTP. Ils reposent sur l'exposition indirect de chaque membre de la CCP vis-à-vis de tous les autres membres de cette même CCP et sur les coûts de financement pour maintenir une position face à la CCP. Si la position d'un membre change, suite au ré-arrangement de ses positions en réponse à une stratégie de gestion de défaut par la CCP, les autres membres de cette CCP vont être impactés aux termes de leurs coûts XVA comme détaillé dans [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#). Pour des fluctuations de portefeuilles supposées gaussiennes, et pour une CCP donnée avant un évènement de défaut, l'équation (1) s'écrit pour le membre 0 sur le marché pré-défaut $E = A \cup B$,

$$\begin{aligned} \mathcal{C}_0 &= \mathcal{C}_0^A + \mathcal{C}_0^B + \mathcal{C}_0^O, \text{ avec} \\ \mathcal{C}_0^A &\stackrel{(1) \& (2)}{=} w_0^A \sum_{a \in A} (1 - J_a) \left[\left(\left(\sum_{b \in B} \mathbf{q}_b^a \right)^\top (\mathbf{p}^E - P) - \text{IM}_a^A \right)^+ + \right. \\ &\quad \left. \left(\mathbf{q}_a^\top (\mathbf{p}^E - P) - \overline{\text{IM}}_a^A \right)^+ - \text{DF}_a^A \right]^+, \\ \mathcal{C}_0^B &= \sum_{b \in B} (1 - J_b) \left((\mathbf{q}_b^0)^\top (\mathbf{p}^E - P) - \text{IM}_0^b \right)^+ \text{ et } \mathcal{C}_0^O = \sum_{o \in O} (1 - J_o) (R_0^o - \text{IM}_0^o)^+. \end{aligned} \quad (12)$$

w_0^A (nul si $0 \notin A$) est le coefficient d'allocation de perte assujéti au membre 0 par la CCP du marché, IM_a^A , $\overline{\text{IM}}_a^A$, et DF_a^A sont les marges initiales et contributions au fonds de défaut comme détaillés dans le **Chapitre I** (résumés dans les équations (1) et (2)). R_0^o est le flux exogène au marché E et reçu du participant 0 de ses contreparties bilatérale o sur les marchés OTC, avec une marge initiale correspondante IM_0^o (qui peut être nulle pour certains engagements de ce type). Après un évènement de défaut, une formulation similaire s'applique en mettant à jour les quantités w_0^A , IM_a^A , $\overline{\text{IM}}_a^A$, DF_a^A et IM_0^b dans (12).

On peut alors calculer les coûts XVA avant et après la gestion par la CCP d'un évènement de défaut, complétant ainsi les coût de marché en utilisant leurs formules explicites (**Table 2** extraite des **Théorème I.4** et **Proposition I.5**). L'ensemble des coûts supportés par les membres survivants s'exprime comme

$$\text{FTP} = \underbrace{\text{LC} + \sum_{E'} \sum_{i \in E'} \Delta \rho_i}_{\text{market cost (MC)}} + \underbrace{\sum_{i \neq d} (\text{XVA}'_i - \text{XVA}_i)}_{\text{credit cost (CC)}} + \text{AC}, \quad (13)$$

avec, pour chaque participant survivant i , $\text{XVA}_i = \text{CVA}_i + \text{FVA}_i + \text{MVA}_i + \text{KVA}_i$, XVA'_i ces montants XVA calculés après la gestion du défaut menée par la CCP et AC le coût dû à la mise aux enchères par les CCPs des portefeuilles de membres ayant fait défaut et incluant leur couverture. Ce coût est un autre coût incrémental en XVA, correspondant au prix de transfert (Fund Transfer Price [FTP]) détaillé dans **III.§7** (qui n'inclut que des coûts de crédit), c'est-à-dire l'impact sur les coûts XVA de la mise aux

enchères de ces portefeuilles de membres en défaut comprenant les instruments de couverture utilisés par les CCPs.

§4 Algorithmes de recuit simulé pour l'identification de ré-arrangements optimaux de réseaux financiers suite à un défaut de clearing member

Dans cette quatrième partie, nous envisageons des scénarios plus catastrophiques où le défaut d'un membre ne se produit pas seulement sur une seule CCP mais sur une multitude d'entre elles. Les illustrations numériques du premier chapitre emploient une approche directe en testant toutes les reprises possibles du portefeuille du membre en défaut sur une unique CCP, ce qui n'est pas possible dans le cas de plusieurs portefeuilles dû au problème combinatoire de tous les scénarios de reprises possibles. Un tel scénario de défaut sur plusieurs CCPs aurait pu réellement se produire au début de l'année 2023 avec les difficultés de la banque Credit Suisse, un acteur majeur sur une trentaine de CCPs, si la Banque Nationale Suisse n'était pas intervenue en lui octroyant un apport de liquidité de l'ordre de €170 mds (CHF 168 mds, voir [Jordan \(2023\)](#)). Un exemple plus réduit, mais bien réel, avec un défaut sur deux CCPs s'est produit aux Etats-Unis avec le gestionnaire d'actifs Ronin Capital et est illustré dans le premier chapitre. En ce qui concerne un réseau de quelques dizaines de CCPs (environ 50 CCPs majeures dans le monde) et centaines de membres (300 à 400), le nombre de combinaisons étant de l'ordre de 10^{130} , même pour des réseaux de petites tailles (comme dix CCPs et 200 membres), il semble ubuesque d'envisager une quelconque exploration de l'espace de ces combinaisons. Le recourt à des techniques d'optimisation stochastique s'impose donc naturellement. Nous illustrons l'application d'algorithmes de recuit simulé discret sur un problème de reprises impliquant quelques CCPs et les comportements numériques attendus pour ce type de technique.

Contributions du Chapitre IV :

- nous décrivons le problème de ré-allocation de portefeuilles d'un membre en défaut sur plusieurs CCPs vis-à-vis de leurs membres survivants; nous le formulons comme solution d'un problème combinatoire de minimisation sur un espace fini d'une fonction non-linéaire et non-convexe de variables aléatoires approximée par des techniques de Monte Carlo; ces variables représentent les flux contractuels futurs de ces portefeuilles et des variables latentes de défaut pour chacun des membres;
- nous résumons les résultats principaux de convergence et de contrôle d'erreurs des algorithmes de recuit simulé discret sur des espaces finis en les mettant dans le contexte de notre problème; pour un nombre donné d'itérations, nous en déduisons des résultats de contrôles d'erreurs entre la valeur de la fonction approximée de l'état retourné par l'algorithme et la valeur minimum de la fonction sans approximation (**Théorème IV.7, Corollaire IV.8**);
- nous obtenons également un contrôle de la probabilité d'erreur entre la combinaison proposée par la version approximative de l'algorithme et la vraie solution du problème non-approximé (**Proposition IV.9**); ces résultats sont obtenus en utilisant des inégalités de concentration résultant des erreurs d'approximation et des erreurs connues des algorithmes de recuit simulé;
- nous spécifions l'application du recuit simulé pour un échantillon permettant d'approximer le vrai problème, avec une phase d'exploration spécifique à la structure financière découlant des CCPs et de leur membres en soulignant le caractère de distribution des pertes de type à queues lourdes que nous pouvons confirmer sur la base de tests statistiques;
- nous illustrons numériquement le bon fonctionnement de l'algorithme sur un ensemble de 4 CCPs

et 11 membres, dont l'un mis en défaut; la performance de l'algorithme ainsi défini surpasse significativement l'approche naïve de suggestions aléatoires successives sans remise de repreneurs.

Le **Chapitre I** aborde l'identification d'un ensemble de repreneurs à moindre coût XVA de portefeuilles, sans subdivision, suite au défaut d'un membre d'une ou plusieurs CCPs en parcourant toutes les reprises possibles, reflétant ainsi une mise-aux-enchères idéalisée où chaque participant propose un prix permettant d'aboutir à la reprise minimisant les coûts XVAs sur l'ensemble du réseau financier. Cette stratégie devient vite très lourde numériquement dans le cas d'un défaut sur plusieurs CCPs avec un nombre considérable de combinaisons de repreneurs. Par ailleurs, bien que les coûts concernés soient de différents types à savoir de crédit (CVA, BCVA et CCVA), de financement (FVA, MVA, BMVA et CMVA) et de capital (KVA), nous considérons une minimisation des seuls coûts de capital. Ce type de coût est en effet prépondérant parmi les autres, comme illustré dans la **Section I.5**.

L'application d'algorithmes de recuit simulé discret (Kirkpatrick et al., 1982, 1983; Aarts and van Laarhoven, 1989; Aarts and Korst, 1989; Catoni, 2004; Duflo, 1996; Delmas and Jourdain, 2006) pour des problèmes impliquant un grand nombre de combinaisons en finance est surtout connu comme une des techniques de la boîte à outils de résolution des problèmes d'optimisation de portefeuilles (Ingber, 1993; Crama and Schyns, 2003). Les dernières avancées sur l'utilisation d'ordinateurs quantiques ont permis des adaptations de ce type d'algorithmes pour ce genre de problème (Lang et al., 2022). Bien qu'il soit possible d'appliquer ces algorithmes pour capturer des scénarios averses menant une institution bancaire à l'état de défaut (Montesi et al., 2020), son application pour identifier une restructuration possible du réseau financier au travers d'une ré-allocation des portefeuilles dans ce réseau semble nouvelle.

Pour tout participant $\ell = 0, \dots, L$, on note $(Y_\ell^1, \dots, Y_\ell^K)$ l'ensemble des variables aléatoires d'un sous-espace vectoriel $\mathcal{X} \subset L^1(\Omega)$, avec $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé, où Y_ℓ^k traduit les flux futurs sur un horizon de temps T de son portefeuille face à la CCP k , pour $k = 1, \dots, K$. Pour $\ell = 1, \dots, L$ et une combinaison de repreneurs $\mathbf{i} = (i_1, \dots, i_K) \in \mathcal{I} = \{1, \dots, L\}^K$, la variable $(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_\ell^k$ traduit la mise à jour du portefeuille du participant ℓ suite à la reprise du portefeuille du membre 0, considéré en défaut, par le repreneur i_k sur la CCP k . Pour $M \in \mathbb{N}^*$, on note \mathcal{R}_ℓ^M la mesure approximative de capital du participant ℓ d'une mesure \mathcal{R}_ℓ sur l'ensemble de ses portefeuilles détenus sur les K CCPs. Cette mesure est appliquée à une fonction de perte g_ℓ qui dépend d'un échantillon de taille M de l'ensemble des variables de variations de portefeuilles après reprise par les repreneurs $\mathbf{i} = (i_1, \dots, i_K)$, soit $\left((\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_1^1, \dots, (\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_1^K, \dots, (\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_L^1, \dots, (\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_L^K \right) =: \mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0$. On peut alors écrire pour une combinaison de repreneurs \mathbf{i} donnée la fonction (l'Hamiltonien) de coût à minimiser comme somme des mesures de risque approximatives sur l'ensemble des participants $(1, \dots, L)$ appliquées aux variables aléatoires après reprise :

$$H_M : \mathcal{I} \longrightarrow \mathbb{R}$$

$$\mathbf{i} \longmapsto \sum_{\ell=1}^L \mathcal{R}_\ell^M \left[g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right]. \quad (14)$$

On cherche ainsi $\arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$, sur la base d'un échantillon de taille M des variables aléatoires de portefeuilles que l'on considère communs pour chaque valeur $H_M(\mathbf{i})$ à calculer (Glasserman and Yao, 1992). Ce problème approxime le vrai problème qui s'écrit de manière similaire à (14) en remplaçant les mesures approximées \mathcal{R}_ℓ^M par leur version non-approximée \mathcal{R}_ℓ , $\ell = 1, \dots, L$.

Etant donnée la forme de l'Hamiltonien H_M à minimiser en fonction de \mathbf{i} , où il est difficile d'identifier une quelconque structure linéaire ou convexe par rapport aux combinaisons $\mathbf{i} \in \mathcal{I}$ et qui

possède de nombreux minima locaux (voir notamment l'exemple dans la **Section IV.§4.E**), l'utilisation d'un algorithme de recuit simulé paraît justifié puisqu'il permet, sous certaines conditions et pour de telles fonctions, une convergence vers une distribution attribuant toute sa masse au minimum de cet Hamiltonien. Cette même distribution, lorsqu'elle est exprimée en fonction d'un paramètre de contrôle convergeant vers 0, s'écrit comme la limite d'une distribution de type Boltzmann-Gibbs qui attribue au minimum de la fonction à minimiser la plus grande probabilité, dépendant de ce paramètre de contrôle.

Pour un nombre fixe d'itérations N , qui indique le budget donné pour l'exécution de l'algorithme, et pour $\varepsilon > 0$, on obtient des bornes d'erreurs, similaires à des inégalités de concentration ([Boucheron et al., 2013](#)) mais avec une composante d'erreur due à l'erreur algorithmique (dans notre cas, celle du recuit simulé, voir [Catoni \(2004\)](#) et [Delmas and Jourdain \(2006\)](#)), entre la valeur de l'Hamiltonien approximatif appliquée à la combinaison obtenue en faisant tourner l'algorithme de recuit simulé et celle non-approximée appliquée à la combinaison optimale du problème non-approximé (**Théorème IV.7**). En notant $\mathbf{i}^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i}) =: \mathcal{I}^{opt}$ et \mathbf{I}_M^N la solution renvoyée par l'algorithme après N itérations pour un échantillon de taille M , on peut ainsi écrire:

$$\mathbb{P} \left(\sqrt{M} |H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)| > \varepsilon \right) \leq B_1(|\mathcal{I}|, \varepsilon, M, N),$$

avec $B_1(|\mathcal{I}|, \varepsilon, M, N)$ une certaine borne déterministe convergeant vers 0 pour $\varepsilon, M, N \rightarrow \infty$ et $|\mathcal{I}| = L^K$. De manière similaire, il est possible d'obtenir une borne de la probabilité d'erreur de d'estimation (**Corollaire IV.8**):

$$\mathbb{P} \left(\sqrt{M} |H(\mathbf{I}_M^N) - H(\mathbf{i}^*)| > \varepsilon \right) \leq B_2(|\mathcal{I}|, \varepsilon, M, N),$$

avec $B_2(|\mathcal{I}|, \varepsilon, M, N)$ une autre borne déterministe convergeant vers 0 pour $\varepsilon, M \rightarrow \infty$.

On peut en déduire des probabilités d'erreur que l'algorithme ne renvoie pas la bonne solution par rapport à celle du problème approximé, et également par rapport à celle du problème non-approximé:

$$\mathbb{P} (I_M^N \notin \mathcal{I}_M^{opt}) \leq B_3(M, N) \quad \text{and} \quad \mathbb{P} (I_M^N \notin \mathcal{I}^{opt}) \leq B_4(M, N, \mathcal{H}_2 - \mathcal{H})$$

avec $\mathcal{I}_M^{opt} := \arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$, $\mathcal{H} = \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i})$, \mathcal{H}_2 la deuxième plus petite valeur des $\{H(\mathbf{i})\}_{\mathbf{i} \in \mathcal{I}}$, $B_3(M, N)$ et $B_4(M, N, \mathcal{H}_2 - \mathcal{H})$ des bornes déterministes convergeant vers 0 pour $M, N \rightarrow \infty$.

Ces bornes d'erreurs peuvent être précisées dans divers cas notamment pour des variables aléatoires de distribution à queues épaisses ([Embrechts et al., 1997](#); [Nair et al., 2022](#)) comme les fonctions de pertes $g_\ell \left(\mathbf{Y} \bigoplus_{\mathbf{i}} \mathbf{Y}_0 \right)$, $\ell = 1, \dots, L$ que nous retrouvons dans notre problème. Cette caractéristique peut être testée pour diverses classes de distributions dont un cadre générique est décrit dans ([Gardes et al., 2011](#)). Elle est en particulier vérifiée pour les fonctions de pertes de notre problèmes qui montrent des queues de distribution de Pareto. Nous pouvons alors préciser les bornes d'erreurs lorsque les mesures de risque sont des espérances, des quantiles ou de type expected shortfall ([Reiss, 1989](#); [Marchina, 2021](#); [Chamakh et al., 2020, 2021](#)). Dans le cas d'une expected shortfall sans connaître la valeur exacte du quantile correspondant, une estimation basée sur un quantile de plusieurs moyennes peut permettre d'assurer la consistance et une certaine robustesse de l'estimateur ([Lugosi and Mendelson, 2019](#); [Bartl and Eckstein, 2024](#)).

Afin de permettre une bonne performance de l'algorithme, la phase d'exploration doit capturer la structure des positions et du réseau financier constitué des CCPs et de leurs membres. Nous proposons une matrice stochastique couvrant partiellement cette exigence mais qui ne retient pas la combinaison de l'étape précédente de la chaîne de Markov correspondante. Bien que les résultats numériques, dans le cas de 4 CCPs et 11 membres dont un en défaut, montrent des performances significativement meilleures qu'une exploration et exploitation naïves de combinaisons de repreneurs sans remises

(Section IV.§4.E), une amélioration devrait être envisagée où l'état précédent influe sur la matrice stochastique de la phase d'exploration. Par ailleurs, le seul choix du coût de capital comme critère de minimisation peut souligner un comportement irrationnel de la part des membres survivants, qui ne devraient pas considérer comme seul critère ce coût de capital mais d'autres contraintes économiques, et ce qui peut être une limite de l'application d'un tel algorithme.

Introduction

Following the subprime crisis, whose roots date back to the American real estate bubble in the 2000s leading to the bankruptcy of Lehman Brothers in September 2008, a crisis of liquidity and solvency of financial institutions, but also of countries in the Euro zone, affected the entire financial sector until the end of 2010. Repercussions followed during 2011 and 2012 via the European debt crisis with Greece's difficulties in paying its debts, the nationalization of several banks in Ireland (Anglo Irish Bank in 2009, Educational Building Society limited and Irish Nationwide Building Society in 2010, and Irish Life and Permanent in 2011, see [Laeven and Valencia \(2012, Table A.3\)](#)) and rescue plans with significant liquidity funding via facilitated loans from the European Union for Portugal, Spain and Cyprus (€77 billion, €41 billion and €7.3 billion respectively, with liquidity funds from EU and IMF funds, see [Gourinchas, Martin, and Messer \(2023, Table 1\)](#)). As a response, international standardization bodies, such as the Basel Committee on Banking Supervision [BCBS] and the Committee on Payments and Market Infrastructures [CPMI] established, in 2008, instructions to banking establishments in order to guarantee better robustness of the financial system. These instructions aim to limit the risks of contagion of defaults of players, mainly banks, in the financial network, both regarding the lack of liquidity of financial establishments and the impact on capital or equity levels required by these establishments. The United States adopted the Dodd-Frank Wall Street Reform and Consumer Protection Act ([The United States Congress, 2010](#)) in 2010, while the European Union adopted regulations of European Market Infrastructure Regulation ([The European Parliament and the Council of the European Union, 2012](#)). This was followed by the updates of the Basel texts that guarantee international agreements governing the solvency ratios of financial institutions. An important component of this regulatory tool box consists in conducting annually stress exercises (also called stress tests) by the authorities of banking institutions such as the European Banking Authority [EBA] and the United States Federal Reserve [FED]. The prospective role of these exercises makes it possible to identify the first signals of weaknesses in the accounting structures of banks recognized as major contributors to systemic financial risk.

The stress test exercises, the principles of which are detailed in [Crouhy, Galai, and Mark \(2014\)](#) and [Bellini \(2017\)](#), are based on a set of adverse projections of economic drivers (GDP, inflation, exchange rates, interest rates, stock indices, etc.) under the assumptions of non-favorable financial market conditions, generally over a three-year time horizon for purely financial exercises, and up to 50 years when these exercises relate to the impact of climate change on the financial system. These economic drivers prospective trajectories are then injected into the default probabilities (for example by using linear regressions of these probabilities time series, or of their latent variables, onto the economic drivers time series) at various future time horizons. The economic drivers can also be linked to the assets composing the portfolios (also referred to as exposures) of the counterparties to which the banks are exposed. Default probabilities and exposure amounts can thus be projected on the basis of this re-interpretation

of macroeconomic variables to deduce the impact on the counterparties risk weighted assets [RWA] and on their capital amounts to be reserved, whether equity capital within the framework of the Capital Requirements Regulation [CRR] instructions or credit provisions within the framework of the International Financial Reporting Standards [IFRS]. A bank institution exposed to various types of counterparties (companies, other banks, pension funds, etc.) will have an even more complex task in carrying out these stress test exercises due to the diversity of positions within its portfolio. Furthermore, if the objective of the regulation is to be able to limit systemic and contagion risks from non-favorable economic and financial events, these exercises must be able to capture and identify the signals of weaknesses due to these risks. This implies in particular, for the banking institution which conducts the exercise, to include the various interactions between the financial and economic actors constituting the network of the financial system. These requests are expressed in particular in article 302 of the regulatory text [The European Parliament and the Council of the European Union \(2013\)](#) (updated in [The European Parliament and the Council of the European Union \(2019\)](#)):

Institutions shall assess, through appropriate scenario analysis and stress testing, whether the level of own funds held against exposures to a CCP, including potential future credit exposures, exposures from default fund contributions and, where the institution is acting as a clearing member, exposures resulting from contractual arrangements [...] adequately relates to the inherent risks of those exposures.

Furthermore, the recommendations in paragraph 105 from [European Bank Authority \(2018\)](#) for conducting stress test exercises regarding clearing activities stipulate:

Institutions should ensure that their institution-wide credit risk stress tests cover all their positions in their banking and trading book, including hedging positions and central clearing house exposures.

The network of the financial system has been profoundly transformed over the last ten years with the increasingly dominant role assigned to central clearinghouse counterparties [CCP], under the incentive of the G20. In September 2009 ([G20 Research Group, 2009](#)), the G20 imposed systematic settlement for over-the-counter [OTC] derivative contracts when they can be standardized through CCPs between two financial players, then recognized as “clearing members” of the CCPs acting as their intermediaries. In other words, the financial flows of a derivative contract between two counterparties pass through an intermediary CCP, the latter becoming the buyer of any selling counterparty and the seller of any buying counterparty of this contract. The CCPs also have the role of guaranteeing the continuity of payments of the contract cash flows even in the event of default by one of the counterparties. They are guaranteed via the collection of collaterals (aka margin calls) from the two parties for which the CCPs act as intermediary, similar to the principles applied by financial exchanges. Derivative contracts are thus negotiated between two counterparties on the exchanges underlying the CCPs to which the counterparties are redirected to settle the financial contracts cash flows. The contracts settled by the CCPs on behalf of the various financial players are said to be *cleared*. When an institution is exposed to various counterparties on derivatives markets, which for a large part of the contracts have been standardized, this exposure is indirect since the CCP is an intermediary. Not all derivative products can be standardized since many of them are specifically designed for the sole needs of clients wishing to protect themselves against certain market fluctuations or simply for speculative purposes. Hence, the structure of financial markets has transformed itself from a structure of bilateral agreements operated between two players in the financial network, into a more complex structure. The latter is made up of tens of CCPs concentrating gigantic

amounts of collaterals collected, a few thousands of financial players most of whom having a significant level of capital to face those CCPs providing them with these collaterals collected from smaller players, financially reduced and unable to directly access the services of a CCP. Although simplistic and schematic with only one CCP represented, the Figure 6 illustrates the change of relationships, i.e. financial positions, from only bilateral ones between six financial actors (left diagram) to financial relationships managed by a single CCP as the sole intermediary between these six actors (middle graph) and a representation more in line with industrial practice consisting of a mixture of bilateral financial cash flows and intermediated ones through the CCP (right graph).

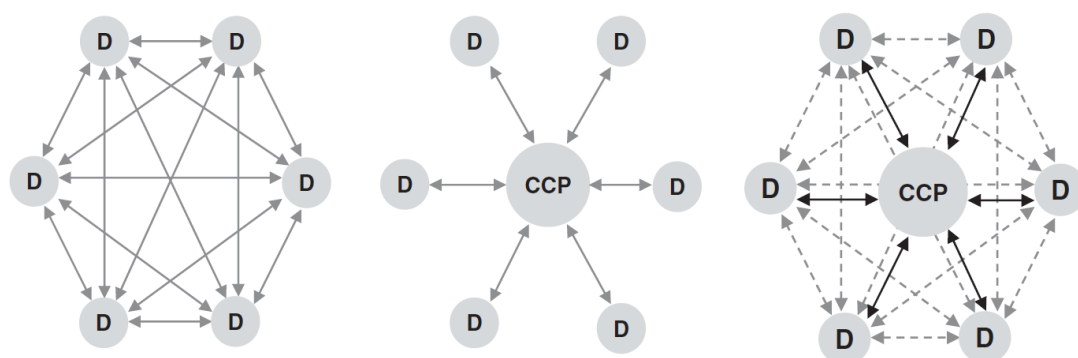


Figure 6: Illustration of a market of financial flows, including collateral, with only bilateral links (left) compared to a market of centralized financial flows through a single CCP (in the center). On the right, illustration of a partly centralized financial flows market (solid links) and containing also bilateral financial flows between members (dotted links) of the CCP. *Source: Gregory (2014, Figures 3.2 and 3.5, pages 28 and 32).*

These amounts of collateral can also be observed via quarterly reports published by most of the CCPs and, for the largest CCPs, amount to hundreds of billions of Euros (these are the quantitative disclosure files). Moreover, positions still originating from certain OTC derivatives products do not settle through the CCPs which cannot manage them and continue to mesh the financial network of bilateral links. Regarding the concentration of collaterals, although these allow a certain guarantee of financial operations continuity for the CCPs and their members following the default of one of the clearing member, they pressure the smallest actors in the network, not members of these CCPs, with significant liquidity needs. However, the number of clearing member default events on a CCP over the last fifteen years is less than ten. It is then recognized that counterparty risk, that is to say the risk that a financial actor defaults and cannot respect its contractual financial commitments towards its counterparties, the latter having to turn to other actors to replace these commitments with additional costs, was a major risk of the financial network when it was made up mainly of bilateral financial links. This counterparty risk has transformed, with the growing importance of CCPs, into a liquidity risk, which consists for a financial actor of the risk of not being able to respect one or more of its financial commitments, including providing collateral additional calls either due to lack of means to ensure them or due to reduced and expensive accessibility for some of these collaterals. Such an event of insufficient liquidity, and therefore failure, has a much lower probability of development given the collaterals posted by the members of the CCPs to guarantee financial flows up to a certain confidence level. This probability is much lower compared to that generated by counterparty risk. However, if the

collateral were to be insufficient — which would then only occur in a more extreme case than one or more defaults of more isolated counterpart(s) in the simpler framework of bilateral financial exchanges —, the liquidity needs would be much more significant for multiple members, some of whom not being able to meet them. A contagion effect is then possible, the defaults of members, and potentially of their clients, can therefore in this rarer but more extreme scenario spread quickly and create a systemic event of devastating intensity for the entire financial network. To illustrate the complexity of the financial network as a whole, an analysis of the links between the members, their clients for whom they operate, and the CCPs, is often favored because it facilitates a certain reading of the network yet simplifies it by omitting the various connections between the members of the CCPs or between the members and other network financial actors that are not members of a CCP as in Rosati and Vacirca (2019, Figure 2), included in the Figure 7.

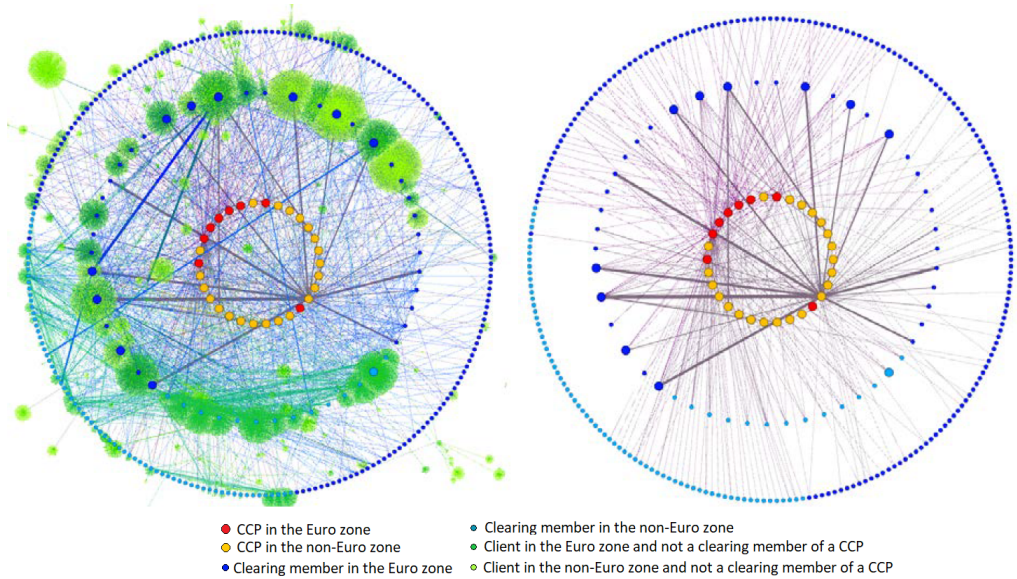


Figure 7: Financial networks indicating the existence of Marked-to-Market positions (solid lines) between two financial actors (dark blue dots for the Euro zone, light blue dots for the non-Euro zone) including 30 CCPs (red and orange dots), their members (190 dark and light blue dots) and non-member clients exposed to these members (dark and light green dots). The network on the right only indicates the Marked-to-Market positions between the CCPs and their members without the clients that are on the other hand included in the graph on the left. The inner blue circles represent clearing members exposed to the highest numbers of CCPs. Source: Rosati and Vacirca (2019, Figure 2).

In practice, it is also possible to generate such networks on a regular basis using annual analyzes carried out on the credit quality of CCPs where the credit analyst, in charge of the internal rating of analysing the credit quality of a such a CCP institution, collects all the names of its clearing members. By crossing this information on various CCPs, it is for instance possible to produce a network as shown in Figure 8. We can observe the relationships that the members have with one or more CCPs but the relationships between the members themselves, within the framework of a bilateral OTC activity, are not indicated so as not to overload the figure.

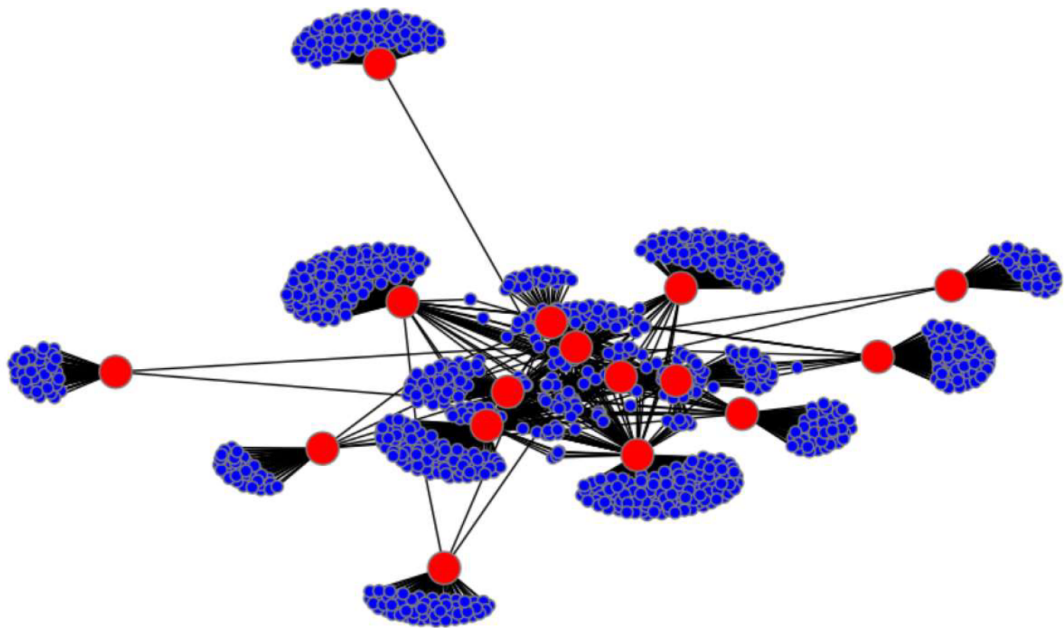


Figure 8: Financial network of the clearing activity in Europe with 16 CCPs in red and their respective members in blue (1059 members in total), of which 112 members have memberships on several CCPs and concentrated at the center of the network, based on data at the end of June 2021.

A more realistic yet more complex representation of the financial network consists in showing both bilateral links of portfolios taken on OTC markets and clearing activity links such as [Fiedor, Lapschies, and Országhová \(2017, Figure 3\)](#), see the Figure 9, but which concentrate, given the complexity of the data to be processed, on the market for interest rate derivatives products (whether cleared or OTC).

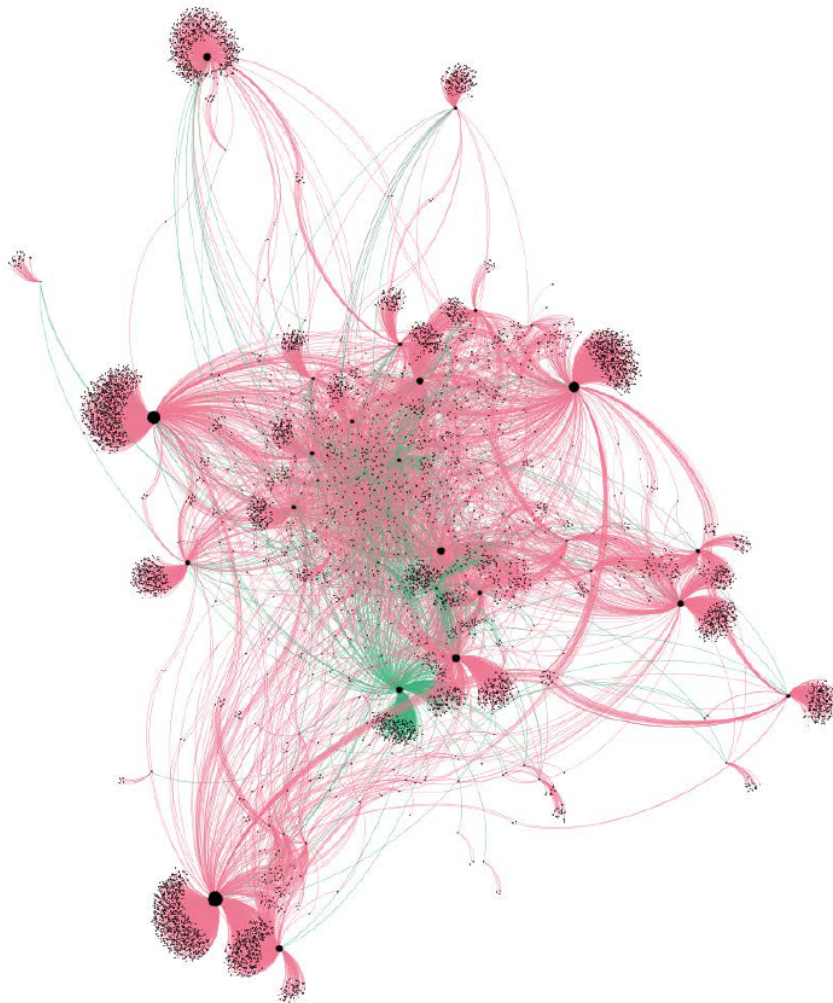


Figure 9: Relationships between counterparties for the clearing activity of interest rate derivatives products as of Dec. 31 2016. Node size is relative to the number of links (degree). The links between CCPs and their members are in green and those between the members and their clients for whom they operate are in red. *Source: Fiedor, Lapschies, and Országhová (2017, Figure 3))*

General problem: The regulatory instructions have taken into account the recent transformations of the financial network and its risks by recommending their adequate representation and quantification for stress test exercises. These transformations also constitute a key point of attention for the executive committee of financial institutions. It requires introspective and prospective exercises of the financial institutions solidity by means of various regulatory exercises to evaluate the adequate processing of their operations and the guarantee of their continuity following adverse events experienced by the financial system. The regulatory exercise frameworks comprised of the Internal Capital Adequacy Assessment Process [ICAAP] and Internal Liquidity Adequacy Assessment Process [ILAAP], both components of the regulatory framework Supervisory Review and Evaluation Process [SREP], are developed for these purposes. They require the analysis of the financial solidity of financial institutions following various

hypothetical economic shocks (whether political, economic and/or climate-related in nature). It is thus critical to develop tools that can capture these financial network mechanisms and its risks, with the support of numerical quantification, having targeted approaches and implementation that can be scalable to a larger and more complete financial network, yet with limited computing power and time.

This thesis offers various contributions on 4 development axis, linked to each other, to provide answers to the general problem:

- a modeling of the various mechanisms and risks of the financial markets for both clearing and bilateral OTC activities with the consideration of the related different actor types and resulting costs, allowing various risk management exercises;
- a theoretical and numerical analysis of the adequacy of this modeling based on the supermodular properties of multi-dimensional functions and random vectors;
- the presentation of an approach based on the concept of Radner's equilibrium to capture the liquidation risk through impact on market prices including the various types of actors as well as both clearing and bilateral OTC activities from the two previous axes;
- the application of an approximate algorithmic resolution for identifying the optimal allocation of the portfolios of a defaulting member on several CCPs, resolved in a naive and computationally expensive manner in the first axis.

We detail these axes below, each being the subject of a chapter in this thesis manuscript.

§5 Derivatives' risks as costs in a one-period network model

In this first part, we present an exhaustive description of the main operating mechanisms of the financial network about the financial derivatives activity, including the resulting costs, taking into account both bilateral and CCPs clearing activity links. The use of elliptical models makes it possible to propose various applications such as a re-arrangement of the financial network (which can also be described as a deformation or a restructuring of this network) following a default by one of the financial actors that is also a CCP clearing member, and with its portfolios takeover at a lower cost by the other members of that same CCP. Another essential application to meet regulatory requirements is the implementation of prospective stress test exercises. Furthermore, although intuitive, monotonicity properties of cost and risk metrics are observed numerically with these models and lead to theoretically validate the relevance of the use of such models.

Contributions in Chapter I: based on [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#),

- we describe the different types of actors and their financial flows, allowing them to maintain their positions in the event their counterparties default, as well as the necessary financing costs accounting for their own default risk (**Lemma I.1**, **Lemma I. 2**);
- we formulate these various types of costs by combining the costs of clearing activity ([Albanese, Armenti, and Crépey, 2020](#)) and bilateral OTC ([Abbas-Turki, Crépey, and Saadeddine, 2023](#)) in a static one-period framework and with multiple CCPs to which a major financial player such as a large bank is exposed (**Theorem I.4** and **Proposition I.5**);
- we illustrate numerically various possible applications such as exploratory stress test or reverse stress test exercises and idealized auctions of defaulting members portfolios.

A major player in the financial network such as a systemic bank holds exposures both to CCPs and to bilateral counterparties for products that cannot be standardized. It therefore incurs costs and liquidity needs on its financial statements whilst holding these positions. The costs are then to be transferred to its clients, the latter being at the origin of holding these financial products in order to meet their economic needs. As for liquidity needs, such as collateral in currency or sufficiently liquid securities whether under adverse market conditions or not, they result in the need for the financial actor bearing those to turn to the markets and other actors dedicated financial institutions in order to have this liquidity, also subject to corresponding costs. Moreover, the interconnection of members through CCPs is reflected in the plurality of accesses of a member to several CCPs, where each covers a certain type of product in its portfolio.

The analysis of these mechanisms for both clearing and bilateral activities and their various formulations goes back to the work of [Brigo, Morini, and Pallavicini \(2013\)](#) and [Gregory \(2015\)](#) for the bilateral OTC part, [Arnsdorf \(2012\)](#), [Gregory \(2014\)](#), [Ghamami \(2015\)](#), [Arnsdorf \(2018\)](#), [Arnsdorf \(2019\)](#), [Andersen and Dickinson \(2019\)](#) and [Andersen and Dickinson \(2020\)](#) for the clearing and CCP part. This is carried out at the level of each contract including their costs in continuous time in [Crépey \(2022\)](#). It can be implemented numerically provided that large computing capacity is available, since large financial players such as systemic banks hold several tens of millions of these transactions in their portfolios.

In [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#), we propose to use the description of these mechanisms through a one-period model taken from the continuous-time version in [Armenti and Crépey \(2017\)](#) for the clearing part and similar to the approach in [Albanese, Crépey, Hoskinson, and Saadeddine \(2021, Section 3\)](#) for the bilateral OTC part. We also propose an alternative to portfolio modeling

where a single random variable describes its future random variations, instead of all the variables thousands of contracts in its portfolio could depend on. The network consisting of financial actors and financial exposures, even if it can contain several thousand nodes, is thus numerically tractable (for similar approaches see for example [Arnsdorf \(2012\)](#) where portfolio random variations are supposed to follow Pareto laws, [Andersen and Dickinson \(2019\)](#) using elliptical distribution and [Barker, Dickinson, Lipton, and Virmani \(2017\)](#) for a contract approach with numerical applications on a network made up of 4 CCPs and their members).

This adaptation also makes it possible to illustrate numerically, via elliptical models for portfolio variation and latent default variables, regulatory stress test exercises and the porting of a defaulting member positions on a CCP to one of its surviving members generating the least cost. This last application is similar to an ideal auction which is resolved with a buyer making it possible to obtain this lower cost for all the surviving members. A stress test approach for a network of CCPs and their members, limited to a few adverse macroeconomic scenarios, but without including counterparty, financing of collateral or capital costs, is presented in [Barker, Dickinson, Lipton, and Virmani \(2017\)](#) and revisited by [Lipton \(2018\)](#). A more complete framework is detailed in [Albanese, Crépey, and Iabichino \(2023\)](#) without focusing on a particular type of activity (clearing or bilateral OTC). As for the analysis of the porting of defaulted portfolios, it makes it possible to understand the possible restructuring of the financial network following one or more defaults in the financial system.

Let J_i, J_c, J_b denote the survival indicators at time 1 of the members i of the CCPs, the clients indexed by c of the reference member 0 for which it operates a clearing service and of its bilateral counterparties indexed by b respectively. A CCP indexed ccp holds exposures to its members, including daily margin calls, denoted Y_i^{ccp} , $i \in I$ with I all members of the CCPs. Hedge positions towards the CCPs are specifically identified by \bar{Y}_i^{ccp} . This distinction will be particularly useful in **Chapter III** for establishing price discoveries in markets and is illustrated in Figure 10.

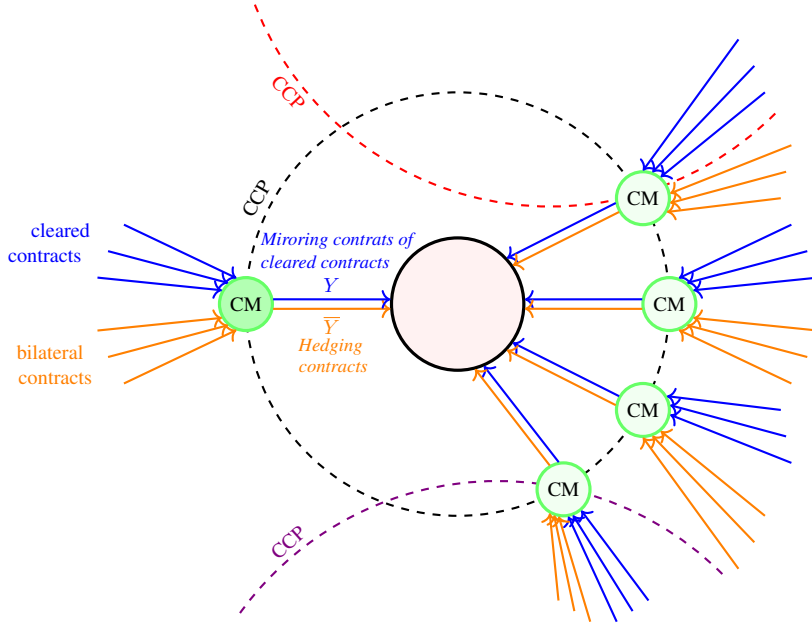


Figure 10: Contractual cash flows between market participants. The blue arrows represent position commitments cleared by members on behalf of their clients, the orange arrows are bilateral commitments and their symmetrical positions facing the CCP as a hedge (also known as *proprietary trading*).

A CCP applies a cascade (*waterfall*) of protection layers $M_i^{ccp} = IM_i^{ccp} + \overline{IM}_i^{ccp} + DF_i^{ccp}$ for each of its members $i \in I$ (see principle 6 in the set of principles issued and recommended by [Bank For International Settlements \(2012\)](#)). Initial margins [IM] are calculated separately between cleared positions (*client account*) and hedging positions (*house account*), and are denoted respectively IM_i^{ccp} and \overline{IM}_i^{ccp} . Member 0 receives from its cleared clients and its bilateral counterparties the cash flows Y_c^{ccp} and Y_b respectively with corresponding initial margins IM_c^{ccp} and IM_b . For bilateral OTC exposures, member 0 can post initial margins \overline{IM}_b , for each of its counterparties b . Most CCPs also require a guarantee fund (or default fund DF) noted DF_i^{ccp} for each member i on the CCP ccp . We ignore an additional layer of capital specific to CCPs (*skin-in-the-game*) given its low amount compared to other layers of protection ([Gregory, 2014](#), Section 8.4.5).

The formulation of the loss \mathcal{C} , supported by the reference member indexed 0, constitutes the starting point of the various costs of holding its portfolios (**Proposition I.5** and equations I.(15) and I.(16))

$$\mathcal{C} = \sum_{ccp,c} (1 - J_c) \underbrace{(Y_c^{ccp} - IM_c^{ccp})^+}_{\text{loss on client } c \text{ cleared by } 0} + \sum_{ccp} \underbrace{\mu^{ccp} \mathcal{L}^{ccp}}_{\text{loss on } ccp \text{ allocated to } 0} + \sum_b (1 - J_b) \underbrace{(Y_b - IM_b)^+}_{\text{loss on counterparty } b}, \quad (15)$$

with

$$\mathcal{L}^{ccp} = \sum_i (1 - J_i) \left(\underbrace{(Y_i^{ccp} - IM_i^{ccp})^+}_{\text{client position on } ccp} + \underbrace{(\overline{Y}_i^{ccp} - \overline{IM}_i^{ccp})^+}_{\text{house position on } ccp} - DF_i^{ccp} \text{ big} \right)^+. \quad (16)$$

The formulations (15) and (16) follow from the following principles:

1. A counterparty in default towards another and having debts beyond the posted collateral, reimburses them up to the posted collateral and not beyond. If this debt is below the collateral, all debts are repaid without any loss materialization for the other counterparty.
2. The incoming and outgoing cash flows to and from any CCP cancel each other out, as do any member facing a CCP. A member is thus considered to be perfectly hedged on its market risks emanating from its own positions, whether for clearing activities or bilateral OTC activities.

Most of the members being large banks, they actually have a relatively well hedged market position, as evidenced by their capital requirements analyzed in [European Bank Authority \(2022\)](#) (estimated as a fraction of the weighted assets reported in this report).

The costs of holding the various types of positions for the member 0 are valued under a probability noted \mathbb{P}^* , with an expectation operator \mathbb{E}^* , and with numeraire a risk-free rate asset making it possible to simplify the valuation equations of all financial flows and collateral accounts. The probability \mathbb{P}^* is a mix of physical and risk-neutral measures, which we will qualify as a “fininsurance” measure, and it is constructed as follows: given a physical probability measure completely defining the model reflected by the σ -algebra \mathcal{A} and equivalent to a risk-neutral measure on the σ -algebra \mathcal{B} reflecting only the market uncertainty and contained in \mathcal{A} , \mathbb{P}^* is equal to the risk-neutral measure on \mathcal{B} and to the physical measure conditionally to \mathcal{B} . These two conditions uniquely characterize \mathbb{P}^* ([Artzner et al., 2023](#), Proposition 4.1). This construction dates back to [Plachky and Rüschemdorf \(1984\)](#) and [Dybvig \(1992\)](#). The default probability of a member i is denoted by $\gamma_i = \mathbb{P}^*(J_i = 0)$. For simplicity, J_0 is denoted simply J , and γ_0 simply γ .

Regarding the funding of the member 0 exposures, and therefore its liquidity needs, the terms IM^{ccp} and $\sum_{b \in B} \overline{IM}_b$, constitute amounts of collateral to be borrowed and posted either to the CCP or to its bilateral counterparties. The member 0 receives in exchange a remuneration rate which is considered below its funding rate taken as its probability of default γ so that the financing cost is based on a diluted rate $\tilde{\gamma} < \gamma$ (member 0 is financed at the rate γ and is remunerated at the rate $\gamma - \tilde{\gamma}$). Additionally, member 0 transfers its aggregate counterparty risk and margin funding costs, summarized by the term CA, to its clients. CA is the acronym for *contra-asset* and plays the role of a liability in the balance sheet of member 0, counterbalancing the value of the derivatives held as assets. Its shareholders must then bear the potential loss $\ell := \mathcal{C} + \mathcal{F} - CA$ for which a capital level EC is calculated as a measure of risk of this loss where the measure is for example an expected shortfall \mathbb{ES}_α^0 under the probability \mathbb{P}^* conditioned by the survival of 0, denoted \mathbb{Q}^0 . This risk measure corresponds, in the case where it is applied to a random variable of continuous distribution, to the average of losses under \mathbb{Q}^0 beyond a certain threshold $\mathcal{Q}_\alpha^0(\ell)$ for a confidence level $\alpha \in (1/2, 1)$ also taken under \mathbb{Q}^0 , i.e. $EC = \mathbb{ES}^0(J\ell) = \mathbb{E}^0[\ell | \ell \geq \mathcal{Q}_\alpha^0(\ell)]$, with $\mathbb{Q}^0(\ell \geq \mathcal{Q}_\alpha^0(\ell)) = 1 - \alpha$. It becomes the measure of choice considered for the regulatory standard for market risk measurements ([Bank for International Settlements, 2013](#)). The executive committee of this member must ensure a dividend, up to a hurdle rate denoted by h , on the risk capital of its shareholders, i.e. $(EC - KVA)^+$ where KVA represents the capital cost also charged to the clients in addition to CA. We then make the assumption that at time 0, the amounts of these costs collected by the member 0 are deposited into its capital reserve and risk capital accounts and can be used for its variation margin funding needs. Thus the expenses due to the financing of the various margins can then be written as the flow (**Proposition I.5** equation (17))

$$\mathcal{F} = \gamma \left(\sum_b V_b - CA - \max(EC, KVA) \right)^+ + \tilde{\gamma} \sum_{ccp} M^{ccp} + \tilde{\gamma} \sum_b \overline{IM}_b, \quad (17)$$

where V_b represents the difference between the collateral owed by member 0 to counterparty b and the

one collected from this same counterparty. CA is decomposed into various terms reflecting credit and margin funding costs of the form $CA = CCVA + BCVA + CMVA + BMVA + FVA$, where the different terms are defined Table 3.

XVA	Expression	Full name and description
KVA	$\mathbb{E}^* [Jh(EC - KVA)^+ + (1 - J)KVA]$, where $EC = \mathbb{E}\mathbb{S}(J(\mathcal{C} + \mathcal{F} - CA))$	capital valuation adjustment
BCVA	$\mathbb{E}^* \left[J \sum_b (1 - J_b)(Y_b - IM_b)^+ + (1 - J)BCVA \right]$	credit valuation adjustment for bilateral expositions
CCVA	$\mathbb{E}^* \left[J \sum_{ccp,c} (1 - J_c)(Y_c^{ccp} - IM_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} + (1 - J)CCVA \right]$	credit valuation adjustment pour clearing activity expositions
BMVA	$\mathbb{E}^* \left[J\tilde{\gamma} \sum_b \overline{IM}_b + (1 - J)BMVA \right]$	margin valuation adjustment for bilateral expositions
CMVA	$\mathbb{E}^* \left[J \sum_{ccp} \tilde{\gamma} (IM^{ccp} + \overline{IM}^{ccp} + DF^{ccp}) + (1 - J)CMVA \right]$	margin valuation adjustment pour clearing activity expositions
FVA	$\mathbb{E}^* \left[J\gamma \left(\sum_b Y_b - CA - \max(EC, KVA) \right)^+ + (1 - J)FVA \right]$	funding valuation adjustment

Table 3: XVA costs definitions (with \mathcal{C} , \mathcal{F} and \mathcal{L}^{ccp} respectively given by (15), (16) et (17)).

We observe that these counterparty risk, margin funding and capital costs are written as a solution of a fixed-point type equation under the probability “fininsurance” and are well defined when they are re-written under the survival probability \mathbb{Q}^0 . We can then obtain explicit formulas for these different costs, which we list in the Table 4 (**Theorem I.4** and **Proposition I.5**).

XVA	Explicit formula
CCVA	$\mathbb{E} \left[\sum_{ccp,c} (1 - J_c)(Y_c^{ccp} - IM_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} \right]$
CMVA	$\tilde{\gamma} \sum_{ccp} M^{ccp}$
BCVA	$\mathbb{E} \left[\sum_b (1 - J_b)(Y_b - IM_b)^+ \right]$
BMVA	$\tilde{\gamma} \sum_b \overline{IM}_b$
EC	$\mathbb{E}\mathbb{S}(J(\mathcal{C} - CVA))$
FVA	$\frac{\gamma}{1 + \gamma} \left(\sum_b Y_b - (CCVA + CMVA + BCVA + BMVA) - EC \right)^+$
KVA	$\frac{h}{1 + h} EC$

Table 4: Explicit formula of XVA costs under the survival probability \mathbb{Q}^0 (with \mathcal{C} and \mathcal{L}^{ccp} given by (15) and (16)).

We propose several numerical examples, based on an elliptic Student model with a degree of freedom of 3 to generate heavy-tailed distributions for the portfolio random variations (amounting to assuming the existence of moments of order 1 and 2 only, as it can be observed in practice on certain portfolios), to highlight the possible application of the XVA metrics framework for exploratory or reverse stress test exercises. One of them relates to the default on two CCPs of the asset manager firm Ronin Capital following the COVID 19 pandemic, in March 2020. Another example highlights the

identification of an optimal re-arrangement that a CCP could carry out following the default of one of its members with the defaulted portfolio takeover, and its corresponding symmetrical positions to the cleared clients, by the surviving member leading to the lowest costs across all surviving members. For each non-analytical XVA metric (e.g. CVA and KVA) and quantiles (taken as exploratory stress test scenario for a given confidence level), we perform simulations via the Monte Carlo technique with non-parametric confidence intervals, and for quantiles estimates based on [Meeker et al. \(2017, Appendix G\)](#) which uses the binomial distribution of order statistics of independent uniform random variables.

§6 Monotonicity of credit provisions and economic capital risk measures

Given the elliptical framework considered for the random variables following Student distributions in the numerical tests mentioned in §1 and Chapter I, we observe that the expectation and expected shortfall measures are nondecreasing compared to the correlation parameters. It leads to the question of having such a property of the modeling framework consisting of the application of convex risk measures to functions of random variables following elliptical distributions. It turns out that elliptical distributions belong to random variables obeying the so-called supermodular property and that the loss functions that we use for bilateral OTC and clearing activities also have the functional property of the same name. By applying convex risk measures to these supermodular functions of such particular random vectors, we can obtain the monotonicity of these risk measures as a function of the covariances and therefore correlation coefficients. These measures are also applied from the point of view of a reference financial actor, typically a systemic bank, also a member of a CCP, subject to bilateral OTC and clearing activities, and which is indexed by 0.

Contributions in Chapter II: In Bastide and Crépey (2024), which constitutes this chapter,

- we demonstrate and illustrate the monotonic behavior of convex risk measures for multidimensional random variables having the supermodular property (**Theorem II.12**);
- we apply these results to elliptical models for the factors governing portfolio movements and latent default variables, thus highlighting the monotonicity of risk measures such as expectation and expected shortfall [ES] operators compared to correlation coefficients between these factors (**Proposition II.13, Proposition II.14, Proposition II.16, Corollary II.17**) and which we illustrate numerically;
- we also extend the results of Cousin and Laurent (2008) concerning the monotonicity of the valuation of equity and senior collateralized debt obligation [CDO] tranches with default latent variable models (**Proposition II.7 and Corollary II.8**).

A supermodular function is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ which, for all $x \in \mathbb{R}^n$, all $\varepsilon > 0$, any $\delta > 0$ and any pair of index $i < j$ in $1..n$, satisfies

$$\begin{aligned} f(x_1, \dots, x_i + \varepsilon, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i + \varepsilon, \dots, x_j, \dots, x_n) \geq \\ f(x_1, \dots, x_i, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_j, \dots, x_n). \end{aligned} \quad (18)$$

The random variables that we consider subsequently in this section are assumed to belong to a closed linear subspace \mathfrak{X} of $L^1(\mathbb{Q})$, defined from a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, with \mathbb{E} the corresponding expectation operator, and considered without any atom, i.e. for all $\omega \in \Omega$, $\mathbb{Q}(\{\omega\}) = 0$. These hypotheses allow the results of Müller (1997); Bäuerle and Müller (1998); Müller and Scarsini (2000); Müller and Stoyan (2002); Bäuerle and Müller (2006) to apply without particularly constraining the modeling framework.

A random vector $\mathbf{X} = (X_1, \dots, X_m)$ is smaller than another random vector $\mathbf{Y} = (Y_1, \dots, Y_m)$ for the supermodular order, also written $\mathbf{X} \leq_{sm} \mathbf{Y}$, if the inequality $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$ holds for all supermodular functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the written expectations exist.

We also define the stop-loss order as follows: \mathcal{X} precedes \mathcal{Y} , both in \mathfrak{X} , for the stop-loss order, written $\mathcal{X} \leq_{sl} \mathcal{Y}$, if $\mathbb{E}[\mathcal{X} - A]^+ \leq \mathbb{E}[\mathcal{Y} - A]^+$ is true for all constant $A \geq 0$, which is equivalent to (Bäuerle and Müller, 1998, Theorem 2.2 b)) that $\mathbb{E}(f(\mathcal{X})) \leq \mathbb{E}(f(\mathcal{Y}))$ holds for any convex increasing

function $f : \mathbb{R} \rightarrow \mathbb{R}$ and such that the written expectations exist (which is the case in particular for $f = id$). This order makes it possible to obtain the result stated in [Bäuerle and Müller \(2006, Theorem 4.4\)](#) concerning the monotonicity of a risk measure ρ defined on a probability space without atom, i.e. $[\mathcal{X} \leq_{sl} \mathcal{Y}] \Rightarrow \rho(\mathcal{X}) \leq \rho(\mathcal{Y})$.

Our considered definition of a risk measure relaxes the translation equivariance assumption commonly used in risk measure theory (see for example [Shapiro \(2013, Introduction \(A3\)\)](#)). It then includes the 4 following properties: (i) the risk measure ρ is assumed to be proper ($\forall \mathcal{X} \in \mathfrak{X}, \rho(\mathcal{X}) > -\infty$, and $\text{dom } \rho := \{\mathcal{X} \in \mathfrak{X}; \rho(\mathcal{X}) < +\infty\} \neq \emptyset$); (ii) ρ is invariant for random variables with the same distribution (aka law-invariant, i.e. $\mathcal{X} \stackrel{d}{=} \mathcal{Y} \Rightarrow \rho(\mathcal{X}) = \rho(\mathcal{Y}), \mathcal{X}, \mathcal{Y} \in \mathfrak{X}$); (iii) ρ is monotonic ($\mathcal{X} \leq \mathcal{Y} \Rightarrow \rho(\mathcal{X}) \leq \rho(\mathcal{Y}), \mathcal{X}, \mathcal{Y} \in \mathfrak{X}$) and (iv) ρ is convex ($\rho(\lambda \mathcal{X} + (1 - \lambda)\mathcal{Y}) \leq \lambda \rho(\mathcal{X}) + (1 - \lambda)\rho(\mathcal{Y}), \forall \lambda \in (0, 1), \mathcal{X}, \mathcal{Y} \in \mathfrak{X}$). The risk measures that we use in [Bastide and Crépey \(2024\)](#) are the expectation and the expected shortfall as defined by [Acerbi and Tasche \(2002, Definition 2.6\)](#) from the probability \mathbb{Q} . These two measurements obey the 4 properties previously mentioned on the space $L^1(\mathbb{Q})$.

The starting argument to obtain our monotonicity results for risk measures according to the super-modular order is based on the [Lemma II.1](#) which we recall below:

Lemma .2. *If $\mathbf{X} \leq_{sm} \mathbf{Y}$ in \mathfrak{X} with \mathbf{X}, \mathbf{Y} with values in \mathbb{R}^m , then, for any supermodular nondecreasing function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, we have $f(\mathbf{X}) \leq_{sl} f(\mathbf{Y})$.*

This lemma directly follows from [Müller and Stoyan \(2002, Theorem 3.9.3 f\), page 113](#) which states that if $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is nondecreasing and super-modular and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and convex, then $\phi \circ f : \mathbb{R}^m \rightarrow \mathbb{R}$ is also supermodular. Consequently, according to [Bäuerle and Müller \(2006, Theorem 4.4\)](#), if $\rho : \mathfrak{X} \rightarrow \mathbb{R}$ is proper, monotonic, convex, and law-invariant and if $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is nondecreasing super-modular, then $\mathbf{X} \leq_{sm} \mathbf{Y} \Rightarrow \rho(f(\mathbf{X})) \leq \rho(f(\mathbf{Y}))$.

Moreover, the result [Müller and Stoyan \(2002, Theorem 3.9.3 f\), page 113](#) makes it possible to complete the results from [Cousin and Laurent \(2008\)](#) regarding the monotonicity properties of the valuations of equity and senior tranches of CDOs under latent default variable models. To do this, it is enough to observe that the corresponding financial cash flows of these tranches are written as convex nondecreasing functions composed with supermodular increasing functions of the latent default variables of the underlying bondholders. We illustrate this result numerically in [II.§3.A](#).

About the monotonicity of the risk measures as previously introduced and applied to loss functions of bilateral OTC and clearing activities, we consider the form of the loss functions over a period of time $[0, T]$ as $\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n) g_i(\mathcal{Y}_i)$ with n a given number of obligors (that can be a member, a bilateral counterparty or a cleared client). The default times τ_i of these obligors are indexed by $i \in 1 \dots n$ and modeled using a latent variable \mathcal{X}_i exceeding a certain level B_i , typically \mathcal{X}_i represents the liabilities level which may exceed the sum of the obligor's assets and capital levels i , and denoted B_i . The variable $g_i(\mathcal{Y}_i)$ represents a positive loss due to the default of obligor i . The functions f_i are positive and nondecreasing with respect to each of its arguments and have the supermodular property (when $n > 1$, as the case $n = 1$ can be handled directly without any particular difficulty). The g_i functions are also positive and nondecreasing.

We thus establish ([Proposition II.11](#)) that $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n) g_i(y_i)$ is a supermodular nondecreasing function on \mathbb{R}^{2n} . For example, we demonstrate in the [Proposition II.16](#) that functions of the form

$$\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \xrightarrow{f} \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}} \mathbb{1}_{\{x_i \geq B_i\}} \times (y_i - m_i)^+ \in \mathbb{R}_+ \quad (19)$$

are nondecreasing and supermodular with $B_1, \dots, B_n \in \mathbb{R}$, $\beta_1, \dots, \beta_n \in \mathbb{R}_+$. These loss function forms are found in clearing activities against CCPs as presented in **§1** and **Chapter I**. The loss function forms for bilateral OTC or cleared client exposures are simpler and expressed as $\sum_b \mathbb{1}_{\{x_b \geq B_b\}} (y_b - m_b)^+$ where the sum is taken over all counterparties and cleared clients b to which the reference bank is exposed, and in which case the form of the loss function is $f_b(x_b)g_b(y_b)$ for a counterparty or cleared client b . So, according to the Lemma .2, if $(\mathcal{X}_1, \dots, \mathcal{X}_n, \mathcal{Y}_1, \dots, \mathcal{Y}_n) \leq_{sm} (\mathcal{X}'_1, \dots, \mathcal{X}'_n, \mathcal{Y}'_1, \dots, \mathcal{Y}'_n)$, we obtain

$$\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i) \leq_{sl} \sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n)g_i(\mathcal{Y}'_i). \quad (20)$$

Bäuerle and Müller (2006, Theorem 4.4) allows us to conclude that $\rho(\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)) \leq \rho(\sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n)g_i(\mathcal{Y}'_i))$.

For $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m) = (X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $\mathbf{X}' = (\mathcal{X}_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m) = (X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ (for $n > 0$) such that $\mathbf{X} \leq_{sm} \mathbf{X}'$, we define the loss functions as

$$L = \sum_{i=1}^n f_i(X_1, \dots, X_n)g_i(Y_i) \text{ and } L' = \sum_{i=1}^n f_i(X'_1, \dots, X'_n)g_i(Y'_i), \quad (21)$$

with supermodular nondecreasing functions f_i and nondecreasing functions g_i . We recall, as specified in **§1** and **Chapter I**, that risk measures are defined under the survival probability measure of the reference financial actor, denoted here \mathbb{Q}^0 , equivalent to a starting probability, for instance with $\mathbb{Q} = \mathbb{P}^*$ the “fininsurance” probability measure introduced in **§1**. More precisely, we can write $\mathbb{Q}^0(\cdot) = \mathbb{Q}^*(\cdot, X_0 < B_0)/(1 - \gamma)$ where γ is the default probability under \mathbb{Q}^* over the period of the static model defined in **§1** and **Chapter I** for this leading financial player. Therefore, this requires that $L, L' \in \mathfrak{X}^0 = L^1(\mathbb{Q}^0)$ while the supermodular order is initially taken under \mathbb{Q}^* . However, the default of this actor 0 stems from the latent variable X_0 so that we can rewrite $\mathbb{Q}^0(\cdot) = \mathbb{E}^*[h(X_0)\mathbb{1}_{\{\cdot\}}]$, with $h(\cdot) = \mathbb{1}_{\cdot < B_0}/(1 - \gamma)$. So as to reproduce the industry practices, we favor supermodular random vectors \mathbf{X}, \mathbf{X}' under \mathbb{Q}^* whose data are observable under this probability. They are also supposed to have the same component \mathcal{X}_0 and verify $[\mathbf{X}_{1:m} | \mathcal{X}_0] \leq_{sm} [\mathbf{X}'_{1:m} | \mathcal{X}_0]$, i.e. the supermodular order on the components other than the component 0 but conditionally on the latter one (which we also write $\mathbb{E}^*[f(\mathbf{X}_{1:m}) | \mathcal{X}_0] \leq \mathbb{E}^*[f(\mathbf{X}'_{1:m}) | \mathcal{X}_0]$ for any supermodular function $f : \mathbb{R}^m \rightarrow \mathbb{R}$). This assumption then makes it possible to demonstrate that for h defining the change of measurement from \mathbb{Q}^* to \mathbb{Q}^0 , we obtain $\mathbf{X}_{1:m} \leq_{sm^0} \mathbf{X}'_{1:m}$ where \leq_{sm^0} is the supermodular order under \mathbb{Q}^0 .

To illustrate these results numerically, we consider the family of random vectors with elliptical distributions under \mathbb{Q}^* with the same characteristic generator, same mean, same first component \mathcal{X}_0 , same variance for all components, but whose non-diagonal covariances $\Gamma_{\mathbf{X}}^{ij} = \text{Cov}^*(\mathcal{X}_i, \mathcal{X}_j)$, $\Gamma_{\mathbf{X}'}^{ij} = \text{Cov}^*(\mathcal{X}'_i, \mathcal{X}'_j)$ (with Cov^* the covariance operator under \mathbb{Q}^*) satisfy $\Gamma_{\mathbf{X}}^{ij} \leq \Gamma_{\mathbf{X}'}^{ij}$, for $i \neq j \in 1..m$ (and $\Gamma_{\mathbf{X}}^{ii} = \Gamma_{\mathbf{X}'}^{ii}$, for $i \in 1..m$ regarding variance terms). These vectors are then supermodular conditionally to \mathcal{X}_0 . The equality of the first components of \mathbf{X} and \mathbf{X}' also requires that $\Gamma_{\mathbf{X}}^{0j} = \Gamma_{\mathbf{X}'}^{0j}$, for all $j \in 0..m$ and reflects the fact of keeping the same point of view of the reference actor through which the risk measures are considered. Under these assumptions, we have $\mathbf{X}_{1:m} \leq_{sm^*} \mathbf{X}'_{1:m}$ where \leq_{sm^*} is the super modular order under \mathbb{Q}^* (**Block and Sampson, 1988, Corollary 2.3**). It follows that $[\mathbf{X}_{1:m} | \mathcal{X}_0] \leq_{sm^*} [\mathbf{X}'_{1:m} | \mathcal{X}_0]$ (**Proposition II.5**) then that $\mathbf{X}_{1:m} \leq_{sm^0} \mathbf{X}'_{1:m}$ (**Lemma II.6**). We can thus apply our **Lemma .2** with f of the form (19) (but also for any nondecreasing supermodular function of more general form $\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)$) for supermodular elliptical random vectors under \mathbb{Q}^* keeping this supermodular property under \mathbb{Q}^0 in order to get a stop loss order relation of the type (20).

By taking a risk measure as previously defined under \mathbb{Q}^0 , we obtain the monotonicity of such a measure as a function of the covariance coefficients of the supermodular vectors (and therefore of the correlation parameters for those expressed as a positive factor of the corresponding covariance coefficients). The expectation and the expected shortfall verify the risk measure hypotheses on $L^1(\mathbb{Q}^0)$. Furthermore, expectation is associated with the accounting metric of current expected credit loss used for the calculations of credit provisions (hence CVA) and the expected shortfall relates to the risk management metric of an economic capital amount (hence of KVA).

§7 Radner Equilibrium and application to Clearing Member Default Management by a CCP

The price impact of the management of one or more defaults by CCPs is covered in a third chapter based on [Bastide, Crépey, Drapeau, and Tadese \(2023b\)](#), along with numerical examples, using the convex analysis and Radner equilibrium which constitute a special case of Nash equilibrium. This price impact relates to the proprietary trading positions of the defaulted members of the CCPs that are not covered in the applications exposed in **Chapter I** dealing only with the resolution of the defaulted members with client positions. This third chapter therefore aims to deal with the proprietary positions of these members not impacted by the porting of client positions from a defaulting member to a surviving member. These default resolutions of defaulting members by CCPs and their proprietary positions are considered with different strategies that a CCP can implement to manage a default of its members, as prescribed by the recommendations of the Bank for International Settlements [BIS] and International Organization of Securities Commissions [IOSCO], namely the hedging and then auctioning of the defaulting member portfolios, or its liquidation on a financial market ([Bank For International Settlements, 2019a, 2020](#)). These organizations explicitly recommend that CCPs analyze the impacts of such strategies on the prices of the corresponding portfolios and markets.

Contributions in Chapter III:

- we express the losses that a participant in one or more markets may suffer on its different types of portfolios that it holds following market prices moves, modeled over a single period of time;
- we formulate the impact of handling defaults resolution by the CCP on the traded prices securities on several markets using Radner equilibria in a default-free network by considering portfolio recoveries following defaults;
- we demonstrate the existence (**Theorem III.2**) and the uniqueness under certain conditions of such equilibria (**Theorem III.3**), adapting results from [Cheridito, Horst, Kupper, and Pirvu \(2015\)](#) to a one-period static framework for unbounded random loss variables, illustrated with several examples covering various possible strategies that a CCP can implement in case of the default of one or more of its members;
- we take the formulations of these price impacts in the expressions of credit, capital and financing costs covered in the **Chapter I** and numerically illustrate these applications.

The main risks in the financial network the players are exposed to, and for which regulators require sufficient amounts of capital from banking institutions with a probability level of 99.9% ([Basel Committee on Banking Supervision, 2005](#)), can be summarized in 4 risk types ([European Bank Authority, 2022](#)): market, counterparty, credit and operational. Liquidity risk, linked to the need for a market participant to honor payment commitments in cash or securities, may indirectly give rise to additional capital requirements. Various indicators allow for testing the solvency of a banking institution regarding its contractual payments and reimbursement commitments. For example, the liquidity coverage ratio [LCR] is defined as the ratio of the amount of very high quality liquid assets divided by the amount of flows to be payed out in the next 30 days citep[Chapters 3 and 8]CGM2014. Poor performance of these indicators for a financial institution can signal disruptions of its operational flows and lead to repressive actions by regulators such as capital surcharges.

When a financial actor exposed to a CCP defaults, its traded portfolio on a financial market must be managed by the CCP, linked to this market. For example, the EUREX Group has its market services for

negotiating the terms of contracts and its own clearing services to settle them once negotiated. The CCP can either fully or partially hedge the defaulting member portfolio and then auction it to its surviving members, possibly inviting other players to the auction, or liquidate it on the financial markets by selling it at a loss or buying it beyond its price depending on whether the CCP initially purchased or sold this portfolio respectively (Bank For International Settlements, 2019a, 2020). These resolution choices may generate losses on this portfolio, which will be borne, beyond the collateral of the defaulting member, by the surviving members: the CCP cannot have a non-zero aggregated position on all of its surviving clients nor support a cost due to losses of the defaulting member (apart from a tiny part of its capital called the *skin-in-the-game*).

To complete the quantification of the credit risk through the XVA costs under the static one-period model detailed in the **Chapter I**, we propose to capture the impact of hedging, full liquidation, or a combination of both, of defaulting members portfolios of a CCP. The use of Radner's equilibrium based on the concepts of Walrassian competitive economic equilibrium with uncertainty appears natural in this context (Radner, 1968, 1972). A Radner equilibrium is defined by a predetermined quantity of each security in the market, traded at a certain unit price minimizing a utility or risk metric applied to the random variables of the future financial loss of each participant in this market. This is a special case of Nash equilibria where a clearing constraint, on top of common securities traded on the considered exchange, connects all the participants and their actions to minimize their respective risk metric. We show that such a Radner equilibrium exists and can be calculated by assuming financial flows with elliptical distributions.

For a given financial market E , composed of $m \geq 1$ securities, we consider that each of its participants i bears a market loss over a period of time $[0, T]$ where T represents the maturity of all negotiated contracts. Each loss is expressed as

$$\underbrace{-R_i}_{\text{random flow exogenous to } E} + \underbrace{q_i^\top (p - P)}_{\text{hedge}}. \quad (22)$$

The cash flow $R_i \in \mathbb{R}$ is promised at maturity T to participant i , and is potentially correlated to the random value market securities which value are $P \in \mathbb{R}^m$ at maturity for a quantity $q_i \in \mathbb{R}^m$, with price $p \in \mathbb{R}^m$ at time 0. The notations $\mathbf{q}_i \in \mathbb{R}^m$, $i \in E$ and $\mathbf{p} \in \mathbb{R}^m$ are reserved for the solutions of a Radner equilibrium on E . The random variables R_i and P defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ belong to a vector subspace $\mathfrak{X} \subset L^1(\Omega)$, containing the constants. We denote by \mathbb{E} , $\mathbb{V}\text{ar}$ and $\mathbb{C}\text{ov}$ the expectation, variance and covariance operators respectively under the probability \mathbb{P} .

The member a portfolios represented by the equilibrium quantities \mathbf{q}_a , facing a CCP settling for a market E , can contain both hedging positions (called also proprietary trading, recorded in the member's owner account, also known as *house*) and client clearing positions of the CCP member (recorded in the member's client account). The hedges are with respect to all positions $R_a = \sum_{o \in O} R_a^o$ (where the financial actor o is outside the market processed by the CCP) taken on OTC markets and not cleared by CCPs. Clearing positions are back-to-back and denoted \mathbf{q}_b^a . This is a consequence of the intermediary role of the CCP between a financial actor b not accessing the CCP, by choice or by constraints due to insufficient resources to claim direct access to the CCP, and the member a which aligns with regulatory requests (G20 Research Group, 2009).

Let us denote $E = A \cup B$, with $A \cap B = \emptyset$, A the set of participants a having direct access to the CCP services for the corresponding market represented by its participants E and B all the participants b that are not members of the CCP and clients to any member in A to indirectly access the CCP. q_a defines a proprietary trading position of member a and q_b^a the position of one of its clients noted b for which a

exercises the role of clearing member. Thus the position q_b^a only transits from b to a then from a to the CCP. So, even if the total position of a towards the CCP is $q_a + \sum_{b \in B} q_b^a$, as a holds $-\sum_{b \in B} q_b^a$ towards its clients it clears for, the positions of a and b involved in the resolution of the Radner equilibrium on E are respectively q_a and $q_b = \sum_{a \in A} q_b^a$.

Prior to any default on the exchange, the equilibrium market loss borne by a member $a \in A$ on the market E is expressed

$$-R_a + \mathbf{q}_a^\top (\mathbf{p}^E - P) = - \sum_{o \in O} R_a^o + \mathbf{q}_a^\top (\mathbf{p}^E - P)$$

and that of a participant b , not a member of the CCP settling positions taken on E , is written

$$-R_b + \mathbf{q}_b^\top (\mathbf{p}^E - P) = -R_b + \left(\sum_{a \in A} \mathbf{q}_b^a \right)^\top (\mathbf{p}^E - P).$$

It is not necessary to know the terms $(\mathbf{q}_b^a)_{a \in A}$ but only \mathbf{q}_b .

When a CCP chooses to hedge the defaulting member's portfolio, it becomes a participant in the exchange within which the hedge is taken. Alternatively it may choose to liquidate the defaulted position on the exchange where it operates on or on another exchange. In both cases, the CCP is not one of the market participants where the position is liquidated. A combination of these two hedging and liquidation strategies on the same exchange or different exchanges is possible. For each market exchange E prior a default event, called pre-default, and whose securities have the initial equilibrium price \mathbf{p} considered at time $0-$, we note E' and we qualify this same market as post-default after the occurrence of one or several instantaneous defaults d at date 0. The quantities and securities prices of E' depend on the resolution procedure considered by the CCP with new prices of securities \mathbf{p}' considered at time 0.

For any member i in a post-default market E' whose set of pre-default participants is E , its incremental share of positions $\Delta \mathbf{q}_i$ decomposes into $\Delta \mathbf{q}_i^l$ due to the liquidation of the defaulting members positions \mathbf{q}_d^l , and into $\Delta \mathbf{q}_i^h$ due to the CCP hedging the positions \mathbf{q}_d^h . We have $\mathbf{q}_d = \mathbf{q}_d^l + \mathbf{q}_d^h$, $\mathbf{q}_d^l = \sum_{i \in E'} \Delta \mathbf{q}_i^l$ and $\mathbf{q}_d^h = \sum_{i \in E'} \Delta \mathbf{q}_i^h$. The post-default equilibrium quantity of participant i is denoted by $\mathbf{q}'_i = \mathbf{q}_i + \Delta \mathbf{q}_i$ with $\mathbf{q}_i = 0$ if i is a new participant, i.e. initially non-participant of E . The liquidation is done on the basis of pre-default and contractual prices \mathbf{p} while the intervention of the CCP in the market to cover part of the portfolio of the defaulting member generates new prices \mathbf{p}' . The post-default market loss for $i \in E'$ is then written (see **III.§3.B**)

$$\underbrace{-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^{E'} - P)}_{\text{market loss post-default at price}} + \underbrace{(\mathbf{q}_i + \Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - \mathbf{p}'^{E'})}_{=:\text{LC}_i \text{ (liquidation cost)}}, \quad (23)$$

where LC represents the cost of liquidating the defaulting members portfolios. It can be interpreted as margin call payments, similar to futures contracts, to move from a pre-default market based on prices \mathbf{p} to a post-default market based on prices \mathbf{p}' . These margin calls, due by any surviving participant $i \in E$, only apply to contracts based on old prices, including those that are liquidated by the CCP.

To solve both the pre and post-default Radner equilibria, the minimization of the risk of any participant $i \in E$ results in a minimization of a convex and homogeneous positive risk measure ρ_i applied to its market loss. This minimization is first done by assuming that no default has occurred, and second following the occurrence of one or more defaults, to establish the equilibrium prices and positions before and after such a default event. The resulting market cost for participant $i \in E'$, denoted MC_i , is defined as the difference between the participant i 's risk measure applied to its market loss after a

default event has occurred and market loss prior any default event, with portfolio allocations and prices taken at both pre-default and post default equilibria:

$$MC_i = LC_i + \underbrace{\rho_i(-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P)) - \rho_i(-R_i + (\mathbf{q}_i)^\top (\mathbf{p}^E - P))}_{=:\Delta\rho_i}.$$

The overall liquidation cost is $LC_E = \sum_{i \in E'} LC_i$ and its overall market cost is $MC_E = \sum_{i \in E'} MC_i$, redistributed across all members of the CCP. If the CCP chooses to implement such a strategy of a mix of liquidation and hedging of one of its defaulting portfolio on several exchanges, the market and liquidation costs are aggregated over all of these exchanges: $MC = \sum_E MC_E$. This cost thus represents the fund transfer price [FTP] since it reflects the cost of transferring defaulted positions to surviving and/or new market participants.

For an exchange E that has not suffered any default, the clearing condition is written $\sum_{i \in E} \mathbf{q}_i = 0$, $\mathbf{q}_i \in \mathbb{R}^m$, corresponding to a quantity of zero for any security in the considered exchange. For each participant i , we consider a risk measure $\rho_i: \mathfrak{X} \rightarrow \mathbb{R}$, law invariant, monotonic, convex, translation equivariant ($\rho_i(L + m) = \rho_i(L) + m$ for all $L \in \mathfrak{X}$ and $m \in \mathbb{R}$) and normalized ($\rho_i(0) = 0$), leading, for a given \mathbf{p} , to the optimality condition

$$\rho_i(-R_i + \mathbf{q}_i^\top (\mathbf{p} - P)) \leq \rho_i(-R_i + q_i^\top (\mathbf{p} - P)), \quad q_i \in \mathbb{R}^m. \quad (24)$$

A Radner equilibrium on an exchange represented by its participants E consists of a pair $\{(\mathbf{q}_i)_{i \in E}, \mathbf{p}\}$ verifying the securities quantities clearing condition and optimality conditions of any participant $i \in E$. Solving a Radner equilibrium relies on the properties of the convex measures ρ_i employed by each of the participants i . These must be convex, monotone (for $L_1, L_2 \in \mathfrak{X}$, $L_1 \leq L_2 \Rightarrow \rho_i(L_1) \leq \rho_i(L_2)$), translation equivariant, law invariant for random variables with the same distribution (for $L_1 \stackrel{d}{=} L_2$, $\rho_i(L_1) = \rho_i(L_2)$), normalized ($\rho_i(0) = 0$) and sensitive to large losses ($\lim_{\lambda \rightarrow \infty} \rho_i(\lambda L) = \infty$ for all $L \in \mathfrak{X}$ such that $\mathbb{P}[L > 0] > 0$). By translation equivariance, we rewrite the optimality condition for any participant $i \in E$ as $\rho_i(-R_i - \mathbf{q}_i^\top P) + \mathbf{q}_i^\top \mathbf{p} \leq \rho_i(-R_i - q_i^\top P) + q_i^\top \mathbf{p}$, $q_i \in \mathbb{R}^m$. Setting $r_i(q_i) = \rho_i(-R_i - q_i^\top P)$, we define the inf-convolution of the participants' risk measures as

$$r(q) = \inf \left\{ \sum_{i \in E} r_i(q_i); \sum_{i \in E} q_i = q \right\}, \quad q \in \mathbb{R}^m. \quad (25)$$

A Radner equilibrium can be expressed equivalently with the following three characteristics based on inf-convolution (11) (**Lemma III.1**): **(i)** $-\mathbf{p} \in \partial r(0)$, **(ii)** $r(0) = \sum_{i \in E} r_i(\mathbf{q}_i)$, and **(iii)** $\sum_{i \in E} \mathbf{q}_i = 0$.

The **Lemma III.1** combined with the assumption of sensitivity to large losses of the risk measures ρ_i guarantees the existence of a Radner equilibrium (**Theorem III.2**). The price uniqueness can be obtained if the risk measures ρ_i expressed as functions of the security quantities held by any participant i are differentiable at their respective equilibrium vector of positions \mathbf{q}_i (the sub-differential $\partial r_i(\mathbf{q}_i)$ at the point \mathbf{q}_i is the singleton gradient $\nabla r_i(\mathbf{q}_i)$ evaluated at this same vector point) while the uniqueness of the equilibrium positions is guaranteed if these same functions are differentiable and strictly convex (**Theorem III.3**). In the case of entropic or expected shortfall risk measures with (P, R_i) taken as vectors with elliptical distributions, it is possible to obtain a unique analytical formula for the equilibrium positions and quantities (see **Proposition III.4**, **Proposition III.5** and **Remark III.5**).

Credit costs which capture counterparty risk in the broad sense, including those induced by funding and capital requirements, must be added to the market cost to constitute the entire FTP. They are based

on the indirect exposure of each member of the CCP to all the other members of that same CCP and on the funding costs to maintain a position towards the CCP. If the position of a member changes, following the rearrangement of its positions in response to a default management strategy by the CCP, the other members of that CCP will be impacted in terms of their XVA costs as detailed in [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#). For portfolio variations assumed to be Gaussian, and for a given CCP before a default event, the equation (1) is written for the member 0 on the pre-default market $E = A \text{ cup } B$,

$$\begin{aligned}
C_0 &= C_0^A + C_0^B + C_0^O, \text{ with} \\
C_0^A &\stackrel{(1) \& (2)}{=} w_0^A \sum_{a \in A} (1 - J_a) \left[\left(\left(\sum_{b \in B} \mathbf{q}_b^a \right)^\top (\mathbf{p}^E - P) - \text{IM}_a^A \right)^+ + \right. \\
&\quad \left. \left(\mathbf{q}_a^\top (\mathbf{p}^E - P) - \overline{\text{IM}}_a^A \right)^+ - \text{DF}_a^A \right]^+, \\
C_0^B &= \sum_{b \in B} (1 - J_b) \left((\mathbf{q}_b^0)^\top (\mathbf{p}^E - P) - \text{IM}_0^b \right)^+ \text{ and } C_0^O = \sum_{o \in O} (1 - J_o) (R_0^o - \text{IM}_0^o)^+.
\end{aligned} \tag{26}$$

w_0^A (zero if $0 \notin A$) is the loss allocation coefficient attributed to member 0 by the CCP, IM_a^A , $\overline{\text{IM}}_a^A$, and DF_a^A are the initial margins and contributions to the default fund as detailed in **Chapter I** (summarized in the equations (1) and (2)).

R_0^o is the cash flow exogenous to the market E and received by the participant 0 from its bilateral counterparties o on the OTC markets, with a corresponding initial margin IM_0^o (which can be zero for certain commitments of this type). After a default event, a similar formulation applies by updating the quantities w_0^A , IM_a^A , $\overline{\text{IM}}_a^A$, DF_a^A and IM_0^b in (26).

We can then calculate the XVA costs before and after the CCP's default event management, thus complementing the market costs using their explicit formulas (**Table 2** taken from **Theorem I.4** and **Proposition I.5**). The total costs borne by the surviving members are expressed as

$$\text{FTP} = \underbrace{\text{LC} + \sum_{E'} \sum_{i \in E'} \Delta \rho_i}_{\text{market cost (MC)}} + \underbrace{\sum_{i \neq d} \underbrace{(XVA_i' - XVA_i)}_{\Delta XVA_i}}_{\text{credit cost (CC)}} + \text{AC}, \tag{27}$$

with, for each surviving participant i , $XVA_i = \text{CVA}_i + \text{FVA}_i + \text{MVA}_i + \text{KVA}_i$, XVA_i' these XVA amounts calculated after the default management carried out by the CCP and AC the cost due to the auctioning by the CCPs of the portfolios of members who have defaulted and including their hedges. This cost is another incremental cost in XVA, corresponding to the funds transfer price detailed in **III.§7** (only including credit costs), i.e. the impact of the auction of these defaulted members' portfolios including the hedging instruments used by the CCPs on XVA costs.

§8 Simulated Annealing Algorithms for identifying optimal financial network reorganization following a clearing member default

In this fourth part, we consider more catastrophic scenarios where the failure of a member does not only occur on a single CCP but on a multitude of them. The numerical illustrations in the first chapter employ a direct approach by testing all possible recoveries of the defaulting member's portfolio on a single CCP, which is not possible in the case of several portfolios due to the combinatorial problem of all possible recoveries scenarios. Such a scenario of default on several CCPs could really have occurred at the beginning of 2023 with the difficulties of Credit Suisse bank, a major player to around thirty CCPs, if the Swiss National Bank had not intervened with granting a liquidity contribution of around €170 billion (CHF 168 billion, see [Jordan \(2023\)](#)). A smaller, but very real, example with a default on two CCPs occurred in the United States with the asset manager Ronin Capital and is illustrated in the first chapter. Regarding a network of a few dozen CCPs (around 50 major CCPs in the world) and hundreds of members (300 to 400), the number of combinations being of the order of 10^{130} , even for small networks (like ten CCPs and 200 members), it seems inconsiderate to envisage the exploration of the entire space of all possible take-over combinations. The use of stochastic optimization techniques thus appears natural. We illustrate the application of discrete simulated annealing algorithms on a network problem involving a few CCPs and the numerical behaviors expected for this type of technique.

Contributions in Chapter IV:

- we describe the re-allocation problem of a defaulting member portfolio on several CCPs towards their surviving members; we formulate it as the solution of a combinatorial minimization problem on a finite space of a non-linear and non-convex function of random variables approximated by Monte Carlo techniques; these variables represent the future contractual flows of these portfolios and latent default variables for each of the members;
- we summarize the main convergence and error control results of discrete simulated annealing algorithms on finite spaces by putting them in the context of our problem; for a given number of iterations, we deduce bounds on errors between the value of the approximated function at the state returned by the algorithm and the minimum value of the function without approximation (**Theorem IV.7, Corollary IV.8**);
- we also obtain a bound of the probability of error between the combination proposed by the approximate version of the algorithm and the ground truth of the non-approximated problem (**Proposition IV.9**); these results are obtained using concentration inequalities resulting from empirical errors and known errors of simulated annealing algorithms;
- we specify the application of simulated annealing for a given sample making it possible to approximate the real problem, with an exploration phase specific to the financial structure arising from the CCPs and their members by emphasizing the heavy tails distribution behaviour of the losses, that we confirm based using dedicated statistical tests;
- we illustrate numerically the adequate behaviour of the algorithm on a set of 4 CCPs and 11 members, among which one is considered in default; the designed algorithm significantly outperforms the naive approach of successive random suggestions takeover combination of takers without replacement.

In **Chapter I**, following the default of a member on one or several CCPs, we address the identification of a member taking over a defaulted portfolio leading to the least XVA costs on all members by going through all the possible recoveries, where the portfolios are taken without subdivision. It thus reflects an idealized auction where each participant proposes a price allowing the takeover to be achieved with the least XVAs costs across the entire financial network. This strategy quickly becomes computationally inefficient in the case of a default on several CCPs with a considerable number of combinations of buyers. Furthermore, although the XVA costs are of different types, namely credit (CVA, BCVA and CCVA), funding (FVA, MVA, BMVA and CMVA) and capital (KVA), we consider the minimization to solely rely on capital costs. This type of cost appears indeed predominant among the others, as illustrated in **Section I.§5**.

The application of discrete simulated annealing algorithms (Kirkpatrick et al., 1982, 1983; Aarts and van Laarhoven, 1989; Aarts and Korst, 1989; Catoni, 2004; Duflo, 1996; Delmas and Jourdain, 2006) to problems involving a large number of combinations in finance is best known as one of the toolbox techniques for solving portfolio optimization problems (Ingber, 1993; Crama and Schyns, 2003). The latest advances in quantum computing have allowed adaptations of this type of algorithms for such portfolio optimization problems (Lang et al., 2022). Although it is possible to apply these algorithms to capture adverse scenarios leading a banking institution to default (Montesi et al., 2020), its application to identify a possible restructuring of the financial network through portfolios re-allocation in the financial network seems new.

For any participant $\ell = 0, \dots, L$, we denote $(Y_\ell^1, \dots, Y_\ell^K)$ the set of random variables defined on a sub-vector space $\mathfrak{X} \subset L^1(\Omega)$, with $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, where Y_ℓ^k expresses the future cash flows over a time horizon T of its portfolio towards the CCP k , $k = 1, \dots, K$. For $\ell = 1, \dots, L$ and a combination of takeover members $\mathbf{i} = (i_1, \dots, i_K) \in \mathcal{I} = \{1, \dots, L\}^K$, the variable $(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_\ell^k$ indicates the ℓ -th participant's portfolio update following the portfolio takeover of the member 0, assumed in default, by the buyer i_k on CCP k . For $M \in \mathbb{N}^*$ and the participant ℓ , we note \mathcal{R}_ℓ^M the empirical measure of a measure \mathcal{R}_ℓ for calculating its required capital on all of its portfolios held on the K CCPs based on a sample of size M . This measure is applied to a loss function g_ℓ which depends on all the portfolio variation variables after takeover by the buyers $\mathbf{i} = (i_1, \dots, i_K)$, i.e. $\left((\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_1^1, \dots, (\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_1^K, \dots, (\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_L^1, \dots, (\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_L^K \right) =: \mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0$. We can then write for a given combination of buyers \mathbf{i} the function (the Hamiltonian) of the cost to be minimized as the sum of the approximated risk measures on all the participants $(1, \dots, L)$ applied to the random variables after takeover:

$$H_M : \mathcal{I} \longrightarrow \mathbb{R}$$

$$\mathbf{i} \longmapsto \sum_{\ell=1}^L \mathcal{R}_\ell^M \left[g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right]. \quad (28)$$

We thus look for $\arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$, based on a sample of size M of the random portfolio variables considered to be common for each value $H_M(\mathbf{i})$ to calculate (Glasserman and Yao, 1992). This problem approximates the real one which is written similarly to (14) by replacing the approximated measurements \mathcal{R}_ℓ^M with their non-approximated version \mathcal{R}_ℓ , $\ell = 1, \dots, L$.

Given the Hamiltonian form H_M to be minimized as a function of \mathbf{i} , where it is difficult to identify any linear or convex structure with respect to the combinations $\mathbf{i} \in \mathcal{I}$ and which has numerous local minima (see in particular the example in **Section IV.§4.E**), the use of a simulated annealing algorithm seems justified since it allows, under certain conditions and for such functions, for a convergence towards a distribution attributing all its mass to the minimum of this Hamiltonian. This same distribu-

tion, when expressed as a function of a control parameter converging towards 0, is written as the limit of a Boltzmann-Gibbs distribution, depending on this control parameter, which attributes the greatest probability to the minimum of the function to be minimized.

For a fixed number of iterations N , which indicates the given budget for the algorithm run, and for $\varepsilon > 0$, we obtain error bounds, similar to concentration inequalities (Boucheron et al., 2013), but with an error component due to the algorithmic error (in our case, that of a simulated annealing algorithm, see Catoni (2004) and Delmas and Jourdain (2006)), for the error between the value of the approximate Hamiltonian applied to the combination obtained by running the algorithm and the non-approximated one applied to the optimal combination of the non-approximated problem (Theorem IV.7). By noting $\mathbf{i}^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i}) =: \mathcal{I}^{opt}$ and \mathbf{I}_M^N the solution returned by the algorithm after N iterations for a sample of size M , we can thus write:

$$\mathbb{P}\left(\sqrt{M}|H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)| > \varepsilon\right) \leq B_1(|\mathcal{I}|, \varepsilon, M, N),$$

with $B_1(|\mathcal{I}|, \varepsilon, M, N)$ a certain deterministic bound converging to 0 for $\varepsilon, M, N \rightarrow \infty$ and $|\mathcal{I}| = L^K$. In a similar way, it is possible to obtain a bound on the probability of the estimation error (Corollary IV.8):

$$\mathbb{P}\left(\sqrt{M}|H(\mathbf{I}_M^N) - H(\mathbf{i}^*)| > \varepsilon\right) \leq B_2(|\mathcal{I}|, \varepsilon, M, N),$$

with $B_2(|\mathcal{I}|, \varepsilon, M, N)$ another deterministic bound converging to 0 for $\varepsilon, M \rightarrow \infty$.

We can deduce bounds on the probabilities of error that the algorithm does not return the correct solution compared to the correct one for the approximated problem, and compared to the correct one of the non-approximate problem:

$$\mathbb{P}\left(\mathbf{I}_M^N \notin \mathcal{I}_M^{opt}\right) \leq B_3(M, N) \quad \text{and} \quad \mathbb{P}\left(\mathbf{I}_M^N \notin \mathcal{I}^{opt}\right) \leq B_4(M, N, \mathcal{H}_2 - \mathcal{H})$$

with $\mathcal{I}_M^{opt} := \arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$, $\mathcal{H} = \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i})$, \mathcal{H}_2 the second smallest value of $\{H(\mathbf{i})\}_{\mathbf{i} \in \mathcal{I}}$, $B_3(M, N)$ and $B_4(M, N, \mathcal{H}_2 - \mathcal{H})$ deterministic bounds converging to 0 for $M, N \rightarrow \infty$.

These error bounds can be specified in various cases, notably for random variables with heavy-tail distributions (Embrechts et al., 1997; Nair et al., 2022) such as the loss functions $g_\ell\left(\mathbf{Y} \bigoplus_{\mathbf{i}} \mathbf{Y}_0\right)$, $\ell = 1, \dots, L$ that we find in our problem. This characteristic can be tested for various classes of distributions, for a generic framework is described in (Gardes et al., 2011). The loss functions of our problems show indeed Pareto distribution tails. We can then specify the error bounds when the risk measures are expectations, quantiles or expected shortfalls (Reiss, 1989; Marchina, 2021; Chamakh et al., 2020, 2021). In the case of an expected shortfall without knowing the exact value of the corresponding quantile, an estimator based on a quantile of several averages can ensure the consistency and a some robustness of such an estimator (Lugosi and Mendelson, 2019; Bartl and Eckstein, 2024).

To ensure good performance of the algorithm, the exploration phase must capture the structure of the positions and the financial network made up of the CCPs and their members. We propose a stochastic matrix partially covering this requirement which does not retain the combination of the previous step of the corresponding Markov chain. Although the numerical results, in the case of 4 CCPs and 11 members including one in default state, show significantly better performance than a naive exploration and exploitation of combinations of takers without resampling (Section IV.§4.E), an improvement should be considered where the previous combination state influences the stochastic matrix of the exploration. Furthermore, the sole choice of the cost of capital as a minimization criterion can lead to irrational behavior from the surviving members, who should not consider this cost of capital as the only criterion but other economic constraints, and that may be a limitation to the application of such an algorithm.

Chapter I

Derivatives' risks as costs in a one-period network model

Note. The results of this chapter are based on the paper¹ [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#).

Abstract. In counterparty credit risk complete markets, collateral and capital requirements would be indifferent to banks. The quantification by banks of market incompleteness based on various XVA metrics ([Crépey \(2022\)](#)) has emerged as the unintended consequence of the FRTB banking reform ([Bank For International Settlements \(2019b\)](#)) and of the more demanding regulatory capital requirements ([The European Parliament and the Council of the European Union \(2013\)](#)). The related risks are in fact reckoned today as the major risks for banks, well ahead market risk ([European Bank Authority, 2022](#), Figure 65 page 67)). The XVA metrics have been introduced and traditionally used by investment banks for pricing and collateral/capital optimization purposes. We demonstrate in this paper that they can be fruitfully used for risk management, suggesting a sound approach to regulatory requirements. We present a one-period cost-of-capital XVA setup encompassing bilateral and centrally cleared trading in a unified framework, with explicit formulas for most quantities at hand. We illustrate possible uses of this framework for running stress test exercises on financial networks with one and two clearinghouses from a clearing member's perspective or for optimizing the porting of the portfolio of a defaulted clearing member using Monte Carlo technique with corresponding confidence errors in elliptical models. A continuous-time extension of this approach is provided in the companion paper [Albanese et al. \(2023\)](#).

§1 Introduction

In the wake of the 2008–09 global financial crisis, clearing through central counterparties (CCPs) has become mandatory for standardized derivatives, other ones remaining under bilateral setup with higher capital requirements.

One role of the CCPs² is to provide to their clearing members fully collateralized hedges of their market risk with their clients. But this comes at a cost to the clearing members, which pass it to their corporate clients in the form of XVA (cross-valuation adjustment) add-ons. Bearing in mind that the

¹*Acknowledgements:* We thank Paul Besson, Head of Quantitative research, Euronext, and Mohamed Selmi, Head of Market Risk, LCH SA, for useful discussions.

²See [Gregory \(2014\)](#) and [Gregory \(2015\)](#) for general CCP and XVA references, as well as [Menkveld and Vuillemeij \(2021\)](#) for a CCP survey.

risks of a hedge are, by definition, of the same magnitudes as the ones of the originating position and that standardized derivatives usable as hedging assets have to be traded through CCPs, the XVA footprint of not only bilateral but also centrally cleared trading is significant and should be analyzed in detail, which is the topic of this paper. (Crépey, 2022, Section 6) provides a continuous-time XVA analysis in the realistic situation of a bank dealing with an arbitrary number of clients and CCPs. For the sake of tractability, this is mimicked here in a stylized one-period setup, fine-tuned to applications including risk assessment in the context of stress test exercises³ or optimizing the porting of the portfolio of defaulted clearing members.

The first type of application is motivated by the default in 2020 of Ronin Capital, a broker/dealer firm that had clearing exposures on both CCP services Fixed Income Clearing Corporation (FICC) GSD⁴ segment (123 members) and CME Futures (56 members of which 24 common with FICC GSD). If all members are assumed to be only exposed to these CCPs and their cleared clients, we can illustrate these relationships by the network depicted in Figure 1. Any common member on those two CCPs needs to ensure conservative risk assessment that can be achieved in the proposed framework by accounting for common memberships on the two CCPs. If such common memberships are ignored, they can lead to lower loss estimates giving wrong risk view on potential losses.

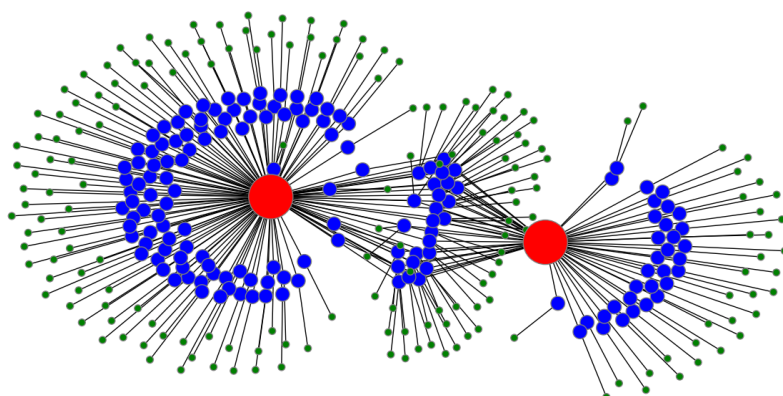


Figure 1: Network consisting of two CCPs (in red), 123 members for CCP1 seen on the left hand side, and 56 members for CCP2 on the right hand side, with 24 common members displayed as the group of members in the middle of the two CCPs (155 members in total, in blue), and with 179 cleared clients (in green).

The second type of application is an illustration of the results of defaulted portfolio porting as it has been the case for the trader Einer Aas on NASDAQ OMX⁵ that has defaulted on 2018 with loss spill-over effect on surviving members.

The paper is outlined as follows. Section §2 sets the stage. Section §3 develops the corresponding

³as required by (The European Parliament and the Council of the European Union, 2013, Article 302).

⁴Government Securities Division.

⁵Optionsmäklarna/Helsinki Stock Exchange.

XVA analysis. Section §4 sets up an elliptical market and credit model amenable to efficient XVA computations. Section §5 introduces the case studies. Section §6 provides numerical results of stress test exercises. Section §7 shows how to optimize the porting of defaulted members portfolios. Section §8 concludes.

§2 General setup

We consider a finite set of market participants, also susceptible to serve as clearing members of CCPs. Derivative transactions can then be concluded between two individual participants, or between a set of participants⁶, pooled in the form of a CCP, and a clearing member of this CCP.

The trades of a clearing member bank with a CCP are partitioned between proprietary trades, which are in effect hedges of the bilateral trading exposure of the bank, and back-to-back hedges of so-called cleared client trades, through which non-member clients gain access to the clearing services of a CCP: see Figure 2. The contractual cash flows from cleared and bilateral clients to a reference clearing member, dubbed the bank hereafter, are promised in successive turns from the bank to the CCP (cash flows denoted by \mathcal{P} and $\bar{\mathcal{P}}$ on Figure 2), from the CCP to other clearing members, and from the latter to their own clients. As a consequence, the CCP is flat in terms of market risk, as is also each of the clearing members. CCPs are typically siloed into different services, each devoted to a specific class of derivatives. We first consider a setup with a single CCP service, the extension to several CCPs being done in Section C.

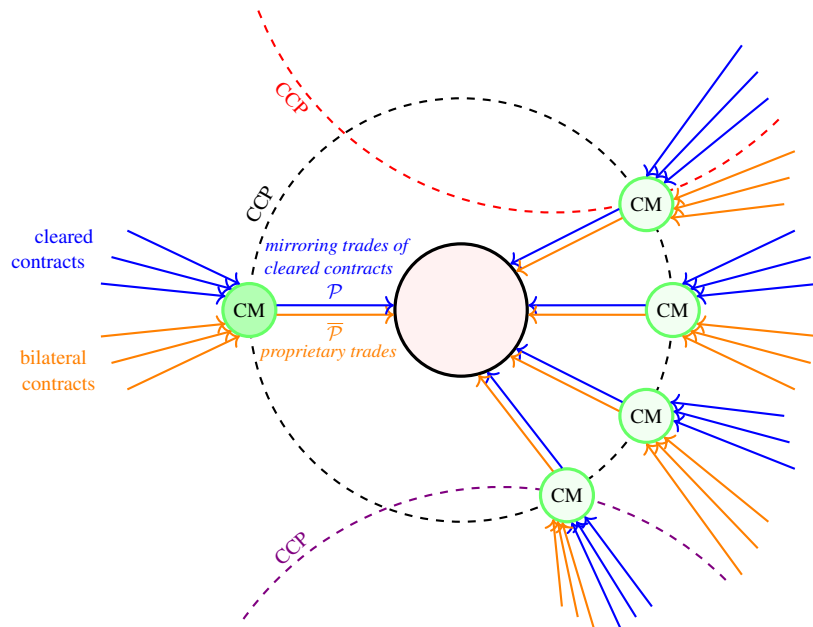


Figure 2: Promised cash flows between market participants. The reference clearing member bank is on the left.

⁶two or more, in practice from a few units to a few hundreds.

A Defaults settlement rule

As reasserted in the wake of the 2008–09 global financial crisis by the Volcker rule, a dealer bank should be hedged as much as possible, at least in terms of market risk⁷. Jump-to-default risk, on the other hand, is hardly hedgeable in practice. Instead it is mitigated through netting and collateralization. Namely, designated netting sets of transactions between two given counterparties (two individual participants or a participant and the CCP) are jointly collateralized, i.e. guaranteed against the default of one or/and the other party. The collateral (or guarantee) comprises a variation margin, which tracks the mark-to-market (counterparty-risk-free value) of the netting set between the two parties, and nonnegative amounts of initial margin posted by each party to the other, which provide a defense against the risk of slippage of the value of the netting set away from its (frozen) variation margin during its liquidation period. In the case of transactions with a CCP, there is an additional layer of collateral in the form of the (funded) default fund contributions of the clearing members, which is meant as a defense against extreme and systemic risk. For each participant, variation margin is rehypothecable and fungible across all its netting sets. Initial margin is segregated at the netting set level. Default fund contributions are segregated at the clearing member level.

The general rule regarding the settlement of contracts of a defaulted netting set, to be instantiated in practical setups on a case by case basis⁸, is that:

Principle 1. If a counterparty in default is indebted toward the other beyond its posted margin, then this debt is only reimbursed at the level of this posted margin (assuming zero recovery rate of the defaulted party for simplicity in this paper); otherwise the debt between the two parties is fully settled.

Here debt is understood on a counterparty-risk-free basis.

Remark 1. One intuitively expects client default cash flows of the form $\mathcal{C} = (1 - R)(D - M)^+$, where the “debt” D represents the pre-default value of the client derivative portfolio to the bank, M the margin posted by the client to the bank, and R the recovery rate of the client. Technically, such an $(1 - R)(D - M)^+$ effectively arises as

$$D - (M + R(D - M)^+ - (D - M)^-) = (1 - R)(D - M)^+,$$

where $M + R(D - M)^+ - (D - M)^-$ is what the bank obtains from the client and D what the bank pays on the hedge of the portfolio. In the special case where case $R = 0$, what the bank obtains from the client simplifies to $M - (D - M)^- = D \wedge M$, in line with Principle 1, and the above expression to

$$D - D \wedge M = (D - M)^+.$$

We emphasize that a counterparty credit default loss $\mathcal{C} = (1 - R)(D - M)^+$ (or simply $(D - M)^+$ if $R = 0$) should not be taken as an *assumption*, but only arises as the result of a *computation* accounting for the cash flows of the portfolio *and its hedge*, derived in a specific market setup under the umbrella of the guiding principle 1 (or the corresponding extension to nonzero recovery, skipped for simplicity in this work). The exact outcome in fact depends on the refined specification of the setup at hand: see e.g. Assumption 3 below and the ensuing formulas (7) (in a single CCP setup) and (15) (under the multiple CCP extension) for the counterparty credit default loss \mathcal{C} in the market setup of this work. Such formulas cannot be safely *guessed*, they should only be *derived* from first principles.

⁷cf. paragraph number 1851 in section 619 from [The United States Congress \(2010\)](#).

⁸cf. e.g. Assumption 3 below.

Remark 2. The above is of course a very crude description of default cash flows. Nonzero, possibly random, recoveries could be introduced at no harm from a theoretical viewpoint, as already pointed out above and done in the continuous-time setup of (Crépey, 2022, Section 3.3). Nonzero recoveries are of course more realistic. But, from a qualitative viewpoint that is our main objective in this work, they only soften the impacts of the defaults. Random recoveries are in line with the uncertainty about the actual level of recovery rates that are only observed a posteriori and can reflect the possibility of liquidating various forms of collateral, account for the output of liquidation procedures, legal resolutions, and other complex and unobservable features. For our purposes in this work, random recoveries could be used for emphasizing some extra dependencies via correlations with other random modeling features. This is all ignored hereafter for avoiding to blur the main features.

Principle 1 also applies to a netting set of transactions between a clearing member and a CCP. However, in our stylized setup, a CCP is nothing but the collection of its clearing members. Our CCP has no resources of its own (in particular, it cannot post any default fund contribution, or “skin-in-the-game”⁹). As long as it is non-default, i.e. as long as at least one of its clearing members is non-default, our CCP can only handle the losses triggered by the defaults of some of its clearing members by redirecting these losses on the surviving ones. This participation of the surviving members to the losses triggered by the defaults of the other members corresponds in our framework to the usage by the CCP of their default fund contributions, both funded (as already introduced above) and unfunded. As will be detailed in equations below, the funded default fund contributions are used for covering losses triggered by the defaults of clearing members over their margins. The unfunded default fund contributions correspond to additional refills that can be required by the CCP, often up to some cap in principle, without bounds in our model, in case the funded default fund contributions of the surviving members are not enough.

B XVA framework

In a nutshell, the main XVAs are the CVA, the FVA/MVA, and the KVA, where:

- i. the CVA is the expected cost for the bank of the default risk of its clients;
- ii. the FVA/MVA is the expected cost for the bank of its own default risk or, more precisely, of the implications of this risk in terms of rehypothecable/segregated collateral funding spreads for the bank;
- iii. the KVA is the cost for the bank of having to remunerate its shareholders at some hurdle rate for their capital at risk, capital which is required by the regulator as a provision against the residual risk left uncovered by i. and ii. (as default risk cannot be hedged by the bank).

Going into details, assume that at time 0 all the banking participants, including the reference clearing member bank¹⁰, with no prior endowments, enter transactions with their clients and hedge their positions, both bilaterally between them and through the CCP. As seen above, the CCP and each bank are flat in terms of market risk. However, as market participants are assumed to be defaultable with zero recovery, in order to account for counterparty credit risk and its funding and capital consequences, the reference bank (and each clearing member bank alike) requires from its corporate clients a pricing rebate

⁹such additional protection layer, though quite common in practice, is of marginal magnitude compared to the other protection layers. By omitting skin-in-the-game component, the obtained results are conservative in terms of risk management and the various formulations are simplified.

¹⁰cf. Figure 2.

(considering conventionally the bank as the “buyer”) with respect to the mark-to-market (counterparty-risk-free) valuation of the deals. The corporate clients of the bank are assumed to absorb the ensuing prices via their corporate business, which is their primary motivation for these deals.

A reference probability measure \mathbb{R}^* , relevant for grounding both stress test exercises and risk management analysis such as economic capital calculation, with corresponding expectation operator denoted by \mathbb{E}^* , is used for the linear valuation of cash flows, using the risk-free asset as our numéraire everywhere. This choice of a numéraire simplifies equations by removing all terms related to the (assumed risk-free) remuneration of all cash and collateral accounts. The funding issue is then refocused on the risky funding side of the problem, i.e. funding costs in what follows really means excess funding costs with respect to a theoretical situation where the bank could equally borrow and lend at the risk-free rate.

More precisely, as suitable for XVA calculations (Albanese et al., 2021, Remark 2.3): given a physical probability measure defined on the full model σ algebra \mathcal{A} and equivalent to a given risk-neutral measure on the financial sub σ algebra \mathcal{B} of \mathcal{A} , we take \mathbb{R}^* equal to the risk-neutral measure on \mathcal{B} and equal to the physical probability measure conditionally on \mathcal{B} ¹¹.

Following the general XVA guidelines of (Crépey, 2022, Section 1), the XVA pricing rebate required by the reference clearing member bank from its corporate clients, dubbed funds transfer price (FTP), comes in two parts: first, the expected counterparty default losses and funding expenditures of the bank, an amount that flows into the bank liabilities and which we refer to as contra-asset valuation (CA = CVA + FVA + MVA as we will see); second, a cost of capital risk premium (KVA), which instead is loss-absorbing¹² and is also used by the management of the bank as retained earnings for remunerating the shareholders of the bank for their capital at risk within the bank. All in one, the bank buys the deals from its clients at the (aggregated) price (MtM – FTP), where MtM is their counterparty-risk-free value and

$$\text{FTP} = \underbrace{\text{CA}}_{\text{Expected costs}} + \underbrace{\text{KVA}}_{\text{Risk premium}}. \quad (1)$$

Let EC denote an economic capital of the bank corresponding to the minimum level of capital at risk that the bank should hold from a regulatory (i.e. solvency) perspective. If $\text{KVA} < \text{EC}$, then the bank shareholders need to provide the missing amount ($\text{EC} - \text{KVA}$) of capital at risk, so that the actual level of capital at risk of the bank is

$$\max(\text{EC}, \text{KVA}),$$

while *shareholder* capital at risk reduces to

$$\max(\text{EC}, \text{KVA}) - \text{KVA} = (\text{EC} - \text{KVA})^+. \quad (2)$$

§3 Theoretical XVA analysis

In this section we detail each term in the equations above, in the realistic setup of a bank involved into an arbitrary combination of bilateral and centrally cleared portfolios, in a tractable one-period setup with period length T . In the one-period XVA model of (Albanese et al., 2021, Section 3), there were no CCPs and the bank was assumed to have access to a “fully collateralized back-to-back hedge of its market risk”, ensuring by definition and for free to the bank a cash-flow ($\mathcal{P} - \text{MtM}$) at time 1,

¹¹these two conditions uniquely characterize \mathbb{R}^* (Artzner et al., 2023, Proposition 4.1).

¹²hence, not a liability.

irrespective of the default status of the bank and its client. There, \mathcal{P} denoted the contractual cash flows from the (assumed unique) client to the bank and MtM was the corresponding counterparty-risk-free value. In the present paper we reveal the mechanism of such a “fully collateralized hedge of the market risk” of the bank, which can be achieved through central clearing, but at a certain cost that we analyze.

All proofs are deferred to Section §9.

A Cash flows

Given disjoint sets of indices $I \ni 0$, C , and B for the clearing members (including the reference bank labeled by 0) and for the respective cleared and bilateral netting sets of the bank with its (individual) counterparties, we denote by:

- J_0 , shortened as J , and $J_i, i \in I \setminus \{0\}$, the survival indicator random variables of the bank and of the other clearing members at time 1; $\gamma = \mathbb{R}^*(J = 0)$, the default probability of the bank;
- $\mathcal{J} = \max_i J_i$, the survival indicator random variable of the CCP (i.e. of at least one clearing member),
- $\mathcal{P}_i, \text{MtM}_i = \mathbb{E}^* \mathcal{P}_i$, and $\text{IM}_i, i \in I$, the contractual cash flows, variation margin, and initial margin from the clearing member i to the CCP corresponding to the cleared clients account of the member i ;
- $\bar{\mathcal{P}}_i, \bar{\text{MtM}}_i = \mathbb{E}^* \bar{\mathcal{P}}_i$, and $\bar{\text{IM}}_i, i \in I$, the contractual cash flows, variation margin, and initial margin from the clearing member i to the CCP corresponding to the proprietary (also dubbed house) account of the clearing member i ;
- $\text{DF}_i, i \in I$, the funded default fund contribution posted by the clearing member i to the CCP;
- $J_b, b \in B$, the survival indicator random variable of the counterparty of the bilateral netting set b of the reference bank; $\mathcal{P}_b, \text{VM}_b$, and IM_b , the associated contractual cash flows, variation margin, and initial margin from the corresponding counterparty to the bank; and $\bar{\text{IM}}_b$, the initial margin from the bank to this counterparty;
- $J_c, c \in C$, the survival indicator random variable of the client of the cleared trading netting set c of the bank, and $\mathcal{P}_c, \text{MtM}_c = \mathbb{E}^* \mathcal{P}_c$ ¹³, and IM_c , the associated contractual cash flows, variation margin, and initial margin from the corresponding client to the bank¹⁴;
- \mathcal{L} , the loss of the CCP, i.e. the loss triggered by the defaults of its clearing members beyond their posted collateral¹⁵, which is borne by the surviving members (if any)¹⁶;
- $\mu = J\mu$, the proportion of these losses allocated to the reference clearing member bank (based on remaining survivors).

Moreover, in case $i = 0$ (so regarding the reference clearing member bank), we typically skip the index i (as in $J_0 = J$).

Assumption 2. $\sum_i (\mathcal{P}_i + \bar{\mathcal{P}}_i) = 0$ (the CCP is flat in terms of market risk), $\sum_c \mathcal{P}_c = \mathcal{P}_0$ (by definition of cleared trades and of their mirroring trades), and $\sum_b \mathcal{P}_b = \bar{\mathcal{P}}_0$ (the reference bank is flat in terms of market risk).

¹³reflecting the fact that members of CCPs are fully collateralized.

¹⁴note that a bank does not post any initial margin on its cleared client netting sets.

¹⁵variation margin, initial margin, and (funded) default fund contributions.

¹⁶see the last paragraph of Section A.

Assumption 2 yields the clearing conditions regarding the contractually promised cash flows, which applies to each banking participant (written there for the reference bank) and to the CCP.

Moreover, in line with Principle 1 that monitors the default cash flows:

Assumption 3. On the CCP survival event $\{\mathcal{J} = 1\}$, the CCP receives from each clearing member i

$$\begin{aligned} & J_i(\mathcal{P}_i + \bar{\mathcal{P}}_i) + (1 - J_i)(\mathcal{P}_i \wedge (\text{MtM}_i + \text{IM}_i) + \bar{\mathcal{P}}_i \wedge (\overline{\text{MtM}}_i + \overline{\text{IM}}_i)) \\ & + ((\mathcal{P}_i - (\text{MtM}_i + \text{IM}_i))^+ + (\bar{\mathcal{P}}_i - (\overline{\text{MtM}}_i + \overline{\text{IM}}_i))^+) \wedge \text{DF}_i. \end{aligned} \quad (3)$$

On the bank survival event $\{J = 1\}$ ($\subseteq \{\mathcal{J} = 1\}$), the bank receives on each cleared netting set c and bilateral netting set b

$$J_c \mathcal{P}_c + (1 - J_c)(\mathcal{P}_c \wedge (\text{MtM}_c + \text{IM}_c)) \text{ and } J_b \mathcal{P}_b + (1 - J_b)(\mathcal{P}_b \wedge (\text{VM}_b + \text{IM}_b)), \quad (4)$$

whereas it pays to the CCP

$$\sum_c \mathcal{P}_c + \sum_b \mathcal{P}_b = \sum_c (J_c \mathcal{P}_c + (1 - J_c) \mathcal{P}_c) + \sum_b (J_b \mathcal{P}_b + (1 - J_b) \mathcal{P}_b). \quad (5)$$

We need one more condition, regarding the funding side of the problem:

Assumption 4. At time 0 the amounts CA and KVA sourced from the corporate clients of the bank are deposited on reserve capital and capital at risk accounts of the bank. The bank can use the amounts CA and $\max(\text{EC}, \text{KVA})$ ¹⁷ on its reserve capital and capital at risk accounts for its variation margin borrowing purposes. Funds needed beyond $\text{CA} + \max(\text{EC}, \text{KVA})$ for variation margin posting purposes are borrowed by the bank at its credit spread γ above OIS. Instead, the bank must borrow entirely the amounts to post as initial margin and funded default fund contributions, but this can be achieved at some blended funding spread $\tilde{\gamma} \leq \gamma$.

The rationale for funding variation margin but not initial margin from $\text{CA} + \max(\text{EC}, \text{KVA})$ is set out before Equation (15) in [Albanese et al. \(2017\)](#). The motivation for the assumption $\tilde{\gamma} \leq \gamma$ is provided in ([Albanese et al., 2020](#), Section 5), along with numerical experiments suggesting that $\tilde{\gamma}$ can be several times lower than γ .

Lemma 1. *The borrowing needs of the bank for reusable and segregated collateral amount to, respectively,*

$$\begin{aligned} & \left(\sum_b (\text{MtM}_b - \text{VM}_b) - \text{CA} - \max(\text{EC}, \text{KVA}) \right)^+ \text{ and} \\ & \text{IM} + \overline{\text{IM}} + \text{DF} + \sum_b \overline{\text{IM}}_b. \end{aligned} \quad (6)$$

Lemma 2. *On the bank survival event $\{J = 1\}$, the counterparty default losses \mathcal{C} and the funding expenses \mathcal{F} of the bank are given by*

$$\mathcal{C} = \sum_b (1 - J_b)(\mathcal{P}_b - \text{VM}_b - \text{IM}_b)^+ + \sum_c (1 - J_c)(\mathcal{P}_c - \text{MtM}_c - \text{IM}_c)^+ + \mu \mathcal{L}, \quad (7)$$

where

$$\mathcal{L} = \sum_i (1 - J_i)((\mathcal{P}_i - \text{MtM}_i - \text{IM}_i)^+ + (\bar{\mathcal{P}}_i - \overline{\text{MtM}}_i - \overline{\text{IM}}_i)^+ - \text{DF}_i)^+, \quad (8)$$

and

$$\mathcal{F} = \tilde{\gamma}(\text{IM} + \overline{\text{IM}} + \text{DF}) + \tilde{\gamma} \sum_b \overline{\text{IM}}_b + \gamma \left(\sum_b (\text{MtM}_b - \text{VM}_b) - \text{CA} - \max(\text{EC}, \text{KVA}) \right)^+. \quad (9)$$

¹⁷where $\max(\text{EC}, \text{KVA}) - \text{KVA} = (\text{EC} - \text{KVA})^+$ is provided by the bank shareholders, cf. (2).

B Valuation

Let \mathbb{E} denote the expectation with respect to the bank survival measure \mathbb{R} associated with \mathbb{R}^* , i.e., for any random variable \mathcal{Y} ,

$$\mathbb{E}\mathcal{Y} = (1 - \gamma)^{-1} \mathbb{E}^*[J\mathcal{Y}]. \quad (10)$$

(expectation of \mathcal{Y} conditional on the survival of the bank). As (readily) seen in (Albanese et al., 2021, Section 3):

Lemma 3. *For any random variable \mathcal{Y} and constant Y , we have*

$$Y = \mathbb{E}^*[J\mathcal{Y} + (1 - J)Y] \iff Y = \mathbb{E}\mathcal{Y}. \quad (11)$$

Under a cost-of-capital XVA approach, the bank charges its future losses to its corporate clients at a CA level making $\ell = J(\mathcal{C} + \mathcal{F} - CA)$, the trading loss of the shareholders of the bank, \mathbb{R}^* centered. In addition, given a target hurdle rate h assumed in $[0, 1]$ (and typically of the order of 10%), the management of the bank ensures to the bank shareholders dividends at the height of h times their capital at risk $(EC - KVA)^+$ (cf. (2)), where we model EC as $\mathbb{E}\mathcal{S}(\ell)$, the expected shortfall of the trading loss ℓ ¹⁸ computed under the bank survival measure \mathbb{R} at a quantile level¹⁹ α (e.g. $\alpha = 99\%$ and $\alpha = 99.75\%$ in our experiments), i.e., under the dual representation of the expected shortfall²⁰:

$$EC = \sup \{ \mathbb{E}[\ell\chi] ; \chi \text{ measurable}, 0 \leq \chi \leq (1 - \alpha)^{-1}, \text{ and } \mathbb{E}[\chi] = 1 \}, \quad (12)$$

which for atomless ℓ coincides²¹ with $\mathbb{E}[\ell | \ell \geq \text{VaR}(\ell)]$, where VaR is the \mathbb{R} value-at-risk (lower quantile) at the level α . Note that, in view of (12), an expected shortfall of a centered random variable is nonnegative.

The definitions of the XVA metrics corresponding to the above specifications are given in Table 1.

¹⁸assumed \mathbb{R} integrable.

¹⁹under normal distribution assumptions, such ES at percentile level 99.75% allows reaching similar loss level as with a VaR (quantile) risk metric at the level 99.9%. In practice, regulatory and economic capital indeed aims at capturing extreme losses that can occur once every 1000 years, cf. paragraph 5.1 from [Basel Committee on Banking Supervision \(2005\)](#) for the detailed instructions.

²⁰see e.g. ([Kaina and Rüschemdorf, 2009](#), Theorem 4.1).

²¹see Corollary 5.3 and representation thanks to expression (3.7) from [Acerbi and Tasche \(2002\)](#).

XVA	Expression	Full name and description
KVA	$\mathbb{E}^* \left[Jh(\text{EC} - \text{KVA})^+ + (1 - J)\text{KVA} \right]$, where $\text{EC} = \mathbb{E}\mathbb{S}(J(\mathcal{C} + \mathcal{F} - \text{CA}))$	capital valuation adjustment
CA	$\text{CVA} + \text{MVA} + \text{FVA}$	contra-asset valuation
CVA	$\text{BCVA} + \text{CCVA}$	credit valuation adjustment
BCVA	$\mathbb{E}^* \left[J \sum_b (1 - J_b)(\mathcal{P}_b - \text{VM}_b - \text{IM}_b)^+ + (1 - J)\text{BCVA} \right]$	credit valuation adjustment for bilateral exposures
CCVA	$\mathbb{E}^* \left[J \sum_c (1 - J_c)(\mathcal{P}_c - \text{MtM}_c - \text{IM}_c)^+ + \mu\mathcal{L} + (1 - J)\text{CCVA} \right]$	credit valuation adjustment for clearing activity exposures
MVA	$\text{BMVA} + \text{CMVA}$	margin valuation adjustment
BMVA	$\mathbb{E}^* \left[J\tilde{\gamma} \sum_b \overline{\text{IM}}_b + (1 - J)\text{BMVA} \right]$	margin valuation adjustment for bilateral exposures
CMVA	$\mathbb{E}^* \left[J\tilde{\gamma}(\text{IM} + \overline{\text{IM}} + \text{DF}) + (1 - J)\text{CMVA} \right]$	margin valuation adjustment for clearing activity exposures
FVA	$\mathbb{E}^* \left[J\gamma \left(\sum_b (\text{MtM}_b - \text{VM}_b) - \text{CA} - \max(\text{EC}, \text{KVA}) \right)^+ + (1 - J)\text{FVA} \right]$	funding valuation adjustment

Table 1: XVA definitions, cf. Section B (with \mathcal{C} , \mathcal{F} and \mathcal{L} given by Lemma 2).

Hence in view of (7) and (9):

$$\text{CA} = \mathbb{E}^* \left[J(\mathcal{C} + \mathcal{F}) + (1 - J)\text{CA} \right], \quad (13)$$

i.e. $\mathbb{E}^* \left[J(\mathcal{C} + \mathcal{F} - \text{CA}) \right] = 0$, as desired²². The terminal cash flows of the form $(1 - J) \times \dots$ in Table 1 expressions and in (13) are thus consistent with the desired shareholder centric perspective. They can also be interpreted as the amounts of reserve capital and risk margin lost by the bank shareholders, hence valued as such by CA, as their property is transferred to the liquidator of the bank if the bank defaults.

Due to these terminal cash flows, the above definition is in fact a fix-point system of equations. The split of the underlying CA equation (13) into the collection of equations in Table 1 is motivated by both interpretation and numerical considerations. From an interpretation viewpoint, it is useful to provide the more granular view on the costs of the bank provided by the split of the global CA amount between, on the one hand, bilateral and centrally cleared trading default risk components BCVA and CCVA and, on the other hand, bilateral and centrally cleared trading funding risk components BMVA and CMVA for segregated initial margin, whereas the FVA cost of funding variation margin is holistic in nature

²²see after Lemma 3.

(can only be apprehended at the level of the bank balance-sheet as a whole), via the feedback impact of $CA + \max(EC, KVA)$ into the FVA. From a numerical viewpoint, the collection of smaller problems in Table 1 may be easier to address than the global equation (13). Each of the smaller problems can also be handled by a dedicated desk of the bank, namely the CVA desk, for the BCVA and CCVA, and the Treasury of the bank, for the BMVA, CMVA and the FVA.

Passing in the above equations to the bank survival measure \mathbb{R} based on Lemma 3 shows that the corresponding fixed point problem is in fact well-posed and yields explicit formulas for all the quantities at hand.

Theorem 4. *The explicit XVA formulas of Table 2 hold and we have*

$$J(\mathcal{C} - CVA) = J\left(\sum_c (1 - J_c)(\mathcal{P}_c - MtM_c - IM_c)^+ + \mu\mathcal{L} - CCVA\right. \\ \left. + \sum_b (1 - J_b)(\mathcal{P}_b - VM_b - IM_b)^+ - BCVA\right). \quad (14)$$

In particular, all the XVA (and also EC) numbers are nonnegative²³.

XVA	Explicit formula
CCVA	$\mathbb{E}\left[\sum_c (1 - J_c)(\mathcal{P}_c - MtM_c - IM_c)^+ + \mu\mathcal{L}\right]$
CMVA	$\tilde{\gamma}(\overline{IM} + \overline{IM} + DF)$
BCVA	$\mathbb{E}\left[\sum_b (1 - J_b)(\mathcal{P}_b - VM_b - IM_b)^+\right]$
BMVA	$\tilde{\gamma}\sum_b \overline{IM}_b$
EC	$\mathbb{E}S(J(\mathcal{C} - CVA))$
FVA	$\frac{\gamma}{1 + \gamma}\left(\sum_b (MtM_b - VM_b) - (CCVA + CMVA + BCVA + BMVA) - EC\right)^+$
KVA	$\frac{h}{1 + h}EC$

Table 2: XVA explicit formulas (with \mathcal{C} , \mathcal{F} and \mathcal{L} given by Lemma 2).

Remark 3. The reason why funding disappears from the bank trading loss, i.e. $J(\mathcal{C} + \mathcal{F} - CA) = J(\mathcal{C} - CVA)$, is because, in a one-period setup, the collateral borrowing requirements (6) of the bank are simply constants. Hence funding triggers no risk to the bank, but only a deterministic cost. In the dynamic setup of [Albanese et al. \(2020\)](#), funding generates both costs and risk.

C Extension to several CCPs or CCP services

In the realistic case where the reference bank is a clearing member of several services of one or several CCPs, we index all the CCP related quantities in the above by an additional index ccp in a finite set disjoint from $I \cup C \cup B$. Then, with $CA = CCVA + CMVA + BCVA + BMVA + FVA$ as before:

²³cf. ([Crépey, 2022](#), Sections 1 and 7.1).

Proposition 5. The counterparty default loss \mathcal{C} across several counterparties and several CCPs is given by

$$\mathcal{C} = \sum_{ccp,c} (1 - J_c)(\mathcal{P}_c^{ccp} - \text{MtM}_c^{ccp} - \text{IM}_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} + \sum_b (1 - J_b)(\mathcal{P}_b - \text{VM}_b - \text{IM}_b)^+, \quad (15)$$

where

$$\mathcal{L}^{ccp} = \sum_i (1 - J_i)((\mathcal{P}_i^{ccp} - \text{MtM}_i^{ccp} - \text{IM}_i^{ccp})^+ + (\overline{\mathcal{P}}_i^{ccp} - \overline{\text{MtM}}_i^{ccp} - \overline{\text{IM}}_i^{ccp})^+ - \text{DF}_i^{ccp})^+. \quad (16)$$

The funding expenses \mathcal{F} across several CCPs and several counterparties are given by

$$\mathcal{F} = \tilde{\gamma} \sum_{ccp} (\text{IM}^{ccp} + \overline{\text{IM}}^{ccp} + \text{DF}^{ccp}) + \tilde{\gamma} \sum_b \overline{\text{IM}}_b + \gamma \left(\sum_b (\text{MtM}_b - \text{VM}_b) - \text{CA} - \max(\text{EC}, \text{KVA}) \right)^+. \quad (17)$$

The only XVA definitions and explicit formulas that change with respect to Tables 1 and 2 (on top of \mathcal{C} and \mathcal{F} generalized as above) are the ones for CCVA and CMVA, the way detailed in Tables 3 and 4. Moreover,

$$J(\mathcal{C} - \text{CVA}) = J \left(\sum_{ccp,c} (1 - J_c)(\mathcal{P}_c^{ccp} - \text{MtM}_c^{ccp} - \text{IM}_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} - \text{CCVA} + \sum_b (1 - J_b)(\mathcal{P}_b - \text{VM}_b - \text{IM}_b)^+ - \text{BCVA} \right). \quad (18)$$

XVA	Expression	Full name and description
CCVA	$\mathbb{E}^* \left[J \sum_{ccp,c} (1 - J_c)(\mathcal{P}_c^{ccp} - \text{MtM}_c^{ccp} - \text{IM}_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} + (1 - J)\text{CCVA} \right]$	credit valuation adjustment for clearing activity exposures
CMVA	$\mathbb{E}^* \left[J \sum_{ccp} \tilde{\gamma} (\text{IM}^{ccp} + \overline{\text{IM}}^{ccp} + \text{DF}^{ccp}) + (1 - J)\text{CMVA} \right]$	margin valuation adjustment for clearing activity exposures

Table 3: CCVA and CMVA definitions with several CCPs (also, \mathcal{C} and \mathcal{F} are now given by Proposition 5, as also \mathcal{L}^{ccp}).

XVA	Explicit formula
CCVA	$\mathbb{E} \left[\sum_{ccp,c} (1 - J_c)(\mathcal{P}_c^{ccp} - \text{MtM}_c^{ccp} - \text{IM}_c^{ccp})^+ + \sum_{ccp} \mu^{ccp} \mathcal{L}^{ccp} \right]$
CMVA	$\sum_{ccp} \tilde{\gamma} (\text{IM}^{ccp} + \overline{\text{IM}}^{ccp} + \text{DF}^{ccp})$

Table 4: CCVA and CMVA explicit formulas with several CCPs (also, \mathcal{C} and \mathcal{F} are now given by Proposition 5, as also \mathcal{L}^{ccp}).

Before passing to the case studies, we specify the calculation of economic capital under the member survival measure.

Lemma 6. If $\mathbb{R}(\ell = \text{VaR}(\ell)) = 0$, where $\ell = J(\mathcal{C} - \text{CVA})$, then

$$\text{EC} = \mathbb{E}^* [\mathcal{C} - \text{CVA} | \mathcal{C} - \text{CVA} \geq \text{VaR}(\ell), J = 1]. \quad (19)$$

§4 Market and credit model

We introduce a market and credit model, written under \mathbb{R}^* , with parameters that can capture dependence between portfolio changes, joint defaults and possible averse exacerbated changes of the portfolio due to their owner default known as wrong-way risk.

For any $j \in I \cup B \cup C$, denoting by F_j the marginal cdf of a financial participant j 's default time τ_j , $\Delta \mathcal{P}_j := \mathcal{P}_j - \text{MtM}_j$, S the Student-t cdf with 3 degrees of freedom, nom_j a (signed) nominal of the portfolio of the market participant j , σ_j its annualized relative volatility, and Δ_l a positive liquidation period accounting for the time taken by the CCP to novate or liquidate²⁴ defaulted portfolios, we define

$$\begin{cases} \tau_j = F_j^{-1}(S(X_j)), \\ \frac{\Delta \mathcal{P}_j}{\text{nom}_j \sigma_j \sqrt{\Delta_l}} = Y_j, \end{cases} \quad (20)$$

where

$$\begin{cases} X_j = \sqrt{\mathcal{K}_j} \left(\sqrt{\rho^{cr}} \mathcal{T} - \sqrt{\rho_j^{wwr}} \mathcal{X}_j + \sqrt{1 - \rho^{cr} - \rho_j^{wwr}} \mathcal{T}_j \right) \\ Y_j = \sqrt{\rho^{mkt}} \mathcal{E} + \sqrt{\mathcal{K}_j} \sqrt{\rho_j^{wwr}} \mathcal{X}_j + \sqrt{1 - \rho^{mkt} - \rho_j^{wwr}} \mathcal{E}_j. \end{cases} \quad (21)$$

Here ρ^{cr} , ρ^{mkt} and the ρ_j^{wwr} are positive credit/credit, market/market and credit/ market correlation coefficients, while \mathcal{T} , \mathcal{T}_j , \mathcal{X}_j are i.i.d. random variables following Normal distributions, \mathcal{E} , \mathcal{E}_j are i.i.d. Student-t distributions with degree of freedom 3, and $3/\mathcal{K}_j$ follows a chi-squared distribution of degree of freedom 3 independent from all other variables, such that:

- \mathcal{T} represents the common systemic factor for default times across members,
- \mathcal{E} represents the common systemic factor for portfolio variations across members,
- $\sqrt{\mathcal{K}_j} \mathcal{X}_j$ is the common factor co-driving portfolio variations and default time of market participant j ,
- \mathcal{T}_j is the idiosyncratic factor for market participant j 's default time,
- \mathcal{E}_j is the idiosyncratic factor for market participant j 's portfolio variations.

Remark 4. In practice, margin computations rely on historical estimates based on several market stressed periods. Our approach, instead, aims at reflecting extreme market shocks with fat tailed Student-t distributions of degree of freedom $\nu = 3$, and volatility level within a reasonable range of [20%, 40%]. Our static formulation depicts stationary increments of the defaulted portfolios' value over the liquidation period.

In view of the above, the setup is well defined if and only if²⁵

$$\rho_j^{wwr} < \min(1 - \rho^{cr}, 1 - \rho^{mkt}). \quad (22)$$

²⁴cf. Section §7.

²⁵otherwise, the model for both default time and portfolio variation factors is undefined due to their idiosyncratic coefficient term $\sqrt{1 - \rho^{cr} - \rho_j^{wwr}}$ and $\sqrt{1 - \rho^{mkt} - \rho_j^{wwr}}$. Also we discard the limit cases where $\rho_j^{wwr} = 1 - \rho^{cr}$ or $\rho_j^{wwr} = 1 - \rho^{mkt}$ as they lead to a zero contribution of the idiosyncratic factors, which would be unrealistic.

The “minus” sign in front of the common credit-market factor $-\sqrt{\rho_j^{ww^T}}$ for the default time component in (21) ensures that the corresponding common factor accelerates defaults, whilst increasing the market exposure due to the $+\sqrt{\rho_j^{ww^T}}$ factor in the second part of (21).

Remark 5. Andersen and Dickinson (2019) introduce a dynamic model locally elliptical in the sense of elliptical on each next time step given the information at the beginning of the time step. Under simplifying assumptions including their equation (32) and Assumption 2, they obtain (in our notation) explicit CMVA and approximate CCVA formulas. In their case, defaults are triggered by $\Delta\mathcal{P}_j$ (in our notation) falling below a Merton-like threshold. In our static setup with extra latent variables for defaults, we do not have such explicit formulas. However, Monte Carlo simulation is quite efficient and required anyway for stress test exercises that aim at identifying scenarios leading to extreme losses with adequate description such as the identification of defaulted members and their corresponding losses.

Hereafter, we describe two possible applications of our XVA framework which will be illustrated by numerical case studies in the above model. To these ends, two networks will be defined to serve the numerical illustrations, one rather educational on the use of the XVA metrics and the other one reflecting the more realistic situation depicted by Figure 1.

In the numerical applications that follow, all members play in turn the role of the reference bank in the theoretical XVA framework of Sections §2-§3. The CVA and KVA computations require a Monte Carlo routine run under \mathbb{R}^* in combination with a rejection technique in order to yield simulations under the survival measures associated with different clearing members. For obtaining confidence intervals regarding the expected shortfalls that are embedded in the KVA computations, the simulations are split into several batches, from which the mean of the (partial) EC estimates yields the final EC estimate, while their standard deviation is used to define a confidence interval.

§5 Case studies setup

In the examples that follow, market participants are identified by a number and can then be included in one of several of the considered CCPs. We restrict ourselves to cleared client trades, so that the nonvanishing XVA metrics reduce to the CCVA, the CMVA, and the KVA.

A Single CCP setup and initial XVA costs

We consider a single CCP service with 20 members labeled by $i \in 0 \dots n = 19$, only trading for cleared clients (i.e. without bilateral or centrally cleared proprietary trading). Each member faces one client. The ensuing financial network is depicted by Figure 3.

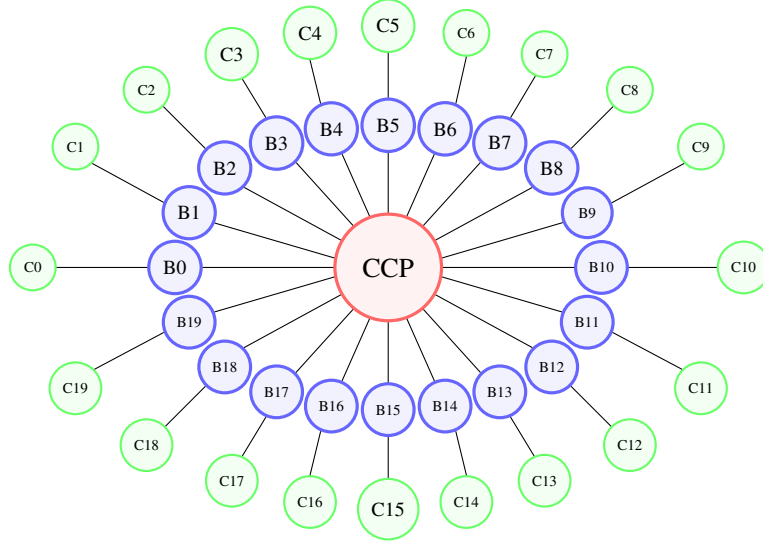


Figure 3: Financial network composed of 1 CCP, its 20 members (labeled by B) and one client per member

All clients are assumed to be risk-free. For any member i , its posted IM to the CCP is calculated based on the idea of a VM call not fulfilled over a time period $\Delta_s < \Delta_l$ at a confidence level $\alpha \in (1/2, 1)$, using a VaR metric²⁶ applied to the non-coverage of VM call taken also to follow a scaled Student-t distribution \mathcal{S}_ν with ν degrees of freedom, with cdf S_ν :

$$\text{IM}_i = \text{VaR}(\text{nom}_i \sigma_i \sqrt{\Delta_s} \mathcal{S}_\nu) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} S_\nu^{-1}(\alpha). \quad (23)$$

The default fund is calculated at the CCP level as

$$\text{Cover2} = \text{SLOIM}_{(0)} + \text{SLOIM}_{(1)}, \quad (24)$$

for the two largest stressed losses over IM (SLOIM_i) among members, identified with subscripts (0) and (1), where SLOIM is calculated as the value-at-risk VaR' at a confidence level $\alpha' > \alpha$ of the loss over IM, i.e.

$$\text{SLOIM}_i = \text{VaR}'(\text{nom}_i \sigma_i \sqrt{\Delta_s} \mathcal{S}_\nu - \text{IM}_i) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} (S_\nu^{-1}(\alpha') - S_\nu^{-1}(\alpha)). \quad (25)$$

The total amount (24) is then allocated between the clearing members to define their (funded) default fund contributions as $\text{DF}_i = \frac{\text{SLOIM}_i}{\sum_j \text{SLOIM}_j} \text{Cover2}$.

The nom_j 's of other clearing members are not observable by a given one. However, following [Murphy and Nahai-Williamson \(2014\)](#) and [Lipton \(2018\)](#), $|\text{nom}|_{(i)}$ denoting the i -th largest absolute nominal amount for $i \in 0 \cdots n = 19$, a parameterization of the form

$$|\text{nom}|_{(i)} = \beta e^{-\beta'(i+1)}, \quad \beta, \beta' > 0 \quad (26)$$

²⁶under the member survival measure.

cm id	0	1	2	3	4	5	6	7	8	9
DP (bps)	50	60	70	80	90	200	190	180	170	160
size	-242	184	139	105	-80	-61	-46	35	26	-20
vol (%)	20	21	22	23	24	25	26	27	28	29

cm id	10	11	12	13	14	15	16	17	18	19
DP (bps)	150	140	130	120	110	100	90	80	70	60
size	-15	-11	-9	-6	5	-4	-3	2	2	-1
vol (%)	30	31	32	33	34	35	36	37	38	39

Table 5: Member characteristics and portfolio parameters, ordered by decreasing member size.

can be fit to the total default fund held by the CCP²⁷ and the sum of its five largest default fund contributions²⁸, made public each quarter for most of the CCPs. The parameters β and β' inferred from the default fund data are used to depict a similar pattern on the absolute nominal sizes²⁹. The participants and portfolios parameter inputs are detailed in Table 5, where id is the identifier of the CM, DP stands for the one year probability of default of the member expressed in percentage points, size represents the overall portfolio size of the member detained within the CCP, and vol is the annual volatility used for the portfolio variations.

The portfolios listed in the Table 5 relate to the members towards the CCP (which are mirroring the ones between the members and their clients). The signs of the nominals are distributed so that $\sum_j \text{nom}_j = 0$, consistent with the clearing condition (first identity in Assumption 2, here without proprietary trades).

The parameters of the XVA costs calculations are summarized in Table 6. Note that the chosen period length of $T = 5$ years covers the bulk (if not the final maturity) of most realistic CCP portfolios.

²⁷item referenced as 4.3.15 in [Bank of International Settlements and OICV-IOSCO \(2015\)](#), *Value of pre-funded default resources (excluding initial and retained variation margin) held for each clearing service in total, post-haircut*. in the quantitative disclosure documents.

²⁸item referenced as 18.4.2 in [Bank of International Settlements and OICV-IOSCO \(2015\)](#):*For each segregated default fund with 25 or more members; Percentage of participant contributions to the default fund contributed by largest five clearing members in aggregate.*; or item referenced 18.4.1 for CCP services with less than 25 members

²⁹as if the default fund amounts are proportional to the portfolio sizes.

³⁰such confidence level at 97% for SLOIM in DF calibration allows for a ratio of default fund over initial margin of about 10% in our calculations, a ratio (of this level or less) often observed in practice.

One-period length T	5 years
Liquidation period at default Δ_l	5 days
Portfolio variations correlation ρ^{mkt}	30%
Credit factors correlation ρ^{cr}	20%
Correlation between credit factors and portfolio variations ρ^{wvr}	20%
IM covering period (MPoR) Δ_s	2 days
IM quantile level	95%
Funding blending ratio $\tilde{\gamma}/\gamma$	25%
SLOIM calculation ³⁰ for DF Cover-2	VaR 97%
Funded DF allocation rule	\propto SLOIM
\mathcal{L} allocation rule ($\ni \mu$)	$\propto DF_i$
Quantile level used for clearing members EC calculation	99.75%
Hurdle rate h used for KVA computations	10.0%
Number of Monte Carlo simulation (for CCVA and KVA computations)	10M
Number of batches (for KVA computations)	100

Table 6: XVAs calculation configuration

For each member, the CCVA, CMVA and KVA costs are calculated and reported in Table 7. For KVA, two calculations have been performed, one based on ES at 99th percentile level and another one based on 99.75th percentile level. The amount in square bracket is the corresponding quantile level from which average is calculated and numbers in parenthesis represent the 95% confidence interval in relative difference from calculated metric for both CCVA and KVA. All the XVA numbers decrease with the member size.

cm id	CMVA	CCVA	KVA (99%)	KVA (99.75%)
0	0.0687	0.1413 (0.3%)	0.2631 [0.1327] (0.7%)	0.8512 [0.5129] (1.0%)
1	0.0656	0.1154 (0.3%)	0.3088 [0.1373] (0.7%)	0.7818 [0.453] (1.1%)
2	0.0604	0.0932 (0.3%)	0.2505 [0.1103] (0.7%)	0.6606 [0.3855] (1.1%)
3	0.0544	0.0750 (0.4%)	0.2023 [0.0886] (0.8%)	0.546 [0.3172] (1.3%)
4	0.0485	0.0604 (0.4%)	0.1562 [0.0646] (0.8%)	0.445 [0.2572] (1.2%)
5	0.0834	0.0440 (0.4%)	0.1303 [0.0540] (0.8%)	0.3578 [0.2108] (1.1%)
6	0.0623	0.0358 (0.4%)	0.1044 [0.0447] (0.8%)	0.2843 [0.1663] (1.1%)
7	0.0467	0.0291 (0.4%)	0.0839 [0.0371] (0.7%)	0.228 [0.1321] (1.4%)
8	0.0341	0.0230 (0.4%)	0.0645 [0.0286] (0.7%)	0.1768 [0.1019] (1.4%)
9	0.0256	0.0187 (0.4%)	0.0514 [0.0227] (0.7%)	0.1425 [0.0811] (1.4%)
10	0.0187	0.0147 (0.4%)	0.0398 [0.0177] (0.7%)	0.1124 [0.0629] (1.8%)
11	0.0132	0.0113 (0.4%)	0.0301 [0.0134] (0.7%)	0.086 [0.0476] (1.5%)
12	0.0104	0.0097 (0.4%)	0.0254 [0.0113] (0.7%)	0.0731 [0.0401] (1.6%)
13	0.0066	0.0068 (0.6%)	0.0174 [0.0077] (0.7%)	0.0536 [0.0276] (2.8%)
14	0.0052	0.0059 (0.6%)	0.0149 [0.0066] (0.7%)	0.0472 [0.0237] (2.8%)
15	0.0039	0.0050 (0.8%)	0.0122 [0.0054] (0.8%)	0.042 [0.0195] (3.4%)
16	0.0027	0.0039 (0.9%)	0.0094 [0.0042] (0.7%)	0.0341 [0.0151] (3.7%)
17	0.0017	0.0028 (1.1%)	0.0064 [0.0029] (0.7%)	0.0251 [0.0103] (4.3%)
18	0.0015	0.0029 (1.4%)	0.0066 [0.0029] (0.7%)	0.0275 [0.0106] (4.9%)
19	0.0007	0.0016 (2.5%)	0.0033 [0.0015] (0.7%)	0.0194 [0.0055] (8.6%)

Table 7: Initial XVA costs: estimates, [value-at-risk underlying the KVA estimate] and (95% confidence level errors).

To assess the average behavior w.r.t. ρ^{cr} , ρ^{mkt} and ρ^{wvr} of the CCVA and KVA, we vary these correlations between 5% and 95%, with 5% step and display in Figures 4 and 5 the corresponding metrics, aggregated over all clearing members successively considered as the reference bank. For such tests, the default correlation ρ^{cr} and ρ^{mkt} are both set to 4% when they are not changed between 5% and 95%. This is to allow for runs with $\rho_i^{wvr} = 95\%$ satisfying the condition (22).

The KVA depicts an increase w.r.t. ρ^{cr} but also w.r.t. ρ^{wvr} and very limited change w.r.t. ρ^{mkt} . The correlation ρ^{wvr} has more impact than ρ^{cr} and ρ^{mkt} (right panels in Figures 4 and 5). As seen on the left panels of Figures 4 and 5, there are very marginal changes for the aggregated CCVA w.r.t. ρ^{cr} and ρ^{mkt} , but a significant positive impact of ρ^{wvr} . This is understandable for the sensitivity to ρ^{cr} and ρ^{mkt} as, apart for modulations of the measure with respect to which each individual CCVA is assessed, the CCVA aggregated over clearing members is essentially an expectation of the CCP loss \mathcal{L} (cf. the first line of Table 2). The individual CCVAs (as per the first line of Table 2) of each clearing member, however, may depend on ρ^{cr} and ρ^{mkt} (on top of ρ^{wvr}) in a strong and nontrivial manner, via the allocation coefficient μ .

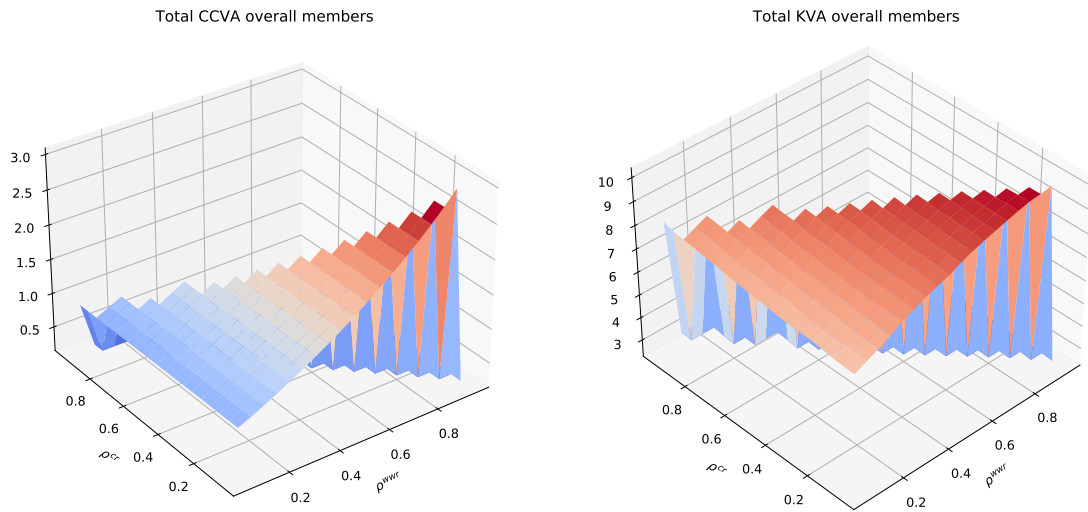


Figure 4: CCVA and KVA w.r.t. credit factors correlation and credit and portfolio variation factors correlation.

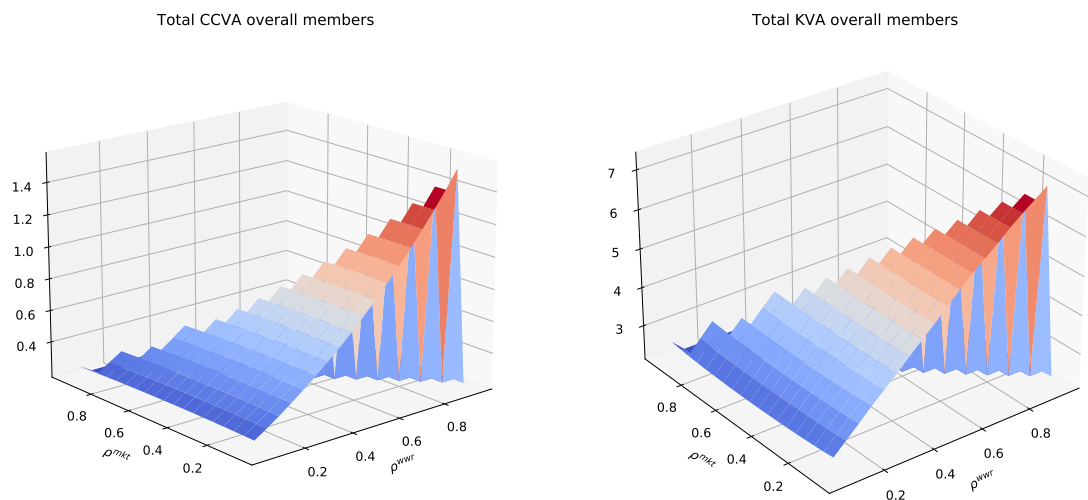


Figure 5: CCVA and KVA w.r.t. market factors correlation and credit and portfolio variation factors correlation.

B Two CCPs network setup

We now consider the case of Figure 1 where there are two CCPs with some common members and stress test is considered from the perspective of one of these common members. The motivation for this case is to provide a realistic example mimicking in a simplified way the default of the trading firm Ronin Capital, which had memberships on both FICC GSD³¹ segment, hereafter denominated by CCP1, and CME Futures segment, hereafter denominated by CCP2, in March 2020. It is well known that a VaR type risk measure is not sub-additive, in particular for credit portfolios as illustrated in Example 5.4 in

³¹Government Securities Division

Acerbi and Tasche (2002) and Example 2.25 in McNeil et al. (2015) for a portfolio of defaultable bonds, so that for a common member adding VaR estimates of trading losses on two CCPs separately can lead to underestimated levels with respect to the actual VaR of the global exposition of the member. As such, stress test exercises accounting for common memberships could reveal a larger value-at-risk compared to the exercise where stress tests are conducted separately on each CCP.

To perform the analysis, the following setup is considered:

- all members have only clearing client positions³², with 123 members on CCP1 and 56 members on CCP2, out of which 24 are common to both CCPs,
- all clients are assumed default free,
- both CCPs use configuration as per Table 6,
- the sizes of the positions are assumed exponentially distributed in the sense that from the most exposed member to the least one, absolute value of positions decrease exponentially with the form in (26) as depicted by Figures 6 and 7 respectively,
- the proportion of the default fund detained by the 5 biggest members is 25% for CCP1 and 61% for CCP2³³,
- the size of the default fund of CCP1 is assumed to be twice the one of the default fund of CCP2.

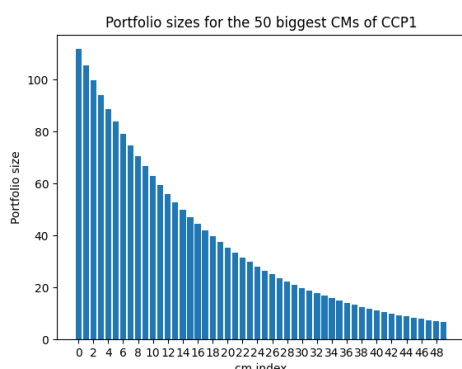


Figure 6: Decreasing absolute nom_i per member for CCP1

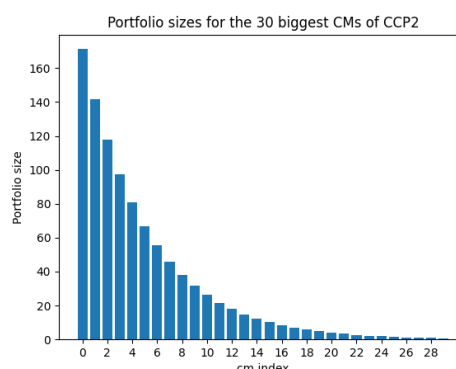


Figure 7: Decreasing absolute nom_i per member for CCP2

All data used are either public sources or have been anonymized, with default intensities ranging from 10 bps to 400 bps and portfolio volatilities ranging from 20 to 30. Similar configuration as given in Table 6 is used, apart from the number of Monte Carlo simulations reduced to 2M for memory capacity reasons.

The clearing conditions are ensured by setting the sum of the portfolio sizes nom_i to zero on each CCP. The situation of member 3, exposed to both CCPs, as the defaulting member, corresponds roughly to the situation of Ronin Capital in 2018. In particular, an annual probability of default of 0.1% corresponds roughly to a BBB rating, that was assigned to Ronin Capital in 2018 for its issuances³⁴.

³²Ronin Capital had in fact only a house account and was thus not clearing any client position.

³³taken from the quantitative disclosure of both CCPs as of third quarter of 2020.

³⁴<https://www.spglobal.com/marketintelligence/en/news-insights/blog/banking-essentials-newsletter-july-edition-2>.

§6 Stress test exercises

As outlined in the capital requirements regulation detailed in [The European Parliament and the Council of the European Union \(2013\)](#) article 290, financial institutions must conduct regular stress test exercises of their credit and counterparty exposures. Paragraph 8 of this article also stipulates the reverse stress test³⁵ requirement to

[...] identify extreme, but plausible, scenarios that could result in significant adverse outcomes.

This is complemented by article 302 on the exposure financial institutions may have towards CCPs:

Institutions shall assess, through appropriate scenario analysis and stress testing, whether the level of own funds held against exposures to a CCP, including potential future credit exposures, exposures from default fund contributions and, where the institution is acting as a clearing member, exposures resulting from contractual arrangements as laid down in Article 304, adequately relates to the inherent risks of those exposures.

In practice, stress test exercises aim at assessing the capacity of financial institutions to absorb financial and economic shocks. In regular exercises, such as the ones conducted by the European Banking Authority, the shocks are usually considered under so called *central* and *baseline* macro-economic scenarios corresponding to a median quantile and *adverse* scenario usually taken as a 90th percentile reflecting severe yet plausible scenario that can occur once every 10 years³⁶. Additionally, extreme scenarios can be considered for measuring the capital adequacy³⁷ for absorbing extremely severe losses around confidence level at 99.9%. From a clearing member perspective, this requires to have the capacity of scanning certain points of its trading loss distribution. In our framework, this boils down to identifying particular levels of the distribution of the trading loss $\ell = J(\mathcal{C} - \text{CCVA} - \text{BCVA})$ of the reference clearing member bank, where the different terms are detailed in Proposition 5.

The other type of stress test exercises, referenced as *reverse stress test*³⁸ [Eichhorn et al. \(2021\)](#), consists in identifying the probability of reaching a given loss level as well as describing the scenario configuration such as projected defaults and loss magnitude leading to such loss levels. The distribution must span a sufficient large spectrum of losses, including the ones targeted by the exercise, but it also has to be sufficiently rich numerically to allow identifying combinations of events leading to such losses.

Confidence intervals of corresponding extreme scenario probabilities should complement the analysis to ensure the reliability of the used model and numerical methods.

Regulators have the ability to challenge financial institutions on these elements and demand for improvements³⁹.

³⁵see dedicated definition p.12 in [Board of Governors of the Federal Reserve System \(2012\)](#) and articles 97, 98 p. 37 in [European Central Bank \(2018\)](#) for official regulatory definitions.

³⁶such confidence levels are suggested by the Federal Reserve outlining p.10 in [Board of Governors of the Federal Reserve System \(2013\)](#) the various recession periods of the United States listed in their Table 1 p. 14. The 2021 instructions in [European Bank Authority \(2021\)](#) also indicate p.72 that stressed market risk factors are based on shocks specified in [European Systemic Risk Board \(2013\)](#), citing ([Dordu et al., 2017](#), p. 29), with the US recessions periods as stressful economic episodes.

³⁷cf. paragraph 5.1 p.11 from [Basel Committee on Banking Supervision \(2005\)](#).

³⁸see also dedicated definition on p.12 in [Board of Governors of the Federal Reserve System \(2012\)](#) and articles 97, 98 p. 37 in [European Central Bank \(2018\)](#) for official regulatory definitions.

³⁹this may entail re-assessment of the Pillar 2 guidance additional capital requirement set in the annual Supervisory Review and Evaluation Process reported by Banks, cf. [European Central Bank \(2021\)](#) for a brief definition

A Scenarios identification for reverse stress test

We now briefly explain how to identify and exploit the scenarios leading to contribute the most to economic capital, in the spirit of [Albanese et al. \(2023\)](#). We denote by M the number of Monte Carlo scenario for which $J = 1$, i.e. survival of the reference bank. Its trading loss $\mathcal{C} - \text{CVA}$ for a simulation m is given by $\mathcal{C}^m - \text{CVA}$, where $m \in 1 \cdots M$ enumerates the simulated scenarios for which the reference member bank ends up in survival state.

To get an estimate of the economic capital based on expected shortfall, relying on ([Acerbi and Tasche, 2002](#), Definition 2.6 and Proposition 4.1), we calculate, for a high confidence level $\alpha \in (\frac{1}{2}, 1)$ and $[x]$ denoting the integer part of any real x ,

$$\widehat{\mathbb{E}\mathbb{S}}(\mathcal{C} - \text{CVA}) := \frac{1}{M - [\alpha M]} \sum_{m=[\alpha M]+1}^M \{\mathcal{C}^{(m)} - \text{CVA}\}, \quad (27)$$

where the $\mathcal{C}^{(m)} - \text{CVA}$'s are the simulated trading losses of the reference bank ranked in increasing order.

To obtain the contribution of any simulated scenario m (with $\mathcal{C}^m \geq \mathcal{C}^{([\alpha M])}$) to the economic capital estimated by (27), we compute

$$\begin{aligned} & \widehat{\mathbb{E}\mathbb{S}}^{-m}(\mathcal{C} - \text{CVA}) \\ & := \frac{1}{M - 1 - [\alpha(M-1)]} \left\{ (M - [\alpha M]) \widehat{\mathbb{E}\mathbb{S}}(\mathcal{C} - \text{CVA}) - (\mathcal{C}^m - \text{CVA}) \right\}. \end{aligned} \quad (28)$$

The contribution $\delta_m \widehat{\mathbb{E}\mathbb{S}}(\mathcal{C} - \text{CVA})$ of scenario m to $\widehat{\mathbb{E}\mathbb{S}}(\mathcal{C} - \text{CVA})$ is then given by:

$$\delta_m \widehat{\mathbb{E}\mathbb{S}}(\mathcal{C} - \text{CVA}) = \widehat{\mathbb{E}\mathbb{S}}(\mathcal{C} - \text{CVA}) - \widehat{\mathbb{E}\mathbb{S}}^{-m}(\mathcal{C} - \text{CVA}). \quad (29)$$

To illustrate the various flavors of stress test exercises that can be conducted by a CCP member, we report numerical results for the two network examples introduced in Section §5. We start with a reverse stress test exercise on example covered by Table 5. For this first illustration, a specific extreme loss is targeted and the corresponding probability of loss reaching at least such target level is estimated. We then consider the example illustrated by Figure 1 where projected loss levels for specific confidence levels are indicated for the members with common memberships on the two CCPs.

B Numerical results

In Table 8, we report, for the example summarized in Table 5, the 99.9th percentile trading loss levels, referenced as extreme quantile, with corresponding (asymmetric) confidence intervals based on the approach proposed in ([Meeker et al., 2017](#), Section G.2). This is done for every clearing member successively playing the role of the reference bank in the setup of Sections §2-§3. We also compute the probabilities of reaching a loss equal to 1.5 times the obtained extreme quantile level, referenced as RST scenario, with corresponding confidence levels⁴⁰.

and use and [Basel Committee on Banking Supervision \(2019\)](#) for more extensive details as well as [Board of Governors of the Federal Reserve System \(2020\)](#) for similar requirements.

⁴⁰the calculation of the latter confidence intervals of the probability of being above a quantile relies on the same numerical approach based on batches used for KVA calculations. Also, the batch approach leads to reasonably tight confidence intervals for the RST scenario probabilities.

cm id	99.9%	$1.5 \times 99.9\%$	RST scenario probability
0	4.7322 (-0.9%, 1.0%)	7.0984	0.0395% (3.2%)
1	5.9679 (-0.9%, 0.9%)	8.9518	0.0382% (3.4%)
2	4.8541 (-0.9%, 1.0%)	7.2812	0.0386% (3.2%)
3	3.9294 (-0.9%, 0.9%)	5.8942	0.0380% (3.7%)
4	3.0931 (-0.9%, 1.1%)	4.6396	0.0388% (3.6%)
5	2.5726 (-0.9%, 1.0%)	3.8590	0.0385% (3.7%)
6	2.0423 (-1.1%, 1.0%)	3.0635	0.0382% (3.6%)
7	1.6224 (-0.9%, 1.0%)	2.4336	0.0380% (3.4%)
8	1.2488 (-0.9%, 1.0%)	1.8732	0.0379% (3.5%)
9	0.9938 (-0.9%, 0.9%)	1.4906	0.0379% (3.6%)
10	0.7682 (-1.0%, 0.9%)	1.1523	0.0379% (3.6%)
11	0.5811 (-1.0%, 0.9%)	0.8716	0.0384% (3.6%)
12	0.4900 (-0.9%, 0.9%)	0.7351	0.0381% (3.6%)
13	0.3353 (-0.9%, 1.0%)	0.5029	0.0379% (3.8%)
14	0.2862 (-0.9%, 0.9%)	0.4294	0.0383% (3.7%)
15	0.2356 (-0.9%, 1.0%)	0.3534	0.0382% (3.8%)
16	0.1809 (-0.8%, 1.0%)	0.2714	0.0382% (3.7%)
17	0.1235 (-0.8%, 1.0%)	0.1852	0.0381% (3.7%)
18	0.1264 (-0.9%, 1.0%)	0.1896	0.0379% (3.6%)
19	0.0645 (-0.9%, 1.0%)	0.0967	0.0382% (3.7%)

Table 8: Stress test (ST) extreme quantile, $1.5 \times$ ST extreme quantile and RST probability to breach 1.5 times the 99.9th quantile loss level, for each member, based on 10M simulations (in parentheses: corresponding 95% confidence intervals).

Our description of the scenarios leading to such losses includes the identified defaulted members, the generated losses and the allocated loss coefficient of the reference clearing member (CM1 in this example). Table 9 provides the description of the 20 worst scenarios, contributing the most to the EC estimation for the second biggest member, that is CM1⁴¹. Most of these scenarios are driven by significant losses stemming from CM0's default, reflecting the highly concentrated position of CM0. We observe that several scenarios illustrate the cases where more than one clearing member default such as 3th to 5th scenarios for which not only CM0 generates most of the loss but other defaulting members generate losses yet of less magnitude compared to CM0.

From CM1 viewpoint (i.e. with CM1 in the role of the reference clearing member), 17 scenarios entail significant losses over the collateral posted by the defaulted CM0 (positive first entries in the last column of Table 9). CM0 bears a very large concentrated position compared to other members. Even if CM0 has more IM and DF requirements than others, this is still not enough: this example highlights that employed DF allocation rules in this example dilute the DF collateral requirements for concentrated positions. It also illustrates that scenarios with multiple defaults do not necessarily lead to extreme losses, due to the fact that members with medium or small positions have large default fund contributions stemming from others' concentrated positions.

In Table 10, we report, for the example illustrated by Figure 1 with 2 CCPs, the trading loss levels

⁴¹its theoretical number of scenarios above the RST loss level should be 3709, i.e. the number of MC simulations of 10M multiplied by CM1's survival probability over 5 years and by CM1's RST loss level probability estimated in Table 8 as 0.0382%, which is of course far too many to report. Nonetheless a focus on the 20 worst ones already illustrates the type of information that can be exploited for such exercises.

Rank	Loss	n	μ	Defaulters	Losses triggered by defaulters
1	10.04	1	0.21	cm0	518
2	8.92	2	0.21	cm0, 19	459.21, 0.18
3	8.67	5	0.23	cm0, 8, 10, 15, 18	411.41, 0, 1.83, 0, 0
4	8.30	3	0.19	cm6, 8, 12	489.31, 0.17, 0.79
5	7.97	1	0.21	cm0	411.19
6	7.81	1	0.21	cm0	403.05
7	7.48	4	0.20	cm3, 7, 12, 17	409.76, 0, 0, 0
8	7.35	2	0.24	cm0, 4	339.10, 0
9	6.96	2	0.23	cm0, 7	340.43, 0
10	6.94	3	0.29	cm0, 2, 4	258.70, 0, 1.01
11	6.91	3	0.24	cm0, 6, 8	318.76, 0, 0
12	6.84	1	0.21	cm0	352.86
13	6.60	2	0.22	cm0, 11	334.17, 0.10
14	6.59	2	0.21	cm2, 5	344.84, 0
15	6.47	1	0.21	cm0	333.87
16	6.27	1	0.21	cm0	323.79
17	5.97	1	0.21	cm0	308.19
18	5.75	2	0.22	cm0, 8	284.80, 0
19	5.70	5	0.28	cm0, 5, 6, 7, 8	222.31, 0, 0.63, 0, 0
20	5.50	3	0.23	cm0, 7, 9	258.51, 1.12, 0.29

Table 9: Economic Capital 20 worst scenarios details for member 1 in decreasing order of total loss where column with header μ indicates allocated coefficient loss to member 1 and n is the number of defaults within the scenario.

(value-at-risks) at confidence levels 90% and 99.9%, for the 24 common members on the two CCPs. The corresponding numbers in the case where the two CCPs would be considered separately is reported in the columns VII and IX. For members with very low size on one of the two CCPs compared to the other, considering the common memberships or not does not affect the loss estimates, as expected⁴². For other members, however, at 90% confidence levels, the value-at-risks are significantly higher (compare columns VII and VIII in Table 10) when the common membership are considered compared to the stand-alone value-at-risks calculation conducted on each CCP and summed, especially for the first ten members. On the contrary, at the confidence level 99.9%, the sum of the stand-alone value-at-risks is well above the value-at-risk when common memberships are taken into consideration (columns IX and X in Table 10). These two situations illustrate that a regulator and the board (top management) of the bank could equally and rightfully criticise a simplistic standalone approach, too aggressive in some cases (making it unacceptable by the regulator) and over-conservative in others (making it unacceptable by the board).

§7 Optimizing the porting of defaulted client portfolios

In case a clearing member defaults, the CCP tentatively novates part of the CCP portfolio of the defaulted member through auctions among the surviving clearing members [Default Risk Management Working Group \(2016\)](#); [Bank For International Settlements \(2019a\)](#), and it liquidates the residual on the market. A natural baseline is that the CCP novates (auctions among surviving members) client trades

⁴²as the CCP with the very low size compared to the other should have marginal impact.

I	II	III	IV	V	VI	VII	VIII	IX	X
3	0.1	19.90	21	-97.48	23	0.0761 (-1.5%, 1.7%)	0.0788 (-1.2%, 1.2%)	3.4710 (-1.8%, 2.0%)	3.1277 (-2.0%, 1.8%)
4	0.1	80.79	24	-18.79	22	0.0679 (-1.6%, 1.5%)	0.0724 (-1.1%, 1.0%)	2.9993 (-2.1%, 2.1%)	2.6544 (-1.9%, 2.2%)
9	3.1	-31.58	29	17.74	23	0.0477 (-1.4%, 1.5%)	0.0619 (-0.8%, 0.8%)	1.8253 (-2.4%, 2.2%)	1.4621 (-2.0%, 2.2%)
12	0.1	17.97	21	-16.75	24	0.0340 (-1.0%, 1.0%)	0.0418 (-0.7%, 0.7%)	0.8706 (-1.9%, 2.1%)	0.5990 (-1.8%, 2.1%)
13	0.1	-14.90	22	15.81	25	0.0328 (-1.0%, 1.1%)	0.0398 (-0.6%, 0.6%)	0.7960 (-1.9%, 2.2%)	0.5434 (-2.3%, 2.0%)
14	0.2	12.34	23	-14.93	26	0.0313 (-1.0%, 1.0%)	0.0379 (-0.6%, 0.6%)	0.7302 (-1.8%, 2.2%)	0.4916 (-1.9%, 2.1%)
15	0.1	-10.23	24	14.09	27	0.0300 (-0.9%, 1.0%)	0.0358 (-0.6%, 0.6%)	0.6683 (-1.9%, 2.2%)	0.4469 (-1.6%, 1.7%)
17	0.3	-7.03	26	-13.30	28	0.0283 (-0.9%, 0.9%)	0.0330 (-0.6%, 0.6%)	0.5767 (-1.8%, 2.0%)	0.3891 (-1.7%, 1.7%)
19	0.2	-4.83	28	12.56	29	0.0267 (-0.9%, 0.9%)	0.0300 (-0.7%, 0.6%)	0.5018 (-1.8%, 2.3%)	0.3495 (-1.5%, 2.0%)
22	3.9	2.75	20	-11.86	30	0.0289 (-0.9%, 0.8%)	0.0303 (-0.7%, 0.7%)	0.4437 (-1.8%, 2.2%)	0.3593 (-2.1%, 1.7%)
26	0.1	1.30	24	11.20	20	0.0152 (-0.8%, 0.8%)	0.0156 (-0.6%, 0.7%)	0.2345 (-1.8%, 2.3%)	0.1930 (-1.9%, 2.4%)
27	0.1	1.07	25	-10.57	21	0.0150 (-0.8%, 0.8%)	0.0152 (-0.7%, 0.7%)	0.2265 (-1.8%, 2.3%)	0.1903 (-1.7%, 2.3%)
28	1.5	0.89	26	9.98	22	0.0157 (-0.7%, 0.8%)	0.0159 (-0.7%, 0.7%)	0.2328 (-1.8%, 2.0%)	0.1997 (-1.8%, 2.1%)
31	0.1	-0.51	29	-9.42	23	0.0143 (-0.7%, 0.8%)	0.0144 (-0.7%, 0.7%)	0.2049 (-1.7%, 2.4%)	0.1845 (-1.9%, 2.4%)
34	0.1	0.29	21	8.89	24	0.0139 (-0.7%, 0.8%)	0.0139 (-0.7%, 0.7%)	0.1902 (-1.8%, 2.4%)	0.1819 (-2.0%, 2.3%)
35	0.1	-0.24	22	-8.40	25	0.0137 (-0.7%, 0.7%)	0.0137 (-0.7%, 0.7%)	0.1859 (-1.7%, 2.4%)	0.1787 (-1.9%, 2.3%)
36	0.1	0.20	23	7.93	26	0.0134 (-0.7%, 0.7%)	0.0134 (-0.7%, 0.7%)	0.1823 (-1.9%, 2.3%)	0.1758 (-1.9%, 2.3%)
39	0.1	-0.11	26	-7.48	27	0.0131 (-0.7%, 0.7%)	0.0131 (-0.7%, 0.7%)	0.1760 (-1.8%, 2.3%)	0.1719 (-1.8%, 2.3%)
40	0.5	0.09	27	7.07	28	0.0131 (-0.7%, 0.8%)	0.0131 (-0.7%, 0.7%)	0.1747 (-1.7%, 2.1%)	0.1713 (-1.8%, 2.0%)
44	0.1	0.04	20	-6.67	29	0.0125 (-0.7%, 0.7%)	0.0125 (-0.7%, 0.7%)	0.1656 (-1.7%, 2.4%)	0.1645 (-1.7%, 2.4%)
49	0.1	-0.02	25	6.30	30	0.0122 (-0.7%, 0.7%)	0.0122 (-0.7%, 0.7%)	0.1614 (-1.7%, 2.4%)	0.1607 (-1.7%, 2.4%)
50	0.1	0.01	26	-5.95	20	0.0077 (-0.7%, 0.7%)	0.0077 (-0.7%, 0.7%)	0.1016 (-1.7%, 2.4%)	0.1012 (-1.7%, 2.4%)
51	0.1	-0.01	27	5.61	21	0.0076 (-0.7%, 0.7%)	0.0076 (-0.7%, 0.7%)	0.1009 (-1.9%, 2.3%)	0.1005 (-1.9%, 2.2%)
55	0.1	-0.01	20	-5.30	22	0.0075 (-0.7%, 0.7%)	0.0075 (-0.7%, 0.7%)	0.0996 (-1.8%, 2.4%)	0.0993 (-1.8%, 2.4%)

Table 10: Quantile loss levels (confidence errors) for 90% and 99.9% confidence levels across members for the example with 2 CCPs and 155 members including 24 common members. Legend for column headers: I. Member Id, II. DP (%), III. Size on CCP1, IV. Volatility on CCP1, V. Size on CCP2, VI. Volatility on CCP2, VII. 90th Perc. stand-alone, VIII. 90th Perc., IX. 99.9th Perc. stand-alone, X. 99.9th Perc.

and their mirroring client account positions, collectively dubbed client positions for brevity hereafter, whereas house account positions are liquidated.

The liquidation side of the procedure cannot be handled in our modeling setup, which does not embed the fundamentals of price formation (our MtM processes are assumed to be exogenously given). On the other hand, an XVA-based procedure can be used for rendering what would be the output of an idealized, efficient auction, assuming a large number of clearing members (Oleschak, 2019, Section 3.3). Namely, supposing that the reference clearing member, labeled by 0 in Sections §2-§3, defaults at time 0, i.e. just after that all portfolios have been settled, for each surviving member CM^* successively envisioned as a potential taker of the defaulted (client) positions of CM_0 , one computes the incremental (Δ) XVAs of porting the defaulted positions to CM^* , for each surviving member (CM^* included⁴³). The corresponding incremental XVA numbers are then summed over metrics and survivors, resulting in the funds transfer price (FTP*) of porting defaulted client positions to CM^* . The effective taker is then the surviving member for which the ensuing FTP* is the smallest⁴⁴. See (Albanese et al., 2020, Section 5.2) for more details on such “XVA Pareto optimally driven” novation procedures.

In what follows, based on the example of Table 5 (which only involves client positions), we analyze from this perspective a first scenario of a single default on the CCP.

Taking the first case with a single default, we first assume the scenario whereby CM_0 defaults at time 0. Table 11 summarizes the total ΔXVA^* aggregated over survivors, across members $*$ from 1 to 19, in increasing order of the FTP* indicated in the last column. Based on the results of Table 11, CM_1 appears to be the potential taker leading to the least overall FTP costs across all surviving members. This is understandable as this member’s portfolio size (184 in Table 5) nets the most the defaulted member’s portfolio size (-242), with volatility and credit default probability similar to⁴⁵ the ones of the defaulted member.

As CM_1 concentrates more risks due in particular to non-perfect offset⁴⁶ between its prior positions and the defaulting one, there is an increase of its IM reflected through an increase of CMVA. But the new risk of CM_1 is less than the sum of the former risks of CM_0 and CM_1 , hence the $\Delta CCVA$ aggregated across surviving members is reduced. This only happens when CM_1 takes over the defaulting portfolio, other potential takers leading to an overall increase of the CCVA. As for the KVA, there is a reduction effect for CM_1 when CM_1 is the taker (see the term in parentheses in Table 11), and an overall decrease in the total KVA (aggregated over all surviving members), which is also the case for most members. Having CM_1 as the taker allows to obtain the most significant decrease in ΔKVA .

As expected, among the three XVA components, KVA is the main determinant of the optimal taker: see Table 12.

Once the CCP has re-allocated all defaulted client positions, the resulting financial network formerly depicted in Figure 3 becomes the network with 19 members shown in Figure 8. The thick lines represent the new portfolio exposures for CM_1 and the pale dashed lines show the defaulted CM_0 positions.

⁴³note that all members are impacted by additional margin to fund due to the re-calibration of their DF by the CCP, whereas only the member taker of the portfolio sees in addition its IM adjusted.

⁴⁴or, indifferently in case of multiple minima, one of the minimizing FTP* members.

⁴⁵in particular, not significantly higher than.

⁴⁶By offset we refer to risk reduction when taking over some additional position. The effect of correlation is such that an opposite sign in portfolio size does not imply an equal offset of the risk of the aggregated positions. For instance, even with opposite sizes and same volatilities but for $\rho^{mkt} \in (0, 1/2)$, the member ends up with more risk.

Surv. member *	Total Δ CMVA*	Total Δ CCVA*	Total Δ KVA*	Total FTP*
1	0.0768 (0.0295)	-0.0428 (-0.0045)	-0.4165 (-0.1065)	-0.3826 (-0.0815)
19	0.1298 (0.0818)	-0.0278 (0.0371)	-0.4284 (0.2790)	-0.3264 (0.3979)
2	0.0921 (0.0428)	-0.0392 (0.0018)	-0.3530 (-0.0391)	-0.3001 (0.0055)
18	0.1417 (0.0939)	-0.0230 (0.0363)	-0.3898 (0.2737)	-0.2710 (0.4038)
3	0.1054 (0.0576)	-0.0323 (0.0080)	-0.2953 (0.0281)	-0.2221 (0.0937)
17	0.1549 (0.1070)	-0.0178 (0.0363)	-0.3284 (0.2759)	-0.1913 (0.4192)
4	0.1525 (0.1022)	-0.0513 (0.0158)	-0.2429 (0.1050)	-0.1416 (0.2230)
16	0.1688 (0.1208)	-0.0145 (0.0359)	-0.2720 (0.2760)	-0.1177 (0.4327)
15	0.1814 (0.1334)	-0.0109 (0.0354)	-0.2262 (0.2739)	-0.0557 (0.4426)
14	0.1903 (0.1426)	-0.0048 (0.0345)	-0.1907 (0.2665)	-0.0052 (0.4437)
13	0.2061 (0.1582)	-0.0030 (0.0344)	-0.1204 (0.2704)	0.0827 (0.4630)
12	0.2171 (0.1692)	-0.0021 (0.0332)	-0.0905 (0.2635)	0.1245 (0.4659)
11	0.2285 (0.1807)	0.0013 (0.0325)	-0.0344 (0.2622)	0.1954 (0.4753)
10	0.2385 (0.1908)	0.0010 (0.0309)	-0.0151 (0.2520)	0.2244 (0.4738)
9	0.2478 (0.2003)	0.0013 (0.0292)	0.0375 (0.2421)	0.2865 (0.4715)
8	0.2340 (0.1881)	0.0149 (0.0262)	0.0515 (0.2057)	0.3004 (0.4200)
7	0.2327 (0.1876)	0.0182 (0.0233)	0.1009 (0.1825)	0.3518 (0.3934)
6	0.2687 (0.2225)	-0.0074 (0.0213)	0.1390 (0.1876)	0.4004 (0.4314)
5	0.2728 (0.2274)	-0.0146 (0.0171)	0.1746 (0.1491)	0.4327 (0.3936)

Table 11: Total Δ XVA* aggregated over survivors corresponding to the different surviving CM*, i.e. for * other than 0, assuming an instant default of CM0 at time 0. In parenthesis, the contributions to Δ XVA* of CM* itself.

Δ CMVA	Δ CCVA	Δ KVA
0.0593	0.0189	0.1964

Table 12: Standard deviation across surviving members * of the Δ XVA* for the example with 1 CCP and 20 members, assuming an instant default of CM0 at time 0.

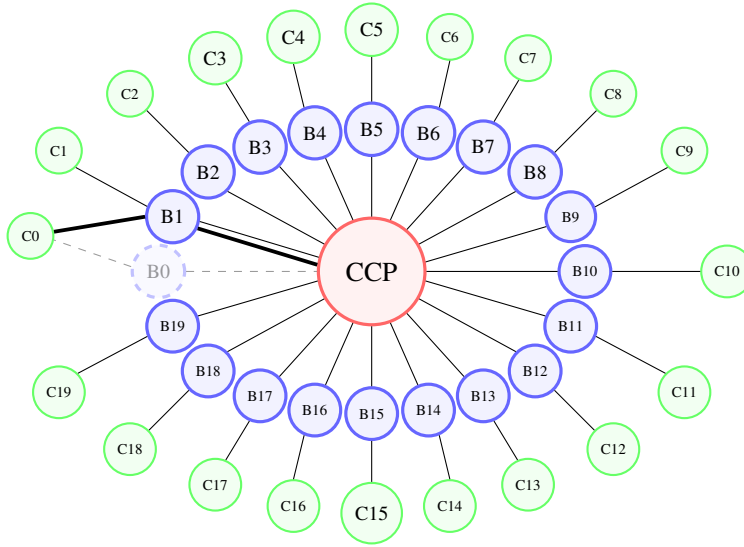


Figure 8: The 1-CCP, former 20-member financial network with 19 members post CM0 default. Defaulted CM0, labeled “B0” in the presented network, is represented as pale dashed node with pale dashed links to reflect former exposures to its client and toward the CCP. The optimal porting of CM0 portfolio with CM1, labeled “B1”, is outlined with bold links to reflect the new exposures for CM1.

§8 Conclusion

We have proposed a fully integrated risk management framework that can be used for stress test analysis, including reverse stress test in line with regulatory requirements, or for optimizing the porting of defaulted portfolios, in a setup encompassing all the trades (bilateral as centrally cleared and their hedges) of a reference bank. The framework includes dependence between financial participants portfolios, joint defaults, and a configurable wrong-way risk feature. This is done in a numerically tractable static setup (although already quite demanding on large financial networks)⁴⁷. A possible improvement would be to incorporate regulatory constraints such as minimum regulatory capital requirements and liquidity leverage ratios. More fundamentally, in this paper, we tackle the derivatives risk problem from a pure counterparty credit risk viewpoint: if members, clients and counterparties are all default free, then in view of Proposition 5 all considered XVAs are zero, so that our setup becomes trivial. Another dimension to the problem is liquidity [Amini et al. \(2020\)](#); [Faruqui et al. \(2018\)](#). Depending on the considered applications⁴⁸, credit or liquidity is the main force at hand. A challenging research project would be to integrate both in a common setup.

⁴⁷The dynamic extension considered in ([Crépey, 2022](#), Section 6) is only workable at a much higher computational burden, using the simulation and learning techniques of [Abbas-Turki et al. \(2023\)](#).

⁴⁸see e.g. the beginning of Section §7.

§9 Appendix

A Proof of Lemma 1 on the borrowing needs

On the bilateral trades of the bank and their hedges, the Treasury of the bank receives $\sum_b \text{VM}_b$ of variation margin from its counterparties and has to post an aggregated amount $\sum_b \text{MtM}_b$ of variation margin. Assumption 4 then leads to (6).

B Proof of Lemma 2 on counterparty loss and funding expense expressions

In view of Lemma 1 and Assumption 4, the (risky) funding expenses of the bank correspond to the formula (9) for \mathcal{F} . Regarding \mathcal{C} , On the CCP survival event $\{\mathcal{J} = 1\}$, the CCP receives, by Assumption 3,

$$\begin{aligned} & \sum_i \left(J_i(\mathcal{P}_i + \bar{\mathcal{P}}_i) + (1 - J_i) \left(\mathcal{P}_i \wedge (\text{MtM}_i + \text{IM}_i) + \bar{\mathcal{P}}_i \wedge (\overline{\text{MtM}}_i + \overline{\text{IM}}_i) \right. \right. \\ & \left. \left. + ((\mathcal{P}_i - (\text{MtM}_i + \text{IM}_i))^+ + (\bar{\mathcal{P}}_i - (\overline{\text{MtM}}_i + \overline{\text{IM}}_i))^+) \wedge \text{DF}_i \right) \right). \end{aligned} \quad (30)$$

By the CCP clearing condition in Assumption 2,

$$0 = \sum_i (\mathcal{P}_i + \bar{\mathcal{P}}_i) = \sum_i (J_i(\mathcal{P}_i + \bar{\mathcal{P}}_i) + (1 - J_i)(\mathcal{P}_i + \bar{\mathcal{P}}_i)).$$

Hence (30) is equal to

$$- \sum_i (1 - J_i) \left((\mathcal{P}_i - \text{MtM}_i - \text{IM}_i)^+ + (\bar{\mathcal{P}}_i - \overline{\text{MtM}}_i - \overline{\text{IM}}_i)^+ - \text{DF}_i \right) = -\mathcal{L}, \quad (31)$$

by definition (8) of \mathcal{L} .

On the bank survival event $\{J = 1\} (\subseteq \{\mathcal{J} = 1\})$, by the respective Assumptions 3 and 2, the bank receives from its clients and counterparties

$$\sum_c \left(J_c \mathcal{P}_c + (1 - J_c) (\mathcal{P}_c \wedge (\text{MtM}_c + \text{IM}_c)) \right) + \sum_b \left(J_b \mathcal{P}_b + (1 - J_b) (\mathcal{P}_b \wedge (\text{VM}_b + \text{IM}_b)) \right), \quad (32)$$

respectively pays to the CCP

$$\sum_c \mathcal{P}_c + \sum_b \mathcal{P}_b = \sum_c (J_c \mathcal{P}_c + (1 - J_c) \mathcal{P}_c) + \sum_b (J_b \mathcal{P}_b + (1 - J_b) \mathcal{P}_b). \quad (33)$$

Subtracting (32) from (33), we obtain

$$\sum_c (1 - J_c) (\mathcal{P}_c - \text{MtM}_c - \text{IM}_c)^+ + \sum_b (1 - J_b) (\mathcal{P}_b - \text{VM}_b - \text{IM}_b)^+.$$

On top of this comes the participation $\mu\mathcal{L}$ of the bank to the CCP default losses, which yields the formula (7) for \mathcal{C} .

C Proof of Theorem 4 on XVA equivalent formulations

By the result recalled after (12), EC is nonnegative as an expected shortfall under \mathbb{R} of the random variable $J(\mathcal{C} + \mathcal{F} - CA)$, which is centered under \mathbb{R}^* and therefore under \mathbb{R} , by (10). The first four formulas in Table 2 directly follow from the definitions of Table 1 and Lemma 3, which also implies that $KVA = \mathbb{E}[h(EC - KVA)^+] = h(EC - KVA)^+$. As h is nonnegative, this KVA semilinear equation is equivalent to

$$(KVA > EC \text{ and } KVA = 0) \text{ or } (KVA \leq EC \text{ and } KVA = \frac{h}{1+h}EC),$$

where $(KVA > EC \text{ and } KVA = 0)$ contradicts the nonnegativity of EC, whereas, for $h \in [0, 1]$ as assumed and $EC \geq 0$, $KVA = \frac{h}{1+h}EC$ implies $KVA \leq EC$, i.e. $\max(EC, KVA) = EC$. This and Lemma 3 yield

$$FVA = \mathbb{E}\left[\gamma\left(\sum_b (\text{MtM}_b - \text{VM}_b) - CA - EC\right)^+\right] = \gamma\left(\sum_b (\text{MtM}_b - \text{VM}_b) - CA - EC\right)^+.$$

As $CA = CCVA + CMVA + BCVA + BMVA + FVA$, this is an FVA semilinear equation, which, as γ is nonnegative, is equivalent to the FVA formula

$$FVA = \frac{\gamma}{1+\gamma}\left(\sum_b (\text{MtM}_b - \text{VM}_b) - (CCVA + CMVA + BCVA + BMVA) - EC\right)^+.$$

Last, we have $EC = \mathbb{E}\mathbb{S}(J(\mathcal{C} + \mathcal{F} - CA))$, where the identity $\mathcal{C} + \mathcal{F} - CA = \mathcal{C} - CVA$ and the formula for $J(\mathcal{C} - CVA)$ in Table 2 are obtained by substituting the already derived XVA formulas in (7) and (9).

D Proof of Proposition 5 on XVA equivalent formulations with several CCPs

In the case of several CCP services, the second line in (6) must be turned into $\sum_{ccp} (\text{IM}^{ccp} + \overline{\text{IM}}^{ccp} + \text{DF}^{ccp}) + \sum_b \overline{\text{IM}}_b$; the terms in the first lines of (7) and (9) must be summed over the various CCP services in which the bank is involved as a clearing member. The rest of the analysis proceeds as before.

E Proof of Lemma 6 on economic capital calculation

If $\mathbb{R}(\ell = \text{VaR}(\ell)) = 0$, then, by the ES formula recalled after (12), we have

$$\begin{aligned} EC &= \mathbb{E}\mathbb{S}(\ell) = \mathbb{E}[J(\mathcal{C} - CVA) | J(\mathcal{C} - CVA) \geq \text{VaR}(\ell)] \\ &= \frac{\mathbb{E}\left[J(\mathcal{C} - CVA) \mathbb{1}_{\{J(\mathcal{C} - CVA) \geq \text{VaR}(\ell)\}}\right]}{\mathbb{R}(J(\mathcal{C} - CVA) \geq \text{VaR}(\ell))} \\ &= \frac{\mathbb{E}^*\left[(\mathcal{C} - CVA) \mathbb{1}_{\{(\mathcal{C} - CVA) \geq \text{VaR}(\ell)\}} \mathbb{1}_{\{J=1\}}\right]}{\mathbb{R}^*((\mathcal{C} - CVA) \geq \text{VaR}(\ell), J=1)}, \end{aligned} \tag{34}$$

using (10) on both numerator and denominator expressed in expectation form, which yields (19).

Chapter II

Provisions and economic capital for credit losses

Note. The results of this chapter are based on the paper¹ [Bastide and Crépey \(2024\)](#).

Abstract. Based on supermodularity ordering properties, we show that convex risk measures of credit losses are nondecreasing w.r.t. credit-credit and, in a wrong-way risk setup, credit-market, covariances of elliptically distributed latent factors. These results support the use of such setups for computing credit provisions and economic capital or for conducting stress test exercises and risk management analysis.

§1 Introduction

Elliptical distributions are largely used in finance modelling, be it for credit latent variables or portfolio positions modelling ([McNeil et al., 2015](#)). They allow to generate, by means of Monte Carlo routines, a span of possible losses. Risk indicators can then be formed so as to inform top management or supervisors about the monitoring of possible future losses borne by a financial institution. In particular, regulatory bodies instruct to rely on value-at-risk and expected shortfall measures, where the last type, which falls under the remit of coherent risk measure², is often preferred to quantiles usage. One example can be found in [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#), where the economic capital of a clearing member bank of a central counterparty (CCP) is based on an expected shortfall risk measure of the bank loss over one year. Such measure of loss depicts numerically a nondecreasing property w.r.t. credit-credit and credit-market dependence parameters, capturing an increase in loss given default amounts as more defaults materialize. [Cousin and Laurent \(2008\)](#) outlined the possible application of supermodular order for comparing CDO tranche premiums w.r.t. a credit correlation parameter of their default latent variables modelled as Brownian motions. [Meyer and Strulovici \(2012\)](#) generalize the use of such notion, citing application to credit losses with bounded support. The supermodular order property finds its root in [Marshall and Olkin \(1979, Definition C.2, page 146\)](#) under the name of lattice-superadditive property. It has attracted subsequent attention with the works of [Müller \(1997\)](#) and [Bäuerle and Müller \(1998\)](#), applied to stop-loss ordering of aggregated losses. [Bäuerle and Müller \(2006\)](#) emphasize the role of several stochastic orders in relation with convex risk measures. In particular, multidimensional

¹*Acknowledgements:* We thank Mekonnen Tadese, postdoctoral researcher at Université Paris Cité / LPSM, for useful references and discussions.

²see e.g. [Acerbi and Tasche \(2002\)](#).

elliptically distributed random variables have the supermodular order property w.r.t. their covariance matrix coefficients (Block and Sampson (1988, Corollary 2.3) recalled in Section B). This result will play a key role in this paper.

In Bastide, Crépey, Drapeau, and Tadese (2023a), the loss takes a more complex form than what is usually found in the credit risk literature such as Cousin and Laurent (2008). This is due to a loss allocation coefficient attributed by the CCP to the surviving members. In this paper, we prove the nondecreasing property of convex risk measures w.r.t. covariance coefficients of portfolio credit losses. Our main motivation is to provide evidence of the soundness of the related approaches for computing credit provisions such as current expected credit loss (CECL, akin to the CVA in the central clearing one-period XVA setup of Bastide, Crépey, Drapeau, and Tadese (2023a)), and economic capital (EC). This is important in justifying model assumptions and design, part of the model development cycle advocated by regulators and supervisors (European Central Bank, 2019). Table 1 details the two main targeted metrics in this work.

Name	Expression	Reference
current expected credit loss (CECL)	$\mathbb{E}^0 \left(\sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \right)$	Definition 5
economic capital (EC)	$\mathbb{E}\mathbb{S}_\alpha^0 \left(\sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \right)$	Definitions 6 and 8

Table 1: Metrics of interest (\cdot^0 relates to the survival probability measure of the reference bank).

The paper is organized as follows. Section §2 introduces the strategy of proof by supermodularity arguments that will be used to establish our main result, Theorem 12 (leading to the CECL and EC monotonicity results of Propositions 13 and 14), in an appropriate elliptical framework with wrong-way risk. Section §3 completes the results from Cousin and Laurent (2008) regarding equity and senior CDO tranches. The properties of Section §2 are used in Section §4 for implying the monotonicity of credit provisions and economic capital metrics w.r.t. covariance coefficients in elliptical models. Section §5 concludes. Supermodular functions, elliptical distributions and risk measures are reviewed in Sections A, B and C.

Hereafter we consider a non-atomic probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, with corresponding expectation, variance and covariance operators denoted by \mathbb{E} , Var and Cov . All the considered random variables are real-valued and taken in a closed linear subspace and sublattice \mathfrak{X} of $L^1(\mathbb{Q})$ that includes the constants.

§2 Supermodular Ordering Properties

A Strategy of Proof

The following classical supermodularity results will be instrumental in establishing our main result Theorem 12.

Müller (1997, Definition 2.6) A random vector $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$ is said to be smaller than the random vector $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ in the supermodular ordering, written $\mathbf{X} \leq_{sm} \mathbf{Y}$, if $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$

holds for all the supermodular functions³ $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the expectations exist. \square

Müller and Stoyan (2002, Definition 3.9.4, page 113) A random vector $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$ is said to be smaller than the random vector $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ in the increasing supermodular ordering, written $\mathbf{X} \leq_{ism} \mathbf{Y}$, if $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$ holds for all the nondecreasing supermodular functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the expectations exist. \square

An equivalent characterization of supermodular vectors is given by

Müller and Stoyan (2002, Theorems 3.9.11 (i) and (ii), page 118) The following statements are equivalent:

- (i) $\mathbf{X} \leq_{sm} \mathbf{Y}$,
- (ii) \mathbf{X} and \mathbf{Y} have the same marginals and $\mathbf{X} \leq_{ism} \mathbf{Y}$. \square

Müller (1997, Theorems 3.2 (c)) If \mathbf{X} , \mathbf{Y} , \mathbf{Z} are random vectors such that any random vectors distributed as \mathbf{X} and \mathbf{Y} conditionally on $\mathbf{Z} = \mathbf{z}$, denoted by $[\mathbf{X}|\mathbf{Z} = \mathbf{z}]$ and $[\mathbf{Y}|\mathbf{Z} = \mathbf{z}]$, verify $[\mathbf{X}|\mathbf{Z} = \mathbf{z}] \leq_{sm} [\mathbf{Y}|\mathbf{Z} = \mathbf{z}]$ for all possible values of \mathbf{z} , then $\mathbf{X} \leq_{sm} \mathbf{Y}$. \square

Müller (1997, Definition 2.1) For \mathcal{X} and \mathcal{Y} in \mathfrak{X} , \mathcal{X} precedes \mathcal{Y} in stop-loss order, written $\mathcal{X} \leq_{sl} \mathcal{Y}$, if $\mathbb{E}(\mathcal{X} - A)^+ \leq \mathbb{E}(\mathcal{Y} - A)^+$ holds for all real constants $A \geq 0$. \square

Equivalently, for the same random variables \mathcal{X} and \mathcal{Y} as in *Müller (1997, Definition 2.1)*:

Bäuerle and Müller (1998, Theorem 2.2 b)) $\mathcal{X} \leq_{sl} \mathcal{Y}$ if and only if $\mathbb{E}(f(\mathcal{X})) \leq \mathbb{E}(f(\mathcal{Y}))$ holds for all the nondecreasing convex functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the expectations exist (e.g. $f = id$). \square

Müller (1997, Theorem 3.1) Let $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$ and $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ be random vectors with $\mathbf{X} \leq_{sm} \mathbf{Y}$ and let $S = \sum_{i=1}^m \mathcal{X}_i$, $S' = \sum_{i=1}^m \mathcal{Y}_i$. Then $S \leq_{sl} S'$. \square

Lemma 1. *If $\mathbf{X} \leq_{sm} \mathbf{Y}$, then, for any nondecreasing supermodular function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, $f(\mathbf{X}) \leq_{sl} f(\mathbf{Y})$.*

Proof. For any $A \in \mathbb{R}$, the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$, $x \mapsto (x - A)^+$ is nondecreasing and convex. By *Müller and Stoyan (2002, Theorem 3.9.3 f), page 113*, $\varphi \circ f$ is nondecreasing supermodular. Hence $\mathbb{E}((f(\mathbf{X}) - A)^+) \leq \mathbb{E}((f(\mathbf{Y}) - A)^+)$, which yields the result by *Müller (1997, Definition 2.1)*. \square

Bäuerle and Müller (2006, Theorem 4.4) Assuming ρ a risk measure as per Definition 7, then $\mathcal{X} \leq_{sl} \mathcal{Y}$ implies $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$. \square

A succinct proof of this result can be found in *Bäuerle and Müller (2006)* prior to its statement. We give a more detailed proof for completeness at the end of Section C. Note that the original statement postulates, instead of the law-invariance property, that the risk measure ρ has the Fatou property, that is, if $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots$ are integrable random variables with $\mathcal{X}_k \xrightarrow{L^1} \mathcal{X}$, then $\rho(\mathcal{X}) \leq \liminf_{k \rightarrow \infty} \rho(\mathcal{X}_k)$. We recall from *Kallenberg (2021, page 832)* that a closed linear subspace of a Banach space is a Banach space and from *Meyer-Nieberg (1991, Definition 1.2.1 i), page 12* that a sublattice of a lattice with the same meet and join operations \wedge and \vee is again a lattice. Therefore, if \mathfrak{X} is a Banach lattice (i.e. an

³see Appendix A.

order lattice that is a complete normed vector space, e.g. L^p space with $p \geq 1$) and $\rho : \mathfrak{X} \rightarrow \mathbb{R}$ is proper, monotonous and convex, then ρ is continuous on the interior of its domain (Ruszczyński and Shapiro, 2006, Proposition 1), thus has the Fatou property on the interior of its domain. Hence the Fatou property requirement is automatically satisfied by ρ as long as it is defined on a Banach lattice. This is the case for both expectation and expected shortfall defined on any sublattice and linear subspace of L^1 .

In Section §4, we will consider financial credit losses over a period of time $[0, T]$ of the form $\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)$ for some given number n of credit obligors. The default times τ_i of the credit obligors indexed by $i \in 1 \dots n$ will be modelled by the latent variable \mathcal{X}_i breaching a certain threshold. $g_i(\mathcal{Y}_i)$ represents nonnegative losses given default of obligor i . The functions f_i are nonnegative nondecreasing w.r.t. each of their arguments and have the supermodular property (when $n > 1$) and the functions g_i are nonnegative and nondecreasing. We will establish in Section §4 that $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i)$ is a nondecreasing supermodular function on \mathbb{R}^{2n} . Hence, by Lemma 1, if $(\mathcal{X}_1, \dots, \mathcal{X}_n, \mathcal{Y}_1, \dots, \mathcal{Y}_n) \leq_{sm} (\mathcal{X}'_1, \dots, \mathcal{X}'_n, \mathcal{Y}'_1, \dots, \mathcal{Y}'_n)$, then

$$\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i) \leq_{sl} \sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n)g_i(\mathcal{Y}'_i). \quad (1)$$

Having assumed the probability space non-atomic⁴, Bäuerle and Müller (2006, Theorem 4.4) allows concluding that $\rho(\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)) \leq \rho(\sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n)g_i(\mathcal{Y}'_i))$. This is the plan for proving our main result Theorem 12.

The following two Lemmas 2 and 3 will be the building blocks for establishing the above-needed supermodularity property.

Lemma 2. *The supermodularity property is satisfied by any function $\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto h(x_1, \dots, x_n) \in \mathbb{R}$ that can be written as $h(x_1, \dots, x_n) = \sum_{i=1}^n h_i(x_i)$ for some functions h_1, \dots, h_n of single arguments. In particular, for any constants B_1, \dots, B_n , the function $(x_1, \dots, x_n) \mapsto -\sum_{i=1}^n \mathbb{1}_{\{x_i \leq B_i\}}$ is supermodular. This function is also nondecreasing w.r.t. each of its arguments x_i .*

Proof. By Yildiz (2010, Corollary 1), recalled in Section A, we can focus on increasing differences. Let $h(x_i, x_j | \mathbf{x}_{-i,j})$ denote the function h applied to x_i and x_j but keeping all other arguments $\mathbf{x}_{-i,j} := (x_k)_{k \neq i,j}$ fixed. Fixing $\delta, \varepsilon > 0$, the difference $h(x_i + \delta, x_j + \varepsilon | \mathbf{x}_{-i,j}) - h(x_i, x_j + \varepsilon | \mathbf{x}_{-i,j}) - h(x_i + \delta, x_j | \mathbf{x}_{-i,j}) + h(x_i, x_j | \mathbf{x}_{-i,j}) = h_i(x_i + \delta) + h_j(x_j + \varepsilon) - h_i(x_i) - h_j(x_j + \varepsilon) - h_i(x_i + \delta) - h_j(x_j) + h_i(x_i) + h_j(y_j)$ simplifies to 0, showing the supermodularity of h . \square

Lemma 3. *If $g : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ are both nondecreasing, then $(x, y) \mapsto g(x)h(y)$ is supermodular.*

Proof. If $x' \geq x$ and $y' \geq y$, then

$$\begin{aligned} & g(x')h(y') - g(x')h(y) - g(x)h(y') + g(x)h(y) \\ &= g(x')(h(y') - h(y)) - g(x)(h(y') - h(y)) \\ &\geq g(x)(h(y') - h(y)) - g(x)(h(y') - h(y)) = 0. \end{aligned} \quad (2)$$

Hence $(x, y) \mapsto g(x)h(y)$ has increasing differences w.r.t. any pair $(x, y) \in \mathbb{R}^2$, i.e. is supermodular, by Yildiz (2010, Corollary 1). \square

⁴see Bäuerle and Müller (2006, Example 3.1) for a counter-example to the monotonicity property for probability space with atom(s).

B Elliptical Setup

We consider a one-period financial market model on (Ω, \mathcal{A}) , assumed arbitrage-free, with related risk-neutral probability measure \mathbb{Q}^* ⁵. We index all the formerly introduced notation by “.” whenever applied in reference to $\mathbb{Q} = \mathbb{Q}^*$. We consider random vectors $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m)$ and $\mathbf{X}' = (\mathcal{X}'_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m)$ following elliptical distributions (see Section B) under \mathbb{Q}^* as per

$$\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z} \text{ and } \mathbf{X}' = \boldsymbol{\mu} + \mathbf{A}'\mathbf{Z}, \quad (3)$$

for constant matrices $\mathbf{A}, \mathbf{A}' \in \mathbb{R}^{m \times k}$ of full rank. $\mathbf{Z} = (Z_0, Z_1, \dots, Z_k)$ follows, under \mathbb{Q}^* , a spherical distribution $S_{k+1}(\psi)$ (see Section B), with characteristic generator ψ . As the components 0 for both vectors \mathbf{X} and \mathbf{X}' must be the same, we also require the first row of \mathbf{A} and \mathbf{A}' to be the same. The rationale for keeping the same reference latent variable \mathcal{X}_0 in \mathbf{X} and \mathbf{X}' in (3) is that the point of view will be the one of a reference bank indexed by 0 (cf. Lemma 4 and Proposition 5).

We denote by $\Gamma_{\mathbf{X}} = \mathbf{A}\mathbf{A}^\top$ and $\Gamma_{\mathbf{X}'} = \mathbf{A}'(\mathbf{A}')^\top$ the \mathbb{Q}^* covariance matrices of \mathbf{X} and \mathbf{X}' , assumed to be positive semi-definite. We write $\Gamma_{\mathbf{X}}^{ij} = \text{Cov}^*(\mathcal{X}_i, \mathcal{X}_j)$ for all $i, j \in 0 \dots m$, and likewise for $\Gamma_{\mathbf{X}'}$.

Assumption 1. $\Gamma_{\mathbf{X}}^{jj} = \Gamma_{\mathbf{X}'}^{jj}$, and $\Gamma_{\mathbf{X}}^{0j} = \Gamma_{\mathbf{X}'}^{0j}$, $j \in 1 \dots m$, and $\Gamma_{\mathbf{X}}^{ij} \leq \Gamma_{\mathbf{X}'}^{ij}$, $i \neq j \in 1 \dots m$. \square

In particular, $\mathbf{X}_{1:m} \leq_{sm^*} \mathbf{X}'_{1:m}$, by [Block and Sampson \(1988, Corollary 2.3\)](#) recalled in Section B.

Let $\boldsymbol{\mu}_{\mathbf{X}_{1:m}} = (\mu_{\mathcal{X}_1}, \dots, \mu_{\mathcal{X}_m})$ and $\Gamma_{\mathbf{X}_{1:m}} = \left(\Gamma_{\mathbf{X}_{1:m}}^{ij} \right)_{1 \leq i, j \leq m}$ be the mean vector and the covariance matrix of $(\mathcal{X}_1, \dots, \mathcal{X}_m)$ under \mathbb{Q}^* . We use similar notations regarding $(\mathcal{X}'_1, \dots, \mathcal{X}'_m)$. Our next result, Lemma 4, makes use of the two following results. Let $U_{\mathbb{S}^{n-1}}$ denote the uniform distribution on the unit sphere $\mathbb{S}^{n-1} := \{\mathbf{s} \in \mathbb{R}^n : \mathbf{s}^\top \mathbf{s} = 1\}$ in \mathbb{R}^n . A radial r.v. is a one-dimensional r.v. with values in $[0, +\infty)$.

[McNeil et al. \(2015, Theorem 6.21, page 197\)](#) \mathbf{Z} has a spherical distribution in \mathbb{R}^n if and only if it has a stochastic representation

$$\mathbf{Z} \stackrel{d}{=} \mathcal{R}\mathbf{S}, \quad (4)$$

where $\mathbf{S} \sim U_{\mathbb{S}^{n-1}}$ and $\mathcal{R} \geq 0$ is a radial r.v. independent of \mathbf{S} . \square

[Fang et al. \(1990, Theorem 2.18, page 45\)](#) Let $\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{S} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ (see Section B) with $\Sigma = \mathbf{A}\mathbf{A}^\top$ positive definite. Let

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}^{(1)} \\ \mathbf{Y}^{(2)} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma^{(1,1)} & \Sigma^{(1,2)} \\ \Sigma^{(2,1)} & \Sigma^{(2,2)} \end{pmatrix}, \quad (5)$$

where $\mathbf{Y}^{(1)}$ and $\boldsymbol{\mu}^{(1)}$ are $m \times 1$ vectors and $\Sigma^{(1,1)}$ is an $m \times m$ matrix, for some $0 < m < n$. Then

$$\begin{aligned} \left(\mathbf{Y}^{(1)} \middle| \mathbf{Y}^{(2)} = \mathbf{y}^{(2)} \right) &\stackrel{d}{=} \boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)} + \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \mathbf{S}^{(m)} \\ &\sim E_m \left(\boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)}, \Sigma_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)}, \psi_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \right), \end{aligned} \quad (6)$$

where

⁵see for instance ([Föllmer and Schied, 2016, Part I](#)).

$$\left\{ \begin{array}{l}
\boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)} = \boldsymbol{\mu}^{(1)} + \Sigma^{(1,2)} (\Sigma^{(2,2)})^{-1} (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)}), \\
\Sigma_{|\mathbf{Y}^{(2)}=\mathbf{x}^{(2)}}^{(1,1)} = \Sigma^{(1,1)} - \Sigma^{(1,2)} (\Sigma^{(2,2)})^{-1} \Sigma^{(2,1)} = \mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \left(\mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \right)^\top, \\
\mathbf{S}^{(m)} \sim U_{\mathbb{S}^{m-1}}, \\
\mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \stackrel{d}{=} \left((\mathcal{R}^2 - q(\mathbf{y}^{(2)}))^{1/2} \middle| \mathbf{y}^{(2)} = \mathbf{y}^{(2)} \right) \text{ and } \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ is independent} \\
\text{of } \mathbf{S}^{(m)}, \\
q(\mathbf{y}^{(2)}) = (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)})^\top (\Sigma^{(2,2)})^{-1} (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)}), \\
\psi_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ is of the form (53) for } n = m, F \text{ given as the c.d.f. of } \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ and} \\
\mathbf{S} \text{ given as } \mathbf{S}^{(m)}. \square
\end{array} \right. \quad (7)$$

Lemma 4. Under the elliptical form (3) satisfying Assumption 1 for \mathbf{X} and \mathbf{X}' , we have, under \mathbb{Q}^* ,

$$\begin{aligned}
[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0] &\sim E_m \left(\boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0}, \boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0}, \psi_{|\mathcal{X}_0} \right), \\
[\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0] &\sim E_m \left(\boldsymbol{\mu}_{\mathbf{X}'_{1:m} | \mathcal{X}_0}, \boldsymbol{\Gamma}_{\mathbf{X}'_{1:m} | \mathcal{X}_0}, \psi'_{|\mathcal{X}_0} \right),
\end{aligned} \quad (8)$$

with

$$\psi_{|\mathcal{X}_0} = \psi'_{|\mathcal{X}_0}, \quad \boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0} = \boldsymbol{\mu}'_{\mathbf{X}'_{1:m} | \mathcal{X}_0} \quad (9)$$

and, for any $i, j \in 1 \dots m$,

$$\boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0}^{ij} \leq \boldsymbol{\Gamma}_{\mathbf{X}'_{1:m} | \mathcal{X}_0}^{ij}. \quad (10)$$

Proof. Applying (7) to $\mathbf{Y}^{(1)} = \mathbf{X}_{1:m}$ and $\mathbf{Y}^{(2)} = \mathcal{X}_0$ yields (8) with

$$\begin{aligned}
\boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0} &= \boldsymbol{\mu}_{\mathbf{X}_{1:m}} + \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \left(\boldsymbol{\Gamma}_{\mathbf{X}}^{01} \dots, \boldsymbol{\Gamma}_{\mathbf{X}}^{0m} \right)^\top (\mathcal{X}_0 - \mu_{\mathcal{X}_0}), \\
\boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0} &= \boldsymbol{\Gamma}_{\mathbf{X}_{1:m}} - \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \sum_{j=1}^m \left(\boldsymbol{\Gamma}_{\mathbf{X}}^{0j} \right)^2
\end{aligned} \quad (11)$$

and, using Assumption 1 and (11),

$$\begin{aligned}
\boldsymbol{\mu}_{\mathbf{X}'_{1:m} | \mathcal{X}_0} &= \boldsymbol{\mu}_{\mathbf{X}_{1:m}} + \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \left(\boldsymbol{\Gamma}_{\mathbf{X}}^{01} \dots, \boldsymbol{\Gamma}_{\mathbf{X}}^{0m} \right)^\top (\mathcal{X}_0 - \mu_{\mathcal{X}_0}) = \boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0} \\
\boldsymbol{\Gamma}_{\mathbf{X}'_{1:m} | \mathcal{X}_0} &= \boldsymbol{\Gamma}_{\mathbf{X}'_{1:m}} - \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \sum_{j=1}^m \left(\boldsymbol{\Gamma}_{\mathbf{X}}^{0j} \right)^2 \geq \boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0}.
\end{aligned} \quad (12)$$

where \geq is meant componentwise.

It remains to show that $\psi_{|\mathcal{X}_0} = \psi'_{|\mathcal{X}_0}$. For all $x_0 \in \mathbb{R}$, by Fang et al. (1990, Theorem 2.18, page 45), which includes (7), the radius $\mathcal{R}_{|\mathcal{X}_0=x_0}$ of $[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0 = x_0]$ is distributed like $\left(\mathcal{R}^2 - \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} (x_0 - \mu_{\mathcal{X}_0})^2 \right)^{1/2}$, where $\mathcal{R} := \|\mathbf{Z}\|$, and so is the radius $\mathcal{R}'_{|\mathcal{X}_0=x_0}$ of $[\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0 =$

$x_0]$ (as both \mathbf{X} and \mathbf{X}' are defined based on the spherically distributed vector \mathbf{Z}). Thus $\mathcal{R}_{|\mathcal{X}_0} \stackrel{d}{=} \mathcal{R}'_{|\mathcal{X}_0}$, with common c.d.f. denoted by $F_{\mathcal{R}_{|\mathcal{X}_0}}$. The corresponding conditional characteristic generator common to $[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0]$ and $[\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0]$ is given by $\psi_{|\mathcal{X}_0}(x) = \int_0^\infty \Omega_m(xr^2) F_{\mathcal{R}_{|\mathcal{X}_0}}(dr)$, where $\Omega_m(u^\top u)$ is the characteristic function of a r.v. $\sim U_{\mathbb{S}^{m-1}}$. \square

Proposition 5. *Under the assumptions of Lemma 4, we have:*

$$[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0] \leq_{sm^*} [\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0]. \quad (13)$$

Proof. By Lemma 4, conditionally on \mathcal{X}_0 , $\mathbf{X}_{1:m}$ and $\mathbf{X}'_{1:m}$ have the same elliptical distribution under \mathbb{Q}^* , except for their covariance matrix coefficients that verify (10). Block and Sampson (1988, Corollary 2.3) recalled in Section B (here applied under \mathbb{Q}^*) then yields the result. \square

C Preserving Supermodularity and Stop-Loss Order Properties Under a Measure Change

The metrics of interest in this work are considered from a reference market participant viewpoint, namely a bank indexed by 0. In this context, it is sometimes useful to introduce a measure \mathbb{Q}^0 defined in terms of a measurable function h of a latent variable \mathcal{X}_0 of the default of the bank, such that

$$h(\mathcal{X}_0) = d\mathbb{Q}^0/d\mathbb{Q}^* \geq 0 \text{ and } \mathbb{E}^*[h(\mathcal{X}_0)] = 1. \quad (14)$$

We index all the formerly introduced notation by “ \cdot^0 ” whenever applied in reference to $\mathbb{Q} = \mathbb{Q}^0$.

Example 1. In the setup of Bastide, Crépey, Drapeau, and Tadese (2023a), financial risk factors are specified under \mathbb{Q}^* , but explicit XVA formulas arise in terms of the related bank survival probability measure \mathbb{Q}^0 .

Assumption 2. $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m)$ and $\mathbf{X}' = (\mathcal{X}_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m)$, with components in $\mathfrak{X}^* = L^1(\mathbb{Q}^*)$ (with the same \mathcal{X}_0 in \mathbf{X} and \mathbf{X}'), satisfy

$$[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0] \leq_{sm^*} [\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0], \quad (15)$$

i.e.

$$\mathbb{E}^*[f(\mathcal{X}_1, \dots, \mathcal{X}_m) | \mathcal{X}_0] \leq \mathbb{E}^*[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m) | \mathcal{X}_0] \quad (16)$$

holds for any supermodular function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the conditional expectations exist.

Lemma 6. *If \mathbf{X} and \mathbf{X}' satisfy Assumption 2, then $(\mathcal{X}_1, \dots, \mathcal{X}_m) \leq_{sm^0} (\mathcal{X}'_1, \dots, \mathcal{X}'_m)$.*

Proof. As $h(\mathcal{X}_0) \geq 0$, for any supermodular function f on \mathbb{R}^m such that both $\mathbb{E}^0[f(\mathcal{X}_1, \dots, \mathcal{X}_m)]$ and $\mathbb{E}^0[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m)]$ exist, (14) and (16) yield

$$\begin{aligned} \mathbb{E}^0[f(\mathcal{X}_1, \dots, \mathcal{X}_m)] &= \mathbb{E}^*[h(\mathcal{X}_0)f(\mathcal{X}_1, \dots, \mathcal{X}_m)] = \mathbb{E}^*[h(\mathcal{X}_0)\mathbb{E}^*[f(\mathcal{X}_1, \dots, \mathcal{X}_m) | \mathcal{X}_0]] \\ &\leq \mathbb{E}^*[h(\mathcal{X}_0)\mathbb{E}^*[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m) | \mathcal{X}_0]] = \mathbb{E}^0[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m)]. \end{aligned} \quad (17)$$

Hence $(\mathcal{X}_1, \dots, \mathcal{X}_m) \leq_{sm^0} (\mathcal{X}'_1, \dots, \mathcal{X}'_m)$. \square

In terms of applications, we start by precisizing the monotonicity result of Cousin and Laurent (2008) for equity and senior CDO tranches default leg w.r.t. the credit correlation. We then explore a more complex counterparty credit risk example pertaining to the risk management of clearing activities.

§3 Teaser: Credit Derivatives

We analyse the monotonicity of default leg and coupon leg of synthetic equity and senior CDO tranches prices w.r.t. credit correlation. Such prices are obtained by taking the expected value under the pricing measure \mathbb{Q}^* of the loss function underlying the CDO tranche contract. The characteristics of the payoff are as follows. There are n obligors, indexed by i . All underlying CDS are assumed to mature at some common time T . For any obligor i , the default time is denoted by τ_i , the recovery rate is $R_i \in [0, 1]$, the underlying notional is $N_i \geq 0$ and the loss given default is $L_i = (1 - R_i)N_i$. The maximum loss is $L_{\max} = \sum_{i=1}^n L_i$. To simplify calculations, we assume that the payments due to the obligors defaults are only made at maturity T and the discounting rates are set to zero (nonzero discounting rates can be included as long as they are independent from the credit risk factors).

Definition 3. The cumulative credit loss at time $t \leq T$ is

$$L(t) = \sum_{i=1}^n L_i \mathbb{1}_{\{\tau_i \leq t\}}. \quad (18)$$

The default leg of an equity tranche with maturity T and detachment point $B \in (0, L_{\max}]$ is

$$D_{\text{eq}}(T) = L(T) - (L(T) - B)^+ = \min(L(T), B). \quad (19)$$

The default leg of a senior tranche with maturity T and attachment point $A \in [0, L_{\max})$ is

$$D_{\text{sen}}(T) = (L(T) - A)^+. \quad (20)$$

Definition 4. The payment leg consists in payments, at K regular times t_k (with $t_K = T$), of a fixed spread s applied to the remaining tranche amount at risk. In the case of the equity tranche, the payment leg writes

$$P_{\text{eq}}(T) = s \frac{T}{K} \sum_{k=1}^K (B - L(t_k))^+. \quad (21)$$

In the case of the senior tranche, it writes

$$\begin{aligned} P_{\text{sen}}(T) &= s \frac{T}{K} \sum_{k=1}^K (L_{\max} - A - (L(t_k) - A)^+) \\ &= sT(L_{\max} - A) - s \frac{T}{K} \sum_{k=1}^K (L(t_k) - A)^+. \quad \square \end{aligned} \quad (22)$$

Putting default and payment payoffs together, we obtain, for the equity tranche,

$$D_{\text{eq}}(T) - P_{\text{eq}}(T) = L(T) - (L(T) - B)^+ - s \frac{T}{K} \sum_{k=1}^K (B - L(t_k))^+, \quad (23)$$

and, for the senior tranche,

$$\begin{aligned} D_{\text{sen}}(T) - P_{\text{sen}}(T) &= \\ &= (L(T) - A)^+ - sT(L_{\max} - A) + s \frac{T}{K} \sum_{k=1}^K (L(t_k) - A)^+. \end{aligned} \quad (24)$$

Specifying $X_i = F_i^{-1}(1 - \gamma_i(\tau_i))$, where F_i is the c.d.f. of X_i , γ_i the \mathbb{Q}^* c.d.f. of τ_i and letting $B_i(t) := F_i^{-1}(1 - \gamma_i(t))$, we have $\{\tau_i \leq t\} = \{X_i \geq B_i(t)\}$ and

$$L(t) = \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}}. \quad (25)$$

The following result precises the outlined application for comparing CDO tranche premiums in [Cousin and Laurent \(2008\)](#) with heterogeneous obligors under our static setup.

Proposition 7. *If $(X_1, \dots, X_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Gamma}, \psi)$ and $(X'_1, \dots, X'_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Gamma}', \psi)$, with $\boldsymbol{\Gamma} \leq \boldsymbol{\Gamma}'$ elementwise except for equal diagonal entries, then*

$$\mathbb{E}^* \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} - A \right)^+ \right] \leq \mathbb{E}^* \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} - A \right)^+ \right], \quad A \in \mathbb{R}. \quad (26)$$

That is, the price of the default leg of a senior CDO tranche is nondecreasing w.r.t. ρ^{cr} ; we also get that the price of the default leg of an equity CDO tranche is nonincreasing w.r.t. ρ^{cr} .

Proof. For any $t \in \mathbb{R}_+$, the function

$$\begin{aligned} f_t : \quad \mathbb{R}^n &\longrightarrow \mathbb{R}_+ \\ (x_1, \dots, x_n) &\longmapsto \sum_{i=1}^n L_i \mathbb{1}_{\{x_i \geq B_i(t)\}} \end{aligned} \quad (27)$$

is nondecreasing w.r.t. each of its arguments and it is supermodular, by Lemma 2. Hence, due to the nondecreasing and convexity properties of $x \mapsto (x - A)^+$, [Müller and Stoyan \(2002, Theorem 3.9.3 f\)](#), page 113), recalled in Section A, implies that $(x_1, \dots, x_n) \mapsto (f_t(x_1, \dots, x_n) - A)^+$ is also nondecreasing supermodular. Moreover, by [Block and Sampson \(1988, Corollary 2.3\)](#), recalled in Section B, $(X_1, \dots, X_n) \leq_{sm^*} (X'_1, \dots, X'_n)$. Applying [Müller \(1997, Definition 2.6\)](#) to (X_1, \dots, X_n) , (X'_1, \dots, X'_n) and $(x_1, \dots, x_n) \mapsto (f_t(x_1, \dots, x_n) - A)^+$ then yields the result for the senior tranche.

As $\mathbb{E}^* \left[\sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right] - \mathbb{E}^* \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} - B \right)^+ \right]$ is the price of an equity tranche default leg, where the left expectation term does not depend on ρ^{cr} , the result for the equity tranche follows. \square

Corollary 8. *Under the same assumptions as in Proposition 7, the price of the payment leg of the CDO equity (resp. senior) tranche is nondecreasing (resp. nonincreasing) w.r.t. ρ^{cr} .*

Proof. By call-put parity,

$$\begin{aligned} \mathbb{E}^* \left[\left(B - \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right)^+ \right] &= \mathbb{E}^* \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} - B \right)^+ \right] \\ &\quad - B + \mathbb{E}^* \left[\sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} \right] \end{aligned} \quad (28)$$

so that, in view of (21)-(22), as a consequence of Proposition 7,

$$\mathbb{E}^* \left[\left(B - \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right)^+ \right] \leq \mathbb{E}^* \left[\left(B - \sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} \right)^+ \right]. \quad \square \quad (29)$$

Obligor id	1	2	3	4	5	6	7	8	9	10
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	30	31	32	33	34	35	36	37	38	39
$\lambda(\%)$	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5

Obligor id	21	222	23	24	25	26	27	28	29	30
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	39	40	30	31	32	33	34	35	36	37
$\lambda(\%)$	12	12.5	13	13.5	14	14.5	15	15.5	16	16.5

Obligor id	11	12	13	14	15	16	17	18	19	20
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	40	30	31	32	33	34	35	36	37	38
$\lambda(\%)$	7	7.5	8	8.5	9	9.5	10	10.5	11	11.5

Table 2: CDO portfolios and obligors parameters.

Remark 2. For mezzanine tranches, such results do not hold. Indeed, the tranching loss default leg payoff function

$$D_{\text{Mezz}}(T) = (L(T) - A)^+ - (L(T) - B)^+, \quad (30)$$

where $A, B \in (0, L_{\max})$, is not a convex function of the cumulative loss nor is the payment leg

$$\begin{aligned} P_{\text{Mezz}}(T) &= s \frac{T}{K} \sum_{k=1}^K \left[B - A - \left((L(t_k) - A)^+ - (L(t_k) - B)^+ \right) \right] \\ &= sT(B - A) - s \frac{T}{K} \sum_{k=1}^K \left[(L(t_k) - A)^+ - (L(t_k) - B)^+ \right]. \end{aligned} \quad (31)$$

A Numerical Results

The results are illustrated in Figure 1 for the equity tranches, varying detachment point from 5% to 95% with 5% steps, i.e. considering the tranches from $[0, 5\%]$ to $[0, 95\%]$. Figure 2 illustrates the results for the senior tranches, varying attachment point from 5% to 95% with 5% steps, i.e. considering the tranches from $[5\%, 100\%]$ to $[95\%, 100\%]$. The correlation ρ^{cr} is varied from 5% to 95% with 5% step for both tranche types. The parameters of the underlying obligors and CDSs are detailed in Table 2, where values have been assigned arbitrarily to ensure heterogeneity of the various obligors. The CDO tranche spread has been set to 10% with a single coupon paid at a maturity of 5 years. The monotonicity patterns are observed for both tranches, with incremental prices between two consecutive credit correlation steps being nonpositive for the CDO equity tranche default leg prices and the CDO senior tranche payment leg prices. Incremental prices between two consecutive credit correlation steps are nonnegative for both the CDO equity tranche payment leg prices and the CDO senior tranche default leg prices. These results are in line with Proposition 7 and Corollary 8. Also, incremental prices between two attachment point steps are nonnegative for both the CDO equity tranche default leg and payment leg prices, as expected from (29). The incremental prices between two attachment point steps are nonpositive for the CDO senior tranche default leg and payment leg prices, as expected from (26).

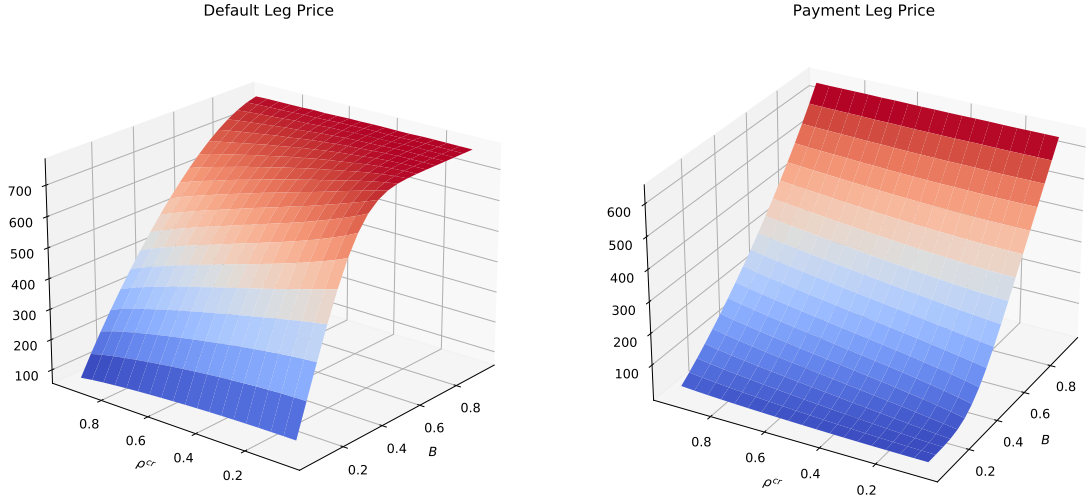


Figure 1: Default leg and Payment leg prices of CDO equity tranches w.r.t. latent variable credit correlation ρ^{cr} and detachment point B .

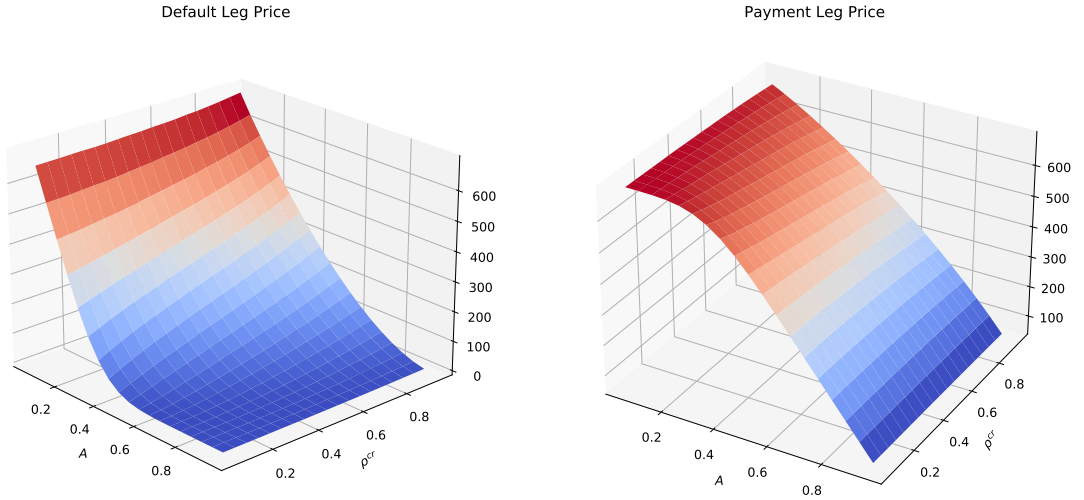


Figure 2: Default leg and Payment leg prices of CDO senior tranches w.r.t. latent variable credit correlation ρ^{cr} and attachment point A . Note that the axes are different from Figure 1, for a better readability.

§4 Main Results: Counterparty Credit Risk

We assume the setup of Sections B and C regarding $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m) = (X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $\mathbf{X}' = (\mathcal{X}'_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m) = (X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ (for some $n > 0$). Let

$$L = \sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \in \mathfrak{X}^0 = L^1(\mathbb{Q}^0), \quad (32)$$

where $Y_i \in \mathfrak{X}^*$ drives a loss that obligor $i \in 1..n$ generates if it defaults, with default of each credit name $j \in 0..n$ (including the reference bank 0 as in Section C) driven by a latent variable $X_j \in \mathfrak{X}^*$. The $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are measurable supermodular (when $n > 1$) functions nondecreasing w.r.t. each of their arguments, and the $g_i : \mathbb{R} \rightarrow \mathbb{R}_+$ are measurable nondecreasing functions. When $f_i(X_1, \dots, X_n) = f_i(X_i)$ as in [Cousin and Laurent \(2008\)](#) or [Meyer and Strulovici \(2013\)](#), $f_i(X_i)g_i(Y_i)$ can represent the loss related to a bilateral counterparty position, i.e. a portfolio position between the reference bank 0 and its client i . As detailed in Section A, the more general case where $f_i(X_1, \dots, X_n)$ depends on several X_j encompasses the financial losses generated by clearing exposures towards a central counterparty (CCP). This is due to the loss allocation coefficients attributed to each surviving member ([Bastide, Crépey, Drapeau, and Tadese, 2023a](#)). It also covers the case of financial resolution funds ([Single Resolution Board, 2021](#)). As outlined in Section A, if we can establish that $\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i) \in \mathbb{R}^+$ is supermodular, then we can conclude that $\rho(L) \leq \rho(L')$ holds for any risk measure ρ on \mathfrak{X}^0 , with $L' = \sum_{i=1}^n f_i(X'_1, \dots, X'_n)g_i(Y'_i) \in \mathfrak{X}^0$.

Remark 3. Wrong-way risk is the potential increase of the exposure a financial actor w.r.t. certain counterparties when their probability of default increase. A risk model should include a wrong-way risk feature in order to ensure conservative treatment. See [Gregory \(2015, Section 8.6.5\)](#) for more detailed explanations and [Brigo, Morini, and Pallavicini \(2013\)](#) for various examples of asset classes models incorporating the wrong-way risk feature. Under the elliptical model (3) and the credit loss form (32), wrong-way risk holds provided that an increase of the covariance between the default latent variable X_i and the potential loss driver Y_i leads to an increase of the loss amount $g_i(Y_i)$. This is the case when f and g are nondecreasing (as assumed) in each of their arguments.

Lemma 9. *If $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a supermodular function nondecreasing w.r.t. each of its arguments and $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing function, $i \in 1..n$, then $\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f(x_1, \dots, x_n)g_i(y_i) \in \mathbb{R}$ has increasing differences w.r.t. any pair $(x_i, y_j) \in \mathbb{R}^2$, $i, j \in 1..n$.*

Proof. Let $f_i(x_j|\mathbf{x}_{-j})$ denote the function f_i applied to x_j but keeping all other arguments $\mathbf{x}_{-j} := (x_k)_{k \neq j}$ fixed. We look at the two cases where we consider either a pair of argument $(x_i, y_i) \in \mathbb{R}^2$, $i \in 1..n$ (i.e. the pair of arguments tested for the increasing difference are part of the same term of the sum), or a pair $(x_i, y_j) \in \mathbb{R}^2$, $i \neq j \in 1..n$, and the corresponding increasing differences.

Case $(x_i, y_i) \in \mathbb{R}^2$, $i \in 1..n$: the function $\mathbb{R}^2 \ni (x_i, y_i) \mapsto f_i(x_i|\mathbf{x}_{-i})g_i(y_i)$, has the increasing differences property by application of Lemma 3 with $g(\cdot) = f_i(\cdot|\mathbf{x}_{-i})$ and $h(\cdot) = g_i(\cdot)$. The increasing difference writes

$$\begin{aligned}
& f_i(x'_i|\mathbf{x}_{-i})g_i(y'_i) + \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - f_i(x'_i|\mathbf{x}_{-i})g_i(y_i) - \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) \\
& - f_i(x_i|\mathbf{x}_{-i})g_i(y'_i) - \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) + f_i(x_i|\mathbf{x}_{-i})g_i(y_i) + \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) \\
& = \underbrace{f_i(x'_i|\mathbf{x}_{-i})g_i(y'_i) - f_i(x'_i|\mathbf{x}_{-i})g_i(y_i) - f_i(x_i|\mathbf{x}_{-i})g_i(y'_i) + f_i(x_i|\mathbf{x}_{-i})g_i(y_i)}_{\geq 0 \text{ by Lemma 3}} \\
& + \underbrace{\sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k)}_{=0}
\end{aligned}$$

$$\begin{aligned}
& - \underbrace{\sum_{k \neq i} f_k(x_i | \mathbf{x}_{-i}) g_k(y_k) + \sum_{k \neq i} f_k(x_i | \mathbf{x}_{-i}) g_k(y_k)}_{=0} \\
& \geq 0,
\end{aligned}$$

hence $(x_i, y_i) \mapsto \sum_{k=1}^n f_k(x_i | \mathbf{x}_{-i}) g_k(y_k)$ has the increasing differences property.

Case $(x_i, y_j) \in \mathbb{R}^2, i \neq j \in 1..n$: we write the increasing difference

$$\begin{aligned}
& \sum_{k \neq j} f_k(x'_i | \mathbf{x}_{-i}) g_k(y_k) + f_j(x'_i | \mathbf{x}_{-i}) g_j(y'_j) - \sum_{k \neq j} f_k(x'_i | \mathbf{x}_{-i}) g_k(y_k) - f_j(x'_i | \mathbf{x}_{-i}) g_j(y_j) \\
& - \sum_{k \neq j} f_k(x_i | \mathbf{x}_{-i}) g_k(y_k) - f_j(x_i | \mathbf{x}_{-i}) g_j(y'_j) + \sum_{k \neq j} f_k(x_i | \mathbf{x}_{-i}) g_k(y_k) + f_j(x_i | \mathbf{x}_{-i}) g_j(y_j) \\
& = f_j(x'_i | \mathbf{x}_{-i}) g_j(y'_j) - f_j(x'_i | \mathbf{x}_{-i}) g_j(y_j) - f_j(x_i | \mathbf{x}_{-i}) g_j(y'_j) + f_j(x_i | \mathbf{x}_{-i}) g_j(y_j) \geq 0,
\end{aligned}$$

by application of Lemma 3 with $g(\cdot) = f_j(\cdot | \mathbf{x}_{-i})$ and $h(\cdot) = g_j(\cdot)$. \square

Proposition 10. *If $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing function and $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing function, $i \in 1..n$, then $\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \xrightarrow{\Lambda} \sum_{i=1}^n f_i(x_i) g_i(y_i) \in \mathbb{R}$ has increasing differences w.r.t. any pair $(x_i, y_j) \in \mathbb{R}^2, i, j \in 1..n$.*

Proof. By Lemma 2, λ has increasing differences w.r.t. any pair $(x_i, x_j) \in \mathbb{R}^2, i \neq j \in 1..n$, as well as any pair $(y_i, y_j) \in \mathbb{R}^2, i \neq j$. For $i \in 1..n, (x_i, y_i) \mapsto f_i(x_i) g_i(y_i)$, has the increasing differences property by application of Lemma 3 with $g(\cdot) = f_i(\cdot)$ and $h(\cdot) = g_i(\cdot)$. Hence, λ has the increasing differences property for any pair $(x_i, y_i) \in \mathbb{R}^2$. Finally, for $i \neq j \in 1..n, (x_i, y_j) \mapsto f_i(x_i) g_i(y_i) + f_j(x_j) g_j(y_j)$ has the increasing differences property by Lemma 2 with $h_i(\cdot) = f_i(\cdot) g_i(y_i)$ and $h_j(\cdot) = f_j(x_j) g_j(\cdot)$. Hence, λ has the increasing differences property for any pair $(x_i, y_j) \in \mathbb{R}^2, i \neq j \in 1..n$. \square

Proposition 11. *Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be a nondecreasing supermodular function and $g_i : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing function, $i \in 1..n$. Then the function*

$$\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \xrightarrow{\Lambda} \sum_{i=1}^n f_i(x_1, \dots, x_n) g_i(y_i) \in \mathbb{R} \quad (33)$$

is supermodular.

Proof. By Lemma 2 with $h_i(\cdot) = f_i(x_1, \dots, x_n) g_i(\cdot)$, for any $i \in 1..n, \Lambda$ has increasing differences with respect to any pair $(y_k, y_l), k, l \in 1..n$. By assumption and closure by addition of the increasing differences property, Λ has increasing differences with respect to any pair $(x_k, x_l), k, l \in 1..n$. Finally, by Lemma 9, Λ has increasing differences with respect to any pair $(x_k, y_l), k, l \in 1..n$. Hence, by Yildiz (2010, Corollary 1)⁶, Λ is supermodular. \square

Theorem 12. *Assume L and L' defined by (32) applied respectively to $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $(X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ s.t. $(X_1, \dots, X_n, Y_1, \dots, Y_n) \leq_{sm^0} (X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$. Then, for any risk measure⁷ ρ on $\mathfrak{X}^0 \ni L, L'$, we have $\rho(L) \leq \rho(L')$. In particular, if $\mathfrak{X}^0 = L^1(\mathbb{Q}^0)$ and that L and L' are \mathbb{Q}^0 integrable, then $\mathbb{E}^0[L] \leq \mathbb{E}^0[L']$.*

⁶see Section A.

⁷see Definition 7.

Proof. By Proposition 11, the function $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i)$ is supermodular on \mathbb{R}^{2n} . The result $\rho(L) \leq \rho(L')$ then follows the way outlined in Section §2. \square

Remark 4. By Lemma 6, the conclusion of Theorem 12 holds, in particular, for $\mathbf{X} = (X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $\mathbf{X}' = (X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ (see beginning of Section §4) satisfying Assumption 2, e.g. (by Proposition 5) in the \mathbb{Q}^* elliptical setup (3) under Assumption 1.

Remark 5. An analogous monotonicity result does not hold for $\check{\rho} = \rho(\cdot - \mathbb{E}(\cdot))$ (see Definition 7). For instance, if $\rho = \mathbb{E}\mathbb{S}_\alpha$, \mathcal{U}_1 is uniform on $[0, 1]$ and \mathcal{U}_2 is uniform on $[1, \theta]$ for some $\theta > 1$, then $\mathcal{U}_1 \leq \mathcal{U}_2$ a.s., $\mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_1) = \frac{1+\alpha}{2} < \mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_2) = 1 + (\theta-1)\frac{1+\alpha}{2}$, but $\mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_1) - \mathbb{E}(\mathcal{U}_1) = \frac{\alpha}{2} > \mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_2) - \mathbb{E}(\mathcal{U}_2) = (\theta-1)\frac{\alpha}{2}$ for $1 < \theta < 2$.

We now take in (14)

$$h(x_0) = \frac{\mathbb{1}_{\{x_0 < B_0\}}}{1 - \gamma}, \quad \text{where } \gamma = \mathbb{Q}^*(X_0 \geq B_0), \quad (34)$$

so that \mathbb{Q}^0 is the survival measure of the reference bank associated with \mathbb{Q}^* (Albanese, Crépey, Hoskinson, and Saadeddine, 2021, Section 3).

Definition 5. The current expected credit loss CECL of the reference bank 0 is $\text{CECL} = \mathbb{E}^0[L]$.

Similarly, we define $\text{CECL}' = \mathbb{E}^0[L']$.

Remark 6. From a financial application point of view, the CECL that considers the lifetime for all related asset is preferred to the IFRS9 expected credit loss (ECL) that segments into stage 1 and stage 2. The CECL allows for a 1-year maturity assumption instead of the full lifetime of the assets considered for the ECL⁸.

Proposition 13. If L and L' are defined by (32) for \mathbf{X} and \mathbf{X}' satisfying Assumption 2, e.g. (by Proposition 5) in the elliptical setup (3) under Assumption 1, then $\text{CECL} \leq \text{CECL}'$ whenever $L, L' \in L^1(\mathbb{Q}^0)$.

Proof. By definition of CECL and application of Theorem 12 and Remark 4 to $\rho = \mathbb{E}^0$. \square

Definition 6. The economic capital of the reference bank 0 is $\text{EC} = \mathbb{E}\mathbb{S}_\alpha^0(L)$, with $\mathbb{E}\mathbb{S}_\alpha^0$ as per Definition 8 assumed under \mathbb{Q}^0 and $L \geq 0$ given by (32).

Similarly, let $\text{EC}' = \mathbb{E}\mathbb{S}_\alpha^0(L')$.

Proposition 14. If L and L' are defined by (32) for \mathbf{X} and \mathbf{X}' satisfying Assumption 2, e.g. (by Proposition 5) in the elliptical setup (3) under Assumption 1, then $\text{EC} \leq \text{EC}'$ whenever $L, L' \in L^1(\mathbb{Q}^0)$.

Proof. By definition of EC and application of Theorem 12 and Remark 5 to $\rho = \mathbb{E}\mathbb{S}_\alpha^0$. \square

Remark 7. The Definition 8 of $\mathbb{E}\mathbb{S}_\alpha^0$ ensures its domain is all $L^1(\mathbb{Q}^0)$, thus is in particular proper. Acerbi and Tasche (2002, Proposition 3.1) outlines $\mathbb{E}\mathbb{S}_\alpha^0$ is subadditive and positively homogeneous (therefore convex) as well as monotonous (in our case where we consider loss variables, for $\mathcal{X} \leq 0$ we have $\rho(\mathcal{X}) \leq 0$ and using subadditivity we get for $\mathcal{X} \leq \mathcal{Y}$, $\rho(\mathcal{X}) = \rho(\mathcal{X} - \mathcal{Y} + \mathcal{Y}) \leq \rho(\mathcal{X} - \mathcal{Y}) + \rho(\mathcal{Y}) \leq \rho(\mathcal{Y})$). Hence it verifies the assumptions of (Ruszczyński and Shapiro, 2006, Proposition 1). Therefore $\mathbb{E}\mathbb{S}_\alpha^0$ is continuous on $L^1(\mathbb{Q}^0)$ and Bäuerle and Müller (2006, Theorem 4.4) applies to $\rho = \mathbb{E}\mathbb{S}_\alpha^0$.

⁸see IFRS (2022, Article 5.3.3) and European Systemic Risk Board (2019).

A CCP Case Study

We denote $\mathbf{Y} = \mathbf{X}_{n+1:2n} = (Y_1, \dots, Y_n)$ and $\mathbf{Y}' = \mathbf{X}'_{n+1:2n} = (Y'_1, \dots, Y'_n)$. Given real constants $\beta_1, \dots, \beta_n \geq 0$, m_1, \dots, m_n and B_1, \dots, B_n , we consider the credit losses

$$L(\mathbf{X}_{1:n}, \mathbf{Y}) = \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{X_j < B_j\}}} \mathbb{1}_{\{X_i \geq B_i\}} \times (Y_i - m_i)^+. \quad (35)$$

$\mathbf{X}_{0:n} = (X_0, X_1, \dots, X_n)$ drives the default events, namely, $\{\tau_i \leq T\} = \{X_i \geq B_i\}$ models the default event of participant $i \in 0..n$, e.g. its liability return taken as the latent factor breaching a certain threshold. $Y_i = \text{nom}_i \sigma_i G_i$, with $\text{nom}_i \in \mathbb{R}$, $\sigma_i > 0$ and G_i spherical (hence $\text{Var}^*(Y_i) = \text{nom}_i^2 \sigma_i^2 \text{Var}^*(G_i)$) represents the loss of the market participant 0 in case of the default of obligors indexed by i , collateralized by a corresponding amount m_i . The weights $\left(1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{X_j < B_j\}}\right)^{-1}$ represent a stylised specification of a default fund allocation in a central counterparty (CCP) setup, with then $\beta_j = \text{DF}_j / \text{DF}$ (see Section B), or of a liability size allocation in a single resolution fund (SRF) setup.

Lemma 15. *For any $i \in 1..n$, the function $\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}} \in \mathbb{R}_+$ is nondecreasing supermodular.*

Proof. By Yildiz (2010, Corollary 1), it is sufficient to show that f_i has increasing differences. Let $k, l \in 1, \dots, n$.

Case $k, l \neq i$: Let $\mathbb{R}^2 \ni (x_k, x_l) \mapsto \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}}} \in \mathbb{R}_+$, with $\Lambda = \sum_{j \neq k, l} \beta_j \mathbb{1}_{\{x_j < B_j\}}$. For $x'_k \geq x_k, x'_l \geq x_l$, we form the increasing difference

$$\begin{aligned} & g(x'_k, x'_l) - g(x'_k, x_l) - g(x_k, x'_l) + g(x_k, x_l) \\ &= \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_1} - \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_2}, \end{aligned} \quad (36)$$

with $\text{denom}_1 = (\Lambda + \beta_k \mathbb{1}_{\{x'_k < B_k\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_k \mathbb{1}_{\{x'_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$ and $\text{denom}_2 = (\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$. If $x_l \leq x'_l < B_l$ or $B_l \leq x_l \leq x'_l$, then the increasing difference (36) is zero as the numerators of both terms are zero. If $x_l < B_l \leq x'_l$, then both numerators in (36) equal $\mathbb{1}_{\{x_i \geq B_i\}} \beta_l$. In this case: (i) if $x_k \leq x'_k < B_k$, then both denominators in (36) equal $(\Lambda + \beta_k)(\Lambda + \beta_k + \beta_l)$ and the increasing difference (36) is zero; (ii) if $B_k \leq x_k \leq x'_k$, then both denominators in (36) equal $\Lambda(\Lambda + \beta_l)$ and the increasing difference (36) is zero; If $x_k < B_k \leq x'_k$, then the increasing difference in (36) writes equivalently

$$\frac{\Lambda \beta_l}{\text{denom}_1 \text{denom}_2} (\text{denom}_2 - \text{denom}_1), \quad (37)$$

with $\text{denom}_2 = (\Lambda + \beta_k + \beta_l)(\Lambda + \beta_k) \geq \Lambda(\Lambda + \beta_l) = \text{denom}_1$ as $\beta_j \geq 0$ for all $j \in 1..n$. Hence the increasing difference (36) is nonnegative.

Case $k = i$ and $l \neq i$: Let $\mathbb{R}^2 \ni (x_i, x_l) \mapsto \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}}} \in \mathbb{R}_+$, with $\Lambda = \sum_{j \neq i, l} \beta_j \mathbb{1}_{\{x_j < B_j\}}$. For $x'_i \geq x_i, x'_l \geq x_l$, we form the increasing difference

$$\begin{aligned} & g(x'_i, x'_l) - g(x'_i, x_l) - g(x_i, x'_l) + g(x_i, x_l) \\ &= \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_1} - \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_2}, \end{aligned} \quad (38)$$

with $\text{denom}_1 = (\Lambda + \beta_i \mathbb{1}_{\{x'_i < B_i\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})$ and $\text{denom}_2 = (\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$. If $x_l \leq x'_l < B_l$ or $B_l \leq x_l \leq x'_l$, then the increasing difference (38) is zero as the numerators of both terms are zero. If $x_l < B_l \leq x'_l$, then the numerator of the first terms in (38) is $\mathbb{1}_{\{x'_i \geq B_i\}}$ and $\mathbb{1}_{\{x_i \geq B_i\}}$ for the second term. In this case: (i) if $x_i \leq x'_i < B_i$, then both numerators in (38) are zero and the increasing difference (38) is zero; (ii) if $B_i \leq x_i \leq x'_i$, then both numerators in (38) are equal to $\mathbb{1}_{\{x_i \geq B_i\}} \beta_l$ and both denominators are equal $\Lambda(\Lambda + \beta_l)$ so the increasing difference (38) is zero; If $x_i < B_i \leq x'_i$, then the second term in (38) is zero and the first term is nonnegative. Hence the increasing difference (38) is nonnegative. \square

Proposition 16. *The function*

$$\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \xrightarrow{f} \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}} \mathbb{1}_{\{x_i \geq B_i\}} \times (y_i - m_i)^+ \in \mathbb{R}_+ \quad (39)$$

is nondecreasing supermodular.

Proof. Let $f_i(x_1, \dots, x_n) = \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}}$ and $g_i(y_i) = (y_i - m_i)^+$ in (33). By Lemma 15, f_i has increasing differences with respect to any pair $(x_k, x_l) \in \mathbb{R}^2$, $k, l \in 1..n$ for any $i \in 1..n$. Hence f verifies all assumptions of Proposition 11, thus it is supermodular. \square

Corollary 17. *If $L(\mathbf{X}_{1:n}, \mathbf{Y}), L(\mathbf{X}'_{1:n}, \mathbf{Y}') \in L^1(\mathbb{Q}^0)$, then,*

$$\mathbb{E}^0 [L(\mathbf{X}_{1:n}, \mathbf{Y})] \leq \mathbb{E}^0 [L(\mathbf{X}'_{1:n}, \mathbf{Y}')], \quad (40)$$

i.e. CECL is nondecreasing w.r.t. $\Gamma_{\mathbf{X}}^{ij}$ for each $i, j \in 1..2n$.

Proof. By Proposition 5 and Lemma 6, $(X_1, \dots, X_n, Y_1, \dots, Y_n) \leq_{sm^0} (X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$. Applying Proposition 13 with f defined by (39), which is supermodular thanks to Proposition 16, then yields the result. \square

Similarly (with Proposition 14 instead of Proposition 13 in the above argument)

Corollary 18. *If $L(\mathbf{X}_{1:n}, \mathbf{Y}), L(\mathbf{X}'_{1:n}, \mathbf{Y}') \in L^1(\mathbb{Q}^0)$, then,*

$$\mathbb{E}\mathbb{S}_\alpha^0 \left(L(\mathbf{X}_{1:n}, \mathbf{Y}) \right) \leq \mathbb{E}\mathbb{S}_\alpha^0 \left(L(\mathbf{X}'_{1:n}, \mathbf{Y}') \right), \quad (41)$$

i.e. EC is nondecreasing w.r.t. $\Gamma_{\mathbf{X}}^{ij}$ for each $i, j \in 1..2n$. \square

B Numerical Setup

We consider a CCP service with 20 members, labeled by $i \in 0..n = 19$, trading for cleared clients (i.e. without bilateral or centrally cleared proprietary trading). Each member faces one client. The corresponding financial network is depicted in Figure 3.

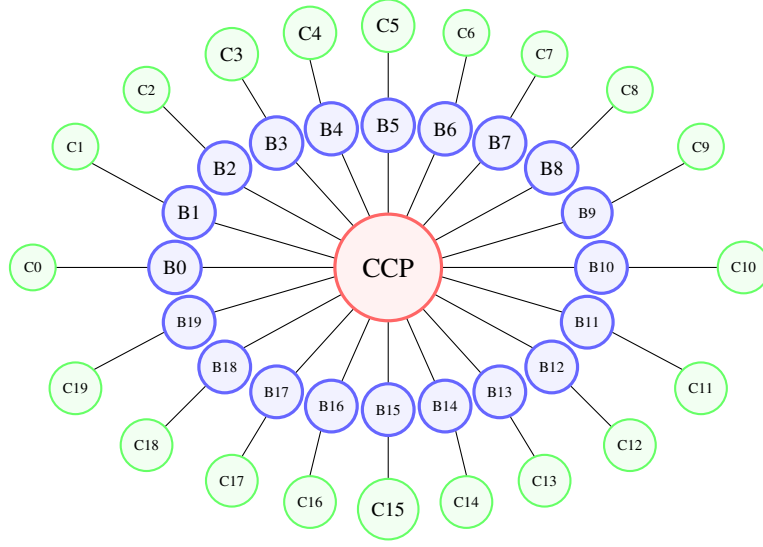


Figure 3: Financial network composed of one CCP service, its 20 members (labeled by B) and one cleared client per member

All clients are assumed to be risk-free. For any member i , its posted initial margin (IM) to the CCP is calculated based on the idea of a variation margin (VM) call not fulfilled over a slippage time period Δ_s at a confidence level $\alpha \in (1/2, 1)$. Such IM uses a VaR metric under the member survival measure applied to the non-coverage of VM call on the cleared portfolio. The latter follows a scaled Student t -distribution S_ν with ν degrees of freedom, with c.d.f. S_ν , and where such scaling reflects both Δ_s , the portfolio nominal size, denoted nom_i , and its standard deviation, denoted σ_i . Namely,

$$\text{IM}_i = \text{VaR}^*(\text{nom}_i \sigma_i \sqrt{\Delta_s} S_\nu) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} S_\nu^{-1}(\alpha). \quad (42)$$

The default fund is calculated at the CCP level as

$$\text{Cover2} = \text{SLOIM}_{(0)} + \text{SLOIM}_{(1)}, \quad (43)$$

for the two largest stressed losses over IM (SLOIM $_i$) among members, identified with subscripts (0) and (1). SLOIM $_i$ is calculated as the value-at-risk $\text{VaR}^{*\prime}$ at a confidence level $\alpha' > \alpha$ of the loss over IM, i.e.

$$\text{SLOIM}_i = \text{VaR}^{*\prime}(\text{nom}_i \sigma_i \sqrt{\Delta_s} S_\nu - \text{IM}_i) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} (S_\nu^{-1}(\alpha') - S_\nu^{-1}(\alpha)). \quad (44)$$

The total amount (43) is then allocated between the clearing members to define their (funded) default fund contributions as $\text{DF}_i = \frac{\text{SLOIM}_i}{\sum_j \text{SLOIM}_j} \times \text{Cover2}$. Finally, the loss function of the reference member 0 with default fund contribution DF_0 is

$$\begin{aligned} L(\mathbf{X}, \mathbf{Y}) &= \sum_{i=1}^n \frac{\text{DF}_0}{1 + \sum_{j=1}^n \text{DF}_j \mathbb{1}_{\{X_j < B_j\}}} \mathbb{1}_{\{X_i \geq B_i\}} \times (Y_i - \text{IM}_i - \text{DF}_i)^+ \\ &= \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \frac{\text{DF}_j}{\text{DF}_0} \mathbb{1}_{\{X_j < B_j\}}} \mathbb{1}_{\{X_i \geq B_i\}} \times (Y_i - \text{IM}_i - \text{DF}_i)^+. \end{aligned} \quad (45)$$

cm	0	1	2	3	4	5	6	7	8	9
λ (bps)	50	60	70	80	90	200	190	180	170	160
size	-242	184	139	105	-80	-61	-46	35	26	-20
vol (%)	20	21	22	23	24	25	26	27	28	29

cm	10	11	12	13	14	15	16	17	18	19
λ (bps)	150	140	130	120	110	100	90	80	70	60
size	-15	-11	-9	-6	5	-4	-3	2	2	-1
vol (%)	30	31	32	33	34	35	36	37	38	39

Table 3: Member characteristics and CCP portfolio parameters, ordered by decreasing member $|\text{size}|$.

Let $\text{sgn}(x) = 1$ if $x > 0$, 0 if $x = 0$, -1 otherwise. An elliptical model is specified under \mathbb{Q}^* as

$$\begin{cases} Y_i = \text{nom}_i \sigma_i \sqrt{\Delta_l} \sqrt{\mathcal{K}} \left(\sqrt{\rho^{mkt}} \mathcal{E} + \sqrt{\rho^{wvr}} \mathcal{W}_i + \sqrt{1 - \rho^{mkt} - \rho^{wvr}} \mathcal{E}_i \right) \\ X_i = \sqrt{\mathcal{K}} \left(\sqrt{\rho^{cr}} \mathcal{T} + \text{sgn}(\text{nom}_i) \sqrt{\rho^{wvr}} \mathcal{W}_i + \sqrt{1 - \rho^{cr} - \rho^{wvr}} \mathcal{T}_i \right) \end{cases} \quad (46)$$

for any $i \in 0..n$, where \mathcal{T} , \mathcal{T}_i , \mathcal{E} , \mathcal{E}_i and \mathcal{W}_i are i.i.d. normal random variables, $3/\mathcal{K}$ follows a chi-squared distribution independent from all random variables (common to all members). In particular, Y_i and X_i follow centered Student t-distributions of degree 3 and $(X_0, Y_0, X_1, Y_1, \dots, X_n, Y_n)$ is a multivariate t-distributed random vector. Δ_l is the period accounting for the time taken by the CCP to novate or liquidate its portfolios in case of defaults (practically, $\Delta_l > \Delta_s$ by a few business days). T represents the final maturity of the clearing members portfolios, assumed to be the same for all members. $B_i = \mathcal{S}_v^{-1}(1 - \text{DP}_i(T))$ where $\text{DP}_i(T)$ is the default probability over the period $[0, T]$ defined from a constant default intensity λ_i given for each member i in Table 3 (which can be obtained from their 1-year \mathbb{Q}^* default probability $\text{DP}_i(1Y)$, inferred either from the agency ratings or the CDS quotes when available, as $\lambda_i = -\ln(1 - \text{DP}_i(1Y))$ so that $\text{DP}_i(T) = 1 - e^{-\lambda_i T}$). The model is well defined if and only if $0 < \rho^{wvr} < \min(1 - \rho^{cr}, 1 - \rho^{mkt})$. Also, note that $\text{Cov}^*(X_i, Y_i) = \text{nom}_i \sigma_i \sqrt{\Delta_l} \sqrt{\rho^{wvr}} \text{sgn}(\text{nom}_i) \sqrt{\rho^{wvr}} = |\text{nom}_i| \sigma_i \sqrt{\Delta_l} \rho^{wvr} \geq 0$, hence increasing ρ^{wvr} leads to an increase of $\text{Cov}^*(X_i, Y_i)$.

The participants and portfolios parameter inputs are detailed in Table 3, where cm is the identifier of the clearing member, λ is the one year \mathbb{Q}^* default intensity of the member expressed in basis points, size represents the overall portfolio size of the member detained within the CCP, and vol is the annual volatility used for the portfolio variations. The portfolios listed in Table 3 relate to the members towards the CCP (which are mirroring the portfolios between the members and their clients). The sizes of the CCP portfolios of members sum up to 0, in line with the CCP clearing condition (without proprietary trades).

Remark 8. The random variables (46) follow Student t-distributions that are continuous. Therefore, 0 is the only possible atom of the nonnegative credit loss (45). Hence, by [Acerbi and Tasche \(2002, Corollary 5.3\)](#), Definition 8 is equivalent to $\mathbb{E}\mathbb{S}_\alpha(\mathcal{X}) = \mathbb{E}(\mathcal{X} | \mathcal{X} \geq \text{VaR}_\alpha(\mathcal{X}))$ ([Acerbi and Tasche, 2002, Eqn. \(3.7\)](#)) whenever $\text{VaR}_\alpha(\mathcal{X}) > 0$, i.e. for $\alpha \in (\frac{1}{2}, 1)$ sufficiently close to 1 so that $\text{VaR}_\alpha(\mathcal{X}) > 0$. In our numerical illustration with $\alpha = 99.75\%$, this is indeed the case.

C Numerical Results

The parameters of the CECL and EC calculations are summarized in Table 4. The confidence level at 97% for SLOIM in DF calibration allows for a ratio of default fund over initial margin of about 10% in our calculations, a ratio (of this level or less) often observed in practice. Note that the chosen period length of $T = 5$ years covers the bulk (if not the final maturity) of most realistic CCP portfolios.

One-period length T	5 years
Liquidation period at default Δ_l	5 days
Portfolio variations correlation ρ^{mkt}	30%
Credit factors correlation ρ^{cr}	20%
Correlation between credit factors and portfolio variations ρ_i^{wwr}	20%
IM covering period (MPoR) Δ_s	2 days
IM quantile level	95%
SLOIM calculation for DF Cover-2	VaR 97%
DF allocation rule	based on IM
Quantile level used for clearing members EC calculation	99.75%
Number of Monte Carlo simulation (for CECL and EC computations)	10M
Number of batches (for EC computations)	100

Table 4: CECL and EC calculation configuration

Figures 4, 5 and 6 show the results of CECL and EC calculated for the members 0, 5, and 10, each under their survival risk measure (i.e. letting them in turn play the role of the reference bank indexed by 0 in previous sections). In each figure, the credit-credit correlation ρ^{cr} and ρ^{wwr} is varied between 5% and 95%, using 5% step. The same nondecreasing pattern is observed for all three members, with nonnegative incremental CECL and EC between two consecutive credit-credit and credit-market correlation steps, in line with Corollaries 17 and 18. The market-market correlation ρ^{mkt} has been kept constant with value 4%. The results of the centered EC, i.e. $EC - CECL$, are also provided for each of these 3 members in Figure 7. As $CECL \ll EC$ holds for all three members, despite Remark 5, the monotonicity is also observed for this centered version of EC.

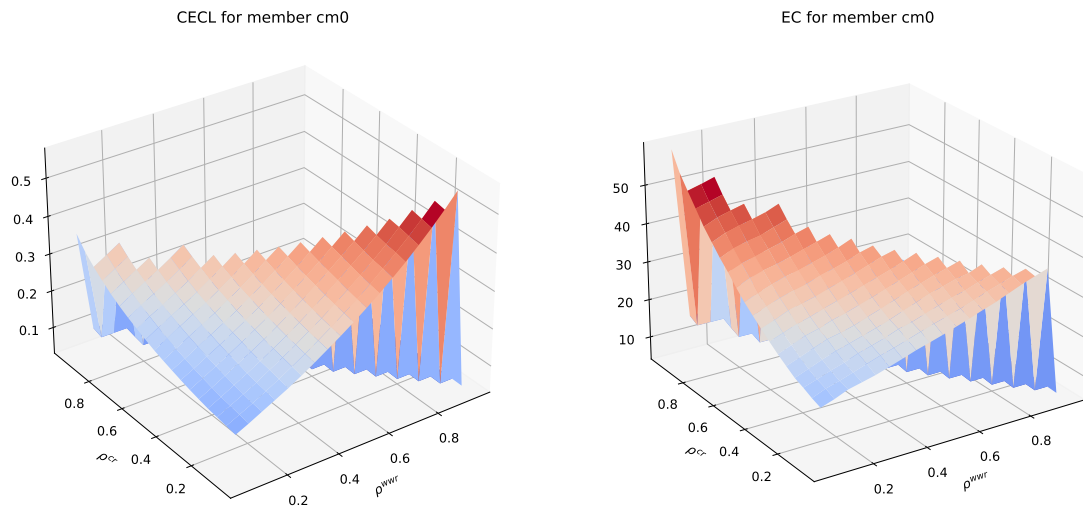


Figure 4: Member 0 CECL and EC w.r.t. credit factors correlation ρ^{CT} and credit and portfolio variation factors correlation ρ^{wvr} .

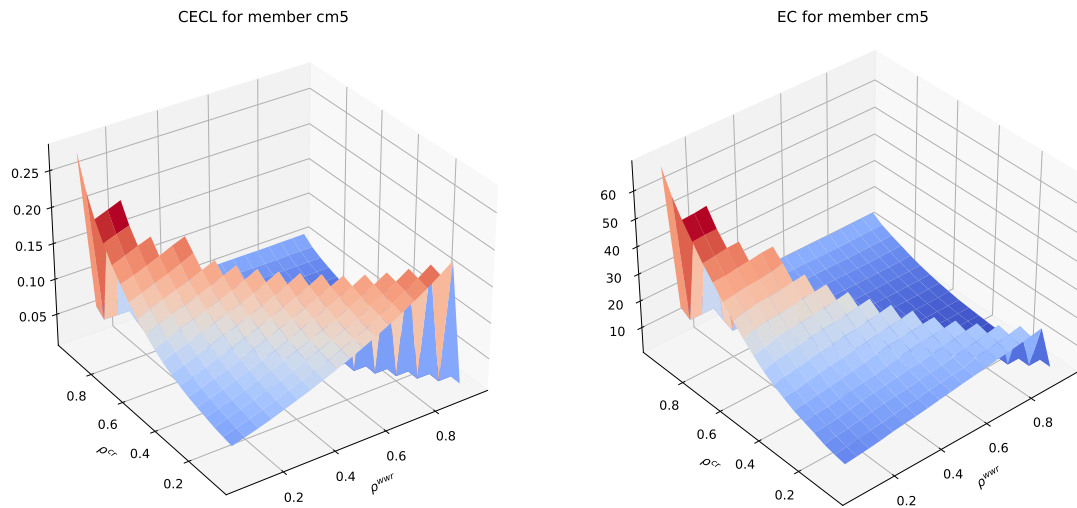


Figure 5: Member 5 CECL and EC w.r.t. credit factors correlation ρ^{CT} and credit and portfolio variation factors correlation ρ^{wvr} .

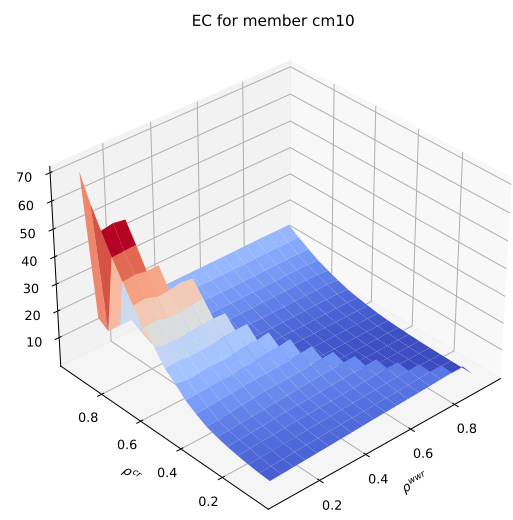
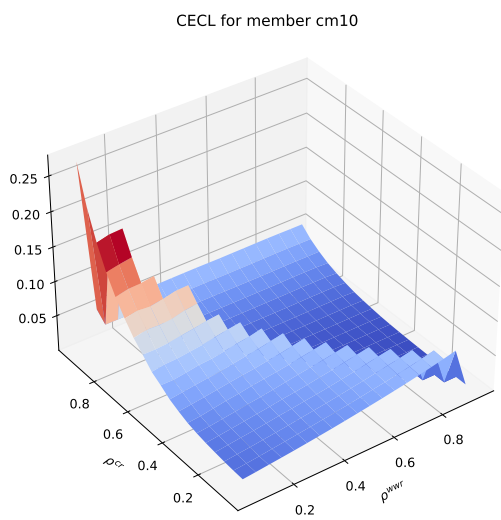


Figure 6: Member 10 CECL and EC w.r.t. credit factors correlation ρ^{CT} and credit and portfolio variation factors correlation ρ^{WWT} .

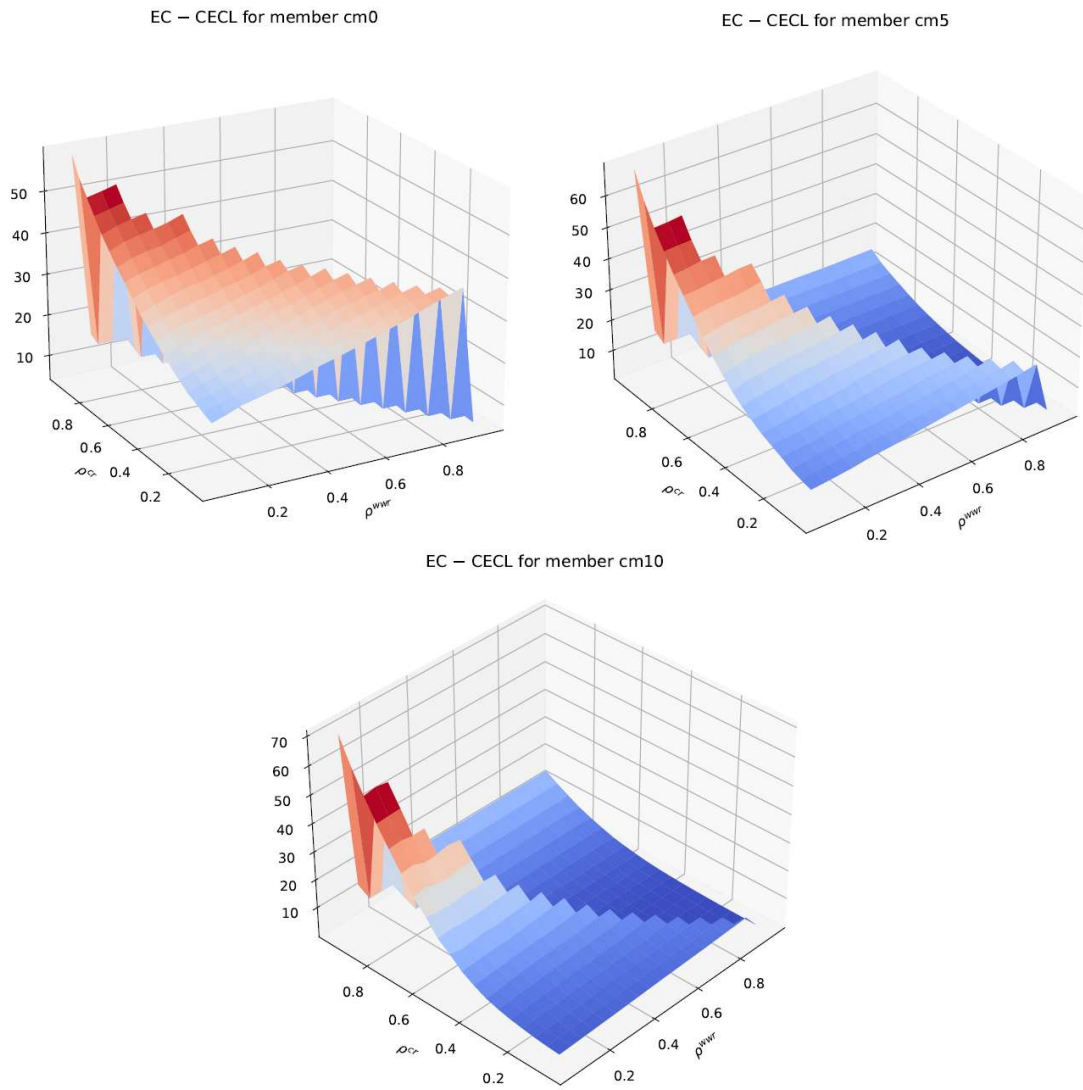


Figure 7: Members 0, 5 and 10 EC – CECL w.r.t. credit factors correlation ρ^{CT} and credit and portfolio variation factors correlation ρ^{wvr} .

Remark 9. In our example, $\text{Cov}^*(Y_i, Y_j) = \text{nom}_i \text{nom}_j \sigma_i \sigma_j \frac{\nu}{\nu-2} \rho^{mkt}$. Hence, depending on the sign of $\text{nom}_i \text{nom}_j$, increasing ρ^{mkt} either increases or decreases $\text{Cov}^*(Y_i, Y_j)$. Thus, we cannot hope to observe a monotonous behaviour of EC or CECL w.r.t. ρ^{mkt} .

§5 Conclusion

The main mathematical results of the paper are summed up in Table 5. In a nutshell, if a participant uses a convex risk measure to assess its credit risk depicted as an aggregation of nonnegative losses driven by elliptically distributed factors, then the measure increases with the covariance coefficients between these factors. These results and their numerical illustrations support the use of such elliptical factor models for both risk management and regulatory credit provision and capital requirement purposes.

Proposition 11	Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be nondecreasing supermodular functions, $g_i : \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative nondecreasing functions, $i \in 1..n$. Then the function $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n) g_i(y_i)$ is supermodular on \mathbb{R}^{2n} .
Theorem 12	If $(\mathbf{X}_{1:n}, \mathbf{Y})^\top \leq_{sm^0} (\mathbf{X}'_{1:n}, \mathbf{Y}')^\top$ and ρ is a risk measure ⁹ on \mathfrak{X}^0 , then $\rho(L) \leq \rho(L')$ holds for any $L, L' \in \mathfrak{X}^0$ of the form (32) applied to $(\mathbf{X}_{1:n}, \mathbf{Y})^\top$ and $(\mathbf{X}'_{1:n}, \mathbf{Y}')^\top$.
Proposition 7 and Corollary 8	The price of the default leg of an equity (resp. senior) tranche is nonincreasing (resp. nondecreasing) w.r.t. the credit correlation ρ^{cr} . The price of the payment leg of the CDO equity (resp. senior) tranche is nonincreasing (resp. nondecreasing) w.r.t. ρ^{cr} .
Proposition 13	Under Assumption 1 on $(X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $(X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ in the roles of \mathbf{X} and \mathbf{X}' there, CECL is a nondecreasing function of the \mathbb{Q}^* covariance coefficients.
Proposition 14	Under Assumption 1 on $(X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $(X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ in the roles of \mathbf{X} and \mathbf{X}' there, EC is a nondecreasing function of the \mathbb{Q}^* covariance coefficients.

Table 5: Main theoretical and applied results of the paper (with risk neutral measure \mathbb{Q}^* , reference participant labelled by 0, related survival measure \mathbb{Q}^0).

§6 Appendix

A Supermodular Functions

Müller (1997, Definitions 2.4 and 2.6) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be supermodular if

$$\begin{aligned} f(x_1, \dots, x_i + \varepsilon, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i + \varepsilon, \dots, x_j, \dots, x_n) \geq \\ f(x_1, \dots, x_i, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \end{aligned} \quad (47)$$

holds for all $x \in \mathbb{R}^n$, $\varepsilon, \delta > 0$ and $1 \leq i < j \leq n$. \square

Müller and Scarsini (2000, Theorem 2.2 (a)) For functions twice differentiable on \mathbb{R}^d , the supermodularity is equivalent to the nonnegativity of its second derivatives. \square

⁹see Definition 7.

More general definitions can be found in [Meyer-Nieberg \(1991\)](#) and [Yildiz \(2010\)](#).

[Meyer-Nieberg \(1991, Section 1.1, page 1\)](#) A partially ordered set (\mathfrak{S}, \geq) is said to be a lattice if and only if any two elements x, y have a greatest common minorant, denoted $x \wedge y$, and a least common majorant, denoted $x \vee y$. \square

[Yildiz \(2010, Definition 6, page 6\)](#) A function $f : \mathfrak{S} \rightarrow \mathbb{R}$ is said to be supermodular on a lattice (\mathfrak{S}, \geq) if

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y) \quad (48)$$

holds for all $x, y \in \mathfrak{S}$. \square

For a family of lattices $(\mathfrak{S}_1, \leq), \dots, (\mathfrak{S}_n, \leq)$, let $\mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$ be endowed with the coordinate-wise order $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ if and only if $\forall i, x_i \leq y_i$. This order makes $\mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$ a lattice. For $x \in \mathfrak{S}$ and any i and j , define $\mathbf{x}_{-(i,j)} = (x_k)_{k \neq i,j}$. For any function $f : \mathfrak{S} \rightarrow \mathbb{R}$, define $f(\cdot | \mathbf{x}_{-(i,j)}) : \mathfrak{S}_i \times \mathfrak{S}_j \rightarrow \mathbb{R}$ as the restriction of f to vectors with entries other than i and j fixed at $\mathbf{x}_{-(i,j)}$ ([Yildiz, 2010, Section 2.4](#)).

[Yildiz \(2010, remark after Definition 7, page 7\)](#) A function $f : \mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n \rightarrow \mathbb{R}$ is said to be pairwise supermodular if

$$\begin{aligned} f((x_i, x_j) \vee (x'_i, x'_j) | \mathbf{x}_{-(i,j)}) + f((x_i, x_j) \wedge (x'_i, x'_j) | \mathbf{x}_{-(i,j)}) \\ \geq f(x_i, x_j | \mathbf{x}_{-(i,j)}) + f(x'_i, x'_j | \mathbf{x}_{-(i,j)}) \end{aligned} \quad (49)$$

holds for all $x_1, \dots, x_n \in \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$. \square

[Yildiz \(2010, Definition 7\)](#) A function $f : \mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n \rightarrow \mathbb{R}$ is said to have increasing differences if

$$\begin{aligned} [x_i \geq x_{i'}, x_j \geq x_{j'}] \implies \\ f(x_{i'}, x_{j'} | \mathbf{x}_{-(i,j)}) - f(x_{i'}, x_j | \mathbf{x}_{-(i,j)}) \geq f(x_i, x_{j'} | \mathbf{x}_{-(i,j)}) - f(x_i, x_j | \mathbf{x}_{-(i,j)}) \end{aligned} \quad (50)$$

holds for any $\mathbf{x} = (x_1, \dots, x_n), i, j, i', j'$. \square

If the partial order is a total order, increasing differences and supermodularity in the sense of Definition A are equivalent. In particular:

[Yildiz \(2010, Corollary 1\)](#) For any $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the following are equivalent:

1. f is supermodular;
2. f has increasing differences;
3. f is pairwise supermodular. \square

[Müller and Stoyan \(2002, Theorem 3.9.3 f\), page 113\)](#) If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is nondecreasing and supermodular and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and convex, then $\phi \circ f : \mathbb{R}^n \rightarrow \mathbb{R}$ is nondecreasing supermodular. \square

B Elliptical Distributions

McNeil et al. (2015, Definition 6.17, page 196) A random vector $\mathbf{Z} = (Z_1, \dots, Z_n)^\top$ has a spherical distribution in \mathbb{R}^n if, for every orthogonal map $\mathbf{A} \in \mathbb{R}^{n \times n}$ ($\mathbf{A}\mathbf{A}^\top = \mathbf{A}^\top\mathbf{A} = \mathbf{I}_n$),

$$\mathbf{AZ} \stackrel{d}{=} \mathbf{Z}. \quad \square \quad (51)$$

McNeil et al. (2015, Theorem 6.18, page 196) The following are equivalent.

- (1) \mathbf{Z} is spherical in \mathbb{R}^n .
- (2) There exists a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{C}$ such that, for all $\mathbf{u} = (u_1, \dots, u_n)^\top \in \mathbb{R}^n$, the characteristic function of \mathbf{Z} is

$$\mathbb{E} \left[e^{i\mathbf{u}^\top \mathbf{Z}} \right] = \psi(\mathbf{u}^\top \mathbf{u}) = \psi(u_1^2 + \dots + u_n^2). \quad (52)$$

- (3) For every $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u}^\top \mathbf{Z} \stackrel{d}{=} \|\mathbf{u}\| Z_1$. \square

ψ is called the characteristic generator of \mathbf{Z} and the notation $\mathbf{Z} \sim S_n(\psi)$ is used (see [Fang, Kotz, and Ng \(1990\)](#) and [McNeil, Frey, and Embrechts \(2015\)](#)). We denote by $\mathbb{S}^{n-1} := \{\mathbf{s} \in \mathbb{R}^n : \mathbf{s}^\top \mathbf{s} = 1\}$ the unit sphere in \mathbb{R}^n , and by $U_{\mathbb{S}^{n-1}}$ the uniform distribution on \mathbb{S}^{n-1} .

Fang et al. (1990, Theorem 2.2, page 29) A function ψ is a generator of an n -dimensional elliptical r.v. if and only if it can be written as

$$\psi(x) = \int_0^\infty \Omega_n(xr^2) F(dr), \quad (53)$$

where $F(\cdot)$ is some c.d.f. over \mathbb{R}^+ and $\Omega_n(\mathbf{u}^\top \mathbf{u})$ is the characteristic generator of a random vector $\mathbf{S} \sim U_{\mathbb{S}^{n-1}}$, namely ([Steerneman and van Perlo-ten Kleij, 2005, Eqn. \(2\)](#))

$$\Omega_n(\mathbf{u}^\top \mathbf{u}) = \mathbb{E} \left(e^{i\mathbf{u}^\top \mathbf{S}} \right) = \frac{\Gamma(m/2)}{\sqrt{\pi} \Gamma((m-1)/2)} \int_{-1}^1 e^{i\mathbf{u}^\top \mathbf{u} t} (1-t^2)^{(m-3)/2} dt, \quad \mathbf{u} \in \mathbb{R}^n. \quad \square \quad (54)$$

McNeil et al. (2015, Definition 6.25, page 200) $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_n)$ is said to have an elliptical distribution in \mathbb{R}^n with parameters $\boldsymbol{\mu}$, Σ , ψ , where Σ is an $n \times n$ square semi-positive definite matrix, if

$$\mathbb{E} \left(e^{i\mathbf{u}^\top (\mathbf{X} - \boldsymbol{\mu})} \right) = \psi(\mathbf{u}^\top \Sigma \mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^n. \quad (55)$$

We then write $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ ¹⁰. \square

McNeil et al. (2015, Proposition 6.27, page 200) $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ if and only if there exist \mathbf{S} , \mathcal{R} and \mathbf{A} satisfying

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{S},$$

where $\mathbf{S} \sim U_{\mathbb{S}^{k-1}}$, \mathcal{R} is a radial r.v. independent of \mathbf{S} , and \mathbf{A} in $\mathbb{R}^{n \times k}$ satisfies $\mathbf{A}\mathbf{A}^\top = \Sigma$. \square

¹⁰see [Fang, Kotz, and Ng \(1990\)](#) and [McNeil, Frey, and Embrechts \(2015\)](#).

Remark 10. As outlined in [McNeil, Frey, and Embrechts \(2015, Eqn. \(6.41\), page 201\)](#)), for Σ positive definite,

$$\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi) \iff \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim S_n(\psi). \quad (56)$$

Following [McNeil, Frey, and Embrechts \(2015, Eqn. \(6.42\), page 201\)](#)), for an elliptical variate $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$, if Σ has full rank n , then, by [McNeil et al. \(2015, Corollary 6.22, page 198\)](#) and (56),

$$\left(\sqrt{(\mathbf{X} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu})}, \frac{\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})}{\sqrt{(\mathbf{X} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu})}} \right) \stackrel{d}{=} (\mathcal{R}, \mathbf{S}). \quad \square \quad (57)$$

[Block and Sampson \(1988, Corollary 2.3\)](#) Let $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a supermodular, bounded and right-continuous function. Then $\mathbb{E}(h(\mathbf{X}))$ is nondecreasing in the off-diagonal elements of Σ . \square

The extension of [Block and Sampson \(1988, Corollary 2.3\)](#) to all supermodular functions follows from [Müller and Scarsini \(2000, Theorem 3.3 and Theorem 3.4\)](#). Hence, if \mathbf{X}, \mathbf{X}' verify Assumption 1 under some probability measure \mathbb{Q} , then $\mathbf{X} \leq_{sm} \mathbf{X}'$. As a direct consequence outlined in [Shaked and Shanthikumar \(1997\)](#) after their Definition 2.1, “it follows that the family of multivariate normal distributions (more generally, the family of elliptically contoured distributions) is increasing in the supermodular stochastic order as the correlations increase.”

C Risk Measures

The following definition of a risk measure relaxes the definition of a convex risk measure in [Shapiro \(2013, Introduction \(A3\)\)](#) by not requiring the translation-equivariance property.

Definition 7. A risk measure is a function $\rho : \mathfrak{X} \rightarrow \mathbb{R}$ satisfying, for $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}$:

- a) properness: $\rho(\mathcal{X}) > -\infty$ and $\text{dom } \rho := \{\mathcal{X} \in \mathfrak{X}; \rho(\mathcal{X}) < +\infty\} \neq \emptyset$;
- b) law-invariance: if $\mathcal{X} \stackrel{d}{=} \mathcal{Y}$ then $\rho(\mathcal{X}) = \rho(\mathcal{Y})$;
- c) monotonicity: if $\mathcal{X} \leq \mathcal{Y}$ then $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$;
- d) convexity: $\rho(\lambda\mathcal{X} + (1 - \lambda)\mathcal{Y}) \leq \lambda\rho(\mathcal{X}) + (1 - \lambda)\rho(\mathcal{Y})$, $\lambda \in (0, 1)$.

$(-\rho(\cdot))$ is named an acceptability functional¹¹. We denote $\check{\rho} = \rho(\cdot - \mathbb{E}(\cdot))$. If ρ is translation-equivariant i.e. $\rho(\mathcal{X} + c) = \rho(\mathcal{X}) + c$, $c \in \mathbb{R}$ (see [Pflug and Römisch \(2007, Definition 2.2 \(i\), page 29\)](#)), then $\check{\rho}(\cdot)$ is known as a deviation risk functional¹² and $\rho(\cdot)$ a risk capital functional¹³. \square

Definition 8. The expected shortfall (ES)¹⁴ at the confidence level (quantile) $\alpha \in (\frac{1}{2}, 1)$ of a loss $\mathcal{X} \in \mathfrak{X} = L^1(\mathbb{Q})$ is $\text{ES}_\alpha(\mathcal{X}) = (1 - \alpha)^{-1} \left(\mathbb{E}(\mathcal{X} \mathbb{1}_{\{\mathcal{X} \geq \text{VaR}_\alpha(\mathcal{X})\}}) + \text{VaR}_\alpha(\mathcal{X})(\mathbb{Q}(\mathcal{X} < \text{VaR}_\alpha(\mathcal{X})) - \alpha) \right)$, with $\text{VaR}_\alpha(\mathcal{X}) = \inf \{x \in \mathbb{R} : \mathbb{Q}(\mathcal{X} \leq x) > \alpha\}$. \square

¹¹see [Pflug and Römisch \(2007, Definition 2.17, pages 35-36\)](#).

¹²see [Pflug and Römisch \(2007, Definition 2.21, page 37\)](#).

¹³see [Pflug and Römisch \(2007, Definition 2.25, pages 38-39\)](#).

¹⁴see [Acerbi and Tasche \(2002, Definition 2.6\)](#).

ES is a standing example of a translation-equivariant risk measure.

We conclude this appendix by a detailed proof of [Bäuerle and Müller \(2006, Theorem 4.4\)](#) stated in Section A. The proof relies on the definition of stochastic order:

[Bäuerle and Müller \(1998, Definition 2.1a\)](#)) For \mathcal{X} and \mathcal{Y} in \mathfrak{X} with respective c.d.f. $F_{\mathcal{X}}$ and $F_{\mathcal{Y}}$, \mathcal{X} precedes \mathcal{Y} in stochastic order, written $\mathcal{X} \leq_{st} \mathcal{Y}$, if $F_{\mathcal{X}}(x) \geq F_{\mathcal{Y}}(x)$, $x \in \mathbb{R}$. \square

Proof of [Bäuerle and Müller \(2006, Theorem 4.4\)](#) (see Section A). If $\mathcal{X} \leq_{st} \mathcal{Y}$, then, by [Müller and Stoyan \(2002, Theorem 1.5.14, page 22\)](#), there exists a r.v. $\mathcal{Z} \in \mathfrak{X}$ s.t. $\mathcal{X} \leq_{st} \mathcal{Z} \leq_{cx} \mathcal{Y}$, where \leq_{cx} is the convex order. By [Müller and Stoyan \(2002, Theorem 1.2.4, page 3\)](#), there exist r.v. \mathcal{X}' and \mathcal{Z}' on a modified probability space $(\Omega', \mathcal{A}', \mathbb{Q}')$, with same respective laws as \mathcal{X} and \mathcal{Z} , such that $\mathcal{X}' \leq \mathcal{Z}'$ holds with certainty i.e., $\forall \omega \in \Omega'$, $\mathcal{X}'(\omega) \leq \mathcal{Z}'(\omega)$ where \leq is the partial order¹⁵ on \mathbb{R}^m if \mathcal{X} and \mathcal{Z} take value in \mathbb{R}^m . The law-invariance and monotonicity of ρ yield

$$\rho(\mathcal{X}) = \rho(\mathcal{X}') \leq \rho(\mathcal{Z}') = \rho(\mathcal{Z}). \quad (58)$$

From [Bäuerle and Müller \(2006, Theorem 4.3\)](#), which requires the convexity and law-invariance of ρ , we also have $\rho(\mathcal{Z}) \leq \rho(\mathcal{Y})$, hence $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$. \square

¹⁵see [Müller and Stoyan \(2002, Remark 1.2.5, page 3\)](#).

Chapter III

Resolving a Clearing Member's Default, a Radner Equilibrium Approach

Note. The results of this chapter are based on the paper¹ [Bastide, Crépey, Drapeau, and Tadese \(2023b\)](#).

Abstract. For vanilla derivatives that constitute the bulk of investment banks' hedging portfolios, central clearing through central counterparties (CCPs) has become hegemonic. A key mandate of a CCP is to provide an efficient and proper clearing member default resolution procedure. When a clearing member defaults, the CCP can hedge and auction or liquidate its positions. The counterparty credit risk cost of auctioning has been analyzed in terms of XVA metrics in [Bastide, Crépey, Drapeau, and Tadese \(2023a\)](#). In this work we assess the costs of hedging or liquidating. This is done by comparing pre- and post-default market equilibria, using a Radner equilibrium approach for portfolio allocation and price discovery in each case. We show that the Radner equilibria uniquely exist and we provide both analytical and numerical solutions for the latter in elliptically distributed markets. Using such tools, a CCP could decide rationally on which market to hedge and auction or liquidate defaulted portfolios.

§1 Introduction

Financial actors can discover counterparties for their transactions by participating in an exchange. In the case of derivatives, an exchange is backed by a central counterparty (CCP). As surveyed in [Gregory \(2014, Sections 13.3.5, 13.3.6 and 13.4\)](#) and [Menkveld and Vuillemey \(2021\)](#), a CCP transforms counterparty credit risk into liquidity risk by netting and managing multiple flows of collateral, including a default fund that is pooled among the clearing members. A CCP is also responsible for rewiring or liquidating the CCP portfolios of a defaulted clearing member within a few days of the default. As emphasized in Table 1,

¹*Acknowledgements:* We thank Yannick Armenti, head of front office risk Europe derivatives execution and clearing at BNP Paribas securities services, and Mohamed Selmi, head of market and liquidity risk at LCH SA, for useful discussions.

	Pros	Cons
Netting	Multilateral netting benefit	Loss of bilateral netting across asset classes
Counterparty credit risk and liquidity risk	Reduced default risk of the CCP itself and reduced “domino effects” between members	Concentration risk if a major CCP were to default. Joint membership and feedback liquidity issues. - about 30 major CCPs today and only a few prominent ones (CME, LCH, Eurex, ICE,...)
Information	Better information of the regulator via access to the repositories of the CCP	Opacity of the default fund for the clearing members, which are not in a position of estimating their risks and the corresponding costs with accuracy
Costs	Default resolution cheaper: - Bilateral trading means a completely arbitrary transaction network. - <i>An orderly default procedure cannot be done manually. It requires an IT network, whether it is CCPs, block-chain technology,...</i>	High cost of raising funding initial margins (at least, if funded by unsecured borrowing)

Table 1: Pros and cons of CCPs (Gregory, 2014; Menkveld and Vuillemeij, 2021; Albanese, Armenti, and Crépey, 2020).

an ordered resolution of clearing members’ defaults might actually be the most tangible benefit of CCPs. However, a CCP is faced with several possibilities for settling the default of a clearing member. This paper provides quantitative tools to assess and compare the costs of alternative default resolution strategies.

So far, the optimal settlement of the portfolio of a clearing member has been mostly addressed from the financial engineering viewpoint of optimal liquidation schedules accounting for exogenous liquidity constraints Avellaneda and Cont (2013); Vicente, Cerezetti, Faria, Iwashita, and Pereira (2015). In this paper, we look at the default resolution procedure from a less dynamic but more endogenous, economic equilibrium perspective, with offer and demand determined by clearing member strategies, positions, and preferences. Our default resolution market costs arise from the perturbation of the market equilibrium triggered by the default resolution procedure. To estimate these costs, one has to know the values of the traded assets as well as the positions of all market participants, in both the pre- and post-default markets. A classical method for valuing a derivative is to compute the expectation of its discounted payoffs under an equivalent martingale measure. But in a realistic, incomplete market, this strategy may not perform properly, because of the non-uniqueness of a pricing measure. An equilibrium approach may then be more appropriate. As surveyed in Magill and Shafer (1991) or Radner (1982),

the concept of competitive or Walras equilibrium in economics was first stated in deterministic and static settings. [Debreu \(1959\)](#) extends it to a stochastic framework based on a sequence of market future conditions, but with trading taking place only at time zero. [Radner \(1968, 1972\)](#) revisit the approach in a multi-period and stochastic setting, well suited to the pricing of financial assets, financial derivatives in particular, with inherently uncertain future cash flows. We use a Radner equilibrium approach to discover prices and reallocate a defaulted derivative portfolio among trading participants, under two (main) possible specifications: either the CCP hedges the portfolio by trading on an exchange, or it liquidates the portfolio on an exchange. The ensuing costs are determined by comparing pre- and post-default Radner equilibria. This is done in a one-period specification of the discrete-time Radner equilibrium approach of [Cheridito, Horst, Kupper, and Pirvu \(2015\)](#). Though the application of Radner equilibrium in static and dynamic problems is not new, using it to estimate CCP close-out costs appears to be a novel approach.

On top of hedging and liquidation, there is a third vertex to the triangle of the possible close-out procedures, namely auctions. Hedging and liquidation are driven by equilibria between the agents, whereby the market decides endogenously the optimal prices and asset allocation. Auctioning, instead, is of a game theoretical nature. Actual default resolution procedures are in fact a combination of the three, with implications not only in terms of price impact liquidity risk as emphasized in the above, but also of counterparty credit risk, the way addressed in the last section of the paper.

Outline

Section §2 provides our Radner equilibrium market model. Section §3 introduces the related comparative statics approach for the analysis of the market costs of hedging or liquidating a defaulted clearing member portfolio, either on the exchange of the CCP of the defaulter, or on an external exchange. Sections §4 and §5 detail these costs in the case of entropic and expected shortfall risk measures. Section §6 analyzes the additional impact of counterparty credit risk, based on XVA specifications detailed in Section A.

Standing notation

Given vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ (understood as column matrices), x^\top is the transpose of x and (x, y) is the vector of \mathbb{R}^{m+n} formed by stacking x above y . We denote by $\mathcal{N}_n(\mu, \Gamma)$, the n -variate Gaussian distribution with mean μ and covariance matrix Γ , and by ϕ and Φ , the standard univariate Gaussian probability density and cumulative density functions; by $\mathcal{E}_n(\mu, \Gamma, \psi)$, the n -variate elliptical distribution with mean μ , covariance matrix Γ , and characteristic generator function ψ , by $\mathcal{T}_n(\mu, \Gamma, \nu)$, the n -variate Student t -distribution of degree of freedom ν with mean μ and covariance matrix Γ , and by t_ν and T_ν , the standard univariate Student t probability density and cumulative density functions of degree ν . Throughout the paper, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes a fixed probability space, with expectation, variance and covariance operators \mathbb{E} , $\mathbb{V}\text{ar}$ and Cov ; L^0 and L^1 respectively denote the space of all the measurable and integrable random variables (identified in the \mathbb{P} almost sure sense), \mathfrak{X} is a linear subspace of L^1 containing the constants. Capital letters denote random vectors with components in \mathfrak{X} . Bold letters refer to the solution of a Radner equilibrium.

For a function $f: \mathbb{R}^m \rightarrow \mathbb{R}$, its directional derivative at x in the direction of y is defined as

$$\mathcal{D}_y f(x) = \lim_{\epsilon \searrow 0} \frac{f(x + \epsilon y) - f(x)}{\epsilon}; \quad (1)$$

for f convex, a point $y \in \mathbb{R}^m$ is said to be a subgradient of f at x , denoted $y \in \partial f(x)$, if

$$f(z) \geq f(x) + y^\top(z - x), \quad z \in \mathbb{R}^m; \quad (2)$$

the convex conjugate f^* of f is defined as

$$f^*(y) = \sup \{y^\top x - f(x); x \in \mathbb{R}^m\}, \quad y \in \mathbb{R}^m. \quad (3)$$

Let $f_i: \mathbb{R}^m \rightarrow \mathbb{R}$ be convex functions, for i in a finite set E . The inf-convolution f of the f_i is defined as

$$f(x) = \inf \left\{ \sum_{i \in E} f_i(x_i); \sum_{i \in E} x_i = x \right\}, \quad x \in \mathbb{R}^m. \quad (4)$$

§2 Radner Equilibrium Market Model

We consider a one-period model of a financial exchange, with trading participants indexed by a finite set E . A unitary position in each of the assets traded on the exchange pays a vector of random payoffs $P \in \mathbb{R}^m$ (with $m \geq 1$) at the terminal time T . We assume that each participant i in E is endowed with a real valued random receivable R_i (i.e. $R_i \geq 0$ means a cash flow promised to the participant i). For each $i \in E$, the participant i hedges² R_i by entering a portfolio $\mathbf{q}_i \in \mathbb{R}^m$ of traded assets. The corresponding market loss of member i is

$$-R_i + \mathbf{q}_i^\top(\mathbf{p} - P), \quad (5)$$

where $\mathbf{p} \in \mathbb{R}^m$ is the vector of prices of the traded assets at initial time: see Table 2, where

$$\Gamma_i = \begin{bmatrix} \text{Var}(R_i) & \text{cov}_i \\ \text{cov}_i & \Gamma \end{bmatrix}. \quad (6)$$

²unhedged market risk can generate significant regulatory capital requirements, rendering the contractual commitments non viable for the trading participant ([Bank For International Settlements, 2019b](#), Section MAR23, pp. 64 and 93).

\mathbf{p}	equilibrium prices of the traded (hedging) assets
\mathbf{q}_i	equilibrium positions of participant i in the traded (hedging) assets
μ	the vector $\mathbb{E}[P]$
Γ	the matrix $\mathbb{Cov}(P)$
R_i	receivable to be hedged by the participant $i \in E$
μ_i	the vector $\mathbb{E}[(R_i, P)]$
cov_i	the vector $\mathbb{Cov}(R_i, P) = (\mathbb{Cov}(R_i, P_1), \dots, \mathbb{Cov}(R_i, P_m))$
cov	$\sum_{i \in E} \text{cov}_i$
Γ_i	the covariance matrix $\mathbb{Cov}((R_i, P))$, i.e. of the vector (R_i, P)
ρ_i	risk measure of the participant i
ϱ_i	risk-aversion parameter of an entropic risk measure of the participant i
ϱ	the number $(\sum_{i \in E} (1/\varrho_i))^{-1}$
α_i	confidence level for an expected shortfall risk measure of the participant i

Table 2: Main notation relative to an exchange E .

For the monetary valuation of the risk of the participant i , we consider a law invariant risk measure $\rho_i : \mathfrak{X} \rightarrow \mathbb{R}$ monotonous, convex³, translation equivariant⁴ in the sense that⁵, and normalized ($\rho_i(0) = 0$). We want to determine the portfolios \mathbf{q}_i and the prices \mathbf{p} endogenously as a Radner equilibrium driven by the offer and demand of all the participants to the exchange:

Definition 1. A matrix of positions $(\mathbf{q}_i)_{i \in E}$ and a price vector $\mathbf{p} \in \mathbb{R}^m$ form a Radner equilibrium on E if

- (optimality condition relative to each market participant $i \in E$)

$$\rho_i(-R_i + \mathbf{q}_i^\top (\mathbf{p} - P)) \leq \rho_i(-R_i + q_i^\top (\mathbf{p} - P)), q_i \in \mathbb{R}^m, \quad (7)$$

- (zero clearing condition)

$$\sum_{i \in E} \mathbf{q}_i = 0. \quad (8)$$

□

Remark 1. Since R_i is assumed to be exogenously given, the price of the corresponding receivable to the participant i is not part of the equilibrium. We say nothing on this price (assumed exogenously given and in fact implicitly part of R_i itself in our setup) in the paper, nor on the way it could be impacted (in our setup it is simply not) by the instant default of a participant to the exchange. □

A Generic Results

For each trading participant i in E , we consider the convex function $r_i : \mathbb{R}^m \rightarrow \mathbb{R}$ defined as

$$r_i(q_i) = \rho_i(-R_i - q_i^\top P), q_i \in \mathbb{R}^m.$$

³in the sense of [McNeil, Frey, and Embrechts \(2015, Definition 2.24, p. 74\)](#).

⁴also known as translation invariance.

⁵see [Pflug and Römisch \(2007, Definition 2.2 \(i\), page 29\)](#) $\rho_i(L + m) = \rho_i(L) + m$ holds for any L in \mathfrak{X} and real constant m .

By translation equivariance of ρ_i , the member i optimality condition (8) can be rewritten as

$$r_i(q_i) \geq r_i(\mathbf{q}_i) + (-\mathbf{p})^\top (q_i - \mathbf{q}_i), \quad q_i \in \mathbb{R}^m,$$

i.e., by (2),

$$-\mathbf{p} \in \partial r_i(\mathbf{q}_i). \quad (9)$$

By [Rockafellar \(1970, Theorem 23.5\)](#), this is in turn equivalent to

$$r_i(\mathbf{q}_i) = -\mathbf{q}_i^\top \mathbf{p} - r_i^*(-\mathbf{p}), \quad (10)$$

where r_i^* is the convex conjugate (3) of r_i . Note that

$$-\mathbf{q}_i^\top p - r_i^*(-p) \leq r_i(q_i), \quad q_i, p \in \mathbb{R}^m, \quad (11)$$

$$\text{with equality if and only if } -p \in \partial r_i(q_i), \quad (12)$$

by [Rockafellar \(1970, Theorem 23.5\)](#).

Lemma 1 and Theorems 2-3 below are variants, for a single period model but with unbounded (R_i, P) (as we want to endorse elliptical factor models later in the paper), of [Cheridito, Horst, Kupper, and Pirvu \(2015, Theorems 1 and 2\)](#). Radner equilibria admit the following dual characterization in terms of the inf-convolution (4) r of the r_i .

Lemma 1. *A matrix of positions $(\mathbf{q}_i)_{i \in E}$ and a price vector $\mathbf{p} \in \mathbb{R}^m$ form a Radner equilibrium on E if and only if*

(i) $-\mathbf{p} \in \partial r(0)$,

(ii) $r(0) = \sum_{i \in E} r_i(\mathbf{q}_i)$, and

(iii) $\sum_{i \in E} \mathbf{q}_i = 0$.

Proof. Let $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$ be a Radner equilibrium as per Definition 1. The zero clearing condition (8) yields (iii). By [Rockafellar \(1970, Theorem 16.4, page 145\)](#), the convex conjugate of the inf-convolution of proper convex functions is the sum of the corresponding conjugates, i.e.

$$r^*(-\mathbf{p}) = \sum_{i \in E} r_i^*(-\mathbf{p}).$$

Summing the expression (10) across all r_i and using (iii) gives

$$\sum_{i \in E} r_i(\mathbf{q}_i) = 0 - r^*(-\mathbf{p}) \leq r(0),$$

where the inequality holds by definition (3) of the convex conjugate of r . By definition of $r(0)$, the above inequality becomes equality, i.e.

$$r(0) = -r^*(-\mathbf{p}) = \sum_{i \in E} r_i(\mathbf{q}_i).$$

Hence (ii) holds and so does also (i), in view of the equivalence between (9) and (10), here applied to r (instead of r_i there).

Conversely, suppose that $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$ satisfies (i)–(iii). (iii) is the zero clearing condition (8), whereas (i) implies via (10) applied to r that

$$r(0) = -r^*(-\mathbf{p}) = \sum_{i \in E} (-\mathbf{q}_i^\top \mathbf{p} - r_i^*(-\mathbf{p})). \quad (13)$$

By (11) and (13), $-\mathbf{p} \notin \partial r_i(\mathbf{q}_i)$ for some $i \in E$ would imply that $r(0) < \sum_{i \in E} r_i(\mathbf{q}_i)$, contradicting (ii). Hence (9), which is equivalent to the member i optimality condition (8), holds for each $i \in E$. \square

Since the subgradient of a real valued convex function is non-empty, Lemma 1 implies that a Radner equilibrium exists if and only if the inf-convolution r is attained at 0. It also implies that, whenever a Radner equilibrium exists, the optimal price is unique if and only if r is differentiable at 0.

Theorem 2. *If ρ_i is sensitive to large losses, i.e.⁶ $\lim_{\lambda \rightarrow \infty} \rho_i(\lambda L) = \infty$ for all $L \in \mathfrak{X}$ such that $\mathbb{P}[L > 0] > 0$, i in E , then there exists a Radner equilibrium on E .*

Proof. Let Q be the set of vector of positions $(q_i)_{i \in E}$ satisfying the zero clearing condition, i.e.

$$Q = \left\{ q \in \mathbb{R}^{m|E|} : q^\top b^k = 0, k = 1, \dots, m \right\},$$

where b^k is a vector in $\mathbb{R}^{m|E|}$ such that, for all $j = 0, \dots, |E| - 1$, the $k + jm$ entries of b^k equal 1 and all the other entries of b^k are 0. Note that Q is a non-empty closed convex polyhedral subset of $\mathbb{R}^{m|E|}$. Let $\mu = \mathbb{E}[P]$. By Dalang, Morton, and Willinger (1990, Theorem 3.3), $q_i^\top (\mu - P) = 0$ almost surely holds or $\mathbb{P}[q_i^\top (\mu - P) > 0] > 0$ holds, for any q_i in \mathbb{R}^m . The closed proper convex function

$$\mathbb{R}^{m|E|} \ni q = (q_1, \dots, q_{|E|}) \xrightarrow{\beta} \sum_{i \in E} \rho_i(-R_i + q_i^\top (\mu - P)) \in \mathbb{R} \quad (14)$$

is such that

$$\inf_{q \in Q} \beta(q) = \inf_{\sum_{i \in E} q_i = 0} \sum_{i \in E} r_i(q_i).$$

In view of the comment preceding the statement of the theorem, it suffices to show that β attains its minimum on Q . Let $B = \{q \in \mathbb{R}^{m|E|} : \beta(q) \leq \beta(0)\}$, with recession cone $0^+B = \{y \in \mathbb{R}^{m|E|} : b + \lambda y \in B, \forall \lambda \geq 0, \forall b \in B\}$. Let $0^+\beta = \{y \in \mathbb{R}^{m|E|} : \beta 0^+(y) \leq 0\}$ denote the recession cone of β , where $\beta 0^+$ is its recession function⁷. By Rockafellar (1970, Theorem 8.7, page 70), $0^+B = 0^+\beta$. Since B is a closed convex set containing the origin, Rockafellar (1970, Corollary 8.3.2, page 64) yields

$$0^+B = \{y \in \mathbb{R}^{m|E|} : \lambda y \in B \quad \forall \lambda > 0\}. \quad (15)$$

Let $y = (y_1, \dots, y_{|E|}) \in \mathbb{R}^{m|E|} \setminus \{0\}$ (where each y_i is in \mathbb{R}^m). (i) If $\mathbb{P}[y_i^\top (\mu - P) > 0] > 0$ holds for some $i \in E$, then, by the sensitivity to large losses condition on ρ_i , $\beta(\lambda y)$ goes to infinity as λ goes to infinity, which implies that $y \notin 0^+B$. (ii) If, instead, $y_i^\top (\mu - P) = 0$ holds for all $i \in E$, then $\pm y \in 0^+B$ hold by definitions (14) of β , B and (15) of 0^+B . In particular, for any $y \in \mathbb{R}^{m|E|} \setminus \{0\}$ such that $y \in 0^+\beta$, we have $-y \in 0^+\beta$. Hence, by Rockafellar (1970, Corollary 8.6.1, page 69), every direction of recession is a direction in which β is constant. Following Rockafellar (1970, Theorem 27.3, page 267), in either case (i) or (ii), β attains its minimum over Q . \square

Remark 2. Let \mathfrak{X} be given as the Orlicz heart corresponding to the Young function $\theta: [0, \infty) \rightarrow [0, \infty)$ given by $\theta(t) = \exp(t - 1) - \exp(-1)$, i.e.

$$\mathfrak{X} = \{L \in L^0 : \mathbb{E}[\theta(c|L|)] < \infty, c > 0\} \subseteq L^1,$$

⁶see Cheridito et al. (2015, Section 2.3).

⁷the recession function $\beta 0^+$ is the map defined on $\mathbb{R}^{m|E|}$ as $\beta 0^+(y) = \inf \{m \in \mathbb{R} : (y, m) \in 0^+ \text{epi } \beta\}$, where $\text{epi } \beta = \{(q, n) \in \mathbb{R}^{m|E|} \times \mathbb{R} : n = g(q)\}$.

where L^0 is the space of all real valued measurable random variables. An entropic risk measure of the form, for some $\varrho_i > 0$,

$$\rho_i(L) = \frac{1}{\varrho_i} \ln(\mathbb{E}[\exp(\varrho_i L)]), \quad L \in \mathfrak{X}, \quad (16)$$

is sensitive to large losses (Cheridito et al., 2015, Section 2.3). \square

Regarding the uniqueness of an optimal solution:

Theorem 3. *Let $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$ be a Radner equilibrium on E .*

(i) If r_i is differentiable at \mathbf{q}_i for some i in E , then \mathbf{p} is unique.

(ii) For any $i \in E$, if r_i is differentiable and strictly convex on \mathbb{R}^m then \mathbf{q}_i is unique.

Proof. Let $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$ be optimal. If r_i is differentiable at \mathbf{q}_i , then Lemma 1(i) together with Rockafellar (1970, Theorems 23.2, p. 216 and 25.2, page 244) yield

$$\mathcal{D}_x r_i(\mathbf{q}_i) = x^\top(-\mathbf{p}) \leq \mathcal{D}_x r(0), \quad (17)$$

where $\mathcal{D}_x r_i(\mathbf{q}_i)$ is the directional derivative (1) of r_i at \mathbf{q}_i along x . Take $\hat{q}_i = \mathbf{q}_i + \epsilon x$ and $\hat{q}_j = \mathbf{q}_j$ for all $j \neq i$. By definition (4) of the inf-convolution,

$$r(\epsilon x) \leq \sum_{j \in E} r_j(\hat{q}_j) = \sum_{j \neq i} r_j(\mathbf{q}_j) + r_i(\mathbf{q}_i + \epsilon x).$$

This together with Lemma 1(ii) yields

$$\mathcal{D}_x r(0) = \lim_{\epsilon \searrow 0} \frac{r(\epsilon x) - r(0)}{\epsilon} \leq \lim_{\epsilon \searrow 0} \frac{\sum_{j \in E} r_j(\hat{q}_j) - r(0)}{\epsilon} = \mathcal{D}_x r_i(\mathbf{q}_i) \leq \mathcal{D}_x r(0),$$

where the second inequality is due to (17). Hence $x \mapsto \mathcal{D}_x r(0)$ is linear. Thus, by Rockafellar (1970, Theorem 25.2, page 244), r is differentiable at 0, i.e. $\partial r(0)$ is a singleton, which, in view of Lemma (1) (ii) implies (i). As for (ii), if r_i is a strictly convex and differentiable on \mathbb{R}^m , then it is closed and proper. Following Rockafellar (1970, Corollary 26.3.1, page 254), $\partial r_i^*(-\mathbf{p})$ is a singleton and $\mathbf{q}_i = \nabla r_i^*(-\mathbf{p})$ is unique, by (10). \square

Remark 3. Let $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$ be optimal. Following Cheridito et al. (2015, Proposition 2), if a risk measure ρ_i is differentiable⁸ at $-R_i - \mathbf{q}_i^\top P$ for some i in E , then r_i is differentiable at \mathbf{q}_i , hence the optimal price \mathbf{p} is unique. \square

B Results Specific to Entropic or Expected Shortfall Risk Measures

With explicit solutions and regulatory standards in view, from now on, ρ_i is either an entropic or an expected shortfall risk measure. In elliptical markets, entropic or coherent⁹ risk measures lead to analytical expressions for equilibria. We first consider the case of entropic ρ_i and normally distributed (R_i, P) .

⁸ ρ_i is differentiable at $L \in \mathfrak{X}$ if there exist a random variable $W \in \mathfrak{X}^*$ (the dual space of \mathfrak{X}) such that (cf. (1))

$$\lim_{\epsilon \searrow 0} \frac{\rho_i(L + \epsilon Y) - \rho_i(L)}{\epsilon} = \mathbb{E}[YW] \quad Y \in \mathfrak{X}.$$

In this case, we write $W = \nabla \rho_i(L)$.

⁹ e.g. expected shortfall.

Proposition 4. Let $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$, i in E , and Γ^{10} be invertible. If $\rho_i(L) = \frac{1}{\varrho_i} \ln(\mathbb{E}[\exp(\varrho_i L)])$ for some $\varrho_i > 0$, i in E , then

$$\mathbf{q}_i = \Gamma^{-1} \left(\frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right), \quad i \in E, \quad \text{and} \quad \mathbf{p} = \mu - \varrho \text{cov}, \quad (18)$$

where $\varrho = \left(\sum_{i \in E} \frac{1}{\varrho_i} \right)^{-1}$ and $\text{cov} = \sum_{i \in E} \text{cov}_i$, is a unique Radner equilibrium.

Proof. By Theorem 2 and Remark 2, there exists a Radner equilibrium. In view of (6),

$$-R_i - \mathbf{q}_i^\top P \sim \mathcal{N}_1 \left(-\mathbb{E}[R_i] - \mathbf{q}_i^\top \mu, \text{Var}(R_i) + 2\mathbf{q}_i^\top \text{cov}_i + \mathbf{q}_i^\top \Gamma \mathbf{q}_i \right). \quad (19)$$

The moment generating function of a standard normal variate L is $\mathbb{R} \ni z \mapsto \mathbb{E}[\exp(zL)] = \exp(z\mathbb{E}[L] + \text{Var}(L)z^2/2)$, hence $\rho_i(L) = \mathbb{E}[L] + \varrho_i \text{Var}(L)/2$. This and (19) yield

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \frac{\varrho_i \text{Var}(R_i)}{2} + \varrho_i q_i^\top \text{cov}_i + \frac{1}{2} \varrho_i q_i^\top \Gamma q_i \quad (20)$$

and

$$\nabla r_i(q_i) = -\mu + \varrho_i \text{cov}_i + \varrho_i \Gamma q_i, \quad i \in E. \quad (21)$$

The optimality condition relative to the participant $i \in E$ yields

$$-\mathbf{p} = \nabla r_i(\mathbf{q}_i) = -\mu + \varrho_i \text{cov}_i + \varrho_i \Gamma \mathbf{q}_i,$$

hence

$$\mathbf{q}_i = \Gamma^{-1} \left(\frac{1}{\varrho_i} (\mu - \mathbf{p}) - \text{cov}_i \right). \quad (22)$$

On the other hand, the clearing condition yields

$$\sum_{i \in E} \mathbf{q}_i = 0 = \Gamma^{-1} \left(\frac{1}{\varrho} (\mu - \mathbf{p}) - \text{cov} \right),$$

which is equivalent to

$$\mathbf{p} = \mu - \varrho \text{cov}.$$

(22) in turn gives (18). \square

We now turn to the case where each ρ_i is an expected shortfall risk measure (McNeil et al., 2015, page 69)

$$\rho_i(L) = \mathbb{E}\mathbb{S}_{\alpha_i}(L) = \frac{1}{1 - \alpha_i} \int_{\alpha_i}^1 q_u(L) du, \quad L \in \mathfrak{X} = L^1, \quad (23)$$

for some $0 \leq \alpha_i < 1$, where $q_u(L)$ is the left u -quantile of L .

Proposition 5. If $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$ and $\rho_i = \mathbb{E}\mathbb{S}_{\alpha_i}$ for some $0 \leq \alpha_i < 1$, $i \in E$, then there exists a Radner equilibrium with a unique equilibrium price. If Γ_i is further positive definite, $i \in E$, then the Radner equilibrium is unique.

Proof. We divide the proof in three steps.

¹⁰see Table 2.

• *Existence:* For a univariate normally distributed L , by [McNeil et al. \(2015, Example 2.14, page 70\)](#), $\mathbb{E}S_\alpha(L) = \mathbb{E}[L] + \sqrt{\text{Var}(L)} \mathbb{E}S_\alpha(Z)$ with $Z \sim \mathcal{N}_1(0, 1)$, $\mathbb{E}S_\alpha(Z) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$. This and (19) yield

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \mathbb{E}S_{\alpha_i}(Z) \sqrt{\text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma_i q_i}, \quad q_i \in \mathbb{R}^m. \quad (24)$$

By [Rockafellar \(1970, Corollary 8.5.2\)](#), the recession function of r_i is given by

$$(r_i 0^+)(y) = \lim_{\lambda \searrow 0} \lambda r_i(y/\lambda) = -y^\top \mu + \sqrt{y^\top \Gamma_i y} \mathbb{E}S_{\alpha_i}(Z).$$

Let $q_1, \dots, q_{|E|}$ be vectors in \mathbb{R}^m such that

$$\sum_{i \in E} (r_i 0^+)(q_i) \leq 0 \quad \text{and} \quad \sum_{i \in E} (r_i 0^+)(-q_i) > 0,$$

i.e.

$$-\mu^\top \left(\sum_{i \in E} q_i \right) < \sum_{i \in E} \sqrt{q_i^\top \Gamma_i q_i} \mathbb{E}S_{\alpha_i}(Z) \leq \mu^\top \left(\sum_{i \in E} q_i \right).$$

Thus $\sum_{i \in E} q_i \neq 0$. By [Rockafellar \(1970, Corollary 9.2.1, page 76\)](#), the inf-convolution of real valued convex functions is a real valued convex function. Hence the inf-convolution r is attained on \mathbb{R}^m and, by Lemma 1, there exists a Radner equilibrium $((\mathbf{q}_i)_{i \in E}, \mathbf{p})$.

• *Unique price:* We know that $-R_i - \mathbf{q}_i^\top P$, $i \in E$, is a continuous random variable. By [Kalkbrenner \(2005, Theorem 4.3 and Section 5.2\)](#), an expected shortfall is differentiable at continuous random variables. Therefore, by Remark 3, the optimal price \mathbf{p} is unique.

• *Unique portfolio:* If Γ_i is positive definite, then,

$$(1, q_i)^\top \Gamma_i (1, q_i) = \text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma_i q_i > 0, \quad q_i \in \mathbb{R}^m.$$

This and (24) implies that r_i is differentiable such that

$$\nabla r_i(q_i) = -\mu + \frac{\mathbb{E}S_{\alpha_i}(Z)}{\sqrt{(1, q_i)^\top \Gamma_i (1, q_i)}} (\text{cov}_i + \Gamma_i q_i), \quad q_i \in \mathbb{R}^m.$$

Following [Rockafellar and Wets \(1998, Theorem 2.14, page 47\)](#), the strict convexity of r_i is equivalent to

$$r_i(y) > r_i(q_i) + \nabla r_i(q_i)^\top (y - q_i), \quad q_i \neq y.$$

A simple computation reduces this first order condition to

$$\sqrt{[(1, q_i)^\top \Gamma_i (1, q_i)] [(1, y)^\top \Gamma_i (1, y)]} > (1, q_i)^\top \Gamma_i (1, y), \quad q_i \neq y. \quad (25)$$

If $y \neq q_i$, then $(1, y)$ is not colinear to $(1, q_i)$. Hence, by [Johnson and Wichern \(2007, Eqn. \(2.49\), page 79\)](#) applied with $\mathbf{b} = (1, q_i)$, $\mathbf{d} = \Gamma_i (1, y)$, and $\mathbf{B} = \Gamma_i$ (hence $(1, q_i) = \mathbf{b} \neq c\mathbf{B}^{-1}\mathbf{d} = c(1, y)$ for any constant c),

$$[(1, q_i)^\top \Gamma_i (1, q_i)] [(1, y)^\top \Gamma_i (1, y)] > [(1, q_i)^\top \Gamma_i (1, y)]^2 \quad \text{holds for any } q_i \neq y.$$

This in turn yields (25). Hence by Theorem 3 there exists a unique equilibrium. \square

Remark 4. If $\text{Var}(R_i) > 0$, $\text{cov}_i = 0$, and Γ is invertible, then $z^\top \Gamma_i z = z_1^2 \text{Var}(R_i) + 2z_1 \hat{z}^\top \text{cov}_i + \hat{z}^\top \Gamma \hat{z} > 0$ holds for any $z = (z_1, \dots, z_{m+1}) \in \mathbb{R}^{m+1} \setminus \{0\}$, where $\hat{z} = (z_2, \dots, z_{m+1})$. Hence Γ_i is positive definite as assumed in the last part of Proposition 5.

Instead, the positive definiteness of Γ_i is not guaranteed when $\text{Var}(R_i) = \text{cov}_i^\top \Gamma^{-1} \text{cov}_i$, because $z^\top \Gamma_i z = 0$ for $z = (-1, \Gamma^{-1} \text{cov}_i)$. This is for instance the case when R_i is in the span of P , i.e. $R_i = a_i^\top P + b_i$ for some constants $a_i \in \mathbb{R}^m$ and $b_i \in \mathbb{R}$, whence $\text{cov}_i = \Gamma a_i$ and $\text{Var}(R_i) = a_i^\top \Gamma a_i = \text{cov}_i^\top \Gamma^{-1} \text{cov}_i$. \square

Remark 5. An n -variate random vector L has an elliptical distribution written as $L \sim \mathcal{E}_n(\mu, \Gamma, \psi)$ if its characteristic function is expressed as

$$\mathbb{E}\left[e^{i z^\top L}\right] = \exp(i z^\top \mu) \psi\left(\frac{1}{2} z^\top \Gamma z\right), \quad z \in \mathbb{R}^n,$$

for $\mu = \mathbb{E}[L]$, $\Gamma = \text{Cov}(L)$, and a function $\psi: [0, \infty) \rightarrow \mathbb{R}$ such that $-\psi'(0) = -1$. As is well known (Landsman and Valdez, 2003; McNeil, Frey, and Embrechts, 2015), if $L \sim \mathcal{E}_n(\mu, \Gamma, \psi)$ (or, more specifically¹¹, $\mathcal{T}_n(\mu, \Gamma, \nu)$), then $a^\top L \stackrel{d}{=} a^\top \mu + \sqrt{a^\top \Gamma a} Z$, where $Z \sim \mathcal{E}_1(0, 1, \psi)$ (specifically, $\mathcal{T}_1(0, 1, \nu)$). Hence, for any coherent risk measure ρ ,

$$\rho(a^\top L) = a^\top \mu + \rho(Z) \sqrt{a^\top \Gamma a}. \quad (26)$$

Assuming $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$ (e.g. $\mathcal{T}_{m+1}(\mu_i, \Gamma_i, \nu)$) with Γ_i positive definite, $i \in E$, the above implies that

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \rho_i(Z) \sqrt{\text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma q_i}. \quad (27)$$

The proof of Proposition 5 thus works for any law invariant and coherent risk measure ρ_i differentiable on the linear space spanned by the components of (R_i, P) , i in E . \square

§3 The Comparative Statics Approach for Default Resolution Analysis

When a clearing member of a CCP defaults, its position is taken over by the CCP. The CCP should then close the defaulter's positions in a way that does not harm the other members or the CCP itself. As outlined in (Bank For International Settlements, 2020), the CCP can settle the defaulter's positions via an auction organized by the CCP between the surviving members (and sometimes invited participants). According to (Bank For International Settlements, 2020, page 7), the chance of a successful auction is increased by hedging the defaulted portfolio's risks prior to the auction:

A CCP should establish a framework for its approach to hedging risks from a defaulted participant's portfolio prior to a default management auction to increase the chance of a successful auction. [...] The goals of a CCP's hedging strategy are generally to minimise the CCP's exposure to the defaulted participant's portfolio and to decrease the overall risk that the portfolio may pose to the CCP and the auction participants. Portfolios with less risk exposure lessen the potential effects of market volatility on the portfolio [...] and time dependency of valuations by auction participants.

¹¹ A Student t -distribution is elliptical (Gaunt, 2021).

A close-out procedure can also involve some liquidation on open markets. As different positions are liquidated separately, hedging prior to liquidation would entail additional costs for liquidating the hedging side of the portfolio. The main default resolution strategies are thus liquidation versus hedging then auctioning. As pointed out in [Oleschak \(2019\)](#),

in cases where the position to be transferred is large in relation to market liquidity or where a central market does not exist, auctions with the surviving agents as bidders is the mechanism of choice.

In any case, the CCP deals with the losses incurred throughout the close-out period by using the collateral of the defaulter, its own resources (skin in the game), and financial resources pooled between the clearing members in the form of a default fund ([Gregory, 2014](#); [Biais et al., 2016](#); [Oleschak, 2019](#)). The CCP should assess the adequacy of these financial resources by a careful estimation of the close-out costs of the defaulters' positions, which is the focus of this paper.

Let an index d represent a clearing member of a CCP defaulting instantaneously at time 0^{12} . We want to analyze and compare different close-out procedures, of the 'hedging or not and liquidation or auctioning' types, for the CCP portfolio of member d . These strategies can be implemented on several possible exchanges E , starting with the one of the CCP itself, denoted hereafter by D . For each (pre-default) exchange E , we denote by E' its advent in the wake of the instant default of d , depending on the settlement procedure implemented by the CCP. We assume that the different exchanges E trade the same assets with terminal payoff P , possibly at different initial prices \mathbf{p} (interpreted in this setup as "time $0-$ ", pre-default prices), reflecting different market equilibria. Hedging procedures involve the CCP itself in the form of a new trading participant, represented for this purpose by a new index c (not involved in any exchange E). We use similar notation for d and c as for participants i of E in Section §2 (see Table 2). Although other choices could be used without methodological change in what follows, fault of suitable calibration data in this regard, we assume that any data other than \mathbf{p} and \mathbf{q}_i in Table 2 are not affected by the instant default of d —with the only exception reflected in (35) below of R_c , which represents the post-default receivable¹³ of the hedging CCP due to the portfolio of the defaulted member d taken over by the hedging CCP, whereas the pre-default receivable of the CCP is zero (a CCP should not bear any positions, except for the ones inherited from defaulted market participants during the close-out period of their portfolios).

The pair $((\mathbf{q}'_i)_{i \in E'}, \mathbf{p}')$ relative to any post-default exchange E' involved in the settlement of the defaulted portfolio is derived using a Radner equilibrium in E' . Note that all the receivables and equilibrium portfolios and prices implicitly depend on the corresponding exchange. Regarding prices, we make this dependence explicit hereafter, denoting by \mathbf{p}^E a pre-default ("time $0-$ ") equilibrium price on E and by $\mathbf{p}'^{E'}$ a post-default ("time 0 ") equilibrium price on E' .

A Price Impact

We define $\mathbf{q}_i = 0, i \in E' \setminus E$, and $\Delta \mathbf{q}_i = \mathbf{q}'_i - \mathbf{q}_i, i \in E' \cup E$, hence

$$\sum_{i \in E' \cap E} \mathbf{q}_i + \sum_{i \in E'} \Delta \mathbf{q}_i = \sum_{i \in E'} \mathbf{q}_i + \sum_{E'} \Delta \mathbf{q}_i = \sum_{i \in E'} \mathbf{q}'_i. \quad (28)$$

¹²Considering several instant defaulters would mainly mean replacing \mathbf{q}_d by $\sum_d \mathbf{q}_d$ hereafter, see e.g. Remark 14. We refrain from doing so for parsimony of notation.

¹³see (42) below.

If the CCP chooses to liquidate a portion \mathbf{q}_d^l of \mathbf{q}_d and hedge the remaining $\mathbf{q}_d^h = \mathbf{q}_d - \mathbf{q}_d^l$, then the incremental positions of the participants to any post-default exchange E' can be split as $\Delta\mathbf{q}_i = \Delta\mathbf{q}_i^l + \Delta\mathbf{q}_i^h$, where $\Delta\mathbf{q}_i^l$ and $\Delta\mathbf{q}_i^h$ are the increments implied by the liquidation and hedging legs of the strategy (see e.g. Sections C-C)–with always in particular

$$\Delta\mathbf{q}_c^l = 0, \quad (29)$$

as a CCP does not take part as a participant to a liquidation. Since the amount demanded should be equal to the amount supplied on both legs of the strategy, we have

$$\sum_{i \in E'} \Delta\mathbf{q}_i^l = \mathbf{q}_d^l \quad \text{and} \quad \sum_{i \in E'} \Delta\mathbf{q}_i^h = 0, \quad \text{hence} \quad \sum_{i \in E'} \Delta\mathbf{q}_i = \mathbf{q}_d^l \quad (30)$$

The first consequence of a default resolution strategy is then a liquidity cost

$$\text{LC} = \sum_E \text{LC}_E, \quad (31)$$

where

$$\text{LC}_E = \sum_{i \in E' \cap E} \mathbf{q}_i^\top (\mathbf{p}^E - \mathbf{p}'^E) + \sum_{i \in E'} (\Delta\mathbf{q}_i^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) \quad (32)$$

$$= \sum_{i \in E'} \underbrace{(\mathbf{q}_i + \Delta\mathbf{q}_i^l)^\top}_{\text{LC}_i} (\mathbf{p}^E - \mathbf{p}'^E) \quad (33)$$

(as $\mathbf{q}_i = 0, i \in E' \setminus E$) corresponds to margin payments (like in futures markets) by market participants at time 0 in response to the default settlement procedure of d , i.e. the price they have to pay for the transition from the pre-default to the post-default exchanges.

As reflected in (32), it is only the contracts $\Delta\mathbf{q}_i^l$ involved in the liquidation leg of the strategy, which are old (“time 0–”) contracts with the pre-default prices \mathbf{p}^E , that deserve margin payments, while the new (“time 0”) contracts $\Delta\mathbf{q}_i^h$ involved in the hedging leg of the strategy are post-default contracts with the new prices \mathbf{p}'^E . However, the following reformulation of LC_E in terms of the $\mathbf{q}_i + \Delta\mathbf{q}_i = \mathbf{q}_i'$ (instead of the $\mathbf{q}_i + \Delta\mathbf{q}_i^l$ natively in (32)) is possible:

Lemma 6. *On each exchange E ,*

$$\text{LC}_E = \sum_{i \in E' \cap E} \mathbf{q}_i^\top (\mathbf{p}^E - \mathbf{p}'^E) + \sum_{i \in E'} \Delta\mathbf{q}_i^\top (\mathbf{p}^E - \mathbf{p}'^E) = \sum_{i \in E'} (\mathbf{q}_i')^\top (\mathbf{p}^E - \mathbf{p}'^E). \quad (34)$$

Proof. By (30), $\sum_{i \in E'} \Delta\mathbf{q}_i^h = 0$. Hence (32) yields

$$\text{LC}_E = \sum_{i \in E'} (\mathbf{q}_i + \Delta\mathbf{q}_i^l + \Delta\mathbf{q}_i^h)^\top (\mathbf{p}^E - \mathbf{p}'^E),$$

where $\Delta\mathbf{q}_i^l + \Delta\mathbf{q}_i^h = \Delta\mathbf{q}_i$. \square

B Funds Transfer Price

Let

$$\Delta\rho_i = \rho_i(-R_i + (\mathbf{q}_i')^\top (\mathbf{p}'^E - P)) - \rho_i(-\mathbf{1}_{i \neq c} R_i + \mathbf{q}_i^\top (\mathbf{p}^E - P)), \quad i \in E'. \quad (35)$$

Using (28) and $\Delta \mathbf{q}_i^l + \Delta \mathbf{q}_i^h = \Delta \mathbf{q}_i$, the post-default market loss of any trading participant $i \in E'$ is

$$\begin{aligned}
& \underbrace{-R_i + \mathbf{q}_i^\top (\mathbf{p}^E - P)}_{\text{pre-default market loss}} + (\Delta \mathbf{q}_i^h)^\top (\mathbf{p}'^E - P) + (\Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - P) \\
&= -R_i + \mathbf{q}_i^\top (\mathbf{p}^E - P) + \Delta \mathbf{q}_i^\top (\mathbf{p}'^E - P) + (\Delta \mathbf{q}_i^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) \\
&= -R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P) + \text{LC}_i,
\end{aligned} \tag{36}$$

for LC_i as per (32). Hence, by translation equivariance of ρ_i , the post-default risk of participant i is

$$\rho_i(-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P)) + \text{LC}_i.$$

The risk incremental of participant i is therefore $\text{LC}_i + \Delta \rho_i$, $i \in E'$. Accordingly, we assess the market cost (MC) of a default resolution strategy by the funds transfer price

$$\text{MC} = \sum_E \text{MC}_E, \text{ where } \text{MC}_E = \text{LC}_E + \sum_{i \in E'} \Delta \rho_i. \tag{37}$$

If the CCP of an exchange D faces the default of a clearing member d , then this CCP can envision different default resolution procedures, impacting possibly different exchanges E' (starting with D' itself), for the CCP portfolio \mathbf{q}_d of the defaulter (in a pre-default equilibrium on D). For each considered default resolution strategy, each of the impacted exchanges E' (or their corresponding CCPs) would compute its corresponding MC_E and communicate it to the CCP of D . The ensuing MC (37) of the strategy is the price that the markets would charge to the CCP of D , should the latter choose this strategy for resolving \mathbf{q}_d . The CCP of D would then choose the most efficient strategy, i.e. the one minimizing MC. This approach is inspired by a notion of Pareto optimality in financial markets, where numerous exchanges (and trading participants themselves in Section §6) compete with one another.

C Examples

The pre-default equilibria $((\mathbf{q}_i)_{i \in E}, \mathbf{p}^E)$ involved in (37) are obtained by direct application of the results of Section §2. We now detail the corresponding post-default Radner equilibria $((\mathbf{q}'_i)_{i \in E'}, \mathbf{p}'^E)$ in eight reference cases (without post-default new invited participants other than the CCP itself in the hedging cases, though; extra new invited participants will only be considered later in the paper). The member optimality condition for the post-default market participant $i \in E'$ is always of the form

$$\rho_i(-R_i + (\mathbf{q}'_i)^\top (\mathbf{p}'^E - P)) \leq \rho_i(-R_i + q_i^\top (\mathbf{p}'^E - P)), \quad q_i \in \mathbb{R}^m. \tag{38}$$

The clearing condition, instead, depends on the considered default resolution strategy.

The CCP fully liquidates on its own exchange

As a first default resolution alternative, the CCP may want to liquidate the defaulter's position \mathbf{q}_d on its own exchange D . Then $\text{MC}_E = 0$, $E \neq D$, and $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = \mathbf{q}_d$. As $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i + \mathbf{q}_d = 0$, we obtain a post-default equilibrium clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = 0 \tag{39}$$

and

$$\text{LC} = \text{LC}_D = 0, \quad \text{MC} = \text{MC}_D = \sum_{D' = D \setminus \{d\}} \Delta \rho_i.$$

The CCP fully liquidates on another exchange

If the CCP liquidates \mathbf{q}_d on some exchange $E \neq d$ (hence $E \neq D$), then $MC_E = 0$, $E \neq E, D$, and $\sum_{i \in E'} \Delta \mathbf{q}_i = \mathbf{q}_d$. As $\sum_{i \in E} \mathbf{q}_i = 0$, the ensuing the post-default equilibrium clearing condition on E' is

$$\sum_{i \in E' = E} \mathbf{q}'_i = \mathbf{q}_d. \quad (40)$$

Remark 6. By change of variables $\mathbf{z}'_i = \mathbf{q}'_i - k_i \mathbf{q}_d$ and $R'_i = R_i + k_i \mathbf{q}_d^\top P$, for reals k_i such that $\sum_{i \in E'} k_i = 1$, the clearing condition (40) and the optimality condition (38) relative to the post-default equilibrium $((\mathbf{q}'_i)_{i \in E'}, \mathbf{p}'^E)$ become $\sum_{i \in E'} \mathbf{z}'_i = 0$ and

$$\rho_i(-R'_i + (\mathbf{z}'_i)^\top (\mathbf{p}'^E - P)) \leq \rho_i(-R'_i + z_i^\top (\mathbf{p}'^E - P)), z_i \in \mathbb{R}^m.$$

On E' , we thus recover a zero clearing condition and member optimality conditions formally similar to Definition 1. \square

On the exchange D of the CCP, we have $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = 0$, whence the post-default clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = -\mathbf{q}_d. \quad (41)$$

Therefore

$$MC = MC_E + MC_D,$$

where

$$MC_E = \underbrace{\mathbf{q}_d^\top (\mathbf{p}^E - \mathbf{p}'^E)}_{LC_E} + \sum_{i \in E' = E} \Delta \rho_i, \quad MC_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{LC_D} + \sum_{i \in D' = D \setminus \{d\}} \Delta \rho_i.$$

Remark 7. By change of variable $\mathbf{z}'_i = \Delta \mathbf{q}_i$ and $R'_i = R_i + \mathbf{q}_i^\top P$, $i \in D' = D \setminus \{d\}$, the clearing condition (41) and optimality condition (38) relative to the post-default equilibrium $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$ become $\sum_{i \in D'} \mathbf{z}'_i = 0$ and

$$\rho_i(-R'_i + (\mathbf{z}'_i)^\top (\mathbf{p}'^D - P)) \leq \rho_i(-R'_i + z_i^\top (\mathbf{p}'^D - P)), z_i \in \mathbb{R}^m. \quad \square$$

The CCP fully hedges on its own exchange

If \mathbf{q}_d is not instantaneously liquidated upon the default of member d at time 0, then the CCP c of d endorses at time 0 the receivable

$$R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D), \quad (42)$$

which it can hedge by holding on its own exchange D a portfolio $\Delta \mathbf{q}_c$ minimizing some risk measure ρ_c . The corresponding member optimality condition (38) for the CCP c , playing the role of a new post-default trading participant, is

$$\rho_c(\mathbf{q}_d^\top (\mathbf{p}^D - P) + (\mathbf{q}'_c)^\top (\mathbf{p}'^D - P)) \leq \rho_c(\mathbf{q}_d^\top (\mathbf{p}^D - P) + q_c^\top (\mathbf{p}'^D - P)), \quad q_c \in \mathbb{R}^d. \quad (43)$$

In this case, $MC_E = 0$, $E \neq D$, and $\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \mathbf{q}_i = 0$ (as, in this hedging case, on the post-default market D' , the amount demanded must be equal to the amount supplied). Since $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i = -\mathbf{q}_d$ and $\Delta \mathbf{q}'_c = \mathbf{q}'_c$, we obtain a post-default equilibrium clearing condition

$$\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \mathbf{q}'_i = -\mathbf{q}_d \quad (44)$$

and

$$\text{MC} = \text{MC}_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{\text{LC}_D} + \sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i.$$

Remark 8. Much like in Remark 7, by change of variable $\mathbf{z}'_i = \Delta \mathbf{q}_i, i \in (D \setminus \{d\}) \cup \{c\}$, and $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}$ and $R'_c = R_c$, the clearing condition (44) relative to the post-default equilibrium $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$ can be converted to a zero clearing condition as per Definition 1 on D' . \square

The CCP fully hedges on another exchange

The considered CCP of d can also hedge the portfolio \mathbf{q}_d that it inherit from member d (if not liquidated) by trading on an exchange $E \neq d$, in which case $E' = E \cup \{c\}$ and $R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D)$ (arising from the pre-default Radner equilibrium on the exchange D of the CCP). In this case, $\text{MC}_E = 0, E \neq E, D$, and $\sum_{E' = E \cup \{c\}} \Delta \mathbf{q}_i = 0$ (the amount demanded must be equal to the amount supplied on the post-default exchange E' where the hedge is implemented). As $\sum_{i \in E} \mathbf{q}_i = 0$ and $\Delta \mathbf{q}'_c = \mathbf{q}'_c$, the ensuing post-default equilibrium clearing condition on E' is

$$\sum_{i \in E' = E \cup \{c\}} \mathbf{q}'_i = 0. \quad (45)$$

The corresponding member optimality condition (38) for the CCP c is

$$\rho_c (\mathbf{q}_d^\top (\mathbf{p}^D - P) + (\mathbf{q}'_c)^\top (\mathbf{p}'^E - P)) \leq \rho_c (\mathbf{q}_d^\top (\mathbf{p}^D - P) + \mathbf{q}_c^\top (\mathbf{p}'^E - P)), \quad q_c \in \mathbb{R}^d.$$

On the own exchange D of the CCP, we have $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = 0$, whence the post-default clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = -\mathbf{q}_d. \quad (46)$$

Therefore $\text{MC} = \text{MC}_E + \text{MC}_D$, where

$$\text{MC}_E = \underbrace{0}_{\text{LC}_E} + \sum_{E' = E \cup \{c\}} \Delta \rho_i, \quad \text{MC}_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{\text{LC}_D} + \sum_{D' = D \setminus \{d\}} \Delta \rho_i.$$

Remark 9. As in Remark 7 again, by change of variable $\mathbf{z}'_i = \Delta \mathbf{q}_i$ and $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}$, the clearing condition (46) relative to the post-default equilibrium $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$ can be converted to a zero clearing condition as per Definition 1 on D' . \square

The CCP fully replicates on its own exchange

By replication, we refer to a default resolution strategy whereby the CCP c replicates the portfolio \mathbf{q}_d (if not liquidated) that the CCP inherits from d by mirroring position $\mathbf{q}'_c = -\mathbf{q}_d$ on its own exchange D . In this case $\text{MC}_E = 0, E \neq D$, and we have $D' = (D \setminus \{d\}) \cup \{c\}, \Delta \mathbf{q}_c = -\mathbf{q}_d$ (in the replication case, the only admissible trading strategy for c as a post-default trading participant is $-\mathbf{q}_d$), $\Delta \rho_c = \rho_c (\mathbf{q}_d^\top (\mathbf{p}^D - P) - \mathbf{q}_d^\top (\mathbf{p}'^D - P)) = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$. On the post-default exchange D' where the hedge is implemented, the amount demanded must be equal to the amount supplied i.e. $\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \mathbf{q}_i = 0$, whence the post-default clearing condition

$$\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \mathbf{q}'_i = \underbrace{\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i}_0 + \underbrace{\mathbf{q}'_c}_{-\mathbf{q}_d} = -\mathbf{q}_d. \quad (47)$$

Therefore

$$MC = MC_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{LC=LC_D} + \sum_{i \in D \setminus \{d\}} \Delta \rho_i + \underbrace{\Delta \rho_c}_{\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)} = \sum_{i \in D \setminus \{d\}} \Delta \rho_i.$$

The market cost is the same as in the liquidation case of C (note that the embedded post-default Radner equilibria are the same), but its split between LC and $\sum_{i \in D'} \Delta \rho_i$ is different, see Table 3.

Remark 10. As in Remark 7, by change of variable $\mathbf{z}'_i = \Delta \mathbf{q}_i, i \in (D \setminus \{d\}) \cup \{c\}$, and $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}$, $R'_c = R_c$, the clearing condition (47) relative to the post-default equilibrium $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$ can be converted to a zero clearing condition as per Definition 1 on D' . \square

The CCP fully replicates on another exchange

Alternatively, the considered CCP of d can replicate the portfolio \mathbf{q}_d (if not liquidated) that it inherits from d by mirroring positions $\mathbf{q}'_c = -\mathbf{q}_d$ on an external exchange $E \neq d$, hence $E' = E \cup \{c\}$. As in Section C, replication means that the only admissible trading strategy for the post-default trading participant c is $-\mathbf{q}_d$. In this case, $MC_E = 0, E \neq E, D$, and $\sum_{E' = E \cup \{c\}} \Delta \mathbf{q}_i = 0$ (the amount demanded must be equal to the amount supplied on the post-default exchange E' where the hedge is implemented), $\sum_{i \in E} \mathbf{q}_i = 0$. The ensuing post-default clearing condition on E' is

$$\sum_{i \in E' = E \cup \{c\}} \mathbf{q}'_i = \underbrace{\sum_{i \in E} \mathbf{q}'_i}_{\mathbf{q}_d} + \underbrace{\mathbf{q}'_c}_{-\mathbf{q}_d} = 0.$$

We also have $\Delta \rho_c = \rho_c (\mathbf{q}_d^\top (\mathbf{p}^D - P) - \mathbf{q}_d^\top (\mathbf{p}'^E - P)) = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^E)$.

On the own exchange D of the CCP, we have $\sum_{i \in D' = D \setminus \{d\}} \Delta \mathbf{q}_i = 0$, whence the post-default clearing condition

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = -\mathbf{q}_d.$$

Therefore $MC = MC_E + MC_D$, where

$$MC_E = \underbrace{0}_{LC_E} + \sum_{i \in E} \Delta \rho_i + \Delta \rho_c \quad \text{and} \quad MC_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{LC_D} + \sum_{i \in D' = D \setminus \{d\}} \Delta \rho_i.$$

Remark 11. As in Remark 7, by change of variable $\mathbf{z}'_i = \Delta \mathbf{q}_i$ and $R'_i = R_i + \mathbf{q}_i^\top P, i \in D \setminus \{d\}$, we recover member optimally and zero clearing conditions as per Definition 1 on D' . \square

The CCP partially liquidates and hedges on its own exchange

The CCP can also liquidate a portion \mathbf{q}_d^l of the defaulted position \mathbf{q}_d and hedge the remaining $\mathbf{q}_d^h = \mathbf{q}_d - \mathbf{q}_d^l$ on its own exchange D . The amount demanded should be equal to the amount supplied on each leg of the strategy, hence $\sum_{i \in D \setminus \{d\}} \Delta \mathbf{q}_i^l = \mathbf{q}_d^l$ and $\sum_{i \in (D \setminus \{d\}) \cup c} \Delta \mathbf{q}_i^h = 0$, thus $\sum_{i \in (D \setminus \{d\}) \cup c} \Delta \mathbf{q}_i = \mathbf{q}_d^l$. As $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i = -\mathbf{q}_d$, the ensuing post-default clearing condition on D' is

$$\sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \mathbf{q}'_i = -\mathbf{q}_d^h, \quad (48)$$

We assume that both liquidation and hedging happen simultaneously at the same price \mathbf{p}^D . Hence each trading participant on the post-default market D' has a single member optimality condition (38) (with, in particular, $R_c = (\mathbf{q}_d^h)^\top (P - \mathbf{p}^D)$). Then

$$\text{MC} = \text{MC}_D = \underbrace{-(\mathbf{q}_d^h)^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{\text{LC}=\text{LC}_D} + \sum_{i \in D' = (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i.$$

Remark 12. By change of variables $\mathbf{z}'_i = \mathbf{q}'_i + k_i \mathbf{q}_d^h$ and $R'_i = R_i - k_i (\mathbf{q}_d^h)^\top P$, for reals k_i such that $\sum_{i \in D'} k_i = 1$, the clearing condition (48) and the optimality conditions (38) relative to the post-default equilibrium $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$ respectively become $\sum_{i \in D'} \mathbf{z}'_i = 0$ and

$$\rho_i (-R'_i + (\mathbf{z}'_i)^\top (\mathbf{p}'^D - P)) \leq \rho_i (-R'_i + z_i^\top (\mathbf{p}'^D - P)), \quad z_i \in \mathbb{R}^m. \quad \square$$

The CCP partially liquidates and hedges on another exchange

The CCP can also liquidate a portion \mathbf{q}_d^l of the defaulted position \mathbf{q}_d and hedge the remaining $\mathbf{q}_d^h = \mathbf{q}_d - \mathbf{q}_d^l$ on another exchange $E \not\equiv d$. In this case, $\text{MC}_E = 0, E \neq E, D$. Since the amount demanded should be equal to the amount supplied on each leg of the strategy, we have $\sum_{i \in E'} \Delta \mathbf{q}'_i = \mathbf{q}_d^l$ and $\sum_{i \in E'} \Delta \mathbf{q}_i^h = 0$, hence $\sum_{i \in E'} \Delta \mathbf{q}_i = \mathbf{q}_d^l$. As $\sum_{i \in E} \mathbf{q}_i = 0$ and $\mathbf{q}_c = 0$, the ensuing post-default equilibrium clearing condition on E' is

$$\sum_{i \in E'} \mathbf{q}'_i = \mathbf{q}_d^l. \quad (49)$$

We assume that both liquidation and hedging happen simultaneously on the exchange E with the same price \mathbf{p}'^E . Hence each trading participant on the post-default market E' has a single member optimality condition (38) (with, in particular, $R_c = (\mathbf{q}_d^h)^\top (P - \mathbf{p}^D)$).

Regarding the own exchange of the CCP, the post-default equilibrium clearing condition on D' is

$$\sum_{i \in D' = D \setminus \{d\}} \mathbf{q}'_i = -\mathbf{q}_d \quad (50)$$

Hence $\text{MC} = \text{MC}_E + \text{MC}_D$, where

$$\text{MC}_E = \underbrace{(\mathbf{q}_d^l)^\top (\mathbf{p}^E - \mathbf{p}'^E)}_{\text{LC}_E} + \sum_{E' = E \cup \{c\}} \Delta \rho_i, \quad \text{MC}_D = \underbrace{-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)}_{\text{LC}_D} + \sum_{D' = D \setminus \{d\}} \Delta \rho_i.$$

Remark 13. Similarly to Remarks 6 and 7, by change of variables, the clearing conditions (49) and (50) relative to the post-default equilibria $((\mathbf{q}'_i)_{i \in E'}, \mathbf{p}'^E)$ and $((\mathbf{q}'_i)_{i \in D'}, \mathbf{p}'^D)$ can be converted to zero clearing conditions on the exchanges E' and D' . \square

	LC	$\sum_E \sum_{i \in E'} \Delta \rho_i$
1.	0	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i$
2.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D) + \mathbf{q}_d^\top (\mathbf{p}^E - \mathbf{p}'^E)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E} \Delta \rho_i$
3.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i$
4.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E \cup \{c\}} \Delta \rho_i$
5.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i$ with $\Delta \rho_c = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$
6.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E \cup \{c\}} \Delta \rho_i$ with $\Delta \rho_c = \mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^E)$
7.	$-(\mathbf{q}_d^h)^\top (\mathbf{p}^D - \mathbf{p}'^D)$	$\sum_{i \in (D \setminus \{d\}) \cup \{c\}} \Delta \rho_i$
8.	$-\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D) + (\mathbf{q}_d^h)^\top (\mathbf{p}^E - \mathbf{p}'^E)$	$\sum_{i \in D \setminus \{d\}} \Delta \rho_i + \sum_{i \in E \cup \{c\}} \Delta \rho_i$

Table 3: Decomposition of the market costs in the eight examples of Section C.

In Sections §4-§5, we provide explicit or numerical solutions regarding the market cost of default resolutions on D, hence

$$\text{LC} = \text{LC}_D = \sum_{i \in D'} (\mathbf{q}'_i)^\top (\mathbf{p}^D - \mathbf{p}'^D), \quad \text{MC} = \text{MC}_D = \text{LC}_D + \sum_{i \in D'} \Delta \rho_i. \quad (51)$$

By translation equivariance of the ρ_i , (35) yields

$$\Delta \rho_i = r_i(\mathbf{q}'_i) - \mathbf{1}_{i \neq c} r_i(\mathbf{q}_i) + (\mathbf{q}'_i)^\top \mathbf{p}'^D - \mathbf{q}_i^\top \mathbf{p}^D, \quad i \in D'.$$

A further computation based on (51) then yields

$$\text{MC} = \text{MC}_D = \sum_{i \in D'} (\Delta \mathbf{q}_i)^\top \mathbf{p}^D + \sum_{i \in D'} (r_i(\mathbf{q}'_i) - \mathbf{1}_{i \neq c} r_i(\mathbf{q}_i)). \quad (52)$$

§4 Market Cost: the Case of Entropic Risk Measures

Throughout this section, we assume that the risk preference of each trading participant i in $D \cup D'$ is an entropic risk measure of the form

$$\rho_i(L) = \frac{1}{\varrho_i} \ln(\mathbb{E}[\exp(\varrho_i L)]), \quad \text{for some } \varrho_i > 0. \quad (53)$$

We also assume that each (R_i, P) is jointly normal, so that $r_i(q_i)$ is given by (20).

A Liquidation on D

Proposition 7. *Let $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$, $i \in D \cup D'$, and Γ be invertible. If the CCP liquidates the defaulter position \mathbf{q}_d on its own exchange D, then*

$$\text{LC}_D = 0 \quad \text{and} \quad (54)$$

$$\text{MC}_D = \frac{1}{2} \varrho' \mathbf{q}_d^\top \Gamma \mathbf{q}_d - \varrho' \left(\Gamma^{-1} \sum_{j \in D' \setminus D} \frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right)^\top \left(\text{cov}' + \frac{1}{2} \Gamma \mathbf{q}_d \right) + \quad (55)$$

$$\sum_{i \in D' \setminus D} \left(\Gamma^{-1} (\varrho \text{cov} - \varrho_i \text{cov}_i) \right)^\top \left(\text{cov}_i + \frac{1}{2} \Gamma \mathbf{q}_i' \right), \quad (56)$$

where $\varrho' = \left(\sum_{i \in D'} \frac{1}{\varrho_i} \right)^{-1}$.

Proof. By the clearing condition $\sum_{i \in D'} \mathbf{q}_i' = 0$ (established like (39)), (51) yields $\text{LC}_D = \sum_{i \in D'} (\mathbf{q}_i')^\top (\mathbf{p}^D - \mathbf{p}'^D) = 0$. Letting D and D' successively play the role of E in Proposition 4, the pre-default and the post-default equilibrium are uniquely given by

$$\mathbf{q}_i = \Gamma^{-1} \left(\frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right), \quad i \in D; \quad \mathbf{p}^D = \mu - \varrho \text{cov}, \quad (57)$$

and

$$\mathbf{q}_i' = \Gamma^{-1} \left(\frac{\varrho'}{\varrho_i} \text{cov}' - \text{cov}_i \right), \quad i \in D'; \quad \mathbf{p}'^D = \mu - \varrho' \text{cov}', \quad (58)$$

where $\varrho = \left(\sum_{i \in D} \frac{1}{\varrho_i} \right)^{-1}$, $\text{cov} = \sum_{i \in D} \text{cov}_i$, $\varrho' = \left(\sum_{i \in D'} \frac{1}{\varrho_i} \right)^{-1}$, $\text{cov}' = \sum_{i \in D'} \text{cov}_i$. From (20), we obtain

$$r_i(\mathbf{q}_i') - r_i(\mathbf{q}_i) = -\Delta \mathbf{q}_i^\top \mu + \varrho_i \Delta \mathbf{q}_i^\top \left[\text{cov}_i + \Gamma \left(\mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i \right) \right], \quad i \in D'. \quad (59)$$

As also $\sum_{i \in D'} \Delta \mathbf{q}_i = \mathbf{q}_d$ holds in a liquidation setup and since $\mathbf{q}_i = 0$ for each trading participant $i \in D' \setminus D$, (52) yields

$$\begin{aligned} \text{MC}_D &= \mathbf{q}_d^\top (\mathbf{p}^D - \mu) + \sum_{i \in D'} \varrho_i (\Delta \mathbf{q}_i)^\top \left[\text{cov}_i + \Gamma \left(\mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i \right) \right] \\ &= -\varrho \mathbf{q}_d^\top \text{cov} + \sum_{i \in D'} \varrho_i (\Delta \mathbf{q}_i)^\top \left[\text{cov}_i + \Gamma \left(\mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i \right) \right], \end{aligned} \quad (60)$$

by the second identity in (57).

To compute $\Delta \mathbf{q}_i$ therein, note that

$$\begin{aligned} \text{cov}' &= \text{cov} - \text{cov}_d + \sum_{i \in D' \setminus D} \text{cov}_i = \frac{\varrho}{\varrho_d} \text{cov} + \left(1 - \frac{\varrho}{\varrho_d} \right) \text{cov} - \text{cov}_d + \sum_{i \in D' \setminus D} \text{cov}_i \\ &= \Gamma \mathbf{q}_d + \frac{\varrho}{\varrho'} \text{cov} - \sum_{i \in D' \setminus D} \left(\frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right), \end{aligned}$$

by the first identity in (57) and the fact that $\frac{1}{\varrho'} = \frac{1}{\varrho} - \frac{1}{\varrho_d} + \sum_{i \in D' \setminus D} \frac{1}{\varrho_i}$. This implies

$$\varrho' \text{cov}' = \varrho' \Gamma \mathbf{q}_d + \varrho \text{cov} - \varrho' \sum_{i \in D' \setminus D} \left(\frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right). \quad (61)$$

The definition of $\Delta \mathbf{q}_i$, (57), (58), and (61) yield

$$\Delta \mathbf{q}_i = \begin{cases} \frac{\varrho'}{\varrho_i} \mathbf{q}_d - \frac{\varrho'}{\varrho_i} \Gamma^{-1} \sum_{j \in D' \setminus D} \left(\frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right) & i \in D \setminus \{d\} \\ \frac{\varrho'}{\varrho_i} \mathbf{q}_d - \frac{\varrho'}{\varrho_i} \Gamma^{-1} \sum_{j \in D' \setminus D} \left(\frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right) + \Gamma^{-1} \left(\frac{\varrho}{\varrho_i} \text{cov} - \text{cov}_i \right) & i \in D' \setminus D. \end{cases} \quad (62)$$

As $\sum_{i \in D \setminus \{d\}} \mathbf{q}_i = -\mathbf{q}_d$ and $\sum_{i \in D'} \Delta \mathbf{q}_i = \mathbf{q}_d$, substituting (62) into (60) yields

$$\begin{aligned} \text{MC}_D = & -\varrho \mathbf{q}_d^\top \text{cov} + \left[\varrho' \mathbf{q}_d - \varrho' \Gamma^{-1} \sum_{j \in D' \setminus D} \left(\frac{\varrho}{\varrho_j} \text{cov} - \text{cov}_j \right) \right]^\top \left(\text{cov}' - \frac{1}{2} \Gamma \mathbf{q}_d \right) + \\ & \sum_{i \in D' \setminus D} \left(\Gamma^{-1} (\varrho \text{cov} - \varrho_i \text{cov}_i) \right)^\top \left(\text{cov}_i + \frac{1}{2} \Gamma \mathbf{q}'_i \right), \end{aligned}$$

whence the expression for MC_D in (54). \square

Remark 14. To cope with the case of several instant defaulters d at time 0, one just needs to replace \mathbf{q}_d , cov_d and $\frac{\varrho}{\varrho_d}$ by $\sum_d \mathbf{q}_d$, $\sum_d \text{cov}_d$ and $\sum_d \frac{\varrho}{\varrho_d}$ in Proposition 7 and its proof. \square

Remark 15. If the CCP liquidates the defaulter's position among the surviving members, i.e. for $D' \setminus D = \emptyset$ in the above, then Proposition 7 yields

$$\text{MC}_D = \frac{1}{2} \varrho' \mathbf{q}_d^\top \Gamma \mathbf{q}_d \geq 0.$$

Using (57), (59), and (61), we obtain $\mathbf{p}^{D'} = \mathbf{p}^D - \varrho' \Gamma \mathbf{q}_d$. Moreover, (62) yields $\Delta \mathbf{q}_i = \frac{\varrho'}{\varrho_i} \mathbf{q}_d$, $i \in D \setminus \{d\}$. In the case $m = 1$ for simplicity, the reason why $\text{MC} \geq 0$ when the CCP liquidates among the surviving member can thus be explained as follows. If $\mathbf{q}_d > 0$, then the CCP replaces the defaulter's contract with each surviving member by selling at a "fire sales" price $\mathbf{p}^{D'} < \mathbf{p}^D$. If $\mathbf{q}_d < 0$, then the CCP buys from each surviving member at a "dear" price $\mathbf{p}^{D'} > \mathbf{p}^D$. In both cases, there is a market cost. \square

Example 16. Let $D = \{1, \dots, 15\}$, $d = \{15\}$, $D' \setminus D = \emptyset$, $\varrho_i = 1$, $m = 1$ and $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$, $i \in D$. Suppose $\text{cov}_i = c_i \sigma \sqrt{\text{Var}(R_i)}$, where $\sigma^2 = \text{Var}(P)$, $c_i = (-1)^{i+1} 0.8$ (the correlation coefficient between R_i and P), and $\text{Var}(R_i) = 0.09i^2$, $i \in D$. Fix $\sigma = 0.2$. The corresponding pre- and post-default optimal positions computed from (57) and (58) are given by Table 4.

CM_i	1	2	3	4	5	6	7	8
cov_i	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
\mathbf{q}_i	-0.56	3.04	-2.96	5.44	-5.36	7.84	-7.76	10.24
\mathbf{q}'_i	-1.80	1.80	-4.20	4.20	-6.60	6.60	-9.00	9.00
CM_i	9	10	11	12	13	14	15	
cov_i	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	
\mathbf{q}_i	-10.16	12.64	-12.56	15.04	-14.96	17.44	-17.36	
\mathbf{q}'_i	-11.40	11.40	-13.80	13.80	-16.20	16.20		

Table 4: Pre- and post-default optimal positions of each clearing member i (CM_i) when the CCP liquidates \mathbf{q}_d on its own exchange in the entropic case.

Note that each \mathbf{q}_i or \mathbf{q}'_i is positive (negative) provided cov_i is negative (positive) (here and again in Table 7 below), in line with the hedging feature of the exchange. For $\mu = 2$, Proposition 7 and its proof yield $\mathbf{p}^D = 1.97$, $\mathbf{p}^{D'} = 2.02$, and $\text{MC}_D = 0.43$ (with $\text{LC}_D = 0$ as per the first line in Table 3). \square

As it can be seen from the analytical expressions (57)-(58) and Table 4, the covariance matrices Γ_i , $i \in D \cup D'$, are the major driving factors for portfolio and price changes. But the number and the risk preferences of the trading participants can also significantly affect these optimal quantities and the market cost:

Example 17. Let $D = \{1, \dots, |D| = n + 1\}$, $d = \{n + 1\}$, $\varrho_i = \varrho_1$, and $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$, $i \in D$. Suppose $\text{cov}_i = 0$, $i \in \{1, \dots, n - 1\}$, and $\text{cov}_n = -\text{cov}_{n+1} = \delta$. We consider two cases.

(i) $D' \setminus D = \emptyset$. By (57) and (58), the pre- and post-default portfolios are then given by

$$\mathbf{q}_i = \begin{cases} 0, & i = 1, \dots, n - 1, \\ -\frac{\delta}{\sigma^2}, & i = n, \\ \frac{\delta}{\sigma^2}, & i = n + 1, \end{cases} \quad \text{and} \quad \mathbf{q}'_i = \begin{cases} \frac{\delta}{n\sigma^2}, & i = 1, \dots, n - 1 \\ -\frac{(n-1)\delta}{n\sigma^2}, & i = n. \end{cases}$$

The pre- and post-default prices are given by

$$\mathbf{p}^D = \mu \quad \text{and} \quad \mathbf{p}'^D = \mu - \frac{\varrho_1 \delta}{n}.$$

A further computation based on (54) yields

$$\text{MC} = \frac{\varrho_1}{2n} \left(\frac{\delta}{\sigma} \right)^2,$$

which decreases to 0 as the number n of surviving members to ∞ .

(ii) $D' \setminus D = \{n + 2\}$ with $\text{cov}_{n+2} = \delta'$. In this case, the pre-default equilibrium is the same as in case (i), while the post-default equilibrium is given by

$$\mathbf{q}'_i = \begin{cases} \frac{\delta + \delta'}{(n+1)\sigma^2}, & i = 1, \dots, n - 1, \\ \frac{\delta' - n\delta}{(n+1)\sigma^2}, & i = n, \\ \frac{\delta - \delta' n}{(n+1)\sigma^2}, & i = n + 2. \end{cases} \quad \text{and} \quad \mathbf{p}'^D = \mu - \frac{\varrho_1(\delta + \delta')}{n + 1}.$$

A further computation based on (54) yields

$$\text{MC} = \frac{2\varrho_1\delta^2 + 2\varrho_1\delta'\delta - \varrho_1(\delta')^2n}{2\sigma^2(n + 1)},$$

the sign of which depends on the value of the parameters. \square

B Hedging on D

We now turn to the ‘‘hedging on own exchange D’’ case of Section C, but with possibly new participants beyond the CCP c in D' .

Proposition 8. Let $(R_i, P) \sim \mathcal{N}_{m+1}(\mu_i, \Gamma_i)$, $i \in D \cup D'$, with invertible covariance matrix Γ of P . When the CCP hedges the defaulter position \mathbf{q}_d on its own exchange D, then

$$\text{LC}_D = -\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D), \quad (63)$$

$$\text{MC}_D = \sum_{i \in D'} \varrho_i (\Delta \mathbf{q}_i)^\top \left[\text{cov}_i + \Gamma(\mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i) \right] + \left(\frac{\varrho_c \text{Var}(R_c)}{2} - \mathbb{E}[R_c] \right), \quad (64)$$

where $\varrho' = \left(\sum_{i \in D'} \frac{1}{\varrho_i} \right)^{-1}$. In the absence of new participants (other than the CCP itself c) to D' ,

$$\text{MC} = \frac{\varrho'^2(\varrho' - \varrho_c)}{2\varrho_c^2} \text{cov}^\top \Gamma^{-1} \text{cov} + \frac{\varrho'}{2} \mathbf{q}_d^\top \Gamma \mathbf{q}_d - \frac{\varrho' \varrho}{\varrho_c} \mathbf{q}_d^\top \text{cov}. \quad (65)$$

Proof. By the clearing condition $\sum_{i \in D'} \mathbf{q}'_i = -\mathbf{q}_d$ (established like (44)), (51) yields $LC_D = \sum_{i \in D'} (\mathbf{q}'_i)^\top (\mathbf{p}^D - \mathbf{p}'^D) = -\mathbf{q}_d^\top (\mathbf{p}^D - \mathbf{p}'^D)$. Applying Proposition 4 to $E = D$, the pre-default equilibrium is uniquely given by (57). As for the post-default equilibrium, if the CCP hedges on D , then, in view of Remark 8, introducing the changed variable $\mathbf{z}'_i = \mathbf{q}'_i - \mathbf{q}_i$ and $R'_i = R_i + \mathbf{q}_i^\top P$ yields

$$r'_i(z_i) := \rho_i(-R'_i - z_i^\top P) = \rho_i(-R_i - (z_i + \mathbf{q}_i)^\top P), \quad i \in D'.$$

Following the proof of Proposition 4 with r'_i here in the role of r_i there, we obtain a unique post-default equilibrium

$$\mathbf{q}'_i = \Gamma^{-1} \left(\frac{\varrho'}{\varrho_i} \text{cov}' - \text{cov}_i \right) - \frac{\varrho'}{\varrho_i} \mathbf{q}_d, \quad i \in D', \quad \text{and} \quad \mathbf{p}'^D = \mu - \varrho' \text{cov}' + \varrho' \Gamma \mathbf{q}_d. \quad (66)$$

Hence, by (20),

$$r_i(\mathbf{q}'_i) - r_i(\mathbf{q}_i) = -\Delta \mathbf{q}_i^\top \mu + \varrho_i (\Delta \mathbf{q}_i)^\top [\text{cov}_i + \Gamma(\mathbf{q}_i + \frac{1}{2} \Delta \mathbf{q}_i)], \quad i \in D'.$$

This and $\sum_{i \in D'} \Delta \mathbf{q}_i = 0$ reduce (52) to (63).

In the absence of new participants other than the CCP itself to D' , (57) and (66) yield

$$\Delta \mathbf{q}_i = \begin{cases} \frac{1}{\varrho_i} \Gamma^{-1} (\varrho' \text{cov}' - \varrho \text{cov}) - \frac{\varrho'}{\varrho_i} \mathbf{q}_d & i \in D \setminus \{d\}, \\ \frac{1}{\varrho_i} \Gamma^{-1} \varrho' \text{cov}' - \mathbf{q}_d - \frac{\varrho'}{\varrho_i} \mathbf{q}_d & i = c. \end{cases}$$

One can check that the value of $\varrho' \text{cov}'$ given by (61) also holds true for the hedging case. Hence, using $\varrho' \text{cov}'$ given by (61) and $\text{cov}_c = \Gamma \mathbf{q}_d$, we obtain

$$\Delta \mathbf{q}_i = \begin{cases} \frac{\varrho'}{\varrho_i} \Gamma^{-1} (\Gamma \mathbf{q}_d - \frac{\varrho}{\varrho_c} \text{cov}), & i \in D \setminus \{d\}, \\ \frac{\varrho'}{\varrho_i} \Gamma^{-1} (\Gamma \mathbf{q}_d - \frac{\varrho}{\varrho_c} \text{cov}) + \frac{\varrho}{\varrho_i} \Gamma^{-1} \text{cov} - \mathbf{q}_d, & i = c. \end{cases} \quad (67)$$

Since $R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D)$, we have $\mathbb{V}\text{ar}(R_c) = \mathbf{q}_d^\top \Gamma \mathbf{q}_d$ and $\mathbb{E}[R_c] = \varrho \mathbf{q}_d^\top \text{cov}$. Further computations using $\Delta \mathbf{q}_i$ given by (67) reduce (63) to (65). \square

Example 18. In the ‘‘CCP hedging on its own exchange’’ case, let $D = \{1, \dots, 15\}$, $d = \{15\}$, $D' \setminus D = \{c\}$, $\varrho_i = 1 = \varrho_c$, and $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$ with $\text{cov}_i = \sigma c_i \sqrt{\mathbb{V}\text{ar}(R_i)}$, $\sigma = 0.2$, $c_i = (-1)^{i+1} 0.8$, and $\mathbb{V}\text{ar}(R_i) = 0.09i^2$, $i \in D$. The corresponding pre- and post-default optimal positions computed from (57) and (66) are given in Table 5.

CM _{<i>i</i>}	1	2	3	4	5	6	7	8
cov _{<i>i</i>}	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
q_{<i>i</i>}	-0.56	3.04	-2.96	5.44	-5.36	7.84	-7.76	10.24
q'_{<i>i</i>}	-1.76	1.84	-4.16	4.24	-6.56	6.64	-8.96	9.04

CM _{<i>i</i>}	9	10	11	12	13	14	15	c
cov _{<i>i</i>}	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	-0.69
q_{<i>i</i>}	-10.16	12.64	-12.56	15.04	-14.96	17.44	-17.36	
q'_{<i>i</i>}	-11.36	11.44	-13.76	13.84	-16.16	16.24		16.80

Table 5: Pre- and post-default optimal positions when the CCP hedges on its own exchange D with $D' \setminus D = \emptyset$ in the entropic case.

For $\mu = 2$, Proposition 8 and its proof yield $\mathbf{p}^D = 1.97$, $\mathbf{p}'^D = 2.02$, and $\text{MC}_D = 0.42$ (with, by the third line in Table 3, $\text{LC}_D = -0.83$). \square

Table 6 displays the impacts of the default resolution of d in terms of LC_i and $\Delta\rho_i$, $i \in D'$. As can be seen from the table, the impact of the default resolution on the $\Delta\rho_i$ is almost the same in the liquidation and hedging cases, whereas its impact on the LC_i is significantly different in the two cases.

	CM _{<i>i</i>}	1	2	3	4	5	6	7	8
Liquidation	LC _{<i>i</i>}	0.09	-0.09	0.21	-0.21	0.33	-0.33	0.45	-0.45
	$\Delta\rho_i$	-0.06	0.1	-0.18	0.24	-0.30	0.36	-0.42	0.48
Hedging	LC _{<i>i</i>}	0.03	-0.15	0.14	-0.26	0.26	-0.38	0.37	-0.49
	$\Delta\rho_i$	-0.06	0.12	-0.17	0.23	-0.29	0.35	-0.40	0.46

	CM _{<i>i</i>}	9	10	11	12	13	14	c
Liquidation	LC _{<i>i</i>}	0.56	-0.56	0.68	-0.68	0.80	-0.80	
	$\Delta\rho_i$	-0.53	0.60	-0.65	0.72	-0.77	0.83	
Hedging	LC _{<i>i</i>}	0.49	-0.61	0.60	-0.72	0.72	-0.84	0
	$\Delta\rho_i$	-0.52	0.58	-0.63	0.69	-0.75	0.81	0.83

Table 6: Impacts of the default resolution on the LC_i and $\Delta\rho_i$ in the liquidation and hedging cases in the entropic risk measure examples 16-18.

§5 Market Cost: the Case of Expected Shortfall

Throughout this section, we assume that the risk preferences of each market participant is an expected shortfall $\mathbb{E}\mathcal{S}_{\alpha_i}$ as per (23); each vector $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$ (or, sometimes, a more specific $\mathcal{T}_{m+1}(\mu_i, \Gamma_i, \nu_i)$); the CCP only operates on its own exchange D. Hence $\text{MC} = \text{MC}_D$ as in (52), with,

by (26),

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \mathbb{E}\mathcal{S}_{\alpha_i}(Z_i) \sqrt{\mathbb{V}\text{ar}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma q_i} \quad (68)$$

$$\text{where } Z_i \sim \mathcal{E}_1(0, 1, \psi) \text{ (or } Z_i \sim \mathcal{T}_1(0, 1, \nu_i)), i \in D \cup D'. \quad (69)$$

A Liquidation on D

Let us first consider the liquidation case of Section C, but with possibly new participants in D' . If $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$ with Γ_i positive definite, $i \in D \cup D'$, then, by Proposition 5, there exists a unique pre-default Radner equilibrium $((\mathbf{q}_i)_{i \in D}, \mathbf{p}^D)$. Following (9) and (27), the pre-default member $i \in D$ optimality condition yields

$$\mathbf{p}^D = \mu - \frac{\mathbb{E}\mathcal{S}_{\alpha_i}(Z)}{\sqrt{\mathbb{V}\text{ar}(R_i) + 2\mathbf{q}_i^\top \text{cov}_i + \mathbf{q}_i^\top \Gamma \mathbf{q}_i}} (\text{cov}_i + \Gamma \mathbf{q}_i), \quad i \in D. \quad (70)$$

The zero clearing condition and (70) imply

$$\begin{cases} \sum_{i \in D} \mathbf{q}_i = 0 \\ \frac{\mathbb{E}\mathcal{S}_{\alpha_1}(Z)}{\mathbb{E}\mathcal{S}_{\alpha_i}(Z)} \sqrt{\frac{\mathbb{V}\text{ar}(R_i) + 2\mathbf{q}_i^\top \text{cov}_i + \mathbf{q}_i^\top \Gamma \mathbf{q}_i}{\mathbb{V}\text{ar}(R_1) + 2\mathbf{q}_1^\top \text{cov}_1 + \mathbf{q}_1^\top \Gamma \mathbf{q}_1}} (\text{cov}_1 + \Gamma \mathbf{q}_1) - \text{cov}_i - \Gamma \mathbf{q}_i = 0, \quad i \in D. \end{cases} \quad (71)$$

Therefore, computing the optimal position is equivalent to finding the root of a vector function from $\mathbb{R}^{m|D|}$ into $\mathbb{R}^{m|D|}$. Likewise, the post-default optimal positions solve

$$\begin{cases} \sum_{i \in D'} \mathbf{q}'_i = 0 \\ \frac{\mathbb{E}\mathcal{S}_{\alpha_1}(Z)}{\mathbb{E}\mathcal{S}_{\alpha_i}(Z)} \sqrt{\frac{\mathbb{V}\text{ar}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}{\mathbb{V}\text{ar}(R_1) + 2\mathbf{q}'_1^\top \text{cov}_1 + \mathbf{q}'_1^\top \Gamma \mathbf{q}'_1}} (\text{cov}_1 + \Gamma \mathbf{q}'_1) - \text{cov}_i - \Gamma \mathbf{q}'_i = 0, \quad i \in D' \end{cases} \quad (72)$$

and the post-default price is equal to

$$\mathbf{p}'^D = \mu - \frac{\mathbb{E}\mathcal{S}_{\alpha_i}(Z_i)}{\sqrt{\mathbb{V}\text{ar}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}} (\text{cov}_i + \Gamma \mathbf{q}'_i), \quad i \in D'.$$

In the remainder of this subsection, we assume that $m = 1$, $\alpha_i = 0.975$ for each market participant, $\mu = 2$, and $\sigma = 0.2$.

Example 19 (Multivariate normal). Let $D = \{1, \dots, 15\}$, $d = \{15\}$, $D' \setminus D = \emptyset$, $\varrho_i = 1$, $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$, $\text{cov}_i = c_i \sigma \sqrt{\mathbb{V}\text{ar}(R_i)}$ with $c_i = (-1)^{i+1} 0.8$, and $\mathbb{V}\text{ar}(R_i) = 0.09i^2$, $i \in D$. Solving (71)-(72) yields the pre- and post-default optimal positions displayed in Table 7.

CM _{<i>i</i>}	1	2	3	4	5	6	7	8
cov _{<i>i</i>}	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
q_{<i>i</i>}	-1.12	2.56	-3.36	5.12	-5.60	7.68	-7.84	10.24
q'_{<i>i</i>}	-1.28	2.24	-3.84	4.48	-6.40	6.72	-8.96	8.96
CM _{<i>i</i>}	9	10	11	12	13	14	15	
cov _{<i>i</i>}	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	
q_{<i>i</i>}	-10.08	12.80	-12.32	15.36	-14.56	17.92	-16.80	
q'_{<i>i</i>}	-11.52	11.20	-14.08	13.44	-16.64	15.68		

Table 7: Pre- and post-default optimal positions when the CCP liquidates on its own exchange D with $D' \setminus D = \emptyset$ in the expected shortfall case.

The above results also yield $\mathbf{p}^D = 1.96$, $\mathbf{p}'^D = 2.04$, and $\text{MC}_D = 0.69$ (with $\text{LC} = 0$ as per the first line in Table 3). \square

*Example 20 (Multivariate Student *t*).* Reconsider Example 19 but for $(R_i, P) \sim \mathcal{T}_2(\mu_i, \Gamma_i, \nu_i)$ with $\nu_i = \nu = 2.5$, $i \in D$. (68) yields

$$r_i(q_i) = -\mathbb{E}[R_i] - q_i^\top \mu + \mathbb{E}\mathcal{S}_{\alpha_i}(Z) \sqrt{\text{Var}(R_i) + 2q_i^\top \text{cov}_i + q_i^\top \Gamma q_i}$$

with $Z \sim \mathcal{T}_1(0, 1, \nu)$, $i \in D$.

By [McNeil et al. \(2015, Example 2.15, page 71\)](#), we obtain $\mathbb{E}\mathcal{S}_{\alpha_i}(Z) = \sqrt{\frac{\nu-2}{\nu}} \frac{t_\nu(T_\nu^{-1}(\alpha_i))[\nu + (T_\nu^{-1}(\alpha_i))^2]}{(1-\alpha)(\nu-1)}$. By inspection, \mathbf{q} and \mathbf{q}' are the same as in Example 19, given by Table 7. The above result also yield $\mathbf{p}^D = 1.94$, $\mathbf{p}'^D = 2.06$, and $\text{MC}_D = 1.06$ (with $\text{LC}_D = 0$ as per the first line in Table 3). \square

B Hedging on D

We now turn to the ‘‘hedging on own exchange D’’ case of Section C, but with possibly new participants beyond the CCP c in D' . Let $(R_i, P) \sim \mathcal{E}_{m+1}(\mu_i, \Gamma_i, \psi)$ with Γ_i positive definite, for each $i \in D \cup (D' \setminus \{c\})$. By Proposition 5, there exists a unique pre-default Radner equilibrium $((\mathbf{q}_i)_{i \in D}, \mathbf{p}^D)$, which can be computed by (70) and (71). Since $R_c = \mathbf{q}_d^\top (P - \mathbf{p}^D)$, letting $z = (z_1, z_2, \dots, z_{m+1}) = (z_1, \hat{z})$, we obtain

$$\mathbb{E}\left[e^{iz^\top (R_c, P)}\right] = \mathbb{E}\left[e^{i(z_1 \mathbf{q}_d + \hat{z})^\top P - z_1 \mathbf{q}_d^\top \mathbf{p}^D}\right] = \exp(i z^\top \mu_c) \psi\left(\frac{1}{2} z^\top \Gamma_c z\right), \quad z \in \mathbb{R}^{m+1},$$

as $(z_1 \mathbf{q}_d + \hat{z})^\top P - z_1 \mathbf{q}_d^\top \mathbf{p}^D \sim \mathcal{E}_1(z^\top \mu_c, z^\top \Gamma_c z, \psi)$. Hence $(R_c, P) \sim \mathcal{E}_{m+1}(\mu_c, \Gamma_c, \psi)$. However, as R_c is in the span of P , Γ_c is not positive definite (see Remark 4). To nevertheless ensure a unique post-default equilibrium (beyond the setup of Proposition 5), we assume that $D' \setminus \{c\} \neq \emptyset$. By Theorem 3, we have a unique post-default price \mathbf{p}^D and a unique post-default portfolio $\mathbf{q}'_i, i \in D' \setminus \{c\}$. Hence by the post-default clearing condition, we also have a unique post-default position \mathbf{q}'_c .

Letting $R'_i = R_i + \mathbf{q}'_i{}^\top P$ play the role of R_i in (24), we obtain

$$\begin{aligned}
r'_i(z_i) &:= \mathbb{E}S_{\alpha_i}(-R'_i - z_i^\top P) \\
&= -\mathbb{E}[R_i] - (z_i + \mathbf{q}_i)^\top \mu + \mathbb{E}S_{\alpha_i}(Z) \sqrt{(1, z_i + \mathbf{q}_i)^\top \Gamma_i (1, z_i + \mathbf{q}_i)}, \quad i \in D'
\end{aligned}$$

with $Z \sim \mathcal{E}_1(0, 1, \psi)$. In view of Remark 8, by change of variable $\mathbf{z}'_i = \mathbf{q}'_i - \mathbf{q}_i$, the proof of Proposition 5 shows that, for $i \in D' \setminus \{c\}$, r'_i is differentiable and strictly convex with

$$\begin{aligned}
-\mathbf{p}'^D &= \nabla r'_i(\mathbf{z}'_i) = -\mu + \frac{\mathbb{E}S_{\alpha_i}(Z)}{\sqrt{(1, \mathbf{z}'_i + \mathbf{q}_i)^\top \Gamma_i (1, \mathbf{z}'_i + \mathbf{q}_i)}} (\text{cov}_i + \Gamma \mathbf{z}'_i + \Gamma \mathbf{q}_i) \\
&= -\mu + \frac{\mathbb{E}S_{\alpha_i}(Z)}{\sqrt{\mathbb{V}\text{ar}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}} (\text{cov}_i + \Gamma \mathbf{q}'_i).
\end{aligned}$$

For the CCP, the member c optimality condition gives $-\mathbf{p}'^D \in \partial r'_c(\mathbf{z}'_c) = \partial r'_c(\mathbf{q}'_c)$, i.e. $(r'_c)^*(-\mathbf{p}'^D) = -\mathbf{q}'_c{}^\top \mathbf{p}'^D - r'_c(\mathbf{q}'_c)$. Hence computing the optimal post-default position reduces to the root-finding problem

$$\begin{cases} \sum_{i \in D'} \mathbf{q}'_i + \mathbf{q}_d = 0 \\ \frac{\mathbb{E}S_{\alpha_i}(Z)}{\mathbb{E}S_{\alpha_i}(Z)} \sqrt{\frac{\mathbb{V}\text{ar}(R_i) + 2(\mathbf{q}'_i)^\top \text{cov}_i + (\mathbf{q}'_i)^\top \Gamma \mathbf{q}'_i}{\mathbb{V}\text{ar}(R_1) + 2\mathbf{q}'_1{}^\top \text{cov}_1 + \mathbf{q}'_1{}^\top \Gamma \mathbf{q}'_1}} (\text{cov}_1 + \Gamma \mathbf{q}'_1) - \text{cov}_i - \Gamma \mathbf{q}'_i = 0, \quad i \in D' \setminus \{c\} \\ r'_c(\mathbf{q}'_c) + \mathbf{q}'_c{}^\top \mathbf{p}'^D + (r'_c)^*(-\mathbf{p}'^D) = 0. \end{cases} \quad (73)$$

Example 21. In the ‘‘hedging on own exchange D’’ case with $D' \setminus D = \{c\}$ (as per Section C) and expected shortfall risk measures, let $D = \{1, \dots, 15\}$, $d = \{15\}$, $\alpha_i = 0.975$ and $(R_i, P) \sim \mathcal{N}_2(\mu_i, \Gamma_i)$ (so $m = 1$) with $\text{cov}_i = \sigma c_i \sqrt{\mathbb{V}\text{ar}(R_i)}$, $\sigma = 0.2$, $c_i = (-1)^{i+1} 0.8$, and $\mathbb{V}\text{ar}(R_i) = 0.09i^2$, $i \in D$. The corresponding pre- and post-default optimal positions computed by (71) and (73) are given in Table 8.

CM_i	1	2	3	4	5	6	7	8
cov_i	0.05	-0.10	0.14	-0.19	0.24	-0.29	0.34	-0.38
\mathbf{q}_i	-1.12	2.56	-3.36	5.12	-5.60	7.68	-7.84	10.24
\mathbf{q}'_i	-1.28	2.24	-3.84	4.48	-6.40	6.72	-8.96	8.96
CM_i	9	10	11	12	13	14	15	c
cov_i	0.43	-0.48	0.53	-0.58	0.62	-0.67	0.72	-0.69
\mathbf{q}_i	-10.08	12.80	-12.32	15.36	-14.56	17.92	-16.80	
\mathbf{q}'_i	-11.52	11.20	-14.08	13.44	-16.64	15.68		16.80

Table 8: Pre- and post-default optimal positions when the CCP hedges on its own exchange D with $D' \setminus D = \{c\}$ in the expected shortfall case.

For $\mu = 2$, we obtain $\mathbf{p}^D = 1.96$, $\mathbf{p}'^D = 2.04$, and $\text{MC}_D = 0.69$ (with, by the third line in Table 3, $\text{LC}_D = -1.39$). \square

Table 9 is the expected shortfall analog of Table 6, with qualitatively similar conclusions.

	CM _{<i>i</i>}	1	2	3	4	5	6	7	8
Liquidation	LC _{<i>i</i>}	0.11	-0.18	0.32	-0.37	0.53	-0.56	0.74	-0.74
	Δρ _{<i>i</i>}	-0.10	0.20	-0.30	0.40	-0.50	0.60	-0.70	0.80
Hedging	LC _{<i>i</i>}	0.09	-0.21	0.28	-0.42	0.46	-0.64	0.65	-0.85
	Δρ _{<i>i</i>}	-0.10	0.20	-0.30	0.40	-0.50	0.60	-0.70	0.80
	CM _{<i>i</i>}	9	10	11	12	13	14	c	
Liquidation	LC _{<i>i</i>}	0.95	-0.93	1.167	-1.11	1.38	-1.30		
	Δρ _{<i>i</i>}	-0.89	0.99	-1.09	1.19	-1.29	1.39		
Hedging	LC _{<i>i</i>}	0.84	-1.06	1.02	-1.27	1.21	-1.48	0	
	Δρ _{<i>i</i>}	-0.89	0.99	-1.09	1.19	-1.29	1.39	1.39	

Table 9: Impacts of the default resolution on the LC_{*i*} and Δρ_{*i*} in the liquidation and hedging cases in the expected shortfall examples 19-21.

§6 Credit Cost

The MC term (37) only addresses the impact of the considered default resolution strategy in terms of mis-hedge of market risk. It remains to address its credit cost (CC), meant to account for counterparty credit risk in a broad sense including the implications of this risk in terms of capital and funding costs. There is in fact empirical evidence that credit risk could actually even dominate market risk (European Bank Authority, 2022, Figure 65 page 67). Accordingly, the overall impact of a clearing member default’s resolution strategy should not be assessed in terms of MC only, but of the following all-inclusive funds transfer price:

$$\text{FTP} = \text{MC} + \text{CC}, \quad (74)$$

which should thus supersede MC in the default resolution approach depicted in the last paragraph of Section B.

A Structure of the Exchanges

As depicted in Figure 1, clearing member bank *a*’s trades with a CCP are divided into proprietary or house trades \mathbf{q}_a , which are in effect hedges of the bank’s OTC bilateral trading exposures $R_a = \sum_{o \in O} R_a^o$ (where the non-cleared, end-clients *o* are “outside” of the exchange), and back-to-back hedges \mathbf{q}_b^a of intermediated cleared client trades, through which non-member clients *b* (simple participants to the exchange) can access the CCP clearing services.

Table 10 shows that both categories are really significant and should therefore be encompassed in the analysis.

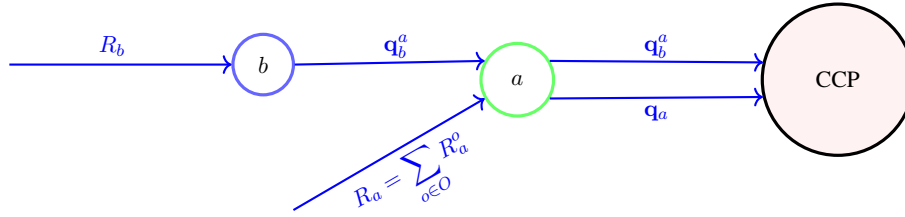


Figure 1: Client clearing (q_b^a transits from b to the CCP via a) versus bilateral hedging (a hedges $R_a = \sum_{o \in O} R_a^o$ by a proprietary trading position q_a with the CCP).

	Interest Rates	Credit	Equity	Commodity
ISDA (USA)	90%	83%	26%	
ESMA (Europe)	71%	41%	2%	1%

Table 10: Percentage of derivative notionals cleared by asset classes. See [International Swaps and Derivatives Association \(2021, pages 2 and 5\)](#) for US as of end of 2020 (no figure for commodity derivatives), [European Securities and Markets Authority \(2021, page 16\)](#) for European interest rates and credit as of end of 2020, and [International Swaps and Derivatives Association \(2018, page 18\)](#) for European equities.

Our next task is to clarify the nature of the Radner equilibrium on an exchange E accounting for this distinction between proprietary and cleared deals. Let $E = A \cup B$, with $A \cap B = \emptyset$, be the split between the set A of those participants a to the exchange that are also clearing members of the CCP and the set B of simple participants (non clearing members) b , having recourse to the clearing members for intermediating trades with the CCP. Let q_a be the proprietary position of member a and q_b^a be the position of client (simple participant to the exchange) b cleared by member a . As depicted in Figure 1, the position q_b^a only transits from b to a and then passes from a to the CCP. Hence, even though the total position of a vis-à-vis the CCP is $q_a + \sum_{b \in B} q_b^a$, as a holds $-\sum_{b \in B} q_b^a$ vis-à-vis the clients it clears for, the respective positions of a and b involved in the Radner equilibrium on E are q_a and $q_b = \sum_{a \in A} q_b^a$. Once $((\mathbf{q}_i)_{i \in E=A \cup B}, \mathbf{p}^E)$ has been obtained as the solution of the corresponding Radner equilibrium (7)-(8), the splits $\mathbf{q}_b = \sum_{a \in A} \mathbf{q}_b^a$ (for each $b \in B$) should follow in a second stage from pure credit risk considerations. Similar comments apply “with \cdot everywhere” to any post-default Radner equilibrium on $E' = A' \cup B'$.

B Credit Costs XVA Framework

The settlement of a CCP portfolio of a defaulted clearing member entails a mixture of market and/or counterparty credit risk, which depends on the nature of this portfolio.

Figure 2 and the rows of Table 11 take up the distinction of Section A between house and intermediation deals. The intermediated client deals of a clearing member are perfectly (back-to-back) hedged by the corresponding intermediating transactions (upper part of Figure 1). If a clearing member d defaults, its client deals and their static hedge are ported as a package to a surviving clearing member (right panel in Figure 2). As market risk is perfectly hedged throughout, such porting has no market impact, but entails a transfer of counterparty credit risk that can be quantified by XVA costs as per [Bastide et al.](#)

(2023a, Section 7) (second row in Table 11).

The columns of Table 11 distinguish delta-one financial assets, often rolled over time with no upfront payment, such as repo market transactions, equity swaps and futures (Crépey, Bielecki, and Brigo, 2014, Section 4.2.1), from upfront derivative assets traded on a primary market, such as multi-legs swaps and options. Delta-one rolled transactions have a nominal maturity that is infinite, but an effective maturity determined by the next reset date, i.e. typically less than one day, whence a very little counterparty credit risk footprint (second column of Table 11), but possibly significant liquidation or hedging costs. The market costs analysis of the above sections can be enough to deal with a portfolio of delta-one assets. In the case of swaps or derivatives portfolios, however (first column of Table 11), the XVA implications of a default resolution strategy also matter, which is the object of this section.

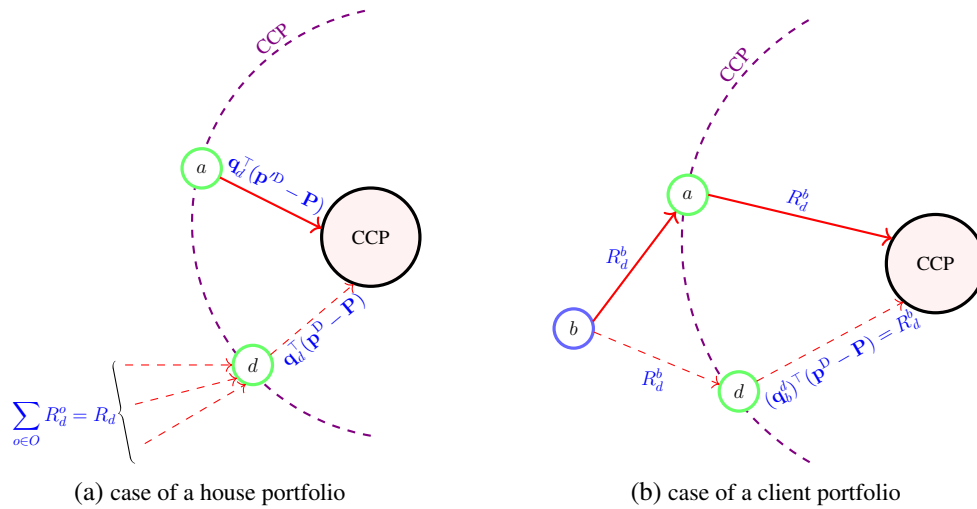


Figure 2: Default management of the CCP portfolios of a defaulted clearing member d . With pre-default positions in red dashed lines and post-default positions in red solid lines: (Left) The receivable R_d of d from its OTC bilateral counterparties (which are outside the exchange) is left aside in the default resolution procedure, while the corresponding house (hedging) portfolio \mathbf{q}_d^d is ported to the surviving clearing member a ; (Right) The client account \mathbf{q}_b^d of the defaulted clearing member d , as well as the corresponding (mirroring) receivable R_d^b from its cleared client b , are ported as a package to the surviving clearing member a .

Notations as detailed below and in Sections §2-§3-A.

	Assets	swaps and options	delta-one
Cleared			
centrally via house account		liquidity and credit	liquidity, no credit
centrally via client account		no liquidity, credit	\emptyset

Table 11: Costs of the CCP for settling a netting set of deals of a defaulted clearing member, depending on the nature of the defaulted portfolio.

The deals corresponding to the lower right cell would entail no liquidity nor credit risk, but this cell is in fact empty, due to the nature of delta-one transactions that do not require intermediation.

Our XVA metrics are computed under the premise that the (random) loss triggered by the default of

a market participant in the future is allocated between the surviving members of its CCP, pro-rata of their default fund contribution to the CCP (see e.g. after (87)). At time “0–”, i.e. “right before” the instant default of the clearing member d at time 0, the participants i to the exchanges charge to their clients the expectation of their ensuing losses, in the form of their CVA $_i$, as well as their collateral funding costs FVA $_i$ and MVA $_i$, and costs of capital KVA $_i$. These costs sum up to XVA $_i = \text{CVA}_i + \text{FVA}_i + \text{MVA}_i + \text{KVA}_i$, computed for each participant i the way detailed for $i = 0$ in Section A, based on the pre-default Radner equilibrium quantities and prices on each exchange. To quantify the XVA impact of a given default resolution procedure for a CCP portfolio of d , we also compute the time 0, post-default XVA' $_i$, for any participant $i \neq d$ to the exchanges. The credit cost of the settlement of the defaulted portfolio, coming on top of the already computed MC, is

$$\text{CC} = \sum_{i \neq d} \underbrace{(\text{XVA}'_i - \text{XVA}_i)}_{\Delta \text{XVA}_i} + \text{AC}, \quad (75)$$

where the auctioning cost AC is another XVA incremental impact corresponding to the FTP in [Bastide et al. \(2023a, Section 7\)](#) (which only involves credit costs), i.e. the XVA impact of auctioning any hedged (as opposed to liquidated) positions. Indeed, a CCP is not inclined to keep the defaulted portfolio and the corresponding hedge on its book, it will look at auctioning them ([Ferrara and Li, 2020](#); [Oleschak, 2019](#)).

In the end, the full FTP (74) that emerges from the present paper (where both market and credit costs are involved) can be detailed as

$$\text{FTP} = \underbrace{\text{LC} + \sum_{E'} \sum_{i \in E'} \Delta \rho_i}_{\text{MC}} + \underbrace{\sum_{i \neq d} \Delta \text{XVA}_i}_{\text{CC}} + \text{AC}. \quad (76)$$

Remark 22. Under the pure auctioning default resolution strategy, only the last term remains. But this AC term could be very expensive in the case of an unhedged portfolio (see the first quotation in Section §3). □

C Example 19 Continued

We complete from the credit costs perspective the “liquidation on own exchange” Example 19. In the case of default resolution procedures implemented on the own exchange D of the CCP, (76) boils down to

$$\text{FTP} = \underbrace{\text{LC}_D + \sum_{i \in D \setminus \{d\}} \Delta \rho_i}_{\text{MC}} + \underbrace{\sum_{i \in D \setminus \{d\}} \Delta \text{XVA}_i}_{\text{CC}} + \text{AC}. \quad (77)$$

We assume risk-free OTC bilateral counterparties (end-users) o and no simple participants to the exchange D , which thus only involves proprietary trading between its clearing members. We endorse the Gaussian latent factors XVA setup of Section B. Table 12 provides the resulting “time 0–” XVA $_i$ (with all members, d included) and time 0 XVA' $_i$ (without d), using the allocated positions of Table 7 and the XVA specifications of Table 21 along with $\text{IM}_i = \text{DF}_i = 0$, whilst Table 13 presents the same results for IM_i and DF_i set at the 75% and 80% confidence level. Note that the chosen period length of $T = 5$ years covers the bulk (if not the final maturity) of most realistic CCP portfolios¹⁴. The aggregated XVA

¹⁴most OTC derivatives have a maturity of less than 5 years ([Bank for International Settlements, 2022](#), Graphs A.2–4).

cost (75) of liquidating the defaulted portfolio¹⁵ is $CC = \sum_{i \in D \setminus \{d\}} \Delta XVA_i$, namely -0.77 for the case $IM_i = DF_i = 0$ and -0.25 with IM_i and DF_i set at the 75% and 80% confidence level, coming on top of the market cost of $MC = 0.70$ already obtained in Example 19.

CM_i	1	2	3	4	5	6	7	8
XVA_i	0.81	1.21	1.36	1.70	1.74	2.05	2.02	2.37
XVA'_i	0.92	1.17	1.47	1.61	1.82	1.93	2.07	2.19
ΔXVA_i	0.11	-0.04	0.12	-0.09	0.08	-0.12	0.05	-0.18
CM_i	9	10	11	12	13	14	15	
XVA_i	2.25	2.62	2.42	2.84	2.57	3.04	2.69	
XVA'_i	2.28	2.42	2.42	2.60	2.54	2.77		
ΔXVA_i	0.03	-0.20	-0.01	-0.24	-0.03	-0.27		

Table 12: The pre- and post-default XVAs computed from (90) when the CCP liquidates d on its own exchange, D , in the expected shortfall case with $IM_i = DF_i = 0$.

CM_i	1	2	3	4	5	6	7	8
XVA_i	0.37	0.59	0.72	0.89	1.00	1.12	1.21	1.34
XVA'_i	0.42	0.56	0.80	0.84	1.07	1.05	1.28	1.22
ΔXVA_i	0.05	-0.03	0.08	-0.06	0.07	-0.07	0.07	-0.11
CM_i	9	10	11	12	13	14	15	
XVA_i	1.42	1.50	1.57	1.64	1.72	1.77	1.85	
XVA'_i	1.49	1.37	1.63	1.49	1.78	1.60		
ΔXVA_i	0.07	-0.12	0.06	-0.15	0.06	-0.17		

Table 13: The pre- and post-default XVAs computed from (90) when the CCP liquidates d on its own exchange, D , in the expected shortfall case with IM_i and DF_i set at the 75% and 80% confidence level.

Instead of liquidation on the CCP's own exchange D , we now consider another default resolution strategy, in the form of an (idealized) auction inducing the taker giving rise to the least auction cost AC among all possible takers $i \in D \setminus \{d\}$ ¹⁶. The results are displayed in Table 14 for the case $IM_i = DF_i = 0$ and in Table 15 for IM_i and DF_i set at the 75% and 80% confidence level.

¹⁵note that there is no auction in this (liquidation) case.

¹⁶This approach developed in (Bastide et al., 2023a, Section 7) can indeed be seen as rendering the output of an idealized, efficient auction used for closing out the account of a defaulted clearing member (cf. Oleschak (2019, Section 3.3)).

CM_i	$\sum_{i \in D \setminus \{d\}} (CVA'_i - CVA_i)$	$\sum_{i \in D \setminus \{d\}} (KVA'_i - KVA_i)$	AC
14	-1.00 (-0.38)	-3.52 (-0.63)	-4.52 (-1.01)
12	-0.85 (-0.34)	-2.93 (-0.57)	-3.78 (-0.91)
10	-0.58 (-0.23)	-2.02 (-0.42)	-2.60 (-0.64)
8	-0.31 (-0.11)	-1.13 (-0.24)	-1.44 (-0.35)
6	-0.04 (0.02)	-0.30 (-0.04)	-0.34 (-0.02)
4	0.23 (0.15)	0.42 (0.20)	0.65 (0.35)
2	0.50 (0.29)	0.95 (0.52)	1.45 (0.81)
1	0.77 (0.43)	1.59 (0.86)	2.35 (1.29)
3	0.77 (0.39)	2.09 (0.62)	2.85 (1.01)
5	0.77 (0.36)	2.47 (0.47)	3.23 (0.82)
7	0.77 (0.32)	2.70 (0.37)	3.46 (0.70)
9	0.77 (0.29)	2.84 (0.30)	3.60 (0.60)
11	0.77 (0.26)	2.92 (0.27)	3.69 (0.53)
13	0.77 (0.23)	2.96 (0.25)	3.73 (0.48)

Table 14: Auctioning costs AC corresponding to the different possible takers of the portfolio of the defaulted member $d = 15$, ranked by increasing value, for $IM_i = DF_i = 0$. In parenthesis, the contributions to AC of the considered possible taker itself.

CM_i	$\sum_{i \in D \setminus \{d\}} (MVA'_i - MVA_i)$	$\sum_{i \in D \setminus \{d\}} (CVA'_i - CVA_i)$	$\sum_{i \in D \setminus \{d\}} (KVA'_i - KVA_i)$	AC
14	-0.06 (-0.15)	-0.49 (-0.13)	-2.33 (-0.42)	-2.88 (-0.71)
12	-0.01 (-0.11)	-0.45 (-0.12)	-2.00 (-0.39)	-2.46 (-0.63)
10	0.06 (-0.04)	-0.36 (-0.08)	-1.47 (-0.27)	-1.77 (-0.40)
8	0.13 (0.03)	-0.27 (-0.04)	-0.93 (-0.14)	-1.07 (-0.16)
6	0.20 (0.10)	-0.17 (0.00)	-0.44 (0.01)	-0.41 (0.11)
4	0.26 (0.17)	-0.07 (0.04)	0.00 (0.19)	0.20 (0.40)
2	0.33 (0.23)	0.03 (0.09)	0.38 (0.41)	0.74 (0.74)
1	0.41 (0.31)	0.11 (0.14)	0.76 (0.64)	1.28 (1.09)
3	0.42 (0.31)	0.10 (0.13)	0.92 (0.51)	1.44 (0.95)
5	0.42 (0.31)	0.09 (0.11)	1.06 (0.42)	1.58 (0.85)
7	0.43 (0.32)	0.08 (0.10)	1.16 (0.37)	1.67 (0.78)
9	0.44 (0.32)	0.07 (0.09)	1.23 (0.31)	1.74 (0.72)
11	0.45 (0.32)	0.06 (0.08)	1.28 (0.28)	1.78 (0.68)
13	0.45 (0.33)	0.05 (0.06)	1.29 (0.26)	1.80 (0.65)

Table 15: Auctioning costs AC corresponding to the different possible takers of the portfolio of the defaulted member $d = 15$, ranked by increasing value, for IM_i and DF_i set at the 75% and 80% confidence level. In parenthesis, the contributions to AC of the considered possible taker itself.

From Tables 14-15, participant 14 is the survivor taker leading to the smallest auctioning cost AC, namely -4.52 and -2.88 , when taking over the defaulted portfolio of CM_{15} (and there are in this case no

additional costs to consider, cf. Remark 22). Such takeover makes intuitive sense given the pre-default position $\mathbf{q}_{15} = -16.8$ of the defaulting member $d = 15$, compared with the position of member 14 at $\mathbf{q}_{14} = 17.92$, an almost offset of \mathbf{q}_{15} . Member 12 shows the second best solution with an auctioning cost AC close to zero, also justifiable by his offsetting position at $\mathbf{q}_{12} = 15.36$.

Finally, we consider one more default resolution strategy, where the CCP hedges (as per Section C) the defaulted portfolio before auctioning all its positions. In the hedging case, with the CCP c contributing to the post-default quantities and price discovery $((\mathbf{q}'_i)_{i \in D' = (D \setminus \{d\}) \cup \{c\}}, \mathbf{p}'^D)$, resolving (38) (for $E = D$) and (42)–(44) in the configuration of Example 19 (except for the new member c) leads to $\mathbf{q}'_c = -\mathbf{q}_d$. Coincidentally, this hedging resolution thus leads to a perfect replication as per Section C. The corresponding $\sum_{i \neq d} \Delta XVA_i$, detailed in Tables 16 and 17, is -14.45 for $IM_i = DF_i = 0$ and -6.80 for IM_i and DF_i set at the 75% and 80% confidence levels. As the residual market risk is null in such a replication case, taking over the defaulted portfolio and its hedge does not generate any market risk. Hence in these cases no additional cost is generated by the auctioning process, i.e. $AC = 0$.

CM_i	1	2	3	4	5	6	7	8
XVA_i	0.81	1.21	1.36	1.7	1.74	2.05	2.02	2.37
XVA'_i	0.19	0.34	0.54	0.63	0.83	0.89	1.07	1.13
ΔXVA_i	-0.62	-0.87	-0.81	-1.06	-0.91	-1.15	-0.94	-1.24
CM_i	9	10	11	12	13	14	15	c
XVA_i	2.25	2.62	2.42	2.84	2.57	3.04	2.69	
XVA'_i	1.28	1.34	1.44	1.54	1.59	1.72		2.39
ΔXVA_i	-0.98	-1.27	-0.98	-1.3	-0.98	-1.32		

Table 16: The pre- and post-default XVAs computed from (90) when the CCP hedges the portfolio of the defaulted member $d = 15$ on its own exchange D, in the expected shortfall case with $IM_i = DF_i = 0$.

CM_i	1	2	3	4	5	6	7	8
XVA_i	0.37	0.59	0.72	0.89	1	1.12	1.21	1.34
XVA'_i	0.13	0.21	0.38	0.4	0.6	0.56	0.79	0.72
ΔXVA_i	-0.24	-0.38	-0.34	-0.5	-0.4	-0.56	-0.42	-0.62
CM_i	9	10	11	12	13	14	15	c
XVA_i	1.42	1.50	1.57	1.64	1.72	1.77	1.85	
XVA'_i	0.97	0.85	1.12	0.98	1.26	1.09		1.30
ΔXVA_i	-0.45	-0.64	-0.45	-0.67	-0.46	-0.68		

Table 17: The pre- and post-default XVAs computed from (90) when the CCP hedges the portfolio of the defaulted member $d = 15$ on its own exchange D, in the expected shortfall case with IM_i and DF_i at the 75% and 80% confidence level.

We sum up in Tables 18 and 19 the FTP of each considered default management scheme without and with collateral (in the sense here of initial margins and default fund contributions), from the cheapest to

the dearest one (again, in this example, hedging then auctioning with $D' \setminus D = \{c\}$ happens to coincide with replicating then auctioning with $D' \setminus D = \emptyset$). The FTPs of the hedging then auctioning scheme provides much larger gains than the pure auctioning strategy, which itself provides more gains than the full liquidation strategy. Coincidentally in this example, at least, this happens to be in line with the BIS recommendations quoted in the beginning of Section §3. However, our approach only endorses the point of view of the participants to the exchange. Indeed, our costs of settling the house portfolio of a defaulted clearing member ignore the damage of the default to the “outer” actors o (end-users external to the exchanges). From this viewpoint (compare the two panels of Figure 2), whenever available, centrally cleared trading is preferable to bilateral trading (but, as per today, centrally cleared trading can only concern the standardized half of the market, cf. Table 10).

	LC_D	$\sum_{i \in D'} \Delta \rho_i$	$\sum_{i \in D \setminus \{d\}} \Delta XVA_i$	AC	FTP
liquidating	0	0.70	-0.77	0	-0.08
auctioning	0	0	0	-4.52	-4.52
hedging then auctioning	-1.39	2.09	-14.45	0	-13.75

Table 18: FTPs of different default management schemes on D split as per (77) for $IM_i = DF_i = 0$.

	LC_D	$\sum_{i \in D'} \Delta \rho_i$	$\sum_{i \in D \setminus \{d\}} \Delta XVA_i$	AC	FTP
liquidating	0	0.70	-0.25	0	0.45
auctioning	0	0	0	-2.88	-2.88
hedging then auctioning	-1.39	2.09	-6.80	0	-6.11

Table 19: FTPs of different default management schemes on D split as per (77) for IM_i and DF_i set at the 75% and 80% confidence level.

In this example, hedging then auctioning with $D' \setminus D = \{c\}$ happens to coincide with replicating then auctioning with $D' \setminus D = \emptyset$.

§7 Appendix

A XVA Gaussian Setup

The purpose of this part is to provide a bridge from the equilibrium setup of Sections §2–§5 to the XVA setup of Bastide et al. (2023a), so that we are able to provide an overall FTP (76) quantifying the market but also credit costs of a given default resolution strategy. We leave for future research the extension of the approach of this paper to a setup where not only the market costs, but also the credit costs, would be treated endogenously as part of a global (or perhaps two-stage¹⁷) equilibrium, ideally in the setup of a dynamic model.

¹⁷accounting for the segregation between the market and credit spheres in banks (Bellini, 2017, Section 6.1) (The European Parliament and the Council of the European Union, 2013, Article 92).

We endorse the structure of exchanges $E = A \cup B$ depicted in Section A. Note that even for those clients clearing through a CCP member and also having OTC positions with that same member, the corresponding exposures must be treated separately, hence $B \cap O = \emptyset$. For each participant i to an exchange E , a comparison of the present setup with [Bastide et al. \(2023a\)](#), Eqns. (15)-(16)) yields the mapping of Table 20.

(Bastide et al., 2023a, Eqns. (15)-(16))	This paper	Description
$\mathcal{P}_c^{ccp} - \text{MtM}_c^{ccp}$	$(\mathbf{q}_b^0)^\top (\mathbf{p}^E - P)$	Cash flows from a cleared client (c in Bastide et al. (2023a) , b in this paper) of a CCP (ccp in Bastide et al. (2023a) , the CCP of exchange E in this paper) to the participant 0 (if the latter is not a clearing member of the CCP, these cash flows are simply zero)
$\overline{\mathcal{P}}_i^{ccp} - \overline{\text{MtM}}_i^{ccp}$	$\sum_{b \in B} (\mathbf{q}_b^a)^\top (\mathbf{p}^E - P)$	Client account cash flows from a clearing member (i in Bastide et al. (2023a) , a in this paper) to his CCP (ccp in Bastide et al. (2023a) , the CCP of exchange E in this paper)
$\overline{\mathcal{P}}_i^{ccp} - \overline{\text{MtM}}_i^{ccp}$	$\mathbf{q}_a^\top (\mathbf{p}^E - P)$	Proprietary account cash flows from a clearing member (i in Bastide et al. (2023a) , a in this paper) to his CCP
$\mathcal{P}_b - \text{VM}_b$	R_0^o	Cash flows from an end-user (b in Bastide et al. (2023a) , o in this paper) to participant 0

Table 20: Some notation adaptation for the cash flows of some market participants after variation margin is subtracted, in the setup of [Bastide et al. \(2023a\)](#) and in this paper.

Let J_i be the survival indicator of participant i to an exchange E , i.e. $J_i = \mathbb{1}_{\{\tau_i > T\}}$, where τ_i is the default time of participant i over the period $[0, T]$, with probability $\gamma_i = \mathbb{P}(J_i = 0)$ of default over $[0, T]$. We denote likewise $J_o = \mathbb{1}_{\{\tau_o > T\}}$ for any end-user $o \in O$. Via the mapping of Table 20, ([Bastide et al., 2023a](#), Eqns. (15)-(16)) yield the following (pre-default equilibrium) credit loss profile \mathcal{C}_0 of a participant $i = 0$ to the exchanges, on which we focus in what follows: $\mathcal{C}_0 = \sum_{E=A \cup B} (\mathcal{C}_0^B + \mathcal{C}_0^A) + \mathcal{C}_0^O$, where

$$\begin{aligned}
\mathcal{C}_0^B &= \sum_{b \in B} (1 - J_b) \left((\mathbf{q}_b^0)^\top (\mathbf{p}^E - P) - \text{IM}_0^b \right)^+, \\
\mathcal{C}_0^A &= w_0^A \sum_{a \in A} (1 - J_a) \left[\left(\left(\sum_{b \in B} \mathbf{q}_b^a \right)^\top (\mathbf{p}^E - P) - \text{IM}_a^A \right)^+ + \right. \\
&\quad \left. \left(\mathbf{q}_a^\top (\mathbf{p}^E - P) - \overline{\text{IM}}_a^A \right)^+ - \text{DF}_a^A \right]^+, \\
\mathcal{C}_0^O &= \sum_{o \in O} (1 - J_o) (R_0^o - \text{IM}_0^o)^+.
\end{aligned} \tag{78}$$

Here IM_0^b is the initial margin (IM) requested by the participant 0 to the simple participant $b \in B$ on the cleared position \mathbf{q}_b^0 (equal to 0 if $0 \notin A$); w_0^A (equal to 0 if $0 \notin A$) is the loss allocation coefficient of the participant 0 w.r.t. the CCP of the exchange $E = A \cup B$; IM_a^A , $\overline{\text{IM}}_a^A$, and DF_a^A are the initial margins for the cleared clients and proprietary accounts as well as the default fund contribution requested by (the CCP of) exchange E to the clearing member a ; R_0^o is the exogenous receivable of the participant 0 from its OTC bilateral counterparty o , with corresponding initial margin IM_0^o (which can be null under OTC agreement) requested by 0 to o (cf. Figure 1).

Likewise, the post-default equilibrium¹⁸ default loss profile \mathcal{C}'_0 of the participant 0 is $\mathcal{C}'_0 = \sum_{E'=A' \cup B'} (\mathcal{C}'_0^{B'} + \mathcal{C}'_0^{A'}) + \mathcal{C}'_0^O$, for \mathcal{C}'_0^O as in (78) and (cf. Table 20 and (36)-(32))

$$\begin{aligned} \mathcal{C}'_0^{B'} &= \sum_{b \in B'} (1 - J_b) \left((\mathbf{q}_b^0)^\top (\mathbf{p}'^E - P) + (\mathbf{q}_b^0 + (\Delta \mathbf{q}_b^0)^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) - \text{IM}_0^b \right)^+, \\ \mathcal{C}'_0^{A'} &\stackrel{(36)}{=} w_0^{A'} \sum_{a \in A'} (1 - J_a) \times \\ &\quad \left(\left(\sum_{b \in B'} (\mathbf{q}_b^a)^\top (\mathbf{p}'^E - P) + \sum_{b \in B'} (\mathbf{q}_b^a + (\Delta \mathbf{q}_b^a)^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) - \text{IM}'_a^A \right)^+ + \right. \\ &\quad \left. \left((\mathbf{q}_a^0)^\top (\mathbf{p}'^E - P) + (\mathbf{q}_a^0 + \Delta \mathbf{q}_a^l)^\top (\mathbf{p}^E - \mathbf{p}'^E) - \overline{\text{IM}}_a^A \right)^+ - \text{DF}'_a^A \right)^+, \end{aligned} \quad (79)$$

where $w_0^{A'}$, IM'_a^A , $\overline{\text{IM}}_a^A$, DF'_a^A , and IM_0^b are the post-default analogs of w_0^A , IM_a^A , $\overline{\text{IM}}_a^A$, DF_a^A , and IM_0^b in (78).

The pre- and post-default CVA of the participant 0 are given by

$$\text{CVA}_0 = \mathbb{E} [\mathcal{C}_0 | J_0 = 1] = (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}_0], \quad \text{CVA}'_0 = (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}'_0], \quad (80)$$

by [Bastide et al. \(2023a, Theorem 3.7\)](#). Denoting by $\overline{\text{IM}}_0^o$ the initial margin from the participant 0 to its OTC bilateral counterparty o , based on Remark 1, such margin remains constant in the post-default equilibrium, hence the pre- and post-default MVA of participant 0 are given by

$$\begin{aligned} \text{MVA}_0 &= \tilde{\gamma}_0 \left(\sum_{o \in O} \overline{\text{IM}}_0^o + \sum_E \left(\text{IM}_0^E + \overline{\text{IM}}_0^E + \text{DF}_0^E \right) \right), \\ \text{MVA}'_0 &= \tilde{\gamma}_0 \left(\sum_{o \in O} \overline{\text{IM}}_0^o + \sum_E \left(\text{IM}_0^{E'} + \overline{\text{IM}}_0^{E'} + \text{DF}_0^{E'} \right) \right), \end{aligned} \quad (81)$$

for some possibly blended funding rate $\tilde{\gamma}_0 \leq \gamma_0$ as detailed in [Albanese, Armenti, and Crépey \(2020, Section 5\)](#). The pre-default KVA, defined for a hurdle rate h , is calculated based on an expected shortfall $\mathbb{ES}_{\tilde{\alpha}_0}^0$ of the participant 0 under its own survival measure, $\mathbb{P}(\cdot | J_0) / (1 - \gamma_0)$ (with $\tilde{\alpha}_0 >$ the confidence level α_0 introduced for the market cost computation in Section §5 when the risk measures used by the hedgers are expected shortfall risk measures¹⁹), as

$$\text{KVA}_0 = \frac{h}{1 + h} \mathbb{ES}_{\tilde{\alpha}_0}^0 (\mathcal{C}_0 - \text{CVA}_0) \quad (82)$$

¹⁸post-default referring as usual in the paper to the instant default at time 0 of a clearing member d , here assumed \neq the reference clearing member 0.

¹⁹regulatory and economic capital aim at capturing extreme losses that can occur once every 1000 years ([Basel Committee on Banking Supervision, 2005, paragraph 5.1](#)), which leads to considering a much higher confidence level $\tilde{\alpha}_0$ for economic capital calculation, such as 0.9975, from which the KVA is defined, than the α_0 used for market risk, set to 0.975, in line with ([Bank for International Settlements, 2013, Section 1.4 \(i\)](#)).

$$= \frac{h}{1+h} \mathbb{E} [J_0 (\mathcal{C}_0 - \text{CVA}_0) | \mathcal{C}_0 - \text{CVA}_0 \geq \text{VaR}_{\tilde{\alpha}_0}^0 (J_0 (\mathcal{C}_0 - \text{CVA}_0)), J_0 = 1], \quad (83)$$

by Bastide et al. (2023a, Theorem 3.7, last row of Table 2), where $\text{VaR}_{\tilde{\alpha}_0}^0$ denotes the value-at-risk at the confidence level $\tilde{\alpha}_0$ under the measure $(1 - \gamma_0)^{-1} \mathbb{P}(\cdot | J_0)$. The post-default KVA has a similar expression substituting \mathcal{C}'_0 to \mathcal{C}_0 and CVA'_0 to CVA_0 in (82):

$$\text{KVA}'_0 = \frac{h}{1+h} \mathbb{E} \mathbb{S}_{\tilde{\alpha}_0}^0 (\mathcal{C}'_0 - \text{CVA}'_0) \quad (84)$$

$$= \frac{h}{1+h} \mathbb{E} [J_0 (\mathcal{C}'_0 - \text{CVA}'_0) | \mathcal{C}'_0 - \text{CVA}'_0 \geq \text{VaR}_{\tilde{\alpha}_0}^0 (J_0 (\mathcal{C}'_0 - \text{CVA}'_0)), J_0 = 1], \quad (85)$$

Finally, by Bastide et al. (2023a, Theorem 3.7, next to last row of Table 2), the pre- and post- FVA of the participant 0 is given by

$$\text{FVA}_0 = \frac{\gamma_0}{1+\gamma_0} \left(\sum_{o \in \mathcal{O}} \mathbb{E} R_o^o - (\text{CVA}_0 + \text{MVA}_0) - \mathbb{E} \mathbb{S}_{\tilde{\alpha}_0}^0 (\mathcal{C}_0 - \text{CVA}_0) \right)^+, \quad (86)$$

$$\text{FVA}'_0 = \frac{\gamma_0}{1+\gamma_0} \left(\sum_{o \in \mathcal{O}} \mathbb{E} R_o^o - (\text{CVA}'_0 + \text{MVA}'_0) - \mathbb{E} \mathbb{S}_{\tilde{\alpha}_0}^0 (\mathcal{C}'_0 - \text{CVA}'_0) \right)^+.$$

B XVA details in the setup of Section C

In the setup of Section C, only clearing members a participate to the only exchange of interest D (so all participants i are clearing members a and there are no cleared clients b) and all the end-users (OTC bilateral counterparties) o are assumed to be default risk-free, Hence the pre-default credit loss (78) of member $0 \in D$ reduces to

$$\mathcal{C}_0 = w_0^D \sum_{j \in D} (1 - J_j) \left(\mathbf{q}_j (\mathbf{p}^D - P) - \overline{\text{IM}}_j^D - \text{DF}_j^D \right)^+, \quad (87)$$

where $w_0^D = \frac{\text{DF}_0^D J_0}{\sum_{j \in D} \text{DF}_j^D J_j}$. Under the post-default equilibrium when the CCP fully liquidates on D (so that $\Delta \mathbf{q}_i^h = 0$), the post-default credit loss (79) of member $0 \in D$ reduces to

$$\begin{aligned} \mathcal{C}'_0 &= w_0^D \sum_{i \in D'} (1 - J_i) \left(\mathbf{q}'_i (\mathbf{p}'^D - P) + \mathbf{q}'_i (\mathbf{p}^D - \mathbf{p}'^D) - \overline{\text{IM}}_i^D - \text{DF}_i^D \right)^+ \\ &= w_0^D \sum_{i \in D'} (1 - J_j) \left(\mathbf{q}'_i (\mathbf{p}^D - P) - \overline{\text{IM}}_i^D - \text{DF}_i^D \right)^+, \end{aligned} \quad (88)$$

whereas, when the CCP fully hedges on D (so that $\Delta \mathbf{q}_i^l = 0$, $\mathbf{q}_j + \Delta \mathbf{q}_j^h = \mathbf{q}'_j$),

$$\mathcal{C}'_0 = w_0^D \sum_{i \in D'} (1 - J_i) \left(\mathbf{q}'_i (\mathbf{p}'^D - P) + \mathbf{q}_i (\mathbf{p}^D - \mathbf{p}'^D) - \overline{\text{IM}}_i^D - \text{DF}_i^D \right)^+. \quad (89)$$

In both cases w_0^D , $\overline{\text{IM}}_i^D$ and DF_i^D , $i \in D'$, are the post-default analogs of w_0^D , $\overline{\text{IM}}_i^D$ and DF_i^D , $i \in D$, based on the post-default updated portfolio positions. The pre- and post-default CVA, MVA and KVA of member $0 \in D \setminus \{d\}$ are given by

$$\text{CVA}_0 = (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}_0], \quad \text{CVA}'_0 = (1 - \gamma_0)^{-1} \mathbb{E} [J_0 \mathcal{C}'_0], \quad (90)$$

$$\text{MVA}_0 = \tilde{\gamma}_0 (\overline{\text{IM}}_0 + \text{DF}_0), \quad \text{MVA}'_0 = \tilde{\gamma}_0 (\overline{\text{IM}}'_0 + \text{DF}'_0), \quad (91)$$

$$\text{KVA}_0 = \frac{h}{1+h} \mathbb{E} [J_0 (\mathcal{C}_0 - \text{CVA}_0) | \mathcal{C}_0 - \text{CVA}_0 \geq \text{VaR}_{\alpha_0}^0 (J_0 (\mathcal{C}_0 - \text{CVA}_0)), J_0 = 1], \quad (92)$$

$$\text{KVA}'_0 = \frac{h}{1+h} \mathbb{E} [J_0 (\mathcal{C}'_0 - \text{CVA}'_0) | \mathcal{C}'_0 - \text{CVA}'_0 \geq \text{VaR}_{\alpha_0}^0 (J_0 (\mathcal{C}'_0 - \text{CVA}'_0)), J_0 = 1]. \quad (93)$$

Moreover, in the setup of Section C, $\mathbb{E}R_0^c$ in (86) corresponds to $\text{MtM}_b - \text{VM}_b$ in the setup of [Bastide et al. \(2023a\)](#), i.e. a difference of received (VM_b) variation margin by the member 0 and posted (MtM_b) variation margin by the member 0 for an OTC bilateral position between the clearing member 0 and the end-user o . We assume, as it is the case in practice, that there is only marginal, if no, difference between the two quantities. Hence we have $\mathbb{E}R_0^c \approx 0$ and, in any case, dominated by $(\text{CVA}_0 + \text{MVA}_0) - \mathbb{E}\mathbb{S}_{\alpha_0}^0 (\mathcal{C}_0 - \text{CVA}_0)$ and $(\text{CVA}'_0 + \text{MVA}'_0) - \mathbb{E}\mathbb{S}_{\alpha_0}^0 (\mathcal{C}'_0 - \text{CVA}'_0)$ in (86), leading to negligible FVA_0 and FVA'_0 , which we therefore simply take as 0 (and do not report) in the numerics of Section C.

Latent Factor Model

For default modeling purposes, we introduce for each member $i \in \mathcal{D}$ a latent variable $X_i \sim \mathcal{N}_1(0, 1)$ such that $\{J_i = 0\} \iff \{X_i \leq \Phi^{-1}(\gamma_i)\}$. These default latent variables are correlated as per $X_i = \sqrt{\varrho^{cr}} \varepsilon + \sqrt{1 - \varrho^{cr}} \varepsilon_i$, where ε and ε_i are i.i.d. $\mathcal{N}_1(0, 1)$, while ϱ^{cr} is a positive credit/credit correlation coefficient. Writing $P = \mu + \sigma Y$ with $Y \sim \mathcal{N}_1(0, 1)$, the IM posted to the CCP by member i , based on the idea of a $\mathbf{q}_i(\mathbf{p}^D - P)$ VM call not fulfilled over its corresponding time period T (versus Δ_s in [Bastide et al. \(2023a\)](#)), is computed by the VaR metric²⁰ at a confidence level $\alpha_{im} \in (1/2, 1)$ as

$$\overline{\text{IM}}_i^D = \text{VaR}_{\alpha_{im}} (\mathbf{q}_i(\mathbf{p}^D - P)) = \mathbf{q}_i(\mathbf{p}^D - \mu) + |\mathbf{q}_i| \sigma \Phi^{-1}(\alpha_{im}). \quad (94)$$

The liquidation time period Δ_l in [Bastide et al. \(2023a\)](#) is also taken as the one-period of time considered in the Radner equilibrium setup of the present paper, so that $\Delta_s = \Delta_l = T$. The default fund is calculated at the level of the considered CCP of d as the sum of the two highest stress loss over IM (SLOIM), where SLOIM is given for each member i as

$$\text{SLOIM}_i^D = \text{VaR}_{\alpha_{df}} (\mathbf{q}_i(\mathbf{p}^D - P) - \overline{\text{IM}}_i^D) = |\mathbf{q}_i| \sigma (\Phi^{-1}(\alpha_{df}) - \Phi^{-1}(\alpha_{im})),$$

for some confidence level $\alpha_{df} > \alpha_{im}$. The default fund contribution of member i is given as

$$\text{DF}_i^D = \frac{\text{SLOIM}_i^D}{\sum_{j \in \mathcal{D}} \text{SLOIM}_j^D} (\text{SLOIM}_{(0)}^D + \text{SLOIM}_{(1)}^D), \quad (95)$$

based on the cover-2 amount given as the sum of two largest stressed losses over IM (SLOIM_i) among its members, identified with subscripts (0) and (1).

²⁰under the member survival measure.

Portfolios maturity T	5 years
Liquidation period at default Δ_l	5 years
Credit factors correlation ρ^{cr}	20%
IM covering period (margin period of risk) Δ_s	5 years
Default probabilities γ_i	39.3%
MVA funding blending ratio $\tilde{\gamma}_i/\gamma_i$	25%
Quantile levels $\tilde{\alpha}_i$ used for clearing members KVA	99.75%
Hurdle rate h used for KVA computations	10.0%
Number of Monte Carlo simulations (for CVA and KVA computations)	10M
Number of batches (for KVA computations)	100

Table 21: XVAs calculation configuration

Chapter IV

Takers identification for defaulted portfolios with simulated annealing algorithms

This chapter corresponds to a work in progress¹ with E. Gobet.

§1 Introduction

This research looks at the problem of identifying multiple takers of portfolios of a member defaulting on several Central Clearinghouse Counterparties [CCPs] using the stochastic optimization algorithm of simulated annealing for discrete combinatorial problems. CCP institutions have become prominent actors within the financial network by transforming counterparty risk into liquidity risk. They allow such transformation by guaranteeing the contracts negotiated on financial markets in exchange of various layers of collaterals posted by their members. The financial network revamp follows from the G20 September 2009 summit where the settlements of transactions through CCPs have been made mandatory for all products that can be standardized including OTC ones ([G20 Research Group, 2009](#)). The considered scenario of particular interest for financial regulators and covered by this research work consists, for each considered CCP, in experiencing the default of a common member to all those CCPs. The corresponding portfolio needs to be taken over by the other surviving members of each CCP. This scenario is of particular interest as it allows identifying optimal takers and assess if such resolution does not put too much pressure on them and other members. As the network of members portfolios have to be re-arranged, corresponding requested collaterals by the CCPs need to be re-calibrated. This can lead to additional collateral calls on certain members that may have difficulty to answer these additional resources requirements thus indicating a possible propagation effect of further defaults. Indeed, members not in the capacity of meeting additional collateral calls are put in default state by the CCPs, resorting then to further portfolios re-allocation needs on the corresponding markets. Such a scenario could have actually occurred at the beginning of 2023 with the difficulties of the bank Credit Suisse, a major member of thirty CCPs across the world, if the Swiss National Bank [SNB] had not intervened. The SNB granted a liquidity contribution of around €170bn (CHF 168bn, see [Jordan \(2023\)](#)) to prevent the

¹*Acknowledgements:* We thank Stéphane Girard, Senior research scientist at Inria Grenoble Rhône-Alpes, for his inputs on extreme value theory.

default of Credit Suisse. A smaller, yet real, example with a default on two CCPs occurred in the United States with the asset management firm Ronin Capital in March 2020 during the Covid-19 pandemic crisis, detailed and illustrated in [Bastide et al. \(2023a\)](#).

The described scenario occurs when looking at the auction results of defaulted members on CCP services. However, we undertake an idealized auction resolution where instead of conducting the auction (see [Oleschak \(2019\)](#) for first price auction application in the CCP portfolio liquidation context with and without additional bidders, incentive for high bids e.g. default fund loss juniorization), each CCP selects the taker among its surviving members leading to minimal costs across all of them, including risk to other CCPs those members detain. Such resolution is considered based on the ideal behavior of members that would agree to take the defaulted portfolio on a CCP leading to the least additional costs for all members across all CCPs. Indeed, any member has to support costs related to all CCPs it has exposure to. In particular, whilst Clearing Credit Value Adjustment [CCVA] ([Armenti and Crépey, 2017](#)) costs are calculated on each CCP independently, the capital costs for any member to ensure their corresponding shareholders adequate remuneration in the form of Capital Value Adjustment [KVA] ([Armenti and Crépey, 2017](#); [Bastide et al., 2023a](#)), must relate to all CCPs it is exposed to, adjusted by all credit costs². As KVA cost is the leading component among other CCP costs that are Credit Margin Value Adjustment ([Armenti and Crépey, 2017](#); [Bastide et al., 2023a](#)) and CCVA for default resolution³, we simplify the cost minimization to the sole capital cost.

Given the high dimension of the problem and the involved high computational capacity needs, we explore the possibility to apply simulated annealing algorithm technics, in particular the discrete version, to identify the set of optimal takers in terms of capital cost minimization across all CCPs and their corresponding surviving members. To the best of our knowledge, this approach, which resembles an idealized auction of defaulted portfolios, has never been attempted so far.

As outlined in [Delahaye, Chaimatanan, and Mongeau \(2019\)](#), the simulated annealing algorithm for combinatorial optimization problems was introduced by [Kirkpatrick, Gelatt, and Vecchi \(1982\)](#), published in [Kirkpatrick, Gelatt, and Vecchi \(1983\)](#), with an implementation based on the Metropolis-Hastings acceptance criteria suggested in [Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller \(1953\)](#). It has established itself as part of the algorithm tool box to solve complex optimization problems with no particular convex features but, potentially, with several local minima, see [Aarts and van Laarhoven \(1989\)](#), [Aarts and Korst \(1989\)](#), [Catoni \(1992\)](#), [Dufflo \(1996\)](#), [Henderson, Jacobson, and Johnson \(2003\)](#), [Moral \(2004\)](#), [Delmas and Jourdain \(2006\)](#), [Gendreau and Potvin \(2010, Chapter 1\)](#) and [Delahaye, Chaimatanan, and Mongeau \(2019\)](#). Part of our problem relates to the use of an approximated energy function rather than the true one through an empirical expected shortfall, mean or quantile with Monte Carlo procedures instead of exact expected shortfall, mean or quantile respectively. The resulting uncertainty can be handled through applying certain adjustments to the empirical energy function as introduced in [Painton and Diwekar \(1995\)](#), with other applications in [Kim and Diwekar \(2002\)](#) and [Tchechmedjiev et al. \(2012\)](#) for instance.

Simulated annealing algorithms for combinatorial problems in finance are mostly known for portfolio assets allocation optimization with discrete number of units of assets ([Ingber, 1993](#); [Crama and Schyns, 2003](#)). The optimization problem formulation is based on Harry Markowitz's portfolio optimization theory through mean-variance formulation ([Markowitz, 1952](#); [Fabozzi et al., 2012](#); [Rubio-García et al., 2022](#)), combined with recent developments of quantum computer technologies such as

²cf Remark 2 in [Bastide et al. \(2023a\)](#)

³See study cases in [Bastide et al. \(2023a\)](#)

quantum annealing algorithm (Lang et al., 2022). It requires the translation of the Hamiltonian of the formulated problem from weighted maximum independent set graph-theoretic approach into Ising form. The latter can be optimized as a quadratic unconstrained binary optimization problem by a quantum computer (Choi, 2011). The simulated annealing algorithm can also be used for identifying optimal capital growth (Luoa et al., 2014) similar to a portfolio allocation with a different Hamiltonian formulation. Another financial application relates to business risk, arising from actions or elements that can diminish the business profits that can be minimized through cost minimization over several safeguards modelled as binary integers under some constraints (Eraña-Díaz et al., 2020). Application of simulated annealing algorithm can also be found in the field of time-series forecasting for optimal portfolio performances based on combining Support Vector Regression and Simulated Annealing for generating a better set of parameters of the Support Vector optimization part (Santamaria-Bonfil et al., 2013; González-Mancha et al., 2018). Finally, the application of identifying combinations of averse macroeconomic and market conditions leading to particular loss magnitude, aka reverse stress test exercises, can be found in Montesi et al. (2020).

We test the algorithm on a complex problem of identifying the optimal combination of takers of a defaulted common member portfolio on a network of 4 CCPs with 10 surviving and common members with non-linear loss calculation depending on all members and their portfolios towards the 4 CCPs. With adequate generation probabilities, we can observe, through several runs, much more frequent and faster selection of the optimal combination of takers w.r.t. the algorithm number of iterations and in comparison to a naive approach where a random combination of takers is suggested without re-sampling. The generation probabilities definition are thus critical in the algorithm performance. This is observed when we compare the runs of the algorithm with the probabilities not suitably defined for the problem structure at hand, showing results even worse than the naive suggestion of combinations of takers.

§2 Motivation: the combinatorial problem with default on several CCPs

We consider an idealized auction situation where, after a default has occurred on the financial network composed of several CCPs and corresponding common members, the combination of surviving members leading to the least costs to the entire network of CCPs and members are the takers of the defaulted portfolios. In such a situation, the CCP where the member 0 has defaulted identifies the optimal taker hence no auction resolution is effectively run. Any potential loss that can materialize following such default scenario and taker-over afterwards is allocated to members pro-rata to their default fund contribution. The latter is calculated based on the position of each member so that any change of the positions following a default resolution impacts the allocation coefficients on top of their portfolios update due to the defaulting member taken out from the network.

A Notation and problem formulation

To probabilistically model all CCP related risks, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-vector space $\mathfrak{B} \subset L^1(\Omega)$ with corresponding set of probability measures noted $\mathcal{P}_{\mathfrak{B}}$. For any scalar r.v. $X \in \mathbb{R}$, we denote by $P_X \in \mathcal{P}_{\mathbb{R}}$ its distribution under \mathbb{P} .

- $L + 1$ is the number of clearing members, indexed by $\ell = 0, 1, \dots, L$, in the network that can

have access to CCPs and K is the number of CCPs⁴, indexed by $k = 1, \dots, K$; the member that will be considered in default state is indexed by 0;

- we denote by τ_ℓ the random default time of member ℓ for any $\ell = 1, \dots, L$ generated by some latent factor X_ℓ breaching a certain constant upper threshold B_ℓ , typically the liability level; we do not need such notation for member 0 as it is considered in default state deterministically; the survival event of member ℓ is described as $\{\tau_\ell > T\} = \{X_\ell < B_\ell\}$ for some maturity time T and with the corresponding survival indicator $J_\ell = \mathbb{1}_{\{\tau_\ell > T\}}$;
- any combination of takers with the assumption of a single defaulter and K corresponding portfolios is denoted $\mathbf{i} = \{i_1, \dots, i_K\} \in \{1, \dots, L\}^K =: \mathcal{I}$; we implicitly assume that the defaulted member 0 is exposed to all CCPs $k = 1, \dots, K$;
- any member ℓ 's portfolio towards CCP k is given by a r.v. $Y_\ell^k \in \mathbb{R}$ with $Y_\ell^{k,m}$ for an m -simulated value among M pre-simulated values for each portfolio;
- $\mathcal{L}^{[k]}$ denotes the overall potential loss across members of the CCP k prior to any *default resolution* and by $\mathcal{L}^{[k, i_k]}$ the corresponding loss post default resolution where member i_k of the k -th CCP takes over the defaulted portfolio;
- $\omega_\ell^{[k]}$ is the allocation coefficient of member ℓ on the k -th CCP prior to any default resolution, and $\omega_\ell^{[k, i_k]}$ is the allocation coefficient post default resolution with member i_k taking over the defaulted portfolio on CCP k ;
- prior to default resolution, we write $\mathcal{L}_\ell^{[k]} = \omega_\ell^{[k]} \mathcal{L}^{[k]}$ for the allocated loss to member ℓ by the CCP k ; $\mathcal{L}_\ell^{[k, i_k]} = \omega_\ell^{[k, i_k]} \mathcal{L}^{[k, i_k]}$ corresponds to the allocated loss to member ℓ by the CCP k in case member i_k takes over the corresponding defaulted portfolio;
- for any member ℓ , its aggregated loss over all CCPs after defaulted portfolio allocation writes $\sum_k \mathcal{L}_\ell^{[k, i_k]}$ and is a nonnegative real-valued r.v.;
- for any member ℓ we define the corresponding survival measure \mathbb{Q}^ℓ and corresponding expectation \mathbb{E}^ℓ so that for any measurable variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{E}^\ell[X] = \mathbb{E}[X \mathbb{1}_{\{\tau_\ell > T\}}] / \mathbb{P}(\tau_\ell > T)$;
- in terms of stochastic factors, and prior to default of member 0, we assume that $\mathcal{L}_\ell^{[k]}$ is a function of each CCP-member payoffs $(Y_0^k, Y_1^k, \dots, Y_L^k)$ that can be easily generated (e.g. elliptical r.v.'s) so that

$$\mathcal{L}_\ell^{[k]} = f_\ell^k(Y_0^k, Y_1^k, \dots, Y_L^k), \quad (1)$$

for some loss function $f_\ell^k : \mathbb{R}^{L+1} \rightarrow \mathbb{R}$ defined prior to default of member 0. For example, in the case of a member $\ell \in 1 \dots L$ exposed to CCP k , with some deterministic real-valued constants b_ℓ^k, c_ℓ^k , its credit loss towards the other members writes

$$f_\ell^k(Y_0^k, Y_1^k, \dots, Y_L^k) = \frac{b_\ell^k \mathbb{1}_{\{\tau_\ell > T\}}}{\sum_{\ell''=0}^L b_{\ell''}^k \mathbb{1}_{\{\tau_{\ell''} > T\}}} \sum_{\ell''=0}^L (Y_{\ell''}^k - c_{\ell''}^k)^+ \mathbb{1}_{\{\tau_{\ell''} \leq T\}}, \quad (2)$$

with the convention $b_{\ell''}^k = c_{\ell''}^k = 0$ as well as $Y_{\ell''}^k = 0$ for members ℓ'' not exposed to the CCP k . We do not consider such loss for member 0 as it will be assumed in default state;

⁴The extreme cases are the network configuration where each of the L members has access to the K CCPs and a more practical case is when each CCP k has a different number of clearing members $L_k \ll L$

- when the defaulted portfolio of the member 0 is taken over by member i_k on CCP k , his payoff variable becomes $Y_{i_k}^k + Y_0^k$ and other payoffs (for other member $i \neq i_k$) remain unchanged. We adopt a short notation for this "taking over" operation: we write $\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0$ the updated version of $\mathbf{Y} := \{Y_\ell^k\}_{\substack{1 \leq \ell \leq L \\ 1 \leq k \leq K}} \in \mathbb{R}^{LK}$ where the components Y_0^1, \dots, Y_0^K are added to $Y_{i_1}^1, \dots, Y_{i_K}^K$ respectively and all other components are kept unchanged; hence

$$\begin{cases} \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_{i_k}^k &= Y_{i_k}^k + Y_0^k, \\ \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_i^k &= Y_i^k, \quad \text{for any } i \notin \{i_1, \dots, i_K\} \text{ and CCP } k; \end{cases} \quad (3)$$

several of the indices among (i_1, \dots, i_K) can represent the same member that can take over a defaulting portfolio on several CCPs. In particular, (2) becomes

$$\begin{aligned} & f_\ell^k \left(\left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_1^k, \dots, \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_L^k \right) \\ &= \frac{b_\ell^k \mathbb{1}_{\{\tau_\ell > T\}}}{\sum_{\ell'=1}^L b_{\ell'}^k \mathbb{1}_{\{\tau_{\ell'} > T\}}} \sum_{\ell''=1}^L \left(\left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_{\ell''}^k - c_{\ell''}^k \right)^+ \mathbb{1}_{\{\tau_{\ell''} \leq T\}}, \end{aligned} \quad (4)$$

where to reduce notations, we have reused f_ℓ^k for defining a function of L arguments instead of $L + 1$ and used the same notations for the constants $b_\ell^k, c_\ell^k, \ell \in 1 \dots L$ though $b_{i_k}^k$ and $c_{i_k}^k$ have to be updated to reflect member i_k has taken-over the defaulted portfolio on CCP k . To alleviate notations, we also omit the dependence of f_ℓ^k w.r.t. latent variables X_1, \dots, X_n that translate the corresponding default events $\{\tau_1 \leq T\}, \dots, \{\tau_L \leq T\}$ as they are assumed to remain unchanged after the default resolution with the combinations of takers \mathbf{i} ;

- we use $g_\ell(\cdot) = \sum_{k=1}^K f_\ell^k(\cdot)$ with $f_\ell^k(\cdot)$ being the post-default loss functions, see for instance (4), $\ell \in 1 \dots L, k \in 1 \dots K$, to express more generically the aggregated trading loss for member ℓ as

$$\mathcal{L}_\ell^{[\mathbf{i}]} := g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right); \quad (5)$$

- each member ℓ manages its overall risk towards the network of CCPs by applying a certain risk measure \mathcal{R}_ℓ on its aggregated loss $\mathcal{L}_\ell^{[\mathbf{i}]} = g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)$, e.g., among other possible choices, expectiles, entropic risk measures, expectations, quantiles (i.e. a VaR-type measure) or expected shortfalls under the survival measure \mathbb{Q}^ℓ . Such risk measure reflects the needed capital to sustain the updated portfolio position for member ℓ w.r.t. all positions it detains against the K CCPs, discarding any other types of positions for simplicity, but that can be added at no additional theoretical cost (e.g. bilateral OTC and other credit portfolios, see (Bastide et al., 2023a, Section 2)).

The ideal takeover across all CCPs should comprise of the set of takers $i_1^*, \dots, i_K^* \in 1 \dots L$ such that the aggregated measure of risks over the CCPs' members is minimal i.e.

$$(i_1^*, \dots, i_K^*) = \arg \min_{\mathbf{i}=(i_1, \dots, i_K)} \sum_{\ell=1}^L \mathcal{R}_\ell \left(\underbrace{\sum_{k=1}^K \mathcal{L}_\ell^{[k, i_k]}}_{=g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)} \right). \quad (6)$$

A risk measure of choice for \mathcal{R}_ℓ , $\ell = 1, \dots, L$, that is coherent (Acerbi and Tasche, 2002), is the expected shortfall (under the survival measure \mathbb{Q}^ℓ). Under certain mild regularity conditions⁵ the expected shortfall applied to the loss $g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0)$, using (Bastide et al., 2023a, Lemma 3.10), writes

$$\begin{aligned} \mathbb{E}\mathcal{S}_\ell^{\alpha_\ell} \left[g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right] &= \frac{\mathbb{E} \left[g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \mathbb{1}_{\left\{ g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \geq \mathcal{Q}_{\alpha_\ell}^\ell \left(g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right) \right\}} \right]}{\mathbb{P} \left(J_\ell = 1, g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \geq \mathcal{Q}_{\alpha_\ell}^\ell \left(g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right) \right)} \\ &= \frac{\mathbb{E} \left[g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \mathbb{1}_{\left\{ g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \geq \mathcal{Q}_{\alpha_\ell}^\ell \left(g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right) \right\}} \right]}{\mathbb{P}(\tau_\ell > T)(1 - \alpha_\ell)}, \end{aligned} \quad (7)$$

with $\mathcal{Q}_{\alpha_\ell}^\ell \left(g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right)$ the quantile level from the distribution of $g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0)$ at confidence level α_ℓ under \mathbb{Q}^ℓ (typically $\alpha_\ell = 99.75\%$) and using $\mathbb{P} \left(g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \geq \mathcal{Q}_{\alpha_\ell}^\ell \left(g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right) \mid J_\ell = 1 \right) = 1 - \alpha_\ell$ by definition. In practice, though the expected shortfall measure has gained momentum in its usage preference by supervisors and regulators for market risk (Bank for International Settlements, 2013; Bank For International Settlements, 2019b), large banking institutions still prefer $\mathcal{R}_\ell = \mathcal{Q}_{\alpha_\ell}^\ell$ rather than $\mathbb{E}\mathcal{S}_\ell^{\alpha_\ell}$.

Capital costs have been the sole cost considered for minimization, excluding costs for funding margins and indirect credit risk to other members in the optimisation problem based on the following observations:

- margin costs are specific to each member and for each CCP a member is exposed to, irrespective from the other exposures in terms of calculation. Moreover, it is not of priority interest, as practically members may require initial margin [IM] on each of their clients for clearing their position whilst members are required by the CCP to post IM on their aggregated positions leading to clients margin levels conservative compared to the CCP requirement made to the member⁶.
- credit costs are calculated for each CCP any member is exposed to and taking into consideration all other members' positions of that same CCP. However, cost calculations are separated i.e. conducted for each CCP irrespective of the other CCPs the member is exposed to. This is due to the use of expectation measure for calculating such cost. Also in practice such cost has an order of magnitude significantly less than the capital costs.
- capital costs, in the form of KVA, is the most complex element as it requires to account for former costs of each member across all CCPs and are calculated considering the overall position of the member across all CCPs.

B Simulated annealing problem formulation

For a member ℓ , considering the takeover of the defaulted portfolio on each CCP k by the combination of takers \mathbf{i} , its individual cost function is $\mathcal{R}_\ell \left[g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right]$. The aggregated cost function across all members is $\sum_{\ell=1}^L \mathcal{R}_\ell \left[g_\ell(\mathbf{Y} \oplus_i \mathbf{Y}_0) \right]$ so that the minimization problem writes as a combinatorial

⁵continuity of the loss variable distribution around the quantile reference level

⁶this is in particular due to the high cost entailed by the client segregated accounts.

problem on the space of combinations of takers $\mathcal{I} = \{1, \dots, L\}^K$

$$\min_{\mathbf{i} \in \{1, \dots, L\}^K} \underbrace{\sum_{\ell=1}^L \mathcal{R}_\ell \left(g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right)}_{=: H(\mathbf{i})}, \text{ with } \mathcal{I}^{opt} = \arg \min_{\mathbf{i} \in \mathcal{I}} \sum_{\ell=1}^L \mathcal{R}_\ell \left(g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right). \quad (8)$$

$H(\mathbf{i}) = \sum_{\ell=1}^L \mathcal{R}_\ell \left(g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right)$ can be interpreted as the Hamiltonian evaluated at state \mathbf{i} in the state space \mathcal{I} .

More generally, for minimizing the aggregated cost across all members of the network, one can recourse to simulated annealing method where, for a given combination of takers \mathbf{i} , the energy function is given by the Hamiltonian

$$H : \mathcal{I} \longrightarrow \mathbb{R} \quad (9)$$

$$\mathbf{i} \longmapsto \sum_{\ell=1}^L \mathcal{R}_\ell \left(g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right). \quad (10)$$

But in our problem, we do not have access to the exact evaluation of (9), only to an approximated version which uses M samples. Let \mathcal{R}_ℓ^M be the empirical version of \mathcal{R}_ℓ for each member ℓ . Using an approximated Hamiltonian, the approximated problem to solve is

$$\text{find } \arg \min_{\mathbf{i} \in \{1, \dots, K\}^L} \underbrace{\sum_{\ell=1}^L \mathcal{R}_\ell^M \left[g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right]}_{=: H_M(\mathbf{i})}. \quad (11)$$

We thus denote by \mathbf{Y}^m and \mathbf{Y}_0^m the m -th simulated value of \mathbf{Y} and \mathbf{Y}_0 respectively. Note that sampling all samples ($\mathbf{Y}^m, \mathbf{Y}_0^m : 1 \leq m \leq M$) will be done once for all before the optimization routine: alternatively, regenerating on the fly new samples would have been more costly; in addition for optimization purposes it is recommended to use *Common Random Numbers*, see [Glasserman and Yao \(1992\)](#). Similarly to the original problem, we define $\mathcal{I}_M^{opt} = \arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$. The problem (11) at hand requires a minimization of a discrete function where the discrete space \mathcal{I} can be computationally tedious and time consuming to explore. Even if such state space is reduced to its most plausible outcomes, one would still have to handle a large number of combinations possibilities in the order of $200^{50} \approx 1.13 \cdot 10^{115}$, 200 being the number of main financial actors and 50 the number of the major CCP services.

Finally, relying on formulation of non-asymptotic concentration inequality (see e.g. [Boucheron et al. \(2013\)](#)), we denote by $B : \mathbb{R}_+^* \times \mathbb{N}^* \rightarrow \mathbb{R}_+^*$, the function such that, for any $\varepsilon > 0$, for any $\mathbf{i} \in \mathcal{I}$, for any $M \in \mathbb{N}^*$,

$$\mathbb{P} \left(\sqrt{M} |H_M(\mathbf{i}) - H(\mathbf{i})| > \varepsilon \right) \leq B(\varepsilon, M). \quad (12)$$

In particular, thanks to a union bound argument (see Lemma 6) below, for any r.v. \mathbf{I} with values in \mathcal{I} , for any $\varepsilon > 0$, for any $M \in \mathbb{N}^*$, we have

$$\mathbb{P} \left(\sqrt{M} |H_M(\mathbf{I}) - H(\mathbf{I})| > \varepsilon \right) \leq |\mathcal{I}| B(\varepsilon, M). \quad (13)$$

In practice, a specific level of confidence δ is considered so that $B(\varepsilon, M) = \delta$. Hence ε should be taken to attain such confidence level i.e. setting $\varepsilon = \varepsilon(\delta, M)$ which leads to

$$\mathbb{P} \left(\sqrt{M} |H_M(\mathbf{i}) - H(\mathbf{i})| > \varepsilon(\delta, M) \right) \leq \delta. \quad (14)$$

We additionally require $B(\varepsilon, M) \xrightarrow{\varepsilon \rightarrow \infty} 0$ for any $M > 0$ and if $B(\varepsilon, M)$ indeed depends on M , $B(\varepsilon, M) \xrightarrow{M \rightarrow \infty} 0$ for any $\varepsilon > 0$.

Example 1. Reiss (1989, Lemma 3.1.1, page 84) Let $U_{1:M} \leq \dots \leq U_{M:M}$ be the sorted order statistics of M i.i.d. uniform random variables on $(0, 1)$, $\alpha_M^r = \frac{r}{M+1}$, $r \in \{1, \dots, M\}$. For every $\varepsilon \geq 0$ and $r \in \{1, \dots, M\}$, we have

$$\mathbb{P} \left(\frac{M^{1/2}}{\sqrt{\alpha_M^r(1-\alpha_M^r)}} (U_{r:M} - \alpha_M^r) \begin{array}{l} \leq -\varepsilon \\ \geq \varepsilon \end{array} \right) \leq \exp \left(-\frac{\varepsilon^2}{3(1+\varepsilon/\sqrt{M\alpha_M^r(1-\alpha_M^r)})} \right).$$

For order statistics $X_{1:M} \leq \dots \leq X_{M:M}$ of M i.i.d. random variables of common c.d.f. F_X , we get

$$\begin{aligned} \mathbb{P} \left(\frac{M^{1/2}g(\alpha_M^r)}{\sqrt{\alpha_M^r(1-\alpha_M^r)}} (X_{r:M} - F_X^{-1}(\alpha_M^r)) \begin{array}{l} \leq -\varepsilon \\ \geq \varepsilon \end{array} \right) &\leq \\ \mathbb{P} \left(\frac{M^{1/2}}{\sqrt{\alpha_M^r(1-\alpha_M^r)}} (U_{r:M} - \alpha_M^r) \begin{array}{l} \leq h(-\varepsilon, M, \alpha_M^r) \\ \geq h(\varepsilon, M, \alpha_M^r) \end{array} \right) &\end{aligned} \quad (15)$$

where $g(\alpha_M^r)$ is a nonnegative constant and $h(\varepsilon, M, \alpha_M^r) = \frac{\sqrt{M}}{\sqrt{\alpha_M^r(1-\alpha_M^r)}} \left[F_X \left(F_X^{-1}(\alpha_M^r) + \varepsilon \frac{\sqrt{\alpha_M^r(1-\alpha_M^r)}}{\sqrt{M}g(\alpha_M^r)} \right) - \alpha_M^r \right]$. We can show that we can still recover a relationship of the type given in (12). Defining $E(\varepsilon, \alpha_M^r, M) := \varepsilon \frac{\sqrt{\alpha_M^r(1-\alpha_M^r)}}{g(\alpha_M^r)}$ and

$B(\varepsilon, \alpha_M^r, M) := \exp \left(-\frac{h(\varepsilon, M, \alpha_M^r)^2}{3(1+h(\varepsilon, M, \alpha_M^r)/(\sqrt{M\alpha_M^r(1-\alpha_M^r)}))} \right)$, (15) writes more compactly

$$\mathbb{P} \left(\sqrt{M} \begin{array}{l} (X_{r:M} - F_X^{-1}(\alpha_M^r)) \leq E(-\varepsilon, \alpha_M^r, M) \\ (X_{r:M} - F_X^{-1}(\alpha_M^r)) \geq E(\varepsilon, \alpha_M^r, M) \end{array} \right) \leq \frac{B(-\varepsilon, M, \alpha_M^r)}{B(\varepsilon, M, \alpha_M^r)}. \quad (16)$$

Denoting $C(M, \alpha_M^r) = \frac{\sqrt{\alpha_M^r(1-\alpha_M^r)}}{g(\alpha_M^r)} > 0$, and $\tilde{B}(\varepsilon, M, \alpha_M^r) := B(\varepsilon/C(M, \alpha_M^r), M, \alpha_M^r)$, (16) can be equivalently expressed for any $\varepsilon > 0$ as

$$\mathbb{P} \left(\sqrt{M} \begin{array}{l} (X_{r:M} - F_X^{-1}(\alpha_M^r)) \leq -\varepsilon \\ (X_{r:M} - F_X^{-1}(\alpha_M^r)) \geq \varepsilon \end{array} \right) \leq \frac{\tilde{B}(-\varepsilon, M, \alpha_M^r)}{\tilde{B}(\varepsilon, M, \alpha_M^r)}. \quad (17)$$

Example 2. Chamakh, Gobet, and Liu (2021, Corollary 2.3) Let $\beta \in (1, +\infty)$. Then for any X_1, \dots, X_M i.i.d. in $L_{\Psi_\beta^{\text{HT}}}(\mathbb{R}) := \left\{ X : \Omega \rightarrow \mathbb{R} \text{ s.t. } \|X\|_{\Psi_\beta^{\text{HT}}} < +\infty \right\}$ with $\|X\|_{\Psi_\beta^{\text{HT}}} = \inf \{ c > 0 : \mathbb{E} [\exp((\ln(|X|/c + 1))^\beta) - 1] \leq 1 \}$,

$$\mathbb{P} \left(\sqrt{M} \left| \frac{1}{M} \sum_{m=1}^M X_m - \mathbb{E}[X] \right| \geq \varepsilon \right) \leq 2 \exp \left(-(\ln(1 + \kappa\varepsilon))^\beta \right), \quad (18)$$

where $\kappa > 0$ is some constant depending on $\|X_1\|_{\Psi_\beta^{\text{HT}}}$ and $\|X_1\|_{L_2(\mathcal{B})}$, but not on M . Hence the r.h.s. of (18) does not depend on M , contrary to (17).

Remark 3. Under the assumptions of (Chamakh et al., 2021, Corollary 2.3), $\frac{1}{M} \sum_{m=1}^M X_m$ is an unbiased estimator of $\mathbb{E}[X]$. Also, if the distribution of X is continuous and strictly increasing around its quantile level $Q_\alpha(X)$ for some confidence level $\alpha \in (0, 1)$, the corresponding expected shortfall $\mathbb{ES}_\alpha(X)$ can be approximated by an empirical unbiased mean estimator as well provided the corresponding quantile $Q_\alpha(X)$ at confidence level α is known (Acerbi and Székely, 2014, Lemma A.1).

Hence (18) applies for an expected shortfall by replacing X with $\frac{1}{1-\alpha}X\mathbb{1}_{\{X \geq \mathcal{Q}_\alpha(X)\}}$ assuming $\mathcal{Q}_\alpha(X)$ is indeed known (where X is a loss random variable and α is a very high confidence level). If the quantile level is unknown, [Bartl and Eckstein \(2024\)](#) proposes a biased but *adversarially robust*⁷ estimator of the expected shortfall for which a concentration inequality can be obtained ([Bartl and Eckstein, 2024](#), Theorem 1.3). Their approach relies on a quantile-of-means estimator (including the well-known median-of-means, see e.g. [Lugosi and Mendelson \(2019\)](#)), for random variables that have a square-integrable positive part with continuous distribution around the quantile level of the considered expected shortfall.

In the case of the use of an approximated algorithm for minimizing $H_M(\mathbf{i})$, it is not straightforward to enforce a certain probability bound level δ from which to infer $\varepsilon(\delta, M)$ as in (14). This is due to the additional error brought by the use of the approximated algorithm.

§3 Simulated annealing algorithm

In this section, we outline main known results on simulated annealing algorithm applied to discrete state space and useful for our combinatorial problem. Less critical yet key known results material are deferred to Appendix A.

A Algorithm construction

For a given energy function H , either given by $\mathcal{Q}_{\alpha_\ell}^\ell \left(g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right)$ or $\mathbb{E}\mathcal{S}_{\alpha_\ell}^\ell \left(g_\ell \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right) \right)$ as per (7), we consider the minimization problem

$$\text{find } \mathbf{i}^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i}), \text{ with } H : \mathcal{I} \longrightarrow \mathbb{R}. \quad (19)$$

The simulated annealing algorithm relies on the Boltzmann-Gibbs distribution. It expresses the probability for a solid subject to a sudden high temperature heat bath to reach a certain thermal equilibrium energy state as the heat bath temperature applied to the solid is slowly reduced. As the temperature applied to the solid reduces slowly, the Boltzmann-Gibbs distribution concentrates most of its mass on the lowest energy states ([Aarts and van Laarhoven, 1989](#), Chapter 2). As such this principle can be used for determining the global minimum of a complex function with no particular regularities.

Definition 1. [Delmas and Jourdain \(2006, Definition 2.3.1, page 37\)](#) The *Boltzmann-Gibbs measure* associated to an energy function H and a temperature parameter $T > 0$ is the probability measure $(\mu_T(\mathbf{i}), \mathbf{i} \in \mathcal{I})$ given by:

$$\mu_T(\mathbf{i}) = \frac{1}{Z_T} e^{-H(\mathbf{i})/T}, \quad (20)$$

with $Z_T = \sum_{\mathbf{i} \in \mathcal{I}} e^{-H(\mathbf{i})/T}$ the normalization constant, also called the partition function. Moreover, for a given $n \in \mathbb{N}^*$ and corresponding real-valued constant c_n such that T is replaced by c_n , we write

$$\mu_n(\mathbf{i}) = \frac{1}{Z_{c_n}} e^{-H(\mathbf{i})/c_n}, \quad (21)$$

with $Z_{c_n} = \sum_{\mathbf{i} \in \mathcal{I}} e^{-H(\mathbf{i})/c_n}$.

⁷i.e. even if a fraction of the sample is changed, it is still possible to have concentration bounds similar to the ones for the unaltered sample ([Bartl and Eckstein, 2024](#), Theorem 1.7)

The steps of the algorithm, running for N iterations and approximating \mathbf{i}^* by \mathbf{i}^N after such number of iterations, are given as per below

- Starting from the former generated state $\mathbf{I}_{n-1} \in \mathcal{I}$ at iteration step $n - 1$ of the algorithm, with a non-null probability, the state \mathbf{I}' is suggested at time n , in some neighborhood of \mathbf{I}_{n-1} noted $\mathcal{O}(\mathbf{I}_{n-1})$ (to be precised later on), based on some given generation (transition) probability distribution – a.k.a. proposal distribution –, for instance for $\mathbf{I}_{n-1} = \mathbf{i}$, by the finite probability coefficient sets $(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{j} \in \mathcal{O}(\mathbf{i})}$ with $(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}$ a stochastic matrix i.e. for any $\mathbf{i} \in \mathcal{I}$, $\sum_{\mathbf{j} \in \mathcal{O}(\mathbf{i})} P_n(\mathbf{i}, \mathbf{j}) = 1$ and $P_n(\mathbf{i}, \mathbf{j}) := \mathbb{P}(\mathbf{I}_n = \mathbf{j} | \mathbf{I}_{n-1} = \mathbf{i})$;
- a temperature schedule $(c_n)_{n \geq 0}$ is given so that at each time n , a low-speed cooling coefficient can be used to accelerate the algorithm convergence, typically $c_n = h / \log(n + 1)$; we recall in Appendix A Theorem 18 (Hajek, 1988, Theorem 1) and Proposition 17 (Delmas and Jourdain, 2006, Theorem 2.3.8) that this logarithm cooling sequence (for h large enough) leads to asymptotic convergence to the minimum (as the number of iteration goes to ∞), but it is frequent to use other cooling sequences that can better find the minimum in a given number of iterations;
- for all $\mathbf{j} \in \mathcal{O}(\mathbf{i})$, assuming the target probability to reach is μ_n given by (21) and given some irreducible Markov chain with probabilities $(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}$ (the generation probabilities), we consider the acceptance probability quantity, depending on n , and noted $A_n(\mathbf{i}, \mathbf{j})$ set to

$$A_n(\mathbf{i}, \mathbf{j}) = \min \left(\frac{\mu(\mathbf{j})P_n(\mathbf{j}, \mathbf{i})}{\mu(\mathbf{i})P_n(\mathbf{i}, \mathbf{j})}, 1 \right), \quad \mathbf{i}, \mathbf{j} \in \mathcal{I}, \quad (22)$$

with the convention $A_n(\mathbf{i}, \mathbf{j}) = 1$ if $\mu(\mathbf{i})P_n(\mathbf{i}, \mathbf{j}) = 0$. The acceptance probability is used with a uniform r.v. $U_n \sim \mathcal{U}(0, 1)$, independent of all other r.v.'s, to form a Bernoulli distributed r.v. of parameter $A_n(\mathbf{i}, \mathbf{j})$. At time n , and assuming $\mathbf{I}_{n-1} = \mathbf{i}$, the value \mathbf{I}' is proposed based on the (discrete and irreducible) generation probability distribution $(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{j} \in \mathcal{O}(\mathbf{i})}$. It allows building a Markov chain verifying the reversible condition w.r.t. μ_n so that the latter is its invariant distribution (Ross, 2010, Section 4.9, p. 260). As outlined in (Nikolaev and Jacobson, 2010, Section 1.2.1), for such reversibility condition to be verified, it is necessary that the acceptance ratio obeys the multiplicative property, necessary for proving the simulated annealing algorithm convergence (Aarts and van Laarhoven, 1989, Theorem 2), (Aarts and Korst, 1989, Theorem 3.3), (Faigle and Kern, 1991, Section 2):

$$\forall \mathbf{i}, \mathbf{l}, \mathbf{j} \in \mathcal{I}, H(\mathbf{i}) \leq H(\mathbf{l}) \leq H(\mathbf{j}) \Rightarrow A_n(\mathbf{i}, \mathbf{j}) = A_n(\mathbf{i}, \mathbf{l})A_n(\mathbf{l}, \mathbf{j}). \quad (23)$$

The particular form of acceptance probabilities, initially considered in Kirkpatrick et al. (1982), and given by

$$A_n(\mathbf{i}, \mathbf{j}) := \exp \left\{ -(H(\mathbf{j}) - H(\mathbf{i}))^+ / c_n \right\}, \quad (24)$$

indeed verifies (23).

- the next iteration of the process \mathbf{I} is defined as:

$$\mathbf{I}_n = \mathbf{I}' \mathbb{1}_{\{U_n \leq A_n(\mathbf{I}_{n-1}, \mathbf{I}')\}} + \mathbf{I}_{n-1} \mathbb{1}_{\{U_n > A_n(\mathbf{I}_{n-1}, \mathbf{I}')\}}. \quad (25)$$

By (Delmas and Jourdain, 2006, Proposition 2.2.1, p. 35), \mathbf{I}_n follows the discrete distribution defined, for all $\mathbf{i} \in \mathcal{I}$ and for all $\mathbf{j} \in \mathcal{O}(\mathbf{i})$, by the probability coefficients

$$Q_n(\mathbf{i}, \mathbf{j}) = \begin{cases} P_n(\mathbf{i}, \mathbf{j})A_n(\mathbf{i}, \mathbf{j}) & \text{if } \mathbf{j} \neq \mathbf{i} \\ 1 - \sum_{\mathbf{j}' \neq \mathbf{i}} Q_n(\mathbf{i}, \mathbf{j}') & \text{if } \mathbf{j} = \mathbf{i}. \end{cases} \quad (26)$$

Now, we can define two Markov chains: first, for each given n , a homogeneous Markov chain with transition probability Q_n ; second, a non-homogeneous Markov chain with a transition probability Q_n at step n (which corresponds to the simulated annealing algorithm). We start by describing a few properties of the fixed- n Markov chain.

Proposition 1. *Delmas and Jourdain (2006, Proposition 2.2.1, page 35) Let n be a fixed integer. The transition probability Q_n in (26) defines a homogeneous Markov Chain $(X_k^{(n)})_{k \geq 0}$ which is reversible w.r.t. μ_n . In particular, μ_n is an invariant probability for the Markov chain $(X_k^{(n)})_{k \geq 0}$.*

The Theorem 2 below outlines the necessary conditions for a homogeneous Markov chain constructed using (26) to result into a Markov chain whose invariant distribution converges towards the distribution putting all mass on the minimizers of the energy function.

Theorem 2. *Aarts and Korst (1989, Theorem 3.3, page 42) Consider the energy function H to minimize on \mathcal{I} with the cooling schedule $(c_n, n \geq 1)$ such that $\lim_{n \rightarrow \infty} c_n = 0$. Assume that for any $n \geq 1$ the acceptance probability function $(A(\mathbf{i}, \mathbf{j}, c_n))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}$ and the generation probabilities $(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}$ are linked through (26) and respectively satisfy*

(i) *the generation transition probability distributions $\{(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}\}_{n \geq 1}$ is irreducible and symmetric*

$$(ii) \quad \forall n \geq 1, \forall \mathbf{i}, \mathbf{j} \in \mathcal{I}, \begin{cases} A(\mathbf{i}, \mathbf{j}, c_n) = 1 & \text{if } H(\mathbf{i}) \geq H(\mathbf{j}) \\ A(\mathbf{i}, \mathbf{j}, c_n) \in (0, 1) & \text{if } H(\mathbf{i}) < H(\mathbf{j}) \end{cases}$$

(iii) $\forall n \geq 1, \forall \mathbf{i}, \mathbf{j}, \mathbf{l} \in \mathcal{I}$ with $H(\mathbf{i}) \leq H(\mathbf{j}) \leq H(\mathbf{l})$, $A(\mathbf{i}, \mathbf{l}, c_n) = A(\mathbf{i}, \mathbf{j}, c_n)A(\mathbf{j}, \mathbf{l}, c_n)$

(iv) $\forall \mathbf{i}, \mathbf{j} \in \mathcal{I}$ with $H(\mathbf{i}) < H(\mathbf{j})$: $\lim_{n \rightarrow \infty} A(\mathbf{i}, \mathbf{j}, c_n) = 0$.

Then the Markov chain $(X_k^{(n)})_{k \geq 0}$ defined by (26) with $A_n(\mathbf{i}, \mathbf{i}) = A(\mathbf{i}, \mathbf{i}, c_n)$ admits an invariant distribution μ_n whose components are given by

$$\mu_n(\mathbf{i}) = \frac{A(\mathbf{i}_0, \mathbf{i}, c_n)}{\sum_{\mathbf{j} \in \mathcal{I}} A(\mathbf{i}_0, \mathbf{j}, c_n)}, \text{ for all } \mathbf{i} \in \mathcal{I}, \quad (27)$$

and for arbitrary $\mathbf{i}_0 \in \mathcal{I}$. Moreover, we have

$$\lim_{n \rightarrow \infty} \mu_n(\mathbf{i}) = \frac{1}{|\mathcal{I}^{opt}|} \mathbb{1}_{\{\mathbf{i} \in \mathcal{I}^{opt}\}}. \quad (28)$$

Remark 4. As outlined in (Henderson et al., 2003, Section 2), the symmetry of the generation probabilities $\{(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}\}_{n \geq 1}$, outlined in (i), is in fact not required as long as the acceptance probabilities verify (iii) in Theorem 3.1 and the generation probabilities reflect a weak reversibility property, introduced in Hajek (1988). According to Faigle and Schrader (1988) and Faigle and Kern (1991), this means that the graph of energy functions of all states is strongly connected in the sense that any pair of vertices $(H(\mathbf{i}), H(\mathbf{j}))$ is connected by a directed graph from \mathbf{i} to \mathbf{j} , $\mathbf{i}, \mathbf{j} \in \mathcal{I}$ and H the energy function, and this assumption suffices for obtaining (28) regarding the homogeneous Markov chain defined by (26).

The case where $A(\mathbf{i}, \mathbf{j}, c_n) = A_n(\mathbf{i}, \mathbf{j})$ is given by (24) leads to μ_n being the Boltzmann-Gibbs distribution (21). The last statement of Theorem 2 indicates that all mass of the limit distribution of the simulated annealing algorithm, if the Doeblin condition is indeed satisfied for the generation probability P , concentrates on the area where the minimum of the energy function is attained. In particular, some Doeblin condition is verified by Q if P satisfies one, cf proof of [Delmas and Jourdain \(2006, Lemma 2.3.6, page 41\)](#).

Remark 5. The reversible property of a homogeneous Markov chain defined by (24) can be obtained by remarking, for $\mathbf{i} \neq \mathbf{j}$, with for any $\mathbf{i}, \mathbf{j} \in \mathcal{I}$, $P_n(\mathbf{i}, \mathbf{j}) > 0$ and as in [Delmas and Jourdain \(2006, proof of Proposition 2.2.1, page 35\)](#),

$$\mu_n(\mathbf{i})Q_n(\mathbf{i}, \mathbf{j}) = \mu_n(\mathbf{i})P_n(\mathbf{i}, \mathbf{j})A_n(\mathbf{i}, \mathbf{j}) \quad (29)$$

$$= \min \left(\mu_n(\mathbf{i})P_n(\mathbf{i}, \mathbf{j}), \mu_n(\mathbf{j})P_n(\mathbf{j}, \mathbf{i}) \right) \quad (30)$$

$$= \mu_n(\mathbf{j})P_n(\mathbf{j}, \mathbf{i}) \min \left(\frac{\mu_n(\mathbf{i})P_n(\mathbf{i}, \mathbf{j})}{\mu_n(\mathbf{j})P_n(\mathbf{j}, \mathbf{i})}, 1 \right) \quad (31)$$

$$= \mu_n(\mathbf{j})P_n(\mathbf{j}, \mathbf{i})A_n(\mathbf{j}, \mathbf{i}) \quad (32)$$

$$= \mu_n(\mathbf{j})Q_n(\mathbf{j}, \mathbf{i}). \quad (33)$$

Remark 6. It is well known that any irreducible homogeneous Markov chain that is defined on \mathcal{I} , which is a finite set, has a unique invariant distribution π , see Theorems 15 and 16 in Appendix A.

We now describe the non-homogeneous Markov chain of the simulated annealing algorithm. In what follows, including our application case, we consider that the generation Markov chain reflected through $\left((P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{j} \in \mathcal{O}(\mathbf{i})} \right)_{n \in \mathbb{N}^*}$ is homogeneous i.e., for all $n \in \mathbb{N}^*$, $(P_n(\mathbf{i}, \mathbf{j}))_{\mathbf{j} \in \mathcal{O}(\mathbf{i})} = (P(\mathbf{i}, \mathbf{j}))_{\mathbf{j} \in \mathcal{O}(\mathbf{i})}$. For very large combinatorial problems, the convergence may be hard to achieve in reasonable amount of time. A finite computational time approach, i.e. the run of the algorithm with a finite given number of steps must be considered instead, as outlined in [Aarts and Korst \(1989, Section 3.3\)](#) and [Catoni \(1992\)](#). Such budget constraint allows to obtain error bounds.

Definition 2. Let \mathbf{i} such that $H(\mathbf{i}) > \mathcal{H}$ where

$$\mathcal{H} := \min_{\mathbf{j} \in \mathcal{I}} H(\mathbf{j})$$

and consider a possible trajectory $\gamma = (\mathbf{i}_0, \dots, \mathbf{i}_n)$ from \mathbf{i} to $\mathbf{i}^* \in \mathcal{I}^{opt}$ with $\mathbf{i}_0 = \mathbf{i}$ and $\mathbf{i}_n = \mathbf{i}^*$ i.e. $\mathbf{i}_n \in \mathcal{I}^{opt}$ and $P(\mathbf{i}_l, \mathbf{i}_{l+1}) > 0$ for $0 \leq l < n$. The energy gap crossed over by this trajectory is

$$\mathbb{H}(\gamma) := \max_{0 \leq l \leq n-1} H(\mathbf{i}_l) - \mathcal{H}.$$

Denote by $\Gamma_{\mathbf{i}}$ the set of all possible trajectories from \mathbf{i} to \mathcal{I}^{opt} and

$$\mathbb{H}_{\mathbf{i}} = \min_{\gamma \in \Gamma_{\mathbf{i}}} \mathbb{H}(\gamma)$$

the minimum of all the energy gap crossed over among all possible trajectories $\Gamma_{\mathbf{i}}$. The quantity

$$\bar{\mathbb{H}} = \max_{\mathbf{i} \in \mathcal{I}} \mathbb{H}_{\mathbf{i}}, \quad (34)$$

is the highest energy gap and depends on both the energy function H and the Markov Chain generation matrix $(P(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}$. We also define

$$\mathbb{H} = \min_{\mathbf{i} \in \mathcal{I}} \mathbb{H}_{\mathbf{i}}$$

the minimum of those energy gaps. Thus for any \mathbf{i} such that $H(\mathbf{i}) > \mathcal{H}$, there exists a trajectory from \mathbf{i} to \mathcal{I}^{opt} for which the energy gap is below or equal to $\overline{\mathbb{H}}$. Let

$$D := \max_{\mathbf{i}: H(\mathbf{i}) > \mathcal{H}} \frac{\overline{\mathbb{H}}_{\mathbf{i}}}{H(\mathbf{i}) - \mathcal{H}} - 1 \geq 0$$

be the difficulty associated to the energy function H .

In particular $D \leq \frac{\overline{\mathbb{H}}}{\mathcal{H}} - 1$ so that D is bounded from both sides for any energy function H . As it can be quantified below, the smaller the gap D , the faster the convergence that can be achieved with a proper cooling schedule.

Theorem 3. *Delmas and Jourdain (2006, Theorem 2.3.9) There are two constants $B_2 \geq B_1 \geq 0$ such that for all $N \geq 1$,*

$$\frac{B_1}{N^{1/D}} \leq \max_{\mathbf{i} \in \mathcal{I}} \inf_{c_0 \geq \dots \geq c_N} \mathbb{P}_{(c_0, \dots, c_N)} (H(\mathbf{I}_N) > \mathcal{H} | \mathbf{I}_0 = \mathbf{i}) \leq \frac{B_2}{N^{1/D}}, \quad (35)$$

where $\mathbb{P}_{(c_0, \dots, c_N)}$ is the probability measure under the considered cooling schedule $c_0 \geq \dots \geq c_N$. And for all $A > 0$, there exists $\delta_A > 0$ such that for all N , the cooling schedule $(c_0^{(N,A)}, \dots, c_N^{(N,A)})$, defined by $c_n^{(N,A)} = \frac{1}{A} \left(\frac{A}{(\log N)^2} \right)^{n/N}$, satisfies

$$\max_{\mathbf{i} \in \mathcal{I}} \mathbb{P}_{(c_0^{(N,A)}, \dots, c_N^{(N,A)})} (H(\mathbf{I}_N) > \mathcal{H} | \mathbf{I}_0 = \mathbf{i}) \leq \delta_A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{1/D}. \quad (36)$$

We recall that the necessary and sufficient conditions for convergence of the simulated annealing algorithm is known (see Proposition 17 and Theorem 18 in Appendix A taken from [Delmas and Jourdain \(2006\)](#) and [Hajek \(1988\)](#) respectively, with logarithm cooling schedule that we refer to as *unlimited budget version*). However, we prefer to focus our analysis on accuracy guarantee at finite range N , using the (triangular) cooling schedule $(c_0^{(N,A)}, \dots, c_N^{(N,A)})$ above that we refer to as *fixed-budget version*. The expression (36) indicates that the worst probability of getting an energy level above the minimum one starting from any $\mathbf{i} \in \mathcal{I}$ is controlled by a bound decreasing in the number of the algorithm number of iterations (but increases with D). As with [Hajek \(1988, Theorem\)](#) in the case where the unlimited budget cooling schedule is of the form $c_n = c / \ln(n + 1)$, [Delmas and Jourdain \(2006, Theorem 2.3.9\)](#) indicates that as N increases, the algorithm based on the fixed-budget cooling schedule $\frac{1}{c_n^{(N,A)}} = A \left(\frac{\log(N)^2}{A} \right)^{n/N}$, $n \leq N$, indeed converges towards the invariant distribution putting all mass on the optimal set \mathbf{I}^{opt} as outlined also by [Aarts and Korst \(1989, Theorem 3.3, page 42\)](#). In particular, whatever the initial distribution of \mathbf{i}_0 is,

$$\begin{aligned} \mathbb{P}(H(\mathbf{I}_N) > \mathcal{H}) &= \sum_{\mathbf{i} \in \mathcal{I}} \mathbb{P}(H(\mathbf{I}_N) > \mathcal{H} | \mathbf{i}_0 = \mathbf{i}) \mathbb{P}(\mathbf{i}_0 = \mathbf{i}) \\ &\leq \max_{\mathbf{i} \in \mathcal{I}} \mathbb{P}(H(\mathbf{I}_N) > \mathcal{H} | \mathbf{i}_0 = \mathbf{i}) \underbrace{\sum_{\mathbf{i} \in \mathcal{I}} \mathbb{P}(\mathbf{i}_0 = \mathbf{i})}_{=1} \\ &\leq \delta_A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{1/D} \xrightarrow{N \rightarrow \infty} 0. \end{aligned} \quad (37)$$

Remark 7. Let $M \in \mathbb{N}$ and consider any empirical estimation H_M of H based on the sample $\mathbf{X}_{1:M} = (X_1, \dots, X_M)$ with X_1, \dots, X_M i.i.d.. Denote by $D_{\mathbf{X}_{1:M}}$ the difficulty associated to the empirical estimate H_M of H with $\mathcal{H}_M = \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$, $\overline{\mathbb{H}}_M$ and $\underline{\mathbb{H}}_M$ the corresponding maximum and minimum gap and $D_{\mathbf{X}_{1:M}}$ the difficulty associated to the function H_M as defined in Definition 2 applied to H_M . From Theorem 3, for any $A > 0$ there exists $\delta_{\mathbf{X}_{1:M}}^A > 0$ such that for all N , using the cooling schedule defined by $c_n^{(N)} = \frac{1}{A} \left(\frac{A}{(\log N)^2} \right)^{n/N}$, $n = 0, \dots, N$, satisfies

$$\mathbb{P}_{\mathbf{X}_{1:M}} (H_M(\mathbf{I}_M^N) > \mathcal{H}_M) \leq \max_{\mathbf{i} \in \mathcal{I}} \mathbb{P}_{\mathbf{X}_{1:M}} (H_M(\mathbf{I}_M^N) > \mathcal{H}_M | \mathbf{I}_0 = \mathbf{i}) \quad (38)$$

$$\leq \delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}, \quad (39)$$

with $\delta_{\mathbf{X}_{1:M}}^A$ and $D_{\mathbf{X}_{1:M}}$ implicitly depending on $\mathbf{X}_{1:M}$. However the considered samples formed of M i.i.d. versions of a r.v. X have the same distributions. To be more specific, for some fixed $A > 0$, for two samples $\mathbf{X}_{1:M}$ and $\mathbf{X}'_{1:M}$ based on two independent sets of i.i.d. versions of a r.v. X , we have that $\mathbf{X}_{1:M}$ and $\mathbf{X}'_{1:M}$ possess the same distribution and:

$$\mathbb{E} \left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}} \right] = \mathbb{E} \left[\delta_{\mathbf{X}'_{1:M}}^A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{\frac{1}{D_{\mathbf{X}'_{1:M}}}} \right]. \quad (40)$$

The latter quantity appears in our bound results in the following section.

B Convergence and error analysis of the approximated problem

We do not have direct access to H but only to an approximated version H_M defined also on \mathcal{I} and which depends on M Monte Carlo samples $\mathbf{X}_{1:M} = (X_1, \dots, X_M)$ (e.g. $X_m = g_\ell \left(\mathbf{Y}^m \bigoplus_{\mathbf{i}} \mathbf{Y}_0^m \right)$, $m \in 1 \dots M$, where $g_\ell(\cdot)$ will be made explicit in Section §4). But the same type of algorithm can find, after N iterations and through a single run, an approximated (random) solution \mathbf{I}_M^N to the approximated problem:

$$\text{find } \mathbf{I}_M^* \in \mathcal{I}_M^{\text{opt}} = \arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i}) \text{ with } H_M : \mathcal{I} \longrightarrow \mathbb{R}. \quad (41)$$

We look at how close the approximated problem algorithm candidate $H_M(\mathbf{I}_M^N)$ to (41) after N iterations of the simulated algorithm is to the true solution $H(\mathbf{i}^*)$ of (19) e.g. how large the difference $|H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)|$ can be with $\mathbf{i}^* \in \mathcal{I}^{\text{opt}}$. Also, we are interested in knowing how likely for the algorithm to be wrong i.e. getting an upper bound for $\mathbb{P}(\mathbf{I}_M^N \neq \mathbf{i}^*)$. We will thus prove the following bounds, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P} \left(\sqrt{M} (H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)) < -\varepsilon \right) &\leq |\mathcal{I}| B(\varepsilon, M), \\ \mathbb{P} \left(\sqrt{M} (H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)) > \varepsilon \right) &\leq \mathbb{E} \left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}} \right] + B \left(\frac{\varepsilon}{2}, M \right), \end{aligned}$$

and

$$\mathbb{P} (\mathbf{I}_M^N \notin \mathcal{I}_M^{\text{opt}}) \leq \mathbb{E} \left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}} \right],$$

$$\begin{aligned} \mathbb{P}(I_M^N \notin \mathcal{I}^{opt}) &\leq |\mathcal{I}|B \left(\frac{\sqrt{M}(\mathcal{H}_2 - \mathcal{H})}{2}, M \right) + \mathbb{E} \left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}} \right] \\ &\quad + B \left(\frac{\sqrt{M}(\mathcal{H}_2 - \mathcal{H})}{4}, M \right), \end{aligned}$$

with $\mathcal{H}_2 := \min_{\mathbf{i} \neq \mathbf{i}^*} H(\mathbf{i})$ the second minimum value of H . These bounds written in these forms only inform about the convergence to zero of an error from both the approximated function value in \mathbb{R} and the approximated solution in \mathcal{I} as M and N tend to zero. Due to the bound constants $\delta_{\mathbf{X}_{1:M}}^A$ and $1/D_{\mathbf{X}_{1:M}}$, it seems delicate to infer an error level for ε from a given probability the way written in (14). This is left to further investigation.

Lemma 4. For any r.v.s $X, Y_i \in \mathfrak{X}$ with $i = 1, \dots, n$ such that $|X| \leq \sum_{i=1}^n |Y_i|$, and any $\varepsilon > 0$, we have

$$\mathbb{P}(|X| > \varepsilon) \leq \sum_{i=1}^n \mathbb{P}\left(|Y_i| > \frac{\varepsilon}{n}\right) \quad (42)$$

Proof. By assumption, $\{|X| > \varepsilon\} \subset \{\sum_{i=1}^n |Y_i| > \varepsilon\}$. Moreover,

$$\left[\sum_{i=1}^n |Y_i| > \varepsilon \right] \Rightarrow \left[|Y_1| > \frac{\varepsilon}{n} \text{ or } \dots \text{ or } |Y_n| > \frac{\varepsilon}{n} \right], \quad (43)$$

which can be proven by contradiction. Hence, we get

$$\mathbb{P}(|X| > \varepsilon) \leq \mathbb{P}\left(\sum_{i=1}^n |Y_i| > \varepsilon\right) \leq \mathbb{P}\left(\bigcup_{i=1}^n |Y_i| > \frac{\varepsilon}{n}\right) \leq \sum_{i=1}^n \mathbb{P}\left(|Y_i| > \frac{\varepsilon}{n}\right), \quad (44)$$

where we have used (43) for the second inequality. \square

Let \mathbf{I}_M^N be some random solution identified by the simulated annealing algorithm after N iterations relying on the samples $\mathbf{X}_{1:M}$ to estimate $H(\cdot)$ from $H_M(\cdot)$. This means that $H_M(\mathbf{I}_M^N)$ writes as a measurable function f of the samples X_1, \dots, X_N , the member proposal on each CCP at each iteration step, based on uniforms $U_1^1, \dots, U_1^K, \dots, U_N^1, \dots, U_N^K$, and acceptance uniform random variables U_1^A, \dots, U_N^A . It may also depend on an initial random combination guess of takers based again on uniforms U_0^1, \dots, U_0^K . Then one can express

$$H_M(\mathbf{I}_M^N) = f(X_1, \dots, X_M, U_0^1, \dots, U_0^K, U_1^1, \dots, U_1^K, \dots, U_N^1, \dots, U_N^K, U_1^A, \dots, U_N^A).$$

If the initial guess is deterministic, as it will be the case in our use case, then the former expression simplifies into

$$H_M(\mathbf{I}_M^N) = f(X_1, \dots, X_M, U_1^1, \dots, U_1^K, \dots, U_N^1, \dots, U_N^K, U_1^A, \dots, U_N^A). \quad (45)$$

Note that \mathbf{I}_M^N should also be understood as a measurable function of $X_1, \dots, X_M, U_0^1, \dots, U_0^K, U_1^1, \dots, U_1^K, \dots, U_N^1, \dots, U_N^K$ (or $X_1, \dots, X_M, U_1^1, \dots, U_1^K, \dots, U_N^1, \dots, U_N^K$ if the initial guess is deterministic).

Lemma 5. Let $\mathbf{i}^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i})$, $\mathbf{I}_M^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$ and \mathbf{I}_M^N be the solution found by the algorithm to minimize $H_M(\cdot)$ over \mathcal{I} after N iterations with $H_M(\mathbf{i})$ an empirical estimation of $H(\mathbf{i})$,

$\mathbf{i} \in \mathcal{I}$, based on M i.i.d. samples X_1, \dots, X_M . We have

$$\underbrace{H_M(\mathbf{I}_M^N) - H(\mathbf{I}_M^N)}_{\text{empirical error}} \leq H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*) \leq \underbrace{H_M(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^*)}_{\text{algorithmic error}} + \underbrace{H_M(\mathbf{i}^*) - H(\mathbf{i}^*)}_{\text{empirical error}}. \quad (46)$$

Proof. Let $\mathbf{I}_M^* \in \mathcal{I}_M^{\text{opt}}$. For the upper bound, we have

$$\begin{aligned} H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*) &= H_M(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^*) + \underbrace{H_M(\mathbf{I}_M^*) - H_M(\mathbf{i}^*)}_{\leq 0} + H_M(\mathbf{i}^*) - H(\mathbf{i}^*) \\ &\leq H_M(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^*) + H_M(\mathbf{i}^*) - H(\mathbf{i}^*). \end{aligned} \quad (47)$$

Moreover, as $H(\mathbf{i}^*) \leq H(\mathbf{I}_M^N)$, we get for the lower bound

$$H_M(\mathbf{I}_M^N) - H(\mathbf{I}_M^N) \leq H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*). \quad (48)$$

Combining (48) with (47) yields (46). \square

Lemma 6. Assume for any $\varepsilon > 0$ that $\max_{\mathbf{i} \in \mathcal{I}} \mathbb{P}(\sqrt{M}|H_M(\mathbf{i}) - H(\mathbf{i})| > \varepsilon) \leq B(\varepsilon, M)$ for some deterministic constant $B(\varepsilon, M)$ depending explicitly on ε and M as introduced with (12) in Section B. Then, for any r.v. \mathbf{I} with values in \mathcal{I} ,

$$\mathbb{P}(\sqrt{M}|H_M(\mathbf{I}) - H(\mathbf{I})| > \varepsilon) \leq |\mathcal{I}|B(\varepsilon, M). \quad (49)$$

Proof. By application of the union bound principle,

$$\begin{aligned} \mathbb{P}(\sqrt{M}|H_M(\mathbf{I}) - H(\mathbf{I})| > \varepsilon) &\leq \sum_{\mathbf{i} \in \mathcal{I}} \mathbb{P}(\sqrt{M}|H_M(\mathbf{i}) - H(\mathbf{i})| > \varepsilon, \mathbf{I} = \mathbf{i}) \\ &\leq \sum_{\mathbf{i} \in \mathcal{I}} \mathbb{P}(\sqrt{M}|H_M(\mathbf{i}) - H(\mathbf{i})| > \varepsilon) \\ &\leq \sum_{\mathbf{i} \in \mathcal{I}} B(\varepsilon, M), \end{aligned}$$

using the assumption for each term of the last inequality, which yields (49). \square

Remark 8. As illustrated in Examples 1, 2 and Remark 3, H_M can be an empirical mean or a quantile estimate. Though empirical quantities can be scaled naturally by \sqrt{M} to allow B to be expressed without M for instance in the case of empirical mean estimate, it may happen that the error level upper bound may still depend on M , like e.g. for the estimates of quantile or expected shortfall without unknown quantile.

Theorem 7. Let $\mathbf{i}^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i})$, $\mathbf{I}_M^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$ and \mathbf{I}_M^N be the solution found by the algorithm to minimize $H_M(\cdot)$ over \mathcal{I} after N iterations with $H_M(\mathbf{i})$ an empirical estimation of $H(\mathbf{i})$, $\mathbf{i} \in \mathcal{I}$, based on M i.i.d. samples X_1, \dots, X_M . For any $\varepsilon > 0$, we have

$$\begin{cases} \mathbb{P}(\sqrt{M}(H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)) < -\varepsilon) \leq |\mathcal{I}|B(\varepsilon, M), \\ \mathbb{P}(\sqrt{M}(H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)) > \varepsilon) \leq \mathbb{E} \left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N) \log(\log(N))}{N} \right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}} \right] + B\left(\frac{\varepsilon}{2}, M\right). \end{cases} \quad (50)$$

Proof. According to Lemma 5, we have

$$H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*) \leq H_M(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^*) + H_M(\mathbf{i}^*) - H(\mathbf{i}^*), \quad (51)$$

$$H_M(\mathbf{I}_M^N) - H(\mathbf{I}_M^N) \leq H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*). \quad (52)$$

Let $\mathcal{E}_M(H)(\mathbf{i}) := H_M(\mathbf{i}) - H(\mathbf{i})$, $\mathbf{i} \in \mathcal{I}$, and $\mathcal{E}^N(H_M)(\mathbf{I}_M^N) := H_M(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^*)$ the empirical error for H evaluated at \mathbf{i}^* and the algorithm error for H_M for N iterations evaluated w.r.t. its optimal value $H_M(\mathbf{I}_M^*)$. Therefore,

$$\mathcal{E}_M(H)(\mathbf{I}_M^N) \leq H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*) \leq \mathcal{E}^N(H_M)(\mathbf{I}_M^N) + \mathcal{E}_M(H)(\mathbf{i}^*). \quad (53)$$

Using the l.h.s. of the first inequality of (53) we get, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\left(\sqrt{M}(H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)) < -\varepsilon\right) &\stackrel{(53) \text{ l.h.s.}}{\leq} \mathbb{P}\left(\sqrt{M}\mathcal{E}_M(H)(\mathbf{I}_M^N) < -\varepsilon\right) \\ &\stackrel{(13)}{\leq} |\mathcal{I}|B(\varepsilon, M). \end{aligned}$$

Using the r.h.s. of the second inequality of (53) we get, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\sqrt{M}(H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)) > \varepsilon\right) \leq \mathbb{P}\left(\sqrt{M}\left(\mathcal{E}^N(H_M)(\mathbf{I}_M^N) + \mathcal{E}_M(H)(\mathbf{i}^*)\right) > \varepsilon\right).$$

Applying Lemma 4 further yields

$$\begin{aligned} \mathbb{P}\left(\sqrt{M}(H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)) > \varepsilon\right) &\leq \mathbb{P}\left(\sqrt{M}\mathcal{E}^N(H_M)(\mathbf{I}_M^N) > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\sqrt{M}\mathcal{E}_M(H)(\mathbf{i}^*) > \frac{\varepsilon}{2}\right) \\ &\leq \mathbb{P}\left(\mathcal{E}^N(H_M)(\mathbf{I}_M^N) > 0\right) + B\left(\frac{\varepsilon}{2}, M\right). \end{aligned} \quad (54)$$

using, for the first term of the r.h.s. of the second inequality, the fact that

$$\begin{aligned} \left\{\sqrt{M}\mathcal{E}^N(H_M)(\mathbf{I}_M^N) > \frac{\varepsilon}{2}\right\} &\subset \left\{\sqrt{M}\mathcal{E}^N(H_M)(\mathbf{I}_M^N) > 0\right\} \\ &\stackrel{\sqrt{M}>0}{=} \left\{\mathcal{E}^N(H_M)(\mathbf{I}_M^N) > 0\right\}, \end{aligned}$$

and by assumption for the second term. Applying Remark 7, and denoting $\mathbb{P}_{\mathbf{X}_{1:M}}$ the conditional probability to the empirical sample $\mathbf{X}_{1:M}$, we get

$$\begin{aligned} \mathbb{P}\left(\mathcal{E}^N(H_M)(\mathbf{I}_M^N) > 0\right) &= \mathbb{E}\left[\mathbb{P}_{\mathbf{X}_{1:M}}\left(\mathcal{E}^N(H_M)(\mathbf{I}_M^N) > 0\right)\right] \\ &\leq \mathbb{E}\left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N)\log(\log(N))}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right]. \quad \square \end{aligned}$$

Corollary 8. (Estimation error) Let $\mathbf{i}^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i})$, $\mathbf{I}_M^* \in \arg \min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$ and \mathbf{I}_M^N some random solution identified by the algorithm after N iterations. For any $\varepsilon > 0$,

$$\mathbb{P}\left(\sqrt{M}(H(\mathbf{I}_M^N) - H(\mathbf{i}^*)) \geq \varepsilon\right) \quad (55)$$

$$\leq |\mathcal{I}|B\left(\frac{\varepsilon}{2}, M\right) + \mathbb{E}\left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N)\log(\log(N))}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right] + B\left(\frac{\varepsilon}{4}, M\right). \quad (56)$$

Proof. We have

$$0 \leq H(\mathbf{I}_M^N) - H(\mathbf{i}^*) = H(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^N) + H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*). \quad (57)$$

By application of Lemma 4, we get

$$\mathbb{P}\left(\sqrt{M}(H(\mathbf{I}_M^N) - H(\mathbf{i}^*)) > \varepsilon\right) \quad (58)$$

$$\leq \mathbb{P}\left(\sqrt{M}\left(H(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^N)\right) > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\sqrt{M}\left(H_M(\mathbf{I}_M^N) - H(\mathbf{i}^*)\right) > \frac{\varepsilon}{2}\right) \quad (59)$$

$$\leq |\mathcal{I}|B\left(\frac{\varepsilon}{2}, M\right) + \mathbb{E}\left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N)\log(\log(N))}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right] + B\left(\frac{\varepsilon}{4}, M\right) \quad (60)$$

where we have used Lemma 6 for bounding the first term of the r.h.s. of the first inequality and Theorem 7 for bounding the second term of the r.h.s. of the first inequality. \square

Proposition 9. Let $\mathcal{H}_2 := \min_{\mathbf{i} \neq \mathbf{i}^*} H(\mathbf{i})$ the second minimum value of H . Under the same assumptions of Theorem 7,

$$\mathbb{P}(I_M^N \notin \mathcal{I}_M^{opt}) \leq \mathbb{E}\left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N)\log(\log(N))}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right], \quad (61)$$

and

$$\mathbb{P}(I_M^N \notin \mathcal{I}^{opt}) \leq |\mathcal{I}|B\left(\frac{\sqrt{M}(\mathcal{H}_2 - \mathcal{H})}{2}, M\right) + \mathbb{E}\left[\delta_{\mathbf{X}_{1:M}}^A \left(\frac{\log(N)\log(\log(N))}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right] \quad (62)$$

$$+ B\left(\frac{\sqrt{M}(\mathcal{H}_2 - \mathcal{H})}{4}, M\right). \quad (63)$$

Proof. Let $\mathbf{I}_M^* \in \mathcal{I}_M^{opt}$. Then $\{\mathbf{I}_M^N \notin \mathcal{I}_M^{opt}\} = \{H_M(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^*) > 0\}$, and therefore,

$$\mathbb{P}(\mathbf{I}_M^N \notin \mathcal{I}_M^{opt}) = \mathbb{P}\left(\sqrt{M}\left(H_M(\mathbf{I}_M^N) - H_M(\mathbf{I}_M^*)\right) > 0\right). \quad (64)$$

Applying Remark 7 and taking expectation w.r.t. the distribution of $\mathbf{X}_{1:M}$, we get the inequality (61). Let $\mathbf{i}^* \in \mathcal{I}^{opt}$. We have $\{\mathbf{I}_M^N \notin \mathcal{I}^{opt}\} = \{H(\mathbf{I}_M^N) - H(\mathbf{i}^*) \geq \varepsilon\}$ for any $\varepsilon \in (0, \mathcal{H}_2 - \mathcal{H}]$, and in particular

$$\mathbb{P}(I_M^N \notin \mathcal{I}^{opt}) = \mathbb{P}\left(\sqrt{M}(H(\mathbf{I}_M^N) - H(\mathbf{i}^*)) \geq \sqrt{M}(\mathcal{H}_2 - \mathcal{H})\right). \quad (65)$$

and applying Corollary 8 with $\varepsilon = \sqrt{M}(\mathcal{H}_2 - \mathcal{H})$, we get the inequality (62). \square

C Tail distribution analysis

The random variables we deal in this work, formalized by (4) is recalled here and develop as:

$$\begin{aligned} f_\ell^k & \left(\left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_1^k, \dots, \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_L^k \right) \\ & = \frac{b_\ell^k \mathbb{1}_{\{\tau_\ell > T\}}}{\sum_{\ell'=1}^L b_{\ell'}^k \mathbb{1}_{\{\tau_{\ell'} > T\}}} \sum_{\ell''=1}^L \left(\left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_{\ell''}^k - c_{\ell''}^k \right)^+ \mathbb{1}_{\{\tau_{\ell''} \leq T\}}. \end{aligned} \quad (66)$$

We assume heavy-tailed distributions (see e.g. (Nair et al., 2022, Chapter 1) or (Embrechts et al., 1997, Chapter 1)) on the risk drivers $(Y_0^k, Y_1^k, \dots, Y_L^k)$, as it should be the common practice in financial applications, and with their updated version post-default resolution with combination of takers \mathbf{i} to $((\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_1^k, \dots, (\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0)_L^k)$, $k = 1, \dots, K$ as per (3). Given the particular form (66) that can be seen as a (stochastic⁸-)weighted sum of semi-linear functions of those risk drivers, we can expect (66) to also have a heavy-tailed distribution. We will see in our numerical examples, though fictitious, yet quite realistic, that the log-log QQ plots depict for the right tail distribution a form similar to a Pareto, log-Weibull or Weibull tail-distribution depending on the assumption of the law of the Y_i 's (see (Resnick, 2003, Section 6.5) or (Embrechts et al., 1997, Section 6.4, page 325)). Under the assumption of Student-t distributions of degree 3 for the portfolio drivers, the statistical tests favor a heavy-tail distribution behaviour for a random variable of the form (66). A generalization of such analysis, finding its root with the Hill-estimator (Hill, 1975) is described in Gardes et al. (2011) that we summarize here for completeness.

Let $\mathfrak{L} : \mathbb{R} \rightarrow \mathbb{R}_+$ be a slowly varying function i.e. $\lim_{x \rightarrow \infty} \frac{\mathfrak{L}(tx)}{\mathfrak{L}(x)} = 1$ for all $t > 0$. For any nondecreasing function $T : \mathbb{R} \rightarrow \mathbb{R}$, with $T(-\infty) = \lim_{x \downarrow -\infty} T(x)$, $T(\infty) = \lim_{x \uparrow \infty} T(x)$, we define its generalized inverse $T^{\leftarrow} : \mathbb{R} \rightarrow \overline{\mathbb{R}} = [-\infty, \infty]$, $y \mapsto \inf \{x \in \mathbb{R} : T(x) \geq y\}$ and with the convention that $\inf \emptyset = \infty$ (such generalized inverse coincides with the classical inverse whenever T is continuous and strictly increasing, see for instance Embrechts and Hofert (2013)). Gardes et al. (2011) consider a family of survival distribution functions $\overline{F}(x) = 1 - F(x)$ with F some c.d.f. and

$$A(\gamma, \xi) : \quad \overline{F}(x) = \exp(-K_\gamma^{-1}(\log T_\xi(x))), \text{ for } x > x_l, \text{ for } x_l > 0, \quad (67)$$

with $K_\gamma(y) = \int_1^y u^{\gamma-1} du$, $\gamma \in [0, 1]$, T_ξ a nondecreasing function such that $T_\xi^{\leftarrow}(t) = t^\xi \mathfrak{L}(t)$, where $\xi > 0$ and \mathfrak{L} a slowly varying function. Let X_1, \dots, X_n be a sample of size n assumed to be drawn from a random variable X with unknown distribution and let $X_{n-k_n+1,n}$ denote the k_n upper order statistics ($X_{n-k_n+1,n} \leq \dots \leq X_{n,n} = \max_{i \in 1..n} X_i$). The (biased) estimator of ξ is defined as

$$\widehat{\xi}_n(k_n) = \frac{1}{\mu_{1,\gamma}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})), \quad (68)$$

with $\mu_{1,\gamma}(t) = \int_0^\infty (K_\gamma(x+t) - K_\gamma(x)) e^{-x} dx$. Under some mild second-order assumption on \mathfrak{L} , it can be shown that $\widehat{\xi}_n(k_n)$ given by (68) converges towards ξ if $k_n \xrightarrow[n \rightarrow \infty]{} \infty$ and $\frac{k_n}{n} \xrightarrow[n \rightarrow \infty]{} 0$ (Gardes et al., 2011, Theorem 1). Considering two quantiles x_u and x_v of \overline{F} with $0 < u < v \leq 1$ and under $A(\gamma, \xi)$ in (67), we have

$$\log x_u - \log x_v = \xi (K_\gamma(-\log u) - K_\gamma(-\log v)) + \log \left(\frac{\mathfrak{L}(\exp K_\gamma(-\log u))}{\mathfrak{L}(\exp K_\gamma(-\log v))} \right). \quad (69)$$

By definition of \mathfrak{L} which is slowly-varying and for u and v close to zero (e.g. $v = k_n/n$ close to zero), the second term of the r.h.s. of (69) is negligible in front of the first term so that we get the approximation

$$\log x_u - \log x_v \simeq \xi (K_\gamma(-\log u) - K_\gamma(-\log v)). \quad (70)$$

⁸due to the allocation coefficients depending on the default events, themselves expressed w.r.t. random latent variables X_1, \dots, X_n

The approximation (70) holds for any $\gamma \in [0, 1]$ hence shared by Pareto-type, Weibull and log-Weibull tail distributions. Such property can be graphically tested on the sample (X_1, \dots, X_n) by plotting the pairs

$$(K_\gamma(\log(n/i)) - K_\gamma(\log(n/k_n)), \log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}))_{1 \leq i \leq k_n} \quad (71)$$

depending on the tested distribution specified by the expression of K_γ . If the graph is approximately linear, there is evidence that the sample may be drawn from the tested distribution type. For particular cases, we consider (Gardes et al., 2011, Proposition 2)

- Weibull tail-distributions $\mathcal{W}(1/\xi, 1)$, with $K_0(t) = \log t$ corresponding to $A(0, \xi)$ in (67),
- log-Weibull tail-distributions with $K_0(t) = \log t$, and for which we define $X = e^Y$ with Y following a Weibull distribution $\mathcal{W}(1/\xi, 1)$, included in the cases $A(0, \xi)$ in (67),
- a Pareto distribution with tail index $1/\xi$, with $K_1(t) = t$, included in the case $A(1, \xi)$ corresponding to distribution tails belonging to the Fréchet Maximum Domain of Attraction.

The type of identified distribution (light, heavy or in-between) for the tail distribution of the loss random variable (66) allows precisising $B(\varepsilon, M)$ in the concentration-like inequality (12). In the case of statistical test in favor of heavy-tailed distribution such as Pareto ones, a Fuk-Nagaev concentration inequality can be envisaged should the risk measure be an expectation or a quantile level (see e.g. (Marchina, 2021, Theorem 2.1) with a class consisting of a single element). If the statistical tests are more in favor of Weibull-like distribution, part of the α -exponential distribution class, a dedicated concentration inequality applies, see (Chamakh et al., 2020, Theorem 1) (with a compact consisting of a single point) whilst a statistical test favoring a log-Weibull type of distribution, also part of the β -heavy tailed distribution class, leads to the concentration inequality that can be found in (Chamakh et al., 2021, Corollary 2.3).

§4 Application to the CCPs defaulted portfolios takeover problem

The calculation of $\mathcal{R}_\ell^M \left[g_\ell \left(\mathbf{Y} \bigoplus_i \mathbf{Y}_0 \right) \right]$ for any $\mathbf{i} \in \mathcal{I}$ in the minimization problem (11) is costly due to the loss variable form (4) applied to (elliptically) heavy-tailed distributed variables for the portfolio drivers and default latent variables, with no analytical form for the employed risk measures \mathcal{R}_ℓ , $\ell = 1, \dots, L$. Moreover, the large financial network at hand, e.g. realistically about 50 CCPs, hundreds of members with many common to several CCPs, and thousands of clearing clients, lead to aggregating thousands of approximated risk measures, each relying at least on hundreds of thousands Monte Carlo simulations for minimal approximated errors. As such the minimization procedure cannot apply on all possible states and must resort to a more strategic research namely based on scenario exploration and exploitation which is the purpose of using the discrete combinatorial version of the simulated annealing algorithm. To this end, we need to define:

- the generation homogeneous Markov kernel for generating a new candidate set of takers, namely $P(\mathbf{i}, \mathbf{j})$;
- an estimate of either the cooling schedule hyper-parameter c (unlimited budget version) or A (fixed-budget version) to ensure adequate algorithm convergence, and, for the fixed-budget version depending on N , to ensure appropriate bound (36).

For the unlimited budget cooling schedule in the form of $c/\ln(n+1)$, the constant c should be set to at least the maximum gap between the energy level of any state \mathbf{i} through a path towards the minimum

energy level, or if unknown, to the difference between an upper bound U_B and lower bound L_B of such energy function.

A Generation probabilities definition

As outlined by [Murphy and Nahai-Williamson \(2014\)](#), the exposure sizes of members decrease exponentially. This implies that the biggest members have portfolio position decreasing when ranked according to any metric relying on their portfolio absolute size whilst smaller members may depict similar characteristics e.g. rating and portfolio size. The collateral layers calculated by the CCPs also depend on the portfolios of the members. Any structural change to these portfolios such as a portfolio update reflecting taking over some defaulter position leads to recalculation of those layers on all members.

As it is not possible to explore the space of combinations of takers directly and as each combination is a proposition consisting of one taker per CCP, we start by considering a taker proposal for each CCP, namely the one minimizing the risk measure on each stand-alone CCP. Then, for each CCP k , one member ℓ out of the surviving L members is suggested based on its portfolio notional size N_ℓ^k w.r.t. to the CCP using the following principles:

- if the notional size of the proposed taker is large compared to the defaulted notional portfolio $N_0^k \neq 0$, that is taking over the defaulted portfolio should not affect significantly the risk of the taker, there should be a significant probability that this taker would be indeed willing to taker over the defaulted position,
- if the defaulted position greatly offsets the taker position, this taker should have a great probability to be willing to take over the defaulted portfolio on that CCP.

One pragmatic and economically sound approach to do so is to look at the netting effect on the portfolio of that member taking over the defaulted portfolio on each CCP. Denoting $P(\mathbf{i}, \mathbf{j})$ the generation (homogeneous) probabilities from state of combination of takers \mathbf{i} to state \mathbf{j} , we start by considering the netting effect properties at each CCP level, independently from each other. For each CCP k , we denote by $P_k(i_k, j_k)$ the proposal generation probability from member proposition i_k to j_k so that $P(\mathbf{i}, \mathbf{j}) = P((i_1, \dots, i_K), (j_1, \dots, j_K)) = \prod_{k=1}^K P_k(i_k, j_k)$ with $\mathbf{i} = (i_1, \dots, i_K)$ and $\mathbf{j} = (j_1, \dots, j_K)$. For any member member ℓ on CCP k , we define the netting benefit ratio

$$\alpha_\ell^k := \left| \frac{N_\ell^k}{N_\ell^k + N_0^k} \right|. \quad (72)$$

There are economic rationale properties stemming from this ratio definition:

- if $N_\ell^k + N_0^k \approx 0$, i.e. there is a strong benefit for member ℓ to take over the defaulted position on CCP k , then the ratio (72) is high and the corresponding probability of picking up such member should be high;
- if $|N_\ell^k| \ll |N_0^k|$, then $\alpha_\ell^k \approx 1$ i.e. there is no strong netting effect when member ℓ acquires the defaulted position on CCP k ;
- if N_ℓ^k and N_0^k are of the same sign, i.e. portfolios have the same direction, with no big magnitude difference between one another, then α_ℓ^k should be smaller i.e. there is an overall position increase for member ℓ when acquiring the defaulted position on CCP k , thus not benefiting from taking over the defaulted position on CCP k ; the corresponding probability of picking up such member should then be low.

Hence the generation probabilities can be set proportional to those netting ratios. As such, by setting the probabilities of suggesting member j_k on CCP k from former member i_k at any iteration as:

$$P_k(i_k, j_k) := \frac{\alpha_{j_k}^k}{\sum_{j'=1}^L \alpha_{j'}^k}, \quad (73)$$

we get the desired features despite losing the information of transiting from i_k to j_k . For a CCP where the notional sizes are not significantly different from one another, a simple homogeneous probability attribution to each surviving member of that CCP is simpler and can be preferred.

At each iteration and for each CCP k , a uniform number $U \sim \mathcal{U}(0, 1)$ is drawn, independently of any other draw, and compared to the cumulative probabilities from (73) to identify the proposed taker i.e. the proposed taker index is $\sum_{\ell=1}^L \ell \mathbb{1}_{s_k(\ell-1) < U \leq s_k(\ell)}$ with $s_k(0) = 0$ and $s_k(\ell) = \sum_{\ell'=1}^{\ell} P_k(i_k, \ell')$, $\ell \in 1..L$, $k \in 1..K$. The resulting generation probability of proposing a combination (j_1, \dots, j_K) from a previous state (i_1, \dots, i_K) is stochastic as required. We indeed observe that

$$\sum_{j_1, \dots, j_K=1}^L \prod_{k=1}^K P_k(i_k, j_k) = \sum_{j_1=1}^L \cdots \sum_{j_K=1}^L P_1(i_1, j_1) \times \cdots \times P_K(i_K, j_K) \quad (74)$$

$$= \underbrace{\sum_{j_1=1}^L P_1(i_1, j_1)}_{=1} \times \cdots \times \underbrace{\sum_{j_K=1}^L P_K(i_K, j_K)}_{=1} \quad (75)$$

$$= 1. \quad (76)$$

B Maximum gap bound based on the problem features

To set the constant c of the unlimited budget cooling schedule $n \mapsto c/\log(n+1)$, and to validate the assumptions on c_n from Theorem 18 (Hajek, 1988, Theorem 1), we look at estimating the maximum gap between the energy levels based on upper and lower bounds of such energy values. For the upper bound, in the ideal case of using a convex risk measure⁹, we can use a diversification argument whereby the risk is bigger when aggregated across each CCP compared to the overall risk of the aggregated exposures to all CCPs. The upper bound U_B is the approximated energy function applied to the combination of takers maximizing the aggregated cost across all members on each CCP k separately. Considering all $\mathcal{R}_\ell^M(\mathcal{L}_\ell^{[k,i]}) \geq 0$, $k = 1, \dots, K$, $\ell, i = 1, \dots, L$, we thus set

$$U_B = H_M \left(\arg \max_{(i_1, \dots, i_K)} \sum_{k=1}^K \sum_{\ell=1}^L \mathcal{R}_\ell^M(\mathcal{L}_\ell^{[k, i_k]}) \right). \quad (77)$$

A simple lower bound L_B can be zero. However, though the unlimited budget cooling schedule constant c should be lower bounded by the maximum energy gap, it can be detrimental to select a too large value as numerical results appear poorer, which is the case $L_B = 0$. Instead, we consider, the approximated energy function applied to the combination of takers minimizing the aggregated cost across all members on each CCP k taken individually, that is

$$L_B := H_M \left(\arg \min_{(i_1, \dots, i_K)} \sum_{k=1}^K \sum_{\ell=1}^L \mathcal{R}_\ell^M(\mathcal{L}_\ell^{[k, i_k]}) \right). \quad (78)$$

⁹which is not the case for VaR measures based on quantiles given the non-elliptical nature of credit losses.

Finally, the constant c in the unlimited budget schedule version of the algorithm is set to

$$c = 2 \times (U_B - L_B), \quad (\text{with } A = \log 2/c, \text{ for fixed-budget cooling schedule}), \quad (79)$$

where the factor 2 in the expression (79) of c allows for a greater magnitude than simply taking $U_B - L_B$. The latter may not be enough for conditions of Theorem 18 (Hajek, 1988, Theorem 1) to be fulfilled. Indeed, L_B is most likely expected not to be below $\min_{\mathbf{i} \in \mathcal{I}} H_M(\mathbf{i})$ as diversification reduces risk when using convex risk measures. Nonetheless, it is expected that (i_1^b, \dots, i_K^b) is not too far from the right solution with an energy level close to the minimum one. Some of the members (i_1^b, \dots, i_K^b) may be even part of the optimal combination of takers and for some CCPs, it should be economically consistent. Hence multiplying by two should ensure c is bounded below by the maximum energy gap $\overline{\mathbb{H}}_M$ given in (34) for the approximated energy function. It also proves to give reasonable good results when looking at numerical examples, see section E. We finally define the initial proposal combination of the algorithm as

$$(i_1^b, \dots, i_K^b) = \arg \min_{(i_1, \dots, i_K)} \sum_{k=1}^K \sum_{\ell=1}^L \mathcal{R}_\ell^M (\mathcal{L}_\ell^{[k, i_k]}). \quad (80)$$

Though in our numerical examples we use a non-convex risk measure quantile approach for the energy function combined with equations (77), (78) and (79) for setting the cooling schedule hyper-parameter, the results appear reasonably good compared to brute force resolution as illustrated in Section E.

C Used simulated algorithm

As there is uncertainty in the risk measure calculation for any member, the algorithm is run based on the upper confidence level of the estimated values of the risk measure for each member. Such upper level, that we denote $H_M^u(\mathbf{i})$ for a combination $\mathbf{i} \in \mathcal{I}$, remains an energy function that depends on both the size of the used samples M and takers combination \mathbf{i} . For instance, in the case of empirical mean, this is simply the Monte Carlo upper confidence level at 97.5% with the corresponding sample standard deviation and the Student-t distribution of degree $M - 1$ (Glasserman, 2003, Appendix A.2, pages 141-144). Another example is quantiles estimate for which a non-parametric upper confidence level is proposed in Meeker et al. (2017, Section G.2). We depict such algorithm in Algorithm 1.

Algorithm 1 Simulated Annealing for CCPs Default resolution

1: **Initialize:**

2: For $k = 1, \dots, K$, $\ell = 0, \dots, L$, generate $Y_0^k, Y_1^k, \dots, Y_L^k$,
 X_1, \dots, X_L , common random numbers to all $H_M(\mathbf{i})$ and $H_M^u(\mathbf{i})$
calculations, $\mathbf{i} \in \mathcal{I} = \{1, \dots, L\}^K$

3: Calculate U_B, L_B, c based on (77), (78) and (79) resp. with H_M^u
deduced from H_M based on (11) with $g_\ell(\cdot) = \sum_{k=1}^K f_\ell^k(\cdot)$ where f_ℓ are
of the form (4)

4: Define (i_1^b, \dots, i_K^b) using (80) with H_M^u from H_M based on (4) and
(11)

5: set $\mathbf{i}^* = \mathbf{i} = (i_1^b, \dots, i_K^b)$, $n = 1$ and evaluate $\mathcal{H}_M^u = H_M^u(\mathbf{i}^*)$ from
 $H_M(\mathbf{i}^*)$ based on (4) and (11)

6: **while** $n \leq N$ **do**

7: $c_n \leftarrow c / \ln(n + 1)$ (or $c_n \leftarrow \frac{1}{A} (A / (\log N)^2)^{n/N}$)

8: **for** $k = 1$ to K **do**

9: Draw i_k based on (72) and (73)

10: or homogeneous probabilities for all surviving members of CCP k

11: **end for**

12: Define $\mathbf{j} = (i_1, \dots, i_K)$ and evaluate $H_M^u(\mathbf{j})$, $H_M^u(\mathbf{i})$ from $H_M(\mathbf{i})$, $H_M(\mathbf{j})$ resp. based
on (4) and (11)

13: Calculate $A_n(\mathbf{i}, \mathbf{j})$ using (24)

14: Draw a uniform r.v. U_n

15: **if** $U_n \leq A_n(\mathbf{i}, \mathbf{j})$ **then**

16: update $\mathbf{i} = \mathbf{j}$, based on (25)

17: **end if**

18: Evaluate $H_M^u(\mathbf{i})$ from $H_M(\mathbf{i})$ based on (4) and (11)

19: **if** $H_M^u(\mathbf{i}) < H_M^u(\mathbf{i}^*)$ **then**

20: $\mathbf{i}^* = \mathbf{i}$ and $\mathcal{H}_M^u = H_M^u(\mathbf{i})$

21: **end if**

22: **end while**

23: Return \mathbf{i}^* and \mathcal{H}_M^u

D Computational complexity

For $\ell = 1, \dots, L$, let $\mathbf{Y}_\ell = \left(\left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_\ell^1, \dots, \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0 \right)_\ell^K \right)$. A one-shot Evaluation cost of $H_M(\mathbf{i})$ consists in:

- updating K position factors among $\mathbf{Y}_1, \dots, \mathbf{Y}_L$, each of size M with Y_0^1, \dots, Y_0^K i.e. $M \times K$ operations,
- for each member, re-evaluating the loss function $g_\ell \left(\mathbf{Y}^m \oplus_{\mathbf{i}} \mathbf{Y}_0^m \right)$ with $\ell = 1, \dots, L$ and $m = 1, \dots, M$ i.e. $\text{Cstt} \times L \times K \times M$ operations with some constant Cstt for the application of the function g_ℓ (similar for all members and depending on all K CCPs),
- for each member, re-evaluating the \mathcal{R}_ℓ such as a quantile or an expected shortfall requiring to read an order statistic from the M simulated values and summing the values above this quantile i.e. an overall cost of $\text{Cstt} \times L \times M \log(M)$ for some other constant Cstt and with $M \log(M)$ the worst case for the sorting algorithm for reading the necessary order statistic.

Considering either $K \approx \log M$ or $K \ll \log M$, we get an overall calculation cost of order $\text{Cstt} \times L \times M \log(M)$ as an incremental evaluation cost for changing from combination \mathbf{i} to $\mathbf{i}' \neq \mathbf{i}$.

E Numerical results

We list below the setup considered for running the approximated algorithm based on given sample of size M .

- Consider 4 CCPs with 11 common members indexed from 0 to 10 among which member 0 has defaulted.
- The portfolio drivers $(Y_\ell^k)_{\substack{0 \leq \ell \leq 10 \\ 1 \leq k \leq 4}}$ are all (correlated) Student r.v.'s with degree of freedom 3. The credit latent variables $(X_\ell)_{1 \leq \ell \leq L}$ are (correlated) standard Gaussian r.v.'s $\mathcal{N}(0, 1)$ (see also Table 2).
- All surviving members are assumed to have a one year default intensity of 1%.
- For convergence numerical illustration and confidence levels, 100 runs of the algorithm has been considered with N varying from 25 to 3000 iteration steps, with a step of 25 iterations.
- A quantile at 99.9% is considered for the risk measure of all members with it upper confidence level as estimated energy levels (Meeker et al., 2017, Appendix G, p. 497).
- The parameters of the various costs calculations are summarized in Table 1:

One-period length for default T	1 year
Portfolio variations correlation ρ_c 's	30%
Credit factors correlation ρ_m 's	20%
Quantile level used for clearing members risk measure calculation	99.9%
Number of Monte Carlo simulation (for risk measure computations)	100,000

Table 1: XVAs calculation configuration

- The (simplified non-centered) loss r.v. for member ℓ is

$$\mathcal{L}_\ell = \sum_{k=1}^4 \frac{\mathbb{1}_{X_\ell < B_\ell}}{1 - \gamma_\ell} \frac{b_\ell^k}{b_\ell^k + \sum_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{10} b_{\ell'}^k \mathbb{1}_{X_{\ell'} < B_{\ell'}}} \sum_{\substack{\ell'=1 \\ (\ell' \neq \ell)}}^{10} \mathbb{1}_{X_{\ell'} \geq B_{\ell'}} \left(\left(\mathbf{Y} \oplus_i \mathbf{Y}_0 \right)_{\ell'}^k \right)^+, \quad (81)$$

where no collateral layers have been considered i.e. $c_\ell = 0$ and $b_\ell^k = |N_\ell^k| \mathbb{1}_{\ell \neq i_k} + |N_\ell^k + N_0^k| \mathbb{1}_{\ell = i_k}$ for $\ell = 1, \dots, 10$ in (4) with i_k the member index taking over the defaulted portfolio on CCP k . For $\ell = 1, \dots, L$, N_ℓ^k corresponds to a deterministic signed size of exposure position member ℓ has towards CCP k .

This example can be run with a brute force ground truth resolution by running the cost of each combination of takers (10000 aggregated cost values $H_M(\mathbf{i})$ to calculate for each combination \mathbf{i} and their corresponding upper confidence levels), pre-generating the empirical energy levels based on the M -sized samples for each possible combination. As such, we can process several tests without having to recompute these energy levels, simply reading them from a saved file.

The characteristics of the positions on each CCP are specified in Table 2.

portfolio id	size	trend	volatility
p20 (cm0)	47.36	4.74	30%
p0 (cm1)	-32.41	-3.24	20%
p1 (cm2)	22.18	2.22	21%
p2 (cm3)	-15.17	-1.52	22%
p3 (cm4)	-10.38	-1.04	23%
p4 (cm5)	-7.1	-0.71	24%
p5 (cm6)	-4.86	-0.49	25%
p6 (cm7)	3.33	0.33	26%
p7 (cm8)	-2.28	-0.23	27%
p8 (cm9)	-1.56	-0.16	28%
p9 (cm10)	0.9	0.09	29%

(a) CCP 1

portfolio id	size	trend	volatility
p21 (cm0)	-3.44	-0.34	30%
p10 (cm1)	-60.54	-6.05	35%
p11 (cm2)	45.88	4.59	36%
p12 (cm3)	-34.77	-3.48	37%
p13 (cm4)	26.35	2.63	38%
p14 (cm5)	19.97	2	39%
p15 (cm6)	15.13	1.51	40%
p16 (cm7)	-11.47	-1.15	39%
p17 (cm8)	-8.69	-0.87	38%
p18 (cm9)	6.59	0.66	37%
p19 (cm10)	4.99	0.5	36%

(b) CCP 2

portfolio id	size	trend	volatility
p31 (cm0)	17.08	1.71	37%
p22 (cm1)	12.94	1.29	30%
p23 (cm2)	-11.27	-1.13	29%
p24 (cm3)	9.81	0.98	28%
p25 (cm4)	-8.54	-0.85	27%
p26 (cm5)	7.43	0.74	26%
p27 (cm6)	10.38	1.04	25%
p28 (cm7)	-25.89	-2.59	40%
p29 (cm8)	22.54	2.25	39%
p30 (cm9)	-19.62	-1.96	38%
p32 (cm10)	-14.87	-1.49	36%

(c) CCP 3

portfolio id	size	trend	volatility
p39 (cm0)	-2.46	-0.25	21%
p33 (cm3)	-4.13	-0.41	20%
p34 (cm2)	3.79	0.38	19%
p35 (cm1)	-3.47	-0.35	18%
p36 (cm4)	3.19	0.32	17%
p37 (cm5)	-2.92	-0.29	16%
p38 (cm6)	2.68	0.27	15%
p40 (cm9)	2.26	0.23	22%
p41 (cm8)	-2.07	-0.21	23%
p42 (cm7)	1.9	0.19	24%
p43 (cm10)	1.24	0.12	25%

(d) CCP 4

Table 2: CCPs and members portfolios with defaulted member and corresponding portfolios, ground truth takers and their corresponding portfolios prior takeover

We start by displaying the energy levels landscape for all possible combinations with the estimated upper and lower confidence levels based on the samples using the approach proposed in (Meeker et al., 2017, Section G.2) in Figure 1. The algorithm will be run on the upper confidence energy levels that can be obtained for each combination of takers.

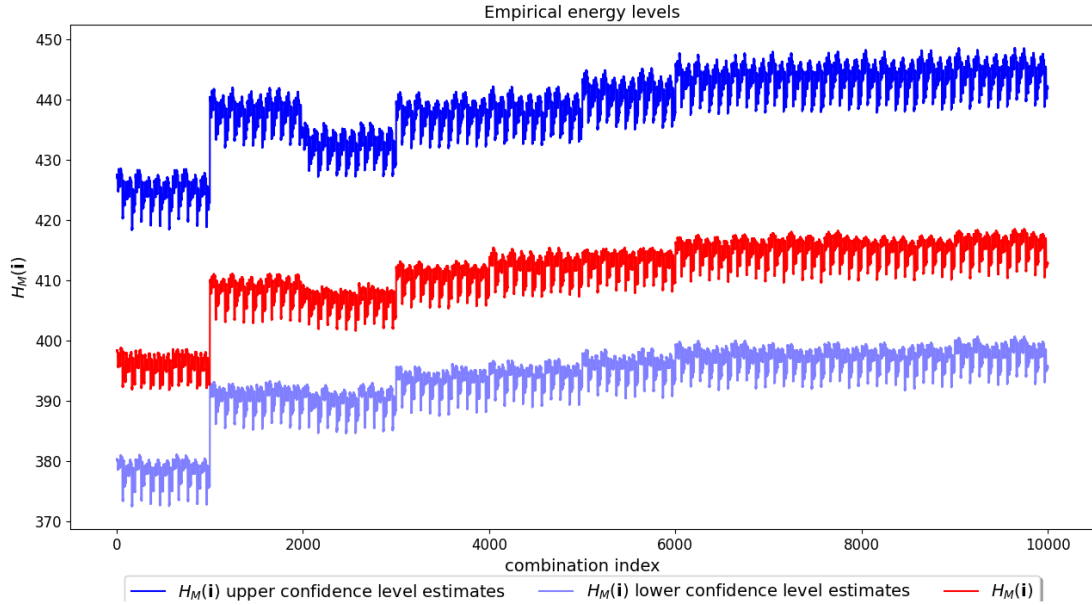


Figure 1: Estimated energy levels including lower and upper confidence levels, per combination indexed from 1 to 10000.

The loss form (81) is suspected to have a heavy-tailed distribution. As we start with Student-t distribution of degree 3 for all Y_ℓ^k and thus their update $\left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0\right)_\ell^k$ that are indeed heavy tailed, the sum of their positive parts is expected to remain heavy-tailed. We thus first conduct heavy-tail tests where we do not consider any allocation factors of the form $\frac{|N_\ell^k|}{|N_\ell| + \sum_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{10} |N_{\ell'}^k| \mathbb{1}_{X_{\ell'} < B_{\ell'}}}$ in (81) nor application of the positive part on the components $\left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0\right)_{\ell'}^k$, $\ell = 1, \dots, 10$, $k = 1, \dots, 4$. The corresponding loss simplifies to

$$\mathcal{L}_\ell = \sum_{k=1}^4 \sum_{\substack{\ell'=1 \\ (\ell' \neq \ell)}}^{10} \mathbb{1}_{X_{\ell'} \geq B_{\ell'}} \left(\mathbf{Y} \oplus_{\mathbf{i}} \mathbf{Y}_0\right)_{\ell'}^k. \quad (82)$$

we test the heavy tailedness on member 4 when the combination of members $\{1, 1, 1, 2\}$ takes over the defaulted portfolio of member 0 on CCP 1, 2, 3 and 4 respectively. To ensure our tests are adequate, we also consider the cases where the portfolio drivers Y_ℓ^k are driven by Student-t distribution of degree 2 (thus heavier than the case of degree of freedom 3) and also standard Gaussian in Appendix B. For each graph, the red line depicts the hypothetical log-log plots based on the tested distribution (Weibull for light-tail test, log-Weibull for in-between light and heavy tail test or fat tail test, Pareto for heavy tail test). We give two performance metrics of the simulated data from (81): the usual R^2 metric and another one based on the percentage of simulated point, taken as a vector departing from 0, remaining

close to the tested theoretical value by less than $5\%\pi$ that we indicate as $n_{\theta \leq 0.05}$ with one unit angle equivalent to π .

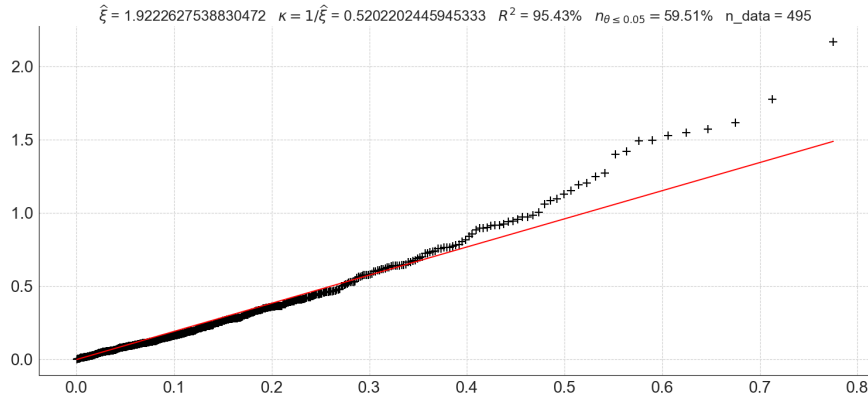


Figure 2: Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 3 degrees of freedom and based on (82).

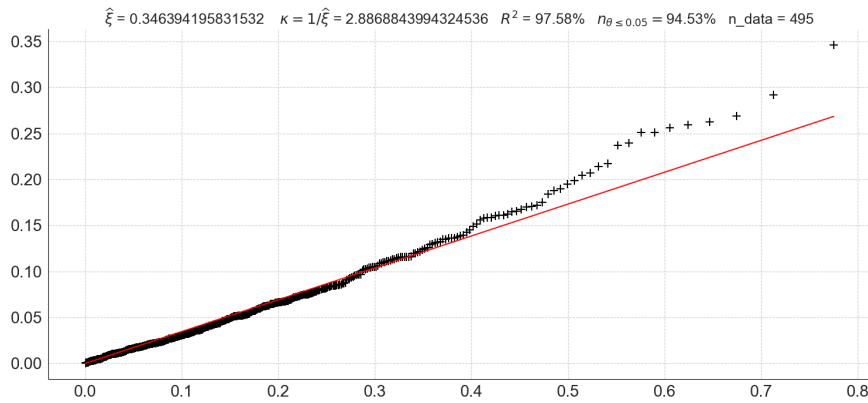


Figure 3: Log-Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 3 degrees of freedom and based on (82).

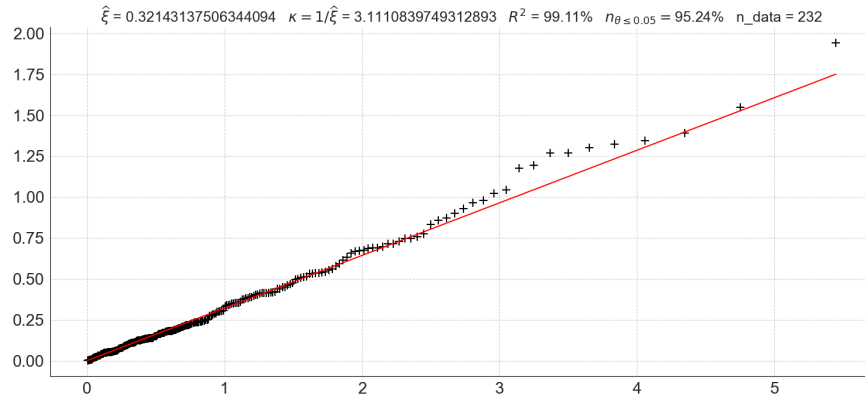


Figure 4: Pareto tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 3 degrees of freedom and based on (82).

As expected, the Pareto test, illustrated in Figure 4, leads to the highest indicator levels (both R^2 and number of angles below $5\% \pi$) suggesting a heavy-tail behaviour for (82), compared to the Weibull and log-Weibull tests in Figures 2 and 3. The estimated parameters in the Weibull and log-Weibull tests (respectively ≈ 0.52 and ≈ 2.89), though giving a poorer fit supports the heavy-tailedness of the considered simplified loss form (82).

We present those same tests in Figures 7, 5 and 6, this time including the allocation coefficients and positive part on the loss definition as per (81)

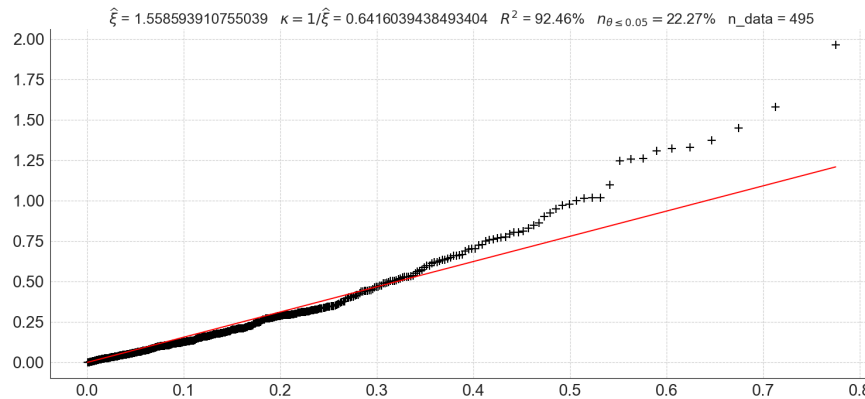


Figure 5: Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 3 degrees of freedom and based on (81).

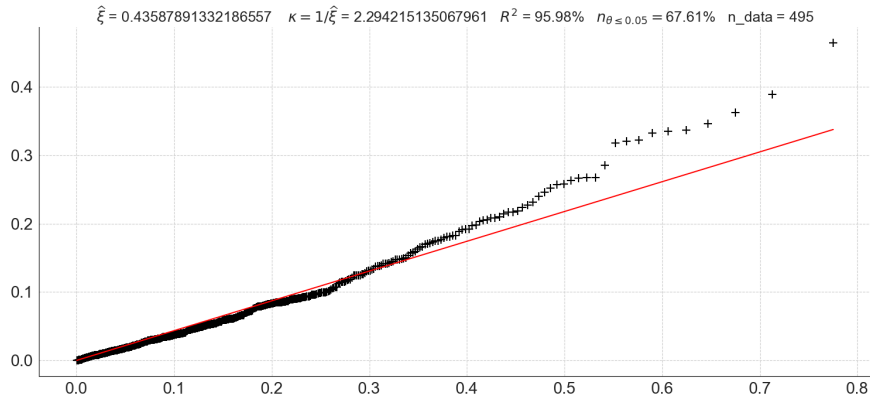


Figure 6: Log-Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 3 degrees of freedom and based on (81).

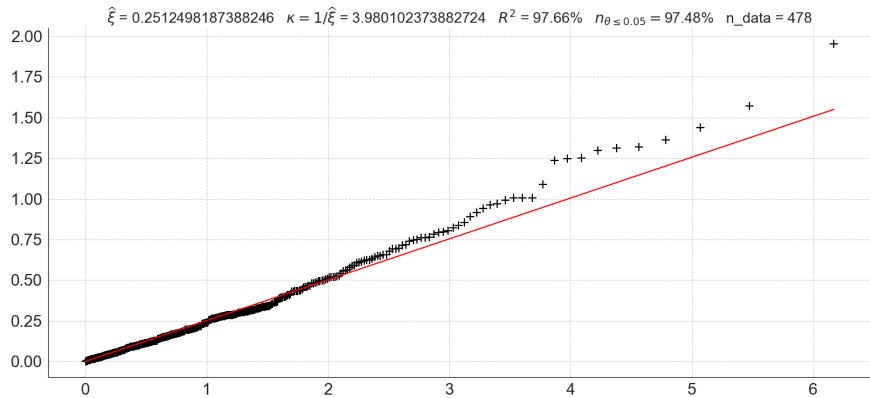


Figure 7: Pareto tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 3 degrees of freedom and based on (81).

Again, the Pareto test, illustrated in Figure 7, depicts the highest indicator levels. We thus have evidence that (81) possesses heavy tails compared to the Weibull and log-Weibull tests in Figures 5 and 6. This implies that the empirical errors, in the case of using expectation, quantiles or expected shortfall with known corresponding quantile level for the risk measure \mathcal{R}_ℓ must rely on the Fuk-Nagaev type of inequalities (Marchina, 2021; Rio, 2017) to be controlled.

To assess the algorithm operating adequacy, we depict some diagnostic indicators of the run algorithm in terms of energy path for those combinations that have been selected, the temperature scheme, the acceptance ratio and the empirical probability of selecting the proposed state across 50 iterations, rolled as the number of iterations increases for a single run of the algorithm. This is the purpose of Figure 8 showing for the first 4 graphs, the diagnosis for the fixed-budget triangular cooling schedule and for the last 4, the same diagnosis but for the logarithmic unlimited budget cooling schedule. As covered in the literature (Aarts and van Laarhoven, 1989; Aarts and Korst, 1989; Delmas and Jourdain, 2006), the fixed-budget cooling schedule is preferable and from the diagnosis, such schedule allows a better exploration of the energy landscape with a reduction of the acceptance ratio kick-in more rapidly

than the logarithmic cooling schedule permitting a more local exploration towards the latest iteration steps.

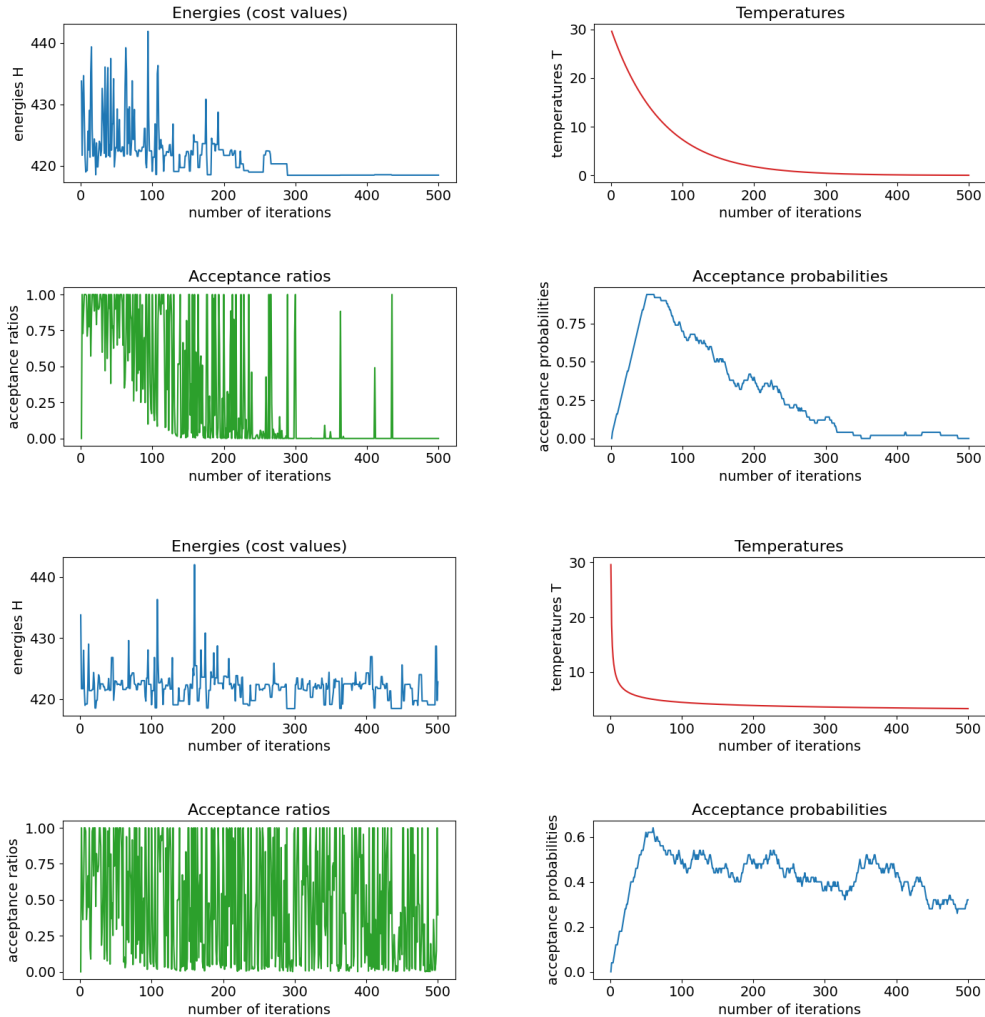


Figure 8: Algorithm diagnosis consisting of the energy path, the temperature cooling schedule function graph, the acceptance ratios and empirical probabilities of acceptance across 50 steps (first 4 panels for the fixed-budget triangular cooling schedule, the last 4 for the unlimited budget schedule).

Conditionally to a set of M samples, and for 100 runs, we can observe the performance of the algorithm based on the fixed-budget triangular cooling schedule where for each of the 100 runs under a fixed-budget schedule, we run the algorithm up to a certain number of iterations N that we make increase with 25 additional iteration steps from 25 to 3000 total iterations. Using a logarithmic unlimited budget cooling schedule does not degrade the result though the exploration is more chaotic when compared to a fixed-budget triangular cooling schedule, see Figure 8. For each tested number of iteration N , we deduce an empirical probability, with its corresponding confidence interval, of not selecting the right combination and observe it decreases significantly w.r.t. N , outperforming a simple greedy search without resampling. This is shown in Figure 9 along with the diagnosis analysis for the first run.

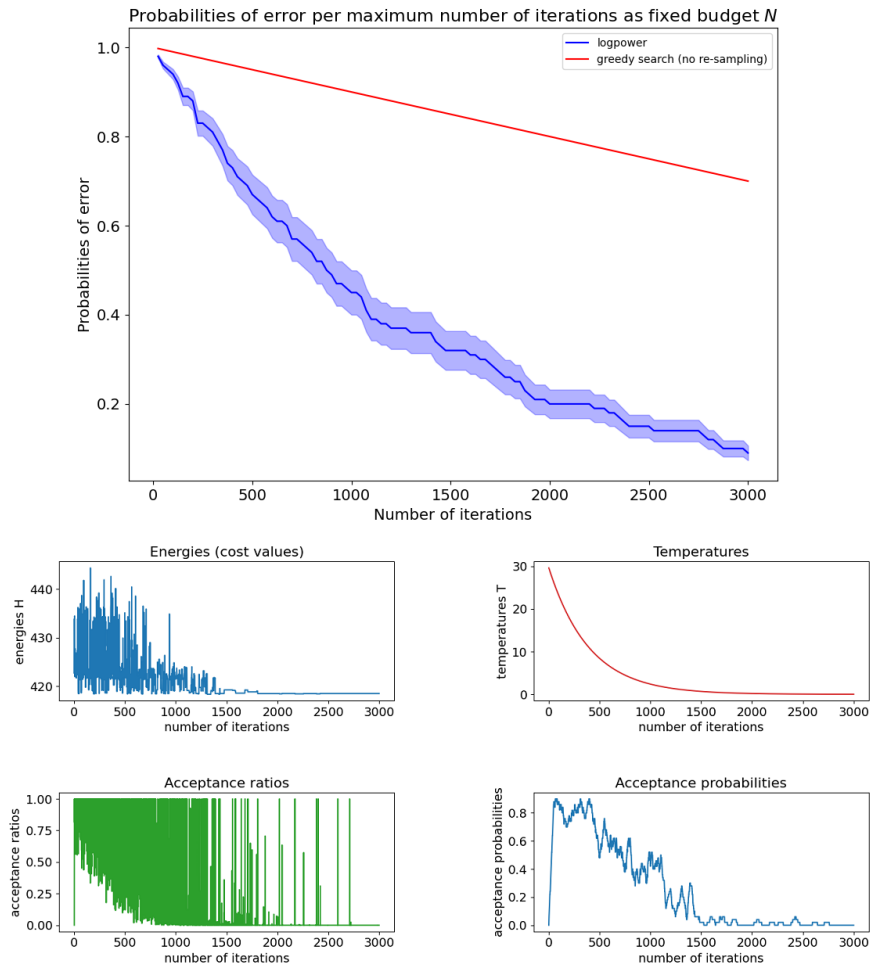


Figure 9: Error probabilities w.r.t. N , using 25 iterations step, using fixed-budget cooling schedule based on 100 runs.

Figure 10 and Figure 11 indicate the most selected taker combinations aggregated across all 100 runs for 3000 iterations for the bar chart 10 and for each run for the heat map 11. The 4 combinations that are the most selected are precised in Table 3 with their rank, a rank of 1 being the combination giving the minimum energy level. The heat map in Figure 11 shows the 4 with most occurrences in accordance with the 4 most selected combination of takers across all runs observable in Figure 10.

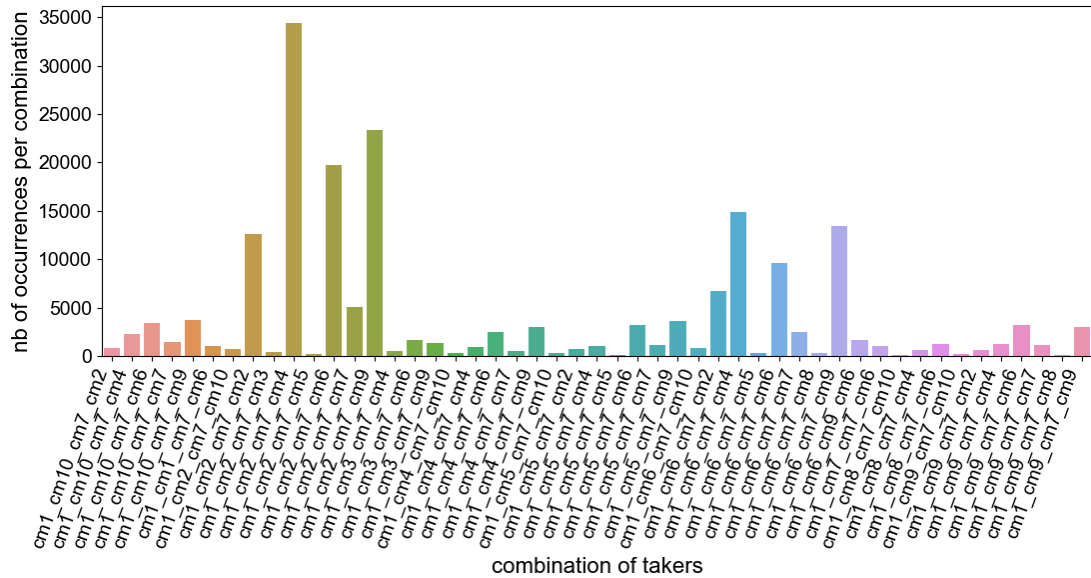


Figure 10: Number of occurrences per combinations aggregated across the 100 runs for those combinations with more than 30 occurrences across the 100 algorithm runs and 3000 iterations.

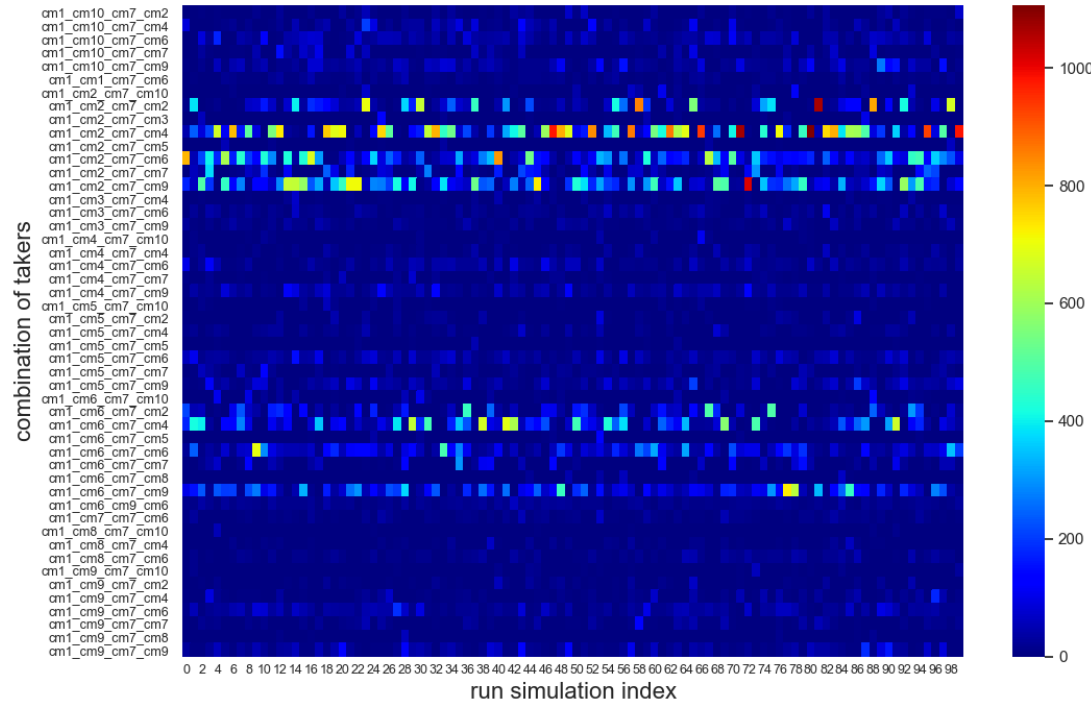


Figure 11: Number of occurrences per combinations for each of the 100 runs for those combinations with more than 30 occurrences across the 100 algorithm runs and 3000 iterations.

combination	number of occurrences	rank
{1, 2, 7, 4}	34431	1
{1, 2, 7, 9}	23402	6
{1, 2, 7, 6}	19728	8
{1, 6, 7, 4}	14859	4

Table 3: Most selected combination of takers for 100 algorithm runs and 3000 iterations and their corresponding ranks.

Another informative analysis consists in looking at the distributions of the iteration step where the optimal combination is selected across the 100 algorithm runs. Figure 12 displays such distribution as we increase the maximum number of iterations N , the budget, to run the algorithm. As expected, the selection of the optimal combination tends to happen slightly more frequently at the beginning of the algorithm run though extending the budget N allows to generate more opportunities for the algorithm to reach this optimal.

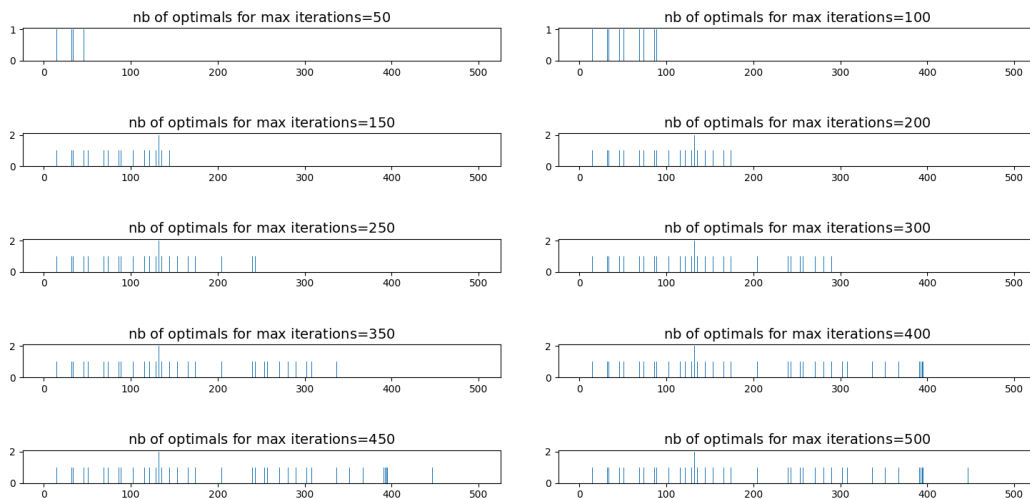


Figure 12: Number of occurrences the optimal combination is reached with respect to the maximum number of iterations based on 100 runs with a fixed-budget cooling schedule. The first figure can be viewed as a focus of the last one for the first 50 iterations.

As a comparison of the similar algorithm set with an inappropriate generation probabilities, such as homogeneous probabilities instead of those formulated in (73), of proposing any combination of takers, we illustrate in Figure 13 the obtained empirical error probabilities with corresponding confidence intervals with respect to the increase of the maximum number of iterations N used to define the fixed-budget schedule.

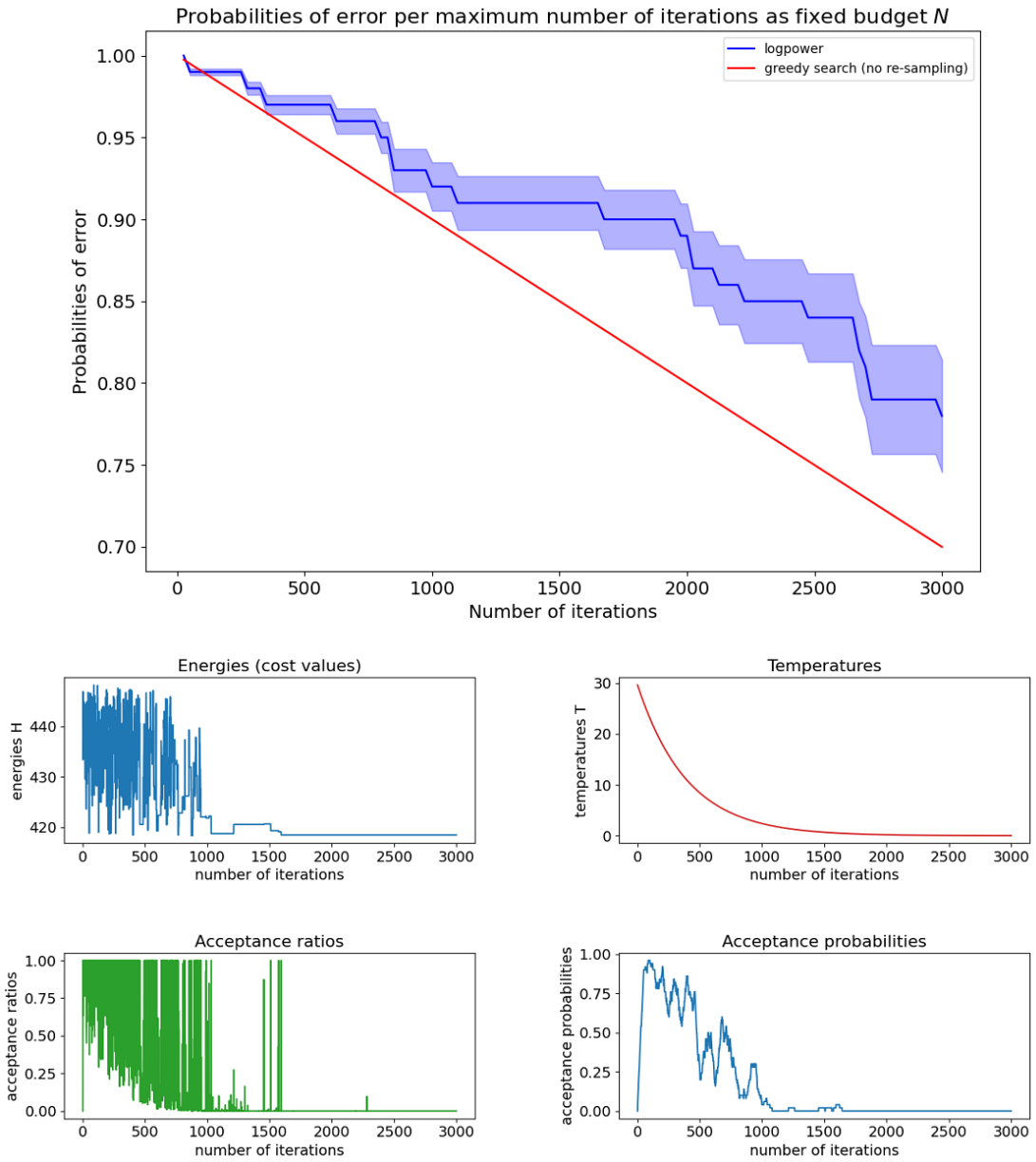


Figure 13: Error probabilities w.r.t. N , using 25 iterations step, using fixed-budget cooling schedule based on 100 runs with homogeneous explorations.

§5 Conclusion

We have tested the use of the simulated annealing algorithm to a financial network problem consisting of a default of a financial actor, member on several CCPs, that can affect all other members towards which the defaulters portfolios must be reallocated. Such reallocation is defined by targeting a minimal overall capital cost as the sum of approximated risk measures across each surviving members, leading to an approximated problem and thus algorithm. If the corresponding generation probabilities are adequately

defined for relevant proposal in the algorithm, numerical examples show it outperforms a simple naive search of the combination of takers. We also explore the error bounds between the solution offered by the approximated algorithm and the true problem algorithm as well as the probability of errors of such approximated algorithm, relying on concentration inequalities and algorithm errors. Given the nature of the problem comprised of financial portfolios represented through risk drivers that should follow heavy-tailed distributions, the use of extreme-value theory appears natural to identify the relevant concentration inequalities. The error bounds, though not fully exploitable, assert of the adequacy for using such approximated algorithm.

§6 Appendix

A Standard results on convergence guarantees

In this section, we consider a discrete state space denoted \mathcal{I} , on which some discrete time stochastic processes take value. All the aforementioned results apply to such state space and such discrete-valued stochastic processes. We start by briefly recalling some characteristics definitions of the Markov chains that will be used in the results on the algorithm convergence.

The case of homogeneous Markov chains

Definition 3. *Delmas and Jourdain (2006, Definition 1.3.1, page 10)* A probability π on \mathcal{I} is called an *invariant* probability, or *stationary* probability, of a homogeneous Markov Chain with transition probability P on \mathcal{I} if $\pi = \pi P$.

Definition 4. *Delmas and Jourdain (2006, Definition 1.3.2, page 11)* The Markov chain of transition matrix P , or simply the transition matrix P , is said to be *reversible* w.r.t. the probability π if for all $\mathbf{i}, \mathbf{j} \in \mathcal{I}$

$$\pi(\mathbf{i})P(\mathbf{i}, \mathbf{j}) = \pi(\mathbf{j})P(\mathbf{j}, \mathbf{i}). \quad (83)$$

Summing on $\mathbf{i} \in \mathcal{I}$, we get the following lemma

Lemma 10. *Delmas and Jourdain (2006, Lemma 1.3.3, page 11)* *If the Markov chain is reversible w.r.t. the probability π , then π is an invariant probability.*

Definition 5. *Delmas and Jourdain (2006, Definition 1.4.1, page 11)* A homogeneous Markov chain $\mathbf{I} = (\mathbf{I}_n)_{n \in \mathbb{N}}$, or its transition matrix P , is said to be *irreducible* if the probability of attaining any state $\mathbf{j} \in \mathcal{I}$ starting from any state $\mathbf{i} \in \mathcal{I}$ in a number of steps $n_{\mathbf{i}, \mathbf{j}}$ is strictly positive i.e.: $\forall \mathbf{i}, \mathbf{j} \in \mathcal{I}$, $\exists n = n_{\mathbf{i}, \mathbf{j}} \geq 1$, s.t. $P^{n_{\mathbf{i}, \mathbf{j}}}(\mathbf{i}, \mathbf{j}) > 0$. Equivalently, for all $\mathbf{i}, \mathbf{j} \in \mathcal{I}$, $\exists \mathbf{i}_0 = \mathbf{i}, \mathbf{i}_1, \dots, \mathbf{i}_{n_{\mathbf{i}, \mathbf{j}}} = \mathbf{j}$ s.t. $\prod_{n=1}^{n_{\mathbf{i}, \mathbf{j}}} P(\mathbf{i}_{n-1}, \mathbf{i}_n) > 0$.

Definition 6. *Delmas and Jourdain (2006, Definition 1.4.2, page 12)* A homogeneous Markov chain is said to be *periodic of period* $D \geq 1$ if we can decompose the state space \mathcal{I} into a partition with D subsets C_1, \dots, C_D such that for any $d \in \{1, \dots, D\}$,

$$\mathbb{P}(\mathbf{I}_1 \in C_d | \mathbf{I}_0 \in C_{d-1}) = 1. \quad (84)$$

The Markov chain is said to be *aperiodic* if its highest period is 1.

The simulated annealing algorithm convergence, relying on the construction of a non-homogeneous Markov chain, subject to Doeblin condition at each algorithm iteration, relies on the following lemma.

Lemma 11. *Delmas and Jourdain (2006, Lemma 2.3.3, page 38)* Let $\Delta H = \max_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i}) - \mathcal{H}$. We have

$$\|\mu_T - \mu_{T'}\| \leq \left| \frac{1}{T} - \frac{1}{T'} \right| \Delta H. \quad (85)$$

Lemma 12. *Delmas and Jourdain (2006, Lemma 2.3.2, page 37)* Let $\mathcal{H} = \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i})$. We have $\lim_{T \rightarrow 0^+} \mu_T(\{\mathbf{i} \in \mathcal{I}; H(\mathbf{i}) > \mathcal{H}\}) = 0$.

Definition 7. *Delmas and Jourdain (2006, Appendix D, page 415)* Given a signed measure π on \mathcal{I} , which is countable, the *variation norm* is defined as

$$\|\pi\| = \frac{1}{2} \sum_{\mathbf{i} \in \mathcal{I}} |\pi(\mathbf{i})| \quad (86)$$

which corresponds to half of the L^1 norm of π seen as a vector of $\mathbb{R}^{\mathcal{I}}$.

Definition 8. *Delmas and Jourdain (2006, Definition D.1, page 415)* A sequence of measures $(\pi_n, n \geq 1)$ converges in variation to π iff $\lim_{n \rightarrow \infty} \|\pi_n - \pi\| = 0$.

Lemma 13. *Delmas and Jourdain (2006, Lemma D.2, page 415)* Let $(\mathbf{I}_n, n \in \mathbb{N}^*)$ be a r.v. sequence in \mathcal{I} . Denote by $\pi_n = (\pi_n(\mathbf{i}) = \mathbb{P}(\mathbf{I}_n = \mathbf{i}), \mathbf{i} \in \mathcal{I})$ the distribution of \mathbf{I}_n . Let \mathbf{I} a r.v. in \mathcal{I} with distribution $\pi = (\pi(\mathbf{i}) = \mathbb{P}(\mathbf{I} = \mathbf{i}), \mathbf{i} \in \mathcal{I})$. The three following statements are equivalent:

- (i) The sequence $(\mathbf{I}_n, n \in \mathbb{N}^*)$ converges in distribution to the distribution of \mathbf{I} .
- (ii) For all $\mathbf{i} \in \mathcal{I}$, we have $\lim_{n \rightarrow \infty} \pi_n(\mathbf{i}) = \pi(\mathbf{i})$
- (iii) The sequence $(\pi_n, n \in \mathbb{N}^*)$ converges in variation to π .

Definition 9. *Delmas and Jourdain (2006, Definition 2.1.1, page 33)* We say that the homogeneous Markov chain $\mathbf{I} = (\mathbf{I}_n, n \geq 0)$ verifies the *Doeblin's condition* iff there exist $n_0 \in \mathbb{N}^*$, $\alpha > 0$ and p a probability on \mathcal{I} such that for all $\mathbf{i}, \mathbf{j} \in \mathcal{I}$

$$P^{n_0}(\mathbf{i}, \mathbf{j}) \geq \alpha p(\mathbf{j}). \quad (87)$$

Theorem 14. *Delmas and Jourdain (2006, Theorem 2.1.2, page 33)* Assume that $\mathbf{I} = (\mathbf{I}_n, n \geq 0)$ verifies the Doeblin's condition. Then, for any distribution μ_0 of \mathbf{I}_0 , the distribution of \mathbf{I}_n converges in variation to a probability measure π . Moreover, this probability π is the unique invariant probability of \mathbf{I} .

The two following results emphasize that any irreducible homogeneous Markov chain that is defined on \mathcal{I} , which is a finite set, has a unique invariant distribution π .

Theorem 15. *Delmas and Jourdain (2006, Theorem 1.4.3, page 12)* An irreducible homogeneous Markov chain has at most one invariant probability π and if so, for all $\mathbf{i} \in \mathcal{I}$, we have $\pi(\mathbf{i}) > 0$. If \mathcal{I} is finite, any irreducible Markov chain has exactly one invariant probability.

Theorem 16. *Delmas and Jourdain (2006, Theorem 1.4.4, page 12)* Let $(\mathbf{I}_n, n \geq 0)$ be an aperiodic and irreducible homogeneous Markov chain. If it has a (unique) invariant probability π , then for all $\mathbf{i} \in \mathcal{I}$, we have $\lim_{n \rightarrow \infty} \mathbb{P}(\mathbf{I}_n = x) = \pi(x)$ i.e. the sequence of the distributions of the r.v. \mathbf{I}_n converges tightly to the unique invariant probability π . If the Markov chain does not possess an invariant probability, then $\lim_{n \rightarrow \infty} \mathbb{P}(\mathbf{I}_n = x) = 0$ for all $\mathbf{i} \in \mathcal{I}$.

Remark 9. Delmas and Jourdain (2006, Remark 2.1.3, page 33) As \mathcal{I} is finite, the Theorem 15 and Theorem 16 imply that the Markov chain $(\mathbf{I}_n, n \geq 0)$ is irreducible and aperiodic iff it verifies the Doeblin's condition with a probability p such that, for all $\mathbf{i} \in \mathcal{I}$, $p(\mathbf{i}) > 0$ (which is not true in the infinite case).

The case of non-homogeneous Markov chains

Equipped with the definitions and results of Section A, we can now state the principal result for the simulated annealing algorithm to converge. We consider the transition matrix $(P(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}$ verifying the Doeblin's condition (87) and with an invariant probability $(\pi(\mathbf{i}), \mathbf{i} \in \mathcal{I})$. We additionally consider the transition probability to be irreducible. We define the Markov chain $(\mathbf{I}_n, n \geq 0)$ with the transition matrix given by

$$Q_n(\mathbf{i}, \mathbf{j}) = \begin{cases} P_n(\mathbf{i}, \mathbf{j})e^{-(H(\mathbf{j})-H(\mathbf{i}))^+/c_n} & \text{if } \mathbf{j} \neq \mathbf{i} \\ 1 - \sum_{\mathbf{j}' \neq \mathbf{i}} Q_n(\mathbf{i}, \mathbf{j}') & \text{if } \mathbf{j} = \mathbf{i} \end{cases} \quad (88)$$

with H given by (7). For $n \geq 1$, the Boltzmann-Gibbs distribution μ_n is given by

$$\mu_n(\mathbf{i}) = \frac{1}{Z_{c_n}} e^{-H(\mathbf{i})/c_n}, \quad \mathbf{i} \in \mathcal{I}, \quad (89)$$

with $Z_{c_n} = \sum_{\mathbf{i} \in \mathcal{I}} e^{-H(\mathbf{i})/c_n}$.

Proposition 17. *Delmas and Jourdain (2006, Proposition 2.3.5, page 40)* Let P be irreducible, satisfying the Doeblin's condition (87). Assuming there is a constant H_0 such that for all $h > H_0$, if the temperature schedule $(c_n, n \geq 1)$ is given by $c_n = h/\log(n+1)$, then for all distribution μ_0 of \mathbf{I}_0 , we have

$$\lim_{n \rightarrow \infty} \|\nu_n - \mu_n\| = 0, \quad (90)$$

where ν_n is the distribution of \mathbf{I}_n . In particular, we have $\lim_{n \rightarrow \infty} \mathbb{P}(H(\mathbf{I}_n) > \mathcal{H}) = 0$ with $\mathcal{H} = \min_{\mathbf{i} \in \mathcal{I}} H(\mathbf{i})$.

Another important convergence result concerns the adequacy of the cooling schedule definition, given by Hajek (1988, Theorem 1) and reformulated in Delmas and Jourdain (2006). As outlined in Delmas and Jourdain (2006), the following results is due to Hajek (1988, Theorem 1):

Theorem 18. *Delmas and Jourdain (2006, Theorem 2.3.8)* We have $\lim_{n \rightarrow \infty} \mathbb{P}(H(\mathbf{I}_n) > \mathcal{H}) = 0$ iff $\lim_{n \rightarrow \infty} c_n = 0$ and $\sum_{n=1}^{\infty} e^{-H^*/c_n} = \infty$.

In particular, for a cooling schedule of the form $c_n = c/\log(n+1)$, Theorem 18 ensures that $\lim_{n \rightarrow \infty} \mathbb{P}(H(\mathbf{I}_n) > \mathcal{H}) = 0$ iff $c \geq H^*$. The important underlying assumptions for these results to be valid are the symmetric and irreducible characteristics of the generation probabilities. Thus a particular attention must be given to these characteristics, in particular for the irreducible characteristic when designing the generation probabilities. All these results are with an infinite exploration time horizon, while for our analysis, we prefer to highlight finite range results.

B Distribution heavy-tail tests

We illustrate the tail distribution behaviour tests for the simplified loss random variable based on (82) when the portfolios drivers Y_ℓ^k , $\ell = 0, \dots, 10$ and $k = 1, \dots, 10$, follow a Student-t distribution with 2 degrees of freedom for member 4 when takeover is conducted by the set of members $\{1, 1, 1, 2\}$ in Figures 14, 15 and 16. As expected for this type of distribution for the portfolio drivers, the Pareto test is the one indicating the highest score. We conduct the same analysis this time with portfolio drivers Y_ℓ^k following standard Gaussian distributions in Figures 17, 18 and 19. Both the Weibull and log-Weibull distribution tests now lead to the highest R^2 and number of small angles scores with for the log-Weibull a low level estimated parameter $\hat{\xi}$. The high level of the tail index in Figure 19 (≈ 11.45) is also indicative of a light tail behaviour when Gaussian random variables are considered for the portfolios drivers.

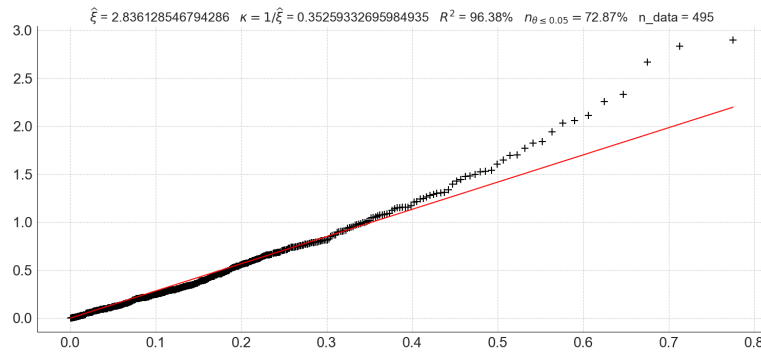


Figure 14: Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 2 degrees of freedom and based on (82).

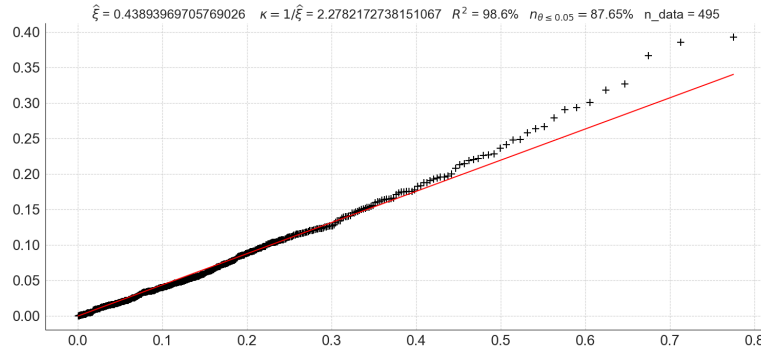


Figure 15: Log-Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 2 degrees of freedom and based on (82).

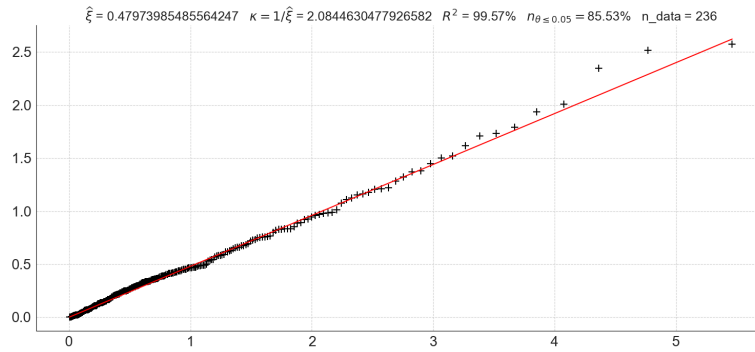


Figure 16: Pareto tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 2 degrees of freedom and based on (82).

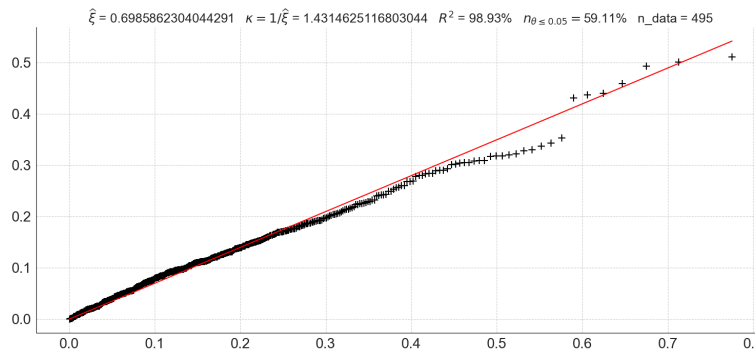


Figure 17: Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Gaussian distributions and based on (82).

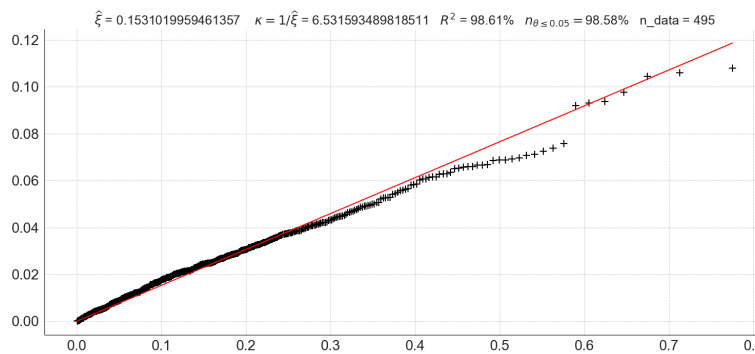


Figure 18: Log-Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Gaussian distributions and based on (82).

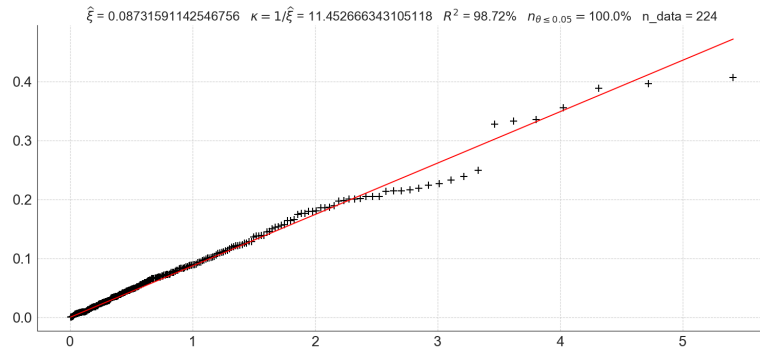


Figure 19: Pareto tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Gaussian distributions and based on (82).

We produce the same analysis for the loss random variable based on (81) with Y_{ℓ}^k , $\ell = 0, \dots, 10$ and $k = 1, \dots, K$ following Student-t distributions with 2 degrees of freedom (Figures 20, 21 and 22) and following standard Gaussian distributions (Figures 23, 24 and 25). Similar observations apply compared to the analysis conducted on the previous simplified form.

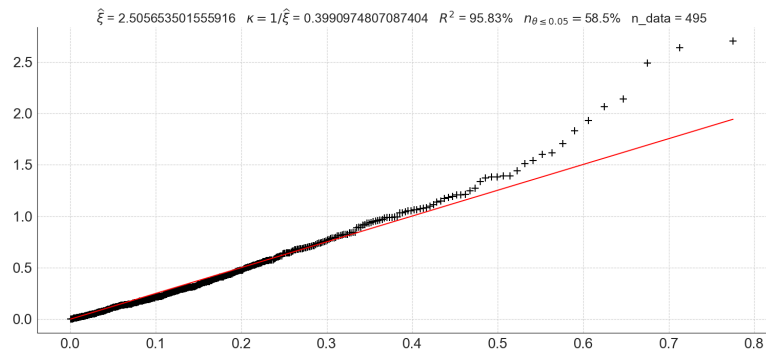


Figure 20: Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 2 degrees of freedom and based on (81).

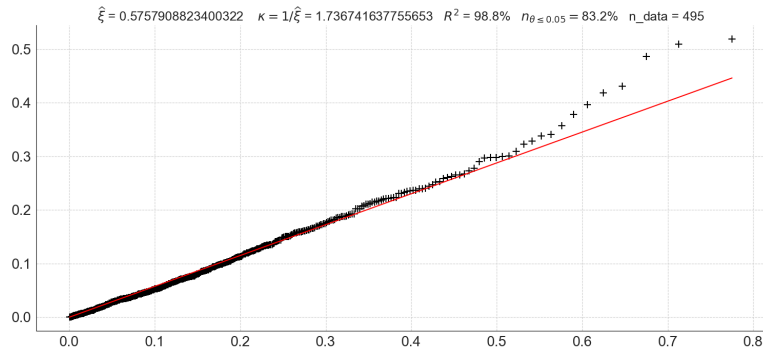


Figure 21: Log-Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 2 degrees of freedom and based on (81).

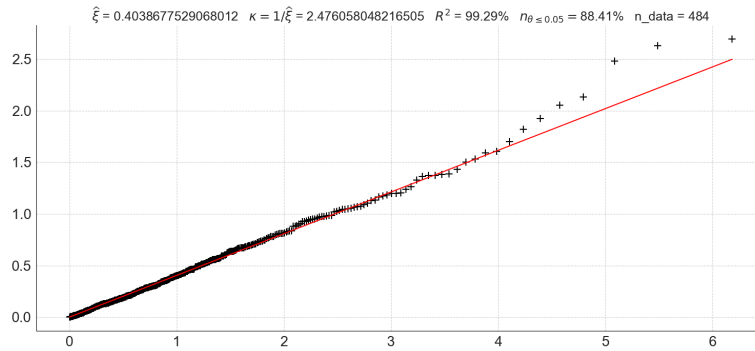


Figure 22: Pareto tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Student-t distributions with 2 degrees of freedom and based on (81).

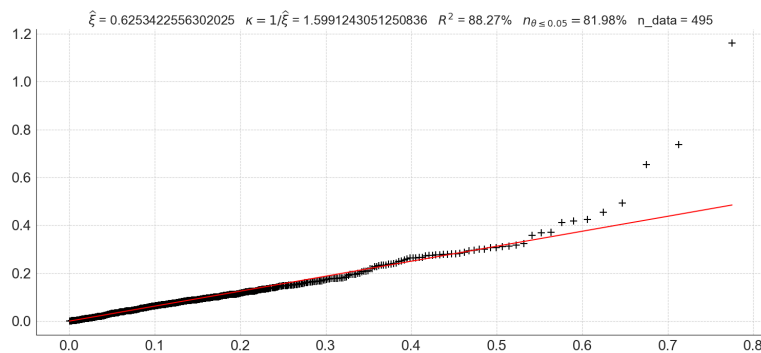


Figure 23: Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Gaussian distributions and based on (81).

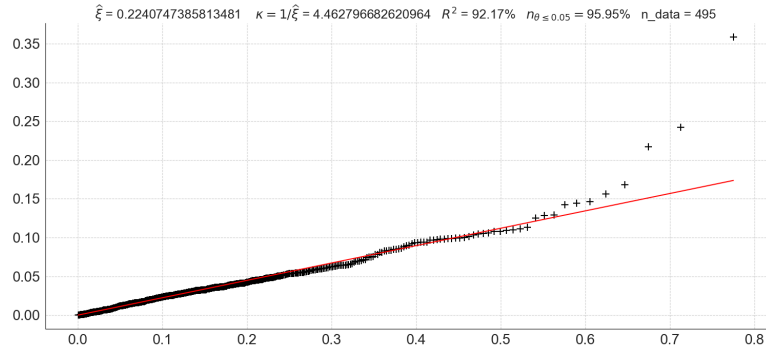


Figure 24: Log-Weibull tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Gaussian distributions and based on (81).

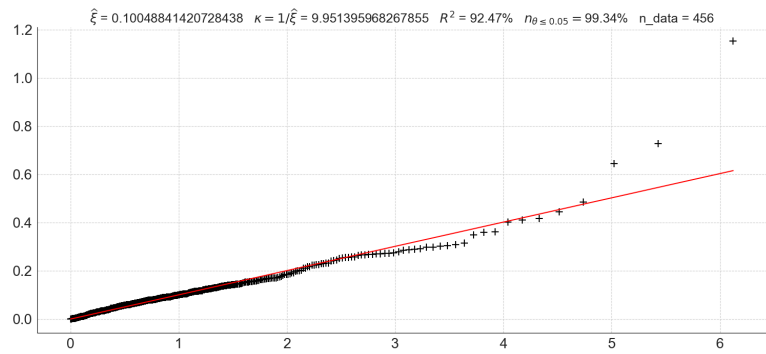


Figure 25: Pareto tail test log-log plot using (71), for member 4 loss r.v. when takers are $\{1, 1, 1, 2\}$ with portfolios drivers following Gaussian distributions and based on (81).

§7 Algorithm diagnosis

We show in Figure 26 a similar analysis as per Figure 12 but in the case where the cooling schedule is based on a logarithmic unlimited budget cooling schedule form. No significant difference can be seen compared to the fixed-budget schedule.

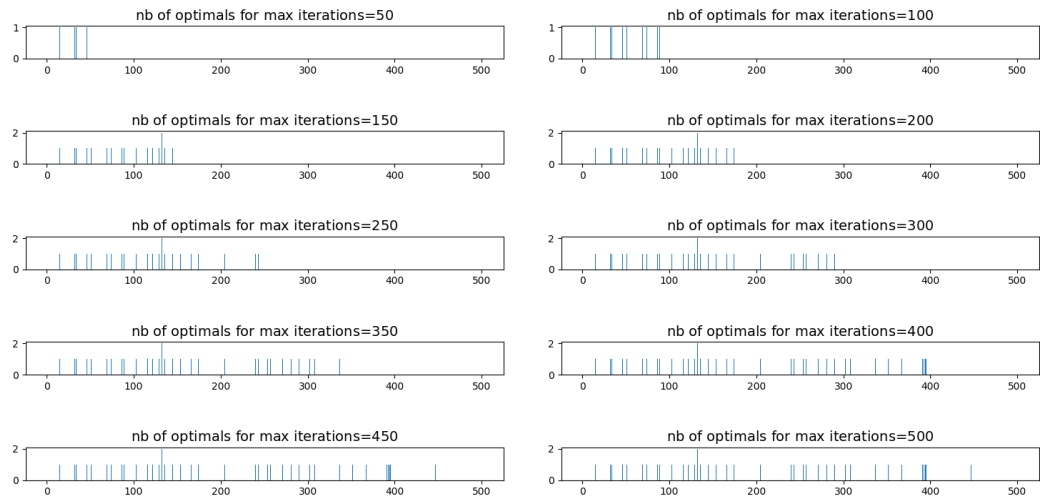


Figure 26: Number of occurrences of the optimal combination of takers reached with respect to the maximum number of iterations based on 100 runs with a logarithmic unlimited budget cooling schedule. The first figure can be regarded as a focus of the last for the first 50 iterations.

Chapter V

Perspectives

All approaches presented in this work are based on static one-period type of models. Though for the portfolio random variations, it can be argued that they can present some stationary increments property, hence the presented models could still prevail for these components, dedicated adjustments are needed for credit default latent variables, e.g. in the vein of [Crépey \(2022\)](#). A dynamic supermodular property analysis could then be considered, based for instance on [Bäuerle et al. \(2008\)](#) and [Kella and Mandjes \(2023\)](#). As for a dynamic version of the Radner equilibrium use for price impact, [Cheridito et al. \(2015\)](#) actually present a dynamic framework for bounded random variables that we adapt in our **Chapter III** to a static one-period setup with non-bounded random variables. A line of research could thus be to go back to their initial dynamic framework and adjust to non-bounded random variables.

Financial actors actions on the market such as a CCP liquidating or hedging large portfolios of defaulting members leading to liquidity drains for certain assets or multiple fire sales from others as a response to such signals, can have detrimental effects on smaller players leading them into default state. Such defaults could exacerbate even more the market with feedback loop effects between financial actors positions and their default. As outlined in **Chapter III**, a future line of research could be the extension of both the approaches detailed in the **Chapters I and III** are combined in a single setup where the market costs and credit costs would be treated endogenously as part of a global equilibrium.

The use of combinatorial simulated annealing algorithms seems promising though requires some refinement regarding the generation Markov chain probabilities to account from the previous proposed combination of takers. Also relying solely on capital costs for identifying the least costly financial network positions rearrangement following a CCP member default may depict abnormal economic situations for certain takers, thus shown to be acting irrationally with respect to other financial constraints (limited funding access, unbearable significant liability increases). The error probabilities bound exposed in **Chapter IV** though informative regarding convergence guarantees like the practicality of usual concentration bounds where by one can fix a certain confidence level of error to reach to infer the parameterization of the hyper-parameters such the sample number or the iterations number. This may be considered for future research. Moreover, we have only covered the case of one single default on several CCPs, but it is worth looking at several defaults, such as two and three big bank institution players using for instance particle models ([Moral, 2004](#)).

As climate risk is also a key topic, and as our setups rely on default latent variables that can include climate risk drivers, see e.g. [Garnier et al. \(2021\)](#), an extension for capturing climate risk aspects should be quite natural and lead to applications for identifying severe climate scenario socks that could rapidly propagate defaults and financial network potential deformations.

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