



HAL
open science

Two problems in constructive stochastic quantisation

Léonard Ferdinand

► **To cite this version:**

Léonard Ferdinand. Two problems in constructive stochastic quantisation. Mathematical Physics [math-ph]. Université Paris-Saclay, 2024. English. NNT : 2024UPASP100 . tel-04735948

HAL Id: tel-04735948

<https://theses.hal.science/tel-04735948v1>

Submitted on 14 Oct 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Two Problems in constructive stochastic Quantisation

*Deux problèmes de quantification stochastique
constructive*

Thèse de doctorat de l'université Paris-Saclay

École doctorale n°564, École doctorale de Physique en Île-de-France (EDPIF)
Spécialité de doctorat: Physique
Graduate School : Physique. Référent : Faculté des sciences d'Orsay

Thèse préparée à l'**IJCLab** (Université Paris-Saclay, CNRS),
sous la direction de **Vincent RIVASSEAU**, Professeur,
et sous le co-encadrement de **Ajay CHANDRA**, Professeur

Thèse soutenue à Paris-Saclay, le 10 Octobre 2024, par

Léonard FERDINAND

Composition du Jury

Membres du jury avec voix délibérative

Colin GUILLARMOU

Directeur de recherche, Université Paris-Saclay Président

Nikolay TZVETKOV

Professeur, ENS Lyon Rapporteur & Examineur

Lorenzo ZAMBOTTI

Professeur, Sorbonne Université Rapporteur & Examineur

Margherita DISERTORI

Professeure, Universität Bonn Examinatrice

Cyril LABBE

Professeur, Université Paris Cité Examineur

Titre : Deux problèmes de quantification stochastique constructive

Mots-clefs : Probabilités, Analyse des EDP, EDP stochastiques, Physique mathématique, Théorie des champs constructive

Résumé : La thèse porte sur l'étude d'équations aux dérivées partielles stochastiques (EDPS) singulières, en connexion avec des questions de physique mathématique et de théorie constructive des champs.

La première partie de la thèse est une introduction à la théorie constructive des champs, à la quantification stochastique, et à la résolution d'EDPS singulières, thématiques dans lesquelles s'insère le reste du manuscrit.

Dans la deuxième partie de la thèse, basée sur un article rédigé en collaboration avec Ajay Chandra, est étudiée la

quantification stochastique de théories des champs euclidiennes non-locales intitulées théories tensorielles des champs, analogues non-locales des mesures Φ^4_2 et Φ^4_3 .

La dernière partie de la thèse porte sur la construction de la mesure Φ^4 sur les variétés fermées tri-dimensionnelles. Cette étude publiée en deux travaux a été réalisée en collaboration avec Ismaël Bailleul, Viet Dang et Tat Dat Tô.

Title: Two Problems in constructive stochastic Quantisation

Keywords: Probabilities, Analysis of PDEs, Stochastic PDEs, Mathematical Physics, Constructive Field Theory

Abstract: The subject of the thesis is the study of singular stochastic partial differential equations (SPDEs), in connection with questions of mathematical physics and constructive field theory.

The first part of the thesis is an introduction to constructive field theory, stochastic quantisation, and the resolution of singular SPDEs, topics in which the rest of the manuscript fits.

The second part of the thesis, based on a paper writ-

ten in collaboration with Ajay Chandra, is about the stochastic quantisation of non-local Euclidean field theories analogous to Φ^4_2 and Φ^4_3 , called tensor field theories.

The last part of the thesis deals with the construction of the Φ^4 measure on three-dimensional closed manifolds. This study, published in two works, was carried out in collaboration with Ismaël Bailleul, Viet Dang and Tat Dat Tô.

Acknowledgements

I am deeply grateful to Vincent for having been my PhD advisor, and for having oriented me both towards constructive physics in general, and stochastic quantisation in particular. Without him, I would have probably never heard about these two subjects, and chosen a different path which I would have enjoyed less. I owe a lot to his advice, and his everlasting curiosity.

The same gratitude goes to Ajay for having accepted so rapidly to supervise me, for his constant and patient guidance and sustain. I also specially thank Ajay for helping me fill the gap between some habits inherited from an education in theoretical physics and proper mathematical redaction.

Finally, albeit informally, Fabien and Viet have also taken a very important part in my supervision, and without all the time they gave me, I would have learnt a lot less during these three years. Many thanks to Viet not only for all the conversations about QFT and stochastic quantisation, but also for introducing me to so many areas of mathematics, and for sharing his communicative passion.

I warmly thank all the other researchers I have collaborated and had long discussions with, in particular I. Bailleul, N. Barashkov, I. Chevyrev, R. Gurau, C. Perez-Sanchez and T.D. Tô. Special thanks also to K. Noui and J. Bengeloun for their involvement in my *comité de suivi* and their close monitoring of my PhD.


I am very thankful to all the members of the jury of my defence, M. Disertori, C. Guillarmou, C. Labbé, N. Tzvetkov, and L. Zambotti. Many thanks to N. Tzvetkov and L. Zambotti for having agreed to refer the present manuscript.

Over the last three years, I also had very exciting conversations with many fellow PhD candidates, including Alberto, Harprit, Killian, Jiasheng, Martin, Parham, Sarah, Valentine and Victor. The simple existence of this manuscript, and the fact that I have pursued the path towards becoming a scientist owes a lot to Antoine, whom I wholeheartedly thank.

I acknowledge the financial assistance provided by the doctoral school EDPIF, and thank all the people who help me do this work, in particular P. Debever who has been of great assistance to tackle all the administrative tasks.

Finally, I would like to thank all the friends and family who have supported me over the last three years.

Contents

Acknowledgements	1
Introduction	4
Summary of the thesis	5
Contribution of the author	6
Résumé détaillé en français	7
1 Constructive Quantum Field Theory and Stochastic Quantisation	12
1.1 The Euclidean approach to Quantum Field Theory	12
1.1.1 From Lorentzian to Euclidean Quantum Field Theory	12
1.1.2 The challenges of Euclidean Quantum Field Theory	17
1.1.3 Perturbative and constructive renormalisation group	21
1.1.4 Osterwalder-Schrader axioms	28
1.2 Some stochastic quantisations of Euclidean QFTs	29
1.2.1 The Langevin dynamic	30
1.2.2 The variational approach	32
1.2.3 The Forward-Backward equations	33
1.2.4 Other PDEs related with EQFTs	35
1.3 Solving singular SPDEs	36
1.3.1 The Da Prato-Debussche trick	38
1.3.2 Paracontrolled calculus	39
1.3.3 Regularity structures	41
1.3.4 The flow approach to singular SPDEs	46
2 Stochastic Quantisation of Tensor Field Theories	49
2.1 Introduction	49
2.1.1 Tensor field theories	50
2.1.2 Main results	51
2.1.3 Notation and conventions	55
2.2 Renormalization and local theory for dynamical T_3^4	57
2.2.1 Random operators for T_3^4	61
2.2.2 Closing the fixed point problem	63
2.3 Renormalization and analysis of dynamical T_4^4	65
2.3.1 Snowball renormalization and stochastic objects	66
2.3.2 Random operators for T_4^4	70
2.3.3 Energy estimates and coming down from infinity	71
2.3.4 Tightness of the invariant measure	74
2.3.5 Local well-posedness: closing the fixed-point problem	76
2.4 The variational approach for tensor field theories	78
2.4.1 Introducing the BG stochastic objects	79
2.4.2 Renormalizing the Boué-Dupuis formula	81
2.5 Kolmogorov arguments for random fields and operators	83
2.5.1 Random fields	83
2.5.2 Random operators	84
2.5.3 Kolmogorov criteria	85
2.5.4 Regularity of the random field \uparrow and operators $\uparrow^{[k]}$	91
2.6 Diagrammatic methods and estimates on larger objects	93
2.6.1 Covariances of the stochastic objects	93
2.6.2 Stranded graphs	94
2.6.3 Bounding renormalized amplitudes with stranded graphs	96
2.6.4 Tensor graphs	102
2.6.5 Multiscale analysis	105
2.6.6 Moment estimates for 	109

2.6.7	Estimates on Ψ , \mathcal{S} , and objects with the rough shift	112
2.G	Besov spaces	116
2.G.1	Bilocal Besov regularity	116
2.H	Facts about the nonlinearity	118
2.I	Controlling mixed terms in $d = 4$	120
3	Construction of the Φ_3^4 Measure on compact Riemannian Manifolds	123
3.1	Introduction	123
3.2	Long time well-posedness and a priori estimate	125
3.2.1	Local in time well-posedness	126
3.2.2	Long time existence and coming down from infinity	132
3.3	Scaling fields, regularity and microlocal extension	137
3.3.1	Scaling fields	137
3.3.2	Function spaces associated with scaling fields	138
3.3.3	The canonical extension	143
3.4	Induction on Feynman amplitudes	151
3.4.1	Configuration space	151
3.4.2	Feynman graphs and Feynman amplitudes	155
3.5	Random fields from renormalisation	163
3.5.1	Kolmogorov type argument	164
3.5.2	The Wick monomials	170
3.5.3	Diagrammatic notation for higher chaoses	173
3.5.4	Modification of the inductive proof	176
3.5.5	Completing the diagrammatic estimates	182
3.6	Invariant measure	189
3.6.1	A Markovian dynamics	189
3.6.2	Non-triviality of the Φ_3^4 measure	192
3.G	Littlewood-Paley-Stein projectors	195
	Bibliography	198

Introduction

The common thread connecting all my research is the mathematical study of *analytically singular* problems. Analytic singularity is the question of making sense of nonlinear operations on distributions, and in particular of the product of two distributions. The space $\mathcal{D}'(\mathbf{R}^d)$ of all distributions on \mathbf{R}^d is the topological dual of $\mathcal{D}(\mathbf{R}^d)$, the space of all smooth and compactly supported functions, or test functions. A distribution Λ is therefore a continuous linear form mapping a function $f \in \mathcal{D}(\mathbf{R}^d)$ to a real number $\langle \Lambda, f \rangle$, and any bounded function $g : \mathbf{R}^d \rightarrow \mathbf{R}$ can be promoted to a distribution $f \mapsto \int f(x)g(x)dx$. Distributions are of interest since many divergent sequences of bounded functions converge as distributions, which is why they can be thought of as generalised functions. Think for instance of

$$\chi_\varepsilon(x) \stackrel{\text{def}}{=} \begin{cases} -1/\varepsilon & \text{on } [-\varepsilon/2, \varepsilon/2], \\ 0 & \text{outside,} \end{cases}$$

which converges as distributions to the Dirac distribution δ , the distribution assigning to any test function f its value at 0. As such, distributions share multiple nice features with test functions, $\mathcal{D}'(\mathbf{R}^d)$ being a nuclear Fréchet space right as $\mathcal{D}(\mathbf{R}^d)$, and distributions can be derived, setting $\langle \partial\Lambda, f \rangle \stackrel{\text{def}}{=} -\langle \Lambda, \partial f \rangle$, and multiplied by any smooth function g , setting $\langle g\Lambda, f \rangle \stackrel{\text{def}}{=} \langle \Lambda, fg \rangle$. Yet, contrary to bounded or test functions, distributions are not an algebra. To see this, just look at the sequence $(\chi_\varepsilon^2)_{\varepsilon>0}$ which fails to converge, ruling out the possibility to define δ^2 . Even bounded functions (that are of course an algebra) viewed as distributions can not be multiplied: indeed, letting

$$\theta(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{on } (0, \infty), \\ 0 & \text{on } (-\infty, 0], \end{cases}$$

in spite of the fact that as bounded functions one has $\theta^n = \theta$, if this relation were to hold as distributions, by differentiating it, one would also have for $n \geq 1$ $n\theta^{n-1}\theta' = n\theta\delta = \theta'$, where we used $\theta' = \delta$, which is a contradiction since the l.h.s. is linear in n while the r.h.s. is independent of n ^[1].

The question of how to multiply distributions therefore seems at best hopeless, at worst ill-posed. Yet a more rigorous analysis reveals that in some favourable cases, this operation remains possible. For instance, when the two distributions have disjoint singular support, or even more precisely micro-support. The micro-support denotes the fact that a distribution, at a point where it is singular, might only be singular in certain directions in Fourier space, think for instance of $\delta(x - y)$, which not only is just singular on the diagonal, but also is singular only in directions normal to the diagonal, and can therefore be multiplied with distributions whose singularities reaches the diagonal tangently. Distributions can also be multiplied with some functions that are rougher than smooth, see Young's products introduced in Lemma [1.14](#).

If nonlinear operations involving distributions sound like an abstract analytic question, analytically singular problems arise in various domains of mathematics and mathematical physics.

A first example comes from Partial Differential Equation (PDE) analysis, since the study of any PDE of the form

$$\begin{aligned} \mathcal{L}u(t, x) &= F(u(t, x)) + f(t, x), \\ u(0, x) &= \phi(x), \end{aligned}$$

where \mathcal{L} is a linear operator (say the l.h.s. of a dissipative or dispersive equation) and F is a (typically polynomial) nonlinearity, in the case where the forcing f or the initial condition ϕ are so rough that the

¹This counterexample is drawn from [\[BDH14\]](#).

solution u can not be expected to be bounded, yields one to try to make sense of $F(u(t, x))$ in spite of the fact that u is merely distributional. Such equations are called *singular PDEs*.

A second example comes from probabilities and stochastic analysis, when trying to define polynomials in a Gaussian process which is almost surely not function valued, such as a white-noise in $d = 1$ or a free field in $d \geq 2$. A paradigmatic example being the fact that, denoting by B a Brownian motion, $\int_s^t B_u dB_u$, despite involving a product of a distribution \dot{B} with a non-smooth function B , when approximated by means of Riemann series, converges as a random variable. Similar constructions can be performed for some other polynomials of a Gaussian free field ϕ , like its square, in which case it can be necessary to remove the expectation which would be divergent, rather looking at

$$:\phi^2 := \phi^2 - \mathbf{E}[\phi^2] \quad (0.1)$$

Note that the above equation is informal, since the l.h.s. is in fact defined after an approximation procedure, and is truly a limit in probability of a sequence of random variables. Such a construction is called a *singular random field*.

A last example comes from mathematical physics, and the rigorous definition of Quantum Field Theories (QFTs). A promising path towards the non-perturbative construction of QFTs indeed consists in trying to construct a measure in the Euclidean space, and to analytically continue its moments in order to define the correlators in Lorentzian signature. Such measures corresponding to QFTs of interest generally take the form of perturbed Gaussian integrals

$$\nu(d\phi) \propto e^{-S^{\text{int}}(\phi)} \mu(d\phi),$$

where μ is a Gaussian supported on distributions but not on bounded functions, and the interaction Euclidean action $S^{\text{int}}(\phi)$ involves performing nonlinear operations on the distribution ϕ . Such a formally ill-defined probability measure is called a *singular measure*.

It turns out that these three examples, in addition to presenting a similar difficulty, are also deeply connected. Indeed, singular measures can be invariant under some suitably chosen singular PDEs, and the solution to some singular PDEs can be formally expanded into some possibly singular random fields that are some arbitrarily large polynomials in a Gaussian (the Picard iterates of the initial condition and/or random driving term of the PDE).

Moreover, in all of the three settings, according to the roughness of the Gaussian/noise, it may become necessary, in order to handle the singularity –that is to say to define the nonlinear operation on some distribution–, to perform a renormalisation procedure generalising the idea of removing the expectation in (0.1). Finally, in all the three settings, such a procedure becomes more and more difficult with the dimension, and there exists a threshold dimension such that it even stops being possible. For instance, the renormalisation made in (0.1) is possible in one dimension (where there is no singularity) and in two and three dimensions, but not in four dimensions and beyond.

Summary of the thesis

My work fully fits into this context, which is why the content of the present manuscript thus lies at the interface between PDE analysis, stochastic analysis, and constructive QFT. Moreover, the purpose of this manuscript is both to introduce and motivate these topics and their relations, and to present the problems that I have studied as a PhD candidate.

Chapter 1 is an introduction to several problems in constructive QFT, with an emphasis on the interest of stochastic analytic and PDE methods to tackle them. There I introduce three different stochastic quantisations, and in particular the Langevin dynamic. This leads me to briefly describe some of the main ideas of the three main approaches to solving singular parabolic SPDEs, that is to say regularity structure, paracontrolled calculus, and the Renormalisation Group (RG) flow approach. The construction of a nonlocal Euclidean QFT measure dubbed tensor field theory by stochastic quantisation is performed in Chapter 2. The nonlocality of the interaction makes regularity structures and paracontrolled calculus inefficient, which is why a new ansatz based on some reinjections is introduced. Chapter 3 is devoted

to the study of the construction of the Φ_3^4 measure on compact closed manifolds using paracontrolled calculus. The estimates on the enhancement of the noise are performed thanks to a new inductive approach inspired by Epstein-Glaser renormalisation. Together these two chapters constitute the main body of the thesis, and are the two problems in constructive stochastic quantisation to which the title refers.

Contribution of the author

This manuscript is based on some of the works in which I have taken part during the 3 years of my PhD:

[CF24a] A. CHANDRA and L. FERDINAND. A flow approach to the generalized KPZ equation. *arXiv e-prints* (2024). <https://arxiv.org/abs/2402.03101>

[BDF⁺23] I. BAILLEUL, N. V. DANG, L. FERDINAND, G. LECLERC and J. LIN. Spectrally cut-off GFF, regularized Φ^4 measure, and reflection positivity. *arXiv e-prints* (2023). <https://arxiv.org/abs/2312.15511>

[BDFT23] I. BAILLEUL, N. V. DANG, L. FERDINAND and T.D. TÔ. Global harmonic analysis for Φ_3^4 on closed Riemannian manifolds. *arXiv e-prints* (2023). <https://arxiv.org/abs/2306.07757>

[CF24b] A. CHANDRA and L. FERDINAND. A Stochastic Analysis Approach to Tensor Field Theories. *arXiv e-prints* (2023). <https://arxiv.org/abs/2306.05305>

[BDFT24] I. BAILLEUL, N. V. DANG, L. FERDINAND and T.D. TÔ. Φ_3^4 measures on compact Riemannian 3-manifolds. *arXiv e-prints* (2023). <https://arxiv.org/abs/2304.10185>

[FGPSVT23] L. FERDINAND, R. GURAU, C.I. PEREZ-SANCHEZ and F. VIGNES-TOURNERET. Borel summability of the $1/N$ expansion in quartic $O(N)$ -vector models. *Annales Henri Poincaré* (2023). <https://link.springer.com/article/10.1007/s00023-023-01350-w>

The article [FGPSVT23] is about a rigorous study of the vectorial $1/N$ limit in the non-singular case (say in $d = 1$). The work [CF24b] is reported in Chapter 2. Chapter 3 is based on the work [BDFT24], which in turn utilises some results proven in [BDFT23]. Since these results are some harmonic analytic technical lemmas –the extension of some paraprodut and kernel estimates to the manifold setting–, we have chosen to separate them from the main body of the proof not to obscure it, and for the same reason I have decided not to report them in the present manuscript. [BDF⁺23] is a short pedagogical note about the fact that Fourier cut-offs are not compatible with proving reflection positivity. Finally, in the work [CF24a], the flow approach to singular PDEs is extended to the case of a non-polynomial interaction, using a new basis for the RG flow based on multiindices indexing the coefficients of a Butcher series.

Résumé détaillé en français

Une *théorie quantique des champs* scalaire sur l'espace de Minkowski $\mathbf{R}^{1,3}$ consiste en un espace de Hilbert $\mathcal{H} = L^2(\mathcal{S}'(\mathbf{R}^3), \nu)$ équipé d'une algèbre d'opérateurs $(\Psi(f))_{f \in \mathcal{S}(\mathbf{R}^3)}$ indexée par $\mathcal{S}(\mathbf{R}^3)$. \mathcal{H} doit également être doté d'une représentation unitaire du groupe de Poincaré, et d'un élément particulier intitulé le vide, qui doit être stabilisé par l'action du groupe de Poincaré. La capacité prédictive de la théorie quantique des champs provient de ce que cette construction peut formellement être utilisée afin de calculer des amplitudes de probabilité de désintégration de particules, quantités qui peuvent *in fine* être mesurées expérimentalement, notamment dans les accélérateurs de particules. Les amplitudes de probabilités sont tout d'abord exprimées en fonction des éléments de matrice de produits d'opérateurs de la forme $\Psi(f)$ entre deux bases correspondant à un passé et un futur lointains où les particules n'interagissent hypothétiquement pas. Il se trouve qu'en dehors de cas exactement solvables, de telles expressions ne peuvent généralement pas être calculés, ni même définies mathématiquement. Parmi les cas exactement solvables d'intérêt, citons notamment celui de particules libres, c'est-à-dire ne subissant absolument aucune désintégration, et n'ayant donc pas d'interactions. La tentative la plus aboutie de donner corps aux éléments de matrice d'intérêt en vue de calculer des probabilités de désintégration est celle qui consiste à les réexprimer à l'aide d'une autre algèbre d'opérateurs, celle-ci correspondant à des particules libres, et pour laquelle certaines quantités peuvent être calculées. Il se trouve que l'on est alors conduit à évaluer des expressions prenant formellement la forme d'intégrales oscillantes de dimension infinie,

$$W^{(n)}(z_1, \dots, z_n) = \frac{\int \prod_{i=1}^n \phi(z_i) e^{iS^{\text{cl}}[\phi]/\hbar} \mathbf{d}\phi}{\int e^{iS^{\text{cl}}[\phi]/\hbar} \mathbf{d}\phi}, \quad (0.2)$$

où S^{cl} est l'action de la théorie classique qui doit être obtenue dans le régime $\hbar \downarrow 0$, et $\mathbf{d}\phi$ est une mesure de Lebesgue formelle sur un hypothétique espace de fonctions sur $\mathbf{R}^{1,3}$.

Si autant calculer que donner une définition mathématique à des expressions de la forme (0.2) reste absolument hors de portée des connaissances actuelles, il existe deux manières d'employer cette expression, toutes deux reposant sur l'idée suivante : les quantités $W^{(n)}$, où *fonctions de Wightman*, peuvent être obtenues comme valeurs au bord des quantités

$$S^{(n)}(x_1, \dots, x_n) = \int_{\mathcal{S}'(\mathbf{R}^4)} \prod_{i=1}^n \phi(x_i) e^{-S^{\text{E,int}}[\phi]} \mu(\mathbf{d}\phi)$$

quand les premières composantes de x_1, \dots, x_n tendent vers des valeurs imaginaires pures. Ici, μ est une mesure Gaussienne sur $\mathcal{S}'(\mathbf{R}^4)$ de covariance C , et $S^{\text{E,int}}[\phi] + \frac{1}{2}\langle \phi, C^{-1}\phi \rangle$ est la quantité obtenue

par continuation analytique de S^{cl}/\hbar , où il est entendu que $S^{\text{E,int}}$ est non-quadratique et en général $C^{-1} = m^2 - \Delta$. Une telle approche consistant à travailler dans l'eulidien et à prendre ensuite un temps imaginaire pur est connue sous le nom de *rotation de Wick*, et elle était déjà pratiquée avant la naissance de la théorie quantique des champs, afin d'extrapoler les propriétés du propagateur des ondes de celles de la fonction de Green du Laplacien. Elle a le grand avantage que la mesure de Lebesgue perturbée par une phase est remplacée par une théorie quantique des champs euclidienne, c'est-à-dire une mesure Gaussienne perturbée par une quantité *a priori* bornée, du moins dans le cas où l'interaction est par exemple un polynôme de degré pair. Pour en revenir aux deux manières d'employer cette expression, la première est le développement perturbatif, qui consiste à écrire

$$S^{(n)}(x_1, \dots, x_n) = \sum_{k \geq 0} \frac{(-1)^k}{k!} \int_{S'(\mathbf{R}^4)} (S^{\text{E,int}}[\phi])^k \prod_{i=1}^n \phi(x_i) \mu(d\phi),$$

permutant formellement la série exponentielle et l'intégrale Gaussienne. Il se trouve que quand la croissance de $S^{\text{E,int}}$ est plus de quadratique, cette permutation est illicite. Cependant, le développement perturbatif permet de calculer des nombres, et de donner des approximations des différents paramètres du système standard de la physique des particules. La seconde manière d'employer cette expression est de tâcher de vérifier qu'elle soit bien définie, afin d'en conclure l'existence de la théorie quantique des champs sous-jacente. Il se trouve qu'en règle générale, construire une mesure Gaussienne perturbée n'est *a priori* pas évident. En effet, pour $C^{-1} = m^2 - \Delta$, μ n'est pas supportée sur des fonctions lisses, mais sur des distributions, ce qui rend la définition de $S^{\text{E,int}}[\phi]$ problématique : il s'agit là du *problème ultraviolet*. D'autre part, si ϕ est tirée au hasard selon μ , elle n'a pas de décroissance à l'infini, ce qui ajoute encore à la difficulté à définir $S^{\text{E,int}}[\phi]$: il s'agit là du *problème infrarouge*. Enfin, si le développement perturbatif permet d'obtenir une expression pour $S_\varepsilon^{\text{E,int}}$ après application d'une régularisation ultraviolette à l'échelle ε qui permette de formellement résoudre le problème ultraviolet, l'*action nue* $S_\varepsilon^{\text{E,int}}$ obtenue à la suite de cette procédure intitulée *renormalisation* perd toute propriété de positivité – et ce car elle est obtenue par ajout de quantités divergeant avec ε , appelées *contre-terme*s : il s'agit enfin là du *problème de grand champ*.

Une approche à la définition des mesures Gaussiennes perturbées à récemment connu une grande activité : il s'agit d'un ensemble de méthodes utilisant les techniques du calcul stochastique, et connues sous le nom de *quantification stochastique*. La principale quantification stochastique est la dynamique de Langevin, qui consiste à construire la théorie quantique des champs euclidienne comme mesure invariante pour la dynamique

$$\partial_t u = -\nabla_\Phi S^{\text{E,int}}[u] + \xi,$$

où ξ est un bruit blanc d'espace-temps. Les travaux présentés dans le présent manuscrit s'inscrivent pleinement dans ce contexte, puisque le premier chapitre [I](#) consiste en une présentation des difficultés

inhérentes à la définition non-perturbative des théories quantiques des champs, et en une introduction à la quantification stochastique. Dans le deuxième chapitre [2] je présente des résultats d'un travail commun avec Ajay Chandra [CF24b], où est construite une théorie quantique des champs non locale de type tensoriel. Quant au troisième chapitre [3], j'y rapporte les principaux arguments d'un travail réalisé en commun avec Ismael Bailleul, Tat Dat Tô, et Viet Dang, et publié en deux parties [BDFT24, BDFT23], portant sur la construction de la mesure Φ_3^4 sur les variétés.

Le deuxième chapitre du présent manuscrit porte donc sur la quantification stochastique d'une théorie quantique des champs non locale, de type tensoriel, dénommée *Théorie tensorielle des champs quartique melonique*, où T_d^4 en dimension d . Les théories tensorielles des champs sont des théories scalaires dont le comptage de puissance perturbatif est lié à la limite à grand N des modèles de tenseurs aléatoires. Cela tient au fait que les mesures T_d^4 sont des mesures gaussiennes perturbées par une interaction qui, dans l'espace de Fourier, correspond à évaluer un invariant de trace tensoriel de la transformée de Fourier du champ $\mathcal{F}(\phi)(m_1, \dots, m_d) \equiv \hat{\phi}_{m_1, \dots, m_d}$ qui peut être vue comme un tenseur de rang d , et de dimension infinie. Tandis que pour les matrices aléatoires, ce sont les graphes planaires qui sont sélectionnés dans la limite à grand N , ce sont des graphes dits *meloniques* (une classe de graphes en bijection avec des arbres) qui subsistent pour les tenseurs : ce sont donc les seuls graphes divergents pour les théories tensorielles des champs. De plus, la non localité du graphe *tadpole* fait qu'en dimension 5, dimension où la théorie est critique, la renormalisation de fonction d'onde de ce graphe l'emporte sur la renormalisation de constante de couplage, ce qui fait que la théorie est asymptotiquement libre dans l'ultraviolet.

En dimension 4, la théorie T^4 présente des divergences ultraviolettes analogues à celles de Φ_3^4 . Des solutions locales à la dynamique de Langevin pour T_4^4 sont d'abord construites, à l'aide d'opérateurs aléatoires permettant de prendre en compte la non localité de la non linéarité, et d'un ansatz en deux composantes : d'une part, de nombreux champs aléatoires au-delà du seul bruit blanc sont éliminés à l'aide de répétitions de la méthode de Da Prato et Debussche, d'autre part, un nouvel ansatz propre à ce modèle est développé pour tenir compte de la divergence d'un opérateur aléatoire. Ce nouvel ansatz se base sur le fait qu'en réinjectant l'équation dans elle-même, l'on crée un opérateur aléatoire ayant de meilleures propriétés, et permettant de résoudre l'équation. Notons que cette méthode ne fonctionnerait bien sûr pas dans le cas local, mais qu'en même temps, les structures de régularité et le calcul para-contrôlé ne s'appliquent pas dans ce contexte non local. La mesure T_4^4 est ensuite construite grâce à des estimées d'énergie L^2 sur la solution de la dynamique de Langevin, et également grâce à la méthode variationnelle introduite par Barashkov et Gubinelli [BG20]. Notons que dans les deux cas, il est nécessaire de tirer avantage de la coercivité de l'interaction afin de contrôler le problème de grand champ. Il se trouve que l'interaction quartique melonique est bien positive et que, bien que plus petite que la norme L^4 , elle permet tout de même de contrôler les termes croisés. Cependant, comme cette interaction est plus petite que la norme L^2 , les meilleures estimées d'énergies qui soient accessibles sont des estimées L^2 , qui ne permettent pas de redémarrer la dynamique de Langevin, ni de prouver que cette équation admet des solutions globales. Pour conclure, tous les champs et opérateurs aléatoires nécessaires

à la construction de solutions locales à la dynamique de Langevin de T_4^4 , ainsi qu'à la construction de la mesure, sont construits à l'aide de l'analyse multi-échelle. Il s'agit d'une construction relativement insensible à la taille des objets, mais qui tire parti de la structure melonique des divergences.

Le troisième et dernier chapitre du présent manuscrit est consacré à la construction de la mesure Φ_3^4 sur les variétés fermées. Cette construction est réalisée en utilisant la dynamique de Langevin, et en démontrant que celle-ci admet des solutions globales, et définit un processus markovien vérifiant la propriété de Feller. Les solutions locales sont d'abord construites grâce à un ansatz multiplicatif, d'abord introduit par Jagannath et Perkowski [JP23]. Il est ensuite démontré que ces solutions sont globales grâce à une estimée d'énergie du type

$$\sqrt{t \vee 1} \|u(t)\|_{L^p(M)} \leq C(\hat{\xi}),$$

où $\hat{\xi}$ désigne l'ensemble des champs aléatoires nécessaires à la résolution de l'équation. Cette estimée, souvent nommée *coming down from infinity*, a en effet la particularité d'être uniforme en la condition initiale, ce qui est déterminant pour montrer le caractère Feller du processus de Markov. Elle est obtenue à l'aide d'estimées d'énergie, suivant la méthode introduite par Mourrat et Weber [MW14] dans le cas d'un ansatz para-contrôlé.

La nouveauté de ce chapitre consiste en une construction de l'ensemble des champs aléatoires sur les variétés. La renormalisation et le contrôle des covariances des champs aléatoires sont inspirés des travaux d'Epstein et Glaser sur la renormalisation, ainsi que des travaux de Brunetti et Fredenhagen. La première étape est de définir des espaces fonctionnels capturant à la fois les propriétés microlocales des noyaux, et leur homogénéité sous dilatation de diagonales. Sont donc définis des espaces $\mathcal{S}_\Gamma^a(X^2)$ de noyaux $K(x, y)$, $x, y \in X$, dont le front d'onde soit supporté dans Γ , et qui soient faiblement homogènes de degré a sous dilatation d'une sous variété $Y \subset X^2$, telle que $N^*(Y) \subset \Gamma$. En pratique, l'on a

$$K \in \mathcal{S}_{N^*(\{(x,t)=(y,s)\})}^a((\mathbf{R} \times M)^2) \Rightarrow |\partial_{x,y,\sqrt{t},\sqrt{s}}^\alpha K((x,t),(y,s))| \lesssim (|x-y| + \sqrt{|t-s|})^{a-|\alpha|},$$

et le noyau de la chaleur, la covariance du champ libre dynamique, ainsi que le noyau $[\odot](x, y, z)$ du terme résonnant appartiennent à des espaces \mathcal{S}_Γ^a . Est ensuite démontré un théorème d'extension de distributions, affirmant qu'une suite Λ_ε de distributions sur X convergent vers $\Lambda \in \mathcal{S}^a(X \setminus Y)$ peut être étendue à une suite convergente de distributions sur la totalité de X , pourvu que l'on ait

$$a > -\text{codim}(Y \subset X).$$

Notons qu'un résultat similaire existe si $\text{codim}(Y \subset X) \geq a > -\text{codim}(Y \subset X) - 1$, cette fois modulo soustraction d'une suite *locale* de distributions supportées sur Y . Grâce à ce résultat, les amplitudes apparaissant lorsque sont évaluées les covariances des objets stochastiques peuvent être construites de manière inductive. Les sous-amplitudes divergentes sont prises en charge grâce à la possibilité

de soustraire une suite locale, correspondant au contre-terme. Enfin, un élément important de cette construction consiste en la gestion des sous-amplitudes convergentes, qui pourraient causer des pertes de régularité pour les champs aléatoires. Cette gestion s'appuie sur une analyse fine des propriétés micro-locales du noyau $[\odot(x, y, z)]$ du terme raisonnant. Ces propriétés correspondent au fait que pour $f \in C^\infty(M)$ et $g \in \mathcal{D}'(M)$, $f \odot g \in C^\infty(M)$, et impliquent que les sous-amplitudes convergentes ne sont pas responsables de pertes de régularité. Il s'agit d'un phénomène analogue à la renormalisation positive dans le cas des structures de régularité.

Chapter 1

Constructive Quantum Field Theory and Stochastic Quantisation

The general mathematical context of the work which is presented in this manuscript is the study of singular stochastic partial differential equations, and the construction of singular Gibbs measures, an area which lies on the boundary between probabilities and stochastic analysis, and the analysis of PDEs. The aim of the present chapter is to introduce several aspects of this field, motivating them by the question of how to define a Quantum Field Theory (QFT). The quest for a non-perturbative definition of QFT dates back from the pioneering works of Nelson [Nel66] in $d = 2$, and Glimm and Jaffe [GJ73] in $d = 3$ based on some discrete renormalisation group techniques or phase space expansions. It was shortly after that Parisi and Wu [PW81] raised the idea that the construction of QFTs could be achieved by solving a formal singular stochastic partial differential equation, a relationship that was dubbed *stochastic quantisation*. If for twenty years, solving such equations has remained out of the scope of the available tools, the recent advances by Da Prato and Debussche [DPD03] followed by the groundbreaking work by Hairer [Hai14] paved the way for a general solution theory for these equations, and for some important progress in the study of singular Gibbs measures. This introduction is organised as follows: in the first section, we introduce the main problems in constructive QFT, before in the second section we discuss several stochastic quantisations, that is to say several ways in which some stochastic analytic tools can be leveraged in order to solve some of these problems. The last section is an introductory overview of some solutions theories to singular SPDEs, presenting some of the main ideas and techniques of this field.

1.1 The Euclidean approach to Quantum Field Theory

In this section, we aim to motivate the so-called “Wick rotation”, which states that the starting point for a rigorous and non-perturbative study of Quantum Field Theory on Lorentzian space-time is a probability measure on a Riemannian manifold. To do so, we start by briefly reviewing what is meant by a QFT, before introducing the difficulties that arise when trying to define it beyond perturbation theory. The connection between QFT and stochastic analysis will be formulated thanks to a heuristic based on the Feynman-Kac formula. We conclude by presenting the renormalisation group flow technique, showing that it is well-suited to a non-perturbative study of the Euclidean Φ^4 measure in $d = 2$, and by shortly discussing the Osterwalder-Schrader axioms allowing to recover the QFT from the measure.

1.1.1 From Lorentzian to Euclidean Quantum Field Theory

1.1.1.1 What is a Quantum Field Theory?

A QFT is primarily the description of a quantum mechanical system with an infinite number of degrees of freedom, indexed by space positions on \mathbf{R}^3 .

Recall that in the case of a finite number of degrees of freedom say p (a paradigmatic example being the p components of the space position of some particle), quantum mechanics is formalised as the datum of some Hilbert space of the form $L^2(\mathbf{R}^p)$, endowed with an algebra of Hermitian operators known as *observables* whose commutation relations reflect the symplectic structure of classical variables. This Hilbert space is furthermore equipped with a distinguished vector called the *vacuum*, and with a unitary representation $(e^{tH})_{t \in \mathbf{R}}$ of time translations that stabilises the vacuum, and whose generator H is called the *Hamiltonian* (here we make the assumption that the underlying classical theory is invariant by

time translations). However, in general, one does not view the time translations as acting on vectors in the Hilbert space, but rather on observables, promoting any observable A to a time-indexed family of operators defined as

$$A(t) \stackrel{\text{def}}{=} e^{tH} A e^{-tH},$$

and known as the *Heisenberg representation* of A .

This picture of quantum mechanics generalises to some extent to the infinite dimensional case of QFT, but in this latter situation, the Hilbert space has to be much bigger, more precisely of the form $\mathcal{H} = L^2(\mathcal{S}'(\mathbf{R}^3))$. This Hilbert space still comes with a vacuum state, along with an algebra of observables, but the algebra now turns out to be indexed by $\mathcal{S}(\mathbf{R}^3)$, and hence infinite dimensional. Denoting by $\Psi = (\Psi(f))_{f \in \mathcal{S}(\mathbf{R}^3)}$ such observables, we will use the informal notation $\Psi(x)$ to denote the operator-valued distribution still requiring a pairing with some smooth function. Moreover, rather than solely representing time translations, one aims to implement at the quantum level the invariance under the full group of isometries of Minkowski space $\mathbf{R}^{1,3}$, so that the Hilbert space must be equipped with a unitary representation of the whole Poincaré group. In the same spirit, one often views the Heisenberg representation of an observable Ψ as a family of operators indexed by space-times Schwartz functions, sometimes called a *quantum field*. Thus, again, given the Hamiltonian H , we define the Heisenberg representation of Ψ by

$$\Psi(t, x) \stackrel{\text{def}}{=} e^{tH} \Psi(x) e^{-tH}.$$

1.1.1.2 S -matrix and time-ordered vacuum expectation values

In scattering experiments, particle physicists have access to the decay of some in-going particles into some other out-going particles, and the predictive power of QFT lies in its capacity to give an approximate value for the probabilities of such events. A first example of QFT is therefore that in which one in-going particle propagates forever with probability one, without ever decaying, called *free theory*. Though idealised, the case of a free theory is of practical importance, since it allows for exact computations. It even turns out to be one of the few cases in which exact computations are possible, which is why it is also viewed as a basis to construct richer QFTs. Indeed, a general assumption often made in scattering theory is that in the far past and future, particles propagate freely, so that there exist Hilbert spaces \mathcal{H}_{in} , \mathcal{H}_{out} isomorphic to \mathcal{H} and corresponding to free theories that can be used to compute the probabilities of decays. Indeed, all these probabilities are gathered in the unitary matrix S called *S -matrix* mapping a basis of \mathcal{H}_{in} on a basis of \mathcal{H}_{out} . It turns out that $S = \lim_{t \uparrow \infty} S(t)$ where $S(t)$ solves

$$dS(t) = -i H^{\text{int}}(\Psi)(t) S(t) dt, \quad \lim_{t \downarrow -\infty} S(t) = 0.$$

Here, H^{int} the *interaction Hamiltonian* stands for the part of the Hamiltonian which is not quadratic in Ψ , and we now have to specify the dependence of the Hamiltonian H on the family of observable Ψ . The particular case of a free theory corresponds to $S = \text{Id}$, and one has $H^{\text{int}} = 0$.

The celebrated LHZ formula states that the knowledge for all $n \geq 1$ and $(z_1, \dots, z_n) \in \mathbf{R}^{1,3}$ of the *time-ordered vacuum expectation values*

$$\langle \Omega_{\text{out}}, T(\Psi(z_1), \dots, \Psi(z_n)) \Omega_{\text{in}} \rangle,$$

is sufficient to recover the coefficients of the S -matrix. Here, T denotes time-ordering and Ω_{in} , Ω_{out} denote the two vacuum states of \mathcal{H}_{in} , \mathcal{H}_{out} . More precisely, when working with the interacting field Ψ , one always computes the expectation values between two a priori different and free vacuums: the incoming and outgoing ones. This corresponds to the aforementioned assumption that the propagation is free in the far past and future. Note that the idea of the LHZ formula stems from the fact that informally S can be written as

$$S = T(e^{-i \int_{-\infty}^{\infty} H^{\text{int}}(\Psi)(t) dt}).$$

Moreover, the Gell-Mann-Low formula expresses these vacuum expectation values in terms of another set of observables Φ corresponding to a free theory with vacuum Ω (in general, just take $\Omega = \Omega_{\text{in}}$):

$$\langle \Omega_{\text{out}}, T(\Psi(z_1), \dots, \Psi(z_n)) \Omega_{\text{in}} \rangle = \frac{\langle \Omega, T(\Phi(z_1), \dots, \Phi(z_n)) e^{i \int_{\mathbf{R}^{1,3}} \mathcal{L}^{\text{int}}(\Phi(z)) dz} \Omega \rangle}{\langle \Omega, T(e^{i \int_{\mathbf{R}^{1,3}} \mathcal{L}^{\text{int}}(\Phi(z)) dz} \Omega \rangle},$$

where $\mathcal{L}^{\text{int}}(\Phi(z))$ is the *interaction Lagrangian density*, that is to say the non-quadratic in Φ part of the Lagrangian density, which we suppose to be local in Φ , and we note that it holds $\int_{\mathbf{R}^{1,3}} \mathcal{L}^{\text{int}}(\Phi(z)) dz = - \int_{-\infty}^{\infty} H^{\text{int}}(\Psi)(t) dt$.

At this point, some additional notation is required. We denote by $Q = (Q_{it})_{t \in \mathbf{R}} \stackrel{\text{def}}{=} (e^{-itH})_{t \in \mathbf{R}}$ the unitary group generated by the total Hamiltonian $H = H^0 + H^{\text{int}}$, and define accordingly Q^0, Q^{int} as the unitary groups generated by H^0, H^{int} (pay attention to the fact that with our notation we rather define Q as a semigroup evaluated at an imaginary time). For simplicity, we assume $t_n \geq t_{n-1} \cdots \geq t_2 \geq t_1$ and enforce it in the sequel. Though computing $\langle \Omega, T(F(\Phi)) \Omega \rangle$ for functionals F of the form

$$\begin{aligned} F(\Phi) &= \prod_{j \in [n]} \Phi(z_j) e^{i \int_{\mathbf{R}^{1,3}} \mathcal{L}^{\text{int}}(\Phi(z)) dz} = \prod_{j \in [n]} \Phi(z_j) e^{-i \int_{\mathbf{R}^{1,3}} H^{\text{int}}(\Phi)(t) dt} \\ &= \prod_{j \in [n]} \Phi(z_j) Q_{i(\infty - (-\infty))}^{\text{int}} = Q_{i(\infty - t_n)}^{\text{int}} \Phi(z_n) Q_{i(t_n - t_{n-1})}^{\text{int}} \cdots Q_{i(t_2 - t_1)}^{\text{int}} \Phi(z_1) Q_{i(t_1 - (-\infty))}^{\text{int}} \end{aligned} \quad (1.1)$$

does not seem particularly straightforward, we can combine the invariance of the (free) vacuum $|\Omega\rangle$ under time translations generated by the free Hamiltonian H^0 (i.e. under the action of the unitary group Q^0) with the fact that writing $z = (t, x)$ we have $\Phi(z) = Q_{-it}^0 \Phi(x) Q_{it}^0$ to reexpress $\langle \Omega, T(F(\Phi)) \Omega \rangle$ in terms of the unitary group Q . We obtain

$$\begin{aligned} \langle \Omega, T(F(\Phi)) \Omega \rangle &= \langle \Omega, Q_{i(\infty - t_n)}^{\text{int}} Q_{-t_n}^0 \Phi(x_n) Q_{i(t_n - t_{n-1})} \cdots Q_{i(t_2 - t_1)} \Phi(x_1) Q_{it_1}^0 Q_{i(t_1 - (-\infty))}^{\text{int}} \Omega \rangle \\ &= \langle \Omega, Q_{i(\infty - t_n)} \Phi(x_n) Q_{i(t_n - t_{n-1})} \cdots Q_{i(t_2 - t_1)} \Phi(x_1) Q_{i(t_1 - (-\infty))} \Omega \rangle, \end{aligned} \quad (1.2)$$

where for $j \in [n]$ we write $z_j = (t_j, x_j)$ and we used the invariance of Ω under Q^0 to sandwich the whole expression with $Q_{i\infty}^0$.

Finally, we conclude this section by making a choice of free Lagrangian/Hamiltonian again dictated by locality, fixing

$$\mathcal{L}^0(\Phi(z), \partial_\mu \Phi(z)) \stackrel{\text{def}}{=} \mathcal{L}(\Phi(z), \partial_\mu \Phi(z)) - \mathcal{L}^{\text{int}}(\Phi(z)) = -\frac{1}{2} \Phi(z) (\square + 1) \Phi(z),$$

where $\square \stackrel{\text{def}}{=} \partial_t^2 - \Delta$. Introducing the conjugate variable $\varpi(z) \stackrel{\text{def}}{=} \partial_t \Phi(z)$ and writing $\varpi(x) \stackrel{\text{def}}{=} \varpi(0, x)$ for the Schrödinger representation of ϖ , the Hamiltonian is given by

$$H = H^0 + H^{\text{int}} = \int_{\mathbf{R}^3} \left(\frac{1}{2} \varpi(x)^2 + \frac{1}{2} \Phi(x) (1 - \Delta) \Phi(x) - \mathcal{L}^{\text{int}}(\Phi(x)) \right) dx.$$

Moreover, in order to represent at the level of the algebra of observables the classical anti-commutation relationships between ϖ and Φ , as in quantum mechanics, we have $\varpi(x) = -i \partial_{\Phi(x)}$. Overall, we have thus expressed the unitary-group Q generated by H as

$$Q_{it} = \exp \left(-it \int_{\mathbf{R}^3} \left(-\frac{1}{2} \partial_{\Phi(x)}^2 + \frac{1}{2} \Phi(x) (1 - \Delta) \Phi(x) - \mathcal{L}^{\text{int}}(\Phi(x)) \right) dx \right). \quad (1.3)$$

What have we learnt? That formally, much of the information about a QFT is contained in the action of the unitary-group of time-translations Q_{it} . While a priori not much more can be said about this unitary-group, thinking of the fact that unitary-groups can sometimes be defined as boundary-values of analytic semigroups gives hope. In the next section, we introduce the Feynman-Kac formula, which states that Markov semigroups can be represented as an infinite-dimensional integral, the expectation over a Markov

stochastic process. Heuristically, extending this principle to the above unitary-group was Feynman's first step towards a means to approximate the scattering probabilities through perturbative computations. On the other hand, taking literally the Feynman-Kac formula, and thus going from a heuristic unitary group to a hopefully well-defined semigroup was the first step towards a rigorous interpretation of QFT, known as Euclidean Quantum Field Theory (EQFT).

1.1.1.3 The Feynman-Kac formula

In the late 40s, Feynman and Kac discovered that they were respectively trying to relate some unitary and semigroups to some functional integrals. The rigorous result that they finally established is a relation between Markov semigroups and Markov stochastic processes that we describe hereafter.

Definition 1.1 A *Markov semigroup* on a Hilbert space \mathcal{H} is a family $P = (P_t)_{t \geq 0}$ of maps $B(\mathcal{H}) \rightarrow B(\mathcal{H})$ satisfying:

- i) the semigroup property $P_{t+s} = P_t P_s$ for $s, t \geq 0$ and $P_0 = \text{Id}$;
- ii) for all $t \geq 0$ and $x \in \mathcal{H}$ there exists a probability measure on \mathcal{H} $\pi_t(x, dy)$ such that for all $F \in B(\mathcal{H})$,

$$P_t F(x) = \int_{\mathcal{H}} F(y) \pi_t(x, dy);$$

- iii) the map $x \mapsto \pi_t(x, A)$ is measurable for every Borel set A and $t \geq 0$.

A Markov semigroup can therefore act both on bounded measurable functions on \mathcal{H} and on Borel measures on \mathcal{H} . For such a measure μ , we denote the latter action as

$$P_t^* \mu(dy) \stackrel{\text{def}}{=} \int_{\mathcal{H}} \pi_t(x, dy) \mu(dx).$$

Definition 1.2 The domain of a Markov semigroup P on \mathcal{H} is the set $D(\mathcal{H})$ of those functions $F \in C(\mathcal{H})$ such that the limit

$$\lim_{t \downarrow 0} \frac{P_t F - F}{t}$$

exists. For $F \in D(\mathcal{H})$, we denote by LF this limit, the *infinitesimal generator* of the Markov semigroup.

Definition 1.3 A stochastic process $(X_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$ of \mathcal{F} verifies the *Markov property* if for every $t \geq s \geq 0$ it holds

$$\mathbf{E}[X_t | \mathcal{F}_s] = \mathbf{E}[X_t | \sigma(X_s)].$$

Such a stochastic process is called a *Markov process*.

Definition 1.4 A \mathcal{H} -valued Markov process is called a *time-homogeneous Markov process with semigroup* P if for every $A \in \mathcal{F}$ and $t \geq s \geq 0$ we have

$$\mathbf{P}(X_t \in A | \sigma(X_s)) = \pi_{t-s}(X_s, A).$$

Definition 1.5 A probability measure μ on \mathcal{H} is *invariant* for the Markov semigroup P if for all $t \geq 0$ and $F \in B(\mathcal{H})$,

$$\int_{\mathcal{H}} P_t F(x) \mu(dx) = \int_{\mathcal{H}} F(x) \mu(dx),$$

that is to say $P_t^* \mu = \mu$.

Lemma 1.6 *If a measure μ is invariant for some Markov semigroup P with infinitesimal generator L it holds*

$$L^* \mu = 0,$$

where L^* denotes the dual action of L on measures.

In (1.3), the generator of the unitary group contains a (formal, functional) Laplacian. We have the following lemma about Markov semigroup with generators of the form Laplacian perturbed by a transport term.

Lemma 1.7 *Suppose that the infinitesimal generator L_0 of a Markov semigroup P on a finite dimensional Hilbert space \mathcal{H} is of the form*

$$L_0 = \mu_i(x, t) \partial_i + \frac{1}{2} \sigma_{ki}(x, t) \sigma_{kj}(x, t) \partial_i \partial_j.$$

Then, the stochastic process X_t solving

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dW_t \quad (1.4)$$

with W_t a \mathcal{H} -valued Wiener process is a time-homogeneous Markov process with semigroup P .

Theorem 1.8 (The Feynman-Kac formula) *Let $V : \mathcal{H} \rightarrow \mathbf{R}$ be bounded from below, X_s^x be the solution to (1.4) on $[0, t]$ with initial condition $x \in \mathcal{H}$, and denote by $Q = (Q_t)_{t \geq 0}$ the Markov semigroup with generator $L_0 - V(x)$. We have*

$$Q_t F(x) = \mathbf{E}[F(X_t^x) e^{-\int_0^t V(X_s^x) ds}].$$

The Feynman-Kac formula therefore states that the action of some semigroup of a function F can be computed by evaluating F at some random walk, and taking the expectation. Feynman's idea was to formally apply this statement in order to view the r.h.s. of (1.2) as an expectation over a random process indexed by space-times points, evaluated at imaginary times. With hopefully suggestive notation, this can be summarised as

$$\langle \Omega, T(F(\Phi)) \Omega \rangle = \mathbf{E} \left[\prod_{j \in [n]} \phi(z_j) e^{\int_{\mathbf{R}^{1,3}} \mathcal{L}^{\text{int}}(\phi(z)) dz} \right],$$

where we recall that $F(\Phi)$ is of the form (1.1). Here, the expectation is taken over an Ornstein-Uhlenbeck process $\phi(t, x)$ which is a Gaussian measure with covariance given by $C = (\square + 1)^{-1}$. Thus, we can formally rewrite the expectation over this process as a Gaussian integral with covariance given by C , which finally yields

$$\langle \Omega, T(F(\Phi)) \Omega \rangle = \int F(\phi) e^{\int_{\mathbf{R}^{1,3}} \mathcal{L}^0(\phi(z), \partial_\mu \phi(z)) dz} d\phi = \int \prod_{j \in [n]} \phi(z_j) e^{\int_{\mathbf{R}^{1,3}} \mathcal{L}(\phi(z)) dz} d\phi. \quad (1.5)$$

Such an expression is known as a *path integral*.

Remark 1.9 The path integral reformulation of QFT of course remains purely formal, since there is no Lebesgue measure $d\phi$ in infinite dimension, and due to the lack of a theory of oscillatory infinite dimensional integrals to make sense of it. Nevertheless, it is extensively used in a perturbative way, expanding the non-Gaussian part in series, permuting the series with the remaining imaginary Gaussian integral, and performing the Gaussian integration. These computations, though they do not prove the existence of the underlying QFT, are sufficient to draw out of it a great predictive power.

To conclude this first presentation of the Feynman-Kac formula, and of its relation with QFT, let us mention that it will turn out later that the Feynman-Kac formula is such an important tool that it will also be at the heart of the following sections about stochastic quantisation, since two of the three approaches we will describe, the Langevin dynamic and the forward-backward equations will be obtained by a Feynman-Kac type argument.

1.1.2 The challenges of Euclidean Quantum Field Theory

1.1.2.1 The scalar Euclidean Quantum Field Theory

While properly defining (1.5) seems hopeless, taking an imaginary time turns out to be a good starting point. Indeed, the ill-defined oscillatory integral based on a formal Lebesgue measure is replaced by a *perturbed Gaussian measure* or *Euclidean QFT* (EQFT) ν formally defined as

$$\nu(d\phi) \propto e^{-\int_{\mathbf{R}^4} \mathcal{L}^E(\phi(z))dz} \mu(d\phi), \quad (1.6)$$

where μ is Gaussian with covariance given by the analytic continuation of the original covariance C , and \mathcal{L}^E is the Euclidean interaction Lagrangian, equally obtained by analytic continuation. In this change of perspective, the fundamental object becomes the measure, and one shall now verify that it is possible to reconstruct the time-ordered vacuum expectation values starting from the moments of ν . While we postpone this discussion to Section 1.1.4, let us already mention that the analytic continuation of the moments yields the vacuum expectation values, or *Wightman functions*, and that a theorem by Wightman ensures that the knowledge of these functions is sufficient to reconstruct the time-ordered vacuum expectation values.

If having got rid of the infinite dimensional Lebesgue measure sounds promising, making sense of ν as defined in (1.6) still poses several problems. Some of them can be read by investigating some properties of the Gaussian measure μ with covariance $(1 - \Delta)^{-1}$, known as the (massive) Gaussian Free Field (GFF), that are presented hereafter.

Lemma 1.10 *Let ϕ denote a random field with law μ given by the GFF on the torus. One has in any dimension d*

$$\begin{aligned} \mu(\mathcal{C}^s(\mathbf{T}^d)) &= 1, \text{ for all } s < 1 - \frac{d}{2}, \\ \mu(\mathcal{C}^s(\mathbf{T}^d)) &= 0, \text{ for all } s \geq 1 - \frac{d}{2}. \end{aligned} \quad (1.7)$$

Moreover, for all $\epsilon, \kappa > 0$ it holds

$$\mathbf{E}[\|\phi\|_{\mathcal{C}^{1-d/2-\epsilon}(\mathbf{T}^d)}^p] \lesssim p^{p/2} \quad (1.8)$$

and

$$\mathbf{E}[\|\langle x \rangle^{-\kappa} \varphi\|_{\mathcal{C}^{1-d/2-\epsilon}(\mathbf{R}^d)}^p] \lesssim p^{p/2}, \quad (1.9)$$

where φ denotes a GFF on the full space.

Remark 1.11 The first two estimates show that μ is not supported on smooth functions, and that in $d \geq 2$, it is not even supported on bounded functions. In particular, (1.7) implies that in $d \geq 2$, Φ is almost surely merely a distribution, and there is no canonical way to define non-linear operations on it. In particular, in $d \geq 2$, one has

$$\int_{\mathbf{T}^d} \mathcal{L}^E(\phi(z))dz = \infty \text{ a.s.},$$

as soon as the interaction is non-linear. This issue is called the *UV problem*. While it is harmless in $d = 1$, i.e. in the context of quantum mechanics, it deteriorates with the dimension. This is the reason why, in the sequel, we no longer keep the dimension $d = 4$, but rather let it vary.

Estimate (1.8) is almost optimal, in the sense that Φ is not a bounded random variable on its underlying probability space. In particular, when integrating a quantity against the Gaussian measure μ , one has to take some care about the convergence of the integral when the value of Φ becomes large. This *large field problem* makes it, even in $d = 1$, impossible to construct a EQFT whose interaction is not integrable

against a Gaussian. If bounded below interactions should a priori not pose any problem, we will see later that the UV and large field problem conspire one against the other, and that treating the UV problem forces one to work with some interactions that are not explicitly bounded below. The large field problem is the reason why, in the sequel, we mostly focus on the quartic case $\mathcal{L}^E(\phi(z)) = \phi(z)^4$.

As for the estimate (I.9), it translates the fact that the GFF, even with a massive covariance, has no decay at infinity. This implies that even if we could make sense of $\int_{\mathbf{T}^d} \mathcal{L}^E(\phi(z)) dz$, this integral would probably become divergent if the volume of the torus were to be taken larger and larger. This fact is known as the *IR problem*.

Remark 1.12 In order to obtain some approximations of the entries of the S -matrix, the QFTs and thus the EQFT measures defined by (I.6) in any dimension d are primarily studied via a perturbative expansion, writing

$$\mathbf{E}_\nu[F(\phi)] = \mathbf{E}_\mu[e^{-g \int_{\mathbf{R}^d} \mathcal{L}^E(\phi(z)) dz} F(\phi)] = \mathbf{E}_\mu\left[\sum_{n \geq 0} \frac{(-g)^n}{n!} \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} \prod_{i \in [n]} \mathcal{L}^E(\phi(z_i)) dz_i F(\phi)\right],$$

and pretending that it is possible to commute the expectation and the integrals. When doing this, one ends up commuting Gaussian expectations of polynomials using Wick's rule. These are the computations that we call *perturbative amplitudes* or *Feynman graphs*.

In this context, the UV and IR problems translate in the facts that the perturbative amplitudes are divergent respectively at short and long distances, and the large field problem is related to the divergence of the perturbative series itself.

1.1.2.2 Beyond scalar theories

In the previous discussion, we pretended for the sake of presentation that all the quantum fields one has to deal with are scalar fields. However, scalar fields turn out to play a rather minor part in the standard model of particle physics, which mostly relies on the quantisation of the Yang-Mills-Dirac theory.

Recall that in QFT, a quantum field is a family of operators, which *do not commute*. More precisely, two fields evaluated at space-like separated points do commute, but this is not the case if they are evaluated at two points that are causally dependent. This non-commutativity therefore translates causality, and vanishes at the Euclidean level – where one can thus rely on *bona fide* commutative probabilities. This is not the case of the non-commutativity of the *Dirac quantum field*, which is intrinsically non-commutative and even in the Euclidean setting requires one to forego probability theory, and to work on non-commutative C^* -algebras, trying to extend some notions of probability to this setting.

Regarding the Yang-Mills quantum field, another difficulty appears. Recall that Euclidean Yang-Mills theory is a model for a random connection on a (say trivial) principal bundle \mathcal{P} over \mathbf{R}^4 with structure group any semi-simple finite dimensional Lie group G with Lie algebra \mathfrak{g} . Even in the case of a trivial principal bundle, one cannot simply think of it as $\mathbf{R}^4 \times G$, since one still has the freedom to rotate G above each point of \mathbf{R}^4 . In other words, the trivial bundle \mathcal{P} is really equal to $\mathbf{R}^4 \times G$ modulo a choice of global section. This freedom to choose a global section is called gauge freedom, and the Yang-Mills action

$$S_{\text{YM}}(A) \stackrel{\text{def}}{=} \int_{\mathbf{R}^4} \langle *d_A A \wedge d_A A \rangle_{\mathfrak{g}} dx$$

is invariant under the action of the *gauge group* $\mathcal{G} \stackrel{\text{def}}{=} C^\infty(\mathcal{P}, \mathfrak{g})$ corresponding to changes of section, that acts on connections as

$$A^g = g^{-1} A g - g^{-1} dg.$$

The Yang-Mills EQFT can therefore a priori not yield a well-defined probability measure, since being invariant under the action of an infinite dimensional group, it can not be normalised. A possible solution would be to work on the quotient \mathcal{A}/\mathcal{G} , where $\mathcal{A} = \Omega^1(\mathcal{P}, \mathfrak{g})$ is the affine space of all smooth connections, but this space has no linear structure, and in general it is unclear whether it can be turned into a linear space by gauge fixing.

1.1.2.3 Renormalisation

In Section [1.1.2.1](#) we gave a first glimpse at the UV problem. In this section, we aim to illustrate the philosophy behind a set of very different ideas, tools and techniques all designed to tackle this difficulty.

In the very same years in which Feynman obtained a way to compute the vacuum expectation values with a path integral, and realised that many of the perturbative amplitude generated by the evaluation of the path integral were divergent, Ito made a significant progress in stochastic analysis, known as *Ito integration*.

Lemma 1.13 (Ito integration) *Fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a Brownian motion $(B_t)_{t \geq 0}$. Then, for any interval $[a, b] \subset \mathbf{R}_{\geq 0}$, we have the convergence in $L^p(\Omega)$ for every $p \geq 1$ (and hence in probability)*

$$\sum_{i=0}^{n-1} (B_{t_i} - B_a)(B_{t_{i+1}} - B_{t_i}) \xrightarrow[n \uparrow \infty]{} \frac{(B_b - B_a)^2}{2} - \frac{b - a}{2},$$

where $a = t_0 < t_1 < \dots < t_n = b$.

Though the connection of Ito integration with the UV problem in EQFT may not be obvious at first, a moment of reflection shows that Ito was actually struggling with the very same problem as Feynman: defining a classically forbidden non-linear operation on a rough Gaussian random function/distribution. By classically ill-defined, we more precisely mean that the operation falls outside the scope of Young's product and Young's integration.

Lemma 1.14 *The product of two smooth functions $f, g : \mathbf{R}^d \rightarrow \mathbf{R}$ extends to a bilinear operator $\mathcal{C}^\alpha(\mathbf{R}^d) \times \mathcal{C}^\beta(\mathbf{R}^d) \rightarrow \mathcal{C}^{\alpha \wedge \beta}(\mathbf{R}^d)$ if and only if $\alpha + \beta > 0$. Moreover, the Riemann integral $\int_a^b f(s)g'(s)ds$ extends to a bilinear operator $\mathcal{C}^\alpha(\mathbf{R}^d) \times \mathcal{C}^\beta(\mathbf{R}^d) \rightarrow \mathbf{R}$ if and only if $\alpha + \beta + 1 > 0$.*

In our case, the Brownian motion being $(\frac{1}{2} - \epsilon)$ -Hölder continuous, the condition $\alpha + \beta + 1 > 0$ to define $\int BdB$ marginally fails. In spite of this failure, Ito was able to make sense in a *probabilistic way* of an *explicit polynomial* non-linearity in a Gaussian process. Ito integration thus paves the way towards constructing more complicated polynomial operations in the GFF.

Moreover, an important remark about the way Ito made sense of $\int BdB$ is in order. As explained above, he showed that a *regularised* version of $\int BdB$ (typically with a finite number of degrees of freedom, such as a Riemann sum) converges in $L^p(\Omega)$ to a limit. The next natural question to ask is whether this limit is universal, i.e. independent of the regularisation. The answer is ambivalent: different regularisations can yield different limits, but all of the form

$$\frac{(B_b - B_a)^2}{2} - c(b - a), \text{ for } c \in \mathbf{R},$$

which means that $\int BdB$ is really defined modulo an ambiguity, or equivalently that one still has some freedom in defining it. Two important cases are of course $c = 0$, in which case the integration which is defined verifies the chain rule (it is called a Stratonovich integral, or geometric rough path lift of the Brownian motion), and the former case $c = 1/2$, in which case one obtains a martingale.

Something even more interesting happens when one tries to replace the first instance of B by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ constructed by convolution of \dot{B} with a $H + \frac{1}{2}$ regularising kernel K , that we denote $C \stackrel{\text{def}}{=} K * \dot{B}$. Indeed, denoting by B_ϵ, C_ϵ some mollified versions of B, C ,

$$\int_a^b C_\epsilon(t)dB_\epsilon(t)$$

fails to converge in $L^p(\Omega)$, but there exists a *divergent sequence* c_ϵ such that

$$\int_a^b B_\epsilon^H(t) dB_\epsilon(t) + c_\epsilon(b - a)$$

does converge as $\epsilon \downarrow 0$ – see [FH14, Chapter 14]. In this case, one can therefore leverage the freedom we have in the definition of the polynomial in \dot{B} in order to make sense of it. This is done modulo a regularisation and a renormalisation procedure, that is to say modulo the insertion of a divergent quantity in the space of freedom which is available to define the polynomial. Note that again, the limiting object is still defined up to the choice of a number, the finite part of the counterterm c_ϵ . Renormalisation of singular SPDEs and QFTs is nothing more than a far-reaching generalisation of this idea.

Going back to the definition of $\int_{\mathbf{T}^d} \phi(z)^4 dz$ when the law of ϕ is the GFF, we can convert the knowledge we drew from Ito integration into the following result.

Lemma 1.15 *Let ϕ be a random field with law the GFF, and define $\phi_\epsilon \stackrel{\text{def}}{=} \varrho_\epsilon * \phi$ where ϱ_ϵ is a sequence of smooth mollifiers. In $d = 2$, then*

$$:\phi_\epsilon^4 \stackrel{\text{def}}{=} \phi_\epsilon^4 - 6\mathbf{E}[\phi_\epsilon^2]\phi_\epsilon^2 + 3\mathbf{E}[\phi_\epsilon^2]^2$$

converges in $L^p(\Omega, \mathcal{C}^{-\kappa}(\mathbf{T}^d))$ for every $\kappa > 0$ to some limit $:\phi^4$: called the fourth Wick power of the GFF. In particular, it can be tested against the constant function, which provides us with a definition of the interaction.

Proof. The first step of the proof is a reduction to a covariance computation. Fix a small $\kappa > 0$ and $p \geq 2/\kappa$ even, and let $\tau_\epsilon = :\phi_\epsilon^4$. By Sobolev embedding,

$$\mathbf{E}[\|\tau_\epsilon\|_{\mathcal{C}^{-2\kappa}}^p] \lesssim \mathbf{E}[\|\tau_\epsilon\|_{B_{p,p}^{-\kappa}}^p] = \sum_{h \geq -1} 2^{-\kappa h p} \int_{\mathbf{T}^2} \mathbf{E}[|\Delta^h \tau_\epsilon(x)|^p] dx = \sum_{h \geq -1} 2^{-\kappa h p} \int_{\mathbf{T}^2} \mathbf{E}[|\sum_{m \sim 2^h} e^{-ix \cdot m} \hat{\tau}_\epsilon(m)|^p] dx,$$

where $(\Delta^h)_{h \geq -1}$ denotes Littlewood-Paley blocks and $\hat{\tau}_\epsilon$ is the Fourier transform of τ_ϵ . The next step is to leverage Gaussian hypercontractivity, which states that the moments of a polynomial of a Gaussian can be controlled by the covariance of the polynomial (extending a result which is trivial for Gaussians). Therefore, one has

$$\mathbf{E}[|\sum_{m \sim 2^h} e^{-ix \cdot m} \hat{\tau}_\epsilon(m)|^p] \lesssim (\mathbf{E}[|\sum_{m \sim 2^h} e^{-ix \cdot m} \hat{\tau}_\epsilon(m)|^2])^{p/2} = (\sum_{m_1, m_2 \sim 2^h} e^{-ix \cdot (m_1 + m_2)} \mathbf{E}[\hat{\tau}_\epsilon(m_1) \hat{\tau}_\epsilon(m_2)])^{p/2}.$$

By stationarity in law of the GFF with respect to space translations, and hence of τ_ϵ , one has $\mathbf{E}[\hat{\tau}_\epsilon(m_1) \hat{\tau}_\epsilon(m_2)] = 0$ unless $m_1 + m_2 = 0$. Thus, by Jensen's inequality and Parseval's theorem, it holds

$$\mathbf{E}[\|\tau_\epsilon\|_{\mathcal{C}^{-\kappa}}^p] \lesssim (\sum_{h \geq -1} 2^{-2\kappa h} \sum_{m \sim 2^h} \mathbf{E}[\hat{\tau}_\epsilon(m) \hat{\tau}_\epsilon(-m)])^{p/2} = (\mathbf{E}[\|\tau_\epsilon\|_{H^{-\kappa}}])^{p/2}.$$

It remains to evaluate the covariance of τ_ϵ . This is done by noting that τ_ϵ is precisely the projection of ϕ_ϵ^4 onto the fourth Gaussian chaos, so that one has

$$\mathbf{E}[\tau_\epsilon(x) \tau_\epsilon(y)] = (C_\epsilon(x, y))^4,$$

where C_ϵ denote the covariance of the regularised GFF. One can then conclude that the $H^{-\kappa}$ norm of τ_ϵ is finite uniformly in ϵ , since C_ϵ is in L^4 uniformly in ϵ , the divergence being merely logarithmic. \square

Remark 1.16 In $d = 2$, similar results hold for every powers of the free field. However, in $d = 3$, only the second Wick power remains well defined, while no Wick power survives until dimension four.

We have learnt from the two dimensional case that a good starting point to make sense of the interacting measure ν in the quadratic case is to consider a sequence of approximations

$$\nu_\epsilon(d\phi) \propto e^{-\int_{\mathbf{R}^4} (g\phi_\epsilon(z)^4 + a_\epsilon(g)\phi_\epsilon(z)^2 + b_\epsilon(g))dz} \mu(d\phi),$$

where $a_\epsilon(g)$ and $b_\epsilon(g)$ are divergent polynomial in g , encoding a finite number of necessary subtraction to be performed at the level of the Lagrangian in order to make all the perturbative amplitudes finite. In the sequel, we therefore denote by \mathcal{L}_ϵ^E the *bare interaction Lagrangian*, and by $V_\epsilon \stackrel{\text{def}}{=} \int_{\mathbf{R}^4} \mathcal{L}_\epsilon^E$ the *bare potential*.

It turns out that indeed in $d = 2, 3$, a finite number of renormalisations are sufficient to obtain a regularised measure whose perturbative contributions are finite. In this regime, one says that the theory is *super-renormalisable*.

This is no longer the case above $d = 4$. However, something very special happens in four dimensions. While the number of necessary subtraction is infinite, these subtractions only affect a finite number of terms in the Lagrangian, more precisely the terms that were originally present, that is to say that renormalisation is limited to vacuum, mass, coupling constant and wave function. In $d = 4$, the theory is therefore said to be *just-renormalisable*. Starting from $d = 5$, not only the number of elementary renormalisations is infinite, but also they would affect more and more terms in the Lagrangian that were originally not present.

To conclude this section, let us point out that the statements of (super)renormalizability in low dimension and (just)renormalizability in $d = 4$ are known as some BPHZ theorems. Their original proofs rely on a careful analysis of the way the perturbative expansion reorganises when renormalisation is performed.

1.1.3 Perturbative and constructive renormalisation group

In this section, we aim to give a bit of the flavour of some of the oldest ideas to make sense of EQFTs. Recall that in the previous section, we showed that in low dimension a finite number of subtractions is sufficient to solve the UV problem, and that in particular in $d = 2$ it is directly solved by Wick ordering. If this procedure fixes the UV problem, the Wick polynomials are no longer bounded below uniformly in the cut-off, so that dealing with the large-field problem has now become a lot more difficult. Fortunately, it was noticed by Nelson [Nel66] that the probability of the event that the GFF takes a large value is sufficiently small to be able to integrate the exponential of the Wick renormalised interaction against the GFF.

Here, we choose to present a different argument based on the discrete renormalisation group, and known a *vertical expansion*. We describe it because there are already very good references about Nelson's argument, and because it will allow us to draw the reader's attention to several interesting facts about EQFTs. Finally, it is also a nice occasion to introduce the renormalisation group (RG) ideas, which will end up being very useful in the next section about stochastic quantisation.

1.1.3.1 The renormalisation group

The renormalisation group techniques all rely on the idea that a splitting of the covariance of some Gaussian μ always induces a splitting of μ into two independent Gaussians. Separating the high and low modes of μ , and integrating over the high modes, one can therefore hope to obtain a quantity which is defined uniformly in the cut-off. Explicitly, denoting for a parametrisation $L \in [0, 1]$ of Fourier scales by $C = C_{\leq L} + C_{>L}$ a split of the covariance into high and low modes, one has the decomposition of Φ as $\Phi = \Phi_{\leq L} + \Phi_{>L}$, and we denote by $\mathbf{E}_{\leq L}$, $\mathbf{E}_{>L}$ the expectation operators associated with $\Phi_{\leq L}$ and $\Phi_{>L}$. For good functionals $F(\Phi)$, there exists a scale L such that $F(\Phi)$ only depends on $\Phi_{\leq L}$. In this case, one has

$$\mathbf{E}_\mu[F(\Phi)e^{-V_{1,\epsilon}(\Phi)}] = \mathbf{E}_{\leq L}[F(\Phi_{\leq L})\mathbf{E}_{>L}[e^{-V_{1,\epsilon}(\Phi_{\leq L} + \Phi_{>L})}]], \quad (1.10)$$

where $V_{1,\epsilon}$ the *bare potential* is typically of the form $V_{1,\epsilon} = V_\epsilon = \int (g\Phi^4 + a_\epsilon(g)\Phi^2 + b_\epsilon(g))$. Letting

$$V_{L,\epsilon}(\Phi_{\leq L}) \stackrel{\text{def}}{=} -\log \mathbf{E}_{>L}[e^{-V_{1,\epsilon}(\Phi_{\leq L} + \Phi_{>L})}], \quad (1.11)$$

be the *effective potential*, (1.10) implies that the knowledge of the effective potential at scales arbitrarily close to one is sufficient to know the whole measure. On the other hand, one can hope that contrary to $V_{1,\epsilon}$, for $L < 1$ $V_{L,\epsilon}$ could be defined uniformly in ϵ . Indeed, representing Gaussian expectations as differential operators implies that $e^{-V_{L,\epsilon}(\Phi)}$ formally solves a functional heat equation, so that $V_{L,\epsilon}$ can be thought of as a heat regularised version of $V_{1,\epsilon}$ in an infinite dimensional setting, and should therefore be smooth as long as $L < 1$. From this functional heat equation, we finally deduce that the effective potential solves the Polchinski equation

$$\partial_L V_{L,\epsilon} + \frac{1}{2} \langle \partial_\Phi, \partial_\Phi \rangle_{\dot{C}_L} V_{L,\epsilon} = \frac{1}{2} \langle \partial_\Phi V_{L,\epsilon}, \partial_\Phi V_{L,\epsilon} \rangle_{\dot{C}_L}, \quad (1.12)$$

where $\dot{C}_L \stackrel{\text{def}}{=} \partial_L C_{\leq L}$. The Polchinski equation was first derived by Polchinski in [Pol84]. To conclude this short presentation of the renormalisation group, let us mention that choosing a well-suited basis for the renormalisation group, the Polchinski equation can be used to prove perturbative renormalisability, i.e. a BPHZ theorem – see for instance [KKS92]. The proof has the nice feature that it is inductive, and insensitive to combinatorial issues. It relies on the fact that most contributions to $V_{\epsilon,L}$ will decay at high energies, and that in the super-renormalisable regime, only a finite number of contributions will blow-up, imposing a blowing-up final condition $V_{1,\epsilon}$.

1.1.3.2 The vertical expansion for Φ_2^4

With the notion of renormalisation group in mind, we are ready to introduce a non-perturbative approach to the Φ_2^4 EQFT based on a discrete version of the renormalisation group. The ideas that we present here are sketched in chapter III.1.D of the book [Riv91], and were carefully explained to me by Fabien Vignes-Tourneret in a private communication. Note that this approach is of course more cumbersome than Nelson's argument, but we chose to include it here in order to give a brief account of old constructive techniques, and to introduce a notion of *weak UV problem* which we discuss at the end of this section. For a modern discussion about Nelson's argument, we refer to [Lin24], where it is applied to prove Segal's axioms for $P(\Phi)_2$.

Definition 1.17 One is interested in showing convergence of the *regularised partition function* of the Φ_2^4 EQFT on the unit torus \mathbf{T}^2 . With the Wick renormalisation described in the previous section already performed, it reads for $I \geq 0$ and $g \in \{z \in \mathbf{C} : \Re z > 0\}$ as

$$Z_I \stackrel{\text{def}}{=} \mathbf{E}_{\mu_I}[e^{-g \int_{\mathbf{T}^2} \phi_I^4}],$$

where the law μ_I of ϕ_I is the heat regularised GFF with covariance

$$C_I \stackrel{\text{def}}{=} \frac{e^{-M^{-2I}(1-\Delta)}}{(1-\Delta)} = \int_{M^{-2I}}^{\infty} e^{-t(1-\Delta)} dt,$$

and in any dimension d

$$: \phi_I^4 :_I \stackrel{\text{def}}{=} \phi_I^4 - 6C_I(0,0)\phi_I^2 + 3C_I(0,0)^2 = (\phi_I^2 - 3C_I(0,0))^2 - 6C_I^2(0,0) \geq -6C_I^2(0,0) = -K_I,$$

where

$$K_I \stackrel{\text{def}}{=} \begin{cases} KI^2 & \text{in } d = 2, \\ KM^{i2(d-2)} & \text{in } d \geq 3, \end{cases}$$

for some universal constant K . In the sequel, we always keep $d = 2$, so that $K_I = KI^2$, but it will be convenient for a future discussion to keep track of the general expression of K_I .

This estimate implies the naive bound

$$\left| e^{-g \int_{\mathbf{T}^2} \phi_I^4} \right| \lesssim e^{K_I \mathfrak{R}g} \quad (1.13)$$

which holds almost surely, and uniformly in $I \geq 0$.

Definition 1.18 We implement the discrete renormalisation group by splitting the field ϕ_I as $\phi_I = \sum_{i=0}^I \phi^i$ where ϕ^i are independent Gaussians with covariances given by

$$C^i \stackrel{\text{def}}{=} \frac{e^{-M^{-2(i-1)}(1-\Delta)} - e^{-M^{-2i}(1-\Delta)}}{(1-\Delta)} = \int_{M^{-2i}}^{M^{-2(i-1)}} e^{-t(1-\Delta)} dt, \text{ and } C^0 \stackrel{\text{def}}{=} \frac{e^{-(1-\Delta)}}{(1-\Delta)} = \int_1^\infty e^{-t(1-\Delta)} dt.$$

Finally, we also introduce for $i \in \{0, \dots, I\}$ the notations

$$\phi_i \stackrel{\text{def}}{=} \sum_{k=0}^I \phi^k, \quad C_i \stackrel{\text{def}}{=} \sum_{k=0}^I C^k, \quad : \phi_i^4 : \stackrel{\text{def}}{=} \phi_i^4 - 6C_i(0,0)\phi_i^2 + 3C_i(0,0)^2 \gtrsim -i^2,$$

and let

$$: (\phi^i)^4 : \stackrel{\text{def}}{=} \phi_i^4 :_i - : \phi_{i-1}^4 :_{i-1}$$

stand for the vertex with at least one field at scale i . With this notation, observe that one has

$$: \phi_I^4 :_I = \sum_{i=0}^I : (\phi^i)^4 :.$$

Definition 1.19 We set, for any $n \in \mathbf{N}$, $\llbracket n \rrbracket \stackrel{\text{def}}{=} \{0, \dots, n\}$, along with $\llbracket -2 \rrbracket = \llbracket -1 \rrbracket = \emptyset$. For any subset $S \subset \llbracket I \rrbracket$ we write $t_S \stackrel{\text{def}}{=} (t_i)_{i \in S}$ and $1_S \stackrel{\text{def}}{=} (t_i = 1)_{i \in S}$, and it is understood that if $S = \llbracket -2 \rrbracket, \llbracket -1 \rrbracket$, then there is no variable in t_S . With this notation in hand, we define a (random) function $f : [0, 1]^{\llbracket I \rrbracket} \rightarrow \mathbf{C}$ by

$$f(t_{\llbracket I \rrbracket}) \stackrel{\text{def}}{=} \prod_{i=0}^I e^{-t_i g \int_{\mathbf{T}^2} (\phi^i)^4},$$

which verifies

$$Z_I = \mathbf{E}_{\mu_I} [f(1_{\llbracket I \rrbracket})].$$

Note that the parameters $t_{\llbracket I \rrbracket}$ are meant to allow us to interpolate between the scales.

Lemma 1.20 Fix for every $i \in \llbracket I \rrbracket$ a large integer $N_i \geq 1$ whose value will be specified later on, and set $N_{-1} = 0$. Then, setting for $i \in \llbracket I \rrbracket$

$$Z_I^{(i)} \stackrel{\text{def}}{=} \sum_{k_{i+1} \in \llbracket N_{i+1}-1 \rrbracket} \dots \sum_{k_I \in \llbracket N_I-1 \rrbracket} \int_0^1 \frac{(-g)^{N_i}}{(N_i-1)!} \prod_{j=i+1}^I \frac{(-g)^{k_j}}{k_j!} (1-t_i)^{N_i-1} \\ \mathbf{E}_{\mu_I} \left[e^{-g \int_{\mathbf{T}^2} t_i : \phi_i^4 :_i + (1-t_i) : \phi_{i-1}^4 :_{i-1}} \left(\int_{\mathbf{T}^2} : (\phi^i)^4 : \right)^{N_i} \prod_{j=i+1}^I \left(\int_{\mathbf{T}^2} : (\phi^j)^4 : \right)^{k_j} \right] dt_i,$$

and

$$Z_I^{(-1)} \stackrel{\text{def}}{=} \sum_{k_0 \in \llbracket N_0-1 \rrbracket} \dots \sum_{k_I \in \llbracket N_I-1 \rrbracket} \prod_{j=0}^I \frac{(-g)^{k_j}}{k_j!} \mathbf{E}_{\mu_I} \left[\prod_{j=i+1}^I \left(\int_{\mathbf{T}^2} : (\phi^j)^4 : \right)^{k_j} \right],$$

it holds

$$Z_I = \sum_{i=-1}^I Z_I^{(i)}.$$

Proof. The proof follows from recursive applications of some Taylor formulas with integral remainder to f .

Denote for any $j \in \llbracket I \rrbracket$ by $T_{t_j}^{N_j-1}$ the Taylor expansion up to order $N_j - 1$ in the variable t_j defined as

$$T_{t_j}^{N_j-1} f(t_{\llbracket I \rrbracket \setminus \{j\}}) \stackrel{\text{def}}{=} \sum_{k=0}^{N_j-1} \frac{1}{k!} \partial_{t_j}^k f(t_1, \dots, 0, \dots, t_I),$$

and by $R_{t_j}^{N_j}$ the Taylor remainder at order N_j in the variable t_j defined as

$$R_{t_j}^{N_j} f(t_{\llbracket I \rrbracket \setminus \{j\}}) \stackrel{\text{def}}{=} \int_0^1 \frac{(1-t_j)^{N_j-1}}{(N_j-1)!} \partial_{t_j}^{N_j} f(t_{\llbracket I \rrbracket}) dt_j.$$

We first interpolate between scales I and $I-1$, which yields

$$f(1_{\llbracket I \rrbracket}) = T_{t_I}^{N_I-1} f(1_{\llbracket I-1 \rrbracket}) + R_{t_I}^{N_I} f(1_{\llbracket I-1 \rrbracket}).$$

The second term of the r.h.s. is now well-prepared, and is equal to $Z_I^{(I)}$. We apply the same Taylor expansion to the variable t_{I-1} in the first term of the r.h.s. Applying this procedure inductively until scale 0 yields

$$Z_I = \mathbf{E}_{\mu_I} [f(1_{\llbracket i \rrbracket})] = \sum_{i=-1}^I \mathbf{E}_{\mu_I} \left[\left(R_{t_i}^{N_i} \prod_{j=i+1}^I T_{t_j}^{N_j-1} \right) f(1_{\llbracket i-1 \rrbracket}) \right],$$

which is the desired result. \square

We now aim to disentangle the perturbative amplitudes that arise falling from the exponentials from the remaining exponentials. To do so, the following piece of notation is required.

Definition 1.21 We let \mathcal{G} be the set of those graphs (G, V, E) such that

- G has *no self-loops*, that is to say that for every edge $\ell \in E$, $\ell = (v_1, v_2)$ with $v_1, v_2 \in V$ and $v_1 \neq v_2$;
- G is *quartic*, that is to say that for each vertex $v \in V$ there are four edges $\ell \in E$ incident to v (of the form $\ell = (v, v') = (v', v)$ for some $v' \in V$); we denote by E_v the set of those edges, and we have $|E_v| = 4$.

In the sequel, we sometimes just write G instead of (G, V, E) .

A graph with *scale assignment* (G, V, E, σ) is a graph $(G, V, E) \in \mathcal{G}$ together with a map $\sigma : E \rightarrow \llbracket I \rrbracket$. Given a graph with scale assignment (G, V, E, σ) , we let, for $v \in V$,

$$e_v(\sigma) \stackrel{\text{def}}{=} \max_{\ell \in E_v} \sigma(\ell),$$

and we define, for $(G, V, E) \in \mathcal{G}$, $\mathcal{S}(G) \stackrel{\text{def}}{=} \llbracket I \rrbracket^E$ the space of all scale assignments to the graph (G, V, E) .

Fix (G, V, E, σ) a graph with scale assignment. We define the *amplitude* $A_\sigma(G)$ of the graph by

$$A_\sigma(G) \stackrel{\text{def}}{=} \prod_{v \in G} \int_{\mathbf{T}^2} dx_v \prod_{\ell=(v_1, v_2) \in E} C^{\sigma(\ell)}(x_{v_1}, x_{v_2}).$$

Fix any $i \in \{-1, \dots, I\}$. Given $k_{i+1} \in \llbracket N_{i+1} - 1 \rrbracket, \dots, k_I \in \llbracket N_I - 1 \rrbracket$, we always write $P_i \stackrel{\text{def}}{=} N_i + \sum_{j=i+1}^I k_j$, and let

$$\mathcal{G} \supset \mathcal{G}_i^{k_{i+1}, \dots, k_I} \stackrel{\text{def}}{=} \{(G, V, E) \in \mathcal{G} : |V| = 2P_i\}.$$

Moreover, for $(G, V, E) \in \mathcal{G}_i^{k_{i+1}, \dots, k_I}$ we say that $\sigma \in \mathcal{S}(G)$ is *compatible* if it holds

$$\#\{v \in V : e_v(\sigma) = j\} = 2k_j \text{ for every } j \in \{i+1, \dots, I\}, \text{ and } \#\{v \in V : e_v(\sigma) = i\} = 2N_i. \quad (1.14)$$

Finally, we denote by $\mathcal{S}^c(G)$ the set of all scale assignments to G that are compatible.

With this notation in hand, we have the following lemma.

Lemma 1.22 *There exists a universal constants $C_1 > 0$ such that it holds*

$$|Z_I| \lesssim \sum_{i=-1}^I \frac{e^{K_i \mathfrak{R}g}}{N_i!} \sum_{k_{i+1} \in \llbracket N_{i+1}-1 \rrbracket} \cdots \sum_{k_I \in \llbracket N_I-1 \rrbracket} \left((C_1 |g|)^{2P_i} \frac{(P_i!)^4}{\prod_{j=i+1}^I (k_j!)^2} \sup_{G \in \mathcal{G}_i^{k_{i+1}, \dots, k_I}} \sum_{\sigma \in \mathcal{S}^c(G)} A_\sigma(G) \right)^{1/2}.$$

Proof. The proof follows in two steps. First, using (1.13), one has

$$\left| e^{-g \int_{\mathbf{T}^2} t_i : \phi_i^4 : i + (1-t_i) : \phi_{i-1}^4 : i-1} \right| \lesssim e^{K_i \mathfrak{R}g}. \quad (1.15)$$

Then, one controls the $L^1(\mathcal{D}'(\mathbf{T}^2), \mu_I)$ norm of the other terms by the $L^2(\mathcal{D}'(\mathbf{T}^2), \mu_I)$ norm. The $L^2(\mathcal{D}'(\mathbf{T}^2), \mu_I)$ norm is computed using Wick's theorem, which produces a sum indexed by $G \in \mathcal{G}_i^{k_{i+1}, \dots, k_I}$ and $\sigma \in \mathcal{S}^c(G)$. Finally, for the enumeration of the graphs in $\mathcal{G}_i^{k_{i+1}, \dots, k_I}$, we can bound the number of quartic graphs with $2P_i$ vertices by $(4 \times 2P_i - 1)!! \lesssim C_1^{2P_i} (P_i!)^4$. \square

We now turn to controlling the perturbative amplitudes, which is done through multiscale analysis.

Lemma 1.23 (multiscale analysis) *Fix $G \in \mathcal{G}$ and $\sigma \in \mathcal{S}(G)$, and let $n = |V|$. There exist a universal constant $C > 0$ such that it holds*

$$|A_\sigma(G)| \lesssim C^n \prod_{v \in V} M^{-2e_v(\sigma)}.$$

Remark 1.24 A similar bound holds for convergent and renormalised graphs of any super-renormalisable Φ^4 theory, with $M^{-2e_v(\sigma)}$ replaced by $M^{-(4-d)e_v(\sigma)}$. In the just-renormalisable case, the decay at a vertex no longer occurs between the highest scale and zero, but between the highest and lowest scales of the propagators attached at the vertex, since the power counting is purely logarithmic. However, to keep the proof short, we refrain from showing the result in dimension $d > 2$, but rather refer to [Riv91, pp. 66-70].

Proof. The proof essentially follows from the following estimate on the propagator in the slice $i \in [I]$, which for the reader's convenience, we state in any dimension d : there exists a universal constant $\delta > 0$ such that

$$|C^i(x, y)| \lesssim M^{(d-2)(i+1)} e^{-\delta M^{i+1} |x-y|}. \quad (1.16)$$

The main idea of the proof is then to benefit the exponential decay in space (which always takes place since we consider graphs with no self-loops) only along the edges of a (rooted) spanning tree of G that also spans all the *high subgraphs* of G , that is to say the subgraphs G_i^k of G that have all internal edges with scale assignment bigger than i and all external edges with have scale assignment smaller than i . More precisely, we first define, for $i \geq 0$, G_i as the subgraph of G with edge set $E_i \stackrel{\text{def}}{=} \sigma^{-1}\{j \in \mathbf{N} : j \geq i\}$, and denote by G_i^k for $k \geq 1$ its connected components. Given such a spanning tree T of G that also spans every G_i^k , denoting its root v_0 , we have

$$\begin{aligned} \prod_{v \neq v_0} \int_{\mathbf{T}^2} dx_v \prod_{\ell=(v_1, v_2) \in E} e^{-\delta M^{\sigma(\ell)+1} |x_{v_1} - x_{v_2}|} &\lesssim \prod_{v \neq v_0} \int_{\mathbf{T}^2} dx_v \prod_{\ell=(v_1, v_2) \in E(T)} e^{-\delta M^{\sigma(\ell)+1} |x_{v_1} - x_{v_2}|} \\ &\lesssim C^n \prod_{v \neq v_0} M^{-2(i_v(T)+1)} = C^n \prod_{v \neq v_0} \prod_{i=0}^{i_v(T)} M^{-2}, \end{aligned}$$

where $E(T)$ is the set of those edges of G that also are edges of T , and where edges of T are ordered starting from the root v_0 , and $i_v(T) \stackrel{\text{def}}{=} \sigma(\ell)$ with ℓ the edge in T which is before v with respect to this

(partial) order. A moment of reflection shows that every vertex $v \in V(G_i^k)$ (the vertex set of G_i^k) but one verify $i_v(T) \geq i$. We denote by v_i^k this one vertex such that $i_{v_i^k}(T) < i$. For example, at scale 0, we have G_0 is connected so that $G_0 = G_0^1 = G$ and $v_0^1 = v_0$. This observation implies that

$$\prod_{v \neq v_0} \prod_{i=0}^{i_v(T)} M^{-2} = \prod_{i \geq 0} \prod_{v: i_v(T) \geq i} M^{-2} = \prod_{i \geq 0} \prod_{k \geq 1} M^{-2(|V(G_i^k)|-1)} = \prod_{i \geq 0} \prod_{k \geq 1} \prod_{v \in V(G_i^k) \setminus \{v_i^k\}} M^{-2}.$$

We can now conclude since any vertex $v \in V$ belongs to exactly one G_i^k for every $i \leq e_v(\sigma)$ and to no G_i^k otherwise. \square

At this stage, we have to make our assumption on the value of N_i more precise. To do so, we first have to fix $\epsilon > 0$ small enough so that $\frac{2-\epsilon}{1+\epsilon} \geq 2 - 2\epsilon$. Then, we assume that, for every $i \in \llbracket I \rrbracket$, N_i verifies the following:

$$N_i \geq \frac{\mathfrak{R}g}{\epsilon \log M} \times \frac{K_i}{i}, \text{ and } N_i \leq M^{(4-d-3\epsilon)i}. \quad (1.17)$$

Lemma 1.25 Fix ϵ and $(N_i)_{i \in \llbracket I \rrbracket}$ as above. Then, we have

$$|Z_I| \lesssim_{|g|} 1.$$

Proof. Recall that for a graph $G \in \mathcal{G}_i^{k_{i+1}, \dots, k_I}$ and $\sigma \in \mathcal{S}^c(G)$, the conditions (1.14) hold. Therefore, we have

$$\prod_{v \in V} M^{-e_v(\sigma)} = M^{-2iN_i} \prod_{j=i+1}^I M^{-2jk_j}.$$

We split this good factor and use a small fraction of it $M^{-\epsilon i N_i} \prod_{j=i+1}^I M^{-\epsilon j k_j}$ to control the sum over $\sigma \in E^{\llbracket I \rrbracket}$ which yields a factor C^{P_i} .

We use another fraction of the good bound to control the bad factors (1.15). The upper bound on N_i in (1.17) implies that

$$M^{-\epsilon i N_i} \leq e^{-K_i \mathfrak{R}g}.$$

Therefore, we have

$$|Z_I| \lesssim \sum_{i=-1}^I \frac{M^{-(2-2\epsilon)iN_i}}{N_i!} \sum_{k_{i+1} \in \llbracket N_{i+1}-1 \rrbracket} \cdots \sum_{k_I \in \llbracket N_I-1 \rrbracket} \left((C_2|g|)^{2P_i} (P_i!)^4 \prod_{j=i+1}^I \frac{M^{-(2-\epsilon)jk_j}}{(k_j!)^2} \right)^{1/2}.$$

Denote $R_i \stackrel{\text{def}}{=} \sum_{j=i+1}^I k_j$, and recall that $P_i = N_i + R_i$. Using the relationship $(p+q)! \leq p!q!$, we have $(P_i!)^4 \leq (N_i!)^4 (R_i!)^4$, so that

$$\begin{aligned} |Z_I| &\lesssim \sum_{i=-1}^I N_i! M^{-(2-2\epsilon)iN_i} \frac{1}{(R_i!)^\epsilon} \sum_{k_{i+1} \in \llbracket N_{i+1}-1 \rrbracket} \cdots \sum_{k_I \in \llbracket N_I-1 \rrbracket} (C_2|g|)^{P_i} \left(\frac{R_i!}{\prod_{j=i+1}^I k_j!} \right)^{2+\epsilon} \prod_{j=i+1}^I (k_j!)^{1+\epsilon} M^{-(2-\epsilon)jk_j} \\ &\lesssim \sum_{i=-1}^I N_i! M^{-(2-2\epsilon)iN_i} (C_2|g|)^{N_i} \times \sum_{R \geq 0} \frac{(C_2|g|)^R}{(R!)^\epsilon} \sum_{\substack{k_0, \dots, k_I \\ \sum_{j=0}^I k_j = R}} \left(\frac{R!}{\prod_{j=0}^I k_j!} \right)^{2+\epsilon} \prod_{j=0}^I (k_j! M^{-(2-2\epsilon)jk_j})^{1+\epsilon}. \end{aligned}$$

Note that on the last line we used the hypothesis that $\frac{2-\epsilon}{1+\epsilon} \geq 2 - 2\epsilon$. We handle both terms in the product on the r.h.s separately.

For the first one, the lower bound on N_i in (1.17) implies that

$$N_i! M^{-(2-3\epsilon)iN_i} \leq 1,$$

so that the first sum is bounded uniformly in I as

$$\sum_{i=-1}^I N_i! M^{-(2-2\epsilon)iN_i} (C_2|g|)^{N_i} \leq 1 + \sum_{i \geq 0} (C_2|g| M^{-\epsilon i})^{N_i} \lesssim_{|g|} 1.$$

For the second sum we use the fact that our upper bound on N_j also implies that

$$k_j! M^{-(2-3\epsilon)jk_j} \leq 1.$$

We can therefore bound the second sum as

$$\begin{aligned} & \sum_{R \geq 0} \frac{(C_2|g|)^R}{(R!)^\epsilon} \sum_{\substack{k_0, \dots, k_I \\ \sum_{j=0}^I k_j = R}} \left(\frac{R!}{\prod_{j=0}^I k_j!} \right)^{2+\epsilon} \prod_{j=0}^I (k_j! M^{-(2-2\epsilon)jk_j})^{1+\epsilon} \lesssim \sum_{R \geq 0} \frac{(C_2|g|)^R}{(R!)^\epsilon} \sum_{\substack{k_0, \dots, k_I \\ \sum_{j=0}^I k_j = R}} \left(\frac{R!}{\prod_{j=0}^I k_j!} \right)^{2+\epsilon} \prod_{j=0}^I M^{-\epsilon(1+\epsilon)jk_j} \\ & \lesssim \sum_{R \geq 0} \frac{(C_2|g|)^R}{(R!)^\epsilon} \left(\sum_{\substack{k_0, \dots, k_I \\ \sum_{j=0}^I k_j = R}} \frac{R!}{\prod_{j=0}^I k_j!} \prod_{j=0}^I M^{-\epsilon(1+\epsilon)jk_j/(2+\epsilon)} \right)^{2+\epsilon} \lesssim \sum_{R \geq 0} \frac{(C_2|g|)^R}{(R!)^\epsilon} \left(\sum_{j=0}^I M^{-\epsilon(1+\epsilon)j/(2+\epsilon)} \right)^{(2+\epsilon)R}, \end{aligned}$$

where in the last inequality we used the multinomial theorem. The series $\sum_{j \geq 0} M^{-\epsilon(1+\epsilon)j/(2+\epsilon)}$ is convergent, so that one finally has that

$$\sum_{R \geq 0} \frac{(C_2|g|)^R}{(R!)^\epsilon} \sum_{\substack{k_0, \dots, k_I \\ \sum_{j=0}^I k_j = R}} \left(\frac{R!}{\prod_{j=0}^I k_j!} \right)^{2+\epsilon} \prod_{j=0}^I (k_j! M^{-(2-2\epsilon)jk_j})^{1+\epsilon} \lesssim \sum_{R \geq 0} \frac{(C_3|g|)^R}{(R!)^\epsilon} \lesssim_{|g|} 1.$$

□

A similar argument would allow us to show that $Z_I - Z_{I-1}$ is decaying exponentially with I , which shows that Z_I converges to some limit Z . The remainder of Z_I minus its Taylor expansion in g can also be controlled similarly, which yields a proof of the Borel summability of the limit in g . Overall, minor modifications of our proof yield the following.

Theorem 1.26 *The limit $\lim_{I \uparrow \infty} Z_I$ exists and is a holomorphic function of g on $\{\Re g > 0\}$. Moreover, though it is not analytic in $g = 0$, it is smooth and Borel summable.*

Remark 1.27 To conclude about the philosophy of old constructive techniques, their purpose is therefore to view the large field problem (once renormalisation is already implemented) as a purely combinatorial problem, expressing the balance between bad factors coming from UV divergences, and good factors coming from scale expansions/the renormalisation group, that is to say from the fact that we integrate a field at some fixed scale, and not at every scales.

It is therefore of utmost importance to understand why integrating at a fixed scale yields some better factors. The answer can be read from the bound on the sliced propagator (1.16): while the sliced propagator diverges at small scales (which corresponds to the factor $M^{i(d-2)}$) it is more convergent at large scales (which corresponds to the exponential decay). However, if one works with the full propagator, and not with the sliced propagator equipped with an IR cut-off, then one does not benefit from this convergence, because

UV converges imply IR divergences.

To rephrase this, in (constructive) EQFT, if a quantity is convergent in the UV then, in general, it can lead to a divergence in the IR coming from the fact that we can not benefit from its good factor, and simply get a constant. Think for instance of the function $x \mapsto 1/x^2$: it is indeed convergent at large scales, and power counting predicts that its integral over $[1, L)$ should give us a good factor $1/L$. However, this is of course not the case, since the integral will just be bounded by one.

Note that these new IR divergences coming from UV convergences are very different from the IR divergences that we described in Remark 1.11. Moreover, they should not be understood as an IR problem, since they truly originate not from a phenomenon taking place in the IR, but in the UV, and are therefore some UV convergences. We will thus refer to them as the *weak UV problem*, while we call the former UV problem (the ill-posedness of the interaction due to the roughness of the free field) the *strong UV problem*.

To conclude about the weak UV problem, let us mention that we already know a way to overcome it. Since the weak UV problem implies IR divergences, it is absent when the propagator is endowed with an IR cut-off, which is precisely what the renormalisation group does. In Section 1.3, we will see other techniques that serve the same purpose, like recentering and paraproductions.

Remark 1.28 One could now wonder where the approach that we presented in this section breaks, and how far in the superrenormalisable regime it can go. The answer can be read from the conditions on N_i in (1.17). Indeed, recall that in general dimension d , $K_i \propto M^{i2(d-2)}$, since the most divergent graph is always the double tadpole. Therefore, it is only possible to fulfil both inequalities in (1.17) provided

$$d < \frac{8}{3}.$$

The fact that this dimension is not an integer should be taken very seriously: it is indeed possible to probe what goes on in fractional dimensions, for instance working with a fractional Laplacian. The question now is, what is so special about dimension $8/3$? It turns out that it is precisely the dimension in which a new renormalisation beyond Wick ordering becomes necessary: the vacuum graphs with two vertices indeed become logarithmically divergent in $d = 8/3$. Moreover, one can see (for instance using the variational approach presented in Section 1.2.2) that this new renormalisation corresponds to the fact that the Φ^4 measure can be expressed as GFF shifted by a quantity which is no longer in H^1 , the Cameron-Martin space of the GFF (see Lemma 1.38). The reader familiar with trees should think of the fact that φ° is of regularity $1-$ in $d = 8/3$. Therefore, in $d = 8/3$, the Φ^4 measure stops being absolutely continuous with respect to the GFF.

This fact definitely rules out the approach that we presented in the present section, that is to say trying to show that the density of the Φ^4 measure with respect to the free field can be controlled uniformly in the UV cut-off. Old constructive studies have overcome this difficulty by making the following observation. The new difficulty arising in $d = 8/3$ is ultimately not related to the strong UV problem, since we still know what counterterms to remove, but with the weak UV problem, and could be avoided if it were possible to extract some more decay from the propagators. It turns out that working on a smaller space, one gains better factors. The spirit of old constructive physics is therefore to integrate fields at scale i in a small box of size M^{-i} , which allows to inductively obtain some control on the partition function at different scales, again rephrasing the large field problem as a combinatorial problem. For more explanations in this direction, we refer to the third section of [Riv91].

1.1.4 Osterwalder-Schrader axioms

At the beginning of this section, we tried to give some motivation to base a rigorous analysis of QFT on its Euclidean counterpart. While this motivation was heuristic, it is important to know whether one can recover a full QFT starting from a EQFT. Osterwalder and Schrader formulated a reconstruction theorem that guarantees that a scalar EQFT ν is sufficient to recover the original QFT provided it verifies a set of axioms. There are several ways to formulate them, but all boil down to requiring that the following four properties hold.

1) the moments of ν must not grow too fast, stretched exponential integrability being sufficient. This property is generally called a *regularity* property.

2) the moments of ν must be *covariant* with respect to the action of the Euclidean group. Moreover, ν should be a commutative probability measure in the bosonic case, or respect the anti-symmetry conditions of fermions in the fermionic case.

3) ν must exhibit a *reflection positivity* in the sense that if F is a functional supported on a half-space (say $x_0 \geq 0$), then it should hold

$$\mathbf{E}_\nu[\overline{\theta F} F] \geq 0,$$

where $\theta F = F(\theta\phi)$ and $\theta\phi$ is the pull-back of ϕ under the reflection preserving the hyper-plane $\{x_0 = 0\}$. Here the fact that the functional is supported on a half-space means that if ϕ is of law ν , then $F(\phi)$ is only sensitive to the fluctuations of ϕ on this half-plane.

4) a last property is necessary to ensure the uniqueness of the vacuum. There are various ways to state it, saying for instance that the measure ν must be *ergodic* with respect to time translations, or that if ϕ is of law ν , then $\phi(x)$ and $\phi(y)$ must decorrelate when y and x become far. This last version is known as *clustering*, and when the theory is massive, one expects the clustering to occur exponentially fast.

Remark 1.29 The crucial property is reflection positivity, since it allows in the first place to construct the Hilbert space, which is defined as follows. One endows the space of all (suitable, say smooth and with exponential growth) functionals supported on the positive half-plane with the inner product

$$(F, G) \stackrel{\text{def}}{=} \mathbf{E}_\nu[\overline{\theta F} G],$$

and let \mathcal{E}_+ be the space of all functionals such that $(F, F) < \infty$, and \mathcal{N} by the null space gathering all functionals in \mathcal{E}_+ such that $(G, G) = 0$. Then, the Hilbert space \mathcal{H} is defined as the completion of $\mathcal{E}_+/\mathcal{N}$.

Time-translation invariance then implies that the time translations act as a semi-group on \mathcal{H} , whose generator is positive and self-adjoint, and that can be analytically continued to a unitary group. The reconstruction of Lorentz boosts is more subtle, since rotations are only local semi-groups, but some theorem by [FOS83] allows to extend some local semi-groups to full analytic semi-groups, which is sufficient for the purpose of reconstruction of QFT.

Remark 1.30 Reflection positivity can be replaced by the stronger requirement that ϕ verifies some spatial Markov property, which in turn should stem from the locality of the Lagrangian. The spatial Markov property roughly states that given an open subdomain A of \mathbf{R}^d , the fluctuations of ϕ on A and A^c should be independent knowing the value of ϕ on the boundary of A . However, there exists some Gaussian measure corresponding to some non-local Lagrangian that are reflection positive, but fail to verify this Markov property, like for instance the Gaussians with covariance $(1 - \Delta)^{-s}$ for $s \in (0, 1)$.

Remark 1.31 Constructions of EQFT generally come with enough information about their integrability. An optimal integrability criterion was proven for Φ_3^4 in [HS21b] using regularity structures. Proving both Euclidean invariance and reflection positivity can turn out to be more challenging, since there is no regularisation that preserves both properties. Indeed, regularisations based on the functional calculus of the Laplacian (such as heat regularisation) are of course Euclidean invariant, but break locality and reflection positivity, while local regularisations preserving say an hyperplane fail to be Euclidean invariant. Regarding the exponential decay of correlators, it was proven in the scalar case by means of the cluster expansion. However, there are less results in this direction in the stochastic analytic context beyond the case of an exponential interaction [VGT22] with convex renormalised potential.

1.2 Some stochastic quantisations of Euclidean QFTs

In the previous section devoted to the presentation of the relationship between infinite dimensional probability measures and Quantum Field Theory, we insisted on some stochastic analytic aspects of the

study of QFT, in particular we established a connection between renormalisation and Ito integration, and we derived the path integral formula with a heuristic based on the Feynman-Kac formula. Moreover, as we saw in Remark [1.28](#), close to the critical dimension $d = 4$, the EQFTs are singular with respect to the GFF, which rules out any naive approach consisting in trying to show that the density is well-defined and integrable with respect to the GFF. We therefore need to come up with a new strategy, better suited to some problems where singularity comes into play. It turns out that stochastic analysis has two powerful tools that are rather insensitive to the singularity of the measure: coupling arguments, and push-forwards. The philosophy of stochastic quantisation is therefore to identify a way to exhibit a coupling of EQFTs with some Gaussian measure, or to show that a EQFT is the pushed-forward of a Gaussian. This section is devoted to introducing several realisations of this idea at a formal level – that is to say working with regularised measures for the moment, without caring about UV or convergence issues. We start by presenting the Langevin dynamic, before we introduce the variational approach and the forward-backward equations.

1.2.1 The Langevin dynamic

The Langevin dynamic turns out to be the original proposal of Parisi and Wu [\[PW81\]](#), which is to view a EQFT ν as the invariant measure of a noisy gradient descent. This approach relies on the idea that to establish a connection between a well-characterised measure μ (in general μ is just a Gaussian) and the EQFT ν one has to come-back to the notion of Markov semigroup introduced in Section [1.1.1.3](#), and to postulate that there exists such a semigroup P acting on smooth functionals such that

$$\mathbf{E}_\nu[F] = \lim_{t \uparrow \infty} \mathbf{E}_\mu[P_t F]. \quad (1.18)$$

If this is the case, then ν must be an invariant measure for the Markov semigroup P . Indeed, applying [\(1.18\)](#) to $P_s F$ for $s \geq 0$ yields

$$\mathbf{E}_\nu[P_s F] = \lim_{t \uparrow \infty} \mathbf{E}_\mu[P_{t+s} F] = \lim_{t \uparrow \infty} \mathbf{E}_\mu[P_t F] = \mathbf{E}_\nu[F].$$

In particular, one must $L^* \nu = 0$, L being the infinitesimal generator of P . The question now boils down to finding a Markov semigroup verifying this property. It turns out that recalling that formally $\nu(d\phi) \propto e^{-S(\phi)} d\phi$ where S is the Euclidean action functional and $d\phi$ is a formal Lebesgue measure, one thus has $L^* \nu = 0$ for the infinitesimal generator L formally defined as

$$L \stackrel{\text{def}}{=} \partial_\Phi^2 - \langle \partial_\Phi S, \partial_\Phi \cdot \rangle = \langle \partial_\Phi, e^{-S} \partial_\Phi (e^S \cdot) \rangle.$$

A good candidate to construct the measure ν starting from μ is therefore the Markov semigroup $P = (P_t)_{t \geq 0}$ with generator L , called *Glauber dynamic*. We can now express its actions on some smooth functional F with the Feynman-Kac formula, writing

$$P_t F(\phi) = \mathbf{E}[F(u_t^\phi)],$$

where $u_t^\phi = u(t, x)$ is the solution to

$$\begin{aligned} du(t, x) &= -\partial_\Phi S(u(t, x)) dt + \sqrt{2} dW_t(x), \\ u(0, x) &= \phi(x). \end{aligned} \quad (1.19)$$

Since it takes a form analogous to the Langevin equation, [\(1.19\)](#) is called the *Langevin dynamic* for the EQFT ν .

Remark 1.32 Let us point out a few interesting facts about the Langevin dynamic.

The first thing one should note is that in the previous section, we saw that there is no hope to construct the density of the Φ^4 EQFT with respect to the GFF. The Langevin dynamic formally allows to construct

the EQFT as a push-forward of the driving Gaussian noise by the solution map, so that the issue with the singularity is absent.

Then, a nice feature of the Langevin dynamic is that it is a parabolic (semi-linear) equation. In particular, a small parameter, the time of existence of solutions, is already present. Contrary to what we saw in the previous section about the way the UV and large field problem combine to generate bigger difficulties, this existence time allows to totally disentangle the UV problem from the large field problem. In the Langevin dynamic context, solving the UV problem just reduces to showing that the equation has local-in-time solutions, while solving the large field problem amounts to showing that these solutions are actually global.

We have thus reexpressed the question of the existence of some EQFT ν as the question whether a Markov semigroup has an invariant measure. In the sequel, we will briefly recall a few key general results in order to prove existence and uniqueness of the invariant measure of a Markov semigroup.

Definition 1.33 A Markov semigroup P on \mathcal{H} verifies the *Feller property* if it maps bounded continuous functions into themselves. Moreover, it has the *strong Feller property* if it maps bounded measurable functions into bounded continuous functions.

Lemma 1.34 (Krylov-Bogolioubov) Consider a Feller Markov semigroup $P = (P_t)_{t \geq 0}$ on \mathcal{H} , and suppose that there exists a Borel measure μ_0 on \mathcal{H} such that $(P_t^* \mu_0)_{t \geq 0}$ is tight. Then, any accumulation point of

$$\frac{1}{t} \int_0^t P_s \mu_0 \, ds$$

is an invariant measure of P .

Definition 1.35 Fix a Markov semigroup $P = (P_t)_{t \geq 0}$ on \mathcal{H} and a point $z \in \mathcal{H}$. We define the resolvent operator ϱ_λ of P by

$$\varrho_\lambda(x, dy) = \lambda \int_0^\infty e^{-\lambda t} \pi_t(x, dy) dt$$

which also generates a Markov semigroup. Then, z is said to be *accessible* for P if for every $x \in \mathcal{H}$, one has $z \in \text{supp } \varrho_\lambda(x, \cdot)$.

Lemma 1.36 Fix a Markov semigroup $P = (P_t)_{t \geq 0}$ on \mathcal{H} which is Strong Feller at an accessible point $x \in \mathcal{H}$ (in the sense that $P_t F$ is continuous in x for every bounded measurable F). Then, it has at most one invariant measure.

Remark 1.37 The proof of this last lemma relies on the structure of the set of invariant measures for P . Indeed, the invariant measures are a convex set whose extremal points are some ergodic measures, that is to say some measures μ_i verifying

$$\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t P_s F \, ds = \int_{\mathcal{H}} F(x) \mu_i(dx).$$

Moreover, the ergodic invariant measures μ_i are all mutually singular. The hypothesis that P is Strong Feller at x ensures that any invariant measure has support in x , which finally implies that no two invariant measures can be singular.

1.2.2 The variational approach

A second stochastic quantisation procedure that we aim to describe is a variational characterisation of a EQFT ν , that was introduced by Barashkov and Gubinelli in [BG20]. This variational approach is based on a property of Gibbs measures of the form

$$\nu(d\phi) = \frac{1}{Z} e^{-S(\phi)} d\phi.$$

Indeed, in finite dimension, denoting by λ the Lebesgue measure, a Gibbs measure satisfies the variational problem

$$-\log Z = \min_{\mu \ll \lambda} \mathbf{E}_\mu [S + \log \frac{d\mu}{d\lambda}]. \quad (1.20)$$

Here, note that the second term in the expectation is the relative entropy of μ with respect to λ , and that the variational problem can thus be interpreted as a balance between energy and entropy. It turns out that (1.20) rewrites nicely when combined with the Girsanov transformation.

Lemma 1.38 (Girsanov) *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space equipped with a Brownian motion $(B_t)_{t \geq 0}$. Then a probability \mathbf{Q} on (Ω, \mathcal{F}) is absolutely continuous with respect to \mathbf{P} if and only if there exists a (random) drift $h = (h_t)_{t \geq 0}$, progressively measurable with respect to the filtration of $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ induced by $(B_t)_{t \geq 0}$ and belonging to $L^2(\mathbf{R}_{\geq 0})$ \mathbf{P} -almost surely, such that it holds*

$$\mathbf{Q} = (T_h)_* \mathbf{P},$$

where $T_h(x) \stackrel{\text{def}}{=} x + h$ is the translation by h . If this is the case, then for any $t \in [0, \infty]$ one has the Girsanov transformation

$$\frac{d\mathbf{Q}}{d\mathbf{P}} \Big|_{\mathcal{F}_t} = \exp \left(\int_0^t h_s dB_s - \frac{1}{2} \int_0^t h_s^2 ds \right).$$

It turns out that the EQFT ν can be constructed as limit of a sequence of finite dimensional approximations $(\nu_t)_{t \geq 0}$ defined as

$$\nu_t(d\phi) = \frac{1}{Z_t} e^{-S_t^{\text{int}}(\phi_t)} \mu(d\phi),$$

where ϕ_t is almost surely smooth, $\lim_{t \uparrow \infty} \phi_t = \phi$, and only a finite number of Fourier modes of ϕ_t are nonzero. Then, fixing a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a process $(W_t)_{t \geq 0}$ depending on a finite number of Brownian motions on $(\Omega, \mathcal{F}, \mathbf{P})$ and such that for every $t \geq 0$ $\text{Law}_{\mathbf{P}} W_t = \text{Law}_{\mu} \phi_t$, by Girsanov's theorem, the sequence $(\nu_t)_{t \geq 0}$ can be promoted to a sequence $(\mathbf{Q}_t)_{t \geq 0}$ of probabilities on (Ω, \mathcal{F}) defined as

$$\frac{d\mathbf{Q}_t}{d\mathbf{P}} \stackrel{\text{def}}{=} \frac{1}{Z_t} e^{-S_t^{\text{int}}(W_t)} = \exp \left(\int_0^\infty \langle u_s^t, dX_s \rangle_{L^2} - \frac{1}{2} \int_0^\infty \|u_s^t\|_{L^2}^2 ds \right). \quad (1.21)$$

In the second equality $(X_t)_{t \geq 0}$ is a cylindrical Brownian motion on $L^2(\mathbf{T}^d)$, and $u^t = (u_s^t)_{s \geq 0}$ is a random drift progressively measurable with respect to the filtration of \mathcal{F} induced by $(X_t)_{t \geq 0}$ and which belongs to $L^2(\mathbf{R}_{\geq 0} \times \mathbf{T}^d)$ \mathbf{P} -almost surely (denote by \mathbf{H}^a the space of such drift). (1.21) therefore rewrites as

$$-\log Z_t = \min_{v \in \mathbf{H}^c} \mathbf{E}_{\mathbf{Q}^v} [S_t^{\text{int}}(W_t^v + I_t(v)) + \frac{1}{2} \int_0^\infty \|v_s\|_{L^2}^2 ds], \quad (1.22)$$

where \mathbf{Q}^v is defined by the second equality in (1.21) with u^t replaced by v , \mathbf{H}^c is the space of those drifts that are such that $\mathbf{Q}^c(\Omega) = 1$, and $I_t(v)$ is defined through Girsanov's transformation by the relation $W_t^v = W_t - I_t(v)$, where $\text{Law}_{\mathbf{Q}^v} W_t^v = \text{Law}_{\mathbf{P}} W_t$.

It turns out that (1.22) is not very convenient, due to the rather intricate definition of \mathbf{H}^c , and to the fact that the expectation operator depends on the drift v . An important contribution of [BG20] has been to use the Boué-Dupuis formula to reformulate the variational problem as follows.

Lemma 1.39 (Boué-Dupuis) *Suppose that S_t^{int} is a tame functional, in the sense that it holds*

$$\mathbf{E}_{\mathbf{P}}[|S_t^{\text{int}}(W_t)|^p] + \mathbf{E}_{\mathbf{P}}[e^{-qS_t^{\text{int}}(W_t)}] \lesssim 1$$

for some $p, q \geq 1$ Hölder conjugate. Then, we have

$$-\log Z_t = \inf_{v \in \mathbf{H}^a} \mathbf{E}_{\mathbf{P}}[S_t^{\text{int}}(W_t + I_t(v)) + \frac{1}{2} \int_0^\infty \|v_s\|_{L^2}^2 ds]. \quad (1.23)$$

Remark 1.40 Let us point out two nice features of the Boué-Dupuis formula. Firstly, in the regime where ν is absolutely continuous with respect to μ , then (1.23) easily gives some upper and lower bounds on $-\log Z_t$ that are uniform in the cut-off t . Secondly, it provides a coupling of ν with μ of the form

$$\nu = \text{Law}_{\mathbf{P}}(W_\infty + I_\infty).$$

Similar results hold when ν is singular with μ but still super-renormalisable, but it becomes necessary to take renormalisation into account, which is done by shifting the drift in (1.23). The Boué-Dupuis formula was used in [BG20] to construct the Φ_3^4 measure. In [BG21], the authors use it to give a proof of the fact that the Φ_3^4 measure is singular with respect to the GFF.

1.2.3 The Forward-Backward equations

We now turn to a third stochastic quantisation, which is known as the Forward-Backward Stochastic Differential Equations (FBSDE) approach. Similarly to the way in which we derived the Langevin dynamic as a Feynman-Kac version of the Glauber dynamic, we would like to use the Feynman-Kac formula to treat the Polchinski renormalisation group introduced in Section 1.1.3.1 by probabilistic means. The FBSDE approach can be thought of as an implementation of the Feynman-Kac formula for the semigroup corresponding to the action of the renormalisation group, that we introduce hereafter.

Definition 1.41 Let Φ be a random field with law the GFF, and following the notation of the beginning of Section 1.1.3.1, denote by $\Phi_{[M,N]}$ a heat regularised version of Φ at scale M in the IR and $N \geq M$ in the UV, and set $\Phi_{\leq M} \stackrel{\text{def}}{=} \Phi_{[0,M]}$, $\Phi_{>M} \stackrel{\text{def}}{=} \Phi_{[M,1]}$. Moreover, denote respectively by $\mathbf{E}_{[M,N]}$, $\mathbf{E}_{\leq M}$ and $\mathbf{E}_{>M}$ their expectation operators. The renormalisation group induces a semigroup $T = (T_{M,N})_{0 \leq M \leq N \leq 1}$ called *Polchinski semigroup*, which is defined for $0 \leq M \leq N \leq 1$ as

$$T_{M,N}F(\Phi_{\leq M}) \stackrel{\text{def}}{=} e^{V_{M,\epsilon}(\Phi_{\leq M})} \mathbf{E}_{[M,N]}[e^{-V_{N,\epsilon}(\Phi_{\leq M} + \Phi_{[M,N]})} F(\Phi_{\leq M} + \Phi_{[M,N]})],$$

where the effective potential $(V_{L,\epsilon})_{L \in [0,1]}$ was defined in (1.11). Note that it is a time-dependent semigroup, and verifies the semigroup property

$$T_{L,M}T_{M,N} = T_{L,N}, \text{ and } T_{L,L} = \text{Id}.$$

Finally, its time-dependent infinitesimal generator is given by

$$L_M F \stackrel{\text{def}}{=} \frac{1}{2} \langle \partial_\Phi, \partial_\Phi \rangle_{\dot{C}_M} F - \langle \partial_\Phi V_{M,\epsilon}, F \rangle_{\dot{C}_M}.$$

The definition of the Polchinski semigroup along with the expression of its infinitesimal generator imply that extending the Feynman-Kac formula to non-time-homogeneous semigroups, one has

$$T_{M,N}F(\Phi_{\leq M}) = \mathbf{E}[F(\varphi_N)],$$

where $(\varphi_L)_{L \in [M,N]}$ solves

$$\begin{aligned} d\varphi_L &= -\dot{C}_L \partial_\Phi V_{L,\epsilon}(\varphi_L) dL + \dot{C}_L^{1/2} dB_L, \\ \varphi_M &= \Phi_{\leq M}. \end{aligned} \quad (1.24)$$

Writing this equation requires prior knowledge of the full effective potential, which amounts to having already solved the Polchinski equation. In particular, solving it a priori ends up being difficult. However, Ito's lemma which we recall for the reader's convenience provides us with an interesting rewriting of (1.24).

Lemma 1.42 (Ito's lemma) *Consider the solution (X_t) to the SDE $dX_t = \mu_t dt + \sigma_t dB_t$. Then, for any C^2 function f , it holds*

$$df(t, X_t) = \left(\partial_t f + \mu_t \partial_x f + \frac{\sigma_t^2}{2} \partial_x^2 f \right) (t, X_t) dt + \sigma_t \partial_x f(t, X_t) dB_t.$$

From now on, we denote by $(\varphi_L)_{L \in [0,1]}$ the solution to (1.24) with null initial condition at $L = 0$. Ito's lemma implies that, denoting $F_{L,\epsilon} \stackrel{\text{def}}{=} \partial_\Phi V_{L,\epsilon}$, one has

$$\begin{aligned} \frac{d}{dL} F_{L,\epsilon}(\varphi_L) &= \partial_L F_{L,\epsilon}(\varphi_L) - \langle \partial_\Phi F_{L,\epsilon}(\varphi_L), F_{L,\epsilon}(\varphi_L) \rangle_{\dot{C}_L} + \frac{1}{2} \langle \partial_\Phi, \partial_\Phi \rangle_{\dot{C}_L} F_{L,\epsilon}(\varphi_L) + \dot{C}_L^{1/2} \partial_\Phi F_{L,\epsilon}(\varphi_L) \frac{dB_L}{dL} \\ &= \dot{C}_L^{1/2} \partial_\Phi F_{L,\epsilon}(\varphi_L) \frac{dB_L}{dL}, \end{aligned}$$

where in the second line we used the equation for $F_{L,\epsilon}$ obtained by taking the functional derivative of (1.12). Therefore, denoting by $(\Omega, \mathcal{F}, \mathbf{P})$ the ambient probability space, $(F_{L,\epsilon}(\varphi_L))_{L \in [0,1]}$ is a martingale with respect to the filtration $(\mathcal{F}_L)_{L \in [0,1]}$ of \mathcal{F} induced by $(B_L)_{L \in [0,1]}$. This last fact implies that one has

$$\mathbf{E}_{>L}[F_{1,\epsilon}(\varphi_1)] = F_{L,\epsilon}(\varphi_L),$$

where we used the fact that $\mathbf{E}_{>L}[\cdot] \equiv \mathbf{E}[\cdot | \mathcal{F}_{>L}]$. Plugging this last equation in (1.24) finally yields the *Polchinski forward-backward SDE*

$$d\varphi_L = -\dot{C}_L \mathbf{E}_{>L}[\partial_\Phi V_{1,\epsilon}(\varphi_1)] dL + \dot{C}_L^{1/2} dB_L. \quad (1.25)$$

Though this equation looks more complicated than (1.24) due to its forward-backward structure, it allows for approximate solutions to the Polchinski equation. In other words, one no longer needs to know the whole solution to the Polchinski equation in order to construct the semigroup. More precisely, letting

$$W_L \stackrel{\text{def}}{=} \int_0^L \dot{C}_L^{1/2} dB_M, \quad Z_L \stackrel{\text{def}}{=} \varphi_L - W_L,$$

and

$$\mathbf{E}_{>L}[\partial_\Phi V_{1,\epsilon}(\varphi_1)] = F_{L,\epsilon}(\varphi_L) + R_{L,\epsilon} = F_{L,\epsilon}(W_L + Z_L) + R_{L,\epsilon},$$

where $R_{L,\epsilon}$ is a remainder term such that $R_{1,\epsilon} = 0$, (1.25) rewrites as

$$\begin{cases} Z_L = - \int_0^L \dot{C}_M(F_{M,\epsilon}(W_M + Z_M) + R_{M,\epsilon}) dM, \\ R_{L,\epsilon} = \mathbf{E}_{>L} \left[\int_L^1 (H_M(W_M + Z_M) + \langle \partial_\Phi F_M(W_M + Z_M), R_{M,\epsilon} \rangle_{\dot{C}_M}) dM \right]. \end{cases}$$

Here, $H_M \stackrel{\text{def}}{=} \partial_M F_M + \frac{1}{2} \langle \partial_\Phi, \partial_\Phi \rangle_{\dot{C}_M} F_{M,\epsilon} - \langle \partial_\Phi F_{M,\epsilon}, F_{M,\epsilon} \rangle_{\dot{C}_M}$ is the Polchinski equation, and solving it approximately thus makes the remainder term non-zero.

Remark 1.43 At this stage, the goal is to find a suitable ansatz for the *effective force* $(F_L)_{L \in [0,1]}$ to take the UV problem/renormalisation into account. If the theory is singular, such an ansatz will necessarily result in a final condition $V_{1,\epsilon}$ which is divergent with the UV cut-off ϵ . On the other hand, the large field problem is now relocated at the level of the equation for the remainder R . Indeed, in general, this equation will only have local solutions close to the UV.

To conclude, the FBSDEs approach was applied in [VFG22] to a purely fermionic model in the full-subcritical regime, and to the sine-Gordon model up to 6π in [GM24].

1.2.4 Other PDEs related with EQFTs

Let us conclude this section by pointing out that the three methods that we described are not the only connections between stochastic analysis and PDEs, and the EQFTs.

Indeed, an argument based on dimensional reduction and supersymmetry and first formulated by Parisi and Sourlas [PS82] suggests that formally the solution to the elliptic problem on \mathbf{R}^{d+2}

$$(1 - \Delta)u + \partial_{\Phi} V(u) = \xi,$$

where ξ is a space-time white noise on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is related to the scalar EQFT ν on \mathbf{R}^d with interaction potential V by the relation

$$\mathbf{E}_{\nu}[F] = \mathbf{E}_{\mathbf{P}}[F(u(x_1, \dots, x_d, 0, 0))].$$

This *elliptic stochastic quantisation* was studied in [ADVG20], where the authors justified this heuristic for a range of potentials including polynomial interactions.

A last relationship between PDEs and EQFTs is through the *Hamiltonian systems*. It roughly states that the scalar EQFT ν on \mathbf{R}^d with interaction potential V should be invariant under the non-linear wave (NLW) equation

$$(\partial_t^2 + 1 - \Delta)u + \partial_{\Phi} V(u) = 0. \quad (1.27)$$

More precisely, setting $v \stackrel{\text{def}}{=} \partial_t u$ and

$$H(u, v) \stackrel{\text{def}}{=} \frac{1}{2} \int v^2 + \frac{1}{2} \int (u(1 - \Delta)u + V(u)),$$

(1.27) rewrites as

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial_v H \\ -\partial_u H \end{pmatrix}.$$

Therefore, provided one can use an approximation argument based on a finite dimensional system, by Liouville theorem, the measure

$$\rho(du, dv) \stackrel{\text{def}}{=} \nu(du) \otimes \xi(dv)$$

is formally invariant under (1.27), where ξ denotes the law of the white noise on \mathbf{R}^d . Moreover, a similar heuristic holds for the non-linear Schrödinger (NLS) equation, the random field u being complex (i.e. \mathbf{R}^2 valued).

Note that this invariance of ν under the flow of some dispersive equation is a very different statement from its invariance under the Langevin dynamic, since solutions to the Langevin dynamic will flow towards the invariant measure, while in the case of dispersive equations, starting from an initial condition which is not drawn according to ν (or to a measure absolutely continuous with respect to ν) yields a solution totally unrelated to ν . Therefore, the Hamiltonian system can not be regarded as a true stochastic quantisation, since it is unlikely that PDE arguments applied to these equations can help to construct the EQFTs.

A second argument confirming this statement is that the dispersive equations are expected to be critical below the dimension in which their invariant measure is just-renormalisable. Indeed, contrarily to the heat operator that, by Schauder estimates, gives two degrees of smoothing, by a naive power counting argument, the wave and Schrödinger operators are expected to give respectively 0 and 1 degree of smoothing. This would rule out the possibility to show well-posedness of the equation started from the Φ^4 measure beyond dimensions respectively 2 and 3. It turns out that the situation is more subtle, since in some cases a *non-linear smoothing effect* can be expected, at least on the torus. This effect can be described using Fourier analysis, and observing that the expressions of the wave and Schrödinger

operators give some restrictions on the volume of the Fourier space which is integrated on, which ultimately entails some improvements in the regularities of the non-linear objects, by a half in the case of NLW, and by one for NLS. This results in the fact that on the torus, the cubic wave equation started from the Φ^4 measure is expected to be critical in $d = 3.5$, while the Schrödinger equation is expected to be critical in $d = 3$.

In [DNY20], leveraging this non-linear smoothing effect, the authors have studied the Schrödinger equation with cubic interaction in the full-subcritical regime. In [BDNY22], the authors have shown that the cubic non-linear wave equation has almost surely global solutions where the null set is a set of measure 0 with respect to ν . Their analysis relies on a sophisticated paracontrolled ansatz (see Section 1.3.2), combined with some random tensors estimates used to prove the non-linear smoothing, and a globalisation argument *à la* Bourgain, leveraging the fact that the Φ^4 measure is invariant under the flow of the equation.

1.3 Solving singular SPDEs

Three main approaches to stochastic quantisation were presented in the previous section, but in a very formal way, since we did not describe any renormalisation procedure allowing for a removal of the regularisation. Moreover, as discussed before, most of the effort in the stochastic quantisation program thus amounts to solving some Stochastic Partial Differential Equations (SPDEs). It turns out that, due to the roughness of their driving noise, these equations are *singular*, in the sense that their solutions can not be obtained by a simple application of the contraction mapping theorem.

In the sequel, we will focus on parabolic equations, with linear term given by $\partial_t - \Delta$. Let us first informally identify three classes of semi-linear singular parabolic SPDEs.

1) the *generalized Kardar–Parisi–Zhang* equation (gKPZ):

$$(\partial_t - \Delta)u = f_1(u)(\nabla u)^2 + f_2(u)\nabla(u) + f_3(u) + g(u)\xi,$$

where ξ is a rough noise and f_1, f_2, f_3 and g are smooth functions.

2) the *exponential Langevin dynamics* (ELD)

$$(\partial_t - \Delta)u = \int_{-\beta}^{\beta} e^{tu} \mu(dt) + \eta, \text{ and } (\partial_t - \Delta)u = \sin(\beta u) + \eta,$$

where η is a Gaussian noise that lies in $\mathcal{C}^{-2-\kappa}$ for every $\kappa > 0$, and μ is a Borel measure on $[-\beta, \beta]$.

3) the *polynomial Langevin dynamic* (PLD)

$$(\partial_t - \Delta)u = P_1(u)\nabla u + P_2(u) + \zeta,$$

where ζ is a rough noise and P_1, P_2 are polynomials.

We will now present a scaling heuristic that will be sufficient to determine some regimes of roughness of the noises in which it should be possible to solve the above three equations.

Regarding gKPZ first, it must be possible to plug the solution into some smooth function, which entails that it has to be of positive (Hölder-Besov) regularity. The application of the parabolic operator increasing the regularity by two, the regularity α of the driving noise ξ must therefore verify

$$\alpha > -2, \tag{1.28}$$

which includes the space-time white noise in $d = 1$.

The situation for ELD is subtle, and relies on some regularity estimates for some (possibly complex) exponentials of the Gaussian, or multiplicative Gaussian chaoses. It turns out that solutions are expected to exist for

$$\beta^2 \in [0, 8\pi). \tag{1.29}$$

Finally, in the case of the PLD, one needs the regularity properties of the two non-linearities to be better than the regularity γ of the noise ζ . Denoting by n_i for $i \in [2]$ the degrees of the polynomial P_i , one has the two constraints

$$(n_1 + 1)(\gamma + 2) - 1 > \gamma, \text{ and } n_2(\gamma + 2) > \gamma,$$

which finally imply

$$\gamma > -\left(\frac{2n_2}{n_2 - 1} \wedge \frac{2n_1 + 1}{n_1}\right). \quad (1.30)$$

In particular, taking $n_1 = 1$ and $n_2 = 3$, which covers the power countings of the stochastic Navier-Stokes equation and of the Langevin dynamic for the Yang-Mills EQFT yields $\alpha > -3$, which includes the space-time white noise in $d \leq 3$ (the white noise fails marginally in $d = 4$).

This very sketchy discussion motivates the definition of subcriticality given in [Hai14].

Definition 1.44 When the conditions (1.28), (1.29) or (1.30) are satisfied respectively for gKPZ, ELD, and PLD, then the equation is *subcritical*. If this condition marginally fails, the equation is *critical*. Otherwise, the equation is *supercritical*. Moreover, the (sub)criticality of the Langevin dynamic of some EQFT ν matches the (super)renormalisability of ν , in the sense that taking the driving noise of PLD or ELD to be a space-time white noise, the equation is subcritical if and only if the measure is superrenormalisable.

Remark 1.45 While subcriticality is an essential condition in order to expect the well-posedness of a singular equation, there is another obstacle to solving singular PDEs. Indeed, subcriticality states that only a finite number of *expectations* of the polynomials in the driving noise generated by the nonlinearity are divergent. However, it does not rule out the fact that higher moments of these polynomials are divergent. It turns out that while the divergence of some moment of order higher than 3 is not compatible with subcriticality, the divergence of some *covariance* can still occur in the subcritical regime. Since a covariance is divergent provided it is not L^1 , this phenomenon becomes less and less likely to occur as the dimension increases, but takes place for instance if one studies the Φ^4 or KPZ equations in $d = 1$ driven by the spatial derivative of the white noise. By appropriately rescaling the noise, Hairer studied in [Hai24] a KPZ equation such that the covariance of the second Picard iteration of the noise is only marginally convergent, and goes to a Dirac. He showed that this object then converges in law to a new white noise, and that the solution converges in law to the solution to the KPZ equation driven by this new noise.

Remark 1.46 Here, the denomination PLD should be understood as purely heuristic, since in full generality, this equation is of course *not* the Langevin dynamic of some invariant measure. It covers for instance the cases of the Navier-Stokes equation with a stochastic forcing, or of the wrong-sign dynamical Φ^4 model with $P_1(u) = 0$ and $P_2(u) = +u^3$. However, this naming is indeed suggestive, since the Langevin dynamics for Yang-Mills and Φ^4 theory fall in that category.

The same discussion holds of the ELD, that covers the Langevin dynamic for the sine-Gordon and Liouville theories, along with many equations which we do not expect to have any invariant measure nor global solutions.

With this heuristic in hand, we are now going to review three approaches to making sense locally in time of singular SPDEs: the theory of regularity structure, paracontrolled calculus, and the flow approach. As alluded to, all the approaches to singular SPDEs that we will present hereafter break in the critical regime. This is why, in the sequel, we will always discuss subcritical equations. Moreover, we will mostly introduce them on the paradigmatic example of the Φ^4 Langevin dynamic driven by a Gaussian noise. Finally, we will not discuss IR divergences, which is why we restrict our study to the case of the torus \mathbf{T}^d , and consider the massive heat operator as linear term.

1.3.1 The Da Prato-Debussche trick

Consider the Langevin dynamic for Φ^4 in $d = 2$ (the case $d = 1$ not being singular):

$$(\partial_t + 1 - \Delta)u = -u^3 + \zeta. \quad (1.31)$$

The singularity makes it necessary to forgo the formulation (1.31), and to rather start with a regularised equation, and try to remove the regularisation. Both the noise or the interaction can be regularised, by convolution with a smooth function called mollifier, or with a Fourier cut-off making the number of Fourier modes undergoing a non-linear evolution finite. For the moment, it is not necessary to specify what regularisation is adopted, but we will assume that we have regularised the noise, substituting it with a smooth noise ζ_ϵ converging to ζ as $\epsilon \downarrow 0$ in $\mathcal{C}^{-2-\kappa}$ for some $\kappa > 0$ arbitrarily small. We also denote by u_ϵ the solution to (1.31) with ζ replaced by ζ_ϵ .

The question now boils down to finding an interpretation of the non-linear term u_ϵ^3 uniformly in ϵ . The crucial observation made by Bourgain in [Bou94] in the context of dispersive equations, and then by Da Prato and Debussche [DPD03] in the present context, is that if u_ϵ simply were the solution to the linear equation (which can be taken stationary in law equal to the GFF), then in view of the discussion taking place in Section 1.1.2.3, modulo renormalisation, we would have a definition of u_ϵ^3 given by Wick ordering. There idea is then to express the solution u_ϵ as

$$u_\epsilon = \mathfrak{I}_\epsilon + v_\epsilon, \text{ with } \mathfrak{I}_\epsilon \stackrel{\text{def}}{=} \underline{\mathcal{L}}^{-1}\zeta_\epsilon, \quad (1.32)$$

and to consider the equation for the remainder v_ϵ . Here, $\underline{\mathcal{L}}^{-1}$ denotes the inverse massive heat operator with null initial condition at $t = -\infty$. Note that this choice ensures that \mathfrak{I}_ϵ is stationary in law. Indeed,

$$\mathfrak{I}_\epsilon(t, x) = \int_{-\infty}^t e^{-(t-s)(1-\Delta)} \xi_\epsilon(s, x) ds,$$

which implies that letting $\mathfrak{I} = \lim_{\epsilon \downarrow 0} \mathfrak{I}_\epsilon$, one has

$$\mathbf{E}[\mathfrak{I}(t, x)\mathfrak{I}(s, y)] = \frac{e^{-|t-s|(1-\Delta)}}{(1-\Delta)}(x, y).$$

The equation for v_ϵ reads

$$(\partial_t + 1 - \Delta)v_\epsilon = -v_\epsilon^3 - 3\mathfrak{I}_\epsilon v_\epsilon^2 - 3\mathfrak{I}_\epsilon^2 v_\epsilon - \mathfrak{I}_\epsilon^3.$$

In the above equation, the products \mathfrak{I}_ϵ^2 and \mathfrak{I}_ϵ^3 are ill-posed, but they now are explicit and polynomial in the noise. This implies that, provided the equation from which one started actually were

$$(\partial_t + 1 - \Delta)u_\epsilon = -u_\epsilon^3 + 3\mathbf{E}[\mathfrak{I}_\epsilon^2]u_\epsilon + \zeta_\epsilon. \quad (1.33)$$

instead of (1.31) driven by ζ_ϵ , then performing the substitution (1.32) one would end up with

$$(\partial_t + 1 - \Delta)v_\epsilon = -v_\epsilon^3 - 3\mathfrak{I}_\epsilon v_\epsilon^2 - 3\mathfrak{V}_\epsilon v_\epsilon - \mathfrak{W}_\epsilon, \quad (1.34)$$

where

$$\mathfrak{V}_\epsilon \stackrel{\text{def}}{=} \mathfrak{I}_\epsilon^2 - \mathbf{E}[\mathfrak{I}_\epsilon^2], \text{ and } \mathfrak{W}_\epsilon \stackrel{\text{def}}{=} \mathfrak{I}_\epsilon^3 - 3\mathbf{E}[\mathfrak{I}_\epsilon^2]\mathfrak{I}_\epsilon.$$

With this choice of renormalisation, we have the following.

Lemma 1.47 *Fix $\kappa > 0$. There exist random fields $\mathfrak{V}, \mathfrak{W}$ such that $\mathfrak{V}_\epsilon, \mathfrak{W}_\epsilon$ converge in probability to $\mathfrak{V}, \mathfrak{W}$ in $C_T \mathcal{C}^{-2\kappa} \times C_T \mathcal{C}^{-3\kappa}$ as $\epsilon \downarrow 0$.*

Remark 1.48 The convergence actually takes place in $L^p(\mathbf{P})$, and the proof relies on some moment computations that are similar to the proof of Lemma 1.15. Throughout the rest of this chapter, we will not prove the stochastic estimates. However, let just mention that they can always be proved by means of a reduction to the evaluation of the covariance using Gaussian hypercontractivity, Sobolev embeddings, and a classical Kolmogorov criterion in time. Covariance computations become more involved than in the proof of Lemma 1.15, but still rely on a decomposition in Gaussian chaoses, and power counting.

With this Lemma in hand, one can conclude that the dynamical Φ^4 equation has local-in-time solutions.

Theorem 1.49 Fix $\kappa > 0$ and $\phi \in \mathcal{C}^{2-\kappa}(\mathbf{T}^2)$. Then there exists a random $T^* > 0$ such that uniformly in $\epsilon > 0$, (1.34) with initial condition ϕ admits a unique maximal solution on $C([0, T^*), \mathcal{C}^{2-\kappa}(\mathbf{T}^2))$, such that $\lim_{t \uparrow T^*} \|v_\epsilon(t)\|_{\mathcal{C}^{2-\kappa}} = \infty$. For every $T < T^*$, the solution on $[0, T]$ depends continuously on ϕ and on the restriction of $\mathcal{V}_\epsilon, \mathcal{V}_\epsilon^\circ$ to $[0, T]$.

The proof relies on a fixed point argument, and makes use of the smoothing effect of the heat kernel, which implies that the inverse heat operator increases the regularity by two. For completeness, we state a classical Schauder estimate about the effect of convolution with the heat kernel.

Lemma 1.50 (Schauder estimate) Fix $d \geq 1$, $\alpha \in \mathbf{R}$, $\beta > 0$, and a smooth function u on \mathbf{T}^d . We have

$$\|e^{t\Delta}u\|_{\mathcal{C}^{\alpha+\beta}} \lesssim t^{-\beta/2}\|u\|_{\mathcal{C}^\alpha}.$$

1.3.2 Paracontrolled calculus

We saw above that the Da Prato-Debussche trick allows to take care of additive stochastic forcings by shifting the solution of the equation. It turns out that even for the sake of stochastic quantisation, if dealing with additive noises is sufficient in $d = 2$, it is necessary starting from $d = 3$ to have in hand a solution theory for equations with multiplicative noise. Indeed, in $d = 3$, the object \mathcal{V} turns out to be of regularity $-1 - \kappa$, which rules out any hope to make sense of the product $\mathcal{V}_\epsilon v_\epsilon$ appearing on the r.h.s. of (1.34) uniformly in ϵ . This is due to the fact that, by Young's estimate for product (see Lemma 1.14), if the product $\mathcal{V}_\epsilon v_\epsilon$ were well-defined, then it would be of regularity $-1 - \kappa$, and by Schauder estimates, v_ϵ would thus be of regularity at most $1 - \kappa$, which in turn would hinder the classical definition of the product of v_ϵ with \mathcal{V}_ϵ .

Remark 1.51 We see that the two UV problems that we identified in Remark 1.27 when discussing the construction of Φ_2^4 with the renormalisation group conspire one against the other. The strong UV problem (UV divergences/singularity) makes it impossible to define the product $\mathcal{V}_\epsilon v_\epsilon$, while the weak UV problem (UV convergences/bad regularity) makes this product too rough, in the sense that we do not benefit from the fact that the rough object \mathcal{V}_ϵ is multiplied by v_ϵ which is more regular. Solution theories to singular SPDEs must therefore face two challenges: the strong UV problem, or making sense of ill-defined products (modulo some renormalisation), and the weak UV problem, or exhibiting some improvement coming from multiplications with smoother terms.

The idea behind paracontrolled calculus is to separate the two difficulties by splitting the product of two distributions into one piece which is rough but not singular and one other piece which is more regular but rough. A way of performing this splitting is by using Fourier analysis.

Lemma 1.52 Let $(\Delta_i)_{i \geq -1}$ denote Littlewood-Paley blocks on \mathbf{T}^d (that is to say $\Delta_i = \chi_i(\sqrt{-\Delta})$ where χ_i is smooth and supported on an annulus of size 2^i for $i \geq 0$ and a ball of size 1 for $i = -1$), and define for $f, g \in C^\infty(\mathbf{T}^d)$

$$f \prec g \stackrel{\text{def}}{=} \sum_{i > j} \Delta_j f \Delta_i g$$

the paraproduct of g with f . Define accordingly $f \succeq g \stackrel{\text{def}}{=} fg - f \prec g$. Then, for all $\alpha, \beta \in \mathbf{R} \setminus \mathbf{N}$, it holds

$$\|f \prec g\|_{\mathcal{C}^{\beta+0 \wedge \alpha}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}.$$

Moreover, provided $\alpha + \beta > 0$, we have

$$\|f \succeq g\|_{\mathcal{C}^{\beta+\alpha}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}.$$

Remark 1.53 The paraproducts precisely disentangle the strong and weak UV problems, in the sense that putting a UV cut-off on the better-behaved function f in $f \prec g$ suppresses the singularity of the product, while putting an IR cut-off on the better-behaved function f in $f \succeq g$ screens the UV convergence/IR divergence of f , so that one can benefit from the regularizing effect coming from the multiplication with f .

The idea of paracontrolled calculus is to treat the rough term $f \prec g$ as an additive noise, and to add it to the ansatz for the solution, while showing independently that the singular term $f \succeq g$ can be made sense of with renormalisation.

Going back to the case of the Φ^4 Langevin dynamic in $d = 3$, we can rewrite (1.34) as

$$(\partial_t + 1 - \Delta)v_\epsilon = -\wp_\epsilon - 3v_\epsilon \prec \wp_\epsilon - 3v_\epsilon \succeq \wp_\epsilon + R_\epsilon,$$

where R_ϵ is a better behaved term, which we do not explicit for clarity. In general, during the following discussion, we will systematically gather under the notation R_ϵ some better terms that we do not want to specify, so that R_ϵ is free to change from line to line.

In the sequel, we let \mathcal{L}^{-1} denote the fundamental solution to the massive heat equation with null initial condition at $t = 0$. We first deal with the term $-\wp_\epsilon$ by means of a second Da Prato-Debussche trick, and write $w_\epsilon \stackrel{\text{def}}{=} v_\epsilon + \wp_\epsilon$, where $\wp_\epsilon \stackrel{\text{def}}{=} \mathcal{L}^{-1}\wp_\epsilon$.

Treating the paraproduct as an additive noise amounts to introducing $X_\epsilon \stackrel{\text{def}}{=} \mathcal{L}^{-1}(-3w_\epsilon \prec \wp_\epsilon)$. When this is done, we focus on the equation for $Y_\epsilon \stackrel{\text{def}}{=} w_\epsilon - X_\epsilon$ that reads

$$\begin{aligned} (\partial_t + 1 - \Delta)Y_\epsilon &= -3\mathcal{L}^{-1}(-3w_\epsilon \prec \wp_\epsilon) \succeq \wp_\epsilon - 3Y_\epsilon \succeq \wp_\epsilon + R_\epsilon \\ &= -3\mathcal{L}^{-1}(-3w_\epsilon \prec \wp_\epsilon) \odot \wp_\epsilon + R_\epsilon, \end{aligned} \tag{1.35}$$

where $f \odot g \stackrel{\text{def}}{=} f \succeq g - f \succ g$. Such a choice of variables is called a *paracontrolled ansatz*.

The renormalisation is implemented by means of the following commutator lemma.

Lemma 1.54 Fix f, g, h three smooth functions on \mathbf{T}^d , and define a trilinear map C as

$$C(f, g, h) \stackrel{\text{def}}{=} \mathcal{L}^{-1}(f \prec g) \odot h - f(\mathcal{L}^{-1}g \odot h).$$

Then, for $\alpha \in (0, 1)$, $\beta + \gamma + 2 < 0$, and $\alpha + \beta + \gamma + 2 > 0$, we have the estimate

$$\|C(f, g, h)\|_{\mathcal{C}^{\alpha+\beta+\gamma+2}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma}.$$

With this lemma in hand, (1.35) can be recast as

$$(\partial_t + 1 - \Delta)Y_\epsilon = 9w_\epsilon(\wp_\epsilon \odot \wp_\epsilon) + R_\epsilon,$$

where $\wp \stackrel{\text{def}}{=} \mathcal{L}^{-1}\wp$. Note that the singular term is now still singular, but we rewrote it as an explicit polynomial in the noise, so that there is hope that it can be defined in the same way we defined \wp, \wp^2 , but modulo a different renormalisation.

Lemma 1.55 Fix $\kappa > 0$. There exists a random fields \wp such that $\wp_\epsilon \odot \wp_\epsilon - \mathbf{E}[\wp_\epsilon \odot \wp_\epsilon]$ converges in probability to \wp in $C_T C^{-4\kappa}$ as $\epsilon \downarrow 0$.

The proof is similar to that of Lemma [1.15](#), and relies on Kolmogorov estimate combined with a covariance computation.

Remark 1.56 More trees are necessary to take care of R_ϵ , but they can all be constructed using the second renormalisation counterterm $\mathbf{E}[\mathfrak{V}_\epsilon \odot \mathfrak{V}_\epsilon]$. In general, the further in the subcritical domain, the more trees are necessary to make sense of the non-linearity. These polynomials in the noise are also known as *enhancement of the noise*. While theoretically they can all be constructed by some covariance computations, it becomes difficult to keep precisely track of the estimates when the trees grow big.

Thanks to the paracontrolled ansatz that we described, it is now possible to make sense of the Φ^4 Langevin dynamic in $d = 3$ uniformly in the regularisation. Moreover, the second renormalisation that we introduced in Lemma [1.55](#) implies that rather than [\(1.33\)](#), in $d = 3$, the starting equation should really be

$$(\partial_t + 1 - \Delta)u_\epsilon = -u_\epsilon^3 + (3\mathbf{E}[\mathfrak{I}_\epsilon^2] - 9\mathbf{E}[\mathfrak{V}_\epsilon \odot \mathfrak{V}_\epsilon])u_\epsilon + \zeta_\epsilon. \quad (1.36)$$

By means of a paracontrolled ansatz, it is therefore possible to show the following.

Theorem 1.57 Fix $\kappa > 0$ and $\phi \in \mathcal{C}^{3/2-\kappa}(\mathbf{T}^3)$. Then, uniformly in $\epsilon > 0$, solutions to [\(1.36\)](#) with initial condition $\mathfrak{I}_\epsilon(0) + \phi$ can be constructed as

$$u_\epsilon = \mathfrak{I}_\epsilon - \mathfrak{V}_\epsilon + X_\epsilon + Y_\epsilon,$$

where (X_ϵ, Y_ϵ) solve a coupled system of (random) PDEs of the form

$$\begin{cases} (\partial_t + 1 - \Delta)X_\epsilon = -3(X_\epsilon + Y_\epsilon) \prec \mathfrak{V}_\epsilon, \\ (\partial_t + 1 - \Delta)Y_\epsilon = 9(X_\epsilon + Y_\epsilon) \mathfrak{R} + R_\epsilon, \end{cases}$$

and there exists a random time $T^* > 0$ such that uniformly in $\epsilon > 0$, the system solved by (X_ϵ, Y_ϵ) admits a unique maximal solution on $C([0, T^*), \mathcal{C}^{1-\kappa}(\mathbf{T}^3)) \times C([0, T^*), \mathcal{C}^{1+\kappa}(\mathbf{T}^3))$.

Remark 1.58 Beyond this local-in-time result, paracontrolled calculus is well-suited to the study of global properties of solutions to singular SPDEs. The first proof of the “coming down from infinity” property (an a priori estimate uniform in the initial condition) was indeed given in [\[MW17a\]](#), combining a paracontrolled ansatz with some energy estimates. In [\[GH19\]](#), the authors use the maximum principle to slightly improve this result. Let us also point out the work [\[HR23\]](#) where paracontrol and energy estimates are used to show some a priori estimates on the solutions to the stochastic Navier-Stokes equation in $d = 2$.

Remark 1.59 To conclude this section about paracontrolled calculus, let us mention that the range of application of the paracontrolled ansatz is wider than singular SPDEs, and applies to other singular problems, like the variational method for Φ_3^4 . Indeed, combining the variational representation [\(1.23\)](#) with a paracontrolled ansatz, Barashkov and Gubinelli were able to construct the Φ_3^4 measure in [\[BG20\]](#), and with the same technique they gave a proof that it is singular with respect to the GFF in [\[BG21\]](#).

1.3.3 Regularity structures

Let us now briefly introduce the main ideas of the theory of regularity structures. Our intention is definitely not to explain in great detail the theory, but to present some of the main ingredients, in connection with the questions and issues arising in the study of constructive QFT.

When discussing the paracontrolled approach used to deal with singular equations involving a multiplicative noise, we saw that the ansatz used to take into account an ill-defined product splits the original equation into two equations. Repeating the operation to handle more singular problems makes the analytic theory of paracontrolled calculus less suitable to situations where the singularity is stronger.

Fortunately, in 2014 Hairer introduced in [Hai14] a general framework allowing to renormalise any singular semi-linear parabolic SPDE. To present it, let us first go back to the discussion of the split of the product of a function f with a distribution g into a rough part inheriting the worst regularity, but which is not singular, and a singular part which captures the good regularity of f . We first performed such a decomposition in Fourier space, using a paraproduct decomposition. The analytic theory underpinning the theory of regularity structures is based on an equivalent decomposition, but taking place in direct space. Indeed, observe that for any *active point* y and *base point* x one has

$$f(y)g(y) = (f(y) - f(x))g(y) + f(x)g(y).$$

Here, the second term is now well-posed, since it actually no longer involves any product, and of course inherits the regularity of g . On the other hand, one can expect that the first term, while being still singular, is now better behaved (at least *locally*), since assuming that f is α -Hölder continuous, the space increment of f vanishes as y becomes close to x .

Note that in this case the weak UV problem is solved by expanding the more regular function f *locally*, so as to only see its short-scale behaviour, and so that in particular its IR divergences are screened.

We deduce from the above discussion that the hope is that close to a fixed base point x , $v_\epsilon(y) - v_\epsilon(x)$ solves a better equation than v_ϵ . Starting again from (1.34) (for simplicity, we just ignore the mass term), we have

$$(\partial_t - \Delta)(v_\epsilon - v_\epsilon(x)) = -3(v_\epsilon - v_\epsilon(x))^{\heartsuit_\epsilon} + 3v_\epsilon(x)^{\heartsuit_\epsilon} + Q_\epsilon,$$

where again, as in the previous section, we gather all the other terms of the r.h.s. in the inexplicit remainder Q_ϵ . Note that importantly, we were able to introduce by hand the term $v_\epsilon(x)$ on the l.h.s.. In the spirit of the Da Prato-Debussche trick, we again treat the new term $3v_\epsilon(x)^{\heartsuit_\epsilon}$ on the r.h.s. as an additive noise which we add to our ansatz, which yields the equation

$$\begin{aligned} (\partial_t + 1 - \Delta)(v_\epsilon - v_\epsilon(x) - 3v_\epsilon(x)(\heartsuit_\epsilon - \heartsuit_\epsilon(x))) \\ = -3(v_\epsilon - v_\epsilon(x) - 3v_\epsilon(x)(\heartsuit_\epsilon - \heartsuit_\epsilon(x)))^{\heartsuit_\epsilon} + 9v_\epsilon(x)(\heartsuit_\epsilon - \heartsuit_\epsilon(x))^{\heartsuit_\epsilon} + Q_\epsilon, \end{aligned}$$

where

$$\heartsuit_\epsilon \stackrel{\text{def}}{=} \mathcal{L}^{-1}\heartsuit_\epsilon,$$

and we took the freedom to *recenter* it by hand, replacing the occurrences of \heartsuit_ϵ by $\Pi_x^\epsilon \heartsuit_\epsilon \stackrel{\text{def}}{=} \heartsuit_\epsilon - \heartsuit_\epsilon(x)$. The reason for this lies in the fact that recentering \heartsuit_ϵ (which is of positive regularity) removes its large scale behaviour, and allows one to fully benefit from its small scale properties. In particular, we have the following counterpart of Lemma 1.55.

Lemma 1.60 *Define*

$$\Pi_x^\epsilon \heartsuit_\epsilon \stackrel{\text{def}}{=} \heartsuit_\epsilon \Pi_x^\epsilon \heartsuit_\epsilon - \mathbf{E}[\heartsuit_\epsilon \heartsuit_\epsilon].$$

Fix a smooth compactly supported function φ on \mathbf{T}^3 and $\lambda \in (0, 1]$, we let $\varphi_x^\lambda \stackrel{\text{def}}{=} \lambda^{-3}\varphi((\cdot - x)/\lambda)$. Fix a small $\kappa > 0$. Then, it holds

$$\langle \Pi_x^\epsilon \heartsuit_\epsilon, \varphi_x^\lambda \rangle \lesssim \lambda^{-\kappa} \tag{1.38}$$

uniformly in $x \in \mathbf{T}^3$, $\lambda \in (0, 1]$, and $\epsilon > 0$. Similarly, there exists a limiting object $\Pi_x \heartsuit_\epsilon$ such that $\Pi_x^\epsilon \heartsuit_\epsilon$ converges in probability to $\Pi_x \heartsuit_\epsilon$ as $\epsilon \downarrow 0$ in the topology induced by (1.38) (which is equivalent to that of $\mathcal{C}^{-\kappa}$ in a neighbourhood of x).

Remark 1.61 Again, one is able to construct some polynomials of the noise having the desired regularity (that is to say the one we infer from the object by power counting). While this was realised by means of a paraproduct decomposition in the paracontrolled context, here, this is the recentering that allows for such a result. The price to pay is that the object now exists only locally, close to a base point. In his work, Hairer calls such a localised enhancement of the noise a *model*. Note that Hairer's model actually also comes with enough information to be able to perform a change of base point.

Regarding the proof, again the result is obtained with a Kolmogorov argument, and some covariance computation. The key step is to read the effect of recentering on kernels, which we present in the following lemma.

Lemma 1.62 ([Hai14][Lemma 10.14]) *Let K_1, K_2 be two compactly supported translation-invariant convolution kernels on $\mathbf{R} \times \mathbf{T}^d$, verifying for some $\alpha, \beta \in (0, d+2)$ such that $\alpha + \beta - (d+2) \in \mathbf{R}_{\geq 0} \setminus \mathbf{N}$*

$$|K_1(x-y)| \lesssim |x-y|^{-(d+2)+\alpha}, \text{ and } |K_2(x-y)| \lesssim |x-y|^{-(d+2)+\beta},$$

along with similar bounds on their derivatives. Then the kernel K defined as

$$K(x-y) \stackrel{\text{def}}{=} K_1 * K_2(x-y) - \sum_{k < \alpha + \beta - (d+2)} \frac{(x-y)^k}{k!} \partial^k (K_1 * K_2)(0)$$

verifies

$$|K(x-y)| \lesssim |x-y|^{-(d+2)+\alpha+\beta},$$

along with similar bounds on its derivatives.

Gathering all the necessary polynomials in the noise/the model yields an abstract algebraic structure that Hairer calls the *regularity structure*. We informally define this concept based on the example of the Φ_3^4 Langevin dynamic.

Definition 1.63 A regularity structure $\mathcal{T} = (A, T, G)$ is a triplet consisting of

- a finite subset A of \mathbf{R} of regularity assignments of elements of the model, such that $0 \in A$.
- the span T of all elements of the model. We have $T = \bigoplus_{\alpha \in A} T_\alpha$ where T_α is the span of all elements with regularity α , and $T_0 = \text{span}(\mathbf{1})$. Denote by $\|\cdot\|_{T_\alpha}$ any norm on T_α .
- the *structure group* G which is the group of those linear operators $G \ni \Gamma : T \rightarrow T$ that are such that $(\Gamma - \text{Id})T_\alpha \subset T_{<\alpha} \stackrel{\text{def}}{=} \bigoplus_{\beta < \alpha} T_\beta$. We have $\Gamma \mathbf{1} = \mathbf{1}$.

In this context, the model $M = (\Gamma, Z)$ is a collection of

- maps $\Gamma_{xy} \in G$ indexed by points $x, y \in \mathbf{R}_{\geq 0} \times \mathbf{T}^d$ encoding the changes of base point, that verify $\Gamma_{xx} = \text{Id}$ and $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$.
- maps $\Pi_x : T \rightarrow \mathcal{S}'(\mathbf{R}_{\geq 0} \times \mathbf{T}^d)$ indexed by points $x \in \mathbf{R}_{\geq 0} \times \mathbf{T}^d$ corresponding to the realisation of the abstract element of the regularity structure, that verify $\Pi_y = \Pi_x \circ \Gamma_{xy}$ for every $x, y \in \mathbf{R}_{\geq 0} \times \mathbf{T}^d$.

The model comes with the topology suggested by (1.38), in the sense that we demand that

$$|\langle \Pi_x \tau, \varphi_x^\lambda \rangle| \lesssim \lambda^\alpha \|\tau\|_{T_\alpha}, \text{ for every } \tau \in T_\alpha,$$

where φ_x^λ is as in Lemma 1.60. Finally, the maps Γ_{xy} must verify

$$\|\Gamma_{xy} \tau\|_{T_\beta} \lesssim |x-y|^{\alpha-\beta} \|\tau\|_{T_\alpha} \text{ for every } \tau \in T_\alpha.$$

Remark 1.64 In the above Definition, and from now on, the distance $|x-y|$ is the parabolic distance, that is to say that, writing $x = (x_0, \mathbf{x}) \in \mathbf{R}_{\geq 0} \times \mathbf{T}^d$ and $y = (y_0, \mathbf{y}) \in \mathbf{R}_{\geq 0} \times \mathbf{T}^d$, $|x-y| \stackrel{\text{def}}{=} \sqrt{|x_0 - y_0|} + |\mathbf{x} - \mathbf{y}|$.

We can now consider the model as a given data of the problem, and turn back to the setting of the analytic framework. We saw that replacing the original unknown v_ϵ by some local expansion indexed by some stochastic objects yields a better equation. More precisely, taking the constant term $-\mathfrak{O}_\epsilon$ into account, we established an ansatz for v_ϵ taking the form of local expansion around the base point x

$$v_\epsilon = v_\epsilon(x) \Pi_x^\epsilon \mathbf{1} - \Pi_x^\epsilon \mathfrak{O}_\epsilon - 3v_\epsilon(x) \Pi_x^\epsilon \mathfrak{Y} + \dots \quad (1.39)$$

where $\Pi_x^\epsilon \mathbf{1} = 1$ and again $\Pi_x^\epsilon \circledast \circledast \stackrel{\text{def}}{=} \mathcal{L}^{-1} \circledast \circledast_\epsilon - \mathcal{L}^{-1} \circledast \circledast_\epsilon(x)$. Such an expansion tells us that the solution can be *modelled* by the enhancement of the noise, in a similar way that a Hölder-continuous function can be modelled by polynomials. Hairer therefore calls such an ansatz a *modelled distribution*, and his idea is to lift the fixed point problem to an abstract space of modelled distributions taking values in the regularity structure.

Definition 1.65 Fix $T > 0$. Given a regularity structure $\mathcal{T} = (A, T, G)$ and a model $M = (\Gamma, \Pi)$, the space \mathcal{D}^γ of modelled distributions of order $\gamma \in \mathbf{R}$ is the space of those functions $[0, T] \times \mathbf{T}^d \rightarrow T_{<\gamma}$ that are such that

$$\sup_{x \in [0, T] \times \mathbf{T}^d} \sup_{\beta < \gamma} \|f(x)\|_{T_\beta} + \sup_{x, y \in [0, T] \times \mathbf{T}^d} \sup_{\beta < \gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_{T_\beta}}{|x - y|^{\gamma - \beta}}$$

is finite.

Remark 1.66 The second condition for f to lie in \mathcal{D}^γ can be read as a *base point continuity*. Indeed, it says that the regularity of the coefficient f_τ in front of some object τ in the expression of the modelled distribution f must "compensate" the lack of regularity of τ , in the sense that if the realisations of τ are only of regularity β , then f_τ has to be of regularity $\gamma - \beta$. It matches with the intuition that we can draw from the case of C^1 functions: a function is C^1 if it is modelled close to x by polynomials of degree one centered in x and if the coefficient in front of X^k is of regularity $1 - k$ in the base point x for $k \in \{0, 1\}$. To rephrase, a function verifying for every x in a domain and every y close to x

$$f(y) \approx f_0(x) + f_1(x)(y - x)$$

is continuously differentiable only if f_0 is differentiable, and if f_1 is continuous, in which case of course it holds $f_k = f^{(k)}/k!$.

Solving the Langevin dynamic for Φ^4 in $d = 3$ now amounts to finding the correct order γ up to which one should model the solution. It turns out that the description (1.39) (for which $\gamma = 1 - \kappa$) is still insufficient. Indeed, \mathcal{V}_ϵ being of regularity $-1 - \kappa$, it is necessary to go beyond one in the modelling to make sense of the product. We reexpress this observation in the following lemma – see [Hai14, Theorem 4.7].

Lemma 1.67 Fix $\alpha_1, \alpha_2 \leq 0$, and $\gamma_1, \gamma_2 \geq 0$. Suppose that $\mathcal{T} = (A, T, G)$ is such that for $i \in [2]$, $T_{\alpha_i}^{\gamma_i} \stackrel{\text{def}}{=} \bigoplus_{\iota=\alpha_i}^{\gamma_i}$ is stable by the action of G . Denote for $i \in [2]$ by $\mathcal{D}_{\alpha_i}^{\gamma_i}$ the set of those modelled distributions of order γ_i that take values in $T_{\alpha_i}^{\gamma_i}$. Furthermore, suppose that for every $\delta_i \in [\alpha_i, \gamma_i]$ such that $\delta_1 + \delta_2 < \gamma$, $\tau \in T_{\delta_1}, \sigma \in T_{\delta_2}$ and $\Gamma \in G$, it holds $\Gamma(\tau \cdot \sigma) = \Gamma\tau \cdot \Gamma\sigma$, where \cdot stands for the product of two elements of T which is defined whenever one can construct via renormalisation the associated random field. Then, for any $f_1 \in \mathcal{D}_{\alpha_1}^{\gamma_1}, f_2 \in \mathcal{D}_{\alpha_2}^{\gamma_2}$ the product f_1, f_2 is a well-defined modelled distribution of order $\gamma \stackrel{\text{def}}{=} (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)$.

Therefore, provided v_ϵ is modelled up to order $1 + \kappa$ for some $\kappa > 0$, then the r.h.s. of the equation can be modelled to a slightly positive order. The importance of this fact will become manifest when we discuss the reconstruction. But before we do so, let us conclude about the modelled distribution for v_ϵ . When modelling v_ϵ , one might want to implement the fact that we extract some local terms by Taylor expanding. Therefore, some polynomials can appear in the model. In particular, in order to lift v_ϵ to $\mathcal{D}^{1+\kappa}$ for some $\kappa > 0$, we must add to T an object \mathbf{X} of regularity assignment 1 which is such that $\Pi_x^\epsilon \mathbf{X}(y) = y - x$. With this new object in hand, we can introduce the modelled distribution

$$\mathcal{V}_\epsilon(x) = v_\epsilon(x)\mathbf{1} - \circledast - 3v_\epsilon(x)\circledast + \nu_\epsilon(x)\mathbf{X},$$

where ν_ϵ is a new C^κ function a priori unrelated to v_ϵ . Now that the equation is lifted to a fixed point problem at the level of modelled distributions, it is possible to solve this fixed point problem in an

appropriate space \mathcal{D}^γ , and ultimately to construct the modelled distribution \mathcal{V}_ϵ or, equivalently, the pair $(v_\epsilon, \nu_\epsilon)$.

It remains to verify that the knowledge of the modelled distribution is sufficient to make sure that a solution actually exists. The answer to this question is provided by the *reconstruction theorem* stated in [Hai14, Theorem 3.10].

Lemma 1.68 (Reconstruction theorem) *Fix $T > 0$, and a regularity structure $\mathcal{T} = (A, T, G)$, a model $M = (\Gamma, \Pi)$ on \mathcal{T} , and let $\alpha = \min A$. For any $\gamma > 0$, there exists a unique linear map $\mathcal{R} : \mathcal{D}^\gamma \rightarrow \mathcal{C}^\alpha([0, T] \times \mathbf{T}^d)$ such that for every $f \in \mathcal{D}^\gamma$ it holds*

$$|\langle \mathcal{R}f - \Pi_x f(x), \varphi_x^\lambda \rangle| \lesssim \lambda^\gamma, \quad (1.40)$$

uniformly in $\lambda \in (0, 1]$, $x \in [0, T] \times \mathbf{T}^d$, and functions φ as in Lemma 1.60.

Remark 1.69 The reconstruction theorem is the reason why, in order to make sense of the product $v_\epsilon \mathfrak{V}_\epsilon$, one needs v_ϵ to be modelled up to order $1 + \kappa$. Indeed, if this is not the case, then $v_\epsilon \mathfrak{V}_\epsilon$ fails to be modelled up to a positive order, and the reconstruction does not hold (there is actually a similar statement in the context of γ negative, but the reconstruction is no longer unique).

Remark 1.70 As long as one still works with the smooth noise ζ_ϵ , it is sufficient to take $\mathcal{R}f : x \mapsto (\Pi_x^\epsilon f(x))(x)$. In particular, in our case, one has $\mathcal{R}\mathcal{V}_\epsilon = v_\epsilon$ so that the bound (1.40) translates into a local representation of the solution close to x

$$v_\epsilon(y) = v_\epsilon(x) - (\mathfrak{V}_\epsilon^\circ(y) - \mathfrak{V}_\epsilon^\circ(x)) - 3v_\epsilon(x)(\mathfrak{V}_\epsilon^\circ(y) - \mathfrak{V}_\epsilon^\circ(x)) + \nu_\epsilon(x)(y - x) + \mathfrak{o}((y - x)^{1+\kappa}). \quad (1.41)$$

Applying ∇_y to the above local representation and evaluating at $y = x$ therefore yields

$$\nu_\epsilon(x) = \nabla(v_\epsilon + \mathfrak{V}_\epsilon^\circ + 3v_\epsilon(x)\mathfrak{V}_\epsilon^\circ)(x),$$

from which we can conclude that ν_ϵ is actually related to v_ϵ but not in a naive way (in the sense that it does not hold $\nu_\epsilon = \nabla v_\epsilon$), and that ν_ϵ should rather be interpreted as a *renormalised gradient* of v_ϵ .

Applying the reconstruction operator to the fixed point problem for the modelled distribution \mathcal{V}_ϵ , we can therefore return to an equation for v_ϵ , and ultimately for u_ϵ . It turns out that, due to the fact that the trees appearing in the local description (1.41) need some renormalisation, the equation solved by v_ϵ keeps track of the necessary counterterms. In particular, the solution v_ϵ also depends on the choices made to fix the freedom we have in defining the renormalised trees, as was described at the beginning of Section 1.1.2.3 in the case of the iterated integral of the Brownian motion. This freedom is known in the theory of regularity structures as the *renormalisation group*. It is a different concept from the notions of renormalisation group flow (be it discrete or continuous), though in the subcritical case, its dimension matches the dimension of the relevant sector of the RG flow.

In the case of the Langevin dynamic for Φ_3^4 , the renormalised equation which is solved by v_ϵ thus includes the necessary counterterms that we had to add at the level of modelled distributions to make sense of the non-linearity. It turns out that one can verify that this equation takes a form which is similar to (1.36), but with $\mathbf{E}[\mathfrak{V}_\epsilon^\circ \odot \mathfrak{V}_\epsilon^\circ]$ in front of the linear term replaced by $\mathbf{E}[\mathfrak{V}_\epsilon^\circ \mathfrak{V}_\epsilon^\circ]$ (both counterterms actually differ by a finite quantity). In this case, the renormalisation group thus acts by finite shift of the linear term (or renormalised mass). This matches with the fact that the Φ^4 measure is really a one-parameter family of measure.

We gather the main conclusion of the analysis in the following theorem, which was proved in [Hai14].

Theorem 1.71 *Fix $\kappa > 0$ and $\phi \in \mathcal{C}^{3/2-\kappa}(\mathbf{T}^3)$. Then, uniformly in $\epsilon > 0$, solutions to (1.36) with initial condition $\mathfrak{V}_\epsilon(0) + \phi$ can be constructed as*

$$u_\epsilon = \mathfrak{V}_\epsilon + v_\epsilon,$$

where v_ϵ solves a (random) PDE of the form

$$(\partial_t + 1 - \Delta)v_\epsilon = -\mathfrak{V}_\epsilon^\rho - 3\mathfrak{V}_\epsilon^\rho v_\epsilon - 3\mathfrak{I}_\epsilon^\rho v_\epsilon^2 - v_\epsilon^3 - 9\mathbf{E}[\mathfrak{Y}_\epsilon^\rho \mathfrak{V}_\epsilon^\rho](\mathfrak{I}_\epsilon + v_\epsilon),$$

and there exists a random time $T^* > 0$ such that uniformly in $\epsilon > 0$, the PDE solved by v_ϵ admits a unique maximal solution on $C([0, T^*), \mathcal{C}^{1/2-\kappa}(\mathbf{T}^3))$.

To conclude, the theory of regularity structures was extended to a black box to solve locally-in-time a wide class of parabolic singular SPDEs in [BHZ19, BCCH21] and, in [CH16], the authors established the convergence of the regularised model using multiscale analysis. A similar statement was also proved in [HS23b] using the spectral gap inequality.

Remark 1.72 Beyond this local-in-time statement, modelled distributions can also be used to establish global existence of the solutions to the dynamical Φ_3^4 model. A ‘‘coming down from infinity’’ result was indeed obtained in [MW20] using regularity structures. It was extended in [CMW19] to the full subcritical regime. In the recent work [CdLFW24], the authors have used similar ideas to show a priori bounds on solutions to the generalised parabolic Anderson model.

To conclude this presentation of regularity structures, and more generally of classical solution theories to singular parabolic PDE, let us observe that both regularity structures and paracontrolled calculus rely on a careful decomposition of the product of a rough distribution with a smoother function. It turns out that usual local products do not cover all the possible interactions that can be considered. In Chapter 2, we present a singular SPDE with a nonlocal interaction, to which both regularity structures and paracontrolled calculus do not apply. We therefore introduce a new solution theory based on a reinjection argument which is adapted to this nonlocal case.

1.3.4 The flow approach to singular SPDEs

To conclude, let us introduce a last approach to solving singular SPDEs, that finds its roots in the renormalisation group *à la* Polchinski which we described in Section 1.1.3.1. Again, the idea is to cure the weak UV problem due to the IR convergences by introducing an IR cut-off on top of the UV cut-off. This *flow approach* was introduced by Duch in [Duc21], where he applied it to some parabolic polynomial Langevin dynamic type equations, before extending it to the case of elliptic equations, still with polynomial interaction in [Duc23]. These works are themselves inspired by a series of papers of Kupiainen [Kup16] and coauthors, in which they set up a similar solution theory for singular SPDEs, but based on a discrete RG flow, and where the polynomials of the noise must be constructed by hand.

Concretely, the IR cut-off on the propagator is implemented at the level of the equation by introducing a second UV cut-off, on top of the usual regularisation. Still considering the example of the Φ^4 Langevin dynamic, say in $d = 3$, (1.31) with initial condition ϕ can be rewritten as

$$u_\epsilon = G(\mathbf{1}_0 F_\epsilon[u_\epsilon] + \delta_0 \otimes \phi), \quad (1.42)$$

where δ_0 is a Dirac distribution in time, $\mathbf{1}_0$ is the indicator function of positive times, $G \equiv \mathcal{L}^{-1}$ is the fundamental solution to the massive heat equation, $F_\epsilon[u_\epsilon] \stackrel{\text{def}}{=} -gu_\epsilon^3 + gC_\epsilon^1 u_\epsilon + gC_\epsilon^2 u_\epsilon + \xi_\epsilon$ and as usual ξ_ϵ is a smooth approximation of as space-time white-noise ξ , for instance using a mollifier. Here C_ϵ^1 and C_ϵ^2 are two counterterms present to take care of the two renormalisations that we saw in the previous sections, and we added a coupling constant g for convenience, which will eventually be put to one at the end of the analysis. In the sequel, we will often refer to F_ϵ as the *force* and, for the sake of clarity of the discussion, we will assume that the initial condition ϕ is equal to zero.

Definition 1.73 In order to construct a solution to (1.42), one introduces a second UV cut-off that makes that the propagator cancels on small times, defining $G_\mu(t, x) \stackrel{\text{def}}{=} \chi_\mu(t)G(t, x)$ where χ_μ is a smooth

increasing function supported outside $[0, \mu^2]$ and which is equal to one on $[2\mu^2, \infty)$. With this notation in hand, one defines for $\mu \geq 0$ the *effective solution* $u_{\varepsilon, \mu}$ as

$$u_{\varepsilon, \mu} \stackrel{\text{def}}{=} G_\mu(\mathbf{1}_0 F_\varepsilon[u_\varepsilon]) = G_\mu \tilde{F}_\varepsilon[u_\varepsilon], \quad (1.43)$$

where we also define $\tilde{F}_\varepsilon[u_\varepsilon] \stackrel{\text{def}}{=} \mathbf{1}_0 F_\varepsilon[u_\varepsilon]$. Then, one postulates the existence of an *effective force* $\tilde{F}_{\varepsilon, \mu}$ which is such that it holds

$$\tilde{F}_\varepsilon[u_\varepsilon] = \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}] \quad (1.44)$$

uniformly in μ .

Since the above equality holds for every $\mu \geq 0$, one has

$$0 = \frac{d}{d\mu}(\tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}]) = \partial_\mu \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}] + \partial_\Phi \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}] \partial_\mu u_{\varepsilon, \mu}.$$

Moreover, we can read from (1.43) that $\partial_\mu u_{\varepsilon, \mu} = \dot{G}_\mu \tilde{F}_\varepsilon[u_\varepsilon] = \dot{G}_\mu \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}]$, so that we have obtained the *flow equation*

$$\partial_\mu \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}] + \partial_\Phi \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}] \dot{G}_\mu \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}] = 0. \quad (1.45)$$

Finding a solution to (1.45) with initial condition $\tilde{F}_{\varepsilon, 0} = \mathbf{1}_0 F_\varepsilon$ and such that $\tilde{F}_{\varepsilon, \mu}[\psi](t, x)$ only depends on the values of ψ before time t is therefore sufficient to realise (1.44). But most importantly, this is also sufficient to solve the original equation, as is shown in the following lemma.

Lemma 1.74 For $T > 0$ let $\mu_T \stackrel{\text{def}}{=} \sqrt{T}$. Then (1.42) on $[0, T]$ with $\phi = 0$ is formally equivalent to

$$u_\varepsilon = G \tilde{F}_{\varepsilon, \mu_T}[0].$$

In other words, the knowledge of $\tilde{F}_{\varepsilon, \mu}$ up to scale μ_T is sufficient to reconstruct the solution u_ε .

Proof. By (1.44), one has $u_\varepsilon = G \tilde{F}_{\varepsilon, \mu_T}[u_{\varepsilon, \mu_T}]$. Moreover, by (1.43), $u_{\varepsilon, \mu_T} = G_{\mu_T}(\mathbf{1}_0 F_\varepsilon[u_\varepsilon])$. The support properties of χ_{μ_T} therefore imply that u_{ε, μ_T} is supported after time T . Finally, by assumption, the effective only depends on the past of its argument, so that $\tilde{F}_{\varepsilon, \mu_T}[u_{\varepsilon, \mu_T}](t, x) = \tilde{F}_{\varepsilon, \mu_T}[0](t, x)$ for every $t \leq T$. \square

Remark 1.75 The solution $\tilde{F}_{\varepsilon, \mu}$ to (1.45) with initial condition $\mathbf{1}_0 F_\varepsilon$ is not going to be stationary in time, and will also imply working with a time-dependent renormalisation. However, it was shown in [Duc21] that it is possible to construct $\tilde{F}_{\varepsilon, \mu}$ starting from the *stationary effective force* $F_{\varepsilon, \mu}$, the solution to (1.45) with initial condition F_ε .

Remark 1.76 A reader familiar with the Polchinski equation that we introduced in Section 1.1.3.1 will identify that the non-linearity of (1.45) is the same as the non-linearity of the equation obtained by taking the gradient of the Polchinski equation (but the linear term does not contain the functional laplacian). Solving (1.45) is therefore as difficult as solving (1.12). To solve this flow equation, it is thus necessary to make an ansatz, a good choice being an ansatz similar to the one used in order to solve the renormalisation group. In the case of (1.31), a reasonable assumption is to choose

$$F_{\varepsilon, \mu}[\psi](z) = \sum_{i \geq 0} \sum_{m \geq 0} g^i \int_{(\mathbf{T}^3 \times \mathbf{R})^m} F_{\varepsilon, \mu}^{i, m}(z, w_1, \dots, w_m) \psi(w_1) \dots \psi(w_m) dw_1 \dots dw_m.$$

Moreover, as was described in the context of forward-backward SDEs, it is also possible to only solve (1.45) approximately, introducing a remainder $R_{\varepsilon, \mu}$ such that

$$\tilde{F}_\varepsilon[u_\varepsilon] = \tilde{F}_{\varepsilon, \mu}[u_{\varepsilon, \mu}] + R_{\varepsilon, \mu}.$$

Now that one has an ansatz for the effective force, it remains to construct the effective *force coefficients* $F_{\varepsilon,\mu}^{i,m}$, that correspond to the enhancement of the noise in paracontrolled calculus/the model in regularity structures, in the sense that each of them is a polynomial in the noise.

A naive way of constructing them could therefore be to write their expression, and to combine a Kolmogorov estimate with a covariance computation. Indeed, their expressions can be inferred from the projection of (1.45) for $F_{\varepsilon,\mu}$ at the level of the force coefficients, which yields a hierarchy of flow equations reading

$$\partial_{\mu} F_{\varepsilon,\mu}^{i,m} = - \sum_{k=0}^m (k+1) \sum_{j=0}^i F_{\varepsilon,\mu}^{i-j,k+1} \dot{G}_{\mu} F_{\varepsilon,\mu}^{j,m-k} . \quad (1.46)$$

However, in [Duc21], Duch observed that their *cumulants* also obey a flow equation, which can be derived from (1.46). This observation paves the way for an inductive construction of the force coefficients, avoiding the tedious moment computations. Indeed, these flow equations for the cumulants are hierarchical in the size of the coefficients, and can therefore be solved by induction, starting from the covariance of the noise.

Chapter 2

Stochastic Quantisation of Tensor Field Theories

2.1 Introduction

As described in the previous chapter, a class of heavily studied and paradigmatic models in constructive quantum field theory are the Φ_d^4 measures. In finite volume, these are non-Gaussian probability measures μ supported on distributions (over space) which are formally written

$$d\mu(\phi) \propto \exp\left(-\frac{\lambda}{4}\|\phi\|_{L^4(\mathbf{T}^d)}^4 + \frac{a}{2}\|\phi\|_{L^2(\mathbf{T}^d)}^2\right)d\mathcal{g}(\phi). \quad (2.1)$$

where $\lambda > 0$ and \mathcal{g} denotes the Gaussian measure with covariance $(1 - \Delta)^{-1}$, that is a massive Gaussian free field on \mathbf{T}^d . When $d = 1$ it is straightforward to make (2.1) rigorous but much less so when $d > 1$ – in this case the measure \mathcal{g} is supported on distributions so rough that the nonlinear expressions $\|\phi\|_{L^4(\mathbf{T}^d)}^4$ and $\|\phi\|_{L^2(\mathbf{T}^d)}^2$ above are ill-defined.

When $d \in \{2, 3\}$ the measure μ can be obtained as a weak limit of regularized and appropriately renormalized measures [Ne173, GJ73, Fe174, FO76] – for instance, the $N \uparrow \infty$ limit of

$$d\mu_N(\phi) \propto \exp\left(-\frac{\lambda}{4}\|\phi\|_{L^4(\mathbf{T}^d)}^4 + \frac{a + a_N}{2}\|\phi\|_{L^2(\mathbf{T}^d)}^2\right)d\mathcal{g}_N(\phi). \quad (2.2)$$

where \mathcal{g}_N is the pushforward of \mathcal{g} under ϱ_N , which is a Fourier multiplier in space with $0 \leq \hat{\varrho} \leq 1$, $\hat{\varrho}_N(k) = 1$ for $|k| \leq N$ and $\hat{\varrho}_N(k) = 0$ for $|k| \geq 2N$.

The constant a_N in (2.2) is a renormalization constant diverging to infinity as $N \uparrow \infty$. One can identify the suitable choices of a_N using perturbation theory, that is formally calculating moments of μ_N by expanding the exponential on the right hand side of (2.2) as a formal series and using Wick's rule for Gaussian moments to integrate out ϕ term by term – one chooses a_N to make these expansions finite order by order in λ as $N \uparrow \infty$. However this series itself is far from convergent and proving μ_N weakly converges as $N \uparrow \infty$ requires more work.

In the case of $d = 2$, one can show that μ is absolutely continuous with respect to \mathcal{g} , the renormalization procedure is simply passing from ill-defined polynomials of a rough Gaussian field to the corresponding well-defined Hermite polynomials/Wick powers. The situation when $d = 3$ is much more complicated due to the presence of so-called “non-local divergences” – it has been shown in [BG21] that μ is singular with respect to \mathcal{g} . Even though a_N is chosen to cancel the divergences of infinitely many terms in the perturbation expansion, when $d = 2, 3$ one can write a_N explicitly as the expectation of a polynomial in ϕ , λ and a under \mathcal{g} . For this reason Φ_2^4 and Φ_3^4 are called *super-renormalizable*.

In the case where $d = 4$ the model is *just-renormalizable*, in particular perturbative renormalization requires including a diverging renormalization counterterm $\lambda_N \|\phi\|_{L^4(\mathbf{T}^4)}^4$ in the exponential in (2.2) along with a wave-function renormalization Z_N . Moreover, perturbation theory gives infinite series (in λ) for λ_N , a_N , and Z_N which do not all have positive radius of convergence. When $d > 4$ the model is *non-renormalizable*, and in addition to the above perturbative renormalization would require inserting infinitely many other counterterms (such as $\|\phi\|_{L^{2j}(\mathbf{T}^d)}^{2j}$) inside of the exponential factor in (2.2).

Turning from questions of perturbative renormalization to those of the convergence of measures, [Aiz82, Frö82] shows that one cannot obtain non-Gaussian limits when $d \geq 5$, while the case of $d = 4$ is more subtle and was only settled recently in [ADC21]. The analysis of renormalization problems in just-renormalizable models is quite subtle and there are other examples [GK86] where one can obtain non-Gaussian limits. An important step of one of the Millenium Problems is investigating whether one

can obtain a non-trivial limit for non-abelian Yang-Mills in 4-dimensions. A major frustration has been the lack of a simple scalar field theory to investigate the construction of a just-renormalizable model. However, a newer class of models, so-called “non-local models” of which tensor field theories are an example, are promising candidates for obtaining non-Gaussian limits in the just-renormalizable case.

2.1.1 Tensor field theories

The regularized, renormalized measures μ_N in (2.2) are non-Gaussian because of the quartic L^4 term. The models we study in this article are T_d^4 models, they are examples of *tensor field models* [GR12] and they are obtained by replacing this quartic L^4 term with a non-local quartic nonlinearity which we introduce now.

For $d \geq 2$ and any fixed $c \in [d] \stackrel{\text{def}}{=} \{1, \dots, d\}$, we define the function

$$\mathbf{T}^d \times \mathbf{T}^d \ni (x, y) \mapsto \chi^c(x, y) = (\chi^c(x, y)_i)_{i=1}^d \in \mathbf{T}^d$$

by setting, for each $1 \leq i \leq d$ and $x = (x_j)_{j=1}^d, y = (y_j)_{j=1}^d \in \mathbf{T}^d$,

$$\chi^c(x, y)_i = \begin{cases} x_i & \text{if } i \neq c, \\ y_c & \text{if } i = c. \end{cases} \quad (2.3)$$

For each $c \in [d]$ we introduce a norm $\|\cdot\|_{M_c^4}$ on scalar functions ϕ on \mathbf{T}^d by setting

$$\|\phi\|_{M_c^4(\mathbf{T}^d)}^4 \stackrel{\text{def}}{=} \int_{\mathbf{T}^d} \int_{\mathbf{T}^d} \phi(x)\phi(y)\phi(\chi^c(x, y))\phi(\bar{\chi}^c(x, y)) \, dx dy, \quad (2.4)$$

where $\bar{\chi}^c(x, y) \stackrel{\text{def}}{=} \chi^c(y, x)$. We also write

$$\|\cdot\|_{M^4(\mathbf{T}^d)} \stackrel{\text{def}}{=} \left(\sum_{c=1}^d \|\cdot\|_{M_c^4(\mathbf{T}^d)}^4 \right)^{1/4}.$$

In Appendix 2.H we confirm that $\|\cdot\|_{M_c^4}$ is indeed a norm and why it may be called a tensor norm.

Formally, the T_d^4 model is given by

$$d\nu(\phi) \propto \exp\left(-\frac{\lambda}{4}\|\phi\|_{M^4(\mathbf{T}^d)}^4 + \frac{a}{2}\|\phi\|_{L^2(\mathbf{T}^d)}^2\right) d\mathcal{G}(\phi). \quad (2.5)$$

Like the local Φ_d^4 models, analysis of higher dimensional T_d^4 models goes via regularisation, introducing renormalization, and then obtaining bounds that are uniform in the regularisation. Heuristically, the level of renormalization needed for a T_d^4 model is comparable to that of the corresponding Φ_{d-1}^4 model.

In particular, T_d^4 is super-renormalizable for $d < 5$ and T_2^4 can be defined without renormalization. For $d \in \{3, 4\}$ the measures ν can be obtained as weak $N \uparrow \infty$ limits of the measures

$$d\nu_N(\phi) \propto \exp\left(-\frac{\lambda}{4}\|\phi\|_{M^4(\mathbf{T}^d)}^4 + \frac{a + a_N}{2}\|\phi\|_{L^2(\mathbf{T}^d)}^2\right) d\mathcal{G}_N(\phi), \quad (2.6)$$

where again a_N diverges as $N \uparrow \infty$.

Moreover, T_3^4 is absolutely continuous with respect to \mathcal{G} and only requires Wick type renormalization while the renormalization of T_4^4 is more difficult and its limit is expected to be singular with respect to \mathcal{G} . The models T_3^4 and T_4^4 [RVT19] are also both super-renormalizable and were constructed and shown to be Borel summable in [DR16] and [RVT19]. However, the technique used in these constructions seems to break-down in $d = 4$, thereby not allowing for an exploration of the full subcritical regime. The model T_5^4 is just-renormalizable and requires an additional counterterm $\lambda_N \|\phi\|_{M^4(\mathbf{T}^4)}^4$ along with a wave-function renormalization Z_N . However, in contrast with Φ_4^4 , there is evidence [RV21] obtaining a non-Gaussian limit for T_5^4 might be possible.

Another feature of the T_d^4 models that makes them seem more tractable is a topological constraint that must be satisfied for a Feynman graph to be divergent, such graphs must be “melonic”. For the T_4^3 and T_4^4 models the melonic constraint plays a role in greatly simplifying the stochastic analysis approach to these models. For T_5^4 the melonic constraint imposes that the infinite collection of divergent Feynman graphs in this model, when organized by the number of their edges, proliferate like trees instead of connected graphs – in particular one could hope that perturbative formula for renormalization constants would be summable for small coupling λ .

Finally, we mention that tensor field theories are not the only examples of toy non-local field theories that have been investigated. For instance, Hartree type nonlinearities can be defined by substituting the L^4 norm with $\|\phi^2\|_{H^{-\beta/2}}^2$ where for the Sobolev space $H^{-\beta/2}$ one takes $\beta \in (0, d)$. However, the Hartree nonlinearity is closer to being a regularized version of the local product while the tensor field theory nonlinearity behaves very differently from the local product in its renormalization. Another non-local model which is closer to our setting and which has been shown to give a non-Gaussian limit in the just-renormalizable case is the Moyal model [GW14].

2.1.2 Main results

Much of the earlier work on both local and non-local field theories constructs the limiting measures by showing convergence of their moments as cut-offs are removed, using expansions and computations of expectations under g . In the stochastic analysis approach one instead tries to prove convergence of a limiting random field through a coupling to an underlying Gaussian field.

In this chapter we construct the T_3^4 and T_4^4 measures, employing two approaches – the first approach is a Langevin-type dynamical stochastic quantization [PW81, GH21] while the other is a variational approach introduced in [BG20] reminiscent of the Gibbs variational principle. The two approaches give two independent ways of constructing the T_3^4 and T_4^4 measures.

We also study the local well-posedness theory of the limiting Langevin dynamic, but

2.1.2.1 Langevin dynamic

A natural Langevin dynamic for the Φ_d^4 model (2.1) is formally given by

$$\partial_t \phi = (\Delta - 1)\phi - \lambda \phi^3 - a\phi + \sqrt{2}\xi, \quad (2.7)$$

where ξ is a d -dimensional space-time white noise on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$ and now ϕ is also a distribution over space and a new “fictitious” time t . When $d \geq 2$, (2.7) is a singular stochastic partial differential equation – the roughness of the noise ξ prevents us from using classical arguments for local well-posedness since ϕ belongs to spaces where ϕ^3 is not well-defined. Local well-posedness for (2.7) was first obtained for $d = 2$ in [DPD03] and was one of the first examples in combining probabilistic estimates with path-wise analysis to solve a singular SPDE. The case $d = 3$ was much more difficult and remained open until the development of more sophisticated path-wise methods such as regularity structures and paracontrolled calculus [Hai14, CC13].

At a formal level, the Φ_d^4 measure should be invariant for the dynamic (2.7) and this can be made rigorous for the regularized dynamic and measure. The corresponding Langevin dynamic for the T_d^4 model is given (with $\lambda = 1 - a = 1$) by

$$\partial_t \phi = (\Delta - 1)\phi - \mathcal{N}(\phi, \phi, \phi) + \sqrt{2}\xi. \quad (2.8)$$

where, ξ is as in (2.7) and, for $f, g, h : \mathbf{T}^d \times \mathbf{R} \rightarrow \mathbf{R}$, we have

$$\mathcal{N}(f, g, h)(x, t) \stackrel{\text{def}}{=} \sum_{c=1}^d \mathcal{N}^c(f, g, h)(x, t), \text{ where}$$

$$\mathcal{N}^c(f, g, h)(x, t) \stackrel{\text{def}}{=} \int_{\mathbf{T}^d} f(\chi^c(x, y), t) g(y, t) h(\bar{\chi}^c(x, y), t) \, dy .$$

The regularized dynamic we study is given by

$$\partial_t \phi = (\Delta - 1)\phi - \Pi_N \mathcal{N}(\phi, \phi, \phi) + a_N \phi + \sqrt{2} \Pi_N \xi , \quad (2.9)$$

where $\Pi_N = \mathbf{1}\{|\nabla|_\infty \leq N\}$ projects onto Fourier modes k with $|k|_\infty \leq N$ and a_N again refers to a renormalization constant. Note if one takes $\varrho_N = \Pi_N$ in (2.6) then the dynamic (2.9) keeps the Π_N marginal of (2.6) invariant – see Remark 2.44.

To show solutions to (2.9) converge is writing, for $N \leq \infty$, $\phi = \mathfrak{I} + v$ where \mathfrak{I} is an explicit, rough (when $N = \infty$) space-time stochastic process and v is the local in time solution to a better behaved and locally well-posedness remainder equation involving renormalized stochastic objects. In $d = 3$, $\mathfrak{I} = \mathfrak{I}^\dagger$ where \mathfrak{I}^\dagger is the space-time stationary solution to the linear problem $(\partial_t + 1 - \Delta)\mathfrak{I}^\dagger = \Pi_N \xi$, while in $d = 4$ the random field \mathfrak{I} is given by (2.50).

By proving uniform in N energy estimates for the dynamic (2.9) at stationarity, we obtain the following tightness result.

Theorem 2.1 *For $d = 3, 4$, any $p \in [1, \infty)$, and any $\epsilon > 0$, the family of measures $(\nu_N)_{N \in \mathbf{N}}$ satisfy*

$$\sup_{N \in \mathbf{N}} \mathbf{E}_{\nu_N} [\|\phi\|_{H^{-\frac{d-2}{2}-\epsilon}(\mathbf{T}^d)}^p] < \infty , \quad (2.10)$$

and it follows that the measures $(\nu_N)_{N \in \mathbf{N}}$ are tight on $H^{-\frac{d-2}{2}-\epsilon}(\mathbf{T}^d)$.

Moreover, for any subsequential limit ν , there exists a coupling (ϕ, \mathfrak{I}) of ν with the law of limiting process \mathfrak{I} (as described under (2.9)) such that, for any $p \in [1, \infty)$,

$$\mathbf{E}[\|\phi - \mathfrak{I}(0, \cdot)\|_{L^2(\mathbf{T}^d)}^p] < \infty . \quad (2.11)$$

While the power counting of \mathbf{T}_4^4 resembles that of Φ_3^4 , it turns out we can bypass the use of more advanced analytic theories for singular SPDE (such as the theory of regularity structures or paracontrolled calculus) when proving the above energy estimates.

In fact, if one sets up the model with the dimension as a real parameter (say by changing the regularity of the noise ξ or using fractional Laplacians) we conjecture that one can control the Langevin dynamic in the entire subcritical/super-renormalizable regime without having to use more advanced singular SPDE theory. In particular, we conjecture as one goes gets closer to the critical dimension $d = 5$, it suffices to just include progressively more terms are included in the ansatz \mathfrak{I} and perform “equation re-injections” – we do not see this second issue for tightness in $d \leq 4$ but it does appear in the $d = 4$ local well-posedness theory.

While the result proven in [RVT19] for \mathbf{T}_4^4 is stronger than ours, the machinery is heavier and it seems unlikely that the arguments of [RVT19] generalize as easily to the full subcritical/super-renormalizable regime.

Another way to use a Langevin dynamic to build a Euclidean quantum field theory measure is to prove local and global well-posedness for arbitrary initial data of the same regularity class expected of the measure along with long-time control of the measure. One can then treat the dynamic as a Markov process use the coming down from infinity estimate to prove a Krylov-Bogoliubov argument. Our analysis appears to be less suited for this approach, our main results in this direction are given below.

Theorem 2.2 *Let $d \in \{3, 4\}$, then there exist a choice of the constants a_N such that one has the $N \uparrow \infty$ convergence of local (in time) solutions to (2.9) when the dynamic is started from $\mathfrak{I}(0) + \mathcal{C}^{\frac{3}{2}-\epsilon}(\mathbf{T}^d)$. See Theorem 2.7 and Proposition 2.52 for a precise statement.*

Moreover, one has an L^2 -coming down from infinity bound for $v(t) = \phi(t) - \mathfrak{I}(t)$, see Proposition 2.33 for a precise statement.

²Note, while \mathfrak{I} is rough when $N = \infty$, we do show it can be evaluated at fixed times.

Note that we cannot start with arbitrary initial data in the expected regularity class of the solution which is $\mathcal{C}^{-(d-2)/2-}$. Much of the improvement in regularity of the non-local product $\mathcal{N}(\phi, \phi, \phi)$ versus the local product ϕ^3 appears to be probabilistic which limits our ability to deal with rough deterministic initial data. However, with extra work one should be able to show that the limiting measure ν has a coupling of the form $\mathfrak{f}(0) + \mathcal{C}^{\frac{3}{2}-\epsilon}(\mathbf{T}^d)$ – in this case the above local well-posedness would allow one to start from the dynamic from the invariant measure.

We also believe the local well-posedness result itself could be upgraded to allow initial data $\mathfrak{f}(0) + L^2(\mathbf{T}^d)$ (see Remark 2.35). With our L^2 -coming down from infinity result, this would mean the remainder equation for v is globally well-posed in L^2 .

We do not pursue either of these directions in this article since our main focus is the construction of the measure. Our main interest in presenting the proof of local well-posedness in (2.2) is presenting a re-injection argument which we believe is of its own interest – it points to an important structural feature of the \mathbf{T}_d^4 Langevin dynamic that we believe could be leveraged to study this models entire subcritical/super-renormalizable regime without using theories like regularity structures or paracontrolled calculus (see Remark 2.36).

The following technical remark, which assumes some familiarity with Feynman diagrams in field theory and perturbative tree expansions for singular SPDE, gives some intuition for why more advanced singular SPDE tools are not needed for the \mathbf{T}_d^4 dynamic.

Remark 2.3 We first give more detail about the power counting for Feynman graphs in Φ_d^4 and \mathbf{T}_d^4 . The superficial degree of divergence of a Feynman graph G in the the local Φ_d^4 theory is given by

$$\omega_{\Phi_d^4}(G) = d - (4 - d)|V(G)| - \frac{d - 2}{2}|L^{ext}(G)|,$$

where $|V(G)|$ and $|L^{ext}(G)|$ denote the number of vertices and external edges of G . Recall that a graph G with $L^{ext}(G) = n$ is called an n -point graph, and a graph G with $L^{ext}(G) = 0$ is called a vacuum graph. In the subcritical/super-renormalizable regime $d < 4$ only a finite number of graphs are superficially divergent.

For Φ_2^4 , apart from the vacuum graphs, only the so-called tadpole graph (the one-vertex two-point graph) is superficially divergent, while in three dimensions, the sunset graph and the snowball graph (the two two-vertex two-point graphs) are superficially divergent.

Moreover, in the case of Φ_3^4 , the snowball graph does not require the introduction of a counterterm, since it already contains a nested tadpole graph whose renormalized amplitude vanishes – that is since the tadpole graph is exactly canceled by its renormalization, this same renormalization also cancels the snowball graph. Therefore, in the local case, one only has to add two counterterms, one for the tadpole and the other for the sunset.

The superficial degree of divergence of the \mathbf{T}_d^4 theories is given by

$$\begin{aligned} \omega_{\mathbf{T}_d^4}(G) &= d - (5 - d)|V(G)| - \frac{d - 3}{2}|L^{ext}(G)| - \delta(G) - C(\partial G) \\ &= \omega_{\Phi_{d-1}^4}(G) - \delta(G) - (C(\partial G) - 1), \end{aligned} \quad (2.12)$$

where $C(\partial G)$, the number of connected components of the boundary graph, and $\delta(G)$, the degree of the graph which is a positive integer, are defined in Section 2.6. This power counting strongly suggests that \mathbf{T}_d^4 is to be compared with Φ_{d-1}^4 . However a difference with Φ_{d-1}^4 is that (apart from the vacuum graphs) only graphs G with $\delta(G) = 0$ can be superficially divergent. Such graphs G are called melonic graphs.

The non-local vertex for \mathbf{T}_d^4 is less symmetric than the local vertex, so in the non-local theory there are many different Feynman graphs corresponding to the local tadpole, sunset, and snowball graphs. Some of these are melonic, some of them are not - in particular there are tadpole and snowball type non-local graphs that can be melonic but none of the sunset type non-local graphs are melonic. Another difference in the non-local case is that the renormalized amplitude of the tadpole is no longer equal to zero (the tadpole is no longer a completely local divergence since the vertex itself is non-local) which is why one

has to add two counterterms, one for the (melonic) tadpole and the other one for the (melonic) snowball. We summarise this in Figure 1 below.




			
	tadpole	snowball	sunset
Φ_3^4	✓	✗	✓
T_4^4	✓	✓	✗
Tree	$\mathbf{E}[\mathfrak{T}^2]$	$\mathbf{E}[\mathfrak{N}]$	$\mathbf{E}[\mathfrak{V}]$

Figure 1: Comparison between the primary divergences of the Φ_3^4 and T_4^4 measures. The top row lists divergent Feynman graphs and the bottom row lists the expectations of the stochastic trees that generate these graphs in the dynamic picture.

The main takeaway is that the divergence \mathfrak{V} , which is the significant non-local divergence of Φ_3^4 , does not pose a problem for T_4^4 and instead we must be careful with \mathfrak{N} (which is not a problem in Φ_3^4 if the tadpole has already been renormalized). Since \mathfrak{N} is “missing” a branch at the root rather than the top internal vertex, we can use an analog of the Da Prato - Debussche argument.

2.1.2.2 The variational approach

Our second approach using the variational method of [BG20] proceeds by representing the Laplace transform of the regularized and renormalized T_3^4 and T_4^4 measures as a stochastic control problem.

In this approach instead of introducing a time that corresponds to evolution under a Langevin dynamic we instead introduce a time that represents scale.

$$-\log \mathbf{E}_{\nu_t}[e^{-f(\phi)}] = \log \mathcal{Z}_t + \inf_{u \in \mathbb{H}_a} \mathbf{E}_{\mathbf{P}} \left[\begin{aligned} & \| \mathfrak{I}_t + I_t(u) \|_{M^4(\mathbf{T}^d)}^4 - a_t \| \mathfrak{I}_t + I_t(u) \|_{L^2(\mathbf{T}^d)}^2 - b_t \\ & + f(\mathfrak{I}_t + I_t(u)) + \text{Ent}(u) \end{aligned} \right]$$

where $\log \mathcal{Z}_t$ is a constant uniformly bounded in t , \mathbb{H}_a is a certain space of adapted random drifts $u : \mathbf{R}_{\geq 0} \times \mathbf{T}^d \rightarrow \mathbf{R}$ which should be thought of as a shift of an underlying white noise process, $I_{\bullet}(u)$ is the corresponding shift of the free field process \mathfrak{I}_t , \mathbf{P} is the law of the entire process $(\mathfrak{I}_t)_{t \in [0, T]}$, $\text{Ent}(u)$ is a relative entropy term which is a quadratic in u , and b_t is a constant independent of f . This variational representation can be used to obtain bounds on the Laplace transform of ν_t uniform in t , which gives the following main result (proven at the end of Section 2.4).

Theorem 2.4 *Let $d \in \{2, 3, 4\}$, $\epsilon > 0$, and $f : \mathcal{C}^{-\frac{d-2}{2}-\epsilon}(\mathbf{T}^d) \rightarrow \mathbf{R}$ have at most linear growth. Then, uniform in $t \geq 0$ one has*

$$\mathbf{E}_{\nu_t}[e^{-f(\phi)}] \lesssim 1, \tag{2.13}$$

Since the embedding $\mathcal{C}^{-\frac{d-2}{2}-2\epsilon} \hookrightarrow \mathcal{C}^{-\frac{d-2}{2}-\epsilon}$ is compact, the family $(\nu_t)_{t \geq 0}$ is tight on $\mathcal{C}^{-\frac{d-2}{2}-2\epsilon}(\mathbf{T}^d)$.

2.1.3 Notation and conventions

Given $n \in \mathbf{N}$ we write $[n] = \{1, \dots, n\}$. For Banach spaces $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, we write $\mathcal{L}(\mathcal{A}, \mathcal{B})$ for the Banach space of bounded linear operators $\mathcal{A} \rightarrow \mathcal{B}$ equipped with the operator norm:

$$\|L\|_{\mathcal{L}(\mathcal{A}, \mathcal{B})} = \sup_{u: \|u\|_{\mathcal{A}}=1} \|L(u)\|_{\mathcal{B}}.$$

For any $\theta \in (0, 1]$, Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, and $T \geq 0$, we write $C_T^\theta \mathcal{B} \stackrel{\text{def}}{=} C^\theta([0, T], \mathcal{B})$ for the Banach space of θ -Hölder continuous functions from $[0, T]$ into \mathcal{B} (with respect to the parabolic scaling) and equip this space with the norm $\|\cdot\|_{C_T^\theta \mathcal{B}} \stackrel{\text{def}}{=} \|\cdot\|_{C_T \mathcal{B}} + \|\cdot\|_{\dot{C}_T^\theta \mathcal{B}}$ where

$$\|u\|_{C_T \mathcal{B}} \stackrel{\text{def}}{=} \sup_{t \in [0, T]} \|u(t)\|_{\mathcal{B}} \quad \text{and} \quad \|u\|_{\dot{C}_T^\theta \mathcal{B}} \stackrel{\text{def}}{=} \sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq 1}} \frac{\|u(t) - u(s)\|_{\mathcal{B}}}{|t-s|^{\frac{\theta}{2}}}.$$

We write $C_T \mathcal{B}$ for the Banach space of bounded continuous \mathcal{B} -valued functions on $[0, T]$ with norm $\|\cdot\|_{C_T \mathcal{B}}$.

For any $d \geq 2$ and $x, y \in \mathbf{R}^d$ we use the scalar product $x \cdot y = \sum_{c=1}^d x_c y_c$, ℓ^2 norm $|x| \stackrel{\text{def}}{=} \sqrt{x \cdot x}$, ℓ^∞ norm $|x|_\infty \stackrel{\text{def}}{=} \max_{i \in [d]} |x_i|$, and the ‘‘bracket’’ norm $\langle x \rangle \stackrel{\text{def}}{=} \sqrt{1 + x \cdot x}$.

We often work on the torus $\mathbf{T}^d = (\mathbf{R}/2\pi\mathbf{Z})^d$ and denote by $|\cdot|_\infty$ the distance on \mathbf{T}^d induced by the ℓ^∞ norm on \mathbf{R}^d . We also write, for $N \in \mathbf{N}$, $\mathbf{Z}_N^d \stackrel{\text{def}}{=} \{k \in \mathbf{Z}^d : |k|_\infty \leq N\} = \{-N, \dots, N\}^d$.

For any spatial function $u : \mathbf{T}^d \rightarrow \mathbf{R}$, we use the (spatial) Fourier transform to define a function \hat{u} on \mathbf{Z}^d by setting

$$\hat{u}_m \equiv \mathcal{F}_x(u)(m) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} e^{ix \cdot m} u(x) dx,$$

whose inverse is given by

$$u(x) = \mathcal{F}_x^{-1}(\hat{u})(x) = \sum_{m \in \mathbf{Z}^d} e^{-ix \cdot m} \hat{u}_m.$$

Note that we only perform Fourier transforms in space, and extend the above notation to space-time functions $u(x, t)$ by writing $\mathcal{F}_x(u(\cdot, t))(m) = \hat{u}_m(t)$.

The Littlewood-Paley blocks $(\Delta^j)_{j \geq -1}$ that we define in Appendix [2.G](#) act in Fourier space by multiplication with $\hat{\Delta}_m^j = \mathbf{1}_{[2^{-j-1}, 1)}(2^{-j}|m|_\infty)$ for $i \geq 0$, which is why for any sequence $v : \mathbf{Z}^d \rightarrow \mathbf{C}$ we introduce the notation

$$\sum_{m \sim 2^i} v_m \stackrel{\text{def}}{=} \sum_{m \in \mathbf{Z}^d} \hat{\Delta}_m^i v_m = \sum_{m \in \mathbf{Z}^d} \mathbf{1}_{[2^{-j-1}, 1)}(2^{-j}|m|_\infty) v_m,$$

with the usual understanding when $i = -1$.

For $p, q \in [1, \infty]$ and $\alpha \in \mathbf{R}$ we denote by $B_{p,q}^\alpha(\mathbf{T}^d)$ the corresponding Besov spaces on \mathbf{T}^d – see Appendix [2.G](#). In the special cases $(p, q) = (\infty, \infty)$ and $(p, q) = (2, 2)$ we denote by $\mathcal{C}^\alpha \stackrel{\text{def}}{=} B_{\infty, \infty}^\alpha$ the corresponding Hölder-Besov space and $H^\alpha \stackrel{\text{def}}{=} B_{2,2}^\alpha$ the Sobolev spaces. We denote the L^2 pairing of two functions $u, v \in L^2(\mathbf{T}^d)$ as

$$(u, v)_{L^2(\mathbf{T}^d)} \stackrel{\text{def}}{=} \int_{\mathbf{T}^d} u(x)v(x) dx,$$

often dropping the subscript $L^2(\mathbf{T}^d)$.

For $c \in [d]$, we allow $\mathcal{N}^c : (f, g, h) \mapsto \mathcal{N}^c(f, g, h)$ to act on a function f (resp. h) depending only on y_c (resp. $y_{\bar{c}}$). When this is the case, $\mathcal{N}^c(f, g, h)(x, t)$ is independent of $x_{\bar{c}}$ (resp. x_c), and we write that $f : \mathbf{T}_c \rightarrow \mathbf{R}$ (resp. $h : \mathbf{T}_{\bar{c}}^{d-1} \rightarrow \mathbf{R}$).

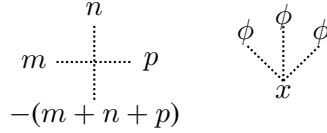


Figure 2: Local vertices

We make frequent use of Kolmogorov estimates that reference various parameters p, T, ϵ where $\epsilon > 0$ is an exponent drop to turn a statement in expectation into a pathwise one, $p \in 2\mathbf{N}$ is a degree of stochastic integrability, and $T \in (0, 1]$ is a finite time cut-off on the relevant process. It is always implied that $p, T > 0$ can be taken arbitrarily large and $\epsilon > 0$ arbitrarily small, with this only changing implicit or explicit constants.

We introduce the shorthand $\mathcal{I}(\cdot) \stackrel{\text{def}}{=} \|\cdot\|_{M^4(\mathbf{T}^d)}^4$ for the tensor nonlinearity and, for $c \in [d]$, write $\mathcal{I}^c(\cdot) \stackrel{\text{def}}{=} \|\cdot\|_{M^c(\mathbf{T}^d)}^4$. We fix $\epsilon > 0$, which later will be taken sufficiently small, and write $\mathcal{K}(\cdot) \stackrel{\text{def}}{=} \|\cdot\|_{H^{1-\epsilon}(\mathbf{T}^d)}^2$ for kinetic term (the Gaussian action).

For $t \geq 0$, we denote by $P_t \stackrel{\text{def}}{=} e^{-t(1-\Delta)}$ the massive heat operator on \mathbf{T}^d , we view P_t as an operator on functions/distributions over space. For nice enough space-time distributions F we write

$$\mathcal{L}^{-1}F(x, t) \stackrel{\text{def}}{=} \int_0^t P_{t-s}(F(\cdot, s))(x)ds, \quad \underline{\mathcal{L}}^{-1}F(x, t) \stackrel{\text{def}}{=} \int_{-\infty}^t P_{t-s}(F(\cdot, s))(x)ds.$$

2.1.3.1 Some graphical notation

We use a variety of graphical representations in the article, some of which might be unfamiliar even if one is familiar with diagrams in local field theories. A more careful explanation of our graphical notation can be found in Section [2.6.2](#).

Suppose $\phi \in C^\infty(\mathbf{T}^d)$, and let us write $\hat{\phi}_m = \hat{\phi}(m)$ viewing the Fourier mode as an index. Then we can write the local nonlinearity as

$$\|\phi\|_{L^4}^4 = \sum_{m,n,p,q \in \mathbf{Z}^d} \hat{\phi}_m \hat{\phi}_n \hat{\phi}_p \hat{\phi}_q \delta_{m+n+p+q,0}. \quad (2.14)$$

In Figure [2](#) we draw the local $\|\phi\|_{L^4}^4$ non-vertex as a diagram on the left (in Fourier space) and the corresponding dynamical/stochastic vertex $\phi^3(x)$ on the right (in direct space).

In Figure [2](#) the dashed lines on the left correspond to fields ϕ , and the vertex in the middle represents the delta function imposed on the sum of incoming momentum from the dashed lines.

For $c \in [d]$ and $m \in \mathbf{Z}^d$, we write $m_{\hat{c}} = \chi^c(m, 0)$. We have the following formula for the monochrome nonlinearity of the non-local theory

$$\mathcal{I}^c(\phi) = \sum_{m,n,p,q \in \mathbf{Z}^d} \delta_{m_c, -n_c} \delta_{p_c, -q_c} \delta_{m_{\hat{c}}, -q_{\hat{c}}} \delta_{n_{\hat{c}}, -p_{\hat{c}}} \hat{\phi}_m \hat{\phi}_n \hat{\phi}_p \hat{\phi}_q. \quad (2.15)$$

The corresponding vertex for the non-local nonlinearity $\mathcal{I}^c(\phi)$ is drawn on the left below (in Fourier space) and we draw the dynamical/stochastic vertex $\mathcal{N}^c(f, g, h)(x)$ on the right (in direct space). For the picture on the right, we have labeled the dashed lines corresponding to fields f, g, h to clarify the asymmetric nature of the vertex.

One way to introduce a diagrammatic approach for the tensor field theory is to start with the graphs of the local theory and replace the local vertices of Figure [2](#) with the colored non-local vertices of Figure [3](#)—we will call the result graphs tensor graphs.

Note that the non-local vertex has less symmetries than the local vertex, in particular a single Feynman graph in the local theory gives rise to many different graphs in the non-local theory.

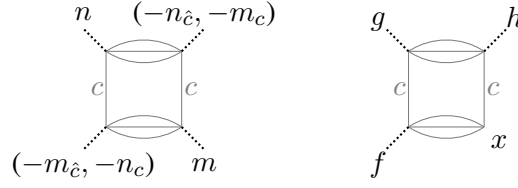


Figure 3: Non-local vertices

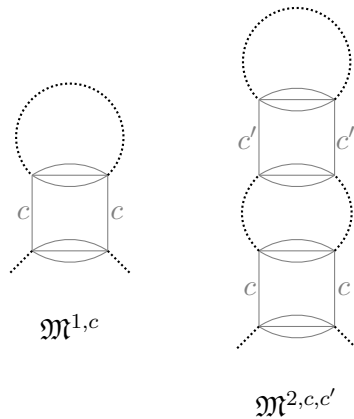
Definition 2.5 We define the *melonic pairing* of ϕ and ψ as $\mathcal{N}^c(\cdot, \phi, \psi)$ – for $\phi, \psi \in C^\infty(\mathbf{T}^d)$, it can be viewed as an operator $C^\infty(\mathbf{T}^d) \ni h \mapsto \mathcal{N}^c(h, \phi, \psi) \in C^\infty(\mathbf{T}^d)$ (which also acts naturally on $h \in C^\infty(\mathbf{T}_c)$). We draw a melonic pairing as


 Figure 4: The melonic pairing of ϕ and ψ

Similarly, $\mathcal{N}^c(\phi, \psi, \cdot)$ is the *non-melonic pairings* of ϕ and ψ that acts on $h \in C^\infty(\mathbf{T}^d)$ (and also $h \in C^\infty(\mathbf{T}_c^{d-1})$) and $\mathcal{N}^c(\phi, \cdot, \psi)$ is the *exterior pairing* that acts on $h \in C^\infty(\mathbf{T}^d)$. Note that contrarily to the melonic and non-melonic pairings, the exterior pairing can be extended to an operator $C^\infty(\mathbf{T}^d) \ni h \mapsto \mathcal{N}^c(\phi, h, \psi) \in \mathcal{D}'(\mathbf{T}^d)$ for $\phi, \psi \in \mathcal{D}'(\mathbf{T}^d)$.

We also talk about melonic pairings in the context of the particular pairings in Wick's formula that lead to divergences, this again refer to pairings that maximize the number of Wick pairings between terms sitting in the second and third arguments of $\mathcal{N}^c(\cdot, \cdot, \cdot)$ – see the discussion above (2.17) and at the beginning of Section 2.3.1.

Definition 2.6 Following the discussion in Figure 1, we define the primary divergent subgraphs for the \mathbb{T}_4^4 measure. For $c \in [d]$, $\mathfrak{M}^{1,c}$ is the melonic tadpole of color c and for $c, c' \in [d], c \neq c'$, $\mathfrak{M}^{2,c,c'}$ is the melonic snowball of colors c, c' . The color index can be dropped when the color of the graph is of no importance. They are pictured as


 Figure 5: The two primary divergent graphs $\mathfrak{M}^{1,c}$ and $\mathfrak{M}^{2,c,c'}$

2.2 Renormalization and local theory for dynamical \mathbb{T}_3^4

In this section we prove the following local (in both time and space) well-posedness result for the parabolic stochastic quantization of the \mathbb{T}_d^4 equation for $d \in \{2, 3\}$. This will also be a warm-up for

describing some of the power-counting and stochastic estimates needed for tensor field theories before we treat the \mathbb{T}_4^4 model.

Recall from the introduction that we define a regularized noise $\xi_N \stackrel{\text{def}}{=} \Pi_N \xi$ where Π_N is the Fourier multiplier that projects onto the Fourier modes of size lower or equal to N . We introduce the stationary (in both space and time) process \mathfrak{I}_N over $\mathbf{T}^d \times \mathbf{R}$ given by

$$\mathfrak{I}_N \stackrel{\text{def}}{=} \sqrt{2} \underline{\mathcal{L}}^{-1} \xi_N.$$

Theorem 2.7 *For $d = 2, 3$, there exist constants a_N such that one has uniform in N control of the local in time solutions to (2.9) with initial condition of the form $\mathfrak{I}_N(0) + \Pi_N v(0)$ with $v(0) \in \mathcal{C}^{\frac{3}{2}-\epsilon}$. In particular, using the ansatz $\phi = \mathfrak{I}_N + v$ for the equation with cut-off N , one has that v converges in $C([0, \bar{T}], \mathcal{C}^{\frac{3}{2}-\epsilon}(\mathbf{T}^3))$ where $\bar{T} \in (0, \infty]$ is a random blow-up time.*

As mentioned earlier, the local theory for the dynamical \mathbb{T}_3^4 is quite similar to that of dynamical Φ_2^4 , and as in the latter model we will be able to apply a Da Prato - Debussche [DPD03] argument. The idea of the argument is that $\lim_{N \uparrow \infty} \mathfrak{I}_N = \mathfrak{I}$ is rough, but explicit and Gaussian, while v is an inexplicit remainder that should have better regularity. One then derives a remainder PDE for v involving Wick powers/Hermite polynomials of \mathfrak{I} (constructed via a probabilistic argument) and shows this PDE for v is locally well-posed (with stability as the regularisation is removed).

We start with the following standard estimate.

Lemma 2.8 *For any fixed $T > 0$ and $\epsilon > 0$, \mathfrak{I}_N converges to a limit $\mathfrak{I} = \mathfrak{I}_\infty$ in $C_T \mathcal{C}^{-\frac{d-2}{2}-\epsilon}$ – see Remark 2.80. Moreover, for any $N \geq 0$ and $t_1, t_2 \in \mathbf{R}_{\geq 0}$,*

$$\mathbf{E}[\mathfrak{I}_N(t_1) \mathfrak{I}_N(t_2)] = \Pi_N \frac{P_{|t_1-t_2|}}{1-\Delta}. \quad (2.16)$$

In particular, for fixed t and as a random field on \mathbf{T}^d , one has $\mathfrak{I}_\infty(t) \stackrel{\text{law}}{=} \mathfrak{g}$.

In what follows we write $\mathcal{N}_N^{(c)} \stackrel{\text{def}}{=} \Pi_N \mathcal{N}^{(c)}$. A natural first question is how to renormalize $\mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \mathfrak{I}_N)$ so it has a meaningful limit as $N \uparrow \infty$. In Φ_d^4 the corresponding (Wick) renormalization is $\mathfrak{I}_N^3(z) \mapsto \mathfrak{I}_N^3(z) - 3\mathbf{E}[\mathfrak{I}_N^2(z)]\mathfrak{I}_N(z)$ which cancels the divergence $E[\mathfrak{I}_N^2(z)]$ produced when any two of the three copies of \mathfrak{I}_N Wick contract.

The product \mathcal{N} is less symmetric, each choice of which two copies of \mathfrak{I}_N Wick contract, that is $\mathcal{N}_N(\bullet, \mathfrak{I}_N, \mathfrak{I}_N)$, $\mathcal{N}_N(\mathfrak{I}_N, \bullet, \mathfrak{I}_N)$ or $\mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \bullet)$, should be treated differently. The second two terms are well-defined without renormalization as $N \uparrow \infty$ in any dimension d , while the first term is melonic – it is well-defined for $d = 2$ but requires renormalization for all $d \geq 3$.

Our “Wick”, or tadpole, renormalization is given by setting, for any $d \geq 2$ and $c \in [d]$,

$$\mathfrak{C}_N^{1,c}(d) \stackrel{\text{def}}{=} \mathbf{E}[\mathcal{N}_N^c(1, \mathfrak{I}_N(t), \mathfrak{I}_N(t))(x)]. \quad (2.17)$$

Note that by stationarity $\mathfrak{C}_N^{1,c}(d)$ does not depend on (t, x) . This is the renormalization constant shown in the first column of Figure 11

Notation 2.9 Recall that we can view \mathcal{N}^c and \mathcal{N} as either acting on functions/distributions over space and returning a function over space, or acting on and producing functions/distributions over space-time. We jump between the two viewpoints often.

We can then write the promised renormalized product by setting, for any $\psi : \mathbf{T}^d \rightarrow \mathbf{R}$,

$$\mathcal{N}_N^c(\psi, \mathfrak{I}_N, \mathfrak{I}_N)(x) = \mathfrak{C}_N^{1,c}(d)\psi(x). \quad (2.18)$$

Note that unlike the local Wick square $\mathfrak{I}_N^2(x) - \mathbf{E}[\mathfrak{I}_N(x)^2]$, (2.18) may not have vanishing expectation since we replaced ψ with 1 in (2.17). Unlike the local case, in (2.18) we are compensating a non-local³ divergence with a local counterterm. The next lemma is an instructive computation⁴ showing how the renormalization cancellation in (2.18) works.

Lemma 2.10 *For $d \geq 2$ and $c \in [d]$, one has*

$$\mathfrak{C}_N^{1,c}(d) = \sum_{m_{\hat{c}} \in \mathbf{Z}_N^{d-1}} \frac{1}{\langle m_{\hat{c}} \rangle^2}.$$

In particular $\mathfrak{C}_N^{1,c}(d)$ is independent of $c \in [d]$. Writing

$$\mathfrak{C}_N^1(d) \stackrel{\text{def}}{=} \sum_{c=1}^d \mathfrak{C}_N^{1,c}(d) = d \mathfrak{C}_N^{1,c}(d), \quad (2.19)$$

we have, for $d \in \{2, 3, 4\}$ and any $\psi, f \in C^\infty(\mathbf{T}^d)$,

$$\begin{aligned} \sup_{N \in \mathbf{N}} \mathbf{E}[(\mathcal{N}_N(\psi, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1 \psi)(f)^2] &< \infty, \\ \sup_{N \in \mathbf{N}} \mathbf{E}[(\mathcal{N}_N(\mathfrak{I}_N, \psi, \mathfrak{I}_N))(f)^2] \vee \mathbf{E}[(\mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \psi))(f)^2] &< \infty. \end{aligned} \quad (2.20)$$

We sometimes suppress d from the notation, just writing \mathfrak{C}_N^1 .

Proof. We start with (2.20), but we only sketch this bound in order to demonstrate the need for renormalization. We state more detailed estimates in Lemma 2.16, steps skipped over here can be reviewed in the proof of that lemma. In what follows we sometimes drop N from our notation and write Fourier variables as subscripts.

Thanks to hypercontractivity and orthogonality of the homogeneous Gaussian chaos it suffices to bound the second moments of projections onto each homogeneous chaos. We first deal with $\mathcal{N}_N(\mathfrak{I}, \psi, \mathfrak{I}) = \mathcal{N}_N(\mathfrak{I}, \psi, \mathfrak{I})^{(2)} + \mathcal{N}_N(\mathfrak{I}, \psi, \mathfrak{I})^{(0)}$ – superscripts indicate the Gaussian chaos. As in the local case, one does not need renormalization to obtain uniform in N bounds on the highest chaos. Regarding the $\mathcal{N}_N(\mathfrak{I}, \psi, \mathfrak{I})^{(0)}$ – the expectation of $\mathcal{N}_N(\mathfrak{I}, \psi, \mathfrak{I})$ – we have

$$\begin{aligned} \mathcal{N}_N^c(\mathfrak{I}(t), \psi, \mathfrak{I}(t))^{(0)}(x) &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \sum_{n \in \mathbf{Z}^d} \hat{\psi}_n \mathbf{E}[\hat{\mathfrak{I}}_{-m_{\hat{c}}, -n_c}(t) \hat{\mathfrak{I}}_{-n_{\hat{c}}, -m_c}(t)] \\ &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \sum_{n \in \mathbf{Z}^d} \hat{\psi}_n \delta_{m, -n} \frac{1}{\langle m \rangle^2} \\ &= \Pi_N(1 - \Delta)^{-1} \psi(x), \end{aligned}$$

which is clearly well behaved as $N \uparrow \infty$. Turning to $\mathcal{N}_N(\mathfrak{I}(t), \mathfrak{I}(t), \psi)$, we have

$$\begin{aligned} \mathcal{N}_N^c(\mathfrak{I}(t), \mathfrak{I}(t), \psi)^{(0)}(x) &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \sum_{n \in \mathbf{Z}^d} \hat{\psi}_{-n_{\hat{c}}, -m_c} \mathbf{E}[\hat{\mathfrak{I}}_{-m_{\hat{c}}, -n_c}(t) \hat{\mathfrak{I}}_n(t)] \\ &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \sum_{n \in \mathbf{Z}^d} \hat{\psi}_{-n_{\hat{c}}, -m_c} \delta_{m_{\hat{c}}, n_{\hat{c}}} \frac{\Pi_N(m_{\hat{c}}, n_c)}{\langle (m_{\hat{c}}, n_c) \rangle^2} \\ &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \hat{\psi}_{-m} \sum_{n_c \in \mathbf{Z}} \frac{\Pi_N(m_{\hat{c}}, n_c)}{\langle (m_{\hat{c}}, n_c) \rangle^2}, \end{aligned}$$

³We recall that non-locality of Wick renormalization has important consequences for our model, see Remark 2.3

⁴In fact we don't use Lemma 2.10 as we need more quantitative regularity estimates, see the sequel.

which poses no problem as $N \uparrow \infty$ since the sum over n_c above is convergent. Now we turn to $\mathcal{N}_N(\psi, \mathfrak{I}(t), \mathfrak{I}(t))$ where we see the need for renormalization.

$$\begin{aligned} \mathcal{N}_N^c(\psi, \mathfrak{I}(t), \mathfrak{I}(t))^{(0)}(x) &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \sum_{n \in \mathbf{Z}^d} \hat{\psi}_{-m_{\hat{c}}, -n_{\hat{c}}} \mathbf{E}[\hat{\mathfrak{I}}_n(t) \hat{\mathfrak{I}}_{-n_{\hat{c}}, -m_c}(t)] \\ &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \sum_{n \in \mathbf{Z}^d} \hat{\psi}_{-m_{\hat{c}}, -n_{\hat{c}}} \delta_{m_c, n_c} \frac{\Pi_N(n_{\hat{c}}, m_c)}{\langle (n_{\hat{c}}, m_c) \rangle^2} \\ &= \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \hat{\psi}_{-m} \sum_{n_{\hat{c}} \in \mathbf{Z}^{d-1}} \frac{\Pi_N(n_{\hat{c}}, m_c)}{\langle (n_{\hat{c}}, m_c) \rangle^2}. \end{aligned} \quad (2.21)$$

Observe that in $d \geq 3$ the sum over $n_{\hat{c}}$ becomes divergent when we take $N \uparrow \infty$. We renormalize to compensate this divergence, and since we want to accomplish this with a local counter-term we only subtract the divergent term evaluated at $m_c = 0$ rather cancelling the whole quantity above. Indeed, subtracting $\mathfrak{C}_N^{1,c} \psi$ yields

$$\mathcal{N}_N^c(\psi, \mathfrak{I}_N, \mathfrak{I}_N)^{(0)}(x) - \mathfrak{C}_N^{1,c} \psi(x) = \sum_{m \in \mathbf{Z}_N^d} e^{-ix \cdot m} \hat{\psi}_{-m} \sum_{n_{\hat{c}} \in \mathbf{Z}^{d-1}} \mathfrak{R}_N^{1,c}(\chi^c(n, m)), \quad (2.22)$$

where

$$\mathfrak{R}_N^{1,c}(m) \stackrel{\text{def}}{=} \frac{\Pi_N(m)}{\langle m \rangle^2} - \frac{\Pi_N(m_{\hat{c}})}{\langle m_{\hat{c}} \rangle^2}. \quad (2.23)$$

We see that (2.22) is uniformly bounded as we take $N \uparrow \infty$ when $d \leq 4$.

Finally, the formula for \mathfrak{C}_N^1 in (2.19) directly comes from inserting $\psi = 1$ in (2.21) since the Fourier transform of 1 is $\delta_{m,0}$. \square

Remark 2.11 \mathfrak{C}_∞^1 is finite when $d = 2$ and we see that the dynamic T_2^4 model, while being singular⁵ does not require renormalization. \mathfrak{C}_N^1 diverges like $\log N$ when $d = 3$ and like N^{d-3} when $d \geq 4$. Viewing \mathfrak{C}_N^1 as the analogue of the ‘‘Wick constant’’ in the local theory, this is consistent with the idea that divergences in the T_d^4 model resemble those in the Φ_{d-1}^4 model for $d \in \{3, 4, 5\}$.

We introduce some symbolic notation, writing, for $N \in \mathbf{N}$,

$$\Psi_N = \mathfrak{W}_N \stackrel{\text{def}}{=} \mathcal{N}(\mathfrak{I}_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1 \mathfrak{I}_N. \quad (2.24)$$

In Lemma 2.12 we prove that, for $d \in \{3, 4\}$, Ψ_N converges to a limiting distribution over space-time as $N \uparrow \infty$. Note that Ψ_N is not in the homogeneous third Gaussian chaos associated to ξ , since only one of the three Wick self-contractions is cancelled, and it is not even fully cancelled.

With this notation at hand, we are going to study the following regularized equation:

$$\mathcal{L}\phi_N = -\mathcal{N}_N(\phi_N, \phi_N, \phi_N) + \mathfrak{C}_N^1 \phi_N + \sqrt{2}\xi_N, \quad \phi_N(0) = \mathfrak{I}_N(0) + \Pi_N v_0. \quad (2.25)$$

Given that we can expect that the noise is the most singular term of the equation, in order to solve the equation, we first have to perform an expansion around the solution to the linear equation, which is known as the Da Prato-Debussche method. Let $v_N \stackrel{\text{def}}{=} \phi_N - \mathfrak{I}_N$ that solves

$$\begin{aligned} \mathcal{L}v_N &= -\mathcal{N}_N(v_N, v_N, v_N) - \mathcal{N}_N(\mathfrak{I}_N, v_N, v_N) - \mathcal{N}_N(v_N, \mathfrak{I}_N, v_N) \\ &\quad - \mathcal{N}_N(v_N, v_N, \mathfrak{I}_N) - \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, v_N) - \mathcal{N}_N(\mathfrak{I}_N, v_N, \mathfrak{I}_N) \\ &\quad - (\mathcal{N}_N(v_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1 v_N) - (\mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1 \mathfrak{I}_N), \end{aligned} \quad (2.26)$$

⁵ $\mathcal{N}(\phi, \phi, \phi)$ is not defined for $\phi \in \mathcal{C}^{0-}$ so the equation is indeed singular, but $\mathcal{N}(\mathfrak{I}, \mathfrak{I}, \mathfrak{I})$ can be constructed with a stochastic estimate without any need for renormalization.

with initial condition $\Pi_N v_0$. In the rest of the section, the cut-off N is often suppressed from notation – for instance we write $\mathcal{N}(v, \mathfrak{I}, v)$ to denote $\mathcal{N}_N(v_N, \mathfrak{I}_N, v_N)$. However, all estimates in this section will be uniform in N . We now state regularity estimates for the stochastic objects appearing in (2.26), all of our proofs of stochastic estimates are deferred to Section 2.5.

Lemma 2.12 (Random fields 1) *Let*

$$\mathfrak{W}_N \stackrel{\text{def}}{=} \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1(d) \mathfrak{I}_N. \quad (2.27)$$

Then, for $d \in \{2, 3, 4\}$, $p \in [1, \infty)$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{W}_N\|_{C_T C^{-\frac{3d-8}{2}-\epsilon}(\mathbf{T}^d)}^p] < \infty. \quad (2.28)$$

Moreover, one has convergence of the above object in the corresponding $L^p(\Omega)$ space as $N \rightarrow \infty$.

Remark 2.13 In (2.28) we see our non-local ‘‘Wick cube’’ has an improvement of regularity by 1 versus the local/pointwise Wick cube. On the other hand, two of our non-local ‘‘Wick squares’’ see a regularity gain of 1/2 versus the local product. More precisely, defining

$$\begin{aligned} \mathfrak{W}_N^c(x_c, y_c) &\stackrel{\text{def}}{=} \int_{\mathbf{T}^{d-1}} \mathfrak{I}_N(y_{\hat{c}}, y_c) \mathfrak{I}_N(y_{\hat{c}}, x_c) dy_{\hat{c}} - \mathfrak{C}_N^{1,c}(d) \delta(x_c - y_c), \\ \mathfrak{K}_N^c(x_{\hat{c}}, y_{\hat{c}}) &\stackrel{\text{def}}{=} \int_{\mathbf{T}} \mathfrak{I}_N(x_{\hat{c}}, y_c) \mathfrak{I}_N(y_{\hat{c}}, y_c) dy_c, \end{aligned} \quad (2.29)$$

we have

$$\begin{aligned} \sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{W}_N^c\|_{C_T C^{-\frac{2d-5}{2}-\epsilon}(\mathbf{T}^2)}^p] &< \infty \text{ for } d \geq 2, \\ \sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{K}_N^c\|_{C_T C^{-\frac{2d-5}{2}-\epsilon}(\mathbf{T}^{2d-2})}^p] &< \infty \text{ for } d \in \{2, 3, 4\}. \end{aligned} \quad (2.30)$$

We present (2.30) to give intuition about the power-counting but in our main proofs we will use define and control these ‘‘Wick squares’’ as random operators⁶, see Lemma 2.16

Notation 2.14 We often use a pictorial representation of the nonlinearity $\mathcal{N}(v, v, v)$ by writing \mathfrak{W}^v . We also often use the same notation for the mixed terms of v and the stochastic objects. For instance, we let \mathfrak{W}^v stand for $\mathcal{N}(\mathfrak{I}, v, v)$, \mathfrak{W}^v for $\mathcal{N}(\mathfrak{I}, v, v)$, \mathfrak{W}^v for $\mathcal{N}(v, \mathfrak{I}, v)$, \mathfrak{W}^v for $\mathcal{N}(v, v, X)$, \mathfrak{W}^v for $\mathcal{N}(\mathfrak{I}, \mathfrak{I}, v)$, and \mathfrak{W}^v for \mathfrak{W}^v . Exceptions to this convention are product for which we need renormalization, that is (2.27) and also $\mathfrak{W}^v = \mathcal{N}(v, \mathfrak{I}, \mathfrak{I}) - \mathfrak{C}^1 v = \sum_{c=1}^d \int_{\mathbf{T}} \mathfrak{W}^c(x_c, y_c) v(x_{\hat{c}}, y_c) dy_c$.

We can then rewrite (2.26) as

$$\mathcal{L}v = -\mathfrak{W}^v - \mathfrak{W}^v - \mathfrak{W}^v - \mathfrak{W}^v - \mathfrak{W}^v - \mathfrak{W}^v - \mathfrak{W}^v, \quad (2.31)$$

with initial condition $v(0) = \Pi_N v_0$.

2.2.1 Random operators for T_3^4

In $d = 3$, \mathfrak{I} is of regularity $-\frac{1}{2}-$ and, by (2.30), we see that \mathfrak{W} and \mathfrak{K} are of the same regularity, so

$$\mathcal{N}(\mathfrak{I}, v, v), \mathcal{N}(v, \mathfrak{I}, v), \mathcal{N}(v, v, \mathfrak{I}), \mathcal{N}(v, \mathfrak{I}, \mathfrak{I}) - \mathfrak{C}^1 v, \text{ and } \mathcal{N}(\mathfrak{I}, \mathfrak{I}, v) \quad (2.32)$$

are all well-defined in the limit as $N \uparrow \infty$ as long as v can be taken to be of regularity better than $\frac{1}{2}$, and are of regularity at worst $-\frac{1}{2}-$. However, the term $\mathcal{N}(\mathfrak{I}, v, \mathfrak{I}) = \mathfrak{W}^v$ remains problematic. We get for

⁶Proofs for (2.30) are contained in our proofs of random operator bounds, see Sections 2.6.6.1 and 2.6.6.2

free that the exterior product of two copies of the free field, denoted $\mathfrak{!} \otimes \mathfrak{!}$, is well-defined without any renormalization but we cannot assume regularity better than $-1-$. Feeding only this into a deterministic argument would give $\mathfrak{!} \mathfrak{!}$ of regularity $-1-$ which means the equation for v could only be solved in $C_T C^{1-\epsilon}$, but pairing v with $\mathfrak{!} \otimes \mathfrak{!}$ to obtain $\mathcal{N}(\mathfrak{!}, v, \mathfrak{!})$ requires v of regularity better than 1.

The same issue is encountered in dynamical Φ_3^4 where the product $\mathfrak{!} v$ is problematic – it requires that v is of regularity greater than 1, which is hindered by the regularity $-1-$ of $\mathfrak{!}$. However we can overcome this issue in \mathbf{T}_3^4 with much less work - since the roughness of the two free fields appearing is (in law) isotropic, we could hope the integration in the non-local product $\mathfrak{!} \mathfrak{!}$ compensates their irregularity. We make this intuition rigorous by promoting $\mathfrak{!} \otimes \mathfrak{!}$ to a random operator, and viewing $\mathfrak{!} \mathfrak{!}$ as this random operator acting on v . To simplify our presentation we write all the mixed terms in (2.32) as random operators.

Definition 2.15 Given $G \in C^\infty(\mathbf{R}_{\geq 0} \times \mathbf{T}^d)$ and $J \subset [d]$, we define operators $G^J : C^\infty(\mathbf{R}_{\geq 0} \times \mathbf{T}^J) \rightarrow C^\infty(\mathbf{R}_{\geq 0} \times \mathbf{T}^J)$, where $\hat{J} = [d] \setminus [J]$, by setting

$$G_N^J(f)(t, y_J) \stackrel{\text{def}}{=} \int_{\mathbf{T}^{\hat{J}}} G(t, y_{\hat{J}}, y_J) f(y_{\hat{J}}) dy_{\hat{J}}. \quad (2.33)$$

For the case of $c \in [d]$ we also write $G^c = G^{\{c\}}$ and $G^{\hat{c}} = G^{[d] \setminus \{c\}}$. We write $\mathfrak{!}_N^J$, $\mathfrak{!}_N^c$, and $\mathfrak{!}_N^{\hat{c}}$ for the above convention applied with $G = \mathfrak{!}_N$.

Lemma 2.16 (Random operators 1) Let $p \in [1, \infty)$.

For any $d \geq 2$, $0 \leq k \leq d$, $J \subset [d]$ with $|J| = k$, and $\alpha > \frac{d-k-2}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{!}_N^J\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-k}), C_T C^{\beta-\epsilon}(\mathbf{T}^k))}^p] < \infty \text{ with } \beta = \min\left(-\frac{k-2}{2}, \alpha - \frac{d-2}{2}\right). \quad (2.34)$$

We also define, for any $c \in [d]$, the three random operators

$$\begin{aligned} \mathfrak{!}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N^c(f, \mathfrak{!}_N, \mathfrak{!}_N) - \mathfrak{C}_N^{1,c}(d)f, \text{ for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}_c \rightarrow \mathbf{R}, \\ \mathfrak{!}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N^c(\mathfrak{!}_N, \mathfrak{!}_N, f), \text{ for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}_c^{d-1} \rightarrow \mathbf{R}, \\ \mathfrak{!}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N(\mathfrak{!}_N, f, \mathfrak{!}_N), \text{ for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}^d \rightarrow \mathbf{R}. \end{aligned} \quad (2.35)$$

Then, in any dimension $d \geq 2$, for all $c \in [d]$ and $\alpha > \frac{d-3}{2}$,

$$\begin{aligned} \sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{!}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-1}), C_T C^{\beta-\epsilon}(\mathbf{T}^{d-1}))}^p] &< \infty \text{ with } \beta = \min\left(-\frac{d-3}{2}, \alpha - \frac{2d-5}{2}\right), \\ \sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{!}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^d), C_T C^{\beta-\epsilon}(\mathbf{T}^d))}^p] &< \infty \text{ with } \beta = \min\left(-\frac{d-3}{2}, \alpha - (d-2)\right). \end{aligned} \quad (2.36)$$

Finally, for $d \in \{2, 3, 4\}$, $c \in [d]$, and $\alpha > -\frac{1}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{!}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}), C_T C^{\beta-\epsilon}(\mathbf{T}))}^p] < \infty \text{ with } \beta = \min\left(\frac{1}{2}, \alpha - \frac{2d-5}{2}\right). \quad (2.37)$$

Moreover, one has convergence of the above objects in the corresponding $L^p(\Omega)$ spaces as $N \rightarrow \infty$.

Notation 2.17 We again omit the superscript c on $\mathfrak{!}_N^c$ and $\mathfrak{!}_N^c$ when the color c is being summed, for instance $\mathfrak{!}_N = \sum_{c \in [d]} \mathfrak{!}_N^c$. Overloading the notation, we will also write

$$\begin{aligned} \|\mathfrak{!}_N\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}), C_T C^{\beta-\epsilon}(\mathbf{T}))} &= \max_{c \in [d]} \|\mathfrak{!}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}), C_T C^{\beta-\epsilon}(\mathbf{T}))}, \\ \|\mathfrak{!}_N\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-1}), C_T C^{\beta-\epsilon}(\mathbf{T}^{d-1}))} &= \max_{c \in [d]} \|\mathfrak{!}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-1}), C_T C^{\beta-\epsilon}(\mathbf{T}^{d-1}))}, \\ \|\mathfrak{!}_N^{(k)}\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-k}), C_T C^{\beta-\epsilon}(\mathbf{T}^k))} &= \max_{J \subset [d], |J|=k} \|\mathfrak{!}_N^J\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-k}), C_T C^{\beta-\epsilon}(\mathbf{T}^k))}. \end{aligned}$$

Remark 2.18 In Remark 2.13 we claim that \mathfrak{V}' and \mathfrak{V}'' see a $+1/2$ regularizing effect versus \mathfrak{V} in the local theory. However, we are only able to obtain stochastic regularisation of the non-local product $\mathcal{N}(f, g, h)$ when two consecutive arguments of \mathcal{N} – that is (f, g) or/and (g, h) – are explicit stochastic objects. We don't obtain a stochastic regularization effect for \mathfrak{V}'' which makes this term more difficult in $d = 4$, more so than \mathfrak{V}' or \mathfrak{V} – see Remark 2.32.

2.2.2 Closing the fixed point problem

We now formulate (2.31) as a fixed point problem in $C_T \mathcal{C}^{\frac{3}{2}-}$. We will use the following deterministic estimates on our nonlinearity.

Notation 2.19 At this point of the article we will start presenting deterministic estimates that will also be useful to apply when \mathfrak{I} is replaced by the more complicated ansatz \mathfrak{I} we use when $d = 4$. For this reason, we extend our definitions of random fields and random operators and corresponding norms (for instance, (2.35) along with Notations 2.14 and 2.17) to the situation where all instances of the cut-off \mathfrak{I} are replaced by a cut-off of an unknown random field \mathfrak{I} . However, we eventually finalize the definition of \mathfrak{I} in our main proof and the statements of our main theorems for the Langevin dynamic setting – choosing $\mathfrak{I} = \mathfrak{I}$ when $d \in \{2, 3\}$ and being given by (2.50) when $d = 4$.

Note that in our definition of renormalized quantities such as \mathfrak{V}^c and \mathfrak{V}''^c , we use the *same* (dimension-dependent) renormalization constants whether we are working with \mathfrak{I} or \mathfrak{I} .

Lemma 2.20 *Let $d \in \{3, 4\}$. Let $\mathfrak{I}, v : \in C^\infty(\mathbf{R}_{\geq 0} \times \mathbf{T}^d)$. We then have the following estimates*

$$\| \mathfrak{V}^v \|_{C_T \mathcal{C}^{-\frac{1}{2}+\epsilon}} \lesssim \| \mathfrak{I}^{(d-1)} \|_{\mathcal{L}(C_T H^{\frac{3}{2}-3\epsilon}(\mathbf{T}), C_T \mathcal{C}^{-\frac{1}{2}-\frac{\epsilon}{4}}(\mathbf{T}^{d-1}))} \| v \|_{C_T L^\infty} \| v \|_{C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}}, \quad (2.38)$$

$$\| \mathfrak{V}^v \|_{C_T \mathcal{C}^{-\frac{1}{2}+\epsilon}} \lesssim \| \mathfrak{I}^{(1)} \|_{\mathcal{L}(C_T H^{\frac{3}{2}-3\epsilon}(\mathbf{T}^{d-1}), C_T \mathcal{C}^{-\frac{1}{2}-\frac{\epsilon}{4}}(\mathbf{T}))} \| v \|_{C_T L^\infty} \| v \|_{C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}}, \quad (2.39)$$

$$\| \mathfrak{V}^v \|_{C_T \mathcal{C}^{-\frac{1}{2}+\epsilon}} \lesssim \| \mathfrak{I}^{(1)} \|_{\mathcal{L}(C_T H^{\frac{3}{2}-3\epsilon}(\mathbf{T}^{d-1}), C_T \mathcal{C}^{-\frac{1}{2}-\frac{\epsilon}{4}}(\mathbf{T}))} \| v \|_{C_T L^\infty} \| v \|_{C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}}, \quad (2.40)$$

$$\| \mathfrak{V}^v \|_{C_T \mathcal{C}^{-\frac{1}{2}+\epsilon}} \lesssim \| \mathfrak{I} \|_{\mathcal{L}(C_T H^{\frac{3}{2}-3\epsilon}, C_T \mathcal{C}^{-\frac{1}{2}-\frac{\epsilon}{4}})} \| v \|_{C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}}, \quad (2.41)$$

$$\| \mathfrak{V}^v \|_{C_T \mathcal{C}^{-\frac{1}{2}+\epsilon}} \lesssim \| \mathfrak{V}^c \|_{\mathcal{L}(C_T H^{\frac{3}{2}-3\epsilon}, C_T \mathcal{C}^{-\frac{1}{2}-\frac{\epsilon}{4}})} \| v \|_{C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}}. \quad (2.42)$$

Remark 2.21 The 2ϵ 's in the regularity of v in the RHS are here for convenience, for a reason that will become clear when we study \mathbb{T}_4^4 .

Proof. We only prove the inequalities for one term quadratic in v , \mathfrak{V}^v , and one term linear in v , \mathfrak{V}''^v . The proofs for the three other terms are very similar (taking into account the renormalization in \mathfrak{V}''^v) and left to the reader. These bounds rely on the random operators estimates from Lemma 2.16 as well as the bilocal Sobolev embeddings given in Appendix 2.G.1, equations (2.122) and (2.122).

We first prove (2.38). Using the embedding $L_{x_c}^\infty \mathcal{C}_{x_{\hat{c}}}^{-\frac{1}{2}-\frac{\epsilon}{4}} \hookrightarrow \mathcal{C}_{x_c, x_{\hat{c}}}^{-\frac{1}{2}+\epsilon}$ given in (2.123), we have

$$\begin{aligned} \| \mathfrak{V}^v \|_{C_T \mathcal{C}^{-\frac{1}{2}+\epsilon}} &\lesssim \sum_{c=1}^d \| \mathcal{N}^c(\mathfrak{I}, v, v)(x_c, x_{\hat{c}}) \|_{C_T \mathcal{C}_{x_c, x_{\hat{c}}}^{-\frac{1}{2}+\epsilon}} \\ &\lesssim \sum_{c=1}^d \| \mathcal{N}^c(\mathfrak{I}, v, v)(x_c, x_{\hat{c}}) \|_{C_T L_{x_c}^\infty \mathcal{C}_{x_{\hat{c}}}^{-\frac{1}{2}-\frac{\epsilon}{4}}} \\ &\lesssim \| v \|_{C_T L^\infty} \sum_{c=1}^d \| \mathfrak{I}(v(\cdot, y_{\hat{c}}))(x_{\hat{c}}) \|_{C_T L_{y_{\hat{c}}}^\infty \mathcal{C}_{x_{\hat{c}}}^{-\frac{1}{2}-\frac{\epsilon}{4}}}. \end{aligned}$$

Here, we can use the estimate (2.34) on the random tensor $\mathfrak{I}^{(d-1)}$:

$$\| \mathfrak{V}^v \|_{C_T \mathcal{C}^{-\frac{1}{2}+\epsilon}} \lesssim \| \mathfrak{I}^{(d-1)} \|_{\mathcal{L}(C_T H^{\frac{3}{2}-3\epsilon}, C_T \mathcal{C}^{-\frac{1}{2}-\frac{\epsilon}{4}})} \| v \|_{C_T L^\infty} \| v(y_c, y_{\hat{c}}) \|_{C_T L_{y_{\hat{c}}}^\infty \mathcal{C}_{y_c}^{\frac{3}{2}-3\epsilon}}.$$

Finally, we can conclude using the embedding $\mathcal{C}_{y_c, y_{\hat{c}}}^{\frac{3}{2}-2\epsilon} \hookrightarrow L_{y_{\hat{c}}}^\infty \mathcal{C}_{y_c}^{\frac{3}{2}-3\epsilon}$ (2.121).

Let us now deal with (2.41). Once again, using the embedding $L_{x_c, x_{\hat{c}}}^\infty \mathcal{C}_{x_{\hat{c}}}^{-\frac{1}{2}-\frac{\epsilon}{4}} \hookrightarrow \mathcal{C}_{x_c, x_{\hat{c}}}^{-\frac{1+\epsilon}{2}}$ yields:

$$\begin{aligned} \|\mathfrak{J}^v\|_{C_T \mathcal{C}^{-\frac{1+\epsilon}{2}}} &\lesssim \sum_{c=1}^d \|\mathcal{N}^c(\mathfrak{I}, \mathfrak{I}, v)(x_c, x_{\hat{c}})\|_{C_T \mathcal{C}_{x_c, x_{\hat{c}}}^{-\frac{1+\epsilon}{2}}} \\ &\lesssim \sum_{c=1}^d \|\mathcal{N}^c(\mathfrak{I}, \mathfrak{I}, v)(x_c, x_{\hat{c}})\|_{C_T L_{x_c}^\infty \mathcal{C}_{x_{\hat{c}}}^{-\frac{1}{2}-\frac{\epsilon}{4}}} \lesssim \sum_{c=1}^d \|\mathfrak{J}^c(v(x_c, \cdot))(x_{\hat{c}})\|_{C_T L_{x_c}^\infty \mathcal{C}_{x_{\hat{c}}}^{-\frac{1}{2}-\frac{\epsilon}{4}}}. \end{aligned}$$

Using (2.54), we have

$$\|\mathfrak{J}^v\|_{C_T \mathcal{C}^{-\frac{1+\epsilon}{2}}} \lesssim \|\mathfrak{J}\|_{\mathcal{L}(C_T H^{\frac{3}{2}-3\epsilon}, C_T \mathcal{C}^{-\frac{1}{2}-\frac{\epsilon}{4}})} \|v(x_c, y_{\hat{c}})\|_{C_T L_{x_c}^\infty \mathcal{C}_{y_{\hat{c}}}^{\frac{3}{2}-3\epsilon}},$$

and we can conclude using (2.121). \square

Define $(\mathbb{X}_{f,3}^{\text{LD}})_N$ to be the tuple of all the random fields defined in Lemma 2.12 and $(\mathbb{X}_{o,3}^{\text{LD}})_N$ to be the tuple of all the random operators defined in Lemma 2.16, both taken at the same $N \in \mathbf{N} \sqcup \{\infty\}$. We again often suppress N from the notation.

We define our enhanced noise in $d = 3$, $\mathbb{X}_3^{\text{LD}} = \mathbb{X}_{f,3}^{\text{LD}} \sqcup \mathbb{X}_{o,3}^{\text{LD}}$ to be the concatenation of these tuples. It is an element of the product of the Banach spaces in which the random fields and random operators live. In particular, for $T > 0$ we introduce the following norm⁷:

$$\|\mathbb{X}_3^{\text{LD}}\|_T = \max \left(\max_{\tau \in \mathbb{X}_{f,3}^{\text{LD}}} \|\tau\|_{C_T \mathcal{C}^{\beta_\tau - \epsilon}}, \max_{\tau \in \mathbb{X}_{o,3}^{\text{LD}}} \|\tau\|_{\mathcal{L}(C_T H^{\alpha_\tau}, C_T \mathcal{C}^{\beta_\tau - \epsilon})} \right), \quad (2.43)$$

where for any stochastic object τ , α_τ and β_τ are the inner and outer regularities of τ as stated in Lemmas 2.12 and 2.16. The stochastic estimates in the above mentioned lemmas imply that, for any $T < \infty$, and for any $1 \leq p < \infty$, \mathbb{X}_3^{LD} converges as $N \rightarrow \infty$ in the stochastic space $L^p(\Omega; \|\cdot\|_T)$.

Notation 2.22 Note that in the discussion around (2.43) we are badly overloading notation by using the same sets and symbols to denote indexing set of a tuple and the actual components of the tuple – we do this to keep notation to a minimum since the meaning should be clear.

We write $A \lesssim_{\mathbb{X}_3^{\text{LD}}} B$ whenever there exists two positive constants C and c such that $A \leq C \|\mathbb{X}_3^{\text{LD}}\|_1^c B$. With this notation, we have the following proposition.

Proposition 2.23 *For any $v_0 \in \mathcal{C}^{\frac{3}{2}-\epsilon}(\mathbb{T}^3)$, there exists random time $\bar{T} \in (0, 1]$ such that equation (2.31) with initial condition $v(0) = \Pi_N v_0$ admits a unique solution v converging as $N \uparrow \infty$ in $C([0, \bar{T}], \mathcal{C}^{\frac{3}{2}-\epsilon}(\mathbb{T}^3))$. If $\bar{T} < 1$, then we have $\lim_{t \uparrow \bar{T}} \|v(t)\|_{\mathcal{C}^{\frac{3}{2}-\epsilon}} = \infty$. Moreover, if $T < \bar{T}$, the solution on $[0, T]$ depends continuously on x and on the enhanced noise set \mathbb{X}_3^{LD} (w.r.t the topology of $\|\cdot\|_T$).*

Proof. We write the mild formulation of (2.31) as a fixed point map for the map

$$\Xi(v) = P_t x - \int_0^t P_{t-s} (v_{\mathfrak{B}^v}^v + v_{\mathfrak{B}^v}^v + v_{\mathfrak{B}^v}^v + v_{\mathfrak{B}^v}^v + v_{\mathfrak{B}^v}^v + v_{\mathfrak{B}^v}^v + v_{\mathfrak{B}^v}^v - v_{\mathfrak{B}^v}^v)(s) ds.$$

It suffices to show that Ξ is a contraction mapping a ball in $C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$ to itself. We first argue that, for any given $R > 0$, one can take $T \in (0, 1]$ small enough to make Ξ a well-defined contraction on the ball of radius R in $C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$. Let $v_1, v_2 \in C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$ with $\|v_1\|_{C_T \mathcal{C}^{\frac{3}{2}-\epsilon}}, \|v_2\|_{C_T \mathcal{C}^{\frac{3}{2}-\epsilon}} \leq R$. Throughout this proof we write $v = v_1 - v_2$.

⁷Technically a semi-norm, but in practice we will only work with these objects up to the given time T

By the Schauder estimate (2.120), one has

$$\|\Xi(v_1) - \Xi(v_2)\|_{C_T C^{\frac{3}{2}-\epsilon}} \lesssim T^{\frac{\epsilon}{4}} \|v_{\mathfrak{B}}^v + \mathfrak{B}^v + \mathfrak{B}^v + v_{\mathfrak{B}}^v + \mathfrak{B}^v + \mathfrak{B}^v + v_{\mathfrak{B}}^v\|_{C_T C^{-\frac{1+\epsilon}{2}}}.$$

For the first term $v_{\mathfrak{B}}^v = \mathcal{N}(v, v, v)$, we have

$$\begin{aligned} \|v_{\mathfrak{B}}^v\|_{C_T C^{-\frac{1+\epsilon}{2}}} &\lesssim \|v_{\mathfrak{B}}^v\|_{C_T L^\infty} \lesssim \|v\|_{C_T L^\infty}^3 \\ &\lesssim (1 + \|v_1\|_{C_T C^{\frac{3}{2}-\epsilon}}^2 + \|v_2\|_{C_T C^{\frac{3}{2}-\epsilon}}^2) \|v_1 - v_2\|_{C_T C^{\frac{3}{2}-\epsilon}}. \end{aligned} \quad (2.44)$$

Using (2.38)- (2.42) for the mixed terms, we obtain

$$\begin{aligned} \|\mathfrak{B}^v + \mathfrak{B}^v + v_{\mathfrak{B}}^v + \mathfrak{B}^v + v_{\mathfrak{B}}^v\|_{C_T C^{-\frac{1+\epsilon}{2}}} &\lesssim_{\mathbb{X}_3^{\text{LD}}} (1 + \|v\|_{C_T L^\infty}) \|v\|_{C_T C^{\frac{3}{2}-\epsilon}} \\ &\lesssim_{\mathbb{X}_3^{\text{LD}}} (1 + \|v_1\|_{C_T C^{\frac{3}{2}-\epsilon}} + \|v_2\|_{C_T C^{\frac{3}{2}-\epsilon}}) \|v_1 - v_2\|_{C_T C^{\frac{3}{2}-\epsilon}}, \end{aligned}$$

Finally, by the random operator estimate (2.37),

$$\|\mathfrak{B}^v\|_{C_T C^{-\frac{1+\epsilon}{2}}} \lesssim \|\mathfrak{B}^v\|_{C_T \mathcal{L}(H^{\frac{3}{2}-\epsilon}, C^{-\frac{1+\epsilon}{2}})} \|v_1 - v_2\|_{C_T C^{\frac{3}{2}-\epsilon}}.$$

We have thus obtained that

$$\|\Xi(v_1) - \Xi(v_2)\|_{C_T C^{\frac{3}{2}-\epsilon}} \lesssim_{\mathbb{X}_3^{\text{LD}}} T^{\frac{\epsilon}{4}} (1 + \|v_1\|_{C_T C^{\frac{3}{2}-\epsilon}}^2 + \|v_2\|_{C_T C^{\frac{3}{2}-\epsilon}}^2) \|v_1 - v_2\|_{C_T C^{\frac{3}{2}-\epsilon}}.$$

Our contractive estimate follows from taking $T < 1$ small enough, depending only on R and $\|\mathbb{X}_3^{\text{LD}}\|_1$. By a standard modification of the above argument that one can take $R > 0$ sufficiently big, depending only on $\|v_0\|_{C^{\frac{3}{2}-\epsilon}}$, so that Ξ maps the ball of radius R in $C_1 C^{\frac{3}{2}-\epsilon}$ to itself.

The Banach fixed point theorem then gives us an existence time T depending only on $\|v_0\|_{C^{\frac{3}{2}-\epsilon}}$ and $\|\mathbb{X}_3^{\text{LD}}\|_1$. Existence up to time to the time \bar{T} and blow up if $\bar{T} < 1$ then follows by a standard iteration argument. Continuity in the data (both the initial data and \mathbb{X}_3^{LD} up to time 1) follows along similar lines to our contractive estimate. The convergence as $N \uparrow \infty$ follows from convergence of the stochastic data and $\Pi_N v_0$. \square

2.3 Renormalization and analysis of dynamical T_4^4

In this section we cover the renormalization of T_4^4 and also give results on energy estimates and local well-posedness.

We now give an overview of the issues that appear in $d = 4$. While writing $\phi = \mathfrak{I} + v$ was enough for $d = 3$, in $d = 4$ such a remainder v would not have positive regularity – the purely stochastic term \mathfrak{V} in (2.31) is of regularity $-2-$.

There are three main arguments that appear in this section.

- We can remove the term \mathfrak{V} by doing a second term in our ansatz for ϕ given by $\underline{\mathcal{L}}^{-1} \mathfrak{V}$, this introduces new stochastic nonlocal products of the form $\mathcal{N}(\cdot, \mathfrak{I}, \underline{\mathcal{L}}^{-1} \mathfrak{V})$ and reveals to us the need to introduce a second renormalization constant (see Lemma 2.25).
- Even after we introduce this additional renormalization to control these new products, we will see the renormalized products of the form $\mathfrak{I} \times \mathfrak{I} \times \underline{\mathcal{L}}^{-1} \mathfrak{V}$ have regularity $-1-$. This turns out to be a problem for our L^2 estimates used for tightness since the pairing of this object with v requires more regularity from v than is given by its H^1 norm – this good term only lets us pair v with something of regularity $-1+$ see the proof of (2.130). To overcome this we introduce a third term in our ansatz (see (2.49)) after which the remaining purely stochastic terms are of regularity $-1/2-$.

- We have one more challenge for local well-posedness, the estimate (2.36) in $d = 4$ shows that if v is of regularity $\alpha \in (\frac{1}{2}, \frac{3}{2})$, then $\mathcal{L}^{-1}\mathcal{N}(\dagger, v, \dagger)$ is of regularity $\alpha - \epsilon$ for $\epsilon > 0$, so that one can not close the fixed point. This is why we also reinject the remainder equation into itself to make it a well-posed remainder equation (see the discussion at the beginning of Section 2.3.5, and in Remark 2.51).

The second and third item below are mostly arguments for the deterministic part of the analysis, while we will see we can avoid the use of regularity structures or paracontrolled calculus, the third item shows we will need more complicated expansions for local well-posedness than we need in order to prove energy estimates. Since our main interest here is using energy estimates on the dynamic at stationary to prove tightness of cut-off T_4^4 measures, we'll discuss energy estimates first and afterwards study local well-posedness.

While we do not use our local well-posedness to construct any measure, the difficulties that we encounter in this result also appear when trying to prove energy estimates in dimensions $d \in (4, 5)$, therefore we still include the argument for local well-posedness for T_4^4 as it sheds light on how the behavior of subcritical T^4 differs from that of T_4^4 .

2.3.1 Snowball renormalization and stochastic objects

The second term of our ansatz is given in the following lemma.

Lemma 2.24 *Define*

$$\Psi_N \stackrel{\text{def}}{=} \mathcal{L}^{-1} \Psi_N.$$

Then Ψ is stationary in space and time, and, for $p \in [1, \infty)$, verifies

$$\sup_{N \in \mathbb{N}} \mathbf{E}[\|\Psi_N\|_{C_T C^{-\epsilon}(\mathbb{T}^4)}^p] < \infty. \quad (2.45)$$

Moreover, one has convergence of the above object in the corresponding $L^p(\Omega)$ space as $N \rightarrow \infty$.

Since we are subtracting both \dagger and Ψ from the solution, we must control cross terms mixing \dagger and Ψ . Indeed, it turns out that nonlinearities of the form $\mathcal{N}(\cdot, \dagger, \Psi)$ or $\mathcal{N}(\cdot, \Psi, \dagger)$ are divergent. We introduce a second renormalization constant to compensate the divergence caused by the melonic pairings of \dagger with Ψ , this renormalization is shown in the second column of Figure 2.3.

We now make precise what we mean by the contribution from the melonic pairing. We write

$$\begin{aligned} \mathbf{E}[\mathcal{N}^c(\phi, \dagger, \Psi^{c'})] &= \mathbf{E}_{(1234)}[\mathcal{N}^c(\phi, \dagger^1, \mathcal{L}^{-1}(\mathcal{N}^{c'}(\dagger^2, \dagger^3, \dagger^4) - \mathfrak{C}^{1,c'} \dagger^2))] \\ &= \mathbf{E}_{(12)(34)}[\mathcal{N}^c(\phi, \dagger, \Psi^{c'})] + \mathbf{E}_{(13)(24)}[\mathcal{N}^c(\phi, \dagger, \Psi^{c'})] + \mathbf{E}_{(14)(23)}[\mathcal{N}^c(\phi, \dagger, \Psi^{c'})], \end{aligned}$$

In the first equality above we view the four instances of the white noise as four different random variables (distinguished by their superscripts) that are almost surely equal. In the second equality, we work with two independent copies of the white noise – in $\mathbf{E}_{(12)(34)}$ we have $\xi^i = \xi^j$ almost surely for $(ij) \in \{(12), (34)\}$ but ξ^1 / ξ^2 is independent of ξ^3 / ξ^4 . The notation $\mathbf{E}_{(13)(24)}$ is defined analogously – the second equality is then just a cumbersome way to write Wick's rule. The term with subscript (12)(34) above is the *melonic part* of $\mathbf{E}[\mathcal{N}^c(\phi, \dagger, \Psi^{c'})]$.

We then define, for $d \geq 2$ and $c \in [d]$,

$$\mathfrak{C}_N^{2,c} \stackrel{\text{def}}{=} \mathbf{E}_{(12)(34)}[\mathcal{N}_N^c(1, \dagger_N(t), \Psi_N(t))(x)] + \mathbf{E}_{(12)(34)}[\mathcal{N}_N^c(1, \Psi_N(t), \dagger_N(t))(x)], \quad (2.46)$$

where for the second term we are using the convention

$$\mathbf{E}_{(12)(34)}[\mathcal{N}_N^c(\phi, \Psi_N(t), \dagger_N(t))(x)] = \mathbf{E}_{(12)(34)}[\mathcal{N}^c(\phi, \mathcal{L}^{-1}(\mathcal{N}^{c'}(\dagger^2, \dagger^3, \dagger^4) - \mathfrak{C}^{1,c'} \dagger^2), \dagger^1)].$$

Note that both terms on the RHS of (2.46) are equal. Moreover, by stationarity $\mathfrak{C}_N^{2,c}(d)$ does not depend on (t, x) . The renormalized products are given by setting for any function ψ of space,

$$\mathcal{N}_N^c(\psi, \mathfrak{I}_N, \Psi_N) - \frac{1}{2} \mathfrak{C}_N^{2,c}(d) \psi \quad \text{and} \quad \mathcal{N}_N^c(\psi, \Psi_N, \mathfrak{I}_N) - \frac{1}{2} \mathfrak{C}_N^{2,c}(d) \psi.$$

Lemma 2.25 For $d \geq 2$ and $c \in [d]$,

$$\mathfrak{C}_N^{2,c}(d) = \sum_{c' \neq c} \sum_{m_{\hat{c}} \in \mathbf{Z}_N^{d-1}} \sum_{n_{\hat{c}} \in \mathbf{Z}^{d-1}} \frac{1}{\langle m_{\hat{c}} \rangle^4} \mathfrak{R}_N^{1,c'}(\chi^{c'}(n, m)),$$

where \mathfrak{R}_N^1 is as defined in equation (2.23). Unlike $\mathfrak{C}_N^{1,c}$, $\mathfrak{C}_N^{2,c}(d)$ does depend on c . Writing

$$\mathfrak{C}_N^2(d) \stackrel{\text{def}}{=} \sum_{c=1}^d \mathfrak{C}_N^{2,c}(d), \quad (2.47)$$

we have, for $d \in \{2, 3, 4\}$ and any $\psi, f \in C^\infty(\mathbf{T}^d)$,

$$\begin{aligned} \sup_{N \in \mathbf{N}} \mathbf{E}[(\mathcal{N}_N(\psi, \mathfrak{I}_N, \Psi_N) - \frac{1}{2} \mathfrak{C}_N^2 \psi)(f)^2] &< \infty, \\ \sup_{N \in \mathbf{N}} \mathbf{E}[(\mathcal{N}_N(\psi, \Psi_N, \mathfrak{I}_N) - \frac{1}{2} \mathfrak{C}_N^2 \psi)(f)^2] &< \infty. \end{aligned}$$

Moreover, the same statement holds for the four other ways of substituting ψ , \mathfrak{I}_N and Ψ_N into \mathcal{N}_N (with no renormalization). We sometimes suppress d from the notation, just writing \mathfrak{C}_N^2 .

Proof. Let us pause one moment to prove that \mathfrak{C}^2 indeed renormalizes products of the form $\mathcal{N}(\psi, \mathfrak{I}, \Psi)$ and $\mathcal{N}(\psi, \mathfrak{I}, \Psi)$. This will be the occasion to verify that it is correctly defined, and that its second expression is correct. In the proof, to lighten the notations, we drop the dependence of the stochastic objects in N . Let us start from

$$\begin{aligned} &\mathbf{E}[\mathcal{N}_N^c(\psi, \Psi(t), \mathfrak{I}(t))(x)] \\ &= \sum_{m, n \in \mathbf{Z}_N^d} e^{-ix \cdot m} \hat{\psi}_{-m_{\hat{c}}, -n_c} \mathbf{E}[\hat{\Psi}_n(t) \hat{\mathfrak{I}}_{-n_{\hat{c}}, -m_c}(t)] \\ &= \sum_{m, n, p \in \mathbf{Z}_N^d} \sum_{c'=1}^d e^{-ix \cdot m} \hat{\psi}_{-m_{\hat{c}}, -n_c} \int_{-\infty}^t ds e^{-(t-s)\langle n \rangle^2} \\ &\quad \mathbf{E}[(\hat{\mathfrak{I}}_{p_{c'}, n_{c'}}(s) \hat{\mathfrak{I}}_{-p}(s) \hat{\mathfrak{I}}_{n_{c'}, p_{c'}}(s) - \mathfrak{C}_N^{1,c'} \hat{\mathfrak{I}}_n(s)) \hat{\mathfrak{I}}_{-n_{\hat{c}}, -m_c}(t)] \\ &= \sum_{m, n, p \in \mathbf{Z}_N^d} \sum_{c'=1}^d e^{-ix \cdot m} \hat{\psi}_{-m_{\hat{c}}, -n_c} \int_{-\infty}^t ds e^{-(t-s)\langle n \rangle^2} (\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3), \end{aligned}$$

We expand the expectation as the sum of three terms \mathfrak{A}_1 , \mathfrak{A}_2 and \mathfrak{A}_3 according to whether the fourth noise contracts with respectively third one, the second one, or the first one. \mathfrak{A}_1 corresponds to the melonic part of the expectation and includes the counterterm $\mathfrak{C}^{1,c'}$. More explicitly,

$$\mathfrak{A}_1 = \delta_{n_c, m_c} \delta_{p_{c'}, n_{c'}} \frac{e^{-(t-s)\langle n \rangle^2}}{\langle n \rangle^2} \left(\frac{\Pi_N(p_{c'}, n_{c'})}{\langle (p_{c'}, n_{c'}) \rangle^2} - \frac{\Pi_N(p_{c'})}{\langle p_{c'} \rangle^2} \right),$$

while \mathfrak{A}_2 and \mathfrak{A}_3 are given by

$$\mathfrak{A}_2 = \delta_{p, -n} \delta_{n_c, m_c} \frac{e^{-(t-s)\langle n \rangle^2}}{\langle n \rangle^4}, \quad \mathfrak{A}_3 = \delta_{p_{c'}, n_{c'}} \delta_{n_c, m_c} \frac{e^{-(t-s)\langle n \rangle^2}}{\langle n \rangle^2} \frac{\Pi_N(n_{c'}, p_{c'})}{\langle (n_{c'}, p_{c'}) \rangle^2}.$$

The contributions from \mathfrak{A}_2 and \mathfrak{A}_3 are convergent while \mathfrak{A}_1 gives

$$\begin{aligned} & \sum_{m \in \mathbf{Z}^d; n_{\hat{c}}, p_{c'} \in \mathbf{Z}^{d-1}} \Pi_N(n_{\hat{c}}, m_c) \sum_{c'=1}^d e^{-ix \cdot m} \hat{\psi}_{-m} \int_{-\infty}^t \frac{e^{-(t-s)\langle (n_{\hat{c}}, m_c) \rangle^2}}{\langle (n_{\hat{c}}, m_c) \rangle^2} \mathfrak{R}_N^{1,c'}(\chi^{c'}(p, n)) \\ &= \frac{1}{2} \sum_{m \in \mathbf{Z}^d; n_{\hat{c}}, p_{\hat{c}} \in \mathbf{Z}^{d-1}} \sum_{c'=1}^d e^{-ix \cdot m} \hat{\psi}_{-m} \frac{\Pi_N(n_{\hat{c}}, m_c)}{\langle (n_{\hat{c}}, m_c) \rangle^4} \mathfrak{R}_N^{1,c'}(\chi^{c'}(p, n)). \end{aligned}$$

As we did with the first renormalisation, we only subtract the value of the divergent expression at zero external Fourier mode, that is $m_c = 0$, since this is sufficient in order to obtain a convergent expression. Thus, the quantity we subtract from $\mathbf{E}[\mathcal{N}_N^c(\psi, \Psi_N(t), \mathfrak{I}_N(t))(x)]$ is given by

$$\frac{1}{2} \psi(x) \sum_{n_{\hat{c}} \in \mathbf{Z}_N^{d-1}} \sum_{p_{\hat{c}} \in \mathbf{Z}^{d-1}} \sum_{c'=1}^d \frac{1}{\langle n_{\hat{c}} \rangle^4} \mathfrak{R}_N^{1,c'}(\chi^{c'}(p, n)).$$

Here, we can observe that when $c' = c$, $\mathfrak{R}_N^{1,c'}$ is evaluated at $(p_{c'}, 0)$ and we recall from (2.23) that $\mathfrak{R}_N^{1,c'}(p_{c'}, 0) = 0$. This means that the first renormalization totally renormalizes the melonic product of \mathfrak{I} times $\Psi^c = \underline{\mathcal{L}}^{-1}(\mathcal{N}^c(\mathfrak{I}, \mathfrak{I}, \mathfrak{I}) - \mathfrak{C}^{1,c}\mathfrak{I})$. Therefore, we have finally obtained that

$$\mathbf{E}_{(12)(34)}[\mathcal{N}_N^c(\psi, \Psi_N(t), \mathfrak{I}_N(t))(x)] = \frac{1}{2} \mathfrak{C}_N^{2,c} \psi(x) + \mathcal{O}(1),$$

where the quantity $\mathcal{O}(1)$ is a convergent sum uniformly in N , while the non melonic part involving $\mathbf{E}_{(13)(24)}$ and $\mathbf{E}_{(14)(23)}$ is finite. The computations for $\mathbf{E}[\mathcal{N}_N^c(\psi, \mathfrak{I}_N(t), \Psi_N(t))(x)]$ are similar. Finally, setting $\psi = 1$ proves (2.47). \square

Inspired by the previous discussion, we define for $d \geq 4$ and $c, c' \in [d]$, $c \neq c'$ the second renormalized amplitude:

$$\mathfrak{R}_N^{2,c,c'}(m, n) \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\Pi_N(m)}{\langle m \rangle^4} - \frac{\Pi_N(m_{\hat{c}})}{\langle m_{\hat{c}} \rangle^4} \right) \mathfrak{R}_N^{1,c'}(n).$$

The renormalized melonic pairing $\sum_{n \in \mathbf{Z}^d} \hat{\psi}_{-m_{\hat{c}}, -n_c} \mathbf{E}[\hat{\Psi}_n^{\mathfrak{I}}(t) \hat{\mathfrak{I}}_{-n_{\hat{c}}, -m_c}(t)] - \frac{1}{2} \mathfrak{C}^2 \hat{\psi}_m$ (as well as its counterpart with \mathfrak{I} and Ψ switched) thus contains

$$\hat{\psi}_m \sum_{n_{\hat{c}} \in \mathbf{Z}^{d-1}} \sum_{p_{\hat{c}} \in \mathbf{Z}^{d-1}} \sum_{c' \neq c} \mathfrak{R}_N^{2,c,c'}(\chi^c(n, m), \chi^{c'}(p, n)).$$

We denote its value at $N = \infty$ by $\mathfrak{R}^{2,c,c'}(m, n) \stackrel{\text{def}}{=} \mathfrak{R}_\infty^{2,c,c'}(m, n)$.

Remark 2.26 \mathfrak{C}_∞^2 is finite when $d = 3$ so we didn't encounter it in Section 2.2. \mathfrak{C}_N^2 diverges like $\log N$ when $d = 4$, while it diverges like N^{2d-8} for $d \geq 5$.

With the second renormalization constant \mathfrak{C}_N^2 defined, our regularized and renormalized Langevin dynamic for T_4^4 is given by

$$\mathcal{L}\phi_N = -\mathcal{N}_N(\phi_N, \phi_N, \phi_N) + (\mathfrak{C}_N^1 - \mathfrak{C}_N^2)\phi_N + \sqrt{2}\xi_N. \quad (2.48)$$

We now give the third contribution to our ansatz along with a stochastic estimate.

Lemma 2.27 (Random fields 2) *Let*

$$\Psi_N \stackrel{\text{def}}{=} (\mathcal{N}_N(\Psi_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1(d)\Psi_N) + (\mathcal{N}_N(\mathfrak{I}_N, \Psi_N, \mathfrak{I}_N) + \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \Psi_N) - \mathfrak{C}_N^2(d)\mathfrak{I}_N),$$

and

$$\mathfrak{I}_N \stackrel{\text{def}}{=} \mathcal{L}^{-1}\Psi_N, \quad (2.49)$$

which is stationary in space and time. Then, in $d = 4$, for $p \in [1, \infty)$

$$\sup_{N \in \mathbb{N}} \mathbf{E}[\|\Psi_N\|_{C_T C^{1-\epsilon}(\mathbb{T}^4)}^p] < \infty \quad \text{and} \quad \sup_{N \in \mathbb{N}} \mathbf{E}[\|\mathfrak{I}_N\|_{C_T C^{1-\epsilon}(\mathbb{T}^4)}^p] < \infty.$$

Moreover, one has convergence of the above objects in the corresponding $L^p(\Omega)$ spaces as $N \rightarrow \infty$.

We now introduce the total ansatz \mathfrak{I} needed for us to solve the T_4^4 dynamic, it is given by

$$\mathfrak{I} \stackrel{\text{def}}{=} \mathfrak{I} - \Psi + \Psi. \quad (2.50)$$

Note that \mathfrak{I} is stationary in space and time. The term \mathfrak{I} is the roughest term in \mathfrak{I} and so one expects \mathfrak{I} and \mathfrak{I} to have the same path-wise regularity. However, for probabilistic estimates, \mathfrak{I} is a more complicated non-Gaussian object.

Below we adopt much of the same notation from Section 2.2 but where we replace instances of \mathfrak{I} with \mathfrak{I} , for instance writing $\mathfrak{I}^{(k)}$, \mathfrak{I} , \mathfrak{I} and \mathfrak{I} – see Lemma 2.31 for explicit formula. We also define the random field

$$\Psi_N \stackrel{\text{def}}{=} \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \mathfrak{I}_N) - (\mathfrak{C}_N^1 - \mathfrak{C}_N^2)\mathfrak{I}_N.$$

They share the regularity properties of $\mathfrak{I}^{(k)}$, \mathfrak{I} , \mathfrak{I} and Ψ stated in Lemmas 2.12 and 2.16. In particular, we recall that in $d = 4$, Ψ is of regularity $-2-$.

Notation 2.28 In the end of the present section, we again use the pictorial notations introduced in the three dimensional case. In particular, recall that \mathfrak{I}_v stands for $\mathcal{N}(v, \mathfrak{I}, \mathfrak{I}) - (\mathfrak{C}^1 - \mathfrak{C}^2)v$ and that we systematically drop the dependencies in N .

Shifting the unknown by \mathfrak{I} and introducing $v = v_N \stackrel{\text{def}}{=} \phi_N - \mathfrak{I}_N$ in (2.48) gives

$$\mathcal{L}v = -\mathfrak{I}_v^v - \mathfrak{I}_v^v - \mathfrak{I}_v^v - \mathfrak{I}_v^v - \mathfrak{I}_v^v - \mathfrak{I}_v^v - \mathfrak{I}_v^v - \mathcal{S}, \quad v(0) = \Pi_N v_0, \quad (2.51)$$

where the \mathcal{S} is a purely stochastic random field that does not involve v , it is of the form

$$\begin{aligned} \mathcal{S}_N &\stackrel{\text{def}}{=} \Psi_N - \Psi_N + \Psi_N \\ &= (\mathcal{N}_N(\Psi_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1\Psi_N) + (\mathcal{N}_N(\Psi_N, \mathfrak{I}_N, \Psi_N) - \frac{1}{2}\mathfrak{C}_N^2\Psi_N) + \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \Psi_N) + 20 \text{ terms}. \end{aligned}$$

\mathcal{S} consists of purely stochastic objects that contains at least seven instances of the noise, for that reason we call it the “septic” object, Lemma 2.29 below shows that \mathcal{S} will not need any additional renormalization.

Lemma 2.29 (Random fields 3) *In $d = 4$ and for $p \in [1, \infty)$,*

$$\sup_{N \in \mathbb{N}} \mathbf{E}[\|\mathcal{S}_N\|_{C_T C^{-\frac{1}{2}-\epsilon}(\mathbb{T}^4)}^p] < \infty.$$

Moreover, one has convergence of the above object in the corresponding $L^p(\Omega)$ space as $N \rightarrow \infty$.

Remark 2.30 The regularity estimate we obtain for \mathcal{S} is worse than what would expect from a direct power counting argument, see Notation 2.65. The reason is analogous to the standard fact that in deterministic estimates, objects of positive regularity don't give a gain in product estimates - recall that for $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ with $\alpha < 0 < -\alpha < \beta$, one has $fg \in \mathcal{C}^\alpha$, there is no gain from β . Similar things happen with our non-local product, and we will not get a gain from objects like \mathfrak{V} which are positive regularity in $d = 4$.

In regularity structures, one can work with a power counting that does see this gain by working with generalized increments of objects of positive regularity but translating this to our non-local setting is unclear. We deal with this issue by mixing stochastic and deterministic estimates, see the proof of Lemma 2.29 in Section 2.6.7.2.

As mentioned in the introduction, we point out that the remainder equation (2.51) involves only stochastic objects that are well-defined in the $N \rightarrow \infty$ limit and no renormalization constants. This is in contrast with dynamic Φ_3^4 where, regardless of how far one goes in the perturbative ansatz, one will always see a term $C_N \mathfrak{V} v$ where v is again the remainder and C_N is a divergent renormalization constant. While this difference makes it possible to use more classical machinery to solve (2.51) versus what is needed it for Φ_3^4 , the argument is still not completely straightforward – see Remark 2.32 below.

2.3.2 Random operators for T_4^4

As in $d = 3$, we view the mixed terms in (2.51) as random operators acting on the solution.

Lemma 2.31 (Random operators 2) *Let $k \in [3]$ and $c \in [4]$. The random operators made with the rough shift \mathfrak{I} are defined as*

$$\begin{aligned} \mathfrak{I}_N^{(k)}(f) &\stackrel{\text{def}}{=} \int_{\mathbf{T}^{4-k}} \mathfrak{I}_N(\cdot, y) f(y) dy, & \text{for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}^{4-k} &\rightarrow \mathbf{R}, \\ \mathfrak{V}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N^c(f, \mathfrak{I}_N, \mathfrak{I}_N) - (\mathfrak{C}_N^{1,c} - \mathfrak{C}_N^{2,c}) f, & \text{for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}_c &\rightarrow \mathbf{R}, \\ \mathfrak{F}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N^c(f, \mathfrak{I}_N, \mathfrak{I}_N), & \text{for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}_c^3 &\rightarrow \mathbf{R}, \\ \mathfrak{B}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N(\mathfrak{I}_N, f, \mathfrak{I}_N) & \text{for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}^4 &\rightarrow \mathbf{R}. \end{aligned}$$

Then, in $d = 4$, for all $\alpha > \frac{2-k}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{I}_N^{(k)}\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{4-k}), C_T C^{\beta-\epsilon}(\mathbf{T}^k))}^p] < \infty \text{ with } \beta = \min(-\frac{k-2}{2}, \alpha - 1). \quad (2.52)$$

Moreover, in $d = 4$, for all $c \in [4]$ and $\alpha > 0$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{V}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}), C_T C^{\beta-\epsilon}(\mathbf{T}))}^p] < \infty \text{ with } \beta = \min(0, \alpha - \frac{3}{2}). \quad (2.53)$$

Finally, in $d = 4$, for all $p \in [1, \infty)$, $c \in [4]$, and $\alpha > \frac{1}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{F}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^3), C_T C^{\beta-\epsilon}(\mathbf{T}^3))}^p] < \infty \text{ with } \beta = \min(-\frac{1}{2}, \alpha - \frac{3}{2}), \quad (2.54)$$

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{B}_N^c\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^4), C_T C^{\beta-\epsilon}(\mathbf{T}^4))}^p] < \infty \text{ with } \beta = \min(-\frac{1}{2}, \alpha - 2). \quad (2.55)$$

Moreover, one has convergence of the above objects in the corresponding $L^p(\Omega)$ spaces as $N \rightarrow \infty$.

Remark 2.32 Lemmas 2.20 and 2.31 control the mixed terms $\mathfrak{V}_N^c v$, $\mathfrak{F}_N^c v$, $\mathfrak{B}_N^c v$ and $\mathfrak{B}_N^c v$ in $\mathcal{C}^{-\frac{1}{2}-\epsilon}$ if v is of regularity $\frac{3}{2} - \epsilon$. However, $\mathfrak{V}_N^c v$ remains problematic. Regardless of our assumption on the regularity of v , the best regularity estimate we can hope for $\mathfrak{V}_N^c v$ is $-\frac{1}{2}-$, which means we can not expect v to better

than $\frac{3}{2} - \epsilon$. However, if $v \in \mathcal{C}^{\frac{3}{2}-\epsilon}$, then $v_{\square} \in \mathcal{C}^{-\frac{1}{2}-2\epsilon}$, and the argument never closes. This term in the \mathbf{T}_4^4 dynamic requires a bigger departure from the approach we used for \mathbf{T}_3^4 , in that sense it is analogous to the product $\forall v$ in dynamical Φ_3^4 .

We now turn to proving energy estimates and tightness, deferring our solution to Remark 2.32 until Section 2.3.5.

2.3.3 Energy estimates and coming down from infinity

In this section, we show two different kinds of estimates on the L^2 norm of the solution to (2.51). In Proposition 2.33 we use the damping effect of the nonlinearity to establish a bound on the L^2 norm of v that is uniform in the initial condition – this is called a “coming down from infinity” estimate. In Proposition 2.40 we use the nonlinearity to prove an energy estimate which is better suited for proving tightness of the invariant measures.

Key ingredients here are estimates on mixed terms of the remainder v with stochastic objects that show they can be bounded by the coercive terms. Since we will want to state and prove these estimates on mixed terms in a way where they can be used for both the Langevin dynamic and the variational approach, we postpone these estimates to Appendix 2.1 and only cite them here.

We first collect the stochastic objects that will appear in our analysis. Define $\mathbb{X}_{f,4}^{\text{LD}}$ to be the tuple of all the random fields defined in Lemmas 2.27 and 2.29, and $\mathbb{X}_{o,4}^{\text{LD}}$ the tuple of all the random operators defined in Lemma 2.31. We define the enhanced noise set $(\mathbb{X}_4^{\text{LD}})_N = \mathbb{X}_4^{\text{LD}} = \mathbb{X}_{f,4}^{\text{LD}} \sqcup \mathbb{X}_{o,4}^{\text{LD}}$ to be the concatenation of these tuples. For any $T > 0$, we equip the space of such objects with the norm $\|\cdot\|_T$ defined as in (2.43) with the maximum over the larger sets $\mathbb{X}_{f,4}^{\text{LD}}$ and $\mathbb{X}_{o,4}^{\text{LD}}$, and with $d = 3$ replaced with $d = 4$ in the computation with of α_τ and β_τ .

The stochastic estimates in the above mentioned lemmas imply that, for any $T < \infty$, and for any $1 \leq p < \infty$, \mathbb{X}_4^{LD} converges as $N \rightarrow \infty$ in the stochastic space $L^p(\Omega; \|\cdot\|_T)$ – however for the present section we only need to know that they are bounded uniformly in N . In this section and Section 2.3.3, we work with $N \in \mathbf{N}$ finite but arbitrary, but these statements would extend to $N = \infty$ without much more effort.

We also use the notation $A \lesssim_{\mathbb{X}_4^{\text{LD}}} B$ as the natural analog of the notation given in Notation 2.22.

Proposition 2.33 (L^2 coming down from infinity) *Recall from Section 2.3.5 the definition of the enhanced noise \mathbb{X}_4^{LD} and its norm $\|\cdot\|_t$ (see equation (2.43)). Pick $T < \bar{T}$ and $v_0 \in \mathcal{C}^{\frac{3}{2}-2\epsilon}(\mathbf{T}^4)$, and let $v_x = X + Y$ with (X, Y) the solution to (2.68) at fixed N with initial condition $(0, \Pi_N v_0)$. There exists a constant $\gamma > 0$ such that for all $t \in [0, T]$, v_x obeys the bound:*

$$\|v(t, v_0)\|_{L^2(\mathbf{T}^4)} \lesssim \|\mathbb{X}_4^{\text{LD}}\|_t^\gamma \max(1, t^{-\frac{3}{2}}). \quad (2.56)$$

Observe that the upper bound on $\|v(t, v_0)\|_{L^2(\mathbf{T}^4)}$ is a positive random variable, uniform in N , that only depends on the randomness on $[0, t]$ (not in the future of t), is in $L^p(\Omega)$ for all $1 \leq p < \infty$, and that it is independent of the initial condition v_0 .

Remark 2.34 Analogous estimates in $L^p(\mathbf{T}^d)$ for $p > 2$ seems to be out of the reach. When pairing the nonlinearity with some higher powers of v , we obtain expressions such as $(v^{p-1}, \mathcal{N}(v, v, v))$, which are neither norms, nor even easily provable to be positive. Moreover, establishing some M^p estimates is not very useful, since we would only obtain them for $p \geq 2$ only, and they would therefore be weaker than the L^2 one. We also note that the non-locality of the nonlinearity prevents us from easily using a maximum principle.

Remark 2.35 The estimate (2.56) would be sufficient to infer that (2.68) has global solutions if it were possible to restart the equation for Y from an L^2 initial condition. While we choose not to dive further into this direction, there is hope that this is possible, paying the price of solving the equation in $H^{\frac{3}{2}-\epsilon}$

rather than $\mathcal{C}^{\frac{3}{2}-\epsilon}$. Indeed, in the same way that $\|v^3\|_{L^\infty} \lesssim \|v\|_{L^\infty}^3$, by definition the nonlinearity satisfies the estimate

$$\|\mathcal{N}(v, v, v)\|_{L^2} \lesssim \|v\|_{M^6}^3 \lesssim \|v\|_{L^2}^3,$$

where we used the embedding $L^2 \hookrightarrow M^6$ stated in (2.124). The second Picard iteration of an L^2 initial condition would therefore be easily controlled in a Sobolev space.

Remark 2.36 In a previous section we pointed out that the term $\mathfrak{v}_{\mathfrak{B}}^v$ will be a challenge for the local in time existence of (2.51), and will require the re-injected system (2.68) in the sequel. However, this re-injection is not necessary when proving tightness or L^2 a priori estimates for dynamic \mathbf{T}_4^4 . When pairing the equation with its solution to derive an L^2 energy estimate, $\mathfrak{v}_{\mathfrak{B}}^v$ becomes $\mathfrak{v}_{\mathfrak{B}_v}^v$, which, by Cauchy Schwarz inequality, is bounded by $\mathfrak{v}_{\mathfrak{B}_v}^v$. The crucial fact is that $\mathfrak{v}_{\mathfrak{B}_v}^v$ can be defined without any new renormalization – after the pairing the term $\mathfrak{v}_{\mathfrak{B}_v}^v$ becomes similar to the better-behaved term $\mathfrak{v}_{\mathfrak{B}}^v$ so the ansatz is not necessary. This observation is also used for treating \mathbf{T}_4^4 in Section 2.4.

While we can use this simplification for $d = 4$, it is not generic in the full subcritical regime. Closer to the critical dimension, formally when d becomes larger than 4.5 – that is to say when the noise is of regularity worse than $-13/4-$, the term $\mathcal{L}^{-1}\mathfrak{v}_{\mathfrak{B}}^v$ (and also $\mathcal{L}^{-1}\mathfrak{v}_{\mathfrak{B}_v}^v$) maps \mathcal{C}^α to $\mathcal{C}^{\alpha-\epsilon}$ for some $\epsilon > 0$. In this case closing tightness and a priori bound estimates will require both a more complicated ansatz and also re-injections similar to what we will use to handle the term $\mathfrak{v}_{\mathfrak{B}}^v$ for local well-posedness of \mathbf{T}_4^4 .

In order to prove Proposition 2.33, we need this a priori estimate on the solution to (2.51):

Proposition 2.37 (A priori estimate) *Let $T < \bar{T} \wedge 1$ and $v = X + Y$ with (X, Y) the solution to (2.68) on $[0, T] \times \mathbf{T}^4$ with initial condition $(0, \Pi_N v_0)$ for $v_0 \in \mathcal{C}^{\frac{3}{2}-2\epsilon}$, and pick $s, t \in [0, T]$, $s < t$. There exists a constant $\kappa > 0$ such that v obeys the following a priori estimate:*

$$\|v(t)\|_{L^2}^2 + \int_s^t \left(\|v(r)\|_{H^1}^2 + \|v(r)\|_{M^4}^4 \right) dr \lesssim \|\mathbb{X}_4^{\text{LD}}\|_t^\kappa + \|v(s)\|_{H^1}^{\frac{3}{2}} + \|v(s)\|_{M^4}^3.$$

Proof. Recall that by the solution theory established in Section 2.3.5, v solves (2.51). Therefore, let us pair Equation (2.51) with the solution v . For $r \in [s, t]$, we obtain

$$\frac{1}{2} \partial_r \|v(r)\|_{L^2}^2 + \|v(r)\|_{H^1}^2 + \|v(r)\|_{M^4}^4 = - \left(3 \mathfrak{v}_{\mathfrak{B}_v}^v + \mathfrak{v}_{\mathfrak{B}_v}^v + \mathfrak{v}_{\mathfrak{B}_v}^v + \mathfrak{v}_{\mathfrak{B}_v}^v + (\mathcal{S}, v) \right) (r), \quad (2.57)$$

where $\mathfrak{v}_{\mathfrak{B}_v}^v \stackrel{\text{def}}{=} (\mathfrak{v}_{\mathfrak{B}_v}^v, v)$, $\mathfrak{v}_{\mathfrak{B}_v}^v \stackrel{\text{def}}{=} (\mathfrak{v}_{\mathfrak{B}_v}^v, v)$, $\mathfrak{v}_{\mathfrak{B}_v}^v \stackrel{\text{def}}{=} (\mathfrak{v}_{\mathfrak{B}_v}^v, v)$ and $\mathfrak{v}_{\mathfrak{B}_v}^v \stackrel{\text{def}}{=} (\mathfrak{v}_{\mathfrak{B}_v}^v, v)$.

We now use the bounds on the mixed terms that are proven in Appendix 2.I. It is at this step that we take advantage of the fact that we have pushed the Da Prato - Debussche expansion far enough so that all the remaining purely stochastic terms on the RHS are of regularity $-1/2-$, this allows us to control the pairing (\mathcal{S}, v) using the H^1 norm of v . By (2.130), (2.131), (2.132), (2.133) and (2.135), we infer the existence of an exponent $\kappa > 0$ such that

$$\frac{1}{2} \partial_r \|v(r)\|_{L^2}^2 + \|v(r)\|_{H^1}^2 + \|v(r)\|_{M^4}^4 \leq 8(\delta^{-1} \|\mathbb{X}_4^{\text{LD}}\|_r)^\kappa + 8\delta(\|v(r)\|_{H^1}^2 + \|v(r)\|_{M^4}^4).$$

Setting $\delta = \frac{1}{16}$ yields

$$\partial_r \|v(r)\|_{L^2}^2 + \|v(r)\|_{H^1}^2 + \|v(r)\|_{M^4}^4 \lesssim \|\mathbb{X}_4^{\text{LD}}\|_r^\kappa, \quad (2.58)$$

which we finally integrate between s and t . We thus have

$$\|v(t)\|_{L^2}^2 + \int_s^t \left(\|v(r)\|_{H^1}^2 + \|v(r)\|_{M^4}^4 \right) dr \lesssim \|\mathbb{X}_4^{\text{LD}}\|_t^\kappa + \|v(s)\|_{L^2}^2.$$

We deduce from the interpolation inequality (2.127) that

$$\|v(s)\|_{L^2}^2 \lesssim \|v(s)\|_{M^4} \|v(s)\|_{H^1} \lesssim \|v(s)\|_{H^1}^{\frac{3}{2}} + \|v(s)\|_{M^4}^3,$$

□

We need the following rewriting of the previous proposition:

Corollary 2.38 Define $F(t) \stackrel{\text{def}}{=} \|v(s)\|_{H^1}^{\frac{3}{2}} + \|v(s)\|_{M^4}^3$, uniform in $T < \bar{T} \wedge 1$, $s, t \in [0, T]$, $s < t$,

$$\|v(t)\|_{L^2}^2 \lesssim \|\mathbb{X}_4^{\text{LD}}\|_t^\kappa + F(s), \quad (2.59)$$

$$\int_s^t F(r)^{\frac{4}{3}} dr \lesssim \|\mathbb{X}_4^{\text{LD}}\|_t^\kappa + F(s). \quad (2.60)$$

Finally, we also need this comparison principle introduced by Mourrat & Weber:

Lemma 2.39 (Comparison: [MW17b], Lemma 7.3) Let $G : [0, T) \rightarrow \mathbf{R}_+$ continuous and such that for all $s, t \in [0, T)$, $s < t$, $\int_s^t G(r)^{\frac{4}{3}} dr \lesssim cG(s)$. Then there exists a sequence of times $0 < t_1 < \dots < t_N = T$ such that for all $n \in \{0, \dots, N-1\}$, we have

$$G(t_n) \lesssim c^3 t_{n+1}^{-3}.$$

With these lemmas at hand, we can now prove Proposition 2.33:

Proof of Proposition 2.33 Pick $t \in [0, T]$ for $T < \bar{T} \wedge 1$. If there exists $s < t$ such that $F(s) \leq \|\mathbb{X}_4^{\text{LD}}\|_t^\kappa$, then using (2.59), we immediately conclude that $\|v(t)\|_{L^2} \lesssim \|\mathbb{X}_4^{\text{LD}}\|_t^{\frac{\kappa}{2}}$. Otherwise, suppose that for all $s \in [0, t]$, we have $F(s) \geq \|\mathbb{X}_4^{\text{LD}}\|_t^\kappa$. Then, by Lemma 2.39, we thus have a sequence $0 < t_1 < \dots < t_N = T$ such that $F(t_n) \lesssim t_{n+1}^{-3}$. On the other hand, there exists a n such that $t \in [t_n, t_{n+1})$, so that taking $s = t_n$ in (2.59) yields $\|v(t)\|_{L^2} \lesssim \max(1, t_{n+1}^{-\frac{3}{2}}) \lesssim \max(1, t^{-\frac{3}{2}})$. Letting $T = 1$, we then iterate the argument on $[1, 2]$, starting the solution from $v(1)$, and then keep iterating in the same way. The proof then follows from the fact that the estimate we have obtained is uniform in the initial condition. □

We also have the weaker bound on the L^2 norm of v , that depends on the initial condition:

Proposition 2.40 Pick $p \geq 2$ and $0 \leq s < t < \bar{T}$. There exists a constant $\gamma > 0$ such that it holds

$$\int_s^t \|v(r)\|_{L^2(\mathbf{T}^d)}^p dr \lesssim (t-s) \|\mathbb{X}_4^{\text{LD}}\|_t^\gamma + \|v(s)\|_{L^2(\mathbf{T}^d)}^p. \quad (2.61)$$

Proof. We start from (2.58) and use the positivity of the M^4 norm to obtain

$$\partial_r \|v(r)\|_{L^2}^2 + \|v(r)\|_{H^1}^2 \lesssim \|\mathbb{X}_4^{\text{LD}}\|_r^\kappa.$$

Multiply this inequality by $\|v(r)\|_{L^2}^{p-2}$ yields for all $\delta \in (0, 1)$

$$\partial_t \|v(r)\|_{L^2}^p + \|v(r)\|_{L^2}^p \lesssim \|\mathbb{X}_4^{\text{LD}}\|_r^\kappa \|v(r)\|_{L^2}^{p-2} \leq C_\delta \|\mathbb{X}_4^{\text{LD}}\|_r^{\frac{p\kappa}{2}} + \delta \|v(r)\|_{L^2}^p,$$

where we used the embedding $H^1 \hookrightarrow L^2$, and then Young's inequality. The L^2 norm can then be reabsorbed on the LHS, and integrating between s and t finally yields the desired result. □

2.3.4 Tightness of the invariant measure

This section is devoted to the proof of Theorem 2.1. We recall from the introduction the definition of ν_N , (2.2) and the fact that (2.48) is chosen in a way such that at fixed $N \in \mathbf{N}$, it leaves ν_N invariant. Combining this with our L^2 estimates for the Langevin dynamic allows us to show tightness of the regularized and renormalized \mathbb{T}_4^4 measures.

We first rewrite (2.48) with $N < \infty$ in Fourier variables. This gives us a system of $(2N + 1)^4$ scalar stochastic ordinary differential equations: for $m \in \mathbf{Z}_N^4$ we have

$$\begin{aligned} d\hat{\phi}_m &= \left(\mathfrak{C}_N \hat{\phi}_m - \langle m \rangle^2 \hat{\phi}_m - \sum_{c=1}^4 \sum_{n \in \mathbf{Z}_N^4} \hat{\phi}_{m_{\hat{e}, n_c}} \hat{\phi}_{-n} \hat{\phi}_{n_{\hat{e}, m_c}} \right) dt + \sqrt{2} dB_t^m \\ &=: b^m(\phi) dt + \sqrt{2} dB_t^m, \end{aligned} \quad (2.62)$$

where $B_t^m \stackrel{\text{def}}{=} \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4 \times \mathbf{R}_{\geq 0}} \mathbf{1}_{[0,t]}(s) e^{ix \cdot m} \xi(dx, ds)$ is a complex Brownian motion, and we write $\mathfrak{C}_N = \mathfrak{C}_N^1 - \mathfrak{C}_N^2$. Note that we will always assume the reality condition $\overline{\hat{\phi}_m} = \hat{\phi}_{-m}$ which means we can view (2.62) as a random dynamic on $\mathbf{R}^{(2N+1)}$.

Since the dynamic is now finite dimensional, we can now use the following explosion criterion of Khasminskii to argue for global-in-time well-posedness.

Lemma 2.41 ([Kha11], Theorem 3.5) *Fix $N \in \mathbf{N}$. Consider a complex random vector valued process $(X^m(t))_{m \in \mathbf{Z}_N^4}$ satisfying the reality condition $X^{-m} = \overline{X^m}$ and solving the system of stochastic differential equations*

$$dX^m(t) = b^m(X(t))dt + \sqrt{2} dB_t^m, \quad m \in I_N, \quad (2.63)$$

where $(B_t^m)_{m \in I_N}$ is a collection of complex Brownian motions such that $B_t^{-m} = \overline{B_t^m}$ and B^m, B^n are independent provided $m + n \neq 0$, and $(b^m)_{m \in I_N}$ is a family of functions $b^m : \mathbf{C}^{I_N} \rightarrow \mathbf{C}$ verifying the reality condition $\overline{b^{-m}} = b^m$.

Then if b is locally Lipschitz, and if there exists a Lyapounov function $V(X)$ and a positive constant C such that $\inf_{x: |x| > R} V(x) \uparrow \infty$ when $R \uparrow \infty$ and $LV(x) \leq CV(x)$ with L the infinitesimal generator of X given by

$$L \stackrel{\text{def}}{=} \frac{1}{2} \sum_{m \in I_N} \left(b^m \partial_m + \overline{b^m} \partial_{-m} \right) + \sum_{m \in I_N} \partial_m \partial_{-m}, \quad (2.64)$$

the solution starting from X_0 at $t = 0$ is a regular almost surely continuous Markov process.

Corollary 2.42 *For fixed $N \in \mathbf{N}$, (2.62) admits global in time solutions for any initial data and the same holds for (2.48) for data of the form $\Pi_N v_0$.*

Remark 2.43 In contrast to Proposition 2.52, we only prove Corollary 2.42 for fixed cut-off N make not claim about the global in time existence of the limiting $N \uparrow \infty$ local in time solution.

Proof. We apply Lemma 2.41 to the system (2.62). We use for a Lyapounov function the L^2 norm:

$$V(\hat{\phi}) \stackrel{\text{def}}{=} 1 + \sum_{m \in \mathbf{Z}_N^4} |\hat{\phi}_m|^2 = 1 + \|\Pi_N \phi\|_{L^2(\mathbb{T}^4)}^2.$$

Whenever $V(\hat{\phi})$ is a function of $(\hat{\phi}_m)_{m \in \mathbf{Z}_N^4}$ we write $\partial_m V \stackrel{\text{def}}{=} \partial_{\hat{\phi}_m} V$ and $\bar{\partial}_m V \stackrel{\text{def}}{=} \partial_{\hat{\phi}_{-m}} V$. With this notation, one has

$$LV(\hat{\phi}) = \frac{1}{2} \sum_{m \in \mathbf{Z}_N^4} \left(b^m(\hat{\phi}) \partial_m V(\hat{\phi}) + \overline{b^m(\hat{\phi})} \bar{\partial}_m V(\hat{\phi}) \right) + \sum_{m \in \mathbf{Z}_N^4} \partial_m \bar{\partial}_m V(\hat{\phi})$$

$$\begin{aligned}
&= \sum_{m \in \mathbf{Z}_N^4} \left((\mathfrak{C}_N - \langle m \rangle^2) |\hat{\phi}_m|^2 - \sum_{c=1}^4 \sum_{n \in \mathbf{Z}_N^4} \hat{\phi}_{-m} \hat{\phi}_{m_\varepsilon, n_c} \hat{\phi}_{-n} \hat{\phi}_{n_\varepsilon, m_c} + 1 \right) \\
&= \mathfrak{C}_N \|\Pi_N \phi\|_{L^2(\mathbf{T}^4)}^2 - \|\Pi_N \phi\|_{H^1(\mathbf{T}^4)}^2 - \mathcal{I}(\Pi_N \phi) + (2N+1)^4 \\
&\leq |\mathfrak{C}_N| \|\Pi_N \phi\|_{L^2(\mathbf{T}^4)}^2 + (2N+1)^4 \lesssim_N V(\hat{\phi}),
\end{aligned}$$

where we used the positivity of $\|\Pi_N \phi\|_{H^1(\mathbf{T}^4)}^2$ and $\mathcal{I}(\Pi_N \phi)$. Above we write \lesssim_N to indicate the inequality holds up to a constant that grows with N .

Lemma 2.41 then shows the explosion time of \hat{u}_m is almost surely infinite. \square

Remark 2.44 Rewriting (2.64) for our dynamic, we can read off that the infinitesimal generator L of the system of stochastic ODEs (2.63), seen as an evolution on $\Pi_N \phi \in \mathbf{R}^{(2N+1)}$, is given by $-\Delta + \nabla S_N \cdot \nabla$ where $S_N(\Pi_N \phi) = \frac{1}{2} \|\Pi_N \phi\|_{H^1(\mathbf{T}^d)}^2 + \frac{1}{4} \|\Pi_N \phi\|_{M^4(\mathbf{T}^d)}^4 - \frac{\mathfrak{C}_N}{2} \|\Pi_N \phi\|_{L^2(\mathbf{T}^d)}^2$. Here Δ and ∇ denote the Laplacian and gradient on $\mathbf{R}^{(2N+1)}$, and in $S_N(\phi)$ we are naturally using our Fourier variables to identify $\mathbf{R}^{(2N+1)}$ with a space of trigonometric polynomials.

This implies that the dynamic does preserve the Π_N marginal of the measure ν_N defined by (2.6) with $a = 0$, $\lambda = 1$ and $a_N = \mathfrak{C}_N$.

We turn to the stationary solution ϕ_N to (2.48) with stationary measure ν_N as defined in (2.6). In particular, $(\hat{\phi}_{N,m})_{m \in \mathbf{Z}_N^4}$ solves (2.62). Note that, for $N < \infty$, we can construct a stationary in time coupling between the laws of ϕ_N , \mathfrak{I}_N , \mathfrak{Y}_N and \mathfrak{Y}_N^* , by using a finite dimensional Krylov-Bogolubov type argument for (2.62). Using this coupling we can then write $v_N = \phi_N - \mathfrak{I}_N$ where v_N is also stationary in time.

Lemma 2.45 For $p \in [1, \infty)$, it holds

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|v_N(0)\|_{L^2(\mathbf{T}^4)}^p] < \infty. \quad (2.65)$$

Proof. Without loss of generality we assume $p \geq 2$. By stationarity,

$$\mathbf{E}[\|v_N(0)\|_{L^2(\mathbf{T}^4)}^p] = \frac{1}{T} \int_0^T \mathbf{E}[\|v_N(r)\|_{L^2(\mathbf{T}^4)}^p] dr.$$

Combining this with (2.61) gives

$$\mathbf{E}[\|v_N(0)\|_{L^2(\mathbf{T}^4)}^p] \lesssim \mathbf{E}[\|\mathbb{X}_4^{\text{LD}}\|_T^\gamma] + \frac{1}{T} \mathbf{E}[\|v_N(0)\|_{L^2(\mathbf{T}^4)}^p].$$

The result follows from taking T sufficiently large which is allowed due to Corollary 2.42. \square

Proof of Theorem 2.1 (2.11) directly follows from (2.65). Regarding (2.10), (2.65) implies

$$\begin{aligned}
&\sup_{N \in \mathbf{N}} \mathbf{E}[\|\phi_N(0)\|_{H^{-1-\varepsilon}(\mathbf{T}^4)}^p] \\
&\leq \sup_{N \in \mathbf{N}} \mathbf{E}[\|\phi_N(0) - \mathfrak{I}_N(0)\|_{H^{-1-\varepsilon}(\mathbf{T}^4)}^p] + \sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{I}_N(0)\|_{H^{-1-\varepsilon}(\mathbf{T}^4)}^p] \\
&\leq \sup_{N \in \mathbf{N}} \mathbf{E}[\|\phi_N(0) - \mathfrak{I}_N(0)\|_{L^2(\mathbf{T}^4)}^p] + \sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{I}_N(0)\|_{C^{-1-\frac{\varepsilon}{2}}(\mathbf{T}^4)}^p] < \infty,
\end{aligned}$$

using the regularity estimate on \mathfrak{I} in Lemma 2.66. The same estimate holds in $H^{-1-2\varepsilon}(\mathbf{T}^4)$, and tightness stems for the fact that the embedding $H^{-1-\varepsilon} \hookrightarrow H^{-1-2\varepsilon}$ is compact. \square

2.3.5 Local well-posedness: closing the fixed-point problem

Our main result on local-wellposedness for $d = 4$ is the following.

Theorem 2.46 *For $d = 4$, there exist constants a_N such that one has uniform in N control of the local in time solutions to (2.9) with initial condition of the form $\mathfrak{I}_N(0) + \Pi_N v(0)$ with $v(0) \in \mathcal{C}^{\frac{3}{2}-\epsilon}$. In particular, using the ansatz $\phi = \mathfrak{I}_N + X + Y$ for the solution to equation with cut-off N , where \mathfrak{I}_N is the explicit stochastic object (2.49) with regularity (uniform in N) of $-1-$, one has that inexplicit remainders (X, Y) converge, as $N \uparrow \infty$, in $C([0, \bar{T}], \mathcal{C}^{\frac{3}{2}-2\epsilon}(\mathbf{T}^4)) \times C([0, \bar{T}], \mathcal{C}^{\frac{3}{2}-\epsilon}(\mathbf{T}^4))$ where $\bar{T} \in (0, \infty]$ is a random existence time.*

Remark 2.47 One way to show that the \mathbf{T}_4^4 measure is invariant under the \mathbf{T}_4^4 equation would be to show that the solution to the equation defines a Markov process on a state space on which the measure is supported - this strategy was adopted in [TW18] in order to construct the Φ_2^4 measure. However, for the deterministic equation $\mathcal{L}\phi = -\mathcal{N}(\phi, \phi, \phi)$, $s = -\frac{2}{3}$ is a critical value below which it is not possible to solve the equation with initial condition in \mathcal{C}^s and this makes it more difficult to adopt the same strategy, and so we instead simply consider an initial condition of the form dictated by our ansatz. Note that similar difficulties appear in the local theory - the threshold $-\frac{2}{3}$ is sufficient to deal with dynamical Φ_2^4 and Φ_3^4 but becomes a problem when trying to cover the full subcritical regime.

To overcome the difficulties mentioned earlier, we rewrite (2.51) using the following ansatz: we split the solution v as $v = X + Y$ where X and Y solve the system

$$\begin{cases} \mathcal{L}X = -\mathfrak{I}_N(X) - \mathfrak{I}_N(Y), & (2.66a) \\ \mathcal{L}Y = -\mathfrak{I}_N^v v - \mathfrak{I}_N^v v - \mathfrak{I}_N^v v - \mathfrak{I}_N^v v - \mathfrak{I}_N^v v - \mathfrak{I}_N^v v - \mathcal{S}, & (2.66b) \end{cases}$$

with initial conditions $(X(0), Y(0)) = (0, \Pi_N v_0)$. Note that v still stands for $X + Y$ in (2.66b). The RHS of (2.66b) gathers all the terms that don't pose any problem for closing the fixed point problem, that is we fix some choice of $X \in C_T \mathcal{C}^{1+}$ then we could close the equation (2.66b) for $Y \in C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$.

Turning to (2.66a), we see that if we fix $Y \in C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$ then we cannot close (2.66a) for X - the map $X \mapsto -\mathcal{L}^{-1} \mathfrak{I}_N(X + Y)$ is well defined for $X \in C_T \mathcal{C}^\alpha$ with $\alpha > \frac{1}{2}$ and that takes $C_T \mathcal{C}^\alpha$ to $C_T \mathcal{C}^{\min(\frac{3}{2}, \alpha)-\epsilon}$. We see that, whatever the value of α , it is mapping $C_T \mathcal{C}^\alpha$ into a larger space - while we can iterate this mapping and we can not use it to post a fixed point problem. While this slightly resembles how $\mathfrak{I}_N v$ poses a problem for the Φ_3^4 dynamic, the difficulty there is different in that it is impossible to compose $v \mapsto \mathcal{L}^{-1}(\mathfrak{I}_N v)$ for generic v of any fixed regularity.

It turns out that we can overcome our difficulty with (2.66a) by injecting this problematic term into itself and then performing a stochastic estimate on this term, see the next lemma.

Lemma 2.48 (Random operators 3) *Defining the operator \mathfrak{I}_N^v acting on $f : \mathbf{R}_{\geq 0} \times \mathbf{T}^4 \rightarrow \mathbf{R}$ as*

$$\mathfrak{I}_N^v(f) \stackrel{\text{def}}{=} \mathfrak{I}_N^v(\mathcal{L}^{-1} \mathfrak{I}_N^v(f)) = \mathcal{N}_N(\mathfrak{I}_N, \mathcal{L}^{-1} \mathcal{N}_N(\mathfrak{I}_N, f, \mathfrak{I}_N), \mathfrak{I}_N),$$

then, in $d = 4$ and for all $\alpha > \frac{1}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{I}_N^v\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^4), C_T \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^4))}^p] < \infty \text{ with } \beta = \min(-\frac{1}{2}, \alpha - 1). \quad (2.67)$$

Notation 2.49 Our use of symbolic notation in \mathfrak{I}_N^v is inconsistent with that of \mathfrak{I}_N so we take a moment to spell out the difference. Recalling that these trees are built with a non-commutative product, \mathfrak{I}_N^v contains only one of the three possible products - the one with the two \mathfrak{I} terms in the first and third positions. This is different then what we refer to with the symbol \mathfrak{I}_N which contains 3 terms for the 3 possible products at the root.

Reinjecting $X = -\mathcal{L}^{-1} \dot{\mathfrak{B}}(X + Y)$ in (2.66a) as described above gives us the system

$$\begin{cases} \mathcal{L}X = \mathfrak{V}(X) + \mathfrak{V}(Y) - \dot{\mathfrak{B}}(Y), \\ \mathcal{L}Y = -\mathfrak{v}^v - \mathfrak{v}^v - \mathfrak{v}^v - \mathfrak{v}^v - \mathfrak{v}^v - \mathfrak{v}^v - \mathcal{S}. \end{cases} \quad (2.68a)$$

$$\quad (2.68b)$$

Remark 2.50 This object was not included in the enhanced data \mathbb{X}_4^{LD} since it does not appear in our tightness estimates – it is only needed for local well-posedness. The estimate in (2.67) is proven by combining the Kolmogorov type estimate for time non-local random operators stated in Lemma 2.77 with the power counting analysis performed in Sections 2.6.6.5 and 2.6.7.3.

Remark 2.51 This reinjection is an analytic argument for proving $N \uparrow \infty$ convergence of solutions as the cut-off. In the smooth setting ($N < \infty$) both the original system (2.66a)+(2.66b) and the modified system (2.68a)+(2.68b) are well-posed and have same solutions. If we write the integral fixed point problem for (2.66a)+(2.66b) as

$$(x, y) \mapsto (\Theta_X(x, y), \Theta_Y(x, y)) \quad (2.69)$$

then the integral fixed point map for (2.68a) + (2.68b) can be written as

$$(x, y) \mapsto \left(\Theta_X(\Theta_X(x, y), y), \Theta_Y(x, y) \right). \quad (2.70)$$

Clearly if (2.69) has a fixed point (x_*, y_*) and we know (2.70) has a unique fixed point, then the fixed point of (2.70) must be given by (x_*, y_*) . However, (2.70) lets us argue convergence as $N \uparrow \infty$.

We define the Banach space $C_T \mathcal{C}^{\frac{3}{2}-2\epsilon} \times C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$ for the solution (X, Y) with norm given by $\|X, Y\|_{\text{sol}_T} \stackrel{\text{def}}{=} \|X\|_{C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}} \vee \|Y\|_{C_T \mathcal{C}^{\frac{3}{2}-\epsilon}}$.

Proposition 2.52 We have the following local well-posedness result for (2.68a)+(2.68b).

For any $v_0 \in \mathcal{C}^{\frac{3}{2}-\epsilon}(\mathbf{T}^4)$, there exists a random blow-up time \bar{T} such that the system (2.68) with initial condition $(X(0), Y(0)) = (0, \Pi_N v_0)$ admits a unique solution (X, Y) converging as $N \uparrow \infty$ in $C([0, \bar{T}], \mathcal{C}^{\frac{3}{2}-2\epsilon}) \times C([0, \bar{T}], \mathcal{C}^{\frac{3}{2}-\epsilon})$. This solution is maximal, and we have $\lim_{t \uparrow \bar{T}} \|X(t)\|_{\mathcal{C}^{\frac{3}{2}-2\epsilon}} \vee \|Y(t)\|_{\mathcal{C}^{\frac{3}{2}-\epsilon}} = +\infty$. Moreover, if $T < \bar{T}$, the solution on $[0, T]$ depends continuously on v_0 and on the enhanced noise set \mathbb{X}_4^{LD} (w.r.t the topology of $\|\mathbb{X}_4^{\text{LD}}\|_T$).

Proof. Let $\Xi(X, Y) = (\Xi_X(X, Y), \Xi_Y(X, Y))$ be the fixed point map of (2.68). We thus have

$$\begin{cases} \Xi_X(X, Y) = \int_0^t P_{t-s} \left(\mathfrak{V}(X) + \mathfrak{V}(Y) - \dot{\mathfrak{B}}(Y) \right)(s) ds, \\ \Xi_Y(X, Y) = P_t \Pi_N v_0 - \int_0^t P_{t-s} \left(\mathfrak{v}^v + \mathfrak{v}^v + \mathfrak{v}^v + \mathfrak{v}^v + \mathfrak{v}^v + \mathfrak{v}^v + \mathcal{S} \right)(s) ds, \end{cases}$$

where $v = X + Y$. We show that Ξ is indeed a contraction on $C_T \mathcal{C}^{\frac{3}{2}-2\epsilon} \times C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$, while the proof that it maps a ball of $C_T \mathcal{C}^{\frac{3}{2}-2\epsilon} \times C_T \mathcal{C}^{\frac{3}{2}-\epsilon}$ into itself is left to the reader. We thus evaluate Ξ at $(X_1 - X_2, Y_1 - Y_2)$. For the sake of notation, throughout this proof, we often write $X = X_1 - X_2$, $Y = Y_1 - Y_2$ and $v = X + Y$.

We first estimate $\Xi_X(X_1 - X_2, Y_1 - Y_2)$ in $C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}$. The Schauder estimate (2.120) gives

$$\|\Xi_X(X, Y)\|_{C_T \mathcal{C}^{\frac{3}{2}-2\epsilon}} \lesssim T^{\frac{\epsilon}{4}} \|\mathfrak{V}(X) + \mathfrak{V}(Y) - \dot{\mathfrak{B}}(Y)\|_{C_T \mathcal{C}^{-\frac{1+3\epsilon}{2}}}.$$

Using (2.55), we obtain

$$\|\dot{\mathfrak{B}}(Y)\|_{C_T \mathcal{C}^{-\frac{1+3\epsilon}{2}}} \lesssim \|\dot{\mathfrak{B}}\|_{\mathcal{L}(C_T \mathcal{C}^{\frac{3}{2}-\epsilon}, C_T \mathcal{C}^{-\frac{1+3\epsilon}{2}})} \|Y\|_{C_T \mathcal{C}^{\frac{3}{2}-\epsilon}},$$

and (2.67) yields

$$\|\mathfrak{V}(X)\|_{C_T C^{-\frac{1+3\epsilon}{2}}} \lesssim \|\mathfrak{V}\|_{\mathcal{L}(C_T C^{\frac{3}{2}-2\epsilon}, C_T C^{-\frac{1+3\epsilon}{2}})} \|X\|_{C_T C^{\frac{3}{2}-2\epsilon}},$$

and

$$\|\mathfrak{V}(Y)\|_{C_T C^{-\frac{1+3\epsilon}{2}}} \lesssim \|\mathfrak{V}\|_{\mathcal{L}(C_T C^{\frac{3}{2}-\epsilon}, C_T C^{-\frac{1+3\epsilon}{2}})} \|Y\|_{C_T C^{\frac{3}{2}-\epsilon}}.$$

We thus have

$$\|\Xi_X(X_1 - X_2, Y_1 - Y_2)\|_{C_T C^{\frac{3}{2}-2\epsilon}} \lesssim_{\mathbb{X}_4^{\text{LD}}} T^{\frac{\epsilon}{4}} \|X_1 - X_2, Y_1 - Y_2\|_{\text{sol}_T}.$$

Turning to the $C_T C^{\frac{3}{2}-\epsilon}$ norm of $\Xi_Y(X_1 - X_2, Y_1 - Y_2)$, by the Schauder estimate (2.120), we have

$$\|\Xi_Y(X, Y)\|_{C_T C^{\frac{3}{2}-\epsilon}} \lesssim T^{\frac{\epsilon}{4}} \left\| \begin{matrix} v_{\mathfrak{B}}^v \\ v_{\mathfrak{B}}^v \\ v_{\mathfrak{B}}^v \\ v_{\mathfrak{B}}^v \\ v_{\mathfrak{B}}^v \\ v_{\mathfrak{B}}^v \end{matrix} \right\|_{C_T C^{-\frac{1+2\epsilon}{2}}}.$$

The estimates (2.52), (2.53) and (2.54) show that the stochastic objects made with the rough shift \mathfrak{I} obey the same estimates than the objects made with \mathfrak{I} . In particular, Lemma 2.20 stills holds in our context, and the terms $v_{\mathfrak{B}}^v$, $v_{\mathfrak{B}}^v$, $v_{\mathfrak{B}}^v$, $v_{\mathfrak{B}}^v$ and $v_{\mathfrak{B}}^v$ can be handled in the same way we did in three dimensions, in the proof of Proposition 2.23. Recalling (2.44),

$$\begin{aligned} \|\Xi_Y(X, Y)\|_{C_T C^{\frac{3}{2}-\epsilon}} &\lesssim T^{\frac{\epsilon}{4}} (1 + \|v_1\|_{C_T C^{\frac{3}{2}-2\epsilon}}^2 + \|v_2\|_{C_T C^{\frac{3}{2}-2\epsilon}}^2) \|v_1 - v_2\|_{C_T C^{\frac{3}{2}-2\epsilon}} \\ &\lesssim T^{\frac{\epsilon}{4}} (1 + \|X_1, Y_1\|_{\text{sol}_T}^2 + \|X_2, Y_2\|_{\text{sol}_T}^2) \|X_1 - X_2, Y_1 - Y_2\|_{\text{sol}_T}, \end{aligned}$$

which gives

$$\|\Xi(X_1 - X_2, Y_1 - Y_2)\|_{\text{sol}_T} \lesssim T^{\frac{\epsilon}{4}} (1 + \|X_1, Y_1\|_{\text{sol}_T}^2 + \|X_2, Y_2\|_{\text{sol}_T}^2) \|X_1 - X_2, Y_1 - Y_2\|_{\text{sol}_T}.$$

Consequently, by choosing T small enough, depending only on $\|\mathbb{X}_4^{\text{LD}}\|_T$ and $\|v_0\|_{C^{\frac{3}{2}-\epsilon}}$, Ξ is indeed a contraction. Iterating this argument yields the desired existence of maximal solutions and showing continuity of this solution in the stochastic data is a standard argument.

Finally, convergence as $N \uparrow \text{inf ty}$ relies on the continuity in the stochastic data and the convergence of the stochastic objects. \square

2.4 The variational approach for tensor field theories

In this section, we follow the approach introduced in [BG20] to establish tightness of approximations of the T_4^4 measure. We follow notations of [BG20] which we briefly recall here. We consider $(\Omega, \mathcal{B}, \mathbf{P})$ a probability space endowed with a collection of complex Brownian motions $(B_t^m)_{m \in \mathbf{Z}^d}$, such that B^m and B^n are independent if $m \neq \pm n$ and $B^{-m} = \overline{B^m}$, and denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration of \mathcal{B} induced by these Brownian motions. \mathbf{E} will denote the expectation with respect to \mathbf{P} .

Let also ϱ be a smooth decreasing function $\mathbf{R}_{\geq 0} \rightarrow \mathbf{R}_{\geq 0}$ such that $\varrho|_{[0, \frac{1}{2}]} = 1$, $\varrho|_{[1, +\infty)} = 0$ and $|\varrho'| \leq 1$ on $[\frac{1}{2}, 1]$. For $t > 0$ we write $\varrho_t(\cdot) \stackrel{\text{def}}{=} \varrho(\langle \cdot \rangle / t)$ and also set for $x \in \mathbf{R}^4$ $\sigma_t(x) \stackrel{\text{def}}{=} \sqrt{\partial_t \varrho_t^2(x)}$. Finally, we define the Fourier multiplier $J_t \stackrel{\text{def}}{=} \sigma_t(\nabla) \langle \nabla \rangle^{-1}$, which verifies the following estimate.

Lemma 2.53 *For $\alpha \in \mathbf{R}$, J_t is continuous from H^α to $H^{\alpha+1}$ uniformly in $t \geq 0$.*

Proof. Using that $\langle x \rangle / t < 1$ and $|\varrho'|(\langle x \rangle / t) \leq 1$ on $\text{supp } \varrho_t$, we have

$$\begin{aligned} \sigma_t(x) &= \sqrt{\partial_t \varrho_t^2(x)} = \frac{1}{\sqrt{t}} \sqrt{2\varrho|\varrho'| \text{Id}(\langle x \rangle / t)} \lesssim t^{-1/2} \mathbf{1}_{\langle x \rangle < t} \\ &\lesssim t^{-1/2} \mathbf{1}_{\langle x \rangle < t} \mathbf{1}_{t < 1} + t^{-1/2} \mathbf{1}_{\langle x \rangle < t} \mathbf{1}_{t \geq 1}. \end{aligned}$$

If $t \geq 1$, then $t^{-1/2} \leq 1$ while if $t < 1$, $\mathbf{1}_{\langle x \rangle < t} \leq \mathbf{1}_{\langle x \rangle < 1} = 0$ (by definition, $\langle x \rangle \geq 1$). This means $\sigma_t(x) \lesssim 1$ uniformly in time so, for $\phi \in H^\alpha$, $J_t(m) \hat{\phi}_m \lesssim \langle m \rangle^{-1} \hat{\phi}_m$ which gives $J_t \phi \in H^{\alpha+1}$. \square

We also define, for $T \geq 0$ and $\phi : [0, T] \times \mathbf{T}^4 \rightarrow \mathbf{R}$, $J(\phi) : [0, T] \times \mathbf{T}^4 \rightarrow \mathbf{R}$, $(t, x) \mapsto J_t \phi(t, x)$ and $I(\phi) : [0, T] \times \mathbf{T}^4 \rightarrow \mathbf{R}$, $(t, x) \mapsto \int_0^t J_s \phi(s, x) ds$, and for $t \in [0, T]$ we sometimes write $I_t(\phi)(x)$ instead of $I(\phi)(t, x)$. I satisfies the following estimate.

Lemma 2.54 ([BG20], Lemma 2) *Let $\alpha \in \mathbf{R}$. Uniformly in $T \geq 0$ we have*

$$\|I(\phi)\|_{L^\infty H^{\alpha+1}(\mathbf{T}^4)} \lesssim \|\phi\|_{L_T^2 H^\alpha(\mathbf{T}^4)}. \quad (2.72)$$

In particular, Lemma 2.53 implies $\|I(J(\phi))\|_{L_T^\infty H^{\alpha+2}(\mathbf{T}^4)} \lesssim \|J(\phi)\|_{L_T^2 H^{\alpha+1}(\mathbf{T}^4)} \lesssim \|\phi\|_{L_T^2 H^\alpha(\mathbf{T}^4)}$ so

$$\|I(J(\phi))\|_{L_T^\infty H^{\alpha+2-\epsilon}(\mathbf{T}^4)} \lesssim \|\phi\|_{L_T^\infty C^\alpha(\mathbf{T}^4)}. \quad (2.73)$$

As mentioned before, in the stochastic processes appearing in Barashkov-Gubinelli approach the time t will play the role of a scale parameter. The most basic stochastic object is then given by $\mathfrak{I}_t \stackrel{\text{def}}{=} \int_0^t J_s dB_s$ where $B_s(x)$ denotes $(s, x) \mapsto \sum_{m \in \mathbf{Z}^4} e^{-ix \cdot m} dB_s^m$. \mathfrak{I}_t should be seen as a cut-off Gaussian free field for $t < \infty$. Indeed, for fixed t , \mathfrak{I}_t is Gaussian with covariance $\varrho_t^2(\nabla) \langle \nabla \rangle^{-2}$ in space so that $\text{Law}_{\mathbf{P}} \mathfrak{I}_t = \varrho_t(\nabla) \# \mathfrak{g}$ – in particular $\text{Law}_{\mathbf{P}} \mathfrak{I}_\infty = \mathfrak{g}$. We denote by \mathbb{H}_a the space of all progressively measurable processes with respect to $(\mathcal{F}_t)_{t \geq 0}$ and \mathbf{P} -almost surely in $L^2(\mathbf{R}_{\geq 0} \times \mathbf{T}^d)$. We can now state the Boué-Dupuis formula.

Theorem 2.55 (Boué-Dupuis formula) *Let $F : C_t \mathcal{C}^{-\frac{d-2}{2}-\epsilon} \rightarrow \mathbf{R}$ be Borel measurable and such that there exist $1 < p, q < \infty$ with $1/p + 1/q = 1$ for which $\mathbf{E}[|F(\mathfrak{I})|^p]$ and $\mathbf{E}[e^{-qF(\mathfrak{I})}]$ are both finite. Then one has the identity*

$$-\log \mathbf{E}[e^{-F(\mathfrak{I})}] = \inf_{u \in \mathbb{H}_a} \mathbf{E}[F(\mathfrak{I} + I(u)) + \frac{1}{2} \|u\|_{L^2([0,t] \times \mathbf{T}^d)}^2]. \quad (2.74)$$

The approach of Barashkov and Gubinelli in [BG20] starts with writing the measure ν_t defined in (2.6) with ϱ_t as above as

$$\mathbf{E}_{\nu_t}[e^{-f(\phi)}] = \frac{1}{\mathcal{Z}_t} \mathbf{E}_{\mathbf{P}}[e^{-f(\mathfrak{I}_t) - \mathcal{I}(\mathfrak{I}_t) + a_t \|\mathfrak{I}_t\|_{L^2}^2 + b_t}],$$

where $\mathcal{Z}_t \stackrel{\text{def}}{=} \mathbf{E}_{\mathbf{P}}[e^{-\mathcal{I}(\mathfrak{I}_t) + a_t \|\mathfrak{I}_t\|_{L^2}^2 + b_t}]$, and to apply the Boué-Dupuis formula to $F(\mathfrak{I}) = V_t^f(\mathfrak{I}_t) \stackrel{\text{def}}{=} f(\mathfrak{I}_t) + \mathcal{I}(\mathfrak{I}_t) - a_t \|\mathfrak{I}_t\|_{L^2(\mathbf{T}^4)}^2 - b_t$. The constant b_t is a “vacuum renormalization” which is explicit and guarantees \mathcal{Z}_t satisfies uniform bounds as $t \rightarrow \infty$. We thus have the following variational representation of ν_t :

$$-\log \mathbf{E}_{\nu_t}[e^{-f(\phi)}] = \log \mathcal{Z}_t + \inf_{u \in \mathbb{H}_a} \mathbf{E}[V_t^f(\mathfrak{I}_t + I_t(u)) + \frac{1}{2} \|u\|_{L^2([0,t] \times \mathbf{T}^4)}^2]. \quad (2.75)$$

Renormalization beyond Wick renormalization (as is needed for Φ_3^4) can be carried out by introducing a shift of the drift u in (2.74) – this shift closely resembles the ansatz used in the analysis of the Langevin dynamic. For \mathbf{T}_4^4 this shift does not belong to the Cameron-Martin space of \mathfrak{g} (the Sobolev space H^1) which suggests the \mathbf{T}_4^4 measure is singular with respect to \mathfrak{g} .

2.4.1 Introducing the BG stochastic objects

To describe this shift, we introduce stochastic objects analogous to those introduced to study the Langevin dynamic – we distinguish these from the former by calling them BG stochastic objects. We first give the BG analogs of the renormalization constants of Lemmas 2.10 and 2.25.

Definition 2.56 (BG renormalization constants)

$$\mathfrak{c}_t^{1,c} \stackrel{\text{def}}{=} \sum_{m_{\hat{c}} \in \mathbf{Z}^{d-1}} \frac{\varrho_t^2(m_{\hat{c}})}{\langle m_{\hat{c}} \rangle^2},$$

$$\mathfrak{C}_t^{2,c} \stackrel{\text{def}}{=} 2 \sum_{c' \neq c} \sum_{m_{\hat{c}} n_{c'} \in \mathbf{Z}^{d-1}} \int_0^t \frac{\varrho_{t \wedge s}^2(m_{\hat{c}}) \partial_s \varrho_s^2(m_{\hat{c}})}{\langle m_{\hat{c}} \rangle^4} \left(\frac{\varrho_s^2(n_{c'}, m_{c'})}{\langle (n_{c'}, m_{c'}) \rangle^2} - \frac{\varrho_s^2(n_{c'})}{\langle n_{c'} \rangle^2} \right) ds,$$

and define $\mathfrak{C}_t^1 \stackrel{\text{def}}{=} \sum_{c=1}^4 \mathfrak{C}_t^{1,c}$ as well as $\mathfrak{C}_t^2 \stackrel{\text{def}}{=} \sum_{c=1}^4 \mathfrak{C}_t^{2,c}$.

We now describe the BG enhancement \mathbb{X}^{BG} of \uparrow consisting of various random fields and operators.

Notation 2.57 In our pictorial representation of the objects in \mathbb{X}^{BG} , it is our convention that for a trees τ different from the noise \bullet , we always have $\overline{\tau}_t = J_t \tau_t$ and $\overline{\tau}_t = \int_0^t J_s^2 \tau_s ds$.

Definition 2.58 (BG random fields) The BG random fields are given by

$$\begin{aligned} \mathfrak{I}_t &\stackrel{\text{def}}{=} \int_0^t J_s dB_s, & \mathfrak{V}_t &\stackrel{\text{def}}{=} 4(\mathcal{N}(\mathfrak{I}_t, \mathfrak{I}_t, \mathfrak{I}_t) - \mathfrak{C}_t^1 \mathfrak{I}_t), & \mathfrak{V}_t &\stackrel{\text{def}}{=} J_t \mathfrak{V}_t, & \mathfrak{V}_t &\stackrel{\text{def}}{=} \int_0^t J_s \mathfrak{V}_s ds, \\ \mathfrak{V}_t^c &\stackrel{\text{def}}{=} 4(\mathcal{N}(\mathfrak{V}_t^c, \mathfrak{I}_t, \mathfrak{I}_t) - \mathfrak{C}_t^1 \mathfrak{V}_t^c) + 4(\mathcal{N}(\mathfrak{I}_t, \mathfrak{V}_t^c, \mathfrak{I}_t) + \mathcal{N}(\mathfrak{I}_t, \mathfrak{I}_t, \mathfrak{V}_t^c) - \mathfrak{C}_t^2 \mathfrak{I}_t), & \mathfrak{V}_t^c &\stackrel{\text{def}}{=} J_t \mathfrak{V}_t^c, \\ \mathfrak{V}_t^c &\stackrel{\text{def}}{=} \int_0^t J_s \mathfrak{V}_s^c ds, & \mathfrak{I}_t &\stackrel{\text{def}}{=} \mathfrak{I}_t - \mathfrak{V}_t + \mathfrak{V}_t^c, & \mathfrak{V}_t^c &\stackrel{\text{def}}{=} 4(\mathcal{N}(\mathfrak{I}_t, \mathfrak{I}_t, \mathfrak{I}_t) - (\mathfrak{C}_t^1 - \mathfrak{C}_t^2) \mathfrak{I}_t), \\ \mathcal{S}_t &\stackrel{\text{def}}{=} \mathfrak{V}_t^c - \mathfrak{V}_t + \mathfrak{V}_t^c, & \mathfrak{I}_t &\stackrel{\text{def}}{=} \mathcal{I}(\mathfrak{I}_t) - 2(\mathfrak{C}_t^1 - \mathfrak{C}_t^2) \|\mathfrak{I}_t\|_{L^2(\mathbf{T}^4)}^2. \end{aligned}$$

Definition 2.59 (BG random operators) The BG random operators are defined as

$$\begin{aligned} \mathfrak{I}_t^{(k)}(f) &\stackrel{\text{def}}{=} 4 \int_{\mathbf{T}^{4-k}} \mathfrak{I}_t(\cdot, y) f(y) dy, & \text{for } k \in [d-1], f : \mathbf{T}^{4-k} &\rightarrow \mathbf{R}, \\ \mathfrak{V}_t^c(f) &\stackrel{\text{def}}{=} 2(\mathcal{N}^c(f, \mathfrak{I}_t, \mathfrak{I}_t) - \mathfrak{C}_t^{1,c} f), & \text{for } c \in [4], f : \mathbf{T}_c &\rightarrow \mathbf{R}, \\ \mathfrak{V}_t^c(f) &\stackrel{\text{def}}{=} 2\mathcal{N}^c(\mathfrak{I}_t, \mathfrak{I}_t, f), & \text{for } c \in [4], f : \mathbf{T}_c^3 &\rightarrow \mathbf{R}, \\ \mathfrak{I}_t^c(f) &\stackrel{\text{def}}{=} 2\mathcal{N}(\mathfrak{I}_t, f, \mathfrak{I}_t), & \text{for } f : \mathbf{T}^4 &\rightarrow \mathbf{R}. \end{aligned}$$

We also define $\mathfrak{I}_t^c, \mathfrak{V}_t^c, \mathfrak{I}_t^c$ and \mathfrak{I}_t^c analogously, substituting every occurrence of \mathfrak{I}_t with \mathfrak{I}_t^c , and replacing $\mathfrak{C}_t^{1,c}$ with $\mathfrak{C}_t^{1,c} - \mathfrak{C}_t^{2,c}$ in the definition of \mathfrak{V}_t^c . Again, we drop the color subscript c on the random operators when there is a sum over c .

Recalling the notation $\mathbb{X} = \mathbb{X}_f \cup \mathbb{X}_o$ from Section 2.3.5, we define the tuple \mathbb{X}_f^{BG} of all the random fields of Definition 2.58 except $\mathfrak{V}_t, \mathfrak{V}_t^c$ and \mathfrak{I}_t^c , and the tuple \mathbb{X}_o^{BG} of all the random operators of Definition 2.59.

Note that all the stochastic objects we have introduced in this section are processes in time taking values in distributions over space or operators on either space distributions or space-time distributions. For $t \in \mathbf{R}_{\geq 0}$, we endow $\mathbb{X}^{\text{BG}}(t) \stackrel{\text{def}}{=} \{\tau(t) | \tau \in \mathbb{X}^{\text{BG}}\}$ with the norm

$$\|\mathbb{X}^{\text{BG}}(t)\| = \max \left(\max_{\tau \in \mathbb{X}_f^{\text{BG}}} \|\tau(t)\|_{\mathcal{C}^{\beta_\tau - \epsilon}}, \max_{\tau \in \mathbb{X}_o^{\text{BG}}} \|\tau(t)\|_{\mathcal{L}(\mathcal{C}_t^{H^{\alpha_\tau}, \mathcal{C}^{\beta_\tau - \epsilon})}} \right), \quad (2.76)$$

where α_τ and β_τ are the inner and outer regularities of the given stochastic objects which are chosen the same as in Lemmas 2.66 and 2.68 for the corresponding LD objects. The next lemma states that we can control the purely stochastic objects appearing in the variational formula.

Lemma 2.60

$$\sup_{t \in \mathbf{R}_{\geq 0}} \mathbf{E}[\|\mathbb{X}^{\text{BG}}(t)\|^p] < \infty.$$

Proof. The statement is proved combining the fixed time Kolmogorov estimates for the random fields (2.86) and for the random operators (2.90) with our bounds on the second moments of the stochastic objects derived in Sections 2.6.3 and 2.6.5. In particular, the second moments are bounded uniformly in the scale parameter in the proof of Lemma 2.85. We conclude by studying individually the power countings of the amplitudes of each object in Sections 2.6.6 and 2.6.7. \square

2.4.2 Renormalizing the Boué-Dupuis formula

We are ready to introduce the shift of the drift that renormalizes the Boué-Dupuis formula, which will be similar to the one that was introduced to deal with the T_4^4 equation. The main estimate for the variational approach to T_4^4 is given in the following proposition.

Proposition 2.61 *Let $f : \mathcal{C}^{-1-\epsilon} \rightarrow \mathbf{R}$ of at most linear growth and $V_t^f(\phi) = f(\phi) + \mathcal{I}(\phi) - a_t \|\phi\|_{L^2(\mathbf{T}^4)}^2 - b_t$ where*

$$a_t \stackrel{\text{def}}{=} 2(\mathfrak{C}_t^1 - \mathfrak{C}_t^2), \quad b_t \stackrel{\text{def}}{=} \mathbf{E}[\mathfrak{I}_t + \|\Psi_t - \Psi_t^v\|_{L^2([0,t] \times \mathbf{T}^4)}^2].$$

Then the change of variable

$$l_s^t(u) \stackrel{\text{def}}{=} u_s + \mathbf{1}_{s \leq t} (\Psi_s - \Psi_s^v),$$

renormalizes the Boué-Dupuis formula, and it holds

$$\begin{aligned} & \mathbf{E}[V_t^f(\mathfrak{I}_t + I_t(u)) + \frac{1}{2} \|u\|_{L^2([0,t] \times \mathbf{T}^4)}^2] \\ &= \mathbf{E}[\mathfrak{M}_t^f(\mathbb{X}^{\text{BG}}, K_t(u)) + \mathcal{I}(K_t(u)) + \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2], \end{aligned} \quad (2.77)$$

where $K_s(u) \stackrel{\text{def}}{=} I_s(l^t(u))$ and

$$\mathfrak{M}_t^f(\mathbb{X}^{\text{BG}}, v) \stackrel{\text{def}}{=} f(\mathfrak{I}_t + v_t) + \left((\mathcal{S}, v) + \mathfrak{I}_v^v + \mathfrak{I}_v^v + \mathfrak{I}_v^v + \mathfrak{I}_v^v \right)(t),$$

is a term collecting all the mixed terms, and that obeys the bound

$$|\mathfrak{M}_t^f(\mathbb{X}^{\text{BG}}, K_t(u))| \leq C(1 + \|\mathbb{X}^{\text{BG}}(t)\|^\gamma) + \frac{1}{2} \left(\mathcal{I}(K_t(u)) + \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^d)}^2 \right), \quad (2.78)$$

where C, γ are two positive constants.

Remark 2.62 Our choice of shift of the drift is made to ensure that, with the above notations, $\mathfrak{M}_t^0(\mathbb{X}^{\text{BG}}, v) = (\mathfrak{N}_t(\mathbb{X}^{\text{BG}}, v), v)$ with $\mathfrak{N}_t(\mathbb{X}^{\text{BG}}, v) \in H^{-1}$ if $v \in H^1$. We make the following observations about our choice of drift shift and why it suffices.

- The first shift by $\mathbf{1}_{s \leq t} \Psi_s^v$ takes care of the term Ψ_t in \mathfrak{N}_t that would be of regularity $-2-$, while the second shift by $-\mathbf{1}_{s \leq t} \Psi_s$ takes care of all the three terms of the form $\Psi_t \times \mathfrak{I}_t \times \mathfrak{I}_t$ that would be of regularity $-1-$.
- The analysis of the Langevin dynamic and the observation that $\mathcal{N}(\mathfrak{I}_t, v, \mathfrak{I}_t)$ is of regularity $-1-$ if $v \in H^1$ strongly suggest that we should cancel this term by introducing the non constant term $J_s \mathcal{N}(\mathfrak{I}_s, I_s(u), \mathfrak{I}_s)$ into the shift of the drift. However, like for the a priori L^2 estimates in Section 2.3.3, we will take advantage of the fact that the non-melonic products do not require any renormalization and we can bound the duality pairing $(v, \mathcal{N}(\mathfrak{I}, v, \mathfrak{I}))$ by the better behaved $(v, \mathcal{N}(\mathfrak{I}, \mathfrak{I}, v))$ thanks to the Cauchy-Schwarz inequality (2.128) (see also the proof of Lemma 2.119). This means $\mathcal{N}(\mathfrak{I}_t, v, \mathfrak{I}_t)$ is harmless even if it is not in H^1 . Note that this simplification appears to only hold for $d \in \{3, 4\}$, in particular, covering the full subcritical regime likely requires a shift that depends on the drift u – see Remark 2.36.

Remark 2.63 Note that our final variational formula differs from that of [BG20] in that the nonlinearity term is evaluated at $K(u)$ instead of $I(u)$. This is due to the fact that our shift of \mathfrak{I}_t does not depend on u , which simplifies our proofs compared to those of [BG20].

Proof. We sketch the proof since it is very similar to the L^2 estimates of Section 2.3.3 – we use the same notation as well. We again use estimates on the mixed terms given in Appendix 2.1. We also write $v \stackrel{\text{def}}{=} K(u)$. We start by observing that we can carry out renormalization cancellations by expanding

$$\begin{aligned} \mathcal{I}(\mathfrak{I}_t + I_t(u)) - a_t \|\mathfrak{I}_t + I_t(u)\|_{L^2(\mathbf{T}^4)}^2 &= \mathcal{I}(\mathfrak{I}_t + K_t(u)) - a_t \|\mathfrak{I}_t + K_t(u)\|_{L^2(\mathbf{T}^4)}^2 \\ &= \mathcal{I}(\mathfrak{I}_t + v_t) - a_t \|\mathfrak{I}_t + v_t\|_{L^2(\mathbf{T}^4)}^2 \\ &= \left(\text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} \right)(t) + \mathcal{I}(v_t) \\ &= \left(\text{diagram} + (\mathcal{S}, v) + (\mathfrak{V} - \mathfrak{V}, v) + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} \right)(t) + \mathcal{I}(v_t). \end{aligned} \quad (2.79)$$

Using the martingale properties of the stochastic objects, the third term in (2.79) rewrites as

$$(\mathfrak{V}_t - \mathfrak{V}_t, v_t) = \int_0^t (\mathfrak{V}_s - \mathfrak{V}_s, \dot{v}_s) ds + M_t = \int_0^t (\mathfrak{V}_s - \mathfrak{V}_s, \dot{K}_s(u)) ds + M_t,$$

where M_t stands for a martingale with vanishing expectation. On the other hand, we have

$$\begin{aligned} \frac{1}{2} \|u\|_{L^2([0,t] \times \mathbf{T}^4)}^2 &= \frac{1}{2} \|l^t(u) - \mathbf{1}_{[0,t]}(\mathfrak{V} - \mathfrak{V})\|_{L^2([0,t] \times \mathbf{T}^4)}^2 \\ &= \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2 + \frac{1}{2} \|\mathfrak{V} - \mathfrak{V}\|_{L^2([0,t] \times \mathbf{T}^4)}^2 - \int_0^t (\mathfrak{V}_s - \mathfrak{V}_s, l_s^t(u)) ds \\ &= \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2 + \frac{1}{2} \|\mathfrak{V} - \mathfrak{V}\|_{L^2([0,t] \times \mathbf{T}^4)}^2 - \int_0^t (\mathfrak{V}_s - \mathfrak{V}_s, \dot{K}_s(u)) ds, \end{aligned}$$

which makes the cancellation with the third term in (2.79) manifest, so that (2.77) follows.

We now have to prove (2.78). In this setting, we still can use (2.130), (2.131), (2.132), (2.133) and (2.135), and the proof is therefore similar to that of Proposition 2.37. Regarding, the term $f(\mathfrak{I}_t + v_t)$, using the fact that f is at most of linear growth we have, by Sobolev embedding,

$$\begin{aligned} |f(\mathfrak{I}_t + v_t)| &\lesssim \|\mathfrak{I}_t + v_t\|_{C^{-1-\epsilon}} \lesssim \|\mathfrak{I}_t\|_{C^{-1-\epsilon}} + \|v\|_{C_t C^{-1-\epsilon}} \lesssim \|\mathfrak{I}_t\|_{C^{-1-\epsilon}} + \|v\|_{C_t H^{1-\epsilon}} \\ &\leq C_\delta (1 + \|\mathbb{X}^{\text{BG}}(t)\|) + \delta \|v\|_{C_t H^{1-\epsilon}}^2, \end{aligned}$$

where we recall that $\|v\|_{C_t H^{1-\epsilon}}^2 = \|K(u)\|_{C_t H^{1-\epsilon}}^2 \lesssim \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2$ by (2.72). \square

We can now prove the main theorem of this section.

Proof Theorem 2.4. We first prove that $\log \mathcal{Z}_t$ is bounded uniformly in t . By the Boué-Dupuis formula (2.74), and then (2.77), $\log \mathcal{Z}_t$ verifies

$$\begin{aligned} -\log \mathcal{Z}_t &= \inf_{u \in \mathbb{H}_a} \mathbf{E}[V_t^0(\mathfrak{I}_t + I_t(u))] + \frac{1}{2} \|u\|_{L^2([0,t] \times \mathbf{T}^4)}^2 \\ &= \inf_{u \in \mathbb{H}_a} \mathbf{E}[\mathfrak{M}_t^0(\mathbb{X}^{\text{BG}}, K_t(u)) + \mathcal{I}(K_t(u)) + \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2]. \end{aligned}$$

For any adapted process $u \in \mathbb{H}_a$, using (2.78), we thus have

$$\begin{aligned} &\mathbf{E}[\mathfrak{M}_t^0(\mathbb{X}^{\text{BG}}, K_t(u)) + \mathcal{I}(K_t(u)) + \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2] \\ &\geq \mathbf{E}[-C(1 + \|\mathbb{X}^{\text{BG}}(t)\|^\gamma) + \frac{1}{2} (\mathcal{I}(K_t(u)) + \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2)] \end{aligned}$$

$$\geq -C(1 + \mathbf{E}[\|\mathbb{X}^{\text{BG}}(t)\|^\gamma]) \geq -C(1 + \sup_{t \in \mathbf{R}_{\geq 0}} \mathbf{E}[\|\mathbb{X}^{\text{BG}}(t)\|^\gamma]),$$

which implies that $-\log \mathcal{Z}_t \geq -C(1 + \sup_{t \in \mathbf{R}_{\geq 0}} \mathbf{E}[\|\mathbb{X}^{\text{BG}}(t)\|^\gamma])$. On the other hand, for any $u \in \mathbb{H}_a$

$$\begin{aligned} -\log \mathcal{Z}_t &\leq \mathbf{E}[V_t^0(\mathfrak{I}_t + I_t(u))] + \frac{1}{2} \|u\|_{L^2([0,t] \times \mathbf{T}^4)}^2 \\ &\leq \mathbf{E}[C(1 + \|\mathbb{X}^{\text{BG}}(t)\|^\gamma) + \frac{1}{2} (\mathcal{I}(K_t(u)) + \frac{1}{2} \|l^t(u)\|_{L^2([0,t] \times \mathbf{T}^4)}^2)] \end{aligned}$$

Taking $u_s = -\mathbf{1}_{s \leq t} (\Psi_s - \Psi_s^{\text{BG}})$ (thus requiring that $l^t(u) = 0$) yields an upper bound on $\log \mathcal{Z}_t$ that only depends on the stochastic objects and is uniformly bounded in time. Finally, we have thus obtained that it holds uniformly in time

$$|\log \mathcal{Z}_t| \lesssim 1 + \sup_{t \in \mathbf{R}_{\geq 0}} \mathbf{E}[\|\mathbb{X}^{\text{BG}}(t)\|^\gamma] < \infty.$$

The representation (2.75) implies that repeating the previous argument with V^f instead of V^0 immediately shows that ν_t verifies the same bound uniform in time, giving us the desired estimate (2.13). The fact that tightness of $(\nu_t)_t$ follows from uniform bounds on the Laplace transforms of $(\nu_t)_t$ is standard. \square

2.5 Kolmogorov arguments for random fields and operators

In our analysis we will need to construct and obtain path-wise regularity estimates on various explicit stochastic objects. For most of these objects (which include both random fields and operators), we proceed by formulating Kolmogorov criteria Lemmas 2.69, 2.72, and 2.77. When these criteria are combined with moment estimates, they will give the results stated in Lemmas 2.12, 2.16, 2.27, 2.29, 2.31, 2.48 and 2.60.

There are differences in both the definition and type of control we obtain on the objects needed for the Langevin dynamic (LD) versus those needed for the Barashkov & Gubinelli approach (BG). On immediate difference is the role played by time, for the case of the free field recall that $\mathfrak{I}^{\text{LD}} = \sqrt{2} \mathcal{L}^{-1} \xi$ while $\mathfrak{I}^{\text{BG}} = \int_0^t J_s dB_s$.

The control of the LD objects is harder – we need to control space *and time* regularity of the LD object – note that uniform space-time control is inside the expectation in the statements for the Lemmas 2.12, 2.16, 2.27, 2.29, 2.31, and 2.48 while for BG objects one needs space-regularity estimates along with control over expectations uniform in time - note that the supremum in time is outside of the expectation. in Lemma 2.60.

For this reason, our discussion will primarily focus on the LD objects, and we point where and how extra considerations appear for the BG objects.

2.5.1 Random fields

We first define the LD random fields and investigate their regularity for $d \in \{2, 3, 4\}$.

Definition 2.64 (LD random fields) Recall that the LD random fields are defined as

$$\begin{aligned} \mathfrak{I}_N &\stackrel{\text{def}}{=} \mathcal{L}^{-1} \xi_N, \quad \Psi_N \stackrel{\text{def}}{=} \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1(d) \mathfrak{I}_N, \quad \Psi_N^{\text{BG}} \stackrel{\text{def}}{=} \mathcal{L}^{-1} \Psi_N, \\ \Psi_N^{\text{LD}} &\stackrel{\text{def}}{=} (\mathcal{N}_N(\Psi_N, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^1(d) \Psi_N) + (\mathcal{N}_N(\mathfrak{I}_N, \Psi_N, \mathfrak{I}_N) + \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \Psi_N) - \mathfrak{C}_N^2(d) \mathfrak{I}_N), \\ \Psi_N^{\text{LD}} &\stackrel{\text{def}}{=} \mathcal{L}^{-1} \Psi_N^{\text{LD}}, \quad \mathfrak{I}_N^{\text{LD}} \stackrel{\text{def}}{=} \mathfrak{I}_N - \Psi_N + \Psi_N^{\text{LD}}, \quad \Psi_N^{\text{LD}} \stackrel{\text{def}}{=} \mathcal{N}_N(\mathfrak{I}_N, \mathfrak{I}_N, \mathfrak{I}_N) - (\mathfrak{C}_N^1(d) - \mathfrak{C}_N^2(d)) \mathfrak{I}_N, \\ \mathcal{S}_N &\stackrel{\text{def}}{=} \Psi_N^{\text{LD}} - \Psi_N + \Psi_N^{\text{LD}}. \end{aligned}$$

Notation 2.65 We associate to each τ a power-counting $|\tau| \in \mathbf{R}$ as a useful notation. We display this in the figure below with τ above the line and $|\tau|$ below it. Note that when τ is a random field, $|\tau|$ is not⁸ always the regularity or order/homogeneity of the random field associated to τ :

$$\begin{array}{cccccccccc}
 \begin{array}{c} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ | \\ \circ \end{array} \\
 \hline
 -\frac{d-2}{2} & -\frac{2d-5}{2} & -\frac{2d-5}{2} & -\frac{3d-8}{2} & -\frac{3d-12}{2} & -(d-2) & -(2d-7) & -\frac{5d-18}{2} & -\frac{5d-22}{2} &
 \end{array}$$

If τ is a random operator acting on argument f , we partition its vertices in four categories: the vertices of the form $\mathcal{N}(\tau_1, \tau_2, \tau_3) = \begin{array}{c} \circ \\ | \\ \circ \end{array}$ for $\tau_1 = \bullet$, $\tau_2 = \bullet$ and $\tau_3 = \bullet$ some stochastic objects, of the form $\mathcal{N}(\tau_1, \tau_2, f) = \begin{array}{c} \circ \\ | \\ \circ \end{array}$, of the form $\mathcal{N}(f, \tau_1, \tau_2) = \begin{array}{c} \circ \\ | \\ \circ \end{array}$ and of the form $\mathcal{N}(\tau_1, f, \tau_2) = \begin{array}{c} \circ \\ | \\ \circ \end{array}$. All the vertices of a random field are of the form $\begin{array}{c} \circ \\ | \\ \circ \end{array}$. With this convention, the general formula is given by

$$\begin{aligned}
 |\tau| \stackrel{\text{def}}{=} & -\frac{d+2}{2}(\# \text{ of noises } \bullet \text{ in } \tau) + (\# \text{ of vertices } \begin{array}{c} \circ \\ | \\ \circ \end{array} \text{ in } \tau) + \frac{1}{2}(\# \text{ of vertices } \begin{array}{c} \circ \\ | \\ \circ \end{array} \text{ in } \tau) \\
 & + \frac{1}{2}(\# \text{ of vertices } \begin{array}{c} \circ \\ | \\ \circ \end{array} \text{ in } \tau) + 2(\# \text{ of edges } | \text{ in } \tau).
 \end{aligned}$$

All the stochastic objects we are considering are some variations of the seven main stochastic objects, either because we rather see them as random operators instead of random fields, or because we study them with the rough shift \dagger instead of \ddagger .

Let us now recall the properties of the random fields:

Lemma 2.66 *In any dimension $d \geq 2$,*

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\dagger_N\|_{C_T \mathcal{C}^{\dagger} |_{-\epsilon}(\mathbf{T}^d)}]^p < \infty, \quad (2.80)$$

Moreover, for $d \in \{2, 3, 4\}$,

$$\begin{aligned}
 & \sup_{N \in \mathbf{N}} \mathbf{E}[\|\begin{array}{c} \circ \\ | \\ \circ \end{array}_N\|_{C_T \mathcal{C}^{\begin{array}{c} \circ \\ | \\ \circ \end{array} |_{-\epsilon}(\mathbf{T}^d)}]^p < \infty, \\
 & \sup_{N \in \mathbf{N}} \mathbf{E}[\|\begin{array}{c} \circ \\ | \\ \circ \end{array}_N\|_{C_T \mathcal{C}^{\begin{array}{c} \circ \\ | \\ \circ \end{array} |_{-\epsilon}(\mathbf{T}^d)}]^p < \infty, \\
 & \sup_{N \in \mathbf{N}} \mathbf{E}[\|\begin{array}{c} \circ \\ | \\ \circ \end{array}_N\|_{C_T \mathcal{C}^{\min(-\frac{d-3}{2}, |\begin{array}{c} \circ \\ | \\ \circ \end{array}|)_{-\epsilon}(\mathbf{T}^d)}}]^p < \infty, \\
 & \sup_{N \in \mathbf{N}} \mathbf{E}[\|\begin{array}{c} \circ \\ | \\ \circ \end{array}_N\|_{C_T \mathcal{C}^{\min(2-\frac{d-3}{2}, |\begin{array}{c} \circ \\ | \\ \circ \end{array}|)_{-\epsilon}(\mathbf{T}^d)}}]^p < \infty, \\
 & \sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathcal{S}_N\|_{C_T \mathcal{C}^{-\frac{d-3}{2}-\epsilon}(\mathbf{T}^d)}]^p < \infty.
 \end{aligned} \quad (2.81)$$

Moreover, all the LD random fields converge in probability as $N \uparrow \infty$ to some limit in the Banach spaces stated above.

Furthermore, for every τ above of regularity β_τ , the corresponding BG random field belongs to $\mathcal{C}^{\beta_\tau - \epsilon}$ for all time $t \in \mathbf{R}_{\geq 0}$, and obeys a bound uniform in time:

$$\sup_{t \in \mathbf{R}_{\geq 0}} \mathbf{E}[\|\tau(t)\|_{\mathcal{C}^{\beta_\tau - \epsilon}}]^p < \infty.$$

2.5.2 Random operators

We state definitions and regularity estimates for the LD random operators.

⁸This is because we do not positively renormalize as in regularity structures or use resonant products as in paracontrolled calculus.

Definition 2.67 (LD random operators) Recall that the LD random operators are defined as

$$\begin{aligned}
\mathfrak{I}_N^{(k)}(f) &\stackrel{\text{def}}{=} \int_{\mathbf{T}^{d-k}} \mathfrak{I}_N(\cdot, y) f(\cdot, y) dy, & \text{for } k \in [d-1], f : \mathbf{R}_{\geq 0} \times \mathbf{T}^{d-k} \rightarrow \mathbf{R}, \\
\mathfrak{V}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N^c(f, \mathfrak{I}_N, \mathfrak{I}_N) - \mathfrak{C}_N^{1,c}(d)f, & \text{for } c \in [d], f : \mathbf{R}_{\geq 0} \times \mathbf{T}_c \rightarrow \mathbf{R}, \\
\mathfrak{J}_N^c(f) &\stackrel{\text{def}}{=} \mathcal{N}_N^c(\mathfrak{I}_N, \mathfrak{I}_N, f), & \text{for } c \in [d], f : \mathbf{R}_{\geq 0} \times \mathbf{T}_{\hat{c}}^{d-1} \rightarrow \mathbf{R}, \\
\mathfrak{B}_N(f) &\stackrel{\text{def}}{=} \mathcal{N}_N(\mathfrak{I}_N, f, \mathfrak{I}_N), & \text{for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}^d \rightarrow \mathbf{R}, \\
\mathfrak{V}_N(f) &\stackrel{\text{def}}{=} \mathfrak{B}_N(\mathcal{L}^{-1} \mathfrak{B}_N(f)), & \text{for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}^d \rightarrow \mathbf{R},
\end{aligned}$$

and \mathfrak{I}_N , \mathfrak{V}_N^c , \mathfrak{J}_N^c , \mathfrak{B}_N and \mathfrak{V}_N are defined accordingly, substituting every occurrence of \mathfrak{I}_N with \mathfrak{I}_N , and replacing $\mathfrak{C}_N^{1,c}(d)$ with $\mathfrak{C}_N^{1,c}(d) - \mathfrak{C}_N^{2,c}(d)$ in the definition of \mathfrak{V}_N^c .

Lemma 2.68 In any dimension $d \geq 2$, for all $\alpha > \frac{d-k-2}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{I}_N^{(k)}\|^p_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-k}), C_T C^{\min(-\frac{k-2}{2}, \alpha+|\cdot|)-\epsilon}(\mathbf{T}^k))}] < \infty. \quad (2.82)$$

For $d \in \{2, 3, 4\}$, $c \in [d]$, and all $\alpha > -\frac{1}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{V}_N^c\|^p_{\mathcal{L}(C_T H^\alpha(\mathbf{T}), C_T C^{\min(\frac{1}{2}, \alpha+|\cdot|)-\epsilon}(\mathbf{T}))}] < \infty. \quad (2.83)$$

In any dimension $d \geq 2$, for all $c \in [d]$ and $\alpha > \frac{d-3}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{J}_N^c\|^p_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-1}), C_T C^{\min(-\frac{d-3}{2}, \alpha+|\cdot|)-\epsilon}(\mathbf{T}^{d-1}))}] < \infty. \quad (2.84)$$

In any dimension $d \geq 2$, for all $\alpha > \frac{d-3}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{B}_N\|^p_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^d), C_T C^{\min(-\frac{d-3}{2}, \alpha+|\cdot|)-\epsilon}(\mathbf{T}^d))}] < \infty. \quad (2.85)$$

In any dimension $d \geq 2$, for all $\alpha > \frac{d-3}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\mathfrak{V}_N\|^p_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^d), C_T C^{\min(-\frac{d-3}{2}, \alpha+|\cdot|)-\epsilon}(\mathbf{T}^d))}] < \infty.$$

The purple objects constructed with the rough shift \mathfrak{I} obey the same bounds as their black counterparts constructed with \mathfrak{I} , but with the restriction that the dimension must be ≤ 4 . Moreover, all the LD random operators converge in probability as $N \uparrow \infty$ to some limit in the Banach spaces stated above.

Furthermore, for τ as above of inner regularity α_τ and outer regularity β_τ , the corresponding BG random operator belongs to $\mathcal{L}(C_t H^{\alpha_\tau}, C^{\beta_\tau-\epsilon})$ for any $t \in \mathbf{R}_{\geq 0}$, and obeys the bound

$$\sup_{t \in \mathbf{R}_{\geq 0}} \mathbf{E}[\|\tau(t)\|^p_{\mathcal{L}(C_t H^{\alpha_\tau}, C^{\beta_\tau-\epsilon})}] < \infty.$$

2.5.3 Kolmogorov criteria

In Lemmas 2.69 and 2.72 we give two estimates, the first being an estimate at a fixed time and the second an estimate on an homogeneous Hölder norm in time, together these give control over the corresponding inhomogeneous Hölder norm in time for the LD objects. For the BG object, only the first estimate of Lemmas 2.69 and 2.72 is relevant.

Lemma 2.77 gives an estimate for an homogeneous norm for a particular class of random operators depending on two times, and is solely used for the LD object \mathfrak{V} .

We also note, since in Section 2.6 we obtain covariance estimates in terms of Fourier variables, it is convenient to do the same for Kolmogorov estimates and so we simplify our estimates using appropriate stationarity of the stochastic objects.

2.5.3.1 Kolmogorov criteria for random fields

We start with a classical Kolmogorov argument for the random fields. Below, T is a fixed positive real number, and we write $\delta_{s,t}\tau \stackrel{\text{def}}{=} \tau(t) - \tau(s)$.

Lemma 2.69 ([MWX17], Proposition 3.6) *For any fixed $k \geq 1$ and $p \in 2\mathbf{N}$, uniform over smooth stationary in space random fields τ over $\mathbf{R}_{\geq 0} \times \mathbf{T}^d$ belonging to the k -th inhomogeneous Gaussian chaos, one has*

$$\mathbf{E}[\|\tau(t)\|_{\mathcal{C}^{\beta-\epsilon}}^p] \lesssim \left(\sum_{m \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \mathbf{E}[\hat{\tau}_m(t)\hat{\tau}_{-m}(t)] \right)^{\frac{p}{2}}. \quad (2.86)$$

Moreover, for any fixed $\theta > \kappa > 0$ and uniform over smooth stationary in space random fields τ over $\mathbf{R}_{\geq 0} \times \mathbf{T}^d$ belonging to the k -th inhomogeneous Gaussian chaos, one has

$$\mathbf{E}[\|\tau\|_{\dot{C}_T^{\theta-\kappa}\mathcal{C}^{\beta-\epsilon}}^p] \lesssim \sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq 1}} |t-s|^{-\frac{\theta p}{2}} \left(\sum_{m \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \mathbf{E}[\delta_{s,t}\hat{\tau}_m\delta_{s,t}\hat{\tau}_{-m}] \right)^{\frac{p}{2}}.$$

Note that the above Lemma can be proved along the lines of Lemma 2.72 below, setting $d_1 = 0$.

2.5.3.2 Kolmogorov criteria for time-local random operators

All the random operators in Definition 2.67 are presented as operators acting on functions of space and time. In this section we give Kolmogorov criteria for $\mathfrak{V}, \mathfrak{J}, \mathfrak{V}^{\text{sto}}$ and \mathfrak{B} .

We separate these cases since \mathfrak{V} is the only operator listed that does not act locally in time and needs a slightly different argument – see the next section.

We note that, instead of being viewed as operators on space-time functions, the remaining operators τ in Definition 2.67 can be viewed as operators $\tilde{\tau}_t$ on space functions that themselves vary in time, by setting

$$\tau(f)(t, x) = \tilde{\tau}_t(f(t, \cdot))(x) \quad (2.87)$$

for $f : \mathbf{R}_{\geq 0} \times \mathbf{T}^{d_1}$ and a time-varying random operator $\tilde{\tau}$. Overloading the notation, we identify $\tilde{\tau}$ with τ , also denoting it τ (or τ_t when it is necessary to distinguish). Moreover, observe that we have

$$\|\tau(t)\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d_1}), \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^{d_2}))} \lesssim \|\tau_t\|_{\mathcal{L}(H^\alpha(\mathbf{T}^{d_1}), \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^{d_2}))},$$

and, for $\theta > 0$,

$$\|\tau\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d_1}), C_T^\theta \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^{d_2}))} \lesssim \|\tau\|_{C_T^\theta \mathcal{L}(H^\alpha(\mathbf{T}^{d_1}), \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^{d_2}))}. \quad (2.88)$$

Our Kolmogorov criteria for local-in-time random operators will estimate the norm on the r.h.s. in both inequalities above.

Our approach to estimating random operators in space is inspired by [GPI17, Sec. 10.2].

Definition 2.70 For any $T > 0$, a *time-varying random smooth operator* over time T will be an element $\tau_t \in C_T \mathcal{L}(H^\alpha(\mathbf{T}^{d_1}), \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^{d_2}))$ which, for $x \in \mathbf{T}^{d_2}$ and $t \in [0, T]$, is of the form

$$\tau_t(f)(x) = \sum_{n \in \mathbf{T}^{d_1}} \mathcal{F}_i \tau_n(x, t) \hat{f}_{-n}$$

where $\tau_t(f)$ denotes the action of τ evaluated at time t on $f \in H^\alpha(\mathbf{T}^{d_1})$, and $\mathcal{F}_i \tau_n(x, t)$ stands for the Fourier transform of τ in the inner space variable evaluated at the mode n , that is to say the Fourier transform in the space variable acting on f . This implies that for $m \in \mathbf{Z}^{d_2}$,

$$\mathcal{F}_x(\tau_t(f))(m) = \sum_{n \in \mathbf{Z}^{d_1}} \hat{\tau}_{m,n}(t) \hat{f}_{-n},$$

for $\hat{\tau}_{m,n}(t)$ rapidly decaying in m and n . We say a time-varying random smooth operator τ is in the inhomogeneous Gaussian chaos of order $k \in \mathbf{N}$ if all the $\hat{\tau}_{m,n}(t)$ are.

For $h \in \mathbf{T}^{d_2}$, we write T_h for the associated translated operator, that is $(T_h g)(\cdot) = g(\cdot - h)$ for functions g on \mathbf{T}^{d_2} . We then say τ , as above is *stationary in space* if for every $h \in \mathbf{T}^{d_2}$, $T_h \circ \tau_t$ is equal in distribution to τ_t as a random element of $C_T \mathcal{L}(H^\alpha(\mathbf{T}^{d_1}), \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^{d_2}))$. Note that stationary in space implies that for $s, t \in [0, T]$ and $n \in \mathbf{Z}^{d_1}$,

$$\mathbf{E}[\hat{\tau}_{m_1, n}(s) \hat{\tau}_{m_2, -n}(t)] = 0 \text{ unless } m_1 + m_2 = 0. \quad (2.89)$$

Remark 2.71 Note that it is easy to verify that, with cut-off $N < \infty$ for the LD objects or a fixed scale/time $t < \infty$ cut-off on the BG objects, the operators \mathfrak{V}_\bullet , \mathfrak{J}_\bullet , $\mathfrak{V}_\bullet^{\text{sto}}$ and $\mathfrak{J}_\bullet^{\text{sto}}$ satisfy Definition 2.70 including the stated stationarity property.

We now state Kolmogorov estimates for τ_t which combined with (2.88) will imply the desired bound for τ . Below, $T > 0$ is fixed, and we extend the notation $\delta_{s,t}\tau = \tau_t - \tau_s$ to a time-varying random operator τ_\bullet .

Lemma 2.72 *For any fixed $k \geq 1$ and $p \in 2\mathbf{N}$, and uniform over smooth time-varying stationary in space random operators τ over time T belonging to the k -th inhomogeneous Gaussian chaos, one has, for any $t \in [0, T]$,*

$$\mathbf{E}[\|\tau_t\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}^p] \lesssim \left(\sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \mathbf{E}[\hat{\tau}_{m,n}(t) \hat{\tau}_{-m,-n}(t)] \right)^{\frac{p}{2}}. \quad (2.90)$$

Moreover, for any fixed $\theta > \kappa > 0$ and uniform over smooth time-varying stationary in space random operators τ over time T belonging to the k -th homogeneous Gaussian chaos, one has

$$\mathbf{E}[\|\tau_\bullet\|_{C_T^{\theta-\kappa} \mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}^p] \lesssim \sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq 1}} |t-s|^{-\frac{\theta p}{2}} \left(\sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \mathbf{E}[\delta_{s,t} \hat{\tau}_{m,n} \delta_{s,t} \hat{\tau}_{-m,-n}] \right)^{\frac{p}{2}}.$$

Proof. We focus on the second statement, the first one will be clear from our argument. Taking $p > d_2/\epsilon$ and using the compact embedding

$$B_{p,p}^\beta(\mathbf{T}^{d_2}) \hookrightarrow \mathcal{C}^{\beta-\epsilon}(\mathbf{T}^{d_2})$$

gives

$$\|\delta_{s,t}\tau(f)\|_{\mathcal{C}^{\beta-\epsilon}}^p \lesssim \|\delta_{s,t}\tau(f)\|_{B_{p,p}^\beta}^p = \sum_{h \geq -1} 2^{\beta h p} \int_{\mathbf{T}^{d_2}} |\Delta^h \delta_{s,t}\tau(f)(x)|^p dx.$$

Then, by Cauchy–Schwarz inequality,

$$\begin{aligned} & \sum_{h \geq -1} 2^{\beta h p} \int_{\mathbf{T}^{d_2}} |\Delta^h \delta_{s,t}\tau(f)(x)|^p dx \\ & \lesssim \sum_{h \geq -1} 2^{\beta h p} \int_{\mathbf{T}^{d_2}} \left(\sum_{n \in \mathbf{Z}^{d_1}} |\Delta^h \delta_{s,t} \mathcal{F}_i \tau_n(x) \hat{f}_{-n}| \right)^p dx \\ & \lesssim \left(\sum_{h \geq -1} 2^{\beta h p} \int_{\mathbf{T}^{d_2}} \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} |\Delta^h \delta_{s,t} \mathcal{F}_i \tau_n(x)|^2 \right)^{\frac{p}{2}} dx \right) \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{2\alpha} |\hat{f}_n|^2 \right)^{\frac{p}{2}}. \end{aligned}$$

We may now divide by

$$\left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{2\alpha} |\hat{f}_n|^2 \right)^{\frac{p}{2}} = \|f\|_{H^\alpha}^p$$

and take the supremum over $f \in H^\alpha$ on the l.h.s. to obtain

$$\|\delta_{s,t}\tau\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}^p \lesssim \sum_{h \geq -1} 2^{\beta hp} \int_{\mathbf{T}^{d_2}} \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} |\Delta^h \delta_{s,t} \mathcal{F}_i \tau_n(x)|^2 \right)^{\frac{p}{2}} dx.$$

We are now ready to take the expectation. Moreover, since τ sits in a finite Gaussian chaos, one has the hypercontractive moment bound

$$\begin{aligned} & \mathbf{E} \left[\sum_{h \geq -1} 2^{\beta hp} \int_{\mathbf{T}^{d_2}} \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} |\Delta^h \delta_{s,t} \mathcal{F}_i \tau_n(x)|^2 \right)^{\frac{p}{2}} dx \right] \\ &= \sum_{h \geq -1} 2^{\beta hp} \int_{\mathbf{T}^{d_2}} \mathbf{E} \left[\left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} |\Delta^h \delta_{s,t} \mathcal{F}_i \tau_n(x)|^2 \right)^{\frac{p}{2}} \right] dx \\ &\lesssim \sum_{h \geq -1} 2^{\beta hp} \int_{\mathbf{T}^{d_2}} \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} \mathbf{E} [\Delta^h \delta_{s,t} \mathcal{F}_i \tau_n(x) \Delta^h \delta_{s,t} \mathcal{F}_i \tau_{-n}(x)] \right)^{\frac{p}{2}} dx \\ &\lesssim \sum_{h \geq -1} 2^{\beta hp} \int_{\mathbf{T}^{d_2}} \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} \sum_{m_1, m_2 \sim 2^h} e^{-ix \cdot (m_1 + m_2)} \mathbf{E} [\delta_{s,t} \hat{\tau}_{m_1, n} \delta_{s,t} \hat{\tau}_{m_2, -n}] \right)^{\frac{p}{2}} dx \\ &\lesssim \sum_{h \geq -1} 2^{\beta hp} \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} \sum_{m \sim 2^h} \mathbf{E} [\delta_{s,t} \hat{\tau}_{m, n} \delta_{s,t} \hat{\tau}_{-m, -n}] \right)^{\frac{p}{2}} \\ &\lesssim \left(\sum_{n \in \mathbf{Z}^{d_1}} \langle n \rangle^{-2\alpha} \sum_{m \in \mathbf{Z}^{d_2}} \langle m \rangle^{2\beta} \mathbf{E} [\delta_{s,t} \hat{\tau}_{m, n} \delta_{s,t} \hat{\tau}_{-m, -n}] \right)^{\frac{p}{2}}. \end{aligned}$$

We can now conclude the proof by means of a standard Kolmogorov argument in time combined with hypercontractive moment estimates, which yields, for any $\kappa > 0$,

$$\begin{aligned} \mathbf{E} [\|\tau\|_{\mathcal{L}(H^\alpha, \dot{\mathcal{C}}_T^{\theta-\kappa} \mathcal{C}^{\beta-\epsilon})}^p] &\lesssim \sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq 1}} |t-s|^{-\frac{\theta p}{2}} \mathbf{E} [\|\delta_{s,t}\tau\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}^2]^{\frac{p}{2}} \\ &\lesssim \sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq 1}} |t-s|^{-\frac{\theta p}{2}} \left(\sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \mathbf{E} [\delta_{s,t} \hat{\tau}_{m, n} \delta_{s,t} \hat{\tau}_{-m, -n}] \right)^{\frac{p}{2}}. \end{aligned}$$

□

2.5.3.3 Kolmogorov criteria for the time non-local random operator \mathfrak{V}

We now provide the Kolmogorov criterion we will use to control the LD random operator $\tau = \mathfrak{V}$. As described before, the feature of \mathfrak{V} that makes it challenging to estimate is that it is non-local in time – this is because of the heat kernel connecting the root of \mathfrak{V} to the internal node pictured right above it.

However, for this operator we can express this non-locality in time through an integral formula, exchanging (2.87) by writing, for $f : [0, T] \times \mathbf{T}^d \rightarrow \mathbf{R}$,

$$\tau(f)(t, \cdot) = \int_0^t \tau_{u,t}(f(u, \cdot)) du. \quad (2.91)$$

More explicitly, for $\tau = \mathfrak{V}$, writing $\mathfrak{I}_s(\cdot) = \mathfrak{I}(s, \cdot)$ for $s \in \mathbf{R}$, we have

$$\tau_{u,t} \equiv \mathfrak{V}_{u,t}(f) \stackrel{\text{def}}{=} \mathcal{N}(\mathfrak{I}_t, P_{t-u} \mathcal{N}(\mathfrak{I}_u, f, \mathfrak{I}_u), \mathfrak{I}_t) \text{ for } f : \mathbf{T}^d \rightarrow \mathbf{R}. \quad (2.92)$$

The following lemma gives us an estimate analogous to (2.88).

Lemma 2.73 Let τ be as in (2.91) and fix $\kappa > 0$ and $\theta \in (\kappa, 2 + \kappa)$. Then, we have

$$\|\tau(t)\|_{\mathcal{L}(C_T H^\alpha, \mathcal{C}^{\beta-\epsilon})} \lesssim \sup_{0 \leq u \leq t} |t - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t}\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}, \quad (2.93)$$

and

$$\begin{aligned} \|\tau\|_{\mathcal{L}(C_T H^\alpha, \dot{\mathcal{C}}_T^{\theta-\kappa} \mathcal{C}^{\beta-\epsilon})} &\lesssim \sup_{0 \leq u \leq t \leq T} |t - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t}(f(u, \cdot))\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})} \\ &\quad + \sup_{0 \leq u \leq s \leq t \leq T} |t - s|^{-\frac{\theta - \kappa}{2}} |s - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t} - \tau_{u,s}\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}. \end{aligned} \quad (2.94)$$

Proof. We start by performing the fixed-time estimate

$$\begin{aligned} \|\tau(f)(t)\|_{\mathcal{C}^{\beta-\epsilon}} &\lesssim \int_0^t \|\tau_{u,t}(f(u, \cdot))\|_{\mathcal{C}^{\beta-\epsilon}} du \lesssim \int_0^t |t - u|^{-1 + \frac{\theta - \kappa}{2}} du \sup_{0 \leq u \leq t \leq T} |t - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t}(f(u, \cdot))\|_{\mathcal{C}^{\beta-\epsilon}} \\ &\lesssim t^{\frac{\theta - \kappa}{2}} \left(\sup_{0 \leq u \leq t \leq T} |t - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t}\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})} \right) \|f\|_{C_T H^\alpha}. \end{aligned}$$

Dividing by $\|f\|_{C_T H^\alpha}$ and taking the supremum over $f \in C_T H^\alpha$ in the l.h.s. gives us

$$\|\tau(t)\|_{\mathcal{L}(C_T H^\alpha, \mathcal{C}^{\beta-\epsilon})} \lesssim \sup_{0 \leq u \leq t \leq T} |t - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t}\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}.$$

Similarly, using the fact that we have

$$\tau(f)(t, x) - \tau(f)(s, x) = \int_s^t \tau_{u,t}(f(u, \cdot)) du + \int_0^s (\tau_{u,t} - \tau_{u,s})(f(u, \cdot)) du,$$

we obtain, denoting $\delta_{s,t}^o \tau_{u,\cdot} \stackrel{\text{def}}{=} \tau_{u,t} - \tau_{u,s}$

$$\begin{aligned} &\|\delta_{s,t}^o \tau(f)\|_{\mathcal{C}^{\beta-\epsilon}} \\ &\lesssim \int_s^t \|\tau_{u,t}(f(u, \cdot))\|_{\mathcal{C}^{\beta-\epsilon}} du + \int_0^s \|\delta_{s,t}^o \tau_{u,\cdot}(f(u, \cdot))\|_{\mathcal{C}^{\beta-\epsilon}} du \\ &\lesssim |t - s|^{\frac{\theta - \kappa}{2}} \left(\sup_{0 \leq u \leq t \leq T} |t - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t}(f(u, \cdot))\|_{\mathcal{C}^{\beta-\epsilon}} \right. \\ &\quad \left. + \sup_{0 \leq u \leq s \leq t \leq T} |t - s|^{-\frac{\theta - \kappa}{2}} |s - u|^{1 - \frac{\theta - \kappa}{2}} \|\delta_{s,t}^o \tau_{u,\cdot}(f(u, \cdot))\|_{\mathcal{C}^{\beta-\epsilon}} \right) \\ &\lesssim |t - s|^{\frac{\theta}{2}} \left(\sup_{0 \leq u \leq t \leq T} |t - u|^{1 - \frac{\theta - \kappa}{2}} \|\tau_{u,t}(f(u, \cdot))\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})} \right. \\ &\quad \left. + \sup_{0 \leq u \leq s \leq t \leq T} |t - s|^{-\frac{\theta - \kappa}{2}} |s - u|^{1 - \frac{\theta - \kappa}{2}} \|\delta_{s,t}^o \tau_{u,\cdot}\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})} \right) \|f\|_{C_T H^\alpha}, \end{aligned}$$

from which the desired estimate follows. \square

Remark 2.74 We allow ourselves the blow-up factor in $|t - u|$ in the estimates of Lemma 2.73 in anticipation of the estimates we expect for the LD object (2.73), our estimates on P_{t-u} blow-up as $(t - u) \downarrow 0$, see the estimate (2.111b).

Our Kolmogorov criterion will be formulated for the quantities on the r.h.s. of (2.94). Before stating this result we state the replacement for Definition 2.70.

Definition 2.75 For any $T > 0$, we say $\tau_{\cdot,\cdot}$ is a 2-parameter time-varying random smooth operator if $\tau_{\cdot,\cdot}$ is a continuous map

$$\text{Simp}_T = \{(u, t) \in [0, T]^2 \ni (u, t) : u < t\} \mapsto \tau_{u,t} \in \mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})$$

which, for $x \in \mathbf{T}^d$, we often write in the form

$$\tau_{u,t}(f)(x) = \sum_{n \in \mathbf{Z}^{d_1}} \mathcal{F}_i \tau_n(x, u, t) \hat{f}_{-n},$$

and for $m \in \mathbf{Z}^d$

$$\mathcal{F}_x(\tau_{u,t}(f))(m) = \sum_{n \in \mathbf{Z}^{d_1}} \hat{\tau}_{m,n}(u, t) \hat{f}_{-n},$$

and $\hat{\tau}_{m,n}(u, t)$ decaying rapidly in m and n .

As before, we say that $\tau_{\cdot, \cdot}$ belongs to the inhomogeneous Gaussian chaos of order k if $\hat{\tau}_{m,n}(u, t)$ always does. We again say $\tau_{\cdot, \cdot}$ is stationary in space if $T_h \circ \tau_{u,t}$ is equal in distribution to $\tau_{u,t}$ for any $h \in \mathbf{T}^d$. In analogy with (2.89), for any $(u, t), (u', t') \in \text{Simp}_T$ and $n \in \mathbf{Z}^d$,

$$\mathbf{E}[\hat{\tau}_{m_1,n}(u, t) \hat{\tau}_{m_2,-n}(u', t')] = 0 \text{ unless } m_1 + m_2 = 0.$$

Remark 2.76 Note that \mathfrak{V}_N satisfies the Definition 2.75.

Finally, to state a Kolmogorov estimate taking the blow up when $u \uparrow t$ into account, for $\theta, \eta \in (0, 1)$, we endow the space of smooth functions $f : \text{Simp}_T \rightarrow \mathbf{R}$ with the Hölder norms with blow up $C_{\text{Simp}_T}^{\theta, \eta}$ defined as

$$\|f\|_{C_{\text{Simp}_T}^{\theta, \eta}} \stackrel{\text{def}}{=} \sup_{(u,t) \in \text{Simp}_T} |t - u|^\eta |f(u, t)| + \|f\|_{C_{\text{Simp}_T}^{\theta, \eta}},$$

where the homogeneous part of the norm is given by

$$\|f\|_{C_{\text{Simp}_T}^{\theta, \eta}} \stackrel{\text{def}}{=} \sup_{((u,t),(r,s)) \in \text{Simp}_T^2} (|t - u| \wedge |s - r|)^\eta (|t - s| + |r - u|)^{-\frac{\theta}{2}} |f(u, t) - f(r, s)|.$$

Note that the r.h.s. of (2.93) and (2.94) are bounded by

$$\|\tau_{\cdot, \cdot}\|_{C_{\text{Simp}_T}^{\theta-\kappa, \eta} \mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}$$

for $\eta = 1 - \frac{\theta-\kappa}{2} \in (0, 1)$.

The above discussion motivates the following lemma. Recall the notation $\delta_{s,t}^o \tau_{u, \cdot} = \tau_{u,t} - \tau_{u,s}$ the variation in the outer time. We also write $\delta_{u,r}^i \tau_{\cdot, t} \stackrel{\text{def}}{=} \tau_{r,t} - \tau_{u,t}$ the variation in the inner time.

Lemma 2.77 For any fixed $k \geq 1$, $p \in 2\mathbf{N}$, $\kappa > 0$ and $\theta \in (\kappa, 2 + \kappa)$, writing $\eta = 1 - \frac{\theta-\kappa}{2} \in (0, 1)$, and uniform over smooth 2-parameter time-varying stationary in space random operators τ over time $T \in (0, 1]$ belonging to the k -th inhomogeneous Gaussian chaos, one has

$$\begin{aligned} & \mathbf{E} \left[\left(\|\tau_{\cdot, \cdot}\|_{C_{\text{Simp}_T}^{\theta-\kappa, \eta} \mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})} \right)^p \right] \\ & \lesssim \sup_{0 \leq u \leq r \leq t \leq T} |t - r|^{\eta p} |r - u|^{-\frac{\theta p}{2}} \left(\sum_{m \in \mathbf{Z}^d, n \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \mathbf{E}[\delta_{u,r}^i \hat{\tau}_{m,n}(\cdot, t) \delta_{u,r}^i \hat{\tau}_{-m,-n}(\cdot, t)] \right)^{\frac{p}{2}} \\ & + \sup_{0 \leq r \leq s \leq t \leq T} |s - r|^{\eta p} |t - s|^{-\frac{\theta p}{2}} \left(\sum_{m \in \mathbf{Z}^d, n \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \mathbf{E}[\delta_{s,t}^o \hat{\tau}_{m,n}(r, \cdot) \delta_{s,t}^o \hat{\tau}_{-m,-n}(r, \cdot)] \right)^{\frac{p}{2}}. \end{aligned} \quad (2.95)$$

Proof. To lighten notation we just write $|\cdot|$ for $\|\cdot\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}$. Then for any random function $f : \text{Simp}_T \rightarrow \mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})$ in the k -th inhomogeneous Gaussian chaos, we again have by a classical Kolmogorov estimate and hypercontractive estimates,

$$\mathbf{E}[\|f\|_{C_{\text{Simp}_T}^{\theta-\kappa, \eta}}^p] \lesssim \sup_{((u,t),(r,s)) \in \text{Simp}_T^2} \left[(|t - u| \wedge |s - r|)^{\eta p} (|t - s| + |r - u|)^{-\frac{\theta p}{2}} \right] \quad (2.96)$$

$$\times \mathbf{E}[|f(u, t) - f(r, s)|^2]^{p/2}].$$

By symmetry, we can take $t \geq s$ in the supremum. Using $f(u, t) - f(r, s) = \delta_{r,u}^i f(\cdot, t) + \delta_{s,t}^o f(r, \cdot)$ we can bound the r.h.s. of (2.96) (up to an inessential constant) by

$$\begin{aligned} & \sup_{\substack{(u,t),(r,s) \in \text{Simp}_T^2 \\ t \geq s}} (|t - u| \wedge |s - r|)^{\eta p} |r - u|^{-\frac{\theta p}{2}} \mathbf{E}[|\delta_{r,u}^i f(\cdot, t)|^2]^{p/2} \\ & + \sup_{\substack{(u,t),(r,s) \in \text{Simp}_T^2 \\ t \geq s}} (|t - u| \wedge |s - r|)^{\eta p} |t - s|^{-\frac{\theta p}{2}} \mathbf{E}[|\delta_{s,t}^o f(r, \cdot)|^2]^{p/2}. \end{aligned}$$

For the first term on the r.h.s. above, observe that $t \geq s$ with $s \geq r$ yields $|t - u| \wedge |s - r| \leq |t - u| \wedge |t - r| = |t - u \vee r|$. For the second term we observe that we have $|t - u| \wedge |s - r| \leq |s - r|$. Together, this allows to estimate the r.h.s of (2.96) by

$$\begin{aligned} & \lesssim \sup_{\substack{(u,t),(r,s) \in \text{Simp}_T^2 \\ t \geq s}} |t - u \vee r|^{\eta p} |r - u|^{-\frac{\theta p}{2}} \mathbf{E}[|\delta_{r,u}^i f(\cdot, t)|^2]^{p/2} \\ & + \sup_{\substack{(u,t),(r,s) \in \text{Simp}_T^2 \\ t \geq s}} |s - r|^{\eta p} |t - s|^{-\frac{\theta p}{2}} \mathbf{E}[|\delta_{s,t}^o f(r, \cdot)|^2]^{p/2}. \end{aligned}$$

The desired estimate then follows by controlling expectations of $\|\cdot\|_{\mathcal{L}(H^\alpha, \mathcal{C}^{\beta-\epsilon})}^2$ by using (2.90) with $p = 2$ to replace them by sums over Fourier modes. \square

Remark 2.78 While the above Lemma gives control on the homogeneous bound, for any fixed $(u, t) \in \text{Simp}_T$ we have

$$\|f\|_{C_{\text{Simp}_T}^{\theta-\kappa, \eta}} \lesssim |t - u|^\eta |f(u, t)| + \|f\|_{\dot{C}_{\text{Simp}_T}^{\theta-\kappa, \eta}},$$

where putting in the factor $|t - u|^\eta |f(u, t)|$ makes the implicit constant above independent of the choice of (u, t) . Therefore, we have

$$\mathbf{E}[\|\tau_{\cdot, \cdot}\|_{C_{\text{Simp}_T}^{\theta-\kappa, \eta}}^p] \lesssim |t - u|^{\eta p} \left(\sum_{m \in \mathbf{Z}^d, n \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \mathbf{E}[\hat{\tau}_{m,n}(u, t) \hat{\tau}_{-m,-n}(u, t)] \right)^{\frac{p}{2}} + \text{r.h.s. of (2.95)}.$$

Controlling the first term of the r.h.s. of the above equation uniformly in $(u, t) \in \text{Simp}_T$ turns out to be similar to controlling the whole r.h.s. of (2.95). We will focus on describing how to estimate this quantity, and point out how the second term above can be obtained by extracting small factors from increments of heat kernels – see Remark 2.81.

Remark 2.79 Those who are less familiar with path-wise approaches to stochastic analysis may be surprised that we need to control Hölder regularity in u , as in the end we only need uniform estimates in u . However, the only way we have to control an expectation of a supremum over u is by proving regularity in u .

2.5.4 Regularity of the random field \uparrow and operators \uparrow^k

As an instructive warm-up before the more involved analysis of bigger trees taking place in Section 2.6, we look at regularity estimates for \uparrow both as a random field and a random operator. Using Lemma 2.69, we see that the key ingredient for (2.80) is estimating, for $\beta = |\uparrow|$,

$$\sum_{m \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \mathbf{E}[\delta_{s,t} \hat{\uparrow}_m \delta_{s,t} \hat{\uparrow}_{-m}] \lesssim \sum_{m \in \mathbf{Z}^d} \langle m \rangle^{2\beta-2+2\theta} |t - s|^\theta, \quad (2.97)$$

where we used (2.16) to obtain

$$\mathbf{E}[\delta_{s,t} \hat{\mathfrak{I}}_m \delta_{s,t} \hat{\mathfrak{I}}_{-m}] = \frac{2}{\langle m \rangle^2} (1 - e^{-|t-s|\langle m \rangle^2}) \lesssim \langle m \rangle^{-2+2\theta} |t-s|^\theta, \quad (2.98)$$

for $\theta \in (0, 1)$. The RHS of (2.97) is convergent if and only if $\beta < -\frac{d-2}{2} - \theta$, and so we obtain the desired statement by taking θ small enough.

For obtaining (2.82), we note that since we have invariance of the law of \mathfrak{I} under permutations of spatial coordinates for the random operator, it suffices to estimate moments of norms of \mathfrak{I}^J for $J = [k] \subset [d]$. Lemma 2.72 tells us the key estimate is

$$\begin{aligned} \sum_{n \in \mathbf{Z}^{d-k}, m \in \mathbf{Z}^k} \langle n \rangle^{-2\alpha} \langle m \rangle^{2\beta} \mathbf{E}[\delta_{s,t} \hat{\mathfrak{I}}_{m,n} \delta_{s,t} \hat{\mathfrak{I}}_{-m,-n}] \\ \lesssim \sum_{n \in \mathbf{Z}^{d-k}, m \in \mathbf{Z}^k} \langle n \rangle^{-2\alpha} \langle m \rangle^{2\beta} \langle (m, n) \rangle^{-2+2\theta} |t-s|^\theta. \end{aligned} \quad (2.99)$$

The situation here is more delicate, with the RHS of (2.99) being convergent if and only if the three sums $\sum_{p \in \mathbf{Z}^d} \langle p \rangle^{-2\alpha+2\beta-2+2\theta}$, $\sum_{n \in \mathbf{Z}^{d-k}} \langle n \rangle^{-2\alpha-2+2\theta}$, and $\sum_{m \in \mathbf{Z}^k} \langle m \rangle^{2\beta-2+2\theta}$ are all convergent. This gives us three conditions:

$$\beta < \alpha - \frac{d-2}{2}, \quad \beta < -\frac{k-2}{2} \quad \text{and} \quad \alpha > \frac{d-k-2}{2}. \quad (2.100)$$

More generally, the regularity we associate to a random field will be determined by how large we can take β while keeping (2.102) convergent. Similarly, for a general random operator τ , we will aim to take α small and β large while keeping (2.102) convergent. The exponents α and β are, respectively, called the *inner* and *outer regularities* of τ . For a random field τ , we also call the supremum over the values of β that we can take the outer regularity of τ .

Finally, since all the τ are polynomials in the Gaussian noise ξ , we can use Wick's rule to compute $C_\tau^{u,t}(m, n)$ as sum of various terms indexed by all the possible contractions of instances of ξ . Each of these contractions will correspond to a (possibly renormalized) amplitude. In the next step, we show that these amplitudes can be indexed by a certain class of graphs, that we define hereafter.

Remark 2.80 To argue the convergence as $N \uparrow \infty$ of \mathfrak{I}_N , note that for $N \geq M$, we have

$$\sum_{m \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \mathbf{E}[(\widehat{\mathfrak{I}}_N - \widehat{\mathfrak{I}}_M)_m (\widehat{\mathfrak{I}}_N - \widehat{\mathfrak{I}}_M)_{-m}] \lesssim \sum_{m \in \mathbf{Z}^d} \mathbf{1}_{[M, N]}(|m|_\infty) \langle m \rangle^{2\beta-2} \lesssim \sum_{m \in \mathbf{Z}^d} \langle m \rangle^{2\beta-2+\epsilon} M^{-\epsilon}$$

By giving up an arbitrarily small amount of regularity in our estimate, we obtain an $o(M)$ factor which we can use to argue convergence.

Remark 2.81 Regarding Hölder estimates in time, recall that the required estimate for the covariance of $\delta_{s,t} \mathfrak{I}$ is straightforward modification of the estimate of $\mathfrak{I}(t)$ for fixed t in (2.98). For computing (2.101) for more complicated stochastic estimates, we can always reorganize the terms by means of some telescoping sums so that we introduce a single time difference of free fields that gives us a good factor of $|t-s|$, see [MWX17].

Moreover, the same technique can be applied to (2.103) to prove Hölder-continuity in the inner time u of the non-local in time random operator: each contribution can be rewritten as a sum over many contribution depending on u and r only through a single term

$$\frac{1}{\langle m \rangle^2} (1 - e^{-|r-u|\langle m \rangle^2}) \lesssim \langle m \rangle^{-2+2\theta} |r-u|^\theta$$

giving us the needed small factor $|r-u|^\theta$. Therefore, in the sequel, we will focus on bounding (2.102) with the understanding that the required estimate on (2.101) and (2.103) can be dealt with similarly. Note that this remark is only relevant for the LD objects.

2.6 Diagrammatic methods and estimates on larger objects

This section is devoted to proofs of regularity estimates for stochastic objects more complicated than \dagger . In Section 2.5.3, we stated some Kolmogorov type estimates for both random fields and operators that allow us to reduce the proofs of stochastic estimates to covariance computations. In Section 2.5.4, we applied this estimates to the case of the dynamical free field \dagger , both seen as a random field and a random operator. When performing covariance computations, the question reduces to verify the values of the regularity parameters for which some sums over Fourier modes are convergent, *along with all their sub-sums* – see for instance (2.99) that gives rise to the three conditions (2.100) coming from the different sub-sums. For bigger objects, these sums become too involved to be directly written and controlled so we develop a more careful diagrammatic framework for obtaining the probabilistic estimates for the random fields and random operators that enter into our analysis, extending the diagrammatic notation introduced in Section 2.1.3.1.

In Section 2.6.1, we recall the expressions of the covariances that we need to control. In Section 2.6.2, we introduce stranded graphs, which are generated by Wick's rule for moments of Gaussians used in Section 2.6.3 to express bounds on covariances of our stochastic objects. A second class of graphs called tensor graphs, which are equivalent to tensor graphs but more suited for power-counting, are introduced in Section 2.6.4. In Section 2.6.5 we present a multiscale method for estimating the contribution from a given tensor graph, here we need to disentangle the behaviours of all the different sub-sums by looking at different sectors indexed by some tensor subgraphs. This analysis is completed in Sections 2.6.6 and (2.6.7) where we derive the stochastic estimates for all the random tensors and operators.

2.6.1 Covariances of the stochastic objects

We introduce notation for covariances that will be used to estimate the r.h.s. of the Kolmogorov estimates, and which in this section we will estimate using sums over graphs.

From now on, we fix $\theta > 0$ sufficiently small and $\eta < 1$ sufficiently close to 1.

Definition 2.82 We introduce a short-hand notation for the covariances of the stochastic objects. For any random field τ , we let

$$\begin{aligned} C_\tau^{u,t}(m, n) &\stackrel{\text{def}}{=} \mathbf{E}[\hat{\tau}_m(t)\hat{\tau}_{-m}(t)]\delta_{n,0}, \\ \tilde{C}_\tau^{r,s,t}(m, n) &\stackrel{\text{def}}{=} \mathbf{E}[\delta_{s,t}\hat{\tau}_m\delta_{s,t}\hat{\tau}_{-m}]\delta_{n,0}, \end{aligned}$$

for all random operators $\tau \neq \mathfrak{V}$, we let

$$\begin{aligned} C_\tau^{u,t}(m, n) &\stackrel{\text{def}}{=} \mathbf{E}[\hat{\tau}_{m,n}(t)\hat{\tau}_{-m,-n}(t)], \\ \tilde{C}_\tau^{r,s,t}(m, n) &\stackrel{\text{def}}{=} \mathbf{E}[\delta_{s,t}\hat{\tau}_{m,n}\delta_{s,t}\hat{\tau}_{-m,-n}], \end{aligned}$$

and for $\tau = \mathfrak{V}$ we let

$$\begin{aligned} C_\tau^{u,t}(m, n) &\stackrel{\text{def}}{=} \mathbf{E}[\hat{\tau}_{m,n}(u, t)\hat{\tau}_{-m,-n}(u, t)], \\ \tilde{C}_\tau^{r,s,t}(m, n) &\stackrel{\text{def}}{=} \mathbf{E}[\delta_{s,t}^o\hat{\tau}_{m,n}(r, \cdot)\delta_{s,t}^o\hat{\tau}_{-m,-n}(r, \cdot)], \\ \hat{C}_\tau^{u,r,t}(m, n) &\stackrel{\text{def}}{=} \mathbf{E}[\delta_{u,r}^i\hat{\tau}_{m,n}(\cdot, t)\delta_{u,r}^i\hat{\tau}_{-m,-n}(\cdot, t)]. \end{aligned}$$

Using Lemmas 2.69, 2.72 and 2.77, the needed space-time regularity estimates on LD stochastic objects reduces to estimating

$$|s-r|^{\mathbf{1}\{\tau=\mathfrak{V}\}2\eta}|t-s|^{-\theta} \sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \tilde{C}_\tau^{r,s,t}(m, n) \quad (2.101)$$

uniformly in the cut-off N and in (r, s, t) , while uniform in time (for the LD objects) and in the cut-off (for both LD and BG objects) control of the space regularity reduces to estimating

$$|t - u|^{1\{\tau = \text{violet}\}2\eta} \sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} C_\tau^{u,t}(m, n) \quad (2.102)$$

uniformly in the cut-off N for the LD objects and in (u, t) . Finally, to handle $\tau = \text{violet}$, it is necessary to control the Hölder-continuity in the inner time u , which corresponds to estimating

$$|t - r|^{2\eta} |r - u|^{-\theta} \sum_{m \in \mathbf{Z}^d, n \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \hat{C}_\tau^{u,r,t}(m, n) \quad (2.103)$$

uniformly in the cut-off N and in (u, r, t) .

Note that with the above notation, when τ is a random field, the term $\delta_{n,0}$ makes that the sum over $n \in \mathbf{Z}^{d_1}$ vanishes, so that one can think of (2.101) and (2.102) with $d_1 = 0$.

2.6.2 Stranded graphs

The first class of diagrams we will work with are called stranded graphs.

Definition 2.83 A *stranded graph* G is given by data $G = (V, L, \mathfrak{c}, \mathfrak{p})$ of the following form.

The data (V, L) specifies a quartic graph, namely where each link $l \in L$ is incident to one or two vertices and each vertex $v \in V$ is incident to four links, in both cases counting multiplicity. We allow a link l to be incident to a vertex v with multiplicity two (which we still call being incident to two vertices), and we call such a link a *tadpole*. In particular, if we say l is a tadpole at v , then v is also incident to l with multiplicity two. If any link l is only incident to one vertex, then we call it an *external link*. Links incident to two vertices (including tadpoles) are called *internal links*. We let $L^{\text{int}}(G)$ and $L^{\text{ext}}(G)$ be the internal link and external link set of G . A vertex which is only incident to internal links is called an *internal vertex* and we denote by $V^{\text{int}}(G)$ the internal vertex set of G . If $|L^{\text{ext}}(G)| = 0$ then G is called a *closed stranded graph* while if $|L^{\text{ext}}(G)| > 0$ we call G an *open stranded graph*.

The data \mathfrak{c} is a “coloring map” $\mathfrak{c} : N \rightarrow [d]$ that indicates which term in $\sum_{c=1}^d \mathcal{I}^c(\phi)$ is being associated to the vertex v . As an example, if we write $[d] = \{1, 2, 3, 4\}$, then if $\mathfrak{c}(v) = 1$, v would be associated to the following picture.

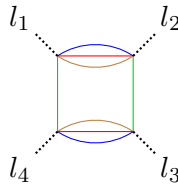


Figure 6: Pictorial representation of a vertex of a stranded graph

The colored edges above are not links in L , only the dashed lines l_1, l_2, l_3, l_4 are. Note that l_1, \dots, l_4 need not all be distinct if we have a tadpole at v . Note also that the four nodes appearing above are not four elements of the set V , but the entire collection of them plus the colored edges correspond to one vertex in V .

The data \mathfrak{p} consists of associating, to each $v \in V$, an ordered pair of partitions of the multi-set of links incident to v , which we write $\mathfrak{p}(v) = (\hat{\mathfrak{p}}(v), \bar{\mathfrak{p}}(v))$. The data $\mathfrak{p}(v)$ tells us how to glue the links incident to v to the non-local vertex, $\hat{\mathfrak{p}}(v)$ is the pairing induced by pairing links that are incident to nodes of the non-local vertex connected by the edges colored $\mathfrak{c}(v)$, while $\bar{\mathfrak{p}}(v)$ is the pairing induced by pairing links that are connected by the other $d - 1$ edges. The data $\mathfrak{p}(v)$ furthermore comes with the

following compatibility condition: we enforce $\hat{p}(v) \neq \bar{p}(v)$, unless $\hat{p}(v) = \bar{p}(v) = \{\{l_t, l_1\}, \{l_t, l_2\}\}$ with $l_t, l_1, l_2 \in L$, and possibly $l_1 = l_2$.

For example, in the picture above one has

$$\hat{p}(v) = \{\{l_1, l_4\}, \{l_2, l_3\}\} \text{ and } \bar{p}(v) = \{\{l_1, l_2\}, \{l_3, l_4\}\}. \tag{2.104}$$

In what follows, we denote by $\{l_1(v), \dots, l_4(v)\}$ the links incident to v such that (2.104) holds.

Finally a *rooted* stranded graph is a stranded graph with either exactly one marked vertex or exactly one marked link.

Note that stranded graphs for the local theory are obtained simply by ignoring the data c and p . We close the section by describing the notion of stranded subgraph.

A stranded graph can be pictured using the representation of Figure 6 for each vertex, which also encodes the data c, p , and representing each link by a dotted black line.

Definition 2.84 A *stranded subgraph* G' of a closed stranded graph G is defined by a subset $L^{int}(G') \subset L(G)$. Its vertex set $V(G')$ is given by the set of all vertices incident to the links in $L^{int}(G')$ while its external link set $L^{ext}(G')$ is given by the set of all the links in $L^{ext}(G)$ that are attached to the vertices in $V(G')$ but are not in $L^{int}(G')$. G' inherits the data c and p of G , so that it is indeed a stranded graph.

2.6.2.1 Drawing stranded graphs

Why we call the objects “stranded graphs” becomes more clear once we draw them by expanding links as d parallel strands. This Section is indeed devoted to the introduction of a second diagrammatic representation of stranded graphs using colored strands.

In Feynman diagrams each link l carries a Fourier variable $m_l = (m_{l,i})_{i=1}^d \in \mathbf{N}^d$ and contributes a factor $\langle m_l \rangle^{-2}$, the total contribution of a given diagram comes from summing over the Fourier variable for every link subject to the constraints imposed by the vertices on Fourier modes of the links incident to them.

At the level of links, drawing a stranded graph using strands involves expanding each link into d -different edges corresponding to the d -different components of its Fourier mode (which we view as carrying different colors $c \in [d]$).

$$l \quad \longleftrightarrow \quad \begin{array}{c} l \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \longleftrightarrow \quad \sum_{m_l=(m_{l,1},m_{l,2},m_{l,3},m_{l,4}) \in \mathbf{Z}^4} \langle m_l \rangle^{-2}$$

Figure 7: A link of a stranded graph representing a Fourier mode m

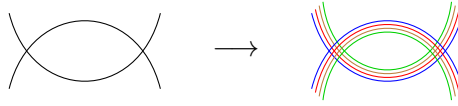
The Fourier relations (2.14) and (2.15) determine how links interact at vertices. We now describe how we write vertices using strands. For the local theory, the relation (2.14) just produces a d -fold reproduction of the original vertex.

$$\begin{array}{c} l_4 \\ \vdots \\ l_1 \cdots l_3 \\ \vdots \\ l_2 \end{array} \quad \longleftrightarrow \quad \delta(\sum_{i=1}^4 m_{e_i}) \quad \longleftrightarrow \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \longleftrightarrow \quad \prod_{c \in \{1,2,3,4\}} \delta(\sum_{i=1}^4 m_{l_i,c})$$

Figure 8: The local vertex, and its trivial stranded representation

For larger Feynman diagrams in the local theory, expanding both links and vertices using strands in

the local theory again just duplicates them, see the example below.



For the tensor field theory, we have a different vertex for each color c . The following picture shows how the relation (2.15) can be pictured in terms of strands for $c = 1$.

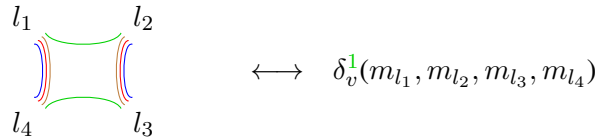
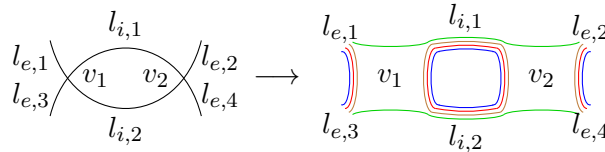


Figure 9: The green vertex representing the green nonlinearity \mathcal{I}^1

where for $c \in [d]$ and Fourier modes $m, n, p, q \in \mathbf{Z}^d$,

$$\delta_v^c(m, n, p, q) \stackrel{\text{def}}{=} \delta_{m_c, -n_c} \delta_{p_c, -q_c} \delta_{m_{\hat{c}}, -q_{\hat{c}}} \delta_{n_{\hat{c}}, -p_{\hat{c}}}.$$

An example of writing both links and vertices with strands in a larger graph is



Going from the left to the right representation uses the data (c, \mathfrak{p}) given by $c(v_a) = \mathbf{1}$, $\hat{\mathfrak{p}}(v_a) = \{\{l_{i,1}, l_{e,a}\}, \{l_{i,2}, l_{e,a+2}\}\}$ and $\bar{\mathfrak{p}}(v_a) = \{\{l_{e,a}, l_{e,a+2}\}, \{l_{i,1}, l_{i,2}\}\}$ for every $a \in \{1, 2\}$. The graph above has three internal strands and eight external strands.

When we draw stranded graphs in their stranded representation, we see that a stranded graph can be associated to a collection of graphs each of which is a single color. Individual colored strands pass through at least one link and one vertex, possibly more.

If a strand runs through at least one external link, it is called an *external strand*. Otherwise, a strand going through only internal links is called an *internal strand*. We let $S(G)$ and $S^{\text{int}}(G)$ be the strand set and internal strand set of G .

Due to the delta functions imposed at vertices, each strand $s \in S(G)$, in addition to carrying a single color, is also associated to a one dimensional Fourier mode which we denote $m_s \in \mathbf{Z}$.

2.6.3 Bounding renormalized amplitudes with stranded graphs

Let us go back to the evaluation of (2.102). In the sequel, we have to partition the stochastic objects we are dealing with into two categories. We call stochastic objects of *type I* all the random fields listed in Definition 2.64 along with stoch and $\text{stoch}^{\text{sto}}$, and stochastic objects of *type II* $\text{stoch}^{\text{stoch}}$ and $\text{stoch}^{\text{stoch}^{\text{sto}}}$ (which are defined in (2.118)). Dealing with these objects is sufficient to prove Lemmas 2.66 and 2.68.

To a stochastic object τ of type I that contains k instances of the noise we associate a collection of closed rooted stranded graphs \mathbb{G}_τ and also define a map $A_{T,N}^{u,t} : \mathbb{G}_\tau \rightarrow \mathbf{R}$ such that

$$|t - u|^{\mathbf{1}\{\tau = \text{stoch}\} 2\eta} \sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} C_\tau^{u,t}(m, n) = \sum_{G \in \mathbb{G}_\tau} A_{T,N}^{u,t}(G). \quad (2.105)$$

Each $G \in \mathbb{G}_\tau$ is determined by choosing one out of all the pairings of the $2k$ noises coming from Wick's rule and one of the colorings of all the vertices in the two copies of τ . Graphs in \mathbb{G}_τ have links, vertices. Moreover, one or two of their links also carry a special labeling reflecting the terms $\langle n \rangle^{-2\alpha}$ and $\langle m \rangle^{2\beta}$ that probe the inner and outer regularities. We denote them as l_α and l_β , and l_β is chosen to be the root of G . Note that $l_\alpha \in L(G)$ if and only if $\tau \in \{ \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \}$.

If τ is of type II, then there is a subtlety related with the definition of l_α and l_β . Indeed, gluing two copies of τ and performing all the possible pairings of the $2k$ noises and choosing all the possible colorings of the vertices again produces terms that are indexed by a collection of closed rooted stranded graphs \mathbb{G}_τ . However, for $G_s \in \mathbb{G}_\tau$, l_α and l_β do not belong to $L(G_s)$ but are defined as follows. Since G_s is representing the gluing of two copies of τ , and τ contains one half-vertex, the gluing produces a new vertex r of color $c \in [d]$, which is chosen to be the root of G_s . If $\tau \in \{ \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}^{\text{sto}} \}$, then r is crossed by two strands s_α and s_β of color c , and we have $\langle m \rangle^{2\beta} = \langle m_{s_\beta} \rangle^{2\beta}$ and $\langle n \rangle^{-2\alpha} = \langle m_{s_\alpha} \rangle^{-2\alpha}$, so that we define $l_\alpha \stackrel{\text{def}}{=} s_\alpha \cap r$, $l_\beta \stackrel{\text{def}}{=} s_\beta \cap r$, and we have $m_{l_\alpha} = m_{s_\alpha} \in \mathbf{Z}$ and $m_{l_\beta} = m_{s_\beta} \in \mathbf{Z}$. The situation is similar if $\tau = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$, but now r is crossed by two beams of $d-1$ strands of all the colors but c , that we denote $f_\alpha \stackrel{\text{def}}{=} (s_{\alpha,i})_{i \in [d] \setminus \{c\}}$ and $f_\beta \stackrel{\text{def}}{=} (s_{\beta,i})_{i \in [d] \setminus \{c\}}$. We also denote $m_{f_\alpha} \stackrel{\text{def}}{=} (m_{s_{\alpha,i}})_{i \in [d] \setminus \{c\}}$ and $m_{f_\beta} \stackrel{\text{def}}{=} (m_{s_{\beta,i}})_{i \in [d] \setminus \{c\}}$, that are such that $\langle m \rangle^{2\beta} = \langle m_{f_\beta} \rangle^{2\beta}$ and $\langle n \rangle^{-2\alpha} = \langle m_{f_\alpha} \rangle^{-2\alpha}$, and again we define $l_\alpha \stackrel{\text{def}}{=} f_\alpha \cap r$, $l_\beta \stackrel{\text{def}}{=} f_\beta \cap r$, so that $m_{l_\alpha} = m_{f_\alpha} \in \mathbf{Z}^{d-1}$ and $m_{l_\beta} = m_{f_\beta} \in \mathbf{Z}^{d-1}$. This definition implies that we do not really see the root r as a vertex, but rather as two parallel links gluing the two copies of τ . From now on, we thus identify G_s with the graph G which is obtained from G_s removing the two times 1 or $d-1$ edges that compose l_α and l_β and replacing them with l_α and l_β . By definition, $L(G) \stackrel{\text{def}}{=} L(G_s) \sqcup \{l_\alpha\} \sqcup \{l_\beta\}$. See Figure 10 for an illustration of the definitions of l_α and l_β in the tensor graph representation. Going from G_s to G then simply involves seeing l_α and l_β as two dotted lines instead of 1 or $d-1$ parallel edges.

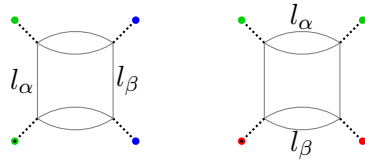


Figure 10: The gluing of two copies of $\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$ and of $\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$, with the positions of l_α and l_β

2.6.3.1 Renormalizing amplitudes

The formula for $A_{T,N}^{u,t}(G)$ must also take into consideration our renormalization coming from the possible divergent subgraphs of the forms $\mathfrak{M}^{1,c}$ and $\mathfrak{M}^{2,c,c'}$ that are introduced in Definition 2.6. For every $G \in \mathbb{G}_\tau$, we thus define the (possibly empty) sets $\mathfrak{M}^{1,c}(G)$ (respectively $\mathfrak{M}^{2,c,c'}(G)$) that contain all the subgraphs of G of the form $\mathfrak{M}^{1,c}$ for $c \in [d]$ (respectively $\mathfrak{M}^{2,c,c'}$ for $c \neq c'$). We write $\mathfrak{M}^1(G) \stackrel{\text{def}}{=} \bigcup_{c=1}^d \mathfrak{M}^{1,c}(G)$, $\mathfrak{M}^2(G) \stackrel{\text{def}}{=} \bigcup_{c \neq c'} \mathfrak{M}^{2,c,c'}(G)$ and $\mathfrak{M}(G) \stackrel{\text{def}}{=} \mathfrak{M}^1(G) \cup \mathfrak{M}^2(G)$.

We distinguish the links and vertices of G that do not belong to some $M \in \mathfrak{M}(G)$, defining $\tilde{V}(G) \stackrel{\text{def}}{=} V(G) \setminus \bigcup_{M \in \mathfrak{M}^2} V^{\text{int}}(M)$ (this definition takes into account the fact that graphs in \mathfrak{M}^1 do not have internal vertices) and $\tilde{L}(G) \stackrel{\text{def}}{=} L(G) \setminus \bigcup_{M \in \mathfrak{M}(G)} L^{\text{int}}(M)$.

Graphs $M \in \mathfrak{M}^1$ only have one internal link $l_i = l_i(M)$, and we define $m_{M,i} \stackrel{\text{def}}{=} m_{l_i}$ the internal Fourier mode of M . They have two external links $l_{e,1} = l_{e,1}(M)$ and $l_{e,2} = l_{e,2}(M)$ that carry the same Fourier mode, and we define $m_{M,e} \stackrel{\text{def}}{=} m_{l_{e,1}} = m_{l_{e,2}}$ the external Fourier mode of M . With this notation, if M is color c it carries the renormalized amplitude $\mathfrak{R}^{1,c}(m_M)$ of (2.23).

Graphs $M \in \mathfrak{M}^2(G)$ have three internal links, among which two are attached to an external node of M (they are thus themselves the two external links of some graph $M_1(M) \in \mathfrak{M}^1(G)$). These two links are called *superficial*, and we denote $l_{s,1} = l_{s,1}(M)$ and $l_{s,2} = l_{s,2}(M)$. We call the remaining internal

link of M the *deep link*, and it is denoted $l_d = l_d(M)$.

The structure of M imposes that the two superficial links of M carry the same Fourier mode which we denote $m_{M,s} \stackrel{\text{def}}{=} m_{l_{s,1}} = m_{l_{s,2}}$ while we denote $m_{M,d} \stackrel{\text{def}}{=} m_{l_d}$ the Fourier mode of the deep link of M_2 (which is also the internal link of $M_1(M_2)$). They also have two external links $l_{e,1} = l_{e,1}(M)$ and $l_{e,2} = l_{e,2}(M)$ that carry the same Fourier mode, and we define $m_{M,e} \stackrel{\text{def}}{=} m_{l_{e,1}} = m_{l_{e,2}}$ the external Fourier mode of M . With this notation, M of colors c, c' bears the amplitude $\mathfrak{R}^{2,c,c'}(m_{M,s}, m_{M,d})$.

Finally, we let $\tilde{\mathfrak{M}}^{1,c}(G)$ be the set of all the graphs in $\mathfrak{M}^{1,c}(G)$ that are not of the form $M_1(M_2)$ for some $M_2 \in \mathfrak{M}^{2,c,c'}(G)$ for $c' \neq c$. We also let $\tilde{\mathfrak{M}}^1(G) \stackrel{\text{def}}{=} \bigcup_{c=1}^d \tilde{\mathfrak{M}}^{1,c}(G)$ and $\tilde{\mathfrak{M}}(G) \stackrel{\text{def}}{=} \tilde{\mathfrak{M}}^1(G) \cup \mathfrak{M}^2(G)$. A link l can link two different graphs $M, M' \in \tilde{\mathfrak{M}}(G)$. If this is the case, we use the freedom we have in the labelling of $l_{e,1}, l_{e,2}$ to enforce the convention that if $l = l_{e,1}(M)$, then $l = l_{e,2}(M')$. See Figure 11 for an illustration of the notations introduced in this paragraph.

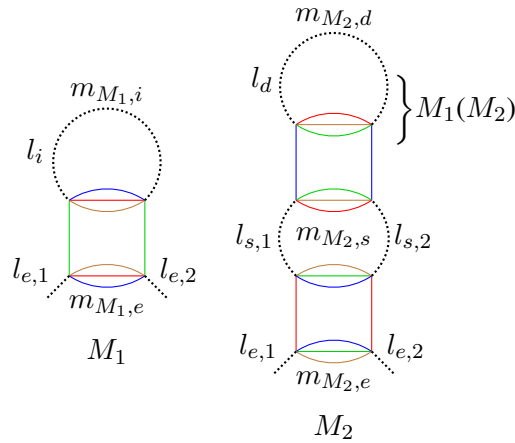


Figure 11: A divergent graph M_1 of type $\mathfrak{M}^{1,c}$ and a divergent graph M_2 of type $\mathfrak{M}^{2,c,c'}$ pictured in the tensor graph representation introduced below

We also need the following partition of $\tilde{L}(G)$ which will enable us to reexpress (2.102). First of all, note that l_α and l_β belong to $\tilde{L}(G)$ since they are attached to two different vertices in a non-melonic way. Links in $\tilde{L}(G)$ are partitioned as $\tilde{L}(G) = \tilde{L}^i(G) \sqcup \tilde{L}^c(G) \sqcup \{l_\alpha\} \sqcup \{l_\beta\}$ (if τ is a random fields, then l_α is absent in this decomposition) where links in $\tilde{L}^i(G)$ stand for *time lines* while links in $\tilde{L}^c(G)$ represent the contraction of two noises. By time lines, we mean the links associated to the inverse heat operator in LD objects (i.e. the time integral of P_{t-s}), and to the time integral of J^2 in BG objects. Note that once all the integrations over the variables of the noises yielding some objects \dagger_t are performed, then the time lines are the only links bearing a time integration. Note also that a graph G can have other links $l \in L(G)$ corresponding to a time integral than those contained in $\tilde{L}^i(G)$, since each subgraph $M \in \mathfrak{M}^2(G)$ has exactly one link corresponding to a time integral (one of its two superficial links) which, since it belongs to a renormalized subgraph, is not contained in $\tilde{L}^i(G)$.

We display below an example of a graph $G \in \mathbb{G}_\tau$ for $\tau = \Psi$. Here, l_α and l_β are indicated by their names, while l_1 and l_2 are the two elements of $\tilde{L}^i(G)$, and all the four remaining links belong to $\tilde{L}^c(G)$. Note that the same graph with l_α replaced by a link $l \in \tilde{L}^c(G)$ belongs to \mathbb{G}_τ for $\tau = \Psi$ (more precisely it would be a contribution coming from the gluing of $\mathcal{N}(\dagger, \Psi, \dagger) - \frac{1}{2}\mathcal{E}^2\dagger$ with itself).

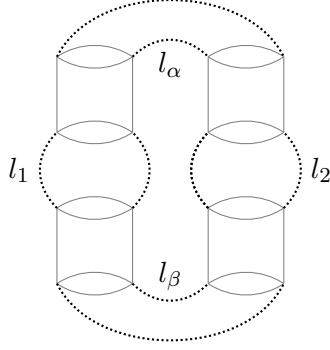


Figure 12: A graph $G \in \mathbb{G}_\tau$ for $\tau = \text{symbol}$ with its links in $\{\tilde{L}^i(G)\} \sqcup \{l_\alpha\} \sqcup \{l_\beta\}$ indicated

With this notation in hand, we are ready to define, for any τ and $G \in \mathbb{G}_\tau$, a labelling of the links of G , $\ell : L(G) \rightarrow \mathbf{R}$:

$$\ell(l) \equiv \ell_\tau(l) \stackrel{\text{def}}{=} 1 - (1 - \alpha)\mathbf{1}\{l = l_\alpha\} - (1 + \beta)\mathbf{1}\{l = l_\beta\} - (1 - \eta)\mathbf{1}\{\tau = \text{symbol} \text{ and } l \in \tilde{L}^i(G)\}. \quad (2.106)$$

The tree structure of τ implies that for every $v \in \tilde{V}(G)$, there exists a unique path \mathfrak{P}_v made with only links in $\tilde{L}^i(G)$ between v and the root (if τ is of type II) or one of the two vertices attached to the root (if τ is of type I). This induces a partial order on $\tilde{V}(G)$, and we say that $v_2 \prec v_1$ if v_1 is on the path \mathfrak{P}_{v_2} . For $l \in \tilde{L}^i(G)$, we denote by $v_1(l)$ and $v_2(l)$ the two extremities of l such that $v_2 \prec v_1$. For $l \in \tilde{L}^c(G)$, we make an arbitrary choice in labelling the two extremities of l as $v_1(l)$ and $v_2(l)$.

2.6.3.2 Explicit formulae and bounds for renormalized amplitudes

We can now give an expression for $A_{T,N}^{u,t}(G)$. Recall that links of G are partitioned as

$$L(G) = \bigsqcup_{M \in \tilde{\mathfrak{M}}(G)} L^{int}(M) \sqcup \tilde{L}^i(G) \sqcup \tilde{L}^c(G) \sqcup \{l_\alpha\} \sqcup \{l_\beta\}.$$

While l_α and l_β bear the kernels $\langle m_{l_\alpha} \rangle^{-2\alpha}$ and $\langle m_{l_\beta} \rangle^{2\beta}$, links l in $\tilde{L}^i(G)$ bear the integral kernel

$$\begin{cases} \int_{a_l}^{t_{v_1(l)}} \mathbf{I}(t_{v_1(l)}, t_{v_2(l)}, m_l) dt_{v_2(l)} & \text{if } \tau \neq \text{symbol}, \\ \mathbf{I}(t_{v_1(l)}, t_{v_2(l)}, m_l) & \text{if } \tau = \text{symbol}. \end{cases}$$

where $a_l \in \{-\infty, 0\}$ and

$$\mathbf{I}(t_1, t_2, m) = \begin{cases} \mathbf{1}\{|m|_\infty \leq N\} e^{-(t_1 - t_2)\langle m \rangle^2} & \text{for a LD object,} \\ \frac{\partial_{t_2}^2 \varrho_{t_2}^2(m)}{\langle m \rangle^2} & \text{for a BG object.} \end{cases}$$

and links l in $\tilde{L}^c(G)$ carry the contraction kernel $\mathbf{C}(t_{v_1(l)}, t_{v_2(l)}, m_l)$ given by

$$\mathbf{C}(t_1, t_2, m) = \begin{cases} \mathbf{1}\{|m|_\infty \leq N\} \frac{e^{-|t_1 - t_2|\langle m \rangle^2}}{\langle m \rangle^2} & \text{for a LD object,} \\ \frac{\varrho_{t_1 \wedge t_2}^2(m)}{\langle m \rangle^2} & \text{for a BG object.} \end{cases}.$$

The remaining links belong to some $M \in \tilde{\mathfrak{M}}(G)$. The subamplitude $M(t_M, m_{M,e})$ of $M \in \tilde{\mathfrak{M}}^{1,c}(G)$ at $t_M = t_{v_2(l_{e,1})}$ is given by

$$M(t_M, m_{M,e}) = \sum_{m_{M,i} \in \mathbf{Z}^d} \delta_{m_{M,i,c}, m_{M,e,c}} (\mathbf{C}(t_M, t_M, m_{M,i}) - \mathbf{C}(t_M, t_M, m_{M,i,\hat{e}})),$$

and the subamplitude $M(t_M, m_{M,e})$ of $M \in \mathfrak{M}^{2,c,c'}(G)$ at $t_M = t_{v_2(l_{e,1})}$ is given by

$$\begin{aligned} M(t_M, m_{M,e}) &= \sum_{m_{M,s}, m_{M,d} \in \mathbf{Z}^d} \delta_{m_{M,s,c}, m_{M,e,c}} \delta_{m_{M,d,c'}, m_{M,s,c'}} \int_{a_{l_s}}^{t_M} ds \\ &\quad (\mathbf{I}(t_M, s, m_{M,s}) \mathbf{C}(t_M, s, m_{M,s}) - \mathbf{I}(t_M, s, m_{M,s,\hat{c}}) \mathbf{C}(t_M, s, m_{M,s,\hat{c}})) \\ &\quad (\mathbf{C}(s, s, m_{M,d}) - \mathbf{C}(s, s, m_{M,d,c'})). \end{aligned}$$

With this notation we have for $\tau \neq \mathfrak{V}$ and $G \in \mathbb{G}_\tau$

$$\begin{aligned} A_{T,N}^{u,t}(G) &= \sum_{m_{l_1}, \dots, m_{l_{|\tilde{L}(G)|}} \in \mathbf{Z}^{d(l)}} \langle m_{l_\alpha} \rangle^{-2\alpha} \langle m_{l_\beta} \rangle^{2\beta} \prod_{l \in \tilde{L}^i(G)} \int_{a_l}^{t_{n_1(l)}} dt_{n_2(l)} \mathbf{I}(t_{n_1(l)}, t_{n_2(l)}, m_l) \\ &\quad \prod_{M \in \mathfrak{M}(G)} M(t_M, m_{M,e}) \prod_{v \in V(G)} \delta_v(m_{l_1(v)}, \dots, m_{l_4(v)}) \prod_{l \in \tilde{L}^c(G)} \mathbf{C}(t_{n_1(l)}, t_{n_2(l)}, m_l), \end{aligned}$$

where $d(l) = d$ except, if τ is of type II, for l_α and l_β . Here, it is understood that the time of the root (if τ is of type II) and the times of the two vertices attached to the root (if τ is of type I) is t .

Moreover, a similar formula holds for $\tau = \mathfrak{V}$ and $G \in \mathbb{G}_\tau$, except that $n_1(l) = t$ and $n_2(l) = u$ for both $l \in \tilde{L}^i(G)$ (so that there is no integration over the variables $(t_{n_2(l)})_{l \in \tilde{L}^i(G)}$), and that the additional factor $|t - u|^{2\eta}$ is present. We state it for completeness:

$$\begin{aligned} A_{T,N}^{u,t}(G) &= \sum_{m_{l_1}, \dots, m_{l_{|\tilde{L}(G)|}} \in \mathbf{Z}^{d(l)}} \langle m_{l_\beta} \rangle^{2\beta} \langle m_{l_\alpha} \rangle^{-2\alpha} \prod_{l \in \tilde{L}^i(G)} |t - u|^\eta \mathbf{I}(t, u, m_l) \\ &\quad \prod_{v \in V(G)} \delta_v(m_{l_1(v)}, \dots, m_{l_4(v)}) \prod_{l \in \tilde{L}^c(G)} \mathbf{C}(t_{n_1(l)}, t_{n_2(l)}, m_l). \end{aligned}$$

Lemma 2.85 *With all the previous notations introduced in Definition 2.83 and in the last paragraph at hand, for every stochastic object τ we have (2.105).*

Moreover, for every $G \in \mathbb{G}_\tau$, we have $A_{T,N}^{u,t}(G) \lesssim \tilde{A}(G)$ where $\tilde{A}(G)$ is independent of u, t, T and N , and is given by

$$\begin{aligned} \tilde{A}(G) \stackrel{\text{def}}{=} &\sum_{m_{s_1}, \dots, m_{s_{|S(G)|}} \in \mathbf{Z}^d} \prod_{l \in \tilde{L}(G)} \langle m_l \rangle^{-2\ell(l)} \prod_{c \in [d]} \prod_{M \in \mathfrak{M}^{1,c}(G)} \frac{\langle m_{M,e} \rangle^2 \wedge \langle m_{M,i} \rangle^2}{\langle m_{M,i} \rangle^2 \langle m_{M,i,\hat{c}} \rangle^2} \\ &\prod_{c \neq c'} \prod_{M \in \mathfrak{M}^{2,c,c'}(G)} \frac{\langle m_{M,e} \rangle^2 \wedge \langle m_{M,s} \rangle^2 \langle m_{M,s} \rangle^2 \wedge \langle m_{M,d} \rangle^2}{\langle m_{M,s} \rangle^2 \langle m_{M,s,\hat{c}} \rangle^4 \langle m_{M,d} \rangle^2 \langle m_{M,d,\hat{c}} \rangle^2}. \end{aligned} \quad (2.107)$$

In other words, we have uniformly in $u, t \in \text{Simp}_T$, $T \geq 0$ and $N \in \mathbf{N}$,

$$|t - u|^{\mathbf{1}\{\tau = \mathfrak{V}\}2\eta} \sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} C_\tau^{u,t}(m, n) \lesssim \sum_{G \in \mathbb{G}_\tau} \tilde{A}(G). \quad (2.108)$$

Moreover, a similar formula holds uniformly in $0 \leq r \leq s \leq t \leq T < \infty$ and $N \in \mathbf{N}$ to control the time continuity of the LD objects:

$$|s - r|^{\mathbf{1}\{\tau = \mathfrak{V}\}2\eta} |t - s|^{-\theta} \sum_{m \in \mathbf{Z}^{d_2}, n \in \mathbf{Z}^{d_1}} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \tilde{C}_\tau^{r,s,t}(m, n) \lesssim \sum_{G \in \mathbb{G}_\tau} \sum_{l_0 \in \tilde{L}^r(G)} \mathring{A}(G, l_0), \quad (2.109)$$

where $\mathring{A}(G, l_0)$ is defined similarly to $\tilde{A}(G)$ but with $\ell(l)$ replaced by

$$\tilde{\ell}(l, l_0) \equiv \tilde{\ell}_\tau(l, l_0) \stackrel{\text{def}}{=} \ell(l) - \theta \mathbf{1}\{l = l_0\}$$

in (2.107), and $\tilde{L}^r(G)$ is the set of all links of G in $\tilde{L}^i(G) \sqcup \tilde{L}^c(G)$ that are attached to the root if τ is of type II or to the two vertices attached to the root if τ is of type I.

Finally, we also have a similar formula holding uniformly in $0 \leq u \leq r \leq t \leq T < \infty$ and $N \in \mathbf{N}$ to control the inner time continuity of \mathfrak{V} :

$$|t-r|^{2\eta}|r-u|^{-\theta} \sum_{m \in \mathbf{Z}^d, n \in \mathbf{Z}^d} \langle m \rangle^{2\beta} \langle n \rangle^{-2\alpha} \hat{C}_\tau^{u,r,t}(m,n) \lesssim \sum_{G \in \mathbb{G}_\tau} \sum_{l_0 \in \tilde{L}^i(G)} \mathring{A}(G, l_0). \quad (2.110)$$

Proof. We have (2.105) by the definition of the quantities on the r.h.s of (2.105) and Wick's rule.

We first show (2.108). In order to deal with both LD and BG objects in our estimates, we denote by \mathbf{I} and \mathbf{C} some general integration and contraction kernels verifying the following assumptions (for $a \in \{0, -\infty\}$):

$$\int_a^t \mathbf{I}(t, s, m) ds \leq \langle m \rangle^{-2} \text{ for all } t \geq 0, \quad (2.111a)$$

$$|t-u|^\eta \mathbf{I}(t, u, m) \lesssim \langle m \rangle^{-2\eta} \text{ for all } t \geq u \geq 0, \quad (2.111b)$$

$$\mathbf{C}(t_1, t_2, m) \leq \langle m \rangle^{-2} \text{ for all } t_1, t_2 \geq 0, \quad (2.111c)$$

$$\mathbf{C}(t, t, m) = \tilde{\mathbf{C}}(t, m) \langle m \rangle^{-2} \text{ with } \tilde{\mathbf{C}}(t) \leq 1 \text{ for all } t \geq 0 \quad (2.111d)$$

$$\text{and } -\partial_{m_i^2} \tilde{\mathbf{C}}(t, m) = |\partial_{m_i^2} \tilde{\mathbf{C}}(t, m)| \lesssim \langle m \rangle^{-2} \text{ for all } t \geq 0.$$

$$\int_a^t \mathbf{I}(t, s, m) \mathbf{C}(t, s, m) ds \lesssim \tilde{\mathbf{C}}(t, m) \langle m \rangle^{-4} \text{ with } \tilde{\mathbf{C}} \text{ as above.} \quad (2.111e)$$

The integral and contraction kernels of both the LD and BG objects verify these assumptions, rather trivially for the LD objects for which $\tilde{\mathbf{C}}(t, m) = 1$. For the BG objects, we have $\tilde{\mathbf{C}}(t, m) = \varrho_t^2(m)$. Note that we have

$$-\partial_{m_i^2} \tilde{\mathbf{C}}(t, m) = 2\varrho_t(m) \partial_{m_i^2} \frac{\langle m \rangle}{t} \left| \varrho' \left(\frac{\langle m \rangle}{t} \right) \right| = \frac{\varrho_t(m)}{t \langle m \rangle} \left| \varrho' \left(\frac{\langle m \rangle}{t} \right) \right|.$$

Recalling that $\text{supp}(\varrho) \subset [\frac{1}{2}, 1]$ and $|\varrho'| \leq 1$, we have $-\partial_{m_i^2} \tilde{\mathbf{C}}(t, m) \lesssim \frac{1}{t \langle m \rangle} \lesssim \frac{1}{\langle m \rangle^2}$. The estimate for $\int_a^t \mathbf{I}(t, s, m) \mathbf{C}(t, s, m) ds$ follows similarly. Regarding (2.111b), it is only necessary in order to handle the LD object \mathfrak{V} , and is easily verified, since

$$|t-u|^\eta \mathbf{I}(t, u, m) = |t-u|^\eta e^{-|t-u| \langle m \rangle^2} = \langle m \rangle^{-2\eta} (|t-u| \langle m \rangle^2)^\eta e^{-|t-u| \langle m \rangle^2},$$

and $r \mapsto r^\eta e^{-r}$ is bounded on $\mathbf{R}_{\geq 0}$.

We first address the case where G does not involve any renormalization. In this case, the proof directly follows from the assumptions (2.111a), (2.111b) and (2.111c) on \mathbf{I} and \mathbf{C} . Indeed, once for all $l \in \tilde{L}^c(G)$, $\mathbf{C}(t_{v_1(l)}, t_{v_2(l)}, m_l)$ has been bounded by $\langle m_l \rangle^{-2}$, there is no dependence left in $t_{v_2(l)}$ for $l \in \tilde{L}^i(G)$ apart in $\mathbf{I}(t_{v_1(l)}, t_{v_2(l)}, m_l)$. If $\tau \neq \mathfrak{V}$, we can therefore perform all the integrations over the parameters $t_{v_2(l)}$, and they are in turn also bounded by $\langle m_l \rangle^{-2}$, while if $\tau = \mathfrak{V}$, we can make use of (2.111b) to bound all the terms $\mathbf{I}(t_{v_1(l)}, t_{v_2(l)}, m_l)$ by $\langle m_l \rangle^{-2\eta}$. Note that if $\tau = \mathfrak{V}$, then $|\tilde{L}^i(G)| = 2$ which matches the fact that we have precisely two factors $|t-u|^\eta$ at our disposal.

We now turn to G that involves renormalization. Below we sometimes abuse notation and write $\sqrt{u} \in \mathbf{R}$ for $\chi^c(0, \sqrt{u}(1, \dots, 1)) \in \mathbf{R}^d$. For $M \in \tilde{\mathfrak{M}}^{1,c}(G)$, by assumption (2.111d), we have

$$\begin{aligned} |\mathbf{C}(t_M, t_M, m_{M,i}) - \mathbf{C}(t_M, t_M, m_{M,i,\hat{c}})| &= \frac{\tilde{\mathbf{C}}(t_M, m_{M,i,\hat{c}})}{\langle m_{M,i,\hat{c}} \rangle^2} - \frac{\tilde{\mathbf{C}}(t_M, m_{M,i})}{\langle m_{M,i} \rangle^2} \\ &= - \int_0^{m_{\hat{c}}^2} \frac{d}{du} \left(\frac{\tilde{\mathbf{C}}(t_M, m_{M,i,\hat{c}} + \sqrt{u})}{(1 + |m_{M,i,\hat{c}}|^2 + u)^{\frac{a}{2}}} \right) du \quad \text{with } a = 2 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{m_c^2} \frac{d}{du} \left(-\tilde{\mathcal{C}}(t_M, m_{M,i,\hat{c}} + \sqrt{u}) \right) \frac{du}{(1 + |m_{M,i,\hat{c}}|^2 + u)^{\frac{a}{2}}} + \int_0^{m_c^2} \frac{\tilde{\mathcal{C}}(t_M, m_{M,i,\hat{c}} + \sqrt{u})}{(1 + |m_{M,i,\hat{c}}|^2 + u)^{\frac{a+2}{2}}} du \\
 &\lesssim \int_0^{m_c^2} \frac{du}{(1 + |m_{M,i,\hat{c}}|^2 + u)^{\frac{a+2}{2}}} = \frac{1}{\langle m_{M,i,\hat{c}} \rangle^2} - \frac{1}{\langle m_{M,i} \rangle^2} \\
 &\lesssim \frac{m_{M,i,c}^2}{\langle m_{M,i} \rangle^2 \langle m_{M,i,\hat{c}} \rangle^2} \lesssim \frac{\langle m_{M,e} \rangle^2 \wedge \langle m_{M,i} \rangle^2}{\langle m_{M,i} \rangle^2 \langle m_{M,i,\hat{c}} \rangle^2}.
 \end{aligned}$$

Regarding $M \in \mathfrak{M}^{2,c,c'}(G)$, we first bound its subamplitude in $\mathfrak{M}^{1,c'}(G)$ as

$$|\mathcal{C}(s, s, m_{M,d}) - \mathcal{C}(s, s, m_{M,d,c'})| \lesssim \frac{\langle m_{M,s} \rangle^2 \wedge \langle m_{M,d} \rangle^2}{\langle m_{M,d} \rangle^2 \langle m_{M,d,c'} \rangle^2}.$$

Then, we can integrate on s which yields, using assumptions (2.111d) and (2.111e) and repeating the argument used to deal with $M \in \tilde{\mathfrak{M}}^{1,c}(G)$ with $a = 4$,

$$\begin{aligned}
 &\int_{a|s}^{t_M} ds |\mathcal{I}(t_M, s, m_{M,s}) \mathcal{C}(t_M, s, m_{M,s}) - \mathcal{I}(t_M, s, m_{M,s,\hat{c}}) \mathcal{C}(t_M, s, m_{M,s,\hat{c}})| \\
 &= \frac{\tilde{\mathcal{C}}(t_M, m_{M,s,\hat{c}})}{\langle m_{M,s,\hat{c}} \rangle^4} - \frac{\tilde{\mathcal{C}}(t_M, m_{M,s})}{\langle m_{M,s} \rangle^4} \lesssim \frac{\langle m_{M,e} \rangle^2 \wedge \langle m_{M,s} \rangle^2}{\langle m_{M,s} \rangle^2 \langle m_{M,s,\hat{c}} \rangle^4}.
 \end{aligned}$$

The proof of (2.109) is similar, but we first have to pre-process a bit the expression of $\delta_{s,t}\tau$. First, observe that there are either two (for $\begin{smallmatrix} \bullet \\ \swarrow \\ \bullet \end{smallmatrix}$ and $\begin{smallmatrix} \bullet \\ \searrow \\ \bullet \end{smallmatrix}$) or three inverse heat operators attached to the root of τ . Using

$$\begin{aligned}
 &\int_a^t P_{t-u_1} du_1 \int_a^t P_{t-u_2} du_2 - \int_a^s P_{s-u_1} du_1 \int_a^s P_{s-u_1} du_2 \\
 &= \left(\int_a^t P_{t-u_1} du_1 - \int_a^s P_{s-u_1} du_1 \right) \int_a^t P_{t-u_2} du_2 + \left(\int_a^t P_{t-u_2} du_2 - \int_a^s P_{s-u_2} du_2 \right) \int_a^t P_{s-u_1} du_1
 \end{aligned}$$

in the first case and a similar telescoping sum in the second case, one can always create a difference of two inverse heat operators. We can now conclude, since proceeding as follows for $a \in \{-\infty, 0\}$, we have that a difference of two inverse heat operators always yields a good factor $|t-s|^\theta$ (losing a bit of the decay in m):

$$\begin{aligned}
 \left| \int_a^t P_{t-u_1}(m) du_1 - \int_a^s P_{s-u_1}(m) du_1 \right| &= \left| \int_s^t P_{t-u_1}(m) du_1 + \int_a^s (P_{t-u_1} - P_{s-u_1})(m) du_1 \right| \\
 &\lesssim \langle m \rangle^{-2} (1 - e^{-|t-s|\langle m \rangle^2}) \lesssim \langle m \rangle^{-2+2\theta} |t-s|^\theta,
 \end{aligned}$$

where for $v \geq 0$ we use the short-hand notation $P_v(m) = e^{-v\langle m \rangle^2}$.

Finally, (2.110) is proved in the same way, pre-processing the expression of $\delta_{u,r}^i \tau$ using

$$P_{t-r}(m) - P_{t-u}(m) = P_{t-r}(m)(1 - P_{r-u}(m)) \lesssim P_{t-r}(m) \langle m \rangle^{2\theta} |r-u|^\theta.$$

□

2.6.4 Tensor graphs

To formulate simple power-counting criteria, we pass from stranded graphs to tensor graphs.

Definition 2.86 A $(d+1)$ -colored graph $G = (N, E, c)$ is a $(d+1)$ -regular properly edge-colored graph. Properly edge-colored means the graph $G = (N, E)$ comes with a map $c: E \rightarrow \{0, \dots, d\}$ such that for every node $n \in N$ and any color $c \in \{0, \dots, d\}$, there exists a unique edge $e \in E(n)$ such that $c(e) = c$ —here $E(n) \subset E$ denotes the set of all edges incident to n .

Definition 2.87 Given a $(d + 1)$ -colored graph G and two colors $0 \leq c_1 < c_2 \leq d$, a *face* of G of color (c_1, c_2) is a (simple) cycle of edges all of which are colored either c_1 or c_2 . The length of a face is the number of edges (or equivalently nodes) in the cycle.

Given a face of color (c_1, c_2) of length 4, $c_1 > 0$ and given $i \in \{1, 2\}$, we call this face a “vertex of color c_i ” if one can label the four nodes the face runs through as n_1, \dots, n_4 so that

- The two pairs of vertices $\{n_1, n_2\}$ and $\{n_3, n_4\}$ are each connected by a single edge of color c_i .
- The two pairs of vertices $\{n_1, n_4\}$ and $\{n_2, n_3\}$ are each connected by $d - 1$ edges of all the colors in $[d] \setminus \{c_i\}$.

Definition 2.88 A *tensor graph* $G = (N, E, c)$ is a $(d + 1)$ -colored graph where we also allow for a new kind of node, called “external nodes”, each of which are incident to precisely one edge of color 0 (which we call an “external edge”). We also enforce the following constraints on tensor graphs:

- Any (c_1, c_2) -face with $c_1 > 0$ has length 2 or 4.
- No two nodes can be connected by d edges of colors $1, \dots, d$.
- Any face of colors (c_1, c_2) with $c_1 > 0$ of length 4 must be a vertex of color c_i for some $i \in \{1, 2\}$

An edge of color 0 is called a *link*, we denote by $L(G)$, $L^{int}(G)$ and $L^{ext}(G)$ the link set, internal link set and external link set of G . We write $V(G)$ for the vertex set of G , a vertex is a set of four nodes $\{n_1, \dots, n_4\}$ forming a vertex of color c as described above. For a vertex $v \in V(G)$ with nodes n_1, \dots, n_4 , we denote by $l_i(v)$ the link attached to n_i .

Faces of color $(0, c)$ that run through at least one external link of G are called “*external strands* of color c ” – by parity they must run through exactly two external links. Faces of color $(0, c)$ that are not external strands are called “*internal strands* of color c ”. We write S , $S^{int}(G)$, and $S^{ext}(G)$ the strand, internal strand, and external strand set of G . For $c \in [d]$, we say that two edges e_1, e_2 of color c belonging to the same external strand s of color c are *consecutive* if there is an internal link of the strand s joining one of the extremities of e_1 to one of the extremities of e_2 .

A *closed tensor graph* is an tensor graph with $|L^{ext}(G)| = 0$, and a tensor graph that is not closed is called *open*. We also use the terminology n -point graph and vacuum graph based on $|L^{ext}(G)|$ as described in Remark 2.3

Definition 2.89 A *tensor subgraph* of a closed tensor graph G is an open tensor graph G' determined by a subset $L^{int}(G')$ of $L(G)$. Its vertex set $V(G')$ is the set of all the extremities of all the links in $L^{int}(G')$, and its external link set $L^{ext}(G')$ are all the other links attached to the vertices in $V(G')$ that don't belong to $L^{int}(G')$. It inherits the data c of G , so that it is indeed a tensor graph. A subgraph G' of a graph G may not be connected, but its connected components are also tensor graphs themselves.

Definition 2.90 An open tensor graph $G = (N, E, c)$ has a *boundary graph* ∂G defined by carrying out the following:

- remove all the internal links of G ;
- for every $c \in [d]$, remove every edge of color c belonging to an internal strand of color c ;
- for every $c \in [d]$ and every external strand of color c , bind any two consecutive edges of color c of the face into a single edge of color c : ultimately, the external strand therefore collapses to one external link, one edge of color c and one external link.

Note that ∂G is not necessarily connected. We denote by $C(\partial G)$ the number of connected components of ∂G .

We extend these notions to any closed tensor graph G in the natural way with the convention that ∂G is empty and $C(\partial G) = 0$.

Note that the boundary graph of a tensor graph G can also be easily identified by looking at the stranded graph associated with G , see Remark 2.94

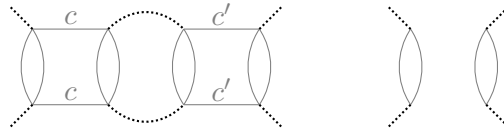


Figure 13: An open tensor graph and its boundary graph for $c \neq c'$. Note that if $c = c'$, the boundary graph would be the node of color c .

Definition 2.91 For a tensor graph G (either closed or open), we define the degree of G to be given by

$$\delta(G) = d - C(\partial G) + (d - 1)|V(G)| - |S^{int}(G)| - \frac{d - 1}{2}|L^{ext}(G)|. \tag{2.112}$$

A tensor graph G is said to be *melonic* if $\delta(G) = 0$

We refer to [Gur16, Ch. 3] for a gentle introduction to this notion of degree, it is introduced in [Gur16, Def. 3.12] and its relation to the number of faces is expressed in [Gur16, Eq. 3.1]. We will only need the following basic facts from [Gur16, Ch. 3].

Lemma 2.92 $\delta(G) \in \mathbf{N}$, and if $\delta(G) \neq 0$ then $\delta(G) \geq d - 2$.

Remark 2.93 Melonic graphs are easily identifiable by their tree-like structure. In particular, melonic tensor graphs are in one-to-one correspondence⁹ with colored planar trees under the transformation that shrinks faces of size $d - 1$ in the original graph to vertices and stretches the vertices of the original graph into edges.

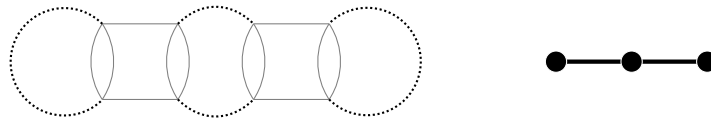


Figure 14: A melonic graph and its tree representation

Remark 2.94 Tensor graphs are in one-to-one correspondence with the stranded graphs introduced in Section 2.6.2 and we often identify them. In particular, the notions of link, vertex and strand match exactly when comparing a stranded graph to the tensor graph it corresponds to.

Writing G for a stranded graph and \tilde{G} for the corresponding tensor graph, note that the boundary graph $\partial\tilde{G}$ of a tensor graph \tilde{G} is the tensor graph corresponding to the stranded graph ∂G obtained by removing all internal strands from G .

In the sequel, using this one-to-one correspondence, tensor graphs are pictured with the two diagrammatic representations introduced for stranded graphs, sometimes drawing each strand and more often contracting the d parallel strands of a link drawing it as a dotted black line.

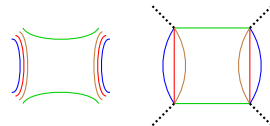


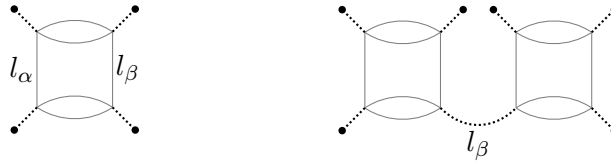
Figure 15: A vertex of a tensor graph represented on the LHS using strands and on the RHS using dotted lines

Thinking in terms of tensor graphs and stranded graphs clarifies the definition of l_α and l_β when τ is of type II. Indeed, these two links are actually one or $d - 1$ of the edges constituting the rooted vertex of the tensor graph, as was shown on Figure 10.

⁹This is another interesting feature of tensor field models in that the proliferation of melonic graphs as one goes to higher orders is exponentially bounded.

Remark 2.95 In two dimensions, stranded graphs are ribbon graphs. It turns out that the degree δ of a tensor graph G is equal to the genus of the ribbon graph corresponding to G , and is thus a topological quantity. In higher dimensions, however, the degree is not a topological invariant, even though melonic graphs are a particular class of tensor graphs that embed in the d -sphere.

We have thus seen that for any object τ , (2.102) is bounded by a sum indexed by \mathbb{G}_τ , a certain class of closed tensor graphs. The number of vertices and some links of the elements of \mathbb{G}_τ are directly fixed by the structure of τ (the links l_α , l_β , and the links corresponding to the time lines), while the remaining links correspond to the possible contractions of the noises. In the sequel, we call the skeleton graph of τ the graph corresponding to $\sum_{n \in \mathbf{Z}^{d_1}, m \in \mathbf{Z}^{d_2}} \langle n \rangle^{-2\alpha} \langle m \rangle^{2\beta} \hat{\tau}_{m,n} \hat{\tau}_{-m,-n}$ before the expectation is taken. Below we present the skeleton graph of Ψ as a random operator on the left and the skeleton graph of Ψ on the right:



We have now gathered all the notations we needed in order to obtain a good upper bound on $\tilde{A}(G)$. This is done through a multiscale analysis, which is introduced in the next section.

2.6.5 Multiscale analysis

Definition 2.96 An open tensor graph $G \in \mathbb{G}_\tau$ furthermore comes with a (*superficial*) *degree of divergence* $\omega(G)$ defined as

$$\omega(G)(\alpha, \beta) = -2 \sum_{l \in L^{int}(G)} \ell(l) + |S^{int}(G)| - 2\mathbf{1}\{G = \mathfrak{M}^1 \text{ or } \mathfrak{M}^2\}.$$

Note that in the case where $\tau = \Psi$, $\omega(G)$ also depends on η , but we choose to drop this dependence since η will be taken arbitrarily close to 1. This definition reflects the fact that each internal strand comes with a sum over \mathbf{Z} , while each internal link brings a factor $\langle m_l \rangle^{-2\ell(l)}$, and each renormalized subgraph has a power counting improved by two. This degree of divergence is “superficial” since it does not take into consideration sub-divergences due to subgraphs.

Moreover, the degree of divergence rewrites as

$$\begin{aligned} \omega(G)(\alpha, \beta) \stackrel{\text{def}}{=} & d - (5 - d)|V(G)| - \frac{d - 3}{2}|L^{ext}(G)| - \delta(G) - C(\partial G) \\ & + 2(1 - \alpha)\mathbf{1}\{l_\alpha \in G\} + 2(1 + \beta)\mathbf{1}\{l_\beta \in G\} + 2(1 - \eta)|\tilde{L}^i(G)|\mathbf{1}\{\tau = \Psi\} \\ & - 2\mathbf{1}\{G = \mathfrak{M}^1 \text{ or } \mathfrak{M}^2\}, \end{aligned} \quad (2.113)$$

and we say that G is *convergent* (resp. *divergent*) if $\omega(G)$ is strictly negative (resp. positive or zero). Note that the second expression is obtained by injecting the definition of the degree (2.112) in (2.113) and using the combinatorial relation $4|V(G)| = 2|L^{int}(G)| + |L^{ext}(G)|$ as well as the definition of ℓ , (2.106). These computations also justify (2.12) in Remark 2.3.

To obtain good upper bounds on amplitudes we slice kernels across scales.

Lemma 2.97 (Multiscale decomposition) *Pick $\ell \in (0, 4]$. There exists a numerical constant $C > 0$ such that for every $m \in \mathbf{Z}^d$,*

$$\langle m \rangle^{-2\ell} \lesssim \sum_{k \geq -1} 2^{-2k\ell} e^{-C2^{-k} \sum_{c \in [d]} |m_c|}. \quad (2.114)$$

Moreover, for $\ell \in [0, 4]$ and $k_0 \geq 0$, we have

$$e^{-2^{-2k_0} \langle m \rangle^2} \langle m \rangle^{-2\ell} \lesssim \sum_{k \leq k_0 - 1} 2^{-2k\ell} e^{-C2^{-k} \sum_{c \in [d]} |m_c|}. \quad (2.115)$$

Proof. We start by writing

$$\begin{aligned} \langle m \rangle^{-2\ell} &\lesssim \int_0^{+\infty} a^{\ell-1} e^{-a \langle m \rangle^2} da \\ &\lesssim \int_1^{+\infty} a^{\ell-1} e^{-a \langle m \rangle^2} da + \sum_{k \geq 0} \int_{2^{-2(k+1)}}^{2^{-2k}} a^{\ell-1} e^{-a \langle m \rangle^2} da. \end{aligned}$$

In the second line, the first term corresponds to the slice -1 and is bounded by a constant while the k -th slice is bounded as

$$\int_{2^{-2(k+1)}}^{2^{-2k}} a^{\ell-1} e^{-a \langle m \rangle^2} da \lesssim 2^{-2k\ell} e^{-2^{-2(k+1)} \langle m \rangle^2} \lesssim 2^{-2k\ell} e^{-C2^{-k} \sum_{c \in [d]} |m_c|},$$

where we used the equivalence of the ℓ^2 and ℓ^1 norms in finite dimension.

The proof of (2.115) goes the same way, starting from the observation that

$$e^{-2^{-2k_0} \langle m \rangle^2} \langle m \rangle^{-2\ell} \lesssim \int_{2^{-2k_0}}^{+\infty} a^{\ell-1} e^{-a \langle m \rangle^2} da.$$

Note that the integration does not include $a = 0$ so it remains finite if $\ell = 0$. \square

Lemma 2.98 (Multiscale analysis) *Let τ a stochastic object and $G \in \mathbb{G}_\tau$, then*

$$\tilde{A}(G) \lesssim \sum_{k_1, \dots, k_{|L(G)|} \geq -1} \prod_{i \geq -1} \prod_{k=1}^{C(\mathcal{G}^i)} 2^{\omega(\mathcal{G}_k^i)},$$

where \mathcal{G}^i is the subgraph of G with internal link set $L^{int}(\mathcal{G}^i) = \{l \in L(G) : k_l \geq i\}$, and we denote by $(\mathcal{G}_k^i : 1 \leq k \leq C(\mathcal{G}^i))$ its connected components.

Proof. Recall the expression of $\tilde{A}(G)$ in (2.107). We first deal with the contribution of the renormalized subgraphs, and begin with the study of the possible subgraphs of the form $\tilde{\mathfrak{M}}^1$. Pick $M \in \tilde{\mathfrak{M}}^{1,c}(G)$. The contribution of $M \cup \{l_{e,1}(M)\}$ to $\tilde{A}(G)$ is given by

$$\begin{aligned} &\frac{\langle m_{M,e} \rangle^2 \wedge \langle m_{M,i} \rangle^2}{\langle m_{M,e} \rangle^{2\ell(l_{e,1})} \langle m_{M,i} \rangle^2 \langle m_{M,i,\hat{c}} \rangle^2} \\ &= \mathbf{1}\{\langle m_{M,e} \rangle^2 < \langle m_{M,i} \rangle^2\} \langle m_{M,e} \rangle^{2(1-\ell(l_{e,1}))} \langle m_{M,i} \rangle^{-2} \langle m_{M,i,\hat{c}} \rangle^{-2} \\ &\quad + \mathbf{1}\{\langle m_{M,i} \rangle^2 \leq \langle m_{M,e} \rangle^2\} \langle m_{M,e} \rangle^{-2\ell(l_{e,1})} \langle m_{M,i,\hat{c}} \rangle^{-2} \\ &\lesssim \mathbf{1}\{\langle m_{M,e} \rangle^2 < \langle m_{M,i} \rangle^2\} \langle m_{M,e} \rangle^{2(1-\ell(l_{e,1}))} \langle m_{M,i,\hat{c}} \rangle^{-4} \\ &\quad + \mathbf{1}\{\langle m_{M,i,\hat{c}} \rangle^2 \leq \langle m_{M,e} \rangle^2\} \langle m_{M,e} \rangle^{-2\ell(l_{e,1})} \langle m_{M,i,\hat{c}} \rangle^{-2} \\ &\lesssim \sum_{k_i \geq -1} 2^{-4k_i} e^{-C2^{-k_i} \sum_{c' \neq c} |m_{M,i,c'}|} \mathbf{1}\{\langle m_{M,e} \rangle^2 < \langle m_{M,i} \rangle^2\} \langle m_{M,e} \rangle^{2(1-\ell(l_{e,1}))} \\ &\quad + \sum_{k_e \geq -1} 2^{-2\ell(l_{e,1})k_e} e^{-C2^{-k_e} \sum_c |m_{M,e,c}|} \mathbf{1}\{\langle m_{M,i,\hat{c}} \rangle^2 \leq \langle m_{M,e} \rangle^2\} \langle m_{M,i,\hat{c}} \rangle^{-2}. \end{aligned}$$

In the first inequality, we use the fact that since $\langle m_{M,i} \rangle^2 \geq \langle m_{M,i,\hat{c}} \rangle^2$, we have

$$\mathbf{1}\{\langle m_{M,i} \rangle^2 \leq \langle m_{M,e} \rangle^2\} \leq \mathbf{1}\{\langle m_{M,i,\hat{c}} \rangle^2 \leq \langle m_{M,e} \rangle^2\},$$

while in going from the first to the second inequality, we used (2.114). To conclude, we bound the two indicator functions as $\mathbf{1}\{\langle m_{M,e} \rangle^2 < \langle m_{M,i} \rangle^2\} \lesssim e^{-2^{-2k_i} \langle m_{M,e} \rangle^2}$ and $\mathbf{1}\{\langle m_{M,i,\hat{c}} \rangle^2 \leq \langle m_{M,e} \rangle^2\} \lesssim e^{-2^{-2k_e} \langle m_{M,i,\hat{c}} \rangle^2}$ and use (2.115). This yields

$$\begin{aligned}
 & \frac{\langle m_{M,e} \rangle^2 \wedge \langle m_{M,i} \rangle^2}{\langle m_{M,e} \rangle^{2\ell(l_{e,1})} \langle m_{M,i} \rangle^2 \langle m_{M,i,\hat{c}} \rangle^{-2}} \\
 & \lesssim \sum_{k_e < k_i} 2^{-4k_i} e^{-C2^{-k_i} \sum_{c' \neq c} |m_{M,i,c'}|} 2^{2(1-\ell(l_{e,1}))k_e} e^{-C2^{-k_e} \sum_{c \in [d]} |m_{M,e}|} \\
 & \quad + \sum_{k_i \leq k_e} 2^{-2\ell(l_{e,1})k_e} e^{-C2^{-k_e} \sum_c |m_{M,e,c}|} 2^{-2k_i} e^{-C2^{-k_i} \sum_{c' \neq c} |m_{M,i,c'}|} \\
 & \lesssim \sum_{k_i, k_e \geq -1} 2^{-2k_i} 2^{-2\ell(l_{e,1})k_e} 2^{2(k_i \wedge k_e - k_i)} \\
 & \quad e^{-C2^{-k_i} \sum_{c' \neq c} |m_{M,i,c'}|} e^{-C2^{-k_e} \sum_{c \in [d]} |m_{M,e}|} \\
 & \lesssim \sum_{k_i, k_e \geq -1} \prod_{l \in \{l_i\} \sqcup \{l_{e,1}\}} 2^{-2\ell(l)k_l} \prod_{s \in S(G)} \prod_{l \in \{l_i\} \sqcup \{l_{e,1}\} | s \in l} e^{-C2^{-k_l} |m_s|} 2^{-2((k_i - k_e) \vee 0)} \\
 & \lesssim \sum_{k_i, k_e \geq -1} \prod_{l \in \{l_i\} \sqcup \{l_{e,1}\}} 2^{-2\ell(l)k_l} \prod_{s \in S(G)} \prod_{l \in \{l_i\} \sqcup \{l_{e,1}\} | s \in l} e^{-C2^{-k_l} |m_s|} \prod_{i=k_e+1}^{k_i} 2^{-2},
 \end{aligned}$$

where we introduced $k_{l_i(M)} = k_i$ for the internal link of M and $k_{l_{e,1}(M)} = k_e$ for the external links of M . Finally, we note that $\prod_{i=k_e+1}^{k_i} 2^{-2} = \prod_{i|\exists k, M=\mathcal{G}_k^i} 2^{-2}$.

The proof for the graphs $M \in \mathfrak{M}^2(G)$ is very similar so we only sketch it. Indeed, for M , one has to introduce three scales k_e , k_s and k_d that correspond to the slicing of the Fourier modes $m_{M_2,e}$, $m_{M_2,s}$ and $m_{M_2,d}$. One then has to consider separately all the six possible orderings of these scales, in the same way we did for $M \in \mathfrak{M}^1(G)$. This yields the following bound over the amplitude of M :

$$\sum_{k_i, k_s, k_e \geq -1} \prod_{l \in L^{int}(M) \sqcup \{l_{e,1}\}} 2^{-2\ell(l)k_l} \prod_{s \in S(G)} \prod_{l \in L^{int}(M) \sqcup \{l_{e,1}\} | s \in l} e^{-C2^{-k_l} |m_s|} \prod_{i=k_e+1}^{k_s \wedge k_d} 2^{-2} \prod_{i=k_s+1}^{k_d} 2^{-2},$$

and we also note that $\prod_{i=k_e+1}^{k_s \wedge k_d} 2^{-2} = \prod_{i|\exists k, M=\mathcal{G}_k^i} 2^{-2}$ and $\prod_{i=k_s+1}^{k_d} 2^{-2} = \prod_{i|\exists k, M_1(M)=\mathcal{G}_k^i} 2^{-2}$.

All the other links that are not in $\bigcup_{M \in \mathfrak{M}(G)} L^{int}(M) \cup \{l_{e,1}(M)\}$ are easily dealt with using (2.114), and for l such a link we have

$$\langle m_l \rangle^{-2\ell(l)} \lesssim \sum_{k_l \geq -1} 2^{-2\ell(l)k_l} e^{-C2^{-k_l} \sum_{c \in [d]} |m_{l,c}|} \lesssim \sum_{k_l \geq -1} 2^{-2\ell(l)k_l} \prod_{s \in S(G) | s \in l} e^{-C2^{-k_l} |m_s|},$$

where we say that $s \in l$ if s runs through l . Collecting all the previous bounds together, we have obtained that

$$\begin{aligned}
 \tilde{A}(G) & \lesssim \sum_{m_{s_1}, \dots, m_{s_{|S(G)|}} \in \mathbf{Z}} \sum_{k_1, \dots, k_{|L(G)|} \geq -1} \prod_{L \in L(G)} 2^{-2\ell(L)k_L} \\
 & \quad \prod_{s \in S(G)} \prod_{l \in L(G) | s \in l} e^{-C2^{-k_l} |m_s|} \prod_{M \in \mathfrak{M}(G)} \prod_{i|\exists k, M=\mathcal{G}_k^i} 2^{-2}.
 \end{aligned}$$

We first rewrite

$$\prod_{M \in \mathfrak{M}(G)} \prod_{i|\exists k, M=\mathcal{G}_k^i} 2^{-2} = \prod_{i \geq -1} \prod_{k=1}^{C(\mathcal{G}^i)} 2^{-2\mathbf{1}\{\mathcal{G}_k^i = \mathfrak{M}^1 \text{ or } \mathfrak{M}^2\}}.$$

Then,

$$\sum_{m_s \in \mathbf{Z}} \prod_{l \in L(G)|s \in l} e^{-C2^{-k_l}|m_s|} \lesssim \sum_{p \in \mathbf{N}} e^{-C \sum_{l \in L(G)|s \in l} 2^{-k_l} p} \lesssim \frac{1}{\sum_{l \in L(G)|s \in l} 2^{-k_l}} \lesssim 2^{k_s},$$

where $k_s \stackrel{\text{def}}{=} \min_{l \in L(G)|s \in l} k_l$. Moreover,

$$\prod_{s \in S(G)} 2^{k_s} \lesssim \prod_{i \geq -1} \prod_{s \in S(G)|k_s \geq i} 2^1,$$

and since the condition $k_s \geq i$ implies that s is an internal strand of some \mathcal{G}_k^i , we finally have

$$\prod_{s \in S(G)} 2^{k_s} \lesssim \prod_{i \geq -1} \prod_{k=1}^{C(\mathcal{G}^i)} 2^{|S^{\text{int}}(\mathcal{G}_k^i)|}.$$

We finish by observing that the condition $k_l \geq i$ is equivalent to $l \in L^{\text{int}}(\mathcal{G}_k^i)$ for some k , so

$$\prod_{l \in L(G)} 2^{-2\ell(l)k_l} = \prod_{l \in L(G)} \prod_{i=1}^{k_l} 2^{-2\ell(l)} \lesssim \prod_{i \geq -1} \prod_{l \in L(G)|k_l \geq i} 2^{-2\ell(l)} \lesssim \prod_{i \geq -1} \prod_{k=1}^{C(\mathcal{G}^i)} 2^{-2 \sum_{l \in L^{\text{int}}(\mathcal{G}_k^i)} \ell(l)}.$$

□

We immediately have the following corollary.

Corollary 2.99 *If $G \in \mathbb{G}_\tau$, then $\tilde{A}(G)$ is finite if for all $i \geq -1$ and $k \in [C(\mathcal{G}^i)]$, we have*

$$\begin{aligned} \max_{\mathcal{G}_k^i | l_\alpha \in \mathcal{G}_k^i \cap l_\beta \notin \mathcal{G}_k^i} \omega(\mathcal{G}_k^i)(\alpha) &< 0, \\ \max_{\mathcal{G}_k^i | l_\alpha \notin \mathcal{G}_k^i \cap l_\beta \in \mathcal{G}_k^i} \omega(\mathcal{G}_k^i)(\beta) &< 0, \\ \max_{\mathcal{G}_k^i | l_\alpha \in \mathcal{G}_k^i \cap l_\beta \in \mathcal{G}_k^i} \omega(\mathcal{G}_k^i)(\alpha, \beta) &< 0. \end{aligned}$$

The proof of the regularities of the different stochastic objects reduces to identifying the worst of the possible subgraphs (including the full graphs) of all the graphs in \mathbb{G}_τ containing l_α and/or l_β . For the smallest objects, we exhibit all the possible contractions. For bigger objects, we will describe the most divergent subgraph, and show that it is indeed maximal. All the worst contributions have a connected boundary graph if they are not a vacuum graph, so that the superficial degree of divergence of a subgraph G is fully characterized by the triplet $(|V(G)|, |L^{\text{ext}}(G)|, \delta(G))$. There we often use these triples to refer to graphs to lighten the notations. For example, a melonic two-vertex four-point graph will be called a $(2, 4, 0)$ graph.

Remark 2.100 The observations of Remarks [2.80](#) and [2.81](#) can also be applied to our estimates for larger stochastic objects. Namely, our argument will focus on developing uniform in N and time estimates of covariances – convergence in N along with time regularity will follow from straightforward post-processing. Covariances of differences of stochastic objects with different scale cut-offs or, for LD objects, evaluated at different times, can also be written as sums indexed by graphs. For each such graph one can rewrite the contribution for that graph as a telescoping sum where in each summand one has a difference of a single kernel at different cut-offs/times, see e.g., [\[MWX17\]](#).

2.6.6 Moment estimates for $\mathfrak{W}, \mathfrak{B}, \mathfrak{V}, \mathfrak{J}, \mathfrak{J}', \mathfrak{W}'$

In this section, we complete the diagrammatic estimates for the listed stochastic objects.

As described in Lemma 2.85, BG objects obey the same power-counting, and using a Kolmogorov estimate for space regularity gives the necessary estimates on the symbols for Lemma 2.60.

The estimate (2.28) from Lemma 2.12 is proven in Section 2.6.6.3. The estimate (2.45) of Lemma 2.24 is proven in Section 2.6.6.3. Lemma 2.16 is proven in Sections 2.6.6.1, 2.6.6.2 and 2.6.6.4. Lemma 2.27 is proven in Section 2.6.7.1, Lemma 2.29 in Section 2.6.7.2 and Lemma 2.31 in 2.6.7.3.

In Section 2.6.7.3 we argue that the relevant estimates in Lemmas 2.31 and 2.48 hold for the random operators defined by replacing instances of \uparrow with the random shift \uparrow . As described in Lemma 2.85, BG objects obey the same power-counting, and using a Kolmogorov estimate for space regularity gives Lemma 2.60.

2.6.6.1 The melonic second Wick power \mathfrak{W}'

The renormalization of \mathfrak{W}' has been described in Lemma 2.10, but now we turn to quantifying its regularity – to do this we study all the possible subgraphs of all the graphs that can be built from its skeleton graph.

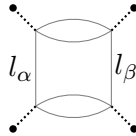


Figure 16: Skeleton graph of the melonic second Wick power

This skeleton graph can give rise to the three following contractions:

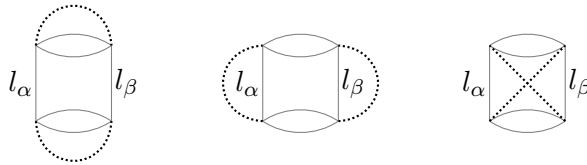


Figure 17: The three contractions: they have respectively degree 0, $d - 2$ and $d - 1$

This object is small enough for the bound (2.107) to be estimated without multiscale analysis, we write the amplitudes of the first two graphs in Figure 17

The melonic contraction $(1, 0, 0)$ is thus the maximal subgraph that contains both l_α and l_β , and its amplitude (2.107) rewrites as

$$\sum_{m \in \mathbf{Z}, a \in \mathbf{Z}^{d-1}, b \in \mathbf{Z}^{d-1}} \langle m \rangle^{-2\alpha+2\beta} \frac{\langle m \rangle^2 \wedge \langle (a, m) \rangle^2}{\langle (a, m) \rangle^2 \langle a \rangle^2} \frac{\langle m \rangle^2 \wedge \langle (b, m) \rangle^2}{\langle (b, m) \rangle^2 \langle b \rangle^2}. \tag{2.116}$$

We see that the sum is convergent if and only if

$$\beta < \alpha + |\mathfrak{W}'|.$$

Returning to our random operator estimate, the only contraction graph with a subgraph containing only l_α/l_β is the graph $(1, 0, d - 2)$, which has a subgraph $(1, 2, d - 2)$ containing l_α/l_β . The amplitude of the larger contraction graph is given by:

$$\sum_{m \in \mathbf{Z}, n \in \mathbf{Z}, a \in \mathbf{Z}^{d-1}} \langle (a, m) \rangle^{-2} \langle (a, n) \rangle^{-2} \langle m \rangle^{-2\alpha} \langle n \rangle^{2\beta}.$$

For this sum to be convergent, the conditions $2\beta - 2\alpha + d - 3 < 0$,

$$\beta < 1/2, \text{ and } \alpha > -1/2$$

have to be fulfilled. Indeed, it may be that the sums over n and a are convergent, even without the factor $\langle(a, m)\rangle^{-2}$, so that we would be left with $\sum_{m \in \mathbf{Z}} \langle m \rangle^{-2-2\alpha}$, which indeed requires $\alpha > -\frac{1}{2}$.

We point out that the estimate (2.30) controlling Ψ as a random field is obtained from taking $\alpha = 0$, which gives the constraint $\beta < |\Psi|$ appearing there.

2.6.6.2 The non-melonic second Wick power Ψ

For the rest of our estimates we leverage the multiscale bound of Corollary 2.99. We identify all the possible graphs made with the skeleton graph, along with all their possible subgraphs, and compute their divergence degree using (2.113).

This object does not involve any renormalization and we turn to its regularity properties.

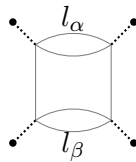


Figure 18: Skeleton graph of the non-melonic second Wick power

As its melonic counterpart, this symbol gives rise to the three one-vertex vacuum graphs.

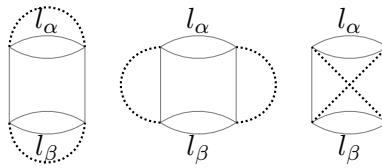


Figure 19: The three contractions

Once again, the melonic contraction (1, 0, 0) is the maximal subgraph that contains both l_β and l_α – this yields the constraint

$$\beta < \alpha + |\Psi|.$$

It turns out that it also has subgraph (1, 2, 0) that only contains l_α/l_β (observe that here, if there is a subgraph that seems to be a melonic tadpole, it is actually not the case, because it contains l_α/l_β , so that it does not require renormalization, and does not benefit from the +2 effect in the power counting coming from renormalization) and yields the constraints

$$\beta < -\frac{d-3}{2} \text{ and } \alpha > \frac{d-3}{2}.$$

This is the only subgraph of the three graphs that contains l_α/l_β . Again, the relevant estimate of (2.30) follows when $\alpha = 0$.

2.6.6.3 The cubic random fields Ψ and Ψ

The renormalization of Ψ is given by a ‘‘Wick’’ renormalization coming from the rightmost two factors of \dagger in this product, corresponding to the graph in \mathfrak{M}^1 and the counterterm \mathfrak{C}^1 . Below we draw the corresponding skeleton graph.

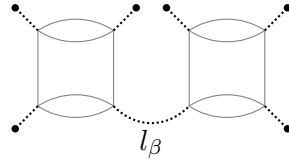


Figure 20: Skeleton graph of the third Wick power

The contractions are all the connected two-vertex vacuum graphs, and the melonic one $(2, 0, 0)$ is therefore maximal – this yields the condition

$$\beta < |\Psi|.$$

Since l_β connects two different vertices, it cannot belong to a $(1, 2, 0)$ subgraph, and $(2, 2, 0)$ graphs have a better power counting than $(2, 0, 0)$ graphs, this handles all the other contractions.

The proof for Ψ follows from the considerations above and observing that the extra integration allows one to replace $\langle m \rangle^{2\beta}$ by $\langle m \rangle^{2\beta-4}$ in estimates for all the graphs appearing above.

2.6.6.4 The quadratic random operator Ψ

This stochastic estimate does not involve any renormalization.

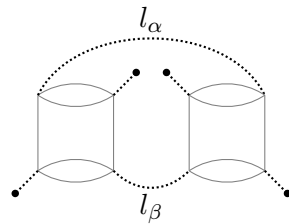


Figure 21: Skeleton graph of the quadratic random operator

We deduce that there are three different contractions with this random operator, but two of them are trivial and have only d faces, and thus no non-trivial subgraphs. The third one is $(2, 0, d - 2)$ which is of maximal degree, and is the maximal graph containing both l_α and l_β since no melonic subgraph can contain both l_α and l_β – this gives the condition

$$\beta < \alpha + |\Psi|.$$

This graph has a $(2, 4, 0)$ subgraph that contains only l_α/l_β and is therefore maximal (since the skeleton graph implies that l_α/l_β cannot belong to any $(2, 2, 0)$ subgraph) – this gives constraints

$$\beta < -\frac{d-3}{2} \text{ and } \alpha > \frac{d-3}{2}.$$

2.6.6.5 The quartic random operator Ψ

This random operator does not involve any renormalization, below we draw the corresponding skeleton graph.

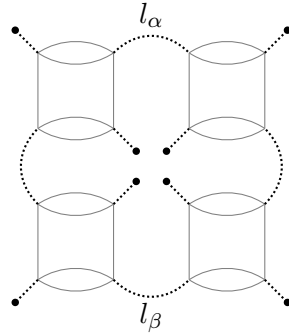


Figure 22: Skeleton graph of the quartic random operator

The vacuum contraction of highest degree $(d - 2)$ yields the following constraint on α and β :

$$\beta < \alpha + |\mathfrak{V}| - 2(1 - \eta).$$

It is maximal since inspecting the skeleton graph shows that any melonic subgraph G containing both l_α and l_β would at least be $(2, 4, 0)$, so that their power counting would be better, since improving $|L^{ext}(G)| =$ by 2 yields a factor $d - 3$ and improving $C(\partial G)$ by one yields a factor 1, which at least compensates the gain of $d - 2$ due to the improvement of the degree.

This contraction has the same $(2, 4, 0)$ subgraph as \mathfrak{B} containing either l_α or l_β , and this subgraph is still maximal since l_β cannot belong to any $(2, 2, 0)$ subgraph. Again, this gives the conditions

$$\beta < -\frac{d-3}{2} \text{ and } \alpha > \frac{d-3}{2}.$$

2.6.7 Estimates on \mathfrak{V} , \mathcal{S} , and objects with the rough shift \mathfrak{I}

We now discuss obtaining estimates on \mathfrak{V} , \mathcal{S} and also estimates on random operators built using the rough shift. For these objects we use a combination of stochastic and deterministic estimates. In particular, in Section 2.6.7.3 we argue that the relevant estimates in Lemmas 2.31 and 2.48 hold for the random operators defined by replacing instances of \mathfrak{I} with the random shift \mathfrak{I} .

2.6.7.1 The quintic random fields \mathfrak{V} and \mathfrak{V}

Recall the product denoted by the root in symbol \mathfrak{V} actually corresponds to three terms due to the three ways to order this product – see Definition 2.64.

The product $\mathcal{N}(\mathfrak{V}, \mathfrak{I}, \mathfrak{I})$ requires a “Wick” renormalization coming from the two factors of \mathfrak{I} in this product to the graph in \mathfrak{M}^1 and counter-term \mathfrak{C}^1 .

The other two products $\mathcal{N}(\mathfrak{I}, \mathfrak{V}, \mathfrak{I})$ and $\mathcal{N}(\mathfrak{I}, \mathfrak{I}, \mathfrak{V})$ contain a melonic pairing of \mathfrak{I} and \mathfrak{V} that is divergent in $d = 4$ and corresponds to graphs in \mathfrak{M}^2 and the counterterm $\frac{1}{2}\mathfrak{C}^2$ (recall Lemma 2.25).

Some care has to be taken here since the regularity of \mathfrak{V} is not determined by its power counting $|\mathfrak{V}| = -\frac{5d-18}{2}$. The skeleton graphs coming from all the three products in \mathfrak{V} have ten leaves, and thus gives rise to 945 graphs. However, we do not need to draw them because some of them are melonic – $(4, 0, 0)$ – this gives the condition

$$\beta < |\mathfrak{V}|.$$

It turns out that these melonic vacuum graphs are maximal only in $d = 4$. Indeed, $(2, 2, 0)$ and $(2, 4, 0)$ subgraphs have a worst power counting, and some of these melonic vacuum graphs have $(2, 2, 0)$ and $(2, 4, 0)$ subgraphs containing l_β . This yields the conditions $\beta < -\frac{(d-3)}{2}$ and $\beta < -(d - 3)$, which

becomes better behaved only when $d \geq 4$ but dominates when $d \in \{2, 3\}$ (however in the latter case we don't need the estimate \mathfrak{V}).

The proof for \mathfrak{V} follows similarly – as in \mathfrak{V} versus \mathfrak{V} , the extra integration involved allows one to replace $\langle m \rangle^{2\beta}$ by $\langle m \rangle^{2\beta-4}$ in the estimates for all the graphs appearing above.

2.6.7.2 The septic random field \mathcal{S}

This symbol requires care, one reason being that its regularity is not given by our usual power counting $|\mathcal{S}| = -\frac{7d-28}{2}$ just as was the case for \mathfrak{V} .

However, a second issue here is that we will not obtain the needed regularity estimate on \mathcal{S} solely through a stochastic estimate. In particular, we split \mathcal{S} as $\mathcal{S} = \mathcal{S}^{\text{det}} + \mathcal{S}^{\text{sto}}$ with

$$\begin{aligned} \mathcal{S}^{\text{det}} &\stackrel{\text{def}}{=} \mathcal{N}(\mathfrak{V}, \mathfrak{I}, \mathfrak{I}) - \mathfrak{C}^1 \mathfrak{V}, \\ \mathcal{S}^{\text{sto}} &\stackrel{\text{def}}{=} (\mathcal{N}(\mathfrak{V}, \mathfrak{I}, \mathfrak{V}) - \frac{1}{2} \mathfrak{C}^2 \mathfrak{V}) + \mathcal{N}(\mathfrak{I}, \mathfrak{I}, \mathfrak{V}) + 20 \text{ terms}. \end{aligned}$$

Here \mathcal{S}^{sto} will be treated as a whole using a stochastic estimate (similarly to the previous stochastic objects). However, trying to use a stochastic estimate to control \mathcal{S}^{det} would yield a regularity estimate with exponent $-(d-3)$ coming from a subgraph $(2, 2, 0)$, which is worse than expected and not enough to close our argument when $d = 4$. We instead write \mathcal{S}^{det} in terms of smaller random operators and combine their estimates, writing

$$\mathcal{S}^{\text{det}}(x) = \sum_{c=1}^d \mathfrak{V}^c(\mathfrak{V}(\cdot, x_c))(x_c).$$

By (2.123) and (2.121), we have

$$\begin{aligned} \|\mathfrak{V}^c(\mathfrak{V})\|_{C_T C^{-\frac{d-3}{2}-3\epsilon}} &\lesssim \|\mathfrak{V}^c(\mathfrak{V})\|_{C_T L_{x_c}^\infty C_x^{-\frac{d-3}{2}-2\epsilon}} \\ &\lesssim \|\mathfrak{V}^c\|_{\mathcal{L}(C_T C^{-(d-5)-2\epsilon}, C_T C^{-\frac{d-3}{2}-2\epsilon})} \|\mathfrak{V}\|_{C_T L_{x_c}^\infty C_{x_c}^{-(d-5)-2\epsilon}} \\ &\lesssim \|\mathfrak{V}^c\|_{\mathcal{L}(C_T C^{-(d-5)-2\epsilon}, C_T C^{-\frac{d-3}{2}-2\epsilon})} \|\mathfrak{V}\|_{C_T C^{-(d-5)-\epsilon}}. \end{aligned}$$

The estimate (2.81) then implies that \mathfrak{V} is of regularity $-(d-5) > -\frac{1}{2}$. We combine this with the random operator estimate (2.83) that states that for $\alpha > -\frac{1}{2}$, \mathfrak{V}^c goes from $C_T H^\alpha$ to $C_T C^{\beta-\epsilon}$ with $\beta = \min(\frac{1}{2}, \alpha - \frac{2d-5}{2}) \geq -\frac{d-3}{2}$ for $\alpha = -(d-5)$. Thus, we can conclude that the right hand side is finite, so that \mathcal{S}^{det} is indeed of regularity $-\frac{d-3}{2}$.

We now turn to showing that \mathcal{S}^{sto} is of regularity $-\frac{d-3}{2}$. \mathcal{S}^{sto} is the sum of 22 terms, but all of them have an important difference with $\mathcal{N}(\mathfrak{V}, \mathfrak{I}, \mathfrak{I}) - \mathfrak{C}^1 \mathfrak{V}$ that we describe here. The skeleton graphs of all these 22 terms give rise to some maximal $(6, 0, 0)$ graph which would yield $\beta < |\mathcal{S}|$. However, one has to take care that the contractions do not contain some $(2, 2, 0)$ or $(2, 4, 0)$ subgraphs, which would yield a worse power counting. It turns out that some of these maximal $(6, 0, 0)$ graph do have a $(2, 4, 0)$ subgraph containing l_β , which thus yields the sharper condition

$$\beta < -\frac{d-3}{2}.$$

However, this constraint is maximal since no contraction can produce a $(2, 2, 0)$ subgraph with l_β inside it, in view of the position of l_β . This is the key difference between the 22 terms in \mathcal{S}^{sto} (like $\mathcal{N}(\mathfrak{I}, \mathfrak{I}, \mathfrak{V})$) and $\mathcal{N}(\mathfrak{V}, \mathfrak{I}, \mathfrak{I}) - \mathfrak{C}^1 \mathfrak{V}$, because the latter can give rise to a $(2, 2, 0)$ subgraph, and would thus be of regularity $-(d-3)$, which is why we had to deal with it deterministically.

2.6.7.3 Random operators built with \dagger

In $d = 4$, we are interested in the previous random operators defined with \dagger substituted with \ddagger . Our regularity estimates for the more basic random fields implies that \ddagger shares the regularity of \dagger . Moreover, going through the proofs of the regularities of the random operators with one or several occurrence of \dagger substituted with Ψ or Ψ always yields better results, because this will give rise to bigger graphs that have a better power counting, which is the key feature of super-renormalizability/subcriticality.

For the sake of completeness, we provide here a short argument that confirms that these objects share the regularity of their counterparts built on the free field. Indeed, like we did above for \mathcal{S} , we can decompose a random operator $\tau \in \{\Psi, \Psi, \Psi, \Psi\}$ made with \dagger as $\tau^{\text{sto}} + \tau^{\text{det}}$ where τ^{sto} will be controlled with a stochastic estimate on the whole object, while τ^{det} will be treated using deterministic estimates.

One key input for this deterministic estimate will be stochastic estimates on the smaller random operators – again by invariance in law under permuting spatial coordinates, we can specialize to $J = [k] \subset [d]$, and then note that $(\dagger - \ddagger)^{[k]}$ defined by setting

$$(\dagger - \ddagger)_N^{[k]}(f) \stackrel{\text{def}}{=} \int_{\mathbf{T}^{d-k}} (-\Psi_N + \Psi_N)(\cdot, y) f(y) dy \quad \text{for } f : \mathbf{R}_{\geq 0} \times \mathbf{T}^{d-k} \rightarrow \mathbf{R}.$$

Denoting $|\dagger - \ddagger| = -\frac{3d-12}{2}$, our key estimate is then

Lemma 2.101 For $d \in \{2, 3, 4\}$ and all $\alpha > \frac{3d-12-k}{2}$,

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|(\dagger - \ddagger)_N^{[k]}\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}^{d-k}), C_T C^\beta(\mathbf{T}^k))}^p] < \infty \quad (2.117)$$

where $\beta = \min\left(-\frac{2d+k-12}{2}, \alpha + |\dagger - \ddagger|\right) - \epsilon$.

Proof. The proof generalizes that for $\dagger^{[k]}$ – see (2.99). When performing the covariance computations for $\dagger^{[k]}$ and $(\dagger - \ddagger)^{[k]}$, one encounters a class of graphs similar to the graphs in G_τ for τ an object of type II, but with a root which is given by a link instead of a vertex. This link is attached to two other links at its two extremities, and its strands divide into k strands corresponding to l_β and $d-k$ strands corresponding to l_α . In the case of $\tau = \dagger^{[k]}$, there is a unique $G \in \mathbb{G}_\tau$ which is the graph for which the root is attached to a single link connected to both its extremities. This graph has three subgraphs (including itself), hence the estimates (2.100).

In the case of $\tau = (\dagger - \ddagger)^{[k]}$, the skeleton graphs obtained when gluing together two copies of Ψ , one of Ψ with one of Ψ , and two copies of Ψ , give rise to several graphs with several subgraphs. We can read from the skeleton graphs the numbers of external links, the numbers of vertices, and the degrees of all the possible graphs in \mathbb{G}_τ and their subgraphs, and therefore estimate their contributions. The most divergent graph is the melonic two-vertex graph arising when gluing two copies of Ψ . It also contains the subgraphs dictating the additional conditions on α and β . □

For every $\tau \in \{\Psi, \Psi, \Psi, \Psi\}$, we take τ^{sto} to be the term containing only \dagger 's (for instance $\Psi^{\text{sto}} = \Psi$) that has been constructed in the previous sections, and τ^{det} the sum of all the remaining cross terms that contain at least one $\dagger - \ddagger$. Regarding Ψ , we chose

$$\Psi^{\text{sto}} : f \mapsto \Psi(f) + \mathcal{N}(f, \Psi, \dagger) + \mathcal{N}(f, \dagger, \Psi) - \mathcal{E}^2 f. \quad (2.118)$$

We have the following lemma about the regularity of Ψ^{sto} .

Lemma 2.102 In $d = 4$, for all $\alpha > -\frac{1}{2}$, it holds

$$\sup_{N \in \mathbf{N}} \mathbf{E}[\|\Psi_N^{\text{sto}} - \Psi_N\|_{\mathcal{L}(C_T H^\alpha(\mathbf{T}), C_T C^{\min(\frac{1}{2}, \alpha - \frac{1}{2}) - \epsilon}(\mathbf{T}))}^p] < \infty.$$

Proof. We deal separately with $f \mapsto \mathcal{N}(f, \Psi, \uparrow) - \frac{\mathfrak{c}^2}{2}f$ and $f \mapsto \mathcal{N}(f, \uparrow, \Psi) - \frac{\mathfrak{c}^2}{2}f$.

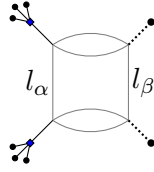


Figure 23: Schematic representation of the skeleton graph of $f \mapsto \mathcal{N}(f, \Psi, \uparrow) - \frac{\mathfrak{c}^2}{2}f$. The one of $f \mapsto \mathcal{N}(f, \uparrow, \Psi) - \frac{\mathfrak{c}^2}{2}f$ is obtained by exchanging l_α and l_β .

Both contributions generate all the vacuum three-vertex graphs, so that the most divergent possible graph or subgraph that contains both l_α and l_β is the melonic graph $(3, 0, 0)$ that brings the constraint $\beta < \alpha - \frac{1}{2}$. The first contribution has a subgraph $(1, 2, d-2)$ (obtained by gluing the two noises coming from the two terms \uparrow) containing only l_β , and the second one the same subgraph containing only l_α , which yields the additional constraints. These subgraphs are maximal because in view of their positions, l_α can not belong to a melonic subgraph that would not also contain l_β . \square

We can turn to τ^{\det} which, as we mentioned earlier, is controlled via deterministic analysis. In particular, for $N < \infty$ we will write the terms in τ^{\det} as compositions of random operators and verify that this composition of random operators is stable in the $N \rightarrow \infty$ limit.

As a first example, we work with the worst term in \mathfrak{J}^{\det} which is given by $f \mapsto \mathcal{N}(\uparrow, \uparrow - \uparrow, f)$ and which we rewrite as

$$\mathcal{N}(\uparrow, \uparrow - \uparrow, f)(x) = \sum_{c \in [4]} \uparrow^{(3)}((\uparrow - \uparrow)^{(1)}(f)(\cdot, x_c))(x_c).$$

If $f \in H^\alpha$ with $\alpha > \frac{1}{2}$, (2.117) implies that $(\uparrow - \uparrow)^{(1)}(f)(y_c, x_c)$ will be of regularity $\min(\frac{1}{2}, \alpha - \frac{1}{2})$ in the direction of y_c , which is bigger than $-\frac{1}{2}$, so that one can use (2.82) to obtain that $\mathcal{N}(\uparrow, \uparrow - \uparrow, f)$ is of regularity $\min(-\frac{1}{2}, \alpha - \frac{3}{2})$. Therefore, $f \mapsto \mathcal{N}(\uparrow, \uparrow - \uparrow, f)$ is indeed better behaved than $\mathfrak{J}(f)$. All the remaining contributions in \mathfrak{J}^{\det} are even better behaved, and can be treated in the same way. The proof is also similar for all the terms in \mathfrak{H}^{\det} and \mathfrak{B}^{\det} , which again we write as some compositions of operators.

For illustration, we detail how this argument works in the case of the biggest operator \mathfrak{V} . The worst terms in \mathfrak{V}^{\det} are

$$f \mapsto \mathcal{N}(\uparrow - \uparrow, \mathcal{L}^{-1} \mathfrak{B}'(f), \uparrow) \quad \text{and} \quad f \mapsto \mathfrak{B}'(\mathcal{L}^{-1} \mathcal{N}(\uparrow - \uparrow, f, \uparrow)).$$

If $f \in H^\alpha$ with $\alpha > \frac{1}{2}$, by the estimate (2.85), we have that $\mathfrak{B}'(f)$ is of regularity $\min(-\frac{1}{2}, \alpha - 2)$, so that $\mathcal{L}^{-1} \mathfrak{B}'(f)$ is of regularity $\min(\frac{3}{2}, \alpha) > \frac{1}{2}$. It follows by the deterministic estimate stated in Lemma 2.104 that one can take the non-melonic pairing of $\mathcal{L}^{-1} \mathfrak{B}'(f)$ with $\uparrow - \uparrow$ which is of regularity $-\epsilon$ as soon as $\epsilon < 1/2$.

On the other hand,

$$\mathcal{N}(\uparrow - \uparrow, \mathcal{L}^{-1} \mathfrak{B}'(f), \uparrow)(x) = \sum_{c \in [4]} \uparrow^{(1)}((\uparrow - \uparrow, \mathcal{L}^{-1} \mathfrak{B}'(f))_{L^2(\mathbf{T})}(x_c, \cdot))(x_c).$$

Since $\mathcal{L}^{-1} \mathfrak{B}'(f)$ is of regularity $> \frac{1}{2}$ (and hence so is $(\uparrow - \uparrow, \mathcal{L}^{-1} \mathfrak{B}'(f))_{L^2(\mathbf{T})}(x_c, y_c)$ in the variable y_c), the estimate (2.82) shows $\mathcal{N}(\uparrow - \uparrow, \mathcal{L}^{-1} \mathfrak{B}'(f), \uparrow)$ is of regularity $\min(0, \alpha - 1)$ as expected.

Regarding $\mathfrak{B}'(\mathcal{L}^{-1} \mathcal{N}(\uparrow - \uparrow, f, \uparrow))$ for $f \in H^\alpha$ with $\alpha > \frac{1}{2}$, we can perform the non-melonic pairing of f with $\uparrow - \uparrow$ using Lemma 2.104, and $\uparrow^{(1)}$ can act on the non-melonic pairing of f with $\uparrow - \uparrow$ since \mathcal{L}^{-1}

increases the regularity by 2 while \mathcal{D}^s decreases it by 2 we can conclude our argument. All the other terms are better behaved, and follow in the same way, which shows the objects made with the rough shift share the analytic properties of the objects made with \mathfrak{I} .

Appendix 2.G Besov spaces

We define Littlewood-Paley blocks, which are Fourier multipliers $(\Delta^j)_{j=-1}^\infty$, by setting $\Delta^{-1} = \mathbf{1}_{[0,2^{-1})}$ ($|\nabla|_\infty$) and, for $j \in \mathbf{N}$, $\Delta^j = \mathbf{1}_{[2^{-j},2^{-j+1})}$ ($2^{-j}|\nabla|_\infty$) (we have $\Pi_N = \sum_{i \leq \log_2 N} \Delta^i$ for N dyadic). We then define two norms on $C^\infty(\mathbf{T}^d)$ by setting

$$\|\cdot\|_{H^\alpha(\mathbf{T}^d)} \stackrel{\text{def}}{=} \|\langle \nabla \rangle^\alpha \cdot\|_{L^2(\mathbf{T}^d)} \text{ and } \|\cdot\|_{B_{p,q}^\alpha(\mathbf{T}^d)} \stackrel{\text{def}}{=} \left\| \left\| 2^{i\alpha} \|\Delta^i \cdot\|_{L^p(\mathbf{T}^d)} \right\|_{\ell_{i \geq -1}^q},$$

where for a countable set A we write $\ell_A^q = \ell^q(\mathbf{R}^A)$. We define $B_{p,q}^\alpha$ to be the completion of $C^\infty(\mathbf{T}^d)$ under the norm $\|\cdot\|_{B_{p,q}^\alpha(\mathbf{T}^d)}$. Note that $B_{2,2}^\alpha = H^\alpha$. We have the following standard facts for Besov spaces

Lemma 2.103 *Pick $\alpha \in \mathbf{R}$, $\delta > 0$ and $p, q, p_+, p_-, q_+, q_- \in [1, \infty]$, $p_- < p_+$, $q_- < q_+$. Then, the following embeddings are compact:*

$$\begin{aligned} B_{p,q}^\alpha &\hookrightarrow B_{p,q}^{\alpha-\delta}, \quad B_{p_+,q}^\alpha \hookrightarrow B_{p_+,q}^\alpha, \quad B_{p_-,q}^\alpha \hookrightarrow B_{p_+,q}^\alpha, \\ B_{p_-,q}^\alpha &\hookrightarrow B_{p_+,q}^{\alpha-d(\frac{1}{p_-}-\frac{1}{p_+})}, \quad B_{p,q_+}^\alpha \hookrightarrow B_{p,q_-}^{\alpha-\delta}. \end{aligned}$$

Lemma 2.104 *Pick $\alpha \in \mathbf{R}$ and $p, p', q, q' \in [1, \infty]$, such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then, the L^2 pairing can be extended to a bilinear map $B_{p,q}^\alpha \times B_{p',q'}^{-\alpha} \rightarrow \mathbf{R}$.*

Finally, we have the following standard heat kernel and Schauder estimates.

Lemma 2.105 *For $\alpha, \beta \in \mathbf{R}$ with $\beta \geq \alpha$ and $t > 0$, one has that $P_t : C^\alpha \rightarrow C^\beta$ is bounded with*

$$\|P_t(u)\|_{C^\beta} \lesssim t^{-\frac{\beta-\alpha}{2}} \|u\|_{C^\alpha}. \tag{2.119}$$

Lemma 2.106 *For any $\alpha \in \mathbf{R}$ and $\epsilon > 0$,*

$$\|\mathcal{L}^{-1}(u)\|_{C_T C^{\alpha+2-\epsilon}} \lesssim T^{\frac{\epsilon}{2}} \|u\|_{C_T C^\alpha}. \tag{2.120}$$

2.G.1 Bilocal Besov regularity

We study mixed terms such as \mathcal{D}^s or \mathcal{D}^s by treating them as distributions over $\mathbf{T}_x^n \times \mathbf{T}_y^m$. We will need anisotropic regularity estimates, for instance allowing terms to be bounded in y but only C^α for $\alpha < 0$ in x . While such distributions certainly belong to $\mathcal{C}_{x,y}^\alpha \stackrel{\text{def}}{=} \mathcal{C}^\alpha(\mathbf{T}_x^n \times \mathbf{T}_y^m)$, it will be better to work in a space like $L_y^\infty C_x^\alpha$. The next lemma gives straightforward estimates for these spaces.

Lemma 2.107 *Let $\alpha \geq 0$, and $\epsilon > 0$. Then uniformly in $u \in C^{\alpha+\epsilon}(\mathbf{T}_x^n \times \mathbf{T}_y^m)$,*

$$\|u\|_{L_y^\infty C_x^\alpha} = \left\| \|u\|_{C_x^\alpha} \right\|_{L_y^\infty} \lesssim \|u\|_{C_{x,y}^{\alpha+\epsilon}}. \tag{2.121}$$

Proof. For a function of two variables (x, y) , we let $\Delta_x^i u(x, y)$ stand for the Littlewood-Paley block Δ^i applied only in the variable x . When no subscript is present, it is understood that $\Delta^j \equiv \Delta_{x,y}^j$ acts on both variables x and y .

$$\left\| \|u\|_{C_x^\alpha} \right\|_{L_y^\infty} = \sup_{y \in \mathbf{T}^m} \|u(y)\|_{C_x^\alpha} = \sup_{x,y \in \mathbf{T}^n \times \mathbf{T}^m} \sup_{i \geq -1} 2^{\alpha i} |\Delta_x^i u(x, y)|$$

$$\begin{aligned}
 &= \sup_{x,y \in \mathbf{T}^n \times \mathbf{T}^m} \sup_{i \geq -1} 2^{\alpha i} |\Delta_x^i \sum_{j \geq i} \Delta^j u(x, y)| \leq \sup_{i \geq -1} 2^{\alpha i} \sum_{j \geq i} \|\Delta_x^i \Delta^j u\|_{L_{x,y}^\infty} \\
 &\leq \sup_{i \geq -1} \sum_{j \geq i} 2^{\alpha j} \|\Delta^j u\|_{L_{x,y}^\infty},
 \end{aligned}$$

where we used $\alpha > 0$ and that boundedness of $\Delta^j u$ implies $\|\Delta_x^i(\Delta^j u)\|_{L^\infty} \lesssim \|\Delta^j u\|_{L^\infty}$. The claim follows by using the embedding $\mathcal{C}^{\alpha+\epsilon} \hookrightarrow B_{\infty,1}^\alpha$ along with the estimate.

$$\left\| \|u\|_{\mathcal{C}_x^\alpha} \right\|_{L_y^\infty} \lesssim \sum_{j \geq -1} 2^{\alpha j} \|\Delta^j u\|_{L_{x,y}^\infty} = \|u\|_{B_{\infty,1}^{\alpha,1}(\mathbf{T}^n \times \mathbf{T}^m)}.$$

□

We also have a Sobolev space version of the previous lemma:

Lemma 2.108 *Let $\alpha \geq 0$, and $\epsilon > 0$. Then uniformly in $u \in H^{\alpha+\epsilon}(\mathbf{T}^n \times \mathbf{T}^m)$,*

$$\|u\|_{L_y^2 H_x^\alpha} = \left\| \|u\|_{H_x^\alpha} \right\|_{L_y^2} \lesssim \|u\|_{H_{x,y}^{\alpha+\epsilon}}. \tag{2.122}$$

Proof. Recall Δ_x^j is a Littlewood- Paley block in x only. We have

$$\begin{aligned}
 \left\| \|u\|_{H_x^\alpha} \right\|_{L_y^2} &= \left(\int_{\mathbf{T}^m} \sum_{i \geq -1} 2^{2\alpha i} \|\Delta_x^i u\|_{L_x^2}^2(y) dy \right)^{\frac{1}{2}} \\
 &= \left(\sum_{i \geq -1} 2^{2\alpha i} \int_{\mathbf{T}^m} \|\Delta_x^i u\|_{L_x^2}^2(y) dy \right)^{\frac{1}{2}} \\
 &= \left(\sum_{i \geq -1} 2^{2\alpha i} \|\Delta_x^i u\|_{L_{x,y}^2}^2 \right)^{\frac{1}{2}} = \left\| 2^{\alpha i} \|\Delta_x^i u\|_{L_{x,y}^2} \right\|_{\ell_i^2} \\
 &= \left\| 2^{\alpha i} \|\Delta_x^i \sum_{j \geq i} \Delta^j u\|_{L_{x,y}^2} \right\|_{\ell_i^2} \leq \left\| 2^{\alpha i} \sum_{j \geq i} \|\Delta_x^i \Delta^j u\|_{L_{x,y}^2} \right\|_{\ell_i^2} \\
 &= \left\| 2^{-\frac{\epsilon}{2}i} \sum_{j \geq i} 2^{(\alpha+\frac{\epsilon}{2})j} \|\Delta_x^i \Delta^j u\|_{L_{x,y}^2} \right\|_{\ell_i^2}.
 \end{aligned}$$

where we used that $\alpha + \frac{\epsilon}{2} > 0$ and that boundedness of $\Delta^j u$ implies $\|\Delta_x^i(\Delta^j u)\|_{L_{x,y}^2} \lesssim \|\Delta^j u\|_{L_{x,y}^2}$. The claim follows using the embedding $H^{\alpha+\epsilon} \hookrightarrow B_{2,1}^{\alpha+\frac{\epsilon}{2}}$ along with the estimate

$$\left\| \|u\|_{H_x^\alpha} \right\|_{L_y^2} \leq \left\| 2^{-\frac{\epsilon}{2}i} \sum_{j \geq -1} 2^{(\alpha+\frac{\epsilon}{2})j} \|\Delta^j u\|_{L_{x,y}^2} \right\|_{\ell_i^2} = \|2^{-\frac{\epsilon}{2}i}\|_{\ell_i^2} \|u\|_{B_{2,1}^{\alpha+\frac{\epsilon}{2}}(\mathbf{T}^n \times \mathbf{T}^m)}.$$

□

Finally, the following lemma deals with distributions of negative bilocal Hölder-Besov regularity:

Lemma 2.109 *For $\alpha \leq 0$ and $\epsilon > 0$, and uniformly in $u \in L_y^\infty \mathcal{C}_x^{\alpha+\epsilon}(\mathbf{T}^n \times \mathbf{T}^m)$:*

$$\|u\|_{\mathcal{C}_{x,y}^\alpha} \lesssim \left\| \|u\|_{\mathcal{C}_x^{\alpha+\epsilon}} \right\|_{L_y^\infty} = \|u\|_{L_y^\infty \mathcal{C}_x^{\alpha+\epsilon}}. \tag{2.123}$$

Proof. Recall Δ_x^j is a Littlewood- Paley block in x only, as opposed to Δ^i . We start from

$$\|u\|_{\mathcal{C}_{x,y}^\alpha} = \sup_{i \geq -1} 2^{\alpha i} \|\Delta^i u\|_{L_{x,y}^\infty} = \sup_{y \in \mathbf{T}^m} \sup_{i \geq -1} 2^{\alpha i} \|\Delta^i \sum_{j \leq i} \Delta^j u(y)\|_{L_x^\infty}$$

$$\lesssim \sup_{y \in \mathbf{T}^m} \sup_{i \geq -1} 2^{\alpha i} \sum_{j \leq i} \|\Delta^i \Delta_x^j u(y)\|_{L_x^\infty} \lesssim \sup_{y \in \mathbf{T}^m} \sup_{i \geq -1} \sum_{j \leq i} 2^{\alpha j} \|\Delta_x^j u(y)\|_{L_x^\infty},$$

where we used that $\alpha < 0$ and that boundedness of $\Delta_x^j u$ in x implies $\|\Delta^i(\Delta_x^j u)(y)\|_{L_x^\infty} \lesssim \|\Delta_x^j u(y)\|_{L_x^\infty}$. The claim follows using the embedding $C^{\alpha+\epsilon} \hookrightarrow B_{\infty,1}^\alpha$ along with the estimate

$$\|u\|_{C_{x,y}^\alpha} \lesssim \sum_{j \geq -1} 2^{\alpha j} \|\Delta^j u\|_{L_{x,y}^\infty} = \|u\|_{B_{\infty,1}^\alpha(\mathbf{T}_x^n \times \mathbf{T}_y^m)}.$$

□

Appendix 2.H Facts about the nonlinearity

In this section we present useful facts about the non-local nonlinearity appearing in the \mathbf{T}_d^4 field theory. We first show that it is related to a specific norm among a wider family of norms over $C^\infty(\mathbf{T}^d)$ that we call M_c^p norms. We then consider the analogue of the Sobolev norms constructed with the M_c^p norms, and establish interpolation inequalities and embeddings for these norms. Finally, we prove two key Cauchy-Schwarz inequalities.

Throughout this section, we fix $d \in \mathbf{N}$ with $d \geq 2$ and always take $c \in [d]$.

Definition 2.110 (M_c^p spaces) Let $u \in C^\infty(\mathbf{T}^d)$. We define a bounded and positive operator $M_c(u) \in \mathcal{L}(\ell^2(\mathbf{Z}))$ on the space of square integrable complex-valued sequences indexed by \mathbf{Z} by setting, for $a_c, b_c \in \mathbf{Z}$, the corresponding matrix entry of $M_c(u)$ to be given by

$$M_c(u)_{a_c, b_c} \stackrel{\text{def}}{=} \sum_{(m_i: i \in [d] \setminus \{c\}) \in \mathbf{Z}^{d-1}} \hat{u}_{m_1, \dots, a_c, \dots, m_d} \hat{u}_{-m_1, \dots, -b_c, \dots, -m_d}.$$

For $p \in (1, \infty)$ we define the Banach space $M_c^p(\mathbf{T}^d)$ as the completion of $C^\infty(\mathbf{T}^d)$ under the norm:

$$\|u\|_{M_c^p} \stackrel{\text{def}}{=} \left(\text{Tr} \left(M_c(u)^{\frac{p}{2}} \right) \right)^{\frac{1}{p}}.$$

Positivity of $\|\bullet\|_{M_c^p}$ is straightforward to check. We also define what we call the M^p norm as

$$\|u\|_{M^p}^p \stackrel{\text{def}}{=} \sum_{c=1}^d \|u\|_{M_c^p}^p.$$

Remark 2.111 The name tensor field theory stems from the fact that a Fourier cut-off distribution u over \mathbf{T}^d can be seen as a tensor $\widehat{\Pi_N u} = (\hat{u}_m)_{m \in \mathbf{Z}_N^d}$. Then, for p an even integer, $\|u\|_{M_c^p}^p$ is a trace invariant of \hat{u} , that is a function of \hat{u} invariant under the action of $O(2N+1)^{\otimes d}$ on the indices of \hat{u} .

Lemma 2.112 Let $1 \leq p < q < \infty$. Then we have the compact embedding

$$\|u\|_{M_c^q} \lesssim \|u\|_{M_c^p}, \quad (2.124)$$

which directly follows from the embedding $\ell^p(\mathbf{Z}) \hookrightarrow \ell^q(\mathbf{Z})$.

Remark 2.113 For any $c \in [d]$, $\text{Tr}(M_c(u)) = \|u\|_{L^2(\mathbf{T}^d)}^2 < \infty$ by Parseval, so the space $M_c^2(\mathbf{T}^d)$ is just $L^2(\mathbf{T}^d)$ —in particular, $\mathcal{I}(u)^{\frac{1}{4}} \lesssim \|u\|_{L^2}$.

Next we state some interpolation and embedding inequalities.

Lemma 2.114 (M_c^4 –Sobolev interpolation 1) For any $\beta \geq \alpha \geq 0$,

$$\|\langle \nabla_c \rangle^\alpha u\|_{M_c^4} \lesssim \|u\|_{M_c^4}^{1-\frac{\alpha}{\beta}} \|\langle \nabla_c \rangle^\beta u\|_{M_c^4}^{\frac{\alpha}{\beta}}. \quad (2.125)$$

Proof. In the proof, we denote $M_c = M_c(u)$. Using Hölder's inequality, we have

$$\begin{aligned} \|\langle \nabla_c \rangle^\alpha u\|_{M_c^4}^4 &= \sum_{p_c, q_c} \langle p_c \rangle^{2\alpha} \langle q_c \rangle^{2\alpha} M_c(p_c, q_c)^2 \\ &= \sum_{p_c, q_c} M_c(p_c, q_c)^{2(1-\frac{\alpha}{\beta})} \left(\langle p_c \rangle^2 \langle q_c \rangle^2 M_c(p_c, q_c)^{\frac{2}{\beta}} \right)^\alpha \\ &\leq \|M_c(p_c, q_c)^{2(1-\frac{\alpha}{\beta})}\|_{\ell_{p_c, q_c}^{\frac{1}{1-\frac{\alpha}{\beta}}}} \left\| \left(\langle p_c \rangle^2 \langle q_c \rangle^2 M_c(p_c, q_c)^{\frac{2}{\beta}} \right)^\alpha \right\|_{\ell_{p_c, q_c}^{\frac{\beta}{\alpha}}} \\ &= \|M_c(p_c, q_c)\|_{\ell_{p_c, q_c}^{2(1-\frac{\alpha}{\beta})}} \|\langle p_c \rangle^\beta \langle q_c \rangle^\beta M_c(p_c, q_c)\|_{\ell_{p_c, q_c}^{\frac{2\alpha}{\beta}}}^{2\frac{\alpha}{\beta}}. \end{aligned}$$

□

Lemma 2.115 (M_c^4 -Sobolev embedding) *Let $\alpha > \frac{1}{4}$, then*

$$\|u\|_{L^2} \lesssim \|\langle \nabla_c \rangle^\alpha u\|_{M_c^4}. \quad (2.126)$$

Proof. Again denoting $M_c = M_c(u)$, by Cauchy-Schwarz we have

$$\begin{aligned} \|u\|_{L^2}^2 &= \text{Tr} M_c = \sum_{p_c} M_c(p_c, p_c) = \sum_{p_c, q_c} M_c(p_c, q_c) \delta_{p_c, q_c} \\ &= \sum_{p_c, q_c} M_c(p_c, q_c) \langle p_c \rangle^\alpha \langle q_c \rangle^\alpha \delta_{p_c, q_c} \langle p_c \rangle^{-\alpha} \langle q_c \rangle^{-\alpha} \\ &\leq \left(\sum_{p_c, q_c} M_c(p_c, q_c)^2 \langle p_c \rangle^{2\alpha} \langle q_c \rangle^{2\alpha} \right)^{\frac{1}{2}} \left(\sum_{p_c, q_c} \delta_{p_c, q_c}^2 \langle p_c \rangle^{-2\alpha} \langle q_c \rangle^{-2\alpha} \right)^{\frac{1}{2}} \\ &\leq \|\langle \nabla_c \rangle^\alpha u\|_{M_c^4}^2 \left(\sum_{p_c} \langle p_c \rangle^{-4\alpha} \right)^{\frac{1}{2}} \lesssim \|\langle \nabla_c \rangle^\alpha u\|_{M_c^4}^2. \end{aligned}$$

□

Corollary 2.116 (M_c^p -Sobolev interpolation 2) *For all $\theta \in (\frac{1}{4}, 1]$, we have*

$$\|u\|_{L^2} \lesssim \|u\|_{M_c^4}^{1-\theta} \|u\|_{H^1}^\theta. \quad (2.127)$$

Proof. Fix $\theta \in (\frac{1}{4}, 1]$, and define $\epsilon \in (0, \frac{3}{8}]$ by the relation $\theta = \frac{\frac{1}{4} + \epsilon}{1 - \epsilon}$. Using (2.126) and then (2.125),

$$\|u\|_{L^2} \lesssim \|\langle \nabla_c \rangle^{\frac{1}{4} + \epsilon} u\|_{M_c^4} \lesssim \|u\|_{M_c^4}^{1-\theta} \|\langle \nabla_c \rangle^{1-\epsilon} u\|_{M_c^4}^\theta \lesssim \|u\|_{M_c^4}^{1-\theta} \|\langle \nabla_c \rangle^{1-\epsilon} u\|_{L^2}^\theta,$$

where on the last inequality we used (2.124) to control the M_c^4 norm by the L^2 norm. We can now conclude since, by (2.122),

$$\|\langle \nabla_c \rangle^{1-\epsilon} u\|_{L^2} = \|u\|_{L_c^2 H_c^{1-\epsilon}} \lesssim \|u\|_{H^1}.$$

□

Lemma 2.117 (Cauchy-Schwarz inequalities) *For any $\phi, \psi \in C^\infty(\mathbf{T}^d)$,*

$$|(\mathcal{N}^c(\phi, \psi, \phi), \psi)_{L^2(\mathbf{T}^d)}| \leq (\mathcal{N}^c(\psi, \psi, \phi), \phi)_{L^2(\mathbf{T}^d)}, \quad (2.128)$$

$$|(\mathcal{N}^c(\phi, \psi, \phi), \phi)_{L^2(\mathbf{T}^d)}| \leq (\mathcal{N}^c(\psi, \psi, \phi), \phi)_{L^2(\mathbf{T}^d)}^{\frac{1}{2}} \|\phi\|_{M_c^4(\mathbf{T}^d)}. \quad (2.129)$$

Proof. For the first estimate, denoting by $\theta(x_c, y_c) = (\phi(x_c, \cdot), \psi(y_c, \cdot))_{L^2(\mathbf{T}_\varepsilon^{d-1})}$ the melonic pairing of ϕ with ψ , we have, by Cauchy-Schwarz inequality,

$$|(\mathcal{N}^c(\phi, \psi, \phi), \psi)_{L^2(\mathbf{T}^d)}| = \left| \int_{\mathbf{T}^2} \theta(x_c, y_c) \theta(y_c, x_c) dx_c dy_c \right| \leq \int_{\mathbf{T}^2} \theta^2(x_c, y_c) dx_c dy_c = (\mathcal{N}^c(\psi, \psi, \phi), \phi)_{L^2(\mathbf{T}^d)}.$$

Similarly, writing $\varphi = (x_c, y_c) = (\phi(x_c, \cdot), \phi(y_c, \cdot))_{L^2(\mathbf{T}_\varepsilon^{d-1})}$, we have

$$|(\mathcal{N}^c(\phi, \psi, \phi), \psi)_{L^2(\mathbf{T}^d)}| = \left| \int_{\mathbf{T}^2} \theta(x_c, y_c) \varphi(y_c, x_c) dx_c dy_c \right| \leq \left| \int_{\mathbf{T}^2} \theta^2(x_c, y_c) dx_c dy_c \right|^{\frac{1}{2}} \left| \int_{\mathbf{T}^2} \varphi^2(x_c, y_c) dx_c dy_c \right|^{\frac{1}{2}}.$$

□

Appendix 2.I Controlling mixed terms in $d = 4$

In this section, we gather the bounds over the mixed term of the solution to the Langevin dynamic/the Barashkov & Gubinelli drift and the stochastic objects needed for Propositions 2.37 and 2.61. For the sake of concreteness, we state these results in $d = 4$ but they also carry over as (much less optimal) estimates for $d < 4$. In this section, we will use notation that allows us to state the needed mixed term estimates for both settings at once since the regularity exponents are the same and the arguments are essentially the same.

Below, v will be a smooth function $[0, T] \times \mathbf{T}^d \rightarrow \mathbf{R}$, and we will write $v_t = v(t, \cdot)$. For such a function v , we use the notation $\mathcal{K}(v_t) = \|v_t\|_{H^{1-\varepsilon}(\mathbf{T}^4)}^2$ and $\mathcal{I}(v_t) = \|v_t\|_{M^4(\mathbf{T}^4)}^4$. Recall that for $d \in \{3, 4\}$, $\mathbb{X}_d^{\text{LD}} = \mathbb{X}_{f,d}^{\text{LD}} \cup \mathbb{X}_{o,d}^{\text{LD}}$ is the collection of all stochastic objects necessary to handle the fixed point problem in dimension d , and that we see it as an element of the product of the Banach spaces in which the objects live. We denote $\mathbb{X}_d^{\text{LD}}(t) \stackrel{\text{def}}{=} \{\tau(t) | \tau \in \mathbb{X}_d^{\text{LD}}\}$ which we endow with the norm:

$$\|\mathbb{X}_d^{\text{LD}}(t)\| = \max \left(\max_{\tau \in \mathbb{X}_{f,d}^{\text{LD}}} \|\tau(t)\|_{\mathcal{C}^{\beta_\tau - \varepsilon}}, \max_{\tau \in \mathbb{X}_{o,d}^{\text{LD}}} \|\tau(t)\|_{\mathcal{L}(\mathcal{C}_t H^{\alpha_\tau}, \mathcal{C}^{\beta_\tau - \varepsilon})} \right),$$

where α_τ and β_τ are the inner and outer regularities of τ as stated in Lemmas 2.66 and 2.68. We also recall that \mathbb{X}_d^{LD} is endowed with the norm $\|\cdot\|_T$ introduced in $d = 3$ in (2.43) and we extended to the case $d = 4$ in Section 2.3. Note that for every $t \in [0, T]$, it holds $\|\mathbb{X}_d^{\text{LD}}(t)\| \lesssim \|\mathbb{X}_d^{\text{LD}}\|_T$.

In the sequel, we often drop the subscript and superscript notation, writing $\|\mathbb{X}(t)\|$ to refer to both $\|\mathbb{X}_d^{\text{LD}}(t)\|$ and $\|\mathbb{X}^{\text{BG}}(t)\|$ as given in (2.76).

Throughout this section, we write $A(t) \lesssim_{\mathbb{X}} B(t)$ whenever there exists constants $C, c > 0$ such that $A(t) \leq C \|\mathbb{X}(t)\|^c B(t)$.

Lemma 2.118 *Recall the random field \mathcal{S}_t defined in Definition 2.64 as an LD object and Definition 2.58 as a BG object. There exist $C, \kappa > 0$ such that, for all $t \in [0, T]$ and $\delta \in (0, 1]$,*

$$|(\mathcal{S}_t, v_t)| \leq C(\delta^{-1} \|\mathbb{X}(t)\|)^\kappa + \delta \mathcal{K}(v_t). \quad (2.130)$$

Proof. This follows from applying Young's inequality to the estimate

$$|(\mathcal{S}_t, v_t)| \leq \|\mathcal{S}(t)\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|v_t\|_{B_{1,1}^{\frac{1}{2}+\varepsilon}} \lesssim_{\mathbb{X}} \|v_t\|_{H^{\frac{1}{2}+2\varepsilon}} \lesssim_{\mathbb{X}} \mathcal{K}(v_t)^{\frac{1}{2}}.$$

□

Lemma 2.119 *There exist $C, \kappa > 0$ such that, for all $t \in [0, T]$ and $\delta \in (0, 1]$,*

$$|\mathfrak{v}_{\mathfrak{v}}^{\circlearrowleft}(t)| \leq C(\delta^{-1} \|\mathbb{X}(t)\|)^\kappa + \delta(\mathcal{I}(v_t) + \mathcal{K}(v_t)), \quad (2.131)$$

$$|\mathfrak{v}_{\mathfrak{v}}^{\circlearrowright}(t)| \leq C(\delta^{-1} \|\mathbb{X}(t)\|)^\kappa + \delta(\mathcal{I}(v_t) + \mathcal{K}(v_t)), \quad (2.132)$$

$$|\mathfrak{v}_{\mathfrak{v}}^{\circlearrowright}(t)| \leq C(\delta^{-1} \|\mathbb{X}(t)\|)^\kappa + \delta(\mathcal{I}(v_t) + \mathcal{K}(v_t)), \quad (2.133)$$

Proof. We first prove (2.131). We view Ψ_t^c as a time indexed space operator, and at fixed $t \geq 0$ we denote its kernel by $\Psi_t^c(x_c, y_c)$. As a first step, observe that

$$\begin{aligned} |v_v^c(t)| &= \int_{\mathbf{T}^5} \Psi_t^c(x_c, y_c) v_t(x_{\hat{c}}, x_c) v_t(x_{\hat{c}}, y_c) dx_{\hat{c}} dx_c dy_c \\ &= \int_{\mathbf{T}^5} \langle \nabla_{x_c} \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}} \langle \nabla_{y_c} \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}} \Psi_t^c(x_c, y_c) \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, x_c) \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, y_c) dx_{\hat{c}} dx_c dy_c \\ &= \int_{\mathbf{T}} \langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}} \circ \Psi_t^c \circ \langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}} \left(\int_{\mathbf{T}^3} \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, x_c) \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, \cdot) dx_{\hat{c}} \right) (x_c) dx_c \\ &= \int_{\mathbf{T}^2} \tilde{\Psi}_t^c(x_c, y_c) V_t(y_c, x_c) dx_c dy_c, \end{aligned}$$

where the notation \circ denotes the composition of operators and, to lighten the notations, we introduced the time-indexed operator on \mathbf{T}

$$\tilde{\Psi}_t^c \stackrel{\text{def}}{=} \langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}} \circ \Psi_t^c \circ \langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}}$$

along with the function $\mathbf{T}^2 \rightarrow \mathbf{R}$

$$V_t(y_c, x_c) \stackrel{\text{def}}{=} \int_{\mathbf{T}^3} \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, x_c) \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, y_c) dx_{\hat{c}}.$$

Therefore,

$$\begin{aligned} |v_v^c(t)| &\leq \int_{\mathbf{T}} \left| \int_{\mathbf{T}} \tilde{\Psi}_t^c(x_c, y_c) V_t(y_c, x_c) dy_c \right| dx_c \lesssim \int_{\mathbf{T}} \left\| \int_{\mathbf{T}} \tilde{\Psi}_t^c(z_c, y_c) V_t(y_c, x_c) dy_c \right\|_{L_{z_c}^\infty} dx_c \\ &= \|\tilde{\Psi}_t^c(V_t(\cdot, x_c))(z_c)\|_{L_{x_c}^1 L_{z_c}^\infty} \lesssim \|\tilde{\Psi}_t^c(V_t(\cdot, x_c))(z_c)\|_{L_{x_c}^2 L_{z_c}^\infty}. \end{aligned}$$

Using

$$\|\tilde{\Psi}_t^c(V_t(\cdot, x_c))(z_c)\|_{L_{z_c}^\infty} \lesssim \|\tilde{\Psi}_t^c\|_{\mathcal{L}(C_T L^2, L^\infty)} \|V_t(y_c, x_c)\|_{L_{y_c}^2},$$

we thus end up with

$$|v_v^c(t)| \lesssim \|\tilde{\Psi}_t^c\|_{\mathcal{L}(C_T L^2, L^\infty)} \|V_t\|_{L^2}.$$

Here, we use the fact that, for every $m \in \mathbf{R}$, $\langle \nabla \rangle^m$ is a continuous operator $L^2 \rightarrow H^{-m}$ and $C^m \rightarrow L^\infty$. Hence,

$$\begin{aligned} \|\tilde{\Psi}_t^c\|_{\mathcal{L}(C_T L^2, L^\infty)} &= \|\langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}} \circ \Psi_t^c \circ \langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}}\|_{\mathcal{L}(C_T L^2, L^\infty)} \\ &\lesssim \|\langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}}\|_{\mathcal{L}(L^2, H^{\frac{3}{4}+\frac{\epsilon}{2}})} \|\Psi_t^c(t)\|_{\mathcal{L}(C_T H^{\frac{3}{4}+\frac{\epsilon}{2}}, C^{-\frac{3}{4}-\frac{\epsilon}{2}})} \|\langle \nabla \rangle^{-\frac{3}{4}-\frac{\epsilon}{2}}\|_{\mathcal{L}(C^{-\frac{3}{4}-\frac{\epsilon}{2}}, L^\infty)} \\ &\lesssim \|\Psi_t^c(t)\|_{\mathcal{L}(C_T H^{\frac{3}{4}+\frac{\epsilon}{2}}, C^{-\frac{3}{4}-\frac{\epsilon}{2}})} \lesssim_{\mathbb{X}} 1. \end{aligned}$$

Moreover, using the interpolation formula (2.125), we have that

$$\begin{aligned} \|V_t\|_{L^2}^{\frac{1}{2}} &= \left\| \int_{\mathbf{T}^3} \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, x_c) \langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t(x_{\hat{c}}, y_c) dx_{\hat{c}} \right\|_{L_{x_c, y_c}^2}^{\frac{1}{2}} \\ &= \|\langle \nabla_c \rangle^{\frac{3}{4}+\frac{\epsilon}{2}} v_t\|_{M_c^4} \lesssim \|v_t\|_{M_c^4}^{\frac{1-10\epsilon}{4-8\epsilon}} \|\langle \nabla_c \rangle^{1-2\epsilon} v_t\|_{M_c^4}^{\frac{3+2\epsilon}{4-8\epsilon}}, \end{aligned}$$

and $\|\langle \nabla_c \rangle^{1-2\epsilon} v_t\|_{M_c^4} \lesssim \|\langle \nabla_c \rangle^{1-2\epsilon} v_t\|_{L^2} = \|v_t\|_{L_{x_{\hat{c}}}^2 H_{x_c}^{1-2\epsilon}} \lesssim \|v_t\|_{H^{1-\epsilon}}$ by (2.122), which yields

$$|v_v^c(t)| \lesssim_{\mathbb{X}} \mathcal{I}(v_t)^{\frac{1-10\epsilon}{16-32\epsilon}} \mathcal{K}(v_t)^{\frac{3+2\epsilon}{8-16\epsilon}}.$$

Since the sum of the exponents of $\mathcal{I}(v_t)$ and $\mathcal{K}(v_t)$ is smaller than 1, the claim follows by Young's inequality.

The proof of (2.132) is very similar and we obtain

$$|\mathfrak{R}_v^v(t)| \lesssim_{\mathbb{X}} \mathcal{I}(v_t)^{\frac{1-10\epsilon}{16-32\epsilon}} \mathcal{K}(v_t)^{\frac{3+2\epsilon}{8-16\epsilon}}, \quad (2.134)$$

while (2.133) follows from (2.132) by the Cauchy-Schwarz inequality (2.128) which implies that

$$|\mathfrak{R}_v^v(t)| \leq |\mathfrak{R}_v^v(t)|.$$

At this stage, we can conclude using the fact that \mathfrak{R} is a stochastic object that can be defined without any renormalization (in the sense that $\mathfrak{R}(f) = \mathcal{N}(\mathfrak{I}, \mathfrak{I}, f)$). \square

Remark 2.120 In the above proof, when applying the Cauchy-Schwarz inequality, it is crucial that $\mathfrak{R}(f) = \mathcal{N}(\mathfrak{I}, \mathfrak{I}, f)$. Indeed, it is of course also true that

$$|\mathfrak{R}_v^v(t)| \leq (\mathcal{N}(v_t, \mathfrak{I}_t, \mathfrak{I}_t), v_t),$$

but $(\mathcal{N}(v_t, \mathfrak{I}_t, \mathfrak{I}_t), v_t) \neq \mathfrak{R}_v^v(t)$ because of renormalization, and we do not expect to the r.h.s of the above inequality uniformly in the cut-off.

Lemma 2.121 *There exist $C, \kappa > 0$ such for all $t \in [0, T]$ and $\delta \in (0, 1]$,*

$$|\mathfrak{R}_v^v(t)| \leq C(\delta^{-1} \|\mathbb{X}(t)\|)^{\kappa} + \delta(\mathcal{I}(v_t) + \mathcal{K}(v_t)). \quad (2.135)$$

Proof. Using the Cauchy-Schwarz inequality (2.129), and again leveraging the fact that \mathfrak{R} can be defined without any new renormalisation, we have

$$|\mathfrak{R}_v^v(t)| \leq |\mathfrak{R}_v^v(t)|^{\frac{1}{2}} |\mathfrak{R}_v^v(t)|^{\frac{1}{2}} = \mathcal{I}(v_t)^{\frac{1}{2}} |\mathfrak{R}_v^v(t)|^{\frac{1}{2}}.$$

Now, using (2.134) yields

$$|\mathfrak{R}_v^v(t)| \lesssim_{\mathbb{X}} \mathcal{I}(v_t)^{\frac{1}{2} + \frac{1-10\epsilon}{32-64\epsilon}} \mathcal{K}(v_t)^{\frac{3+2\epsilon}{16-32\epsilon}},$$

and since the sum of the exponents of $\mathcal{I}(v)$ and $\mathcal{K}(v)$ is smaller than one, the claim follows by Young's inequality. \square

Chapter 3

Construction of the Φ_3^4 Measure on compact Riemannian Manifolds

3.1 Introduction

Let (M, g) stand for a closed 3-dimensional Riemannian manifold. This chapter is dedicated to constructing the Φ_3^4 measure over M , formally the ill-defined functional integral probability measure

$$\nu(\mathrm{d}u) \propto e^{-\frac{1}{2} \int_M (|\nabla u|^2 + u^2) - \frac{1}{4} \int_M u^4 \mathrm{d}u},$$

as an invariant probability measure of the dynamics

$$(\partial_t + P)u = -u^3 + \sqrt{2}\xi \tag{3.1}$$

where ξ stands for a spacetime white noise and $P \stackrel{\text{def}}{=} (1 - \Delta_g)$.

As summarised in Chapter 1, the Φ_3^4 measure was first constructed in the 70s by Glimm & Jaffe [GJ73], on \mathbf{R}^3 . The recent breakthroughs by Hairer [Hai14] and Gubinelli, Perkowski & Imkeller [GIP15] allowed several authors to recover the results of Glimm & Jaffe using only PDE and probabilistic techniques without the intricate combinatorial methods from the constructive school.

The dynamical construction of the Φ_3^4 measure was first done in finite volume on the flat 3-dimensional torus by Mourrat & Weber [MW17a], Hairer & Mattingly [HM18], Hairer & Schönbauer [HS21a], Albeverio & Kusuoka [AK20], then in the infinite volume 3-dimensional Euclidean space by different authors – Albeverio & Kusuoka [AK22], Moinat & Weber [MW20], Gubinelli & Hofmanová [GH18, GH21], Barashkov & Gubinelli [BG20, BG21], using different methods. An integrability property of the Φ_3^4 measure was proved in Hairer & Steele’s work [HS21b].

Yet most results in constructive quantum field theory are proved in the geometric settings of either \mathbf{R}^3 or the flat torus \mathbf{T}^3 . On manifolds, some constructive results were obtained on *compact surfaces* in [Pic08] by Pickrell and [DIM04] by Dimock for the $P(\varphi)_2$ theories, in [Lev01] by Lévy for the $2d$ Yang-Mills theory and in [GKRV21] by Guillarmou, Kupiainen, Rhodes & Vargas for the Liouville field theory on Riemann surfaces.

From the PDE side, the constructions of Gibbs measures on Riemannian surfaces that we are aware of, come from [BTT14] by Burq, Thomann & Tzvetkov for dynamical $P(\Phi)_2$ and from [ORTW20] by Oh, Robert, Tzvetkov & Wang for the dynamical Liouville model. Let us also cite the works [BB16, BBF17, BBI19] of Bailleul & Bernicot, and the works [DDD17], [Mou21b] and [BDM23] of Dahlqvist, Diehl & Driver, Mouzard and Bailleul, Dang & Mouzard on the Anderson operator on a 2-dimensional Riemannian manifold. We refer to [Mou21a] for an exposition of some of the above results. Finally, Hairer & Singh [HS23a] have developed a generalisation of the original Theory of Regularity Structures which is able to treat SPDEs on manifolds with values in vector bundles in full generality.

It is therefore a longstanding open problem in both constructive quantum field theory and field theory on curved spaces to construct the Φ_3^4 measure on an arbitrary closed, compact 3-dimensional Riemannian manifold. Let us mention that both the $2d$ Yang-Mills and Liouville theories are *integrable*. This work seems to give the first construction of a non-integrable, interacting quantum field theory on 3-manifolds.

Local well-posedness of (3.1) was first proved by Hairer in [Hai14] – see also Catellier & Chouk’s work [CC13] for a proof of that result with the tools of paracontrolled calculus. Mourrat & Weber proved in [MW14] an a priori estimate that gives the long time existence (and well-posedness) of the

solution to (3.1) and the existence of an invariant probability measure. The uniqueness of this invariant probability measure comes from the works of Hairer & Mattingly [HM18] on the strong Markov property of transition semigroups associated to singular stochastic PDEs, and Hairer & Schönbauer [HS21a] on the support of the laws of solutions to singular stochastic PDEs. See Hairer & Steele’s work [HS21b] for more references. Note that none of the previous works are readily available in a manifold setting.

The aim of the present chapter, together with the work [BDFT23], is to develop in a self-contained way all the tools needed to run the analysis in a 3 dimensional closed Riemannian manifold. On the purely analytical side

- We follow Jagannath & Perkowski’s simple approach [JP23] of equation (3.1) to prove that this equation is locally well-posed. Their formulation of the problem avoids the use of regularity structures or paracontrolled calculus. We freely use several tools of pseudodifferential and paradifferential calculus on manifolds which are developed in detail in [BDFT23].
- We give a simple and short proof of an L^p “coming down from infinity” property satisfied by the solution to equation (3.1) using energy methods. The longtime existence of a unique solution to equation (3.1) follows as a consequence.
- As usual in the study of singular stochastic PDEs we need to feed the analytic machinery with a number of random distributions whose formal definitions involve some ill-defined products and whose actual definitions involve some probabilistic constructions based on regularisation and renormalisation. Our approach to the renormalisation problem relies on microlocal analysis. It is a far reaching generalisation of the Epstein-Glaser point of view where we benefit from the many improvements contained in [PS16, BF00, HW01, HW02]. We reduce the problem of renormalisation to an extension problem for distributions on a configuration space defined outside all the diagonals, for which we develop a general machinery. To control analytically the Feynman amplitudes appearing in the stochastic bounds we feed our renormalisation machine with several microlocal estimates of distributional kernels which are done separately in [BDFT23].

We note that there is also a new approach to SPDE’s relying on the Epstein-Glaser renormalisation in the works [DDRZ21] by Dappiaggi, Drago, Rinaldi & Zambotti and [BDR23] by Bonicelli, Dappiaggi & Rinaldi. However it seems that these authors work only at a perturbative level whereas our results are non-perturbative.

One remarkable feature of our approach is that we are able to renormalise Eq. (3.1) using universal counterterms – they do not depend on the metric on M . We emphasise that fact in the following statement where $\epsilon \in (0, 1/8)$ is a positive constant and $\xi_r \stackrel{\text{def}}{=} e^{-rP}\xi$ stands for a *space regularisation* of ξ by the heat operator – so ξ_r is still white in time. Set

$$\begin{aligned} a_r &\stackrel{\text{def}}{=} \frac{r^{-1/2}}{4\sqrt{2}\pi^{3/2}}, \\ b_r &\stackrel{\text{def}}{=} \frac{|\log r|}{32\pi^2}. \end{aligned} \tag{3.2}$$

Theorem 3.1 *Pick $\phi \in C^{-1/2-\epsilon}(M)$. The equation*

$$(\partial_t - \Delta + 1)u_r = \sqrt{2}\xi_r - u_r^3 + 3(a_r - b_r)u_r \tag{3.3}$$

with initial condition ϕ has a unique solution over $[0, \infty) \times M$ in some appropriate function space. For any $0 < T < \infty$ this random variable converges in probability in $C([0, T], C^{-1/2-\epsilon}(M))$ as $r > 0$ goes to 0 to a limit u .

The function u is what we define as the solution to Equation (3.1) and it turns out to be a Markov process. The a priori estimate encoded in the coming down from infinity property provides a compactness statement from which the existence of an invariant probability measure for the Markovian dynamics follows.

Theorem 3.2 *The dynamics of u is Markovian and its associated semigroup on $C^{-1/2-\epsilon}(M)$ has an invariant non-Gaussian probability measure.*

A Φ_3^4 measure over M is that invariant measure. Note that the constants a_r, b_r in (3.3) are universal in the sense that they do not depend on the Riemannian metric on M . They depend however on the regularisation scheme we use, here the heat regularisation in space which is *fully covariant* with respect to the Riemannian structure.

The study of Equation (3.1) on an arbitrary fixed time interval $[0, T]$ is the object of Section 3.2. We follow [JP23] which yields a robust formulation of Equation (3.1) which avoids the use of regularity structures or paracontrolled calculus. The local in time well-posedness of (3.1) is proved in Section 3.2.1. We get the longtime existence from the r -uniform L^p ‘coming down from infinity’ property satisfied by u_r , proved in Section 3.2.2. This property is proved in the formulation of Equation (3.1) as in [JP23]. The results of Section 3.2 show that u_r depends continuously on a finite family $\hat{\xi}_r$ of multilinear functionals of ξ_r . The functional setting needed to prove their convergence in some appropriate space as r goes to 0 is detailed in Section 3.3. A crucial role is played here by a set of distributions with given wavefront sets and a certain scaling property with respect to some submanifolds. The notion of scaling field is introduced in Section 3.3.1 and the preceding set of distributions is introduced in Section 3.3.2. We prove our main workhorse in Section 3.3.3, Theorem 3.19. It provides a numerical criterion for a distribution defined outside a submanifold, with some wavefront set bound and some scaling property with respect to that submanifold, to have a possibly unique extension to the whole manifold. We draw consequences of this general statement for the particular case of a configuration space in Section 3.4.1. The relevance of Theorem 3.19 to the convergence problem for $\hat{\xi}_r$ stems from the fact that we can formulate the latter as a quantitative extension problem. This point of view is inspired by the Epstein-Glaser approach of renormalisation. The convergence of $\hat{\xi}_r$ in an appropriate space is the object of Section 3.5. At that point of the analysis of Equation (3.1) one can make sense of its $r = 0$ version as a Markovian Feller dynamics in the state space $C^{-1/2-\epsilon}(M)$, for any fixed sufficiently small $\epsilon > 0$. The r -uniform L^p coming down property from Section 3.2.2 is used to get the existence of an invariant probability measure for this dynamics. The non-triviality of this invariant probability measure is proved in Section 3.6.2. To keep the presentation relatively short, the detailed proofs of a number of basic tools from microlocal and harmonic analysis are given in [BDFT23]. They are mainly put to work in our analysis of the convergence of the enhanced noise $\hat{\xi}_r$ in Section 3.5.

Notation 3.3 Given $0 < T < \infty$ and a Banach space B we write $C_T B$ for $C([0, T], B)$. The parabolic Besov-Hölder spaces on space time are denoted by $C^\gamma((a, b) \times M)$, the Besov spaces on the manifold M are denoted by $B_{p,q}^a(M)$, these spaces being defined in Definition 3.43. The cotangent space to M is denoted by T^*M , the conormal to a submanifold E of M is denoted by $N^*(E)$, and we denote by $dv(x)$ the volume form on M . Throughout this chapter we are given a finite cover of M with some open charts $M = \cup_{i \in I} U_i$, where I is a fixed finite set. By localising on these charts, we define in Appendix 3.G using the flat Laplacian some Littlewood-Paley blocks $(P_j^i, \tilde{P}_j^i)_{j \geq -1}^{i \in I}$ on M – see (3.62), along with some paraproducts and resonant term $(\prec_i, \succ_i, \odot_i)_{i \in I}$ such that for any smooth functions $a, b \in \mathcal{D}(M)$ one has

$$ab = \sum_{i \in I} (a \prec_i b + a \odot_i b + a \succ_i b) = \sum_{i \in I} \left(\sum_{j \leq k-1} P_j^i a \tilde{P}_k^i b + \sum_{|j-k| \leq 1} P_j^i a \tilde{P}_k^i b + \sum_{j \geq k+1} P_j^i a \tilde{P}_k^i b \right).$$

Remark 3.4 The properties of the Littlewood-Paley decomposition defined in (3.62) and of the subsequent paraproducts along with the bounds on the kernels stated in Lemma 3.50 are proven in the work [BDFT23]. These proofs are mostly technical, and the results do not differ from what can be proven in the flat case using Fourier analysis.

3.2 Long time well-posedness and a priori estimate

We prove the existence of a unique solution to equation (3.1) over any fixed time interval $[0, T]$, for an arbitrary initial condition in $C^{-1/2-\epsilon}(M)$, for $\epsilon > 0$ small enough. We adopt here the robust approach

of [JP23]. They use a clever change of variable to reformulate the equation as a non-singular parabolic partial differential equation (3.7) with random coefficients of regularity no worse than $-1/2 - \epsilon$. This allows to solve the equation locally in time by a fixed point argument set in a classical functional space without resorting to regularity structures or paracontrolled calculus. Section 3.2.2 is dedicated to proving an L^p estimate on the solution to equation (3.7) that is independent of the initial condition. This plays a crucial role in proving the existence of an invariant probability measure for (3.1) by an argument using compactness.

3.2.1 Local in time well-posedness

This section is dedicated to proving the local well-posedness of a solution to equation (3.1), uniformly over $r > 0$. In [JP23], the authors noticed that a clever reformulation of the equation brings its study back to the study of a nonsingular stochastic PDE for which local in time well-posedness follows from an elementary fixed point argument.

Some distributions in the list (3.4) below involve an operator \odot , called *resonant operator*, that we introduce formally in Appendix 3.G; its precise definition here does not matter other than the fact that it is well-defined and continuous from $B_{p_1, q_1}^{\alpha_1}(M) \times B_{p_2, q_2}^{\alpha_2}(M)$ into some Besov space if and only if $\alpha_1 + \alpha_2 > 0$, in which case it takes values in $B_{p, q}^{\alpha_1 + \alpha_2}(M)$, for some integrability exponents p, q whose precise value does not matter here. For $\Lambda_1 \in B_{p_1, q_1}^{\alpha_1}(M)$ and $\Lambda_2 \in B_{p_2, q_2}^{\alpha_2}(M)$, the product $\Lambda_1 \Lambda_2$ is well-defined if and only if $\Lambda_1 \odot \Lambda_2$ is well-defined.

3.2.1.1 The enhanced noise

We regularise ξ in space (only) using the heat kernel and set

$$\xi_r \stackrel{\text{def}}{=} e^{-rP}(\xi);$$

this is a white noise in time with values in a space of regular functions. The fact that ξ appears in an additive form in (3.1) does not make it necessary to regularise it in time. Regularising ξ only in space makes clear the Markovian character of the renormalised equation (3.3). Denote by \mathcal{L}^{-1} , respectively $\underline{\mathcal{L}}^{-1}$, the resolvent operator of $\partial_t + P$ with null initial condition at time 0, respectively at time $-\infty$. Explicitly, $\mathcal{L}^{-1}f(t, \cdot) = \int_0^t e^{-(t-s)P} f(s, \cdot) ds$ and $\underline{\mathcal{L}}^{-1}f(t) = \int_{-\infty}^t e^{-(t-s)P} f(s, \cdot) ds$ for any function f on $\mathbf{R} \times M$. The operator $\underline{\mathcal{L}}^{-1}$ provides stationary solutions. Recall from (3.2) the definitions of the constants a_r and b_r and set

$$\mathfrak{I}_r \stackrel{\text{def}}{=} \underline{\mathcal{L}}^{-1}(\xi_r)$$

and

$$\mathfrak{V}_r \stackrel{\text{def}}{=} (\mathfrak{I}_r)^2 - a_r, \quad \mathfrak{Y}_r \stackrel{\text{def}}{=} \underline{\mathcal{L}}^{-1}(\mathfrak{V}_r)$$

$$\mathfrak{W}_r \stackrel{\text{def}}{=} (\mathfrak{I}_r)^3 - 3a_r \mathfrak{I}_r, \quad \mathfrak{Z}_r \stackrel{\text{def}}{=} \underline{\mathcal{L}}^{-1}(\mathfrak{W}_r)$$

and

$$\widehat{\xi}_r \stackrel{\text{def}}{=} \left(\xi_r, \mathfrak{V}_r, \mathfrak{W}_r, \mathfrak{Z}_r \odot \mathfrak{I}_r, \mathfrak{Y}_r \odot \mathfrak{V}_r - \frac{b_r}{3}, |\nabla \mathfrak{Y}_r|^2 - \frac{b_r}{3}, \mathfrak{Z}_r \odot \mathfrak{V}_r - b_r \mathfrak{I}_r \right). \quad (3.4)$$

One has $\xi_r \in \mathcal{C}^{-5/2-\epsilon}([0, T] \times M)$ and the restriction to any time interval $[0, T]$ of the other components of $\widehat{\xi}_r$ is seen as an element of the product space

$$C_T C^{-1-2\epsilon}(M) \times C^{-3/2-3\epsilon}([0, T] \times M) \times C^{-4\epsilon}([0, T] \times M)^3 \times C^{-1/2-5\epsilon}([0, T] \times M). \quad (3.5)$$

Note that only \mathfrak{V}_r is an element of a space of the form $C_T C^\gamma(M)$, the other terms in $\widehat{\xi}_r$ are elements of a parabolic space of negative regularity, which is less precise than being an element in a space of the form $C_T C^\gamma(M)$ for a negative exponent γ . This is sufficient for our needs. The enhancement ξ_r can be seen as

a placeholder for a number of products that are not well-defined in the zero regularisation limit. We will see in Section 3.5 that $\widehat{\xi}_r$ converges in all the $L^p(\Omega)$ spaces, $1 \leq p < \infty$, as $r > 0$ goes to 0, to a limit that does not depend on the mollification used to define ξ_r from ξ . Using the operator $\underline{\mathcal{L}}^{-1}$ rather than the operator \mathcal{L}^{-1} in the definitions of \mathfrak{Y}_r and $\mathfrak{Y}_r^{\circ\circ}$ builds some random distributions that are stationary in time. This property will be useful in Section 3.6 to get a compactness statement on the family of laws of the solutions to (3.3).

3.2.1.2 Jagannath & Perkowski's formulation of equation (3.3).

Set

$$v_{r,\text{ref}} \stackrel{\text{def}}{=} 3 \underline{\mathcal{L}}^{-1} \left(e^{3\mathfrak{Y}_r} \left\{ \mathfrak{Y}_r^{\circ\circ} \mathfrak{V}_r - b_r(\mathfrak{I}_r + \mathfrak{Y}_r) \right\} \right).$$

This is an element of $C_T C^{1-\epsilon}(M)$. The starting point for the analysis of the renormalized form (3.3) of equation (3.1) is that u_r is a solution to (3.3) if and only if

$$v_r \stackrel{\text{def}}{=} e^{3\mathfrak{Y}_r} \left(u_r - \mathfrak{I}_r + \mathfrak{Y}_r \right) - v_{r,\text{ref}} \quad (3.6)$$

is a solution of a particular equation of the form

$$(\partial_t + P)v_r = -6\nabla\mathfrak{Y}_r \cdot \nabla v_r - e^{-6\mathfrak{Y}_r} v_r^3 + Z_{2,r} v_r^2 + Z_{1,r} v_r + Z_{0,r}, \quad (3.7)$$

where $Z_{2,r}, Z_{1,r}, Z_{0,r}$ are elements of $C_T C^{-1/2-\eta}(M)$, for all $\eta > 0$, that depend continuously on $\widehat{\xi}_r$ – see equation (2.4) in [JP23] (We deduce the regularity properties of the Z_i from the fact that \mathcal{L}^{-1} sends continuously $C^\gamma([0, T] \times M)$ into $C_T C^{\gamma+2}(M)$ when $-2 < \gamma < 0$).

We now solve equation (3.7) with an arbitrary initial condition in $C^{-1/2-\epsilon}(M)$ – [JP23] only considered the case of an initial condition that differs from $\mathfrak{I}_r(0)$ by an element of $C^{3/2-\epsilon}(M)$. For that purpose, and for exponents $\alpha > 0, \beta \in \mathbf{R}$, we introduce the spaces (α, β) made up of all functions $v \in C((0, T], C^\beta(M))$ such that

$$t^\alpha \|u(t)\|_{L^\infty} \xrightarrow[t \downarrow 0]{} 0$$

and

$$\|v\|_{(\alpha, \beta)} \stackrel{\text{def}}{=} \max \left\{ \sup_{0 < t \leq T} t^\alpha \|v(t)\|_{C^\beta}, \sup_{0 \leq t \neq s \leq T} \frac{\|t^\alpha v(t) - s^\alpha v(s)\|_{L^\infty}}{|s - t|^{\beta/2}} \right\} < \infty.$$

(The use of such weighted spaces is suggested in [JP23]; we use here the same spaces as in Section 6 of Gubinelli & Perkowski's work [GPI6].) The free propagation map

$$(\mathcal{F}a)(t) \stackrel{\text{def}}{=} e^{-tP} a$$

sends for instance $C^\beta(M)$ into $(\alpha, \beta + 2\alpha)$, for all $\beta \in \{\mathbf{R} \setminus \mathbf{N}\}$ and $\alpha > 0$, and one has for all $0 \leq \delta < \min(\beta, 2\alpha)$

$$\|\mathcal{L}^{-1}(f)\|_{(\alpha-\delta/2, \beta-\delta)} \lesssim \|f\|_{(\alpha, \beta-2)} \quad (3.8)$$

This inequality allows to trade some explosion rate against some regularity. We also have for the same range of exponents and all $f \in (\alpha, \beta - 2)$

$$\|f\|_{(\alpha-\delta/2, \beta-\delta)} \lesssim \|f\|_{(\alpha, \beta)}. \quad (3.9)$$

These statements correspond in our setting to Lemma 6.6 and Lemma 6.8 in Gubinelli & Perkowski's work [GPI6] – a proof is given in [BDFT23, Lemma 2.3]. Note that since the different components of the enhanced noise are stationary they do not take value 0 at time 0. The initial condition for v_r is thus

different from the initial condition for u_r . We keep the notation ϕ for the initial condition for u_r and write ϕ' for the initial condition for v_r . We will repeatedly use the estimate

$$\|fg\|_{C^{\alpha\wedge\beta}} \leq \|f\|_{C^\alpha} \|g\|_{C^\beta}, \quad (3.10)$$

if $\alpha + \beta > 0$ which follows immediately from Proposition 3.61

Proposition 3.5 *Pick $\epsilon' = 4\epsilon$ and set $\alpha_0 \stackrel{\text{def}}{=} 3/4 + (\epsilon + \epsilon')/2$. For any $\phi' \in C^{-1/2-\epsilon}(M)$ there exists a positive time T^* such that for all $0 < T < T^*$ equation (3.7) has a unique solution*

$$v_r \in C_T C^{-1/2-\epsilon}(M) \cap \langle \alpha_0, 1 + \epsilon' \rangle$$

with initial condition ϕ' . This solution depends continuously on $\widehat{\xi}_r$ and $\phi' \in C^{-1/2-\epsilon}(M)$, and for any small positive λ these exist $T_\lambda \in (\lambda, T^*)$ such that $u \in C([\lambda, T_\lambda], C^{3/2-4\epsilon}(M))$.

Proof. First, remark that $\lim_{t \downarrow 0} t^{\alpha_0} \mathcal{F}(\phi') = 0$, since $\|\mathcal{F}(\phi')\|_{L^\infty} \leq t^{-1/4-\epsilon/2} \|\phi'\|_{-1/2-\epsilon}$, so $\mathcal{F}(\phi') \in C_T C^{-1/2-\epsilon}(M) \cap \langle \alpha_0, 1 + \epsilon' \rangle$. We use a standard Picard iteration argument for the map

$$F(v) \stackrel{\text{def}}{=} \mathcal{F}(\phi') + \mathcal{L}^{-1} \left(-6\nabla \mathfrak{Y}_r \cdot \nabla v - e^{-6\mathfrak{Y}_r} v^3 + Z_{2,r} v^2 + Z_{1,r} v + Z_{0,r} \right). \quad (3.11)$$

Denote B_R the ball of radius $R = 4\|\phi'\|_{C^{-1/2-\epsilon}}$ in $C_T C^{-1/2-\epsilon}(M) \cap \langle \alpha_0, 1 + \epsilon' \rangle$. Let $v_1, v_2 \in B_R$.

Our first goal is to get a bound of the form

$$\|F(v_2) - F(v_1)\|_{\langle \alpha_0, 1 + \epsilon' \rangle} \lesssim_{\widehat{\xi}_r} T^{\epsilon/2} (R + R^2) \|v_2 - v_1\|_{\langle \alpha_0, 1 + \epsilon' \rangle}.$$

meaning F is a contraction for the $\langle \alpha_0, 1 + \epsilon' \rangle$ norm by choosing T small enough. We have

$$\begin{aligned} \|F(v_2) - F(v_1)\|_{\langle \alpha_0, 1 + \epsilon' \rangle} &\leq \|\mathcal{L}^{-1}(6\nabla \mathfrak{Y}_r \cdot \nabla(v_2 - v_1))\|_{\langle \alpha_0, 1 + \epsilon' \rangle} + \|\mathcal{L}^{-1}(e^{-6\mathfrak{Y}_r}(v_2^3 - v_1^3))\|_{\langle \alpha_0, 1 + \epsilon' \rangle} \\ &\quad + \|\mathcal{L}^{-1}(Z_{2,r}(v_2^2 - v_1^2))\|_{\langle \alpha_0, 1 + \epsilon' \rangle} + \|\mathcal{L}^{-1}(Z_{1,r}(v_2 - v_1))\|_{\langle \alpha_0, 1 + \epsilon' \rangle}. \end{aligned}$$

Since $\nabla \mathfrak{Y}_r \in C_T C^{-\eta}(M)$ for all $\eta > 0$, we first use the estimate (3.8) with $\delta = 0$ and (3.10) to get

$$\begin{aligned} \|\mathcal{L}^{-1}(\nabla \mathfrak{Y}_r \cdot \nabla(v_2 - v_1))\|_{\langle \alpha_0, 1 + \epsilon' \rangle} &\lesssim \sup_{0 < t \leq T} t^{\alpha_0} \|(\nabla \mathfrak{Y}_r \cdot \nabla(v_2 - v_1))\|_{C^{-\eta}} \\ &\lesssim_{\widehat{\xi}_r} \sup_{0 < t \leq T} t^{\alpha_0} \|\nabla(v_2 - v_1)\|_{C^{2\eta}} \\ &\lesssim T^{\alpha_0 - \alpha_1} \|v_2 - v_1\|_{\langle \alpha_1, 1 + 2\eta \rangle}, \quad (\alpha_1 = 3/4 + (\epsilon + 2\eta)/2) \\ &\lesssim T^{\epsilon'/2 - \eta} \|v_2 - v_1\|_{\langle \alpha_1, 1 + 2\eta \rangle}, \quad (\eta = \epsilon'/4) \\ &\lesssim T^{\epsilon'/4} \|v_2 - v_1\|_{\langle \alpha_0, 1 + \epsilon' \rangle}, \quad (\text{by (3.9) with } \delta = \epsilon' - 2\eta). \end{aligned}$$

Now using again (3.8), (3.9), (3.10) and the fact that $\exp(-6\mathfrak{Y}_r) \in C^{1-\eta}(M)$ for all $\eta > 0$, we have, for $\epsilon' = 4\epsilon$

$$\begin{aligned} \|\mathcal{L}^{-1}(\exp(-6\mathfrak{Y}_r)(v_2^3 - v_1^3))\|_{\langle \alpha_0, 1 + \epsilon' \rangle} &\lesssim_{\widehat{\xi}_r} \sup_{0 < t \leq T} t^{\alpha_0} \|(v_2 - v_1)(v_2^2 + v_1^2 + v_2 v_1)(t)\|_{C^\eta} \\ &\lesssim T^{\alpha_0 - 3\alpha'_0} \|v_2 - v_1\|_{\langle \alpha'_0, \eta \rangle} (\|v_2\|_{\langle \alpha'_0, \eta \rangle}^2 + \|v_1\|_{\langle \alpha'_0, \eta \rangle}^2 + \|v_2\|_{\langle \alpha'_0, \eta \rangle} \|v_1\|_{\langle \alpha'_0, \eta \rangle}), \quad (\alpha'_0 = 1/4 + (\epsilon + \eta)/2) \\ &\lesssim T^{(\epsilon' - 2\epsilon - 3\eta)/2} \|v_2 - v_1\|_{\langle \alpha'_0, \eta \rangle} (\|v_2\|_{\langle \alpha'_0, \eta \rangle}^2 + \|v_1\|_{\langle \alpha'_0, \eta \rangle}^2 + \|v_2\|_{\langle \alpha'_0, \eta \rangle} \|v_1\|_{\langle \alpha'_0, \eta \rangle}) \\ &\lesssim T^{\epsilon/2} \|v_2 - v_1\|_{\langle \alpha_0, 1 + \epsilon' \rangle} (\|v_2\|_{\langle \alpha_0, 1 + \epsilon' \rangle} + \|v_1\|_{\langle \alpha_0, 1 + \epsilon' \rangle}^2 + \|v_2\|_{\langle \alpha_0, 1 + \epsilon' \rangle} \|v_1\|_{\langle \alpha_0, 1 + \epsilon' \rangle}), \end{aligned}$$

choosing $\eta = \epsilon/3$ and using (3.9) in the last inequality. Next, with the same argument we have for $\epsilon' = 4\epsilon$, and setting $\alpha'_0 = \frac{1}{2} + (\epsilon + \eta)/2$, $\alpha''_0 = \frac{1}{4} + (\epsilon + \eta)/2$ in the third inequality

$$\begin{aligned}
\|\mathcal{L}^{-1}(Z_{2,r}(v_2^2 - v_1^2))\|_{(\alpha_0, 1+\epsilon')} &\lesssim_{\widehat{\xi}_r} \sup_{0 < t \leq T} t^{\alpha_0} \|(v_2^2 - v_1^2)(t)\|_{C^{1/2+\eta}} \\
&\lesssim \sup_{0 < t \leq T} t^{\alpha_0} \|v_2(t) - v_1(t)\|_{C^{1/2+\eta}} (\|v_2(t)\|_{\eta} + \|v_1(t)\|_{\eta}) \\
&\leq T^{\alpha_0 - \alpha'_0 - \alpha''_0} \|v_2 - v_1\|_{(\alpha'_0, 1/2+\eta)} (\|v_2\|_{(\alpha''_0, \eta)} + \|v_1\|_{(\alpha''_0, \eta)}) \\
&\leq T^{(\epsilon' - 2\epsilon - 2\eta)/2} \|v_2 - v_1\|_{(\alpha_0, 1+\epsilon')} (\|v_2\|_{(\alpha_0, 1+\epsilon')} + \|v_1\|_{(\alpha_0, 1+\epsilon')}) \\
&\leq T^{\epsilon/2} \|v_2 - v_1\|_{(\alpha_0, 1+\epsilon')} (\|v_2\|_{(\alpha_0, 1+\epsilon')} + \|v_1\|_{(\alpha_0, 1+\epsilon')}),
\end{aligned}$$

using (3.9) in the fourth inequality and choosing $\eta = \epsilon/2$ in the last inequality. Similarly, we get

$$\|\mathcal{L}^{-1}(Z_{1,r}(v_2 - v_1))\|_{(\alpha_0, 1+\epsilon')} \lesssim_{\widehat{\xi}_r} T^{\epsilon/2} \|v_2 - v_1\|_{(\alpha_0, 1+\epsilon')}.$$

Therefore

$$\|F(v_2) - F(v_1)\|_{(\alpha_0, 1+\epsilon')} \lesssim_{\widehat{\xi}_r} T^{\epsilon/2} (R + R^2) \|v_2 - v_1\|_{(\alpha_0, 1+\epsilon')}.$$

The rest is to estimate $F(v_1) - F(v_2)$ in $C_T C^{-1/2-\epsilon}$. Since $v_1, v_2 \in B_R$, for any $s \in [0, T]$, $i = 1, 2$ and $0 \leq \delta < 1 + \epsilon'$, it follows from (3.9) that $\|v_i\|_{(\alpha_0 - \delta/2, 1+\epsilon' - \delta)} \leq R$, hence we have

$$\|v_i(s)\|_{C^{-1/2-\epsilon}} \leq R \text{ and } \|v_i\|_{C^{1+\epsilon' - \delta}} \leq R s^{-(\alpha_0 - \delta/2)}. \quad (3.12)$$

Recall the following estimate which is used many times below: if $\|u(s)\|_{C^\beta} \lesssim s^{-\gamma}$, for $\gamma < 1$, then

$$\|\mathcal{L}^{-1}u\|_{C^\beta} \lesssim \int_0^T s^{-\gamma} \lesssim T^{1-\gamma}. \quad (3.13)$$

Using (3.12) and the fact that $\nabla^{\mathfrak{Y}_r} \in C_T C^{-\epsilon'/4}(M)$, we have

$$\|\nabla^{\mathfrak{Y}_r} \cdot \nabla(v_2 - v_1)(s)\|_{C^{-1/2-\epsilon}} \lesssim_{\widehat{\xi}_r} \|\nabla(v_2 - v_1)(s)\|_{C^{\epsilon'}} \lesssim \|v_2 - v_1\|_{C^{1+\epsilon'}} \lesssim s^{-\alpha_0} \|v_2 - v_1\|_{(\alpha_0, 1+\epsilon')},$$

hence by (3.13)

$$\|\mathcal{L}^{-1}(\nabla^{\mathfrak{Y}_r} \cdot \nabla(v_2 - v_1))\|_{C^{-1/2-\epsilon}} \lesssim_{\widehat{\xi}_r} T^{1-\alpha_0} \|v_2 - v_1\|_{(\alpha_0, 1+\epsilon')} = T^{1/4-\epsilon/2-\epsilon'/2} \|v_2 - v_1\|_{(\alpha_0, 1+\epsilon')}.$$

Again, by (3.12) and the fact that $\mathfrak{Y}_r \in C_T C^{1-\eta}(M)$, and $\|uv\|_{C^\beta} \leq \|u\|_{L^\infty} \|v\|_{C^\beta} + \|v\|_{L^\infty} \|u\|_{C^\beta}$, for $\beta \in (0, 1)$,

$$\begin{aligned}
\|(\exp(-6^{\mathfrak{Y}_r})(v_1^3 - v_2^3))(s)\|_{-1/2-\epsilon} &\lesssim_{\widehat{\xi}_r} \|(v_2 - v_1)(v_2^2 + v_1^2 + v_2 v_1)(t)\|_{C^{-1/2-\epsilon}} \\
&\lesssim \|v_2 - v_1\|_{C^{-1/2-\epsilon}} \|v_2^2 + v_1^2 + v_2 v_1\|_{C^{1/2+2\epsilon}} \\
&\lesssim \|v_2 - v_1\|_{C^{-1/2-\epsilon}} \sum_{i,j=1}^2 \|v_i\|_{C^\epsilon} \|v_j\|_{C^{1/2+2\epsilon}} \\
&\lesssim \|v_2 - v_1\|_{C^{-1/2-\epsilon}} s^{-3/4-5\epsilon/2} R^2
\end{aligned}$$

where we use (3.12) for both $\|v_i\|_{C^\epsilon}$ and $\|v_j\|_{C^{1/2+2\epsilon}}$ in the last inequality. Hence by (3.13) we have

$$\|\mathcal{L}^{-1}((\exp(-6^{\mathfrak{Y}_r})(v_1^3 - v_2^3)))\|_{-1/2-\epsilon} \lesssim_{\widehat{\xi}_r} T^{1/4-5\epsilon/2} R^2 \|v_2 - v_1\|_{C^{-1/2-\epsilon}}.$$

Now, with the same argument and (3.9), for $\eta > \epsilon$, we have

$$\begin{aligned}
\|(Z_{2,r}(v_2^2 - v_1^2))(s)\|_{C^{-1/2-\epsilon}} &\lesssim_{\widehat{\xi}_r} \|(v_2^2 - v_1^2)(s)\|_{C^{1/2+\eta}} \\
&\lesssim (\|v_1\|_{C^\eta} + \|v_2\|_{C^\eta}) \|v_1 - v_2\|_{1/2+\eta}
\end{aligned}$$

$$\begin{aligned} &\lesssim s^{-(\alpha_0-(1+\epsilon'-\eta)/2)} R s^{-1/2-\epsilon/2-\eta/2} \|v_1 - v_2\|_{(\frac{1}{2}+\frac{\epsilon+\eta}{2}, \frac{1}{2}+\eta)} \\ &\lesssim s^{-(3/4+\epsilon+\eta)} R \|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')}, \end{aligned}$$

hence, choosing $\eta = 2\epsilon$ yields

$$\|\mathcal{L}^{-1}(Z_{2,r}(v_2^2 - v_1^2))\|_{-1/2-\epsilon} \lesssim_{\widehat{\xi}_r} T^{1/4-3\epsilon} R \|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')}.$$

Similarly, we get

$$\|\mathcal{L}^{-1}(Z_{1,r}(v_2 - v_1))\|_{C^{-1/2-\epsilon}} \lesssim_{\widehat{\xi}_r} T^{1/2-3\epsilon/2} \|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')}.$$

Therefore

$$\begin{aligned} \|F(v_2) - F(v_1)\|_{C^{-1/2-\epsilon}} &\lesssim_{\widehat{\xi}_r} T^{1/4-5\epsilon/2} R^2 \|v_2 - v_1\|_{C^{-1/2-\epsilon}} \\ &\quad + (T^{1/4-\epsilon/2-\epsilon'/2} + T^{1/4-3\epsilon} R + T^{1/2-3\epsilon/2}) \|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')}. \end{aligned}$$

Combining the estimates above we infer that for $T > 0$ sufficiently small, depending on $\|\phi'\|_{C^{-1/2-\epsilon}}$, $\widehat{\xi}_r$, the map F is a contraction on the ball of radius $4\|\phi'\|_{C^{-1/2-\epsilon}}$ in $C_T C^{-1/2-\epsilon}(M) \cap (\alpha_0, 1+\epsilon')$. The unique fixed point is our solution on $[0, T]$. Taking the supremum of all such T gives the maximal existence time T^* . Once we know that v takes values in $(\alpha_0, 1+\epsilon')$ we can restart the fixed point procedure from a positive time, with an initial condition that is now of Hölder regularity $(1+\epsilon')$. It is elementary to adapt the preceding estimates to see that now the solution will take values in $C^{3/2-4\epsilon}(M)$.

For the continuous dependence on $\widehat{\xi}_r$ and the initial data, we define

$$F(\widehat{\xi}_r, \phi, v) \stackrel{\text{def}}{=} e^{-tP}(\phi) + \mathcal{L}^{-1}\left(-6\nabla^{\mathfrak{P}} \mathfrak{P}_r(\xi_r) \cdot \nabla v - e^{-6\mathfrak{P}} \mathfrak{P}_r(\xi_r) v^3 + Z_{2,r}(\xi_r) v^2 + Z_{1,r}(\xi_r) v + Z_{0,r}(\xi_r)\right).$$

Let $K > 0$ be a uniform constant satisfying

$$\|e^{-tP} \phi\|_{C^{-1/2-\epsilon}} \leq K \|\phi\|_{C^{-1/2-\epsilon}} \quad \text{and} \quad \|e^{-tP} \phi\|_{(\alpha_0, 1+\epsilon')} \leq K \|\phi\|_{C^{-1/2-\epsilon}}.$$

Take the ball B_R in $C^{-1/2-\epsilon}(M)$. Since F depends linearly on $\widehat{\xi}_r$ and $\exp(-6\mathfrak{P}_r)$, by the same arguments above, for any $\phi \in B_R$ and we can choose $T = T(R, \widehat{\xi}_r, \widehat{\xi}_r')$ small enough such that $C(T) < 1/2$ and

$$\|F(\widehat{\xi}_r, \phi, v_1) - F(\widehat{\xi}_r', \phi, v_2)\|_{C_T C^{1/2-\epsilon}} \leq C(T) (\|v_1 - v_2\|_{C_T C^{1/2-\epsilon}} + \|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')} + \|\widehat{\xi}_r - \widehat{\xi}_r'\|)$$

and

$$\|F(\widehat{\xi}_r, \phi, v_1) - F(\widehat{\xi}_r', \phi, v_2)\|_{(\alpha_0, 1+\epsilon')} \leq C(T) (\|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')} + \|\widehat{\xi}_r - \widehat{\xi}_r'\|).$$

Now for $\phi_1, \phi_2 \in B_R$ we have

$$\begin{aligned} &\|v_r(\widehat{\xi}_r, \phi_1) - v_r(\widehat{\xi}_r', \phi_2)\|_{C_T C^{1/2-\epsilon}} = \|F(\widehat{\xi}_r, \phi_1, v_r(\phi_1)) - F(\widehat{\xi}_r', \phi_2, v_r(\phi_2))\|_{C_T C^{1/2-\epsilon}} \\ &\leq \|F(\widehat{\xi}_r, \phi_1, v_r(\phi_1)) - F(\widehat{\xi}_r, \phi_2, v_r(\phi_1))\|_{C_T C^{1/2-\epsilon}} + \|F(\widehat{\xi}_r, \phi_2, v_r(\phi_1)) - F(\widehat{\xi}_r', \phi_2, v_r(\phi_2))\|_{C_T C^{1/2-\epsilon}} \\ &\leq K \|\phi_1 - \phi_2\|_{C^{-1/2-\epsilon}} + C(T) \left(\|v_r(\phi_1) - v_r(\phi_2)\|_{C_T C^{1/2-\epsilon}} + \|v_r(\phi_1) - v_r(\phi_2)\|_{(\alpha_0, 1+\epsilon')} + \|\widehat{\xi}_r - \widehat{\xi}_r'\| \right). \end{aligned}$$

Similarly we have

$$\begin{aligned} \|v_r(\phi_1) - v_r(\phi_2)\|_{(\alpha_0, 1+\epsilon')} &= \|F(\widehat{\xi}_r, \phi_1, v_r(\phi_1)) - F(\widehat{\xi}_r', \phi_2, v_r(\phi_2))\|_{(\alpha_0, 1+\epsilon')} \\ &\leq K \|\phi_1 - \phi_2\|_{C^{-1/2-\epsilon}} + C(T) (\|v_r(\phi_1) - v_r(\phi_2)\|_{(\alpha_0, 1+\epsilon')} + \|\widehat{\xi}_r - \widehat{\xi}_r'\|), \end{aligned}$$

so

$$\|v_r(\phi_1) - v_r(\phi_2)\|_{C_T C^{1/2-\epsilon}} + \|v_r(\phi_1) - v_r(\phi_2)\|_{(\alpha_0, 1+\epsilon')}$$

$$\begin{aligned} &\leq 2K\|\phi_1 - \phi_2\|_{C_T C^{1/2-\epsilon}} + C(T)\|v_r(\phi_1) - v_r(\phi_2)\|_{C_T C^{-1/2-\epsilon}} \\ &\quad + 2C(T)\|v_r(\phi_1) - v_r(\phi_2)\|_{(\alpha_0, 1+\epsilon')} + 2\|\widehat{\xi}_r - \widehat{\xi}'_r\|, \end{aligned}$$

and we read on the estimate

$$\begin{aligned} &\|v_r(\phi_1) - v_r(\phi_2)\|_{C_T C^{1/2-\epsilon}} + \|v_r(\phi_1) - v_r(\phi_2)\|_{(\alpha_0, 1+\epsilon')} \\ &\leq \frac{1}{1 - 2C(T)} \left(2K\|\phi_1 - \phi_2\|_{C^{-1/2-\epsilon}} + 2C(T)\|\widehat{\xi}_r - \widehat{\xi}'_r\| \right) \end{aligned}$$

the continuous dependence of the solution on $\widehat{\xi}$ and the initial data. \square

We see from the proof that T^* depends only on $\widehat{\xi}_r$ and $\phi' \in C^{-1/2-\epsilon}(M)$. The following additional piece of information will be useful when proving the coming down from infinity property by energy methods in the next section.

Lemma 3.6 *Let $0 < t_0 < t_1$. For $\beta = 3/2 - \epsilon$ and any $\kappa \leq \beta/2$, then $t \mapsto v_r(t, x)$ is κ -Hölder continuous as a function from $[t_0, t_1]$ to L^∞ .*

Proof. By the change of variable $t \mapsto t - t_0$, we can assume $t_0 = 0, t_1 = T > 0$ and $v_r \in C_T C^{3/2-\epsilon}(M)$. We now show the Hölder regularity of v_r at time 0, the adaptation to arbitrary times is straightforward. We have

$$v_r(t, \cdot) = e^{-tP} v_r(0) + \int_0^t e^{-(t-s)P} \left(-6\nabla\mathfrak{Y}_r \cdot \nabla v_r - e^{-6\mathfrak{Y}_r} v_r^3 + Z_{2,r} v_r^2 + Z_{1,r} v_r + Z_{0,r} \right) (s) ds.$$

We first remark that

$$|v_r(t, \cdot) - v_r(0)| \leq |e^{-tP} v_r(0) - v_r(0)| + \left| \int_0^t e^{-(t-s)P} (f_r(s)) ds \right|, \quad (3.14)$$

where $f_r = -6\nabla\mathfrak{Y}_r \cdot \nabla v_r - e^{-6\mathfrak{Y}_r} v_r^3 + Z_{2,r} v_r^2 + Z_{1,r} v_r + Z_{0,r}$. It follows from the time regularity of the heat flow that $\|(1 - e^{-tP})h\|_{L^\infty} \lesssim t^{\beta/2} \|h\|_{C^\beta}$ for $0 \leq \beta \leq 2$, hence

$$\|e^{-tP} v_r(0) - v_r(0)\|_{L^\infty} \lesssim t^{\beta/2} \|v_r(0)\|_{C^\beta}.$$

Since $v_r \in C_T C^\beta$, with $\beta = 3/2 - \epsilon$, $\nabla\mathfrak{Y}_r \in C_T C^{-\epsilon}$, $e^{-6\mathfrak{Y}_r} \in C_T C^{1-\epsilon}$, $\mathbf{Z}_r \stackrel{\text{def}}{=} \{Z_{0,r}, Z_{1,r}, Z_{2,r}\} \in (C_T C^{-\alpha})^3$ with $\alpha = 1/2 + \epsilon'$, using $\|gh\|_{C^{\alpha' \wedge \beta'}} \lesssim \|g\|_{C^{\alpha'}} \|h\|_{C^{\beta'}}$ for $\alpha' + \beta' > 0$ we have

$$\|f_r(s, \cdot)\|_{C^{-\alpha}} \lesssim \mathbf{Z}_r \|v_r\|_{C_T C^\beta} \lesssim 1.$$

Then the estimate $\|e^{-tP} h\|_{C^{\eta+\gamma}} \lesssim t^{-\gamma/2} \|h\|_{C^\eta}$, for $\gamma \geq 0$ implies

$$\left\| \int_0^t e^{-(t-s)P} (f_r(s)) ds \right\|_{C^{\epsilon'}} \leq \int_0^t (t-s)^{-(\alpha-\epsilon')/2} \|f_r(s)\|_{C^{-\alpha}} ds \lesssim t^{1-\alpha/2-\epsilon'/2}.$$

By choosing $\epsilon' \geq \epsilon/2$ we have $\kappa \stackrel{\text{def}}{=} 1 - \alpha/2 - \epsilon'/2 \leq \beta/2$, therefore u is κ -Hölder from $[0, T]$ to L^∞ . \square

As in Proposition 6.8 of Mourrat & Weber's work [\[MW17c\]](#) it follows from this property that the function

$$t \in (0, T] \mapsto \|v_r(t)\|_{L^p}^p$$

satisfies the equation

$$\begin{aligned} \frac{1}{p} (\|v_r(t)\|_{L^p}^p - \|v_r(s)\|_{L^p}^p) &= \int_s^t (v_r^{p-1}, \Delta v_r) - \int_s^t \left(\int_M e^{-6\mathfrak{Y}_r(s_1)} v_r^{p+2}(s_1) \right) ds_1 \\ &\quad + \int_s^t \left(\frac{1}{p} (-6\nabla\mathfrak{Y}_r, \nabla v_r^p) + (Z_2, v_r^{p+1}) + (Z_1, v_r^p) + (Z_0, v_r^{p-1}) \right). \end{aligned} \quad (3.15)$$

3.2.2 Long time existence and coming down from infinity

We show in this section that the superlinear attractive drift $-\exp(-6\mathfrak{Y}_r)v_r^3$ in equation (3.7) entails an a priori bound on the $L^p(M)$ norm of the solution away from the initial time that is independent of the initial condition. This bound entails the long time existence of the solution v_r to (3.7) and is the key to proving the existence of an invariant probability measure for the dynamics (3.1) via a compactness argument. This point will be developed in Section 3.6.

We rewrite equation (3.7) in the form

$$(\partial_t + P + B_r \nabla)v_r = -A_r v_r^3 + Z_{2,r} v_r^2 + Z_{1,r} v_r + Z_{0,r}, \quad (3.16)$$

with

$$B_r \stackrel{\text{def}}{=} 6\nabla\mathfrak{Y}_r \in C_T C^{-\eta}(M), \quad A_r \stackrel{\text{def}}{=} e^{-6\mathfrak{Y}_r} \in C_T C^{1-\eta}(M),$$

and $Z_{i,r} \in C_T C^{-1/2-\eta}(M)$, for all $\eta > 0$.

Theorem 3.7 *The solution $v_r(t) \in C^{3/2-\epsilon}(M)$ exists for all times $t > 0$. Pick an even integer $p \geq 8$. There is a random variable $C(p, \widehat{\xi}_r|_{[0,t]})$ that depends only on the restriction to the interval $[0, t]$ of $\widehat{\xi}_r$ such that one has*

$$\|v_r(t)\|_{L^p(M)} \leq C(p, \widehat{\xi}_r|_{[0,t]}) \max \left\{ \frac{1}{\sqrt{t}}, 1 \right\} \quad (3.17)$$

for all $t > 0$, independently of the initial condition $\phi' \in C^{-1/2-\epsilon}(M)$.

The upper bound in (3.17) is in particular independent of the initial condition in (3.16); this phenomenon is called *coming down from infinity*. We note for later use that keeping track of the implicit constants in the computations below gives an upper in (3.17) takes for $1 \leq t \leq 2$ the form

$$(1 + \|\widehat{\xi}_r\|)^\gamma \left(\exp(\gamma' \|\mathfrak{Y}_r\|_{L^\infty([0,2] \times M)}) + 1 \right) \quad (3.18)$$

for some positive constants $\gamma = \gamma(p)$, $\gamma' = \gamma'(p)$, up to a multiplicative constant. We denoted here by $\|\widehat{\xi}_r\|$ the norm of $\widehat{\xi}_r$ seen as an element of the product space where $\widehat{\xi}_r$ takes its values. We use a priori energy estimates to prove Theorem 3.7, following the strategy initiated by Mourrat & Weber in their proof of a similar result in [MW14], Theorem 7.1 therein. Gubinelli & Hofmanová also used energy estimates in their work [GH18] on the Φ_3^4 measure on \mathbf{R}^3 . See also the proof of Proposition 3.7 in the work [TW18] of Tsatsoulis & Weber for an implementation of that strategy in the 2-dimensional torus.

We will use in the remainder of this section the shorthand notation

$$B_p^\gamma(M) \stackrel{\text{def}}{=} B_{p,\infty}^\gamma(M)$$

for any $\gamma \in \mathbf{R}$ and $1 \leq p \leq \infty$. Set

$$F_r(t) \stackrel{\text{def}}{=} \|v_r(t)\|_{p+2}^p + \|v_r(t)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p}{3}}.$$

We prove below that one has for all $0 < T_0 \leq s < t \leq T < T^* \wedge 1$

$$\|v_r(t)\|_{L^p}^p + \int_s^t F_r(s_1)^{\frac{p+2}{p}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s). \quad (3.19)$$

This inequality shows that, all $0 < T_0 \leq s < t \leq T$

$$\int_s^t F_r(s_1)^{\frac{p+2}{p}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s).$$

It then follows from a modified version of Mourrat & Weber's comparison test recalled in Proposition 3.62 of Appendix 3.G that there is an integer $N \geq 1$ and sequence of times $T_0 = t_0 < t_1 < t_2 < \dots < t_N = T$ such that for all $n \in \{0, \dots, N-1\}$,

$$F_r(t_n) \lesssim_{\widehat{\xi}_r} 1 + t_{n+1}^{-\frac{p}{2}},$$

for an implicit constant that does not depend on T_0, T . Pick $t \in [T_0, T]$. There exists $n \in \{0, \dots, N-1\}$ such that $t \in [t_n, t_{n+1}]$. Moreover, by (3.19) with $s = t_n$, we have

$$\|v_r(t)\|_{L^p}^p \lesssim_{\widehat{\xi}_r} 1 + F_r(t_n) \lesssim_{\widehat{\xi}_r} 1 + t_{n+1}^{-\frac{p}{2}} \lesssim_{\widehat{\xi}_r} 1 + t^{-\frac{p}{2}}.$$

This bound holds for T_0 arbitrarily small and $T = 1$, and can be repeated on $[1, 2]$, etc, so that the uniform estimate (3.17) follows. Recall $p > 6$ so the space $L^p(M)$ is continuously embedded into the space $C^{-1/2-\epsilon}(M)$. Given that T^* depends only on the restriction to $[0, T^*]$ of $\widehat{\xi}_r$ and the initial condition $\phi' \in C^{-1/2-\epsilon}(M)$ the uniform estimate (3.17) and the continuous injection of $L^p(M)$ into $C^{-1/2-\epsilon}(M)$ imply we can extend the solution though $T^* \wedge 1$, hence $T^* > 1$. Then, we can repeat the same argument on the interval $[1, 2]$ and so on to get the long time existence of v_r .

On a technical level, our proof of Theorem 3.7 will only use the fractional Leibniz rule from Proposition 3.63 and the elementary interpolation result from Proposition 3.64, both recalled in Appendix 3.G. Last, recall Young inequality that gives the existence for any positive δ of a constant C_δ such that one has

$$ab \leq \delta a^{p'} + \delta^{-\frac{q'}{p'}} b^{q'},$$

for all positive a, b and exponent $1 < p' < \infty$ with conjugate exponent q' . The proof of (3.19) requires two intermediate results stated as lemmas.

Lemma 3.8 *For every $0 < s < t \leq T$, we have*

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} ds_1 \lesssim_{\widehat{\xi}_r} 1 + \|v_r(s)\|_{L^{p+2}}^p + \int_s^t \|v_r(s_1)\|_{B^{\frac{p+2}{3}, \frac{p+2}{3}}}^{p+2} ds_1. \quad (3.20)$$

Proof. Pairing the equation with v^{p-1} with respect to the L^2 scalar product yields identity (3.15). As A is positive and bounded below and $(v^{p-1}, -\Delta v)$ is positive, since p is an even integer greater than 4, we obtain

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r\|_{L^{p+2}}^{p+2} \lesssim_{\widehat{\xi}_r} \|v_r(s)\|_{L^p}^p + \int_s^t (B_r, \nabla v_r^p) + (Z_{2,r}, v_r^{p+1}) + (Z_{1,r}, v_r^p) + (Z_{0,r}, v_r^{p-1}), \quad (3.21)$$

where the implicit constant is $p \left(\exp(6\|\mathfrak{V}_r\|_{L^\infty([0,2] \times M)}) + 1 \right)$.

We bound the different terms in the right hand side of (3.21). Recall that since B_r is an element of $B_{\infty, \infty}^{-\epsilon'}(M)$ for all $\epsilon' > 0$ it is an element of $B_{1, \infty}^{-\epsilon}(M)$. By the fractional Leibniz rule from Proposition 3.63 and Young inequality we have for $|(B_r, \nabla v_r^p)|$, up to a $\widehat{\xi}_r$ -dependent multiplicative constant, the upper bound

$$\|\nabla v_r^p\|_{B_1^\epsilon} \lesssim \|v_r^p\|_{B_1^{1+\epsilon}} \lesssim \|v_r^{p-1}\|_{L^{\frac{p+2}{p-1}}} \|v_r\|_{B_1^{1+\epsilon}} \lesssim \|v_r\|_{L^{p+2}}^{p-1} \|v_r\|_{B_1^{1+\epsilon}} \lesssim \delta \|v_r\|_{L^{p+2}}^{p+2} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B_1^{1+\epsilon}}^{\frac{p+2}{3}},$$

where $\delta > 0$ is arbitrarily small. For the other terms, we have first

$$|(Z_{2,r}, v_r^{p+1})| \lesssim_{\widehat{\xi}_r} \|v_r^{p+1}\|_{B_1^{\frac{1+\epsilon}{2}}} \lesssim \|v_r^p\|_{L^{\frac{p+2}{p}}} \|v_r\|_{B_1^{\frac{1+\epsilon}{2}}} \lesssim \|v_r\|_{L^{p+2}}^p \|v_r\|_{B_1^{\frac{1+\epsilon}{2}}}.$$

Here we interpolate the last term to obtain

$$\|v_r\|_{B_1^{\frac{1+\epsilon}{2}}} \lesssim \|v_r\|_{L^{p+2}}^{\frac{1}{2}} \|v_r\|_{B_1^{\frac{1+\epsilon}{2}}}^{\frac{1}{2}},$$

and we deduce that

$$|(Z_{2,r}, v_r^{p+1})| \lesssim_{\widehat{\xi}_r} \|v_r\|_{L^{p+2}}^{p+\frac{1}{2}} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{1}{2}} \leq \delta \|v_r\|_{L^{p+2}}^{p+2} + C_\delta \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{p+2}{3}},$$

using Young inequality in the second inequality, here $C_\delta = \delta^{-\frac{2p+1}{6}}$.

Similar estimates hold for the $Z_{1,r}$ and $Z_{0,r}$ terms. We have

$$\begin{aligned} |(Z_{1,r}, v_r^p)| &\lesssim_{\widehat{\xi}_r} \|v_r^p\|_{B_1^{1+\epsilon}} \lesssim \|v_r^{p-1}\|_{L^{\frac{p+2}{p-1}}} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}} \lesssim \|v_r\|_{L^{p+2}}^{p-1} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}} \\ &\lesssim \delta \|v_r\|_{L^{p+2}}^{p+2} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{p+2}{3}}. \end{aligned}$$

and

$$\begin{aligned} |(Z_{0,r}, v_r^{p-1})| &\lesssim_{\widehat{\xi}_r} \|v_r^{p-1}\|_{B_1^{1+\epsilon}} \lesssim \|v_r^{p-2}\|_{L^{\frac{p+2}{p-2}}} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}} \lesssim \|v_r\|_{L^{p+2}}^{p-2} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}} \\ &\lesssim \delta \|v_r\|_{L^{p+2}}^{\frac{(p+2)(p-2)}{p-1}} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{p+2}{3}} \lesssim 1 + \delta^{\frac{p-1}{p-2}} \|v_r\|_{L^{p+2}}^{p+2} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{p+2}{3}}. \end{aligned}$$

One can then absorb the δ terms of these upper bounds in the corresponding L^{p+2} term in the left hand side of (3.21) to get the result by integrating in time on the interval (s, t) . Since we choose $\delta \lesssim (1 + \|\widehat{\xi}_r\|)^{-1}$, then $C_\delta, \delta^{-\frac{p-1}{3}} \gtrsim (1 + \|\widehat{\xi}_r\|)^\gamma$ for some $\gamma > 0$ depending on p . Combining with the implicit constant in (3.21) we obtain that the implicit constant in (3.20) is of form

$$(1 + \|\widehat{\xi}_r\|)^\gamma \left(\exp(\gamma' \|\mathcal{V}_r\|_{L^\infty([0,2] \times M)}) + 1 \right)$$

for some $\gamma, \gamma' > 0$ depending on p . The implicit constants in the next steps will be obtained in the same way. \square

Lemma 3.9 For $0 \leq s < t < T < T^* \wedge 1$ we have

$$\int_s^t \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s) + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} ds_1. \quad (3.22)$$

Note that we have a $B_{\frac{p+2}{3}}^{1+\epsilon}$ norm involved in (3.20):

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} ds_1 \lesssim \|v_r(s)\|_{L^{p+2}}^p + \int_s^t \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{p+2}{3}} ds_1$$

while we estimate in (3.22) a stronger $B_{\frac{p+2}{3}}^{1+2\epsilon}$ norm. We postpone for a second the proof of Proposition 3.9 and explain now how we get the estimate (3.19)

$$\|v_r(t)\|_{L^p}^p + \int_s^t F_r(s_1)^{\frac{p+2}{p}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s)$$

from (3.20) and (3.22). Since

$$F_r(s_1)^{\frac{p+2}{p}} \lesssim \|v_r(s_1)\|_{L^{p+2}}^{p+2} + \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}}$$

we see as a consequence of (3.22) that one gets (3.19) if one proves that

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s). \quad (3.23)$$

We start from the inequality (3.20) and use the interpolation estimate

$$\|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}} \lesssim \|v_r\|_{L^{\frac{p+2}{3}}}^{1-\theta_\epsilon} \|v_r\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\theta_\epsilon} \leq \delta \|v_r\|_{L^{\frac{p+2}{3}}}^3 + C_\delta \|v_r\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\sigma_\epsilon} \leq \delta \|v_r\|_{L^{p+2}}^3 + C_\delta \|v_r\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\sigma_\epsilon},$$

with

$$\theta_\epsilon \stackrel{\text{def}}{=} \frac{1+\epsilon}{1+2\epsilon} < 1, \quad \sigma_\epsilon \stackrel{\text{def}}{=} \frac{3\theta_\epsilon}{2+\theta_\epsilon} < 1.$$

We feed this upper bound inside (3.20); the contribution of the small factor involving the L^{p+2} norm of v_r can be absorbed in the corresponding term in the left hand side of (3.20), so we have

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} \, ds_1 \lesssim \|v_r(s)\|_{L^{p+2}}^p + \int_s^t \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\sigma_\epsilon \frac{p+2}{3}} \, ds_1. \quad (3.24)$$

We use Young inequality once more to bound

$$\|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\sigma_\epsilon \frac{p+2}{3}} \lesssim \delta \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} + C_\delta.$$

By choosing δ small enough we can absorb the L^{p+2} term that comes from (3.22) in the left hand side of (3.24) and use that $\|v_r(s)\|_{L^{p+2}}^p \leq F_r(s)$ to get (3.23) from (3.24).

Proof. We proceed in two steps.

Step 1. We first prove that one has

$$\|v_r(t)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} \lesssim_{\widehat{\xi}_r} \|e^{-(t-s)P} v_r(s)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} + \left(\int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} \, ds_1 \right)^{\frac{3}{p+2}} + \left(\int_s^t \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} \, ds_1 \right)^{\frac{3}{p+2}}. \quad (3.25)$$

We look at each term in the expression for $v_r(t) - e^{-(t-s)P} v_r(s)$

$$\int_s^t e^{-(t-s_1)P} \left(-A_r(s_1)v_r(s_1)^3 + B_r(s_1)\nabla v_r(s_1) + Z_{2,r}(s_1)v_r(s_1)^2 + Z_{1,r}(s_1)v_r(s_1) + Z_{0,r}(s_1) \right) ds_1.$$

One has

$$\begin{aligned} \left\| \int_s^t e^{-(t-s_1)P} (A_r(s_1)v_r(s_1)^3) \, ds_1 \right\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} &\lesssim \int_s^t \|e^{-(t-s_1)P} (A_r(s_1)v_r(s_1)^3)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} \, ds_1 \\ &\lesssim \int_s^t (t-s_1)^{-\frac{1+2\epsilon}{2}} \|A_r(s_1)v_r(s_1)^3\|_{L^{\frac{p+2}{3}}} \, ds_1 \\ &\lesssim_{\widehat{\xi}_r} \int_s^t (t-s_1)^{-\frac{1+2\epsilon}{2}} \|v_r(s_1)\|_{L^{p+2}}^3 \, ds_1 \\ &\lesssim \left(\int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} \right)^{\frac{3}{p+2}}, \end{aligned}$$

where we used Hölder inequality, the integrability in time of $(t-s_1)^{-\frac{(1+2\epsilon)(p+2)}{2(p-1)}}$ and the fact that $s \leq s_1 \leq t < T < 1$. Similarly, we have

$$\left\| \int_s^t e^{-(t-s_1)P} (B_r(s_1)\nabla v_r(s_1)) \, ds_1 \right\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} \lesssim \int_s^t \|e^{-(t-s_1)P} (B_r(s_1)\nabla v_r(s_1))\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} \, ds_1$$

$$\begin{aligned}
&\lesssim \int_s^t (t-s_1)^{-\frac{1+3\epsilon}{2}} \|B(s_1)\nabla v_r(s_1)\|_{B_{\frac{p+2}{3}}^{-\epsilon}} \, ds_1 \\
&\lesssim_{\widehat{\xi}_r} \int_s^t (t-s_1)^{-\frac{1+3\epsilon}{2}} \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+\epsilon}} \, ds_1 \\
&\lesssim \left(\int_s^t \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{p+2}{3}} \right)^{\frac{3}{p+2}}.
\end{aligned}$$

Next we have

$$\begin{aligned}
\left\| \int_s^t e^{-(t-s_1)P} (Z_{2,r}(s_1) v_r(s_1)^2) \, ds_1 \right\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} &\lesssim \int_s^t \left\| e^{-(t-s_1)P} (Z_{2,r}(s_1) v_r(s_1)^2) \right\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}} \, ds_1 \\
&\lesssim \int_s^t (t-s_1)^{-\frac{1+2\epsilon+\frac{1+\epsilon}{2}}{2}} \|Z_{2,r}(s_1) v_r^2(s_1)\|_{B_{\frac{p+2}{3}}^{-\frac{1+\epsilon}{2}}} \, ds_1.
\end{aligned}$$

Using the interpolation result from Proposition [3.64](#) and Young inequality we have

$$\|Z_{2,r} v_r^2\|_{B_{\frac{p+2}{3}}^{-\frac{1+\epsilon}{2}}} \lesssim_{\widehat{\xi}_r} \|v_r^2\|_{B_{\frac{p+2}{3}}^{\frac{1+\epsilon}{2}}} \lesssim \|v_r\|_{L^{p+2}} \|v_r\|_{B_{\frac{p+2}{3}}^{\frac{1+\epsilon}{2}}} \lesssim \|v_r\|_{L^{p+2}}^{\frac{3}{2}} \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{1}{2}} \lesssim \|v_r\|_{L^{p+2}}^{p+2} + \|v_r\|_{B_{\frac{p+2}{3}}^{1+\epsilon}}^{\frac{p+2}{3}},$$

and the desired estimate follows as in the previous terms. One proceeds in exactly the same way to prove similar estimates on the $Z_{1,r}$ and $Z_{0,r}$ terms. We leave the details to the interested reader.

Step 2. We first rewrite [\(3.25\)](#) at $t = s_1$ and raise this inequality to the power $\frac{p+2}{3}$. It yields the upper bound

$$\begin{aligned}
\|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} &\lesssim_{\widehat{\xi}_r} \|e^{-(s_1-s)P} v_r(s)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} + \int_s^{s_1} \|v_r\|_{L^{p+2}}^{p+2} + \int_s^{s_1} \|v_r\|_{B_{\frac{p+2}{3}}^{\frac{p+2}{3}}}^{\frac{p+2}{3}} \\
&\lesssim_{\widehat{\xi}_r} \|e^{-(s_1-s)P} v_r(s)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} + \int_s^t \|v_r\|_{L^{p+2}}^{p+2} + \int_s^t \|v_r\|_{B_{\frac{p+2}{3}}^{\frac{p+2}{3}}}^{\frac{p+2}{3}},
\end{aligned}$$

which holds for $t \geq s_1$, using the fact that $\int_s^{s_1} \|v_r\|^\alpha$ is increasing in s_1 whatever the exponent α the norm on v_r . Integrating on $s_1 \in [s, t]$, we obtain

$$\begin{aligned}
\left(\int_s^t \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} \, ds_1 \right)^{\frac{3}{p+2}} &\lesssim_{\widehat{\xi}_r} \left(\int_s^t \|e^{-(s_1-s)P} v_r(s)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} \, ds_1 \right)^{\frac{3}{p+2}} \\
&+ (t-s)^{\frac{3}{p+2}} \left[\left(\int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} \, ds_1 \right)^{\frac{3}{p+2}} + \left(\int_s^t \|v_r(s_1)\|_{B_{\frac{p+2}{3}}^{\frac{p+2}{3}}}^{\frac{p+2}{3}} \, ds_1 \right)^{\frac{3}{p+2}} \right],
\end{aligned} \tag{3.26}$$

as the function $[\dots]$ after $(t-s)^{\frac{3}{p+2}}$ in the right hand side is increasing. We bound the first term in [\(3.26\)](#) by $\|v_r(s)\|_{B_{\frac{p+2}{3}}^{\frac{p+2}{3}}}$ using the fact that the linear continuous map

$$e^{-(s_1-s)P} : B_{\frac{p+2}{3}}^{(1+2\epsilon)(1-\frac{3}{p+2})}(M) \rightarrow B_{\frac{p+2}{3}}^{1+2\epsilon}(M)$$

has a norm bounded above by $(s_1-s)^{-\frac{3(1+2\epsilon)}{2(p+2)}} \leq (s_1-s)^{-1/2}$, a quantity that is integrable over the interval (s, t) . The interpolation estimate

$$\|v_r(s)\|_{B_{\frac{p+2}{3}}^{\frac{p+2}{3}}} \lesssim \|v_r(s)\|_{L^{\frac{p+2}{3}}} \|v(s)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}-1} \lesssim \|v_r(s)\|_{L^{\frac{p+2}{3}}}^p + \|v_r(s)\|_{B_{\frac{p+2}{3}}^{1+2\epsilon}}^{\frac{p+2}{3}} = F_r(s),$$

gives $F_r(s)$ as a final upper bound for this term. Now, with v_r evaluated at time s_1 and $\theta_\epsilon = \frac{1+\epsilon}{1+2\epsilon}$, we have

$$\|v_r\|_{B^{\frac{1+\epsilon}{3}}_{\frac{p+2}{3}}} \lesssim \|v_r\|_{L^{\frac{p+2}{3}}}^{1-\theta_\epsilon} \|v_r\|_{B^{\frac{\theta_\epsilon}{3}}_{\frac{p+2}{3}}} \lesssim \|v_r\|_{B^{\frac{2+\theta_\epsilon}{3}}_{\frac{p+2}{3}}} + \|v_r\|_{L^{p+2}}^3 \leq \delta \|v_r\|_{B^{\frac{1+2\epsilon}{3}}_{\frac{p+2}{3}}} + C_\delta(1 + \|v_r\|_{L^{p+2}}^3),$$

for some $C_\delta > 0$ and δ small enough so that the term related to $\delta \|v_r\|_{B^{\frac{1+2\epsilon}{3}}_{\frac{p+2}{3}}}$ can be absorbed by the left hand side of (3.26). This gives inequality (3.22). \square

3.3 Scaling fields, regularity and microlocal extension

We state in this section an extension result, Theorem 3.19, that provides conditions under which a distribution on a manifold defined outside a submanifold can be extended to the whole manifold. The quantification of this extension result involves the notion of scaling field that is introduced in Section 3.3.1. Such vector fields are also known as Euler vector fields. Some function spaces associated with a given scaling field are introduced in Section 3.3.2, they allow us to measure the singularity of distributions when we scale along certain submanifolds of some given ambient space. They generalise the weakly homogeneous distributions introduced by Meyer in [Mey98]. An extension Theorem 3.19 is proven in Section 3.3.3, given a distribution whose blow up is moderate along a given submanifold, it allows to extend canonically the distribution to the whole ambient space in the spirit of the definition of the principal value. This statement is put to work in the particular setting of a configuration space in Section 3.4.1 to give a useful extension result for a class of Feynman amplitudes – see Theorem 3.40. Moreover in Theorem 3.40, we will need to control the blow up of our Feynman amplitudes in two steps, first when all points collapse on the deepest diagonal, then when all points collapse to a single given point – this difficulty comes from the absence of translation invariance. This requires a variant of the extension Theorem 3.19 stated in Theorem 3.20, in which we scale w.r.t a given submanifold first, and then w.r.t. a given point, in order to control the blow up of our distributions w.r.t. both scalings.

We work in Sections 3.3.1 to 3.3.3 in the setting of a smooth manifold \mathcal{X} where a smooth submanifold $\mathcal{Y} \subset \mathcal{X}$ is given.

3.3.1 Scaling fields

Definition 3.10 Let $\mathfrak{I}_{\mathcal{Y}}$ be the ideal of smooth real valued functions on \mathcal{X} that vanish on \mathcal{Y} . Set for $k \geq 1$:

$$\mathfrak{I}_{\mathcal{Y}}^k \stackrel{\text{def}}{=} \{f_1 \dots f_k; (f_1, \dots, f_k) \in \mathfrak{I}_{\mathcal{Y}} \times \dots \times \mathfrak{I}_{\mathcal{Y}}\}.$$

A vector field ϱ defined on a neighbourhood of \mathcal{Y} is called an isotropic *scaling field* for $\mathcal{Y} \subset \mathcal{X}$ if for all $f \in \mathfrak{I}_{\mathcal{Y}}$

$$f \in \varrho f + \mathfrak{I}_{\mathcal{Y}}^2.$$

This type of vector field is also called an *Euler vector field* in the literature. Denote by n the dimension of \mathcal{X} and by d the dimension of \mathcal{Y} . If ϱ is a scaling field for $\mathcal{Y} \subset \mathcal{X}$ there exists a neighbourhood of \mathcal{Y} that is stable by the backward semiflow $(e^{-s\varrho})_{s \geq 0}$ of ϱ and every point $y \in \mathcal{Y}$ has a neighbourhood U_y in \mathcal{X} on which coordinates

$$h = (h_1, \dots, h_n) : U \rightarrow \mathbf{R}^n$$

are defined and such that

$$U_y \cap \mathcal{Y} = h^{-1}(\mathbf{R}^d)$$

with $\mathbf{R}^d \subset \mathbf{R}^n$, and

$$\varrho = \sum_{i=d+1}^n h^i \partial_{h^i}.$$

A proof of existence of a stable neighbourhood can be found in Lemma 2.4 of [DW22] and the normal form theorem can be found in Proposition 2.5 of [DW22] – see also Lemma 2.1 in [Mei20]. The example of the configuration space of ℓ points in \mathbf{R}^k will be particularly relevant for us. The scaling field ϱ whose flow reads

$$e^{-t\varrho}(x_1, \dots, x_\ell) = \left(x_1, e^{-t}(x_2 - x_1) + x_1, \dots, e^{-t}(x_\ell - x_1) + x_1 \right)$$

will move all points towards the deepest diagonal and its dynamics is tangent to all the larger diagonals. In the sequel, we will only work with product (sub)manifolds of the form

$$\begin{aligned} \mathcal{X} &= \mathbf{R}^p \times X, \\ \mathcal{Y} &= (\{0\} \times \mathbf{R}^q) \times Y \end{aligned} \quad (3.27)$$

with $(\{0\} \times \mathbf{R}^q) \subset \mathbf{R}^p$ and $Y \subset X$, and some non-isotropic scaling fields, of the form

$$\varrho = \sum_{j=1}^q 2t_j \partial_{t_j} + \varrho_Y \quad (3.28)$$

for the canonical coordinates $(t_j)_{1 \leq j \leq q}$ on \mathbf{R}^q and a scaling field ϱ_Y for $Y \subset X$. This is an example of a *weighted vector field on a weighted manifold*. We give a formal definition in the case where our submanifold \mathcal{Y} is the transverse intersection $\mathcal{Y}_1 \cap \mathcal{Y}_2$ where $\mathcal{Y}_1 = (\{0\} \times \mathbf{R}^q) \times X$ and $\mathcal{Y}_2 = (\mathbf{R}^p \times Y)$. Then one has a description of the ideal $\mathfrak{I}_{\mathcal{Y}}$ as the product $\mathfrak{I}_{\mathcal{Y}_1} \mathfrak{I}_{\mathcal{Y}_2}$ so the ideal $\mathfrak{I}_{\mathcal{Y}}$ has a bifiltration

$$\mathfrak{I}_{\mathcal{Y}_1}^m \mathfrak{I}_{\mathcal{Y}_2}^n \subset \dots \subset \mathfrak{I}_{\mathcal{Y}_1} \mathfrak{I}_{\mathcal{Y}_2} = \mathfrak{I}_{\mathcal{Y}}.$$

Now assume we want to put a weight 2 to powers of $\mathfrak{I}_{\mathcal{Y}_1}$ and weight 1 to powers of $\mathfrak{I}_{\mathcal{Y}_2}$. We want to give an intrinsic characterisation for vector fields of the form given by Equation (3.28).

Definition 3.11 A vector field ϱ defined on a neighbourhood of \mathcal{Y} is called a *parabolic scaling field for $\mathcal{Y} \subset \mathcal{X}$* if for all (m, n) and for all $f \in \mathfrak{I}_{\mathcal{Y}_1}^m \mathfrak{I}_{\mathcal{Y}_2}^n$ one has

$$\varrho f - (2m + n)f \in \left(\mathfrak{I}_{\mathcal{Y}_1}^{m+1} \mathfrak{I}_{\mathcal{Y}_2}^n + \mathfrak{I}_{\mathcal{Y}_1}^m \mathfrak{I}_{\mathcal{Y}_2}^{n+1} \right).$$

In the sequel we will simply call ‘scaling fields’ some parabolic scaling fields as we will only work with such fields. The *weighted co-dimension* of \mathcal{Y} is defined here as

$$\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) \stackrel{\text{def}}{=} 2(p - q) + \dim(X) - \dim(Y).$$

3.3.2 Function spaces associated with scaling fields

We assume from now on that \mathcal{X} has a Riemannian structure and we denote by $K_r(x, y)$ its heat kernel. We also let $U \subset \mathcal{X}$ be an open set and Γ be a closed conic set in $T^*U \setminus \{0\}$.

Definition 3.12 We denote by $\mathcal{D}'_{\Gamma}(U)$ the space of distributions on U whose wave front set is contained in Γ . This is a locally convex topological vector space endowed with a natural *normal* topology invented by Y. Dabrowski – see [Dan14, p. 823] and [BDH16] for results about why this topology is well-behaved with respect to some natural operations on distributions. The seminorms defining its topology are

- given a chart $\kappa : \Omega \subset U \mapsto \mathbf{R}^{\dim(\mathcal{X})}$, an integer $N \in \mathbf{N}$, $\chi \in C_c^\infty(\kappa(\Omega))$, and a cone $V \subset \mathbf{R}^{n^*}$ such that

$$\text{supp}(\chi) \times V \cap \kappa_* \Gamma = \emptyset, \quad \text{where} \quad \kappa_* \Gamma = \left\{ (\kappa(x); ({}^t d\kappa)_x^{-1}(\xi)); (x; \xi) \in \Gamma \right\},$$

we have the norm

$$\|\Delta\|_{N, V, \chi, \kappa} = \sup_{\xi \in V} (1 + |\xi|)^N |\widehat{(\kappa_* \Gamma)} \chi(\xi)|;$$

- the seminorms of the strong topology of distributions

$$\sup_{\chi \in \mathcal{B}} |\langle \Lambda, \chi \rangle|,$$

where \mathcal{B} is a bounded set of $C_c^\infty(\mathcal{X})$ which means that there is some compact K such that $\text{supp}(f) \subset K$ for all $f \in \mathcal{B}$, and for any differential operator Q one has $\sup_{\chi \in \mathcal{B}} \|Q\chi\|_{L^\infty(K)} < \infty$.

To be bounded in $\mathcal{D}'_\Gamma(U)$ will always mean that all the above seminorms are bounded.

The following elementary example will play an important role in the sequel. Assume $k \in \mathbf{N}$ is of the form $d_1 + d_2 + d_3$ with $d_i \in \mathbf{N} \setminus \{0\}$ and $\mathbf{R}^k = \mathbf{R}^{d_1} \times \mathbf{R}^{d_2} \times \mathbf{R}^{d_3}$, and denote here by ϱ the linear vector field on \mathbf{R}^k whose restriction to \mathbf{R}^{d_1} is null, whose restriction to \mathbf{R}^{d_2} is the identity and whose restriction to \mathbf{R}^{d_3} is twice the identity. So for $z = (x, y, t) \in \mathbf{R}^{d_1} \times \mathbf{R}^{d_2} \times \mathbf{R}^{d_3}$ one has

$$\varrho = y\partial_y + 2t\partial_t.$$

The third component $t \in \mathbf{R}^{d_3}$ has the meaning of a time coordinates. This vector field over \mathbf{R}^k will be our model scaling field in a parabolic setting.

Lemma 3.13 *The family of distribution*

$$\delta(z' - e^{-\ell\varrho}z) \quad (1 \leq \ell \leq +\infty)$$

on $\mathbf{R}^k \times \mathbf{R}^k$ is bounded in $\mathcal{D}'_{\Gamma_\varrho}(\mathbf{R}^k \times \mathbf{R}^k)$, where

$$\Gamma_\varrho = \bigcup_{1 \leq \ell \leq +\infty} \left\{ ((z, e^{-\ell\varrho}z), (\lambda, e^{\ell\varrho}\lambda)); (z, \lambda) \in T^*\mathbf{R}^k \right\} \subset T^*(\mathbf{R}^k \times \mathbf{R}^k).$$

This estimate can also be used to give an upper bound on the wave front set of the resolvent $(\varrho + z)^{-1}$ which implies the radial type estimates for ϱ . This is very similar in spirit to the radial estimates from the works of Melrose [Mel20], Vasy [AV13] or Dyatlov & Zworski [DZ16].

Proof. Note that the distributions $\delta(z' - e^{-\ell\varrho}z) \in \mathcal{D}'(\mathbf{R}^k \times \mathbf{R}^k)$ are nothing but the Schwartz kernels of the transfer operator $\varphi \in C^\infty(\mathbf{R}^k) \mapsto e^{-\ell\varrho^*}\varphi \in C^\infty(\mathbf{R}^k)$, so we will use the identification $[e^{-\ell\varrho^*}] = \delta(z' - e^{-\ell\varrho}z)$. The fact that this family of distributions is bounded (weak boundedness implies strong boundedness by uniform boundedness) automatically follows from the continuity of the pull-back of a distribution by a smooth family of diffeomorphisms and the strong convergence of $\delta(z' - e^{-\ell\varrho}z)$ to $\delta(z' - (x, 0, 0))$ when ℓ goes to infinity, for $z = (x, y, t)$.

Fix an arbitrary compact subset $K \subset \mathbf{R}^{d_1+d_2+d_3}$ that is stable by the scaling maps

$$(x, y, t) \mapsto (x, e^{-\ell}y, e^{-2\ell}t) \quad (\ell \geq 0).$$

Then we shall restrict the Schwartz kernel $[e^{-\ell\varrho}]$ to $K \times K$. It means we estimate this wave front set near the diagonal but for arbitrary large ℓ . Choose some test functions χ_1, χ_2 in $C_K^\infty(\mathbf{R}^{d_1+d_2+d_3})$, supported in K . In local coordinates we have

$$\begin{aligned} & \int_{\mathbf{R}^{d_1+d_2+d_3}} e^{i\xi_2 \cdot x + i\eta_2 \cdot y + i\tau_2 \cdot t} \chi_2(x, y, t) e^{-t\varrho^*} (\chi_1 e^{i\xi_1 \cdot x + i\eta_1 \cdot y + i\tau_1 \cdot t}) dx dy dt \\ &= e^{-\ell(d_2+2)} \int_{\mathbf{R}^{d_1+d_2+d_3}} e^{i\xi_2 \cdot x + i\eta_2 \cdot y + i\tau_2 \cdot t} \chi_2(x, y, t) (\chi_1(x, e^{-\ell}y, e^{-2\ell}t) e^{i\xi_1 \cdot x + i\eta_1 \cdot y + i\tau_1 \cdot t}) dx dy dt \\ &= \widehat{\chi}_\ell \left(\xi_1 + \xi_2, \eta_2 + e^{-\ell}\eta_1, \tau_2 + e^{-2\ell}\tau_1 \right) \end{aligned}$$

where

$$\chi_\ell(x, y, t) = \chi_2(x, y, t)\chi_1(x, e^{-\ell}y, e^{-2\ell}t)$$

is a bounded family of smooth compactly supported functions (this is crucial) when $\ell \in [0, +\infty)$. We then have for any $N \geq 1$ the upper bound

$$|\widehat{\chi}_\ell(\xi, \eta, \tau)| \leq C_N(1 + |\xi| + |\eta| + |\tau|)^{-N} \quad (3.29)$$

where the constant C_N does not depend on $\ell \in [0, +\infty)$. Hence in any closed conic set V which does not meet the subset

$$\Lambda = \left\{ \left(\xi, -\xi, \eta, -e^{-\ell}\eta, \tau, -e^{-2\ell}\tau \right) \in (\mathbf{R}^k \times \mathbf{R}^k)^*, \ell \geq 1 \right\},$$

there exists some $\varepsilon > 0$ such that for all $(\xi_1, \xi_2, \eta_1, \eta_2, \tau_1, \tau_2) \in V \subset (\mathbf{R}^k \times \mathbf{R}^k)^*$ we have for all $\ell \geq 1$ the inequality

$$\left| (\xi_1 + \xi_2, e^{-s}\eta_1 + \eta_2, e^{-2s}\tau_1 + \tau_2) \right| \geq \varepsilon \left(|\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| + |\tau_1| + |\tau_2| \right).$$

This implies the following Fourier bound

$$\begin{aligned} \left| \int_{\mathbf{R}^k} e^{i\xi_2 \cdot x + i\eta_2 \cdot y + i\tau_2 \cdot t} \chi_2(x, y, t) e^{-\ell \varrho^*} (\chi_1 e^{i\xi_1 \cdot x + i\eta_1 \cdot y + i\tau_1 \cdot t}) dx dy dt \right| \\ \leq C_N \left(1 + |\xi_1 + \xi_2| + |\eta_2 + e^{-s}\eta_1| + |\tau_2 + e^{-2s}\tau_1| \right)^{-N} \\ \leq C_N \varepsilon^{-N} \left(1 + |\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| + |\tau_1| + |\tau_2| \right)^{-N} \end{aligned}$$

for all $\ell \geq 1$ and $(\xi_1, \xi_2, \eta_1, \eta_2, \tau_1, \tau_2) \in V \subset (\mathbf{R}^k \times \mathbf{R}^k)^*$. The previous bound analyzes the wave front set of the family $\delta(z' - e^{-\ell \varrho}(z))$ near $T^*(K \times K) \subset T^*(\mathbf{R}^k \times \mathbf{R}^k)$. Since K is arbitrary the family of distributions $\delta(z' - e^{-\ell \varrho}(z))$ is bounded in $\mathcal{D}'_{\Gamma_\varrho}(\mathbf{R}^k \times \mathbf{R}^k)$, with $\Gamma_\varrho \subset T^*(\mathbf{R}^k \times \mathbf{R}^k)$ given by

$$\begin{aligned} \Gamma_\varrho = \underbrace{\left\{ (x, x, 0, 0, 0, 0; \xi, -\xi, 0, \eta_2, 0, \tau_2); (\xi, \eta_2, \tau_2) \neq (0, 0, 0) \right\}}_{\text{the radial set which is the conormal of the singular set of } \varrho} \\ \cup \left\{ (x, y, t, x, e^{-\ell}y, e^{-2\ell}t; \xi, \eta, \tau, -\xi, -e^\ell\eta, -e^{2\ell}\tau); \ell \geq 1, (\xi, \eta, \tau) \neq (0, 0, 0) \right\}. \end{aligned}$$

This concludes the proof. \square

Set

$$\pi(z) \stackrel{\text{def}}{=} (x, 0, 0). \quad (3.30)$$

Lemma 3.13 is useful to give a description of the Taylor subtraction operation. The Taylor subtractors of order 0 and 1 read

$$\begin{aligned} R_0 : \varphi &\mapsto \varphi - \varphi \circ \pi, \\ R_1 : \varphi &\mapsto \varphi - \varphi(x, 0, 0) - y \cdot \partial_y \varphi(x, 0, 0) - t \partial_t \varphi(x, 0, 0), \end{aligned}$$

We call these operators R_0 and R_1 , with the letter R chosen for ‘remainder’. Denote generically by $[\Lambda]$ the Schwartz kernel of an operator Λ .

Proposition 3.14 *The operators R_0, R_1 have Schwartz kernel*

$$[R_0] = \int_0^\infty [\varrho e^{-a\varrho^*}] da,$$

$$[R_1] = \int_0^\infty \left[(1 - e^a + e^a \varrho) \varrho e^{-a\varrho^*} \right] da - [\varrho],$$

and

$$\begin{aligned} [e^{-\ell\varrho^*} R_0] &= \int_0^\infty [\varrho e^{-(\ell+a)\varrho^*}] da, \\ [e^{-\ell\varrho^*} R_1] &= \int_0^\infty [(1 - e^a + e^a \varrho) \varrho e^{-(\ell+a)\varrho^*}] da - [e^{-\ell\varrho^*} \varrho]. \end{aligned}$$

and the families of distributions $([e^{-\ell\varrho^*} R_0])_{0 \leq \ell \leq +\infty}$ and $([e^{-\ell\varrho^*} R_1])_{0 \leq \ell \leq +\infty}$ are bounded in $\mathcal{D}'_{\Gamma_e}(\mathbf{R}^k \times \mathbf{R}^k)$.

Proof. We write a detailed proof for R_0 ; the proof for R_1 is very similar and left to the reader. For a test function χ with compact support on $\mathbf{R}^k \times \mathbf{R}^k$ write $\varrho_1 \chi$ for the action of the vector field on the first component of χ . We have

$$\langle [e^{-\ell\varrho^*}], \chi \rangle = - \int_{\mathbf{R}^k} (\varrho_1 \chi)(z, e^{-\ell\varrho}(z)) dz$$

and since $(\varrho_1 \chi)(z, e^{-\ell\varrho} z)$ vanishes along the singular set $\{y = 0, t = 0\}$ the integrand is of order $e^{-\ell}$, so the integral is convergent. The wavefront bound follows from the wave front bound on the propagator $[e^{-\ell\varrho^*}]$ and the fact that the wave front of a distribution is stable under the action of the vector field ϱ . \square

We come back to the general setting of an open subset $U \subset \mathcal{X}$ and assume we are given a closed conic set Γ in $T^*U \setminus \{0\}$. It is a classical fact that for $\alpha < 0$ the Besov space $C^\alpha(\mathcal{X}) = B_{\infty, \infty}^\alpha(\mathcal{X})$ can be characterised as the set of distributions Λ on \mathcal{X} such that

$$\sup_{x \in \mathcal{X}} \sup_{0 < r \leq 1} r^{-\frac{\alpha}{2}} |\langle \Lambda, K_r(x, \cdot) \rangle| < \infty.$$

A distribution $\Lambda \in \mathcal{D}'(U)$ is an element of $\mathcal{D}'_\Gamma(U)$ iff for all pseudodifferential operators Q with Schwartz kernel compactly supported in $U \times U$ and whose symbol vanishes on Γ one has for every compact subset C of \mathcal{X}

$$\sup_{x \in C} \sup_{0 < r \leq 1} |\langle \Lambda, QK_r(x, \cdot) \rangle| < \infty. \quad (3.31)$$

One can describe an element of $C_{\text{loc}}^\alpha(U)$, with $\alpha < 0$, with wave front set in Γ in terms similar to (3.31) as the set of distributions $\Lambda \in C_{\text{loc}}^\alpha(U)$ iff for all pseudodifferential operators Q with Schwartz kernel compactly supported in $U \times U$ and whose symbol vanishes on Γ one has for every compact subset C of \mathcal{X}

$$\sup_{x \in C} \sup_{0 < r \leq 1} r^{-\frac{\alpha}{2}} |\langle \Lambda, (I + Q)K_r(x, \cdot) \rangle| < \infty.$$

The I element ensures that Λ is Hölder whereas the operator Q is here to test the smoothness of Λ outside of the wave front set.

We now come back to the parabolic setting $\mathcal{Y} \subset \mathcal{X}$ of (3.27) in Section 3.3.1 and denote by ϱ a parabolic scaling field for this embedding. Let U stand for an open set of \mathcal{X} which is stable by the backward semiflow of ϱ , in the sense that $e^{-\ell\varrho}(U) \subset U$ for all $\ell \geq 0$.

Definition 3.15 Given some closed conic set $\Gamma \subset T^*U \setminus \{0\}$ we denote by $(e^{-\ell\varrho})^* \Gamma$ the set defined as

$$(e^{-\ell\varrho})^* \Gamma \stackrel{\text{def}}{=} \left\{ \left(e^{\ell\varrho} x; (t \text{de}^{\ell\varrho})_x^{-1}(\xi) \right); (x; \xi) \in \Gamma \right\}.$$

Then we assume that the lifted flow of $e^{-\ell\varrho}$ leaves the conic set Γ stable i.e.

$$(e^{-\ell\varrho})^* \Gamma \subset \Gamma$$

for all $\ell \geq 0$.

Having described the necessary geometric framework, we are ready to state the definition of the scaling spaces that will be used extensively in the present work.

Definition 3.16 For $\alpha < 0$ and $a \in \mathbf{R}$ we define the *scaling space* $\mathcal{S}_\Gamma^{a,\varrho;\alpha}(U)$ of distributions $\Lambda \in \mathcal{D}'(U)$ with the following property. For all pseudodifferential operators Q with Schwartz kernel compactly supported in $U \times U$ and whose symbol vanishes on Γ , for each compact set $C \subset U$

$$\sup_{\ell \geq 1} \sup_{x \in C} \sup_{0 < r \leq 1} e^{a\ell} r^{-\frac{\alpha}{2}} |\langle (e^{-\ell\varrho})^* \Lambda, (I + Q)K_r(x, \cdot) \rangle| < \infty.$$

We define $\mathcal{S}_\Gamma^a(U)$ as the union over α of all the spaces $\mathcal{S}_\Gamma^{a,\varrho;\alpha}(U)$, for $a \in \mathbf{R}$ fixed and ϱ a parabolic scaling field for the inclusion $\mathcal{Y} \subset \mathcal{X}$ whose backward semiflows leave Γ fixed. This implies in particular that $N^*(\mathcal{Y} \subset \mathcal{X}) \subset \Gamma$. The letter ‘ \mathcal{S} ’ is chosen for *scaling*. The exponent a retains the scaling property and Γ retains some information on the wavefront set. Note that the space $\mathcal{S}_\Gamma^a(U)$ is a priori larger than the space of conormal distributions with wavefront set in $N^*(\mathcal{Y} \subset U)$ since elements in $\mathcal{S}_\Gamma^a(U)$ might have some wavefront set contained in the cone Γ which is not necessarily included in $N^*(\mathcal{Y} \subset U)$. The notation $\mathcal{S}_\Gamma^a(U)$ does not emphasise the dependence of this space on the inclusion $\mathcal{Y} \subset \mathcal{X}$. This will always be clear for us from the context. An elementary example is given by the principal value of $1/|x|$ in \mathbf{R} , where $\mathcal{Y} = \{0\} \subset \mathbf{R}$ and it has scaling exponent $a = -1$ and wavefront set $\Gamma = T_0^*\mathbf{R}$. Note the fact that for all element $\Lambda \in \mathcal{S}_\Gamma^a(U)$ the family of scaled distributions $(e^{a\ell} e^{-\ell\varrho^*} \Lambda)_{\ell \geq 0}$ is bounded in \mathcal{D}'_Γ . The next proposition gives an example of an element of some space $\mathcal{S}^{a,\varrho;0}(U)$ for some scaling exponent a and some scaling field ϱ . For $n \geq 2$, denote by \mathbf{d}_n the diagonal of M^n . Denote by ϱ_n a scaling field on M^n for the inclusion $\mathbf{d}_n \subset M^n$ and define on $M^2 \times \mathbf{R}^2$ the parabolic scaling field

$$\varrho = 2(t - s)\partial_s + \varrho_2.$$

Denote also by

$$\pi_{\leq 2} : (x_1, \dots, x_n) \in M^n \rightarrow (x_1, x_2) \in M^2$$

the canonical projection on the first two components.

Lemma 3.17 *Let M be a closed manifold and $A_r(x, y)$ be a smooth kernel on $M^2 \setminus \mathbf{d}_2$ such that one can associate to any small enough open set U a coordinate system in which one has for all multiindices α, β*

$$|\partial_{s,t}^\alpha \partial_{x,y}^\beta A_{|t-s|}(x, y)| \lesssim (\sqrt{t-s} + |y-x|)^{a-2|\alpha|-|\beta|}. \quad (3.32)$$

Then the family

$$\left(e^{\ell a} (e^{-\ell\varrho})^* \pi_{\leq 2}^* A_{|t-\cdot|}(\cdot, \cdot) \right)_{\ell \geq 0}$$

is bounded in

$$\mathcal{D}'_{N^*(\{s=t\})} \left((M^n \times \mathbf{R}) \setminus (\pi_{\leq 2}^* \mathbf{d}_2 \cap \{s=t\}) \right),$$

that is

$$\pi_{\leq 2}^* A_{|t-\cdot|}(\cdot, \cdot) \in \mathcal{S}_{N^*(\{s=t\})}^{a,\varrho;0} \left((M^n \times \mathbf{R}) \setminus (\pi_{\leq 2}^* \mathbf{d}_2 \cap \{s=t\}) \right).$$

In the sequel we denote by \mathcal{K}^a the C^∞ -module of kernels $A_r(x, y)$ as above depending on two variables endowed with the weakest topology containing the $C^\infty([0, +\infty) \times (M^2 \setminus \mathbf{d}_2))$ topology and which makes all the seminorms defined by the estimates (3.32) continuous.

Proof. We first localize in a neighbourhood $U \times U$ of the diagonal since K is smooth off-diagonal. It is enough to prove the claim for $A_{|t-s|}(x, y)\chi_1(y)\chi_2(x)$ where $\chi_i \in C_c^\infty(U)$ and use a partition of unity to get the global result. In $U \times U$ we pull-back everything to the configuration space, which we write with a slight abuse of notations

$$\pi_{\leq 2}^*(A\chi_1\chi_2)(t, s, x_1, \dots, x_n) = A_{|t-s|}(x_1, x_2)\chi_1(x_1)\chi_2(x_2).$$

We already know that this kernel satisfies some bound of the form

$$|A(t, s, x_1, x_2)\chi_1(x_1)\chi_2(x_2)| \lesssim \left(\sqrt{|t-s|} + |x_1 - x_2|\right)^a.$$

Somehow we would like to flow both sides of the inequality by the parabolic dynamics $(e^{-\ell\varrho})^*$ and bound the term $e^{-\ell\varrho^*} \left(\sqrt{|t-s|} + |x_1 - x_2|\right)^a$ asymptotically when ℓ goes to $+\infty$. We use for that purpose the Normal Form Theorem for the space part of the isotropic scaling fields

$$\varrho_{[n]} \stackrel{\text{def}}{=} \sum_{k=2}^n h_k \cdot \partial_{h_k},$$

for some new coordinates $(h_k)_{k=2}^n$ that vanish at order 1 along the deep space diagonal \mathbf{d}_n . The fact that $x_1 - x_2$ vanishes at first order along \mathbf{d}_n implies by Taylor expansion at first order that

$$x_1 - x_2 = L(h) + \mathcal{O}(|h|^2) \quad (3.33)$$

where $L(h)$ is a linear function of $(h_k)_{k=2}^n$. One then has

$$(e^{-\ell\varrho_{[n]}})^*(x_1 - x_2) = (e^{-\ell\varrho_{[n]}})^*L(h) + \mathcal{O}(e^{-2\ell}|h|^2) = L(e^{-\ell}h) + \mathcal{O}(e^{-2\ell}|h|^2),$$

and an exponential lower bound of the form

$$e^{-\ell}|x_1 - x_2| \lesssim |(e^{-\ell\varrho_{[n]}})^*(x_1 - x_2)|$$

which yields the desired bound

$$\left| \partial_t^\alpha \partial_x^\beta e^{-\ell\varrho^*} \pi^*(A\chi_1\chi_2)(t, s, x_1, \dots, x_n) \right| \lesssim e^{\ell a} \left(\sqrt{|t-s|} + |x_1 - x_2|\right)^{a-2|\alpha|-|\beta|}$$

and proves the claim. The above bound allows for instance to justify that the singularities when $x_1 \neq x_2$ are conormal along the equal time region $t = s$ since we are smooth on each half region $t \geq s$ and $s > t$. \square

3.3.3 The canonical extension

3.3.3.1 The basic extension result

Let us use a unique notation $\mathbf{0}$ for the zero section of any vector bundle.

Definition 3.18 Let \mathcal{X} be a smooth manifold and $\mathcal{Y} \subset \mathcal{X}$. A closed conic set $\Gamma \subset T^*(\mathcal{X} \setminus \mathcal{Y}) \setminus \mathbf{0}$ is said to satisfy the **conormal landing condition** if its closure $\tilde{\Gamma}$ in $T^*(\mathcal{X}) \setminus \mathbf{0}$ satisfies $\tilde{\Gamma} \subset (\Gamma \cup N^*(\mathcal{Y}))$.

In what follows, we will use the terminology *smooth weighted manifolds* for smooth weighted manifolds of product type as introduced in Section [3.3.1](#).

Theorem 3.19 Let \mathcal{X} be a smooth weighted manifold and $\mathcal{Y} \subset \mathcal{X}$ and $\Gamma \subset T^*(\mathcal{X} \setminus \mathcal{Y}) \setminus \mathbf{0}$ be a closed conic set that satisfies the conormal landing condition. Assume we are given a family $(\Lambda_\epsilon)_{0 < \epsilon \leq 1}$ of distributions on \mathcal{X} that converge in $\mathcal{D}'(\mathcal{X} \setminus \mathcal{Y})$ as ϵ goes to 0 to an element $\Lambda \in \mathcal{S}_\Gamma^a(\mathcal{X} \setminus \mathcal{Y})$.

(a) If

$$-\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) < a$$

then Λ has a unique extension into a distribution over \mathcal{X} such that the convergence of Λ_ϵ to Λ occurs in $\mathcal{S}_{\Gamma \cup N^*(\mathcal{Y})}^a(\mathcal{X})$.

(b) If

$$-\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1 \leq a < -\text{codim}_w(\mathcal{Y} \subset \mathcal{X})$$

there exists a family $\Lambda_{\mathcal{Y},\epsilon}$ of distributions supported on \mathcal{Y} , with wavefront set in $N^*(\mathcal{Y})$ such that $\Lambda_\epsilon - \Lambda_{\mathcal{Y},\epsilon}$ has a limit in $\mathcal{D}'(\mathcal{X})$ and the convergence occurs in $\mathcal{S}'_{\Gamma \cup N^*(\mathcal{Y})}(\mathcal{X})$ for all

$$a' < a.$$

The particular case where $\mathcal{Y} = \{p\}$ is a point will be used in Theorem 3.20 below. We lose any information on the wavefront set of the extension at p in that case, so the convergence of Λ_ϵ to Λ happens in $\mathcal{S}'_{\Gamma \cup T_p^*M}(\mathcal{X})$, for $b \in \{a, a'\}$, depending on the situation.

Proof. We follow the proof of similar results proved in an elliptic setting in [Dan14] – see Theorem 1.10, Theorem 4.4 and Section 6 therein. We give here the main arguments to emphasize the differences with [Dan14] that come from our parabolic setting. The main idea of the proof is to start from a continuous partition of unity which approximates the constant function 1 but vanishes near \mathcal{Y} . Then we multiply the distribution Θ by this partition of unity to product an approximation of Θ outside \mathcal{Y} and use the assumption on the weak homogeneity of Θ near \mathcal{Y} to conclude that the approximation genuinely converges to some given distribution Θ^+ which is the desired extension.

Let ϱ be a scaling field for the inclusion $\mathcal{Y} \subset \mathcal{X}$ such that $\Lambda \in \mathcal{S}'_{\Gamma \cup N^*(\mathcal{Y})}^{a,\varrho;\alpha}(\mathcal{X})$ for some $\alpha \in \mathbf{R}$, and let χ be a smooth function equal to 1 in a neighbourhood of \mathcal{Y} stable by the backward semiflow of ϱ and such that χ vanishes outside some larger neighbourhood.

(1) We first use the normal form theorem to reduce our problem to the model case of a distribution on \mathbf{R}^k , with $k = d_1 + d_2 + d_3$ with coordinates (x, y, t) , the scaling field $\varrho = y\partial_y + 2t\partial_t$ is linear and globally defined, and the extension is done with respect to the linear subspace $\mathbf{R}^{d_1} \subset \mathbf{R}^k$. We work in that setting in the remainder of the proof. Let then $(\Theta_\epsilon)_{0 < \epsilon \leq 1}$ be a family of distributions on \mathbf{R}^k . We assume that the Θ_ϵ converge as ϵ goes to 0 to an element $\Theta \in \mathcal{S}'_{\Gamma}(\mathbf{R}^k \setminus \mathbf{R}^{d_1})$ where $a > -\text{codim}_w(\mathcal{Y} \subset \mathcal{X})$.

(2) Pick $0 < \ell_0$ and think of it as being large. We use the continuous partition of unity

$$\text{Id} - (e^{\ell_0\varrho})^*\chi = \text{Id} - \chi + \int_0^{\ell_0} (e^{\ell\varrho})^*(-\varrho\chi) d\ell$$

to define an extension of our distribution Θ . We set for convenience $\bar{\chi} \stackrel{\text{def}}{=} -\varrho\chi$; its support does not meet \mathcal{Y} . We have for any test function $f \in \mathcal{D}(\mathbf{R}^k)$

$$\langle \Theta(1 - (e^{\ell_0\varrho})^*\chi), f \rangle = \langle \Theta(1 - \chi), f \rangle + \int_0^{\ell_0} e^{-\ell(d_2+2d_3)} \langle \bar{\chi}(e^{-\ell\varrho})^*\Theta, (e^{-\ell\varrho})^*f \rangle d\ell.$$

The exponential factor $e^{-\ell(d_2+2d_3)}$ comes from the Jacobian of the flow of $e^{-\ell\varrho}$. Note that $d_2 + 2d_3 = \text{codim}_w(\mathcal{Y} \subset \mathcal{X})$. If $\Gamma \subset T^*\mathbf{R}^k \setminus \mathbf{0}$ stands for a closed conic set invariant by the lifted dynamics of $(e^{-\ell\varrho})^*$ such that $\Gamma \cap T^*\mathbf{R}^{d_1} \subset N^*(\mathbf{R}^{d_1})$, our choice of scaling exponent a ensures that the family

$$(\Theta^{(\ell)})_{\ell \geq 0} \stackrel{\text{def}}{=} (e^{a\ell}(e^{-\ell\varrho})^*\Theta)_{\ell \geq 0}$$

is bounded in $\mathcal{D}'_{\Gamma}(\mathbf{R}^k)$. One then has for the Schwartz kernels

$$[\Theta(1 - (e^{\ell_0\varrho})^*\chi)](z, z') = [\Theta(1 - \chi)](z, z') + \int_0^{\ell_0} e^{-\ell(a+d_2+2d_3)} (\bar{\chi}\Theta^{(\ell)})(z)\delta(z' - e^{-\ell\varrho}(z)) d\ell.$$

We know from the hypocontinuity theorem on the Hörmander product of distributions [BDH16, theorem 6.1 p. 219] that the family

$$(\bar{\chi}\Theta^{(\ell)})(z)\delta(z' - e^{-\ell\varrho}(z))$$

with $\ell \geq \ell_1$ large enough, is bounded in $\mathcal{D}'_{\bar{\Gamma}}(\mathbf{R}^k)$, where

$$\bar{\Gamma} \stackrel{\text{def}}{=} (\Gamma \times \mathbf{0}) \cup \Gamma_\varrho \cup ((\Gamma \times \mathbf{0}) + \Gamma_\varrho).$$

For the moment this means that the ℓ_0 -dependent family of distributions associated with the kernels

$$(\star) \stackrel{\text{def}}{=} \int_0^{\ell_0} e^{-\ell(a+d_2+2d_3)} (\overline{\chi} \Theta^{(\ell)})(z) \delta(z' - e^{-\ell\varrho}(z)) d\ell$$

is bounded in $\mathcal{D}'_{\Gamma}(\mathbf{R}^k \times \mathbf{R}^k)$ uniformly in $\ell_0 \geq \ell_1$. In particular the integral converges in $\mathcal{D}'_{\Gamma}(\mathbf{R}^k \times \mathbf{R}^k)$ when ℓ_0 goes to $+\infty$. Now we interpret the integral over the variable z as a push-forward along the fibers of the linear projection

$$p_2 : (z, z') \mapsto z'.$$

The pushforward Theorem yields that $p_{2*}(\cdot)$ is bounded in $\mathcal{D}'_{p_{2*}\overline{\Gamma}}(\mathbf{R}^k)$ where

$$p_{2*}\overline{\Gamma} = (p_{2*}\Gamma_{\varrho}) \cup p_{2*}(\Gamma \times \mathbf{0} + \Gamma_{\varrho})$$

and

$$\begin{aligned} p_{2*}\Gamma_{\varrho} &= \{(x, 0, 0), (0, \eta, \tau)\} \\ p_{2*}(\Gamma \times \mathbf{0} + \Gamma_{\varrho}) &= \left\{ \left(e^{-\ell\varrho}(z), (e^{-\ell\varrho})^*(\lambda) \right); (z, \lambda) \in \Gamma, 0 \leq \ell \leq +\infty \right\} \subset \Gamma \cup N^*(\mathcal{Y}) \end{aligned}$$

since the cone Γ is invariant by the lifted flow of $e^{-\ell\varrho}$ provided $\ell < +\infty$ and the limit points of the form $\lim_{\ell \uparrow \infty} (e^{-\ell\varrho}(z), (e^{-\ell\varrho})^*(\lambda))$ for $(z, \lambda) \in \Gamma$ must belong to the conormal $N^*(\mathcal{Y})$ by the conormal landing condition on Γ . It is at this precise place that we are using the conormal landing condition assumption on Γ . The distributions $\Theta(1 - (e^{\ell\varrho})^*\chi)$ are thus converging in $\mathcal{D}'_{\Gamma \cup N^*\mathbf{R}^{d_1}}(\mathbf{R}^k)$ to

$$\langle \Theta^+, f \rangle = \langle \Theta(1 - \chi), f \rangle + \int_0^{\infty} e^{-\ell(d_2+2d_3)} \langle \overline{\chi}(e^{-\ell\varrho})^*\Theta, f \circ e^{-\ell\varrho} \rangle d\ell.$$

The uniqueness of the extension Θ^+ follows from the continuity of all the operations involved in above. To see the scaling property of the extension we note that the family

$$(\Theta^{(\ell)})_{\ell \geq 0} = (e^{a\ell}(e^{-\ell\varrho})^*\Theta)_{\ell \geq 0}$$

is bounded in $\mathcal{D}'_{\Gamma}(\mathbf{R}^k)$, since $\Theta \in \mathcal{S}'_{\Gamma}(\mathbf{R}^k)$ means that the family $(e^{a\ell}(e^{-\ell\varrho})^*\Theta)_{\ell \geq 0}$ is bounded in $\mathcal{D}'_{\Gamma}(\mathbf{R}^k)$ by definition of $\mathcal{D}'_{\Gamma}(\mathbf{R}^k)$ and $\mathcal{S}'_{\Gamma}(\mathbf{R}^k)$, and observe that

$$e^{a\ell}(e^{-\ell\varrho})^*\Theta^+ = p_{2*} \left(\int_0^{\infty} e^{-\ell(a+d_2+2d_3)} \left((\Theta^{\ell+a})' \overline{\chi} \otimes 1 \right) [e^{-\ell\varrho}] d\ell \right).$$

(3) In the borderline case our proof follows closely [Dan14, Prop 4.9 p. 841] except we work in a parabolic setting. We proceed as above with $(e^{-\ell\varrho})^*$ replaced by $(e^{-\ell\varrho})^*R_0$ if $-\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1 < a \leq -\text{codim}_w(\mathcal{Y} \subset \mathcal{X})$ and $(e^{-s\varrho})^*R_1$ if $a = -\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1$. So our extension reads

$$\langle \Theta^+, f \rangle = \langle \Theta(1 - \chi), f \rangle + \int_0^{\infty} e^{-\ell(d_2+2d_3)} \langle \overline{\chi}(e^{-\ell\varrho})^*\Theta, (R_i f) \circ e^{-\ell\varrho} \rangle d\ell,$$

where $R_i f, i = 0, 1$ is obtained from f by Taylor subtraction. The integral converges absolutely since $[e^{-\ell\varrho}R_i] = \mathcal{O}_{\mathcal{D}'_{\Gamma}}(e^{-\ell(1+i)})$ and the map

$$\Theta \mapsto \Theta^+$$

is continuous from $\mathcal{S}'_{\Gamma}(\mathcal{X} \setminus \mathcal{Y})$ to $\mathcal{S}'_{\Gamma \cup N^*(\mathcal{Y})}(\mathcal{X})$ for all $a' < a$ as we will see below when we check the weak homogeneity of the extension Θ^+ . This shows that when

$$-\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1 < a \leq -\text{codim}_w(\mathcal{Y} \subset \mathcal{X})$$

one can take

$$\Lambda_{\mathbf{R}^{d_1}, \epsilon}(f) = \int_0^{\infty} e^{-\ell(d_2+2d_3)} \langle \Theta_{\epsilon}, \overline{\chi} \circ e^{-\ell\varrho} \Pi(f) \rangle d\ell$$

and when

$$a = -\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1$$

one can take

$$\Lambda_{\mathbf{R}^{d_1}, \epsilon}(f) = \int_0^\infty e^{-\ell(d_2+2d_3)} \langle \Theta_\epsilon, \bar{\chi} \circ e^{-\ell\varrho} \Pi(f) + t(\partial_t f)(\cdot, 0, 0) + y \cdot (\partial_y f)(\cdot, 0, 0) \rangle d\ell.$$

For simplicity, in the remainder of the proof we shall specialise to the case $-\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1 < a \leq -\text{codim}_w(\mathcal{Y} \subset \mathcal{X})$. By the wave front set condition on Θ_ϵ one can always decompose $\Lambda_{\mathbf{R}^{d_1}, \epsilon}$ under the product form

$$\Lambda_{\mathbf{R}^{d_1}, \epsilon} = c_\epsilon \Lambda_{\mathbf{R}^{d_1}}$$

where

$$\Lambda_{\mathbf{R}^{d_1}} = \Pi$$

is a distribution independent of ϵ , supported on \mathbf{R}^{d_1} , with wavefront set contained in $N^*(\mathbf{R}^{d_1})$, and the function c_ϵ is given by

$$c_\epsilon(x) = \int_0^\infty e^{-\ell(d_2+2d_3)} \Theta_\epsilon(x, y, t) \bar{\chi}(e^{-\ell\varrho}(x, y, t)) d\ell dy dt.$$

To check the weak homogeneity bound write

$$e^{\ell' a} \langle e^{-\ell'\varrho} \Theta^+, f \rangle = \int_0^\infty \langle \Theta^{(\ell)} \bar{\chi}, e^{-(\ell-\ell')\varrho} \varphi - \Pi(\varphi) \rangle d\ell$$

and observe that the support of $e^{-(\ell-\ell')\varrho} \varphi$ meets the support of $\bar{\chi}$ only if $\ell \geq C + \ell'$ for a constant C that depends only on the support of $\bar{\chi}$. So the integral can be split in

$$- \int_0^{C+\ell'} \langle \Theta^{(\ell)} \bar{\chi}, \Pi(\varphi) \rangle d\ell + \int_{C+\ell'}^\infty \langle \Theta^{(\ell)} \bar{\chi}, e^{-(\ell-\ell')\varrho} \varphi - \Pi(\varphi) \rangle d\ell$$

A change of variable shows that the second term is uniformly bounded in ℓ' whereas the first term is bounded above by $(C + \ell') \|\varphi\|_{C^0}$. This concludes the proof that $e^{\ell a} e^{-t\varrho} \bar{U} = \mathcal{O}_{\mathcal{D}'}(a)$. \square

In the case (b), note that the extension is no longer unique. Any two extensions differ by some conormal distribution supported on \mathcal{Y} whose order is 0 if $-\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1 < a \leq -\text{codim}_w(\mathcal{Y} \subset \mathcal{X})$ and of order 1 if $a = -\text{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1$.

3.3.3.2 Two step canonical extensions

We need the following refinement of the canonical extension. It involves a two step extension procedure.

Theorem 3.20 *Let $\mathcal{Y} \subset \mathcal{X}$ be a closed embedded submanifold in the ambient smooth manifold \mathcal{X} , $p \in \mathcal{Y}$ a point in \mathcal{Y} , a_1, a_2 some real numbers. Assume we are given a smooth function χ such that $\chi = 1$ near \mathcal{Y} and $\chi = 0$ outside some larger neighbourhood. Set*

$$\Omega \stackrel{\text{def}}{=} \left\{ m \in \mathcal{X} : \text{dist}(m, p) > 2\text{dist}(m, \mathcal{Y}) \right\}.$$

Let Θ be a distribution in $\mathcal{D}'(\mathcal{X} \setminus \mathcal{Y})$ such that there exists a scaling field ϱ_1 with respect to $\{p\}$ and a scaling field ϱ_2 with respect to \mathcal{Y} , such that both flows of ϱ_1 and ϱ_2 preserve the inclusion $p \in \mathcal{Y}$. Furthermore assume that

– for every test function $\varphi \in C_c^\infty(\Omega \setminus \mathcal{Y})$,

$$e^{\ell_2 a_2 + \ell_1 a_1} \left\langle e^{-\ell_2 \varrho_2^*} e^{-\ell_1 \varrho_1^*} (\chi \Theta), \varphi \right\rangle \quad (\ell_1, \ell_2 \geq 0)$$

is bounded;

– for every test function $\varphi \in C_c^\infty(\mathcal{X} \setminus \mathcal{Y})$,

$$e^{\ell_1 a_1} \left\langle e^{-\ell_1 \varrho_1^*}(\chi \Theta), \varphi \right\rangle \quad (\ell_1 \geq 0)$$

is bounded.

If

$$\begin{aligned} a_1 &> -\dim_w(\mathcal{X}), \\ a_2 &> -\text{codim}_w(\mathcal{Y} \subset \mathcal{X}), \end{aligned}$$

then there exists a canonical extension of Θ in $\mathcal{D}'(\mathcal{X})$.

Note that we no longer need any microlocal control in the above statements, only weak topology statements. Let us emphasise that our double scaling assumption only applies when our test function is in the domain $\Omega \setminus \mathcal{Y}$ whereas the second simple scaling assumption applies only when we scale toward the point p . In the Feynman amplitude context of Section 3.4.1 both scaling fields will be admissible in the sense of definition 3.24 hence the fact that the inclusion $p \in \mathcal{Y}$ is preserved by both dynamics will be immediate.

Proof. We reduce the proof to some normal form. We cover the whole of \mathcal{X} by some locally finite open cover, use charts and a subordinated partition of unity $\sum \chi_i = 1$. It suffices to prove the same claim for $\Theta \chi$ where χ supported in some open chart $\kappa : \mathcal{X} \supset U \rightarrow \mathbf{R}^k$ containing p , where $\kappa(\mathcal{Y} \cap U) \subset \mathbf{R}^{d_1+d_2} \times \{0, 0\}$ and $\kappa(p) = 0 \in \mathbf{R}^k$. So we are reduced to study the distribution $\kappa_*(\Theta \chi)$ which is compactly supported and satisfies the assumption of our Theorem on $\mathcal{X} = \mathbf{R}^k$ with $k = d_1 + d_2 + d_3 + d_4$ with coordinates (x, t, y, s) such that

$$\begin{aligned} \{p\} &= \{x = 0, t = 0, y = 0, s = 0\}, \quad \text{and} \quad \mathcal{Y} = \{y = 0, s = 0\} = \mathbf{R}^{d_1+d_2} \times \{0\}, \\ \Omega &= \left\{ \sqrt{|x|^2 + |t|^2 + |y|^2 + |s|^2} > 2\sqrt{|y|^2 + |s|^2} \right\}. \end{aligned}$$

Near p we have from the Linearisation Theorem proved in [Dan14, Prop 2.3] or [Mei20] the identities

$$e^{-\ell E_1} = e^{-\ell \varrho_1} \circ U_1(\ell), \quad \text{and} \quad e^{-\ell E_2} = e^{-\ell \varrho_2} \circ U_2(\ell)$$

where E_1, E_2 are the linear scaling fields reading

$$\begin{aligned} E_1 &= 2t \cdot \partial_t + x \cdot \partial_x + 2s \cdot \partial_s + y \cdot \partial_y, \\ E_2 &= 2s \cdot \partial_s + y \cdot \partial_y, \end{aligned}$$

and $U_1(\ell)$ and $U_2(\ell)$ are two smooth germs of diffeomorphisms near p depending smoothly on $e^{-\ell}$ when $\ell \uparrow \infty$, and both U_1, U_2 have limits when $\ell \uparrow \infty$ that also are smooth germs of diffeomorphisms near $p \in \kappa(U)$. Then, for every test function $\varphi \in C_c^\infty(\Omega \setminus \mathcal{Y})$, we have

$$\begin{aligned} \left\langle e^{-\ell E_1^*} e^{-r E_2^*}(\Theta \chi), \varphi \right\rangle &= \left\langle U_1(\ell) e^{-\ell \varrho_1^*} e^{-r E_2^*}(\Theta \chi), \varphi \right\rangle \\ &= \left\langle e^{-\ell \varrho_1^*} e^{-r E_2^*}(\Theta \chi), U_1(\ell)^{-1*} \varphi \right\rangle \\ &= \left\langle e^{-\ell \varrho_1^*} U_2(r)^* e^{-r \varrho_2^*}(\Theta \chi), U_1(\ell)^{-1*} \varphi \right\rangle \\ &= \left\langle \left(e^{-\ell \varrho_1^*} U_2(r)^* e^{u \varrho_1^*} \right) e^{-\ell \varrho_1^*} e^{-r \varrho_2^*}(\Theta \chi), U_1(\ell)^{-1*} \varphi \right\rangle \\ &= \left\langle e^{-\ell \varrho_1^*} e^{-r \varrho_2^*}(\Theta \chi), \Psi_{\ell, r}^{-1*} U_1(\ell)^{-1*} \varphi \right\rangle, \end{aligned}$$

where

$$\Psi_{\ell, r} \stackrel{\text{def}}{=} e^{-\ell \varrho_1^*} U_2(r)^* e^{\ell \varrho_1^*}.$$

At this stage we make the observation that the families $(\Psi_{u,r}^{-1*}U_1(u)^{-1*}\varphi)_{u \geq 0, r \geq 0}$ are bounded families of test functions since $(\Psi_{u,r})_{u \geq 0, r \geq 0}$ is a bounded family of smooth germs of diffeomorphisms. By the Banach-Steinhaus Theorem the family of distributions

$$(e^{\ell a_1 + r a_2} e^{-\ell \varrho_1^*} e^{-r \varrho_2^*} \Theta \chi)_{\ell, r \geq 0}$$

is weakly, hence strongly, bounded. From this we deduce that

$$e^{\ell a_1 + r a_2} \left\langle e^{-\ell \varrho_1^*} e^{-r \varrho_2^*} \Theta \chi, \Psi_{\ell, r}^{-1*} U_1(\ell)^{-1*} \varphi \right\rangle$$

is bounded for all test functions $\varphi \in C_c^\infty(\Omega \setminus \mathcal{Y})$. Therefore the family $e^{\ell a_1 + r a_2} e^{-\ell E_1^*} e^{-r E_2^*} (\Theta \chi)$ is also bounded in $\mathcal{D}'(\Omega \setminus \mathcal{Y})$. From the Linearisation Theorem, without loss of generality, we may thus choose some linear scaling fields ϱ_1 and ϱ_2 with respect to $\{p\}$ and \mathcal{Y} , respectively, that read

$$\begin{aligned} \varrho_1 &= 2t \cdot \partial_t + x \cdot \partial_x + 2s \cdot \partial_s + y \cdot \partial_y, \\ \varrho_2 &= 2s \cdot \partial_s + y \cdot \partial_y. \end{aligned}$$

Since Θ is a distribution on $\mathcal{D}'(\Omega \setminus \mathcal{Y})$ with scaling degree a_2 with respect to \mathcal{Y} and since $a_2 > -\text{codim}_w(\mathcal{Y} \subset \mathcal{X})$, the extension theorem [3.19](#) implies that we can extend Θ to a distribution $\mathcal{R}\Theta$ on Ω , recall that $p \notin \Omega$. The first extension $\mathcal{R}\Theta$ for Θ extending from $\Omega \setminus \mathcal{Y}$ to the larger space Ω satisfies

$$\mathcal{R}\Theta = \Theta(1 - \chi) + \underbrace{\int_0^\infty (e^{r \varrho_2^*} \psi) \Theta dr}_{\text{converges in } \mathcal{D}'(\Omega)}$$

for every χ such that $\chi = 1$ near \mathcal{Y} , $\chi = 0$ outside some larger neighbourhood (recall that here $\psi = -\varrho_2 \chi$) where the integral underbraced converges in $\mathcal{D}'(\Omega)$ by the assumption on the scaling degree of Θ with respect to \mathcal{Y} . Indeed for any test function $\varphi \in C_c^\infty(\Omega)$ one has

$$\left\langle \int_0^\infty (e^{r \varrho_2^*} \psi) \Theta dr, \varphi \right\rangle = \int_0^\infty e^{-r(d_3 + 2d_4 + a_2)} \langle (e^{r a_2} e^{-r \varrho_2^*} \Theta), \psi e^{-r \varrho_2^*} \varphi \rangle dr$$

where it is immediate that the integrand converges since $d_3 + 2d_4 + a_2 > 0$, the family $(\psi e^{-r \varrho_2^*} \varphi)_{r \geq 0}$ forms a bounded family of test functions in $C_c^\infty(\Omega \setminus \mathcal{Y})$ and $(e^{r a_2} e^{-r \varrho_2^*} \Theta)_{r \geq 0}$ is a bounded family of distributions in $\mathcal{D}'(\Omega \setminus \mathcal{Y})$. We also have from the identity for $T > 0$

$$1 - \chi + \int_0^T e^{r \varrho_2^*} \psi dr = 1 - e^{T \varrho_2^*} \chi,$$

and the absolute convergence of the integral $\int_0^\infty (e^{r \varrho_2^*} \psi) \Theta dr$, that $\mathcal{R}\Theta$ can be defined as the limit

$$\mathcal{R}\Theta = \lim_{T \uparrow \infty} \Theta(1 - e^{T \varrho_2^*} \chi).$$

Here we make the observation that for every isomorphism $f : \mathbf{R}^k \mapsto \mathbf{R}^k$ that stabilizes \mathcal{Y} and commutes with ϱ_2 we have the identity

$$\mathcal{R}\Theta = \Theta(1 - f^* \chi) + \underbrace{\int_0^\infty (e^{r \varrho_2^*} f^* \psi) \Theta dr}_{\text{converges in } \mathcal{D}'(\Omega)}.$$

Since $f^* \chi$ still satisfies the same technical assumptions as χ and $-\varrho_2 f^* \chi = -f^* \varrho_2 \chi = f^* \psi$, the integral underbraced also converges in $\mathcal{D}'(\Omega)$. The key observation is the continuous partition of unity identity for $T > 0$

$$1 - \chi + \int_0^T e^{r \varrho_2^*} \psi dr = 1 - e^{T \varrho_2^*} \chi,$$

which implies

$$\begin{aligned} 1 - \chi + \int_0^T e^{r\varrho_2^*} \psi \, dr - \left(1 - f^* \chi + \int_0^T e^{r\varrho_2^*} f^* \psi \, dr \right) \\ = 1 - e^{T\varrho_2^*} \chi - (1 - e^{T\varrho_2^*} f^* \chi) = e^{T\varrho_2^*} f^* \chi - e^{T\varrho_2^*} \chi = \int_T^\infty e^{r\varrho_2^*} (\psi - f^* \psi) \, dr, \end{aligned}$$

where the last equality holds true in $C^\infty(\Omega \setminus \mathcal{Y})$. Then, we can control the difference

$$\Theta(1 - e^{T\varrho_2^*} \chi) - \Theta(1 - e^{T\varrho_2^*} f^* \chi) = \Theta(e^{T\varrho_2^*} (f^* \chi - \chi)) = \int_T^\infty e^{r\varrho_2^*} (\psi - f^* \psi) \Theta \, dr.$$

Repeating the above estimate, for every test function $\varphi \in C_c^\infty(\Omega \setminus \mathcal{Y})$

$$\left| \left\langle \int_T^\infty e^{r\varrho_2^*} (\psi - f^* \psi) \Theta \, dr, \varphi \right\rangle \right| \lesssim \int_T^\infty e^{-r(d_2 + 2d_3 + a_2)} \, dr \longrightarrow 0$$

as $T \uparrow \infty$.

The whole point is that we start from the information that Θ defined on $\Omega \setminus \mathcal{Y}$ is weakly homogeneous with respect to $\{p\}$ of degree a_1 and we need to make sure that the same property still holds true for the first extension $\mathcal{R}\Theta$. Therefore we need to check that the partial extension $\mathcal{R}\Theta \in \mathcal{D}'(\Omega)$ (recall $p \notin \Omega$) is still weakly homogeneous of degree a_1 when we scale with respect to $\{p\}$.

We scale the renormalized $\mathcal{R}\Theta$ using $e^{-\ell\varrho_1}$ which gives

$$\begin{aligned} e^{\ell a_1} e^{-\ell\varrho_1^*} \mathcal{R}\Theta &= e^{\ell a_1} e^{-\ell\varrho_1^*} \left(\Theta(1 - f^* \chi) + \underbrace{\int_0^\infty (e^{r\varrho_2^*} f^* \psi) \Theta \, dr}_{\text{underbraced term}} \right) \\ &= e^{\ell a_1} \left(e^{-\ell\varrho_1^*} \Theta \right) (1 - \chi) + \underbrace{\int_0^\infty (e^{r\varrho_2^*} \psi) \left(e^{\ell a_1} e^{-\ell\varrho_1^*} \Theta \right) \, dr}_{\text{underbraced term}}, \end{aligned}$$

choosing $f = e^{\ell\varrho_1^*}$ and using the fact the flows $e^{-r\varrho_2}$, $e^{-\ell\varrho_1}$ commute and the absolute convergence of the underbraced term. Then we make the second observation that the family $(e^{t a_1} e^{-\ell\varrho_1^*} \Theta)_{\ell \geq 0}$ satisfies the assumptions of the extension Theorem 3.19 applied to the scaling with respect to $\{p\}$ **uniformly** in the parameter $\ell \geq 0$. Using the boundedness of the extension map \mathcal{R} , this concludes that $\mathcal{R}\Theta \in \mathcal{D}'(\Omega)$ is weakly homogeneous of degree a_1 with respect to $\{p\}$.

We can now conclude, choose a partition of unity $1 = \chi_\Omega + (1 - \chi_\Omega)$ where $\chi_\Omega \in C^\infty(\mathbf{R}^k \setminus \{0\})$ is scale invariant with respect to ϱ_1 , $\chi_\Omega|_\Omega = 1$ near \mathcal{Y} and $\chi_\Omega = 0$ outside Ω , then note that the two families

$$(e^{\ell a_1} e^{-\ell\varrho_1^*} \chi_\Omega \mathcal{R}\Theta)_{\ell \geq 0}$$

and

$$(e^{\ell a_1} e^{-\ell\varrho_1^*} (1 - \chi_\Omega) \Theta)_{\ell \geq 0}$$

are bounded in $\mathcal{D}'(\mathbf{R}^k \setminus \{0\})$. Hence applying the extension Theorem 3.19 to the sum $\chi_\Omega \mathcal{R}\Theta + (1 - \chi_\Omega) \Theta$ yields that $\Theta = \chi_\Omega \mathcal{R}\Theta + (1 - \chi_\Omega) \Theta \in \mathcal{D}'(\mathbf{R}^k \setminus \{0\})$ has a canonical extension in $\mathcal{D}'(\mathbf{R}^k)$ which is the result we wanted. \square

Example 3.21 On \mathbf{R}^2 , fix $p = (p_1, p_2) \in \mathbf{R}^2$ and let

$$\varphi(x_1, x_2) \stackrel{\text{def}}{=} |x_1 - x_2|^{a_2} (|x_1 - p_1| + |x_2 - p_2|)^{(a_1 - a_2)}.$$

It is weakly homogeneous of degree a_2 with respect to the diagonal $\{x_1 = x_2\}$ and of degree a_1 with respect to the point p . Theorem 3.20 therefore applies to φ provided $a_2 > -1$ and $a_1 > -2$.

3.3.3.3 Invariance properties of scaling spaces

We end this section by stating some invariance results which will be useful later, in the proofs of Theorems 3.40 and Theorem 3.20, when we need to make a particular choice of scaling field in the proof and then use the fact that the blow-up does not depend on this choice. In the next proposition all the scaling fields are relative to the closed embedding $\mathcal{Y} \subset \mathcal{X}$.

Proposition 3.22 *Assume $a \in \mathbf{R}$, $\mathcal{Y} \subset \mathcal{X}$ and the distribution $\Lambda \in \mathcal{D}'(\mathcal{X})$, resp. $\mathcal{D}'(\mathcal{X} \setminus \mathcal{Y})$, is weakly homogeneous of degree a for some scaling field ϱ , in the sense that the family*

$$\left(e^{\ell a} e^{-\ell \varrho^*} (\chi \Lambda) \right)_{\ell \geq 0}$$

is bounded in $\mathcal{D}'(\mathcal{X})$, resp. $\mathcal{D}'(\mathcal{X} \setminus \mathcal{Y})$, for some scaling field ϱ and some χ such that $\chi = 1$ near \mathcal{Y} and ϱ is well-defined near $\text{supp}(\chi)$. Then Λ is weakly homogeneous of degree a for every scaling fields. More precisely, for any scaling field $\tilde{\varrho}$ with respect to \mathcal{Y} , every function $\tilde{\chi}$ supported in the domain of $\tilde{\varrho}$ such that $\tilde{\chi} = 1$ near \mathcal{Y} ,

$$\text{the family of distributions } \left(e^{t a} e^{-t \tilde{\varrho}^*} (\tilde{\chi} \Lambda) \right)_{t \geq 0} \text{ is bounded in } \mathcal{D}'(\mathcal{X}), \text{ resp. } \mathcal{D}'(\mathcal{X} \setminus \mathcal{Y}). \quad (3.34)$$

In the sequel, we shall denote by $\mathcal{S}^a(\mathcal{X})$ (resp. $\mathcal{S}^a(\mathcal{X} \setminus \mathcal{Y})$) the space of distributions such that (3.34) holds. Consequently, assume we are given an open subset $\mathcal{U} \subset \mathcal{X}$ such that \mathcal{U} is stable by ϱ_1 and ϱ_2 . Then every $\Lambda \in \mathcal{D}'(\mathcal{U})$ which is weakly homogeneous of degree a for ϱ_1 is also weakly homogeneous of the same degree a for ϱ_2 . In the sequel, we shall denote by $\mathcal{S}^a(\mathcal{U})$ the space of such distributions.

We note that the space $\mathcal{S}^a(\mathcal{X})$, resp. $\mathcal{S}^a(\mathcal{X} \setminus \mathcal{Y})$, is intrinsic, defined independently from the choice of scaling fields. These spaces do not contain microlocal information hence the topologies of the spaces \mathcal{S}^a are weaker than the topologies of spaces \mathcal{S}_Γ^a .

Proof. The proof can be found in [Dan14, theorem 3.3 p. 828] and relies on the linearisation Theorem for scaling fields proved in [Dan14] and also in the exposition of Meinrenken [Mei20, Lemma 2.1 p. 226]. We sketch a proof for completeness. We localize near $x \in \mathcal{Y}$, we choose some chart $\kappa : U \ni x \mapsto \mathbf{R}^{d_1+d_2+d_3}$ in which $\kappa(\mathcal{Y} \cap U) = (\mathbf{R}^{d_1} \times \{0\}) \cap \kappa(U)$. Then once we push ϱ by the linear chart κ we still get a scaling field that we abusively denote by ϱ . In these coordinates the vector field ϱ has the form $\varrho = 2t \cdot \partial_t + y \cdot \partial_y + R(t, y, \partial_t) + H(t, y, \partial_y)$ where R and H vanish at order 2 when (t, y) go to $(0, 0) \in \mathbf{R}^{d_2+d_3}$. Choose χ supported in the open chart U , $\kappa_*(\Lambda \chi)$ is a distribution in $\mathbf{R}^{d_1+d_2+d_3}$ weakly homogeneous of degree a under scaling by ϱ . With no loss of generality it suffices to prove the weak homogeneity of $\kappa_*(\Lambda \chi)$ when we scale with a different scaling field ϱ_2 , the general claim can be deduced by localising plus gluing with a partition of unity without problem. Then the linearisation proof [Dan14, Prop 2.3 p. 826] tells us that we have an equation of the form

$$e^{-\ell \varrho} \circ U(\ell) = e^{-\ell \varrho_2}$$

where $U(\ell)$ is a family of diffeomorphism germs near $(x, 0, 0)$ which has a limit when $\ell \uparrow \infty$, the limit is still a diffeomorphism germ. So for every test function $\varphi \in C_c^\infty(\kappa(U))$ one has

$$e^{\ell a} \left\langle e^{-\ell \varrho_2^*} \kappa_*(\Lambda \chi), \varphi \right\rangle = e^{\ell a} \left\langle U(\ell)^* e^{-\ell \varrho_2^*} \kappa_*(\Lambda \chi), \varphi \right\rangle = e^{\ell a} \left\langle e^{-\ell \varrho^*} \kappa_*(\Lambda \chi), U(\ell)^{-1*} \varphi \right\rangle,$$

where the last quantity on the right hand side is bounded when $\ell \geq 0$ since $(U(\ell)^{-1*} \varphi)_{\ell \geq 0}$ is a bounded family of test functions and $(e^{\ell a} e^{-\ell \varrho^*} \kappa_*(\Lambda \chi))_{\ell \geq 0}$ is a bounded family of distributions by assumption. Then we just proved that $(e^{\ell a} e^{-\ell \varrho_2^*} \kappa_*(\Lambda \chi))_{\ell \geq 0}$ is a weakly, hence strongly, bounded family of distributions which concludes the proof. \square

3.4 Induction on Feynman amplitudes

In the present section we describe a general induction to control analytically the Feynman amplitudes which appear when we study the regularities of the stochastic trees appearing in our equation. This is the content of the main Theorem 3.40. To state this Theorem precisely, we need to introduce several layers of formalism. We first describe correctly the configuration spaces on which we work which is the content of Section 3.4.1 then we give the definition of the Feynman graphs in Section 3.4.2. There is a subtlety in the stochastic estimates, since we work in the non stationary setting (our manifold M is not a flat torus or \mathbf{S}^3 with the round metric) we need to control the size of the pointwise value of Littlewood-Paley blocks, we refer the reader to Lemma 3.45 and Definition 3.46 where we discuss precisely these matters. In terms of Feynman amplitudes, this forces us to introduce a certain distinguished propagator \mathcal{Q}_x^η in which depends on some point $x \in M$, and scale the whole amplitude when all points collapse to the given point $x \in M$, we control the size of the amplitude uniformly in $x \in M$. Mathematically, this requires the definition of some pointed scaling space in Section 3.4.2.1. Finally the section concludes with the inductive proof of Theorem 3.40 which involves a double scaling, where we need to control the growth of the amplitude when all points collide and also when all points collide on a given $x \in M$, uniformly in $x \in M$. This makes the proof much more involved than usual Weinberg convergence like Theorems in usual Quantum Field Theories. It is also because of these double scalings that we need the double extension Theorem 3.20. For several graphs that appear in our stochastic estimates which are enumerated in the first paragraph in Section 3.5.4, we need to feed the induction Theorem 3.40 with more information, this uses subtle smoothing properties of the Schwartz kernel of the resonant product which is discussed in Section 3.5.4.1 and also some control of certain renormalised subamplitude in Section 3.5.4.2 we refer the reader to the relevant parts for more details but this can be read after one is used to the general inductive machinery.

3.4.1 Configuration space

The Feynman amplitudes which arise from the stochastic estimates involve the analysis of products of distributional kernels on space-time configuration spaces. In the present section we introduce the necessary general formalism for analysing these amplitudes.

3.4.1.1 Diagonals and scaling fields

In the sequel we write

$$\mathcal{M} \stackrel{\text{def}}{=} \mathbf{R} \times M$$

and work on

$$\mathbf{R}^p \times M^{p+q} \simeq \mathcal{M}^p \times M^q$$

for $p, q \geq 0$. Given a fixed $t \in \mathbf{R}$ we always view some elements of $\mathbf{R}^p \times M^{p+q}$ as some elements of \mathcal{M}^{p+q} using the mapping

$$\mathcal{M}^p \times M^q \simeq \mathcal{M}^p \times (\{t\} \times M)^q \subset \mathcal{M}^{p+q}, \quad (3.35)$$

that is to say attributing the time t to the purely spatial points. The configuration space \mathcal{M}^n of $n = p + q$ points in \mathcal{M} will play a particular role in the sequel. Given a distinguished point $(t, x) \in \mathcal{M}$, writing $m_i = (t_i, x_i) \in \mathcal{M}$, for $I \subset \{1, \dots, n\}$ we denote by

$$\mathbf{d}_I \stackrel{\text{def}}{=} \left\{ m = (m_1, \dots, m_n); x_i = x_j \text{ and } t_i = t_j = t \text{ for } i \neq j \text{ if } (i, j) \in I^2 \right\}, \quad (3.36)$$

when space-time points labelled by I collide;

$$\mathbf{d}_{I, (t, x)} \stackrel{\text{def}}{=} \left\{ m = (m_1, \dots, m_n), m_i = (t, x) \text{ if } i \in I \right\}, \quad \text{the marked diagonal}$$

where all space-time points labelled by I collide to (t, x) ,

$$\mathcal{T}_I \stackrel{\text{def}}{=} \left\{ m = (m_1, \dots, m_n), t_i = t_j = t \text{ for } i \neq j \text{ if } (i, j) \in I^2 \right\},$$

when only time components labelled by I coincide with t

the corresponding diagonals in the product spaces. Let us make several important observations. First the marked diagonal $\mathbf{d}_{[n],(t,x)}$ is in reality just **one point** $((t, x), \dots, (t, x))$ in the configuration space \mathcal{M}^n . Second one has the natural inclusion relations

$$\mathbf{d}_I \subset \mathcal{T}_I \text{ and } \mathbf{d}_I \subset \mathbf{d}_J \text{ and } \mathcal{T}_I \subset \mathcal{T}_J \text{ if } J \subset I.$$

We denote by

$$\mathbf{d}_n \stackrel{\text{def}}{=} \left\{ ((t, x_1), \dots, (t, x_n)) \in \mathcal{M}^n : x_1 \in M \right\},$$

and

$$\mathbf{d}_{n,(t,x)} \stackrel{\text{def}}{=} \left\{ ((t, x), \dots, (t, x)) \right\} \subset \mathcal{M}^n.$$

In the sequel, to take (3.35) into account, we will work on some submanifold $\mathcal{M}^p \times (\{t\} \times M)^q = \mathcal{T}_J \subset \mathcal{M}^n$ for some fixed $J \subset \{1, \dots, n\}$, $|J| = q$, where all time variables indexed by J coincide and equal t .

Example 3.23 We give two examples of stochastic estimates we will meet in the sequel so that the reader can see in what type of space-time domains we need to integrate our Feynman amplitudes.

In the sequel we will calculate $\mathbf{E}[\mathcal{Q}_x^\gamma(\mathfrak{Q}^\varphi(t), \mathfrak{Q}^\varphi(t))]$ which can be represented as a Feynman amplitude (underbraced below) tested against the constant function 1 that reads

$$\begin{aligned} & \mathbf{E}[\mathcal{Q}_x^\gamma(\mathfrak{Q}^\varphi(t), \mathfrak{Q}^\varphi(t))] \\ &= \int_{(-\infty, t]^2 \times M^4} \underbrace{G^{(3)}(s_a - s_b, x_a, x_b) \underline{\mathcal{L}}^{-1}(t - s_a, y_*, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_*, x_b) \mathcal{Q}_x^\gamma(y_*, z_*)}_{\text{Feynman amplitude}} \end{aligned}$$

for some function $G^{(3)}$. The integration is with respect to $dv(x_a, x_b, y_*, z_*) ds_a ds_b$. In this case we integrate a certain Feynman amplitude underbraced on some domain $\mathcal{T}_{\{c,d\}} \subset \mathcal{M}^4$ with two space-time variables (s_a, x_a) , (s_b, x_b) and two space variable (s_c, y_*) , (s_d, z_*) promoted to space-time variables by writing $s_c = s_d = t$. On the other hand, when we want to prove the continuity in time using a Kolmogorov type argument we will also consider

$$\begin{aligned} & \mathbf{E}[\mathcal{Q}_x^\gamma(\mathfrak{Q}^\varphi(t_1), \mathfrak{Q}^\varphi(t_2))] \\ &= \int_{(-\infty, t]^2 \times M^4} \underbrace{G^{(3)}(s_a - s_b, x_a, x_b) \underline{\mathcal{L}}^{-1}(t_1 - s_a, y_*, x_a) \underline{\mathcal{L}}^{-1}(t_2 - s_a, z_*, x_b) \mathcal{Q}_x^\gamma(y_*, z_*)}_{\text{Feynman amplitude}}. \end{aligned}$$

In this case we integrate over $\mathcal{T}_{\{c,d\}} \subset \mathcal{M}^4$ where the last two time variables are taken equal to some fixed times t_1, t_2 . With this example we see that we also need to fix the time variables.

We will also use a **particular class of scaling fields that will leave all \mathcal{T}_J and \mathbf{d}_I stable**, such class of scaling fields will be called *admissible*.

Definition 3.24 (Admissible scaling fields) Pick some open chart $U \subset M, \kappa : U \mapsto \mathbf{R}^d$ such that $\kappa(U) \subset \mathbf{R}^d$ is an open convex ball – this is always possible up to making things smaller.

We define a scaling field $\varrho_{[p]}$ in U from its flow given by for any $(x_1, \dots, x_p) \in \kappa(U)^p$ by

$$(x_1, \dots, x_p) \in U^p \mapsto (x_1, e^{-t}(x_2 - x_1) + x_1, \dots, e^{-t}(x_p - x_1) + x_1) \in \kappa(U)^p;$$

it means $\varrho_{[p]}$ reads $\sum_{j=2}^p (x_j - x_1) \partial_{x_j}$ in the above coordinate chart. We define the local scaling field on $(\mathbf{R} \times U)^p$, with local coordinates $(s_i, x_i)_{1 \leq i \leq p}$, setting

$$\varrho = 2 \left(\sum_{j=2}^p (s_j - s_1) \partial_{s_j} \right) + \varrho_{[p]} \stackrel{\text{def}}{=} 2\varrho_{\text{times}} + \varrho_{[p]}.$$

We obtain global scaling fields by gluing together the above local objects. Consider a cover $\cup_i U_i^p$ of some neighborhood of the space diagonal and choose $\chi \in C_c^\infty(\cup_i U_i^p)$ such that $\chi = 1$ near the space diagonal. For a subordinated partition of unity $\sum \chi_i = 1$ of $\text{supp}(\chi)$, we set $\varrho = 2\varrho_{\text{times}} + \chi \sum_i \chi_i \varrho_i$ where each local scaling field $\varrho_i \in C^\infty(T(U_i^p))$ is constructed in some charts as above.

The same construction also works when we scale towards some marked points, in all charts $\kappa : U \mapsto M$ containing x , we decide that we scale with the flow

$$(x_1, \dots, x_p) \in U^p \mapsto \left(e^{-t}(x_1 - x) + x, e^{-t}(x_2 - x) + x, \dots, e^{-t}(x_p - x) + x \right) \in \kappa(U)^p,$$

it means $\varrho_{[p]}$ reads $\sum_{j=1}^p (x_j - x) \partial_{x_j}$ in the above coordinate chart. If the chart does not contain x , we decide our flow is trivial (the generator of the flow is the zero vector field) and $\varrho_{[p]} = 0$ in such a chart. We define the local scaling field on $(\mathbf{R} \times U)^p$, with local coordinates $(s_i, x_i)_{1 \leq i \leq p}$, setting

$$\varrho_{(t,x)} = 2 \left(\sum_{j=1}^p (s_j - t) \partial_{s_j} \right) + \varrho_{[p]} \stackrel{\text{def}}{=} 2\varrho_{\text{times}} + \varrho_{[p]}.$$

We obtain global scaling fields by gluing together the above local objects. Consider a cover $\cup_i U_i^p$ of some neighborhood of the space diagonal and choose $\chi \in C_c^\infty(\cup_i U_i^p)$ such that $\chi = 1$ near the space diagonal. For a subordinated partition of unity $\sum \chi_i = 1$ of $\text{supp}(\chi)$, we set $\varrho = 2\varrho_{\text{times}} + \chi \sum_i \chi_i \varrho_i$ where each local scaling field $\varrho_i \in C^\infty(T(U_i^p))$ is constructed in charts as above.

We make the observation that in all the above situations the constructed vector fields generate some dynamics which preserve all the diagonals $\mathbf{d}_I, \mathcal{T}_I$ or all the marked diagonals $\mathbf{d}_{I,(t,x)}$ in the case of scaling fields $\varrho_{(t,x)}$ on marked points **by construction** (they preserve these diagonals in the charts locally, hence globally by gluing). The above definition shows that *admissible* scaling fields are abundant. Admissible scaling fields enjoy another remarkable property. The cotangent lift of $e^{-u\varrho}$ stabilizes all the conormal bundles of all the partial diagonals.

Lemma 3.25 *For all $I \subset \{1, \dots, p\}$ and $\ell \geq 0$, for all admissible scaling fields ϱ , one has*

$$e^{-\ell\varrho^*}(N^*(\mathbf{d}_I \subset \mathcal{M}^p)) \subset N^*(\mathbf{d}_I \subset \mathcal{M}^p).$$

Proof. If $v \in T_x \mathbf{d}_I$, $\text{de}_x^{\ell\varrho}(v) \in T_{e^{\ell\varrho}(x)} \mathbf{d}_I$ (since the flow stabilizes the diagonals) implies that if $\xi \in T_{e^{\ell\varrho}(x)} \mathbf{d}_I^\perp$ then $\langle \xi, \text{de}_x^{\ell\varrho}(v) \rangle = 0$ for all $v \in T_x \mathbf{d}_I$ hence ${}^t \text{de}_x^{\ell\varrho}(v)(\xi) \in T_x \mathbf{d}_I^\perp = N_x^*(\mathbf{d}_I)$, which concludes our proof. \square

We will typically be given a family $(\Lambda_\epsilon)_{0 < \epsilon \leq 1}$ of distributions on $\mathcal{T}_J \setminus (\cup_{I \subset \{1, \dots, p\}} \mathbf{d}_I)$ that converge to a limit as a distribution outside all the diagonals of \mathcal{T}_J . We will use Theorem 3.19 to extend it to the whole of \mathcal{T}_J by an inductive procedure under some scaling-type assumptions. The inductive structure of the extension procedure will come from the geometric form of *Popineau & Stora's lemma*, which we recall here. We associate to $I \subset \{1, \dots, p\}$, the open set:

$$\mathcal{O}_I \stackrel{\text{def}}{=} \left\{ m = (m_1, \dots, m_p) \in \mathcal{M}^p; m_i \neq m_j \forall (i, j) \in I \times I^c \right\} \subset \mathcal{M}^p.$$

Lemma 3.26 *One has*

$$\mathcal{M}^p \setminus \mathbf{d}_p = \bigcup_{I \subset \{1, \dots, p\}} \mathcal{O}_I$$

and there is an associated smooth partition of the unity, $1 = \sum_{I \subset \{1, \dots, p\}} \eta_I \in C^\infty(\mathcal{M}^p \setminus \mathbf{d}_p)$, with the family $(\eta_I \circ e^{-\ell\varrho})_{\ell \geq 0}$ bounded in $C^\infty(\mathcal{M}^p \setminus \mathbf{d}_p)$ for every admissible scaling field ϱ with respect to the deepest diagonal $\mathbf{d}_p \subset \mathcal{M}^p$.

The proof is simple and can be found in [DH19, Lemma 6.3]. The proof of the claim on the family $(\eta_I \circ e^{-\ell\varrho})_{\ell \geq 0}$ can be found in [Dan13, Lemma 6.3.1 p. 131]. Since $\mathcal{T}_J \subset \mathcal{M}^p$ for $J \subset \{1, \dots, p\}$, the above partition of unity induces naturally a partition of unity on $\mathcal{T}_J \setminus \mathbf{d}_p$ with the same properties.

3.4.1.2 Hörmander product of distributions

In the simplest cases the distributions Λ_ϵ will be given as some products of distributions, with each factor depending possibly only on a subset of the variables \mathcal{M}^p . The easiest case in which to make sense of such products relies on Hörmander's product theorem [BDHI6, theorem 6.1 p. 219] and gives the following statement.

Lemma 3.27 *If $\Lambda_1 \in \mathcal{D}'(\mathcal{M}^p)$ depends only on the first $1 \leq k < p$ components of \mathcal{M}^p and $\Lambda_2 \in \mathcal{D}'(\mathcal{M}^p)$ depends only on the last $p - (k - 1)$ components, so they have only one component in common, and*

$$\begin{aligned} WF(\Lambda_1) &\subset \bigcup_{I \subset \{1, \dots, k\}} N^*(\mathbf{d}_I) \cup N^*(\mathcal{T}_I), \\ WF(\Lambda_2) &\subset \bigcup_{J \subset \{k, \dots, p\}} N^*(\mathbf{d}_J) \cup N^*(\mathcal{T}_J), \end{aligned}$$

then the product $\Lambda_1 \Lambda_2$ is well-defined in $\mathcal{D}'(\mathcal{M}^p)$ and

$$WF(\Lambda_1 \Lambda_2) \subset (WF(\Lambda_1) + WF(\Lambda_2)) \cup WF(\Lambda_1) \cup WF(\Lambda_2).$$

Proof. Denote by λ a generic element of $T^*\mathcal{M}$. If $(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$ and $(0, \dots, 0, \mu_k, \mu_{k+1}, \dots, \mu_p)$ stand for some non-null elements of $T^*(\mathcal{M}^p)$ such that

$$\sum \lambda_i = 0, \quad \sum \mu_j = 0,$$

then the convex sum

$$(\lambda_1, \dots, \lambda_k, 0, \dots, 0) + (0, \dots, 0, \mu_k, \mu_{k+1}, \dots, \mu_p) = (\lambda_1, \dots, \lambda_k + \mu_k, \mu_{k+1}, \dots, \mu_p)$$

cannot vanish. This implies that $WF(\Lambda_1) + WF(\Lambda_2)$ does not meet the zero section $\{0\}$ and one can apply Hörmander's Theorem [BDHI6, theorem 6.1 p. 219] which yields the existence of the distributional product $\Lambda_1 \Lambda_2$ together with a bound on the wave front set $WF(\Lambda_1 \Lambda_2)$ of the product. \square

We give another important consequence of Theorem 3.19 before talking about Feynman amplitudes.

Proposition 3.28 *Let $\mathcal{Y} \subset \mathcal{X}$ be a closed embedding and let ϱ stand for a parabolic scaling field for the inclusion $\mathcal{Y} \subset \mathcal{X}$. Assume we are given some closed conic sets Γ_1, Γ_2 in $T^*\mathcal{X} \setminus \mathcal{Y}$ such that*

$$\mathbf{0} \notin (\Gamma_1 + \Gamma_2).$$

and such that $e^{-\ell\varrho^*}(\Gamma_i) \subset \Gamma_i$, $\forall \ell \geq 0$ and $i \in \{1, 2\}$. Assume also that we are given two distributions

$$\Lambda_1 \in \mathcal{S}_{\Gamma_1}^{s_1, \varrho}(\mathcal{X} \setminus \mathcal{Y}) \cap \mathcal{D}'(\mathcal{X}), \quad \Lambda_2 \in \mathcal{S}_{\Gamma_2}^{s_2, \varrho}(\mathcal{X}) \cap \mathcal{D}'(\mathcal{X}),$$

such that the product $\Lambda_1 \Lambda_2$ is well-defined on $\mathcal{U} \subset \mathcal{X} \setminus \mathcal{Y}$. If

$$s_1 + s_2 > -\text{codim}_w(\mathcal{Y})$$

then this product has a unique extension as an element of $\mathcal{S}_{\Gamma}^{s_1+s_2, \varrho}(\mathcal{X})$ with

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2) \cup N^*(\mathcal{Y} \subset \mathcal{X}).$$

The condition $\mathbf{0} \notin (\Gamma_1 + \Gamma_2)$ ensures that the distributional product $\Lambda_1 \Lambda_2$ is well-defined at least on $\mathcal{X} \setminus \mathcal{Y}$ and the stability of Γ_1, Γ_2 under the lifted cotangent flow ensures that the convex sum $(\Gamma_1 + \Gamma_2) \subset T^*\mathcal{X} \setminus \mathcal{Y}$ satisfies the conormal landing condition for the inclusion $\mathcal{Y} \subset \mathcal{X}$. The statement of the above proposition means that for any mollification $\Lambda_1^\epsilon, \Lambda_2^\epsilon$ of these distributions which converge in the respective functional spaces the product $\Lambda_1^\epsilon \Lambda_2^\epsilon$ is converging in $\mathcal{S}_{\Gamma}^{s_1+s_2, \varrho}(\mathcal{M}^p)$ to a limit independent of the mollification.

3.4.2 Feynman graphs and Feynman amplitudes

We are now ready to define the Feynman graphs we are using in order to control the Hölder-Besov norms of the tree appearing in the construction of the Φ_3^4 measure. For an oriented finite graph (V, E) with vertex set V and edge set E we denote by $v(e)_-, v(e)_+$ its two vertices, according to its orientation.

Definition 3.29 A Feynman graph for Φ_3^4 is an oriented finite graph $\mathcal{G}_{(t,x)} = (V, E)$ with p vertices in V and an edge set E with no two edges joining a given pair of vertices, along with

- a distinguished edge $e_{\text{ref}} \in E$,
- a subset J of V which indicates which times are set equal to the fixed $t \in \mathbf{R}$,
- for each vertex $v \in V$, a variable $z_v = (t_v, x_v) \in (\mathbf{R} \times M)$, with the restriction that for the two vertices $v(e_{\text{ref}})_-, v(e_{\text{ref}})_+$ attached to e_{ref} , one has $z_{v(e_{\text{ref}})_\pm} = (t, x_{v(e_{\text{ref}})_\pm})$ for some $x_{v(e_{\text{ref}})_\pm} \in M$.

We set

$$\tilde{E} \stackrel{\text{def}}{=} E \setminus \{e_{\text{ref}}\}.$$

We furthermore assume that the following facts hold.

- The set V of vertices can be partitioned as

$$V = V' \sqcup V_A$$

where V' is a disjoint union of singletons and V_A is a disjoint union of

$$n_{\mathcal{G}} \stackrel{\text{def}}{=} |A|$$

triples of vertices indexed by a finite set A , with each triple made up of a distinguished pair of vertices and another vertex. For $j \in [n_{\mathcal{G}}]$ such a triple in V_A reads $(v_*^j, (v_1^j, v_2^j))$ where (v_1^j, v_2^j) is the distinguished pair and v_*^j the remaining vertex.

- For every $j \in [n_{\mathcal{G}}]$ there is no edge in the graph relating v_1^j to v_2^j or one of these points to v_*^j .
- We are given for each edge $e \in E$ a kernel $K_e \in \mathcal{K}^{a_e}$ for some scaling exponent $a_e \in \mathbf{R}$. (The space \mathcal{K}^{a_e} was defined in Lemma 3.17). Moreover for the distinguished edge $e_{\text{ref}} \in E$ the corresponding kernel is given by $K_{e_{\text{ref}}} = \mathcal{Q}_x^\gamma$ and e_{ref} is the only edge in E whose kernel K_e is of the form \mathcal{Q}_y^γ for some $y \in M$.

See figure 3.4.2 below for an illustration. In the sequel we often omit the base point (t, x) , writing \mathcal{G} instead of $\mathcal{G}_{(t,x)}$. Since renormalisation also involves the analysis of singularities of Feynman subgraphs we also need a notion of Feynman subgraphs adapted to our specific setting.

Definition 3.30 A Feynman subgraph $\mathcal{G}_1 = (V_1, E_1) \subset \mathcal{G}_{(t,x)} = (V, E)$ is the data of

- some subset V_1 of the vertices V of \mathcal{G} ,
- some subset E_1 of the edges E of \mathcal{G} such that V_1 can be partitioned as $V_1 = V'_1 \sqcup V'_A$ and $V'_A \subset V_A$ respects the partitioning of V_A , any triple $(v_*^j, (v_1^j, v_2^j))$, $j \in n_{\mathcal{G}_1}$ in V'_A corresponds to a triple in V_A ;

along with the conditions that every edge $e \in E_1$ has its bounding vertices $v(e)_-, v(e)_+$ in V_1 , and that the subgraph \mathcal{G}_1 does not necessarily contain the distinguished edge e_{ref} of \mathcal{G} . A (sub)graph \mathcal{G} is said to be irreducible if it cannot be disconnected by removing exactly one edge $e \in E$.

We also need to recall the notion of loops for the Feynman graphs we consider, this is given by the usual Euler formula.

Definition 3.31 Given a Feynman graph \mathcal{G} whose rule is defined above we define the *number of loops for our exotic Feynman graph \mathcal{G}* as

$$b_1(\mathcal{G}) \stackrel{\text{def}}{=} |E(\mathcal{G})| - |V'(\mathcal{G})| - n_{\mathcal{G}} + 1.$$

We are given some distributions

$$\begin{aligned} [\odot](\cdot, \cdot, \cdot) &\in \mathcal{S}_{N^*(\mathbf{d}_3)}^{-6}(M^3), \\ \mathcal{Q}_x^\gamma &\in \mathcal{S}_{T^*(x,x)M^2}^{-6-2\gamma}(M^2), \end{aligned}$$

where the scaling of $[\odot]$ is with respect to the deepest diagonal $\mathbf{d}_3 = \{(y, y, y) : y \in M\} \subset M^3$ and the scaling of \mathcal{Q}_x^γ is with respect to the point (x, x) . Here $[\odot]$ is a general notation for the kernel $[\odot_i]$ of the resonant product in some chart $i \in I$ which is defined by

$$[\odot_i](x, y, z) \stackrel{\text{def}}{=} \sum_{|k-\ell| \leq 1} P_k^i(x, y) \tilde{P}_\ell^i(x, z),$$

where P_k^i, \tilde{P}_k^i are the Littlewood-Paley blocks introduced in Appendix 3.G, and \mathcal{Q}_x^γ is the kernel probing the regularity of the trees, introduced below in Definition 3.46. The weak homogeneity exponent -6 for $[\odot_i]$ comes from the fact that $[\odot_i] \in \mathcal{D}'(M^3)$ is the Schwartz kernel of the resonant product and that our manifold M has dimension 3.

As a last piece of notation let us introduce a function space \mathcal{R}^γ adapted to the kernels varying with a parameter $x \in M$, such as the kernel \mathcal{Q}_x^γ . Note that we already know that given $x \in M$ the kernel $\mathcal{Q}_x^\gamma \in \mathcal{D}'(M \times M)$ is singular at (x, x) and smooth everywhere else. Also, the singular locus is moving with $x \in M$. Our goal is to define a correct functional space \mathcal{R}^γ that measures the singularities of \mathcal{Q}_x^γ uniformly in $x \in M$. This is done in the following definition.

Definition 3.32 Fix $x_0 \in M$, along with a pair of closed neighbourhoods $U_{x_0} \Subset \Omega_{x_0} \subset M$ of x_0 , and define a conical neighbourhood

$$\mathcal{C}_{x_0} \stackrel{\text{def}}{=} \bigcup_{x \in \Omega_{x_0}} T^*(x,x)M^2$$

of $T^*(x_0, x_0)M^2$. Then we define \mathcal{R}^γ as the space of all families $(\mathcal{Q}_x)_{x \in U_{x_0}}$ of distributions

$$\mathcal{Q}_x \in \mathcal{D}'(M^2) \cap C^\infty(M^2 \setminus \{(x, x)\})$$

such that

$$\left(e^{-(6+2\gamma)\ell} e^{-\ell \varrho_x} \mathcal{Q}_x \right)_{\ell \geq 0, x \in U_{x_0}}$$

is bounded in $\mathcal{D}'_{\mathcal{C}_{x_0}}(M^2)$ for every family of scaling fields ϱ_x scaling with respect to (x, x) .

We can now define the Feynman amplitudes.

Definition 3.33 We view $[\odot]$ and \mathcal{Q}_x^γ as some distributions on \mathcal{M}^3 and \mathcal{M}^2 by pull-back by the canonical projection from \mathcal{M}^p to M^p . Denote by $\mathbf{d}_{V'}$ the diagonals of $(\mathbf{R} \times M)^p$, for $V' \subset V$. The amplitude $\mathcal{A}^{\mathcal{G}}$ associated with the graph $\mathcal{G}_{(t,x)}$ is the distribution on $\mathcal{T}_J \setminus \bigcup_{V' \subset V} \mathbf{d}_{V'}$ defined by the product

$$\mathcal{A}_{\mathcal{G}}(z_1, \dots, z_p) \stackrel{\text{def}}{=} \mathcal{Q}_x^\gamma(x_{v(e_{\text{ref}})_-}, x_{v(e_{\text{ref}})_+}) \prod_{e \in \tilde{E}} K_e(z_{v(e)_-}, z_{v(e)_+}) \prod_{1 \leq j \leq n_{\mathcal{G}}} [\odot](x_{v_*^j}, x_{v_1^j}, x_{v_2^j})$$

with the second product corresponding to all of V_A , see Figure 3.4.2 for a fully detailed example. We talk of $\mathcal{A}_{\mathcal{G}}$ as the *Feynman amplitude associated with \mathcal{G}* .

The following fact is a direct consequence of Lemma 3.27. It allows two things in the analysis of Feynman amplitudes

1. If a Feynman graph is a tree, i.e. it contains no loops, then the corresponding Feynman amplitude $\mathcal{A}_{\mathcal{G}}$ is always well-defined as a distribution on the corresponding configuration space.

2. We can reduce the analysis to **irreducible graphs** which contain at least one loop since joining two subgraphs by a bridge is always well-defined microlocally.

These two facts are detailed in the next two statements. The following Lemma states that Feynman trees are always well defined.

Lemma 3.34 *If $\mathcal{G} = (V, E)$ is a tree, for every $e \in \tilde{E}$, each two point kernel K_e belongs to the module \mathcal{K}^{a_e} , $a_e \in \mathbf{R}$ endowed with the topology of Lemma 3.17 each three point kernel*

$$[\odot](\cdot, \cdot, \cdot) \in \mathcal{S}_{N^*(\mathbf{d}_3 \subset M^3)}^{-6}(M^3)$$

where the scaling is with respect to $\mathbf{d}_3 \subset M^3$ and the eventual marked edge e_{ref} is associated with the propagator \mathcal{Q}_x^γ which belongs to the topological space \mathcal{R}^γ from Definition 3.32. Then the multilinear map

$$\left([\odot], (K_e)_{e \in E}, \mathcal{Q}_x^\gamma\right) \in \mathcal{S}_{N^*(\mathbf{d}_3 \subset M^3)}^{-6}(M^3) \times \prod_{e \in E} \mathcal{K}^{a_e} \times \mathcal{R}^\gamma \longmapsto \mathcal{A}_{\mathcal{G}} \in \mathcal{D}'_\Gamma(\mathcal{T}_J)$$

where

$$\Gamma = \bigcup_{V' \subset V} N^*(\mathbf{d}_{V'}) \cup N^*(\mathcal{T}_{V'})$$

is continuous.

The simplest example which illustrates the above claim is the composition of pseudo-differential kernels. If we represent each kernel by an edge then composition can be interpreted in terms of Feynman rules as gluing the edges at one common vertex and this is always perfectly well defined, the diagonal singularities of the kernels do not matter. We now state a useful corollary of Lemma 3.34 which allows to restrict the analysis of Feynman amplitudes to *connected irreducible subgraphs*.

Corollary 3.35 *Let $\mathcal{A}_{\mathcal{G}}$ be a Feynman amplitude which is obtained by joining two irreducible amplitudes $\mathcal{A}_{\mathcal{G}_1}, \mathcal{A}_{\mathcal{G}_2}(z_j, \mathbf{z}_2)$ by a propagator $K \in \mathcal{D}'(M^2)$ whose wave front set is in the conormal $N^*(\mathbf{d}_2 \subset M^2)$ to the diagonal*

$$\mathcal{A}_{\mathcal{G}_1}(\mathbf{z}_1, z_i)K(z_i, z_j)\mathcal{A}_{\mathcal{G}_2}(z_j, \mathbf{z}_2)$$

then if $\mathcal{A}_{\mathcal{G}_1}$ and $\mathcal{A}_{\mathcal{G}_2}(z_j, \mathbf{z}_2)$ are some well defined distributions with wave front set in

$$\begin{aligned} WF(\mathcal{A}_{\mathcal{G}_1}) &\subset \bigcup_{V'_1 \subset V_1} N^*(\mathbf{d}_{V'_1}) \cup N^*(\mathcal{T}_{V'_1}), \\ WF(\mathcal{A}_{\mathcal{G}_2}) &\subset \bigcup_{V'_2 \subset V_2} N^*(\mathbf{d}_{V'_2}) \cup N^*(\mathcal{T}_{V'_2}), \end{aligned} \tag{3.37}$$

then the global amplitude $\mathcal{A}_{\mathcal{G}}$ is a well-defined distribution with wave front set included in

$$\bigcup_{V' \subset V} N^*(\mathbf{d}_{V'}) \cup N^*(\mathcal{T}_{V'}).$$

The weak homogeneity of $\mathcal{A}_{\mathcal{G}}$ is then the sum of the weak homogeneities of the subamplitudes and of the kernel K .

Proof. Since the amplitude of a reducible graph reads

$$\mathcal{A}_{\mathcal{G}_1}(\mathbf{z}_1, z_i)K(z_i, z_j)\mathcal{A}_{\mathcal{G}_2}(z_j, \mathbf{z}_2)$$

for some collective variables $(\mathbf{z}_1, z_i), (z_j, \mathbf{z}_2)$ partitioning $\{z_v\}_{v \in V}$ and corresponding to a partition $V = V_1 \cup V_2$ of V , where $K(z_i, z_j)$ has wave front contained in $N^*(\mathbf{d}_2 \subset (\mathbf{R} \times M)^2) \cup N^*(\{t_i = t_j\})$ and (3.37) holds, one can apply Lemma 3.27 twice to

$$\mathcal{A}_{\mathcal{G}_1}(\mathbf{z}_1, z_i)K(z_i, z_j)$$

then to

$$(\mathcal{A}_{\mathcal{G}_1}(\mathbf{z}_1, z_i)K(z_i, z_j))\mathcal{A}_{\mathcal{G}_2}(z_j, \mathbf{z}_2).$$

It shows that the product is well defined so the only difficulty is to treat the amplitudes $\mathcal{A}_{\mathcal{G}_1}$ and $\mathcal{A}_{\mathcal{G}_2}$. \square

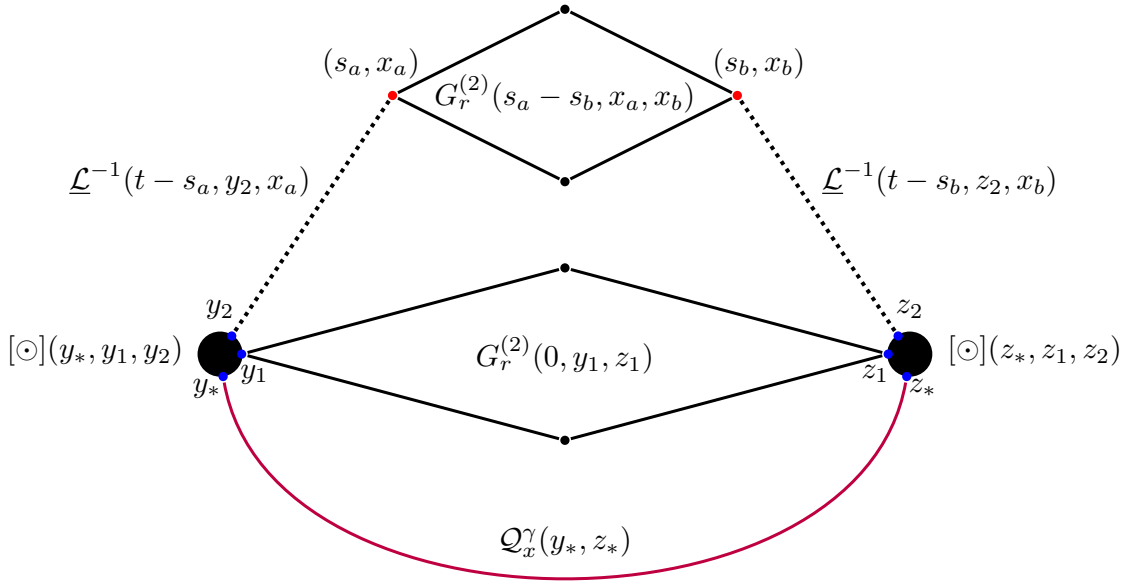


Figure 24: An example of Feynman graph $\mathcal{G}(= \mathcal{G}_{24})$ with $n_{\mathcal{G}} = 2$ along with the contributions to $\mathcal{A}_{\mathcal{G}}$ of its edges and vertices. Space-time vertices are pictured by red nodes, purely space vertices (with time set to the value t) are pictured by blue nodes, and black nodes represent the noises.

3.4.2.1 Pointed scaling spaces

We next define a parameterised version of the functional scaling spaces \mathcal{S}_Γ^a in configuration space which generalises Definition 3.32. The elements in our new functional space depend on some space time point $z = (t, x) \in \mathcal{M}$. We will use these functional spaces when we will scale the whole Feynman amplitude $\mathcal{A}_{\mathcal{G}}$ with respect to the marked diagonal $(z, \dots, z) \subset \mathcal{M}^p$ and test that everything is uniform in $z \in \mathcal{M}$. We start by describing the geometrical setting. We are given:

- For every $z_0 = (t_0, x_0)$ an open neighborhood U_{z_0} of z_0 in \mathcal{M} so that $U_{z_0}^p$ is a neighborhood of $(z_0, \dots, z_0) \in \mathcal{M}^p$;
- a continuous family of scaling fields $\varrho_z \in C^\infty(T(\mathcal{M}^p))$, $z \in U_{z_0}$ on \mathcal{M}^p such that ϱ_z scales with respect to $\{(z, \dots, z)\} \subset U_{z_0}^p$ and $e^{-\ell\varrho_z}U_{z_0}^p \subset U_{z_0}^p$ for every $z_0 \in U_{z_0}$, $\ell \geq 0$ and the flow of ϱ_z preserves all the diagonals \mathbf{d}_I and marked diagonals $\mathbf{d}_{I,z}$.

We need to give an important example which shows that such a geometric setting is non-empty and that one can always produce such a setting.

Example 3.36 *In a local product chart $(a, b) \times U$, $U \subset M$, for every $z = (t, x) \in (a, b) \times U$ in the chart, the typical example of such a vector field reads*

$$\varrho_z = 2(t_1 - t)\partial_{t_1} + \dots + 2(t_p - t)\partial_{t_p} + (x_1 - x) \cdot \partial_{x_1} + \dots + (x_p - x) \cdot \partial_{x_p}.$$

Then we define the functional data. As above we define the functional spaces associated to distributions which are weakly homogeneous at a space-time diagonal $(z, \dots, z) \in \mathcal{M}^I$, in a way which is uniform in $z \in \mathcal{M}$. Here is an example that may help fix the setting.

Example 3.37 *The function*

$$(\mathbf{R}^3)^p \ni (x_1, \dots, x_p) \mapsto \left(|x_1 - x| \cdots |x_p - x| \right)^{-\frac{1}{2}}$$

is a function on the configuration space $(\mathbf{R}^3)^p$ with a singular locus $\{x_1 = \cdots = x_p = x\}$ which is moving with x .

Recall from (3.36) the definitions of the sets \mathbf{d}_I and \mathcal{T}_I .

Definition 3.38 Fix $a \in \mathbf{R}$, $J \subset \{1, \dots, p\}$ and $z_0 = (t_0, x_0) \in \mathcal{M}$. Choose some neighbourhood U_{z_0} of z_0 . Then define $\mathcal{S}^a(\mathcal{T}_J; U_{z_0})$ as the set of families $(T_z)_{z \in U_{z_0}}$ of distributions $T_z \in \mathcal{D}'(\mathcal{T}_J \setminus \mathbf{d}_{p,z})$ such that

$$\left(e^{\ell a} e^{-\ell \varrho_z} T_z \right)_{\ell \geq 0, z \in U_{z_0}}$$

is bounded in $\mathcal{D}'(\mathcal{T}_J)$ for every ϱ_z as above.

Implicitly, in the definition of \mathcal{T}_J there is a time variable t that we view as parameter where some time variables are taken to be equal to this time parameter t , t is not equal to t_0 but $(t, x) \in U_{t_0, x_0}$.

The next statement makes explicit a construction of such families; it is concerned with recentering.

Proposition 3.39 *Assume we are given a distribution $T \in \mathcal{S}_\Gamma^a(\mathcal{T}_J)$ with $\Gamma = N^*(\mathbf{d}_I \subset \mathcal{T}_J)$ for some $I \subset \{1, \dots, p\}$. Then for any admissible ϱ_z scaling with respect to $\mathbf{d}_{V(\mathcal{G}),z}$ the family*

$$\left(e^{\ell a} e^{-\ell \varrho_z} T \right)_{\ell \geq 0}$$

is bounded in $\mathcal{D}'(\mathcal{T}_J)$ uniformly in z .

Let $I \subset \{1, \dots, p\}$ and denote by $\pi : \mathcal{M}^p \mapsto \mathcal{M}^I$ the canonical projection map. Given $T \in \mathcal{S}_\Gamma^a(\mathcal{M}^I)$ where $\Gamma = \cup_{I' \subset I} N^* \mathbf{d}_{I'}$ and we scale with respect to \mathbf{d}_I . Then $\pi^* T \in \mathcal{S}_{\pi^* \Gamma}^a(\mathcal{M}^p)$ where we scale with respect to the deepest diagonal \mathbf{d}_p using admissible scaling fields.

Roughly speaking, in practice, it means you have a subamplitude $\mathcal{A}_{\mathcal{G}'}$ with a certain scaling degree with respect to its deepest diagonal $\mathbf{d}_{V(\mathcal{G}'})$, then since we would like to see how this subamplitude scales inside a bigger graph, we need to lift this to a bigger configuration space $\mathcal{M}^{V(\mathcal{G})}$, and we would like to examine the scaling degree with respect to the deepest diagonal $\mathbf{d}_{V(\mathcal{G})}$ of the larger graph or the marked diagonal $\mathbf{d}_{V(\mathcal{G}),z}$ of the larger graph. We need to ensure the scaling degree is unaffected under changes of the scaling dynamics.

Proof. The proof is very similar to the proof of [Dan13, Lemma 6.4.5 p. 144]. By the invariance results under scalings from Proposition 3.22, the proof reduces to the comparison of linear scalings w.r.t. different points. So we are reduced to the following generic situation, we work on (\mathbf{R}^{1+d}) and we would like to compare the two linear scaling flows :

$$e^{-s \varrho_{z_0}} : (t, x) \in \mathbf{R}^{1+d} \mapsto (e^{-2s}(t - t_0) + t_0, e^{-s}(x - x_0) + x_0) \in \mathbf{R}^{1+d}$$

which scales w.r.t. $z_0 = (t_0, x_0)$ and the second scaling

$$e^{-s \varrho_{z_1}}(t, x) \in \mathbf{R}^{1+d} \mapsto (e^{-2s}(t - t_1) + t_1, e^{-s}(x - x_1) + x_1) \in \mathbf{R}^{1+d}$$

which scales w.r.t. a different point $z_1 = (t_1, x_1)$.

Then just observe the identity $e^{-s \varrho_{z_1}} = (e^{-s \varrho_{z_1}} e^{s \varrho_{z_0}}) e^{-s \varrho_{z_0}}$ where the composition $\Phi(s) \stackrel{\text{def}}{=} e^{-s \varrho_{z_1}} e^{s \varrho_{z_0}}$ has a smooth linear limit when $s \uparrow \infty$ as can be easily inspected by a direct calculation: $\Phi(s) : (t, x) \mapsto (t + (e^{-2s} - 1)(t_1 - t_0), x + (e^{-s} - 1)(x_1 - x_0))$. \square

In spite of the fact that these spaces are rather ad hoc and not really intrinsic, they are sufficient to capture uniformity in $x \in M$ and therefore to control the size of the Feynman amplitudes uniformly in x .

3.4.2.2 The inductive theorem for convergent amplitudes

Our next goal is to describe a recursive algorithm that controls the convergence as a distribution *over the space* \mathcal{T}_J . For a given graph \mathcal{G} with marked edges, given a fixed $J \subset V(\mathcal{G})$, we prove in the next statement that for every $x_0 \in M$ and every compact neighbourhood U_{x_0} of x_0 , the following multilinear Feynman map is continuous under suitable conditions on the weak homogeneity a_e of the two point kernels $(K_e)_{e \in E}$

$$\left([\odot], (K_e)_{e \in E}, \mathcal{Q}_x^\gamma \right) \in \mathcal{S}_{N^*(\mathbf{d}_3 \subset M^3)}^{-6}(M^3) \times \prod_{e \in E} \mathcal{K}^{a_e} \times \mathcal{R}^\gamma \longmapsto \mathcal{A}_{\mathcal{G}, x} \in \mathcal{S}_\Gamma^{a_1}(\mathcal{T}_J) \cap \mathcal{S}^{a_2}(\mathcal{T}_J; U_{\mathfrak{z}}) \quad (3.38)$$

where

$$\begin{aligned} \Gamma &= \bigcup_{V' \subset V(\mathcal{G})} \left(N^*(\mathbf{d}_{V'} \subset \mathcal{T}_J) \cup N^*(\mathcal{T}_{V'} \subset \mathcal{T}_J) \right) \\ a_1 &= -6n_{\mathcal{G}} - \sum_{e \in E \setminus \{e_{\text{ref}}\}} a_e \\ a_2 &= a_1 - 6 - 2\gamma. \end{aligned}$$

The space $\mathcal{S}^{a_2}(\mathcal{T}_J; U_{x_0})$ which appears in Definition 3.38 accounts for the fact that our estimates should be uniform in $x \in U_{x_0}$ when we scale with respect to the marked diagonal $\mathbf{d}_{V(\mathcal{G}), x}$. Recall also that we work on $\mathcal{T}_J \subset \mathcal{M}^{V(\mathcal{G})}$ for some $J \subset V(\mathcal{G})$ because we take into account that our amplitudes are integrated on regions where certain time variables coincide. If the graph \mathcal{G} has no distinguished edge e_{ref} , then we do not need to test the regularity in the space $\mathcal{S}^{a_2}(\mathcal{T}_J; U_{x_0})$ and the target functional space is just $\mathcal{S}_\Gamma^{a_1}(\mathcal{T}_J)$ for $a_1 = -6n_{\mathcal{G}} - \sum_{e \in E} a_e$.

We only consider below some subgraphs $\mathcal{G}' = (V', E')$ of \mathcal{G} which contain all the points of a given triple if ever they contain one of them. Recall all our analysis takes place in the submanifold \mathcal{T}_J of \mathcal{M}^p . With a slight abuse of notation we will also denote by $\mathbf{d}_{V'}$ the diagonal $\mathbf{d}_{V'} \cap \mathcal{T}_J$.

For any marked subgraph \mathcal{G}' we set

$$\begin{aligned} a_{\mathcal{G}', 1} &\stackrel{\text{def}}{=} -6n_{\mathcal{G}'} - \sum_{e \in E' \setminus \{e_{\text{ref}}\}} a_e \\ a_{\mathcal{G}', 2} &\stackrel{\text{def}}{=} a_{\mathcal{G}', 1} - 6 - 2\gamma, \end{aligned}$$

while for any subgraph \mathcal{G}' with no marked/reference edge we set $a_{\mathcal{G}'} \stackrel{\text{def}}{=} -6n_{\mathcal{G}'} - \sum_{e \in E'} a_e$ and

$$\Gamma_{\mathcal{G}'} \stackrel{\text{def}}{=} \bigcup_{V'' \subset V'} \left\{ N^*(\mathbf{d}_{V''} \subset \mathcal{T}_J) \cup N^*(\mathcal{T}_{V''} \subset \mathcal{T}_J) \right\}.$$

A given Feynman graph \mathcal{G} will be said to *have a loop* if $b_1(\mathcal{G}) > 0$.

Theorem 3.40 *The following holds.*

(a) *If every connected irreducible subgraph $\mathcal{G}' = (V', E')$ of \mathcal{G} with a loop satisfies*

$$\begin{aligned} a_{\mathcal{G}', 1} + \text{codim}_w(\mathbf{d}_{V'}) &> 0, \\ a_{\mathcal{G}', 2} + \text{codim}_w(\mathbf{d}_{V', (t, x)}) &> 0, \end{aligned} \quad (3.39)$$

then the Feynman map defined by Equation 3.38 is continuous.

(b) *If $\mathcal{G} = (V, E)$ is a graph that contains no distinguished edge q , every connected irreducible strict subgraph of \mathcal{G} with a loop satisfies Condition (3.39) and \mathcal{G} satisfies*

$$a_{\mathcal{G}} + \text{codim}_w(\mathbf{d}_V \subset \mathcal{T}_J) > -1,$$

then there exists a family $\Lambda_{\mathcal{Y}, \epsilon}$ of distributions supported on \mathbf{d}_V , with wavefront set in $N^(\mathbf{d}_V \subset \mathcal{T}_J)$ such that $\Lambda_\epsilon - \Lambda_{\mathcal{Y}, \epsilon}$ has a limit in $\mathcal{D}'(\mathcal{T}_J)$ and the convergence occurs in $\mathcal{S}'_{\Gamma_{\mathcal{G}} \cup N^*(\mathbf{d}_V)}(\mathcal{T}_J)$ for all*

$$a' < -\text{codim}_w(\mathbf{d}_V \subset \mathcal{T}_J).$$

Before we give the proof of Theorem 3.40 an important Remark is in order.

Remark 3.41 We aim to point an important fact. The above result is general, but relies of the fact that the irreducible subgraphs are convergent. It turns out that we took care to implement the fact that for any of the Feynman graphs that we need to study, the most divergent subgraph is the graph itself. Note that this would not be the case if we were to replace all the cubic kernels $[\odot](x, y, z)$ by $\delta_x(y)\delta_x(z)$. Indeed while $[\odot]$ and the product of two δ s have the same weak homogeneity (-6) with respect to the deep diagonal $\{x = y = z\}$ they do not share the same microlocal properties. We will further elaborate on this point later in Remark 3.53. In particular when we deal with the resonant products $\mathcal{V} \odot \mathcal{V}$ and $\mathcal{V} \odot \mathcal{V}^\circ$ in Section 3.5, this will imply that the subgraph containing the covariance $G^{(2)}(z_1, z_2)$ of \mathcal{V} and the probe operator $\mathcal{Q}_x^\gamma(y_1, y_2)$ is very convergent, since these two operators are linked by $[\odot](y_1, z_1, w_1)$ and $[\odot](y_2, z_2, w_2)$ for w_i far from y_i, z_i . Conversely, replacing $[\odot](y_i, z_i, w_i)$ by $\delta_{y_i}(z_i)\delta_{y_i}(w_i)$, then this subgraph would be more divergent than the graph itself, which corresponds to the fact that while $\mathcal{V} \odot \mathcal{V}$ and $\mathcal{V} \odot \mathcal{V}^\circ$ are of regularities 0^- and $(-1/2)^-$, the products $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{V}^\circ$ are both of regularity $(-1)^-$.

Proof. Our proof proceeds by induction on subgraphs for the inclusion relation. For a subset $V_1 = V_1' \cup V_A' \subset V(\mathcal{G})$ of vertices and $E' \subset E(\mathcal{G})$ of edges, we denote by $\mathcal{G}_{V_1, E'}$ the corresponding subgraph.

The initialisation of the induction is immediate, the simplest subgraphs are just the propagators or the kernel of the resonant product which have their respective wave front sets on the conormals of the diagonals.

Given a graph \mathcal{G} constructed according to our rules, assume that all subgraphs \mathcal{G}' of \mathcal{G} have well-defined Feynman amplitudes $\mathcal{A}_{\mathcal{G}'}$ with wave front set in $\Gamma_{\mathcal{G}'}$, without loss of generality, we may assume that \mathcal{G} has p vertices which means that $V(\mathcal{G})$ is in bijection with $\{1, \dots, p\}$ hence $\mathcal{M}^{V(\mathcal{G})} \simeq \mathcal{M}^p$. For every $I \subsetneq V(\mathcal{G}) \simeq \{1, \dots, p\}$ we consider the open subset \mathcal{U}_I defined as follows

$$\mathcal{U}_I \stackrel{\text{def}}{=} \left\{ (m_1, \dots, m_p) \in \mathcal{M}^{V(\mathcal{G})}; i \in I, j \in V(\mathcal{G}) \setminus I, m_i \neq m_j \right\}.$$

By the Popineau-Stora covering Lemma 3.26 we have the covering $\mathcal{T}_J \setminus \mathbf{d}_{V(\mathcal{G})} = \cup_{I \subsetneq V(\mathcal{G})} \mathcal{U}_I$. The idea is to restrict $\mathcal{A}_{\mathcal{G}}|_{\mathcal{U}_I}$, then to factor the restricted amplitude as a product of subamplitudes which are well-defined by the inductive assumption and a product of smooth kernels.

For given a subset $I \subsetneq V(\mathcal{G})$ of vertices, we set

$$I' \stackrel{\text{def}}{=} V(\mathcal{G}) \setminus I, \quad (I, I') \stackrel{\text{def}}{=} (I \times I') \cup (I' \times I).$$

If K is another subset of $V(\mathcal{G})$, we write $K \not\subset I^{(\prime)}$ to say that K is neither a subset of I nor of I' . With this notation the amplitude $\mathcal{A}_{\mathcal{G}}$ factors as

$$\mathcal{A}_{\mathcal{G}} = \mathcal{A}_{\mathcal{G}_I} \mathcal{A}_{\mathcal{G}_{I'}} \left(\prod_{e: (v(e)_-, v(e)_+) \in (I, I')} K_e(z_{v(e)_-}, z_{v(e)_+}) \prod_{j: \{v_1^j, v_2^j, v_*^j\} \not\subset I^{(\prime)}} [\odot](x_{v_*^j}, x_{v_1^j}, x_{v_2^j}) \right),$$

where the subgraph \mathcal{G}_I , resp. $\mathcal{G}_{I'}$, is some subgraph of \mathcal{G} with only vertices in I , resp. I' , and edges $e \in E$ such that both vertices $v(e)_-, v(e)_+ \in I$, resp. $v(e)_-, v(e)_+ \in I'$, bounding e belong to I , resp. I' . More importantly, the product on the resonant kernels indexed by integers j runs over the kernels $[\odot]$ whose vertices $(v_*^j, (v_1^j, v_2^j))$ are not entirely contained neither in I nor I' : some vertices amongst them are in I and some others are in I' . Note the fact that the product on resonant kernels might well be empty, for instance if all distinguished triples are contained either in I or I' .

Now observe that when we restrict the amplitude $\mathcal{A}_{\mathcal{G}}$ on the open subset \mathcal{U}_I , this yields

$$\mathcal{A}_{\mathcal{G}}|_{\mathcal{U}_I} = \underbrace{\mathcal{A}_{\mathcal{G}_I} \mathcal{A}_{\mathcal{G}_{I'}}}_{\text{exterior product}} \underbrace{\left(\prod_{e: (v(e)_-, v(e)_+) \in (I, I')} K_e(z_{v(e)_-}, z_{v(e)_+}) \prod_{j: \{v_1^j, v_2^j, v_*^j\} \not\subset I^{(\prime)}} [\odot](x_{v_*^j}, x_{v_1^j}, x_{v_2^j}) \right)}_{\in C^\infty},$$

where the restriction to open subset \mathcal{U}_I of the product inside the parenthesis is smooth, and the product $\mathcal{A}_{\mathcal{G}_I} \mathcal{A}_{\mathcal{G}_{I'}}$ denotes an exterior tensor product of distributions which is always well-defined. We know by induction that the two amplitudes $\mathcal{A}_{\mathcal{G}_I}((z_j)_{j \in I})$ and $\mathcal{A}_{\mathcal{G}_{I'}}((z_j)_{j \in I'})$ are well-defined distributions with wave front sets

$$WF(\mathcal{A}_{\mathcal{G}_I}) \subset \Gamma_I \stackrel{\text{def}}{=} \bigcup_{K \subset I} N^*(\mathbf{d}_K \subset \mathcal{T}_J) \cup N^*(\mathcal{T}_K \subset \mathcal{T}_J),$$

and

$$WF(\mathcal{A}_{\mathcal{G}_{I'}}) \subset \Gamma_{I'} \stackrel{\text{def}}{=} \bigcup_{K' \subset I'} N^*(\mathbf{d}_{K'} \subset \mathcal{T}_J) \cup N^*(\mathcal{T}_{K'} \subset \mathcal{T}_J).$$

Analysing the graph \mathcal{G} , there are two situations, either \mathcal{G} contains some distinguished edge \mathcal{Q}_x^γ in which case we should study the scaling behaviour in two steps, first scale on the partial diagonals $\mathbf{d}_{V(\mathcal{G})}$ where all points labelled by $V(\mathcal{G})$ collide, then study the scaling on the marked diagonals $\mathbf{d}_{V(\mathcal{G}), (t,x)}$ (recall the marked diagonal is just one point in $\mathcal{M}^{|V(\mathcal{G})|}$) where all points labelled by $V(\mathcal{G})$ collide uniformly on (t, x) , we use the spaces of Definition 3.38 to control the uniformity in the parameter (t, x) . If the graph contains no distinguished edge, then we just scale in one step with respect to $\mathbf{d}_{V(\mathcal{G})}$.

Assume we are in the first situation where \mathcal{G} contains a distinguished edge, the second case is simpler to handle. Choosing any admissible scaling field ϱ scaling with respect to the deep diagonal $\mathbf{d}_{V(\mathcal{G})}$ and using the stability of \mathcal{U}_I by such scaling field (a consequence of admissibility of ϱ , we made an essential use of the fact that the flow by scaling fields ϱ from Definition 3.24 preserves both time and space-time diagonals $\mathcal{T}_J, \mathbf{d}_I$ hence every $\mathcal{U}_I, I \subsetneq V(\mathcal{G})$). The weak homogeneity of each term in factor does not depend on the choice of scaling field with respect to $\mathbf{d}_{V(\mathcal{G})}$ by Proposition 3.22. Moreover, we also use the property that the symplectic lifts of these scaling fields preserve the conormals of all time and space-time diagonals $N^*(\mathcal{T}_I), N^*(\mathbf{d}_I)$ so that the cones $\Gamma_I, \Gamma_{I'}$ containing microsingularities of $\mathcal{A}_{\mathcal{G}_I}, \mathcal{A}_{\mathcal{G}_{I'}}$ are stable by the lifted flow: $e^{-\ell\varrho^*} \Gamma_I \subset \Gamma_I, e^{-\ell\varrho^*} \Gamma_{I'} \subset \Gamma_{I'}$ by Lemma 3.25. Using the fact that each element $\eta_I \in C^\infty(M^{d_{V(\mathcal{G})}} \setminus \mathbf{d}_{V(\mathcal{G})})$ of the partition of unity belongs to $\mathcal{S}_\emptyset^0(M^{V(\mathcal{G})} \setminus \mathbf{d}_{V(\mathcal{G})})$, it has scaling degree 0 in $C^\infty(M^{d_{V(\mathcal{G})}} \setminus \mathbf{d}_{V(\mathcal{G})})$ in the sense that

$$\left(e^{-\ell\varrho^*} \eta_I \right)_{\ell \geq 0}$$

is a bounded family in $C^\infty(M^{V(\mathcal{G})} \setminus \mathbf{d}_{V(\mathcal{G})})$. We can thus define the renormalized product

$$\eta_I \mathcal{A}_{\mathcal{G}_I} \mathcal{A}_{\mathcal{G}_{I'}} \left(\prod_{e: (v(e)_-, v(e)_+) \in (I, I')} K_e(z_{v(e)_-}, z_{v(e)_+}) \prod_{j: \{v_1^j, v_2^j, v_*^j\} \not\subset I'} [\odot](x_{v_*^j}, x_{v_1^j}, x_{v_2^j}) \right)$$

on $\mathcal{T}_J \setminus \mathbf{d}_{V(\mathcal{G}), (t,x)}$ by applying Proposition 3.28 twice and we find that the product is well-defined since the weak homogeneity of the above product is $> -\text{codim}_w(\mathbf{d}_{V(\mathcal{G})})$ by assumption. We insist that when we scale with respect to $\mathbf{d}_{V(\mathcal{G})}$ the weak homogeneity of \mathcal{Q}_x^γ is 0 by Lemma 3.42. By summation over I we conclude that $\mathcal{A}_{\mathcal{G}}$ is well-defined on $\mathcal{T}_J \setminus \mathbf{d}_{V(\mathcal{G}), (t,x)}$.

To define an extension on the whole configuration space \mathcal{T}_J we need the two step extension from Theorem 3.20. We now need to apply Proposition 3.28 and now Theorem 3.20 a second time to the expression

$$\eta_I \mathcal{A}_{\mathcal{G}_I} \mathcal{A}_{\mathcal{G}_{I'}} \left(\prod_{e: (v(e)_-, v(e)_+) \in (I, I')} K_e(z_{v(e)_-}, z_{v(e)_+}) \prod_{j: \{v_1^j, v_2^j, v_*^j\} \not\subset I'} [\odot](x_{v_*^j}, x_{v_1^j}, x_{v_2^j}) \right)$$

but for the scaling fields with respect to the distinguished diagonal $\mathbf{d}_{V(\mathcal{G}), (t,x)}$. *It is at this point that we are using our second inductive assumption on the weak homogeneity of all subgraphs containing the distinguished edge*, equivalently all subamplitudes containing the propagator \mathcal{Q}_x^γ . We make an induction on all marked subgraphs whose amplitude contains \mathcal{Q}_x^γ , the induction starts with \mathcal{Q}_x^γ itself. We use the fact that the scaling degree of \mathcal{Q}_x^γ with respect to $\mathbf{d}_{V(\mathcal{G})} \setminus \mathbf{d}_{V(\mathcal{G}), (t,x)}$ equals 0, while when scaling with

respect to $\mathbf{d}_{V(\mathcal{G}), (t, x)}$, the scaling degree of \mathcal{Q}_x^γ is now given by $-6 - 2\gamma$. This last claim is proven in Lemma 3.42 below.

At this stage given a graph \mathcal{G} , assume by induction that all subgraphs \mathcal{G}' containing the distinguished edge have scaling degree with respect to the marked diagonal $> -\text{codim}_w(\mathbf{d}_{V(\mathcal{G}'), (t, x)})$. Then every product defined on $\mathcal{T}_J \setminus \mathbf{d}_{V(\mathcal{G})}$

$$\eta_I \mathcal{A}_{\mathcal{G}_I} \mathcal{A}_{\mathcal{G}_{I'}} \left(\prod_{e: (v(e)_-, v(e)_+) \in (I, I')} K_e(z_{v(e)_-}, z_{v(e)_+}) \prod_{j: \{v_1^j, v_2^j, v_*^j\} \not\subset I'} [\odot](x_{v_*^j}, x_{v_1^j}, x_{v_2^j}) \right)$$

will be weakly homogeneous of degree $a_{\mathcal{G}, 2} = \sum_e a_e - 6n_{\mathcal{G}} - 6 - 2\gamma$ when we scale with respect to the marked diagonal $\mathbf{d}_{V(\mathcal{G}), (t, x)}$ where we need to include the scaling degree of \mathcal{Q}_x^γ which equals $-6 - 2\gamma$. So each piece above satisfies the assumptions of Theorem 3.20 hence extends to \mathcal{T}_J .

The uniform estimates in x essentially follow from the stability assumptions of the cones by change of scaling in Proposition 3.39. \square

Lemma 3.42 Fix $d \geq 2$ and denote by $\pi_{\leq 2} : \mathcal{M}^d \rightarrow \mathcal{M}^2$ the canonical projection. For all values of $\gamma \in \mathbf{R}$ the kernel $\pi_{\leq 2}^* \mathcal{Q}_x^\gamma$ is a smooth germ near $\mathbf{d}_d \setminus \mathbf{d}_{d, x}$ in $\mathbf{R}^2 \times M^d \setminus \mathbf{d}_{d, x}$. We define

$$\mathcal{O} \stackrel{\text{def}}{=} \left\{ m \in \mathbf{R}^2 \times M^d : \text{dist}(m, \mathbf{d}_{d, x}) > 2\text{dist}(m, \mathbf{d}_d) \right\}.$$

Then $\pi_{\leq 2}^* \mathcal{Q}_x^\gamma$ is weakly homogeneous of degree 0 when scaling with respect to $\mathbf{d}_d \setminus \mathbf{d}_{d, x}$ in $\Omega \setminus \mathbf{d}_d$ in the following sense: For every compact $K \Subset \Omega$, for all scaling field ϱ scaling with respect to \mathbf{d}_d , there exists $\ell_K > 0$ such that the family

$$\left(\{e^{-\ell\varrho} \pi_{\leq 2}^* \mathcal{Q}_x^\gamma\} |_{K} \right)_{\ell \geq \ell_K}$$

is bounded in $C^\infty(K)$.

Proof. For any element $m \in \mathcal{O}$ we know that the limit point $m_\infty = \lim_{\ell \uparrow \infty} e^{-\ell\varrho}(m) \in \mathbf{d}_d \setminus \mathbf{d}_{d, x}$ has the form $m_\infty = (y, \dots, y, t, t)$ for $y \neq x$ and therefore $\pi_{\leq 2}^* \mathcal{Q}_x^\gamma(\cdot)$ is smooth near the limit point $m_\infty = (y, \dots, y, t, t)$ since $\pi_{\leq 2}^* \mathcal{Q}_x^\gamma(m_\infty) = \mathcal{Q}_x^\gamma(y, y)$ for $y \neq x$. Now by a continuity argument, we know that for any element $m \in \mathcal{O}$ there exists a neighbourhood $V_m \subset \Omega$ and $\ell_m > 0$, such that for all $\ell \geq \ell_m$ one has $e^{-\ell\varrho}(V_m) \subset \Omega$ and $\text{dist}(\pi_{\leq 2}(e^{-\ell\varrho}(V_m)), (x, x)) \geq \delta > 0$ hence $\pi_{\leq 2}^* \mathcal{Q}_x^\gamma|_{e^{-\ell\varrho}(V_m)}$ is smooth uniformly in $\ell \geq \ell_m$ which yields the claim by compactness. \square

3.5 Random fields from renormalisation

Recall the definition (3.4) of the enhancement $\hat{\xi}_r$ of the regularized noise ξ_r

$$\hat{\xi}_r \stackrel{\text{def}}{=} \left(\xi_r, \mathcal{V}_r, \mathcal{W}_r, \mathcal{Y}_r \odot \mathcal{I}_r, \mathcal{Y}_r \odot \mathcal{V}_r - \frac{b_r}{3}, |\nabla \mathcal{Y}_r|^2 - \frac{b_r}{3}, \mathcal{Y}_r \odot \mathcal{V}_r - b_r \mathcal{I}_r \right).$$

We use the index r to emphasize the regularisation of the noise. We show in this section the convergence of $\hat{\xi}_r$ to some limit enhanced noise in its natural space (3.5). The random fields are constructed by means of some covariance computations, which finally yield the existence of the stochastic objects in some Hölder-Besov spaces by a Kolmogorov type argument. In this section we need to extract a Littlewood-Paley block of our stochastic trees, then control the size of the Littlewood-Paley block evaluated at some point $x \in M$, uniformly in $x \in M$. We need some uniform control because of the loss of translation invariance, which is the main difficulty of this section and of SPDE on non-homogeneous Riemannian manifolds. We start the section by defining abstractly in Section 3.5.1 the framework of Kolmogorov estimates in our non stationary setting where we need pointwise bounds. This also requires the introduction of some distinguished propagator \mathcal{Q}_x^γ which depends on a base point $x \in M$. Then we put our definition in action by discussing the stochastic estimates of Wick monomials where things are

simple. The reader who wants to get some familiarity with our approach is urged to first read how we handle the regularities of the simple Wick monomials \mathfrak{I} , \mathfrak{V} , \mathfrak{O} in Lemma 3.48 and sections 3.5.2.2 and 3.5.2.3. A novelty of our approach for the Wick monomials is that we renormalize in a **locally covariant way** which is the only way to ensure some form of locality. We refer the reader to Section 3.5.2.1 for a more detailed discussion. The estimates on Wick monomials should also be useful for readers interested in simpler Quantum Field Theories on surfaces.

Then, starting from Section 3.5.3, we deal with more complicated graphs coming from the higher chaoses, as enumerated in the first paragraph in Section 3.5.4. While some of the amplitude arising from these computations can be directly handled using Theorem 3.40, for several graphs that appear in our stochastic estimates we need to feed the induction Theorem 3.40 with more information, this uses subtle smoothing properties of the Schwartz kernel of the resonant product which is discussed in Section 3.5.4.1 and also some control of certain renormalized subamplitude in Section 3.5.4.2. We refer the reader to the relevant parts for more details but this can be read after one is used to the general inductive machinery. The amplitude containing a more singular subamplitude require a more careful analysis, taking place in Section 3.5.4. We also devote Section 3.5.5.3 to the exact calculation of the counterterms from the divergent subamplitude; this uses subtle arguments involving heat asymptotics, stationary phase and the borderline case of our extension Theorem 3.19 where we need a renormalisation.

3.5.1 Kolmogorov type argument

We start by giving the definitions of the Besov spaces that we need for our analysis.

Definition 3.43 In the sequel we fix I a finite set and $(U_i, \kappa_i)_{i \in I}$ a cover of M with $\kappa_i : U_i \rightarrow \mathbf{R}^3$, along with a quadratic partition of unity $(\chi_i)_{i \in I}$ on the closed manifold M subordinated to $(U_i)_{i \in I}$. For any $\alpha \in \mathbf{R}$ and $p, q \in [1, \infty]$ we define the Besov space $B_{pq}^\alpha(M)$ on the closed manifold M as the completion of $C^\infty(M)$ with the norm

$$\|f\|_{B_{pq}^\alpha(M)} \stackrel{\text{def}}{=} \max_{i \in I} \|\kappa_{i*}(\chi_i f)\|_{B_{pq}^\alpha(\mathbf{R}^3)} \stackrel{\text{def}}{=} \max_{i \in I} \left\| 2^{k\alpha} \|\Delta_k(\kappa_{i*}(\chi_i f))\|_{L^p(\mathbf{R}^3)} \right\|_{\ell_k^q}, \quad (3.40)$$

where the operators Δ_k for $k \geq -1$ are Littlewood-Paley blocks on the flat space \mathbf{R}^3 .

We slightly rewrite this norm as follows. We fix a collection $(\psi_i)_{i \in I}$ of functions on \mathbf{R}^3 such that $\text{supp}(\psi_i) \subset \kappa_i(U_i)$ and $\psi_i|_{\kappa_i(\text{supp}(\chi_i))} = 1$; there is enough room, since the support of χ_i does not fill U_i . It turns out that the norm defined in (3.40) is equivalent to

$$\max_{i \in I} \left\| 2^{k\alpha} \|\psi_i \Delta_k(\kappa_{i*}(\chi_i f))\|_{L^p(\mathbf{R}^3)} \right\|_{\ell_k^q}. \quad (3.41)$$

We therefore define for $i \in I$ and $k \geq -1$ some Littlewood-Paley blocks on the manifold M by

$$P_k^i(f) \stackrel{\text{def}}{=} \kappa_i^*[\psi_i \Delta_k(\kappa_{i*}(\chi_i f))]. \quad (3.42)$$

With this notation in hand $\|f\|_{B_{pq}^\alpha(M)}$ therefore equivalent to

$$\max_{i \in I} \left\| 2^{k\alpha} \|P_k^i(\bullet)\|_{L^p(M)} \right\|_{\ell_k^q}. \quad (3.43)$$

In the sequel, to lighten the notation, we often let $B_{p,q}^\alpha$ stand for $B_{pq}^\alpha(M)$ and define

$$C^\alpha \stackrel{\text{def}}{=} B_{\infty\infty}^\alpha(M).$$

Remark 3.44 The equivalence of (3.40) with (3.41) is deduced from some argument which we derive in the proof of [BDET23, Lemma 2.10], where we control on \mathbf{R}^3 the commutator of a Littlewood-Paley block with multiplication by a smooth and compactly supported function. Let us explain it in more detail. For any smooth function $a \in C^\infty(\mathbf{R}^3)$, we denote by M_a the multiplication operator by a . On

\mathbf{R}^3 , we would like to compare $M_\psi \Delta_k M_{\kappa_{i^*} \chi_i}$ with $\Delta_k M_{\kappa_{i^*} \chi_i}$. Note that $\Delta_k M_{\kappa_{i^*} \chi_i} = \Delta_k M_\psi M_{\kappa_{i^*} \chi_i}$ since $\psi = 1$ on the support of $\kappa_{i^*} \chi_i$ hence the difference $M_\psi \Delta_k M_{\kappa_{i^*} \chi_i} - \Delta_k M_{\kappa_{i^*} \chi_i}$ can be written as a commutator

$$[M_\psi, \Delta_k] M_{\kappa_{i^*} \chi_i}.$$

Therefore, by the argument in the proof of [BDFT23, Lemma 2.10], this composition is a smoothing operator in the semiclassical sense (setting $\hbar = 2^{-k}$). It follows that

$$\|\Delta_k \kappa_{i^*}(\chi_i f) - \psi_i \Delta_k \kappa_{i^*}(\chi_i f)\|_{L^p} \lesssim 2^{-kN} \|f\|_{C^{-N}}$$

for every $N \geq 1$. Finally, we can therefore absorb this error term in the semi norms, which indeed shows that the two norms are equivalent.

In the sequel we denote by $\psi \circ |\cdot| \in \mathcal{S}(\mathbf{R})$ the Schwartz kernel of Δ_0 in \mathbf{R}^3 ; its Fourier transform η is supported in an annulus. Note that for any $k \geq 0$, the Schwartz kernel of Δ_k is given by

$$\Delta_k(x - y) = 2^{3k} \psi(2^k |x - y|).$$

We are now ready to state our Kolmogorov type lemma. It turns out that due to the lack of stationarity of the stochastic objects on manifolds (as opposed to the flat case, or the sphere), the inequalities stated in Lemma 3.45 below involve a pointwise bound requiring to be uniform in some base-point $x \in M$. The mechanics involved here are classical. We give the details for the reader's convenience.

Lemma 3.45 *Fix $T > 0$, $n \in \mathbf{N}$ and let u be a random distribution in $\mathcal{D}'([0, T] \times M)$ which belongs to the Wiener chaos of degree n .*

– Assume one has

$$\mathbf{E}[(P_k^i u)(t, x)^2] \lesssim 2^{-2k\gamma} \quad (3.44)$$

uniformly in $x \in M$, $i \in I$ and $k \geq 0$ for some $\gamma \in \mathbf{R}$, for some time $0 \leq t \leq T$. Then for every $s < \gamma$ and $p > 1$ we have

$$\mathbf{E}[\|u(t)\|_{\mathcal{C}^s}^p] \lesssim_p 1.$$

– If, in addition to (3.44), the distribution u further satisfies

$$\mathbf{E}[(P_k^i u)(t_1, x) - (P_k^i u)(t_2, x)]^2 \lesssim 2^{-2k\gamma} |t_1 - t_2|^{2\beta} \quad (3.45)$$

uniformly in $t_1, t_2, x \in [0, T] \times [0, T] \times M$ and $(i, k) \in I \times \mathbf{N}$ for some $\gamma \in \mathbf{R}$, $\beta > 0$ then for every $s < \gamma$ and $\theta < \beta$ we have

$$u \in C_T^\theta \mathcal{C}^s.$$

Proof. We concentrate on the second item as the first item is proved by a similar reasoning. By Sobolev embedding we have

$$\begin{aligned} \mathbf{E}[\|u(t_1) - u(t_2)\|_{\mathcal{C}^s}^p] &\lesssim \mathbf{E}[\|u(t_1) - u(t_2)\|_{B_{p,p}^{s+3/p}}^p] \\ &\lesssim \max_{i \in I} \sum_{k \geq 0} 2^{pk(s+3/p)} \int_M \mathbf{E}[|(P_k^i u)(t_1, x) - (P_k^i u)(t_2, x)|^p] dx. \end{aligned}$$

Then the Gaussian hypercontractivity estimate implies that

$$\mathbf{E}[\|u(t_1) - u(t_2)\|_{\mathcal{C}^s}^p] \lesssim \max_{i \in I} \sum_{k \geq 0} 2^{pk(s+3/p)} \int_M \mathbf{E}[\{(P_k^i u)(t_1, x) - (P_k^i u)(t_2, x)\}^2]^{\frac{p}{2}} dx$$

$$\lesssim \max_{i \in I} \sum_{k \geq 0} \sup_{x \in M} \left(2^{2k(s+3/p)} \mathbf{E} \left[\{(P_k^i u)(t_1, x) - (P_k^i u)(t_2, x)\}^2 \right] \right)^{\frac{p}{2}}.$$

Therefore using the hypothesis (3.45) we end up with

$$\mathbf{E} \left[\|u(t_1) - u(t_2)\|_{\mathcal{C}^s}^p \right] \lesssim \sum_{k \geq 0} 2^{pk(s+3/p-\gamma)} |t_1 - t_2|^{p\beta} \lesssim |t_1 - t_2|^{p\beta},$$

which holds provided we choose p large enough to have $s < \gamma - 3/p$. The proof now follows using a classical Kolmogorov argument in time. \square

We denote by $P_k^i(x, y)$ the smooth Schwartz kernel of the operator P_k^i .

Definition 3.46 We introduce a bilinear kernel depending on a point $x \in M$ and $\gamma \in \mathbf{R}$ allowing to probe the regularities of objects as follows

$$\mathcal{Q}_x^\gamma(y_1, y_2) \stackrel{\text{def}}{=} \sum_{i \in I} \sum_{k \geq -1} 2^{2k\gamma} P_k^i(x, y_1) P_k^i(x, y_2). \quad (3.46)$$

Note that if there exists $\gamma \in \mathbf{R}$ such that one has

$$\mathbf{E} \left[\mathcal{Q}_x^\gamma(u(t, \cdot), u(t, \cdot)) \right] = \mathbf{E} \left[\sum_{k \geq -1} 2^{2k\gamma} (P_k^i u)(t, x)^2 \right] = \sum_{k \geq -1} 2^{2k\gamma} \mathbf{E}[(P_k^i u)(t, x)^2] < \infty$$

uniformly in $x \in M$, then we have

$$\mathbf{E}[(P_k^i u)(t, x)^2] \lesssim 2^{-2k\gamma}$$

uniformly in $k \geq -1$.

The microlocal properties of the kernel \mathcal{Q}_x^γ are stated in the following lemma.

Lemma 3.47 Fix $x \in M$ and $\gamma \in \mathbf{R}$. The series defining \mathcal{Q}_x^γ converges in $C^\infty(M^2 \setminus \{(x, x)\})$ and in the space $\mathcal{D}'_{\Gamma_x}(M \times M)$ of distributions with wave front set

$$\Gamma_x = T_{(x,x)}^* M^2.$$

Moreover it is weakly homogeneous of degree $-6 - 2\gamma$ with respect to the scaling towards $\mathcal{Y} = (x, x)$, that is to say we have $\mathcal{Q}_x^\gamma \in \mathcal{S}_{\Gamma_x}^{-6-2\gamma}$ with a control uniform in x .

Proof. For the convergence in $C^\infty(M^2 \setminus \{(x, x)\})$, in a local chart indexed by $i \in I$, the kernel of \mathcal{Q}_x^γ is given by

$$\sum_{k \geq 0} 2^{k(6+2\gamma)} \psi_i^2(\tilde{x}) (\kappa_{i*} \chi_i)^{\otimes 2}(y_1, y_2) \psi(2^k |y_1 - \tilde{x}|) \psi(2^k |y_2 - \tilde{x}|),$$

where $\psi \circ |\cdot|$ is a Schwartz function on \mathbf{R} with Fourier transform η supported on an annulus, and $\tilde{x} \stackrel{\text{def}}{=} \kappa_i(x)$. For $y_i \neq \tilde{x}$ we have for any $N \geq 0$ and any $\alpha \in \mathbf{N}^3$

$$|\partial_{y_i}^\alpha \psi(2^k |y_i - \tilde{x}|)| \lesssim 2^{k|\alpha|} (1 + 2^k |y_i - \tilde{x}|)^{-N} \lesssim 2^{k|\alpha| - Nk}.$$

Therefore choosing N large enough ensures the convergence of the series in C^∞ .

We now turn to the proof of the convergence of the series in the sense of distributions in $\mathcal{D}'_{\Gamma_x}(M \times M)$. We use different arguments depending on the sign of γ .

• Suppose that $\gamma < 0$. Then we have $\|2^{3k}\psi(2^k|\cdot - \tilde{x}|\|_{L^1} = \|\psi\|_{L^1}$ and the series (3.46) converges absolutely in L^1 since

$$\left\| \sum_{k \geq 0} 2^{k(6+2\gamma)} \psi_i^2(\tilde{x})(\kappa_{i*}\chi_i)^{\otimes 2}(y_1, y_2) \psi(2^k|y_1 - \tilde{x}|) \psi(2^k|y_2 - \tilde{x}|) \right\|_{L^1_{y_1, y_2}} \lesssim \sum_{k \geq 0} 2^{k\gamma} \|\psi\|_{L^1}^2 \|\psi_i\|_{L^\infty}^2 \|\chi_i\|_{L^\infty}^2 \lesssim 1.$$

• Suppose now that $\gamma \geq 0$. We must prove that the series converges in the distributional sense. For every test function $\varphi \in C_c^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$ we must prove the convergence of the series

$$\begin{aligned} & \sum_{k \geq 0} 2^{k(6+2\gamma)} \int_{(\mathbf{R}^3)^2} \varphi(y_1, y_2) \psi_i^2(\tilde{x})(\kappa_{i*}\chi_i)^{\otimes 2}(y_1, y_2) \psi(2^k|y_1 - \tilde{x}|) \psi(2^k|y_2 - \tilde{x}|) dy_1 dy_2 \\ &= \sum_{k \geq 0} 2^{2k\gamma} \int_{(\mathbf{R}^3)^2} \varphi(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}) \\ & \quad \times \psi_i^2(\tilde{x})(\kappa_{i*}\chi_i)^{\otimes 2}(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}) \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) dy_1 dy_2. \end{aligned}$$

We rename by $\tilde{\varphi}$ the test function

$$\tilde{\varphi}(y_1, y_2) \stackrel{\text{def}}{=} \varphi(y_1, y_2)(\kappa_{i*}\chi_i)^{\otimes 2}(y_1, y_2).$$

The series therefore rewrites as

$$\sum_{k \geq 0} 2^{2k\gamma} \int_{(\mathbf{R}^3)^2} \tilde{\varphi}(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}) \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) \psi_i^2(\tilde{x}) dy_1 dy_2.$$

Note that the product $\psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) \psi_i^2(\tilde{x})$ is a Schwartz function in y_1, y_2 uniformly in \tilde{x} . The idea is to Taylor expand $\tilde{\varphi}$ about (\tilde{x}, \tilde{x}) at order N with integral remainder. This yields

$$\tilde{\varphi} = P_N + R_N, \quad R_N(y_1, y_2) = \mathcal{O}(|y_1 - \tilde{x}| \vee |y_2 - \tilde{x}|^{N+1}),$$

where P_N is a polynomial function of (y_1, y_2) centered at (\tilde{x}, \tilde{x}) . The rescaled $\tilde{\varphi}$ reads

$$\begin{aligned} & \tilde{\varphi}\left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}\right) \\ &= P_N\left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}\right) + R_N\left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}\right), \end{aligned}$$

where $R_N = \mathcal{O}(2^{-k(N+1)})$ depends on the jets of order $N + 1$ of $\tilde{\varphi}$. Injecting the above decomposition in our series yields

$$\sum_{k \geq 0} 2^{2k\gamma} \int_{(\mathbf{R}^3)^2} (P_N + R_N)\left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}\right) \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) \psi_i^2(\tilde{x}) dy_1 dy_2.$$

Now we make use of the fact that we have

$$\int_{(\mathbf{R}^3)^2} P_N\left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}\right) \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) \psi_i^2(\tilde{x}) dy_1 dy_2 = 0,$$

since the support of η is included in an annulus, which implies that its integral against all polynomials vanish. The series therefore simplifies as

$$\sum_{k \geq 0} 2^{2k\gamma} \int_{(\mathbf{R}^3)^2} \underbrace{R_N\left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}\right)}_{\mathcal{O}(2^{-k(N+1)})} \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) \psi_i^2(x) dy_1 dy_2$$

$$\lesssim \sum_{k=0}^{\infty} 2^{k(2\gamma-N-1)} \sum_{|\alpha|=N+1} \|\partial^\alpha \tilde{\varphi}\|_{L^1},$$

where the series on the right hand side converges as soon as $N + 1 > 2\gamma$ and we control the convergence from the $N + 1$ derivatives of $\tilde{\varphi}$ hence η . We just proved that \mathcal{Q}_x^γ converges as a distribution of order $N + 1$ for all $N + 1 > 2\gamma$.

Finally we need to control the weak homogeneity when we scale towards (x, x) . Working in a chart $i \in I$ this reduces to estimate the weak homogeneity near (x, x) of the series

$$U \stackrel{\text{def}}{=} \sum_{k \geq 0} 2^{k(6+2\gamma)} \psi(2^k |y_1 - \tilde{x}|) \psi(2^k |y_2 - \tilde{x}|) \in \mathcal{S}'(\mathbf{R}^3 \times \mathbf{R}^3).$$

By duality it suffices to estimate the weak homogeneity at ∞ of its Fourier transform

$$\begin{aligned} \widehat{U}(\xi_1, \xi_2) &= \sum_{k \geq 0} 2^{k(6+2\gamma)} \mathcal{F}[\psi(2^k |\cdot - \tilde{x}|)](\xi_1) \mathcal{F}[\psi(2^k |\cdot - \tilde{x}|)](\xi_2) \\ &= \sum_{k \geq 0} 2^{2k\gamma} e^{i\tilde{x} \cdot (\xi_1 + \xi_2)} \eta(2^{-k} \xi_1) \eta(2^{-k} \xi_2). \end{aligned}$$

Here we can use the fact that $\eta(2^{-k} \xi)$ is zero unless $2^{k-1} \leq |\xi| \leq 2^{k+1}$, so that both ξ_1 and ξ_2 are localised at scale 2^j for some $j \geq 0$, which yields the bound

$$\left| \sum_{k \geq 0} 2^{2k\gamma} e^{i\tilde{x} \cdot (\xi_1 + \xi_2)} \eta(2^{-k} \xi_1) \eta(2^{-k} \xi_2) \right| = \mathcal{O}(|\xi_1|^{2\gamma} \vee |\xi_2|^{2\gamma})$$

uniform in \tilde{x} , hence the Fourier transform \widehat{U} is such that the family of tempered distributions $(\lambda^{-2\gamma} \widehat{U}(\lambda \cdot))_{\lambda \geq 1}$ is bounded. By Plancherel this entails that the family of distributions

$$(\lambda^{-2\gamma-6} U(\lambda \cdot + (\tilde{x}, \tilde{x})))_{\lambda \in (0,1]}$$

is bounded too, which concludes the proof. \square

As an example we discuss below the regularity of the dynamical free field.

Lemma 3.48 *Fix $T > 0$ and $\epsilon > 0$. Then, we have $\mathfrak{f} \in C_T \mathcal{C}^{-1/2-\epsilon}$.*

Proof. We start by performing the estimate at some fixed time $t > 0$. By stationary in time of \mathfrak{f} one has for any $x \in M$ the t -independent bound

$$\mathbf{E}[\mathcal{Q}_x^\gamma(\mathfrak{f}(t, \cdot), \mathfrak{f}(t, \cdot))] \lesssim \int_{M^2} \mathcal{Q}_x^\gamma(y_1, y_2) P^{-1}(y_1, y_2) dv(y_1) dv(y_2).$$

Next we use the representation (3.42) of the Schwartz kernel of P_k^i , which implies that pushing the estimate in the chart $\kappa_i(U_i)$ we have

$$\mathbf{E}[(P_k^i \mathfrak{f})(t, x)^2] \lesssim 2^{6k} \int_{(\mathbf{R}^3)^2} \psi_i(\tilde{x})^2 \psi(2^k |y_1 - \tilde{x}|) \psi(2^k |y_2 - \tilde{x}|) (\kappa_{i*} \chi_i)^{\otimes 2}(y_1, y_2) \kappa_{i*}^{\otimes 2} P^{-1}(y_1, y_2) dy_1 dy_2$$

where we denote $\tilde{x} \stackrel{\text{def}}{=} \kappa_i(x)$. Also, denoting by $\kappa_{i*}^{\otimes 2} P^{-1} =: P_{i,i}^{-1}$ the pulled-back Green function, we obtain

$$\begin{aligned} \mathbf{E}[(P_k^i \mathfrak{f})(t, x)^2] &\lesssim \int_{(\mathbf{R}^3)^2} \psi_i(\tilde{x})^2 \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) (\kappa_{i*} \chi_i)^{\otimes 2} \left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x} \right) \\ &\quad \times P_{i,i}^{-1}(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}) dy_1 dy_2. \end{aligned}$$

Since \tilde{x} is chosen uniformly in a compact set, $\psi(|y_1 - \tilde{x}|), \psi(|y_2 - \tilde{x}|)$ are Schwartz in y_1, y_2 and $P_{i,i}^{-1}(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x})$ is integrable in y_1, y_2 near $y_1 = y_2$, so the second integral is well-defined. Using the bound on the rescaled Green function

$$P_{i,i}^{-1}\left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}\right) \lesssim 2^k |y_1 - y_2|^{-1},$$

we can now conclude that we have

$$\mathbf{E}[(P_k^{i\uparrow})(t, x)^2] \lesssim 2^k \int_{(\mathbf{R}^3)^2} |\psi(|y_1 - \tilde{x}|)\psi(|y_2 - \tilde{x}|)| |y_1 - y_2|^{-1} dy_1 dy_2 \lesssim 2^k.$$

We now show that \uparrow is indeed continuous in time. To do so, let us first introduce the notation

$$G_{i,i}^{(1)}(t_1 - t_2, x, y) \stackrel{\text{def}}{=} \kappa_{i*}^{\otimes 2} \left(\frac{e^{-|t_1 - t_2|P}}{P} \right) (x, y).$$

The quantity $\mathbf{E}[(P_k^{i\uparrow})(t_1, x) - P_k^{i\uparrow}(t_2, x)]^2$ is bounded above by

$$\left| \int_{(\mathbf{R}^3)^2} \psi_i(\tilde{x})^2 \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) (\kappa_{i*} \chi_i)^{\otimes 2} \left(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x} \right) \right. \\ \left. \times \left(P_{i,i}^{-1}(2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}) - G_{i,i}^{(1)}(t_1 - t_2, 2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}) \right) \right|$$

for an integral with respect to $dy_1 dy_2$. Our goal is to bound the difference of kernels $P_{i,i}^{-1} - G_{i,i}^{(1)}(r, \bullet)$. To do so, we use the identity

$$\frac{1 - e^{-rP}}{P} = \int_0^r e^{-sP} ds$$

which expresses the difference as an integral of the heat operator. Then we can inject in the integral expression the bound on the heat kernel on the product chart

$$|e^{-sP}(y_1, y_2)| \lesssim s^{-3/2} e^{-s} e^{-C|y_1 - y_2|^2/s},$$

where $C > 0$. We obtain the estimate for $\beta \in (0, 1)$

$$\begin{aligned} \left| P_{i,i}^{-1}(y_1, y_2) - G_{i,i}^{(1)}(r, y_1, y_2) \right| &\lesssim \int_0^r s^{-3/2} e^{-C|y_1 - y_2|^2/s-s} ds \lesssim \int_{r^{-1}}^\infty e^{-Cs|y_1 - y_2|^2} e^{-\frac{1}{s}} s^{-\frac{1}{2}} ds \quad (3.47) \\ &\lesssim |y_1 - y_2|^{-1} \int_{r^{-1}|y_1 - y_2|^2}^\infty e^{-Cs - \frac{1}{s}} s^{-\frac{1}{2}} ds \lesssim |y_1 - y_2|^{-1} \left(e^{-\frac{|y_1 - y_2|^2}{r}} \right) \\ &\lesssim |y_1 - y_2|^{-1} \left(1 + \frac{|y_1 - y_2|^2}{r} \right)^{-N} \lesssim |y_1 - y_2|^{-1-\beta} r^{\frac{\beta}{2}}. \end{aligned}$$

Replacing in the difference estimate finally yields

$$\mathbf{E} \left[\left((P_k^{i\uparrow})(t_1, x) - (P_k^{i\uparrow})(t_2, x) \right)^2 \right] \lesssim 2^{k(1+\beta)} |t_1 - t_2|^{\frac{\beta}{2}}.$$

□

Together with Kolmogorov's classical regularity theorem what we said above justifies that \uparrow takes almost surely its values in $C_T^\eta C^{-1/2-\epsilon}(M)$, for all small enough $\eta > 0$ and all $\epsilon > 0$, and its norm in the corresponding space has moments of any finite order.

Remark 3.49 In the sequel we will not dwell on the proof of the estimate (3.45) for larger trees. However it is by now classical that when computing in charts the covariance of $(P_k^i u)(t_1, x) - (P_k^i u)(t_2, x)$ the amplitudes appearing can be reorganised by means of some telescoping sums in order to introduce a terms $P_{i,i}^{-1} - G_{i,i}^{(1)}(r, \bullet)$. In view of (3.47) this term always yields a good factor in time modulo a small loss in spatial regularity.

3.5.2 The Wick monomials

3.5.2.1 Locally covariant Wick renormalisation

In the sequel we will systematically first estimate regularities using Wick renormalisation. Then we shall compare the usual Wick renormalisation which is not *locally covariant* with a *locally covariant* renormalisation in which we only subtract *universal quantities* at the cost of producing objects which do not belong to homogeneous Wiener chaoses. Let us discuss the notion of *local covariance* with some simple example which also explains why the usual Wick renormalisation fails to be locally covariant. A *function valued invariant of Riemannian structure* is a function c which assigns to each manifold M and Riemannian structure g on M some function $c[g]$ on M which is *locally covariant* in the following sense: given $f : M' \hookrightarrow M$ a diffeomorphism onto an open submanifold of M , and a Riemannian structure g on M , then c must satisfy the equation $f^*c[g] = c[f^*g]$. Moreover, we require that c depends smoothly on the Riemannian metric g . In fact, one can prove such locally covariant function c should depend at every point on finite jets of the metric and c has an invariance property under changes of coordinates [CF12, section 8 p. 76]. This definition is partly motivated by [Gi18, 2.4 p. 160] and [MA73, p. 282] on local index theory and also by the notion of local covariance arising in algebraic Quantum Field Theory [Rej16, KMI16, BFV03]. In the usual Wick renormalisation for some massive Gaussian free field ϕ on some Riemannian 3-manifold (M, g) with covariance P^{-1} , one first mollifies ϕ via heat regularisation. This yields a random smooth function $\phi_r \stackrel{\text{def}}{=} e^{-rP}\phi$. To renormalize the square ϕ_r^2 , one subtracts to the square $(\phi_r)^2(x)$ of the mollified field at x , the counterterm $a_r(x) = e^{-2rP}P^{-1}(x, x)$ and it is well-known that the difference $(\phi_r)^2(\cdot) - a_r(\cdot)$ will converge as random distribution when $r \downarrow 0$. However, the counterterm $a_r(x) = e^{-2rP}P^{-1}(x, x)$ that we subtracted is **nonlocal** in the metric g at x , it depends on the global Riemannian geometry of (M, g) and not on finite jets of the metric g at x . Hence such a_r is not *locally covariant* in the above sense. Now we observe that the diagonal value $e^{-2rP}P^{-1}(x, x)$ has an asymptotic expansion of the form:

$$e^{-2rP}P^{-1}(x, x) \sim \frac{1}{4\pi^{\frac{3}{2}}r^{\frac{1}{2}}} + \mathcal{O}(1).$$

If instead of subtracting the diagonal value of $e^{-2rP}P^{-1}$ one subtracted its **singular part**: $(\phi_r)^2(\cdot) - \frac{1}{4\pi^{\frac{3}{2}}r^{\frac{1}{2}}}$, then one would still get a random distribution at the limit when $r \downarrow 0$ but the covariantly renormalized Wick square : ϕ^2 : would no longer have zero expectation. So one may wonder why is it so important to subtract only locally covariant quantities ? The answer lies in the deep notion of locality in quantum field theory. It is a folklore result in quantum fields on curved backgrounds that subtracting non locally covariant counterterms is incompatible with locality in the sense of Atiyah-Segal. Let us quote the beautiful discussion on the regularisation of tadpoles and the relation with Atiyah-Segal gluing in [KMW21, 1.2 p. 1852]: ”In various treatments of scalar theory, tadpole diagrams were set to zero (this corresponds to a particular renormalisation scheme – in flat space, this is tantamount to normal ordering, ...). However, in our framework this prescription contradicts locality in Atiyah-Segal sense, ... One good solution is to prescribe to the tadpole diagrams the zeta-regularized diagonal value of the Green’s function. We prove that assigning to a surface its zeta-regularized tadpole is compatible with locality, ... However there are other consistent prescriptions (for instance, the tadpole regularized via point-splitting and subtracting the singular term, ...). This turns out to be related to Wilson’s idea of RG flow in the space of interaction potentials, ...” We refer the interested reader to [KMW21, Section 5 p. 1885] for further details on this central topic of quantum fields on curved backgrounds. So in the present chapter, we follow a similar strategy as in the previous example and try to subtract only locally covariant quantities, in fact we shall see that we subtract universal quantities that do not even depend on the metric g .

3.5.2.2 The quadratic term

The argument used for the study of \mathfrak{I} , without regularisation, works for the study of \mathfrak{V}_r , with regularisation. Since ξ_r is regularized in space \mathfrak{I}_r is here a function on $[0, T] \times M$. Set

$$a_r(z) \stackrel{\text{def}}{=} \mathbf{E}[\mathfrak{I}_r^2(z)], \quad \overline{\mathfrak{V}}_r(z) \stackrel{\text{def}}{=} \mathfrak{I}_r^2(z) - a_r(z).$$

To treat the regularity of $\overline{\mathfrak{V}}$, we will follow a similar strategy as for \mathfrak{I} and try to control

$$\mathbf{E}[(P_k^i \overline{\mathfrak{V}})(t, x)^2]$$

when i becomes large and uniformly in $x \in M$. Denote by $G_r^{(2)}(t-s)$ the operator with kernel

$$\left(\frac{e^{-(2r+|t-s|)P}}{P}(x, y) \right)^2.$$

The operators $G_r^{(2)}(0)$ take values in $\Psi^{-1}(M)$ and $G_r^{(2)}(t)$ is of order $|t|^{\gamma/2}$ in $\Psi^{-1+\gamma}(M)$ since $e^{-|t|P} P^{-1}$ is of order $|t|^\gamma$ in $\Psi^{-2+2\gamma}(M)$. This holds uniformly in $r \in [0, 1]$. By definition, and using the Wick formula, we can compute the quantity $\mathbf{E}[P_k^i \overline{\mathfrak{V}}(t, x)^2]$, which is equal to

$$\begin{aligned} & \mathbf{E}[(P_k^i \overline{\mathfrak{V}})(t, x)^2] \\ &= \int_{M^2} G^{(2)}(t_1 - t_2, y_1, y_2) P_k^i(x, y_1) P_k^i(x, y_2) dv(y_1) dv(y_2) \\ &\lesssim \int_{(\mathbf{R}^3)^2} G_{i,i}^{(2)}(t_1 - t_2, y_1, y_2) \psi_i(\tilde{x})^2 \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) (\kappa_{i*} \chi_i)^{\otimes 2} (2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}) \\ &\lesssim 2^{2k} \int_{(\mathbf{R}^3)^2} |y_1 - y_2|^{-2} \psi_i(\tilde{x})^2 \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) (\kappa_{i*} \chi_i)^{\otimes 2} (2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x}), \end{aligned}$$

for some integrals over $(\mathbf{R}^3)^2$ with respect to $dy_1 dy_2$, and where again $\tilde{x} = \kappa_i(x)$ and $G_{i,i}^{(2)} = \kappa_{i*}^{\otimes 2} G^{(2)}$. We used the estimate

$$|G_{i,i}^{(2)}(t_1 - t_2, x_1, x_2)| \lesssim |x_1 - x_2|^{-2}$$

and the changes of variables

$$y_1 \mapsto 2^{-i}(y_1 - \tilde{x}) + \tilde{x}, \quad y_2 \mapsto 2^{-i}(y_2 - \tilde{x}) + \tilde{x}.$$

We can now conclude since the integral

$$\int_{(\mathbf{R}^3)^2} |y_1 - y_2|^{-2} \psi_i(\tilde{x})^2 \psi(|y_1 - \tilde{x}|) \psi(|y_2 - \tilde{x}|) (\kappa_{i*} \chi_i)^{\otimes 2} (2^{-k}(y_1 - \tilde{x}) + \tilde{x}, 2^{-k}(y_2 - \tilde{x}) + \tilde{x})$$

with respect to $dy_1 dy_2$ is bounded uniformly in $x \in \text{supp}(\psi_i)$ because $\psi(|y - \tilde{x}|)$ is a Schwartz function of y .

We conclude as above from Kolmogorov's regularity theorem that $\overline{\mathfrak{V}}_r \in C_T^\eta C^{-1-\epsilon}(M)$ for all $\eta > 0, \epsilon > 0$, uniformly in $r \in (0, 1]$, and we further get from the r -uniform and continuity of the different quantities as functions of r the convergence of $\overline{\mathfrak{V}}_r$ in $C_T^\eta C^{-1-\epsilon}(M)$ to a limit which we denote by $\overline{\mathfrak{V}}$.

In the present chapter, we choose to define the **renormalisation in a locally covariant way** with respect to the Riemannian metric g . Therefore, we shall subtract from \mathfrak{I}_r^2 only the **singular part** of the function $a_r(z) = \mathbf{E}[\mathfrak{I}_r^2(z)]$. We actually prove that this singular part a_r is actually some universal constant. The function $a_r(z)$ differs from the constant a_r , but for $z = (t, x)$ we have

$$a_r(z) = \frac{e^{-2rP}}{P}(x, x) = \int_{2r}^1 e^{-sP}(x, x) ds + b(x),$$

where b is a smooth function. The small time asymptotic of the heat kernel then tells us that $e^{-sP}(x, x) = \frac{1}{(4\pi s)^{\frac{3}{2}}} + \mathcal{O}(s^{-\frac{1}{2}})$, so that the function $a_r(\cdot) - a_r$ is indeed bounded, uniformly in $r \in (0, 1]$. To prove that the difference $a_r(\cdot) - a_r$ is smooth, we rely on the description of the heat kernel of [Gri04, def 2.1 p. 6]. In local coordinates in some open subset U , the heat kernel can be represented as $s^{-\frac{3}{2}}A(a, \frac{x-y}{\sqrt{s}}, y)$ where $A \in C^\infty([0, +\infty)_{\frac{1}{2}} \times \mathbf{R}^3 \times U)$. Hence $a_r(x) - a_r = \int_{2r}^1 s^{-\frac{3}{2}}(A(\sqrt{s}, 0, x) - A(0, 0, x))ds$ converges together with all its derivatives in x since $\partial_x^\beta s^{-\frac{3}{2}}(A(\sqrt{s}, 0, x) - A(0, 0, x)) = \mathcal{O}(s^{-\frac{1}{2}})$ for all multi-indices β . Hence the convergent integral defining $a_r(x) - a_r$ depends smoothly on $x \in U$.

The convergence of $\varphi_r \stackrel{\text{def}}{=} \varphi_r^2 - a_r$ in $C_T^\eta C^{-1-\epsilon}(M)$ to a limit which we denote by φ follows as a consequence. We note that as a random variable, φ is not a homogeneous element in the chaos of order 2 of the Gaussian noise since we did not subtract the full expectation. However it differs from a homogeneous element by a deterministic smooth function hence φ has moments of any order $1 \leq r < \infty$ that are equivalent to its second moment. In the sequel, we will always prove stochastic estimates for homogeneous elements in Wiener chaoses and then justify why the locally covariant renormalisation, subtracting only universal local quantities, still yields a stochastic object with the same analytic properties.

3.5.2.3 The cubic term

For the stochastic term φ° , we do not bound the regularity by hands anymore and use the microlocal machinery. Recall that our aim is to find the range of $\gamma \in \mathbf{R}$ so that

$$\mathbf{E}[\mathcal{Q}_x^\gamma(\varphi^\circ(t, \cdot), \varphi^\circ(t, \cdot))] < \infty$$

uniformly in $x \in M$. By application of Wick's Theorem we have

$$\begin{aligned} \mathbf{E}[\mathcal{Q}_x^\gamma(\varphi^\circ(t, \cdot), \varphi^\circ(t, \cdot))] &= \int_{(-\infty, t]^2 \times M^4} G^{(3)}(s_1 - s_2, z_1, z_2) e^{-|t-s_1|P}(y_1, z_1) e^{-|t-s_2|P}(y_2, z_2) \mathcal{Q}_x^\gamma(y_1, y_2), \end{aligned}$$

for an integral with respect to $dv(y_1, y_2, z_1, z_2)ds_1ds_2$, where we denote by $G_r^{(3)}(t_1 - t_2)$ the operator with kernel

$$\left(\frac{e^{-(2r+|t_1-t_2|)P}}{P}(x, y) \right)^3.$$

This operator belongs to $\mathcal{S}_\Gamma^{-3}(\mathbf{R}^2 \times M^2)$ with

$$\Gamma = N^*(\{t = s\} \times \mathbf{d}_2 \subset \mathbf{R}^2 \times M^2) \cup N^*(\{t = s\} \subset \mathbf{R}^2 \times M^2).$$

By the general theory described above, the only irreducible subgraph of the Feynman graph is the graph itself, therefore the Feynman amplitude is well-defined on $(-\infty, t]^2 \times M^4 \setminus \Sigma$ where the singular locus Σ corresponds to the generalized diagonal

$$\underbrace{\{y_1 = y_2 = z_1 = z_2 = x\}}_{\text{codim}_w=12} \cap \underbrace{\{s_1 = s_2 = t\}}_{\text{codim}_w=4} \subset (-\infty, t]^2 \times M^4$$

which has weighted codimension 16. The sum of the weak homogeneities of the kernels in the Feynman amplitude is given by

$$-3 + 2(-3) - 6 - 2\gamma,$$

so that is finite for $\gamma < \frac{1}{2}$ by Theorem 3.40.

To conclude this section, let us now discuss the general strategy for larger random fields. First, a general object τ_m lying in the p -th inhomogeneous Gaussian chaos can be expended onto the p first

homogeneous chaoses, so that its covariance can be controlled by the covariances of the projections and subsequently by some amplitudes indexed by Feynman graphs, denoted \mathcal{A}_{mn} for $n = p, p - 2$, etc. The next step is to localise the amplitudes in local charts using the partition of unity $1 = \sum_{i \in I} \chi_i$ subordinated to the cover $M = \cup_{i \in I} U_i$, and to control for every $i \in I$ the localised version \mathcal{A}_{mn}^i . The amplitudes \mathcal{A}_{mn} correspond to products of kernels, and the aim is to identify the range of γ for which they are well defined using item **(a)** of Theorem 3.40. However, two difficulties occur:

1. Because it contains a divergent subgraph, the term coming from the lowest chaos, \mathcal{A}_{m0} or \mathcal{A}_{m1} , can not be directly handled using item **(a)** of Theorem 3.40, and rather requires the use of item **(b)**: this amounts to subtracting a local counterterm.
2. The terms \mathcal{A}_{mn} coming from higher chaoses do not contain any divergent subgraph and could therefore be defined using item **(a)** of Theorem 3.40. However, due to the fact that \mathcal{A}_{mn} contains a subgraph including \mathcal{Q}_x^γ with worse weak homogeneity than the whole graph, this would deteriorate the value of γ . This last case require a special treatment leveraging the precise microlocal properties of $[\odot]$, which is performed in Section 3.5.4.

3.5.3 Diagrammatic notation for higher chaoses

To handle the quartic and quintic terms, we first need to introduce more notation. From Appendix 3.G, we define a family $(P_k^i, \tilde{P}_k^i)_{k \in \mathbb{N}, i \in I}$ of generalized Littlewood-Paley-Stein projectors indexed by the frequency 2^k and the chart index $i \in I$. The localized resonant product \odot_i , where i is the chart index, is defined in the appendix as $u \odot_i v = \sum_{|k-\ell| \leq 1} P_k^i(u) \tilde{P}_\ell^i(v)$. The goal of the present section is to deal with the localized objects

$$\tau_1 = \mathfrak{Y}_r^\circ \odot_i \mathfrak{I}_r, \quad \bar{\tau}_2 = \mathfrak{Y}_r \odot_i \mathfrak{V}_r - \chi_i \frac{b_r}{3}, \quad \bar{\tau}_3 = \chi_i |\nabla \mathfrak{Y}_r^\circ|^2 - \chi_i \frac{b_r}{3}, \quad \bar{\tau}_4 = \mathfrak{Y}_r^\circ \odot_i \mathfrak{V}_r - \chi_i b_r \mathfrak{I}_r,$$

where $\chi_i \in C_c^\infty(U_i)$ is the cut-off function used to define \odot_i , and find for each of them the range of exponents γ for which our test criterion is verified.

Note that the trees are defined with the localized resonant term \odot_i made with these projectors and recall from Appendix 3.G that these resonant terms are not commutative, in the sense that $A \odot_i B \neq B \odot_i A$. However, we do not have to worry about the definitions of τ_1 and $\bar{\tau}_2$, since the analytic properties of P_k^i and \tilde{P}_k^i are similar, which entails that for every A, B , the construction of the renormalized part of $B \odot_i A$ is totally equivalent to that of $A \odot_i B$ which we provide here.

Denote by $[\odot_i](x, y, z)$ the kernel of the localized resonant operator \odot_i on the chart $U_i \subset M$. Recall that we write

$$\underline{\mathcal{L}}^{-1}(t - s, x, y) = \mathbf{1}_{(-\infty, t]}(s) e^{-(t-s)P}(x, y)$$

and

$$\begin{aligned} G_r^{(p)}(t - s, x, y) &= \left(\left\{ e^{-|t-s|P} P^{-1} \right\} (x, y) \right)^p, & 1 \leq p \leq 3, \\ [\odot_i](x, y, z) &= \sum_{|k-\ell| \leq 1} P_k^i(x, y) \tilde{P}_\ell^i(x, z), & i \in I, \\ \mathcal{Q}_x^\gamma(y, z) &= \sum_{i \in I} \sum_{k \geq -1} 2^{2k\gamma} P_k^i(x, y) P_k^i(x, z), & \gamma \in \mathbf{R}, x \in M. \end{aligned}$$

The followings microlocal bounds on the kernels $\underline{\mathcal{L}}^{-1}$, $G_r^{(p)}$ and $[\odot_i]$ are proven in [BDF123, Theorem 1.2], which is proven in Section 7 thereof. We recall them, along with the bounds for \mathcal{Q}_x^γ proven in Lemma 3.47.

Lemma 3.50 (Weak homogeneities and wave fronts)

– The kernel $\underline{\mathcal{L}}^{-1}(t - s, x, y)$ belongs to \mathcal{S}_Γ^{-3} with wavefront set

$$\Gamma = N^*(\{t = s\} \times \mathbf{d}_2 \subset \mathbf{R}^2 \times M^2).$$

– The kernel $G_r^{(p)}(t - s, x, y)$ belongs to \mathcal{S}_Γ^{-p} with wavefront set

$$\Gamma = N^* (\{t = s\} \subset \mathbf{R}^2 \times M^2) \cup N^* (\{t = s\} \times \mathbf{d}_2 \subset \mathbf{R}^2 \times M^2).$$

– The kernel $[\odot_i](x, y, z)$ belongs to \mathcal{S}_Γ^{-6} with wavefront set

$$\Gamma = N^* (\{x = y = z\} \subset M^3).$$

– The kernel $\mathcal{Q}_x^\gamma(y_1, y_2)$ belongs to $\mathcal{S}_{\Gamma_x}^{-6-2\gamma}$ with wavefront set

$$\Gamma_x = T_{(x,x)}^*(M^2).$$

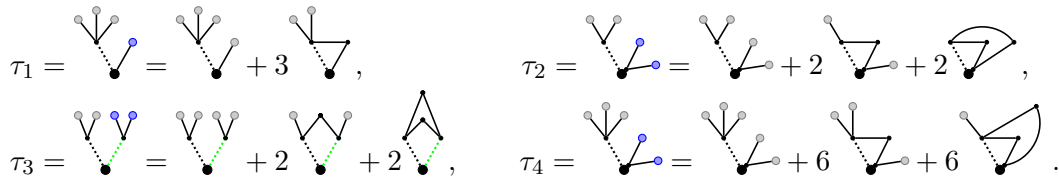
Beware that in the case of \mathcal{Q}_x^γ , the singular locus depends on the marked point $x \in M$.

In this list the kernels on $(\mathbf{R} \times M)^2$ satisfy a local diagonal bound of the form

$$|\partial_{\sqrt{t}, \sqrt{s}, y_1, y_2}^\alpha K| \lesssim \left(\sqrt{|t - s|} + |y_1 - y_2| \right)^{a - |\alpha|}$$

for the corresponding scaling exponent a . We also often see the kernels $\underline{\mathcal{L}}^{-1}$, \mathcal{Q}_x^γ and $G_r^{(1)}$ as some time dependent space operators $X(t_1 - t_2)$ whose kernels are then given by $X(t_1 - t_2, y_1, y_2)$.

We use a pictorial representation of the τ_i in which the black dot \bullet represents a resonant operator and the noises are coloured circles. In a given graph noises of the same colour are integrated outside all diagonals of the corresponding set of variables. In the present section, it will be convenient to first discuss stochastic estimates for Wick ordered elements which live in homogeneous Wiener chaoses, then explain why our locally covariant renormalisation yields stochastic elements that differ from the Wick renormalized ones only up to higher regularity elements. The Wiener chaos decomposition of the τ_i is



where black edges stand for the kernel of $\underline{\mathcal{L}}^{-1}$ and green edges stand for the kernel of $\Delta \underline{\mathcal{L}}^{-1}$. The reason for the presence of the Laplacian operators in τ_3 will be made clear in Section 3.5.5.4.

We write for $m \in \{1, 2, 3\}$ $\tau_m = \tau_{m4} + \tau_{m2} + \tau_{m0}$ with τ_{mn} in the homogeneous chaos of degree $n \in \{0, 2, 4\}$ and accordingly $\tau_4 = \tau_{45} + \tau_{43} + \tau_{41}$. By orthogonality

$$\mathbf{E} \left[\mathcal{Q}_x^\gamma(\tau_m(t, \cdot), \tau_m(t, \cdot)) \right] = \sum_{n \in \mathbf{N}} \mathbf{E} \left[\mathcal{Q}_x^\gamma(\tau_{mn}(t, \cdot), \tau_{mn}(t, \cdot)) \right].$$

Remember that we are not interested in the τ_m themselves but in their renormalised versions generically written $\bar{\tau}_m = \tau_m - c_{m,r}$, with $c_{m,r}$ for ‘counterterm’, for which we still have the orthogonality relation

$$\begin{aligned} & \mathbf{E} \left[\mathcal{Q}_x^\gamma(\bar{\tau}_m(t, \cdot), \bar{\tau}_m(t, \cdot)) \right] \\ &= \sum_{n \geq 2} \mathbf{E} \left[\mathcal{Q}_x^\gamma(\tau_{mn}(t, \cdot), \tau_{mn}(t, \cdot)) \right] + \mathbf{E} \left[\mathcal{Q}_x^\gamma((\tau_{mp}(t, \cdot) - c_{k,r}), (\tau_{mp}(t, \cdot) - c_{m,r})) \right], \end{aligned}$$

where $p = 0$ for $m \in \{1, 2, 3\}$ and $p = 1$ for $m = 4$. Also, note that each element τ_{mn} in the Wiener chaos can be written as $F_n(\cdot; \xi^{\otimes n} \cdot)$ where each $F_n : C^\infty(\mathbf{R} \times M)^{\otimes n} \rightarrow \mathcal{D}'(\mathbf{R} \times M)$ is a multilinear functional valued in distributions involving iterated integrals of the heat operator, products represented by the trees. The following elementary well-known result tells us that we only need to bound very symmetric Feynman diagrams to bound the preceding expectations.

Lemma 3.51 *Let $F \in L^2(M^n)$ be a function of n variables on some compact Riemannian manifold (M, g) , ξ the white noise on (M, g) . Then:*

$$\mathbf{E} \left[\langle F, : \xi^{\otimes n} : \rangle^2 \right] \leq \|F\|_{L^2(M^n)}^2.$$

Proof. We define the symmetrisation operator S_n as

$$S_n(\varphi)(x_1, \dots, x_n) \stackrel{\text{def}}{=} \frac{1}{n!} \sum_{\sigma \in S_n} \varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Note the important fact that S_n is self-adjoint on $L^2(M^n)$ hence S_n is the orthogonal projector on symmetric L^2 functions. By the Itô isometry property,

$$\mathbf{E} \left(\langle F, : \xi^{\otimes n} : \rangle^2 \right) = \mathbf{E} \left(\langle S_n F, : \xi^{\otimes n} : \rangle^2 \right) = \|S_n F\|_{L^2(M^n)}^2$$

hence to prove the Lemma it suffices to note that $\|S_n F\|_{L^2(M^n)} \leq \|F\|_{L^2(M^n)}$ which is obvious since S_n is an orthogonal projector. \square

Each expectation can therefore be bounded by a quantity of the form

$$\int_{(\mathbf{R} \times M)^{n+2}} F_n(x_1; y_1, \dots, y_n) F_n(x_2; y_1, \dots, y_n) \mathcal{Q}_x^\gamma(x_1, x_2) dy dx,$$

which can be represented by some mirror symmetric Feynman diagram. Using a **purple edge** for the kernel of the operator \mathcal{Q}_x^γ , a dotted edge for $\underline{\mathcal{L}}^{-1}$, a black edge for $G_r^{(1)}$, and a **green edge** for $\Delta \underline{\mathcal{L}}^{-1}$, we have

$$\begin{aligned} \mathbf{E} \left[\mathcal{Q}_x^\gamma(\bar{\tau}_1(t, \cdot), \bar{\tau}_1(t, \cdot)) \right] &\lesssim \text{diagram 1} + \text{diagram 2} =: \mathcal{G}_{14} + \mathcal{G}_{12}, \\ \mathbf{E} \left[\mathcal{Q}_x^\gamma(\bar{\tau}_2(t, \cdot), \bar{\tau}_2(t, \cdot)) \right] &\lesssim \text{diagram 3} + \text{diagram 4} + \mathbf{E} \left[\mathcal{Q}_x^\gamma((\tau_{20}(t, \cdot) - \chi_i \frac{b_r}{3}), (\tau_{20}(t, \cdot) - \chi_i \frac{b_r}{3})) \right] \\ &\stackrel{\text{def}}{=} \mathcal{G}_{24} + \mathcal{G}_{22} + \mathcal{G}_{20}, \\ \mathbf{E} \left[\mathcal{Q}_x^\gamma(\bar{\tau}_3(t, \cdot), \bar{\tau}_3(t, \cdot)) \right] &\lesssim \text{diagram 5} + \text{diagram 6} + \mathbf{E} \left[\mathcal{Q}_x^\gamma((\tau_{30}(t, \cdot) - \chi_i \frac{b_r}{3}), (\tau_{30}(t, \cdot) - \chi_i \frac{b_r}{3})) \right] \\ &\stackrel{\text{def}}{=} \mathcal{G}_{34} + \mathcal{G}_{32} + \mathcal{G}_{30}, \\ \mathbf{E} \left[\mathcal{Q}_x^\gamma(\bar{\tau}_4(t, \cdot), \bar{\tau}_4(t, \cdot)) \right] &\lesssim \text{diagram 7} + \text{diagram 8} + \mathbf{E} \left[\mathcal{Q}_x^\gamma((\tau_{41}(t, \cdot) - \chi_i b_r \mathfrak{I}_r), (\tau_{41}(t, \cdot) - \chi_i b_r \mathfrak{I}_r)) \right] \\ &\stackrel{\text{def}}{=} \mathcal{G}_{45} + \mathcal{G}_{43} + \mathcal{G}_{41}. \end{aligned}$$

We will use the notation \mathcal{G}_{mn} for positive quantity represented by the mirror graph associated with τ_{mn} and the notation \mathcal{A}_{mn} for the associated distribution on the corresponding configuration space ; we use Theorem 3.40 to prove their convergence. We use the word ‘*amplitude*’ to talk about any of the \mathcal{A}_{mn} . The terms involving τ_{20} and τ_{30} are treated differently from the mirror graphs. Recall that the vertex set is partitioned as $V = V' \sqcup V_A$. We always denote by x_a, x_b , etc. the space points associated with the vertices in V' , while in our setting V_A will always contain two triplets $((v_1^j, v_2^j, v_*^j))$ whose associated space points are denoted $((y_1, y_2), y_*)$ and $((z_1, z_2), z_*)$ for $n_{\mathcal{G}} = 2$.

Our aim is now to control all of the eleven amplitudes \mathcal{A}_{mn} using Theorem 3.40. It turns out that this naive approach will only work for \mathcal{A}_{12} and, after renormalisation, \mathcal{A}_{20} and \mathcal{A}_{30} . All the remaining eight amplitudes need a special care for one of their subamplitudes. This analysis is performed in the next section. Once the problematic subamplitudes are constructed, it will then suffice to plug this knowledge inside the recursion of the proof of Theorem 3.40, and this will yield the existence of \mathcal{A}_{mn} .

3.5.4 Modification of the inductive proof

As anticipated, while Theorem 3.40 turns out to be sufficient to handle some graphs and subgraphs of the graphs introduced in Section 3.5.3, yet most of the total amplitudes need a particular care in order to be controlled:

1. The amplitudes of the graphs \mathcal{G}_{14} , \mathcal{G}_{24} , and \mathcal{G}_{45} are given for $(m, n) \in \{(1, 4), (2, 4), (4, 5)\}$ by

$$\mathcal{A}_{mn} = [\odot](y_*, y_1, y_2) [\odot](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(k)}(0, y_1, z_1) \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(\ell)}(s_a - s_b, x_a, x_b),$$

where $(k, \ell) = (1, 3)$ for \mathcal{G}_{14} , $(k, \ell) = (2, 2)$ for \mathcal{G}_{24} , and $(k, \ell) = (2, 3)$ for \mathcal{G}_{45} , and have a subgraph with amplitude reading

$$\mathcal{C}^{(k)}(y_*, y_1, y_2, z_*, z_1, z_2) \stackrel{\text{def}}{=} [\odot](y_*, y_1, y_2) [\odot](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(k)}(0, y_1, z_1)$$

which is *less convergent* (in the sense of the power counting from Theorem 3.40) than the whole graph: since this is a one loop subgraph, naively applying the power counting criterion from Theorem 3.40 on the deepest diagonal $\{y_* = y_1 = y_2 = z_* = z_1 = z_2 = x\}$ would result in a potential loss in the regularity γ . The exact same phenomenon also occurs for the amplitude (3.55) of the graph \mathcal{G}_{34} whose subamplitude

$$\begin{aligned} \tilde{\mathcal{C}}^{(2)}(y_*, y_1, y_2, z_*, z_1, z_2, x_c, x_d, s_c, s_d) &\stackrel{\text{def}}{=} [\odot](y_*, y_1, y_2) [\odot](z_*, z_1, z_2) \\ &\times \mathcal{Q}_x^\gamma(y_*, z_*) \Delta \underline{\mathcal{L}}^{-1}(t - s_c, y_1, x_c) \Delta \underline{\mathcal{L}}^{-1}(t - s_d, z_1, x_d) G_r^{(2)}(s_c - s_d, x_c, x_d) \end{aligned}$$

is also *less convergent* than the whole amplitude, which equally forbids the use of a naive power counting argument.

2. The amplitudes of the graphs \mathcal{G}_{22} and \mathcal{G}_{43} are given for $(m, n) \in \{(2, 2), (4, 3)\}$ by

$$\mathcal{A}_{mn} = [\odot](y_*, y_1, y_2) [\odot](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(1)}(0, y_1, z_1) \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(1)}(t - s_a, y_1, x_a) G_r^{(1)}(t - s_b, y_2, x_b) G_r^{(\ell)}(s_a - s_b, x_a, x_b),$$

where $\ell = 1$ for \mathcal{G}_{22} and $\ell = 2$ for \mathcal{G}_{43} , and have a subgraph with amplitude $\mathcal{C}^{(1)}$ which is *less convergent* than the whole graph (marginally for \mathcal{G}_{43}). Again the same phenomenon occurs for the amplitude (3.56) of the graph \mathcal{G}_{32} that has a subamplitude $\tilde{\mathcal{C}}^{(1)}$ defined as $\tilde{\mathcal{C}}^{(2)}$ above with $G_r^{(2)}$ replaced with $G_r^{(1)}$ which is *less convergent* than the whole amplitude.

3. The amplitude of the graph \mathcal{G}_{41} , which is given by

$$\mathcal{A}_{41} = \mathcal{Q}_x^\gamma(y_*, z_*) \mathcal{B}(y_*, y_1, y_2, x_a, s_a) \mathcal{B}(z_*, z_1, z_2, x_b, s_b) G_r^{(1)}(s_a - s_b, x_a, x_b), \quad (3.48)$$

where

$$\mathcal{B}(y_*, y_1, y_2, x_a, s_a) \stackrel{\text{def}}{=} [\odot](y_*, y_1, y_2) G_r^{(2)}(t - s_a, y_1, x_a) \underline{\mathcal{L}}^{-1}(t - s_a, y_1, x_a) - b_r \delta_{\mathbf{d}_4}, \quad (3.49)$$

requires a renormalisation to define the subamplitudes

$$\mathcal{B}(y_*, y_1, y_2, x_a, s_a) \text{ and } \mathcal{B}(z_*, z_1, z_2, x_b, s_b).$$

Fortunately, for the items (a) and (b), the presence of the kernel $[\odot]$ does prevent this loss in the value of γ from occurring, due to a subtle mechanism which is not taken into account by the recursion of Theorem 3.40. Indeed, while the kernel $[\odot](y_*, y_1, y_2)$ is divergent on the deep diagonal $\{y_* = y_1 = y_2\}$, it is *smooth* as long as one of the three points is distinct from the two others and applying any smoothing operator at one of its input variables, indexed by (y_1, y_2) , makes the *whole kernel smooth*. In particular, provided the kernel $[\odot]$ is tested against a smooth kernel in one of its two input variables (y_1, y_2) , then it is smooth in the two remaining variables – see Lemma 3.52. In terms of multilinear operators, this translates the important fact that

for every $f \in \mathcal{D}'(M)$ and $g \in C^\infty(M)$, $f \odot g \in C^\infty(M)$.

To take both this effect and the renormalisation of the divergent subamplitudes of \mathcal{A}_{41} into account, we introduce in this section a minor modification of the induction in the proof of Theorem 3.40. More precisely, the strategy is to first construct the subamplitudes by hand, and then to inject our knowledge of the subamplitudes into the induction yielding Theorem 3.40. Indeed, if an amplitude \mathcal{A} contains the subamplitude \mathcal{B} , $\mathcal{C}^{(k)}$ or $\tilde{\mathcal{C}}^{(k)}$, then we will add the estimates on the subamplitude to the hypothesis of Theorem 3.40 when applying it to \mathcal{A} , which will yield control over \mathcal{A} avoiding the issue which would naively be caused by the subamplitude.

3.5.4.1 The smoothing effect from the kernel of the resonant product

This section is devoted to items (a) and (b) above, that is to say with the construction of the subamplitudes $\mathcal{C}^{(k)}$ and $\tilde{\mathcal{C}}^{(k)}$ for $k \in \{1, 2\}$. Note that by a naive application of Theorem 3.19, it would be possible to construct these subamplitudes for every $\gamma < -k/2$. However, to avoid losing some regularity, the aim is to actually construct them for any $\gamma \in \mathbf{R}$. This is possible thanks to the lemma below, which expresses the smoothing property of the kernel $[\odot]$. Informally, in the sequel, we denote the smoothing effect described in the following lemma as *shielding effect*.

Lemma 3.52 *Fix a chart label $i \in I$, and define a collection of scaling fields $(\varrho_{i,x})_{x \in M}$ as follows: for $x \in U_i$, the scaling field $\varrho_{i,x}$ is such that $\kappa_i(e^{-t\varrho_{i,x}z}) = e^{-t(\kappa_i(z) - \kappa_i(x))} + \kappa_i(x)$, while if $x \notin U_i$ we choose for $\varrho_{i,x}$ any scaling field with respect to $x \in M \setminus U_i$. Also fix $\Xi_i \in C_c^\infty(M^3)$ such that $\Xi_i(y_*, y_1, y_2)$ has support in y_2 contained in the open chart U_i .*

Then, the following holds: for any $x \in M$, $\varphi \in \mathcal{D}(M)$, and any scaling vector field $\varrho_{i,x}$ scaling with respect to x chosen as above, the partial integration

$$I_u^i(y_*, y_1) \stackrel{\text{def}}{=} e^{-6u} \int_M (e^{-u\varrho_{i,x^*}[\odot]})(y_*, y_1, y_2) \varphi(y_2) \Xi_i(y_*, y_1, y_2) dv(y_2)$$

is an element of $C^\infty(M^2)$ bounded uniformly in $u \in [0, +\infty)$ and $x \in M$.

Proof. Recall that in the chart $i \in I$, we have

$$[\odot_i](y_*, y_1, y_2) = \sum_{|k-l| \leq 1} 2^{3k+3l} \frac{\psi_i(y_*) \tilde{\psi}_i(y_*)}{|g_i|(y_1) |g_i|(y_2)} \psi(2^k |y_1 - y_*|) \psi(2^l |y_2 - y_*|) \kappa_{i^*}^{\otimes 2}(\chi_i \otimes \tilde{\chi}_i)(y_1, y_2),$$

where $|g_i| \stackrel{\text{def}}{=} \sqrt{\det(\kappa_{i^*} g)}$, and that the family

$$e^{-6u} (e^{-u\varrho_{i,x^*}[\odot_i]}), \quad u \in [0, +\infty), \quad x \in M,$$

is bounded in $\mathcal{D}'_{N^*(\mathbf{d}_3)}(M^3)$. We first assume without loss of generality that $x \in U_i$ then we choose a **very specific scaling vector field** adapted to our chart structure as

$$e^{-u\varrho_{i,x}}(y_*, y_1, y_2) \stackrel{\text{def}}{=} (e^{-u(y_* - x)} + x, e^{-u(y_1 - x)} + x, e^{-u(y_2 - x)} + x),$$

so the scaling is linear in the chart $\kappa_i : U_i \rightarrow \mathbf{R}^3$. Here and throughout the proof, with a slight abuse of notation, for any $a \in M$ we also use the notation a to denote $\kappa_i(a)$. With the same above of notation, we identify I_u^i with its localisation on U_i , which thus reads

$$\begin{aligned} I_u^i(y_*, y_1) &= \sum_{|k-l| \leq 1} 2^{3k+3l} e^{-6u} \int_{\mathbf{R}^3} \psi_i(e^{-u}(y_* - x) + x) \tilde{\psi}_i(e^{-u}(y_* - x) + x) \\ &\quad \times \frac{\psi(2^k e^{-u} |y_1 - y_*|) \psi(2^l e^{-u} |y_2 - y_*|)}{|g_i|(e^{-u}(y_1 - y_*) + y_*) |g_i|(e^{-u}(y_2 - y_*) + y_*)} \end{aligned}$$

$$\times \kappa_{i_*}^{\otimes 2}(\chi_i \otimes \tilde{\chi}_i)(e^{-u}(y_1 - x) + x, e^{-u}(y_2 - x) + x)\varphi(y_2)\Xi_i(y_*, y_1, y_2)dy_2,$$

Our goal is to prove that the above function I_u^i is smooth in the variables (y_*, y_1) uniformly in the scale parameter $u \geq 0$ and in $x \in \mathbf{R}^3$. At this point in our proof, it is crucial that the scaling by $\varrho_{i,x}$ is the same scaling as the one used to define the Littlewood-Paley blocks. We consider the series with $k = l$ in I_u^i , other series are treated similarly, which reads

$$\sum_{k=0}^{\infty} 2^{6k} e^{-6u} \psi(2^k e^{-u}|y_1 - y_*|) \psi(2^k e^{-u}|y_2 - y_*|) \in \mathcal{S}'(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3),$$

and write $e^{-u} = 2^{-\ell}$. The above family rewrites

$$\sum_{k=0}^{\infty} 2^{6(k-\ell)} \psi(2^{k-\ell}|y_1 - y_*|) \psi(2^{k-\ell}|y_2 - y_*|) = \sum_{k=-\ell}^{\infty} 2^{6k} \psi(2^k|y_1 - y_*|) \psi(2^k|y_2 - y_*|),$$

the summation getting shifted. Moreover, introduce the family of smooth functions

$$F_{\ell,x}(y_*, y_1, y_2)$$

defined by the formula

$$\frac{\psi_i(2^{-\ell}(y_* - x) + x) \tilde{\psi}_i(2^{-\ell}(y_* - x) + x) \kappa_{i_*}^{\otimes 2}(\chi_i \otimes \tilde{\chi}_i)(2^{-\ell}(y_1 - x) + x, 2^{-\ell}(y_2 - x) + x) \varphi(y_2) \Xi_i(y_*, y_1, y_2)}{|g_i|(2^{-\ell}(y_1 - y_*) + y_*) |g_i|(2^{-\ell}(y_2 - y_*) + y_*)}.$$

They are bounded in C^∞ as functions of the variables y_*, y_1, y_2 uniformly in ℓ, x , and have compact support in the variable y_2 in some compact $K \Subset \mathbf{R}^3$. By Plancherel's theorem applied to the variable y_2 , we have

$$\begin{aligned} & \sum_{k=-\ell}^{\infty} 2^{6k} \int_{\mathbf{R}^3} \psi(2^k|y_1 - y_*|) \psi(2^k|y_2 - y_*|) F_{\ell,x}(y_*, y_1, y_2) dy_2 \\ &= \sum_{k=-\ell}^{\infty} 2^{3k} \int_{\mathbf{R}^3} e^{i\xi_2 \cdot y_*} \psi(2^k|y_1 - y_*|) \eta(2^{-k}\xi_2) \mathcal{F}_{y_2} F_{\ell,x}(y_*, y_1, \xi_2) d\xi_2, \end{aligned}$$

where we recall that η denotes the Fourier transform of $\psi \circ |\cdot|$. Using the fact that $F_{\ell,x}$ is smooth, which entails the bound $\partial_{y_*, y_1}^\alpha \mathcal{F}_{y_2} F_{\ell,x}(y_*, y_1, \xi_2) = \mathcal{O}(\langle \xi_2 \rangle^{-2N})$, we obtain

$$\sum_{k=1}^{\infty} 2^{3k} \int_{\mathbf{R}^3} e^{i\xi_2 \cdot y_*} \psi(2^k|y_1 - y_*|) \eta(2^{-k}\xi_2) \mathcal{F}_{y_2} F_{\ell,x}(y_*, y_1, \xi_2) d\xi_2 \lesssim \sum_{k=1}^{\infty} 2^{(3-N)k} \int_{\mathbf{R}^3} \langle \xi_2 \rangle^{-N} d\xi_2,$$

which choosing N large enough is absolutely convergent. Uniform in $x \in \mathbf{R}^3$ bounds on the derivatives can be obtained in a similar fashion. Regarding the sum over k negative we have

$$\begin{aligned} & \left| \sum_{k=-\ell}^0 2^{3k} \int_{\mathbf{R}^3} e^{i\xi_2 \cdot y_*} \psi(2^k|y_1 - y_*|) \eta(2^{-k}\xi_2) \mathcal{F}_{y_2} F_{\ell,x}(y_*, y_1, \xi_2) d\xi_2 \right| \\ & \lesssim \sum_{k=0}^{\ell} 2^{-3k} \int_{\mathbf{R}^3} \eta(2^k \xi_2) \langle \xi_2 \rangle^{-N} d\xi_2 \lesssim 1, \end{aligned}$$

uniformly in $\ell \geq 0$. Indeed $\sum_{k=0}^{\ell} 2^{-3k} \eta(2^k \xi_2) \leq \sum_{k=0}^{\ell} \eta(2^k \xi_2) \lesssim 1$ by definition the Littlewood-Paley blocks. This concludes the proof of the claim. \square

Remark 3.53 We note that this shielding phenomenon is second microlocal in nature and cannot be captured by classical (“conical”) wave front set analysis. In fact, it can be only captured semiclassically. Let us just explain things in the flat case. Indeed, the wave front set of both $[\odot](y_*, y_1, y_2)$ and $\delta_{y_*}(y_1)\delta_{y_*}(y_2)$ are contained in the conormal of the deepest diagonal

$$\{y_1 = y_2 = y_*; \eta_1 + \eta_2 + \eta_* = 0\}$$

but we see that the Fourier support of $[\odot]$ is smaller than the hyperplane $\{\eta_1 + \eta_2 + \eta_* = 0\}$, it is contained in the subset $\{\eta_1 + \eta_2 + \eta_* = 0, |\eta_1| \simeq |\eta_2|\}$ which is not a closed conical subset ($\{\eta_1 + \eta_2 + \eta_* = 0, \eta_1 \neq 0, \eta_2 \neq 0\}$ is not a closed conical subset) and is responsible for the shielding. The purpose of the above lemma is therefore to capture such phenomenon on manifolds using local charts.

With the above shielding lemma in hand we are ready to construct the subamplitudes $\mathcal{C}^{(k)}$ and $\tilde{\mathcal{C}}^{(k)}$. Note that since these amplitudes contain \mathcal{Q}_x^γ , we aim to construct them both on $\mathbf{d}_6 \setminus \mathbf{d}_{6,x}$ and $\mathbf{d}_{6,x}$, where it has a different weak homogeneity.

Lemma 3.54 Fix $k \in \{1, 2\}$. For any $\gamma \in \mathbf{R}$, the distribution $\mathcal{C}^{(k)}$ is well-defined on M^6 . More precisely, we have

$$\mathcal{C}^{(k)} \in \mathcal{S}^{-18-k-2\gamma}(M^6; U_{x_0}) \cap \mathcal{S}_\Gamma^{-12-k}$$

uniformly in $x \in U_{x_0}$, where

$$\begin{aligned} \Gamma = & N^*(\mathbf{d}_6) \cup N^*\{y_* = y_1 = y_2\} \cup N^*\{z_* = z_1 = z_2\} \cup N^*\{y_1 = z_1\} \\ & \cup N^*\{y_* = y_1 = y_2 = z_1\} \cup N^*\{z_* = z_1 = z_2 = y_1\}. \end{aligned}$$

Moreover, a similar statement holds for $\tilde{\mathcal{C}}^{(k)}$, which is well-defined as a distribution on $\mathbf{R}^2 \times M^8$ and belongs to $\mathcal{S}^{-18-k-2\gamma}(\mathbf{R}^2 \times M^8; U_{x_0}) \cap \mathcal{S}_{\tilde{\Gamma}}^{-12-k}$ where $\tilde{\Gamma}$ is a union of conormals defined accordingly.

Remark 3.55 With this lemma in hand, we will be able to inject our knowledge on $\mathcal{C}^{(k)}$ and $\tilde{\mathcal{C}}^{(k)}$ inside the recurrence of the proof of Theorem 3.40, when applied to an amplitude \mathcal{A} containing these subamplitudes. Since they exist for every $\gamma \in \mathbf{R}$, they will therefore not contribute to the upper bound on γ , which will actually come from the scaling of the whole amplitude \mathcal{A} itself as desired.

Proof. We prove the statement for $\mathcal{C}^{(k)}$, the proof for $\tilde{\mathcal{C}}^{(k)}$ being a straightforward modification ((3.54) shows that $\Delta \underline{\mathcal{L}}^{-1}$ actually scales like a Dirac, so that its presence does not affect the argument). Recall that this subamplitude reads

$$\mathcal{C}^{(k)}(y_*, y_1, y_2, z_*, z_1, z_2) = [\odot](y_*, y_1, y_2)[\odot](z_*, z_1, z_2)\mathcal{Q}_x^\gamma(y_*, z_*)G_r^{(1)}(0, y_1, z_1).$$

We first claim that $\mathcal{C}^{(k)}$ is well-defined as a distribution on M^6 for all $\gamma \in \mathbf{R}$ (and not just $\gamma < -1/2$). Indeed, when testing it against some $\varphi \in \mathcal{D}(M^6)$, one can first perform the partial integration

$$\int_M [\odot](y_*, y_1, y_2)\varphi(y_*, y_1, y_2, z_*, z_1, z_2)dv(y_2).$$

An application of Lemma 3.52 shows that this expression is a smooth function on M^6 , and testing it against $[\odot](z_*, z_1, z_2)\mathcal{Q}_x^\gamma(y_*, z_*)G_r^{(1)}(0, y_1, z_1)$ proves the claim.

Regarding the scaling on Γ , observe that

$$\underbrace{\int_M e^{-6u} (e^{-u\varrho_i^*} [\odot_i]) (y_*, y_1, y_2)\varphi(y_*, y_1, y_2, z_*, z_1, z_2)dv(y_2)}$$

$$\times e^{-(6+k)u} e^{-u\varrho_i^*} \left([\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(k)}(0, y_1, z_1) \right)$$

is bounded since $[\odot](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(k)}(0, y_1, z_1)$ is a tree and the underbraced term is bounded by Lemma 3.52. Note that we used the fact that \mathcal{Q}_x^γ is weakly homogeneous of degree 0, which is proven in Lemma 3.42. It follows that

$$e^{-(12+k)u} e^{-u\varrho_i^*} \left([\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(k)}(0, y_1, z_1) \right)$$

is bounded in $\mathcal{D}'_r(M^6)$. Then, by Proposition 3.22, the same holds with any scaling field ϱ with respect to \mathbf{d}_6 , which concludes after summation over $i \in I$ that $\mathcal{C}^{(k)} \in \mathcal{S}_\Gamma^{-12-k}$.

It remains to scale on the deep marked diagonal $\mathbf{d}_{6,x}$. This directly stems from Theorem 3.20, and from an easy modification of the above discussion, since the only propagator whose scaling degree changes when we scale with respect to \mathbf{d}_6 or $\mathbf{d}_{6,x}$ is \mathcal{Q}_x^γ , as discussed in Lemma 3.42. Here we need to rely on Proposition 3.39 which allows to keep the same scaling degrees for the new scaling field ϱ_x . \square

3.5.4.2 Renormalising the divergent subamplitude

This section is devoted to the construction of the subamplitude \mathcal{B} , which is performed in the following Lemma.

Lemma 3.56 *For any $\gamma < 0$, the distribution \mathcal{B} is well-defined on $M^4 \times \mathbf{R}$. More precisely, we have*

$$\mathcal{B} \in \mathcal{S}_\Gamma^{-11},$$

where Γ is equal to

$$\begin{aligned} & N^*({s_a = t}) \cup N^*({s_a = t, y_1 = x_a}) \cup N^*({s_a = t, y_2 = x_a}) \cup N^*({y_* = y_1 = y_2}) \\ & \cup N^*({s_a = t, y_1 = y_2 = x_a}) \cup N^*({s_a = t, y_* = y_1 = y_2}) \cup N^*({s_a = t, y_* = y_1 = y_2 = x_a}). \end{aligned}$$

Remark 3.57 Again, with this lemma in hand, we will be able to inject our knowledge on \mathcal{B} inside the recurrence of the proof of Theorem 3.40, when applied to an amplitude \mathcal{A}_{41} which contains two copies of this subamplitude.

Proof. Recall the definition (3.49) of \mathcal{B} and note that in (3.49), $\delta_{\mathbf{d}_4}$ is the unique distribution on $M^4 \times \mathbf{R}$ supported on $\mathbf{d}_4 = \{s_a = t, y_* = y_1 = y_2 = x_a\}$ and such that for all $\varphi \in C_c^\infty(M^4 \times \mathbf{R})$, one has

$$\begin{aligned} \langle \delta_{\mathbf{d}_4}, \varphi \rangle_{\mathbf{R} \times M^4} &\stackrel{\text{def}}{=} \int_{M^4 \times \mathbf{R}} \delta_{\mathbf{d}_4}(y_*, y_1, y_2, x_a, t, s_a) \varphi(y_*, y_1, y_2, x_a, t, s_a) dv_{M^4 \times \mathbf{R}} \\ &= \int_{y_* \in M} \varphi(y_*, y_*, y_*, y_*, t, t) dv_M(y), \end{aligned}$$

where both $dv_{M^4 \times \mathbf{R}}$ and dv_M are the natural volume forms on the respective Riemannian manifolds, so that $\delta_{\mathbf{d}_4}$ only depends on the choice of Riemannian volume form on M . We denote by $\tilde{\mathcal{B}}$ the unrenormalized amplitude of \mathcal{B} which reads

$$\tilde{\mathcal{B}}(y_*, y_1, y_2, x_a, s_a) \stackrel{\text{def}}{=} [\odot](y_*, y_1, y_2) G_r^{(2)}(t - s_a, y_1, x_a) \underline{\mathcal{L}}^{-1}(t - s_a, y_1, x_a),$$

and verifies $\mathcal{B} = \tilde{\mathcal{B}} - b_r \delta_{\mathbf{d}_4}$. Moreover, to simplify the notation, we write y_a rather than x_a , and therefore focus on $\tilde{\mathcal{B}}(y_*, y_1, y_2, y_a, s_a)$. Finally, throughout the proof, given $y_* \in M$ we use the short-hand notation $z = (y_*, y_*, y_*, y_*, t) \in M^4 \times \mathbf{R}$.

The definition of \mathcal{B} is a consequence of item (b) in Theorem 3.40, and of the discussion performed in Section 3.5.5.3. Indeed, naively, we know that \mathcal{B} belongs to $\mathcal{S}_\Gamma^{-11}(M^4 \times \mathbf{R}^2 \setminus \mathbf{d}_4)$. But the weighted codimension of \mathbf{d}_4 also equals 11, so we are in case (b) of Theorem 3.19 and our extension requires

a renormalisation as in Hadamard's finite parts. We thus aim to show that one can renormalize $\tilde{\mathcal{B}}$ by subtracting some explicit distributional counterterm $\chi_i c_r$ proportional to $\delta_{\mathbf{d}_4}$. A priori $c_r = c_r(y_*, t)$ is a function on $M \times \mathbf{R}$, but we also want to show that one can actually take c_r constant equal to b_r .

The first step in our proof is to implement in real conditions the abstract extension Theorem 3.19 with counterterms, also controlling the wave front of the extension. The second step is to explicitly compute the abstract counterterm c_r whose existence is given by Theorem 3.19 in terms of trace densities of some operators, this computation is similar to the one which is done for the quartic term $\bar{\tau}_2$ in section 3.5.5.3.

Let us now study the renormalisation problem for $\tilde{\mathcal{B}}(y_*, y_1, y_2, y_a, s_a)$ locally in $U_i^4 \times \mathbf{R}$ for $i \in I$, since the diagonal $\mathbf{d}_4 = \{y_* = \dots = y_a, s_a = t\}$ can be covered by such sets, so that we can recover the global extension just from working with the local extensions. Fix $\chi_i : M^4 \rightarrow \mathbf{R}_{\geq 0}$ with support in U_i^4 , identically equal to 1 in a neighbourhood of $\{y_* = \dots = x_a\} \cap U_i^4$. Here and from now on, abusing notation, use the same notation to denote both elements of U_i and their coordinates. As in the extension Theorem 3.19, we use the parabolic scaling defined by the scaling field $\varrho = 2(t-s)\partial_s + \sum_{j \in \{1,2,a\}} (y_j - y_*) \cdot \partial_{y_j}$ whose semiflow $(e^{-t\varrho})_{t \geq 0}$ leaves U_i^4 stable. Then we have the continuous partition of unity formula $\chi_i = \int_0^\infty e^{u\varrho^*} w_i du$ where we set $w_i \stackrel{\text{def}}{=} -\varrho \chi_i \in C^\infty(U_i^4)$ which vanishes near $\{y_* = \dots = x_a\} \cap U_i^4$. Moreover, we denote for $k \in \mathbf{N}_{\geq 1}$ by

$$\text{dens}_k(y_1, \dots, y_k) \stackrel{\text{def}}{=} \prod_{j=1}^k \sqrt{\det(g)_{y_j}}$$

the density of the Riemannian volume on U_i^k endowed with the product metric with respect to the Lebesgue measure $\prod_{j=1}^k dy_j$.

Fix a test function $\varphi \in C_c^\infty(U_i^4 \times \mathbf{R})$. We decompose the pairing of $\tilde{\mathcal{B}}$ with φ as

$$\begin{aligned} \langle \tilde{\mathcal{B}}, \varphi \rangle &= \langle \tilde{\mathcal{B}}(1 - \chi_i), \varphi \rangle \\ &+ \int_0^\infty \int_{-\infty}^t \int_{U_i^4} \left(w_i e^{-u\varrho^*} \tilde{\mathcal{B}} e^{-u\varrho^*} (\Pi_z(\varphi \text{dens}_4)) \right) (y_*, y_1, y_2, y_a, s_a) e^{-11u} \prod_{j \in \{*,1,2,a\}} dy_j ds_a du \\ &+ \int_0^\infty \int_{-\infty}^t \int_{U_i^4} (\tilde{\mathcal{B}} e^{u\varrho^*} w_i) (y_*, y_1, y_2, y_a, s_a) (\varphi \text{dens}_4)(z) \prod_{j \in \{*,1,2,a\}} dy_j ds_a du, \end{aligned}$$

where we used the notation $\Pi_z(\varphi \text{dens}_4) \stackrel{\text{def}}{=} \varphi \text{dens}_4 - (\varphi \text{dens}_4)(z)$ to denote the recentering at the point $z = (y_*, y_*, y_*, y_*, t)$, and the factor e^{-11u} comes from the Jacobian of the semi-flow generated by ϱ . Moreover, observe that $e^{-u\varrho^*} \tilde{\mathcal{B}} = \mathcal{O}_{\mathcal{D}'}(e^{11u})$ and that $\Pi_z(\varphi \text{dens}_4)$ vanishes on $\mathbf{d}_4 \cap (U_i^4 \times \mathbf{R})$, which implies that

$$e^{-u\varrho^*} (\Pi_z(\varphi \text{dens}_4)) = \mathcal{O}(e^{-u}).$$

The integral over u in the second term of the r.h.s. is therefore absolutely convergent (in fact there is a subtlety related to compactness, but it is immediate to check that the product $\psi_i e^{-u\varrho^*} (\Pi_z(\varphi \text{dens}_4))$ forms a bounded family of test functions). We have thus proved that the first two terms of the r.h.s. have well-defined limits as $r \downarrow 0$. It remains to identify the counterterm as the divergent part of the third term, which we denote by $\mathfrak{D}(\cdot)$. Note that all the quantities $f = f(r)$ that we compute and which depend on the cut-off r are contained in some algebra $\mathcal{O}_{>0}$ of functions of the cut-off r which are smooth functions of $r \in (0, 1]$ and have log-polyhomogeneous expansions as $r \downarrow 0$, $f(r) = ar^{-1/2} + b \log r^{-1} + c + o(1)$ where $(a, b, c) \in \mathbf{R}^3$. We define the divergent part of any element $f \in \mathcal{O}_{>0}$ as $\mathfrak{D}(f) \stackrel{\text{def}}{=} ar^{-1/2} + b \log r^{-1}$.

With this definition in hand, the divergent part of the third term verifies

$$\mathfrak{D} \left(\int_0^\infty \int_{-\infty}^t \int_{U_i^4} (\tilde{\mathcal{B}} e^{u\varrho^*} w_i) (y_*, y_1, y_2, y_a, s_a) (\varphi \text{dens}_4)(z) \prod_{j \in \{*,1,2,a\}} dy_j ds_a du \right)$$

$$\begin{aligned}
&= \int_{U_i} \mathfrak{D} \left(\int_{-\infty}^t \int_{U_i^3} (\tilde{\mathcal{B}}\chi_i)(y_*, y_1, y_2, y_a, s_a) \text{dens}_3(y_*, y_*, y_*) \prod_{j \in \{1,2,a\}} dy_j ds_a \right) \varphi(z) dv(y_*) \\
&= \langle c_r \delta_{\mathbf{d}_4}, \varphi \rangle,
\end{aligned}$$

which implies by definition of $\delta_{\mathbf{d}_4}$ that the counterterm reads

$$\begin{aligned}
c_r(y_*, t) &= \mathfrak{D} \left(\int_{-\infty}^t \int_{U_i^3} (\tilde{\mathcal{B}}\chi_i)(y_*, y_1, y_2, y_a, s_a) \text{dens}_3(y_*, y_*, y_*) \prod_{j \in \{1,2,a\}} dy_j ds_a \right) \quad (3.50) \\
&= \mathfrak{D} \left(\int_{-\infty}^t \int_{U_i^3} (\tilde{\mathcal{B}}\chi_i)(y_*, y_1, y_2, y_a, s_a) \text{dens}_3(y_1, y_2, y_a) \prod_{j \in \{1,2,a\}} dy_j ds_a \right).
\end{aligned}$$

To go from the first to the second line of this last equality, again we used the fact that

$$\text{dens}_3(y_1, y_2, y_a) - \text{dens}_3(y_*, y_*, y_*)$$

vanishes on $\{y_* = \dots = y_a\}$, so that it does not contribute to the singular part of the above integral, which gives us the freedom to change the evaluation point of the volume elements. The reader might also wonder why we could interchange the extraction of singular parts with the integration: this is due to the fact that the integrand admits an asymptotic expansion. To conclude about the expression of $c_r(y_*, t)$, since in (3.50) we reconstructed the volume form $\prod_{j \in \{1,2,a\}} dv(y_j) = \text{dens}_3(y_1, y_2, y_a) \prod_{j \in \{1,2,a\}} dy_j$, it remains to plug the expression of $\tilde{\mathcal{B}}$, and we obtain

$$c_r(y_*, t) = \mathfrak{D} \left(\int_{-\infty}^t \sum_{|k-\ell| \leq 1} P_k^i(y_*, \cdot) \circ \tilde{\chi}_i \circ (G_r^{(2)}(t-s) \circ e^{-(t-s)P}) \circ \tilde{\chi}_i \circ \tilde{P}_\ell^i(\cdot, y_*) ds \right),$$

where the P_k^i, \tilde{P}_ℓ^i are the Littlewood-Paley-Stein projectors, the functions $\tilde{\chi}_i \in C_c^\infty(U_i)$ are arbitrary test functions such that $\tilde{\chi}_i = 1$ near y_* and where we used the explicit definition of $[\odot_i]$ in terms of the Littlewood-Paley-Stein projectors. This is indeed the desired expression (3.51), since this last term is precisely a trace density whose singular part matches the divergent part of τ_{20} up to a combinatorial factor (see Section 3.5.5.3). In particular, c_r can ultimately be chosen independent of y_*, t , and factors out of the pairing.

It remains to prove why the renormalized amplitude

$$\mathcal{B}_0 \stackrel{\text{def}}{=} \lim_{r \downarrow 0} (\tilde{\mathcal{B}}_r - c_r(y_*, t) \delta_{\mathbf{d}_4})$$

has the correct wave front set in Γ . To do so, it suffices to prove the property for $\mathcal{B}\chi_i R$ where R is the Taylor subtraction operator from Proposition 3.14. This follows from the proof of Theorem 3.19 where one replaces the wave front bound on $[e^{-u\varrho}]$ by the identical wave front bound on $[e^{-u\varrho} R]$ which coincide by Proposition 3.14. \square

3.5.5 Completing the diagrammatic estimates

3.5.5.1 Bounds for τ_1

The amplitude $\mathcal{G}_{14} = \langle \text{diagram} \rangle$ contains the subgraph with subamplitude $\mathcal{C}^{(1)}$, and thus needs the special treatment performed in Section 3.5.4 (see Lemma 3.54), using the smoothing effect of the kernel of \odot . To complete the argument, we verify the scalings of all the other subgraphs. \mathcal{A}_{14} has four closed, irreducible, connected subgraphs, for each of which we need to compute the weak homogeneity. We do


the computation for \mathcal{G}_{14} itself and let the reader deal with its three subgraphs. The amplitude \mathcal{A}_{14} of \mathcal{G}_{14} is

$$[\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(1)}(0, y_1, z_1) \\ \times \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(3)}(s_a - s_b, x_a, x_b).$$

By summing the weak homogeneity of all analytical objects appearing in the amplitude \mathcal{A}_{14} , the kernels $[\odot_i], G_r^{(1)}, \underline{\mathcal{L}}^{-1}, G_r^{(3)}, \mathcal{Q}_x^\gamma$, we get for any $\gamma < 0$

$$\sum \text{weak homogeneities} = 2(-6) + (-1) + 2(-3) - 3 - 6 - 2\gamma = -28 - 2\gamma \\ > -\text{codim}_w(\{s_a = s_b = t, y_* = \dots = x_b = x\}) = -4 - 24 = -28,$$

which is sharp. One get the same condition on γ when checking the condition on the subgraphs of \mathcal{G}_{14} so Theorem 3.40 entails that the contribution of \mathcal{G}_{14} is finite for all $\gamma < 0$.

The verification for the graph $\mathcal{G}_{12} =$  is similar. We have

$$\mathcal{A}_{12} = [\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(1)}(t - s_a, y_1, x_a) G_r^{(1)}(t - s_b, z_1, x_b) \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \\ \times \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(2)}(s_a - s_b, x_a, x_b).$$

For instance the subamplitude $[\odot_i](y_*, y_1, y_2) G_r^{(1)}(t - s_a, y_1, x_a) \underline{\mathcal{L}}^{-1}(t - s_a, z_1, x_a)$ associated with a triangle attached to a \bullet is weakly homogeneous of degree

$$-6 - 3 - 1 = -10 > -\text{codim}_w(\{s_a = t, y_* = y_1 = z_1 = x_a\}) = -2 + 3(-3) = -11.$$

It yields the same range of Sobolev regularity $\gamma < 0$. We invite the reader to check all the subgraphs by themselves. In the locally covariant renormalisation picture, the random distribution τ_1 will differ from the Wick renormalized one by a quantity of the form

$$f((\underline{\mathcal{L}}^{-1}\mathfrak{I}) \odot_i \mathfrak{I}) + f\left(\int [\odot_i](y_*, y_1, y_2) \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) G_r^{(1)}(t - s_a, y_1, x_a) ds_a dv(y_1, dy_2, dx_a)\right)$$

where f is some deterministic smooth function. The resonant product between $(\underline{\mathcal{L}}^{-1}\mathfrak{I}) \in \mathcal{C}^{\frac{3}{2}-\varepsilon}, \forall \varepsilon > 0$ and $\mathfrak{I} \in \mathcal{C}^{-\frac{1}{2}-\varepsilon}, \forall \varepsilon > 0$ is well-defined in $\mathcal{C}^{1-\varepsilon}, \forall \varepsilon > 0$. The second term, which rewrites as $f\left(\sum_{|k-\ell|\leq 1} \int_{-\infty}^t \left(P_k^i \circ e^{-(t-s)P} \circ G^{(1)}(t-s) \circ \tilde{P}_\ell^i\right)(y_*, y_*) ds\right)$ is a well-defined smooth function that we can control by the following argument. The operator $\underline{\mathcal{L}}^{-1} \in \Psi_H^{-1}$ and $G^{(1)} \in \Psi_P^{-2}$ where the heat calculus Ψ_H and parabolic calculus Ψ_P are defined in [BDF23]. Then the composition $\left(\int_{-\infty}^t \underline{\mathcal{L}}^{-1}(t-s) \circ G^{(1)}(t-s) ds\right)$ belongs to Ψ_P^{-3} , hence it belongs to $\Psi^{-4}(M)$ uniformly in t and is trace class on M . The series of pseudodifferential operators

$$\sum_{|k-\ell|\leq 1} \int_{-\infty}^t \left(P_k^i \circ e^{-(t-s)P} \circ G^{(1)}(t-s) \circ \tilde{P}_\ell^i\right)(y_*, y_*) ds = \\ \sum_{|k-\ell|\leq 1} \int_{-\infty}^t \left\{ \left(P_k^i \circ \tilde{P}_\ell^i \circ e^{-(t-s)P} \circ G^{(1)}(t-s)\right) + \left(P_k^i \circ [e^{-(t-s)P} \circ G^{(1)}(t-s), \tilde{P}_\ell^i]\right) \right\}(y_*, y_*)$$

will also converge in $\Psi^{-4}(M)$ since the series $\sum_{|k-\ell|\leq 1} P_k^i \circ \tilde{P}_\ell^i$ converges in $\Psi_{0,1}^0(M)$ and the commutator $[e^{-(t-s)P} \circ G^{(1)}(t-s), \tilde{P}_\ell^i]$ is bounded in $\Psi^{-5}(M)$ uniformly in (ℓ, i) since the sequence $(\tilde{P}_\ell^i)_{\ell,i}$ is bounded in $\Psi^0(M)$. Therefore the second series involving the commutator term converges in $\Psi^{-5}(M)$. We use for that purpose a commutator identity that says that for every pseudodifferential operator $A \in \Psi^m(M)$, the series

$$\sum_{|k-\ell|\leq 1} \left(P_k^i A \tilde{P}_\ell^i - A P_k^i \tilde{P}_\ell^i\right)$$

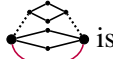
converges as a pseudodifferential operator of order $m - 1$ that we prove in our companion work. Finally this implies that the term

$$\int [\odot_i](x_*, x_1, x_2) \underline{\mathcal{L}}^{-1}((t, x_2), (s, x_a)) G_r^{(1)}((t, x_1), (s, x_a)) ds dx_1 dx_2 dx_a$$

is a well-defined smooth function.

3.5.5.2 Elementary bounds for τ_2

Again, \mathcal{A}_{24} and \mathcal{A}_{22} need the special treatment from Section 3.5.4 since they respectively have the subamplitudes $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(1)}$: Lemma 3.5.4 states that these subamplitudes are indeed well-defined for every $\gamma \in \mathbf{R}$.


To complete the argument, we check for the reader's convenience the weak homogeneities of the amplitudes of all the other subgraphs. The amplitude \mathcal{A}_{24} of $\mathcal{G}_{24} =$  is

$$\begin{aligned} [\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(2)}(0, y_1, z_1) \\ \times \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(2)}(s_a - s_b, x_a, x_b). \end{aligned}$$

Here again this graph has four closed, irreducible, connected subgraphs and we calculate the weak homogeneity of \mathcal{A}_{24} itself by summing the weak homogeneity of all analytical objects, the kernels $G_r^{(2)}$, $[\odot_i]$, $\underline{\mathcal{L}}^{-1}$, \mathcal{Q}^γ , appearing in the amplitude

$$\begin{aligned} \sum \text{weak homogeneities} &= 2(-2) + 2(-6) + 2(-3) - 6 - 2\gamma = -28 - 2\gamma \\ &> -\text{codim}_w(\{s_a = s_b = t, y_* = \dots = x_b\}) = -4 - 24 = -28, \end{aligned}$$

hence $\gamma < 0$. Repeating this verification for all subgraphs yields the result that $\mathcal{G}_{24} < \infty$ for all $\gamma < 0$.

The verification for the graph $\mathcal{G}_{22} =$  of amplitude

$$\begin{aligned} \mathcal{A}_{22} &= [\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(1)}(t - s_a, y_1, x_a) G_r^{(1)}(t - s_b, z_1, x_b) \\ &\times \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(1)}(s_a - s_b, x_a, x_b) \end{aligned}$$

is similar, for instance the subamplitude

$$[\odot_i](y_*, y_1, y_2) G_r^{(1)}(t - s_a, y_1, x_a)$$

is weakly homogeneous of degree $-6 - 1 = -7 > \text{codim}_w(\{s_a = t, y_* = y_1 = y_2 = x_a\}) = -2 - 9 = -11$ and yields the same range of regularity exponent. In a similar way as what we did for τ_1 , the locally covariant renormalized τ_2 will differ from the Wick renormalized τ_2 by homogeneous terms in Wiener chaoses of order 2 and 0 which have the form $P_1 \mathfrak{Y}^\circ + P_2 \mathfrak{V}^\circ + f_3$ where P_1, P_2 are smoothing operators and f_3 is a smooth function. We used the fact that both \mathfrak{Y}° and \mathfrak{V}° differ from their Wick renormalized version by a smooth function and also that for any smooth function $f \in C^\infty(M)$, the multiplication operator by the localized resonant product $u \in \mathcal{D}'(M) \mapsto f \odot_i u \in C^\infty(M)$ is smoothing.

3.5.5.3 The divergent part of τ_{20}

Recall that we denote by $[\odot_i]$ the kernel of \odot_i . An immediate calculation yields for $z = (t, y_*)$

$$\begin{aligned} \tau_{20}(z) &= \mathbf{E}[\tau_2(z)] = 2 \int_{-\infty}^t \int_{M^3} [\odot_i](y_*, y_1, y_2) \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) G_r^{(2)}(t - s_a, y_1, x_a) dv(y_1, y_2, x_a) ds_a \\ &= 2 \sum_{|k-\ell| \leq 1} \int_{-\infty}^t P_k^i(y_*, \bullet) \circ G_r^{(2)}(s - t) \circ e^{-(t-s)P} \circ \tilde{P}_\ell^i(\bullet, y_*) ds, \end{aligned} \quad (3.51)$$

with the shorthand notation $dv(y_1, y_2, x_a) \stackrel{\text{def}}{=} dv(y_1)dv(y_2)dv(x_a)$ for the Riemannian volume form on M^3 , with dv the corresponding volume form on M .

§1. *Preliminary analysis.* Recall that

$$G_r^{(2)}(t-s, x, y) = \left(\left\{ e^{-(|t-s|+2r)P} P^{-1} \right\} (x, y) \right)^2.$$

We reformulate the integral

$$\int_{-\infty}^t G_r^{(2)}(t-s) \circ e^{-(t-s)P} ds \quad (3.52)$$

as the composition of two operators in the parabolic calculus $(\Psi_P^\alpha)_{\alpha \in \mathbf{R}}$ defined in [BDFT23, Definition 6.1]. This calculus extends the heat calculus $(\Psi_H^\alpha)_{\alpha \in \mathbf{R}}$ as defined in Grieser's note [Gri04] in order to include the kernels $G_r^{(p)}$, $p \in \{1, 2, 3\}$ needed to define Φ_3^4 amplitudes. Since $G_r^{(2)} \in \Psi_P^{-\frac{3}{2}}$ uniformly in $r > 0$ and $e^{-tP} \in \Psi_H^{-1}$, we prove in [BDFT23, Theorem 8.1] that the composition $\int_{-\infty}^t G_r^{(2)}(s, t) \circ e^{-(t-s)P} ds$ defines an element in $\Psi_P^{-\frac{5}{2}+\gamma}$ for any $\gamma > 0$, uniformly in $r > 0$. Hence it is a pseudodifferential operator depending continuously on t of order $-3 + 2\gamma$, by [BDFT23, Proposition 6.14] that allows to view parabolic operators as parameter dependent pseudodifferential. It follows that the t -indexed family of operators (3.52) is bounded in $\Psi^{-3+2\gamma}(M)$ and fails to be trace class when the regularisation parameter r tends to 0. Our goal in the sequel of this section is to extract the singular part of this operator.

First we need to disentangle this operator in (3.51) from the Littlewood-Paley-Stein projectors P_k^i, \tilde{P}_ℓ^i . We use for that purpose a commutator identity that says that for every pseudodifferential operator $A \in \Psi^m(M)$, the series

$$\sum_{|k-\ell| \leq 1} \left(P_k^i A \tilde{P}_\ell^i - A P_k^i \tilde{P}_\ell^i \right)$$

converges as a pseudodifferential operator of order $m - 1$. The operator

$$\sum_{|k-\ell| \leq 1} \left(P_k^i G_r^{(2)}(t-s) e^{-(t-s)P} \tilde{P}_\ell^i - G_r^{(2)}(t-s) e^{-(t-s)P} P_k^i \tilde{P}_\ell^i \right) \quad (3.53)$$

is in particular in $\Psi^{-4+2\gamma}(M)$ uniformly in $r \in [0, 1]$, for any $0 < \gamma < 1/2$, so it is trace class.

Denote by δ_{y_*} is the unique distribution depending on the Riemannian volume form $dv(y)$ such that

$$\int_M \delta_{y_*}(z) f(z) dv(z) = f(y_*),$$

for all bounded measurable functions f . With this notation one has

$$\tau_{20}(z) = 2 \int_{-\infty}^t \sum_{|k-\ell| \leq 1} \left\langle P_k^i(\delta_{y_*}), G_r^{(2)}(t-s) e^{-(t-s)P} \tilde{P}_\ell^i(\delta_{y_*}) \right\rangle ds$$

and we see from the preceding regularity result for the commutator (3.53) that the singular part $\mathfrak{D}\tau_{20}(z)$ of $\tau_{20}(z)$ coincides with the singular part of

$$\left\langle \left(\left\{ \int_{-\infty}^t G_r^{(2)}(t-s) e^{-(t-s)P} ds \right\} \delta_{y_*} \right) \odot_i \delta_{y_*}, 1 \right\rangle.$$

Given that the localized paraproduct of any two distributions is always well-defined, and that (cf Appendix 3.G)

$$\chi_i(uv) = u \odot_i v + u \prec_i v + u \succ_i v,$$

this localized singular part coincides with the singular part of

$$\chi_i(y_*) \left\{ \int_{-\infty}^t G_r^{(2)}(t-s) e^{-(t-s)P} ds \right\} (y_*, y_*).$$

§2. *Explicit computation of the divergent part.* We proceed in two steps by localising first in space and time and then by using the heat kernel asymptotic.

§2.1. *Space and time localisation.* Fix $z = (t, y_*)$. We start by localising in space near y_* . Let χ stand for a smooth indicator function of a neighbourhood U_{y_*} of y_* in M – without loss of generality the domain of a chart. Since the quantity

$$\int_{-\infty}^t \int_M G_r^{(2)}(t-s)(y_*, y) (1 - \chi(y e^{-(t-s)P}(y, y_*))) dv(y) ds$$

has a well-defined limit when $r \downarrow 0$, we concentrate on

$$\begin{aligned} (\star) &\stackrel{\text{def}}{=} \int_{-\infty}^t \int_{U_{y_*}} G_r^{(2)}(t-s)(y_*, y) e^{-(t-s)P}(y, y_*) \chi(y) dv(y) ds \\ &= \int_0^\infty \left(\int_{[a+2r, \infty)^2} \int_{U_{y_*}} e^{-s_1 P}(y_*, y) e^{-s_2 P}(y_*, y) e^{-aP}(y_*, y) \chi(y) dv(y) ds_1 ds_2 \right) da. \end{aligned}$$

§2.2. *Use the heat kernel asymptotics.* We have near y_*

$$e^{-sP}(y_*, y) = \frac{1}{(4\pi s)^{\frac{3}{2}}} e^{-\frac{|y_* - y|_g^2(y_*)}{4s}} e^{-s} + R(s, y_*, \frac{y_* - y}{\sqrt{s}}) =: K^0(s, y_*, y) + R\left(s, y_*, \frac{y_* - y}{\sqrt{s}}\right)$$

for a remainder term $R \in \Psi_H^{-2}(M)$ in the heat calculus as defined in [Gri04], that is $R(s, y_*, \frac{y_* - y}{\sqrt{s}})$ has the same estimates as s times the heat kernel itself. It follows from that fact that replacing any of the heat kernels $e^{-s_1 P}$, $e^{-s_2 P}$, e^{-aP} by a remainder term R gives a contribution to (\star) that remains uniformly bounded for $r \in [0, 1]$. We can therefore keep in our computations only the leading term of the heat expansion. Integrating first with respect to y the stationary phase in U_{y_*} gives the asymptotics

$$\begin{aligned} &\int_{U_{y_*}} K^0(s_1, y_*, y) K^0(s_2, y_*, y) \chi(y) K^0(a, y_*, y) dv(y) \\ &= \frac{e^{-(s_1+s_2+a)} (4\pi)^{-\frac{9}{2}}}{(s_1 s_2 a)^{\frac{3}{2}}} \int_{U_{y_*}} e^{-\frac{|y_* - y|_g^2(y_*)}{4} (s_1^{-1} + s_2^{-1} + a^{-1})} \chi(y) \sqrt{\det(g)_y} dy \\ &= \frac{e^{-(s_1+s_2+a)} \chi(y_*) \sqrt{\det(g)_{y_*}} (4\pi)^{-3}}{(s_1^{-1} + s_2^{-1} + a^{-1})^{\frac{3}{2}} (s_1 s_2 a)^{\frac{3}{2}} \sqrt{\det \text{Hess}(| \cdot - y_* |_g^2(y_*))}(y_*)}} \\ &\quad + \mathcal{O}\left((s_1 s_2 a)^{-\frac{3}{2}} (s_1^{-1} + s_2^{-1} + a^{-1})^{-\frac{5}{2}}\right) \\ &= \frac{e^{-(s_1+s_2+a)} (4\pi)^{-3}}{(s_1^{-1} + s_2^{-1} + a^{-1})^{\frac{3}{2}} (s_1 s_2 a)^{\frac{3}{2}}} + \mathcal{O}\left((s_1 s_2 a)^{-\frac{3}{2}} (s_1^{-1} + s_2^{-1} + a^{-1})^{-\frac{5}{2}}\right), \end{aligned}$$

with an error term $\mathcal{O}((s_1 s_2 a)^{-\frac{3}{2}} (s_1^{-1} + s_2^{-1} + a^{-1})^{-\frac{5}{2}})$ bounded uniformly in the y_* variable. Note that to go from the second to the last line we used $\chi(y_*) = 1$, along with the fact that the determinant of the Hessian equals the determinant of g evaluated at y_* .

Here, note that the divergence of the integral in a only takes place when $a \downarrow 0$, so that in order to seek the divergent part, one can restrict the integration over a to $[0, 1]$. Moreover, the exponential decay $e^{-(s_1+s_2+a)}$ will not help, so that we can discard it when computing the divergent part. In the end, we are therefore left with computing the singular part of

$$\begin{aligned}
\frac{1}{(4\pi)^3} \int_0^1 \int_{[a+2r, \infty)^2} \frac{ds_1 ds_2 da}{(s_2 a + s_1 a + s_1 s_2)^{\frac{3}{2}}} &= \frac{1}{(4\pi)^3} \int_0^1 \frac{da}{a+2r} \int_{[1, \infty)^2} \frac{d\alpha d\beta}{(\alpha + \beta + \alpha\beta - \frac{2r(\alpha+\beta)}{a+2r})^{\frac{3}{2}}} \\
&= -\frac{1}{(4\pi)^3} \int_{[1, \infty)^2} \frac{d\alpha d\beta}{(\alpha + \beta + \alpha\beta)^{\frac{3}{2}}} \log r + \mathcal{O}(1) \\
&= -\frac{1}{3} \times \frac{\log r}{32\pi^2} + \mathcal{O}(1).
\end{aligned}$$

On the first line, we did the two changes of variables $s_1 = \alpha(a + 2r)$ and $s_2 = \beta(a + 2r)$, while on the second line the integral over α, β can be computed explicitly. This finally tells us that the diverging part of τ_{20} is exactly given by $\chi_i \frac{\log r}{3}$ for every $i \in I$. To conclude, note that what is left of τ_{20} after the subtraction of the divergent part defines a smooth function of y_* .

3.5.5.4 Bounds on $\bar{\tau}_3$

We first briefly discuss the counterterm for $\bar{\tau}_3$ and we will do the stochastic estimates in a second step. Note that for any test function φ we have from an integration by parts

$$\int_M \chi_i |\nabla \mathring{Y}_r^\circ|^2 \varphi = - \int_M (\Delta \mathring{Y}_r^\circ) \mathring{Y}_r^\circ \chi_i \varphi - \int_M \langle \nabla \mathring{Y}_r^\circ, \nabla(\chi_i \varphi) \rangle \mathring{Y}_r^\circ.$$

The singular part of $|\nabla \mathring{Y}_r^\circ|^2$ as a random distribution is thus the same as the singular part of $(\Delta \mathring{Y}_r^\circ) \mathring{Y}_r^\circ$. The latter is also the singular part of

$$\begin{aligned}
\mathbf{E} \left[\left(P \mathring{\varphi}_r \right) \mathring{Y}_r^\circ \right] &= \int_{(-\infty, t]^2} \text{tr}_{L^2} \left(P e^{-(t-s_1)P} \circ G_r^{(2)}(s_1 - s_2) \circ e^{-(t-s_2)P} \right) ds_1 ds_2 \\
&= \int_{(-\infty, t]^2} \text{tr}_{L^2} \left(P e^{-(t-s_1)P} \circ G_r^{(2)}(s_1 - s_2) \circ e^{-(t-s_2)P} \right) ds_1 ds_2 \\
&= \int_{(-\infty, t]^2} \text{tr}_{L^2} \left(G_r^{(2)}(s_1 - s_2) \circ P e^{-(2t-s_1-s_2)P} \right) ds_1 ds_2
\end{aligned}$$

so changing variables for $s'_1 = t - s_1$ and $s'_2 = s_1 - s_2$ gives

$$\begin{aligned}
\int_{(-\infty, t]^2} \text{tr}_{L^2} \left(G_r^{(2)}(s'_2) \circ P e^{-(2s'_1+s'_2)P} \right) ds'_1 ds'_2 &= \int_{(-\infty, t]^2} \text{tr}_{L^2} \left(G_r^{(2)}(s'_2) \circ \frac{d}{s'_1} e^{-(2s'_1+s'_2)P} \right) ds'_1 ds'_2 \\
&= \int_{-\infty}^t \text{tr}_{L^2} \left(e^{-s'_2 P} \circ G_r^{(2)}(s'_2) \right) ds'_2,
\end{aligned}$$

where we use time permutation symmetry of $G_r^{(2)}$ and cyclicity of the L^2 -trace. This quantity is equal to the divergent part of τ_{20} . We next discuss the regularity of $\bar{\tau}_3$. First, we need to isolate the resonant part in the scalar product. Since we work in the manifold setting, note that we need to define carefully the resonant scalar product of two vector fields in $C^\infty(TM)$. For $s_1, s_2 \in C^\infty(TM)^2$, using the notations and conventions of the Appendix [3.G](#), we define $\langle s_1 \odot_i s_2 \rangle_{TM}$ as:

$$\langle s_1 \odot_i s_2 \rangle_{TM} \stackrel{\text{def}}{=} \kappa_i^* \left((\kappa_{i*} g)^{\mu\nu} \kappa_{i*} \psi_i (\kappa_{i*} (\chi_i s_1)_\mu \odot \kappa_{i*} (\tilde{\chi}_i s_2)_\nu) \right)$$

where $\kappa_i, \psi_i, \chi_i, \tilde{\chi}_i$ come from our definition of resonant product, $(\kappa_{i*} g)$ is the metric g induced by the charts $\kappa_i : U_i \mapsto \kappa_i(U_i) \subset \mathbf{R}^d$. Similarly we have

$$\langle s_1 \prec_i s_2 \rangle_{TM} \stackrel{\text{def}}{=} \kappa_i^* \left((\kappa_{i*} g)^{\mu\nu} \kappa_{i*} \psi_i (\kappa_{i*} (\chi_i s_1)_\mu \prec \kappa_{i*} (\tilde{\chi}_i s_2)_\nu) \right)$$

and we recover the usual decomposition

$$\langle s_1, s_2 \rangle_{TM} = \sum_i \langle s_1 \odot_i s_2 \rangle_{TM} + \langle s_1 \prec_i s_2 \rangle_{TM} + \langle s_1 \succ_i s_2 \rangle_{TM}$$

for the scalar product on sections of TM . We need to prove that such resonant scalar product satisfies some approximate integration by parts identity and we are careful since the Laplacian is no longer translation invariant. For every $Y \in C^\infty(M)$ a calculation relying on the definitions of both resonant product and resonant scalar product:

$$\langle \nabla Y \odot_i \nabla Y \rangle_{TM} = \chi_1(Y \odot_1 Y) + (PY) \odot_2 Y + Y \odot_i \Delta Y$$

where χ_1 is a smooth function, P is a differential operator of order 1 on M with smooth coefficients, \odot_1, \odot_2 are localized resonant type products which might differ from the original \odot_i in the choice of smooth cut-off functions involved in the definition but have the exact same analytical properties from Proposition 3.61 and the last term involves the localized resonant product of Y with ΔY . From the point of view of regularities, for $Y = \mathring{Y}_r^\circ \in \mathcal{C}^{1-2\varepsilon}([0, T] \times M)$, $\chi_1(\mathring{Y}_r^\circ \odot_1 \mathring{Y}_r^\circ) \in C_T \mathcal{C}^{2-4\varepsilon}(M)$ and $(P\mathring{Y}_r^\circ) \odot_2 \mathring{Y}_r^\circ \in C_T \mathcal{C}^{1-4\varepsilon}(M)$ and finally we are reduced to the study of $\mathring{Y}_r^\circ \odot_i \Delta \mathring{Y}_r^\circ$. This is now really very similar to what we did for the graph τ_2 except there is an extra propagator $\Delta \underline{\mathcal{L}}^{-1}$ in all the Feynman amplitudes. However microlocal estimates proven in [BDFT23] actually show that

$$\Delta \underline{\mathcal{L}}^{-1} \in \mathcal{S}_{N^*}^{-5}(\{t=s\} \times \mathbf{d}_2 \subset \mathbf{R}^2 \times M^2) (\mathbf{R}^2 \times M^2). \quad (3.54)$$

Now we repeat the stochastic estimates on $\mathbf{R}^4 \times M^{10}$ taking this new kernel and its weak homogeneity into account. For instance, the amplitude controlling the homogeneous chaos of orders 2 and 4 in the chaos decomposition of $\mathring{Y}_r^\circ \odot_i \Delta \mathring{Y}_r^\circ$ now read

$$\begin{aligned} \mathcal{A}_{34} = & [\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) \quad (3.55) \\ & G_r^{(2)}(s_a - s_b, x_a, x_b) \Delta \underline{\mathcal{L}}^{-1}(t - s_c, y_1, x_c) \Delta \underline{\mathcal{L}}^{-1}(t - s_d, z_1, x_d) G_r^{(2)}(s_c - s_d, x_c, x_d), \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{32} = & [\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) \quad (3.56) \\ & G_r^{(1)}(s_a - s_b, x_a, x_b) \Delta \underline{\mathcal{L}}^{-1}(t - s_c, y_1, x_c) \Delta \underline{\mathcal{L}}^{-1}(t - s_d, z_1, x_d) G_r^{(1)}(s_c - s_d, x_c, x_d) \\ & G_r^{(1)}(s_a - s_c, x_a, x_c) G_r^{(1)}(s_b - s_d, x_b, x_d). \end{aligned}$$

These two amplitudes respectively have the subamplitudes $\tilde{\mathcal{C}}^{(2)}$ and $\tilde{\mathcal{C}}^{(1)}$, which are dealt with in Section 3.5.4 (see Lemma 3.54). Here again, it remains to calculate the weak homogeneities of the amplitudes and all the other subamplitudes, by summing the weak homogeneity of all analytical objects, the kernels $G_r^{(2)}, [\odot_i], \underline{\mathcal{L}}^{-1}, \Delta \underline{\mathcal{L}}^{-1}, \mathcal{Q}_x^\gamma$. For \mathcal{A}_{34} , this yields

$$\begin{aligned} \sum \text{weak homogeneities} &= 2(-2) + 2(-6) + 2(-3) - 6 - 2\gamma + 2(-5) = -38 - 2\gamma \\ &> -\text{codim}_w(\{s_a = s_b = s_c = s_d = t, y_* = \dots = x_d = x\}) \\ &= -8 - 30 = -38, \end{aligned}$$


hence we obtain $\gamma < 0$. Repeating this verification for all subgraphs and for all amplitudes controlling the term homogeneous of order 2 in the chaos decomposition yields the result that $\mathring{Y}_r^\circ \odot_i \Delta \mathring{Y}_r^\circ \in \mathcal{C}^{-4\varepsilon}([0, T] \times M)$ almost surely.

The locally covariant renormalisation for $\overline{\tau}_3$ differs from the Wick one by $f_1 \cdot \nabla^{\circ\mathring{Y}} + f_2$ where f_1 is a smooth vector and f_2 a smooth function which has higher regularity than $|\nabla^{\circ\mathring{Y}}|^2$.


3.5.5.5 The quintic term

The first two graphs respectively have the subgraphs with amplitude $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(1)}$, which are dealt with in Section 3.5.4 (see Lemma 3.54). To complete the argument, one extracts all the other closed, connected

irreducible graphs and finds the range of parameter γ so that they satisfy the criterion of Theorem [3.40](#).

For the reader's convenience, recall that the amplitude \mathcal{A}_{45} of $\mathcal{G}_{45} =$  is given by

$$[\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(2)}(0, y_1, z_1) \\ \times \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(3)}(s_a - s_b, x_a, x_b),$$

and the amplitude \mathcal{A}_{43} of $\mathcal{G}_{43} =$  is

$$\mathcal{A}_{43} = [\odot_i](y_*, y_1, y_2) [\odot_i](z_*, z_1, z_2) \mathcal{Q}_x^\gamma(y_*, z_*) G_r^{(1)}(0, y_1, z_1) G_r^{(1)}(t - s_a, y_1, x_a) G_r^{(1)}(t - s_b, z_1, x_b) \\ \times \underline{\mathcal{L}}^{-1}(t - s_a, y_2, x_a) \underline{\mathcal{L}}^{-1}(t - s_b, z_2, x_b) G_r^{(2)}(s_a - s_b, x_a, x_b).$$

The reader can check that the scaling degrees of the subgraphs yield the range $\gamma < -\frac{1}{2}$.

It remains to handle the last contribution, which verifies

$$\mathbf{E}[\mathcal{Q}_x^\gamma((\tau_{41}(t, \cdot) - \chi_i b_r \mathfrak{I}_r), (\tau_{41}(t, \cdot) - \chi_i b_r \mathfrak{I}_r))] = \langle \mathcal{A}_{41}, 1 \rangle,$$

where \mathcal{A}_{41} is given by [\(3.48\)](#). The amplitude \mathcal{A}_{41} is indeed well-defined as a distribution of $M^8 \times \mathbf{R}^2$ by Theorem [3.40](#) for $\gamma < -1/2$, taking the renormalized subamplitude \mathcal{B} (that is constructed in Lemma [3.56](#)) into account in the induction over all subgraphs of \mathcal{A}_{41} .

We conclude our discussion by pointing out the difference between the locally covariant and Wick renormalisations. The locally covariant renormalisation of $\mathfrak{Y}^\circ \odot_i \mathfrak{V}$ differs from the Wick renormalisation by $(\underline{\mathcal{L}}^{-1}(f \mathfrak{I})) \odot_i \mathfrak{V} + P(\mathfrak{Y}^\circ)$ where f is a smooth function and P a smoothing operator and therefore $(\underline{\mathcal{L}}^{-1}(f \mathfrak{I})) \odot_i \mathfrak{V} \in \mathcal{C}^{\frac{1}{2}-\varepsilon}$, $\forall \varepsilon > 0$ which is absorbed in the $\mathcal{C}^{-\frac{1}{2}-0}$ regularity of $\mathfrak{Y}^\circ \odot_i \mathfrak{V}$.

The results of this section justify the convergence in $L^2(\Omega)$, hence in any $L^p(\Omega)$ with $p < \infty$, of $\widehat{\xi}_r$ to a limit random variable $\widehat{\xi}$ in its natural space. The convergence *in probability* of $v_r \in (\alpha_0, 1 + \varepsilon')$ to a limit v in that space follows as a consequence of the pathwise continuity of v_r with respect to $\widehat{\xi}_r$ obtained from the fixed point construction of v_r . Formula [\(3.6\)](#) relating v_r to u_r shows the convergence *in probability* of $u_r \in C_T C^{-1/2-\varepsilon}(M)$ to a limit u in that space.

3.6 Invariant measure

We prove in Section [3.6.1](#) that the dynamics generated by equation [\(3.1\)](#) is Markovian and that its semigroup has the Feller property. The existence of an invariant measure is obtained from a compactness argument building on the L^p coming down from infinity property of Theorem [3.7](#). We prove in Section [3.6.2](#) that the invariant probability measure is non-Gaussian.

3.6.1 A Markovian dynamics

Denote by \mathcal{F}_t the usual augmentation of the σ -algebra generated by the random variables $\xi(f)$, for functions $f \in L^2(\mathbf{R} \times M)$ that are null on $[t, \infty) \times M$. Since ξ_r is white in time the dynamics

$$(\partial_t + P)u_r = -u_r^3 + 3(a_r - b_r)u_r + \sqrt{2}\xi_r$$

generates an $(\mathcal{F}_t)_{t \geq 0}$ -Markov process. For looking at the restriction to a finite time interval $[0, T]$ of this process it is convenient to extend functions on $[0, T]$ into functions on $[0, +\infty)$ that are constant on $[T, +\infty)$. For $t \in \mathbf{R}$ and any $(s, x) \in \mathbf{R} \times M$ set

$$\tau_t(s, x) \stackrel{\text{def}}{=} (s - t, x).$$

Denote by $\theta_s : \Omega \rightarrow \Omega$, $s \geq 0$ a family of measurable shifts on (Ω, \mathcal{F}) such that one has

$$(\xi \circ \theta_s, f) = (\xi, f \circ \tau_s)$$

for all s and all L^2 test functions f . The Markov property for

$$u_r : \Omega \times [0, T] \times C^{-1/2-\epsilon}(M) \rightarrow C^{-1/2-\epsilon}(M)$$

writes

$$\mathbf{E}[F(u_r(s + \cdot, \phi))\mathbf{1}_E] = \mathbf{E}[F(u_r \circ \theta_s(\cdot, u_r(s, \phi)))\mathbf{1}_E], \quad (3.57)$$

for any bounded measurable cylindrical functional F on $C((0, T], C^{-1/2-\epsilon}(M))$ and all events $E \in \mathcal{F}_s$, with $0 \leq s \leq T$ arbitrary. We need the following quantitative stability result to pass to the zero r limit in (3.57).

Lemma 3.58 *Fix some positive times $t_1 < \dots < t_k$. There exists two positive constants γ, γ' such that the restriction of the functions*

$$\phi \in C^{-1/2-\epsilon}(M) \mapsto u_r(t_i, \phi) \in C^{-1/2-\epsilon}(M), \quad (1 \leq i \leq k)$$

to any centered ball of $C^{-1/2-\epsilon}(M)$ with radius $R > 0$ is Lipschitz continuous, with Lipschitz constant bounded above by an explicit function of R and $\hat{\xi}_r$.

Proof. This result is obtained from the exact same statement for the functions $v_r(t_i, \cdot)$. The relation

$$u_r(t, \phi_1) - u_r(t, \phi_2) = e^{-3\mathfrak{Y}_r(t)}(v_r(t, \phi'_1) - v_r(t, \phi'_2)),$$

with

$$\phi_i = \mathfrak{I}_r(0) - \mathfrak{Y}_r(0) + e^{-3\mathfrak{Y}_r(0)}(\phi'_i + v_{\text{ref}}(0))$$

allows to transport the locally Lipschitz character of v_r to u_r . It suffices to prove the statement with $k = 1$ and $t_1 = 1$; we prove in that case that $\phi \in C^{-1/2-\epsilon}(M) \mapsto v_r(1, \phi) \in C^{-1/2-\epsilon}(M)$ is locally Lipschitz. Define

$$F(\phi, v) \stackrel{\text{def}}{=} e^{-tP}(\phi) + \mathcal{L}^{-1}\left(-6\nabla\mathfrak{Y}_r \cdot \nabla v - e^{-6\mathfrak{Y}_r}v^3 + Z_{2,r}v^2 + Z_{1,r}v + Z_{0,r}\right). \quad (3.58)$$

Let $K > 0$ be a uniform constant satisfying

$$\|e^{-tP}\phi\|_{C^{-1/2-\epsilon}} \leq K\|\phi\|_{C^{-1/2-\epsilon}} \quad \text{and} \quad \|e^{-tP}\phi\|_{(\alpha_0, 1+\epsilon')} \leq K\|\phi\|_{C^{-1/2-\epsilon}}.$$

Take the ball B_R in $C^{-1/2-\epsilon}(M)$. It follows from the proof of Proposition 3.5 that for any $\phi \in B_R$, there exists $T = T(\hat{\xi}_r|_{[0,2]}, R)$ and a constant $C(T) < 1/2$ only depending on T such that

$$\|F(\phi, v_1) - F(\phi, v_2)\|_{C_T C^{-1/2-\epsilon}} \leq C(T)(\|v_1 - v_2\|_{C_T C^{-1/2-\epsilon}} + \|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')})$$

and

$$\|F(\phi, v_1) - F(\phi, v_2)\|_{(\alpha_0, 1+\epsilon')} \leq C(T)\|v_1 - v_2\|_{(\alpha_0, 1+\epsilon')}.$$

Now by the same argument in the proof of Proposition 3.5, we infer that for $\phi_1, \phi_2 \in B_R$,

$$\begin{aligned} & \|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{C_T C^{-1/2-\epsilon}} \\ & \leq K\|\phi_1 - \phi_2\|_{C^{-1/2-\epsilon}} + C(T)(\|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{C_T C^{-1/2-\epsilon}} + \|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{(\alpha_0, 1+\epsilon')}) \end{aligned}$$

and

$$\|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{(\alpha_0, 1+\epsilon')} \leq K\|\phi_1 - \phi_2\|_{C^{-1/2-\epsilon}} + C(T)\|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{(\alpha_0, 1+\epsilon')}.$$

This implies that

$$\begin{aligned} & \|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{C_T C^{-1/2-\epsilon}} + \|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{(\alpha_0, 1+\epsilon')} \\ & \leq 2K \|\phi_1 - \phi_2\|_{C_T C^{-1/2-\epsilon}} + C(T) \|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{C_T C^{-1/2-\epsilon}} \\ & \quad + 2C(T) \|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{(\alpha_0, 1+\epsilon')}, \end{aligned}$$

hence

$$\|v_r(\cdot, \phi_1) - v_r(\cdot, \phi_2)\|_{C_T C^{-1/2-\epsilon}} \leq \frac{2K}{1-C} \|\phi_1 - \phi_2\|_{C^{-1/2-\epsilon}}.$$

By the L^p a priori estimate of Theorem 3.7, if $T < 1$ then

$$\|v_r(t, \phi)\|_{C^{-1/2-\epsilon}} \leq R' \stackrel{\text{def}}{=} \frac{C(\widehat{\xi}_r|_{[0,2]})}{T^{1/2}}, \quad (T/2 \leq t \leq 1).$$

Then as above we can get a short time $T' = T'(\widehat{\xi}_r|_{[0,2]}, R')$ such that the map $F(\cdot, \cdot)$ is contracting with a constant $C(T') < 1/2$ for any initial condition in $B_{R'} \subset C^{-1/2-\epsilon}(M)$. Since $v_r(t) \in B_{R'}$ for all $t \in [T/2, 1]$ we can divide the interval $[T/2, 1]$ into subintervals $[t_j, t_j + T']$ and repeat our process above to get

$$\|v_r(t, \phi_1) - v_r(t, \phi_2)\|_{C^{-1/2-\epsilon}} \leq \frac{2K}{1-C(T')} \|v_r(t_j, \phi_1) - v_r(t_j, \phi_2)\|_{C^{-1/2-\epsilon}},$$

on each small interval $[t_j, t_j + T']$. Combining all yields that $\phi \mapsto v_r(1, \phi)$ is locally Lipschitz. \square

Proposition 3.59 *The dynamics of u is Markovian and its associated semigroup $(\mathcal{P}_t)_{t \geq 0}$ on $C^{-1/2-\epsilon}$ has the Feller property.*

Proof. Given any $\eta > 0$ it follows from Lemma 3.58 there is an $R(\eta) > 0$ such that outside an event of probability η the random variables $(u_r \circ \theta_s(\cdot, \cdot))_{0 < r \leq 1}$ are (r -uniformly) uniformly continuous in their second argument on the centered ball of $C^{-1/2-\epsilon}(M)$ of radius $R(\eta)$ and $u_r(s, \phi)$ is converging to $u(s, \phi)$ with $|u(s, \phi)|_{C^{-1/2-\epsilon}} \leq R(\eta)$. The process $u_r \circ \theta_s(\cdot, u_r(s, \phi))$ is thus converging in probability to $u \circ \theta_s(\cdot, u(s, \phi))$, so one can get the Markov property of the limit process u by passing to the zero r limit in (3.57) along a subsequence r_k where the convergence of u_{r_k} is almost sure, using dominated convergence.

The Feller property of the semigroup $(\mathcal{P}_t)_{t \geq 0}$, that is the fact that it sends the space of continuous functions on $C^{-1/2-\epsilon}(M)$ into itself, is a direct consequence of the pathwise continuous dependence of the solution u to (3.1) with respect to the initial condition ϕ and dominated convergence in the expression $(\mathcal{P}_t f)(\phi) = \mathbf{E}[f(u(t, \phi))]$. \square

Proposition 3.60 *The semigroup $(\mathcal{P}_t)_{t \geq 0}$ has an invariant probability measure.*

Proof. Recall we turned equation (3.3) on u_r into equation (3.7) on v_r , with abstract form (3.16). Coming back to

$$u = \mathfrak{I} - \mathfrak{Y}^\circ + e^{-3\mathfrak{Y}^\circ}(v + v_{\text{ref}}), \quad (3.59)$$

seen as an element of $C_T C^{-1/2-\epsilon}(M)$, one can write for any fixed time

$$\begin{aligned} & \{\|u(t)\|_{C^{-1/2-\epsilon}} > 3m\} \\ & \subset \left\{ \|\mathfrak{I}(t)\|_{C^{-1/2-\epsilon}} > m \right\} \cup \left\{ \|\mathfrak{Y}^\circ(t)\|_{C^{-1/2-\epsilon}} > m \right\} \cup \left\{ \|e^{-3\mathfrak{Y}^\circ}(v + v_{\text{ref}})\|_{C^{-1/2-\epsilon}} > m \right\}, \end{aligned}$$

with

$$\mathbf{P}(\|\mathfrak{I}(t)\|_{C^{-1/2-\epsilon}} > m) + \mathbf{P}(\|\mathfrak{Y}^\circ(t)\|_{C^{-1/2-\epsilon}} > m) = o_m(1)$$

uniformly in $t \geq 0$ by stationarity. We also have

$$\begin{aligned} \mathbf{P}\left(\|e^{-3\mathfrak{Y}^\circ(t)}(v + v_{\text{ref}})(t)\|_{C^{-1/2-\epsilon}} > m\right) \\ \leq \mathbf{P}\left(\|e^{-3\mathfrak{Y}^\circ(t)}\|_{C^{-1/2-\epsilon}} \geq c\right) + \mathbf{P}\left(\|(v + v_{\text{ref}})(t)\|_{C^{-1/2-\epsilon}} > \frac{m}{c}\right) \\ \leq o_c(1) + \mathbf{P}\left(\|v(t)\|_{C^{-1/2-\epsilon}} > \frac{m}{2c}\right) + \mathbf{P}\left(\|v_{\text{ref}}(t)\|_{C^{-1/2-\epsilon}} > \frac{m}{2c}\right) \\ \leq o_c(1) + o_{m/c}(1). \end{aligned}$$

The $o_m(1)$ function does not depend on t by stationarity. In the last step we used the ϕ -independent the estimate (3.18) quantifying the upper bound (3.17) in the coming down from infinity property together with the stationarity of v_{ref} . This gives the t -uniform and ϕ -independent estimate

$$\mathbf{P}(\|u(t)\|_{C^{-1/2-\epsilon}} > 3m) = o_m(1). \quad (3.60)$$

We have been cautious to construct an enhanced noise whose law is stationary in time. This property together with the independence of the estimate (3.60) with respect to the initial condition allows then to propagate (3.60) uniformly in time by restarting fictively the dynamics every integer time while keeping an upper bound $o_m(1)$ that does not depend on the interval considered. The family of laws $\mathcal{L}(u_r(t, \phi))$ of $u_r(t, \phi)$ is thus tight in $C^{-1/2-2\epsilon}(M)$, independently of the regularisation parameter $r \in [0, 1]$ and the initial condition $\phi \in C^{-1/2-\epsilon}(M)$, uniformly in $t \geq 1$. It follows that for any $\phi \in C^{-1/2-2\epsilon}(M)$ the probability measures on $C^{-1/2-\epsilon}(M)$

$$\frac{1}{T-1} \int_1^T \delta_{\mathcal{L}(u(t, \phi))} dt, \quad (T \geq 2)$$

have a weak limit along a subsequence of times T tending to infinity. The Feller property of the semigroup generated by (3.1) ensures that this weak limit is an invariant probability measure of the dynamics. \square

3.6.2 Non-triviality of the Φ_3^4 measure

Some care is needed when working with Jagannath & Perkowski's representation

$$u_r = \mathfrak{I}_r - \mathfrak{Y}_r^\circ + e^{-3\mathfrak{Y}_r^\circ}(v_r + v_{r,\text{ref}})$$

of u_r when it comes to taking the expectation of some quantities. This is related to the fact that the random variable \mathfrak{Y} being a quadratic polynomial of a Gaussian noise the random variable $\exp(-3\mathfrak{Y}^\circ)$ may not be integrable. This a priori makes tricky to say anything about the integrability of $u_r(t)$ from its description in terms of $v_r(t)$. To circumvent this problem we follow Jagannath & Perkowski' suggestion to trade $\exp(-\mathfrak{Y}^\circ)$ for $\exp(-P_{\geq n} \mathfrak{Y}^\circ)$ in their change of unknown (3.6). The operator

$$P_{\geq n} = \sum_{i \in I} \sum_{|k| \geq n} P_k^i$$

removes a number of initial terms of a Littlewood-Paley expansion. One can thus choose n random so that

$$\|P_{\geq n} \mathfrak{Y}^\circ\|_{C_T C^{1-\eta}} \leq 1.$$

Set

$$f_n \stackrel{\text{def}}{=} \sum_{i \in I} \sum_{|k| \leq n-1} P_k^i(\mathfrak{Y}^\circ) \in C_T C^\infty(M).$$

This change of unknown adds a term $f_n v$ into the equation for v , which only changes Z_1 for a new Z_1 that is still an element of $C_T C^{-1/2-\epsilon}(M)$ and is a polynomial of the noise. As $P_k^i P_\ell^j = 0$ for $|k - \ell|$ greater than a fixed constant we have

$$\|f_n\|_{C_T C^{1-2\epsilon}} \lesssim \|\mathfrak{Y}^\circ\|_{C_T C^{1-2\epsilon}}$$

independently of our definition of the random integer n . So the new Z_1 has finite moments of any order. We get from the estimates (3.17) and (3.18) quantifying of the coming down, with $\exp(-\mathfrak{Y})$ now replaced by $\exp(-P_{\geq n}\mathfrak{Y})$, and the formula (3.6) relating u and v the fact that

$$u(1) = \mathfrak{I}(1) + \mathfrak{Y}^{\circ\circ}(1) + (\star) \quad (3.61)$$

for an element $(\star) \in C^{1-\eta}(M)$ whose norm belongs to all the $L^q(\Omega)$ spaces, $1 \leq q < \infty$, uniformly with respect to the initial condition ϕ of u . We now see clearly that $u(1) \in C^{-1/2-\epsilon}(M)$ belongs to all the $L^q(\Omega)$ spaces, $1 \leq q < \infty$, uniformly with respect to the initial condition ϕ . We assume in the remainder of this section that we work with this version of Jagannath & Perkowski's equation.

The mechanics of the proof that the Φ_3^4 measure is non-Gaussian is well-known. We write it here for completeness and follow for that purpose the work by Gubinelli & Hofmanová's [GH21]. Assume ϕ is random, with law the invariant measure of the dynamics, so $u(1)$ itself has the same law. Consider the heat regularisation $e^{r\Delta}u(1)$ of our solution u at time 1. In this section, for simplicity, we shall assume that we used true Wick ordering for the renormalisation which simplifies the discussion and allows us to use true orthogonality properties of the Wiener chaos decomposition. Our argument is of semiclassical nature, we will use the small r asymptotic behaviour of heat kernels to justify nontriviality – so r somehow plays the role of a semiclassical parameter. If the Φ_3^4 measure were Gaussian the random variable $e^{-rP}(u(1))$ would also be Gaussian uniformly when $r > 0$ goes to 0. So its truncated four point function

$$\begin{aligned} C_4^r &= C_4 \left(e^{-rP}(u(1)), e^{-rP}(u(1)), e^{-rP}(u(1)), e^{-rP}(u(1)) \right) \\ &\stackrel{\text{def}}{=} \mathbf{E}[e^{-rP}(u(1))^4] - 3 \mathbf{E}[e^{-rP}(u(1))^2]^2 \end{aligned}$$

would be identically null uniformly in $r \in (0, 1]$. The sufficient integrability of the different elements of the decomposition

$$u(1) = \mathfrak{I}(1) - \mathfrak{Y}^{\circ\circ}(1) + e^{-3P_{\geq n}\mathfrak{Y}}(1)(v(1) + v_{\text{ref}}(1)),$$

allows to plug it inside the formula for the fourth order cumulant and use Wick's Theorem to get

$$\begin{aligned} &C_4 \left(e^{-rP}(u(1)), e^{-rP}(u(1)), e^{-rP}(u(1)), e^{-rP}(u(1)) \right) \\ &= 24 \int_{-\infty}^t G_r^{(3)}(t-s) \circ e^{-(t-s+r)P}(x, x) ds + 216 \int_{(-\infty, t]^2} \int_{y_1, y_2 \in U^2} G_0^{(2)}(s_1 - s_2, y_1, y_2) \\ &\quad \times e^{-(r+t-s_1)P}(y_1, x) e^{-(r+t-s_2)P}(y_2, x) G_r^{(1)}(t-s_1, y_1, x) G_r^{(1)}(t-s_2, y_2, x) dy_{12} ds_{12} \\ &\quad + \mathbf{E} \left[\left(e^{-rP}\mathfrak{I}(1) P \left(e^{-rP}\mathfrak{Y}^{\circ\circ}(1), e^{-rP}e^{-3P_{\geq n}\mathfrak{Y}}(1)(v(1) + v_{\text{ref}}(1)) \right) \right) \right] \end{aligned}$$

where P is some polynomial functional in its stochastic arguments. We have many cancellations in the above expression since Gaussian cumulants only retain connected Feynman graphs and we also use orthogonality of homogeneous Wiener chaoses of different degrees. The remainder has the corresponding decay

$$\mathbf{E} \left[\mathfrak{I}(1) P \left(e^{-rP}\mathfrak{Y}^{\circ\circ}(1), e^{-rP}e^{-3P_{\geq n}\mathfrak{Y}}(1)(v(1) + v_{\text{ref}}(1)) \right) \right] = \mathcal{O}(r^{-\frac{1}{4}})$$

since we just need to recall that the remainder only involves the terms,

$$e^{-rP}\mathfrak{I}(1) = \mathcal{O}(r^{-\frac{1}{4}}), \quad e^{-rP}\mathfrak{Y}^{\circ\circ}(1) = \mathcal{O}(1), \quad e^{-rP}e^{-3P_{\geq n}\mathfrak{Y}}(1)(v(1) + v_{\text{ref}}(1)) = \mathcal{O}(1)$$

since they are Hölder regular in $\mathcal{C}^{\frac{1}{2}-0}$ and \mathcal{C}^{1-0} respectively. (We used the fact that we can probe the space Hölder regularity by testing against heat kernels: $\sup_{\epsilon \in (0, 1]} \epsilon^{-\frac{s}{2}} \|e^{-\epsilon P}u\|_{L^\infty(M)} \lesssim \|u\|_{C^s(M)}$ and also we made an implicit use of Besov embeddings, $\forall \delta > 0, \|\cdot\|_{C^{s-\frac{d}{p}-\delta}} \lesssim \|\cdot\|_{B_{p,p}^s}$ together with hypercontractive estimates which allows us to consider expectations of Hölder norms.)

Let us study in detail the asymptotics of the first term on the right hand side of the equation for C_4^r which has a Feynman integral interpretation. For every $x \in U$, choose some cut-off function $\chi \in C_c^\infty(U)$ which equals 1 near x that we use to localize the asymptotics as in the calculation of counterterms, then we can extract the small r leading asymptotics as

$$\begin{aligned} \int_{-\infty}^t G_r^{(3)}(t-s) \circ e^{-(t-s+r)P}(x, x) ds \\ \simeq \int_0^\infty \int_{y \in U} \chi(y) \left(\int_{[a+r, +\infty)^3} \prod_{i=1}^3 e^{-s_i P}(x, y) ds_i \right) e^{-(a+r)P}(y, x) da. \end{aligned}$$

We compute the integral with respect to y first; this reads

$$\begin{aligned} \int \left(\prod_{i=1}^3 K^0(s_i, x, y) \right) K^0(a+r, x, y) \chi(y) dy \\ \simeq \frac{e^{-(s_1+s_2+s_3+a+r)} (4\pi)^{-6}}{(s_1 s_2 s_3 (a+r))^{\frac{3}{2}}} \int_U e^{-\frac{|x-y|g(x)}{4} (s_1^{-1}+s_2^{-1}+s_3^{-1}+(a+r)^{-1})} \chi(y)_{\det(g)}^{\frac{1}{2}} dy \\ \simeq \frac{(4\pi)^{-\frac{9}{2}} e^{-(s_1+s_2+s_3+a+r)}}{(s_1 s_2 s_3 (a+r))^{\frac{3}{2}} (s_1^{-1} + s_2^{-1} + s_3^{-1} + (a+r)^{-1})^{\frac{3}{2}}} \\ + \mathcal{O}\left((s_1 s_2 s_3 (a+r))^{-\frac{3}{2}} (s_1^{-1} + s_2^{-1} + s_3^{-1} + (a+r)^{-1})^{-\frac{5}{2}} \right) \end{aligned}$$

where we only keep the leading terms in the heat asymptotic expansion and use a stationary phase estimate. It is possible, as we did for the counterterms, to show that the integral with respect to a, s_1, s_2, s_3 of the $\mathcal{O}(\dots)$ term gives subleading asymptotics compared to the leading term. We are reduced after a change of variables to the asymptotics of the following integral

$$\int_0^1 (a+r)^{-\frac{3}{2}} \left(\int_{[1, +\infty)^3} (a_2 a_3 + a_1 a_3 + a_1 a_2 + a_1 a_2 a_3)^{-\frac{3}{2}} da_{123} \right) da \simeq cr^{-\frac{1}{2}}$$

for some non-null constant c . The next term in the formula for C_4^r is

$$\begin{aligned} \int_{(-\infty, t]^2} \int_{y_1, y_2 \in U^2} G_0^{(2)}(s_1 - s_2, y_1, y_2) e^{-(r+t-s_1)P}(y_1, x) e^{-(r+t-s_2)P}(y_2, x) \\ \times G_r^{(1)}(t - s_1, y_1, x) G_r^{(1)}(t - s_2, y_2, x) dy_1 dy_2 ds_{12} \end{aligned}$$

which is bounded by a constant multiple of the integral over $(-\infty, t]^2 \times U^2$ of

$$\left(\sqrt{|s_1 - s_2|} + |y_1 - y_2| \right)^{-2} \left(\sqrt{|r+t-s_1|} + |y_1 - x| \right)^{-4} \left(\sqrt{|r+t-s_2|} + |y_2 - x| \right)^{-4}.$$

Making first the change of variables $s_i \mapsto r^2(s_i - t) + s_i, y_i \mapsto r(y_i - x) + x$ and then using polar coordinates gives $\mathcal{O}(|\log r|)$ as an upper bound for this integral, therefore the cumulant C_4^r blows-up like $24cr^{-\frac{1}{2}}$ when $r > 0$ goes to 0. This shows that C_4^r does not vanish asymptotically and that the Φ_3^4 measure is non-Gaussian.

For a Φ_3^4 measure obtained as above as weak limit of Birkhoff averages, the covariance property under Riemannian isometries is clear from its construction and the fact that the renormalisation constants a_r, b_r do not depend on which Riemannian metric is used: given a field ϕ on (M, g) whose law is a Φ_3^4 measure, let $f : M' \mapsto M$ be a smooth diffeomorphism, then the pulled-back field $f^*\phi$ on (M', f^*g) will have the law of a Φ_3^4 measure of the SPDE (3.1) for the metric f^*g . Such measure gives for the first time a non-perturbative, non-topological interacting quantum field theory on 3-dimensional curved Riemannian spaces. We prove in [Bai23] that the semigroup on $C^{-1/2-\epsilon}(M)$ generated by the dynamics (3.1) has a unique invariant probability measure. This uniqueness result yields a stronger notion of covariance.

Appendix 3.G Littlewood-Paley-Stein projectors

There are several ways of defining some Littlewood-Paley type projectors on function spaces over a manifold – see [KR03, BB16, BBF17, GdP22, Mou21b] for a sample. We choose here an intermediate road and use the classical Littlewood-Paley projectors over \mathbf{R}^d to define a number of operators on function spaces over M using local charts. This allows to import at low cost some known regularity properties of the corresponding objects from the flat to the curved setting. We denote as usual by $B_{p,q}^\gamma(M)$ the Besov spaces over M and by $C^\gamma(M)$ the Besov-Hölder space $B_{\infty,\infty}^\gamma(M)$, with associated norm denoted by $\|\cdot\|_{C^\gamma}$.

Let then denote by

$$a' \prec b' \stackrel{\text{def}}{=} \sum_{-1 \leq j < k-1} (\Delta_j a') (\Delta_k b')$$

the paraproduct of some distributions a' and b' on \mathbf{R}^d , and write

$$a' \odot b' \stackrel{\text{def}}{=} \sum_{|j-k| \leq 1} (\Delta_j a') (\Delta_k b')$$

for the resonance of a' and b' whenever the latter is defined. Let $(U_i, \kappa_i)_i$ denote a finite open cover of M by some charts, with κ_i a smooth diffeomorphism between $U_i \subset M$ and $\kappa_i(U_i) \subset \mathbf{R}^d$. Let $(\chi_i)_i$ be a partition of unity subordinated to $(U_i)_i$, so $\sum_i \chi_i = 1$, with $\chi_i \in C_c^\infty(U_i)$. Choose also for every index i a function $\tilde{\chi}_i \in C_c^\infty(U_i)$ such that $\tilde{\chi}_i$ equals 1 on the support of χ_i and some function $\psi_i \in C_c^\infty(\kappa_i(U_i))$ which equals 1 on the support of $\kappa_{i*}(\tilde{\chi}_i)$. Given some smooth functions a, b on M we have the decomposition

$$\begin{aligned} ab &= \sum_{i \in I} (a \chi_i) (b \tilde{\chi}_i) = \sum_{i \in I} \kappa_i^* [\kappa_{i*}(a \chi_i)] \kappa_i^* [\kappa_{i*}(b \tilde{\chi}_i)] \\ &= \sum_{i \in I} \kappa_i^* [(\kappa_{i*}(a \chi_i)) (\kappa_{i*}(b \tilde{\chi}_i))] = \sum_{i \in I} \kappa_i^* [\psi_i \kappa_{i*}(a \chi_i) \kappa_{i*}(b \tilde{\chi}_i)] \\ &= \sum_{i \in I} \kappa_i^* [\psi_i (\kappa_{i*}(a \chi_i) \prec \kappa_{i*}(b \tilde{\chi}_i))] + \sum_{i \in I} \kappa_i^* [\psi_i (\kappa_{i*}(a \chi_i) \odot \kappa_{i*}(b \tilde{\chi}_i))] \\ &\quad + \sum_{i \in I} \kappa_i^* [\psi_i (\kappa_{i*}(a \chi_i) \succ \kappa_{i*}(b \tilde{\chi}_i))]. \end{aligned}$$

Actually for arbitrary $\chi_i, \tilde{\chi}_i \in C_c^\infty(U_i)^2$ such that $\tilde{\chi}_i = 1$ on the support of χ_i , we set the generalised Littlewood-Paley-Stein projectors

$$P_k^i(a) \stackrel{\text{def}}{=} \kappa_i^* [\psi_i \Delta_k (\kappa_{i*}(a \chi_i))], \text{ and } \tilde{P}_k^i(a) \stackrel{\text{def}}{=} \kappa_i^* [\tilde{\psi}_i \Delta_k (\kappa_{i*}(\tilde{\chi}_i a))], \quad (3.62)$$

where $\tilde{\psi}_i \in C_c^\infty(\kappa_i(U_i))$ equals 1 on the support of ψ_i . We do not necessarily require that $\sum_{i \in I} \tilde{\chi}_i = 1$.

On the manifold M , recall $i \in I$ denotes a chart index, we define generalised chart localised operations as:

$$a \prec_i b \stackrel{\text{def}}{=} \sum_{-1 \leq j < k-1} P_j^i a \tilde{P}_k^i b, \quad a \succ_i b \stackrel{\text{def}}{=} \sum_{i \in I} \sum_{-1 \leq k < j-1} P_j^i a \tilde{P}_k^i b, \quad \text{and } a \odot_i b \stackrel{\text{def}}{=} \sum_{i \in I} \sum_{|j-k| \leq 1} P_j^i a \tilde{P}_k^i b.$$

In particular when $\sum_{i \in I} \chi_i = 1$, the above operations decompose the product ab on M as:

$$ab = \sum_{i \in I} (a \prec_i b + a \odot_i b + a \succ_i b).$$

Note the important fact that the definition of the resonant product and paraproducts are asymmetrical, therefore they are noncommutative meaning that $a \prec b \neq b \succ a$, however all the regularity properties are similar as in the flat case. We collect in the next two statements some regularity properties of these operators and refer the reader to [BDFT23, Proposition 2.6] for their proofs.

Proposition 3.61 *One has the following continuity estimates. For every chart index i ,*

– For p, p_1, p_2, q, q_1, q_2 in $[1, +\infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$

1. For $\gamma_2 \in \mathbf{R}$

$$\|a \prec_i b\|_{B_{p,q_2}^{\gamma_2}} \lesssim \|a\|_{L^{p_1}} \|b\|_{B_{p_2,q_2}^{\gamma_2}},$$

2. For $\gamma_1 < 0$ and $\gamma_2 \in \mathbf{R}$

$$\|a \prec_i b\|_{B_{p,q}^{\gamma_1+\gamma_2}} \lesssim \|a\|_{B_{p_1,q_1}^{\gamma_1}} \|b\|_{B_{p_2,q_2}^{\gamma_2}}$$

3. For any $\gamma_1, \gamma_2 \in \mathbf{R}$ with $\gamma_1 + \gamma_2 > 0$ one has

$$\|a \odot_i b\|_{B_{p,q}^{\gamma_1+\gamma_2}} \lesssim \|a\|_{B_{p_1,q_1}^{\gamma_1}} \|b\|_{B_{p_2,q_2}^{\gamma_2}}.$$

We recall from Lemma 7.2 of Mourrat & Weber’s work [MW14] the following comparison test that we used in our proof of Theorem 3.7.

Proposition 3.62 *Let a continuous function $F : [0, T] \rightarrow [0, +\infty)$ that satisfies the inequality*

$$\int_s^t F(s_1)^\lambda ds_1 \leq c(F(s) + 1) \quad (3.63)$$

for all $0 \leq s \leq t \leq T$, for some exponent $\lambda > 1$ and some positive constant c . Then there is a sequence of times $t_0 = 0 < t_1 < \dots < t_N = T$ such that one has

$$F(t_n) \leq 1 + 2^{\frac{\lambda}{\lambda-1}} \left(\frac{c}{1 - 2^{-(\lambda-1)}} \right)^{\frac{1}{\lambda-1}} t_{n+1}^{-\frac{1}{\lambda-1}},$$

for all $0 \leq n \leq N - 1$.

Proof. We include a proof following closely Mourrat-Weber’s comparison test in a slightly different setting compared to them since we have $F(s) + 1$ rather than $F(s)$ on the right hand side of the inequality (3.63). We first define $t_0 = 0$, then given some time t_n , consider $t_{n+1}^* = t_n + c2^\lambda(1 + F(t_n))^{1-\lambda}$, if $t_{n+1}^* \geq T$ we stop the algorithm and set $N = n + 1$, $t_{n+1} = T$ and verify that the conclusion of the statement holds. Otherwise, choosing t_{n+1} such that $F(t_{n+1}) = \inf_{t_n < s < t_{n+1}^*} F(s)$ yields a bound of the form $F(t_{n+1}) \leq \frac{1+F(t_n)}{2}$. By iteration, this yields a bound of the form $F(t_{n+1}) \leq \frac{F(t_0)-1}{2^{n+1}} + 1$. Note that for n large enough, since $\lambda > 1$

$$t_{n+1}^* - t_n = c2^\lambda(1 + F(t_n))^{1-\lambda} \geq c2^\lambda(2 + \frac{F(t_0)-1}{2^n})^{1-\lambda} \geq c2^\lambda(2 + 1/3)^{1-\lambda} \geq 2c,$$

hence the algorithm must terminate for n large enough after finite number of iterations. Now we need to check the conclusion $t_n = \sum_{k=0}^{n-1} (t_{k+1} - t_k) \leq c2^\lambda \sum_{i=0}^{n-1} (1 + F(t_i))^{1-\lambda}$. Note that since $F(t_i) \geq (F(t_n) - 1)2^{n-i} + 1$ then $(1 + F(t_i))^{1-\lambda} \leq ((F(t_n) - 1)2^{n-i} + 2)^{1-\lambda}$, so we have

$$\begin{aligned} t_{n+1} &\leq c2^\lambda \sum_{i=0}^n ((F(t_n) - 1)2^{n-i} + 2)^{1-\lambda} \leq c2^\lambda (F(t_n) - 1)^{1-\lambda} \sum_{i=0}^n 2^{(n-i)(1-\lambda)} \\ &\leq c2^\lambda (F(t_n) - 1)^{1-\lambda} \frac{1}{1 - 2^{1-\lambda}}, \end{aligned}$$

which yields the estimate from the statement. \square

Last we recall the fractional Leibniz rule and an elementary interpolation result used in the proof of the coming down property in Section 3.2.2, we prove these results in [BDFT23, Propositions 2.1 and 2.2].

Proposition 3.63 *Let $\alpha > 0$, $r \in \mathbf{N}$ and $p, p_1, p_2, q \in [1, \infty]$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then

$$\|u^{r+1}\|_{B_{p,q}^\alpha} \lesssim \|u^r\|_{L^{p_1}} \|u\|_{B_{p_2,q}^\alpha}.$$

Proposition 3.64 *Let $\alpha_1, \alpha_2 \in \mathbf{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$ and $\theta \in [0, 1]$. Define $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$, and $p, q \in [1, \infty]$ by*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Then

$$\|u\|_{B_{p,q}^\alpha} \lesssim \|u\|_{B_{p_1,q_1}^{\alpha_1}}^\theta \|u\|_{B_{p_2,q_2}^{\alpha_2}}^{1-\theta}.$$

Bibliography

- [ADC21] M. AIZENMAN and H. DUMINIL-COPIN. Marginal triviality of the scaling limits of critical 4D Ising and $\lambda\phi_4^4$ models. *Annals of Mathematics* **194**, no. 1, (2021), 163 – 235. [doi:10.4007/annals.2021.194.1.3](https://doi.org/10.4007/annals.2021.194.1.3).
- [ADVG20] S. ALBEVERIO, F. C. DE VECCHI, and M. GUBINELLI. Elliptic stochastic quantization. *The Annals of Probability* **48**, no. 4(2020). [doi:10.1214/19-aop1404](https://doi.org/10.1214/19-aop1404).
- [Aiz82] M. AIZENMAN. Geometric analysis of φ^4 fields and Ising models. I, II. *Comm. Math. Phys.* **86**, no. 1, (1982), 1–48.
- [AK20] S. ALBEVERIO and S. KUSUOKA. The invariant measure and the flow associated to the ϕ_3^4 -quantum field model. *ANNALI SCUOLA NORMALE SUPERIORE - CLASSE DI SCIENZE* (2020), 1359–1427. [doi:10.2422/2036-2145.201809_008](https://doi.org/10.2422/2036-2145.201809_008).
- [AK22] S. ALBEVERIO and S. KUSUOKA. Construction of a non-gaussian and rotation-invariant ϕ^4 -measure and associated flow on \mathbb{S}^3 through stochastic quantization. *arXiv preprint* (2022).
- [AV13] S. D. A. VASY. Microlocal analysis of asymptotically hyperbolic and kerr-de sitter spaces. *Inventiones Math.* (2013).
- [Bai23] I. BAILLEUL. Uniqueness of the ϕ_3^4 measures on closed riemannian 3-manifolds. *arXiv preprint* (2023). [arXiv:2306.07616](https://arxiv.org/abs/2306.07616).
- [BB16] I. BAILLEUL and F. BERNICOT. Heat semigroup and singular pdes. *J. Funct. Anal.*, **270**:3344–3452 (2016). [arXiv:1501.06822](https://arxiv.org/abs/1501.06822).
- [BB19] I. BAILLEUL and F. BERNICOT. High order paracontrolled calculus. *Forum of Mathematics, Sigma* **7**, (2019), e44. [doi:10.1017/fms.2019.44](https://doi.org/10.1017/fms.2019.44).
- [BBF17] I. BAILLEUL, F. BERNICOT, and D. FREY. Space-time paraproducts for paracontrolled calculus, 3d-pam and multiplicative burgers equations. *Ann. Sc. Éc. Norm. Sup.*, **51**:1399–1457 (2017). [arXiv:1506.08773](https://arxiv.org/abs/1506.08773).
- [BCCH21] Y. BRUNED, A. CHANDRA, I. CHEVYREV, and M. HAIRER. Renormalising SPDEs in regularity structures. *J. Eur. Math. Soc. (JEMS)* **23**, no. 3, (2021), 869–947. [doi:10.4171/jems/1025](https://doi.org/10.4171/jems/1025).
- [BDF⁺23] I. BAILLEUL, N. V. DANG, L. FERDINAND, G. LECLERC, and J. LIN. Spectrally cut-off gff, regularized ϕ^4 measure, and reflection positivity. *arXiv preprint* (2023). [arXiv:2312.15511](https://arxiv.org/abs/2312.15511).
- [BDFT23] I. BAILLEUL, N. V. DANG, L. FERDINAND, and T. D. TÔ. Global harmonic analysis for ϕ_3^4 on closed riemannian manifolds. *arXiv preprint* (2023). [arXiv:2306.07757](https://arxiv.org/abs/2306.07757).
- [BDFT24] I. BAILLEUL, N. V. DANG, L. FERDINAND, and T. D. TÔ. ϕ_3^4 measures on compact riemannian 3-manifolds. *arXiv preprint* (2024). [arXiv:2304.10185](https://arxiv.org/abs/2304.10185).
- [BDH14] C. BROUDER, N. V. DANG, and F. HÉLEIN. A smooth introduction to the wavefront set. *Journal of Physics A: Mathematical and Theoretical* **47**, no. 44, (2014), 443001. [doi:10.1088/1751-8113/47/44/443001](https://doi.org/10.1088/1751-8113/47/44/443001).
- [BDH16] C. BROUDER, N. V. DANG, and F. HÉLEIN. Continuity of the fundamental operations on distributions having a specified wave front set (with a counterexample by semyon alesker). *Studia Mathematica* (2016), 1–26. [doi:10.4064/sm8316-3-2016](https://doi.org/10.4064/sm8316-3-2016).
- [BDM23] I. BAILLEUL, N. V. DANG, and A. MOUZARD. Analysis of the anderson operator. *arXiv preprint* (2023). [arXiv:2201.04705](https://arxiv.org/abs/2201.04705).
- [BDNY22] B. BRINGMANN, Y. DENG, A. R. NAHMOD, and H. YUE. Invariant gibbs measures for the three dimensional cubic nonlinear wave equation, 2022. [arXiv:2205.03893](https://arxiv.org/abs/2205.03893).
- [BDR23] A. BONICELLI, C. DAPPIAGGI, and P. RINALDI. An algebraic and microlocal approach to the stochastic nonlinear schrödinger equation. *Annales Henri Poincaré* **24**, no. 7, (2023), 2443–2482. [doi:10.1007/s00023-023-01291-4](https://doi.org/10.1007/s00023-023-01291-4).
- [BF00] R. BRUNETTI and K. FREDENHAGEN. Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Communications in Mathematical Physics* **208**, no. 3, (2000), 623–661. [doi:10.1007/s002200050004](https://doi.org/10.1007/s002200050004).

- [BFV03] R. BRUNETTI, K. FREDENHAGEN, and R. VERCH. The generally covariant locality principle – a new paradigm for local quantum field theory. *Communications in Mathematical Physics* **237**, no. 1, (2003), 31–68. [doi:10.1007/s00220-003-0815-7](https://doi.org/10.1007/s00220-003-0815-7).
- [BG20] N. BARASHKOV and M. GUBINELLI. A variational method for Φ_3^4 . *Duke Math. J.* **169**, no. 17, (2020), 3339–3415. [doi:10.1215/00127094-2020-0029](https://doi.org/10.1215/00127094-2020-0029).
- [BG21] N. BARASHKOV and M. GUBINELLI. The Φ_3^4 measure via Girsanov’s theorem. *Electronic Journal of Probability* **26**, no. none, (2021), 1 – 29. [doi:10.1214/21-EJP635](https://doi.org/10.1214/21-EJP635).
- [BHZ19] Y. BRUNED, M. HAIRER, and L. ZAMBOTTI. Algebraic renormalisation of regularity structures. *Invent. Math.* **215**, no. 3, (2019), 1039–1156. [doi:10.1007/s00222-018-0841-x](https://doi.org/10.1007/s00222-018-0841-x).
- [Bou94] J. BOURGAIN. Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* **166**, no. 1, (1994), 1–26.
- [BTT14] N. BURQ, L. THOMANN, and N. TZVETKOV. Remarks on the gibbs measures for nonlinear dispersive equations. *arXiv preprint* (2014). [arXiv:1412.7499](https://arxiv.org/abs/1412.7499).
- [CC13] R. CATELLIER and K. CHOUK. Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation. *ArXiv e-prints* (2013). To appear in *Ann. Probab.* [arXiv:1310.6869](https://arxiv.org/abs/1310.6869).
- [CdLFW24] A. CHANDRA, G. DE LIMA FELTES, and H. WEBER. A priori bounds for 2-d generalised parabolic anderson model, 2024. [arXiv:2402.05544](https://arxiv.org/abs/2402.05544).
- [CF12] C. R. G. C. FEFFERMAN. *The ambient metric*. Princeton University Press, 2012.
- [CF24a] A. CHANDRA and L. FERDINAND. A flow approach to the generalized kpz equation. *arXiv preprint* (2024). [arXiv:2402.03101](https://arxiv.org/abs/2402.03101).
- [CF24b] A. CHANDRA and L. FERDINAND. A stochastic analysis approach to tensor field theories. *arXiv preprint* (2024). [arXiv:2306.05305](https://arxiv.org/abs/2306.05305).
- [CH16] A. CHANDRA and M. HAIRER. An analytic BPHZ theorem for regularity structures. *ArXiv e-prints* (2016). [arXiv:1612.08138](https://arxiv.org/abs/1612.08138).
- [CMW19] A. CHANDRA, A. MOINAT, and H. WEBER. A priori bounds for the ϕ^4 equation in the full sub-critical regime, 2019. [arXiv:1910.13854](https://arxiv.org/abs/1910.13854).
- [Dan13] N. DANG. *Renormalization of quantum field theory on curved space-times, a causal approach*. Ph.D. thesis, Université Pierre et Marie Curie, 2013.
- [Dan14] N. V. DANG. The extension of distributions on manifolds, a microlocal approach. *Annales Henri Poincaré*, **17**(4) (2014).
- [DDD17] A. DAHLQVIST, J. DIEHL, and B. DRIVER. The parabolic anderson model on riemann surfaces. *arXiv preprint* (2017). [arXiv:1702.02965](https://arxiv.org/abs/1702.02965).
- [DDRZ21] C. DAPPIAGGI, N. DRAGO, P. RINALDI, and L. ZAMBOTTI. A microlocal approach to renormalization in stochastic pdes. *Communications in Contemporary Mathematics* **24**, no. 07(2021). [doi:10.1142/S0219199721500759](https://doi.org/10.1142/S0219199721500759).
- [DH19] N. V. DANG and E. HERSCOVICH. Renormalization of quantum field theory on riemannian manifolds. *Reviews in Mathematical Physics* **31**, no. 06, (2019), 1950017. [arXiv:https://doi.org/10.1142/S0129055X1950017X](https://arxiv.org/abs/https://doi.org/10.1142/S0129055X1950017X), [doi:10.1142/S0129055X1950017X](https://doi.org/10.1142/S0129055X1950017X).
- [DIM04] J. DIMOCK. Markov quantum fields on a manifold. *Reviews in Mathematical Physics* **16**, no. 02, (2004), 243–255. [doi:10.1142/S0129055X04001947](https://doi.org/10.1142/S0129055X04001947).
- [DNY20] Y. DENG, A. R. NAHMOD, and H. YUE. Random tensors, propagation of randomness, and nonlinear dispersive equations, 2020. [arXiv:2006.09285](https://arxiv.org/abs/2006.09285).
- [DPD03] G. DA PRATO and A. DEBUSSCHE. Strong solutions to the stochastic quantization equations. *Ann. Probab.* **31**, no. 4, (2003), 1900–1916. [doi:10.1214/aop/1068646370](https://doi.org/10.1214/aop/1068646370).
- [DR16] T. DELEPOUVE and V. RIVASSEAU. Constructive tensor field theory: the T_3^4 model. *Communications in Mathematical Physics* **345**, no. 2, (2016), 477–506. [doi:10.1007/s00220-016-2680-1](https://doi.org/10.1007/s00220-016-2680-1).
- [Duc21] P. DUCH. Flow equation approach to singular stochastic PDEs, 2021. [arXiv:2109.11380](https://arxiv.org/abs/2109.11380).
- [Duc23] P. DUCH. Renormalization of singular elliptic stochastic PDEs using flow equation, 2023. [arXiv:2201.05031](https://arxiv.org/abs/2201.05031).

- [DW22] N. V. DANG and M. WROCHNA. Dynamical residues of lorentzian spectral zeta functions. *arXiv preprint* (2022). [arXiv:2108.07529](https://arxiv.org/abs/2108.07529).
- [DZ16] S. DYATLOV and M. ZWORSKI. Dynamical zeta functions for anosov flows via microlocal analysis. *Ann. Sc. Éc. Norm. Sup.* (2016).
- [Fel74] J. FELDMAN. The $\lambda\varphi_3^4$ field theory in a finite volume. *Comm. Math. Phys.* **37**, (1974), 93–120.
- [FGPSVT23] L. FERDINAND, R. GURAU, C. I. PEREZ-SANCHEZ, and F. VIGNES-TOURNERET. Borel summability of the $1/n$ expansion in quartic $o(n)$ -vector models. *Annales Henri Poincaré* **25**, no. 3, (2023), 2037–2064. [doi:10.1007/s00023-023-01350-w](https://doi.org/10.1007/s00023-023-01350-w).
- [FH14] P. K. FRIZ and M. HAIRER. *A course on rough paths*. Universitext. Springer, Cham, 2014, xiv+251. With an introduction to regularity structures. [doi:10.1007/978-3-319-08332-2](https://doi.org/10.1007/978-3-319-08332-2).
- [FO76] J. S. FELDMAN and K. OSTERWALDER. The Wightman axioms and the mass gap for weakly coupled $(\Phi^4)_3$ quantum field theories. *Ann. Physics* **97**, no. 1, (1976), 80–135.
- [FOS83] J. FROHLICH, K. OSTERWALDER, and E. SEILER. On Virtual representations of symmetric spaces and their analytic continuation. *Annals Math.* **118**, (1983), 461–489. [doi:10.2307/2006979](https://doi.org/10.2307/2006979).
- [Frö82] J. FRÖHLICH. On the triviality of $\lambda\varphi_d^4$ theories and the approach to the critical point in $d > 4$ dimensions. *Nuclear Phys. B* **200**, no. 2, (1982), 281–296. [doi:10.1016/0550-3213\(82\)90088-8](https://doi.org/10.1016/0550-3213(82)90088-8).
- [GdP22] C. GUILLARMOU and T. DE POYFERRÉ. A paradifferential approach for hyperbolic dynamical systems and applications. *Tunisian Journal of Mathematics* **4**, no. 4, (2022), 673–718. [doi:10.2140/tunis.2022.4.673](https://doi.org/10.2140/tunis.2022.4.673).
- [GH18] M. GUBINELLI and M. HOFMANOVA. A PDE construction of the Euclidean Φ_3^4 quantum field theory. *ArXiv e-prints* (2018). [arXiv:1810.01700](https://arxiv.org/abs/1810.01700).
- [GH19] M. GUBINELLI and M. HOFMANOVÁ. Global solutions to elliptic and parabolic Φ^4 models in Euclidean space. *Comm. Math. Phys.* **368**, no. 3, (2019), 1201–1266. [doi:10.1007/s00220-019-03398-4](https://doi.org/10.1007/s00220-019-03398-4).
- [GH21] M. GUBINELLI and M. HOFMANOVÁ. A PDE Construction of the Euclidean Φ_3^4 Quantum Field Theory. *Communications in Mathematical Physics* **384**, no. 1, (2021), 1–75. [doi:10.1007/s00220-021-04022-0](https://doi.org/10.1007/s00220-021-04022-0).
- [Gil18] P. B. GILKEY. *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*. CRC press, 2018.
- [GIP15] M. GUBINELLI, P. IMKELLER, and N. PERKOWSKI. Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3**, (2015), e6, 75. [arXiv:1210.2684v3](https://arxiv.org/abs/1210.2684v3) [doi:10.1017/fmp.2015.2](https://doi.org/10.1017/fmp.2015.2).
- [GJ73] J. GLIMM and A. JAFFE. Positivity of the ϕ_3^4 Hamiltonian. *Fortschr. Physik* **21**, (1973), 327–376.
- [GK86] K. GAWEDZKI and A. KUPIAINEN. Continuum limit of the hierarchical $O(N)$ non-linear σ -model. *Communications in Mathematical Physics* **106**, no. 4, (1986), 533–550. [doi:10.1007/BF01463394](https://doi.org/10.1007/BF01463394).
- [GKRV21] C. GUILLARMOU, A. KUPIAINEN, R. RHODES, and V. VARGAS. Segal’s axioms and bootstrap for liouville theory. *arXiv preprint* (2021). [arXiv:2112.14859](https://arxiv.org/abs/2112.14859).
- [GM24] M. GUBINELLI and S.-J. MEYER. The fbsde approach to sine-gordon up to 6π , 2024. [arXiv:2401.13648](https://arxiv.org/abs/2401.13648).
- [GP16] M. GUBINELLI and N. PERKOWSKI. Kpz reloaded. *Communications in Mathematical Physics* **349**, no. 1, (2016), 165–269. [doi:10.1007/s00220-016-2788-3](https://doi.org/10.1007/s00220-016-2788-3).
- [GP17] M. GUBINELLI and N. PERKOWSKI. KPZ reloaded. *Comm. Math. Phys.* **349**, no. 1, (2017), 165–269. [doi:10.1007/s00220-016-2788-3](https://doi.org/10.1007/s00220-016-2788-3).
- [GR12] J. B. GELOUN and V. RIVASSEAU. A renormalizable 4-dimensional tensor field theory, 2012. [arXiv:1111.4997](https://arxiv.org/abs/1111.4997).
- [Gri04] D. GRIESER. Notes on heat kernel asymptotics. *online preprint* (2004).
- [Gur16] R. G. GURĂU. *Random Tensors*. Oxford University Press, 2016. [doi:10.1093/acprof:oso/9780198787938.001.0001](https://doi.org/10.1093/acprof:oso/9780198787938.001.0001).

- [GW14] H. GROSSE and R. WULKENHAAR. Construction of the Φ_4^4 -quantum field theory on non-commutative Moyal space. *arXiv e-prints* (2014), arXiv:1402.1041. [arXiv:1402.1041](https://arxiv.org/abs/1402.1041). [doi:10.48550/arXiv.1402.1041](https://doi.org/10.48550/arXiv.1402.1041).
- [Hai14] M. HAIRER. A theory of regularity structures. *Inventiones mathematicae* **198**, no. 2, (2014), 269–504. [doi:10.1007/s00222-014-0505-4](https://doi.org/10.1007/s00222-014-0505-4).
- [Hai24] M. HAIRER. Renormalisation in the presence of variance blowup. *arXiv preprint* (2024). [arXiv:2401.10868](https://arxiv.org/abs/2401.10868).
- [HM18] M. HAIRER and J. MATTINGLY. The strong Feller property for singular stochastic PDEs. *Ann. Inst. Henri Poincaré Probab. Stat.* **54**, no. 3, (2018), 1314–1340. [doi:10.1214/17-AIHP840](https://doi.org/10.1214/17-AIHP840).
- [HR23] M. HAIRER and T. ROSATI. Global existence for perturbations of the 2d stochastic navier-stokes equations with space-time white noise, 2023. [arXiv:2301.11059](https://arxiv.org/abs/2301.11059).
- [HS21a] M. HAIRER and P. SCHÖNBAUER. The support of singular stochastic pdes. *Forum Math. Pi* (2021).
- [HS21b] M. HAIRER and R. STEELE. The Φ_3^4 measure has sub-Gaussian tails. *arXiv e-prints* (2021). [arXiv:2102.11685](https://arxiv.org/abs/2102.11685).
- [HS23a] M. HAIRER and H. SINGH. Regularity structures on manifolds and vector bundles. *arXiv preprint* (2023). [arXiv:2308.05049](https://arxiv.org/abs/2308.05049).
- [HS23b] M. HAIRER and R. STEELE. The bphz theorem for regularity structures via the spectral gap inequality. *arXiv preprint* (2023). [arXiv:2301.10081](https://arxiv.org/abs/2301.10081).
- [HW01] S. HOLLANDS and R. M. WALD. Local wick polynomials and time ordered products of quantum fields in curved spacetime. *Communications in Mathematical Physics* **223**, no. 2, (2001), 289–326. [doi:10.1007/s002200100540](https://doi.org/10.1007/s002200100540).
- [HW02] S. HOLLANDS and R. M. WALD. Existence of local covariant time ordered products of quantum fields in curved spacetime. *Communications in Mathematical Physics* **231**, no. 2, (2002), 309–345. [doi:10.1007/s00220-002-0719-y](https://doi.org/10.1007/s00220-002-0719-y).
- [JP23] A. JAGANNATH and N. PERKOWSKI. A simple construction of the dynamical ϕ_3^4 model. *Transactions of the American Mathematical Society* **376**, no. 3, (2023), 1507–1522. [doi:10.1090/tran/8724](https://doi.org/10.1090/tran/8724).
- [Kha11] R. KHASHMINSKII. *Stochastic Stability of Differential Equations*. Springer Berlin, Heidelberg, 2011. [doi:https://doi.org/10.1007/978-3-642-23280-0](https://doi.org/10.1007/978-3-642-23280-0).
- [KKS92] G. KELLER, C. KOPPER, and M. SALMHOFER. Perturbative renormalization and effective Lagrangians in ϕ^4 in four-dimensions. *Helv. Phys. Acta* **65**, (1992), 32–52.
- [KM16] I. KHAVKINE and V. MORETTI. Analytic dependence is an unnecessary requirement in renormalization of locally covariant qft. *Communications in Mathematical Physics* **344**, no. 2, (2016), 581–620. [doi:10.1007/s00220-016-2618-7](https://doi.org/10.1007/s00220-016-2618-7).
- [KMW21] S. KANDEL, P. MNEV, and K. WERNLI. Two-dimensional perturbative scalar qft and atiyah–segal gluing. *Advances in Theoretical and Mathematical Physics* **25**, no. 7, (2021), 1847–1952. [doi:10.4310/atmp.2021.v25.n7.a5](https://doi.org/10.4310/atmp.2021.v25.n7.a5).
- [KR03] S. KLAINERMAN and I. RODNIANSKI. A geometric approach to the littlewood-paley theory. *Geom. Funct. Anal.*, **16**(1):126–163 (2003). [arXiv:math/0309463](https://arxiv.org/abs/math/0309463).
- [Kup16] A. KUPIAINEN. Renormalization group and stochastic PDEs. *Annales Henri Poincaré* **17**, no. 3, (2016), 497–535. [arXiv:1410.3094](https://arxiv.org/abs/1410.3094). [doi:10.1007/s00023-015-0408-y](https://doi.org/10.1007/s00023-015-0408-y).
- [Lev01] T. LEVY. Yang-mills measure on compact surfaces. *American Mathematical Soc.* (2001).
- [Lin24] J. LIN. The bayes principle and segal axioms for $p(\phi)_2$, with application to periodic covers, 2024. [arXiv:2403.12804](https://arxiv.org/abs/2403.12804).
- [MA73] V. K. P. M. ATIYAH, R. BOTT. On the heat equation and the index theorem. *Inventiones Mathematicae* (1973).
- [Mei20] E. MEINRENKEN. Euler-like vector fields, normal forms, and isotropic embeddings. *Indagationes Mathematicae* (2020).
- [Mel20] R. B. MELROSE. Spectral and scattering theory for the laplacian on asymptotically euclidian spaces. *Spectral and scattering theory* (2020).

- [Mey98] Y. MEYER. *Wavelets, vibrations and scalings*. American Mathematical Soc., 1998.
- [Mou21a] A. MOUZARD. *A paracontrolled calculus for singular stochastic PDEs on manifold : to infinity and beyond*. Ph.D. thesis, Université de Rennes, 2021.
- [Mou21b] A. MOUZARD. Weyl law for the anderson hamiltonian on a two-dimensional manifold. *Ann. Institut H. Poincaré* (2021). [arXiv:2009.03549](https://arxiv.org/abs/2009.03549).
- [MW14] J.-C. MOURRAT and H. WEBER. Convergence of the two-dimensional dynamic Ising-Kac model to ϕ_2^4 . *ArXiv e-prints* (2014). [arXiv:1410.1179](https://arxiv.org/abs/1410.1179).
- [MW17a] J.-C. MOURRAT and H. WEBER. The dynamic Φ_3^4 model comes down from infinity. *Communications in Mathematical Physics* **356**, no. 3, (2017), 673–753. [doi:10.1007/s00220-017-2997-4](https://doi.org/10.1007/s00220-017-2997-4).
- [MW17b] J.-C. MOURRAT and H. WEBER. The dynamic Φ_3^4 model comes down from infinity. *Comm. Math. Phys.* **356**, no. 3, (2017), 673–753. [arXiv:1601.01234](https://arxiv.org/abs/1601.01234). [doi:10.1007/s00220-017-2997-4](https://doi.org/10.1007/s00220-017-2997-4).
- [MW17c] J.-C. MOURRAT and H. WEBER. Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.* **45**, no. 4, (2017), 2398–2476. [doi:10.1214/16-AOP1116](https://doi.org/10.1214/16-AOP1116).
- [MW20] A. MOINAT and H. WEBER. Space-time localisation for the dynamic Φ_3^4 model. *Comm. Pure Appl. Math.* **73**, no. 12, (2020), 2519–2555. [doi:10.1002/cpa.21925](https://doi.org/10.1002/cpa.21925).
- [MWX17] J.-C. MOURRAT, H. WEBER, and W. XU. Construction of ϕ_3^4 diagrams for pedestrians, 2017. [arXiv:1610.08897](https://arxiv.org/abs/1610.08897).
- [Nel66] E. NELSON. A quartic interaction in two dimensions. *Mathematics Theory of Elementary Particles* (1966).
- [Nel73] E. NELSON. Construction of quantum fields from markoff fields. *Journal of Functional Analysis* **12**, no. 1, (1973), 97–112. [doi:https://doi.org/10.1016/0022-1236\(73\)90091-8](https://doi.org/10.1016/0022-1236(73)90091-8).
- [ORTW20] T. OH, T. ROBERT, N. TZVETKOV, and Y. WANG. Stochastic quantization of liouville conformal field theory. *arXiv preprint* (2020). [arXiv:2004.04194](https://arxiv.org/abs/2004.04194).
- [Pic08] D. PICKRELL. $p(\phi)_2$ quantum field theories and segal’s axioms. *Communications in Mathematical Physics* **280**, no. 2, (2008), 403–425. [doi:10.1007/s00220-008-0467-8](https://doi.org/10.1007/s00220-008-0467-8).
- [Pol84] J. POLCHINSKI. Renormalization and effective lagrangians. *Nuclear Physics B* **231**, no. 2, (1984), 269–295. [doi:10.1016/0550-3213\(84\)90287-6](https://doi.org/10.1016/0550-3213(84)90287-6).
- [PS82] G. PARISI and N. SOURLAS. Supersymmetric Field Theories and Stochastic Differential Equations. *Nucl. Phys. B* **206**, (1982), 321–332. [doi:10.1016/0550-3213\(82\)90538-7](https://doi.org/10.1016/0550-3213(82)90538-7).
- [PS16] G. POPINEAU and R. STORA. A pedagogical remark on the main theorem of perturbative renormalization theory. *Nucl. Phys. B* **912**, (2016), 70–78. [doi:10.1016/j.nuclphysb.2016.04.046](https://doi.org/10.1016/j.nuclphysb.2016.04.046).
- [PW81] G. PARISI and Y. S. WU. Perturbation theory without gauge fixing. *Sci. Sinica* **24**, no. 4, (1981), 483–496. [doi:10.1360/ya1981-24-4-483](https://doi.org/10.1360/ya1981-24-4-483).
- [Rej16] K. REJZNER. *Perturbative algebraic quantum field theory*. *Math. Phys. Stud.*, Springer (2016).
- [Riv91] V. RIVASSEAU. *From Perturbative to Constructive Renormalization*. Princeton University Press, 1991.
- [RV21] V. RIVASSEAU and F. VIGNES-TOURNERET. Can we make sense out of “Tensor Field Theory”? *arXiv e-prints* (2021), arXiv:2101.04970. [arXiv:2101.04970](https://arxiv.org/abs/2101.04970). [doi:10.48550/arXiv.2101.04970](https://doi.org/10.48550/arXiv.2101.04970).
- [RVT19] V. RIVASSEAU and F. VIGNES-TOURNERET. Constructive tensor field theory: The \mathbf{T}_4^4 model. *Communications in Mathematical Physics* **366**, no. 2, (2019), 567–646. [doi:10.1007/s00220-019-03369-9](https://doi.org/10.1007/s00220-019-03369-9).
- [TW18] P. TSATSOULIS and H. WEBER. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *Annales de L’Institut Henri Poincaré Section (B) Probability and Statistics* **54**, no. 3, (2018), 1204–1249. [doi:10.1214/17-AIHP837](https://doi.org/10.1214/17-AIHP837).
- [VFG22] F. C. D. VECCHI, L. FRESTA, and M. GUBINELLI. A stochastic analysis of subcritical euclidean fermionic field theories, 2022. [arXiv:2210.15047](https://arxiv.org/abs/2210.15047).
- [VGT22] F. C. D. VECCHI, M. GUBINELLI, and M. TURRA. A singular integration by parts formula for the exponential euclidean qft on the plane, 2022. [arXiv:2212.05584](https://arxiv.org/abs/2212.05584).