Involutions and Real Flexible Curves on Complex Surfaces

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RÉPUBLIQUE FRANÇAISE
Liberté Égalité Fraternité

Université de Toulouse

Doctorat de l'Université de Toulouse

préparé à l'Université Toulouse III - Paul Sabatier

Involutions et courbes flexibles réelles sur des surfaces complexes

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Thèse présentée et soutenue, le 5 juillet 2024 par

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Spécialité
Mathématiques et Applications

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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Préparé à : *L'Université Toulouse III Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *05/07/2024* **par**

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À Lucile,
Sébastien,
Momo et Lulu

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Remerciements

Je voudrais tout d'abord exprimer ma profonde gratitude à mes directeurs de thèse, Thomas et Delphine. Danke schön pour le temps, la patience et l'aide que vous m'avez accordés tout au long de cette aventure académique. Votre expertise, vos conseils avisés et votre encouragement constant m'ont été essentiels, et sans votre dévouement et votre implication, ce projet n'aurait pas été possible. Je vous en suis reconnaissant. Un grand merci aussi à vous, Séverine, pour votre accueil et votre générosité.

À ma compagne, Lucile, je tiens à exprimer toute ma reconnaissance pour les innombrables sacrifices que tu as faits et pour ta patience infinie tout au long de ces années. Ton soutien a été essentiel à la réussite de cette thèse. Malgré les nombreux défis que nous avons affrontés, tu as toujours été là pour me soutenir et m'encourager. Ta présence à mes côtés, tes encouragements et ta foi en moi m'ont donné le courage et la motivation de persévérer et d'arriver au bout. Tu as su me pousser vers le haut, même dans les moments les plus difficiles, et sans ton amour et ta compréhension, je n'aurais pas pu surmonter les défis que j'ai rencontrés.

Je souhaite aussi exprimer toute ma reconnaissance à ma famille, dont l'appui a été inestimable au cours de ces trois dernières années. À mes grands-parents, Momo et Lulu, merci pour votre amour et votre aide inconditionnelle. Votre présence et vos encouragements m'ont donné la force de passer toutes les difficultés. À mon frère, Sébastien, et à ma mère, merci pour votre présence et les moments de joie et de détente que nous avons partagés. Ces instants précieux m'ont permis de me ressourcer et de retrouver l'énergie nécessaire pour avancer. Un grand merci aussi à Annie et Francis, et à Thierry.

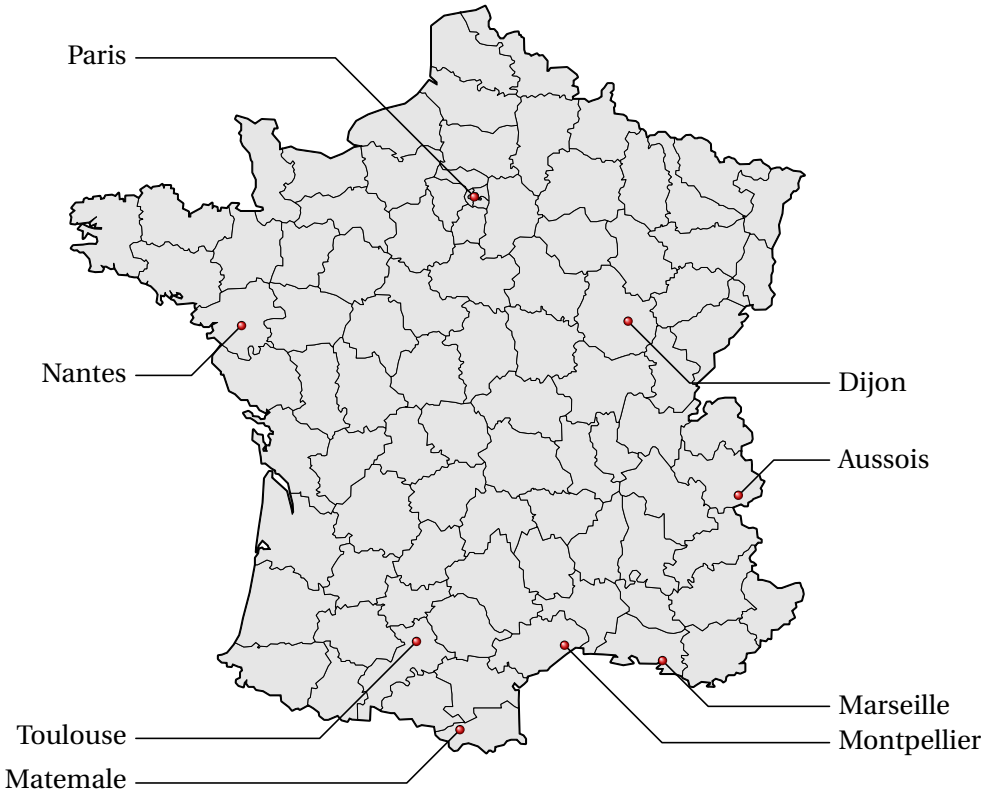
Спасибо, Олег Виро, за ваши подробные и отличные лекционные материалы по топологии вещественных алгебраических многообразий. Ваши детальные объяснения значительно способствовали моему пониманию и прогрессу в этой области. Grazie mille, Marco Golla, per aver risposto con precisione e pazienza a tutte le mie domande, anche le più banali. Le tue risposte puntuali sono state di grande aiuto durante il mio percorso di ricerca.

Je souhaite également remercier mes amis rencontrés au cours de mes années universitaires. Benjamin, Pablo, Lucie, Adrien, votre amitié a rendu ce parcours beaucoup plus agréable et enrichissant. Nos discussions et les moments que nous avons partagés ont été essentiels pour traverser les hauts et les bas de cette épreuve. Je tiens aussi à remercier chaleureusement certains de mes enseignants qui ont joué un rôle clé dans ma formation. Guillaume, Fanny, Joseph, Stéphane, Pascale, votre passion

et votre disponibilité m'ont inspiré, et vos conseils ont largement marqué mon parcours.

Je tiens à exprimer ma sincère reconnaissance à toutes les personnes de l'IMT que j'ai eu le privilège de rencontrer au fil des années. À mes co-bureaux, Florian, Adrien, Irène, vos encouragements et notre quotidien ont été cruciaux dans mon expérience. J'adresse mes remerciements les plus chaleureux à Candice, Sophia, Benjamin, Paul, William, Athmane, Florian, Bruno, Romain, Julien, Matthieu, Anthony, ainsi qu'à toutes celles et ceux que j'ai eu le plaisir de côtoyer (et que je m'excuse sincèrement de ne pas nommer ici). Votre amitié, vos conseils avisés et nos échanges enrichissants m'ont aidé à mieux affronter le quotidien que nous partagions tous. 卜天，祝你在接下来的一年顺利，我知道会有挑战，但你一定能应对。

Over the course of the past three years, I have had the chance to travel around the world, visiting numerous places around three continents and seven countries (my own, as well as the US, Canada, Italy, Norway, the UK and even Namibia). In France, I have had the opportunity to visit the following places.



I am deeply thankful to the following individuals, in no particular order, who have enriched my journey: Manu, Laura, Livio, Valentina, Gaëtan, Jenni, Edwin, Léo, Giulio, Stavroula, Manos, Filippo, Diego, Quentin, and many more that I must be forgetting. Your friendship, support, and insights have made a profound impact on my personal and professional growth during these travels.

Introduction

Involutions and Real Flexible Curves on Complex Surfaces

The origins of D. Hilbert's 16th problem date prior to its formulation at the Paris conference of the International Congress of Mathematicians of 1900. Indeed, A. Harnack found in 1876 the first breakthrough in the classification of non-singular real plane algebraic curves. He gives the complete list of possible number of connected components of such a curve in terms of its degree. That is: for any algebraic curve of degree m, its number b of connected components must satisfy the bounds

$$\frac{1+(-1)^{m+1}}{2} \leqslant b \leqslant \frac{(m-1)(m-2)}{2}+1,$$

and any such number b can be attained. This is based on a clever application of E. Bézout's famous curve theorem from 1779 (which itself was partially first stated by I. Newton in 1687).

In more modern terms, we can formulate Hilbert's 16th problem as follows. Given a homogeneous polynomial $F \in \mathbf{R}[x_0, x_1, x_2]$ such that $\nabla F \neq 0$ on $\mathbf{RP}^2$, denote as $\mathbf{R}F$ the projective curve defined by F:

$$\mathbf{R}F = \left\{ [x_0 : x_1 : x_2] \in \mathbf{RP}^2 \;\middle|\; F(x_0, x_1, x_2) = 0 \right\} \subset \mathbf{RP}^2.$$

If $m \geqslant 1$ is fixed, then what are the possible topological types of the pair $(\mathbf{RP}^2, \mathbf{R}F)$ for curves F of degree m?

Traditionally, the approach to the classification is two-fold. First, find restrictions on the possible arrangements of a fixed degree. Once sufficiently many have been shown unrealizable, try to construct the remaining ones. The case of curves of degree $1 \leqslant m \leqslant 5$ is handled entirely by the Bézout curve theorem (and Harnack's bounds). The case of degree 6 was settled by D. Gudkov in 1969, and that of degree 7 came a few years later, where the definitive answer was given by O. Viro in 1980.

Currently, the situation is still unsettled for curves of degree 8. Recent work by S. Orevkov (2002) reduces the classification of curves with the maximal number of 22 connected components to uncertainty regarding six possible arrangements.

Typically, restrictions are of topological nature. This motivated Viro to introduce flexible curves in 1984, a wider class of curves that encapsulate the most important topological features of algebraic curves.

A real curve F of degree m gives rise to its complexification $\mathbf{C}F \subset \mathbf{CP}^2$, which is a knotted surface of genus $\frac{(m-1)(m-2)}{2}$. Moreover, this surface comes with a natural orientation-reversing ambient involution, namely the complex conjugation $\mathrm{conj} : \mathbf{CP}^2 \to \mathbf{CP}^2$. Therefore, the surface $\mathbf{C}F/\mathrm{conj}$ has boundary diffeomorphic to $\mathbf{R}F$ the real curve.

There is a clear topological distinction between even degree curves and odd degree ones. Indeed, in $H_1(\mathbf{RP}^2;\mathbf{Z}/2) \cong \mathbf{Z}/2$, the homology class of the real curve is congruent to its degree mod 2. This is also true for the homology class of $\mathbf{C}F$ in $H_2(\mathbf{CP}^2;\mathbf{Z}/2) \cong \mathbf{Z}/2$. If the curve has an even degree, then it is possible to glue the part of $\mathbf{RP}^2$ which is bounded by $\mathbf{R}F$ to the surface $\mathbf{C}F/\mathrm{conj}$ and obtain a *closed* surface. This was first introduced by V. Arnold, and now bears its name: this is the Arnold surface of the curve F. By studying the normal bundle of this surface, one can derive strong restrictions, e.g. depending on the Euler characteristic of the half of $\mathbf{RP}^2$ used to close $\mathbf{C}F/\mathrm{conj}$.

The counterpart of quotienting by complex conjugation is taking the double branched cover of $\mathbf{CP}^2$, with ramification locus the surface $\mathbf{C}F$. Again, this construction is only feasible when the degree of the curve is even.

In the case of odd degree curves, Viro and V. Zvonilov considered a cyclic branched cover of $\mathbf{CP}^2$ ramified along the surface $\mathbf{C}F$. The number of sheets was a prime number that divides the degree. They managed to derive an upper bound on a certain number of the contractible components that form $\mathbf{R}F$.

More precisely, a contractible component in $\mathbf{RP}^2$ bounds a disc and a Möbius band, respectively called the interior and the exterior of that component. We set ℓ^0 and $\ell^\pm$ to be the numbers of contractible components of $\mathbf{R}F$ which bound from the outside a subset of $\mathbf{RP}^2 \smallsetminus \mathbf{R}F$ of zero, positive or negative Euler characteristic. The bound that they obtained was the following.

Theorem 3.27 ([VZ92, Theorem 2])**.** *Let F be a flexible curve of odd degree m, and let $h(m)$ denote the biggest prime power that divides m. Then:*

$$\ell^0 + \ell^- \leqslant \frac{(m-3)^2}{4} + \frac{m^2 - h(m)^2}{4h(m)^2}.$$

The lack of an Arnold surface in the odd degree case means that some restrictions coming from taking a *double* branched cover are not available.

In this work, we define an analogue of the Arnold surface in the odd degree case. A well-known result (due to Arnold, N. Kuiper and W. Massey) states that the quotient of $\mathbf{CP}^2$ by the $\mathbf{Z}/2$ action of complex conjugation is the standard 4-sphere (that is, the natural smooth structure on the quotient of $\mathbf{CP}^2 \smallsetminus \mathbf{RP}^2$ under the *free* action of conj extends to the standard one on $\mathbf{S}^4$). We consider the Fermat conic Q, which is an algebraic conic with $\mathbf{R}Q = \varnothing$ and which relates to the geometry of $(\mathbf{CP}^2, \mathrm{conj})$ in a particularly nice way. The quotient $\mathbf{C}Q/\mathrm{conj}$ is a real projective plane embedded in $\mathbf{S}^4$, and its self-intersection is opposite to that of the image of $\mathbf{RP}^2 = \mathrm{Fix}(\mathrm{conj})$ in the quotient.

The standard complex projective plane $\mathbf{CP}^2$ is the double branched cover of $\mathbf{S}^4$ ramified along $\mathbf{RP}^2/\mathrm{conj}$. We study the "dual" scenario, where we take the double branched cover of $\mathbf{S}^4$ ramified along $\mathbf{C}Q/\mathrm{conj}$. We show that this is diffeomorphic to $\overline{\mathbf{CP}}^2$, and we investigate the lift of $\mathbf{C}F/\mathrm{conj}$ to that $\overline{\mathbf{CP}}^2$. In Definition 3.17, we consider a potential analogue of the Arnold surface of an odd degree curve. This retains enough topological information about $\mathbf{R}F$ so that we can derive the following, by

methods analogous to Viro–Zvonilov's.

Theorem 3.20. *Let F be a totally flexible curve of odd degree $m = 2k+1$. Then*

$$\ell^0 + \ell^- \leqslant k^2.$$

If equality holds, then the curve is type I.

In fact, the main topic of interest for us is the topological origins of the restrictions on the possible arrangements of real plane curves. Therefore, we work in Viro's wider class of flexible curves. In our main result, *totally* flexible means that we impose an additional condition on F, namely to intersect the Fermat conic transversely and only in positive points (i.e. like an algebraic curve would generically).

We further investigate our construction for curves on real quadric surfaces (that is, complex surfaces in $\mathbf{CP}^3$ given by a real non-degenerate degree two equation). There are two such surfaces of interest: the hyperboloid quadric, and the ellipsoid.

On the hyperboloid, real curves lie on a 2-torus, and they have a *bidegree* $(a,b) \in \mathbf{N}^\star \times \mathbf{N}^\star$ rather than a degree $m \in \mathbf{N}^\star$. We show the following.

Theorem 3.31. *Let F be a totally flexible curve on (X, c_{hyp}) of bidegree (a,b) where both a and b are odd. Then:*

$$\ell^0 + \ell^- \leqslant \frac{ab+1}{2}.$$

The situation is very similar to that of plane curves in this setting. However, the novelty of our approach is that we can derive an upper bound even when $\gcd(a,b) = 1$, contrary to Zvonilov's results from 2022. In particular, it seems that even in the algebraic case, our method yields new restrictions not previously known.

For curves on the ellipsoid, the real curve lies on a 2-sphere $\mathfrak{R}$ this time. This means that the situation is slightly different.

Theorem 3.41. *Let $F \subset (X, c_{\mathrm{ell}})$ be a totally flexible curve of bidegree (m,m) with m odd, and let $\lambda^\pm$ and λ^0 denote the numbers of connected components of $\mathfrak{R} \smallsetminus \mathbf{R}F$ with positive, negative or zero Euler characteristic, respectively. Then:*

$$\lambda^0 + \lambda^- \leqslant \frac{m^2+1}{2}.$$

The homological situation differs greatly from the previous two cases. This further lays down the difficulties that one may encounter to generalize our method to *arbitrary* complex surfaces, as discussed in §3.4.4.

The benefit of only considering *double* branched covers is that one can entirely disregard the orientability of the ramification locus. This eventually lead us to digress around the new notion of non-orientable flexible curves, for which our method still works and an analogue of Theorem 3.20 is produced. In doing so, it became clear that non-orientable genus bounds are important, and we investigate an analogue of the Thom conjecture.

If $m \in \mathbf{Z}$ is an integer, consider the collection $\Sigma(m)$ of all closed, connected and non-orientable surfaces smoothly embedded in $\mathbf{CP}^2$ and whose self-intersection equals m. We consider the following non-orientable genus function:

$$\begin{array}{rccc} \widetilde{g} & : & \mathbf{Z} & \longrightarrow & \mathbf{Z}_{\leqslant 1} \\ & & m & \longmapsto & \max\limits_{F \in \Sigma(m)} \chi(F). \end{array}$$

We almost entirely compute this function.

Theorem 2.46. *Let $k \in \mathbf{N}^\star$ be a non-negative integer.*

(1) We have $\widetilde{g}(0) = 0$.

(2) Let $\ell \in \{0,1\}$ have the same parity as k. Then:

$$\widetilde{g}(-k) = 2 - \frac{k+\ell}{2}.$$

(3) On even positive integers, we have:

$$\widetilde{g}(4k) = 4 - 2k \text{ for } k \geqslant 2, \text{ and } \widetilde{g}(4k+2) = 3 - 2k,$$

together with the special values $\widetilde{g}(2) = 1$ and $\widetilde{g}(4) = 0$.

(4) On odd positive integers, we have lower bounds:

$$\widetilde{g}(4k+1) \geqslant 2 - 2k \text{ and } \widetilde{g}(4k+3) \geqslant 1 - 2k,$$

with some special values: $\widetilde{g}(1) = 0$, $\widetilde{g}(3) = 1$, $\widetilde{g}(5) = 0$, $\widetilde{g}(7) = -1$ and $\widetilde{g}(9) = -2$.

In the above case, we consider *smoothly* embedded surfaces. It is only natural to ask whether this function differs from the case of *locally flat* surfaces in $\mathbf{CP}^2$. In fact, our proof of the previous shows the following.

Proposition 2.47. *We have $\widetilde{g}_{\text{diff}} = \widetilde{g}_{\text{top}}$ on every negative integer and every non-negative even integer.*

Finally, we further digress around the topological aspects of Hilbert's 16th problem with the notion of flexible *symmetric* curves, introduced by T. Fiedler. We give a few results regarding the disposition of the curve and the elements of symmetry, and we propose a method to generalize Fiedler's refinement of a congruence of Gudkov and V. Rokhlin, holding for symmetric curves with the maximal number of connected components, to one for curves with one or two less components than that maximum.

This manuscript is organized in four major parts. The first one focuses on some well-known results in the theory of real plane algebraic curves. We review proofs of the classical results, both from the real and the complex points of view, and we finish by surveying which main results hold for flexible curves.

The second part deals with elements of the general theory of knotted surfaces in 4-manifolds. We discuss the Euler class of the normal bundle of surfaces, and we tackle the notion of double branched

covers, an important tool useful throughout this whole work. In particular, we put it into use to compute the non-orientable genus function of $\mathbf{CP}^2$.

In the third part, we state and prove our main result for curves in the plane and on quadrics. We also propose a further research track by defining conjugations on symplectic 4-manifolds and by investigating our construction in this setting.

The fourth and final part is dedicated to both digressions of non-orientable flexible curves and of flexible symmetric curves.

In an appendix, we provide some computer code that was extensively used in investigating examples and constructions, as well as example plots. We also propose our source code for studying the geometry of $(\mathbf{CP}^2, \mathrm{conj})$ and of real algebraic curves.

Most of the work from this thesis has been published in article form at the Pacific Journal of Mathematics in [Sai24].

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Involutions et courbes flexibles réelles sur des surfaces complexes

Le seizième problème de Hilbert prend ses origines avant la conférence tenue par D. Hilbert à Paris en 1900 à l'occasion du Congrès International de Mathématiques. En effet, A. Harnack découvrit en 1876 la première avancée dans la classification des courbes algébriques réelles planes sans singularité. Il donna la liste complète du nombre de composantes connexes d'une telle courbe en fonction de son degré. Plus précisément, pour toute telle courbe algébrique de degré m, son nombre b de composantes connexes satisfait les bornes

$$\frac{1+(-1)^{m+1}}{2} \leqslant b \leqslant \frac{(m-1)(m-2)}{2} + 1,$$

et chacun de ces nombres b est atteint. Cela se base sur une application astucieuse du célèbre théorème de E. Bézout de 1779 concernant les courbes (ce résultat ayant également été partiellement formulé par I. Newton en 1687).

En termes plus modernes, nous formulons le 16ème problème de Hilbert comme suit. Étant donné un polynôme homogène $F \in \mathbf{R}[x_0, x_1, x_2]$ tel que $\nabla F \neq 0$ sur $\mathbf{RP}^2$, nous notons $\mathbf{R}F$ la courbe projective définie par F, à savoir

$$\mathbf{R}F = \left\{ [x_0 : x_1 : x_2] \in \mathbf{RP}^2 \;\middle|\; F(x_0, x_1, x_2) = 0 \right\}.$$

Si $m \geqslant 1$ est fixé, quels sont les types topologiques possibles pour la paire $(\mathbf{RP}^2, \mathbf{R}F)$ étant données des courbes F de degré m ?

Typiquement, l'approche à la classification se fait en deux temps. D'abord, il s'agit de trouver des restrictions sur les arrangements possibles en degré fixé. Ensuite, une fois que suffisamment de configurations sont connues irréalisables, il est question d'essayer de construire les restantes. Le cas des courbes de degré $1 \leqslant m \leqslant 5$ est entièrement traité par le théorème de Bézout sur les courbes (ainsi que la borne de Harnack). Le cas des courbes de degré 6 a été conclu par D. Gudkov en 1969, et celui du degré 7 vint quelques années ensuite, avec les travaux de O. Viro en 1980.

Aujourd'hui, la situation est toujours indéterminée pour les courbes de degré 8. Des travaux récents de S. Orevkov (2002) ont réduit la classification des courbes ayant le nombre maximal de 22 composantes connexes à la réalisabilité de six arrangements.

Traditionnellement, les restrictions obtenues sont de nature topologique. Ceci motiva Viro à introduire la notion de courbe flexible en 1984, une classe plus large d'objets qui capture les propriétés topologiques majeurs d'une courbe algébrique.

Une courbe réelle F de degré m donne lieu à sa complexification $\mathbf{C}F \subset \mathbf{CP}^2$, qui est une surface nouée de genre $(m-1)(m-2)/2$. De plus, cette surface arrive avec une involution ambiante naturelle qui renverse l'orientation, à savoir la conjugaison complexe $\mathrm{conj} : \mathbf{CP}^2 \to \mathbf{CP}^2$. Ainsi, la surface $\mathbf{C}F/\mathrm{conj}$ est à bord difféomorphe à $\mathbf{R}F$ la courbe réelle.

Il y a une distinction topologique claire entre les courbes de degré pair et celles de degré impair. En effet, la classe d'homologie de la courbe dans $H_1(\mathbf{RP}^2;\mathbf{Z}/2) \cong \mathbf{Z}/2$ est congruente à son degré modulo 2. Ceci reste également vrai pour la classe d'homologie décrite par $\mathbf{C}F$ dans $H_2(\mathbf{CP}^2;\mathbf{Z}/2)$. Si la courbe a un degré pair, alors il est possible de recoller une partie de $\mathbf{RP}^2$ bordant $\mathbf{R}F$ à $\mathbf{C}F/\mathrm{conj}$ afin d'obtenir une surface *fermée*. Cette construction a été introduite pour la première fois par V. Arnold, et porte désormais son nom : la surface d'Arnold de la courbe F. En étudiant le fibré normal de cette surface, nous pouvons déduire des restrictions fortes, par exemple sur la caractéristique d'Euler de la moitié de $\mathbf{RP}^2$ utilisée pour recoller à $\mathbf{C}F/\mathrm{conj}$.

La construction opposée à quotienter par la conjugaison complexe est de considérer le revêtement double de $\mathbf{CP}^2$ ramifié le long de $\mathbf{C}F$. De même, cette construction n'est possible que lorsque le degré de F est pair.

Dans le cas des courbes de degré impair, Viro et V. Zvonilov ont considéré un revêtement cyclique de $\mathbf{CP}^2$ ramifié le long de la surface $\mathbf{C}F$. Le nombre de feuillets était un nombre premier divisant le degré. Ils ont ainsi pu obtenir une borne supérieure sur un certain nombre de composantes contractiles formant $\mathbf{R}F$.

Plus précisément, une composante contractile borde un disque et une bande de Möbius, respectivement appelés son intérieur et son extérieur. Nous notons ℓ^0 et $\ell^{\pm}$ les nombres de composantes contractiles qui bordent de l'extérieur une composante de $\mathbf{RP}^2 \smallsetminus \mathbf{R}F$ dont la caractéristique d'Euler est nulle, positive ou négative. La borne qu'ils ont alors obtenue est la suivante.

Théorème 3.27 ([VZ92, Theorem 2])**.** *Soit F une courbe flexible de degré impair m, et soit $h(m)$ la plus grande puissance d'un nombre premier divisant m. Alors :*

$$\ell^0 + \ell^- \leqslant \frac{(m-3)^2}{4} + \frac{m^2 - h(m)^2}{4h(m)^2}.$$

L'absence de surface d'Arnold en degré impair signifie que bon nombre de restrictions obtenues en considérant un revêtement ramifié *double* ne sont pas accessibles.

Dans ce travail, nous définissos un analogue de la surface d'Arnold en degré impair. Un résultat bien connu (dû à Arnold, N. Kuiper et W. Massey) stipule que le quotient de $\mathbf{CP}^2$ par la $\mathbf{Z}/2$-action de la conjugaison complexe est la 4-sphère standard (à savoir que le quotient de $\mathbf{CP}^2 \smallsetminus \mathbf{RP}^2$ par l'action *libre* de conj admet une structure lisse naturelle qui se prolonge en celle standard sur $\mathbf{S}^4$). Nous considérons la conique de Fermat Q, qui est une conique algébrique avec $\mathbf{R}Q = \varnothing$, et qui se relie particulièrement

bien à la géométrie de $(\mathbf{CP}^2, \mathrm{conj})$. Le quotient $\mathbf{C}Q/\mathrm{conj}$ est un plan projectif réel plongé dans $\mathbf{S}^4$, et dont le nombre d'auto-intersection est l'opposé de celui de l'image de $\mathbf{RP}^2 = \mathrm{Fix}(\mathrm{conj})$ dans le quotient.

Le plan projectif complexe usuel $\mathbf{CP}^2$ est donc le revêtement double de $\mathbf{S}^4$ ramifié le long de $\mathbf{RP}^2/\mathrm{conj}$. Nous étudions le scénario "dual", où nous prenons le revêtement double de $\mathbf{S}^4$ ramifié le long de $\mathbf{C}Q/\mathrm{conj}$. Nous montrons que cette 4-variété est difféomorphe à $\overline{\mathbf{CP}}^2$, et nous étudions le relevé de $\mathbf{C}F/\mathrm{conj}$ à ce $\overline{\mathbf{CP}}^2$. Dans la Définition 3.17, nous considérons un analogue potentiel de la surface d'Arnold pour une courbe de degré impair. Celle-ci préserve suffisamment d'information topologique à propos de $\mathbf{R}F$ pour que nous puissions obtenir le résultat suivant, par des méthodes similaires à celle de Viro–Zvonilov.

Théorème 3.20. *Soit F une courbe totalement flexible de degré impair $m = 2k+1$. Alors :*

$$\ell^0 + \ell^- \leqslant k^2.$$

De plus, en cas d'égalité, la courbe est de type I.

En réalité, le sujet qui nous intéresse est celui des origines topologiques des restrictions sur les arrangements des courbes planes. Ainsi, nous étudions la classe de Viro des courbes flexibles. Dans notre résultat principal, une courbe *totalement* flexible signifie que nous ajoutons une condition supplémentaire sur F, à savoir d'intersecter la conique de Fermat de manière transverse et uniquement en des points positifs (*i.e.* comme une courbe algébrique le ferait génériquement).

Nous observons également notre construction pour des courbes sur des surfaces quadriques réelles (c'est-à-dire des surfaces complexes dans $\mathbf{CP}^3$ données par des équations réelles non dégénérées de degré 2). Il y a deux telles surfaces qui nous intéressent : la quadrique hyperboloïde, et la quadrique ellipsoïde.

Sur l'hyperboloïde, une courbe réelle se situe à la surface d'un 2-tore, et possède un *bi-degré* $(a,b) \in \mathbf{N}^\star \times \mathbf{N}^\star$ au lieu d'un degré $m \in \mathbf{N}^\star$. Nous montrons le résultat suivant.

Théorème 3.31. *Soit F une courbe flexible sur (X, c_{hyp}) de bi-degré (a,b) avec a et b tous les deux impairs. Alors :*

$$\ell^0 + \ell^- \leqslant \frac{ab+1}{2}.$$

La situation est très similaire à celle des courbes planes. En revanche, la nouveauté de cette approche est que nous arrivons à obtenir une borne indépendamment d'une condition de type $\mathrm{PGCD}(a,b) = 1$, contrairement à des travaux récents de Zvonilov (2022). En particulier, il semble que même dans le cas des courbes algébriques, notre méthode produise une nouvelle restriction non connue auparavant.

Pour les courbes sur l'ellipsoïde, les courbes réelles vivent à la surface d'une 2-sphère $\mathfrak{R}$. Cela signifie que la situation est légèrement différente.

Théorème 3.41. *Soit $F \subset (X, c_{\mathrm{ell}})$ une courbe totalement flexible de bi-degré (m,m) avec m impair, et soient $\lambda^\pm$ et λ^0 les nombres de composantes connexes de $\mathfrak{R} \smallsetminus \mathbf{R}F$ de caractéristique d'Euler nulle,*

positive ou négative. Alors :

$$\lambda^0 + \lambda^- \leqslant \frac{m^2+1}{2}.$$

Cependant, la situation homologique est très différente des deux cas précédents. Ceci expose d'autant plus les difficultés qui pourraient se présenter si l'on souhaitait généraliser notre approche à des surfaces complexes *arbitraires*, comme discuté en §3.4.4.

L'avantage de ne considérer que des revêtements ramifiés *doubles* est que l'on peut complètement ignorer l'orientabilité du lieu de ramification. Cela nous amène à digresser autour de la nouvelle notion de courbes flexibles non-orientables, pour lesquelles notre méthode fonctionne toujours et un résultat semblable au Théorème 3.20 est obtenu. Ce faisant, il apparaît clairement que des bornes sur le genre de surfaces non-orientables sont importantes, et nous analysons un analogue de la conjecture de Thom.

Si $m \in \mathbf{Z}$ est un entier relatif, soit $\Sigma(m)$ la collection de toutes les surfaces fermées, connexes et non-orientables plongées de manière lisse dans $\mathbf{CP}^2$ et dont l'auto-intersection est égale à m. Nous définissons la fonction de genre non-orientable suivante :

$$\begin{array}{rccc} \widetilde{g} & : & \mathbf{Z} & \longrightarrow & \mathbf{Z}_{\leqslant 1} \\ & & m & \longmapsto & \max\limits_{F\in\Sigma(m)} \chi(F). \end{array}$$

Nous calculons quasi-entièrement cette fonction.

Théorème 2.46. *Soit $k \in \mathbf{N}^\star$ un entier strictement positif.*

(1) Nous avons $\widetilde{g}(0) = 0$.

(2) Soit $\ell \in \{0,1\}$ ayant la même parité que k. Alors :

$$\widetilde{g}(-k) = 2 - \frac{k+\ell}{2}.$$

(3) Sur les entiers pairs positifs, nous avons :

$$\widetilde{g}(4k) = 4-2k \text{ pour } k \geqslant 2, \text{ et } \widetilde{g}(4k+2) = 3-2k,$$

ainsi que les valeurs spéciales $\widetilde{g}(2) = 1$ et $\widetilde{g}(4) = 0$.

(4) Sur les entiers impairs positifs, nous avons les bornes inférieures

$$\widetilde{g}(4k+1) \geqslant 2-2k \text{ et } \widetilde{g}(4k+3) \geqslant 1-2k,$$

ainsi que certaines valeurs spéciales : $\widetilde{g}(1) = 0$, $\widetilde{g}(3) = 1$, $\widetilde{g}(5) = 0$, $\widetilde{g}(7) = -1$ et $\widetilde{g}(9) = -2$.

Dans le cas précédent, nous considérons les surfaces plongées de manière *lisse*. Il est alors naturel de se poser la question de savoir si cette fonction diffère dans le cas des surfaces *localement plates* dans $\mathbf{CP}^2$. En réalité, notre preuve du résultat ci-dessus montre le fait suivant.

Proposition 2.47. *Nous avons $\widetilde{g}_{\text{diff}} = \widetilde{g}_{\text{top}}$ sur tous les entiers négatifs et tous les entiers positifs pairs.*

Enfin, nous digressons davantage autour des aspects topologiques du seizième problème de Hilbert avec la notion de courbe flexible *symétrique*, introduite par T. Fiedler. Nous énonçons quelques résultats à propos de la disposition d'une telle courbe vis-à-vis des éléments de symétrie, et nous proposons une méthode pour généraliser le raffinement de Fiedler d'une congruence de Gudkov et V. Rokhlin, vérifiée par les courbes ayant le nombre maximal de composantes connexes, en une congruence pour les courbes symétriques ayant une ou deux composantes de moins que ce maximum.

Ce manuscrit s'articule en quatre parties majeures. La première se focalise sur des résultats bien connus en théorie des courbes planes réelles. Nous étudions des preuves des résultats classiques, tant du point de vue réel que du point de vue complexe, et nous finissons par donner un aperçu des résultats principaux qui subsistent pour les courbes flexibles.

La deuxième partie traite des éléments de la théorie générale des surfaces nouées dans les 4-variétés. Nous discutons de la classe d'Euler du fibré normal de telles surfaces, et nous considérons la notion de revêtement double ramifié, un outil important qui sera utile tout au long de ce travail. En particulier, nous mettons à l'usage cette théorie pour calculer la fonction de genre non-orientable de $\mathbf{CP}^2$.

Dans une troisième partie, nous énonçons et démontrons notre résultat principal à propos des courbes planes et des courbes sur les quadriques. Nous proposons également une méthode pour de la recherche future en définissant les conjugaisons sur des 4-variétés symplectiques, et en analysant notre construction dans ce cadre.

La quatrième et dernière partie se consacre aux deux digressions conernant les courbes flexibles non-orientables et les courbes flexibles symétriques.

Dans un appendice, nous fournissons des extraits de code informatique qui a été utilisé tout au long de ce travail pour étudier exemples et constructions, ainsi que des exemples de graphes obtenus par ce biais. Nous proposons également notre code source permettant d'étudier la géométrie de $(\mathbf{CP}^2, \mathrm{conj})$ et des courbes algébriques réelles.

La plupart des résultats de ce manuscrit ont été publiés sous la forme d'un article paru au journal Pacific Journal of Mathematics dans [Sai24].

Notations

$\mathbf{N}$, $\mathbf{N}^\star$	Integers $\{0,1,2,\dots\}$, non-zero integers $\{1,2,3,\dots\}$.
$a \equiv b\ [n]$	The integer a is congruent to b mod n.
$\mathbf{i}$	The complex unit: $\mathbf{i}^2 = -1$.
$\mathbf{RP}^2$, $\mathbf{CP}^2$	Real and complex projective planes, respectively.
$K\ell$	Klein bottle.
$\Sigma_{g,b}$, Σ_g	Compact orientable connected surface of genus g and b boundary component; $\Sigma_g = \Sigma_{g,0}$.
#	Connected sum operation.
$\pitchfork$	Transverse intersection.
$\cong$	Diffeomorphism (if manifolds) or isomorphism (if groups or rings).
$b_k(X)$	k-th Betti number of X: $b_k(X) = \operatorname{rank}_{\mathbf{Z}} H_k(X;\mathbf{Z})$.
$b_k(X;\mathbf{Z}/2)$	k-th mod 2 Betti number of X: $b_k(X;\mathbf{Z}/2) = \operatorname{rank}_{\mathbf{Z}/2} H_k(X;\mathbf{Z}/2)$.
$\chi(X)$	Euler characteristic of X: $\chi(X) = b_0(X) - b_1(X) + \cdots = b_0(X;\mathbf{Z}/2) - b_1(X;\mathbf{Z}/2) + \cdots$.
$b_*(X)$	Total Betti number: $b_*(X) = b_0(X) + b_1(X) + \cdots$.
$b_*(X;\mathbf{Z}/2)$	Total mod 2 Betti number: $b_*(X;\mathbf{Z}/2) = b_0(X;\mathbf{Z}/2) + b_1(X;\mathbf{Z}/2) + \cdots$.

Hilbert's 16th Problem

1

Was eine Kurve ist, glaubt jeder Mensch zu wissen, bis er so viel Mathematik gelernt hat, daß ihn die unzähligen möglichen Abnormitäten verwirrt gemacht haben.[†]

F. Klein

Outline of this chapter

[†] *Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.*

1.1 Non-singular Real Plane Algebraic Curves

In this section, we will survey the general theory of real plane algebraic curves. More specifically, we will only focus on the *real* approach to the problem. We re-expose Harnack's proof of his bounds, which are derived from the Bézout theorem. This is enough to fully classify the arrangements of the real part of curves of degree up to 5.

To this end, we will have to make a necessary detour into the world of *singular curves* a little. One of the effects this has is the enabling of construction techniques, mainly by perturbing singular curves into smooth ones.

In a sense, the only tool necessary to achieve the classification in low degree is the Bézout theorem. However, as anyone could guess, this will not suffice in higher degrees. Therefore, the goal of this section is also to realize that there might be a more appropriate framework to attack the problem.

Oleg Viro's lecture notes [Vir00] and survey [Vir84] were a real source of inspiration in the writing of this section, as well as Wilson's survey [Wil78].

1.1.1 Basic Definitions and First Results

Given an integer $m \in \mathbf{N}^\star$, a **real plane algebraic curve** (or plane curve, for short) is a three-variable homogeneous polynomial $A \in \mathbf{R}[x_0, x_1, x_2]$ of degree m, regarded up to a multiplicative constant. We can therefore consider the unambiguously defined subset

$$\mathbf{R}A = \left\{ [x_0 : x_1 : x_2] \in \mathbf{RP}^2 \;\middle|\; A(x_0, x_1, x_2) = 0 \right\} \subset \mathbf{RP}^2$$

of the real projective plane. We shall also call $\mathbf{R}A$ the real plane curve. A point $[x_0 : x_1 : x_2] \in \mathbf{RP}^2$ is said to be a **singular point** for A if $\nabla A(x_0, x_1, x_2) = 0$. The plane curve is said to be **non-singular** if it has no singular points.

In the case of a non-singular plane curve A, the implicit function theorem[†] ensures that $\mathbf{R}A$ is a smooth 1-submanifold of $\mathbf{RP}^2$. In this case, this means that $\mathbf{R}A$ is a collection of embedded circles.

To represent a curve, we view $\mathbf{RP}^2$ as a disc whose boundary points are identified pairwise by symmetry along the center of that disc. In Figure 1.1, we depict such an example.

One of the fundamental results about algebraic curves is the so-called Bézout theorem.

Theorem 1.1. *Let A and B be two non-singular real plane algebraic curves of respective degrees m_1 and m_2, and such that $\mathbf{R}A \cap \mathbf{R}B$ is finite. Then $\mathbf{R}A \cap \mathbf{R}B$ contains at most $m_1 \times m_2$ points. If moreover $\mathbf{R}A \pitchfork \mathbf{R}B$, then:*

$$\#\mathbf{R}A \pitchfork \mathbf{R}B \equiv m_1 \times m_2 \;\; [2].$$

Note that if $\mathbf{R}A$ does not meet $\mathbf{R}B$ transversely, then it is always possible to find a small perturbation B' of B such that $\mathbf{R}A \pitchfork \mathbf{R}B'$. By a small perturbation, we mean that for any regular neighborhood $\mathcal{U} \subset \mathbf{RP}^2$ of $\mathbf{R}B$, there exists $\varepsilon \in \mathbf{R}[x_0, x_1, x_2]$ homogeneous of the same degree as that of B, such that

[†] The $\mathbf{R}$-valued map $\hat{A} : \mathbf{RP}^2 \to \mathbf{R}$ is ill-defined; only its vanishing locus makes sense. Therefore, one needs to look at $\mathbf{R}A$ inside affine charts.

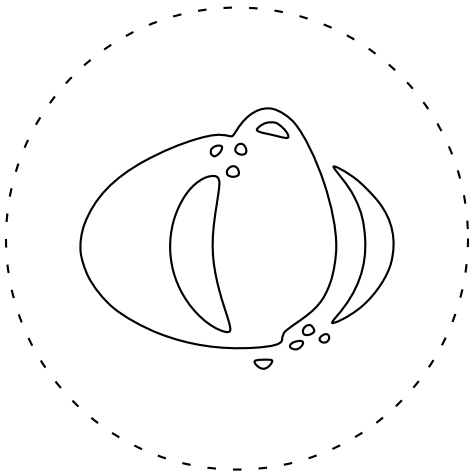


Figure 1.1. An example of a non-singular real plane algebraic curve of degree 6.

$B' = B + \varepsilon$ is a non-singular curve and $\mathbf{R}B' \subset \mathcal{U}$. However, the numbers of intersection points in $\mathbf{R}A \cap \mathbf{R}B$ or $\mathbf{R}A \pitchfork \mathbf{R}B'$ need not agree in general.

The previous result allows us to compute the homology class of a real curve in $H_1(\mathbf{RP}^2;\mathbf{Z})$ depending on the degree. Recalling that $H_1(\mathbf{RP}^2;\mathbf{Z}) \cong \mathbf{Z}/2$, any non-zero element is also a generator of the homology group. We call such a generator a **pseudo-line**.

Proposition 1.2. *Let A be a non-singular curve of degree m. Then $[\mathbf{R}A] = 0 \in H_1(\mathbf{RP}^2;\mathbf{Z})$ if and only if m is even, in which case all the connected components of $\mathbf{R}A$ are contractible in $\mathbf{RP}^2$. Moreover, if m is odd, then $\mathbf{R}A$ has one and exactly one of its connected components that is a pseudo-line, the others being contractible.*

Proof. Let $\mathcal{J} = \mathbf{R}B$ be a pseudo-line which is realized by a non-singular curve B of degree 1 and such that $\mathbf{R}A \pitchfork \mathcal{J}$. The Bézout theorem therefore ensures that

$$\#\mathbf{R}A \pitchfork \mathcal{J} \equiv m \ [2].$$

It happens that the intersection form

$$q : H_1(\mathbf{RP}^2;\mathbf{Z}/2) \times H_1(\mathbf{RP}^2;\mathbf{Z}/2) \to \mathbf{Z}/2$$

is such that if $\xi \in H_1(\mathbf{RP}^2;\mathbf{Z})$, then $\xi = 0 \iff q(\xi, [\mathcal{J}]) = 0$. This implies the claim that

$$[\mathbf{R}A] = 0 \iff m \in 2\mathbf{N},$$

and that in this case the set $\mathbf{R}A$ is contractible as a whole (it may also be empty). In the case where the degree is odd, note that there is an odd number of pseudo-lines in $\mathbf{R}A$ (because $[\mathbf{R}A] \neq 0$), but there cannot be more than one, since two pseudo-lines always intersect, and $\mathbf{R}A$ is a submanifold of $\mathbf{RP}^2$. ∎

We call a contractible circle in $\mathbf{RP}^2$ an **oval**. For instance, the curve of Figure 1.1 has 11 ovals and no pseudo-line.

Hilbert asked the following question in his famous 1900 list.

Question 1.3. *If a degree m is fixed, what are the possible relative positions of each individual components of a non-singular curve of degree m?*

In more modern terms, Hilbert's 16th problem asks about a classification of the topological types of the pairs[†] $(\mathbf{RP}^2, \mathbf{R}A)$ where A is a non-singular curve of degree m. To this day, this is still a very much open question, and the complete classification is only fully known in degrees up to 7. There are two steps to proceed:

(1) find restrictions, and
(2) try to construct what was not obstructed.

Proposition 1.2 is already a restriction in itself:

$$b_0(\mathbf{R}A) \geqslant \frac{1+(-1)^{m+1}}{2}.$$

Indeed, an empty collection cannot be realized by an odd degree curve! Of course, this bound is sharp, as one can check that the curve

$$A(x_0, x_1, x_2) = x_0^m + x_1^m + x_2^m$$

realizes $b_0(\mathbf{R}A) = 0$ or 1.

It may be more convenient to start introducing some more terminology. Note that a pseudo-line is one-sided and an oval is two-sided, meaning that their complement in $\mathbf{RP}^2$ has one or two connected components respectively. In the case of an oval $o \subset \mathbf{RP}^2$, we obtain that $\mathbf{RP}^2 \smallsetminus o$ has one orientable component, diffeomorphic to an open disc, and one non-orientable one, diffeomorphic to an open Möbius band. We call the disc the **interior** of o, denoted as $\mathrm{Int}(o)$, and the Möbius band its **exterior**.

Two disjoint ovals are said to be **nested** if one of them is included in the interior of the other. For instance, in Figure 1.1, there is one oval surrounding 5 others, which also has 5 ovals in its exterior not making nested pairs. A **nest** of ovals is a collection $o_1, \dots, o_d \subset \mathbf{RP}^2$ with $d \geqslant 2$ of disjoint ovals such that o_{i+1} is in the interior of o_i for all $1 \leqslant i \leqslant d-1$. The integer d is called the **depth** of the nest.

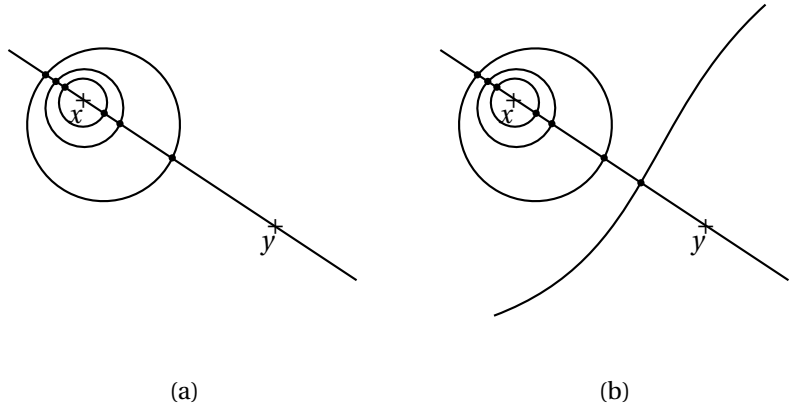
The Bézout theorem is a useful tool to rule out certain arrangements to be realizable by an algebraic curve.

Proposition 1.4. *Let A be a non-singular curve of degree m. If $\mathbf{R}A$ contains a nest of ovals, then its depth cannot exceed $\left\lfloor \frac{m}{2} \right\rfloor$. Moreover, if there is a nest with that depth, then there is no other oval in $\mathbf{R}A$.*

Proof. Assume that there is a nest of depth d. Pick a point $x \in \mathbf{RP}^2$ in the interior of the deepest oval, and pick a point $y \in \mathbf{RP}^2$ outside the outermost one (see Figure 1.2). Those points are necessarily distinct, and as such there is a unique line going through both. This defines an algebraic curve L of

[†] By the topological type of a pair $(\mathbf{RP}^2, \mathbf{R}A)$, we mean the isotopy type of $\mathbf{R}A$ inside $\mathbf{RP}^2$, or equivalently, the homeomorphism type of $\mathbf{RP}^2 \smallsetminus \mathbf{R}A$.

degree one with $\mathbf{R}A \pitchfork \mathbf{R}L$ finite (up to taking a small perturbation of L, which will not change the final argument).



(a) (b)

Figure 1.2. Showing the "deep nest property". (a) An even degree curve with only ovals. (b) An odd degree curve with a pseudo-line.

The Bézout theorem means that $\mathbf{R}A \pitchfork \mathbf{R}L$ has at most m points. However, depending on whether $\mathbf{R}A$ contains a pseudo-line or not, we see that, by construction, $\mathbf{R}A \pitchfork \mathbf{R}L$ contains at least $2d$ or $2d+1$ points. This implies the inequality.

In the case where there is a nest of depth $d = \lfloor \frac{m}{2} \rfloor$, then assume by contradiction that there is another oval somewhere. Pick a point z inside that other oval, consider the line joining x and z. This line must cross $\mathbf{R}A$ in at least $2d+2$ points, but this contradicts Bézout's $2d+2 \leqslant m$. ■

There is an importance, however, in fixing the degree in Question 1.3. Indeed, it is easy to show the following (which also has the consequence that a curve with a maximal nest as in Proposition 1.4 always exists in every degree).

Proposition 1.5. *Let $\mathscr{C} \subset \mathbf{RP}^2$ be a collection of embedded circles.*

(1) If $\mathscr{C}$ only consists of ovals, then there exists a non-singular curve A of degree $m = 2b_0(\mathscr{C})$ such that $\mathbf{R}A$ and $\mathscr{C}$ are isotopic in $\mathbf{RP}^2$.

(2) If $\mathscr{C}$ has one pseudo-line, then there exists a non-singular curve A of degree $m = 2b_0(\mathscr{C}) - 1$ such that $\mathbf{R}A$ and $\mathscr{C}$ are isotopic.

Proof. In the case where $\mathscr{C}$ contains only contractible components, we search for a curve of the form

$$A(x_0, x_1, x_2) = \prod_{i=1}^{b_0(\mathscr{C})} \left[(x_0 - a_i)^2 + (x_1 - b_i)^2 - R_i^2 x_2 \right].$$

That is, we consider the curve which is built by the union of the circles of centers the (a_i, b_i) and radii the R_i. This will be singular, but taking a small perturbation resolves that issue, and one can ensure that nesting is respected as in $\mathscr{C}$ by choosing the centers and radii of each individual circle.

In the case where $\mathscr{C}$ contains a pseudo-line, it suffices to pick a curve defined by a union of circles together with a real line disjoint from them. Again, up to a small perturbation, this will be non-singular and with prescribed topological type. ■

For instance, the topological type described in Figure 1.1 contains 11 ovals, and therefore there exists a curve of degree 22 which realizes it. However, as we will see later, this arrangement is also realizable in degree 6. Therefore, Question 1.3 can be reformulated into the following equivalent problem.

Question 1.6. *Given any collection of embedded circles $\mathscr{C} \subset \mathbf{RP}^2$, what is the minimum $m \in \mathbf{N}^\star$ such that there exists a non-singular curve A of degree m and with $(\mathbf{RP}^2, \mathbf{R}A)$ having the same topological type as $(\mathbf{RP}^2, \mathscr{C})$?*

We call the topological type of $(\mathbf{RP}^2, \mathbf{R}A)$ associated to an algebraic curve A its **real scheme**. We are now interested in classifying all real schemes of A when $\deg(A) \leqslant 5$. A refined version of the Bézout theorem (that accounts for the algebraic properties of polynomial equations) will be useful. Assume that A and B are two curves such that $\mathbf{R}A \cap \mathbf{R}B$ is finite. Let $x \in \mathbf{R}A \cap \mathbf{R}B$. There are two possibilities.

(1) The intersection $\mathbf{R}A \cap \mathbf{R}B$ is transverse at x; that is, there is a small neighborhood $x \in \mathcal{U}$ such that $(\mathcal{U} \cap \mathbf{R}A) \pitchfork (\mathcal{U} \cap \mathbf{R}B) = \{x\}$. We then say it has multiplicity 1.
(2) The intersection is not transverse, and we say that it has multiplicity $\geqslant 2$.

We will detail slightly how to define multiplicities in §1.1.3. The underlying idea is that it should mimick the one-dimensional phenomenon of multiple roots of polynomials. We obtain the following.

Theorem 1.7. *Let A and B be curves of respective degrees m_1 and m_2 such that $\mathbf{R}A \cap \mathbf{R}B$ is finite. Then $\#(\mathbf{R}A \cap \mathbf{R}B) \leqslant m_1 \times m_2$, with intersection points in $\mathbf{R}A \cap \mathbf{R}B$ counted with multiplicites. More specifically, if m_x denotes the multiplicity of $x \in \mathbf{R}A \cap \mathbf{R}B$, then:*

$$\sum_{x \in \mathbf{R}A \cap \mathbf{R}B} m_x \leqslant m_1 \times m_2.$$

This allows to prove one of the earliest restrictions about algebraic plane curves, which is due to Harnack. We present his original proof from [Har76].

Theorem 1.8. *Let A be a non-singular plane curve of degree m. Then:*

$$\frac{1 + (-1)^{m+1}}{2} \leqslant b_0(\mathbf{R}A) \leqslant \frac{(m-1)(m-2)}{2} + 1.$$

Moreover, given any b respecting those bounds, there exists a degree m curve A with $b_0(\mathbf{R}A) = b$.

Proof. The lower bound was already computed. We will not prove the existence part of the theorem just yet; this will come in §1.1.4.

We set $g = (m-1)(m-2)/2$, and we assume by contradiction that there exists a degree m curve A with $b_0(\mathbf{R}A) \geqslant g+2$. Then, at least $g+1$ of those components of $\mathbf{R}A$ are ovals, which we denote as $o_1, \dots, o_{g+1}$. Pick a point x_i lying *on* each oval o_i, and pick $m-3$ points $y_1, \dots, y_{m-3}$ on any remaining $(g+2)^{\text{nd}}$ component. Note that we need to assume $m \geqslant 3$, but the cases $m \leqslant 2$ can be dealt with by

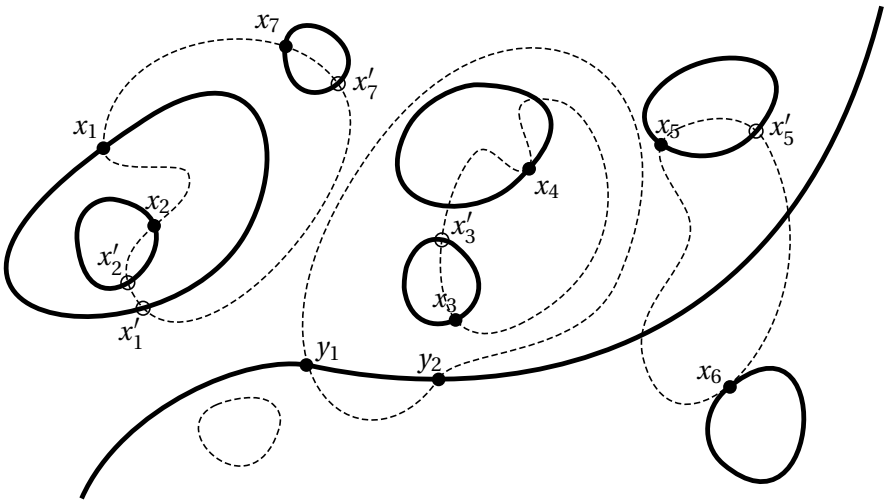


Figure 1.3. In bold, the curve $\mathbf{R}A$, and in dashed line, the curve $\mathbf{R}B$. The points x_4 and x_6 have multiplicities at least 2.

hand using the Bézout theorem and a well-chosen line. We refer the reader to Figure 1.3 for a visual reference.

It is a known fact[†] that (given $d \geqslant 1$): through $\binom{d+2}{2} - 1 = \frac{(d+2)(d+1)}{2} - 1$ points there always passes a degree d curve. From

$$\#\{x_1, \dots, x_{g+1}\} \cup \{y_1, \dots, y_{m-3}\} = \frac{(m-1)(m-2)}{2} + 1 + m - 3 = \frac{m(m-1)}{2} - 1 = \binom{m}{2} - 1,$$

we see that there exists a degree $m-2$ curve B with $\mathbf{R}B$ passing though all the points x_i and y_j. Moreover, by a small perturbation of B (which can be realized by moving the points x_i and y_j slightly while remaining on their respective components), we can assume that $\mathbf{R}A \cap \mathbf{R}B$ is finite. The Bézout theorem therefore provides:

$$\sum_{x \in \mathbf{R}A \cap \mathbf{R}B} m_x \leqslant m(m-2).$$

It now suffices to find a lower-bound for the term on the left.

There are two possibilities for an intersection point x_i.

(1) The intersection $\mathbf{R}A \cap \mathbf{R}B$ is not transverse at x_i, which means that $m_{x_i} \geqslant 2$.

(2) This intersection *is* transverse. But the oval o_i being closed, because $\mathbf{R}B$ enters its interior, it also has to exit it at another point x'_i.

This means that

$$\sum_{x \in \mathbf{R}A \cap \mathbf{R}B} m_x \geqslant \sum_{\substack{i=1 \\ \mathbf{R}A \pitchfork \mathbf{R}B \text{ at } x_i}}^{g+1} \underbrace{\left(m_{x_i} + m_{x'_i}\right)}_{=2} + \sum_{\substack{i=1 \\ \mathbf{R}A \not\pitchfork \mathbf{R}B \text{ at } x_i}}^{g+1} \underbrace{m_{x_i}}_{\geqslant 2} + \sum_{j=1}^{m-3} \underbrace{m_{y_j}}_{\geqslant 1} \geqslant 2(g+1) + (m-3) = (m-1)^2.$$

[†] Which will be proved in §1.1.3

This yields $(m-1)^2 \leqslant m(m-2)$, which is a contradiction! ■

Now, Proposition 1.4 and Theorem 1.8 are sufficient to classify all curves of degrees less than 5. In essence, this means that the classification is entirely a consequence of Bézout's theorem. The classification is given in Figure 1.4 and Figure 1.5.

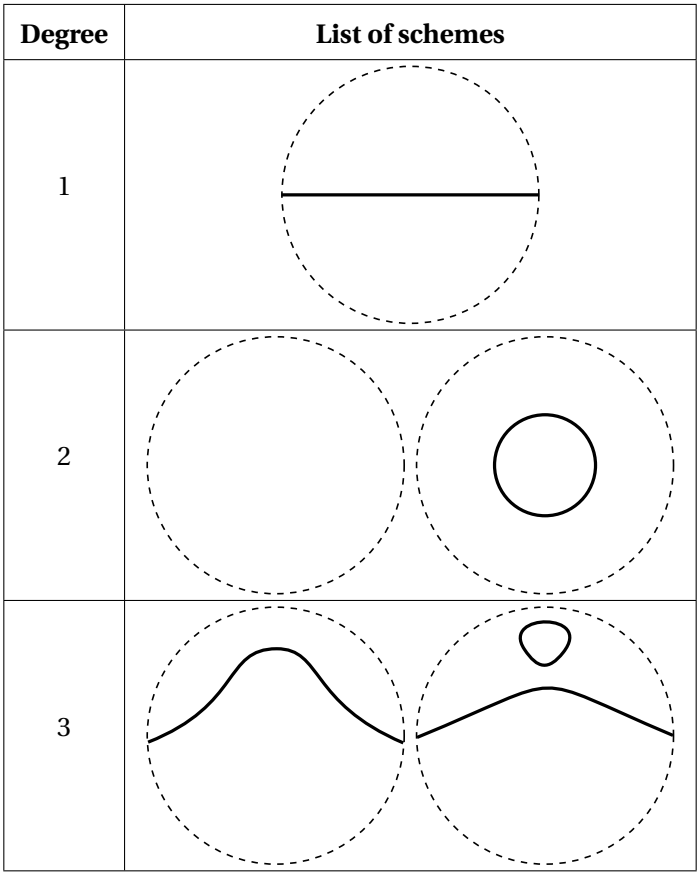


Figure 1.4. List of real schemes of non-singular real algebraic curves of degrees 1, 2 and 3.

We prove the classification degree by degree, by studying the maximal depth of possible nestings of ovals.

(1) In degrees 1 and 2, there is at most one component, so the classification is straight-forward.
(2) In degree three, there are between 1 and 2 components, and both cases are realizable by elliptic curves.
(3) In degree 4, there are between 0 and 4 ovals, and there is at most one nesting of two ovals. In the case of nesting, there is no other oval. This restricts the possibilities, and all are realizable by taking a small perturbation of a product of conics (the scheme with 3 ovals can also be obtained as a perturbation of a singular folium).

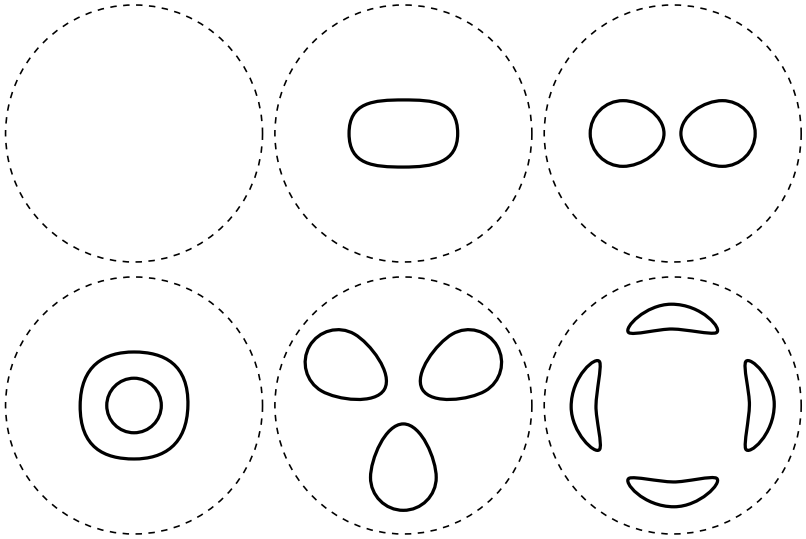
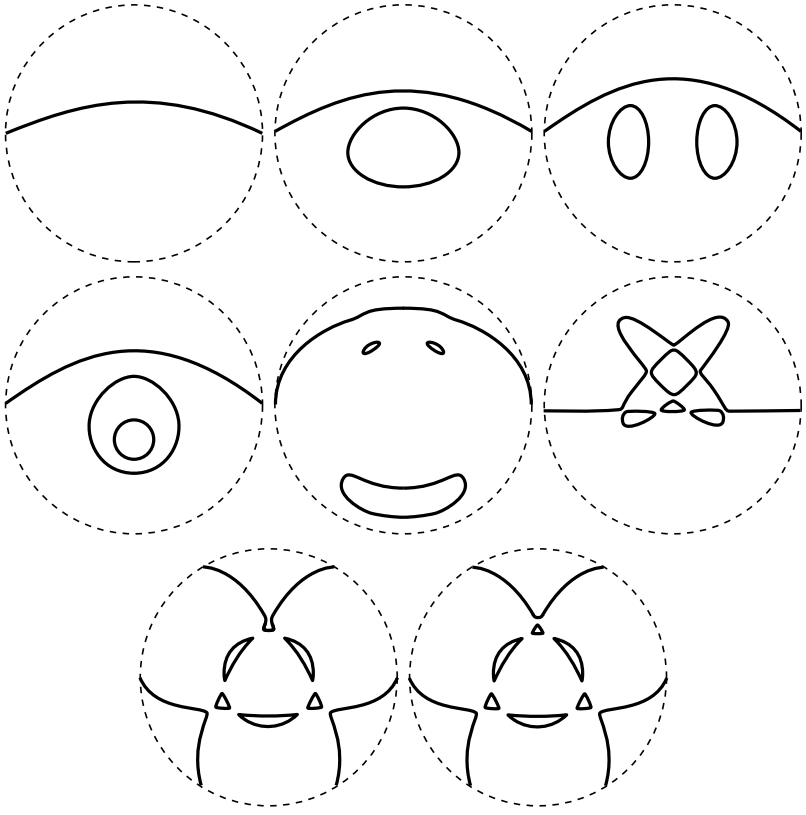
Degree	List of schemes
4	
5	

Figure 1.5. List of real schemes of non-singular real algebraic curves of degrees 4 and 5.

(4) In degree 5, there are at most 6 ovals accompanying the pseudo-line. The only possibility for a nesting is with a pair of ovals, and there is no other oval in this case. Again, this restricts the possibilities to the ones depicted in Figure 1.5, and they are all realizable by taking suitable perturbations of a line and two conics or of three lines and a conic.

Of course, one might wonder what happens in the case of degree 6 curves. Harnack's theorem tells us that there are bewteen 0 and 11 ovals, and the deepest nest has depth 3 (in which case there is no other oval). However, in the case of nests of depth 2, there may be other ovals elsewhere.

Proposition 1.9. *A real scheme of degree* 6 *contains at most one nest.*

Proof. If there were two nests, picking two points x and y in the inner-most oval of each, this defines a line which would intersect the curve in at least 8 points, contradicting Bézout's theorem. ■

This still leaves room for a lot of freedom, and Bézout's theorem is not sufficient to eliminate all non-realizable possibilities. Gudkov was the one to complete the classification in [Gud69], and in the end, there are 56 real schemes in total. In degree 7, Viro finished the classification in [Vir80], and found that there are 121 isotopy types.

1.1.2 Collections of Circles in $\mathbf{RP}^2$

We need a more compact way to encode the real schemes combinatorically. Otherwise, listing the 56 degree 6 schemes alone would take a few pages. We describe Viro's notation, introduced in [Vir80]. This is done recursively, in the following manner.

(1) The set consisting of one oval is denoted as $\langle 1\rangle$, and the set consisting of one pseudo-line is denoted as $\langle \mathcal{J}\rangle$.
(2) If $\langle A\rangle$ is the code for some set of ovals, then the collection obtained by adding one outer oval enveloping those in A is denoted as $\langle 1\langle A\rangle\rangle$.
(3) If $\langle A\rangle$ and $\langle B\rangle$ denote the codes for two non-intersecting collections of ovals, and such that no oval of one set is contained in the interior of an oval of the other, then the collection obtained by taking the union of both is denoted as $\langle A \sqcup B\rangle$.
(4) If a curve is one-sided, and the code for its collection of ovals is $\langle A\rangle$, then the code for the whole curve is $\langle \mathcal{J} \sqcup A\rangle$.
(5) We abbreviate $A \sqcup \cdots \sqcup A$ as $n \times A$, and we further abbreviate $n \times 1$ as n.

We refer the reader to Figure 1.6 for an illustrative example.

We can reformulate the classification of schemes of degree less than 5 more compactly in Figure 1.7, and we give the classifications of degree 6 and 7 curves in Figure 1.8 and Theorem 1.10, respectively.

Theorem 1.10 ([Vir80, Theorem 1])**.** *There exist non-singular curves of degree* 7 *of the following isotopy types:*

(1) $\langle \mathcal{J} \sqcup \alpha \sqcup 1\langle\beta\rangle\rangle$ *with* $\alpha+\beta \leqslant 14$, $0 \leqslant \alpha \leqslant 13$, $1 \leqslant \beta \leqslant 13$*;*
(2) $\langle \mathcal{J} \sqcup \alpha\rangle$ *with* $0 \leqslant \alpha \leqslant 15$*;*
(3) $\langle \mathcal{J} \sqcup 1\langle 1\langle 1\rangle\rangle\rangle$.

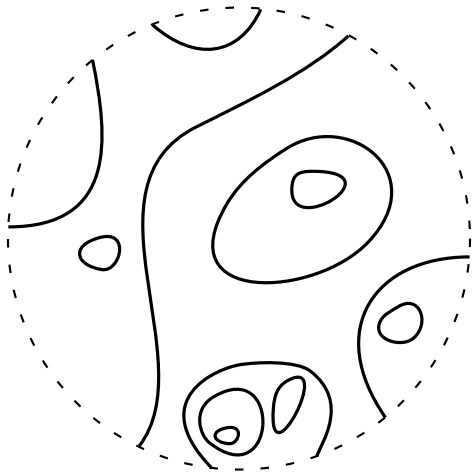


Figure 1.6. This scheme has Viro notation $\langle \mathcal{J} \sqcup 1 \sqcup 2\langle 1\rangle \sqcup 1\langle 1 \sqcup 1\langle 1\rangle\rangle\rangle$.

Any non-singular curve of degree 7 *belongs to one of these* 121 *types.*

1.1.3 The Space of Real Curves

A curve (possibly singular) is determined by a homogeneous polynomial $A \in \mathbf{R}[x_0 : x_1 : x_2]$ up to a multiplicative constant. A degree m polynomial is of the form

$$A(x_0, x_1, x_2) = \sum_{i+j \leqslant m} \lambda_{i,j} x_0^i x_1^j x_2^{m-i-j},$$

and there are

$$\sum_{i=0}^{m} \sum_{j=0}^{i} 1 = \frac{m(m+3)}{2} + 1$$

degrees of freedom in choosing the coefficients. This means that the space $\mathscr{C}_m$ of curves of degree m forms a real projective space

$$\mathscr{C}_m \cong \mathbf{RP}^{m(m+3)/2}.$$

This observation implies the following result (which was used previously in Harnack's proof of his theorem).

Lemma 1.11. *Through* $\binom{m+2}{2} - 1$ *points there always passes a degree* m *curve.*

Proof. It suffices to observe that $\binom{m+2}{2} - 1 = \frac{m(m+3)}{2}$, that the condition that a point is on a curve defines a hyperplane in $\mathbf{RP}^{m(m+3)/2}$, and that the intersection of $m(m+3)/2$ such hyperplanes is non-empty. ■

Degree	List of schemes
1	$\langle \mathcal{J} \rangle$
2	$\langle \varnothing \rangle$, $\langle 1 \rangle$
3	$\langle \mathcal{J} \rangle$, $\langle \mathcal{J} \sqcup 1 \rangle$
4	$\langle \varnothing \rangle$, $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 1\langle 1 \rangle \rangle$, $\langle 3 \rangle$, $\langle 4 \rangle$
5	$\langle \mathcal{J} \rangle$, $\langle \mathcal{J} \sqcup 1 \rangle$, $\langle \mathcal{J} \sqcup 2 \rangle$, $\langle \mathcal{J} \sqcup 1\langle 1 \rangle \rangle$, $\langle \mathcal{J} \sqcup 3 \rangle$, $\langle \mathcal{J} \sqcup 4 \rangle$, $\langle \mathcal{J} \sqcup 5 \rangle$, $\langle \mathcal{J} \sqcup 6 \rangle$

Figure 1.7. All real algebraic schemes of degree less than 5.

We shall describe a decomposition of the space $\mathscr{C}_m$. Define $\mathrm{Sing}^0(\mathscr{C}_m) = \mathscr{C}_m$. For $k \in \mathbf{N}^\star$, set

$$\pi_k : \left\{ (x_1, \ldots, x_k, A) \in (\mathbf{RP}^2)^k \times \mathscr{C}_m \;\middle|\; \begin{array}{c} \nabla A(x_1) = \cdots = \nabla A(x_k) = 0 \\ x_1, \ldots, x_k \text{ distinct} \end{array} \right\} \subset (\mathbf{RP}^2)^k \times \mathscr{C}_m \to \mathscr{C}_m$$

the projection onto $\mathscr{C}_m$, and define $\mathrm{Sing}^k(\mathscr{C}_m)$ to be the image of π_k. This means that $\mathrm{Sing}^k(\mathscr{C}_m)$ is the set of degree m curves with at least k singular points. Finally, set

$$(\mathscr{C}_m)_k = \mathrm{Sing}^k(\mathscr{C}_m) \smallsetminus \mathrm{Sing}^{k+1}(\mathscr{C}_m)$$

the subset of curves with *exactly* k critical points. In particular, we find that $(\mathscr{C}_m)_0$ is the set of non-singular curves. Moreover, it happens that $(\mathscr{C}_m)_k$ is dense in $\mathrm{Sing}^k(\mathscr{C}_m)$. The decomposition

$$\mathscr{C}_m = \mathrm{Sing}^0(\mathscr{C}_m) \supset \mathrm{Sing}^1(\mathscr{C}_m) \supset \mathrm{Sing}^2(\mathscr{C}_m) \supset \cdots$$

is a *stratification* (in the sense of [Whi65]), and the $(\mathscr{C}_m)_k$ are the associated *strata*.

Naturally, two curves that lie in the same connected component of $(\mathscr{C}_m)_0$ determine isotopic real schemes. However, if converse is true in degrees $\leqslant 4$, it is known not to hold in degrees $\geqslant 5$, as Rokhlin discussed in [Rok78, §4.2].

Definition 1.12. *Two non-singular real plane algebraic curves that belong to the same component of $(\mathscr{C}_m)_0$ are called* ***rigid isotopic****.*

There is a rigid isotopy analog of Hilbert's 16th problem, asking to classify all non-singular real plane algebraic curves of a fixed degree up to rigid isotopy. This problem is known to have an answer in degrees $\leqslant 6$. For instance, there are 64 rigid isotopy classes of non-singular sextics ([Nik79, Remark 3.10.9]).

We shall describe singular events with more care now. There are two notions from which one can approach the situation:

(1) the multiplicity of a singular point on a curve, and

(2) the multiplicity of an (isolated) intersection point between two curves.

Number of ovals	List of schemes
11	$\langle 9 \sqcup 1\langle 1\rangle\rangle$, $\langle 5 \sqcup 1\langle 5\rangle\rangle$, $\langle 1 \sqcup 1\langle 9\rangle\rangle$
10	$\langle 10\rangle$, $\langle 8 \sqcup 1\langle 1\rangle\rangle$, $\langle 5 \sqcup 1\langle 4\rangle\rangle$, $\langle 4 \sqcup 1\langle 5\rangle\rangle$, $\langle 1 \sqcup 1\langle 8\rangle\rangle$, $\langle 1\langle 9\rangle\rangle$
9	$\langle 9\rangle$, $\langle 7 \sqcup 1\langle 1\rangle\rangle$, $\langle 6 \sqcup 1\langle 2\rangle\rangle$, $\langle 5 \sqcup 1\langle 3\rangle\rangle$, $\langle 4 \sqcup 1\langle 4\rangle\rangle$, $\langle 3 \sqcup 1\langle 5\rangle\rangle$, $\langle 2 \sqcup 1\langle 6\rangle\rangle$, $\langle 1 \sqcup 1\langle 7\rangle\rangle$, $\langle 1\langle 8\rangle\rangle$
8	$\langle 8\rangle$, $\langle 6 \sqcup 1\langle 1\rangle\rangle$, $\langle 5 \sqcup 1\langle 2\rangle\rangle$, $\langle 4 \sqcup 1\langle 3\rangle\rangle$, $\langle 3 \sqcup 1\langle 4\rangle\rangle$, $\langle 2 \sqcup 1\langle 5\rangle\rangle$, $\langle 1 \sqcup 1\langle 6\rangle\rangle$, $\langle 1\langle 7\rangle\rangle$
7	$\langle 7\rangle$, $\langle 5 \sqcup 1\langle 1\rangle\rangle$, $\langle 4 \sqcup 1\langle 2\rangle\rangle$, $\langle 3 \sqcup 1\langle 3\rangle\rangle$ $\langle 2 \sqcup 1\langle 4\rangle\rangle$, $\langle 1 \sqcup 1\langle 5\rangle\rangle$, $\langle 1\langle 6\rangle\rangle$
6	$\langle 6\rangle$, $\langle 4 \sqcup 1\langle 1\rangle\rangle$, $\langle 3 \sqcup 1\langle 2\rangle\rangle$, $\langle 2 \sqcup 1\langle 3\rangle\rangle$, $\langle 1 \sqcup 1\langle 4\rangle\rangle$, $\langle 1\langle 5\rangle\rangle$
5	$\langle 5\rangle$, $\langle 3 \sqcup 1\langle 1\rangle\rangle$, $\langle 2 \sqcup 1\langle 2\rangle\rangle$, $\langle 1 \sqcup 1\langle 3\rangle\rangle$, $\langle 1\langle 4\rangle\rangle$
4	$\langle 4\rangle$, $\langle 2 \sqcup 1\langle 1\rangle\rangle$, $\langle 1 \sqcup 1\langle 2\rangle\rangle$, $\langle 1\langle 3\rangle\rangle$
3	$\langle 3\rangle$, $\langle 1 \sqcup 1\langle 1\rangle\rangle$, $\langle 1\langle 2\rangle\rangle$, $\langle 1\langle 1\langle 1\rangle\rangle\rangle$
2	$\langle 2\rangle$, $\langle 1\langle 1\rangle\rangle$
1	$\langle 1\rangle$
0	$\varnothing$

Figure 1.8. Gudkov's theorem ([Gud69, §4]) states that the 56 isotopy types listed here, and no other, can be realized by non-singular plane curves of degree 6.

We shall see that they disagree in general.

Let $A \in \mathscr{C}_m$ be a curve, and let $p \in \mathbf{R}A$ be a point on that curve. First, we describe how to move the problem back to the affine world : take a projective transformation $\varphi : \mathbf{RP}^2 \to \mathbf{RP}^2$ such that $\varphi([0:0:1]) = p$, and consider the associated curve $\varphi(\mathbf{R}A) = \mathbf{R}(A \circ \varphi)$. We obtain that $[0:0:1] \in \mathbf{R}(A \circ \varphi)$, and we may look only at the affine part:

$$f(x,y) = A \circ \varphi(x,y,1),$$

where $f(0,0) = 0$. This function f is a two-variable polynomial, which can be decomposed in the basis of symmetric polynomials:

$$f = f_1 + \cdots + f_m,$$

with f_i homogeneous of degree i. We define the **multiplicity** of p on A as the least integer i such that $f_i \neq 0$, and we denote it as

$$\mathrm{Mult}_A(p).$$

By definition, we have $\mathrm{Mult}_A(p) \geqslant 1$. There remains to check that this does not depend on the choice of the projective transformation φ, which is an easy exercise. One can make the following quick observations.

(1) $\mathrm{Mult}_A(p) = 1$ if and only if $\nabla A(p) \neq 0$.

(2) $\mathrm{Mult}_A(p) \leqslant \deg(A)$.

(3) $\mathrm{Mult}_A(p) \geqslant 2$ if and only if $\nabla A(p) = 0$.

(4) If $\nabla A(p) = 0$ and $\nabla^2 A(p)$ is non-singular, then $\mathrm{Mult}_A(p) = 2$, but the converse is false.

The case of multiplicity two needs more description. There are indeed two possibilities for the quadratic form $\nabla^2 A(p)$ to be non-singular.

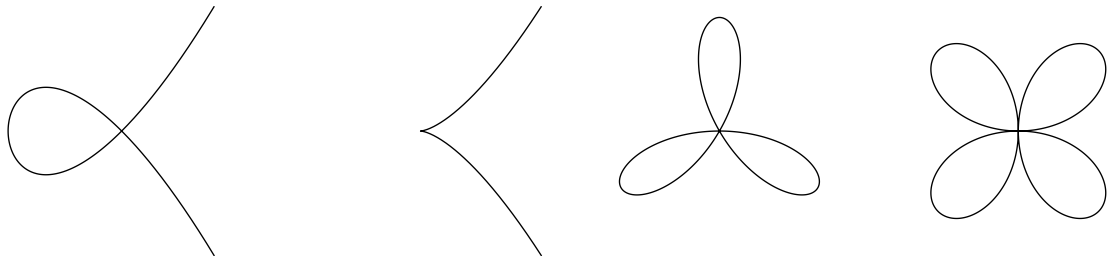
(1) It can be definite (positive or negative), in which case the curve $\mathbf{R}A$ has an isolated point, called an **acnode**. It corresponds to the fact that the surface

$$\Sigma = \bigcup_{-\varepsilon<t<\varepsilon} \{f(x,y) = t\} \times \{t\} \subset \mathbf{R}^2 \times]-\varepsilon, \varepsilon[$$

has a minimum or a maximum at $t = 0$ (depending on the signature of the quadratic form).

(2) It can be indefinite, where it has signature zero. In this case, the curve $\mathbf{R}A$ is locally homeomorphic around p to two arcs intersecting transversely at p. We call such a singular point a **double point** (sometimes the old terminology of a *crunode* is still employed). This time, it corresponds to the fact that the surface Σ has a saddle point at $t = 0$.

We now give some examples that come from Fulton's book [Ful89, §3.1], where we directly regard the affine curve. We refer to Figure 1.9 for a depiction of the curves.



(a) (b) (c) (d)

Figure 1.9. (a) A double point $f_1(x,y) = y^2 - x^3 - x^2$. (b) A cusp $f_2(x,y) = y^2 - x^3$. (c) A triple point $f_3(x,y) = (x^2+y^2)^2 + 3x^2y - y^3$. (d) A quadruple point $f_4(x,y) = (x^2+y^2)^3 - 4x^2y^2$.

For the first and the second curves f_1 and f_2, we have $\mathrm{Mult}_{f_1}((0,0)) = \mathrm{Mult}_{f_2}((0,0)) = 2$. We can compute the Hessians for both cases and we obtain:

$$\nabla^2 f_1(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \nabla^2 f_2(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

We see that $\nabla^2 f_1(0,0)$ is non-singular indefinite, and $\nabla^2 f_2(0,0)$ is singular. For the other curves f_3 and f_4, we have that the lowest degree homogeneous terms are respectively $3x^2y - y^3$ and $-4x^2y^2$, so that $\mathrm{Mult}_{f_3}((0,0)) = 3$ and $\mathrm{Mult}_{f_4}((0,0)) = 4$. For f_3 and f_4, the surface Σ defined above undergoes a transformation between $t < 0$ and $t > 0$ which is not of Morse type (for f_3, this is the so-called "Monkey saddle" at $t = 0$ for the surface Σ).

We further extend the definition of multiplicity to be $\mathrm{Mult}_A(p) = 0$ if $p \notin \mathbf{R}A$. In the case where A is a reducible curve, we let

$$A = \prod_{i=1}^{n} A_i^{e_i}$$

be its factorization into irreducible components. It is easy to verify the relation

$$\mathrm{Mult}_A(p) = \sum_{i=1}^{N} e_i \mathrm{Mult}_{A_i}(p).$$

We denote as $(\mathscr{C}_m)_{1,\mathrm{reg}}$ the subset of $(\mathscr{C}_m)_1$ of curves having exactly one critical point which is a non-degenerate critical point (that is, $\nabla^2 A(p)$ is non-singular). By a careful examination of the (algebraic) equations that this condition yields on the coefficients on a curve, we can check that this is open and dense subset of $\mathrm{Sing}^1(\mathscr{C}_m)$, since it has codimension two in $\mathscr{C}_m$.

Defining intersection multiplicities is slightly more delicate. Again, given two curves $A, B \in \mathscr{C}_m$ intersecting at a point p, we may switch back to the affine world with two polynomials f and g such that $f(0,0) = g(0,0) = 0$, by use of a suitable projective transformation. We consider the local ring

$$\mathcal{O} = \mathbf{R}[x,y]_{(0,0)} = \left\{ \frac{a(x,y)}{b(x,y)} \;\middle|\; a, b \in \mathbf{R}[x,y],\ b(0,0) \neq 0 \right\}.$$

The **intersection multiplicity** of A and B at the point p is defined to be

$$\mathrm{Mult}_{A,B}(p) = \dim \mathcal{O}/\langle f, g \rangle.$$

This multiplicity number can be characterized (see [Ful89, §3.3]) as the unique number satisfying a number of properties, one of them being the following:

$$\mathrm{Mult}_{A,B}(p) \geqslant \mathrm{Mult}_A(p) \times \mathrm{Mult}_B(p),$$

with equality if and only if A and B have no common tangent line at p. In particular, two curves intersect with multiplicity one if and only if their images intersect transversely.

Surprisingly, if one considers two curves A and B intersecting at p, and then looks at the (reducible) curve $A \times B$, then $\mathrm{Mult}_{A,B}(p) \neq \mathrm{Mult}_{A\times B}(p)$ in general. Indeed, taking a transverse intersection for example, we see that $\mathrm{Mult}_{A,B}(p) = 1$ whereas $\mathrm{Mult}_{A\times B}(p) = 2$.

1.1.4 The Classical Perturbation Theorem

We now discuss the most classical method for constructing algebraic curves realizing a given real scheme.

Theorem 1.13 (Small Perturbation Theorem)**.** *Let A be a singular degree m real algebraic curve whose only singularities are transverse double points. Let B be a (possibly singular) degree m curve such that $\mathbf{R}B$ does not contain any of the singular points of A. Let $\mathcal{U} \subset \mathbf{RP}^2$ be a neighborhood of $\mathbf{R}A$, which decomposes into $\mathcal{U} = \mathcal{U}_{\mathrm{reg}} \cup \mathcal{U}_{\mathrm{sing}}$, with $\mathcal{U}_{\mathrm{sing}}$ the union of neighborhoods of the singular points of A and $\mathcal{U}_{\mathrm{sing}} \cap \mathbf{R}B = \varnothing$, and $\mathcal{U}_{\mathrm{reg}}$ a regular neighborhood of $\mathbf{R}A \setminus \mathcal{U}_{\mathrm{sing}}$. Then there exists $\varepsilon > 0$ such that for all $0 < t \leqslant \varepsilon$, the curve $F = A + tB$ is non-singular and satisfies the following:*

(1) $\mathbf{R}F \subset \mathcal{U} \cap \{AB \leqslant 0\}$*;*

(2) for any connected component $\mathcal{V}$ *of* $\mathcal{U}_{\text{sing}}$*, there exists a diffeomorphism* $h : \mathcal{V} \to [-1,1] \times [-1,1]$ *such that* $h(\mathbf{R}A \cap \mathcal{V}) = [-1,1] \times \{0\} \cup \{0\} \times [-1,1]$ *and* $h(\mathbf{R}F \cap \mathcal{V}) = \{xy = 1/2\}$ *a hyperbola;*

(3) $\mathbf{R}F \smallsetminus \mathcal{U}_{\text{sing}}$ *is a section of the unit normal bundle* $\mathcal{U}_{\text{reg}} \to \mathbf{R}A \smallsetminus \mathcal{U}_{\text{sing}}$*;*

(4) $\mathbf{R}A \cap \mathbf{R}F = \mathbf{R}B \cap \mathbf{R}F = \mathbf{R}A \cap \mathbf{R}B$*; if* $x \in \mathbf{R}A \cap \mathbf{R}B$ *is a non-singular point of* B *and* $\mathbf{R}A \pitchfork \mathbf{R}B$ *at* x*, then* $\mathbf{R}A \pitchfork \mathbf{R}A$ *at* x *too.*

We depict an example application of Theorem 1.13 in Figure 1.10. In this example, the perturbation of the singular curve A becomes a degree 4 non-singular curve with real scheme $\langle 2 \rangle$.

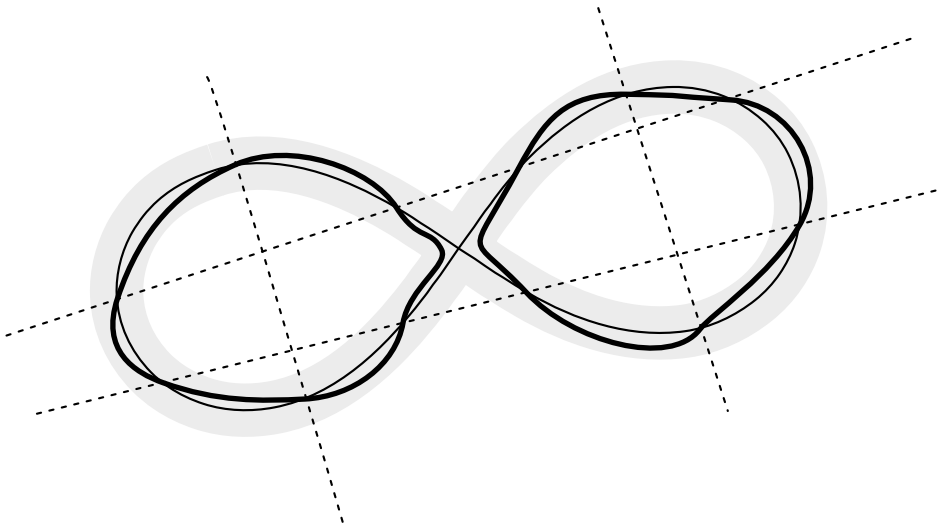


Figure 1.10. A is the lemniscate $\{x^4 + 2x^2y^2 - x^2 + y^4 + y^2 = 0\}$ (affine equation), and B is the union of the four dashed lines. The neighborhood $\mathcal{U}$ is in light gray, and the new curve F is in bold line.

This theorem itself is sufficient to conclude the proof of Harnack's theorem (see Theorem 1.8).

Corollary 1.14. *For any integer* k *such that*

$$\frac{1 + (-1)^{m+1}}{2} \leqslant k \leqslant \frac{(m-1)(m-2)}{2} + 1,$$

there exists a non-singular real plane algebraic curve A *whose real scheme satisfies* $b_0(\mathbf{R}A) = k$*.*

Proof. We shall start by proving that there always exists a curve whose real scheme has the maximal number of connected components. To this end, we explain how Harnack constructed examples

in every degrees recursively. We will rather construct an affine curve, but this technicality is of no importance.

It is trivial to construct curves with the maximal number of components in degrees $m \leqslant 2$. We shall construct a family in each degree recursively. Fix a horizontal line L, and pick a conic C which intersects it twice. The curve A_3 is built by considering a small perturbation of $C \times L$ using three vertical lines located to the right of the conic. The family $(A_m)_{m\geqslant 3}$ is built recursively as follows. Consider the (singular) curve $A_m \times L$ of degree $m+1$, and take a A_{m+1} to be small perturbation of it by using $m+1$ vertical lines located to the right of all previously considered vertical lines. We refer the reader to Figure 1.11 for a depiction of the construction of the curves A_3, A_4, A_5 and A_6.

By all the considerations we made, we see that $b_0(\mathbf{R}A_{m+1}) = b_0(\mathbf{R}A_m) + m - 1$, which by induction, gives $b_0(\mathbf{R}A_m) = \frac{(m-1)(m-2)}{2} + 1$.

Finally, to show that there are curves whose real schemes realize each intermediate numbers k, pick a curve $A_{\min}$ and a curve $A_{\max}$ whose real schemes have the minimal and maximal number of connected components, respectively. Note that we need to assume that $m \geqslant 2$ in what follows, as the case $m = 1$ has $\mathscr{C}_1 = (\mathscr{C}_1)_0$ which is path-connected. Choose a path γ in $\mathscr{C}_m$ connecting $A_{\min}$ to $A_{\max}$. By codimension reasons, we may assume that this path satisfies

$$\gamma \subset (\mathscr{C}_m)_0 \cup (\mathscr{C}_m)_{1,\mathrm{reg}},$$

as well as that there are only finitely many times t for which $\gamma(t) \in (\mathscr{C}_m)_{1,\mathrm{reg}}$. This means that at each such singular time t, the curve $\gamma(t-\varepsilon)$ undergoes a Morse modification to become the curve $\gamma(t+\varepsilon)$. This corresponds to either the merging of two components (or even of the same component), which changes b_0 by at most one, or the birth or death of an oval, which changes b_0 by ± 1. Therefore, going from $A_{\min}$ to $A_{\max}$ ensures that we meet curves A with all possible values for $b_0(\mathbf{R}A)$ in between. ■

Definition 1.15. *A non-singular plane curve A of degree m with $b_0(\mathbf{R}A) = \frac{(m-1)(m-2)}{2} + 1$ is called an M-**curve**. A curve with $b_0(\mathbf{R}A) = \frac{(m-1)(m-2)}{2} + 1 - k$ is called an $(M-k)$-**curve**.*

We finish by stating a theorem of Brusotti, whose content is basically that resolution of double point singularities are independent of each other. A more modern proof of this result can be found in [BR90, §5.5].

Theorem 1.16 ([Bru21])**.** *Let A be a curve with only double point singularities $x_1, \dots, x_s$. Then there is a small perturbation A' of A with double point singularities only $x_1, \dots, x_{s-1}$, which is obtained by resolution of the double point x_s. Moreover, there is choice in which of the two ways the singularity x_s can be resolved.*

Using the previous result inductively for every double point $x_1, \dots, x_s$, one sees that it is possible to resolve each of those singularities in a chosen way, independently of the others. We illustrate this in Figure 1.12, where we construct degree 4 real schemes from the union of two conics.

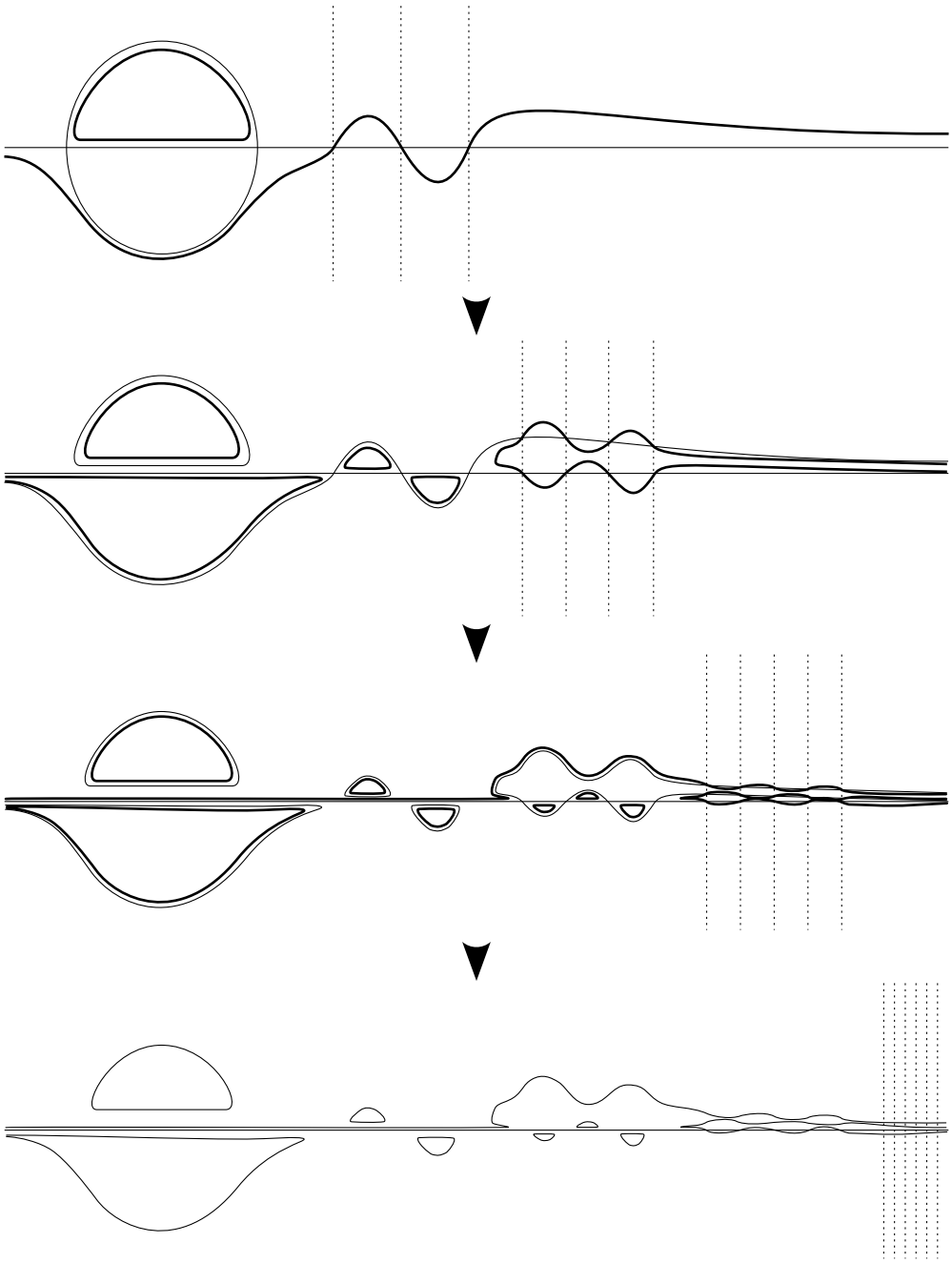


Figure 1.11. Harnack's construction of curves of degrees 3 through 6 with the maximal number of components. The singular curve $A_m \times L$ is in thinner stroke than the resulting curve A_{m+1}. The curve A_6 is not depicted, only $A_5 \times L$ is. These curves will have the real schemes $\langle \mathcal{J} \sqcup 1 \rangle$, $\langle 4 \rangle$, $\langle \mathcal{J} \sqcup 6 \rangle$ and $\langle 9 \sqcup 1\langle 1 \rangle \rangle$, respectively.

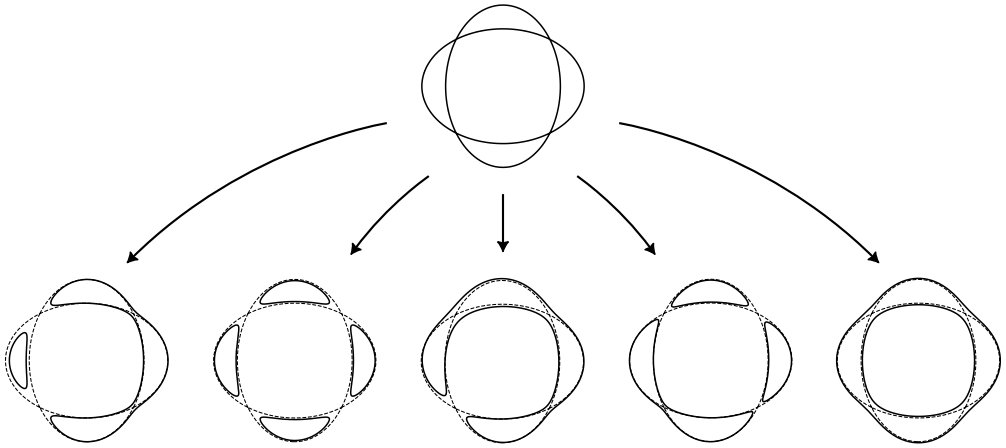


Figure 1.12. The construction of all non-empty real schemes of degree 4 by using Brusotti's theorem. In total, there are $2^4 = 16$ possible choices of resolutions of the 4 singularities, but many pairs of choices give the same real schemes.

1.2 Curves From the Complex Viewpoint

Typically, the main caveat with the field of real numbers is the lack of algebraic closure. Namely, some algebraic equations need not have the correct number of solutions, if they have any in the first place. The study of real curves is no exception, and there may be curves A which are *purely imaginary*, in the sense that $\mathbf{R}A = \varnothing$. The same is true for *intersections* between curves: the Bézout theorem only gives an upper bound in the real case, whereas it becomes an equality (if counting with multiplicities) in the complex case.

It therefore comes as no surprise that one can leverage restrictions on a real curve by regarding it as the *real part* of the associated complex curve. Some results, such as the Harnack upper bound, become clear if one has this context in mind. Moreover, restrictions of an entirely new type can be stated, such as the Rokhlin–Mishachev orientation formulas. It also becomes possible to take profit from geometric considerations, which were not previously available to us.

1.2.1 The Complex Projective Plane

Given a homogeneous polynomial $A \in \mathbf{R}[x_0, x_1, x_2]$, one can look at it over $\mathbf{C}[z_0, z_1, z_2]$ instead. This induces the so-called **complexification** of the real curve:

$$\mathbf{C}A := \left\{ [z_0 : z_1 : z_2] \in \mathbf{CP}^2 \;\middle|\; A(z_0, z_1, z_2) = 0 \right\}.$$

If $\nabla A \neq 0$ on the whole $\mathbf{CP}^2$, then, by a standard submersion argument, we see that $\mathbf{C}A$ is a closed connected surface smoothly embedded in $\mathbf{CP}^2$. In fact, it is a complex submanifold of $\mathbf{CP}^2$, and thus it is even orientable (and oriented). One may ask whether this surface is connected, and what is its genus. We re-prove the following well-known fact.

Proposition 1.17. *If $A \in \mathbf{C}[z_0, z_1, z_2]$ is homogeneous of degree $m \geqslant 1$ with $\nabla A \neq 0$ on $\mathbf{CP}^2$, then $\mathbf{C}A$ is connected and has genus $g = \frac{(m-1)(m-2)}{2}$.*

Proof. One may look, as in the real case, at the space $\mathscr{C}_m^{\mathbf{C}}$ of *complex* curves of degree m. By the very same arguments as before, this is a $\mathbf{CP}^{m(m+3)/2}$, and the subspace $\mathrm{Sing}^1(\mathscr{C}_m^{\mathbf{C}})$ of singular curves has *complex* codimension one. In particular, since it has real codimension two, the set

$$(\mathscr{C}_m^{\mathbf{C}})_0 = \mathscr{C}_m^{\mathbf{C}} \smallsetminus \mathrm{Sing}^1(\mathscr{C}_m^{\mathbf{C}})$$

is path-connected. As such, any two non-singular curves A and B connected, and this induces an isotopy from $\mathbf{C}A$ to $\mathbf{C}B$. Therefore, it only suffices to comupute the topological type of $\mathbf{C}A$ for any one *specific* choice of $A \in \mathscr{C}_m^{\mathbf{C}} \smallsetminus \mathrm{Sing}^1(\mathscr{C}_m^{\mathbf{C}})$.

First pick A to be the union of m distinct lines[†] in generic position (that is: any pair of lines have exactly one intersection point, and no three line intersect together). There are $\binom{m}{2} = \frac{m(m-1)}{2}$ such intersection points. Taking a small perturbation of A into A' turns it into a curve $A' \in (\mathscr{C}_m^{\mathbf{C}})_0$. Topologically, it corresponds to removing a ball at each intersection between two lines and replacing it with a small tube connecting both spheres. This yields an embedded surface which is diffeomorphic to a sphere with $\binom{m-1}{2} = \frac{(m-1)(m-2)}{2}$ handles attached. ∎

[†] It is known that if L is a line, then $\mathbf{C}L$ is an embedded 2-sphere which spans $H_2(\mathbf{CP}^2; \mathbf{Z})$.

In fact, the proof of the previous fact also has two implications. First, the real question of finding the topological type of $\mathbf{R}A$, given $A \in (\mathscr{C}_m)_0$, was entirely answered by Harnack's theorem, and it only depends on $b_0(\mathbf{R}A)$. The complex analogue of that question is different in that the topological type of $\mathbf{C}A$ only depends on the *degree* of A. The second consequence is the following.

Proposition 1.18. *If $A \in \mathbf{C}[z_0, z_1, z_2]$ is homogeneous of degree m and $\nabla A \neq 0$ on $\mathbf{CP}^2$, then the homology class of $\mathbf{C}A$ in $H_2(\mathbf{CP}^2; \mathbf{Z})$ is m times the generator (given as the homology class of any complex line):*

$$[\mathbf{C}A] = m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2; \mathbf{Z}).$$

Proof. The union of m lines realizes that homology class, and taking a perturbation of it (by tubing the $\binom{m}{2}$ double points) does not change this fact. ■

Of course, the condition that $\nabla A \neq 0$ on the whole $\mathbf{CP}^2$ seems more restrictive than its real counterpart. But it turns out that the following holds.

Proposition 1.19 ([Gil92, Corollary 2.5])**.** *Suppose that $A \in \mathbf{R}[x_0, x_1, x_2]$ verifies $\nabla A \neq 0$ on $\mathbf{RP}^2$. Then there is a small (real) perturbation A' of A such that $\mathbf{R}A$ and $\mathbf{R}A'$ are isotopic and $\nabla A' \neq 0$ on $\mathbf{CP}^2$.*

Proof. The proper real algebraic subset $\mathscr{C}_m \subset \mathscr{C}_m^{\mathbf{C}}$ is nowhere dense. ■

There is an even stronger version of the previous claim.

Proposition 1.20 ([Gil92, Proposition 2.6])**.** *Let $(\Lambda_i)_{i\in I}$ be a finite collection of real curves (possibly singular, and not necessarily of the same degree), and let A be a real curve with $\nabla A \neq 0$ on $\mathbf{RP}^2$. Suppose that $\mathbf{R}A$ intersects each $\mathbf{R}\Lambda_i$ transversely in $\mathbf{RP}^2$ at non-singular points of Λ_i and which are not intersection points with another $\mathbf{R}\Lambda_j$. Then there is an arbitrarily small real perturbation A' of A so that $\nabla A' \neq 0$ on $\mathbf{CP}^2$, $\mathbf{R}A$ and $\mathbf{R}\Lambda_i$ intersect transversely, and without changing the set $\mathbf{R}A \cup \bigcup_{i\in I} \mathbf{R}\Lambda_i$ up to ambient isotopy. Moreover, it is possible to do so in a way that $\mathbf{C}A'$ also avoids a finite collection of given points $(q_j)_{j\in J}$.*

Therefore, from now on, when we mean a non-singular real plane algebraic curve of degree m, we really mean a curve $A \in \mathscr{C}_m \cap (\mathscr{C}_m^{\mathbf{C}})_0$.

It is interesting to see how the real curve $\mathbf{R}A$ sits on its complexification $\mathbf{C}A$. First, looking at complex conjugation

$$\begin{array}{rccc} \mathrm{conj} & : & \mathbf{CP}^2 & \longrightarrow & \mathbf{CP}^2 \\ & & [z_0 : z_1 : z_2] & \longmapsto & [\overline{z_0} : \overline{z_1} : \overline{z_2}], \end{array}$$

we obtain $\mathbf{RP}^2 = \mathrm{Fix}(\mathrm{conj})$. Moreover, $\mathrm{conj}(\mathbf{C}A) = \mathbf{C}A$ since A is a real polynomial. In fact, we have:

$$\mathbf{R}A = \mathrm{Fix}\left(\mathrm{conj}|_{\mathbf{C}A}\right).$$

There are two mutually-exclusive possibilities for the set $\mathbf{C}A \smallsetminus \mathbf{R}A$: either it is path-connected, or it has exactly two connected components. Indeed, if one looks at a connected component C of $\mathbf{C}A \smallsetminus \mathbf{R}A$, takes its closure $\overline{C}$ and then the union $\overline{C} \cup \mathrm{conj}(\overline{C})$, this defines a non-empty closed and open subset

of $\mathbf{C}A$, which is connected. As such, $\overline{C} \cup \mathrm{conj}(\overline{C}) = \mathbf{C}A$, and thus there are at most two connected components in $\mathbf{C}A \smallsetminus \mathbf{R}A$.

Definition 1.21. *A real non-singular curve is called* ***type I****,* ***separating*** *or* ***dividing****, if* $\mathbf{C}A \smallsetminus \mathbf{R}A$ *is disconnected. In the case where it is connected, the curve is said to be* ***type II****, or* ***non-dividing****.*

It is to be noted that this is a property of the *curve*, and not merely of its real scheme. In this case, the scheme is said to be of **indefinite type**. On the contrary, if a real scheme can only be realized as a type I or type II curve, it is said to be of **definite type**. Indeed, there are real schemes which can be realized both as type I and type II curves. For instance, the degree 5 scheme $\langle \mathcal{J} \sqcup 4 \rangle$ is of indefinite type[†] (and it is the only degree 5 one with this property). Another example is (see [Kai+19, Figure 1]) that there are exactly 8 degree 6 real schemes of indefinite type, which are the following:

$$\begin{array}{ccc} \langle 1\langle 8\rangle\rangle & \langle 4 \sqcup 1\langle 4\rangle\rangle & \langle 9\rangle \\ \langle 1 \sqcup 1\langle 5\rangle\rangle & \langle 3 \sqcup 1\langle 3\rangle\rangle & \langle 5 \sqcup 1\langle 1\rangle\rangle \\ \langle 1\langle 4\rangle\rangle & \langle 2 \sqcup 1\langle 2\rangle\rangle. & \end{array}$$

Thinking of $\mathbf{R}A$ as sitting on $\mathbf{C}A$ allows to simplify the proof of Harnack's upper bound. Klein's proof thinks of $\mathbf{R}A$ as the fix-point set of an orientation-reversing involution on a genus g surface, and this must have at most $g+1$ components[‡].

We are interested in studying the topological differences between type I and type II curves. We now summarize some of them.

Proposition 1.22. *Let A be a non-singular real curve of degree m. We denote as $\Sigma = \mathbf{C}A/\mathrm{conj}$ the orbit space of the action of $\mathbf{Z}/2 \cong \{\mathrm{id}, \mathrm{conj}\}$ on $\mathbf{C}A$.*

(1) Σ has boundary diffeomorphic to $\mathbf{R}A$.

(2) A is type I if and only if Σ is orientable.

(3) ***(Klein's congruence)*** *If A is type I, then $b_0(\mathbf{R}A) \equiv \left\lfloor \frac{m+1}{2} \right\rfloor$ [2].*

(4) The orbit space Σ is a sphere with holes if and only if A is an M-curve. It is a projective plane with holes if and only if A is an $(M-1)$-curve.

(5) An M-curve is type I, and an $(M-1)$-curve is type II.

(6) If $m \geqslant 3$ and $b_0(\mathbf{R}A) = \frac{1+(-1)^{m+1}}{2}$ the minimal number of components, then A is type II.

Proof.

(1) The boundary of Σ corresponds exactly to $\mathrm{Fix}\left(\mathrm{conj}|_{\mathbf{C}A}\right) = \mathbf{R}A$.

(2) If A is type I, then $\mathbf{C}A \smallsetminus \mathbf{R}A$ has two diffeomorphic components which are swapped by conj. As such, the orbit space Σ is diffeomorphic to each of those two halves, which are orientable since $\mathbf{C}A$ is. Conversely, if A is type II, then taking a path which connects a point $q \in \mathbf{C}A \smallsetminus \mathbf{R}A$ to its image $\mathrm{conj}(q)$, and taking the image of this path in the orbit space, this gives an orientation-reversing loop (since $\mathrm{conj}|_{\mathbf{C}A}$ is orientation-reversing).

[†] We will describe how to construct both realizations in Figure 1.17.

[‡] One can make use of the Smith–Floyd inequalities (which will be stated and proved in §2.2.1) to derive this fact.

(3) If A is type I, then $\chi(\Sigma) = \frac{1}{2}\chi(\mathbf{C}A) = 1-g$, Σ is orientable and has $b_0(\mathbf{R}A)$ boundary components. Gluing discs to those components yields an orientable closed surface Σ' with:

$$\chi(\Sigma') = 1 - g + b_0(\mathbf{R}A).$$

Its Euler characteristic being an even number gives the relation.

(4) The surface Σ' is a sphere if and only if $\chi(\Sigma') = 2$. That is, if and only if $b_0(\mathbf{R}A) = g+1$. Similarly, Σ' is a projective plane if and only if $\chi(\Sigma') = 1$, which gives $b_0(\mathbf{R}A) = g$.

(5) For M-curves, this is a direct consequence of (2) and (4), and for $(M-1)$-curves, this follows from (2) or (3).

(6) If m is even, then a curve with $\mathbf{R}A = \varnothing$ cannot be type I (the empty set cannot be dividing). If m is odd, this is not immediate and will follow from §1.2.2 (see Corollary 1.28). ■

We also have a special case of scheme of definite type.

Proposition 1.23. *If A is a non-singular real curve of degree m whose real scheme is the maximal nest (that is, a nesting of k ovals with $k = \left\lfloor \frac{m}{2} \right\rfloor$), then A is type I.*

Proof. Pick a point $q_0 \in \mathbf{RP}^2$ located in the inner-most oval of the nest. The subspace $\mathscr{C}_1^{\mathbf{C}}(q_0)$ of $\mathscr{C}_1^{\mathbf{C}}$ of complex lines that pass through the point q_0 is a $\mathbf{CP}^1$ (topologically, a 2-sphere) inside $\mathscr{C}_1^{\mathbf{C}} \cong \mathbf{CP}^2$. Its subspace $\mathscr{C}_1^{\mathbf{R}}(q_0) \cong \mathbf{RP}^1$ of real lines is a great circle of that 2-sphere. Define a map

$$\varphi : \mathbf{CP}^2 \smallsetminus \mathbf{RP}^2 \to \mathscr{C}_1^{\mathbf{C}}(q_0)$$

by setting the image $\varphi(q)$ of a point to be the unique line connecting q to q_0. It is straight-forward to see that φ commutes with the conjugation, and that conjugation on $\mathscr{C}_1^{\mathbf{C}}(q_0)$ is the antipodal map.

Because any real line in $\mathscr{C}_1^{\mathbf{R}}(q_0)$ intersects the curve $\mathbf{R}A$ in exactly m points, there are no intersection points in $\mathbf{C}A \smallsetminus \mathbf{R}A$. As such, the image of $\mathbf{C}A \smallsetminus \mathbf{R}A$ under φ is included in $\mathscr{C}_1^{\mathbf{C}}(q_0) \smallsetminus \mathscr{C}_1^{\mathbf{R}}(q_0)$. But since a point $q \in \mathbf{C}A \smallsetminus \mathbf{R}A$ and its conjugate are both in that image, we see that $\mathbf{C}A \smallsetminus \mathbf{R}A$ cannot possibly be connected, as $\mathscr{C}_1^{\mathbf{C}}(q_0) \smallsetminus \mathscr{C}_1^{\mathbf{R}}(q_0)$ is disconnected (it is the disjoint union of two open 2-discs). This means that A is indeed type I. ■

Definition 1.24. *The real scheme prescribed by Proposition 1.23 is called the* ***hyperbolic scheme****.*

1.2.2 Complex Orientations

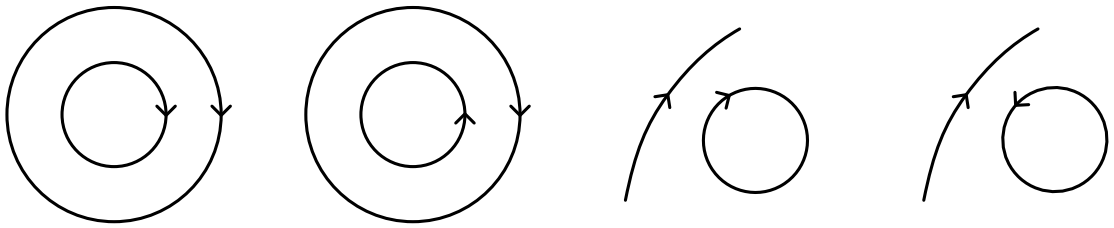
With type I curves comes a whole new world of restrictions. Indeed, if A is type I, then $\mathbf{R}A$ comes naturally equipped with a pair of opposite orientations, induced from both halves of $\mathbf{C}A \smallsetminus \mathbf{R}A$. We define a **semi-orientation** on any real scheme $\mathscr{C} \subset \mathbf{RP}^2$ to be a pair of opposite orientations on $\mathscr{C}$. A semi-orientation is said to **come from topology** if $\mathscr{C}$ is the real scheme of a type I curve A and that semi-orientation corresponds to the one induced from $\mathbf{C}A \smallsetminus \mathbf{R}A$.

Of course, just like there are real schemes of indefinite type, the real scheme alone is not sufficient to determine entirely a semi-orientation coming from topology. For instance, the real degree 7 $(M-4)$-scheme $\langle \mathcal{J} \sqcup 9 \sqcup 1\langle 1\rangle\rangle$ can be endowed with two distinct semi-orientations both coming from

topology (explicit constructions can be found in [Bru07, §5.2]). See Figure 1.14(c) and Figure 1.14(d) for a representation of both those realizations.

Just like we had a way to encode real schemes, we now describe Viro's extended notation to describe schemes endowed with a semi-orientation. First, two ovals are said to form an **injective pair** if one is located in the interior of the other. An injective pair of ovals is said to be **positive** (*resp.* **negative**) if their orientations are induced from an orientation of the annulus they co-bound (*resp.* their orientations disagree with any orientation of the annulus). We denote the numbers of positive and negative injective pairs as Π^+ and Π^-, respectively.

Now, in the special case where the real scheme has a pseudo-line $\mathcal{J}$, it is possible to give a sign to each oval relatively to that pseudo-line. Indeed, if o denotes the oval, then $\mathbf{RP}^2 \smallsetminus \mathrm{Int}(o)$ is a Möbius strip, and $\mathcal{J}$ (together with its orientation) determines a unique homology class in $H_1(\mathbf{RP}^2 \smallsetminus \mathrm{Int}(o); \mathbf{Z})$. If that class agrees with twice the one determined by o (with its orientation), the oval o is said to be **negative**. Otherwise, if they are opposite, the oval is said to be **positive**. The numbers of positive and negative ovals are denoted as Λ^+ and Λ^-, respectively. We refer the reader to Figure 1.13 for a visual reference.



(a) (b) (c) (d)

Figure 1.13. (a) A negative injective pair of ovals. (b) A positive injective pair. (c) A negative oval. (d) A positive oval.

It is now possible to describe a semi-oriented real scheme. If the real scheme has a pseudo-line, every oval comes with a sign. Otherwise, any non-outer oval comes with a sign, depending on whether it makes a positive or a negative injective pair with its immediate surrounding oval. Outermost ovals do not come equipped with a sign. To the code describing the real scheme we append a subscript $\pm$ to each oval, depending on its sign. We describe several examples in Figure 1.14.

Of course, one may ask, at least in the case of even degree schemes where outermost ovals do not come equipped with a sign, if this extended notation for semi-orientations is actually enough to recover those. We give a heuristic as to why the following fact holds in Figure 1.15.

Proposition 1.25. *If two semi-oriented real schemes have the same code, then there exists a homeomorphism of* $\mathbf{RP}^2$ *carrying one of them to the other while preserving semi-orientations.*

If A is a non-singular plane curve of degree m, there are two possibilities.

(1) The curve A is type II. In this case, if $\langle \mathscr{A} \rangle$ is the code for its real scheme, we further enhance it to $\langle \mathscr{A} \rangle_{II}^m$ to describe this fact and precise the degree.

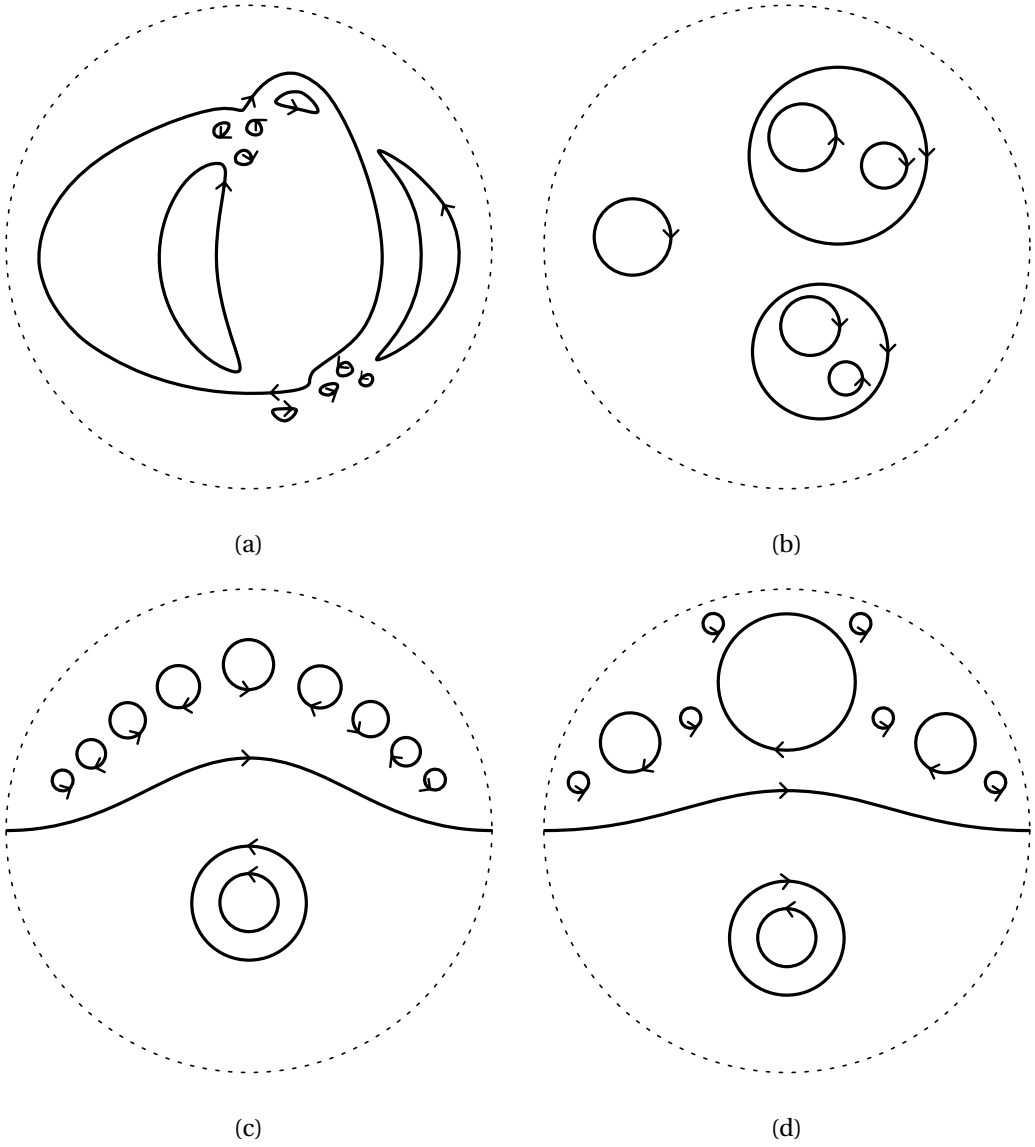


Figure 1.14. Four examples of semi-oriented schemes in $\mathbf{RP}^2$. The schemes (c) and (d) are examples of a real schemes with two distinct semi-orientations that come from topology. (a) $\langle 5 \sqcup 1\langle 3_+ \sqcup 2_- \rangle\rangle$. (b) $\langle 1 \sqcup 2\langle 1_+ \sqcup 1_- \rangle\rangle$. (c) $\langle \mathcal{J} \sqcup 4_+ \sqcup 5_- \sqcup 1_+\langle 1_+ \rangle\rangle$. (d) $\langle \mathcal{J} \sqcup 3_+ \sqcup 6_- \sqcup 1_-\langle 1_+ \rangle\rangle$.

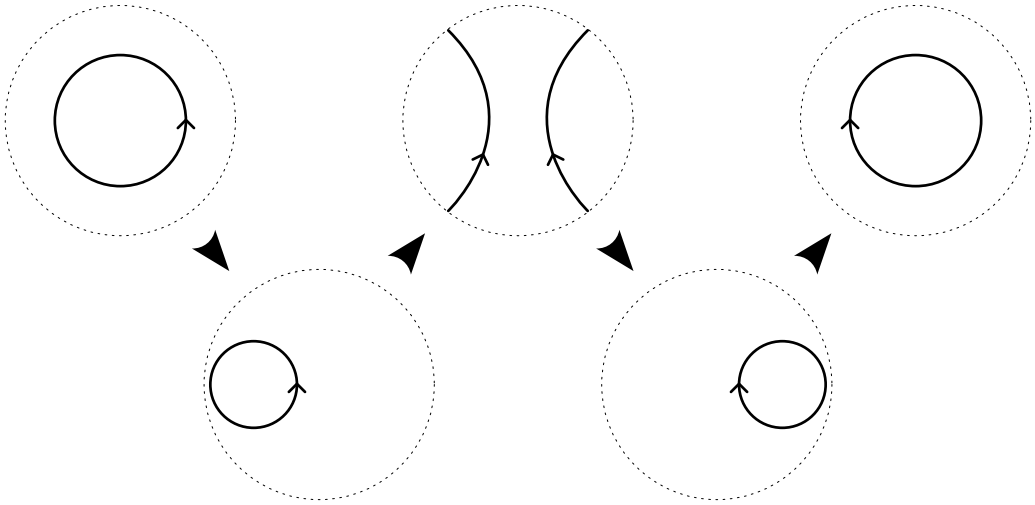


Figure 1.15. "Clock-wise" and "anti-clockwise" actually bear no meaning in the case of an oriented circle embedded in $\mathbf{RP}^2$.

(2) The curve has type I. In this case, we enhance the code $\langle \mathscr{A} \rangle$ into a code $\langle \mathscr{A}' \rangle$ for its semi-orientation that comes from topology, and we further indicate this by $\langle \mathscr{A}' \rangle_I^m$.

We call this enhanced information the **complex scheme** of the curve A. There is a whole new question which can now be asked.

Question 1.26. *Can we classify all complex schemes of non-singular real plane algebraic curves of a fixed degree?*

That question is straight-forward in degrees 1 and 2. Indeed, a degree 1 curve necessarily has $\mathbf{R}A$ composed of one pseudo-line, and thus the only complex scheme of degree 1 is $\langle \mathcal{J} \rangle_I^1$. Now, a degree 2 curve has either no component or one oval, and those are respectively type II and type I. Therefore, there are exactly two complex schemes of degree 2, mainly $\langle \varnothing \rangle_{II}^2$ and $\langle 1 \rangle_I^2$.

Already in degree three, there is a slight issue. The real scheme can be either $\langle \mathcal{J} \rangle$ or $\langle \mathcal{J} \sqcup 1 \rangle$. The first one is an $(M-1)$-curve, and the second is an M-curve. Therefore, they are respectively types II and I, by Proposition 1.22(5). But the second scheme could be either $\langle \mathcal{J} \sqcup 1_+ \rangle$ or $\langle \mathcal{J} \sqcup 1_- \rangle$. We need new restrictions to the semi-orientations that come from topology to be able to decide.

The complex orientation formula, due to Rokhlin [Roh74, §2] and Mishachev [Miš75, §3], is the first source of restrictions on complex schemes of curves. Their proofs were originally formulated for M-curves, but work for all type I curves. We give the formulation from [Rok78, 2.3 and 2.4]. We denote as ℓ the number of *ovals* of a curve. That is, $\ell = b_0(\mathbf{R}A)$ if the degree is even, and $\ell = b_0(\mathbf{R}A) - 1$ if the degree is odd.

Theorem 1.27 (Complex orientation formula)**.** *Let A be a degree m type I non-singular plane curve.*

(1) If $m = 2k$ is even, then $2(\Pi^+ - \Pi^-) = \ell - k^2$.

(2) If $m = 2k+1$ is odd, then $\Lambda^+ - \Lambda^- + 2(\Pi^+ - \Pi^-) = \ell - k(k+1)$.

This allows to show that the only complex M-scheme of degree 3 is $\langle \mathcal{J} \sqcup 1_- \rangle_I^3$. Indeed, if $\langle \mathcal{J} \sqcup 1_+ \rangle$ were to be a degree 3 complex scheme, then this would have $\Lambda^+ = 1$ and $\Lambda^- = \Pi^+ = \Pi^- = 0$, which violates the complex orientation formula. We state a corollary of the complex orientation formula.

Corollary 1.28. *If A is a curve of odd degree $m = 2k+1$ with $k \geqslant 1$ and $b_0(\mathbf{R}A) = 1$, then A is type II.*

Proof. If it were type I, on the one hand we would have $\Lambda^+ = \Lambda^- = \Pi^+ = \Pi^- = 0$. On the other hand however, we have $\ell - k(k+1) < 0$, which violates the complex orientation formula. ■

In fact, we can give bounds for the number of ovals of a type I curve. We denote as $\Pi = \Pi^+ + \Pi^-$ the total number of injective pairs of ovals. The following is just a writing game and a matter of using the triangle inequality.

Corollary 1.29. *Let A be a type I curve of degree m.*

(1) If $m = 2k$ is even, then $2\Pi \geqslant |\ell - k^2|$.

(2) If $m = 2k+1$ is odd, then $2(\Pi + \ell) \geqslant k(k+1)$.

There is another important fact which is implied by the complex orientation formula. Recall that the depth of a nest of oval is at most $k = \lfloor \frac{m}{2} \rfloor$, and that in this case there are no other ovals. We also saw that this scheme is always type I. We now show that there is only one possible complex orientation of this real scheme.

Corollary 1.30. *Let A be a curve of degree m whose real scheme is the hyperbolic scheme (that is, the maximal nesting of k ovals, with $k = \lfloor \frac{m}{2} \rfloor$). Then all ovals are negative, except possibly the outermost one in the case of even degree, which does not come with a sign.*

Proof. We see that $\ell = k$ and $\Pi = \frac{k(k-1)}{2}$. In the even degree case, we obtain:

$$\begin{cases} \Pi^+ - \Pi^- = \frac{-k(k-1)}{2} \\ \Pi^+ + \Pi^- = \frac{k(k-1)}{2}, \end{cases}$$

from which we readily see that $\Pi^+ = 0$. In the odd degree case, we have:

$$\begin{cases} \Lambda^+ - \Lambda^- + 2(\Pi^+ - \Pi^-) = -k^2 \\ \Lambda^+ + \Lambda^- + 2(\Pi^+ + \Pi^-) = k^2, \end{cases}$$

which yields $\Lambda^+ = \Pi^+ = 0$. ■

We wish to finish the classification of complex schemes of degree $m \leqslant 5$. We give the list in Figure 1.16. For degree 4, the scheme $\langle \varnothing \rangle$ is necessarily type II. The schemes $\langle 1 \rangle$ and $\langle 3 \rangle$ are also type II, by Proposition 1.22(3). The scheme $\langle 1\langle 1 \rangle \rangle$ is that of the maximal nest, and is therefore type I. The complex orientation formula provides that it must have the complex scheme $\langle 1\langle 1_- \rangle \rangle_I^4$. Finally, the remaining one $\langle 4 \rangle$ is that of an M-quartic, and thus is type I.

Degree	List of schemes
1	$\langle \mathcal{J} \rangle_I^1$
2	$\langle \varnothing \rangle_{II}^2$, $\langle 1 \rangle_I^2$
3	$\langle \mathcal{J} \rangle_{II}^3$, $\langle \mathcal{J} \sqcup 1_- \rangle_I^3$
4	$\langle \varnothing \rangle_{II}^4$, $\langle 1 \rangle_{II}^4$, $\langle 2 \rangle_{II}^4$, $\langle 1\langle 1_- \rangle \rangle_I^4$, $\langle 3 \rangle_{II}^4$, $\langle 4 \rangle_I^4$
5	$\langle \mathcal{J} \rangle_{II}^5$, $\langle \mathcal{J} \sqcup 1 \rangle_{II}^5$, $\langle \mathcal{J} \sqcup 2 \rangle_{II}^5$, $\langle \mathcal{J} \sqcup 1\langle 1_- \rangle \rangle_I^5$, $\langle \mathcal{J} \sqcup 3 \rangle_{II}^5$, $\langle \mathcal{J} \sqcup 4 \rangle_{II}^5$, $\langle \mathcal{J} \sqcup 3_- \sqcup 1_+ \rangle_I^5$, $\langle \mathcal{J} \sqcup 5 \rangle_{II}^5$, $\langle \mathcal{J} \sqcup 3_+ \sqcup 3_- \rangle_I^5$

Figure 1.16. All complex algebraic schemes of degree less than 5.

For degree 5, the scheme $\langle \mathcal{J} \rangle$ is that of a minimal curve, and is thus type II. The three schemes $\langle \mathcal{J} \sqcup \alpha \rangle$ with $\alpha = 1,3,5$ are type II by Proposition 1.22(3). The M-quintic $\langle \mathcal{J} \sqcup 6 \rangle$ is type I, and the complex orientation formula (noting that $\Pi = 0$) yields $\Lambda^+ = \Lambda^- = 3$, and thus it has the complex scheme $\langle \mathcal{J} \sqcup 3_+ \sqcup 3_- \rangle$. The scheme $\langle \mathcal{J} \sqcup 1\langle 1 \rangle \rangle$ is that of a maximal nest. This is type I, and the complex orientation formula ensures it must be $\langle \mathcal{J} \sqcup 1_- \langle 1_- \rangle \rangle_I^5$. The scheme $\langle \mathcal{J} \sqcup 2 \rangle$ has no possible semi-orientation that satisfies the complex orientation formula, and is thus type II. There is one remaining scheme, mainly $\langle \mathcal{J} \sqcup 4 \rangle$. It turns out that it is of indefinite type (it can be realised both by a type I or a type II curve). The type I realization only has the semi-orientation $\langle \mathcal{J} \sqcup 3_- \sqcup 1_+ \rangle_I^5$ satisfying the orientation formula. However, to construct this scheme, we need an improved version of the small perturbation theorem.

The following observation is due to Fiedler (see [Rok78, §3.7]).

Theorem 1.31. *Let $A_1, \dots, A_r$ be non-singular real curves of respective degrees $m_1, \dots, m_r$. Assume that triple intersections are all empty, and that each $\mathbf{R}A_i$ intersects each $\mathbf{R}A_j$ transversely in $m_i m_j$ double points. Let A be a curve obtained by a classical small perturbation of the curve $A_1 \times \cdots \times A_r$. The following are equivalent:*

(1) A is type I;

(2) each of the A_i is type I, and the perturbation of $\mathbf{R}A_1 \cup \cdots \cup \mathbf{R}A_r$ induces a well-defined orientation on the whole $\mathbf{R}A$.

In this case, the orientation on $\mathbf{R}A$ induced by the perturbation of $\mathbf{R}A_1 \cup \cdots \cup \mathbf{R}A_r$ comes from topology, and is in fact one of the two complex orientations induced by each component of $\mathbf{C}A \smallsetminus \mathbf{R}A$.

We now realize the two complex schemes $\langle \mathcal{J} \sqcup 4 \rangle_{II}^5$ and $\langle \mathcal{J} \sqcup 3_- \sqcup 1_+ \rangle_I^5$ by a perturbation of the union of two conics and a line.

To obtain a construction of the scheme $\langle \mathcal{J} \sqcup 3_- \sqcup 1_+ \rangle_I^5$, we fix the orientations on both conics and the line as depicted in Figure 1.17. Now, Theorem 1.31 ensures that the oriented resolutions (which are possible to choose, by Brusotti's theorem) of each double point indeed yield a type I curve. For the construction of the scheme $\langle \mathcal{J} \sqcup 4 \rangle_{II}^5$, this time, we resolve each double point (again, this is possible

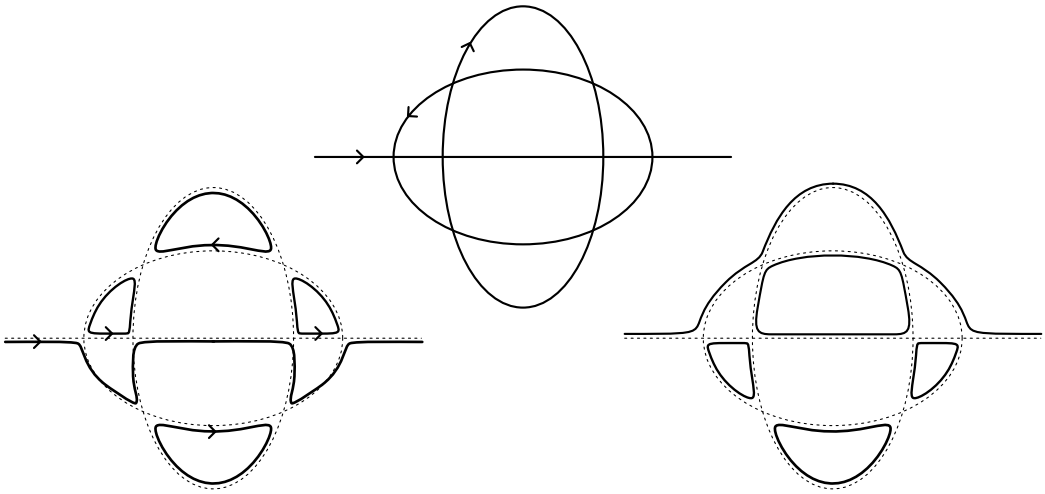


Figure 1.17. The real degree 5 scheme $\langle \mathcal{J} \sqcup 4 \rangle$ is of indefinite type. Both the type I and type II realizations can be obtained by perturbing a union of two conics and a line.

by Brusotti's theorem) as in the right of Figure 1.17. By contradiction, assume that the resulting curve was type I. Then any orientation of the pseudo-line determines a unique orientation on both conics and of the line of the singular degree 5 curve, by Theorem 1.31. But it is easy to see that taking those orientations back to orientations on the ovals of the perturbed curve, there is a mismatch.

1.2.3 New Restrictions Coming From the Complex Viewpoint

Because the topology of knotted surfaces in $\mathbf{CP}^2$ is richer than that of embedded circles in $\mathbf{RP}^2$, this allows for more freedom in finding new restrictions on real schemes of algebraic curves. So far, all restrictions[†] we have seen have taken the form of (in)equalities. New restrictions in the case of even degree curves take the form of *congruences*.

If A is an even degree $m = 2k$ curve, ovals can be partitioned into **even** or **odd** ones if they are included in an even or odd number of other ovals accordingly. We denote as p (*resp.* n) the number of even (*resp.* odd) ovals.

Theorem 1.32. *Let A be an even degree $m = 2k$ curve.*

(1) ***(Arnold)*** *[Arn71, Theorem 1] If A is type I, then $p - n \equiv k^2$ [4].*

(2) ***(Gudkov–Rokhlin)*** *[Rok78, (1)] If A is an M-curve, then $p - n \equiv k^2$ [8].*

(3) ***(Gudkov–Krakhnov–Kharlamov)*** *[GK73], [Kha75] If A is an $(M-1)$-curve, then $p - n \equiv k^2 \pm 1$ [8].*

(4) ***(Kharlamov–Marin)*** *[Rok78, §3.4] If A is an $(M-2)$-curve and $p - n \equiv k^2 + 4$ [8], then A is type I.*

[†] Except the Klein congruence Proposition 1.22(3).

The complex point of view may also shed light on new restrictions on the real scheme, even when the curve is not necessarily of type I. One such instance is about bounding the number of non-empty ovals of a curve. We denote as ℓ^+, ℓ^0 and ℓ^- the number of ovals which bound from the outside a component of the complement of the curve in $\mathbf{RP}^2$ which has positive, zero or negative Euler characteristic, respectively. Note that an oval is empty if and only if it contributes to ℓ^+, in which case the Euler characteristic of that oval is one. In particular, the number of non-empty ovals of a curve is $\ell^- + \ell^0$. Similarly, we set p^+, p^0, p^-, n^+, n^0 and n^- to be the number of even or odd ovals whose Euler characteristic of their interior has its sign prescribed.

Theorem 1.33. *Let A be a non-singular real curve of degree m.*

(1) ***(Arnold inequalities)*** *[Rok78, (7) and (8)] If $m = 2k$ is even, then:*

$$p^- + p^0 \leqslant \frac{(k-1)(k-2)}{2} + \frac{1+(-1)^k}{2} \text{ and } n^- + n^0 \leqslant \frac{(k-1)(k-2)}{2}.$$

Moreover, we have the following extremal properties:

(a) if k is even and $p^- + p^0 = \frac{(k-1)(k-2)}{2} + 1$, then $p^- = p^0 = 0$;

(b) if k is odd and $n^- + n^0 = \frac{(k-1)(k-2)}{2}$, then $n^- = n^+ = 0$.

(2) ***(Viro–Zvonilov inequalities)*** *[VZ92, Theorem 2] If $m = 2k+1$ is odd, then*

$$\ell^- \leqslant \frac{(m-3)^2}{4} \text{ and } \ell^- + \ell^0 \leqslant \frac{(m-3)^2}{4} + \frac{m^2-h^2}{4h^2},$$

where $h = \max p^{\nu_p(m)}$ is the biggest prime power dividing m. Moreover, in the case where equality holds, letting $h = p^\alpha$, then there exist components $B_1, \dots, B_r$ of $\mathbf{RP}^2 \smallsetminus \mathbf{R}A$ and scalars $\alpha_1, \dots, \alpha_r \in \mathbf{Z}/p$ such that the boundary of the chain $\alpha_1[B_1] + \cdots + \alpha_r[B_r] \in C_2(\mathbf{RP}^2; \mathbf{Z}/p)$ is $[\mathbf{R}A] \in C_1(\mathbf{RP}^2; \mathbf{Z}/p)$.

We now state Fiedler's alternation rule for complex orientations. This restrictions is of a whole new type, in the sense that it works for both even and odd degrees, and that it does not take the form of neither an (in)equality or a congruence (and rather really takes the *geometry* into account).

Theorem 1.34 ([Fie82, Theorem 1])**.** *Let A be a non-singular type I real curve, and let Δ be a pencil of lines passing through a point $q \in \mathbf{RP}^2$ not lying on $\mathbf{R}A$, on one of the inflectional tangents of $\mathbf{R}A$ nor on a real line tangent to $\mathbf{C}A \smallsetminus \mathbf{R}A$. Let τ_1 and τ_2 be two tangency points between two lines L_1 and L_2 of Δ and $\mathbf{R}A$. Assume that τ_1 and τ_2 are connected by a path in $\mathbf{C}A \smallsetminus \mathbf{R}A \cup \{\tau_1, \tau_2\}$. Fix an orientation on L_1 that agrees to that of $\mathbf{R}A$ at the point τ_1, and bring this orientation to one on L_2 through Δ. Then this orientation of L_2 agrees with that of $\mathbf{R}A$ at the point τ_2.*

A proof of this result can also be found in [Vir83, Theorem 1.3.A]. Instead of presenting it, we will rather show how one may apply this fact and derive[†] the following.

Proposition 1.35 ([Fie82, Theorem 4])**.** *There are no M-curves of degree 8 with the real scheme $\langle 1\langle 1\rangle \sqcup 1\langle \alpha\rangle \sqcup 1\langle \beta\rangle$ where $\alpha, \beta \geqslant 1$, $\alpha + \beta = 18$ and α and β are both even.*

[†] There was a small translation error from Russian to English in the proof of that fact, but the result itself is correct.

Proof. We assume the contrary. Like all M-curves, such a curve will be type I. Choose a basepoint for the pencil of lines inside the inner oval of the subscheme $\langle 1\langle 1\rangle\rangle$. By the Bézout theorem, we see that the curve is arranged as in Figure 1.18.

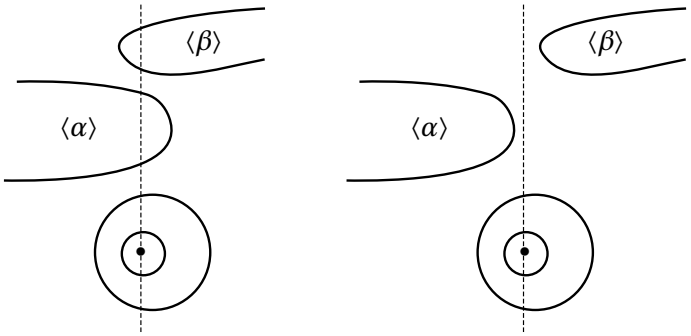


Figure 1.18. The two possible dispositions of the degree 8 curve with respect to a line of the pencil.

Therefore, the α ovals together form what is called a *chain* of ovals. That is: their orientations alternate in the manner depicted in Figure 1.19. The same can be said about the β ovals.

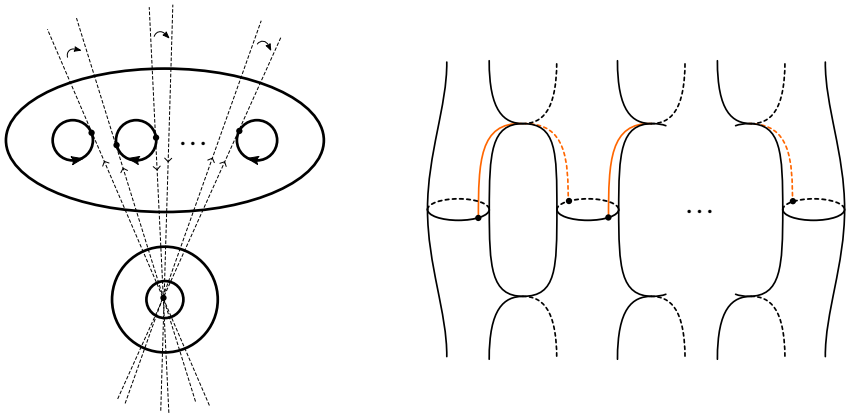


Figure 1.19. Using Theorem 1.34 in succession yields that the α ovals have alternating signs. To the right, in orange, we depict paths in $\mathbf{C}A \smallsetminus \mathbf{R}A$ connecting two successive tangency points.

Therefore, the number of positive and negative pairs formed by the α ovals are equal (α is even by assumption), and thus their contribution to $\Pi^+ - \Pi^-$ is zero. The same holds for the β ovals. It remains only one injective pair, mainly the $\langle 1\langle 1\rangle\rangle$ subscheme, which contributes to ± 1 to $\Pi^+ - \Pi^-$. This altogether provides $\Pi^+ - \Pi^- = \pm 1$. Now, because $\ell = 22$ and $m^2/4 = 16$, the complex orientation formula gives $\Pi^+ - \Pi^- = 3$, a contradiction. ∎

1.3 Towards the Topological Approach: Flexible Curves

In general, many of the known restrictions on the topology of real plane algebraic curves are of topological nature. For instance, the Harnack upper bound can be deduced solely from the study of involutions on orientable closed surfaces, without ever using the Bézout theorem.

In this spirit, Viro proposed in 1984 to study a wider class of curves, namely *flexible* curves. As the name suggests, they allow for more flexibility, as opposed to the rigid nature of algebraic equations. Of course, algebraic curves will also be flexible, and thus any result regarding constructions will hold.

Restrictions on algebraic curves that also hold for flexible ones are said to be of topological nature. The main focus of the present work is to study restrictions of topological nature. We therefore give a brief survey of what restrictions are known to be true for flexible curves.

1.3.1 A Few Words About the Definition

The following definition is due to Viro [Vir84, §1].

Definition 1.36. *Let $F \subset \mathbf{CP}^2$ be a smoothly embedded closed, connected and orientable surface. F is called a **flexible degree m curve** if:*

(1) $\mathrm{conj}(F) = F$;
(2) $[F] = m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2;\mathbf{Z})$;
(3) $\chi(F) = -m^2 + 3m$;
(4) for all $x \in \mathbf{R}F$, $T_xF = T_x\mathbf{R}F \oplus \mathbf{i}\cdot T_x\mathbf{R}F$, where $\mathbf{R}F = F \cap \mathbf{RP}^2$.

Each point of this definition deserves some explaining. Of course, asking that F is closed, connected and orientable are reasonable[†] requirements.

(1) The condition that $\mathrm{conj}(F) = F$ is analogous to considering a *real* algebraic curve. In particular, if we restrict complex conjugation to F, we see that $\mathrm{Fix}\left(\mathrm{conj}|_F\right) = F \cap \mathbf{RP}^2$.

(2) The way to topologically *see* the degree of an algebraic curve is through its homology class; therefore, it is natural that they should satisfy $[F] = m[\mathbf{CP}^1]$.

(3) The condition $\chi(F) = -m^2 + 3m$ is equivalent to asking that F has genus $g = (m-1)(m-2)/2$. If this can be thought as merely demanding that F satisfies some genus-degree formula, it really means that F realizes some extremal genus bound. Indeed, the Thom conjecture, proved by Kronheimer and Mrowka in [KM94, Theorem 1], states that any (closed, connected and orientable) smoothly embedded surface $\Sigma \subset \mathbf{CP}^2$ with $[\Sigma] = d[\mathbf{CP}^1]$ satisfies $\chi(\Sigma) \leqslant -d^2 + 3|d|$. In this sense, algebraic curves are genus-minimizing in their own homology classes, hence that ought to be true for flexible curves too.

(4) The last condition is maybe more geometric. It has the consequence that $\mathbf{R}F$ is a collection of embedded circles in $\mathbf{RP}^2$ (there are no isolated points, nor self-intersections). One can think of it as requiring that a flexible curve intersects $\mathbf{RP}^2$ "like an algebraic curve would". Indeed, what we ultimately care about is the topology of the pair $(\mathbf{RP}^2, \mathbf{R}F)$ for those algebraic curves, so it

[†] Despite this, we will still consider the possibility to allow F being non-orientable in Chapter 4.1.

makes sense that we want this to resemble a real plane curve as much as possible. We will also see in §2.1.1 that this condition is important to compute homological invariants.

The most important difference between algebraic and flexible curves is the lack of the Bézout theorem. A restriction on the scheme of an algebraic curve is said to be of **topological nature** if it also holds for flexible curves. On the other hand, those restrictions that are known not to hold for flexible curves could be called of *algebraic nature*. The difficulty lies in between. Indeed, algebraic restrictions having no apparent reason of holding for flexible curves sometimes do, at least in low degrees.

In fact, the Bézout theorem itself does *not* hold for flexible curves. Indeed, take an algebraic conic, and deform it slightly (the deformation happens to the whole surface) as in Figure 1.20. Then a real line (let alone a flexible curve of degree 1) can intersect it transversely in an arbitrarily large even number of points.

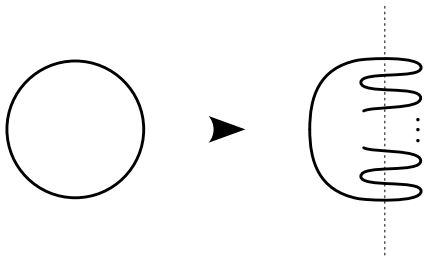


Figure 1.20. An isotopy of an algebraic curve turning it into a flexible one needs not respect the Bézout theorem anymore.

The main ingredient to start classifying real schemes of *flexible* curves is still going to be Harnack's bounds. Because Klein's proof is purely topological, the upper bound

$$b_0(\mathbf{R}F) \leqslant g+1$$

still holds for flexible curves. Regarding the lower bound, it suffices to see that $[\mathbf{R}F]$ has its homology class in $H_1(\mathbf{RP}^2;\mathbf{Z})$ prescribed by its degree, just as before. Consider a projective line L such that $\mathbf{C}L$ and F intersect transversely. Therefore, we have $Q_{\mathbf{CP}^2}(F,\mathbf{C}L) = m$, where $Q_{\mathbf{CP}^2}$ is the intersection form of $\mathbf{CP}^2$ and m denotes the degree of F. In particular, we see that $Q_{\mathbf{CP}^2}(F,\mathbf{C}L)$ has the same parity as m. Moreover, any intersection points in $(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}L$ come in conjugate pairs, so that

$$Q_{\mathbf{CP}^2}(F,\mathbf{C}L) \equiv \#\mathbf{R}F \cap \mathbf{R}L \ [2].$$

This implies that $[\mathbf{R}F] = 0$ or 1 in $H_1(\mathbf{RP}^2;\mathbf{Z}) \cong \mathbf{Z}/2$, with the same parity as the degree.

This is sufficient to classify all flexible schemes of degree $m \leqslant 3$. Indeed, the bounds are so restrictive that there are not so many possible real schemes, and all were already realized as algebraic ones (thus flexibly too).

We need finer restrictions in degrees $m \geqslant 4$. The classification of flexible schemes of degrees $m \leqslant 5$ agrees with that of algebraic curves. To this end, we state which results hold for flexible curves.

- Proposition 1.2: the real curve $\mathbf{R}F$ has the homology class its degree mod 2 in $H_1(\mathbf{RP}^2;\mathbf{Z})$.

- Theorem 1.8: the Harnack bounds $\varepsilon \leqslant b_0(\mathbf{R}F) \leqslant g+1$, and the existence of schemes realizing every possible integer in between.
- Figure 1.7: the classification of algebraic and flexible real schemes coincides in degrees $m \leqslant 5$.
- Figure 1.8: the classification also coincides in degree $m = 6$.
- Definition 1.21: the dichotomy between separating and non-separating also makes sense for flexible curves, and semi-orientations do too.
- Proposition 1.22: in particular, what also holds is orientability of the orbit space F/conj, Klein's congruence and the fact that M-curves and $(M-1)$-curves are type I and II respectively.
- Theorem 1.27: Rokhlin–Mishachev's complex orientation formula is topological. In particular, Corollary 1.28 and Corollary 1.29 hold as well.
- Corollary 1.30: if we add the assumption that a flexible realization of the maximal nesting of ovals has no other oval and is type I, then it has all ovals that are negative.
- Figure 1.16: the classification of algebraic and flexible complex schemes of degrees $m \leqslant 5$ agree.
- Theorem 1.32: the congruences for M-curves, $(M-1)$-curves and $(M-2)$-curves are of topological nature.
- Theorem 1.33: the Arnold and the Viro–Zvonilov inequalities are topological too.

Of course, all the constructions of algebraic curves also produce constructions of flexible ones. Moreover, all scheme notations introduced by Viro are well-defined even for flexible schemes.

On the other hand, we also list the results whose proofs do not work for flexible curves.

- Proposition 1.4, that the maximal depth of a nest of ovals is k, and such a scheme is unique and contains no other ovals.
- Proposition 1.23: *a priori*, since the proof of the fact that the maximal nesting of ovals is type I relies heavily on the study of the spaces $\mathscr{C}_m$ and $\mathscr{C}_m^{\mathbf{C}}$.

Now, for degree $m = 4$, the Arnold inequalities imply that $n^- + n^0 \leqslant 0$, and thus the curve cannot have a nest of depth $\geqslant 3$. Moreover, if the curve has a nest of depth 2, then $p^- + p^0 = 1$, and the extremal property of the Arnold inequalities provide that $p^+ = 0$, so that there are no other ovals.

In degree $m = 5$, the Viro–Zvonilov inequality gives $\ell^- + \ell^0 \leqslant 1$, so that the curve has at most one non-empty oval. Moreover, the extramal property of that inequality ensures that if the curve does have a non-empty oval, then it has no other empty oval.

1.3.2 Pseudo-Holomorphic Curves and Inclusions

We will denote as $\mathscr{A}_m$ and $\mathscr{F}_m$ the collections of *isotopy types* of $\mathbf{R}F \subset \mathbf{RP}^2$ for F an algebraic or flexible plane curve of degree m, respectively. Among flexible curves, there are some of a particular type, with additional geometric properties that resemble more algebraic curves, yet retaining some of the topological freedom. Recall that $\mathbf{CP}^2$ comes with a complex structure $\mathbf{i}$, mainly complex multiplication by the imaginary unit.

Definition 1.37. *Let J be an almost complex structure on $\mathbf{CP}^2$. A J-**holomorphic curve** in $(\mathbf{CP}^2, J)$ is a smooth map $\varphi : (\Sigma, j) \to (\mathbf{CP}^2, J)$ which is J-holomorphic; that is:*

$$\forall x \in \Sigma,\ \mathrm{d}_x\varphi \circ j_x = -J_x \circ \mathrm{d}_x\varphi,$$

where Σ *is a closed oriented, connected and surface and* j *is an almost-complex structure on* Σ.

Note that we do not necessarily require the map φ to be an embedding (or even an immersion). In the case of real dimension two, any almost-complex structure j on a closed connected oriented surface Σ is integrable. That is: any almost-complex surface is a Riemann surface.

In the set $\mathscr{F}_m$ comes the subset of real schemes of flexible curves of the form $\varphi(\Sigma)$ where $\varphi : \Sigma \to \mathbf{CP}^2$ is a J-holomorphic *embedding*[†]. We denote as $\mathscr{J}_m$ that subset, and we call such a curve a **pseudo-holomorphic curve**. Clearly, we have:

$$\mathscr{A}_m \subset \mathscr{J}_m \subset \mathscr{F}_m.$$

One can naturally ask whether those inclusions are strict. We have proved in §1.3.1 that we have $\mathscr{A}_m = \mathscr{F}_m$ for $m \leqslant 5$. In fact, this is also true for $m = 6$. However, those occurrences are, in a sense, *pure luck*, since many restrictions for schemes in $\mathscr{A}_m$ do not have reasons to hold for schemes in $\mathscr{J}_m$ or $\mathscr{F}_m$. Mainly, those restrictions come directly from applications of the Bézout theorem. For instance, the facts that a nest has depth at most $\left\lfloor \frac{m}{2} \right\rfloor$, or that if a curve has two nests of depth d_1 and d_2, then $d_1 + d_2 \leqslant \left\lfloor \frac{m}{2} \right\rfloor$, come directly from using an auxiliary real line and counting real intersection points.

For a partial answer to this question we refer the reader to:

(1) Orevkov's construction [Ore21] of an *oriented* type I scheme of degree 9 which is realizable by a pseudo-holomorphic curve but not by an algebraic one;

(2) Fiedler-Le Touzé and Orevkov's work [FO02] and [FOS20] on the classification of *affine* sextics (sextics arranged in a certain way with respect to a real line), where they construct pseudo-holomorphic affine sextics which are not realizable algebraically;

(3) Brugallé's classification [Bru07] of symmetric curves of degree 7, which also exhibits pseudo-holomorphic symmetric curves not realizable by algebraic symmetric curves.

[†] Here, we ask that it is J-holomorphic for *some* almost-complex structure J which is tamed by the Fubini–Study symplectic form of $\mathbf{CP}^2$.

1.4 A Glossary of Standard Notations

Unless context is clear, we will reserve some letters to denote certain integer values associated to a real curve, and stick to those notations. They are standard in literature. When in doubt, the reader can always refer to this list.

Notation	Description
m	Degree of the curve.
k	$k = \lfloor \frac{m}{2} \rfloor$, so that $m = 2k$ or $m = 2k+1$ whenever m is even or odd.
g	$g = \frac{(m-1)(m-2)}{2}$ the genus of the curve.
ℓ	Number of ovals of the curve. $\ell = b_0(\mathbf{R}F)$ if m is even, and $\ell = b_0(\mathbf{R}F) - 1$ if m is odd.
p	Number of positive ovals; those that are situated inside an even number of other ovals.
n	Number of negative ovals; those situated inside an odd number of ovals.
Π	Number of injective pairs of ovals (pairs of ovals where one is situated inside the other).
$\ell^{\pm}$, ℓ^0	Number of ovals which bound from the outside a component of $\mathbf{RP}^2 \smallsetminus \mathbf{R}F$ of positive, negative or zero Euler characteristic.
$p^{\pm}$, p^0	Number of positive ovals which bound from the outside a component of $\mathbf{RP}^2 \smallsetminus \mathbf{R}F$ of positive, negative or zero Euler characteristic.
$n^{\pm}$, n^0	Number of negative ovals which bound from the outside a component of $\mathbf{RP}^2 \smallsetminus \mathbf{R}F$ of positive, negative or zero Euler characteristic.
$\Pi^{\pm}$	For a Type I curve only. Number of positive or negative injective pairs of ovals, respectively. $\Pi = \Pi^+ + \Pi^-$.
$\Lambda^{\pm}$	For an odd degree curve of type I only; number of positive or negative ovals (with respect to the pseudo-line $\mathcal{J}$), respectively. $\ell = \Lambda^+ + \Lambda^-$.
d	For a Type I curve only. Number of disorienting ovals; those negative ovals which form a negative injective pair with the positive oval immediately surrounding them.
$D^{\pm}$	For a Type I curve only. Number of positive or negative injective pairs whose outer oval is disorienting, respectively.

The Topology of Knotted Surfaces

2

As a simple example, currently there are no tools for studying smooth manifolds homeomorphic to the 4-dimensional sphere: there might be infinitely many distinct such creatures, or just good old S^4. It's a wide and wild world out there.

A. Scorpan

Outline of this chapter

2.1 Intersection Theory

If X is a closed 4-manifold and Σ_1, Σ_2 are two closed oriented surfaces embedded in X, then by codimension reasons, they can be perturbed slightly to ensure that their intersection $\Sigma_1 \cap \Sigma_2$ is transverse, and is a collection of finitely many points. The number of such transverse intersection points is a well-defined map on homology if it is regarded mod 2, and it gives a bilinear map

$$Q_{X,\mathbf{Z}/2} : H_2(X;\mathbf{Z}/2) \times H_2(X;\mathbf{Z}/2) \to \mathbf{Z}/2.$$

There is, in fact, an integral lift of this map to $H_2(X;\mathbf{Z})$: consider, for $x \in \Sigma_1 \pitchfork \Sigma_2$, the local orientations o_1 and o_2 of Σ_1 and Σ_2 at x, and compare the orientation (o_1, o_2) with the ambient orientation of X at x. Set $\varepsilon(x)$ to be ± 1 according to whether those agree or not, and consider the following sum:

$$Q_X(\Sigma_1, \Sigma_2) = \sum_{x \in \Sigma_1 \pitchfork \Sigma_2} \varepsilon(x).$$

This again gives a well-defined symmetric bilinear map

$$Q_X : H_2(X;\mathbf{Z}) \times H_2(X;\mathbf{Z}) \to \mathbf{Z},$$

which is called the **intersection form** of X. This is an integral lift of $Q_{X,\mathbf{Z}/2}$ in the following sense: if $\beta : H_2(X;\mathbf{Z}) \to H_2(X;\mathbf{Z}/2)$ denotes reduction mod 2, then:

$$\forall \xi_1, \xi_2 \in H_2(X;\mathbf{Z}),\ Q_{X,\mathbf{Z}/2}(\beta(\xi_1), \beta(\xi_2)) \equiv Q_X(\xi_1, \xi_2)\ [2].$$

It is also possible to define Q_X algebraically, by means of Poincaré duality. For instance, it corresponds to the cohomological pairing

$$(\alpha, \beta) \in H^2(X) \times H^2(X) \mapsto \langle \alpha \smile \beta, [X] \rangle.$$

It turns out that Q_X disregards torsion in $H_2(X;\mathbf{Z})$, so that the map

$$Q_X : H_2(X;\mathbf{Z})/\mathrm{Tors} \times H_2(X;\mathbf{Z})/\mathrm{Tors} \to \mathbf{Z}$$

is non-degenerate.

The special case of **self-intersections** $Q_X(\Sigma, \Sigma)$ is also very interesting. This can be further extended to when Σ is non-orientable. Indeed, consider a transverse perturbation Σ' of Σ in the normal direction. Then, for $x \in \Sigma \pitchfork \Sigma'$, pick a local orientation o of Σ at x, and push this orientation through the section of the normal bundle used to perturb Σ into Σ'. This gives o' an orientation at x for Σ'. As before, define $\varepsilon(x) = \pm 1$ according to whether (o, o') agrees with the orientation of X at x, and define the self-intersection to be

$$\Sigma \cdot \Sigma = \sum_{x \in \Sigma \pitchfork \Sigma'} \varepsilon(x) \in \mathbf{Z}.$$

In the case where Σ is orientable, this agrees with $Q_X(\Sigma, \Sigma)$. However, it is not possible to define an intersection number $\Sigma_1 \cdot \Sigma_2$ when at least one of the Σ_i is non-orientable.

Self-intersections will be an important tool for us to study the topology of plane curves. Indeed, from a general knotted surface-theoretic point of view, knowing the self-intersection of an embedded surface gives rise to constraints on its topology.

Details regarding intersection forms of 4-manifolds can be found in [GS99, §1.2] or [Kir89, Chapter II]. For characteristic classes, we refer the reader to the famous and unavoidable [MS74]. Regarding twisted (co-)homology, constructions are found in [Ste43] or [Hat02, Appendix 3.H].

2.1.1 Normal Euler Numbers

We discuss the possible interpretations of the self-intersection number of a surface, and the generalizations to surfaces with boundary. We fix X to be a connected, closed and oriented smooth 4-manifold. In particular, for the first Stiefel–Whitney class, we observe that $w_1(TX) = 0$. We also consider X to be endowed with a Riemannian metric. One has to keep in mind that normal bundles do depend on this metric, although this dependence will be omitted throughout.

Pick F to be a closed connected surface. Then, denoting as νF the normal bundle of F, we observe that

$$TX|_F = TF \oplus \nu F.$$

From the Whitney sum formula

$$w_1(TX|_F) = w_1(TF) + w_1(\nu F),$$

we obtain $w_1(\nu F) = w_1(TF)$, and thus νF is an orientable bundle over F if and only if TF is, which is the case only when F is orientable. In this case, νF therefore comes with an Euler class $e(\nu F) \in H^2(F;\mathbf{Z})$ which is an integral lift of the second Stiefel–Whitney class $w_2(\nu F) \in H^2(F;\mathbf{Z}/2)$. We recall (see [Šar73, §1.3]) that this Euler class can be constructed in the following manner. Denote as $u \in H^2(\nu F, (\nu F)_0;\mathbf{Z})$ the fundamental class, and consider the projection and inclusion maps

$$p : \nu F \to F \text{ and } j : (\nu F, \varnothing) \to (\nu F, (\nu F)_0),$$

where $(\nu F)_0 = \nu F \smallsetminus F$. Then the Euler class is given by

$$e(\nu F) = (p^*)^{-1} j^* u.$$

Evaluating $e(\nu F)$ over the fundamental class $[F]$, we obtain an integer

$$e(X,F) = \langle e(\nu F), [F] \rangle \in \mathbf{Z}.$$

If we drop the condition that F is orientable, this time the bundle νF is non-orientable, but there is still a *twisted* Euler class

$$e(\nu F) \in H^2(F;\mathbf{Z}_w),$$

where $\mathbf{Z}_w$ stands for $\mathbf{Z}$-coefficients twisted by $w_1(\nu F) = w_1(TF)$. The same construction as before can be applied by replacing all $\mathbf{Z}$-coefficients with $\mathbf{Z}_w$, where $w_1(TF)$ induces a representation of $\pi_1(\nu F)$ *via* the inclusion $p_*(\pi_1(\nu F)) \subset \pi_1(F)$. Twisted Poincaré duality provides:

$$H^2(F;\mathbf{Z}_w) \cong H_0(F;\mathbf{Z}_w) \cong \mathbf{Z}.$$

Therefore, evaluating $e(\nu F)$ over the twisted fundamental class $[F] \in H^2(F;\mathbf{Z}_w)$ also gives an integer:

$$e(X,F) = \langle e(\nu F), [F] \rangle \in \mathbf{Z}.$$

Note that in the case where F *is* orientable, then $\mathbf{Z}_w$ is isomorphic to $\mathbf{Z}$, and thus twisted cohomology corresponds to regular cohomology.

We call $e(X,F)$ the **normal Euler number** of the embedding $F \subset X$. Note that this number *does* depend on how F is embedded in X, and not just of the topological type of F (indeed, one can consider a null-homologous sphere compared to a complex line in $\mathbf{CP}^2$ for instance). The following result is standard and can be derived from [BT82, Proposition 12.8].

Proposition 2.1. *The normal Euler number $e(X,F)$ agrees with the self-intersection number $F\cdot F$. In particular, if F is orientable, then $e(X,F) = Q_X(F,F)$.*

In the case of surfaces with boundary, one needs to fix a section on that boundary. More precisely, let $F \subset X$ be a compact and connected surface with boundary $\partial F \neq \varnothing$. Choose a *non-vanishing* section $\theta : \partial F \to \nu F|_{\partial F}$ of the normal bundle on the boundary. Then there is an associated *relative* (twisted) Euler class

$$e_\theta(\nu F) \in H^2(F, \partial F; \mathbf{Z}_w).$$

This can be defined in the same manner as before: set $V = \theta(\partial F) \subset (\nu F)_0$, and denote the projection and inclusion maps as $p : (\nu F, V) \to (F, \partial F)$ and $j : (\nu F, V) \to (\nu F, (\nu F)_0)$ respectively. If $u \in H^2(\nu F, (\nu F)_0; \mathbf{Z}_w)$ is the fundamental class, then

$$e_\theta(\nu F) = (p^*)^{-1} j^* u \in H^2(F, \partial F; \mathbf{Z}_w).$$

Again, evaluating this on the (twisted) relative fundamental class $[F, \partial F]$, we obtain an integer

$$e_s(X,F) = \langle e_s(\nu F), [F, \partial F] \rangle \in \mathbf{Z}.$$

This time, there is a dependence both on the embedding *and* on the choice of the section s on the boundary. By the same arguments as in the closed case, one can obtain the following.

Proposition 2.2. *Let $F \subset X$ be a surface with boundary $\partial F \neq \varnothing$, and let $s : \partial F \to \nu F$ be a non-vanishing section. Extend this section into a section $\sigma : F \to \nu F$ which is transverse to the zero-section. For any intersection point $x \in F \pitchfork \sigma(F)$, set $\varepsilon(x) = \pm 1$ according to the same principle as in the closed case. Then:*

$$e_s(X,F) = \sum_{x \in F \pitchfork \sigma(F)} \varepsilon(x).$$

In particular, this does not depend on the choice of the extension σ.

It will be possible to use relative normal Euler numbers and to glue surfaces with boundary together to compute the self-intersection of the closed surface thus obtained. We need the following definition.

Definition 2.3. *Let F_1 and F_2 be two surfaces (possibly with boundary) in X. We say that F_1 and F_2 intersect **neatly** if the normal bundles νF_1 and νF_2 satisfy:*

$$\forall x \in F_1 \cap F_2,\ \mathrm{rk}(\nu_x F_1 \cap \nu_x F_2) = 1.$$

We will denote as $F_1 \pitchfork\!\!\!\!\circ\ F_2$ a neat intersection.

In particular, if F_1 and F_2 intersect neatly, then $F_1 \pitchfork\!\!\!\!\circ\ F_2$ is a 1-submanifold of both F_1 and F_2. To give a collection of examples, if $F \subset \mathbf{CP}^2$ is a flexible curve, then F intersects $\mathbf{RP}^2$ neatly, and $F \pitchfork\!\!\!\!\circ\ \mathbf{RP}^2 = \mathbf{R}F$. Indeed, this is the content of the last condition in Definition 1.36, which implies that for all $x \in \mathbf{R}F$, we have[†]:

$$\nu_x F = \nu_x \mathbf{R}F \oplus \mathbf{i}\cdot\nu_x \mathbf{R}F \text{ and } \mathbf{i}\cdot\nu_x \mathbf{R}F \subset \mathbf{i}\cdot T_x \mathbf{RP}^2 = \nu_x \mathbf{RP}^2.$$

[†] The fact that $\nu\mathbf{RP}^2 = \mathbf{i}\cdot T\mathbf{RP}^2$ will be re-proved in §3.4.3.

In the special case where a closed surface F is obtained as $F = F_1 \cup F_2$ where F_1 and F_2 have boundary, then one may wonder if it is possible to have a relation between $e(X,F)$ and the relative numbers $e_{s_1}(X,F_1)$ and $e_{s_2}(X,F_2)$. In this direction, we have the following.

Proposition 2.4. *Let F_1 and F_2 be two surfaces in X with common boundary $\partial F_1 = \partial F_2$ which satisfies $\partial F_i = F_1 \pitchfork F_2$. Further assume that there exists a non-vanishing section $s : \partial F_i \to \nu F_1 \cap \nu F_2$, which can be seen as a section $s : \partial F_i \to \nu F_i|_{\partial F_i}$. Then:*

$$e(X, F_1 \cup F_2) = e_s(X,F_1) + e_s(X,F_2).$$

Proof. Let $F = F_1 \cup F_2$. The normal Euler number $e(X,F)$ does not depend on the choice of a transverse section on a submanifold of F. Therefore, extend the section $s : \partial F_i \to \nu F$ into a (possibly-vanishing) transverse section $\sigma : F \to \nu F$, and use this section to compute the self-intersection $F \cdot F$. On the other hand, this quantity also computes the sum $e(X,F_1) + e(X,F_2)$, by Proposition 2.2. ∎

In general, if F_1 and F_2 are two surfaces with common boundary $\partial F_1 = \partial F_2$ such that $\partial F_i = F_1 \pitchfork F_2$, then $\Lambda = \nu F_1 \cap \nu F_2$ is a line bundle over ∂F_i, which can also be seen as a line sub-bundle of both $\nu F_i|_{\partial F_i}$. However, this line bundle needs not have a non-vanishing section (in fact, it will have one if and only if it contains no component isomorphic to a Möbius strip).

Let $F \subset X$ be a surface with boundary $\partial F \neq \varnothing$. We see that Proposition 2.2 means that if $s : \partial F \to \nu F$ is a non-vanishing section, then $e_s(X,F)$ is the integer obstruction to extending this section into a global non-vanishing one. In this fashion, consider any $\Lambda \subset \nu F|_{\partial F}$ line sub-bundle, and define

$$\widetilde{e}_\Lambda(X,F) \in \mathbf{Z}$$

to be the integer obstruction to extending this field of lines into a global field of lines $\widetilde{\Lambda} \subset \nu F$. The following result is contained in [GM86, §III].

Proposition 2.5. *Let F_1 and F_2 be two surfaces in X with common boundary $\partial F_1 = \partial F_2$ and with $\partial F_i = F_1 \pitchfork F_2$. Let $\Lambda = \nu F_1 \cap \nu F_2$.*

(1) If Λ admits a non-vanishing section $s : \partial F_i \to \Lambda$, then $\widetilde{e}_\Lambda(X,F_i) = 2e_s(X,F_i)$.

(2) If $F = F_1 \cup F_2$, then $2e(X,F) = \widetilde{e}_\Lambda(X,F_1) + \widetilde{e}_\Lambda(X,F_2)$.

This means that, in practice, to compute the self-intersection of a surface obtained by the union of two surfaces neatly intersecting along their common boundary, we do not need that the corresponding line bundle admits a non-vanishing section.

Another way of decomposing surfaces is by tubing or taking connected sums. We make the following observation.

Proposition 2.6.

(1) Let $F_1, F_2 \subset X$ be two closed connected surfaces which are disjoint: $F_1 \cap F_2 = \varnothing$. Consider a path γ connecting F_1 to F_2, and denote as F the surface obtained from F_1 and F_2 by tubing F_1 to F_2 along γ. Then:

$$e(X,F) = e(X,F_1) + e(X,F_2).$$

(2) Let $F_1 \subset X_1$ and $F_2 \subset X_2$ be two closed connected surfaces. Form the connected sum $(X_1\#X_2, F_1\#F_2)$. Then:

$$e(X_1\#X_2, F_1\#F_2) = e(X_1, F_1) + e(X_2, F_2).$$

Proof. The second claim is an immediate consequence of the first applied to the 4-manifold $X = X_1\#X_2$. For the first, let D_1 and D_2 be small discs which are neighborhoods of the endpoints of γ in F_1 and F_2, respectively, and let T be a small cylinder around γ with $\partial T = \partial D_1 \sqcup \partial D_2$. We can ensure that:

(1) $T \cap F_1 = \partial D_1$ and $T \cap F_2 = \partial F_2$;
(2) $\partial D_1 = (F_1 \smallsetminus D_1) \curlywedge (T \cup D_2)$, and there is a nowhere vanishing section $s : (T \cup D_2) \to \nu(T \cup D_2)$ (indeed, topologically, $T \cup D_2$ is a disc, which is contractible);
(3) similarly, $\partial D_2 = (F_2 \smallsetminus D_2) \curlywedge (T \cup D_1)$, and thre is a nowhere vanishing section $s : (T \cup D_1) \to \nu(T \cup D_1)$.

Now, by noting that:

$$F = [(F_1 \smallsetminus D_1) \cup T] \cup (F_2 \smallsetminus D_2),$$

we can apply Proposition 2.4 twice. ∎

One final comment we can make is that whenever surfaces are disconnected, we consider their self-intersection (and all associated Euler classes and integer obstructions) to be the sum of those of each individual components. Everything that was said so far generalizes immediately to this setting.

2.1.2 Surfaces in $\mathbf{S}^4$ and the Whitney–Massey Theorem

The case of surfaces embedded in the 4-sphere is of particular interest. Since there is no second homology, we obtain that for any closed surface $F \subset \mathbf{S}^4$, the intersection form satisfies

$$F \cdot F \equiv Q_{\mathbf{S}^4,\mathbf{Z}/2}(F, F) \equiv 0 \ [2].$$

This means that any self-intersection number is even. Moreover, in the case where F is orientable, then $[F] = 0 \in H_2(\mathbf{S}^4;\mathbf{Z})$, and thus $F \cdot F = 0$. In particular, only when a surface is non-orientable can it have a non-zero self-intersection number.

Whitney then proved the following.

Theorem 2.7 ([Whi41, Equation (8.4)])**.** *Let $F \subset \mathbf{S}^4$ be a closed connected surface. Then:*

$$e(\mathbf{S}^4, F) + 2\chi(F) \equiv 0 \ [4].$$

He also conjectured the following fact in [Whi41, Page 113], which Massey later proved.

Theorem 2.8 ([Mas69, Theorem 1])**.** *Let $F \subset \mathbf{S}^4$ be a closed connected non-orientable surface. Then:*

$$e(\mathbf{S}^4, F) \in \{2\chi(F) - 4, 2\chi(F), 2\chi(F) + 4, \ldots, 4 - 2\chi(F)\}.$$

Moreover, any of these possible values can be attained by an appropriate embedding of F in $\mathbf{S}^4$.

Theorem 2.8 will be referred to as the Whitney–Massey theorem. Moreover, it calls for the following definition, which will be used later.

Definition 2.9. *Let $\chi \leqslant 1$ be an integer, and let $e \in \{2\chi-4, 2\chi, 2\chi+4, \dots, 4-2\chi\}$. Let X be a closed connected and oriented 4-manifold. A surface $F \subset X$ which is embedded in a contractible ball inside X such that $e(X,F) = e$ and $\chi(F) = \chi$ will be called a* ***local surface*** *of* ***parameters*** *(e,χ).*

By the Whitney–Massey theorem, local surfaces of any (admissible) parameter always exist in any 4-manifold. Indeed, consider the corresponding surface in $\mathbf{S}^4$, which necessarily avoids a point. Removing an open neighborhood of that point means that the surface embeds in a 4-ball, which itself can be mapped into X in a chart. Both the Euler characteristic and the normal Euler number do not change under these operations. In fact, the existence part of the Whitney–Massey theorem follows from Proposition 2.6 and the existence of real projective planes embedded in $\mathbf{S}^4$ with normal Euler numbers ± 2. Those will be re-constructed explicitly in §3.1.2.

2.1.3 Surfaces in Other 4-Manifolds

In general, if X is a closed, connected and oriented 4-manifold, and if $F \subset X$ is a non-orientable surface smoothly embedded in it, then one cannot expect bounds on $e(X,F)$ in terms of $\chi(F)$. Indeed, X may have a lot of not null-homologous spheres and non-zero self-intersection. However, the congruence itself still holds.

Assume that X has $H_1(X;\mathbf{Z}/2) = 0$. Then there is a well-defined map (see [Yam95])

$$q : H_2(X;\mathbf{Z}/2) \to \mathbf{Z}/4$$

if one sets

$$q(\xi) = Q_X(\widetilde{\xi},\widetilde{\xi}) \ [4]$$

for $\widetilde{\xi} \in H_2(X;\mathbf{Z})$ any choice of an integral lift of $\xi \in H_2(X;\mathbf{Z}/2)$. We have the following.

Theorem 2.10 ([Yam95, Theorems 1.2 and 1.4])**.** *Let $F \subset X$ be a closed connected (but not necessarily orientable) surface in X a closed, connected and oriented 4-manifold.*

(1) If $H_1(X;\mathbf{Z}) = 0$, then

$$e(X,F) + 2\chi(F) \equiv q([F]) \ [4],$$

where $[F] \in H_2(X;\mathbf{Z}/2)$.

(2) Without the assumption on $H_1(X;\mathbf{Z})$, the map $q' : H_2(X;\mathbf{Z}/2) \to \mathbf{Z}/4$ defined by

$$q'([F]) = e(X,F) + 2\chi(F) \bmod 4$$

is a well-defined $\mathbf{Z}/4$ quadratic map.

If one studies Yamada's proof of (1), it quickly comes out that the requirement $H_1(X;\mathbf{Z}) = 0$ cannot be replaced by the weaker $H_1(X;\mathbf{Z}/2) = 0$. Indeed, one has to perform surgery on elements of the fundamental group of X and reduce to the case of a simply-connected X. However, we realized that when we needed to apply the congruence, we were in a slightly more favorable case.

Proposition 2.11. *Let X be a smooth, closed, connected and oriented 4-manifold with $H_1(X;\mathbf{Z}/2)=0$ and which is negative definite. Then the conclusion of (1) in Theorem 2.10 still holds.*

Proof. By Donaldson's diagonalization theorem, Q_X is equivalent to $\langle -1\rangle^{\oplus b_2(X)}$. We know by (2) of Theorem 2.10 that the assignment

$$q' : [F] \in H_2(X;\mathbf{Z}/2) \mapsto e(X,F) + 2\chi(F) \in \mathbf{Z}/4$$

is quadratic. Considering a basis $F_1,\dots,F_k$ of $H_2(X;\mathbf{Z})$ which diagonalizes Q_X, and reducing those basis elements mod 2, one obtains a basis of $H_2(X;\mathbf{Z}/2)$ by the assumption that $H_1(X;\mathbf{Z}/2)=0$, and on which the computation of q is easy. Indeed:

$$e(X,F_k) = Q_X(F_k,F_k) = -1 \text{ and } \chi(F_k) \equiv 0 \ [2] \text{ since } F_k \text{ is orientable.}$$

In particular, on each of those basis elements, it *does* correspond to the self-intersection of an integral lift, and this completely characterizes the quadratic form q', therefore $q'=q$. ∎

We have a stronger congruence modulo 16 for *characteristic* surfaces; that is, surfaces $F\subset X$ such that $[F]\in H_2(X;\mathbf{Z}/2)$ is Poincaré dual to $w_2(X)\in H^2(X;\mathbf{Z}/2)$. By Wu's formula, an equivalent condition is that for all $\xi\in H_2(X;\mathbf{Z}/2)$, we have $Q_{X,\mathbf{Z}/2}(\xi,[F]) = Q_{X,\mathbf{Z}/2}(\xi,\xi)$. We state the very useful Guillou–Marin congruence.

Theorem 2.12 ([GM86, Theorem 1])**.** *Let $F\subset X$ be a characteristic surface in a closed, connected and oriented 4-manifold with $H_1(X;\mathbf{Z})=0$. Then:*

$$\sigma(X) - e(X,F) \equiv 2\beta(X,F) \ [16],$$

where $\sigma(X)$ is the signature of X and $\beta(X,F)$ the Brown invariant of the embedding.

We shall describe what $\beta(X,F)$ is. Note that this is an invariant of the *embedding* $F\subset X$, and not merely of the topological type of the surface F.

We re-expose Matsumoto's description of the Brown invariant from [Mat86, §5]. Let $F\subset X$ be a closed connected surface in X. We first describe the Guillou–Marin form $\varphi : H_1(F;\mathbf{Z}/2)\to\mathbf{Z}/4$. For an immersed circle $\mathscr{C}\looparrowright F$, we have, since $H_1(X;\mathbf{Z})=0$, that $\mathscr{C}$ bounds an immersed orientable surface $\mathscr{D}\looparrowright X$. We may also impose that $\mathscr{D}$ is nowhere tangent to F. The normal bundle $\nu\mathscr{D}$ is trivial[†], and induces a global trivialization $\nu\mathscr{D}|_{\mathscr{C}}\cong\mathscr{C}\times\mathbf{R}^2$ on the boundary $\mathscr{C}=\partial\mathscr{D}$. Moreover, the normal bundle $\nu\mathscr{C}$ of $\mathscr{C}$ in F defines a line sub-bundle of $\nu\mathscr{D}|_{\mathscr{C}}$. Denote as $n(\mathscr{D})$ the number of right-handed half-twists of $\nu\mathscr{C}$ with respect to the trivialization $\nu\mathscr{D}|_{\mathscr{C}}\cong\mathscr{C}\times\mathbf{R}^2$ above (see Figure 2.1).

Define:

$$\varphi(\mathscr{C}) = n(\mathscr{D}) + 2\mathscr{D}\cdot F + 2\mathscr{C}\cdot\mathscr{C} \bmod 4, \qquad (*)$$

where $\mathscr{D}\cdot F$ is the algebraic intersection number between F and $\mathscr{D}$ (alternatively, it equals the number of transverse intersection points between F and $\mathscr{D}$ mod 2), and $\mathscr{C}\cdot\mathscr{C}$ is the self-intersection of $\mathscr{C}$

[†] The surface $\mathscr{D}$ has one boundary component. Therefore, considering a cellular decomposition of $\mathscr{D}$ to which we abstractly glue a disc, and puncturing that disc, we obatin that $\mathscr{D}$ deformation retracts to a wedge of circles. Isomorphism classes of rank 2 bundles are invariant under homotopy equivalence, and 2-plane *orientable* bundles over $\mathbf{S}^1$ are trivial. In particular, one really needs orientability of $\mathscr{D}$, and therefore the condition that $H_1(X;\mathbf{Z}/2)$ is not sufficient.

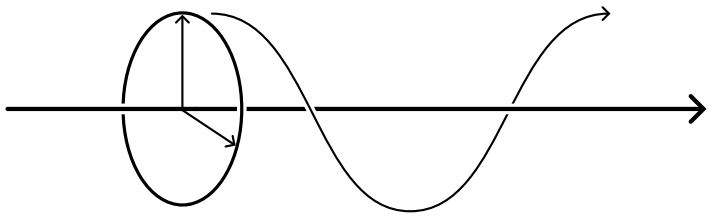


Figure 2.1. A right-handed *full* twist.

inside F. This definition of φ does not depend on the choice of $\mathscr{D}$. It only depends on the homology class $[\mathscr{C}] \in H_1(F;\mathbf{Z}/2)$, and induces a $\mathbf{Z}/4$-quadratic map

$$\varphi : H_1(F;\mathbf{Z}/2) \to \mathbf{Z}/4;$$

see [Mat86, Lemma 5.1]. The Brown invariant of the embedding is that of φ:

$$\beta(X,F) \stackrel{\text{def.}}{=} \left(\frac{1}{\sqrt{2}}\right)^{b_1(F;\mathbf{Z}/2)} \sum_{x \in H_1(F;\mathbf{Z}/2)} e^{\mathbf{i}\varphi(x)\pi/2} \in \mathbf{Z}/8. \qquad (**)$$

More precisely, $\beta(X,F)$ is an eight root of unity, but those are identified with elements of $\mathbf{Z}/8$ under

$$e^{\mathbf{i}\pi/4} \leftrightarrow 1 \in \mathbf{Z}/8.$$

We give the following example of computation.

Example 2.13. *If $K\ell \subset X$ is an embedding of a Klein bottle inside a closed, connected and oriented 4-manifold X with $H_1(X;\mathbf{Z}) = 0$, then $\beta(X,K\ell) \in \{0,2,6\}$.*

Indeed, consider the basis for $H_1(K\ell;\mathbf{Z})$ depicted in Figure 2.2, which implies that

$$H_1(K\ell;\mathbf{Z}/2) = \{0,a,b,a+b\},$$

with $b_1(K\ell;\mathbf{Z}/2) = 2$.

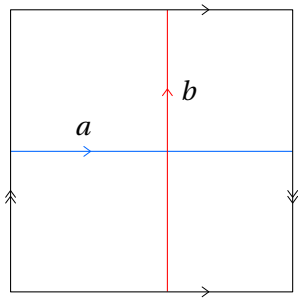


Figure 2.2. The standard basis for $H_1(K\ell;\mathbf{Z})$.

Because φ is $\mathbf{Z}/4$-quadratic, it suffices to compute $\varphi(a)$ and $\varphi(b)$ to derive:

$$\varphi(a+b) = \varphi(a) + \varphi(b) + 2a \cdot b = \varphi(a) + \varphi(b) + 2.$$

There is only a handful of possibilities, which can be computed by inspecting each term from (∗). Plugging this back into (∗∗), we obtain the following.

$\varphi(a) = \dots$	$\varphi(b) = \dots$	$\beta(\mathbf{CP}^2, K\ell) = \dots$
1	0	0
1	2	2
3	0	0
3	2	6

By similar arguments (the computations are easier), we also obtain the following.

Example 2.14. *If $\mathbf{RP}^2 \subset X$ is an embedded real projective plane, then $\beta(X, \mathbf{RP}^2) = \pm 1$.*

Alternatively, Example 2.13 and Example 2.14 can be derived from the classification of low-dimensional $\mathbf{Z}/4$-quadratic spaces; see [KV88, §5.4]. Indeed, for the case of $\mathbf{RP}^2$, we have a one-dimensional space, and thus there are only two possible $\mathbf{Z}/4$-quadratic forms for the Guillou–Marin form, whose Brown invariants are ± 1. For the Klein bottle, this is a two-dimensional vector space, but since it has an element a such that $a \cdot a \neq 0$, it is necessarily decomposable as the orthogonal sum of two rank one spaces, and thus the possible Guillou–Marin forms limit the Brown invariant values to 0 or ± 2.

2.1.4 The Case of Nodal Immersions

We will not be merely interested in embedded surfaces, but also *immmersed* ones with generic singularities.

Definition 2.15. *An immersed closed connected surface $F \subset X$ is said to be **nodal** if all of its singular points are transverse double points (also called nodes).*

Nodal surfaces $F \subset X$ also have self-intersection numbers, which are obtained by considering a normal push-off F' of F and counting transverse intersection points with signs. It happens that this push-off can be chosen so that transverse intersections between F and F' are situated away from the double points of F and of F'. In fact, some of the results stated before (Yamada's generalization of the Whitney congruence for instance; see [Yam95, Corollary 1.5]) also hold in this special case of nodally-immersed surfaces.

Of course, if Σ_1 and Σ_2 are two surfaces which intersect transversely, then $\Sigma_1 \cup \Sigma_2$ is nodally immersed, and we have:

$$e(X, \Sigma_1 \cup \Sigma_2) = e(X, \Sigma_1) + e(X, \Sigma_2).$$

We are now interested in transforming a nodal immersion into an embedding and understanding how the self-intersection might change.

Consider $\Sigma \subset X$ to be an nodally immersed surface with only one transverse double point $x \in \Sigma$. Pick $B \subset X$ to be an arbitrarily small ball around x. The intersection $\partial B \cap \Sigma$ gives a link in the 3-sphere ∂B, and this is known to be a Hopf link. There are two possibilities to resolve this singularity:

(1) blow-up the manifold X at the point x;
(2) remove $B \cap \Sigma$ from Σ, and glue a Hopf band embedded in B.

The first case is easy to work with.

Proposition 2.16. *Let Σ be as above, and denote as $\Sigma' \subset X'$ the embedded surface which results from performing the blow-up X' of X at the double point $x \in \Sigma$. Then, with respect to the decomposition $H_2(X';\mathbf{Z}/2) = H_2(X;\mathbf{Z}/2) \oplus H_2(\overline{\mathbf{CP}}^2;\mathbf{Z}/2)$, we have:*

$$[\Sigma'] = [\Sigma] \oplus [\mathbf{CP}^1] \in H_2(X';\mathbf{Z}/2),\ \chi(\Sigma') = \chi(\Sigma) + 1 \text{ and } e(X',\Sigma') = e(X,\Sigma) - 1.$$

Proof. This simply results from the fact that topologically, blowing-up amounts to taking the connected sum of (X,Σ) with $(\overline{\mathbf{CP}}^2, L)$ at x, where $L \subset \overline{\mathbf{CP}}^2$ is a line whose integral homology class spans $H_2(\overline{\mathbf{CP}}^2;\mathbf{Z})$. In particular, $L \cdot L = -1$.

The claim regarding the Euler characteristic follows from the fact that we are topologically removing a wedge of two discs from Σ to glue back two disjoint discs to it. This means that

$$\chi(\Sigma') = \chi(\Sigma) - 1 + 2,$$

as claimed. ∎

For the other resolution of the singularity, we have the following.

Proposition 2.17. *Let $\Sigma \subset X$ be an immersed surface with one nodal singularity only at $x \in \Sigma$. Let $B \subset X$ be an arbitrarily small ball around x. Consider the surface Σ' which is obtained by removing $B \cap \Sigma$ from Σ and gluing back a Hopf band bounding $\partial B \cap \Sigma$. Then:*

$$[\Sigma'] = [\Sigma] \in H_2(X;\mathbf{Z}/2),\ \chi(\Sigma') = \chi(\Sigma) - 1 \text{ and } e(X,\Sigma') = e(X,\Sigma) \pm 2.$$

Moreover, we have freedom in the choice for the ± 2.

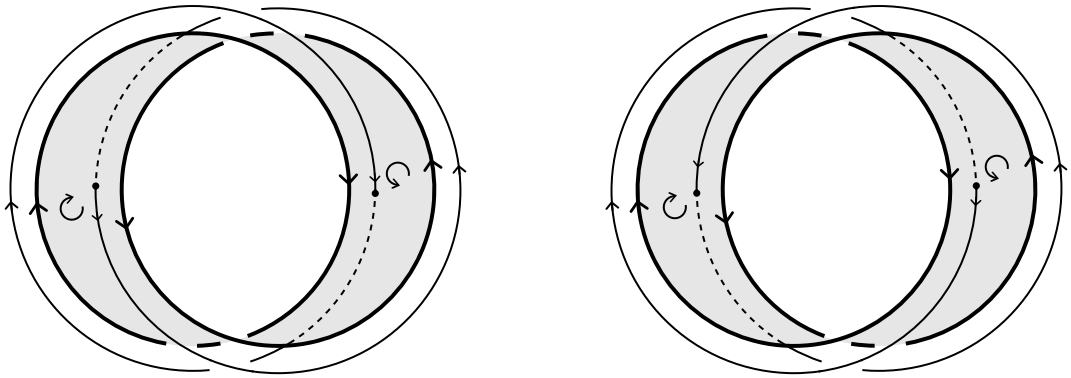
Proof. The computation of the Euler characteristic follows from the fact that we are removing a wedge of two discs to glue back a Hopf band, and thus:

$$\chi(\Sigma') = \chi(\Sigma) - 1 + 0.$$

Moreover, since the perturbation is a local construction, the homology class does not change.

For the claim regarding the normal Euler numbers, we repeat the arguments from [Yam95, §5]. The transverse push-off $s(\Sigma)$ can be assumed to be parallel to Σ near x. Let $H \subset B$ be a Hopf band bounding $\partial B \cap \Sigma$. Let $\Sigma' = (\Sigma \smallsetminus B) \cup H$. Then the intersection $s(\Sigma) \cap \Sigma'$ is two points with the same sign, and

all other points in $s(\Sigma) \cap \Sigma$ are preserved with sign in $s(\Sigma) \cap \Sigma'$. This means that $e(X, \Sigma') = e(X, \Sigma) \pm 2$. Finally, if one wants to *choose* the ± 2, note that the Hopf link $\partial B \cap \Sigma$ comes with two possible orientations, each coming from a choice of an oriented Hopf band inside B. It is easy to verify that those choices give rise to both choices of signs for the pairs of intersection points created; see Figure 2.3. ■



$e(\Sigma') = e(\Sigma) \pm 2$ $\qquad$ $e(\Sigma') = e(\Sigma) \mp 2$

Figure 2.3. The two possible choices of a smoothing of a singularity of a nodal immersion, given by both choices of orientations of the associated Hopf link.

Proposition 2.17 can be used successively at all double points of a nodal immersion. We have the choice at each double point in the ± 2 that occurs in the self-intersection. If we pick-up a $+2$ each time, we obtain the following.

Corollary 2.18. *Let $\Sigma \subset X$ be a nodal immersion with r double points. Then it is possible to resolve each of those r nodes to obtain an embedded surface $\Sigma' \subset X$ with:*

$$[\Sigma'] = [\Sigma] \in H_2(X; \mathbf{Z}/2),\ \chi(\Sigma') = \chi(\Sigma) - r \text{ and } e(X, \Sigma') = e(X, \Sigma) + 2r.$$

Moreover, this construction is local in the sense that it happens in arbitrarily small neighborhoods of the nodes.

2.2 Double Branched Covers

We introduce rudiments of the theory of double branched covers in the case of surfaces and of 4-manifolds; the full story is actually richer. If Fox's exposition [Fox57] is very complete, it is starting to date, and it is difficult to find a modern survey. Hence, Marco Golla's lecture notes [Gol22] were a real treasure, especially in that they focus on branched covers in dimensions three and four.

2.2.1 Smith–Floyd Theory

We will start with some results from Smith–Floyd theory (see [Smi41] and [Flo52]). We will only focus on the case of *involutions*, although the Smith exact sequence and the Floyd inequalities hold for actions on cyclic groups of prime order.

Consider a cellular space Y and a cellular involution $\tau : Y \to Y$. That is: an involution which is also a cellular map with respect to some cellular decomposition of Y. We set $F = \mathrm{Fix}(\tau)$, and we let $X = Y/\tau$ be the quotient space, with the quotient map denoted as $p : Y \to X$. We first state the so-called *topological Riemann–Hurwitz formula*.

Theorem 2.19 ([Flo52, Theorem 4.2])**.** *We have* $\chi(Y) = 2\chi(X) - \chi(F)$.

Proof. Consider a cellular decomposition of $p(F) \cong F$, and complete it into a cellular decomposition of X. Lifting all the cells through a cellular decomposition of Y *via* p, we obtain the desired formula. ∎

Corollary 2.20. *We have* $\chi(F) \equiv \chi(Y)$ [2].

By using cellular decompositions, we can give a simplified proof of the Floyd inequalities in this special case of involutions.

Theorem 2.21 ([Flo52, Theorem 4.4])**.** *For all* $n \in \mathbf{N}$*, we have*

$$\sum_{k=n}^{+\infty} b_k(F;\mathbf{Z}/2) \leqslant \sum_{k=n}^{+\infty} b_k(Y;\mathbf{Z}/2).$$

In particular, for the total Betti number, we obtain $b_*(F;\mathbf{Z}/2) \leqslant b_*(Y;\mathbf{Z}/2)$.

Sketch of proof. The author is thankful to C. Scaduto for having presented this simplified proof in the $\mathbf{Z}/2$ case.

Consider a cellular decomposition of $p(F)$, and lift it to one of F. Moreover, fill the cell structure of $p(F)$ into one on X, and lift that one as well into one on Y. This means that we have two types of cells on Y:

(1) *fixed* cells: those cells e fixed by τ ($\tau(e) = e$);
(2) *free* cells: those coming in pairs $e_1 \neq e_2$ with $\tau(e_1) = e_2$.

On the cellular complexes, this induces a direct sum decomposition

$$C_\bullet(Y;\mathbf{Z}/2) = C_\bullet^{\mathrm{free}}(Y;\mathbf{Z}/2) \oplus C_\bullet^{\mathrm{fix}}(Y;\mathbf{Z}/2).$$

Moreover, the induced map $\tau_\# : C_\bullet(Y;\mathbf{Z}/2) \to C_\bullet(Y;\mathbf{Z}/2)$ preserves this decomposition:

$$\tau_\# : C_\bullet^{\text{free}}(Y;\mathbf{Z}/2) \to C_\bullet^{\text{free}}(Y;\mathbf{Z}/2) \text{ and } \tau_\# : C_\bullet^{\text{fix}}(Y;\mathbf{Z}/2) \to C_\bullet^{\text{fix}}(Y;\mathbf{Z}/2).$$

Since free cells come in pairs, it is possible to choose a subset $\mathcal{E}$ of free cells such that $\mathcal{E} \cup \tau(\mathcal{E})$ is the whole collection of free cells and $\mathcal{E} \cap \tau(\mathcal{E}) = \varnothing$. This means that

$$C_\bullet^{\text{free}}(Y;\mathbf{Z}/2) = \mathbf{Z}/2\langle \mathcal{E} \rangle \oplus \mathbf{Z}/2\langle \tau(\mathcal{E}) \rangle. \tag{$*$}$$

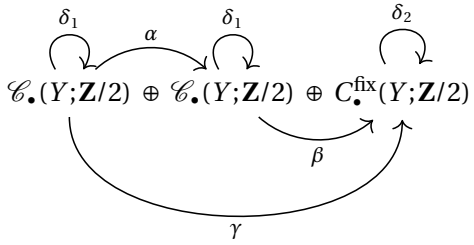
Set $\varrho = \mathrm{id} + \tau_\#$. Since τ is involutive, we obtain $\varrho^2 = 0$. Let $\mathscr{C}_\bullet(Y;\mathbf{Z}/2)$ be the image of $C_\bullet^{\text{free}}(Y;\mathbf{Z}/2)$ by ϱ. Then there is an isomorphism between $C_\bullet^{\text{free}}(Y;\mathbf{Z}/2)$ and $\mathscr{C}_\bullet(Y;\mathbf{Z}/2) \oplus \mathscr{C}_\bullet(Y;\mathbf{Z}/2)$ given by

$$(x, \tau_\# y) \mapsto (\varrho(x+y), \varrho(y)),$$

where we use the decomposition $(*)$. The question is now to regard the differential on the decomposition

$$C_\bullet(Y;\mathbf{Z}/2) \cong \mathscr{C}_\bullet(Y;\mathbf{Z}/2) \oplus \mathscr{C}_\bullet(Y;\mathbf{Z}/2) \oplus C_\bullet^{\text{fix}}(Y;\mathbf{Z}/2).$$

A computation yields that it decomposes in the direct sum as the following, where zero maps are not displayed.



Again, a verification provides the following complex isomorphisms:

$$(\mathscr{C}_\bullet(Y;\mathbf{Z}/2), \delta_1) \cong C_\bullet(X, p(F);\mathbf{Z}/2) \text{ and } (C_\bullet^{\text{fix}}(Y;\mathbf{Z}/2), \delta_2) \cong C_\bullet(F,\mathbf{Z}/2).$$

The short exact sequence

$$0 \longrightarrow 0 \oplus \mathscr{C}_\bullet(Y;\mathbf{Z}/2) \oplus C_\bullet^{\text{fix}}(Y;\mathbf{Z}/2) \longrightarrow C_\bullet(Y;\mathbf{Z}/2) \to \mathscr{C}_\bullet(Y;\mathbf{Z}/2) \longrightarrow 0$$

of simplicial complexes induces the following long exact sequence of homology groups:

$$\cdots \longrightarrow H_{k+1}(X, p(F);\mathbf{Z}/2) \longrightarrow H_k(X, p(F);\mathbf{Z}/2) \oplus H_k(F;\mathbf{Z}/2) \longrightarrow H_k(Y;\mathbf{Z}/2) \longrightarrow \cdots.$$

Exactness of the previous sequence gives $b_k(X, p(F);\mathbf{Z}/2) + b_k(F;\mathbf{Z}/2) \leqslant b_{k+1}(X, p(F);\mathbf{Z}/2) + b_k(Y;\mathbf{Z}/2)$, which re-arranges into

$$b_k(F;\mathbf{Z}/2) \leqslant b_{k+1}(X, p(F);\mathbf{Z}/2) - b_k(X, p(F);\mathbf{Z}/2) + b_k(Y;\mathbf{Z}/2). \tag{$**$}$$

Summing $(**)$ for $k \geqslant n$ provides the result. ■

2.2.2 Definition, Existence and Uniqueness

The general idea is that if a usual (unbranched) double covering is the quotient map of some space by the action of a free involution, *branched* double coverings are the case where the involution is allowed to have fixed points. We quantify this with the actual definition.

Definition 2.22. *Let $p : \widetilde{X} \to X$ be a smooth, orientation-preserving map between two smooth, connected and oriented manifolds. Let $A \subset X$ be a closed codimension two smooth submanifold of X (not necessarily connected nor orientable), and let $\widetilde{A} = p^{-1}(A) \subset \widetilde{X}$. The map p is called a* ***double branched cover*** *with* ***ramification locus*** *A if:*

(1) the restriction $p' : \widetilde{X} \smallsetminus \widetilde{A} \to X \smallsetminus A$ is a usual unbranched double covering, which is called the ***principal part*** *of p, and*

(2) if a tubular neighborhood of A is foliated by discs $(\mathscr{D}_x)_{x\in A}$ transverse to A, then for any $x \in A$, the preimage $p^{-1}(\mathscr{D}_x)$ is a disc transverse to $\widetilde{A}$, and the restriction $p : p^{-1}(\mathscr{D}_x) \to \mathscr{D}_x$ is topologically the map $z \mapsto z^2$ of the complex unit disc to itself.

In particular, the map $p : \widetilde{A} \to A$ is a diffeomorphism, and the principal part is a two-to-one local diffeomorphism. We denote a branched cover as a map of pairs $p : (\widetilde{X}, \widetilde{A}) \to (X, A)$, and we say that X is the[†] branched cover of X ramified over A.

We will try to show the following fundamental result step by step, by performing homological computations along the way.

Theorem 2.23. *Let X be a closed, connected and oriented smooth 4-manifold with $H_1(X;\mathbf{Z}/2) = 0$. Let $A \subset X$ be a closed and connected surface smoothly embedded (A needs not be orientable). Then there exists a double branched cover $p : (\widetilde{X}, \widetilde{A}) \to (X, A)$ if and only if $[A] = 0 \in H_2(X;\mathbf{Z}/2)$. In this case, this is unique up to diffeomorphism.*

This has the following consequence.

Corollary 2.24. *There always exists a unique double branched cover of $\mathbf{S}^4$ ramified along any connected surface, and there exists a unique double branched cover of $\mathbf{CP}^2$ ramified along a connected surface F if and only if $e(\mathbf{CP}^2, F)$ is even.*

The starting point to prove Theorem 2.23 is to try to extend an unbranched covering of the complement of a surface. That is, given an unbranched double covering on a surface complement, is it the principal part of a branched covering?

Proposition 2.25. *Let $A \subset X$ be a codimension two connected submanifold of an n-manifold X, and let $q : Y \to X \smallsetminus A$ be a connected double unbranched covering be given. Then there exists an n-manifold $\widetilde{X}$ inside which Y embeds and a double branched covering $p : \widetilde{X} \to X$ such that q is the principal part of p.*

Sketch of proof. For every point $x \in A$, choose a neighborhood $x \in \mathcal{U} \subset X$ such that $\mathcal{U} \cong \mathbf{D}^{n-2} \times \mathbf{D}^2$ and

[†] It is not clear at this point that this is unique. In fact, it has no reason to be; a fact which is addressed in Theorem 2.23.

$\mathcal{U} \cap A \cong \mathbf{D}^{n-2} \times \{0\}$. The map q restricts to an unbranched double covering $q^{-1}(\mathcal{U} \setminus A) \to \mathcal{U} \setminus A$, which in turn gives a double covering of $\mathbf{D}^2 \setminus \{0\}$. This being unique, the map q can therefore be identified with $z \mapsto z^2$ on the punctured complex unit disc.

It remains to check that locally (*e.g.* in charts), it is possible to extend by gluing a 2-disc to fill the punctures, and that the map q therefore extends. This gives a smooth manifold $\widetilde{X}$, as well as a smooth extension $p : \widetilde{X} \to X$. ■

Proposition 2.26. *A double branched cover $p : (\widetilde{X}, \widetilde{A}) \to (X, A)$ exists if and only if there is a homomorphism $H_1(X \setminus A, \mathbf{Z}) \to \mathbf{Z}/2$ mapping a meridian of A to the non-trivial element.*

Proof. By Proposition 2.25, it is equivalent to asking that there exists a connected unbranched covering of $X \setminus A$. This is the case if and only if $\pi_1(X \setminus A)$ has an index two subgroup, or equivalently, if and only if there exists a surjective group homomorphism $\pi_1(X \setminus A) \to \mathbf{Z}/2$. Since $\mathbf{Z}/2$ is abelian, this factors through $H_1(X \setminus A; \mathbf{Z}) \cong \pi_1(X \setminus A)^{\mathrm{ab}}$, thus this is equivalent to asking that there is a surjective homomorphism $H_1(X \setminus A; \mathbf{Z}/2) \to \mathbf{Z}/2$. ■

It becomes clear that the proof of Theorem 2.23 rests on a computation of the first homology of the complement $H_1(X \setminus A; \mathbf{Z})$. We will review the computations done in [Nag00, §2.2].

Definition 2.27. *Let $\Lambda = F \oplus T$ be a finite rank $\mathbf{Z}$-module whose free part is F and torsion part T.*

*(1) An element $x = f \oplus \tau \in \Lambda$ is **primitive** if f is not divisible by any integer other than ± 1.*

(2) If $x \in \Lambda$ has infinite order, then there exists a unique integer $\alpha(x) \in \mathbf{N}$ and two elements $f \in F$ and $\tau \in T$ such that f is primitive and $x = \alpha(x) f + \tau$. If x has infinite order, set $\alpha(x) = 0$.

Proposition 2.28 ([Nag00, Proposition 2.8])**.** *Let $A \subset X$ be a closed, connected surface smoothly embedded in X a closed, connected and oriented 4-manifold with $H_1(X; \mathbf{Z}/2) = 0$. Let $\widetilde{H}$ be the subgroup of $H_1(X \setminus A; \mathbf{Z})$ generated by an oriented meridian of A.*

(1) Assume A is oriented, so that $[A] \in H_2(X; \mathbf{Z})$. If $\alpha([A]) \neq 0$, then $\widetilde{H}$ has order $\alpha([A])$. Otherwise, it is infinite cyclic. In both cases, $\widetilde{H}$ has an odd index in $H_1(X \setminus A; \mathbf{Z})$.

(2) Assume that A is non-orientable. If $[A] = 0 \in H_2(X; \mathbf{Z}/2)$, then $\widetilde{H}$ has order 2 and odd index inside $H_1(X \setminus A; \mathbf{Z})$. Otherwise, if $[A] \neq 0 \in H_2(X; \mathbf{Z}/2)$, then $H_1(X \setminus A; \mathbf{Z})$ has odd order.

Proof. Start with the long exact sequence in homology for the pair $(X, X \setminus A)$:

$$H_2(X; \mathbf{Z}) \xrightarrow{f_*} H_2(X, X \setminus A; \mathbf{Z}) \xrightarrow{\partial} H_1(X \setminus A; \mathbf{Z}) \longrightarrow H_1(X; \mathbf{Z}). \qquad (*)$$

Set $\widetilde{H} = \partial H_2(X, X \setminus A; \mathbf{Z}) \subset H_1(X \setminus A; \mathbf{Z})$. Take D to be a small disc transverse to A, whose boundary ∂A is a meridian of A. Denote as $u : (D, \partial D) \to (X, X \setminus A)$ the inclusion map. If F is oriented, then both D and ∂D come with natural orientations, and we require that u preserves those. By excising the complement of a tubular neighborhood $Tub(A)$ (and by noting that $Tub(A) \setminus A$ deformation retracts to $\partial Tub(A)$), we obtain that

$$H_2(X, X \setminus A, \mathbf{Z}) \cong H_2(Tub(A), \partial Tub(A); \mathbf{Z}) \cong \mathbf{Z} \text{ or } \mathbf{Z}/2,$$

depending on whether A is orientable or not, respectively. Moreover, $H_2(Tub(A), \partial Tub(A); \mathbf{Z})$ is clearly spanned by the homology class of u, so we see that $\widetilde{H}$ is cyclic and spanned by $\partial[u] = [\partial D]$ the homology class of a meridian of A. By exactness of $(*)$ at $H_1(X \smallsetminus A; \mathbf{Z})$, it now only suffices to compute the order of $\widetilde{H}$, since $H_1(X; \mathbf{Z}/2) = 0$ means that $H_1(X; \mathbf{Z})$ is finite and has only odd torsion, and thus the index

$$[H_1(X \smallsetminus A) : \widetilde{H}] = |H_1(X; \mathbf{Z})|$$

is necessarily finite and odd.

(1) Assume that A is oriented. In the first case where $\alpha([F]) \neq 0$, let $[A] = \alpha([A])\xi + \tau$ for some choice of $\xi \in H_2(X; \mathbf{Z})$ primitive and $\tau \in H_2(X; \mathbf{Z})$ torsion. The map $Q_X(\xi, -)$ is invertible and ξ is primitive, so there exists some $\zeta \in H_2(X; \mathbf{Z})$ such that $\xi \cdot \zeta = 1$. For this element, we see that $\zeta \cdot [A] = \alpha([A])$. Letting $n = |\widetilde{H}|$, we have, since $\widetilde{H}$ is cyclic:

$$n = \min\left\{k \in \mathbf{N}^\star \;\middle|\; k[\partial D] = 0\right\},$$

with the convention $\min \varnothing = +\infty$. Because $f_*(\zeta) = \alpha([A])[u]$ and $(*)$ is exact at $H_2(X, X \smallsetminus A; \mathbf{Z})$, we derive $\alpha([A])[\partial u] = 0$. In particular, n is finite and divides $\alpha([A])$. Conversely, we can construct a surface representing $z \in H_2(X; \mathbf{Z})$ such that $z \cdot [A] = n$. Therefore, n is also divisible by $\alpha([A])$, and this yields $\widetilde{H} \cong \mathbf{Z}/\alpha([A])$.

In the second case where $\alpha([A]) = 0$, the map f_* is necessarily zero. Indeed, if it were not, there would exist a homology class $z \in H_2(X; \mathbf{Z})$ such that $z \cdot [A] \neq 0$, which is prevented by $\alpha([A]) = 0$. This implies that $\widetilde{H} \cong H_2(X, X \smallsetminus A; \mathbf{Z}) \cong \mathbf{Z}$.

(2) Assume that A is non-orientable. We aim to show that $\widetilde{H} \cong \mathbf{Z}/2$ if and only if $[A] = 0 \in H_2(X; \mathbf{Z}/2)$, since $H_2(X, X \smallsetminus A; \mathbf{Z}) \cong \mathbf{Z}/2$. We consider the analogue of $(*)$ with coefficients in $\mathbf{Z}/2$:

$$H_2(X; \mathbf{Z}/2) \xrightarrow{f_*} H_2(X, X \smallsetminus A; \mathbf{Z}/2) \xrightarrow{\partial} H_1(X \smallsetminus A; \mathbf{Z}/2) \longrightarrow H_1(X; \mathbf{Z}/2) = 0. \qquad (**)$$

First suppose that $[A] = 0 \in H_2(X; \mathbf{Z}/2)$. If the map f_* in $(**)$ is non-zero, then we can find a homology class $y \in H_2(X; \mathbf{Z}/2)$ such that $y \cdot [A] = 1$, a contradiction. Therefore, we obtain $H_1(X \smallsetminus A; \mathbf{Z}/2) \cong H_2(X, X \smallsetminus A; \mathbf{Z}/2) \cong \widetilde{H} \cong \mathbf{Z}/2$.

Conversely, assume that $[A] \neq 0 \in H_2(X; \mathbf{Z}/2)$. Then there *does* exist a homology class $y \in H_2(X; \mathbf{Z}/2)$ such that $y \cdot [A] = 1$, and therefore $f_*(y) \neq 0 \in H_2(X, X \smallsetminus A; \mathbf{Z}/2) \cong \mathbf{Z}/2$. This means that f_* is surjective, and therefore ∂ is the zero map in $(**)$. Therefore, $H_1(X \smallsetminus A; \mathbf{Z}/2) = 0$. ■

Proof of Theorem 2.23. It suffices to notice that a homomorphism prescribed as in Proposition 2.26 exists if and only if $[F] = 0 \in H_2(X; \mathbf{Z}/2)$.

The uniqueness part of the theorem comes from the uniqueness of the principal part of the branched cover (this is the traditional theory of covering spaces), and from the uniqueness of the "filling" operation (Proposition 2.25). ■

We will now see the connection between involutions and branched covers.

Proposition 2.29. *Let $\tau : \widetilde{X} \to \widetilde{X}$ be a smooth orientation-preserving involution and whose fix-point set* $\mathrm{Fix}(\tau)$ *is a codimension two submanifold of $\widetilde{X}$. Then The quotient space $\widetilde{X}/\tau$ can be endowed with the structure of a smooth manifold such that the quotient map $p : \widetilde{X} \to \widetilde{X}/\tau$ is a smooth double branched cover.*

Proof. The quotient $\widetilde{X}/\tau$ does not come with an automatic smooth structure, only the quotient of the principal part does. More precisely, let $\widetilde{A} = \mathrm{Fix}(\tau) \subset \widetilde{X}$, let $X = \widetilde{X}/\tau$ and let $A = p(\widetilde{A}) \subset X$ where $p : \widetilde{X} \to X$ is the quotient map. Then $\widetilde{X} \setminus \widetilde{A}/\tau = X \setminus A$ has an automatic smooth structure, since it is the quotient of a smooth manifold under a smooth, free and properly discontinuous group action (the group being $\mathbf{Z}/2$, where the generator is acting on $\widetilde{X}$ *via* the involution τ).

Locally around any point $x \in \widetilde{A}$ we have a local chart $\mathcal{U}$ of $\widetilde{X}$ such that $(\mathcal{U}, \mathcal{U} \cap \widetilde{A}) \cong (\mathbf{R}^n, \mathbf{R}^{n-2})$ and the action of τ in this chart is rotation by $180°$ in the first two coordinates. This gives charts $\mathcal{U}' = \mathcal{U} \setminus \widetilde{A} \cong \mathbf{R}^n \setminus \mathbf{R}^{n-2} \cong \mathbf{R}^{n-2} \times \mathbf{C}^\star$ for $\widetilde{X} \setminus \widetilde{A}$ around each $x \in \mathrm{Fix}(\tau)$. The quotient of this chart is

$$\mathcal{U}'/\tau \cong \mathbf{R}^{n-2} \times \mathbf{C}^\star,$$

where the projection is $\mathrm{id}_{\mathbf{R}^{n-2}} \times \mathrm{Sq}$, with $\mathrm{Sq} : z \in \mathbf{C}^\star \mapsto z^2$. These charts on $X \setminus A$ can be extended smoothly to charts on the whole X, and the projection $\widetilde{X} \to X$ agrees with the local model of a double branched cover. ∎

We will now focus on the converse.

Definition 2.30. *Let $p : (\widetilde{X}, \widetilde{A}) \to (X, A)$ be a double branch cover. A **deck transformation** for p is a diffeomorphism $\tau : \widetilde{X} \to \widetilde{X}$ such that $p \circ \tau = p$. The collection of deck transformations is denoted as $\mathrm{Aut}(p)$ and is called the **automorphism group** of p.*

Proposition 2.31. $\mathrm{Aut}(p)$ *is a group isomorphic to $\mathbf{Z}/2$. It is spanned by a smooth orientation-preserving involution τ whose fix-point set is the lift of the branch locus by p.*

Proof. The fact that $\mathrm{Aut}(p)$ is a group is immediate. Denote as $p' : \widetilde{X} \setminus \widetilde{A} \to X \setminus A$ the principal part. The restriction map

$$\begin{array}{rccc} R & : & \mathrm{Aut}(p) & \longrightarrow & \mathrm{Aut}(p') \\ & & \tau & \longmapsto & \tau|_{\widetilde{X} \setminus \widetilde{A}} \end{array}$$

describes a group isomorphism. That is: the non-trivial involution spanning $\mathrm{Aut}(p)$ is obtained from the free involution spanning $\mathrm{Aut}(p')$ and extending it onto the whole $\widetilde{X}$ in a straight-forward manner. ∎

2.2.3 Fundamental Examples

We now give a few crucial examples of double branched covers that will be helpful throughout this thesis. We start with double branched covers of orientable surfaces.

Proposition 2.32. *Let Σ_g be a closed, connected and orientable surface of genus g. Then there exists a double branch cover $p : \Sigma_g \to \mathbf{S}^2$ whose branch locus is $2g+2$ points. The involution $\tau : \Sigma_g \to \Sigma_g$ spanning $\mathrm{Aut}(p)$ is depicted in Figure 2.4.*

Proof. The involution $\tau : \Sigma_g \to \Sigma_g$ depicted in Figure 2.4 clearly preserves orientation and has $2g+2$ fixed points. Therefore, the quotient is a smooth surface and the quotient map is a double branched

cover, by Proposition 2.29. Next, the topological Riemann–Hurwitz formula (Theorem 2.19) allows to compute the Euler characteristic of the quotient:

$$2-2g = \chi(\Sigma_g) = 2\chi(\Sigma_g/\tau) - \chi(\mathrm{Fix}(\tau)) = 2\chi(\Sigma_g/\tau) - (2g+2).$$

This readily gives $\chi(\Sigma_g/\tau) = 2$, which means Σ_g/τ is a 2-sphere. ■

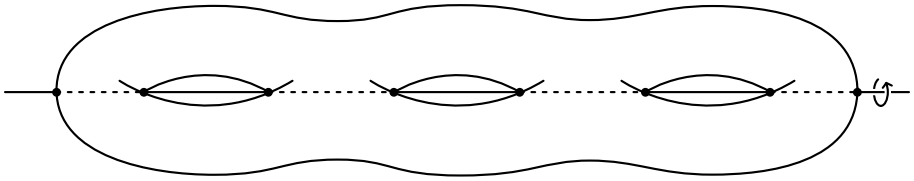


Figure 2.4. The hyperelliptic involution on Σ_g has $2g+2$ isolated fixed points, and the quotient is the 2-sphere. This is a depiction of the involution for $g = 3$: rotation by 180° along an axis of symmetry of the surface.

We are also interested in double branched covers of surfaces *with boundary*. More specifically, the case of the disc is of particular interest.

Proposition 2.33. *Let $p : \mathscr{D} \to \mathbf{D}^2$ be a double branched cover of the 2-disc, with branch locus n interior points to $\mathbf{D}^2$. Then $\mathscr{D}$ is a surface with boundary $\mathscr{D} \cong \Sigma_{g,b}$, where:*

(1) $b \in \{1,2\}$ and $b \equiv n$ [2];
(2) $g = \left\lfloor \frac{n-1}{2} \right\rfloor$.

In particular, if $n = 1$, then $\mathscr{D}$ is a 2-disc, and if $n = 2$, then $\mathscr{D}$ is an annulus. We depict those two examples in Figure 2.5.

Proof. Letting $\mathscr{D} \cong \Sigma_{g,b}$, the topological Riemann–Hurwitz formula provides

$$2-2g-b = \chi(\mathscr{D}) = 2\chi(\mathbf{D}^2) - n = 2-n,$$

which can be re-written as $2g+b = n$.

(1) The map p restricts to an *unbranched* double covering on the boundaries $\partial\mathscr{D} \to \partial\mathbf{D}^2$. This means that $\partial\mathscr{D}$ is a double covering of a circle, which is either connected (in which case $b = 1$) or trivial (in which case $b = 2$). Reducing the relation $2g+b = n$ modulo 2 finishes proving the first claim.
(2) We obtain $g = \frac{n-b}{2}$, which equals $\left\lfloor \frac{n-1}{2} \right\rfloor$ by studying both cases where $n = 2p$ and $n = 2p+1$ individually. ■

We now switch to examples of branched covers of 4-manifolds. The following fact is well-known, and was proved incrementally: first a homotopy equivalence was obtained, then a homeomorphism, and lastly a diffeomorphism.

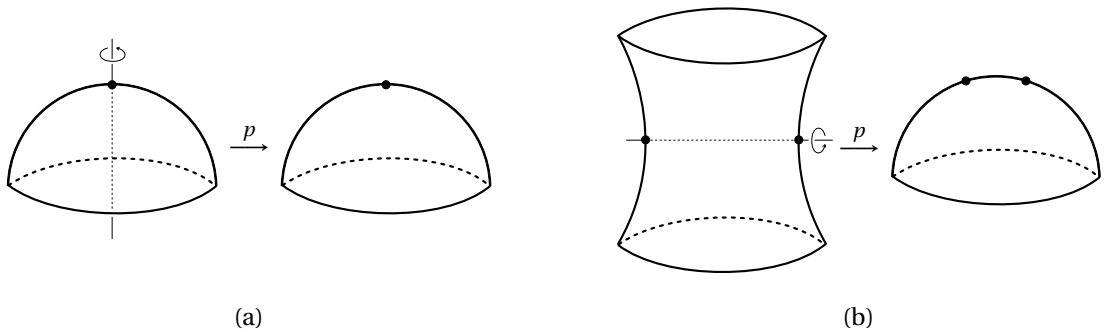


Figure 2.5. The double branched covers of a disc with branch locus either one or two interior points. In each case, we depict the involution spanning the group of deck transformations.

Theorem 2.34 (Arnold–Kuiper–Massey, [Kui74])**.** *The quotient of* $\mathbf{CP}^2$ *by complex conjugation is diffeomorphic (in the sense of Proposition 2.29) to the standard* 4*-sphere* $\mathbf{S}^4$*.*

We will rely heavily on this previous fact, as well as the following in the case of curves on quadrics.

Theorem 2.35 ([Let84, §3])**.** *Define the following involutions on* $\mathbf{CP}^1 \times \mathbf{CP}^1$*:*

$$c_{\text{hyp}} : (x, y) \mapsto (\overline{x}, \overline{y}) \text{ and } c_{\text{ell}} : (x, y) \mapsto (\overline{y}, \overline{x}).$$

We have diffeomorphisms $\mathbf{CP}^1 \times \mathbf{CP}^1 / c_{\text{hyp}} \cong \mathbf{S}^4$ *and* $\mathbf{CP}^1 \times \mathbf{CP}^1 / c_{\text{ell}} \cong \overline{\mathbf{CP}}^2$*.*

Of course, those diffeomorphisms also hold in the sense of Proposition 2.29, since they are both orientation-preserving involutions whose fix-point sets are the surfaces

$$\text{Fix}(c_{\text{hyp}}) = \mathbf{RP}^1 \times \mathbf{RP}^1 \subset \mathbf{CP}^1 \times \mathbf{CP}^1 \text{ and } \text{Fix}(c_{\text{ell}}) = \left\{ (x, \overline{x}) \;\middle|\; x \in \mathbf{CP}^1 \right\} \subset \mathbf{CP}^1 \times \mathbf{CP}^1.$$

They fit in the more general setting of conjugations on 4-manifolds, which will be discussed in §3.4.3.

2.2.4 Computing Homological Invariants

The topological Riemann–Hurwitz formula allowed us to compute the most basic homological invariant of a double branched cover: the Euler characteristic. In the special case of 4-manifolds, another invariant is of major importance too: the signature.

Theorem 2.36 (Hirzebruch's signature formula, [Hir69, §3(iii)])**.** *Let* $p : (\widetilde{X}, \widetilde{A}) \to (X, A)$ *be a double branched cover of a closed, connected and oriented smooth* 4*-manifold* X*, which is ramified along a closed surface (not necessarily connected nor orientable). Then:*

$$\sigma(\widetilde{X}) = 2\sigma(X) - \frac{e(X, A)}{2}.$$

Here, it is to be noted that in the case where $H_1(X;\mathbf{Z}/2) = 0$, we have $[A] = 0 \in H_2(X;\mathbf{Z}/2)$ by Theorem 2.23, and thus $e(X,A)$ is necessarily even. In fact, it is true in the more general case that the self-intersection of a branch locus is even, by Lemma 2.38.

This means that we can derive $2-2b_2+b_2$ and $b_2^+ - b_2^-$ for a double branch cover. In most cases, we will be in the favourable situation where the first homology of the base space with $\mathbf{Z}/2$-coefficients vanishes, and this will be transported to the branch cover.

Proposition 2.37. *Let $p : (\widetilde{X}, \widetilde{A}) \to (X, A)$ be a double branched cover of 4-manifolds ramified along a connected closed surface. If $b_1(X;\mathbf{Z}/2) = 0$, then $b_1(\widetilde{X};\mathbf{Z}/2) = 0$.*

Proof. Assume that $H_1(X;\mathbf{Z}/2) = 0$. We use the following generalization of the Gysin sequence, from [LW95, Theorem 1]:

$$H_1(X,A;\mathbf{Z}/2) \longrightarrow H_1(\widetilde{X},*;\mathbf{Z}/2) \longrightarrow H_1(X,A;\mathbf{Z}/2). \tag{$*$}$$

Take the long exact sequence in homology for the pair (X,A):

$$0 = H_1(X;\mathbf{Z}/2) \longrightarrow H_1(X,A;\mathbf{Z}/2) \longrightarrow H_0(A;\mathbf{Z}/2) \xrightarrow{\sim} H_0(X;\mathbf{Z}/2).$$

This provides $H_1(X,A;\mathbf{Z}/2) = 0$, which can be re-introduced into $(*)$ to yield

$$0 \cong H_1(\widetilde{X},*;\mathbf{Z}/2) \cong \widetilde{H}_1(\widetilde{X};\mathbf{Z}/2) \cong H_1(\widetilde{X};\mathbf{Z}/2),$$

where $\widetilde{H}_1$ stands for reduced homology. ∎

We finish by stating a crucial lemma allowing us to relate the self-intersection numbers of a surface and its lift or image through a double branched cover. This is a fact that was already known to Massey [Mas69].

Lemma 2.38. *Let $p : (\widetilde{X}, \widetilde{A}) \to (X, A)$ be a double branched cover of 4-manifolds, and let $F \subset X$ be a closed surface. Denote as $\widetilde{F} = p^{-1}(F) \subset \widetilde{X}$ the lift of F.*

(1) If $F \pitchfork A$, possibly allowing $F \cap A = \varnothing$, then $e(\widetilde{X},\widetilde{F}) = 2e(X,F)$.

(2) If $F \subset A$, then $e(X,F) = 2e(\widetilde{X},\widetilde{F})$.

Proof. We treat each case individually. For the first, note that lifting a perturbation F' of F gives a perturbation $\widetilde{F}'$ of the lift $\widetilde{F}$. We can ensure that all the intersection points in $F \pitchfork F'$ occur away from the ramification locus A. Therefore, each such point lifts to two intersection points in $\widetilde{F}' \pitchfork \widetilde{F}$, and the signs agree since p is orientation-preserving. This can be used to compute that

$$e(\widetilde{X},\widetilde{F}) = \sum_{x \in \widetilde{F}' \pitchfork \widetilde{F}} \varepsilon(x) = \sum_{x \in F' \pitchfork F} 2\varepsilon(x) = 2e(X,F).$$

For the second case, we can deduce it from the first. Let $\widetilde{F}'$ be a small transverse push-off of $\widetilde{F}$. Denote as $\tau : \widetilde{X} \to \widetilde{X}$ the involution spanning $\mathrm{Aut}(p)$. Set $F' = p(\widetilde{F}')$. We see that F' is a perturbation of F, and $\widetilde{F}' \cup \tau(\widetilde{F}')$ is its lift. By the first case, we have:

$$e(\widetilde{X},\tau(\widetilde{F}')) = e(\widetilde{X},\widetilde{F}') = 2e(X,F') = 2e(X,F).$$

Moreover, we have

$$2e(\widetilde{X},\widetilde{F}) = e(\widetilde{X},\widetilde{F}' \cup \tau(\widetilde{F}')) = e(\widetilde{X},\widetilde{F}') + e(\widetilde{X},\tau(\widetilde{F}')) = 4e(X,F),$$

which yields the claim. ■

We will also use the previous lemma in the following cases:

(1) if F is a surface with boundary and $\partial F = F \pitchfork A$, and
(2) if $F \subset A$ has boundary.

This means that we are actually making the computations with the formalism of §2.1.1 in the relative case. In order to ease out the exposition when we are in need to make such computations, we will allow ourselves to work with half-integer normal Euler numbers and use the generalization of Lemma 2.38 accordingly. For more details, we refer the reader to [GM86] or [Mar80]. Indeed, we will utimately always glue the surfaces with boundary we consider in order to obtain *closed* surfaces, so the details regarding the choices of sections on the boundary and of relative Euler classes do not matter, and the results are genuine self-intersection numbers.

We finish this section with one last statement regarding characteristic surfaces and double branched covers.

Lemma 2.39 ([Nag00, Lemma 3.4])**.** *Let $p : (\widetilde{X},\widetilde{A}) \to (X,A)$ be a double branched covers of 4-manifolds with A a connected surface, and where $H_1(X;\mathbf{Z}/2) = 0$. Let $F \subset X$ be a mod 2 characteristic surface (that is, $[F] \in H_2(X;\mathbf{Z}/2)$ is Poincaré dual to $w_2(X)$). Then a characteristic surface in $\widetilde{X}$ is $\widetilde{A} \cup p^{-1}(F)$.*

2.3 Genus Functions of 4-Manifolds

Given an integral homology class $\xi \in H_2(X;\mathbf{Z})$ with X a smooth, closed, connected and simply-connected 4-manifold, what is the smallest genus of an orientable surface smoothly embedded in X and representing that homology class? Thom stated his famous conjecture for the complex projective plane: non-singular algebraic curves are genus-minimizing in their homology classes.

We will be interested in the non-orientable analogue of this statement. Of course, a non-orientable surface in X does not give rise to an integral homology class, but merely a $\mathbf{Z}/2$ one. Because classes in $H_2(\mathbf{CP}^2;\mathbf{Z}/2)$ have integral lifts representable by spheres, either the situation is much less interesting, or the question needs to be addressed differently.

2.3.1 The Thom Conjecture and the Adjunction Formulas

We will use the formulation in terms of the Euler characteristic instead of the genus. First, recall the statement of the Thom conjecture.

Theorem 2.40 ([KM94, Theorem 1])**.** *Let $F \subset \mathbf{CP}^2$ be a closed, connected and oriented surface smoothly embedded, and assume that $[F] = m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2;\mathbf{Z})$, where $m \in \mathbf{Z}$. Then:*

$$\chi(F) \leqslant -m^2 + 3|m|.$$

Of course, the genus-degree formula ensures that this bound is sharp, since algebraic curves realize it. In fact, not just algebraic curves, but pseudo-holomorphic ones too.

We consider a closed connected almost-complex 4-manifold (X, J). Let $\varphi : (\Sigma, j) \to (X, J)$ be a J-holomorphic curve (where Σ is a Riemann surface). The curve φ is called **simple** if it does not factor through a holomorphic branched cover[†]. That is: there does not exist a Riemann surface (Σ', j'), a holomorphic branched cover $\pi : \Sigma \to \Sigma'$ of degree $d \geqslant 2$ and a J-holomorphic curve $\varphi' : (\Sigma', j') \to (X, J)$ such that $\varphi = \varphi' \circ \pi$.

Roughly-speaking, simple curves are generic, in the sense that non-simple curves are singular points in the moduli space of all J-holomorphic curves. Given two J-holomorphic curves $\varphi_1 : (\Sigma_1, j_1) \to (X, J)$ and $\varphi_2 : (\Sigma_2, j_2) \to (X, J)$ which are simple, define:

$$\delta(\varphi_1, \varphi_2) = \#\Big\{(z_1, z_2) \in \Sigma_1, \Sigma_2 \;\Big|\; \varphi_1(z_1) = \varphi_2(z_2)\Big\}.$$

There is no immediate reason why this number should be finite. In fact, this is the content of the following; see [MS04, Theorem 2.6.3].

Theorem 2.41 (Positivity of intersections)**.** *Let φ_1 and φ_2 be two simple J-holomorphic curves, as above. Assume further that their union is simple, and denote as ξ_1 and ξ_2 the integral homology classes of $\varphi_1(\sigma_1)$ and $\varphi_2(\Sigma_2)$ in $H_2(X;\mathbf{Z})$, respectively. Then*

$$\delta(\varphi_1, \varphi_2) \leqslant Q_X(\xi_1, \xi_2),$$

with equality if and only if all intersections are transverse.

[†] We only reviewed the case of double branched covers, which were always simple and cyclic. Here, we do not impose such a condition.

A consequence is that in the case where all intersections are transverse, then:

$$\delta(\varphi_1,\varphi_2) = Q_X(\xi_1,\xi_2) = \sum_{x\in\varphi_1(\Sigma_1)\pitchfork\varphi_2(\Sigma_2)} \varepsilon(x) \leqslant \sum_{x\in\varphi_1(\Sigma_1)\pitchfork\varphi_2(\Sigma_2)} 1 = \delta(\varphi_1,\varphi_2),$$

so that $\varepsilon(x) = 1$ for all $x \in \varphi_1(\Sigma_1) \pitchfork \varphi_2(\Sigma_2)$. That is: all intersections between $\varphi_1(\Sigma_1)$ and $\varphi_2(\Sigma_2)$ are of positive sign.

In the case where we only pick *one* simple J-holomorphic curve $\varphi : (\Sigma, j) \to (X, J)$, define:

$$\delta(\varphi) = \frac{1}{2}\#\left\{(z_1,z_2)\in\Sigma \,\middle|\, z_1\neq z_2 \text{ and } \varphi(z_1)=\varphi(z_2)\right\}.$$

Again, this number is not obviously finite, but this is contained in [MS04, Theorem 2.6.4].

Theorem 2.42 (Adjunction inequality)**.** *Let $\varphi : (\Sigma, j) \to (X, J)$ be a simple J-holomorphic curve, and denote as $\xi \in H_2(X;\mathbf{Z})$ the integral homology class represented by $\varphi(\Sigma)$. Then*

$$2\delta(\varphi) - \chi(\Sigma) \leqslant Q_X(\xi,\xi) - \langle c_1(X),\xi\rangle,$$

with equality if and only if φ is a nodal immersion.

In particular, in the case of a J-holomorphic *embedding*, we have:

$$\chi(\Sigma) = \langle c_1(X), [\Sigma]\rangle - e(X,\Sigma).$$

Applying this to smooth algebraic curves in $\mathbf{CP}^2$ for instance (or even to nodal curves), one recovers the genus-degree formula.

Observe that if $\Sigma \subset X$ is a J-holomorphic embedding with (X,ω) symplectic and J tamed by ω (that is, $\omega(v, Jv) > 0$ for all $v \neq 0$), then Σ is also a symplectic submanifold. Indeed, the tangent spaces $T_x\Sigma$ are J-invariant, and thus define complex lines of TX on which ω restricts to a positive form. Conversely, if $\Sigma \subset X$ is a symplectic submanifold, then it is possible to define a tame almost-complex structure J such that Σ is J-holomorphic. Therefore, the adjunction inequality is crucial in understanding the analogue of Thom's conjecture in *symplectic* 4-manifolds.

Theorem 2.43 ([OS00, Theorem 1.1])**.** *An embedded symplectic surface in a closed, symplectic 4-manifold is genus-minimizing in its homology class.*

2.3.2 The Non-Orientable Genus Function of $\mathbf{CP}^2$

As outlined before, asking the question of the minimal genus of a surface representing an integral lift of a homology class $\xi \in H_2(\mathbf{CP}^2;\mathbf{Z}/2)$ is very poor, since both classes $0 \in H_2(\mathbf{CP}^2;\mathbf{Z}/2)$ and $[\mathbf{RP}^2] \in H_2(\mathbf{CP}^2;\mathbf{Z}/2)$ are representable by embedded 2-spheres (respectively *via* a sphere embedded in a chart of $\mathbf{CP}^2$ and a complex line). However, we have a characterization of integral homology classes solely in terms of the Euler class of their representatives.

Proposition 2.44. *Let $F \subset \mathbf{CP}^2$ be a closed, connected and oriented surface smoothly embedded. Then:*

$$[F] = \pm m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2;\mathbf{Z}) \iff e(\mathbf{CP}^2,F) = m^2.$$

Proof. The group $H_2(\mathbf{CP}^2;\mathbf{Z})$ has rank one, and $Q_{\mathbf{CP}^2}$ is definite positive. ■

In particular, we could consider the collection of all smoothly embedded non-orientable surfaces $F \subset \mathbf{CP}^2$ with $e(\mathbf{CP}^2, F) = m^2$, and ask what is the minimal genus of those. In fact, self-intersection numbers of non-orientable surfaces need not be perfect squares anymore.

Definition 2.45. *Given $m \in \mathbf{Z}$, denote as $\Sigma(m)$ the set of all closed, connected and non-orientable surfaces F smoothly embedded in $\mathbf{CP}^2$ and such that $e(\mathbf{CP}^2, F) = m$. The function*

$$\begin{array}{rccc} \widetilde{g} & : & \mathbf{Z} & \longrightarrow & \mathbf{Z}_{\leqslant 1} \\ & & m & \longmapsto & \max\limits_{F\in\Sigma(m)} \chi(F) \end{array}$$

will be called the ***non-orientable genus function*** *of $\mathbf{CP}^2$.*

This is well-defined since, for any $m \in \mathbf{Z}$, the collection $\Sigma(m)$ is non-empty (by considering local surfaces for even m, and local surfaces connected *via* a tube to a complex line for odd m for instance). We will be interested in proving the following computation of $\widetilde{g}$.

Theorem 2.46. *Let $k \in \mathbf{N}^\star$ be a non-negative integer.*

(1) We have $\widetilde{g}(0) = 0$.

(2) Let $\ell \in \{0,1\}$ have the same parity as k. Then:

$$\widetilde{g}(-k) = 2 - \frac{k+\ell}{2}.$$

(3) On even positive integers, we have:

$$\widetilde{g}(4k) = 4-2k \text{ for } k \geqslant 2, \text{ and } \widetilde{g}(4k+2) = 3-2k,$$

together with the special values $\widetilde{g}(2) = 1$ and $\widetilde{g}(4) = 0$.

(4) On odd positive integers, we have lower bounds:

$$\widetilde{g}(4k+1) \geqslant 2-2k \text{ and } \widetilde{g}(4k+3) \geqslant 1-2k,$$

with some special values: $\widetilde{g}(1) = 0$, $\widetilde{g}(3) = 1$, $\widetilde{g}(5) = 0$, $\widetilde{g}(7) = -1$ and $\widetilde{g}(9) = -2$.

Before we dive into the proof of Theorem 2.46, we will make a few comments about this result. Looking at the values taken by $\widetilde{g}$ on squares, we obtain:

$$\begin{cases} \widetilde{g}(m^2) = \frac{8-m^2}{2} & \text{if } m \text{ is even,} \\ \widetilde{g}(m^2) \geqslant \frac{5-m^2}{2} & \text{if } m \text{ is odd.} \end{cases}$$

This is very far from the bound

$$\chi(F) \leqslant -m^2 + 3m$$

in the orientable case, as the quadratic term is off by 50%.

If one inspects carefully the proof, one sees that $\widetilde{g}$ does not detect the smoothness of the embedding; More specifically, if we set $\Sigma_{\text{diff}}(m)$ and $\Sigma_{\text{top}}(m)$ the respective collections of surfaces smoothly or locally flatly embedded in $\mathbf{CP}^2$ which are connected, non-orientable and with[†] $e(\mathbf{CP}^2, F) = m$, then we can consider the associated functions $\widetilde{g}_{\text{diff}}$ and $\widetilde{g}_{\text{top}}$. The precise statement is the following.

Proposition 2.47. *We have $\widetilde{g}_{\text{diff}} = \widetilde{g}_{\text{top}}$ on every negative integer and every non-negative even integer.*

Proof. In order to give upper bounds, we will only use the following results:

(1) the computation of the homology class in $H_2(\mathbf{CP}^2; \mathbf{Z}/2)$, which is only topological;
(2) the signature of a 4-manifold, and the rank of the associated subgroups on which the intersection form is maximally definite positive/negative are both topological invariants;
(3) the Hirzebruch signature formula to compute the signature of the double branched cover of $(\mathbf{CP}^2, F)$. Note that this formula was stated for *smooth* branched cover, but recent work [GKS21] has shown that this is also true if the branched cover is only topological and the branch locus is a locally flat surface;
(4) Yamada's generalization of the Whitney congruence is stated for locally flat embeddings, and so is the Guillou–Marin congruence;
(5) [LRS15, Theorem 10.1] holds for locally flat surfaces.

For constructions, they are all based on local surfaces, which are smoothly embedded, and thus locally flat too. ■

As a consequence, if one wants to distinguish an exotic $\mathbf{CP}^2$, studying the function $\widetilde{g}$ is not going to work.

We now switch to the proof of Theorem 2.46. The proof goes in two steps :

(1) find an upper bound for $\widetilde{g}(m)$, and
(2) construct a non-orientable surface $F \subset \mathbf{CP}^2$ with $e(\mathbf{CP}^2, F) = m$ and $\chi(F)$ realizing that upper bound.

To derive upper bounds for $\widetilde{g}(-k)$ where $k \in \mathbf{N}^\star$, we use the following.

Theorem 2.48 ([LRS15, Theorem 10.1])**.** *Let X be a closed, connected, oriented, positive definite 4-manifold with $H_1(X; \mathbf{Z}) = 0$, and let $F \subset X$ be a closed, connected, nonorientable surface with nonorientable genus $h(F) = 2 - \chi(F)$. Denote as $\ell(F)$ the minimal self-intersection of an integral lift of $[F] \in H_2(X; \mathbf{Z}/2)$. Then*

$$e(X, F) \geqslant \ell(F) - 2h(F).$$

Proposition 2.49. *Let $k \in \mathbf{N}^\star$, and let $\ell \in \{0, 1\}$ have the same parity as k. We have*

$$\widetilde{g}(-k) \leqslant 2 - \frac{k + \ell}{2}.$$

[†] In both cases, the surface F comes with a well-defined normal bundle, so self-intersection makes sense even if the embedding is not smooth but only locally flat.

Proof. If $e(\mathbf{CP}^2, F) = -k < 0$, then $\ell(F) = \ell$, since $[F] \neq 0 \in H_2(\mathbf{CP}^2; \mathbf{Z}/2)$ if and only if k is odd, in which case the homology class of a complex line is an integral lift for $[F]$, and this has minimal self-intersection. ■

We now describe how to obtain bounds on $\widetilde{g}(m)$ when $m \geqslant 0$ is even.

Proposition 2.50. *Let $F \subset \mathbf{CP}^2$ be a closed, connected (not necessarily orientable) surface with $[F] = 0 \in H_2(\mathbf{CP}^2; \mathbf{Z}/2)$. Then:*

$$\chi(F) \leqslant 4 - \frac{e(\mathbf{CP}^2, F)}{2}.$$

Proof. Denote as Y the double branched cover of $\mathbf{CP}^2$ ramified over F. A computation using Theorem 2.19 and Theorem 2.36 gives:

$$\chi(Y) = 6 - \chi(F) \text{ and } \sigma(Y) = 2 - \frac{e(\mathbf{CP}^2, F)}{2}.$$

Moreover, by Proposition 2.37, we have $b_1(Y; \mathbf{Z}/2) = 0$, and thus

$$b_2(Y) = \chi(Y) - 2 = 4 - \chi(F).$$

Solving for $b_2^+(Y) = b_2(Y) + \sigma(Y)$, we obtain

$$b_2^+(Y) = 2 - \frac{2\chi(F) + e(\mathbf{CP}^2, F)}{4}.$$

Consider any integral nonzero homology class $\chi \in H_2(\mathbf{CP}^2; \mathbf{Z})$ which is represented by an embedded surface $\Sigma \subset \mathbf{CP}^2$ transverse to F. Then, letting $\widetilde{\Sigma}$ be the lift of Σ to Y, Lemma 2.38 implies that $e(Y, \widetilde{\Sigma}) = 2e(\mathbf{CP}^2, \Sigma) > 0$. This yields $b_2^+(Y) \geqslant 1$, which is the bound we claimed. ■

This has the following consequence.

Proposition 2.51. *We have*

$$\widetilde{g}(4k) \leqslant 4 - 2k \text{ and } \widetilde{g}(4k+2) \leqslant 3 - 2k$$

for all $k \geqslant 2$ and $k \geqslant 1$, respectively.

Proof. This is simply a matter of applying the previous result, noting that $[F] = 0 \in H_2(\mathbf{CP}^2; \mathbf{Z}/2)$ if and only if $e(\mathbf{CP}^2, F)$ is even. ■

The cases of $\widetilde{g}(0)$, $\widetilde{g}(2)$ and $\widetilde{g}(4)$ have not been covered. More precisely, the bounds we obtain are valid, but are vacuous.

Note that this strategy will not work to derive bounds on $\widetilde{g}(m)$ for *odd* m, since the only branched covers whose ramification locus is a non-orientable surface are necessarily two-sheeted. We will, however, provide bounds for small odd values of m.

Proposition 2.52. *Let $F \subset \mathbf{CP}^2$ be a closed, connected surface (not necessarily orientable). Then:*

(1) *if* $e(\mathbf{CP}^2, F) \equiv 0$ *or* 1 $[4]$*, then* $\chi(F)$ *is even;*
(2) *if* $e(\mathbf{CP}^2, F) \equiv 2$ *or* 3 $[4]$*, then* $\chi(F)$ *is odd.*

Proof. This is simply a matter of applying Yamada's generalization of the Whitney congruence. More precisely, we use Theorem 2.10(1). Indeed:

(1) if $e(\mathbf{CP}^2, F) = 4k$ or $4k+2$, then $q([F]) \equiv 0$ $[4]$;
(2) if $e(\mathbf{CP}^2, F) = 4k+1$ or $4k+3$, then $q([F]) \equiv 1$ $[4]$.

Plugging this back into the congruence

$$e(\mathbf{CP}^2, F) + 2\chi(F) \equiv q([F]) \ [4]$$

yields the claim. ∎

Notice that in the case where F *is* orientable, we have that $e(\mathbf{CP}^2, F)$ is a square, and thus is necessarily 0 or 1 mod 4.

Proposition 2.53. *We have:*

$$\widetilde{g}(1) \leqslant 0,\ \widetilde{g}(3) \leqslant 1,\ \widetilde{g}(5) \leqslant 0,\ \widetilde{g}(7) \leqslant -1 \text{ and } \widetilde{g}(9) \leqslant -2,$$

as well as

$$\widetilde{g}(0) \leqslant 0,\ \widetilde{g}(2) \leqslant 1 \text{ and } \widetilde{g}(4) \leqslant 0.$$

Proof. From Proposition 2.52, we see that if $F \subset \mathbf{CP}^2$ has $e(\mathbf{CP}^2, F) \in \{1,5,9\}$, then $\chi(F)$ is even. In particular, since $\chi(F) \leqslant 1$ in the case of a non-orientable surface, we obtain the bounds $\widetilde{g}(1) \leqslant 0$ and $\widetilde{g}(5) \leqslant 0$. We also obtain $\widetilde{g}(9) \leqslant 0$.

By a similar argument, if $e(\mathbf{CP}^2, F) \in \{3,7\}$, then $\chi(F)$ is odd, and thus $\widetilde{g}(3) \leqslant 1$ and $\widetilde{g}(5) \leqslant 5$.

Again, a parity argument gives the upper bounds for $\widetilde{g}$ on 0, 2, and 4.

It only suffices to see that $\widetilde{g}(7) \neq 1$ and $\widetilde{g}(9) \neq 0$.

(1) For $\widetilde{g}(7) \neq 1$, assume by contradiction that there exists a projective plane $F \subset \mathbf{CP}^2$ with $e(\mathbf{CP}^2, F) = 7$. Then $[F] \neq 0 \in H_2(\mathbf{CP}^2; \mathbf{Z}/2)$, which meas that F is mod 2 characteristic. The Guillou–Marin congruence Theorem 2.12 gives:

$$1 - 7 \equiv 2\beta(\mathbf{CP}^2, F) \ [16].$$

That is: we have $\beta(\mathbf{CP}^2, F) = 5$, which contradicts Example 2.14.

(2) For $\widetilde{g}(9) \neq 0$, assume that there is an embedded Klein bottle $F \subset \mathbf{CP}^2$ with $e(\mathbf{CP}^2, F) = 9$. Again, the Guillou–Marin congruence gives

$$1 - 9 \equiv 2\beta(\mathbf{CP}^2, F) \ [16],$$

which gives $\beta(\mathbf{CP}^2, F) = 4$. This contradicts Example 2.13. ∎

To obtain bounds on higher values, the Guillou–Marin congruence is not going to be enough anymore. For instance, we can derive

$$\widetilde{g}(11) \leqslant -1$$

by the same methods. However, all possible values for $\beta(\mathbf{CP}^2, F)$ are realizable by a non-orientable surface of genus $h \geqslant 3$, so it is not possible to rule out that a surface $F \subset \mathbf{CP}^2$ with $\chi(F) = -1$ and $e(\mathbf{CP}^2, F) = 11$ exists.

We now switch to constructions. The main (and only) tool is local surfaces, introduced in Definition 2.9.

Proposition 2.54. *All upper bounds obtained so far are sharp. Furthermore, we have*

$$\widetilde{g}(4k+1) \geqslant 2-2k \text{ and } \widetilde{g}(4k+3) \geqslant 1-2k$$

for all $k \in \mathbf{N}^\star$.

Proof. It suffices to construct surfaces with prescribed self-intersection and whose Euler characteristic realize that upper bound.

(1) To construct surfaces realizing the bounds for $\widetilde{g}(-k)$, we distinguish two cases. In the first, assume that $k = 2p$, and take F to be a local surface in $\mathbf{CP}^2$ with self-intersection $-2p$ and non-orientable genus p. Then $\chi(F) = 2 - k/2$ and $e(\mathbf{CP}^2, F) = -k$. In the second case, let $k = 2p+1$, and pick F to be a local surface with $e(\mathbf{CP}^2, F) = -2(p+1)$ and non-orientable genus $p+1$. Embed this surface in a small ball inside $\mathbf{CP}^2$ situated away from a complex line L, and form the connected sum $F\#L$ by puncturing F and L and connecting them via a tube. We obtain that $\chi(F\#L) = \chi(F) = 2-(k+1)/2$ and $e(\mathbf{CP}^2, F\#L) = -2(p+1)+1 = -k$.

(2) For $\widetilde{g}(4k) \geqslant 4-2k$ where $k \geqslant 2$, consider F a local surface with non-orientable genus $2(k-1)$ and self-intersection $4(k-1)$. Embed this in a ball away from a conic Q, and form the connected sum $F\#Q$ by using a tube. We obtain:

$$\chi(F\#Q) = \chi(F) = 4-2k \text{ and } e(\mathbf{CP}^2, F\#Q) = 4(k-1)+4 = 4k.$$

Note that in the case where $k = 1$, the surface we obtain is the conic Q itself, which is orientable.

(3) For $\widetilde{g}(4k+2) \geqslant 3-2k$ where $k \geqslant 1$, consider a local surface F with non-orientable genus $2k-1$ and self-intersection $4k-2$, and form the connected sum $F\#Q$ as in the previous case. This yields:

$$\chi(F\#Q) = 3-2k \text{ and } e(\mathbf{CP}^2, F\#Q) = 4k+2.$$

(4) For $\widetilde{g}(4k+1) \geqslant 2-2k$ and $\widetilde{g}(4k+3) \geqslant 1-2k$, take F to be a local surface of non-orientable genus $2k$ (*resp.* $2k+1$) and self-intersection $4k$ (*resp.* $4k+2$), and form the connected sum with a complex line L via a small tube. This gives a surface $F\#L$ with $\chi(F\#L) = 2-2k$ (*resp.* $\chi = 1-2k$) and $e(\mathbf{CP}^2, F\#L) = 4k+1$ (*resp.* $e = 4k+3$).

(5) It remains to treat the special values not obtained thus far. That is, we want to construct surfaces realizing the bounds for $\widetilde{g}$ obtained on $\{0,1,2,3,4,5,6,7,9\}$. The cases of $\widetilde{g}(5)$, $\widetilde{g}(7)$ and $\widetilde{g}(9)$ are covered by the previous constructions. For $\widetilde{g}(0) = 0$, pick a local Klein bottle with zero self-intersection. For $\widetilde{g}(1) = 0$, take a local Klein bottle with zero self-intersection and form the

connected sum with a complex line. For $\widetilde{g}(2) = 1$, a local projective plane with self-intersection $+2$ works. For $\widetilde{g}(3) = 1$, a conic with empty real part can be tubed to $\mathbf{RP}^2 = \mathrm{Fix}(\mathrm{conj})$. For $\widetilde{g}(4) = 0$, a local Klein bottle with self-intersection $+4$ works. ■

In view of how the use of local surfaces proved to be optimal in all cases where we *could* obtain upper bounds for $\widetilde{g}$, we formulate the following conjecture, which is supported by values of $\widetilde{g}$ at 5, 7 and 9.

Conjecture 2.55. *We have*

$$\widetilde{g}(4k+1) = 2-2k \text{ and } \widetilde{g}(4k+3) = 1-2k$$

for all $k \geqslant 1$.

Naturally, when we first tried to compute bounds for $\widetilde{g}$, we turned our attention to Kronheimer and Mrowka's proof [KM94] of the Thom conjecture. The two main ingredients for the proof are the following.

(1) Given a smoothly embedded orientable surface $F \subset \mathbf{CP}^2$ with $[F] = m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2;\mathbf{Z})$, form the blow-up m^2 times $X = \mathbf{CP}^2 \# m^2\overline{\mathbf{CP}}^2$ at points on F, and consider the blown-up surface $F' \subset X$. This surface now has $e(X,F) = 0$. This operation increases the complexity of the ambient 4-manifold, but the gain is that its normal bundle has been trivialized. The first Chern class of X is still easy to compute.

(2) Consider (A,Φ) to be a special solution to the Seiberg–Witten equations, and use the Weitzenböck formula to obtain *a priori* bounds on $|\Phi|^2$ in terms of the scalar curvature of X ([KM94, Lemma 2]). Then, using Chern–Weil theory and a Gauss–Bonnet argument, this can be used to derive the genus bound ([KM94, Lemma 9]).

Of course, one really sees the need for asking that embeddings are smooth and not merely locally flat in how the Seiberg–Witten equations play a role. We have seen that in the non-orientable case, this distinction is invisible. Still, we can inspect where Kronheimer and Mrowka's proof fails to work in this setting. Evaluating the first Chern class on a non-integral homology class is going to be problematic, but can probably be remedied through the use of *twisted* homology, and twisted Chern–Weil theory[†]. The Gauss–Bonnet formula also has a non-orientable formulation.

The proof really gets stuck when one tries to trivialize the normal bundle of the surface. Indeed: non-orientable surfaces will have non-orientable normal bundle, and thus will never be trivial, no matter how much one blows-up the ambient 4-manifold. This step cannot be avoided, since the vanishing argument [KM94, §5] works over a cylinder.

2.3.3 Non-Orientable Genus Functions in 4-Manifolds

Let X be a smooth, closed, connected and oriented 4-manifold. Given $m \in \mathbf{Z}$, define $\Sigma_X(m)$ to be the collection of closed, connected and non-orientable surfaces $F \subset X$ smoothly embedded such that $e(X,F) = m$. In particular, by using local surfaces inside X, we see that if $m \in \mathbf{Z}$ is even, then $\Sigma_X(m) \neq \varnothing$.

[†] However, we were not able to find a source exposing this topic.

Definition 2.56. *The **non-orientable genus function** of X is the application*

$$\begin{array}{rccc} \widetilde{g}_X & : & \mathbf{Z} & \longrightarrow & \mathbf{Z}_{\leqslant 1} \cup \{-\infty\} \\ & & m & \longmapsto & \max\limits_{F \in \Sigma_X(m)} \chi(F), \end{array}$$

where we make the convention that $\max \varnothing = -\infty$.

Of course, we have the following immediate observation.

Proposition 2.57. *If $-X$ denotes X with the opposite orientation, then:*

$$\forall\, m \in \mathbf{Z},\ \widetilde{g}_{-X}(m) = \widetilde{g}_X(-m).$$

Proof. This comes from the fact that if $x \in F \pitchfork F'$ is an intersection point with sign ± 1 in X, then it has the sign ∓ 1 in $-X$. ■

This means that the function $\widetilde{g}_{\overline{\mathbf{CP}}^2}$ is almost completely computed by Theorem 2.46.

Recall that a 4-manifold is called **even** if $Q_X(\xi,\xi)$ is an even integer for all $\xi \in H_2(X;\mathbf{Z})$, and is called odd otherwise[†]. Being even or odd is a topological invariant of 4-manifolds (for instance, it allows to distinguish $\mathbf{S}^2 \times \mathbf{S}^2$ from $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$). We have the following characterization of parity in terms of $\widetilde{g}_X$.

Proposition 2.58. *The 4-manifold X is even if and only if it $\widetilde{g}_X$ assumes the value $-\infty$. If it does, then $\widetilde{g}(m) = -\infty$ for all odd $m \in \mathbf{Z}$.*

Proof. If X is even, then any surface $F \subset X$ has an even self-intersection, thus $\Sigma_X(m) = \varnothing$ if m is odd.

If X is odd however fix $m \in \mathbf{Z}$ and let $F \subset X$ be a connected surface representing a homology class $\xi \in H_2(X;\mathbf{Z})$ such that $Q_X(\xi,\xi)$ is odd. Consider a local surface G with self-intersection $m - Q_X(\xi,\xi)$, which can be embedded in a 4-ball away from F. Performing the connected sum $F\#G$ *via* a small tube joining both surfaces, we obtain a non-orientable connected surface $F\#G \in \Sigma_X(m)$, meaning that $\Sigma_X(m) \neq \varnothing$. ■

Example 2.59. *We have:*

$$\widetilde{g}_{\mathbf{S}^4}(0) = 0 \text{ and } \widetilde{g}_{\mathbf{S}^4}(2k) = 2 - |k|.$$

This is a direct consequence of The Whitney–Massey theorem (Theorem 2.8).

In general, however, it will be difficult to compute upper bounds for $\widetilde{g}_X$. Lower bounds can still be obtained from constructions using local surfaces (and a minimum of knowledge about the integral second homology of X), but the strategy used in Proposition 2.50 is not going to work in all generality. Indeed, for the case of $X = \mathbf{CP}^2 \# \mathbf{CP}^2$ for instance, there are surfaces $F \subset X$ with $e(X,F) \equiv 0\ [2]$ but $[F] \neq 0 \in H_2(X;\mathbf{Z}/2)$, contrary to the case of $\mathbf{CP}^2$. In fact, considering a double branched cover of X will only prove to be useful if $b_1(X;\mathbf{Z}/2) = 0$.

Lastly, without assuming that X is definite (positive or negative), we cannot apply Theorem 2.48.

We finish with a small observation that allows to derive *lower* bounds for $\widetilde{g}$ on a connected sum.

[†] We do *not* ask that *all* homology classes have an odd self-intersection, just that *exists* one.

Proposition 2.60. *Let X and Y be two closed, connected and oriented smooth 4-manifolds, and let $m \in \mathbf{Z}$. We have:*

$$\widetilde{g}_{X\#Y}(m) \geqslant \max_{k\in\mathbf{Z}}\Big[\widetilde{g}_X(k) + \widetilde{g}_Y(m-k) - 2\Big].$$

Proof. Let $k \in \mathbf{Z}$. If $\widetilde{g}_X(k) = -\infty$ or $\widetilde{g}_Y(m-k) = -\infty$, then $\widetilde{g}_X(k) + \widetilde{g}_Y(m-k) - 2 = -\infty \leqslant \widetilde{g}_{X\#Y}(m)$. Therefore, assume that $\widetilde{g}_X(k) > -\infty$ and $\widetilde{g}_Y(m-k) = -\infty$. Let $F_1 \in \Sigma_X(k)$ and $F_2 \in \Sigma_Y(m-k)$ such that $\chi(F_1) = \widetilde{g}_X(k)$ and $\chi(F_2) = \widetilde{g}_Y(m-k)$. Form the surface $F = F_1\#F_2 \subset X\#Y$. Then:

$$e(X\#Y, F) = e(X, F_1) + e(Y, F_2) = m \text{ and } \chi(F) = \chi(F_1) + \chi(F_2) - 2 = \widetilde{g}_X(k) + \widetilde{g}_Y(m-k) - 2,$$

which proves the claim. ∎

The Arnold Surface of an Odd Degree Flexible Curve

3

Our brain has two halves: one is responsible for the multiplication of polynomials and languages, and the other half is responsible for orientation of figures in space and all the things important in real life. Mathematics is geometry when you have to use both halves.

V. I. Arnold

Outline of this chapter

3.1 The Geometry of the Complex Projective Plane

We endow $\mathbf{CP}^2$ with its Fubini–Study metric, which makes it a Kähler manifold. Recall that in particular, it is a symplectic 4-manifold, and $\mathbf{RP}^2 = \mathrm{Fix}(\mathrm{conj})$ is a Lagrangian and totally geodesic surface[†]. Moreover, the complex conjugation is an anti-holomorphic involution and an isometry for the associated Riemannian metric.

3.1.1 The Focal Conic and Totally Flexible Curves

Consider the distance function

$$\begin{array}{rccc} f & : & \mathbf{CP}^2 & \longrightarrow & \mathbf{R} \\ & & z & \longmapsto & \mathrm{dist}(z, \mathbf{RP}^2) = \inf\limits_{x \in \mathbf{RP}^2} \mathrm{dist}(z, x). \end{array}$$

By compactness, this infimum is, in fact, a minimum. We can even say more about geodesics realizing this minimum.

Theorem 3.1 ([BC64, §8.1 Theorem 3])**.** *Let L_1 and L_2 be two submanifolds of a Riemannian manifold X, and let γ be a geodesic with endpoints in L_1 and L_2 and whose length is equal to* $\mathrm{dist}(L_1, L_2)$*. Then γ is perpendicular to both L_1 and L_2.*

Recall that we denote the exponential map of $\mathbf{CP}^2$ at a point z as

$$\exp_z : T_z\mathbf{CP}^2 \to \mathbf{CP}^2,$$

and that this is defined by the property that the map $t \mapsto \exp_z(tv)$ is the (unique) geodesic starting at z and directed by $v \in T_z\mathbf{CP}^2$. This has the following consequence.

Proposition 3.2. *Let $z \in \mathbf{CP}^2$. Then there exists $(x, v) \in T\mathbf{RP}^2$ such that $z = \exp_x(\mathbf{i}v)$ and $f(z) = \|v\|$.*

Proof. From Theorem 3.1, if $x \in \mathbf{RP}^2$ is such that $\mathrm{dist}(z, x) = f(z)$, then there exists $w \in \nu_x\mathbf{RP}^2$ such that $z = \exp_x(w)$. From $\nu\mathbf{RP}^2 = \mathbf{i} \cdot T\mathbf{RP}^2$, we get that $w = \mathbf{i}v$ for some $v \in T_x\mathbf{RP}^2$. Finally, it is a standard fact that the length of the geodesic

$$t \in [0, 1] \mapsto \exp_x(\mathbf{i}v)$$

is the norm $\|\mathbf{i}v\| = \|v\|$. ∎

In particular, if one looks at the *global* exponential map $\exp : T\mathbf{CP}^2 \to \mathbf{CP}^2$ and restricts it into the normal exponential map

$$\varepsilon : \nu\mathbf{RP}^2 \to \mathbf{CP}^2,$$

then this map is surjective and smooth.

[†] We will re-prove this in §3.4.3.

By compactenss of $\mathbf{CP}^2$, we have that f has a maximum it realizes. Call $\alpha = \max(f) > 0$ this maximum. We are interested in the following set:

$$Q = \left\{ z \in \mathbf{CP}^2 \,\middle|\, f(z) = \alpha \right\}.$$

We will relate Q to the following notions.

Definition 3.3. *Let $\vec{n} \in \nu\mathbf{RP}^2$ be a normal vector to $\mathbf{RP}^2$. $\vec{n}$ is called a **focal vector** of $\mathbf{RP}^2$ if the map*

$$\mathrm{d}_{\vec{n}}\varepsilon : T_{\vec{n}}\nu\mathbf{RP}^2 \to T_{\varepsilon(\vec{n})}\mathbf{CP}^2$$

*fails to be injective. A point $z \in \mathbf{CP}^2$ is called a **focal point** of $\mathbf{RP}^2$ if $z = \varepsilon(\vec{n})$ for some focal vector $\vec{n}$.*

Using [BH84, Proposition 1], we give the following characterization of focal points: $z \in \mathbf{CP}^2$ is a focal point if and only if the squared distance function

$$\begin{array}{rccc} \varphi_z & : & \mathbf{RP}^2 & \longrightarrow & \mathbf{R} \\ & & x & \longmapsto & \mathrm{dist}(x,z)^2 \end{array}$$

admits a degenerate critical point. We denote as $\mathrm{Foc}(\mathbf{RP}^2)$ the set of focal points of $\mathbf{RP}^2$.

Theorem 3.4. *We have*

$$\mathrm{Foc}(\mathbf{RP}^2) = Q = \left\{ [z_0 : z_1 : z_2] \in \mathbf{CP}^2 \,\middle|\, z_0^2 + z_1^2 + z_2^2 = 0 \right\}$$

the Fermat conic, and $\alpha = \pi/4$.

Proof. We re-expose the computations made in [Ber97, §6], where the author computes the focal set of $\mathbf{CP}^{m+1}$ inside $\mathbf{HP}^{m+1}$. We therefore take the group $O(m+2)$ instead of $SU(m+2)$, and we take $m = 1$. We denote as $\tau : \mathbf{S}^5 \to \mathbf{CP}^2$ the Hopf map. Then, if $\mathbf{x}, \mathbf{y} \in \mathbf{S}^2$ are such that $\langle \mathbf{x} \,|\, \mathbf{y} \rangle_{\mathbf{R}} = 0$, we set:

$$\gamma_{\mathbf{x},\mathbf{y}}(t) = \tau(\cos(t)\mathbf{x} + \mathbf{i}\sin(t)\mathbf{y}).$$

The geodesics normal to $\mathbf{RP}^2$ are exactly all the $\gamma_{\mathbf{x},\mathbf{y}}$. Now, if one sets

$$N_r = \left\{ \gamma_{\mathbf{x},\mathbf{y}}(r) \,\middle|\, \mathbf{x}, \mathbf{y} \in \mathbf{S}^2,\ \langle \mathbf{x} \,|\, \mathbf{y} \rangle_{\mathbf{R}} = 0 \right\}$$

for $r \geqslant 0$, then N_r is the set of points in $\mathbf{CP}^2$ which can be reached from $\mathbf{RP}^2$ by following a normal geodesic for the time r. Then:

$$\mathbf{CP}^2 = \bigsqcup_{0 \leqslant r \leqslant \pi/4} N_r.$$

This already implies that $\alpha = \pi/4$ and $Q = N_{\pi/4}$. Now, it is easy to see that if $z \in N_r$ with $0 < r < \pi/4$, then the squared distance function φ_z has no critical points on $\mathbf{RP}^2$ (and in fact, there is a *unique* $x \in \mathbf{RP}^2$ such that $f(z) = \mathrm{dist}(z,x)$ in this case). Therefore, we also have $\mathrm{Foc}(\mathbf{RP}^2) = N_{\pi/4}$. In particular:

$$N_{\pi/4} = \left\{ \tau\left(\frac{\mathbf{x} + \mathbf{i}\mathbf{y}}{\sqrt{2}} \right) \,\middle|\, \mathbf{x}, \mathbf{y} \in \mathbf{S}^2,\ \langle \mathbf{x} \,|\, \mathbf{y} \rangle_{\mathbf{R}} = 0 \right\}.$$

In fact, if one sees $\mathbf{S}^5$ as the unit sphere of $\mathbf{C}^3$ (that is, the set of triples $(z_0,z_1,z_2)\in\mathbf{C}^3$ such that $|z_0|^2+|z_1|^2+|z_2|^2=1$), then if $\mathbf{x},\mathbf{y}\in\mathbf{S}^2$ are such that $\langle\mathbf{x}\,|\,\mathbf{y}\rangle_{\mathbf{R}}=0$, we have $\mathbf{x}+\mathbf{i}\mathbf{y}\in\mathbf{S}^5$. Moreover, the map $\tau:\mathbf{S}^5\to\mathbf{CP}^2$ corresponds to the map $\tau(z_0,z_1,z_2)=[z_0:z_1:z_2]$. Therefore, we obtain:

$$\gamma_{\mathbf{x},\mathbf{y}}(\pi/4)=[x_0+\mathbf{i}y_0:x_1+\mathbf{i}y_1:x_2+\mathbf{i}y_2].$$

Finally, the computation

$$(x_0+\mathbf{i}y_0)^2+(x_1+\mathbf{i}y_1)^2+(x_2+\mathbf{i}y_2)^2=\|\mathbf{x}\|^2-\|\mathbf{y}\|^2+2\mathbf{i}\langle\mathbf{x}\,|\,\mathbf{y}\rangle_{\mathbf{R}}\qquad(*)$$

ensures that

$$N_{\pi/4}\subset\left\{[z_0:z_1:z_2]\in\mathbf{CP}^2\;\middle|\;z_0^2+z_1^2+z_2^2=0\right\}.$$

Conversely, if $[z_0:z_1:z_2]$ is a point on the Fermat conic, then setting $z_i=x_i+\mathbf{i}y_i$, we indeed have that $[z_0:z_1:z_2]=\gamma_{\mathbf{x}/\|\mathbf{x}\|,\mathbf{y}/\|\mathbf{y}\|}(\pi/4)$, where $\mathbf{x}\neq 0$ and $\mathbf{y}\neq 0$ since $Q\cap\mathbf{RP}^2=\varnothing$, and one checks that $\langle\mathbf{x}\,|\,\mathbf{y}\rangle_{\mathbf{R}}=0$ by using $(*)$ again. ■

Definition 3.5. *The set Q of focal points of $\mathbf{RP}^2$ will be called the* ***focal conic****.*

In particular, this is a non-singular conic with empty real part $\mathbf{R}Q=\varnothing$, and is diffeomorphic to a 2-sphere. It represents the degree 2 homology class, $[Q]=2[\mathbf{CP}^1]\in H_2(\mathbf{CP}^2;\mathbf{Z})$, and its normal Euler number is therefore $e(\mathbf{CP}^2,Q)=4$.

For $0<r<\pi/4$, we see that $f^{-1}([0,r])$ is a regular neighborhood of $\mathbf{RP}^2$ and $f^{-1}([r,\pi/4])$ is one for Q. Moreover, the submanifold $f^{-1}(r)$ is the common boundary of those neighborhoods, and can be identified with either of the unit normal bundles of $\mathbf{RP}^2$ or Q. From $\nu\mathbf{RP}^2\cong -T\mathbf{RP}^2$, we obtain that the unit normal bundle of $\mathbf{RP}^2$ is identified with its unit tangent bundle. This is doubly-covered by $SO(3)$ the unit tangent bundle of S^2, and thus

$$f^{-1}(r)\cong SO(3)/\{\pm 1\}\cong\mathbf{S}^3/\{\pm 1,\pm\mathbf{i}\}=\mathbf{S}^3/(\mathbf{Z}/4),$$

which is a lens space $L(4,1)$ or $L(4,3)=-L(4,1)$. In fact, depending on whether one orients $f^{-1}(r)$ as the boundary of $f^{-1}([0,r])$ or of $f^{-1}([r,\pi/4])$, one obtains either choices. These orientation considerations are made exact in [Gil92, §4].

Thinking of $\mathbf{RP}^2$ and Q as "two ends" of $\mathbf{CP}^2$, if a flexible curve is required to look like an algebraic curve near $\mathbf{RP}^2$, it might make sense to also ask a similar thing near Q.

Definition 3.6. *Let $F\subset\mathbf{CP}^2$ be a flexible curve of degree m. F is called* ***totally flexible*** *if $F\pitchfork Q$ is $2m$ points.*

In [Sai24], those were called *Q-flexible* curves instead.

In general, we know that if F is a flexible curve of degree m and if $F\pitchfork Q$, then the intersection is some even number $2m'$ of points, with $m'\geqslant m$, since

$$2m=|Q_{\mathbf{CP}^2}(F,Q)|\leqslant\#F\pitchfork Q.$$

This means that we are requiring the intersection between F and Q to be *minimal*.

Question 3.7. *If $F \subset \mathbf{CP}^2$ is a flexible curve, is there a totally flexible curve $F' \subset \mathbf{CP}^2$ such that $\mathbf{R}F$ and $\mathbf{R}F'$ are isotopic?*

This legitimate question probably has, in fact, a negative answer to it. However, the fact that the focal conic has empty real part might still be of some help, compared to the analogue question with a real line in place of Q. A way to construct a surface F' would be by using the Whitney trick.

Suppose that $F \pitchfork Q$ possesses a negative intersection point z_1. Then there is also a positive one z_2. Consider two arcs $\gamma_F \subset F$ and $\gamma_Q \subset Q$ whose endpoints are z_1 and z_2. Since $\pi_1(\mathbf{CP}^2) = 1$, we can consider a smoothly immersed disc $D \looparrowright \mathbf{CP}^2$ with $\partial D = \gamma_F \cup \gamma_Q$. In the favorable case where D is embedded and the interior of D does not intersect F nor Q, then we can[†] *push* F following D to construct a surface F' with $F' \pitchfork Q = F \pitchfork Q \smallsetminus \{z_1, z_2\}$ (see Figure 3.1).

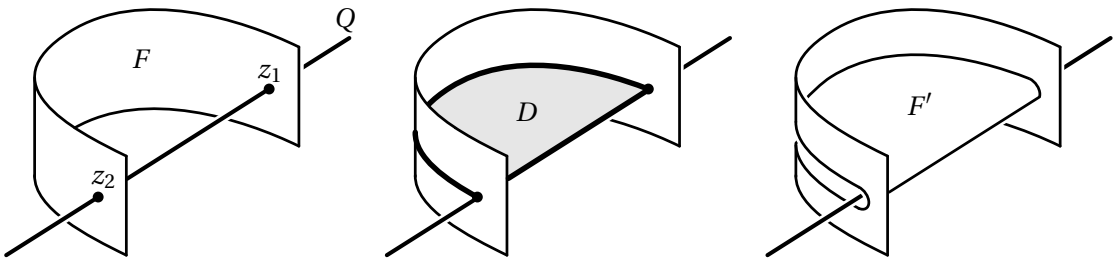


Figure 3.1. The Whitney trick allows to simplify a pair of intersection points with opposite signs.

If the disc D were to intersect F, that can still be resolved at a cost on the topological type of F. Indeed, this will create an *immersed* surface F' with two nodal points, with $\chi(F') = \chi(F) - 2$ and with $e(\mathbf{CP}^2, F') = e(\mathbf{CP}^2, F)$. Those nodal points will be of opposite signs. Therefore, they can both be resolved by performing the operation described in Proposition 2.17, and this will have the effect of creating a new surface F'' with $\chi(F'') = \chi(F') - 2 = \chi(F) - 4$ and $e(\mathbf{CP}^2, F'') = e(\mathbf{CP}^2, F') \pm 2 \pm 2$ (with a choice in both ± 2).

There are still many issues:

(1) this construction must also be performed with the points $\mathrm{conj}(z_1)$ and $\mathrm{conj}(z_2)$ to obtain a new surface F' invariant under conjugation;

(2) this construction must happen away from $\mathbf{RP}^2$ (that is, we need $D \cap \mathbf{RP}^2 = \varnothing$), in order for $\mathbf{R}F$ to remain unchanged;

(3) we need to make sure we can choose a Whitney disc that does not intersect Q, otherwise this will have no effect.

Another possibility would be to consider the four points z_1, z_2, $\mathrm{conj}(z_1)$ and $\mathrm{conj}(z_2)$, as well as four arcs $\gamma_{F,1}$, $\gamma_{F,2}$, $\gamma_{Q,1}$ and $\gamma_{Q,2}$ joining them on F and Q, and use the Whitney *rectangle* R with $\partial R = \gamma_{F,1} \cup \gamma_{F,2} \cup \gamma_{Q,1} \cup \gamma_{Q,2}$ to deform F into F', as depicted in Figure 3.2.

[†] There is also a condition on the normal bundle of D to perform this action, but we shall not digress with this here.

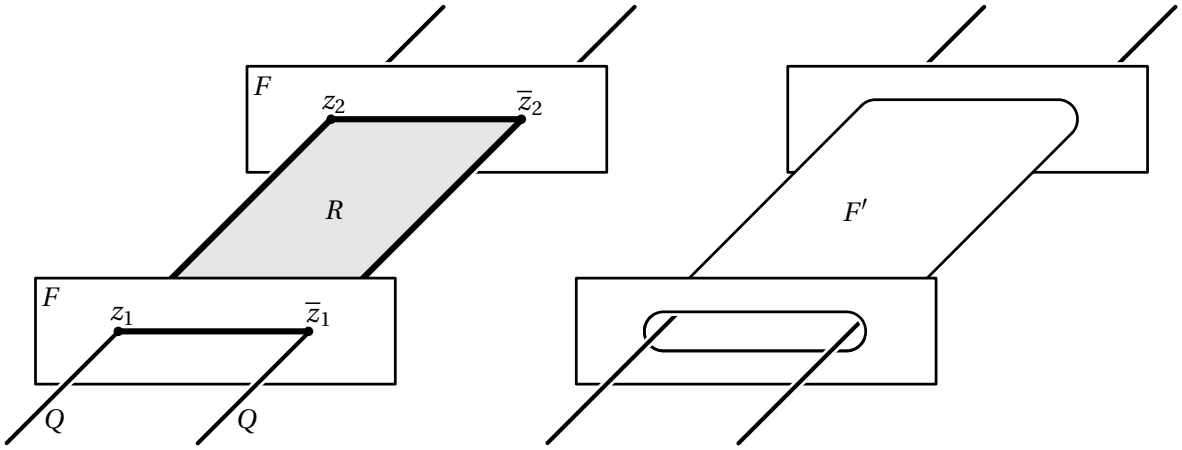


Figure 3.2. Using a Whitney rectangle to deform the surface F.

However, most of the above problems remain, in one form or another. Furthermore, we are changing the topological type of F by creating genus, which is not desirable.

Anticipating on what will come next, what we *really* care about is that F intersects *some flexible* conic C with $\mathbf{R}C = \varnothing$ minimally; it needs not be Q specifically. Indeed, in the end, we only care about the fact that the double branched cover of $\mathbf{S}^4$ ramified along $p(C)$ is a *homology* $\overline{\mathbf{CP}}^2$.

Question 3.8. *Given a flexible curve F, does there exist a purely imaginary flexible conic C and a suitable deformation F' of F such that $\mathbf{R}F'$ and $\mathbf{R}F$ are isotopic and F' intersects C minimally?*

This question is probably more feasible, since this time we are allowed to also perturb the conic. It happens to have a positive answer for *pseudo-holomorphic* curves. Indeed, consider the union of two complex conjugated J-holomorphic lines (where J is the tamed almost-complex structure making F pseudo-holomorphic), and take the conic C to be a J-holomorphic perturbation of this union. Since all intersection points between F and any of the two lines are positive, by Theorem 2.41, and since this fact remains true after the perturbation, the pseudo-holomorphic conic C intersects F in positive points only.

In particular, Theorem 3.20, which will be stated in terms of totally flexible curves, really holds for:

(1) flexible curves which intersect a purely imaginary flexible conic in positive points only, and
(2) pseudo-holomorphic curves.

3.1.2 Double Branched Covers of the 4-Sphere

Since $\mathbf{R}Q = Q \cap \mathbf{RP}^2 = \varnothing$, we obtain that $p : \mathbf{CP}^2 \to \mathbf{S}^4$ induces a double *unbranched* covering $p|_Q : Q \to p(Q)$. We set:

$$\mathcal{R} = p(\mathbf{RP}^2) \text{ and } \mathcal{Q} = p(Q).$$

In particular, $\mathcal{R}$ and $\mathcal{Q}$ are both projective planes embedded in $\mathbf{S}^4$. From Lemma 2.38, we obtain that

$$e(\mathbf{S}^4, \mathcal{R}) = -2 \text{ and } e(\mathbf{S}^4, \mathcal{Q}) = +2.$$

In particular, those are the only two possible embeddings of real projective planes, by Theorem 2.8, and they can be used to construct any local surface (see Definition 2.9).

Lemma 3.9. *There exists an orientation reversing involution $\psi : \mathbf{S}^4 \to \mathbf{S}^4$ such that $\psi(\mathcal{R}) = \mathcal{Q}$ and $\psi(\mathcal{Q}) = \mathcal{R}$. This involution does* not *lift into a map $\Psi : \mathbf{CP}^2 \to \mathbf{CP}^2$ such that $p \circ \Psi = \psi \circ p$.*

Proof. We use the notations of the proof of Theorem 3.4 again. Because of the relation

$$\mathrm{conj}(\gamma_{\mathbf{x},\mathbf{y}}(r)) = \gamma_{\mathbf{x},-\mathbf{y}}(r),$$

we see that N_r is invariant under conj for each $0 \leqslant r \leqslant \pi/4$. In particular, this yields:

$$\mathbf{S}^4 = \bigsqcup_{0 \leqslant r \leqslant \pi/4} N_r/\mathrm{conj}.$$

Moreover, $N_0/\mathrm{conj} = \mathcal{R}$ and $N_{\pi/4}/\mathrm{conj} = \mathcal{Q}$, and each N_r/conj is[†] diffeomorphic to each other for any $0 < r < \pi/4$.

Set $U = \mathbf{S}^4 \smallsetminus (N_0/\mathrm{conj} \cup N_{\pi/4}/\mathrm{conj})$. Then any $z \in U$ has exactly two pre-images $z_1 \neq z_2$ in $\mathbf{CP}^2$ under p:

$$p(z_1) = z = p(z_2) \text{ and } z_1 = \mathrm{conj}(z_2).$$

Therefore, there exist unique $0 < r < \pi/4$ and $\mathbf{x}, \mathbf{y} \in \mathbf{S}^2$ with $\langle \mathbf{x} \mid \mathbf{y} \rangle_{\mathbf{R}} = 0$ such that $z_1 = \gamma_{\mathbf{x},\mathbf{y}}(r)$ and $z_2 = \gamma_{\mathbf{x},-\mathbf{y}}(r)$. Define:

$$\psi(z) = p(\gamma_{\mathbf{x},\mathbf{y}}(\pi/4 - r)).$$

Therefore, the map $\psi : U \to U$ is clearly involutive, and it extends to the whole $\mathbf{S}^4$.

The map ψ cannot be lifted into $\Psi : \mathbf{CP}^2 \to \mathbf{CP}^2$, because otherwise we would have

$$\Phi(N_r) = N_{\pi/4-r} \text{ for all } 0 \leqslant r \leqslant \pi/4,$$

and thus N_0 and $N_{\pi/4}$ would be swapped by Ψ despite them not being diffeomorphic. However, the map $\psi|_U : U \to U$ *does* lift into a map $\Psi : \mathbf{CP}^2 \smallsetminus (\mathbf{RP}^2 \cup Q) \to \mathbf{CP}^2 \smallsetminus (\mathbf{RP}^2 \cup Q)$ in an obvious way. ■

A consequence of that is the following.

Proposition 3.10. *The double branched cover of $(\mathbf{S}^4, \mathcal{Q})$ is diffeomorphic to $\overline{\mathbf{CP}}^2$.*

Proof. The map $\psi \circ p : \mathbf{CP}^2 \to \mathbf{S}^4$ satisfies all the properties of a double branched cover, *except* the fact it is orientation-reversing. Therefore, reversing orientation on $\mathbf{CP}^2$ produces a genuine double branched cover $\widetilde{p} : \overline{\mathbf{CP}}^2 \to \mathbf{S}^4$ defined again by $\widetilde{p} = \psi \circ p$. ■

[†] In fact, $N_r/\mathrm{conj} = L(4,1)/\mathrm{conj}$ is identified with the quotient $(SO(3)/\{\pm 1, \pm \mathbf{i}\})/\mathrm{conj} = SO(3)/Q_8$, where Q_8 is the quaternion group. This can be obtained, for instance, by a surgery presentation for N_r/conj; see [Gil92, §8].

We set $\overline{\mathbf{RP}}^2 = \widetilde{p}^{-1}(\mathcal{Q})$ and $\overline{Q} = \widetilde{p}^{-1}(\mathcal{R})$. We justify the terminology with the following.

Proposition 3.11. *The surfaces $\overline{\mathbf{RP}}^2$ and $\overline{Q}$ are a projective plane and a 2-sphere, respectively. Moreover, we have:*

$$e(\overline{\mathbf{CP}}^2, \overline{\mathbf{RP}}^2) = +1 \text{ and } e(\overline{\mathbf{CP}}^2, \overline{Q}) = -4.$$

Proof. The claim regarding the normal Euler numbers is yet another application of Lemma 2.38. The fact that $\overline{\mathbf{RP}}^2$ is a projective plane is simply because $\widetilde{p} : \overline{Q} \to \mathcal{Q}$ is a diffeomorphish on the ramification loci, and $\mathcal{Q}$ was already a projective plane. Finally, $\widetilde{p}$ induces a double *unbranched* covering of $\mathcal{R}$, which means that there are two possibilities: either $\overline{Q} = \widetilde{p}^{-1}(\mathcal{R})$ is disconnected, in which case it is a disjoint union of two projective planes, or it is connected and thus a 2-sphere. To show it is connected, it suffices to find a loop $\gamma \subset \mathcal{R}$ such that $\widetilde{p}^{-1}(\gamma)$ is a connected loop rather than two disjoint copies of it.

Take L to be a generic real line, and consider $\gamma = p(\mathbf{R}L) \subset \mathcal{R}$. Then $D = p(\mathbf{C}L)$ is a 2-disc with boundary γ, and $D \pitchfork \mathcal{Q}$ is one single point, coming from the two complex conjugate transverse intersections $\mathbf{C}L \pitchfork Q$. Consider $\widetilde{D} = \widetilde{p}^{-1}(D)$. We therefore obtain that $\widetilde{p} : \widetilde{D} \to D$ is a branched cover of a disc, whose ramification locus is a single point. By Proposition 2.33, we obtain that necessarily, $\widetilde{D}$ is a disc as well, and thus $\widetilde{p}^{-1}(\gamma)$ is connected. ∎

In Figure 3.3 we represent all the relevant data regarding the two double branched covers of $\mathbf{S}^4$ we are considering.

$$\begin{array}{lccccccc}
e(\mathbf{CP}^2, \mathbf{RP}^2) = -1 & \mathbf{RP}^2 & \subset & \mathbf{CP}^2 & \supset & Q & & e(\mathbf{CP}^2, Q) = +4 \\
& \| & & \downarrow p & & \downarrow\!\!\downarrow & & \\
e(\mathbf{S}^4, \mathcal{R}) = -2 & \mathcal{R} & \subset & \mathbf{S}^4 & \supset & \mathcal{Q} & & e(\mathbf{S}^4, \mathcal{Q}) = +2 \\
& \uparrow\!\!\uparrow & & \uparrow \widetilde{p} & & \| & & \\
e(\overline{\mathbf{CP}}^2, \overline{Q}) = -4 & \overline{Q} & \subset & \overline{\mathbf{CP}}^2 & \supset & \overline{\mathbf{RP}}^2 & & e(\overline{\mathbf{CP}}^2, \overline{\mathbf{RP}}^2) = +1
\end{array}$$

Figure 3.3. The two possible double branched covers of the 4-sphere ramified along a real projective plane. Arrows $\twoheadrightarrow$ indicate a double unbranched cover, and arrows $=$ indicate a diffeomorphism induced on the ramification loci.

Given a flexible curve $F \subset \mathbf{CP}^2$ of degree m, we are interested in the following:

$$\widetilde{p}^{-1}(p(F)).$$

This is a surface in $\overline{\mathbf{CP}}^2$ with boundary. Given an oval $o \subset \mathbf{R}F$, denote as

$$\widetilde{o} = \widetilde{p}^{-1}(p(o))$$

the lift of o to $\overline{Q}$, and given a pseudo-line $\mathcal{J}$, denote as

$$\widetilde{\mathcal{J}} = \widetilde{p}^{-1}(p(\mathcal{J}))$$

its lift to $\overline{Q}$. The map $\widetilde{p}$ induces *unbranched* double covers

$$\widetilde{p} : \widetilde{o} \to p(o) \text{ and } \widetilde{p} : \widetilde{\mathcal{J}} \to p(\mathcal{J}).$$

This means that they are either the non-trivial double covering of the circle (given by the map $e^{\mathbf{i}\theta} \mapsto e^{2\mathbf{i}\theta}$) or the trivial one.

Proposition 3.12. *If o is an oval, then $\widetilde{o}$ is two circles and $\widetilde{p} : \widetilde{o} \to p(o)$ is the trivial cover. If $\mathcal{J}$ is a pseudo-line, then $\widetilde{\mathcal{J}}$ is connected, and $\widetilde{p} : \widetilde{\mathcal{J}} \to p(\mathcal{J})$ is the non-trivial double covering of a circle.*

Proof. Given $\gamma \subset \mathcal{R}$ a loop, the map $\widetilde{p} : \overline{Q} \to \mathcal{R}$ induces a trivial covering $\widetilde{p} : \widetilde{p}^{-1}(\gamma) \to \gamma$ if and only if the homotopy class of the loop γ belongs to the subgroup $\widetilde{p}_*(\pi_1(\overline{Q}))$; that is, if and only if γ is null-homotopic. ■

Recall from §1.1.1 that if o is an oval, then $\mathbf{RP}^2 \smallsetminus o$ is the disjoint union of an open disc and an open Möbius strip. The disc component is called the interior of o, and is denoted as $\mathrm{Int}(o)$. An oval o_1 is **included** in an other oval o_2, and we denote it as $o_1 \subset o_2$, if $o_1 \subset \mathrm{Int}(o_2)$.

Proposition 3.13. *Let o_1 and o_2 be ovals of $\mathbf{R}F$, and let $\mathcal{J}$ be the pseudo-line of $\mathbf{R}F$ if m is odd.*

(1) The set $\overline{Q} \smallsetminus \widetilde{\mathcal{J}}$ is the union of two open discs, each containing one of the two components of $\widetilde{o}_1$.

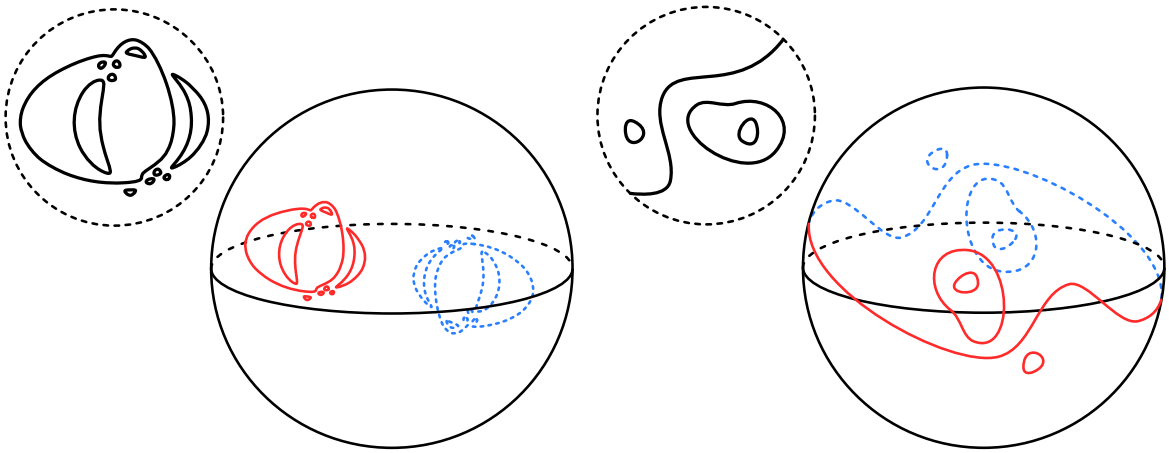
(2) Assume that $o_1 \subset o_2$. Then $\widetilde{o}_1 \subset \widetilde{o}_2$ in the following sense. The three components of $\overline{Q} \smallsetminus \widetilde{o}_2$ are two open discs and an open annulus. Then each disc component contains one of the two components of $\widetilde{o}_1$. Moreover, if m is odd, then the annulus component contains $\widetilde{\mathcal{J}}$.

Proof. This is, in fact, immediate, if one thinks of $\widetilde{p} : \overline{Q} \to \mathcal{R}$ as the quotient of the 2-sphere by the action of a fixed-point free and non-trivial involution (namely the antipodal map). That is: if $\tau : \overline{Q} \to \overline{Q}$ is the involution that spans $\mathrm{Aut}(\widetilde{p})$ the group of deck transformations, then $\mathcal{R} = \overline{Q}/\tau$. ■

We denote the lift of $p(\mathbf{R}F)$ as $\overline{\mathbf{R}}F$. That is:

$$\overline{\mathbf{R}}F = \widetilde{p}^{-1}(p(\mathbf{R}F)) = \bigsqcup_{o \text{ oval}} \widetilde{o} \sqcup \widetilde{\mathcal{J}}.$$

This can be thought of as the fact that ovals are "doubled" when going from $\mathbf{CP}^2$ to $\overline{\mathbf{CP}}^2$, whereas the pseudo-line is not, and is travelled at twice the speed instead. We refer the reader to Figure 3.4 for a depiction of the situation.



(a) (b)

Figure 3.4. (a) The set $\overline{\mathbf{R}}F \subset \overline{Q}$ for an even degree curve. (b) The same for an odd degree curve. It is understood that the red part is on the "front side" of the 2-sphere, and the blue part is on its "back side".

3.2 The Even Degree Case

In this section, we review how the traditional Arnold surface of an even degree curve can be used to derive restrictions on the real scheme of the curve. Recall that we denote as $p : \mathbf{CP}^2 \to \mathbf{S}^4$ the branched cover given by taking the quotient $\mathbf{S}^4 = \mathbf{CP}^2/\mathrm{conj}$.

Definition 3.14. *Let $F \subset \mathbf{CP}^2$ be a flexible curve of even degree $m = 2k$. Let $\mathbf{RP}^2_\pm$ be the closures of both choices of components of $\mathbf{RP}^2 \smallsetminus \mathbf{R}F$ which bound $\mathbf{R}F$, with the convention that $\mathbf{RP}^2_+$ is orientable. The* ***Arnold surface*** *of F is the closed surface*

$$\mathcal{A}(F) = p(F) \cup p(\mathbf{RP}^2_+) \subset \mathbf{S}^4.$$

Recall the notations relevant to Hilbert's 16$^{\text{th}}$ problem from §1.4. We have, from Theorem 2.19:

$$\chi(\mathcal{A}(F)) = \frac{\chi(F)}{2} + \chi(\mathbf{RP}^2_+) = -2k^2 + 3k + (p-n).$$

Moreover, Lemma 2.38 yields

$$e(\mathbf{S}^4, \mathcal{A}(F)) = 2k^2 - 2(p-n),$$

where we notice that $e(\mathbf{CP}^2, \mathbf{RP}^2_+) = -(p-n)$ since $\nu\mathbf{RP}^2$ is anti-isomorphic to $T\mathbf{RP}^2$, and this holds for its subsets as well. Note that the surface $\mathcal{A}(F)$ is not orientable in general, even if F is type I (and thus $p(F)$ is), and even if F is an M-curve. In fact, it is orientable if and only if F is type I and the number d of disorienting ovals of the curve is zero. Moreover, in this case, the formula

$$k^2 - (p-n) = 4(d - D^+ + D^-)$$

of Rokhlin [Rok78, (19)], valid for type I curves, yields that $e(\mathbf{S}^4, \mathcal{A}(F)) = 0$ when $\mathcal{A}(F)$ is orientable, which recovers the fact that an orientable surface in $\mathbf{S}^4$ has trivial normal bundle.

Since $H_2(\mathbf{S}^4; \mathbf{Z}) = H_2(\mathbf{S}^4; \mathbf{Z}/2) = 0$, all surfaces are characteristic, and the Guillou–Marin congruence Theorem 2.12 applies. For the surface $\mathcal{A}(F)$, this therefore yields:

$$\sigma(\mathbf{S}^4) - e(\mathbf{S}^4, \mathcal{A}(F)) \equiv 2\beta(\mathbf{S}^4, \mathcal{A}(F)) \text{ [16]}.$$

Since $\sigma(\mathbf{S}^4) = 0$ and $e(\mathbf{S}^4, \mathcal{A}(F))$ is even, this gives:

$$p - n \equiv k^2 + \beta(\mathbf{S}^4, \mathcal{A}(F)) \text{ [8]}. \qquad (*)$$

In particular, it is important to understand the Brown invariant of $\mathcal{A}(F)$.

Proposition 3.15 ([Mar80, Proposition 1])**.** *Let $L \subset H_1(\mathcal{A}(F); \mathbf{Z}/2)$ be the subspace spanned by the homology classes of all components of $p(\mathbf{R}F)$. Then the Guillou–Marin form $\varphi : H_1(\mathcal{A}(F); \mathbf{Z}/2) \to \mathbf{Z}/4$ vanishes on L.*

Proof. First, notice that L is spanned by the boundary of the *orientable* components of $p(\mathbf{RP}^2_-)$. Indeed, those correspond to non-outer ovals of $\mathbf{R}F$, where:

(1) an empty outer oval has zero homology class in $H_1(\mathcal{A}(F); \mathbf{Z}/2)$, since it is bounded by a disc in $\mathbf{RP}^2_+$, and

(2) the homology class in $H_1(\mathcal{A}(F);\mathbf{Z}/2)$ of a non-empty outer oval equals the sum of those of its inner ovals, since they bound a component of $\mathbf{RP}^2_+$ together.

Now, we will unravel each piece of the definition of the Guillou–Marin form from §2.1.3. Given such an oval $o \subset \mathbf{R}F$ where $o = \partial\Omega$ with $\Omega \subset \mathbf{RP}^2_-$ and Ω is orientable, then $p(\Omega)$ is an orientable surface in $\mathbf{S}^4$ bounding $p(o)$. Moreover, this does not intersect $\mathcal{A}(F)$ other than on $p(o) = \partial p(\Omega)$. In particular, we have

$$\varphi([p(o)]) = n(p(\Omega)) + 2p(\Omega)\cdot\mathcal{A}(F) + 2p(o)\cdot p(o) \ [4],$$

where:

(1) the number $n(p(\Omega))$ of half-twists of $vp(o)$ is zero,

(2) the intersection number $p(\Omega)\cdot\mathcal{A}(F)$ is zero (again, $p(\Omega)$ does not intersect $\mathcal{A}(F)$ other than at its boundary), and

(3) the self-intersection $p(o)\cdot p(o)$ in $\mathcal{A}(F)$ is zero, since there is a nowhere-vanishing section of $T\mathbf{RP}^2_+$ normal to o (for instance, an inwards-facing vector field). ■

As a consequence, we can prove the congruences stated in §1.2.3.

Proof of Theorem 1.32. We refer the reader to [KV88, §5.3 and §6.1] for details. Let $\varphi : H_1(\mathcal{A}(F);\mathbf{Z}/2) \to \mathbf{Z}/4$ be the Guillou–Marin form. From $\varphi|_L = 0$ and $\mathcal{A}(F) = p(F) \cup_\partial p(\mathbf{RP}^2_+)$, we obtain:

$$\beta(\mathbf{S}^4, \mathcal{A}(F)) = \beta(\varphi_1) + \beta(\varphi_2),$$

where $\beta(\varphi_1)$ and $\beta(\varphi_2)$ are the Brown invariants of the following $\mathbf{Z}/4$-quadratic forms:

$$\begin{array}{ccc} H_1(\mathbf{RP}^2_+;\mathbf{Z}/2) & \xrightarrow{(\mathrm{in}\circ p)_*} & H_1(\mathcal{A}(F);\mathbf{Z}/2) \\ & \searrow^{\varphi_1} & \downarrow{\scriptstyle \varphi} \\ & & \mathbf{Z}/4 \end{array} \quad \text{and} \quad \begin{array}{ccc} H_1(p(F);\mathbf{Z}/2) & \xrightarrow{\mathrm{in}_*} & H_1(\mathcal{A}(F);\mathbf{Z}/2) \\ & \searrow^{\varphi_2} & \downarrow{\scriptstyle \varphi} \\ & & \mathbf{Z}/4 \end{array} .$$

In fact, we also have $\varphi_1 = 0$ for the same reasons as in Proposition 3.15, and thus:

$$\beta(\mathbf{S}^4, \mathcal{A}(F)) = \beta(\varphi_2).$$

It suffices to inspect each case individually. For instance, in the case of an M-curve, $p(F)$ is a punctured sphere, so $\varphi_2 = 0$. In the case of an $(M-1)$-curve, $p(F)$ is a punctured projective plane, and one verifies that $\beta(\varphi_2) = \pm 1$. ■

The Arnold surface can also be used to derive Theorem 1.33(1). We will need both choices of Arnold surfaces. That is, set

$$\mathcal{A}_\pm(F) = p(F) \cup p(\mathbf{RP}^2_\pm) \subset \mathbf{S}^4.$$

Then:

$$\begin{cases} \chi(\mathcal{A}_+(F)) = -2k^2 + 3k + (p-n), \\ e(\mathbf{S}^4, \mathcal{A}_+(F)) = 2k^2 - 2(p-n) \end{cases} \quad \text{and} \quad \begin{cases} \chi(\mathcal{A}_-(F)) = -2k^2 + 3k + 1 + (p-n), \\ e(\mathbf{S}^4, \mathcal{A}_-(F)) = 2k^2 - 2 + 2(p-n). \end{cases}$$

From Theorem 2.23, denote as $Y_\pm$ the smooth 4-manifold which is obtained by taking the double branched cover of $\mathbf{S}^4$ ramified along $\mathcal{A}_\pm(F)$. Using Theorem 2.19 and Theorem 2.36, we derive:

$$\begin{cases}\chi(Y_+) = 2k^2-3k+2-(p-n),\\ \sigma(Y_+) = -k^2+(p-n)\end{cases} \quad \text{and} \quad \begin{cases}\chi(Y_-) = -2k^2+3k+1-(p-n),\\ \sigma(Y_-) = -k^2+1+(p-n).\end{cases}$$

Finally, from Proposition 2.37, we have $b_2(Y_\pm) = \chi(Y_\pm)-2$, from which we can solve for b_2^+ and obtain:

$$b_2^+(Y_+) = b_2^+(Y_-) = \frac{(k-1)(k-2)}{2}.$$

We will use the following lemma[†], where all coefficients in the homology groups are $\mathbf{Z}/p$.

Lemma 3.16 ([VZ92, Lemma 1.3])**.** *Let $h = p^r$ be a prime power. Let $\nu : Y \to X$ be an h-sheeted cyclic covering between two n-manifolds, branched over a codimension-two subset $A \subset X$. Let $B \subset X$ be a membrane[‡], let b be the class in $H_k(X,A)$ determined by B, and let β be the class in $H_k(Y)$ determined by $\nu^{-1}(B)$, oriented coherently with B. Let $\tau : Y \to Y$ be a generator of* $\mathrm{Aut}(\nu)$*, and let $\varrho = 1-\tau \in (\mathbf{Z}/p)[\mathrm{Aut}(\nu)]$. Recall the Smith long exact sequence in homology (with coefficients in $\mathbf{Z}/p$):*

$$\cdots \to H^\varrho_{k+1}(Y) \xrightarrow{\partial} H_k(X,A) \oplus H_k(A) \xrightarrow{\alpha_k} H_k(Y) \xrightarrow{\varrho_*} H^\varrho_k(Y) \to \cdots.$$

Then, the restriction $\widetilde{\alpha}_k : H_k(X,A) \to H_k(Y)$ maps b to β, and

(1) α_{n-1} is monic if $H^\varrho_n(Y) = 0$;

(2) $\widetilde{\alpha}_{n-2}$ is monic if X is connected and $H_{n-1}(Y) = 0$;

(3) if $\lfloor (n+1)/2 \rfloor \leqslant k < n-2$, then α_k is monic if X and A are connected and if $H_i(Y) = 0$ for all $k+1 \leqslant i \leqslant n-1$.

In order to derive bounds on p^0+p^- and n^0+n^-, we therefore need to construct homology classes in $H_2(\mathbf{S}^4, \mathcal{A}_\pm(F); \mathbf{Z}/2)$ associated to ovals contributing to p^-, p^0, n^- and n^0.

(1) If $o \subset \mathbf{R}F$ is an even oval which is non-empty (that is, contributing to p^0+p^-), then let $\Omega_p(o)$ be the connected component of $\mathbf{RP}^2_-$ which is bounded by o and by its inner ovals (see Figure 3.5(a)).

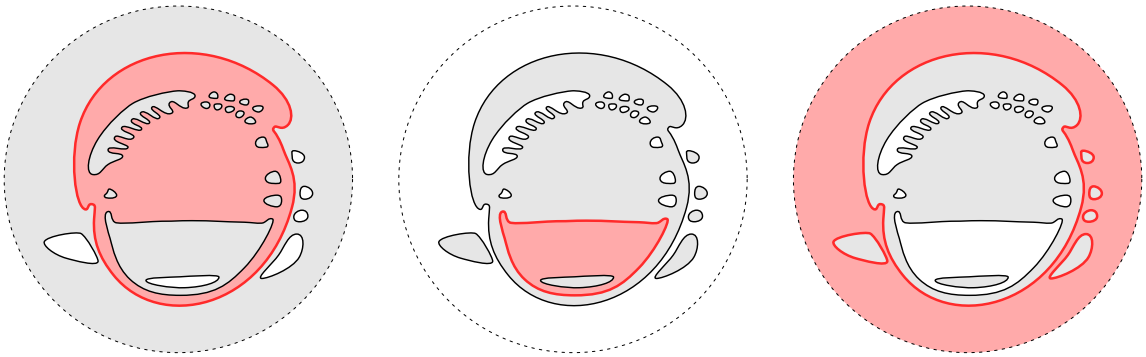
(2) If $o \subset \mathbf{R}F$ is an odd oval which is non-empty, let $\Omega_n(o)$ be the connected component of $\mathbf{RP}^2_+$ which is bounded by o and by its inner ovals (see Figure 3.5(b)).

(3) If there is at least one oval in $\mathbf{R}F$ (if $\mathbf{R}F = \varnothing$, the bounds are all trivial), denote as $\Omega_n(\infty)$ the only non-orientable component of $\mathbf{RP}^2_-$, which is bounded by all outer-most ovals of $\mathbf{R}F$ (see Figure 3.5(c)).

It is now clear that the collection of the $\Omega_p(o)$ for o a non-empty even oval describes (*via* their image under p) a set of relative homology classes in $H_2(\mathbf{S}^4, \mathcal{A}_-(F); \mathbf{Z}/2)$. Similarly, the collection of the $\Omega_n(o)$ for o a non-empty odd oval, together with $\Omega_n(\infty)$, defines a collection of relative homology classes in $H_2(\mathbf{S}^4, \mathcal{A}_+(F); \mathbf{Z}/2)$.

[†] We only described what *double* branched covers are, which are covered by the case $h = 2^1$ and are always cyclic. We state the result in its full generality, but the reader unfamiliar with the terminology may imagine that $h = 2$ here.

[‡] A membrane is simply a submanifold $B \subset X$ with boundary $\partial B \subset A$ and such that $B \cap A = \partial B$.



(a) (b) (c)

Figure 3.5. A depiction of the sets $\Omega_p(o)$, $\Omega_n(o)$ and $\Omega_n(\infty)$ in the case of an M-curve of degree 8 with real scheme $\langle 5 \sqcup 1\langle 14 \sqcup 1\langle 1\rangle\rangle\rangle$. (a) The set $\Omega_p(o)$, in red, for the outer-most non-empty oval, in thick red. The set $\mathbf{RP}^2_-$ is in gray. (b) The set $\Omega_n(o)$, in red, for the only odd non-empty oval, in thick red. The set $\mathbf{RP}^2_+$ is in gray. (c) The set $\Omega_n(\infty)$, in red, whose boundary is the outer-most ovals, in thick red.

Sketch of proof of Theorem 1.33(1). An application of Lemma 3.16 ensures that the lifts of the $\Omega_p(o)$ form a collection of homology classes whose rank is at least $p^0 + p^- - 1$. We can also compute the self-intersection of those lifts by using Lemma 2.38, and check that they are negative. This gives:

$$p^0 + p^- \leqslant b_2^+(Y_-) + 1.$$

Doing the same for the collection of the $\Omega_n(o)$ and $\Omega_n(\infty)$, we obtain:

$$n^0 + n^- \leqslant b_2^+(Y_+).$$

We omit a lot of details, since the same method will be used to derive Theorem 3.20. ■

Note that the upper bound for $n^0 + n^-$ that we obtain is the one from Theorem 1.33(1), but the one for $p^0 + p^-$ is off by 1 when k is odd.

3.3 The Odd Degree Case

In the case of a curve of odd degree, we do not have a possibility to construct the Arnold surface. More precisely, given the surface with boundary $p(F) \subset \mathbf{S}^4$, there is no planar surface $\Omega \subset \mathbf{RP}^2$ with $\partial\Omega = \mathbf{R}F$ which can be used to glue, since $[\mathbf{R}F] \neq 0 \in H_1(\mathbf{RP}^2;\mathbf{Z}/2)$. We will use the construction that was described in §3.1.2.

3.3.1 The Arnold Surface of an Odd Degree Flexible Curve

Recall that if $p : \mathbf{CP}^2 \to \mathbf{S}^4$ denotes the quotient map under the action of complex conjugation, then $\mathcal{R} = p(\mathbf{RP}^2)$ and $\mathcal{Q} = p(Q)$, with Q the Fermat conic. Moreover, letting $\widetilde{p} : \overline{\mathbf{CP}}^2 \to \mathbf{S}^4$ be the double branched cover of $\mathbf{S}^4$ with ramification locus $\mathcal{Q}$, then $\overline{Q} = \widetilde{p}^{-1}(\mathcal{R})$ and $\overline{\mathbf{RP}}^2 = \widetilde{p}^{-1}(\mathcal{R})$. We refer the reader to Figure 3.3 for a schematic of the situation.

Let $\mathcal{R}F = p(\mathbf{R}F)$, and let $\overline{\mathbf{R}}F = \widetilde{p}^{-1}(\mathcal{R}F) \subset \overline{Q}$. Now, we *do* have $[\overline{\mathbf{R}}F] = 0 \in H_1(\overline{Q};\mathbf{Z}/2)$, and therefore $\overline{\mathbf{R}}F$ *does* bound two subsets $\overline{Q}_\pm \subset \overline{Q}$ with $\partial\overline{Q}_\pm = \overline{\mathbf{R}}F$.

Definition 3.17. *The **Arnold surface** of a flexible curve of odd degree is the surface*

$$\mathcal{A}(F) = \widetilde{p}^{-1}(p(F)) \cup \overline{Q}_+$$

for some choice of a labeling of $\overline{Q}_\pm$.

In fact, the choice in the labeling of $\overline{Q}_\pm$ is not relevant (at least regarding the process of obtaining Theorem 3.20). Indeed, both subsets $\overline{Q}_+$ and $\overline{Q}_-$ are diffeomorphic, and exchanged by the involution spanning $\mathrm{Aut}(\widetilde{p})$ (which is the antipodal map on the 2-sphere $\overline{Q}$). See Figure 3.6 for a depiction of $\overline{Q}_+$ on an example.

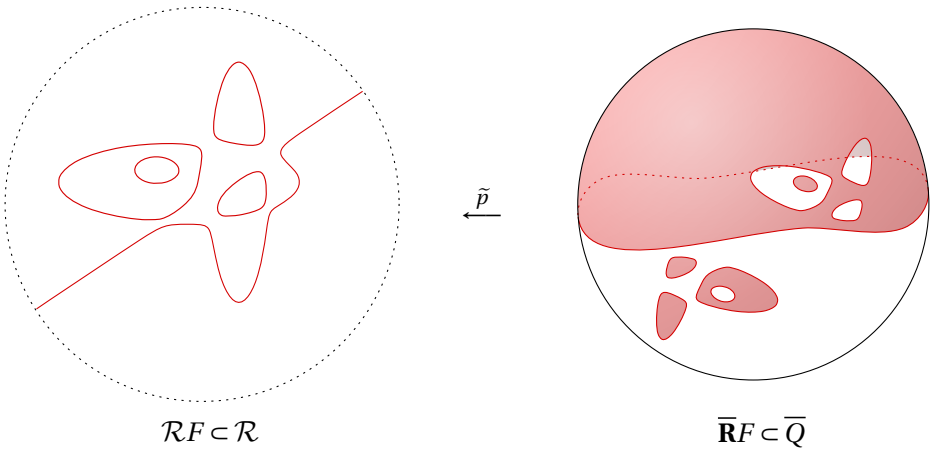


Figure 3.6. A choice of the set $\overline{Q}_+$, in red, for an algebraic curve of degree 7 with real scheme $\langle \mathcal{J} \sqcup 2 \sqcup 1\langle 1 \rangle \rangle$.

Proposition 3.18. *Let F be a flexible curve of odd degree m. Then $e(\overline{\mathbf{CP}}^2, \mathcal{A}(F)) = m^2 - 2$.*

Proof. Using Lemma 2.38 twice, we can derive that $e(\overline{\mathbf{CP}}^2, \widetilde{p}^{-1}(p(F))) = m^2$. Moreover, we have $e(\overline{\mathbf{CP}}^2, \overline{Q}) = -4$. If $\tau \in \mathrm{Aut}(\widetilde{p})$ is the orientation-preserving involution such that $\overline{Q}_- = \tau(\overline{Q}_+)$, we obtain $e(\overline{\mathbf{CP}}^2, \overline{Q}_+) = e(\overline{\mathbf{CP}}^2, \overline{Q}_-)$. Moreover, from $\overline{Q} = \overline{Q}_+ \cup \overline{Q}_-$, this yields:

$$\begin{cases} -4 = e(\overline{\mathbf{CP}}^2, \overline{Q}_+) + e(\overline{\mathbf{CP}}^2, \overline{Q}_-) \\ e(\overline{\mathbf{CP}}^2, \overline{Q}_+) = e(\overline{\mathbf{CP}}^2, \overline{Q}_-) \end{cases} \quad \Longrightarrow \quad e(\overline{\mathbf{CP}}^2, \overline{Q}_\pm) = -2.$$

Finally, from $\mathcal{A}(F) = \widetilde{p}^{-1}(p(F)) \cup \overline{Q}_+$, we obtain the claim. ∎

Proposition 3.19. *Let F be a totally flexible of odd degree m. Then $\chi(\mathcal{A}(F)) = -m^2 + 2m + 1$.*

Proof. By a similar argument as in the proof of Proposition 3.18, we derive that $\chi(\overline{Q}_\pm) = 1$, and we can use Theorem 2.19 twice. However, one needs to be careful in the second application:

$$\chi(\widetilde{p}^{-1}(p(F)) = 2\chi(p(F)) - \chi(p(F) \cap \mathcal{Q}),$$

where $p(F) \pitchfork \mathcal{Q}$ is a collection of m isolated points (under the assumption of being totally flexible). Indeed, an intersection point in $p(F) \pitchfork \mathcal{Q}$ corresponds to a pair of complex conjugate points in $F \pitchfork Q$. Therefore, $\chi(p(F) \cap \mathcal{Q}) = m$. ∎

In the general case of a (not necessarily totally) flexible curve of odd degree m, because $e(\overline{\mathbf{CP}}^2, \mathcal{A}(F))$ is odd, we have that $[\mathcal{A}(F)] \neq 0 \in H_2(\overline{\mathbf{CP}}^2; \mathbf{Z}/2) \cong \mathbf{Z}/2$, and thus $\mathcal{A}(F)$ is a characteristic surface. However, applying the Guillou–Marin congruence gives

$$-1 - (m^2 - 2) \equiv 2\beta(\overline{\mathbf{CP}}^2, \mathcal{A}(F)) \ [16],$$

which readily gives, setting $m = 2k + 1$:

$$\beta(\overline{\mathbf{CP}}^2, \mathcal{A}(F)) = -2k(k+1).$$

In particular, the Brown invariant of the surface $\mathcal{A}(F)$ is not constrained by the real scheme $\mathbf{R}F$, only by the degree of the curve, contrary to in the even degree case, where we could derive restrictions on $\mathbf{R}F$ in the form of congruences.

However, not all hope is lost, as the other usage of the Arnold surface will still yield restrictions in the form of bounds on $\ell^0 + \ell^-$, with the same method exposed in §3.2.

3.3.2 Bounding the Number of Non-Empty Ovals

We now prove one of the main results of this thesis.

Theorem 3.20. *Let F be a totally flexible curve of odd degree $m = 2k + 1$. Then*

$$\ell^0 + \ell^- \leqslant k^2.$$

If equality holds, then the curve is type I.

Therefore, throughout the rest of this section, F will be a fixed totally flexible of odd degree $m = 2k+1$. In order to proceed, we want to:

(1) consider the double branched cover Y of $\overline{\mathbf{CP}}^2$ ramified along $\mathcal{A}(F)$,
(2) compute $b_2^+(Y)$, and
(3) construct surfaces describing relative homology classes in $H_2(\overline{\mathbf{CP}}^2, \mathcal{A}(F); \mathbf{Z}/2)$ and that can be lifted to closed surfaces in Y spanning a subspace of $H_2(Y;\mathbf{Z})$ on which Q_Y is definite positive.

However, we have that $\mathcal{A}(F)$ is characteristic, instead of null-homologous mod 2, and thus Theorem 2.23 gets us stuck at step one. We can fix this issue with the following observation: both $\mathcal{A}(F)$ and $\overline{\mathbf{RP}}^2$ realize the unique non-zero homology class in $H_2(\overline{\mathbf{CP}}^2; \mathbf{Z}/2)$.

Consider $\Sigma(F) = \mathcal{A}(F) \cup \overline{\mathbf{RP}}^2$, which is a nodally immersed surface inside $\overline{\mathbf{CP}}^2$, and there are $\#\mathcal{A}(F) \pitchfork \overline{\mathbf{RP}}^2 = m$ double points. Therefore:

$$\chi(\Sigma(F)) = \chi(\mathcal{A}(F)) + \chi(\overline{\mathbf{RP}}^2) - m \text{ and } e(\overline{\mathbf{CP}}^2, \Sigma(F)) = e(\overline{\mathbf{CP}}^2, \mathcal{A}(F)) + e(\overline{\mathbf{CP}}^2, \overline{\mathbf{RP}}^2).$$

From Proposition 3.18 and Proposition 3.19, we therefore obtain:

$$\chi(\Sigma(F)) = -m^2 + m + 2 \text{ and } e(\overline{\mathbf{CP}}^2, \Sigma(F)) = m^2 - 1.$$

Finally, consider $\mathcal{X}(F)$ to be the embedded surface obtained from Σ by the smoothing of singularities prescribed by Corollary 2.18. Therefore, we obtain:

$$\chi(\mathcal{X}(F)) = \chi(\Sigma(F)) - m = -m^2 + 2 \text{ and } e(\overline{\mathbf{CP}}^2, \mathcal{X}(F)) = e(\overline{\mathbf{CP}}^2, \Sigma(F)) + 2m = m^2 + 2m - 1.$$

We have the following.

Proposition 3.21. *The surface $\mathcal{X}(F)$ is smoothly embedded in $\overline{\mathbf{CP}}^2$ and null-homologous in $H_2(\overline{\mathbf{CP}}^2; \mathbf{Z}/2)$. Moreover, we have*

$$\mathcal{X}(F) \cap \overline{Q} = \mathcal{A}(F) \cap \overline{Q}.$$

Proof. The self-intersection of $\mathcal{X}(F)$ is an even integer, hence $[\mathcal{X}(F)] = 0 \in H_2(\overline{\mathbf{CP}}^2; \mathbf{Z}/2)$. For the other claim, notice that $\overline{\mathbf{RP}}^2 \cap \overline{Q} = \varnothing$, and because the smoothing of the singularities of $\Sigma(F)$ is a purely local construction, we have $\Sigma(F) \cap \overline{Q} = \mathcal{X}(F) \cap \overline{Q}$. Finally, for the same reason, we have $\Sigma(F) \cap \overline{Q} = \mathcal{A}(F) \cap \overline{Q}$. ∎

We can now consider the 4-manifold Y which is the double branched cover of $\overline{\mathbf{CP}}^2$ ramified along $\mathcal{X}(F)$. We denote as $\Theta : Y \to \overline{\mathbf{CP}}^2$ this double branched cover.

Proposition 3.22. *We have:*

$$b_2^+(Y) = \frac{(m-1)^2}{4} = k^2.$$

Proof. Using Theorem 2.19, we derive that

$$\chi(Y) = 2\chi(\overline{\mathbf{CP}}^2) - \chi(\mathcal{X}(F)) = m^2 + 4.$$

Additionally, Proposition 2.37 and Poincaré duality provide $b_1(Y;\mathbf{Z}/2) = b_3(Y;\mathbf{Z}/2) = 0$, and therefore

$$b_2(Y) = \chi(Y) - 2 = m^2 + 2.$$

Lastly, the Hirzebruch signature formula Theorem 2.36 gives

$$\sigma(Y) = 2\sigma(\overline{\mathbf{CP}}^2) - \frac{e(\overline{\mathbf{CP}}^2, \mathcal{X}(F))}{2} = \frac{-m^2 - 2m - 3}{2}.$$

This can be solved for $2b_2^+(Y) = b_2(Y) + \sigma(Y)$, which proves the claim. ■

This takes care of the first two steps of the proof, and we now need to construct relative homology classes associated to non-empty ovals. Fix $o \subset \mathbf{R}F$ to be a non-empty oval, and denote as $\Omega(o)$ the connected component of $\mathbf{RP}^2 \smallsetminus \mathbf{R}F$ which is bounded by o and by its inner ovals.

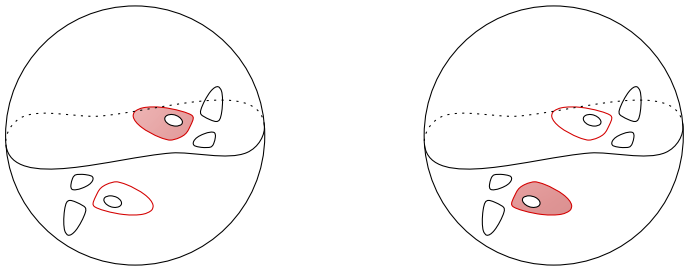
Definition 3.23. *Given a non-empty oval $o \subset \mathbf{R}F$, define $C(o)$ to be the closure of the image of $\Omega(o)$ by p. Moreover, set*

$$\chi(o) \overset{def.}{=} \chi(C(o))$$

the ***Euler characteristic*** *of the oval o.*

In particular, we see that an oval contributes to ℓ^0 if and only if $\chi(o) = 0$, and contributes to ℓ^- if and only if $\chi(o) \leqslant -1$.

By Proposition 3.13, observe that the lift $\widetilde{p}^{-1}(C(o))$ is diffeomorphic to two disjoint copies of $C(o)$, each lying inside either $\overline{Q}_+$ or $\overline{Q}_-$. Denote as $C_\pm(o)$ the component of $\widetilde{p}^{-1}(C(o))$ which lies inside $\overline{Q}_\pm$, respectively. See Figure 3.7 for an example.



$C_-(o) \subset \overline{Q}_-$ $C_+(o) \subset \overline{Q}_+$

Figure 3.7. Using the same scheme $\langle \mathcal{J} \sqcup 2 \sqcup 1\langle 1\rangle\rangle$ and the same choice of $\overline{Q}_\pm$ as in Figure 3.6, we take o to be the only non-empty oval in $\mathbf{R}F$. In red, we represent both subsets $C_\pm(o) \subset \overline{Q}_\pm$, where part of their boundary is one of the components of $\widetilde{p}^{-1}(p(o))$.

The surfaces $C_-(o)$ for all non-empty ovals are the analogue of the $\Omega_p(o)$ or $\Omega_n(o)$ as in §3.2. In particular: $C_+(o)$ is entirely included in the ramification locus of the branched cover $\Theta : Y \to \overline{\mathbf{CP}}^2$, and $C_-(o)$ intersects that ramification locus only at its boundary.

Denote as $\widetilde{C}(o) = \Theta^{-1}(C_-(o)) \subset Y$ the lift of $C_-(o)$ by the covering Θ. The restriction

$$\Theta : \widetilde{C}(o) \to C_-(o)$$

is not a branched cover, but it is quite close (we may call it a "pseudo" branched cover). The map

$$\Theta : \Theta^{-1}(\partial C_-(o)) \to \partial C_-(o)$$

is one-to-one, and it is a genuine (orientation-reversing) unbranched cover on the interior of $C_-(o)$. This way, the surface $\widetilde{C}(o)$ is closed and oriented, since it is obtained by gluing two copies of the same planar surface along its boundary. In fact, if $\tau : Y \to Y$ denotes the involution spanning $\mathrm{Aut}(\Theta)$, we see that τ fixes $\widetilde{C}(o)$ set-wise and acts on it with fix-point set $\mathrm{Fix}(\tau|_{\widetilde{C}(o)}) = \Theta^{-1}(\partial C_-(o))$, and the restriction of Θ to $\widetilde{C}(o)$ is the quotient of $\widetilde{C}(o)$ by this involution; see Figure 3.8 for a depiction of this restriction.

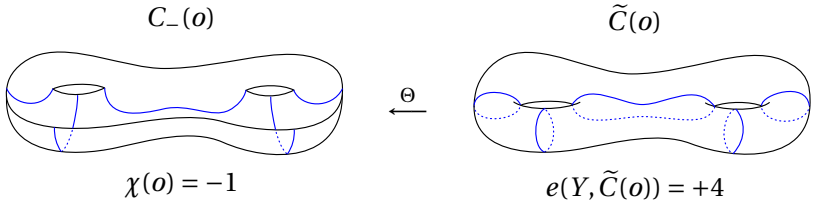


Figure 3.8. The "pseudo" branched cover $\Theta : \widetilde{C}(o) \to C_-(o)$.

The construction can be extended to also give an analogue of $\Omega_n(\infty)$ in the even degree case §3.2. That is, let $\mathcal{J}$ denote the pseudo-line of $\mathbf{R}F$, and assume that $\mathbf{R}F$ contains at least one oval (otherwise Theorem 3.20 is evidently true). There are exactly two connected components of $\overline{Q} \smallsetminus \overline{\mathbf{R}}F$ which are diffeomorphic and whose closure have $\widetilde{p}^{-1}(p(\mathcal{J}))$ as part of their boundary, and they each lie in one of $\overline{Q}_\pm$. Denote as $D_\pm(\mathcal{J})$ the one which is included inside $\overline{Q}_\pm$, respectively. Define $\chi(\mathcal{J}) = \chi(D_\pm(J))$. Then:

$$\chi(\mathcal{J}) = 1 - e,$$

with e the number of exterior ovals of the curve (those which do not lie inside any other oval). See Figure 3.9 for a depiction.

Set $\widetilde{D}(\mathcal{J}) = \Theta^{-1}(D_-(\mathcal{J})) \subset Y$. From the same reasoning as in for $\widetilde{C}(o)$, we obtain that $\widetilde{D}(\mathcal{J})$ is an oriented surface in Y, and the restriction $\Theta : \widetilde{D}(\mathcal{J}) \to D_-(\mathcal{J})$ is a pseudo-branched cover in the same sense.

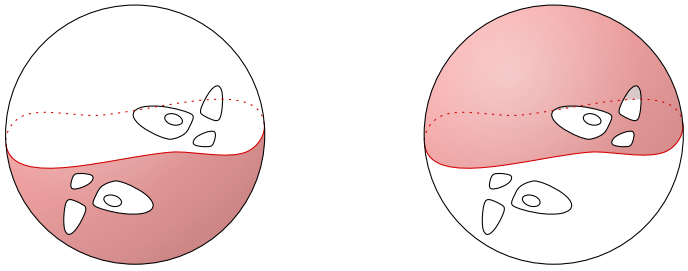
Lemma 3.24. *Let $X \subset \overline{Q}$ be a 2-dimensional submanifold of $\overline{Q}$. Then $e(\overline{\mathbf{CP}}^2, X) = -2\chi(X)$.*

Proof. This follows from the facts that the normal bundle of $\mathbf{RP}^2$ in $\mathbf{CP}^2$ is anti-isomorphic to its tangent bundle and that $\widetilde{p}$ is an *unbranched* in a regular neighborhood of $\overline{Q}$, whence $\nu\overline{Q} \cong -2T\overline{Q}$. ∎

A consequence of this fact is the following.

Proposition 3.25. *Let $o, o' \subset \mathbf{R}F$ be ovals, $o \neq o'$, and let $\mathcal{J}$ denote the pseudo-line of $\mathbf{R}F$.*

(1) We have $Q_Y(\widetilde{C}(o), \widetilde{C}(o)) = -4\chi(o)$ and $Q_Y(\widetilde{D}(\mathcal{J}), \widetilde{D}(\mathcal{J})) = -4\chi(J)$.



$D_-(o) \subset \overline{Q}_-$ $D_+(o) \subset \overline{Q}_+$

Figure 3.9. Again with the same curve as in Figure 3.6, the sets $D_\pm(\mathcal{J})$ are depicted in red.

(2) We have $Q_Y(\widetilde{C}(o), \widetilde{C}(o')) = 0$ *and* $Q_Y(\widetilde{C}(o), \widetilde{D}(\mathcal{J})) = 0$.

Proof. For the first claim, note that $e(\overline{\mathbf{CP}}^2, C_-(o)) = -2\chi(o)$ by Lemma 3.24 and $e(\overline{\mathbf{CP}}^2, D_-(\mathcal{J})) = -2\chi(\mathcal{J})$. The self-intersection of the surfaces $\widetilde{C}(o)$ and $\widetilde{D}(\mathcal{J})$ can therefore be computed using Lemma 2.38, and by orientability, they agree with the evaluation of the intersection form Q_Y on their homology classes.

For the other claim, note that distinct ovals o and o' cannot satisfy $C_-(o) \cap C_-(o') \neq \varnothing$, even if one is situated inside the other oval (however, it may be possible that $C_-(o) \cap C_+(o') \neq \varnothing$). The same goes for $C_-(o) \cap D_-(\mathcal{J}) = \varnothing$. In particular, the surfaces $\widetilde{C}(o)$ and $\widetilde{C}(o')$ are disjoint, and so are $\widetilde{C}(o)$ and $\widetilde{D}(\mathcal{J})$. ■

Proposition 3.26. *Consider the collection of the integral homology classes of the* $\widetilde{C}(o)$ *where* $\chi(o) \leqslant 0$ *and of* $\widetilde{D}(\mathcal{J})$. *Then this collection spans a subspace of* $H_2(Y;\mathbf{Z})$ *which has rank at least* $\ell^0 + \ell^-$.

Proof. The proof is analogous to Viro and Zvonilov's [VZ92, Corollary 1.5.C], so we will use the same notations as theirs. Consider the homology classes $[C_-(o)] \in H_2(\overline{\mathbf{CP}}^2, \mathcal{X}(F); \mathbf{Z}/2)$ and $[D_-(\mathcal{J})] \in H_2(\overline{\mathbf{CP}}^2, \mathcal{X}(F); \mathbf{Z}/2)$. We denote this collection as $(b_0, b_1, \ldots, b_r)$, where $r = \ell^0 + \ell^-$, $b_0 = [D_-(\mathcal{J})]$ and the b_i are the other classes. Through the boundary morphism

$$H_2(\overline{Q}, \overline{\mathbf{R}}F; \mathbf{Z}/2) \xrightarrow{\partial} H_1(\overline{\mathbf{R}}F; \mathbf{Z}/2)$$

we can look at their images, which will form a basis of $H_1(\overline{\mathbf{R}}F; \mathbf{Z}/2)$. The inclusion morphism

$$H_1(\overline{\mathbf{R}}F; \mathbf{Z}/2) \to H_1(\mathcal{X}(F); \mathbf{Z}/2))$$

is either injective (if $\mathcal{X}(F) \smallsetminus \overline{\mathbf{R}}F$ is connected) or has a one-dimensional kernel (if $\mathcal{X}(F) \smallsetminus \overline{\mathbf{R}}F$ has two connected components). Therefore, the rank of the span of the classes $(b_0, b_1, \ldots, b_r)$ is either r or $r+1$, depending on whether $\mathcal{X}(F) \smallsetminus \overline{\mathbf{R}}F$ is connected or disconnected.

Letting β_0 denote the homology class $[\widetilde{D}(\mathcal{J})] \in H_2(Y; \mathbf{Z}/2)$ and β_i that of $[\widetilde{C}(o)] \in H_2(Y; \mathbf{Z}/2)$ if $b_i = [C_-(o)]$, we obtain from Lemma 3.16 that the map

$$\widetilde{\alpha}_2 : H_2(\overline{\mathbf{CP}}^2, \mathcal{X}(F); \mathbf{Z}/2) \to H_2(Y; \mathbf{Z}/2)$$

is injective and maps b_i to β_i, since $\overline{\mathbf{CP}}^2$ is connected and $H_3(Y;\mathbf{Z}/2) = 0$ (Proposition 2.37). Therefore, the rank of the span of $(\beta_0, \beta_1, \ldots, \beta_r)$ is at least r.

Finally, the rank of the *integral* homology classes $[\widetilde{C}(o_i)]$ and $[\widetilde{D}(\mathcal{J})]$ is at least that of their mod 2 reduction, and thus the claim follows. ■

Proof of Theorem 3.20. Because of Proposition 3.25, we see that the homology classes $[\widetilde{C}(o_i)]$ and $[\widetilde{D}(\mathcal{J})]$ span a subspace on which Q_Y is positive. Therefore, we obtain that the rank of this family is bounded above by the maximal dimension $b_2^+(Y)$ on which Q_Y is positive definite, from which the inequality follows by Proposition 3.26.

For the case where equality is attained, this follows from the computation of the kernel of the inclusion $H_1(\overline{\mathbf{R}}F;\mathbf{Z}/2) \to H_1(\mathcal{X}(F);\mathbf{Z}/2)$ in the proof of Proposition 3.26. ■

3.3.3 Further Comments

The original paper by Viro and Zvonilov managed to obtain the following upper bound.

Theorem 3.27 ([VZ92, Theorem 2])**.** *Let F be a flexible curve of odd degree m, and let $h(m)$ denote the biggest prime power that divides m. Then:*

$$\ell^0 + \ell^- \leqslant \frac{(m-3)^2}{4} + \frac{m^2 - h(m)^2}{4h(m)^2}.$$

The method we used here is of course based on theirs. The key difference is that they consider the $h(m)$-sheeted *cyclic* branched cover of $\mathbf{CP}^2$ with ramification locus F directly. Of course, taking the 2-sheeted branched covering directly is not possible, since $[F] \neq 0 \in H_2(\mathbf{CP}^2, \mathbf{Z}/2)$.

We set:

$$Vz(m) = \frac{(m-3)^2}{4} + \frac{m^2 - h(m)^2}{4h(m)^2} \text{ and } S(m) = \frac{(m-1)^2}{4}.$$

When m is a prime power, we obviously have $h(m) = m$, and thus $VZ(m) < S(m)$. In fact, we have the following observation.

Proposition 3.28. *If F is a totally flexible curve of odd degree m which is 2 less than a prime power, then the conclusion of Theorem 3.20 can be derived solely from that of Theorem 3.27.*

Proof. Consider the nodally immersed surface $\Sigma = F \cup Q$, which has

$$\chi(\Sigma) = -m^2 + 2m + 2 \text{ and } e(\mathbf{CP}^2, \Sigma) = m^2 + 4.$$

Smooth the singularities in a conj-invariant way as in Corollary 2.18. This gives a new surface $\Sigma' \subset \mathbf{CP}^2$ which is embedded, invariant under complex conjugation, and has

$$\chi(\Sigma') = -m^2 + m + 2 = -(m+1)^2 + 3(m+1) \text{ and } e(\mathbf{CP}^2, \Sigma') = m^2 + 2m + 2 = (m+1)^2.$$

Moreover, if T denotes a sufficiently small tubular neighborhood of $\mathbf{RP}^2$ in $\mathbf{CP}^2$, then since $Q \cap \mathbf{RP}^2 = \varnothing$ and the construction $\Sigma \rightsquigarrow \Sigma'$ is local, we have $F \cap T = \Sigma \cap T = \Sigma' \cap T$. Therefore, the surface Σ' is a

flexible curve of degree $m+2$ and with $\mathbf{R}\Sigma' = \mathbf{R}F$. Applying Theorem 3.27 to it, we derive

$$\ell^0 + \ell^- \leqslant \frac{(m-1)^2}{4} + \frac{(m+2)^2 - h(m+2)^2}{4h(m+2)^2}.$$

Since $h(m+2) = m$ by assumption, the conclusion of Theorem 3.20 follows. ■

Recreatively, if the twin prime conjecture holds (or more generally if the twin prime *powers* conjecture holds), then there are infinitely many degrees m for which $VZ(m) < S(m)$ and for which the conclusion of Theorem 3.20 is a direct consequence of Theorem 3.27.

However, we also have the converse.

Proposition 3.29. *There are infinitely many odd degrees m for which $S(m) < VZ(m)$ and such that neither m nor $m+2$ is a prime power.*

Proof. The difference between the upper bounds is

$$VZ(m) - S(m) = \frac{1}{4}\left(\left[\frac{m}{h(m)}\right]^2 - 4m + 7\right).$$

It suffices to find an infinite family $(m_p)_{p\in\mathbf{N}}$ of suitable degrees such that

$$VZ(m_p) - S(m_p) \underset{p\to+\infty}{\longrightarrow} +\infty.$$

Set $m_p = 1287 \times 429^{12p+1}$. Because $1287 = 3^2 \times 11 \times 13$, we see that m_p is never a prime power. Additionally, we have

$$\begin{aligned} m_p + 2 &\equiv 2 \times 4^{12p+1} + 2 \ [5] \\ &\equiv 2 \times \left(1 + 4 \times (4^{5-1})^{3p}\right) \ [5] \\ &\equiv 2 \times \left(1 + 4 \times 1^{3p}\right) \ [5] \qquad \text{by Fermat's little theorem} \\ &\equiv 0 \ [5] \end{aligned}$$

and

$$\begin{aligned} m_p + 2 &\equiv 6 \times 2 \times 2^{12p} + 2 \ [7] \\ &\equiv 5 \times (2^{7-1})^{2p} + 2 \ [7] \\ &\equiv 5 \times 1 + 2 \ [7] \qquad \text{by Fermat's little theorem} \\ &\equiv 0 \ [7]. \end{aligned}$$

This means that $5 \mid m_p + 2$ and $7 \mid m_p + 2$, so that $m_p + 2$ is not a prime power.

Lastly, it remains to prove that the difference diverges to $+\infty$ on the degrees m_p. In fact, we know exactly the prime decomposition of m_p:

$$m_p = 3^2 \times 11 \times 13 \times (3 \times 11 \times 13)^{12p+1} = 3^{12p+3} \times 11^{12p+2} \times 13^{12p+2}.$$

Therefore, we obtain $h(m_p) = 13^{12p+2}$. Let $\varepsilon > 0$. We have

$$\frac{h(m_p)}{m_p^\varepsilon} = \frac{13^{(1-\varepsilon)(12p+2)}}{3^{(12p+3)\varepsilon} \times 11^{(12p+2)\varepsilon}},$$

hence:

$$\log\left(\frac{h(m_p)}{m_p^\varepsilon}\right) = 12\Big[\log(13) - \log(3 \times 11 \times 13)\varepsilon\Big]\, p + \text{const.}$$

Therefore, if

$$\varepsilon < \frac{\log(13)}{\log(3) + \log(11) + \log(13)} \approx 0.423,$$

we obtain

$$\frac{h(m_p)}{m_p^\varepsilon} \underset{p\to+\infty}{\longrightarrow} 0.$$

Taking $\varepsilon = \frac{4}{10}$, we get $2(1-\varepsilon) > 1$, and thus:

$$VZ(m_p) - S(m_p) = \frac{1}{4}\left(\underbrace{\left[\frac{m_p^\varepsilon}{h(m_p)}\right]^2}_{\geqslant 5} \times \underbrace{m_p^{2(1-\varepsilon)}}_{\geqslant m_p} - 4m_p + 7\right) \underset{p\to+\infty}{\longrightarrow} +\infty.$$

This statement is stronger than what was required, but works just fine. ■

Naturally, one may wonder what the effect is of using other methods to resolve the singularities that arose when taking the union $\mathcal{A}(F) \cup \overline{\mathbf{RP}}^2$ in the proof of Theorem 3.20. In fact, in the gluing of Hopf bands in Corollary 2.18, we picked the gluing that increased the self-intersection the most, but there was room for choice. Doing other choices results in having a 4-manifold Y with a bigger b_2^+ than the one we obtained.

Using blow-ups as in Proposition 2.16 is also another possibility. However[†], the resulting surface will *not* be null-homologous, and rather will have a *large* homology class in the sense that in $\overline{\mathbf{CP}}^2 \# m\overline{\mathbf{CP}}^2$, the homology class of the blown-up surface will be $(0, 1, 1, \ldots, 1)$.

[†] There is a slight mistake in [Sai24, §5], where the reason invoked to rule out the blowing-up was that the surface cannot be connected after m blow-ups, and we needed to perform at most $m-1$ of them. It turns out that if this *was* working, doing the computation with $m-1$ blow-ups and one Hopf band gluing (assuming that Y exists) will indeed yield $b_2^+ = (m-1)^2/4$ as before.

3.4 Generalizing to Other 4-Manifolds?

There are three families of real non-singular quadric surfaces in $\mathbf{CP}^3$, but only two of them are of interest to us. Consider the following involutions on $\mathbf{CP}^1 \times \mathbf{CP}^1$:

$$c_{\text{hyp}} : (x, y) \mapsto (\overline{x}, \overline{y}) \text{ and } c_{\text{ell}} : (x, y) \mapsto (\overline{y}, \overline{x}).$$

The pairs $(\mathbf{CP}^1 \times \mathbf{CP}^1, c_{\text{hyp}})$ and $(\mathbf{CP}^1 \times \mathbf{CP}^1, c_{\text{ell}})$ will respectively be called the **hyperboloid** and the **ellipsoid quadrics**. They are indeed real quadrics, since both can be embedded in $\mathbf{CP}^3$ *via* the algebraic equations

$$z_0^2 + z_1^2 \pm z_2^2 - z_3^2 = 0,$$

and their respective involutions are the restriction of the complex conjugation of $\mathbf{CP}^3$ onto them. They are also distinct, since:

$$\text{Fix}(c_{\text{hyp}}) = \mathbf{RP}^1 \times \mathbf{RP}^2 \text{ and } \text{Fix}(c_{\text{ell}}) = \left\{ (x, \overline{x}) \;\middle|\; x \in \mathbf{CP}^1 \right\}$$

are repectively a torus and a 2-sphere.

The last quadric is embedded in $\mathbf{CP}^3$ *via* the equation

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0,$$

and this is distinct from the others since the conjugation of $\mathbf{CP}^3$ has no real fixed points on it. In fact, the fact that the real part of this quadric is empty means that the study of real curves on it is *empty* as well.

The hyperboloid and the ellipsoid quadrics are two examples of complex surfaces to study real curves on, and they fit in the more general setting of *conjugations*; see Definition 3.42. The third example does not fit into this category, as the fix-point set of the conjugation is empty.

We will first investigate the previous methods for curves of odd degree on $\mathbf{CP}^2$ and give an analogue of Theorem 3.20 for curves on quadrics. There is a study of algebraic curves on quadrics; we refer the reader to [Gil91] or [Mat91] for elementary results. We will focus on *flexible* curves and *totally flexible* ones, and we will recall the results needed along the way.

The inequalities we obtain in Theorem 3.31 and Theorem 3.41 are to be compared with Zvonilov's work from [Zvo22]. In particular, Proposition 3.29 should remain true by the same arguments. However, it is to be noted that Zvonilov imposes an implicit condition that $\gcd(a, b) \neq 1$ in the case of the hyperboloid, which we do not. To the best of our knowledge, this means that Theorem 3.31 yields new restrictions, even in the algebraic case.

3.4.1 Curves on the Hyperboloid Quadric

We first treat the case of curves on the hyperboloid. That is: set $X = \mathbf{CP}^1 \times \mathbf{CP}^1$, and consider the involution c_{hyp} on it. Set $\mathfrak{R} = \text{Fix}(c_{\text{hyp}})$, and consider $\mathfrak{Q}$ to be a bidegree $(2,2)$ curve with $\mathbf{R}\mathfrak{Q} = \varnothing$. We consider the basis for $H_2(X; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$ which is spanned by the homology classes of a bidegree $(1,0)$ algebraic curve and a bidegree $(0,1)$ algebraic curve.

Definition 3.30. *A closed, connected and oriented surface $F \subset X$ smoothly embedded is called a* ***bidegree*** (a,b) ***flexible curve*** *if:*

(1) $c_{\text{hyp}}(F) = F$*;*

(2) $[F] = (a,b) \in H_2(X;\mathbf{Z})$*;*

(3) $\chi(F) = 2-2(a-1)(b-1)$*;*

(4) if $\mathbf{R}F = F \cap \mathfrak{R}$*, then for all* $x \in \mathbf{R}F$*,* $T_xF = T_x\mathbf{R}F \oplus \mathbf{i}\cdot T_x\mathbf{R}F$*.*

If, additionally, $F \pitchfork \mathfrak{Q}$ *consists of* $2(a+b)$ *points, then the curve is called* ***totally flexible****.*

Recall that $\mathfrak{R}$ is a 2-torus inside X. A connected component of $\mathbf{R}F \subset \mathfrak{R}$ is called an **oval** if it is contractible. Recall that if both a and b are odd, then $\mathbf{R}F$ is some number of ovals together with some non-zero number of parallel copies of a curve on $\mathfrak{R}$ with homology class $(\alpha,\beta) \in H_1(\mathfrak{R};\mathbf{Z}) \cong \mathbf{Z}\oplus\mathbf{Z}$, with $1 \leqslant \alpha \leqslant a$ and $1 \leqslant \beta \leqslant b$ both odd and coprime. If $o \subset \mathbf{R}F$ is an oval, then $\mathfrak{R} \smallsetminus o$ has two connected components. One of them is diffeomorphic to an open disc, called the **interior** of o and denoted $\text{Int}(o)$, and the other diffeomorphic to a punctured torus and called the **exterior**. We say that o bounds $\text{Int}(o)$ **from the outside**.

Just as in the traditional case of plane curves, ovals may be **nested**, and ovals may be **non-empty** if they are surrounding at least one oval. We will denote as ℓ the number of ovals of $\mathbf{R}F$, as well as $\ell^{\pm}$ and ℓ^0 the number of ovals which bound from the outside a component of $\mathfrak{R} \smallsetminus \mathbf{R}F$ of positive, negative or zero Euler characteristic, respectively.

We will show the following analogue of Theorem 3.20.

Theorem 3.31. *Let F be a totally flexible curve on (X, c_{hyp}) of bidegree (a,b) where both a and b are odd. Then:*

$$\ell^0 + \ell^- \leqslant \frac{ab+1}{2}.$$

For this, we will describe how to construct the Arnold surface of such a curve. From Theorem 2.35, we have that the quotient X/c_{hyp} is diffeomorphic to $\mathbf{S}^4$. Denote as $p : X \to \mathbf{S}^4$ the quotient map, which is a double branched cover, and set $\mathcal{R} = p(\mathfrak{R})$ and $\mathcal{Q} = p(\mathfrak{Q})$.

We have $\chi(\mathfrak{R}) = 0$ and $\chi(\mathfrak{Q}) = 0$, both being tori. Moreover:

$$e(X,\mathfrak{R}) = 0 \text{ and } e(X,\mathfrak{Q}) = Q_X(\mathfrak{Q},\mathfrak{Q}) = 8,$$

since $\mathfrak{R}$ is Lagrangian (see §3.4.3) and since $\mathfrak{Q}$ is an oriented surface with homology class $(2,2)$. Therefore, we have that $\mathscr{R}$ is also a torus, whereas $\mathcal{Q}$ is a Klein bottle, since $p : \mathfrak{Q} \to \mathcal{Q}$ is a double *unbranched* covering. Moreover, from Lemma 2.38, we obtain:

$$e(\mathbf{S}^4,\mathcal{R}) = 0 \text{ and } e(\mathbf{S}^4,\mathcal{Q}) = 4.$$

Finally, let $\overline{X}$ be the double branched cover of $\mathbf{S}^4$ ramified along $\mathcal{Q}$. Denote the covering map as $\widetilde{p} : \overline{X} \to \mathbf{S}^4$, and set $\overline{\mathfrak{R}} = \widetilde{p}^{-1}(\mathcal{Q})$ and $\overline{\mathfrak{Q}} = \widetilde{p}^{-1}(\mathcal{R})$. One last application of Lemma 2.38 gives $e(\overline{X},\overline{\mathfrak{R}}) = 2$ and $e(\overline{X},\overline{\mathfrak{Q}}) = 0$. We represent the situation in Figure 3.10.

Note that Theorem 2.36, Theorem 2.19 and Proposition 2.37 yield that $\overline{X}$ is a $\mathbf{Z}/2$-homology $\overline{\mathbf{CP}}^2\#\overline{\mathbf{CP}}^2$. More precisely, we have:

$$b_1(\overline{X};\mathbf{Z}/2) = b_3(\overline{X};\mathbf{Z}/2) = 0,\ b_2(\overline{X};\mathbf{Z}/2) = 2 \text{ and } \sigma(\overline{X}) = -2.$$

$$\begin{array}{lccccccl}
e(X,\mathfrak{R})=0 & \mathfrak{R} & \subset & X & \supset & \mathfrak{Q} & & e(X,\mathfrak{Q})=+8 \\
 & \| & & \downarrow p & & \downarrow & & \\
e(\mathbf{S}^4,\mathcal{R})=0 & \mathcal{R} & \subset & \mathbf{S}^4 & \supset & \mathcal{Q} & & e(\mathbf{S}^4,\mathcal{Q})=+4 \\
 & \uparrow & & \uparrow \tilde{p} & & \| & & \\
e(\overline{X},\overline{\mathfrak{Q}})=0 & \overline{\mathfrak{Q}} & \subset & \overline{X} & \supset & \overline{\mathfrak{R}} & & e(\overline{X},\overline{\mathfrak{R}})=+2
\end{array}$$

Figure 3.10. The geometric situation in the case of the quadric hyperboloid.

We now need to understand the covering $\tilde{p}:\overline{\mathfrak{Q}}\to\mathcal{R}$ in order to describe the lift of the real curve.

Proposition 3.32. *The covering $\tilde{p}:\overline{\mathfrak{Q}}\to\mathcal{R}$ is connected and corresponds to the subgroup*

$$G=\left\{(x,y)\in\mathbf{Z}^2 \;\middle|\; x+y\equiv 0\ [2]\right\}$$

of $\pi_1(\mathcal{R})\cong\mathbf{Z}^2$, where $\pi_1(\mathcal{R})$ is spanned by real curves of bidegree $(1,0)$ and $(0,1)$.

Proof. There are only three subgroups of index two in $\mathbf{Z}^2$:

$$\mathbf{Z}\oplus 2\mathbf{Z},\ 2\mathbf{Z}\oplus\mathbf{Z} \text{ and } G.$$

To show that the covering is connected and corresponds to the subgroup G, it suffices to show that the pre-images of a $(1,0)$-curve and a $(0,1)$-curve are connected.

The arguments are exactly the same for both, so we treat the case of a $(0,1)$-curve. Let γ be such a curve, which can be given as $\gamma=p(\mathbf{R}G)$ with G a generic bidegree $(0,1)$ real curve. In particular, $\gamma=\partial p(\mathbf{C}G)$, and $\mathbf{C}G\pitchfork\mathfrak{Q}$ is 2 conjugate points, thus $p(\mathbf{C}G)\pitchfork\mathcal{Q}$ is a single point. The covering $\tilde{p}:\tilde{p}^{-1}(p(\mathbf{C}G))\to p(\mathbf{C}G)$ is a double branched covering of the 2-disc $p(\mathbf{C}G)$ with one single interior branch point. By Proposition 2.33, we obtain that $\tilde{p}^{-1}(p(\mathbf{C}G))$ is also a 2-disc, and it induces the non-trivial double covering on the boundaries. Since $\gamma=\partial p(\mathbf{C}G)$, we have that $\tilde{p}^{-1}(\gamma)=\partial\tilde{p}^{-1}(p(\mathbf{C}G))$ is connected, thus the claim. ∎

We depict the covering $\tilde{p}:\overline{\mathfrak{Q}}\to\mathcal{R}$ in Figure 3.11.

In particular, if a curve $\gamma\subset\mathcal{R}$ has homology class (α,β) with α and β coprime and both odd, then its pre-image $\tilde{p}^{-1}(\gamma)$ is two parallel copies of the same curve on $\overline{\mathfrak{Q}}$, since $[\gamma]\in G$ in this case. This gives the following observations:

(1) each oval is doubled, and each copy of the (α,β) curve is doubled;

(2) the pre-image respects mutual position of components, as in Proposition 3.13.

In particular, letting $\overline{\mathbf{R}}F=\tilde{p}^{-1}(p(\mathbf{R}F))\subset\overline{\mathfrak{Q}}$, we derive that $\overline{\mathfrak{Q}}\smallsetminus\overline{\mathbf{R}}F$ has two diffeomorphic subsets $\overline{\mathfrak{Q}}_{\pm}$ with the property that $\partial\overline{\mathfrak{Q}}_{\pm}=\overline{\mathbf{R}}F$. We give such an example in Figure 3.12.

Definition 3.33. *The **Arnold surface** of a flexible curve of bidegree (a,b) curve in (X,c_{hyp}) is the surface*

$$\mathcal{A}(F)=\overline{\mathfrak{Q}}_{+}\cup\tilde{p}^{-1}(p(F)).$$

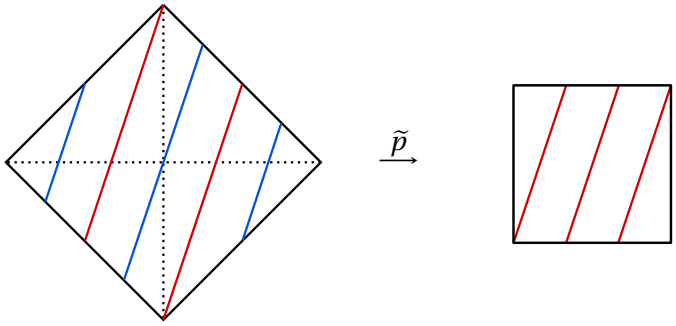


Figure 3.11. The unbranched covering $\widetilde{p} : \overline{\mathfrak{Q}} \to \mathcal{R}$ corresponding to the subgroup G. On the left, the pre-image of the $(3,1)$ curve is two parallel copies of a $(1,-2)$ curve. It is understood that the two tori are represented by the two squares, whose opposite sides are identified.

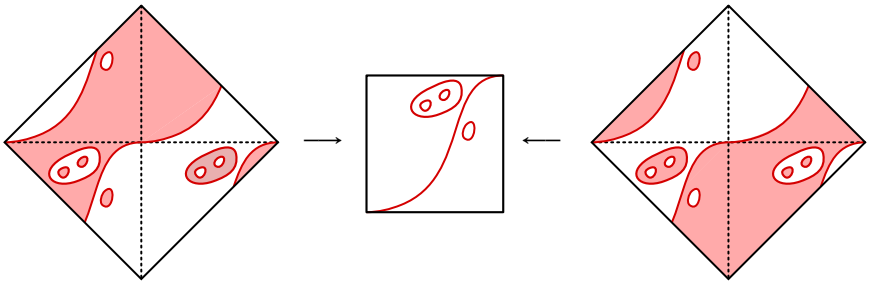


Figure 3.12. In the middle, a curve with real scheme $\langle (1,1), 1 \sqcup 1\langle 2\rangle\rangle$. On the left and on the right, the two possible choices $\overline{\mathfrak{Q}}_{\pm}$.

Proposition 3.34. *We have* $\chi(\mathcal{A}(F)) = -2ab$ *and* $e(\overline{X}, \mathcal{A}(F)) = 2ab + 2$.

Proof. Using the same arguments as before, we have:

$$\chi(\overline{\mathfrak{Q}}_{\pm}) = 0 \text{ and } e(\overline{X}, \overline{\mathfrak{Q}}_{\pm}) = 0.$$

The rest is simply a matter of applying Lemma 2.38 and Theorem 2.19, by noting that $p(F) \pitchfork \mathcal{Q}$ is a collection of $a+b$ isolated points. ∎

As in the case of Theorem 3.20, we would like to consider the double branched cover of $\overline{X}$ ramified along a suitable deformation of $\mathcal{A}(F)$. For this, we need to understand the homology class of the surfaces $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$.

Proposition 3.35. *The surfaces* $\mathcal{A}(F)$ *and* $\overline{\mathfrak{R}}$ *are both mod* 2 *characteristic surfaces in* $\overline{X}$.

Proof. There are only four homology classes in $H_2(\overline{X}; \mathbf{Z}/2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$, which we denote as $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ in an obvious manner. Because $e(\overline{X}, \mathcal{A}(F))$ and $e(\overline{X}, \overline{\mathfrak{R}})$ are both even, this rules out the classes $(1,0)$ and $(0,1)$, since:

$$e(\overline{X}, \Sigma) \equiv Q_{\overline{X}, \mathbf{Z}/2}(\Sigma, \Sigma) \ [2].$$

Now, $\overline{X}$ is definite negative, and thus Proposition 2.11 provides the following:

$$e(\overline{X},\mathcal{A}(F))+2\chi(\mathcal{A}(F))\equiv q([\mathcal{A}(F)])\ [4] \text{ and } e(\overline{X},\overline{\mathfrak{R}})+2\chi(\overline{\mathfrak{R}})\equiv q([\overline{\mathfrak{R}}])\ [4].$$

This yields:

$$q([\mathcal{A}(F)])\equiv 2ab\ [4] \text{ and } q([\overline{\mathfrak{R}}])\equiv 2\ [4].$$

In particular, $[\overline{\mathfrak{R}}]\neq 0\in H_2(\overline{X};\mathbf{Z}/2)$ since $2\not\equiv 0\ [4]$, and because a and b are both odd, we have $2ab\not\equiv 0\ [4]$, which yields that $[\mathcal{A}(F)]\neq 0\in H_2(\overline{X};\mathbf{Z}/2)$. ■

Define $\mathcal{X}(F)\subset\overline{X}$ to be the embedded surface otained by smoothing the singularities of the nodally immersed surface $\mathcal{A}(F)\cup\overline{\mathfrak{R}}$ accordingly to Corollary 2.18. We therefore observe that

$$\chi(\mathcal{X}(F))=-2ab-a-b \text{ and } e(\overline{X},\mathcal{X}(F))=2ab+2a+2b+2.$$

Moreover, by Proposition 3.35, we obtain that $[\mathcal{X}(F)]=0\in H_2(\overline{X};\mathbf{Z}/2)$. Let Y denote the double branched cover of $\overline{X}$ ramified along $\mathcal{X}(F)$, and let $\Theta: Y\to\overline{X}$ be the covering map.

Proposition 3.36. *We have*

$$b_2^+(Y)=\frac{ab+1}{2}.$$

Proof. This is just a matter of using the Hirzebruch signature formula, the topological Riemann–Hurwitz formula and Proposution 2.37. ■

As before, if $o\subset\mathbf{R}F$ is a non-empty oval, let $\Omega(o)$ denote the connected component of $\mathfrak{R}\smallsetminus\mathbf{R}F$ which is bounded by o and its inner ovals, and let $C_\pm(o)\subset\overline{\Omega}_\pm$ be the connected components of the lift of $\widetilde{p}^{-1}(p(\Omega(o)))$. Then, set $\widetilde{C}(o)=\Theta^{-1}(C_-(o))$.

For the analogue of $\widetilde{D}(\mathcal{J})$ in this setting, there is a small subtlety that needs addressing. Indeed, there may be several (at least one) copies of an (α,β)-curve in $\mathbf{R}F$, and each of these curves will lift to two copies inside $\overline{\Omega}$. However, in the case where $\mathbf{R}F$ contains at least one oval (in the case where it does not, the bound is trivial), there is at least one component $D_-\subset\overline{\Omega}_-$ of $\overline{\Omega}\smallsetminus\overline{\mathbf{R}}F$ that has one of those lifts of the (α,β)-curves as a boundary component, together with some exterior ovals of $\overline{\mathbf{R}}F$, and at least one. Again, set $\widetilde{D}=\Theta^{-1}(D_-)$. See Figure 3.13 for a depiction of the sets $C_-(o)$ and D_-.

Proof of Theorem 3.31. By computations strictly analogous to the case of planar curves, we have:

(1) the surfaces $\widetilde{C}(o)$ and $\widetilde{D}$ are oriented in Y;
(2) if $o\subset\mathbf{R}F$ is an oval, then $Q_Y(\widetilde{C}(o),\widetilde{C}(o))=-4\chi(o)$, and $Q_Y(\widetilde{D},\widetilde{D})\leqslant 0$;
(3) if $o,o'\subset\mathbf{R}F$ are distinct ovals, then $Q_Y(\widetilde{C}(o),\widetilde{C}(o'))=0$, and $Q_Y(\widetilde{C}(o),\widetilde{D})=0$.

In particular, applying the same reasoning as in the proof of Theorem 3.20, we derive that the rank of the family spanned by the homology classes of those surfaces in Y is less than $b_2^+(Y)$. ■

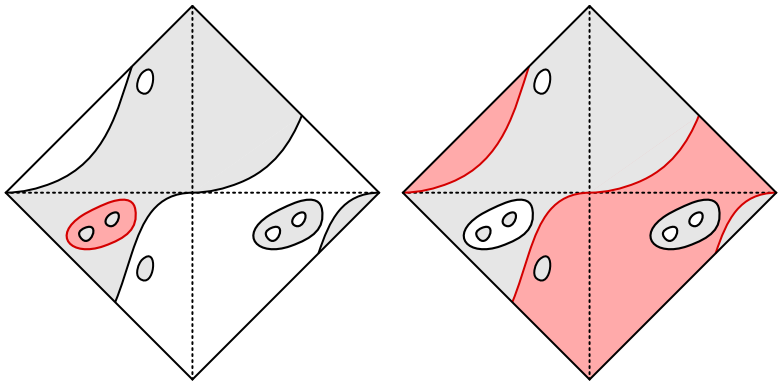


Figure 3.13. We use the same curve as in Figure 3.12, and the choice of $\overline{\Omega}_+$ is depicted in gray. On the left: the set $C_-(o)$ for the only non-empty oval. On the right: a choice of a set D_- (note that in this case, there is only one such choice possible).

3.4.2 Curves on the Ellipsoid Quadric

We now switch to the involution c_{ell} on $X = \mathbf{CP}^1 \times \mathbf{CP}^1$. This time, if $\mathfrak{R} = \text{Fix}(c_{\text{ell}})$, then $\mathfrak{R}$ is an embedded 2-sphere inside X. Again, let Ω denote an imaginary bidegree $(2,2)$-curve.

Definition 3.37. *Let $F \subset X$ be a closed, connected and oriented surface smoothly embedded. F is called a* ***bidegree***[†] (m,m) ***flexible curve*** *if:*

(1) $c_{\text{ell}}(F) = F$;

(2) $[F] = (m,m) \in H_2(X;\mathbf{Z})$;

(3) $\chi(F) = 2-2(m-1)^2$;

(4) if $\mathbf{R}X = F \cap \mathfrak{R}$, *then for all* $x \in \mathbf{R}F$, $T_xF = T_x\mathbf{R}F \oplus \mathbf{i}\cdot T_x\mathbf{R}F$.

If additionally $F \pitchfork \Omega$ consists of $4m$ points, then the curve is called ***totally flexible****.*

By Theorem 2.35, the quotient X/c_{ell} is $\overline{\mathbf{CP}}^2$, and we denote as $p : X \to \overline{\mathbf{CP}}^2$ the quotient map, and $\mathcal{R} = p(\mathfrak{R})$ and $\mathcal{Q} = p(\Omega)$ as usual. Because the self-intersection $e(\overline{\mathbf{CP}}^2, \mathcal{Q}) = 4$ is even, we are allowed to consider $\overline{X}$ the double branched cover of $\overline{\mathbf{CP}}^2$ ramified over $\mathcal{Q}$. We denote as $\widetilde{p} : \overline{X} \to \overline{\mathbf{CP}}^2$ the covering map, as well as $\overline{\Omega} = \widetilde{p}^{-1}(\mathcal{R})$ and $\overline{\mathfrak{R}} = \widetilde{p}^{-1}(\mathcal{Q})$.

Proposition 3.38. *The 4-manifold $\overline{X}$ is a $\mathbf{Z}/2$-homology $4\overline{\mathbf{CP}}^2$. The different self-intersection numbers are represented in Figure 3.14. The surfaces $\mathfrak{R}$ and $\mathcal{R}$ are 2-spheres, and $\overline{\Omega}$ is a disjoint union of two 2-spheres. The surface Ω is a 2-torus, and the surfaces $\mathcal{Q}$ and $\overline{\mathfrak{R}}$ are Klein bottles. The covering $\widetilde{p} : \overline{\Omega} \to \mathcal{R}$ is the trivial double covering of the 2-sphere $\mathcal{R}$.*

Proof. The surface $\mathfrak{R}$ is Lagrangian, thus $e(X,\mathfrak{R}) = -\chi(\mathfrak{R})$. The rest of the computations of self-

[†] In the case of curves on the hyperboloid, curves invariant under c_{ell} necessarily have a bidegree of the form (m,m).

$$\begin{array}{lccccccl}
e(X,\mathfrak{R})=-2 & \mathfrak{R} & \subset & X & \supset & \mathfrak{Q} & & e(X,\mathfrak{Q})=+8 \\
& \| & & \downarrow p & & \downarrow & & \\
e(\overline{\mathbf{CP}}^2,\mathcal{R})=-4 & \mathcal{R} & \subset & \overline{\mathbf{CP}}^2 & \supset & \mathcal{Q} & & e(\overline{\mathbf{CP}}^2,\mathcal{Q})=+4 \\
& \uparrow & & \tilde{p}\uparrow & & \| & & \\
e(\overline{X},\overline{\mathfrak{Q}})=-8 & \overline{\mathfrak{Q}} & \subset & \overline{X} & \supset & \overline{\mathfrak{R}} & & e(\overline{X},\overline{\mathfrak{R}})=+2
\end{array}$$

Figure 3.14. The geometric situation in the case of the quadric ellipsoid.

intersection numbers is simply a matter of using Lemma 2.38 a couple times.

The fact that $\mathcal{Q}$ is a Klein bottle comes from the double *unbranched* covering $p:\mathfrak{Q}\to\mathcal{R}$. However, there is no non-trivial double covering of a 2-sphere, and therefore $\tilde{p}:\overline{\mathfrak{Q}}\to\mathcal{R}$ is that trivial covering, implying that $\overline{\mathfrak{Q}}$ is a disjoint union of two 2-spheres.

The claim regarding $\overline{X}$ comes from Theorem 2.19, Theorem 2.36 and Proposition 2.37. In particular, note that $\overline{X}$ is negative definite with $H_2(X;\mathbf{Z})$ having rank four. ■

We will set $\overline{\mathfrak{Q}}=Q_1\sqcup Q_2$ with Q_i a 2-sphere, and where if $\tau:\overline{X}\to\overline{X}$ spans $\mathrm{Aut}(\tilde{p})$, then $\tau(Q_1)=Q_2$ and *vice-versa*. Denote as R_1 and R_2 the two subsets of $\mathcal{R}\smallsetminus p(\mathbf{R}F)$ with $\partial R_i=p(\mathbf{R}F)$, and define:

$$\begin{cases}\overline{\mathfrak{Q}}_+=Q_1\cap\tilde{p}^{-1}(R_1)\sqcup Q_2\cap\tau(\tilde{p}^{-1}(R_1)),\\ \overline{\mathfrak{Q}}_-=Q_2\cap\tilde{p}^{-1}(R_1)\sqcup Q_1\cap\tau(\tilde{p}^{-1}(R_1)).\end{cases}$$

We refer the reader to Figure 3.15.

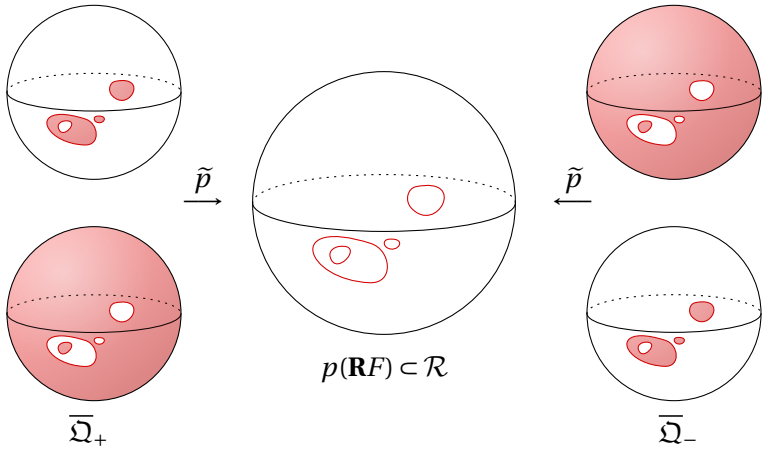


Figure 3.15. The two possible subsets $\overline{\mathfrak{Q}}_\pm$, shaded. It is understood that the two spheres in the first row are Q_1, and the two in the second are Q_2.

Definition 3.39. *Let $F\subset(X,c_{\mathrm{ell}})$ be a totally flexible curve of bidegree (m,m) with m odd. The* ***Arnold***

surface *of F is the surface*

$$\mathcal{A}(F) = \widetilde{p}^{-1}(p(F)) \cup \overline{\Omega}_+.$$

By using the exact same arguments as usual, we derive:

$$\chi(\mathcal{A}(F)) = -2m^2 + 2m + 2 \text{ and } e(\overline{X}, \mathcal{A}(F)) = 2m^2 - 4.$$

However, one major difference is that this time, the second homology group of $\overline{X}$ is a lot bigger:

$$H_2(\overline{X}; \mathbf{Z}/2) = \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2,$$

where we pick a basis of $H_2(\overline{X}; \mathbf{Z}/2)$ which is the mod 2 reduction of a basis of $H_2(X; \mathbf{Z})$ in which the intersection form $Q_{\overline{X}}$ is $-I_4$ (we use Donaldson's diagonalization theorem here, since $\overline{X}$ is negative definite). In the previous situations, the surfaces $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ were both mod 2 characteristic and homologous to one another. The situation differs slightly more in this case.

Proposition 3.40. *The surfaces $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ are homologous mod 2 and not null-homologous, but are not characteristic surfaces.*

Proof. We first need to describe geometrically a suitable basis for $H_2(\overline{X}; \mathbf{Z}/2)$.

Consider a complex line $\Lambda \subset \overline{\mathbf{CP}}^2 = X / c_{\text{ell}}$ with $\Lambda \cap \mathcal{Q} = \varnothing$. This is possible, since $\mathcal{Q}$ is a Klein bottle with $e(\overline{\mathbf{CP}}^2, \mathcal{Q}) = +4$, and therefore this is a local surface. Hence, $\widetilde{p}^{-1}(\Lambda) = S_1 \sqcup S_2$ is a disjoint union of two 2-spheres, each with $e(\overline{X}, S_i) = -1$. Moreover, since they are disjoint, they satisfy $Q_{\overline{X}, \mathbf{Z}/2}(S_1, S_2) \equiv 0$ [2], *i.e.* they are linearly independent in $H_2(\overline{X}; \mathbf{Z}/2)$. We need two more generators.

The surface $\mathcal{Q}$ is a Klein bottle, which can be seen as the desingularization of two real projective planes R_1 and R_2. More precisely, by Proposition 2.17, we can find two embedded real projective planes $R_1, R_2 \subset \overline{\mathbf{CP}}^2$ such that $R_1 \pitchfork R_2$ is a single point and $e(\overline{\mathbf{CP}}^2, R_i) = 1$ and such that $\mathcal{Q}$ is obtained by the smoothing of the only singularity of the nodally immersed union $R_1 \cup R_2 \subset \overline{\mathbf{CP}}^2$. Note that in this case, we would have $\Lambda \pitchfork R_1 = \Lambda \pitchfork R_2 = R_1 \pitchfork R_2$, but after the smoothing, we do obtain $\Lambda \cap \mathcal{Q} = \varnothing$ (the surface Λ is locally knotted with the Hopf band glued to smooth the singularity; it "goes through its middle").

Denote as x the unique point inside $R_1 \pitchfork R_2$, and let $D_i \subset R_i$ be a small open disc centered at x. Each of $R_i \smallsetminus D_i$ is a Möbius strip, whose core ℓ_i, a curve spanning $\pi_1(R_i \smallsetminus D_i)$, can be seen as a real flexible line in R_i; see Figure 3.16.

This real ℓ_i line separates a complex line $L_i \subset \overline{\mathbf{CP}}^2$ into two disc components $L_i = L_i^+ \cup_{\ell_i} L_i^-$ with $\partial L_i^{\pm} = \ell_i$. Define $\Sigma_i = \widetilde{p}^{-1}(L_i^+)$.

Since $\partial L_i^+ = \ell_i \subset \mathcal{Q}$, we see that Σ_i is obtained by gluing two discs along their common boundary (the restriction $\widetilde{p} : \Sigma_i \to L_i^+$ is a "pseudo branched cover" in the same sense as in Figure 3.8). Therefore, Σ_i is a 2-sphere inside $\overline{X}$. We can compute its self-intersection using Lemma 2.38:

$$e(\overline{X}, \Sigma_i) = 2e(\overline{\mathbf{CP}}^2, L_i^+) = -1,$$

since $L_i = L_i^+ \cup L_i^-$ and $e(\overline{\mathbf{CP}}^2, L_i^+) = e(\overline{\mathbf{CP}}^2, L_i^-)$.

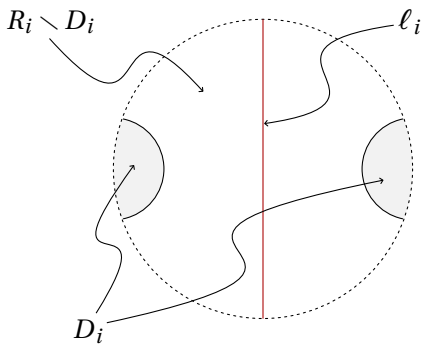


Figure 3.16. The core of a Möbius strip can be seen as a real line in the associated real projective plane.

Next, we need to compute $Q_{\overline{X},\mathbf{Z}/2}(\Sigma_1,\Sigma_2)$. If the two discs $L_1^+ \cap L_2^+$ intersect in their interior, these intersections gets lifted to two intersection points in $\Sigma_1 \pitchfork \Sigma_2$. In particular, $\Sigma_1 \pitchfork \Sigma_2$ must be an even number of points, which yields $Q_{\overline{X},\mathbf{Z}/2}(\Sigma_1,\Sigma_2) \equiv 0$ [2].

It finally suffices to verify that $Q_{\overline{X},\mathbf{Z}/2}(\Sigma_i,S_j) \equiv 0$ [2] for all i,j. Each intersection point in $\Lambda \pitchfork L_i^+$ gives rise to one intersection point in $S_1 \pitchfork \Sigma_i$ and one in $S_2 \pitchfork \Sigma_i$. Moreover, we have

$$Q_{\overline{\mathbf{CP}}^2}(\Lambda, L_i) = -1,$$

since both Λ and L_i were taken to be complex lines. Moreover, since $\Lambda \cap \mathcal{Q} = \varnothing$, we can ensure that $\ell_i \cap \Lambda = \varnothing$. Because $Q_{\overline{\mathbf{CP}}^2}(\Lambda, L_i)$ is odd, we have that $\Lambda \pitchfork L_i$ is too, and we can therefore ensure that, up to a choice of labelling, $\Lambda \pitchfork L_i^-$ is an odd number of points and $\Lambda \pitchfork L_i^+$ is an even number of points. This finally gives:

$$\#\Sigma_i \pitchfork S_j = \#\Lambda \pitchfork L_i^+ \equiv 0 \text{ [2]},$$

q.e.d.

We have obtained a $Q_{\overline{X},\mathbf{Z}/2}$-orthogonal basis $([S_1],[S_2],[\Sigma_1],[\Sigma_2])$ for $H_2(\overline{X};\mathbf{Z}/2)$. We will shorten the notations by writing

$$(a,b,c,d) = a[S_1] + b[S_2] + c[\Sigma_1] + d[\Sigma_2],$$

where $a,b,c,d \in \{0,1\}$. We therefore let $[\mathcal{A}(F)] = (a_1,a_2,a_3,a_4)$ and $[\overline{\mathfrak{R}}] = (b_1,b_2,b_3,b_4)$.

(1) We have $a_1+a_2+a_3+a_4 \equiv 2$ [4] and $b_1+b_2+b_3+b_4 \equiv 2$ [4]. That is: two of the a_i are non-zero, and the others are, and the same goes for the b_i. Indeed, in both cases, we have that the Euler characteristic is even, whereas the normal Euler number is congruent to 2 mod 4. Plugging this into Proposition 2.11 gives:

$$q([\mathcal{A}(F)]) \equiv 2 \text{ [4] and } q([\overline{\mathfrak{R}}]) \equiv 2 \text{ [4]}.$$

On the other hand, we have

$$q((a,b,c,d)) \equiv a+b+c+d \text{ [4]}.$$

This proves the claim, and already rules out the zero homology class as well as the characteristic class $(1,1,1,1)$.

(2) By Lemma 2.39, we obtain that $\widetilde{p}^{-1}(\Lambda) \cup \overline{\mathfrak{R}} = S_1 \cup S_2 \cup \overline{\mathfrak{R}}$ is characteristic in $\overline{X}$. This implies[†] that $[\mathfrak{R}] = (0,0,1,1)$. To show that $[\mathcal{A}(F)] = [\overline{\mathfrak{R}}]$, it finally suffices to verify that $[\mathcal{A}(F)] = (0,0,1,1)$, or equivalently that

$$Q_{\overline{X},\mathbf{Z}/2}(\mathcal{A}(F), S_1) \equiv Q_{\overline{X},\mathbf{Z}/2}(\mathcal{A}(F), S_2) \equiv 0 \ [2],$$

by using the orthogonal basis and the previous claim. For this, it suffices to check that $\mathcal{A}(F) \pitchfork S_i$ is an even number of points. Those intersection points come in two flavours.
The first kind of intersections are points in $\widetilde{p}^{-1}(p(F)) \pitchfork S_i$. From

$$\#S_1 \cap \widetilde{p}^{-1}(p(F)) + \#S_2 \cap \widetilde{p}^{-1}(p(F)) = 2\#p(F) \pitchfork \Lambda \text{ and } \#S_1 \cap \widetilde{p}^{-1}(p(F)) = \#S_2 \cap \widetilde{p}^{-1}(p(F)),$$

we obtain that $\#S_i \cap \widetilde{p}^{-1}(p(F)) = \#p(F) \pitchfork \Lambda$. Now, because $e(\overline{\mathbf{CP}}^2, p(F)) = 2m^2$ is even, we obtain that $\#p(F) \pitchfork \Lambda$ is even as well.
The other kind of intersections are points in $\overline{\Omega}_+ \pitchfork S_i$. By a reasoning similar to the previous case, we have

$$\#\overline{\Omega}_+ \pitchfork S_i = \#\mathcal{R} \pitchfork \Lambda.$$

Indeed, the map $\widetilde{p} : \overline{\Omega}_+ \to \mathcal{R}$ is one-to-one on the complement of $\overline{\mathbf{R}}F$. Because $e(\overline{\mathbf{CP}}^2, \mathcal{R}) = -4$ is even, we also have that $\#\mathcal{R} \pitchfork \Lambda$ is even. ■

Theorem 3.41. *Let $F \subset (X, c_{\text{ell}})$ be a totally flexible curve of bidegree (m,m) with m odd, and let $\lambda^{\pm}$ and λ^0 denote the numbers of connected components of $\mathfrak{R} \smallsetminus \mathbf{R}F$ with positive, negative or zero Euler characteristic, respectively. Then:*

$$\lambda^0 + \lambda^- \leqslant \frac{m^2+1}{2}.$$

Proof. By Proposition 3.40, this means that considering the surface $\mathcal{X}(F) \subset \overline{X}$ obtained from $\mathcal{A}(F) \cup \overline{\mathfrak{R}}$ and smoothing the singularities, we get:

$$\chi(\mathcal{X}(F)) = -2m^2 - 2m + 2,\ e(\overline{X}, \mathcal{X}(F)) = 2m^2 + 4m - 2 \text{ and } [\mathcal{X}(F)] = 0 \in H_2(\overline{X}; \mathbf{Z}/2).$$

Once again, set Y to be the double branched cover of $\overline{X}$ ramified over $\mathcal{X}(F)$. We obtain, after performing the computations:

$$b_2^+(Y) = \frac{m^2-3}{2}.$$

The difference with previously is that we consider all possible connected components of $\mathfrak{R} \smallsetminus \mathbf{R}F$, instead of merely non-empty ovals to which we could create one last membrane $\Omega_n(\infty)$, $D_-(\mathcal{J})$ or D_- in the previous proofs. This means that we obtain:

$$\lambda^0 + \lambda^- \leqslant b_2^+(Y) + 1,$$

which is the desired upper bound. ■

[†] There is a slight typo in the proof of [Sai24, Proposition 4.7]. In the article, it should read that the homology class of $\overline{\mathfrak{R}}$ is $[\Sigma_1] + [\Sigma_2]$ instead of $[S_1] + [S_2]$. This typo propagated until the end of the proof, where we claim to be counting intersection points in $\mathcal{A}(F) \pitchfork \Sigma_i$ where in reality we count intersection points in $\mathcal{A}(F) \pitchfork S_i$. The arguments themselves are correct, and so is the result (fortunately!).

3.4.3 Conjugations on 4-Manifolds

Consider (X,ω) a symplectic closed and connected 4-manifold with a compatible almost complex structure J. That is: the assignment

$$g(u,v) = \omega(Ju,v)$$

defines a Riemannian metric on X.

Definition 3.42. *A **conjugation** on (X,ω,J) is an involution $c : X \to X$ with non-empty fixed-point set and which is*

(1) anti J-holomorphic; that is: for all $x \in X$, we have $J_{c(x)} \circ \mathrm{d}_x c = -\mathrm{d}_x c \circ J_x$, and
(2) g-isometric; that is, for all $x \in X$ and all $u, v \in T_x X$, we have $g_{c(x)}(\mathrm{d}_x c(u), \mathrm{d}_x c(v)) = g_x(u,v)$.

If (X,c) is a such 4-manifold with a conjugation, we denote as $\mathbf{R}X = \mathrm{Fix}(c)$ the fix-point set of that conjugation, and we call it the **real part** of X. In particular, we have:

(1) if $X = \mathbf{CP}^2$ and $c = \mathrm{conj}$, then $\mathbf{R}X = \mathbf{RP}^2$;
(2) if $X = \mathbf{CP}^1 \times \mathbf{CP}^1$ and $c = c_{\mathrm{hyp}}$, then $\mathbf{R}X = \mathbf{RP}^1 \times \mathbf{RP}^1$;
(3) if $X = \mathbf{CP}^1 \times \mathbf{CP}^1$ and $c = c_{\mathrm{ell}}$, then $\mathbf{R}X = \left\{ (x, \overline{x}) \;\middle|\; x \in \mathbf{CP}^1 \right\}$.

Theorem 3.43. *The real part $\mathbf{R}X \subset X$ is a closed surface which is totally geodesic and Lagrangian.*

For this, we will need two things:

(1) the differential $\mathrm{d}_p c : T_p X \to T_p X$ of c at a point $p \in \mathbf{R}X$, and
(2) the exponential map $\exp_p : T_p X \to T_p X$.

Moreover, note that if $f : X \to X$ is an isometry, if $p \in X$ and if $v \in T_p X$, then:

$$f(\exp_p(v)) = \exp_{f(p)}(\mathrm{d}_p f(v)).$$

Indeed, one checks that the map $t \mapsto f(\exp_p(tv))$ defines a geodesic starting at $f(p)$ and directed by $\mathrm{d}_p f(v)$. We would like to prove a few key lemmas first. Throughout those, we assume that $p \in \mathbf{R}X$ is a given fixed point under c.

Lemma 3.44. *The map $\mathrm{d}_p c : T_p X \to T_p X$ is not the identity map.*

Proof. Consider a point $z \in X \smallsetminus \mathbf{R}X$ arbitrarily close to p. This exists, since otherwise, c would be *J-holomorphic* in a neighborhood of p, instead of *anti* J-holomorphic. This point z can be taken so as to be in the image under $\exp_p$ of the ball whose radius is the injectivity radius at p (that is, z is in the image under $\exp_p$ of an open ball on which this exponential map realizes a local diffeomorphism). In particular, there exists $u \in T_p X$ such that $z = \exp_p(u)$, and $c(z) \neq z$. Therefore, we obtain:

$$c(z) = c(\exp_p(u)) = \exp_p(\mathrm{d}_p c(u)),$$

which yields that $\mathrm{d}_p c(u) \neq u$, otherwise $c(z) = z$. ∎

Lemma 3.45. *We have $\dim(T_p \mathbf{R}X) = \dim(\nu_p \mathbf{R}X) = 2$, and thus $\mathbf{R}X$ is indeed a surface.*

Proof. From the facts that $(\mathrm{d}_p c)^2 = \mathrm{id}_{T_pX}$ and $\det(\mathrm{d}_p c) = 1$ since c is orientation-preserving, basic linear algebra tells us that necessarily, the linear map $\mathrm{d}_p c$ has two eigenspaces Λ^+ and Λ^- associated to the eigenvalues ± 1, and both have even dimension (remark that $\dim(T_pX) = 4$). Moreover, we have $\Lambda^- \neq \{0\}$, from Lemma 3.44, since $\Lambda^+ = \mathrm{Fix}(\mathrm{d}_p c)$. In particular, we obtain:

$$\dim(\Lambda^+) \in \{0,2\} \text{ and } \dim(\Lambda^-) \in \{2,4\}.$$

Finally, from $T_p\mathbf{R}X \subset \mathrm{Fix}(\mathrm{d}_p c)$, we have $\dim(T_p\mathbf{R}X) \leqslant 2$, and thus $\dim(\nu_p\mathbf{R}X) \geqslant 2$. Lastly, because $\mathrm{d}_p c$ is an isometry of T_pX, it preserves orthogonality, and since $d_p c(T_p\mathbf{R}X) \subset T_p\mathbf{R}X$, we also have that $\nu_p\mathbf{R}X$ is stabilized:

$$\mathrm{d}_p c(\nu_p\mathbf{R}X) \subset \nu_p\mathbf{R}X.$$

Again, basic linear algebra from looking at $T_pX = T_p\mathbf{R}X \oplus \nu_p\mathbf{R}X$ ensures that we therefore have

$$\mathrm{d}_p c(T_p\mathbf{R}X) = T_p\mathbf{R}X \text{ and } \mathrm{d}_p c(\nu_p\mathbf{R}X) = \nu_p\mathbf{R}X,$$

and both spaces have dimension two. ■

Lemma 3.46. *Let $v \in T_pX$. Then:*

$$v \in T_p\mathbf{R}X \iff \mathrm{d}_p c(v) = v \text{ and } v \in \nu_p\mathbf{R}X \iff \mathrm{d}_p c(v) = -v.$$

Proof. It suffices to check that we have $T_p\mathbf{R}X = \Lambda^+$ and $\nu_p\mathbf{R}X = \Lambda^-$, with the notations of the proof of Lemma 3.45. ■

Lemma 3.47. *We have $J_p(T_p\mathbf{R}X) = \nu_p\mathbf{R}X$ and $J_p(\nu_p\mathbf{R}X) = T_p\mathbf{R}X$.*

Proof. We only need to verify the two inclusions $J_p(T_p\mathbf{R}X) \subset \nu_p\mathbf{R}X$ and $J_p(\nu_p\mathbf{R}X) \subset T_p\mathbf{R}X$. We use the characterization of Lemma 3.46.

(1) If $v \in T_p\mathbf{R}X$, then $\mathrm{d}_p c(v) = v$, and thus $J_p v \in \nu_p\mathbf{R}X$, because $\mathrm{d}_p c(J_p v) = -J_p \mathrm{d}_p c(v) = -J_p v$.
(2) If $v \in \nu_p\mathbf{R}X$, then $\mathrm{d}_p c(v) = -v$, which gives $\mathrm{d}_p c(J_p v) = -J_p \mathrm{d}_p c(v) = -J_p(-v) = J_p v$, whence $J_p v \in T_p\mathbf{R}X$. ■

We can now prove the first statement about $\mathbf{R}X$.

Proof of Theorem 3.43. We have already obtained that $\mathbf{R}X$ is a surface from Lemma 3.45.

Firstly, we show that it is totally geodesic. That is: if a geodesic $t \mapsto \gamma(t)$ has $(\gamma(0), \dot\gamma(0)) \in T\mathbf{R}X$, then $\gamma \subset \mathbf{R}X$. Consider $p \in \mathbf{R}X$ and $v \in T_p\mathbf{R}X$. The corresponding geodesic is $\gamma(t) = \exp_p(tv)$, and thus:

$$c(\gamma(t)) = c(\exp_p(tv)) = \exp_p(t\mathrm{d}_p c(v)) = \exp_p(v) = \gamma(t),$$

which is indeed $\gamma \subset \mathrm{Fix}(c)$.

Lastly, to show that $\mathbf{R}X$ is Lagrangian, it only suffices to check that it is isotropic, since it has dimension half that of X. For this, let $v, w \in T_p\mathbf{R}X$ for some $p \in \mathbf{R}X$. Then:

$$\omega_p(v,w) = -g_p(J_p v, w) = 0,$$

since $J_p v \perp w$. ■

Corollary 3.48. *The normal Euler number of* $\mathbf{R}X$ *is given by:*

$$e(X,\mathbf{R}X) = -\chi(\mathbf{R}X) = -2 - \mathrm{Tr}(c_* : H_2(X;\mathbf{Q}) \to H_2(X;\mathbf{Q})).$$

Proof. The only thing which is not an immediate consequence is the statement involving the trace of the operator c_* on $H_2(X;\mathbf{Q})$. This is the content of the Lefschetz fixed-point theorem, stating that for the isometry c, we have:

$$\chi(\mathrm{Fix}(c)) = \sum_{k=0}^{4} (-1)^k \mathrm{Tr}(c_* : H_k(X;\mathbf{Q}) \to H_k(X;\mathbf{Q})),$$

where $\mathrm{Tr}(c_* : H_0(X;\mathbf{Q}) \to H_0(X;\mathbf{Q})) = 1$ since X is connected and $\mathrm{Tr}(c_* : H_4(X;\mathbf{Q}) \to H_4(X;\mathbf{Q})) = 1$ since c preserves orientation. ■

3.4.4 Flexible and Totally Flexible Curves in This Setting

We propose the following definition of flexible curve, which englobes all examples seen so far.

Definition 3.49. *Let* (X,c) *be a* 4*-manifold with a conjugation. A* ***flexible curve*** *in* X *is a closed, connected and oriented surface* $F \subset X$ *smoothly embedded such that:*

(1) $c(F) = F$*;*

(2) $[F] \neq 0 \in H_2(X;\mathbf{Z})$*;*

(3) F *satisfies the Adjunction formula (Theorem 2.42):* $\chi(F) = \langle c_1(X), [F]\rangle - Q_X(F,F)$*;*

(4) if $\mathbf{R}F = F \cap \mathbf{R}X$*, then for all* $x \in \mathbf{R}F$*, we have* $T_xF = T_x\mathbf{R}F \oplus J \cdot T_x\mathbf{R}F$*.*

In order to find restrictions on the isotopy type of $\mathbf{R}F$ in $\mathbf{R}X$, it may be a good starting point to look at the quotient X/c. By Proposition 2.29, the quotient X/c can be endowed with the structure of a smooth manifold, and the quotient map $p : X \to X/c$ is a double branched cover. We set $S_X = X/c$ and $\mathcal{R} = p(\mathbf{R}X)$. We will further assume that $H_1(X;\mathbf{Z}/2) = 0$. Of course, we can re-do all the computations we did so far to derive the following. Note that

$$\chi(\mathbf{R}X) \equiv \chi(X) \equiv b_2(X) \equiv \sigma(X) \ [2]$$

by Corollary 2.20, and thus the quantities involved are indeed integers.

(1) $b_1(S_X;\mathbf{Z}/2) = b_3(S_X;\mathbf{Z}/2) = 0$.

(2) $\chi(S_X) = \frac{\chi(X)+\chi(\mathbf{R}X)}{2}$.

(3) $b_2(S_X) = \frac{b_2(X)+\chi(\mathbf{R}X)}{2} - 1$.

(4) $e(S_X,\mathcal{R}) = 2e(X,\mathbf{R}X) = -2\chi(\mathbf{R}X)$.

(5) $\sigma(S_X) = \frac{\sigma(X)-\chi(\mathbf{R}X)}{2}$.

(6) $b_2^+(S_X) = \frac{b_2^+(X)-1}{2}$.

We see that $b_2^+(X)$ is therefore odd, since $b_2^+(S_X)$ is an integer, by all those computations. This recovers the following more general fact.

Proposition 3.50. *Let (X, J) be an almost complex 4-manifold with $b_1(X) = 0$ (and in particular if $b_1(X; \mathbf{Z}/2) = 0$). Then $b_2^+(X)$ is odd.*

Proof. We use the generalization of the Hodge numbers to almost complex manifolds described in [CW22]. From their Proposition 3.4, we obtain:

$$\sigma(X) = \widetilde{\sigma} + 4(Td - \chi_h),$$

with Td the top Todd class, $\chi_h = h^{0,2} - h^{0,1} + h^{0,0}$ the holomorphic Euler characteristic and

$$\widetilde{\sigma} = \sum_{p,q=0}^{2} (-1)^q h^{p,q}.$$

By using the so-called *degeneration property*

$$b_k(X) = \sum_{p+q=k} h^{p,q}$$

and Serre duality $h^{2-p,2-q} = h^{p,q}$, this gives, after simplifications:

$$\widetilde{\sigma} = 2 + 2h^{1,0} + 2h^{2,0} - 2h^{0,1} - h^{1,1} \text{ and } b_2(X) = 2h^{2,0} + h^{1,1}.$$

Solving for $2b_2^+(X) = b_2(X) + \sigma(X)$, we obtain:

$$b_2^+(X) = 2h^{2,0} + 1 + h^{1,0} - h^{0,1} + 2(Td - \chi_h).$$

From $b_1(X) = h^{0,1} + h^{1,0} = 0$, the claim follows. ∎

As a remark, a corollary of this fact is that the connected sum $n\mathbf{CP}^2 \# m\overline{\mathbf{CP}}^2$ does not admit an almost-complex structure unless n is odd.

In particular, in the cases where $X = (\mathbf{CP}^2, \text{conj})$, $(\mathbf{CP}^1 \times \mathbf{CP}^1, c_{\text{hyp}})$ or $(\mathbf{CP}^1 \times \mathbf{CP}^1, c_{\text{ell}})$, we had $b_2^+(X) = 1$, so $b_2^+(S_X) = 0$, meaning that S_X was definite negative. However, for X a K3 surface for instance, we have $b_2^+(X) = 3$, and thus $b_2^+(S_X) = 1$.

Example 3.51. *Given $n \in \mathbf{N}$, the n-th Hirzebruch surface Σ_n is a complex surface given as a projective bundle:*

$$\Sigma_n = \mathbf{P}^1(\mathcal{O} \oplus \mathcal{O}(-n)).$$

It is a simply-connected 4-manifold with $b_2(\Sigma_n) = 2$. In some basis $H_2(\Sigma_n; \mathbf{Z}) = \mathbf{Z}\langle f, s\rangle$, we have:

$$Q_{\Sigma_n} = \begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix}.$$

The first Chern class is given by:

$$\langle c_1(\Sigma_n), f\rangle = 2 \text{ and } \langle c_1(\Sigma_n), s\rangle = n + 2.$$

We consider the actual complex conjugation on Σ_n. A flexible curve $F \subset \Sigma_n$ whose homology class is $(a, b) = af + bs$ has

$$\chi(F) = 2a + 2b - 2ab + nb - nb^2 \text{ and } e(\Sigma_n, F) = 2ab + nb^2.$$

We have $b_2^+(\Sigma_n) = 1$ and $\sigma(\Sigma_n) = 0$, so the quotient manifold $S_{\Sigma_n} = \Sigma_n/\text{conj}$ is a $\mathbf{Z}/2$-homology sphere.

Note that as complex manifolds, they are all pairwise distinct, but as smooth manifolds, we have

$$\Sigma_m \cong \Sigma_n \iff m \equiv n \ [2],$$

and $\Sigma_0 \cong \mathbf{CP}^1 \times \mathbf{CP}^1$ and $\Sigma_1 \cong \mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$.

We may want to also consider *totally* flexible curves. When we looked at the situation in $\mathbf{CP}^2$, we turned our attention to the focal set of $\mathbf{RP}^2$. This happened to be an algebraic (let alone, a flexible) conic with empty real part, and this is the feature that we actually cared about. If one looks at the focal set of $\mathbf{R}X$ when $X = (\mathbf{CP}^1 \times \mathbf{CP}^1, c_{\text{hyp}})$ for instance, we find that this focal set is four isolated points. Enlarging the definition, we could simply look at the set of singular points for the (squared) distance function to $\mathbf{R}X$. In the case of the hyperboloid again, we see that this time, we get a singular curve with bidegree $(2,2)$, empty real part and four double points (those double points being the focal set of $\mathbf{RP}^1 \times \mathbf{RP}^1$). If we smooth the singularities, we obtain a smooth curve $\mathfrak{Q}$ with $\mathbf{R}\mathfrak{Q} = \varnothing$ and bidegree $(2,2)$.

Definition 3.52. *Let (X,c) be a 4-manifold with a conjugation and such that $H_1(X;\mathbf{Z}/2) = 0$. A **focal conic**[†] for X is a flexible curve $\mathfrak{Q} \subset X$ such that:*

(1) $\mathbf{R}\mathfrak{Q} = \varnothing$;

(2) letting F be the free part of $H_2(X;\mathbf{Z})$, where $r = b_2(X) = \mathrm{rk}(F)$, there exists a basis $(\xi_1,\dots,\xi_r)$ of F such that $[F] = 2\xi_1 + \cdots + 2\xi_r$.

Example 3.53. *In the case of the Hirzebruch surfaces Σ_n, introduced previously, we can pick a $\mathfrak{Q}$ to be a bidegree $(2,2)$ curve, whose homology class will therefore be $2f+2s$. In this case, we have:*

$$\chi(\mathfrak{Q}) = -2n \text{ and } e(\Sigma_n, \mathfrak{Q}) = 4n+8.$$

We also have, letting $\mathcal{Q} = p(\mathfrak{Q})$:

$$\chi(\mathcal{Q}) = -n \text{ and } e(S_{\Sigma_n}, \mathcal{Q}) = 2n+4.$$

Since $H_2(S_{\Sigma_n};\mathbf{Z}/2) = 0$, we can always consider the double branched cover $\overline{\Sigma}_n$ of S_{Σ_n} ramified along $\mathcal{Q}$. We have that $\mathfrak{R} = \mathbf{R}\Sigma_n = \mathrm{Fix}(\mathrm{conj})$ is either a torus (when n is even) or a Klein bottle (when n is odd). Therefore: $\chi(\mathfrak{R}) = e(\Sigma_n, \mathfrak{R}) = 0$. We obtain the situation depicted in Figure 3.17.

The 4-manifold $\overline{\Sigma}_n$ has $b_1(\overline{\Sigma}_n;\mathbf{Z}/2) = 0$ by Proposition 2.37, has $b_2(\overline{\Sigma}_n) = n+2$ by Theorem 2.19, and has $\sigma(\overline{\Sigma}_n) = -(n+2)$ by Theorem 2.36. In particular, it is definite negative (and this its intersection form is diagonalizable, by Donaldson's theorem), and hence is a $\mathbf{Z}/2$-homology $(n+2)\overline{\mathbf{CP}}^2$. Moreover, by Lemma 2.39, the surface $\overline{\mathfrak{R}}$ is a characteristic surface for $\overline{\Sigma}_n$.

Definition 3.54. *Let (X,c) be a 4-manifold with a conjugation, with $H_1(X;\mathbf{Z}/2) = 0$ and with a fixed focal conic $\mathfrak{Q}$. A flexible curve $F \subset X$ is called **totally flexible** if*

$$\#F \pitchfork \mathfrak{Q} = |Q_X(F,\mathfrak{Q})|.$$

[†] The terminology can be slightly misleading, since $\mathfrak{Q}$ is neither a conic nor the focal set of $\mathbf{R}X$! But it seems that no confusion is possible, so we stick to it, for lack of a better name.

$$\begin{array}{lccccccl}
e(\Sigma_n,\mathfrak{R})=0 & \mathfrak{R} & \subset & \Sigma_n & \supset & \mathfrak{Q} & & e(X,\mathfrak{Q})=4n+8\\
& \| & & \downarrow p & & \downarrow & & \\
e(S_{\Sigma_n},\mathcal{R})=0 & \mathcal{R} & \subset & S_{\Sigma_n} & \supset & \mathcal{Q} & & e(S_{\Sigma_n},\mathcal{Q})=2n+4\\
& \uparrow\uparrow & & \tilde{p}\uparrow & & \| & & \\
e(\overline{\Sigma}_n,\overline{\mathfrak{Q}})=0 & \overline{\mathfrak{Q}} & \subset & \overline{\Sigma}_n & \supset & \overline{\mathfrak{R}} & & e(\overline{\Sigma}_n,\overline{\mathfrak{R}})=n+2
\end{array}$$

Figure 3.17. The geometric situation in the case of the Hirzebruch surface.

Example 3.55. *For the Hirzebruch surfaces Σ_n with the focal conic being a bidegree $(2,2)$ curve, we have that a flexible curve F of bidegree (a,b) is totally flexible if and only if $F \pitchfork \mathfrak{Q}$ is $2(a+b)+2nb$ points. This means that $p(F)\pitchfork\mathcal{Q}$ is $a+b+n$ points.*

We would like to understand the unbranched covering $\tilde{p}:\overline{\mathfrak{Q}}\to\mathcal{R}$. For this, we do what we did so far: consider either $L_{(1,0)}$ a generic $(1,0)$ curve, or $L_{(0,1)}$ a $(0,1)$ curve. We have:

$$\#L_{(1,0)}\pitchfork\mathfrak{Q}\equiv Q_X(L_{(1,0)},\mathfrak{Q})\equiv 2\ [2] \text{ and } \#L_{(0,1)}\pitchfork\mathfrak{Q}\equiv Q_X(L_{(0,1)},\mathfrak{Q}\equiv=2n+2\ [2].$$

Moreover, $L_\bullet$ is topologically a 2-sphere, and $\mathbf{R}L_{(1,0)}$ and $\mathbf{R}L_{(0,1)}$ are two curves which span $H_1(\mathfrak{R};\mathbf{Z})$ together. Looking at the quotient, we have that $p(L_\bullet)$ is a disc with boundary $\partial p(L_\bullet)=p(\mathbf{R}L_\bullet)$, together with:

$$\#p(L_{(1,0)})\pitchfork\mathcal{Q}\equiv 1\ [2] \text{ and } \#p(L_{(0,1)})\pitchfork\mathcal{Q}\equiv n+1\ [2].$$

In particular, by Proposition 2.33, we see that for the lifts, we have:

(1) $\tilde{p}^{-1}(p(\mathbf{R}L_{(1,0)}))$ is always connected;

(2) $\tilde{p}^{-1}(p(\mathbf{R}L_{(0,1)}))$ is connected when n is even (that is, when $\mathfrak{R}$ is a torus) and disconnected when n is odd (that is, when $\mathfrak{R}$ is a Klein bottle).

This means that the covering $\tilde{p}:\overline{\mathfrak{Q}}\to\mathcal{R}$ is always path-connected, and is either the same as in the case of the hyperboloid when n is even, or the oriented cover of a Klein bottle when n is odd.

However, it seems that in all generality, knowing what type of double covering $\tilde{p}:\overline{\mathfrak{Q}}\to\mathcal{R}$ is will prove to be difficult. Indeed, so far, we relied on:

(1) the existence of flexible curves L_i whose real parts $\mathbf{R}L_i$ were connected and spanned the whole $H_1(\mathbf{R}X;\mathbf{Z})$ together;

(2) the fact that those flexible curves were topologically 2-spheres, and thus in the quotient, that the surfaces $p(L_i)$ were 2-discs;

(3) the fact that those discs intersected $\mathcal{Q}$ in their interior in a number of points with suitable parity.

Again, in the case of the Hirzebruch surface, this worked just fine, but required a special study of this specific family of complex surfaces. In fact, another problem arises that was invisible in the case of Σ_n: is it always true that $[\mathcal{Q}]=0\in H_2(S_X;\mathbf{Z}/2)$? This step is required to show that the double branched cover $\tilde{p}:(\overline{X},\overline{\mathfrak{R}})\to(S_X,\mathcal{Q})$ actually exists.

Sometimes, we will be in luck, and:

(1) the double unbranched covering $\widetilde{p}:\overline{\Omega}\to\mathcal{R}$ is connected;
(2) the subset $p(\mathbf{R}F)\subset\mathcal{R}$ is not null-homologous, but belongs to the subgroup corresponding to the $\widetilde{p}:\overline{\Omega}\to\mathcal{R}$.

In this case, there will be two subsets $\overline{\Omega}_\pm$ of $\overline{\Omega}$ which have $\partial\overline{\Omega}_\pm=\widetilde{p}^{-1}(p(\mathbf{R}F))$. We therefore define the **Arnold surface** of the associated curve to be $\mathcal{A}(F)=\overline{\Omega}_+\cup\widetilde{p}^{-1}(p(F))\subset\overline{X}$.

One last important step is to understand the homology classes of $\mathcal{A}(F)$ and of $\overline{\mathfrak{R}}$ in $H_2(\overline{X};\mathbf{Z}/2)$. In the lucky scenario where they are homologous, we can consider their union, perform a smoothing of the singularities to obtain a null-homologous surface $\mathcal{X}(F)\subset\overline{X}$ and take the double branched cover $\Theta:Y\to(\overline{X},\mathcal{X}(F))$. Computing $b_2^+(Y)$ should be easy at this point, and there should be an inequality appearing. In the case of Σ_n, assuming that all goes right, we obtain

$$b_2^+(Y)=\frac{ab+1}{2}+\frac{n(b+1)^2}{4}.$$

However, showing that $\mathcal{A}(F)$ is also homologous to $\overline{\mathfrak{R}}$ in $\overline{\Sigma}_n$ is probably difficult, simply because the rank of $H_2(\overline{\Sigma}_n;\mathbf{Z}/2)$ is quite large.

Question 3.56. *Can the above method be made to work and derive that for a totally flexible curve $F\subset\Sigma_n$ of bidegree (a,b), where a and b are both odd, we have*

$$\ell^0+\ell^-\leqslant\frac{ab+1}{2}+\frac{n(b+1)^2}{4}?$$

We now summarize the method in a step-by-step guide. What is given is the 4-manifold X together with its conjugation $c:X\to X$.

(1) Pick a focal conic $\mathfrak{Q}$ for (X,c).
(2) Prove that $p(\mathfrak{Q})$ is null-homologous in $H_2(S_X;\mathbf{Z}/2)$.
(3) Denote as $\widetilde{p}:(\overline{X},\overline{\mathfrak{Q}})\to(S_X,\mathcal{Q})$ the double branched covering. Describe fully the unbranched covering $\widetilde{p}:\overline{\Omega}\to\mathcal{R}$.
(4) Show that, up to a condition on $[F]$, we have that $\widetilde{p}^{-1}(p(\mathbf{R}F))\subset\overline{\Omega}$ is the boundary of two diffeomorphic subsets $\overline{\Omega}_\pm$.
(5) Defining the Arnold surface as $\mathcal{A}(F)=\overline{\Omega}_+\cup\widetilde{p}^{-1}(p(F))$, show that $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ are homologous mod 2 and not null-homologous.
(6) Consider the surface $\mathcal{X}(F)$ obtained from $\mathcal{A}(F)\cup\overline{\mathfrak{R}}$, and compute $b_2^+(Y)$ for Y the double branched cover of $\overline{X}$ ramified along $\mathcal{X}(F)$ to derive an upper bound.

Example 3.57. *We studied the case of plane curves and of curves on a quadric. A natural continuation of the work would be about curves on a cubic. There are five families of such real cubic surfaces (see for instance Viro's lecture notes [Vir00, §5.4]).*

(1) The first one is obtained by blowing-up $\mathbf{CP}^2$ at 6 (distinct) real points. This defines a smooth 4-manifold $X_{6,0}\cong\mathbf{CP}^2\#6\overline{\mathbf{CP}}^2$ together with a conjugation $c_{6,0}:X_{6,0}\to X_{6,0}$ with $\mathrm{Fix}(c_{6,0})=\mathbf{RP}^2\#6\mathbf{RP}^2=7\mathbf{RP}^2$.

(2) *The second one is obtained by blowing up* $\mathbf{CP}^2$ *in* 4 *real points and* 2 *complex conjugate ones. We obtain a conjugation* $c_{4,1}$ *with* $\mathbf{R}X_{4,1} = 5\mathbf{RP}^2$.

(3) *The third is* $X_{2,2}$ *the blow-up of* $\mathbf{CP}^2$ *in two real points and in two more pairs of complex conjugate points. The associated conjugation* $c_{2,2}$ *has* $\mathrm{Fix}(c_{2,2}) = 3\mathbf{RP}^2$.

(4) *The fourth* $X_{0,3}$ *is the blow-up of* $\mathbf{CP}^2$ *in three pairs of complex conjugate points, and the conjugation* $c_{0,3}$ *obtained is such that* $\mathbf{R}X_{0,3} = \mathbf{RP}^2$.

(5) *The fifth and last one* X_2 *is defined by an algebraic equation. The affine equation is* $2x^3 + y^3 + z^3 - x^2 + y^2 + z^2 + \lambda = 0$, *where* $0 < \lambda < 1/27$. *The associated conjugation induced by* $\mathrm{conj} : \mathbf{CP}^3 \to \mathbf{CP}^3$ *is* $c_2 : X_2 \to X_2$, *and it has* $\mathrm{Fix}(c_2) \cong \mathbf{S}^2 \sqcup \mathbf{RP}^2$.

Letizia's work [Let84] also proves that the quotient of a real cubic surface by its induced conjugation is either the smooth 4-sphere, or some number of copies of $\overline{\mathbf{CP}}^2$. This means that in this case, the associated 4-manifold S_X is not merely a $\mathbf{Z}/2$-homology $n\overline{\mathbf{CP}}^2$, but an *actual* one.

More generally, given r and s two integers, it is possible to consider the complex surface $X_{r,s}$ obtained by blowing-up $\mathbf{CP}^2$ in r distinct real points and in s pairs of complex conjugate points. The induced conjugation $c_{r,s} : X_{r,s} \to X_{r,s}$ has $\mathrm{Fix}(c_{r,s}) = (r+1)\mathbf{RP}^2$, and $X_{r,s}$ has $b_2(X_{r,s}) = 2s + r + 1$. The cubics described before correspond to the cases $r + 2s = 6$.

Curves on quartic surfaces, on quintic surfaces or on Del Pezzo surfaces are all natural classes of examples we did not have time to investigate.

Some Digressions

4

Многие, которым никогда не представлялось случая более узнать математику, смешивают ее с арифметикой и считают ее наукой сухой и бесплодной. В сущности же это наука, требующая наиболее фантазии, и... нельзя быть математиком, не будучи в то же время и поэтом в душе.[†]

S. Kovalevskaya

Outline of this chapter

[†] *Many who have never had the opportunity to learn more about mathematics confuse it with arithmetic and consider it a dry and sterile science. In essence, this is a science that requires the most imagination, and... you cannot be a mathematician without at the same time being a poet at heart.*

4.1 Non-orientable Flexible Curves

In this section, we discuss the possibility of a new topological object to study: non-orientable flexible curves.

Definition 4.1. *Let $F \subset \mathbf{CP}^2$ be a closed, connected and non-orientable smoothly embedded. F is called a* ***non-orientable flexible curve*** *of* ***degree*** *m and* ***genus*** *g if:*

(1) $\mathrm{conj}(F) = F$*;*

(2) $e(\mathbf{CP}^2, F) = m^2$*;*

(3) $\chi(F) = 2-2g$*;*

(4) if $\mathbf{R}F = F \cap \mathbf{RP}^2$*, then for all* $x \in \mathbf{R}F$*,* $T_xF = T_x\mathbf{R}F \oplus \mathbf{i}\cdot T_x\mathbf{R}F$*.*

In contrast with the usual definition of a flexible curve, we cannot ask the degree to be related to the integral homology class of F, since there is none. However, we *do* have the equivalence, when $\Sigma \subset \mathbf{CP}^2$ is orientable:

$$[\Sigma] = \pm m[\mathbf{CP}^1] \iff e(\mathbf{CP}^2, \Sigma) = m^2.$$

Since the right-hand side still makes sense for non-orientable surfaces as well, it seems like a natural generalization.

Regarding the genus-degree formula, we saw in §2.3.2 that the orientable Thom conjecture and its non-orientable counterpart disagree considerably. However, Proposition 2.52 ensures that if $e(\mathbf{CP}^2, F) = m^2$, then $\chi(F)$ is necessarily even. This justifies asking that $\chi(F) = 2-2g$. Note that the non-orientable genus (the number of cross-caps) of F is actually $2g$.

This also ensures that the fourth property regarding $TF|_{\mathbf{R}F}$ makes sense. Indeed, this implies that $\mathbf{R}F$ is a collection of embedded circles. However, as will be emphasized in §4.2.2, there may be isolated fixed points for involutions on non-orientable surfaces (the complex conjugation *is* such an involution on F). In fact, if $\chi(F)$ is odd, Corollary 2.20 affirms that there has to be an odd number of isolated fixed points, and thus at least one. On the contrary, if $\chi(F)$ is even, then there are involutions with no isolated fixed points[†].

In fact, we have the following.

Proposition 4.2. *There exist non-orientable flexible curves in all degrees.*

Proof. Consider a usual flexible curve of degree m. Take a local Klein bottle $K\ell \subset \mathbf{CP}^2$ embedded away from $F \cup \mathbf{RP}^2$ and with $e(\mathbf{CP}^2, K\ell) = 0$, and form the connected sum $F' = F\#K\ell\#\mathrm{conj}(K\ell)$ by using small tubes. This satisfies all items of the definition, with $\chi(F') = \chi(F) - 4$ (and thus the genus of F' as a flexible curve is that of F plus two). ∎

Another possibility to construct non-orientable flexible curve with prescribed real scheme could be:

(1) consider that real scheme Λ on $\mathcal{R} = p(\mathbf{RP}^2)$;

[†] For instance, set $F \cong k\mathbf{RP}^2\#k\mathbf{RP}^2$, where $\chi(F) = 2-2k$, and define the involution that swaps both $k\mathbf{RP}^2$ summands. We obtain only one circle of fixed points which is the equator of the sphere used to form the connected sum.

(2) find a non-orientable surface Σ with $\partial\Sigma = \Lambda$ and such that the intersection between Σ and $\mathcal{R}$ is composed of circles of Λ (and locally Σ intersects $\mathcal{R}$ neatly along those circles) and isolated points (and locally Σ intersects $\mathcal{R}$ transversely);
(3) lift the surface Σ into $\widetilde{\Sigma} = p^{-1}(\Sigma) \subset \mathbf{CP}^2$, which will be nodally immersed if there are transverse intersection points between $\mathcal{R}$ and Σ;
(4) resolve those nodes by gluing a Hopf band in a way that removes all intersections with $\mathbf{RP}^2$ (locally, the situation is the same as an algebraic acnode curve being desingularized into a conic with empty real part).

In particular, the construction $F\#K\ell\#\mathrm{conj}(K\ell)$ described previously is of this form, where we take $\Sigma = (F/\mathrm{conj})\#K\ell \subset \mathbf{S}^4$ with no transverse intersection points between Σ and $\mathcal{R}$.

The self-intersection of F we obtain will be related to that of Σ and to the number of smoothings we perform. More precisely, if $\Sigma \cap \mathcal{R}$ consists of r isolated points, then:

$$e(\mathbf{CP}^2, F) = 2e(\mathbf{S}^4, \Sigma) + 2r.$$

Moreover, we also have

$$\chi(F) = 2\chi(\Sigma) - 2r.$$

This means that this construction is costly regarding to $\widetilde{g}_{\mathbf{CP}^2}$. In particular, we were not able to find non-orientable flexible curves of a fixed degree with lower genus than those obtained by $F\#K\ell\#\mathrm{conj}(K\ell)$. This rises the following question.

Question 4.3. *Let*

$$\widetilde{h}(m) = \max_F \chi(F),$$

where the maximum ranges over all non-orientable flexible curves of degree m. Then what is $\widetilde{h}(m)$ compared to $\widetilde{g}_{\mathbf{CP}^2}(m^2)$?

In particular, do we have $\widetilde{h}(m) = -m^2 + 3m - 4$? Knowing this is important, and it should be part of the definition, in order to prevent unwanted behaviour such as the following.

Example 4.4. *There exists a non-orientable flexible cubic F with $b_0(\mathbf{R}F) = 7$. It has flexible genus $g = 10$. Indeed, consider an algebraic M-quintic F_0, and take Σ to be a local surface embedded away from $\mathbf{C}F_0 \cup \mathbf{RP}^2$ with $e(\mathbf{CP}^2, \Sigma) = -8$ and $\Sigma \cong 2K\ell$. Form the connected sum $F = F_0\#\Sigma\#\mathrm{conj}(\Sigma)$.*

This construction was possible also in the usual case of flexible curves, by taking the connected sum with null-homologous torus and its image under conjugation. This means that the condition that $\chi(F) = \widetilde{h}(m)$ should be part of the definition.

Despite all those genus considerations, we can investigate what results hold for non-orientable curves, and which ones do no longer.

(1) $\mathbf{R}F \subset \mathbf{RP}^2$ is contractible if and only if m is even, and if m is odd, then $\mathbf{R}F$ contains exactly one pseudo-line.
(2) The Harnack upper bound becomes $b_0(\mathbf{R}F) \leqslant g + 1$, which follows from Theorem 2.21.

(3) Type I curves are defined by the property that $F \smallsetminus \mathbf{R}F$ is disconnected, whereas type II curves are defined by the property that $F \smallsetminus \mathbf{R}F$ is connected. There is no longer the interpretation that F is type I if and only if F/conj is orientable. Indeed, taking $F' = F\#K\ell\,\mathrm{conj}(K\ell)$ for F a type I algebraic curve gives that F' is type I, but the quotient F'/conj is diffeomorphic to $(F/\mathrm{conj})\#K\ell$, which is non-orientable.

(4) Type I curves do no longer come with a semi-orientation.

(5) The Arnold surfaces of even and odd degree curves are still well-defined.

(6) The Viro–Zvonilov inequality does not hold for odd degree curves, since there is no branched cover other than the doubly-sheeted one if the ramification locus is non-orientable. However, Theorem 3.20 still holds and becomes

$$\ell^0 + \ell^- \leqslant -\frac{\chi(F)}{2} - \frac{m^2-1}{4} + m.$$

(7) If F is an M-curve of even degree $m = 2k$ (that is, $b_0(\mathbf{R}F) = g+1$), then F/conj is still a 2-sphere (indeed, it has $\chi(F/\mathrm{conj}) = 1-g$ and $g+1$ boundary components, so filling those with discs we obtain a surface with $\chi = 2$). This means that the Gudkov–Rokhlin congruence $p - n \equiv k^2$ [8] from Theorem 1.32(2) still holds.

(8) Similarly, if F is an even degree $(M-1)$-curve, the surface F/conj is a punctured projective plane (there is only one surface with $\chi = 1$), and thus the Gudkov–Krakhnov–Kharlamov congruence $p - n \equiv k^2 \pm 1$ [8] also holds.

Non-orientable flexible curves on other 4-manifolds might pose even more problems. Indeed, in the orientable case, there is no characterization of the integral homology class solely in terms of the self-intersection of the surface. For instance, if $a \neq b$, then any pair of curves of bidegree (a,b) and (b,a) on the hyperboloid quadric have the same self-intersection equal to $2ab$. On the ellipsoid quadric, we still have the equivalence

$$[F] = \pm(m,m) \iff e(\mathbf{CP}^1 \times \mathbf{CP}^1, F) = 2m^2.$$

We would need a better way to "select" integral homology classes in terms of characteristic classes of the normal bundle of the surface[†].

[†] Just like we only defined the non-orientable genus function $\tilde{g}_X$ of a 4-manifold X solely in terms of the self-intersection of non-orientable surfaces, where we would have liked a finer invariant

4.2 Classification of Involutions on Surfaces

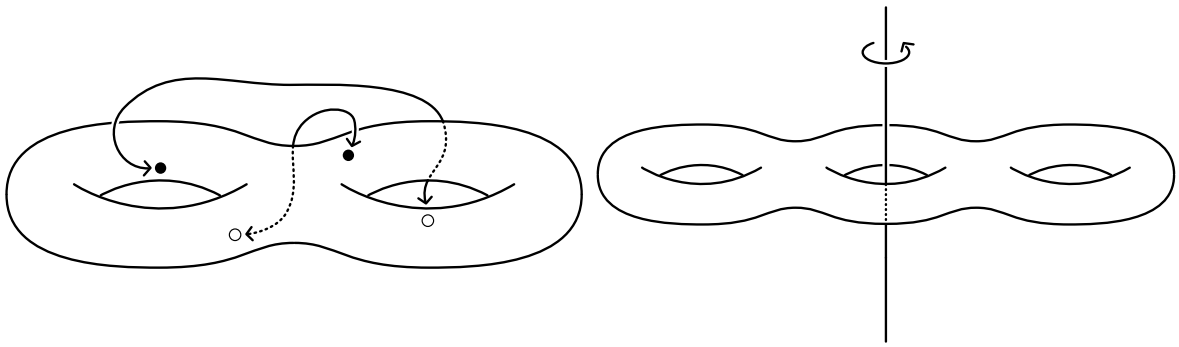
In order to study symmetric curves, we will need a better understand of all possible involutions on closed connected surfaces (orientable or not). We will reformulate the statements from [Dug19], which surveys modern proofs using equivariant surgery, as well as interpret them in terms of the fix-point set of those involutions.

We will denote as Σ_g the closed connected orientable surface of genus g (that is, $\chi(\Sigma_g) = 2-2g$). Similarly, $\widetilde{\Sigma}_h$ will denote the non-orientable surface of genus h (that is, a sphere with h cross-caps, such that $\chi(\widetilde{\Sigma}_h) = 2-h$).

4.2.1 Orientable Surfaces

We start with the case of orientable surfaces. It is well-known that the 2-sphere has 4 involutions: the identity, the antipodal map, rotation along an axis and reflection along an equatorial plane. We focus to the case of orientable surfaces Σ_g, $g \geqslant 1$.

Consider the standard embedding $\Sigma_g \subset \mathbf{R}^3$, which is symmetric along reflection by any of the xy, xz and yz planes, by rotation around any of the x, y and z axes, as well as the antipodal map $x \mapsto -x$. The origin of $\mathbf{R}^3$ is therefore the barycenter of Σ_g, and this barycenter is "inside" the surface if and only if g is even (otherwise, it is "outside"). The antipodal map $x \mapsto -x$ restricts to a free involution $T_g^{\text{anti}} : \Sigma_g \to \Sigma_g$. In the case where g is odd, the rotation along the z-axis also restricts to a free involution $T_g^{\text{rot}} : \Sigma_g \to \Sigma_g$. We refer the reader to Figure 4.1 for a depiction of those involutions.

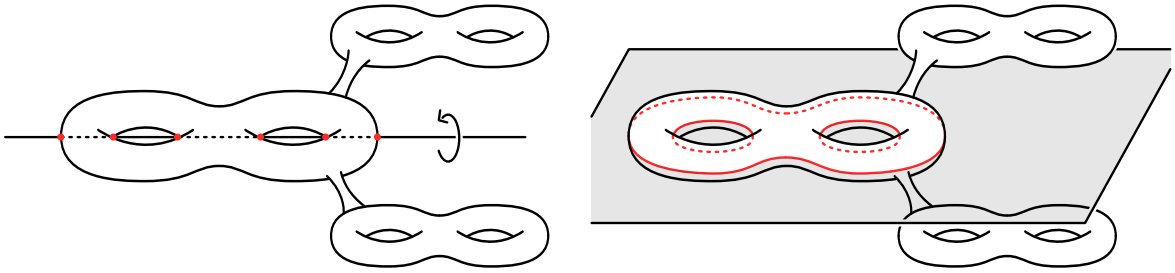


(a) (b)

Figure 4.1. (a) The involution T_g^{anti} for $g = 2$. (b) The involution T_g^{rot} for $g = 3$.

For any $1 \leqslant r \leqslant \left\lfloor \frac{g}{2} \right\rfloor$, we consider the involutions $T_{g,r}^{\text{spit}} : \Sigma_g \to \Sigma_g$ and $T_{g,r}^{\text{refl}} : \Sigma_g \to \Sigma_g$ given respectively by rotation along an axis and reflection along a plane; see Figure 4.2. The fix-point sets $\text{Fix}(T_{g,r}^{\text{spit}})$ and $\text{Fix}(T_{g,r}^{\text{refl}})$ consist respectively of $2+2g-4r$ isolated points and of $1+g-2r$ circles.

Finally, for $1 \leqslant k \leqslant g$, there is an involution $T_{g,k}^{\text{tube}} : \Sigma_g \to \Sigma_g$ which is constructed in the following manner. Start with the antipodal involution $T_{g-k}^{\text{anti}} : \Sigma_{g-k} \to \Sigma_{g-k}$, and choose k discs $D_1, \ldots, D_k \subset \Sigma_{g-k}$



(a) (b)

Figure 4.2. (a) The involution $T^{\text{spit}}_{g,r}$ for $g = 6$ and $r = 2$. (b) The involution $T^{\text{refl}}_{g,r}$ for $g = 6$ and $r = 2$. In both cases, the fix-point set is colored in red.

such that:

$$\forall i \neq j,\ D_i \cap D_j = D_i \cap T^{\text{anti}}_{g-k}(D_j) = D_i \cap T^{\text{anti}}_{g-k}(D_i) = \varnothing.$$

Consider the cylinder $C = \mathbf{S}^1 \times I$ together with the involution

$$\begin{array}{rccc} s & : & C & \longrightarrow & C \\ & & (e^{\mathbf{i}\theta}, x) & \longmapsto & (e^{\mathbf{i}\theta}, 1-x). \end{array}$$

Then, for each $i \in \{1, \dots, k\}$, remove the discs D_i and $T^{\text{anti}}_{g-k}(D_i)$ from Σ_{g-k}, and glue a copy of C together with the action of the involution s. This constructs an involution $T^{\text{tube}}_{g,k} : \Sigma_g \to \Sigma_g$, and $\text{Fix}(T^{\text{tube}}_{g,k})$ consists of k circles. This time, the involutions $T^{\text{tube}}_{g,k}$ do *not* extend to involutions of $\mathbf{R}^3$ in which Σ_g embeds (simply because such an involution would necessarily have a fix-point set separating Σ_g), so it is not possible to make a depiction of their action on Σ_g.

Theorem 4.5 ([Dug19, Theorem 1.7])**.** *There are $2g+4$ isomorphism types of involutions on Σ_g, given by:*

(1) free involutions: the identity, the antipodal map T^{anti}_g, and if g is even, the rotation T^{rot}_g;

(2) the rotations $T^{\text{spit}}_{g,r}$ for $1 \leqslant r \leqslant \lfloor \frac{g}{2} \rfloor$;

(3) the reflections $T^{\text{refl}}_{g,r}$ for $1 \leqslant r \leqslant \lfloor \frac{g}{2} \rfloor$;

(4) the involutions $T^{\text{tube}}_{g,k}$ for $1 \leqslant k \leqslant g$.

We are interested in the following consequences for the fix-point set of involutions on orientable surfaces.

Corollary 4.6. *Let $\tau : \Sigma_g \to \Sigma_g$ be an involution with non-empty fix-point set.*

(1) *If* $\mathrm{Fix}(\tau)$ *contains an isolated fixed point, then* $\mathrm{Fix}(\tau)$ *is only composed of isolated fixed points, and we have:*

$$\#\mathrm{Fix}(\tau) \leqslant 2g+2 \text{ and } \#\mathrm{Fix}(\tau) \equiv 2g+2 \ [4].$$

(2) *If* $\mathrm{Fix}(\tau)$ *contains a circle, then* $\mathrm{Fix}(\tau)$ *is a collection of circles, and* $b_0(\mathrm{Fix}(\tau)) \leqslant g+1$. *Moreover, if* $\mathrm{Fix}(\tau)$ *separates* Σ_g, *then* $b_0(\mathrm{Fix}(\tau)) \equiv g+1$ [2].

We see that the classification recovers Klein's results proving the Harnack bound as well as the Klein congruence. In fact, Theorem 4.5 was already known to him (see [Kle93]).

4.2.2 Non-Orientable Surfaces

We describe the procedure to construct all possible involutions on all possible surfaces, orientable or not. The non-orientable classification is a lot more convoluted than its orientable counterpart, but we are merely concerned with the fix-point sets of the involutions.

Given a surface with a non-trivial involution (Σ, τ), we can construct a new surface Σ' from Σ and an involution τ' on it by the following five distinct operations.

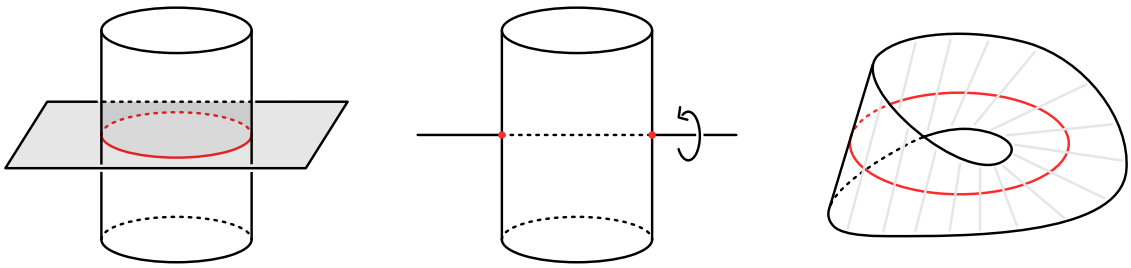
(1) Consider a small disc $\mathcal{D} \subset \Sigma$ such that $\tau(\mathcal{D}) \cap \mathcal{D} = \varnothing$. The surface Σ' is obtained from $(\Sigma \smallsetminus \mathcal{D} \cup \tau(\mathcal{D}))$ by gluing two Möbius bands, and the involution τ' is defined on Σ' by extending τ into the involution freely acting on the two Möbius bands glued and swapping them.

(2) Perform the same operation as above, but glue two conjugate punctured tori instead of two Möbius bands.

(3) If τ has an isolated fixed point $x \in \Sigma$, consider $\mathcal{D} \subset \Sigma$ a small disc centered at x. Denote as M a Möbius band, and let $r : M \to M$ be the involution which is reflection along the core of M (so that $\mathrm{Fix}(r)$ is that core). The surface Σ' is obtained from $\Sigma \smallsetminus \mathcal{D}$ by gluing (M, r) on its boundary.

(4) Let $\mathcal{D} \subset \Sigma$ be a small disc such that $\tau(\mathcal{D}) \cap \mathcal{D} = \varnothing$. Glue an annulus to $\Sigma \smallsetminus \mathcal{D} \cup \tau(\mathcal{D})$, and extend the involution τ by either reflection of the annulus or rotation on it.

In Figure 4.3 we depict the involutions on the annulus and the Möbius band that we use to glue to Σ.



(a) (b) (c)

Figure 4.3. (a) The reflection on the annulus. (b) The rotation. (c) The reflection on the Möbius band. In all cases, the fix-point set is depicted in red.

From [Dug19, §1.6], we have that all possible involutions on surfaces are obtained by performing a sequence of those operations, starting with the 2-sphere with either its antipodal involution, its reflection involution or its rotation involution.

This can be re-interpreted in saying that:

(1) the cost of creating two isolated fixed points is reducing the Euler characteristic by two;
(2) the cost of creating a circle of fixed points is either reducing χ by two, or reducing it by one and trading one isolated fixed point for it;
(3) the cost of transforming an orientable surface into a non-orientable one is reducing the Euler characteristic by two;
(4) the Euler characteristic can be reduced by any even number without changing the topological type of the fix-point set.

We therefore have the following.

Proposition 4.7. *Let $\tau : \Sigma \to \Sigma$ be a non-trivial involution on a non-orientable surface Σ.*

(1) If $\Sigma \cong \mathbf{RP}^2$, then $\mathrm{Fix}(\tau)$ can only be[†] a disjoint union of a circle and an isolated point.
(2) If $\Sigma \cong K\ell$, then $\mathrm{Fix}(\tau)$ can only be: empty, one circle, two circles, two points or two points and one circle.
(3) If $\Sigma \cong 3\mathbf{RP}^2$, then $\mathrm{Fix}(\tau)$ can be either one point and one circle, one point and two circles, or three points and one circle.

[†] In this case, there is only one *involution* rather than merely one possibility for the fix-point set.

4.3 Symmetric Curves and Flexible Symmetric Curves

We consider the following (holomorphic) involution on $\mathbf{CP}^2$:

$$\begin{array}{rccc} \sigma & : & \mathbf{CP}^2 & \longrightarrow & \mathbf{CP}^2 \\ & & [z_0 : z_1 : z_2] & \longmapsto & [-z_0 : z_1 : z_2]. \end{array}$$

This is a real involution in the following sense:

$$\sigma \circ \mathrm{conj} = \mathrm{conj} \circ \sigma.$$

We set $\widetilde{\mathrm{conj}} = \sigma \circ \mathrm{conj}$, which is a conjugation in the sense of §3.4.3, and we let $\widetilde{\mathbf{RP}}^2 = \mathrm{Fix}(\widetilde{\mathrm{conj}})$. Note that

$$\widetilde{\mathbf{RP}}^2 = \left\{ [\mathbf{i}x_0 : x_1 : x_2] \,\middle|\, [x_0 : x_1 : x_2] \in \mathbf{RP}^2 \right\},$$

and if $g_0 = [1:0:0]$ and G_0 is the line $\{z_0 = 0\}$, then:

$$\mathrm{Fix}(\sigma) = \mathbf{RP}^2 \cap \widetilde{\mathbf{RP}}^2 = \mathbf{C}G_0 \sqcup \{g_0\}.$$

In particular, the involution $\sigma|_{\mathbf{RP}^2} : \mathbf{RP}^2 \to \mathbf{RP}^2$ represents the only non-trivial isomorphism type of involutions on $\mathbf{RP}^2$.

Definition 4.8. *The line $\mathbf{C}G_0$ will be called the **axis of symmetry**, and the point g_0 the **center of symmetry**.*

We now give the definition of a symmetric curve, due to Fiedler.

Definition 4.9. *An non-singular real plane algebraic curve A is called **symmetric** if one of the following equivalent conditions hold:*

(1) for all $x_0, x_1, x_1 \in \mathbf{R}$, we have $A(-x_0, x_1, x_2) = A(x_0, x_1, x_2)$;

(2) $\sigma(\mathbf{C}A) = \mathbf{C}A$;

(3) A is a real curve for both real structures conj *and* $\widetilde{\mathrm{conj}}$.

*In this case, we denote as $\widetilde{\mathbf{R}}A = \mathbf{C}A \cap \widetilde{\mathbf{RP}}^2$, and we call $\widetilde{\mathbf{R}}A$ the **mirror curve** of A.*

Of course, in order to generalize to *flexible* curves, one cannot make sense of the first condition. However, it turns out that *a priori*, the other two conditions are not equivalent, and the third is stronger. We therefore propose the following.

Definition 4.10. *Let $F \subset \mathbf{CP}^2$ be a closed, connected and oriented surface. F is said to be a **flexible symmetric curve** of degree m if:*

(1) $\mathrm{conj}(F) = F = \widetilde{\mathrm{conj}}(F)$;

(2) $\chi(F) = -m^2 + 3m$;

(3) $[F] = m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2; \mathbf{Z})$;

(4) if $\mathbf{R}F = F \cap \mathbf{RP}^2$, then for all $x \in \mathbf{R}F$, $T_xF = T_x\mathbf{R}F \oplus \mathbf{i} \cdot T_x\mathbf{R}F$;

(5) *if* $\widetilde{\mathbf{R}}F = F \cap \widetilde{\mathbf{RP}}^2$, *then for all* $y \in \widetilde{\mathbf{R}}F$, $T_yF = T_y\widetilde{\mathbf{R}}F \oplus \mathbf{i}\cdot T_y\widetilde{\mathbf{R}}F$;

(6) $\mathbf{R}F \pitchfork \mathbf{R}G_0$.

In this case, merely asking that a flexible curve satisfies $\sigma(F) = F$ will not ensure that condition (5) is satisfied. Additionally, condition (6) is a technicality which will avoid some technicalities later on. In fact, it only matters in the case of odd degrees and ensures that $\mathbf{R}G_0$ is not a component[†] of $\mathbf{R}F$. In the even degree case, this condition is always satisfied (it is not possible to make a point of tangency between $\mathbf{R}F$ and $\mathbf{R}G_0$ in a symmetric way without self-intersecting).

We will denote as $\mathscr{C}_m^{\mathrm{sym}}$ the space of (possibly-singular) real plane symmetric curves of degree m. In terms of the defining polynomial

$$A(x_0,x_1,x_2) = \sum_{i+j\leqslant m} \lambda_{i,j}\, x_0^i x_1^j x_2^{m-i-j},$$

the condition that A is symmetric corresponds to:

$$A \in \mathscr{C}_m^{\mathrm{sym}} \iff A(-x_0,x_1,x_2) = A(x_0,x_1,x_2) \iff \forall\, i \equiv 1\ [2],\ \lambda_{i,j} = 0.$$

Therefore, one can compute that $\mathscr{C}_m^{\mathrm{sym}} \cong \mathbf{RP}^{\delta}$, where

$$\delta = \left(\sum_{\substack{i+j\leqslant m,\\ i \text{ even}}} 1\right) - 1 = \begin{cases} k(k+2) & \text{if } m = 2k, \\ k(k+3)+1 & \text{if } m = 2k+1. \end{cases}$$

As was the case in §1.1.3, the subspace $\mathrm{Sing}^1(\mathscr{C}_m^{\mathrm{sym}})$ of curves with at least one singularity has codimension one, and the subspace $(\mathscr{C}_m^{\mathrm{sym}})_{1,\mathrm{reg}}$ of symmetric curves with exactly one singularity which is non-degenerate is dense in $\mathrm{Sing}^1(\mathscr{C}_m^{\mathrm{sym}})$. We prove the symmetric version of Harnack's theorem.

Proposition 4.11. *Let F be a non-singular symmetric real plane algebraic curve of degree $m \geqslant 1$. Then:*

$$\frac{1+(-1)^{m+1}}{2} \leqslant b_0(\mathbf{R}F), b_0(\widetilde{\mathbf{R}}F) \leqslant \frac{(m-1)(m-2)}{2} + 1.$$

Moreover, for any k respecting those bounds, there exist symmetric curves with either $b_0(\mathbf{R}F) = k$ or $b_0(\widetilde{\mathbf{R}}F) = k$.

Proof. The bounds for $b_0(\mathbf{R}F)$ were already obtained in Theorem 1.8. The same bounds can be derived for $b_0(\widetilde{\mathbf{R}}F)$, by considering only the curve in $(\mathbf{CP}^2, \widetilde{\mathrm{conj}})$. Moreover, if we can construct a symmetric curve such that $b_0(\mathbf{R}F) = k$, then we can also construct another symmetric one F' with $b_0(\widetilde{\mathbf{R}}F') = k$, simply by considering $F' = \sigma(F)$.

As was the case in the proof Corollary 1.14, it only suffices to construct a symmetric curve with $b_0(\mathbf{R}F)$ maximal, by the fact that $(\mathscr{C}_m^{\mathrm{sym}})_{1,\mathrm{reg}}$ is dense in $\mathrm{Sing}^1(\mathscr{C}_m^{\mathrm{sym}})$. If Trilles states in [Tri03] that the Harnack curve can be constructed in all degrees, it is not clear how the classical perturbation

[†] In the algebraic case, this would imply that the defining polynomial is not irreducible, and thus singular. However, flexibly, we cannot really prevent this quirk.

theorem can be applied to perform such a construction. We rather resort to Viro's patchworking; see [Vir06, Theorem 1.1.D].

Consider the standard triangulation of the triangle

$$\Delta = \left\{ (x, y) \in \mathbf{R}^2 \,\middle|\, x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant m \right\},$$

where Harnack signs are given:

$$\operatorname{sign}(i, j) = +1 \iff i \equiv j \equiv 0\ [2].$$

See Figure 4.4 for a depiction of the signed triangulation and of the result of the combinatorial patchworking.

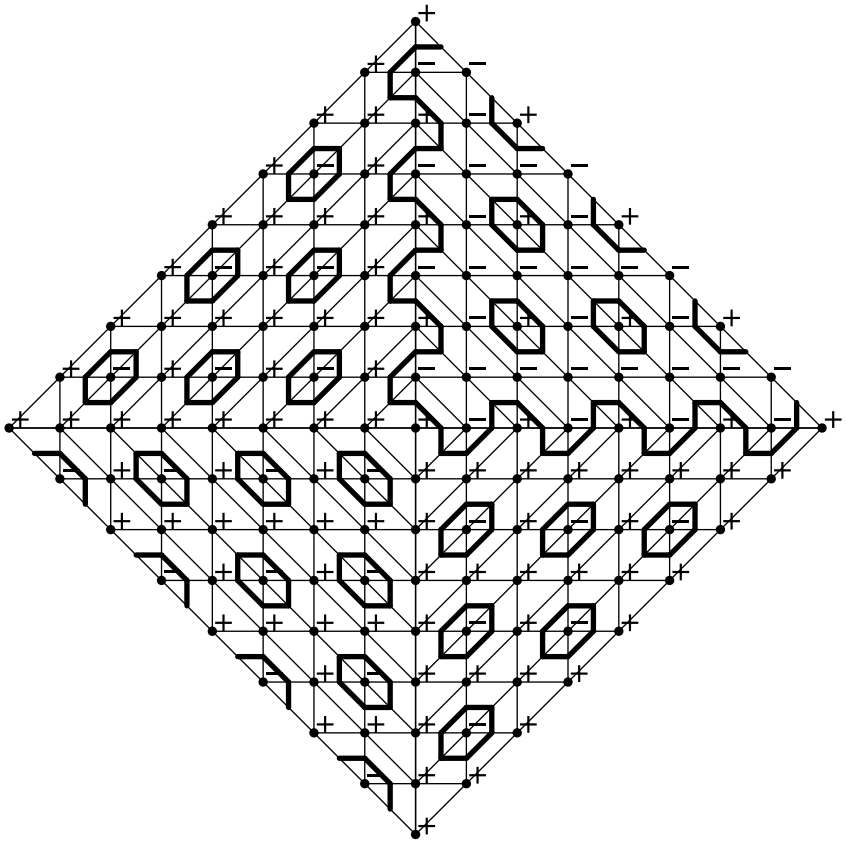


Figure 4.4. Viro's combinatorial patchworking of a degree 8 Harnack curve with real scheme $\langle 18 \sqcup 1\langle 3\rangle\rangle$.

Consider the *convexification* function $\nu : \Delta \to \mathbf{R}$ which is piecewise-linear and defined at vertices of $\mathbf{T}$ by:

$$\nu(i, j) = (i + 1)^2 + (j + 1)^2.$$

Viro's patchworking theorem states that for $\varepsilon > 0$ sufficiently small, the polynomial

$$A(x_0, x_1, x_2) = \sum_{(i,j) \in \mathbf{T}} \operatorname{sign}(i, j) \varepsilon^{\nu(i,j)} x_0^i x_1^j x_2^{m-i-j}$$

will be non-singular, and the pair $(\mathbf{RP}^2, \mathbf{R}A)$ will have the same topological type as the associated combinatorial patchworking $(\overline{\Delta}, \overline{L})$; in this case, a Harnack M-curve.

Now, the curve A is not symmetric *per se*. However, it is close enough: because ν satisfies $\nu(i,j) = \nu(j,i)$, and because the distribution of signs also satisfies $\mathrm{sign}(i,j) = \mathrm{sign}(j,i)$, we see that:

$$A(x_0, x_1, x_2) = A(x_1, x_0, x_2).$$

In particular, considering the curve

$$B(x_0, x_1, x_2) = A(x_0 + x_1, -x_0 + x_1, x_2),$$

we obtain that B is a symmetric M-curve. ■

This almost implies the following result (note that this holds both for algebraic and flexible symmetric curves).

Proposition 4.12. *All non-singular real plane curves of degree $m \leqslant 5$ are realizable by a symmetric curve.*

Proof. This is true in degrees $m \leqslant 3$, by Proposition 4.11. Indeed, for any number of components, there is only one curve with that number of components.

In degree 4, the only case where there are two possible arrangements with a given number of components are the schemes $\langle 1\langle 1\rangle\rangle$ and $\langle 2\rangle$. Both are easily realized by a symmetric curve (their usual construction works).

In degree 5, the only pair of arrangements with the same number of components are $\langle \mathcal{J} \sqcup 1\langle 1\rangle\rangle$ and $\langle \mathcal{J} \sqcup 3\rangle$. Again, both are realized by symmetric curves, by their usual constructions. ■

Note that Figure 1.4 and Figure 1.5 depicted examples of such symmetric realizations of those real schemes.

We will, however, not solely concerned with the question of determining if a given scheme can be realized by a symmetric curve. Instead, we will focus on classifying both topological types of $(\mathbf{RP}^2, \mathbf{R}F)$ and $(\widetilde{\mathbf{RP}}^2, \widetilde{\mathbf{R}}F)$ at the same time. In fact, we also care about the disposition of the curve with respect to the axis of symmetry, and about the position of the center of symmetry.

Question 4.13. *Given $m \geqslant 1$, what are the possible pairs of topological types of $(\mathbf{RP}^2, \mathbf{R}F \cup \mathbf{R}G_0 \cup \{g_0\})$ and $(\widetilde{\mathbf{RP}}^2, \widetilde{\mathbf{R}}F \cup \widetilde{\mathbf{R}}G_0 \cup \{g_0\})$?*

We will now prove a few small observations that go in the direction of answering Question 4.13.

Proposition 4.14. *Let F be a flexible symmetric curve of degree m. Then $\widetilde{\mathbf{R}}F \pitchfork \widetilde{\mathbf{R}}G_0$, and $g_0 \in \mathbf{R}F$ if and only if m is odd, in which case g_0 is located on the pseudo-line of $\mathbf{R}F$. Moreover, if m is odd, then $\mathbf{R}F \cap \mathbf{R}G_0 \neq \varnothing$.*

Proof. The condition that $\mathbf{R}F \pitchfork \mathbf{R}G_0$ ensures that, if m is odd, then the pseudo-line of $\mathbf{R}F$ must contain g_0. In the case of even degree, it is not possible to have $g_0 \in \mathbf{R}F$ without self-intersecting.

The claim about the fact that $\widetilde{\mathbf{R}}F \pitchfork \widetilde{\mathbf{R}}G_0$ is always verified in the case of even degrees. Moreover, if m is odd, then the pseudo-line of $\mathbf{R}F$ contains $g_0 \in \widetilde{\mathbf{RP}}^2$, and thus $g_0 \in \widetilde{\mathbf{R}}F$ is located on the pseudo-line of $\widetilde{\mathbf{R}}F$.

Finally, if the degree is odd, then the pseudo line must intersect any other non-contractible component of $\mathbf{RP}^2$, and in particular intersects $\mathbf{R}G_0$. ■

The following fact was shown a couple times in the literature (see Brugallé [Bru07, Proposition 2.11] or Trilles [Tri03, Lemma 6]), but is first attributed to Fiedler (unpublished).

Lemma 4.15. *Let F be a flexible symmetric curve of degree m and type I. Then either $\mathbf{R}F \cap \mathbf{R}G_0 = \varnothing$, or $\mathbf{R}F \pitchfork \mathbf{R}G_0$ is m points, all positive, and $(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0 = \varnothing$. In particular, $\mathbf{R}F \cap \mathbf{R}G_0 = \varnothing$ cannot occur if m is odd.*

Proof. Assume that $\mathbf{R}F \cap \mathbf{R}G_0 \neq \varnothing$, and consider the symmetry $\sigma : \mathbf{CP}^2 \to \mathbf{CP}^2$. Because σ fixes F and $\mathbf{R}F$, it restricts to an orientation-preserving involution

$$\sigma' : F \smallsetminus \mathbf{R}F \to F \smallsetminus \mathbf{R}F.$$

This means that around any point $x \in F$ fixed by σ, the involution σ' locally acts like rotation by 180°. Setting $F \smallsetminus \mathbf{R}F = F_1 \sqcup F_1$, since it is assumed that F is type I, we can see that rotation by 180° around a point $x \in \mathbf{R}F \cap \mathbf{R}G_0$ necessarily swaps the two half-surfaces F_1 and F_2, meaning that

$$F_1 = \sigma'(F_2) \text{ and } F_2 = \sigma'(F_1).$$

Therefore, there cannot be an imaginary intersection point $y \in (F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0$, since, assuming that $y \in F_1$, we would obtain that $\mathrm{conj}(y) \in F_2$ but $\mathrm{conj}(y) = \mathrm{conj}(\sigma'(y)) \in \mathrm{conj}(F_2) = F_1$. ■

Note that both scenarii can occur. For instance, the hyperbolic scheme $\langle 1\langle 1 \cdots \langle 1 \rangle \cdots \rangle\rangle$ can be realized symmetrically with both $\mathbf{R}F \cap \mathbf{R}G_0$ empty or m points, by considering a (perturbed) union of concentric circles, centered around either a point on $\mathbf{R}G_0$ or g_0.

In Figure 4.5, we depict a representation of an example of a symmetric curve.

We finish with one last terminology comment, which will be motivated throughout the rest of this chapter.

Definition 4.16. *A (flexible) symmetric curve F is called* ***exceptional*** *if $\mathbf{R}F \cap \mathbf{R}G_0 = \varnothing$. Necessarily, an exceptional curve has even degree.*

4.3.1 The Quotient Involution

In the general case of flexible curves, it was clear that the surface $F/\mathrm{conj} \subset \mathbf{S}^4$ was a tool of great importance. Symmetric curves are no exception, and carry more structure to it. Indeed, the map $\sigma : \mathbf{CP}^2 \to \mathbf{CP}^2$ commutes with complex conjugation, and therefore induces an involution

$$s : \mathbf{S}^4 \to \mathbf{S}^4$$

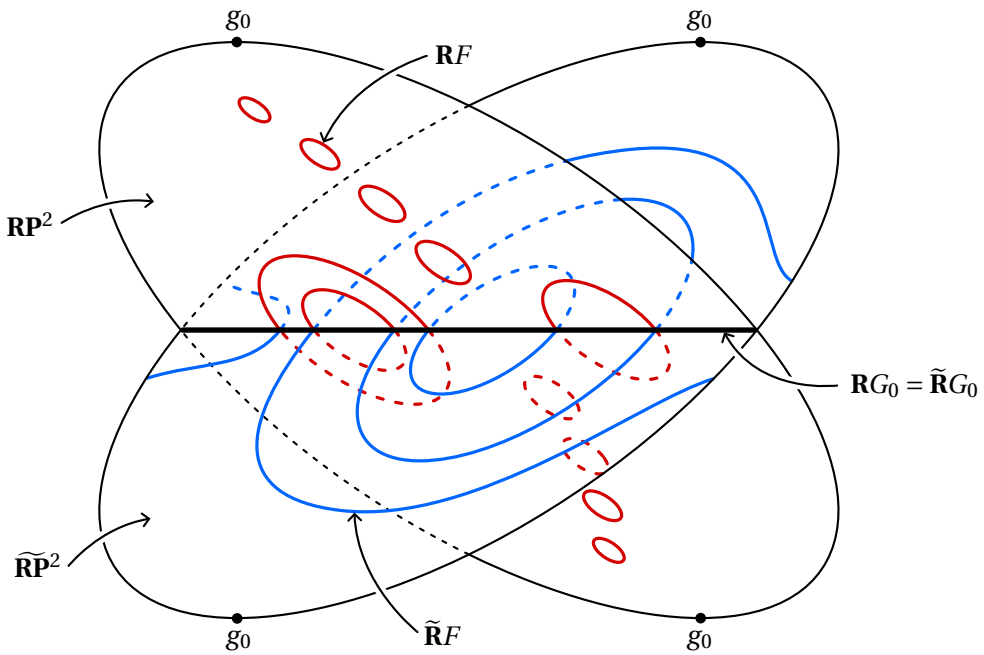


Figure 4.5. An example of a symmetric M-sextic, in red, together with its associated mirror curve, in blue. It is understood that $\mathbf{RP}^2$ and $\widetilde{\mathbf{RP}}^2$ are represented by discs whose boundary antipodal points are pairwise identified.

on the quotient. Moreover, we observe that

$$\mathrm{Fix}(s) = p(\widetilde{\mathbf{RP}}^2) \cup p(\mathbf{C}G_0),$$

where $p : \mathbf{CP}^2 \to \mathbf{S}^4$ is the projection map. Notice that both $p(\widetilde{\mathbf{RP}}^2)$ and $p(\mathbf{C}G_0)$ are 2-discs, meeting at their common boundary $\partial p(\widetilde{\mathbf{RP}}^2) = \partial p(\mathbf{C}G_0) = p(\mathbf{R}G_0)$. This implies that $\mathrm{Fix}(s)$ is an embedded 2-sphere inside $\mathbf{S}^4$. Denote as $S = \mathbf{S}^4/s$ the quotient manifold, and as $\pi : \mathbf{S}^4 \to S$ the projection map, which is a double branched cover. We can compute, using Proposition 2.37 and Theorem 2.19, that:

$$H_1(S;\mathbf{Z}/2) = 0 \text{ and } \chi(S) = 2.$$

This implies that S is a $\mathbf{Z}/2$-homology sphere, since $H_3(S;\mathbf{Z}/2) = 0$ also, by Poincaré duality, and thus $b_2(S;\mathbf{Z}/2) = \chi(S) - 2 = 0$ in this case.

The surface (with boundary) F/conj is invariant under s, and so is its boundary $\partial F/\mathrm{conj}$. We consider the restriction

$$s' : \partial F/\mathrm{conj} \to \partial F/\mathrm{conj},$$

which is an involution on a collection of circles. No restriction of s' on any connected component of $\partial F/\mathrm{conj}$ is the identity map, as no component of $\mathbf{R}F$ was fixed point-wise by the symmetry σ. Therefore, there are three ways of "filling" this involution s'.

(1) There is a pair of circles $C_1, C_2 \subset \partial F/\mathrm{conj}$ swapped by s'. Consider two (abstract) discs D_1 and D_2 bounding C_1 and C_2, and extend s' into $\widetilde{s}' : D_1 \sqcup D_2 \to D_1 \sqcup D_2$ by swapping those whole discs. The extension will be a free involution on $D_1 \sqcup D_2$.

(2) There is a circle $C \subset \partial F/\mathrm{conj}$ fixed by s' on which it acts freely. This means that s' is topologically the map

$$\begin{array}{ccc} \mathbf{S}^1 & \longrightarrow & \mathbf{S}^1 \\ e^{\mathbf{i}\theta} & \longmapsto & e^{2\mathbf{i}\theta}, \end{array}$$

which is rotation by 180°. Consider a disc D which bounds C, and extend s' into $\widetilde{s}' : D \to D$ by also performing rotation by 180° on D. The extension has an isolated fixed point at the origin of D.

(3) There is a circle $C \subset \partial F/\mathrm{conj}$ fixed by s' on which it acts with two fixed points, in which case s' is topologically the map

$$\begin{array}{ccc} \mathbf{S}^1 & \longrightarrow & \mathbf{S}^1 \\ e^{\mathbf{i}\theta} & \longmapsto & e^{-\mathbf{i}\theta}, \end{array}$$

a reflection along an axis. Again, extend this into $\widetilde{s}' : D \to D$, where D is a disc bounding C, by performing reflection along that axis. The extension will have an interval of fixed points, bounded by $\mathrm{Fix}(s'|_C)$.

Performing this filling operation for each (pair of) boundary components, we obtain an extension $\widetilde{s}'$ of s' on a collection $\mathscr{D}$ of discs bounding $\partial F/\mathrm{conj}$. Now, this extension can be glued to $s|_{F/\mathrm{conj}}$ in order to obtain a closed surface $\Sigma(F) = F/\mathrm{conj} \cup \mathscr{D}$ and an involution

$$s_F = s \cup \widetilde{s}' : \Sigma(F) \to \Sigma(F)$$

such that $s_F|_{F/\mathrm{conj}} = s|_{F/\mathrm{conj}}$.

Definition 4.17. *The map $s_F : \Sigma(F) \to \Sigma(F)$ is called the* ***quotient involution*** *associated to F.*

Regarding the fix-point set of s_F, it is possible to get information solely from the sets $\mathbf{R}F$ and $\widetilde{\mathbf{R}}F$.

Lemma 4.18. *The surface $\Sigma(F)$ is orientable if and only if F is type I. Moreover,* $\mathrm{Fix}(s_F)$ *is a non-empty collection of isolated points and circles, where:*

(1) each oval $o \subset \mathbf{R}F$ which has g_0 in its interior gives rise to an isolated point in $\mathrm{Fix}(s_F)$*;*

(2) each oval $\widetilde{o} \subset \widetilde{\mathbf{R}}F$ which has g_0 in its interior gives rise to a circle in $\mathrm{Fix}(s_F)$*;*

(3) each pair of distinct ovals $\widetilde{o}_1, \widetilde{o_2} \subset \widetilde{\mathbf{R}}F$ such that $\widetilde{o}_2 = \sigma(\widetilde{o_1})$ gives rise to a circle in $\mathrm{Fix}(s_F)$*;*

(4) if $\mathbf{R}F \cap \mathbf{R}G_0 = \widetilde{\mathbf{R}}F \cap \widetilde{\mathbf{R}}G_0 \neq \varnothing$, then this gives rise to at least one circle in $\mathrm{Fix}(s_F)$ *(possibly more), which contain (the images of) all those intersection points;*

(5) each pair of conjugate imaginary points $y, \mathrm{conj}(y) \in (F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0$ gives rise to an isolated point in $\mathrm{Fix}(s_F)$*.*

Lastly, any circle or isolated point in $\mathrm{Fix}(s_F)$ *is obtained in exactly one of the above possible ways.*

Proof. Let F be a flexible symmetric curve. The surface $\Sigma(F)$ is orientable if and only if F/conj is, and this is the case if and only if F is separating. For the first claim, we inspect each case individually.

(1) An oval $o \subset \mathbf{R}F$ which has g_0 in its interior is necessarily fixed by σ on which it acts freely, and thus induces a component of $\partial F/\mathrm{conj}$ itself fixed under s'. The action of s' on that component is rotation by 180°, which will be filled into an isolated fixed point.

(2) An oval $\widetilde{o} \subset \widetilde{\mathbf{R}}F$ with g_0 in its interior is also fixed by σ on which it acts freely. However, when taking the quotient by complex conjugation, pairs of points x and $\sigma(x)$ get identified, and thus s acts as identity on the circle $\widetilde{o}/\mathrm{conj}$, situated in the interior of F/conj.

(3) The situation where $\widetilde{o}_1, \widetilde{o}_2 \subset \widetilde{\mathbf{R}}F$ is a pair of ovals swapped by σ is very similar: they get identified into $\widetilde{o}_1/\mathrm{conj} = \widetilde{o}_2/\mathrm{conj}$ a circle in the interior of F/conj on which s acts as the identity.

(4) If $\mathbf{R}F \cap \mathbf{R}G_0 \neq \varnothing$, then this will give rise to circles in $\partial F/\mathrm{conj}$ on which s' acts like reflection with two fixed points. Note that if m is odd, then there will be an odd number of intersection points in $\mathbf{R}F \cap \mathbf{R}G_0$, but an extra point is obtained from the fact that $g_0 \in \mathbf{R}F$.

(5) Pairs of complex conjugate intersection points in $(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0$ are identified when taking the quotient by complex conjugation, and both points were already fixed by σ. This will give an isolated fixed point for s situated in the interior of F/conj.

The last claim is obtained easily by noting that all those cases are mutually exclusive, and by the fact that any fixed point for s_F (being isolated or situated on a circle) is either obtained by the filling operation, or must come from either a real fixed point for σ or a pair of complex conjugate points $y, z \in \mathbf{CP}^2$ such that $z = \mathrm{conj}(y) = \sigma(y)$. ∎

Conversely, knowledge about this involution can be carried over to restrictions on the disposition of $\mathbf{R}F$ with respect to g_0 and $\mathbf{R}G_0$ and to restrictions on the mirror curve $\widetilde{\mathbf{R}}F$. For this, we need the topological type of the surface $\Sigma(F)$, which is characterized (since it is closed) by its orientability and its genus. The former is prescribed by whether F is type I or II, and the latter by the topological Riemann–Hurwitz formula:

$$\chi(\Sigma(F)) = \frac{-m^2 + 3m}{2} + b_0(\mathbf{R}F).$$

This allows to obtain the following.

Proposition 4.19. *Let F be a flexible symmetric curve of type I and even degree $m = 2k$.*

(1) If $\mathbf{R}F \cap \mathbf{R}G_0 \neq \varnothing$, then g_0 is situated in the exterior of the curve.

(2) If $\mathbf{R}F \cap \mathbf{R}G_0 = \varnothing$, then $\widetilde{\mathbf{R}}F = \varnothing$, and if r denotes the number of ovals that contain g_0 in interior (or equivalently, the depth of the nest centered at g_0), then

$$r \equiv k \ [2] \text{ and } r \leqslant 2(k-1)^2 + 2 - b_0(\mathbf{R}F).$$

Proof. In the case where $\mathbf{R}F \cap \mathbf{R}G_0$ is non-empty, it is necessarily $2k$ points, by Lemma 4.15. In particular, by (4) of Lemma 4.18, we obtain that $\mathrm{Fix}(s_F)$ contains at least one circle. From the classification of involutions (more specifically from Corollary 4.6), we obtain that there are no isolated points[†] in $\mathrm{Fix}(s_F)$. Finally, by (1) of Lemma 4.18, we obtain the claim.

[†] Note that this observation gives another proof of Lemma 4.15, by using (5) of Lemma 4.18.

In the case where $\mathbf{R}F \cap \mathbf{R}G_0 = \varnothing$ however, the situation is opposite. Indeed, we have :

$$|Q_{\mathbf{CP}^2}(F, \mathbf{C}G_0)| = 2k \leqslant \#F \cap \mathbf{C}G_0.$$

Hence we obtain, in total, at least $r + k$ isolated points in $\mathrm{Fix}(s_F)$, and no fixed circles. Necessarily, $\widetilde{\mathbf{R}}F = \varnothing$, otherwise there would be a circle in $\mathrm{Fix}(s_F)$. By Corollary 4.6, we obtain :

$$\#\mathrm{Fix}(s_F) \leqslant 2\gamma + 2 \text{ and } \#\mathrm{Fix}(s_F) \equiv 2\gamma + 2 \ [4],$$

where $\chi(\Sigma(F)) = 2 - 2\gamma$. By the previous computation, we obtain:

$$\gamma = k^2 + 1 - \frac{3k + b_0(\mathbf{R}F)}{2}.$$

Note that the Klein congruence for type I curves ensures that this is indeed an integer. We therefore obtain:

$$\#\mathrm{Fix}(s_F) \leqslant r + k \leqslant 2\gamma + 2 = 2k^2 + 2 - 3k - b_0(\mathbf{R}F),$$

from which the bound follows. For the congruence, we derive it from $\#\mathrm{Fix}(s_F) \equiv 2\gamma + 2 \equiv 0 \ [2]$ and from the fact that

$$\#\mathrm{Fix}(s_F) = r + \frac{\#F \cap \mathbf{C}G_0}{2}.$$

Indeed, we necessarily have $\#F \cap \mathbf{C}G_0 \equiv m \ [4]$, because in this case there are no real intersections, and $y \in (F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0$ is a negative intersection point if and only if $\mathrm{conj}(y)$ is one too. ■

The fact that r might be non-zero for even degree curves is very different from the odd degree case. However, we have the following.

Proposition 4.20. *Let F be a flexible symmetric curve of odd degree. Then $b_0(\mathbf{R}F) \equiv j \ [2]$.*

Proof. It suffices to note that $b_0(\mathbf{R}F) = r + 2q$. ■

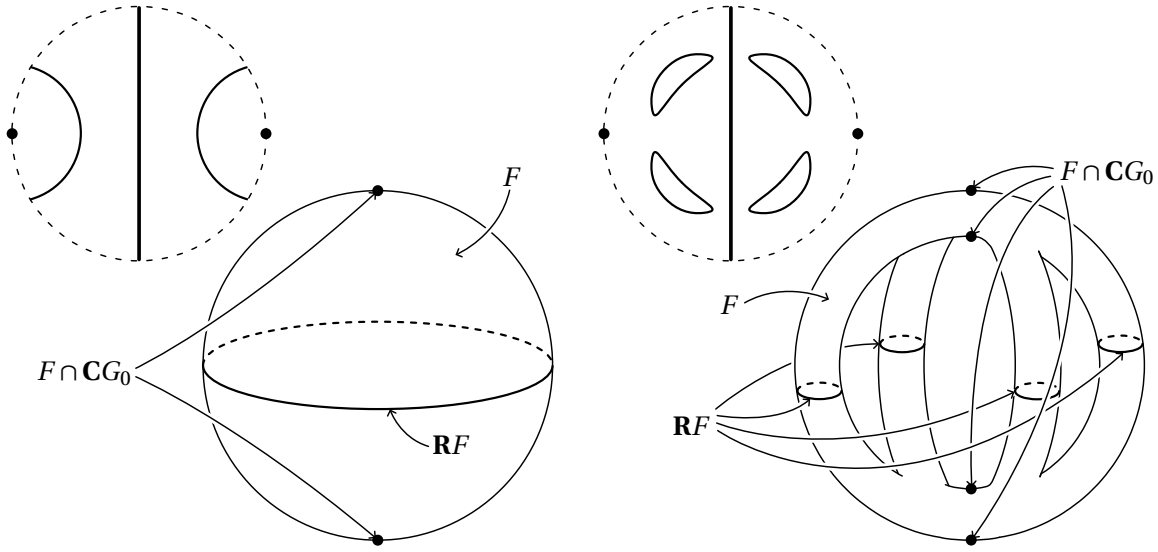
The bound obtained on r is not sharp in general. However, one can look what happens in special cases, for instance in the case of M-curves. We obtain the following.

Proposition 4.21. *If F is an exceptional flexible symmetric M-curve of degree m, then $k \in \{2, 4\}$. Moreover, $m = 2$ if and only if $g_0 \in \mathbf{RP}^2_+$, in which case it is located inside the only oval, and $m = 4$ if and only if $g_0 \in \mathbf{RP}^2_-$, in which case it is located in the exterior of the curve.*

Proof. If $\mathbf{R}F \cap \mathbf{R}G_0 = \varnothing$ and $m = 2k$ (it is necessarily even), then the bound obtained in Proposition 4.19 provides, if r denotes the number of ovals containing g_0 in their interior:

$$r \leqslant 2(k-1)^2 + 2 - [(k-1)(2k-1) + 1] = -k + 2.$$

Because $r \geqslant 0$, we obtain $k \in \{1, 2\}$. The rest follows from the congruence of Proposition 4.19 and from the classification of M-conics and M-quartics. ■



(a) (b)

Figure 4.6. (a) The exceptional M-conic. The involution s_F on $\Sigma(F)$ has two fixed points, one coming from $F \cap \mathbf{C}G_0$ and the other from filling with a disc. (b) The exceptional M-quartic. The involution s_F has two fixed points, both coming from $F \cap \mathbf{C}G_0$.

The previous statement can be reformulated as follows: if F is a flexible symmetric M-curve of degree at least 5, then it is not exceptional. We depict the exceptional M-curves in Figure 4.6. In particular, if F is a symmetric flexible M-curve of even degree $m \geqslant 6$, then g_0 is situated in the exterior of the curve.

We give an analogue of Proposition 4.19 in the case of non-dividing curves.

Proposition 4.22. *Let F be a flexible symmetric curve of even degree $m = 2k$ and type II, and let $j = \frac{1}{2}\#\mathbf{R}F \cap \mathbf{R}G_0$. If r denotes the number of ovals of $\mathbf{R}F$ that contain g_0 in their interior, then:*

$$r \equiv j + b_0(\mathbf{R}F) \ [2] \text{ and } r \leqslant 2(k-1)^2 + 2 - j - b_0(\mathbf{R}F).$$

Proof. The proof is similar as before. For the bound, one needs to note that this time, we have

$$k - j \leqslant \frac{\#(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0}{2},$$

and thus, if s denotes the number of isolated fixed points in $\mathrm{Fix}(s_F)$, then:

$$r + k - j \leqslant s \leqslant b_*(\mathrm{Fix}(s_F)) \leqslant b_*(\Sigma(F)) = 4 - \chi(\Sigma(F)),$$

by using Theorem 2.21. For the congruence, we have, from Corollary 2.20:

$$\chi(\mathrm{Fix}(s_F)) = s \equiv \chi(\Sigma(F)) \ [2],$$

and we also have $s \equiv m - 2j$ [4], for the same reason as in the proof of Proposition 4.19. ■

In fact, some knowledge about the mirror curve can be used in order to improve the bounds of Proposition 4.19 and Proposition 4.22. We use the following notation.

Definition 4.23. *Let F be a flexible symmetric curve of degree $m = 2k + \varepsilon$, $\varepsilon \in \{0, 1\}$.*

(1) We let j denote the number of components of $\mathbf{R}F$ that intersect $\mathbf{R}G_0$. In particular, if j_0 denotes the number of ovals *of $\mathbf{R}F$ that intersect $\mathbf{R}G_0$, then $j = j_0 + \varepsilon$. We also let $\widetilde{\jmath}$ denote the number of components of $\widetilde{\mathbf{R}}F$ that intersect $\widetilde{\mathbf{R}}G_0$. In fact, $j = \widetilde{\jmath}$, and a curve is exceptional if and only if $j = 0$.*

(2) We set r and $\widetilde{r}$ to be the number of ovals of $\mathbf{R}F$ and $\widetilde{\mathbf{R}}F$ that contain g_0 in their interior, respectively. Alternatively, an oval contains g_0 in its interior if and only if it is fixed by σ and does not intersect $\mathbf{R}G_0 = \widetilde{\mathbf{R}}G_0$. Note that if $\varepsilon = 1$ (that is, m is odd), then $r = \widetilde{r} = 0$.

(3) We define q and $\widetilde{q}$ to be the number of pairs of distinct ovals of $\mathbf{R}F$ and $\widetilde{\mathbf{R}}F$ swapped by σ, respectively. An oval contributes to q or $\widetilde{q}$ if and only if it does not intersect $\mathbf{R}G_0 = \widetilde{\mathbf{R}}G_0$ and does not contain g_0 in its interior.

By definition, we immediately see that

$$b_0(\mathbf{R}F) = 2q + r + j \text{ and } b_0(\widetilde{\mathbf{R}}F) = 2\widetilde{q} + \widetilde{r} + j.$$

For instance:

(1) if F is the exceptional M-conic, then $r = 1$, $j = q = 0$ and $\widetilde{r} = \widetilde{q} = 0$;

(2) if F is the exceptional M-quartic, then $j = r = 0$, $q = 2$ and $\widetilde{r} = \widetilde{q} = 0$;

(3) if F is the curve depicted in Figure 4.5, then $r = 0$, $j = 3$, $q = 4$ and $\widetilde{r} = \widetilde{q} = 0$.

We also need to define an object associated to a symmetric curve. Recall that $p : \mathbf{CP}^2 \to \mathbf{S}^4$ denoted the quotient map under the action of conj, and that $\pi : \mathbf{S}^4 \to S$ is the quotient under the action of s. The union

$$\Omega = \pi \circ p(\mathbf{RP}^2) \cup \pi \circ p(\widetilde{\mathbf{RP}}^2)$$

is that of two 2-discs, glued along their boundary and a single isolated point. This can also be thought as a pinched torus. We let Γ_F denote the subset

$$\Gamma_F = \pi \circ p(\mathbf{R}F \cup \widetilde{\mathbf{R}}F),$$

and we consider the subset Γ'_F of Γ_F of its components that intersect $\pi \circ p(\mathbf{R}G_0)$. Then both Γ_F and Γ'_F are closed 1-manifold, possibly disconnected. For instance, if F is the sextic depicted in Figure 4.5, then Γ'_F is a single circle containing all the blue half-circles, and Γ_F is a collection of 5 circles. An immediate observation is that Γ'_F contains at least one circle if $j \neq 0$, and $\Gamma_F \smallsetminus \Gamma'_F$ contains exactly $r + q + \widetilde{r} + \widetilde{q}$ circles. We refer the reader to Figure 4.11 for a depiction of the sets Ω and Γ_F.

Putting all this together, Lemma 4.18 has the consequence that $\mathrm{Fix}(s_F)$ is a collection of

(1) $r + \frac{1}{2}\#(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0$ isolated points, and

(2) $\delta + r + \widetilde{r} + \widetilde{q}$ circles, where $0 \leqslant \delta \leqslant j$.

Moreover, in the case of an *algebraic* symmetric curve, we obtain that:

$$\frac{1}{2}\#(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0 = k + \varepsilon - j.$$

We now make the bounds of Proposition 4.19 and Proposition 4.22 more precise.

Proposition 4.24. *Let F be a flexible symmetric curve of degree m. Set $\delta = 0$ if $j = 0$ and $\delta = 1$ if $j \geqslant 1$. If $m = 2k$ is even, then:*

$$r + \widetilde{r} + q + \widetilde{q} + \delta \leqslant (k-1)^2 + 1.$$

If $m = 2k+1$ is odd, then:

$$q + \widetilde{q} \leqslant k(k-1).$$

Proof. By Lemma 4.18, we see that $\mathrm{Fix}(s_F)$ consists of

(1) at least $r + k - j$ isolated points and $\delta + \widetilde{q} + \widetilde{r}$ circles in the even degree case, and
(2) at least $k + 1 - j$ points and $1 + \widetilde{q}$ in the odd case.

Recall that in the odd degree case, we have $j \geqslant 1$, and thus $\delta = 1$, as well as $r = \widetilde{r} = 0$. The bound therefore follows from applying the Smith–Floyd inequality $b_*(\mathrm{Fix}(s_F)) \leqslant 4 - \chi(\Sigma(F))$. ∎

So far, the surface $\Sigma(F)$ and its involution s_F were obtained by quotienting *once* by one of the involutions conj and σ on $\mathbf{CP}^2$. We now investigate what information we can derive from considering the double quotient $(F/\mathrm{conj})/s \subset S$. Firstly, this allows to answer a very natural question. A consequence is that, for instance, a symmetric conic cannot have both the curve and its mirror that are empty.

Proposition 4.25. *Let F be a flexible symmetric curve of degree m such that $\mathbf{R}F = \varnothing$ and $\widetilde{\mathbf{R}}F = \varnothing$. Then $m \equiv 0$ [4].*

Proof. Assume that $\mathbf{R}F = \widetilde{\mathbf{R}}F = \varnothing$. First, note that necessarily the degree is even, since otherwise both $\mathbf{R}F$ and $\widetilde{\mathbf{R}}F$ contain a pseudo-line. Denote as $m = 2k$ that degree. We see that F/conj is already closed, and thus $\Sigma(F) = F/\mathrm{conj}$ is embedded in $\mathbf{S}^4$. Therefore, we have $e(\mathbf{S}^4, \Sigma(F)) = 2k^2$, by Lemma 2.38. Finally, $s : \mathbf{S}^4 \to \mathbf{S}^4$ restricts on $\Sigma(F)$ to s_F, which has k isolated fixed points, and nothing else, by Lemma 4.18. Furthermore, we have $\Sigma(F) \pitchfork \mathrm{Fix}(s)$, since this intersection is the image of $F \pitchfork \mathbf{C}G_0$ under $p : \mathbf{CP}^2 \to \mathbf{S}^4$. Using Lemma 2.38 again, we obtain that

$$e(S, \Sigma(F)/s) = k^2.$$

Because $H_2(S; \mathbf{Z}/2) = 0$, this self-intersection is necessarily even, which implies that k itself is even. ∎

We can even show that the previous restriction is optimal, by observing the Fermat curves. Consider the curve $F_{2k} \in \mathscr{C}^{\mathrm{sym}}_{2k}$ defined by:

$$F_{2k}(x_0, x_1, x_2) = x_0^{2k} + x_1^{2k} + x_2^{2k}.$$

Then:

(1) if $k \equiv 0$ [2], then both $\mathbf{R}F_{2k}$ and $\widetilde{\mathbf{R}}F_{2k}$ are empty;

(2) if $k \equiv 1$ [2], then $\mathbf{R}F_{2k} = \varnothing$ and $\widetilde{\mathbf{R}}\mathcal{F}_{2k}$ is a single oval, containing g_0 in its interior.

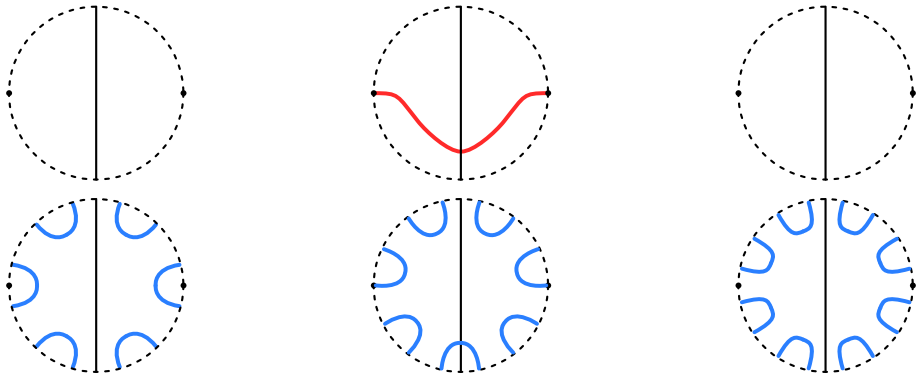
One can also consider the Fermat curves up to a projective transformation. That is: for all $m \in \mathbf{N}^\star$, set

$$F'_m(x_0, x_1, x_2) = (x_0 + x_1)^m + (-x_0 + x_1)^m + x_2^m \in \mathscr{C}_m^{\mathrm{sym}}.$$

Then:

(1) if m is odd, then $\mathbf{R}F'_m$ and $\widetilde{\mathbf{R}}F'_m$ are both a single pseudo-line;
(2) if $m = 2k$, then $\mathbf{R}F'_m = \varnothing$, $\widetilde{\mathbf{R}}F'_m$ has the real scheme $\langle k \rangle$, and $g_0 \in \mathbf{RP}^2_-$ if and only if $k \equiv 0$ [2].

See Figure 4.7 for a depiction of those twisted Fermat curves.



(a) (b) (c)

Figure 4.7. Top row: curves inside $\mathbf{RP}^2$. Bottom row: curves inside $\widetilde{\mathbf{RP}}^2$.
(a) The curve F'_6. (b) The curve F'_7. (c) The curve F'_8.

Lastly, we can further improve the bounds obtained in Proposition 4.24 by considering the double quotient surface. This allows to alleviate the uncertainty on the number of circles of the form given by (4) of Lemma 4.18.

Proposition 4.26. *Let F be a flexible symmetric curve of degree m.*

(1) If $m = 2k$ is even, then

$$r + \widetilde{r} + q + \widetilde{q} + b_0(\Gamma'_F) \leqslant (k-1)^2 + 1.$$

(2) If $m = 2k+1$ is odd, then

$$q + \widetilde{q} + b_0(\Gamma'_F) \leqslant k(k-1) + 1.$$

Proof. We treat each case individually. We may refer the reader to Figure 4.8 for a visual aid (for instance, the arcs fixed by $s|_{F/\mathrm{conj}}$ are the blue arcs coming from the j ovals touching $\widetilde{\mathbf{R}}G_0$ in $\widetilde{\mathbf{R}}F$).

(1) If $m = 2k$, then $F/\mathrm{conj} \cap \mathrm{Fix}(s)$ is composed of j arcs, $r + \widetilde{r} + \widetilde{q}$ circles and of $\frac{1}{2}\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0$ isolated points. The involution s swaps $2q$ boundary components of F/conj, and acts like

reflection on the j others. This means that using Theorem 2.19, we obtain:

$$\chi((F/\mathrm{conj})/s) = \frac{1}{2}\left(\chi(F/\mathrm{conj}) + j + \frac{1}{2}\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0\right).$$

Moreover, the surface $(F/\mathrm{conj})/s$ has exactly $b_0(\Gamma'_F) + r + q + \widetilde{r} + \widetilde{q}$ boundary components. This means that adding this number of disc components to it, we can bound the Euler characteristic of the closed surface by 2 to obtain

$$\chi((F/\mathrm{conj})/s) + r + q + \widetilde{r} + \widetilde{q} + b_0(\Gamma'_F) \leqslant 2.$$

On the other hand, we have $\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0 \geqslant 2(k-j)$, so that

$$\chi((F/\mathrm{conj})/s) \geqslant \frac{1}{2}\big(-2k^2 + 3k + j + (k-j)\big) = -k^2 + 2k.$$

Comparing both sides yields the inequality.

(2) If $m = 2k+1$, this time we have that $F/\mathrm{conj} \cap \mathrm{Fix}(s)$ consists of $j+1$ arcs, $\widetilde{q}$ circles and $\frac{1}{2}\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0$ isolated points, with $\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0 \geqslant 2(k-j)$, and s swaps $2q$ boundary components and acts like reflection on $j+1$ others. The rest of the argumentation is the same, only the computations differ slightly. ■

4.3.2 The Case of M-Curves

In order to answer Question 4.13, we must first understand better the situation at least for M-curve, and possibly find new restrictions in this setting. We state two results due to Fiedler (unpublished).

The first was explained before in the literature (see [Bru07, Theorem 2.12]), and describes exactly the mirror curve in the case of a symmetric M-curve. We will merely reproduce the proof in detail, by noting that this works in the flexible case.

Theorem 4.27 (Fiedler)**.** *Let F be a flexible symmetric M-curve which is not exceptional. Then $\widetilde{\mathbf{R}}F$ is the hyperbolic scheme, and all its ovals intersect $\mathbf{R}G_0$.*

Proof. If F a non-excetional flexible symmetric M-curve of degree $m = 2k+\varepsilon$, where $k = \lfloor \frac{m}{2} \rfloor$ and $\varepsilon \in \{0,1\}$, then there are $j = k+\varepsilon$ components of $\mathbf{R}F$ intersecting $\mathbf{R}G_0$, one of which is a pseudo-line only when $\varepsilon = 1$. The surface $\Sigma(F)$ is a 2-sphere, and $\mathrm{Fix}(s_F)$ contains at least one circle, by (4) of Lemma 4.18. In particular, s_F can only be the reflection along an equatorial plane, and $\mathrm{Fix}(s_F)$ is exactly one circle. This implies that $\widetilde{\mathbf{R}}F$ contains no other components than those intersecting $\widetilde{\mathbf{R}}G_0$, by (2) and (3) of Lemma 4.18, and therefore contains exactly $k+\varepsilon$ components.

The surface F must be obtained by gluing two copies of F/conj along its boundary in an equivariant way. More precisely, denote as F' one of the two possible choices of closures of a connected component of $F \smallsetminus \mathbf{R}F$, so that $F = F' \underset{\mathbf{R}F}{\cup} \mathrm{conj}(F')$ and $p : \mathbf{CP}^2 \to \mathbf{S}^4$ induces two diffeomorphisms

$$p|_{F'} : F' \to F/\mathrm{conj} \text{ and } p|_{\mathrm{conj}(F')} : \mathrm{conj}(F') \to F/\mathrm{conj}$$

such that $p|_{F'} \circ \sigma = s|_{F/\mathrm{conj}} \circ p$ and $p|_{\mathrm{conj}(F')} \circ \sigma = s|_{F/\mathrm{conj}} \circ p$.

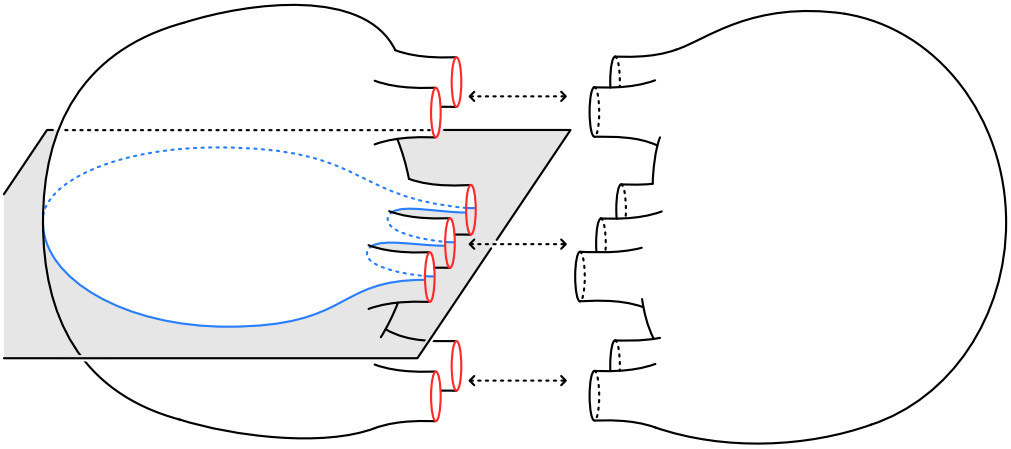


Figure 4.8. Each surface with boundary is one half of $F \smallsetminus \mathbf{R}F$. On the left, reflection along the horizontal plane is the action of s on F/conj. The red part is $\mathbf{R}F/\mathrm{conj}$, and the blue part is $\widetilde{\mathbf{R}}F/\mathrm{conj}$. There are j red circles that touch the reflection plane, and there are q pairs of other circles that are swapped by reflection.

As Figure 4.8 depicts, this implies that, since $r = 0$ and $(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0 = \varnothing$, the $k+\varepsilon$ components of $\widetilde{\mathbf{R}}F$ must separate the surface F, and thus the curve F is type I in $(\mathbf{CP}^2, \widetilde{\mathrm{conj}})$.

Finally, the Rokhlin–Mishachev complex orientation formula (Theorem 1.27) applied to $\widetilde{\mathbf{R}}F$ gives that $\widetilde{\mathbf{R}}F$ is the hyperbolic scheme. Indeed, there are two cases to treat. We add a tilde to the usual notations from §1.4 to denote the corresponding quantities for the real scheme $\widetilde{\mathbf{R}}F$.

(1) If $m = 2k$ is even, then the formula gives, since $\widetilde{\ell} = k$:

$$2(\widetilde{\Pi}^+ - \widetilde{\Pi}^-) = k - k^2.$$

On the other hand, since there are k ovals, thus

$$\widetilde{\Pi} = \widetilde{\Pi}^+ + \widetilde{\Pi}^- \leqslant \binom{k}{2} = \frac{k(k-1)}{2}.$$

This means that we have equality in the complex orientation formula, and thus $\widetilde{\Pi}^+ = 0$, $\widetilde{\Pi}^- = \widetilde{\Pi} = k(k-1)/2$ and the scheme of $\widetilde{\mathbf{R}}F$ is the hyperbolic one.

(2) If $m = 2k+1$ is odd, the same reasoning gives $\widetilde{\Pi} = k(k-1)/2$ and $\widetilde{\Lambda}^+ = 0$. ■

A consequence of the proof of the previous result is the following.

Proposition 4.28. *Let F be a flexible symmetric M-curve which is not exceptional. Then Γ'_F is a connected 1-manifold.*

Proof. It suffices to see that on Figure 4.8, Γ'_F is identified with the collection of blue arcs on F_+ together with the red half-circles of F_+ intersecting $\mathrm{Fix}(\sigma)$ and situated "above" that plane.

Alternatively, this can be derived from Proposition 4.26, since in this case, we have $r = \widetilde{r} = \widetilde{q} = 0$, $j = k + \varepsilon$ and $q = \frac{b_0(\mathbf{R}F) - j - r}{2}$. ∎

The second main result is a generalization of the Gudkov–Rokhlin congruence to symmetric curves (see (2) of Theorem 1.32).

Theorem 4.29 (Fiedler)**.** *Let F be a flexible symmetric M-curve of even degree $m = 2k$. Then*

$$p - n \equiv k^2 \ [16].$$

This could be interpreted as saying that at most half of the M-schemes are realizable by a symmetric curve.

If $F \subset \mathbf{CP}^2$ if a flexible symmetric curve of type I and even degree $m = 2k$ which is not exceptional, we can consider the Arnold surface $\mathcal{A}(F) = F/\mathrm{conj} \cup p(\mathbf{RP}^2_+) \subset \mathbf{S}^4$. The involution $s : \mathbf{S}^4 \to \mathbf{S}^4$ fixes $\mathcal{A}(F)$, and we have:

(1) $\mathcal{A}(F) \cap \mathrm{Fix}(s)$ consists of $b_0(\Gamma'_F)$ circles;
(2) there are q pairs of circles on $\mathcal{A}(F)$ which are swapped by s;
(3) there are $\widetilde{q} + \widetilde{r}$ circles on $\mathcal{A}(F)$ on which s acts like rotation by 180°.

Recall that we denoted as $S = \mathbf{S}^4/s$, which is a $\mathbf{Z}/2$-homology sphere, and as $\pi : \mathbf{S}^4 \to S$ the associated double branched cover. There is exactly one of the two subsets $p(\widetilde{\mathbf{RP}}^2_\pm)$ which intersects $p(\mathbf{RP}^2_+)$ in arcs, and the other intersects it in isolated points. We denote the former as $\widetilde{\mathcal{R}}_+$. However, it is to be noted that we need *not* have that $\widetilde{\mathcal{R}}_+ = p(\widetilde{\mathbf{RP}}^2_+)$.

Definition 4.30. *The **Arnold surface of the second kind** of F is the surface*

$$\mathcal{B}(F) = \pi(\mathcal{A}(F)) \cup \pi(\widetilde{\mathcal{R}}_+) \subset S.$$

We can compute $\chi(\mathcal{B}(F))$ and $e(S, \mathcal{B}(F))$, and Fiedler then computed that the Brown invariant of $\mathcal{B}(F)$ vanishes, and thus obtained the congruence.

4.3.3 The Case of $(M-1)$-Curves and $(M-2)$-Curves

Recall that if F is an $(M-1)$-curve, then $\Sigma(F)$ is a real projective plane, and s_F is a non-trivial involution on it. By Proposition 4.7, we see that $\mathrm{Fix}(s_F)$ consists of exactly one isolated point and one circle.

Proposition 4.31. *Let F be a flexible symmetric $(M-1)$-curve of degree m.*

(1) If F is exceptional, then $m = 2$, $\mathbf{R}F = \varnothing$ and $\widetilde{\mathbf{R}}F$ consists of a single oval containing g_0 in its interior.
(2) If F is not exceptional, then $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 \in \{m, m-2\}$. In the case where m is even, it equals $m-2$ if and only if $g_0 \in \mathbf{RP}^2_-$, in which case g_0 is situated in the exterior of the curve, and in the other case where $g_0 \in \mathbf{RP}^2_+$, g_0 is situated inside exactly one oval of $\mathbf{R}F$.

Proof. If F is exceptional, then m must be even. Moreover, any pair of complex conjugate intersection points in $(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0$ will give rise to one isolated fixed point in $\mathrm{Fix}(s_F)$. We therefore have:

$$\frac{m}{2} = \frac{|Q_{\mathbf{CP}^2}(F, \mathbf{C}G_0)|}{2} \leqslant \frac{\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0}{2} \leqslant 1,$$

which readily gives $m = 2$. In this case, we obtain that $\mathbf{R}F = \varnothing$. Now, s_F also contains one circle, which must therefore come from $\widetilde{\mathbf{R}}F$. This means that $\widetilde{\mathbf{R}}F$ is an exceptional M-conic, and is thus as prescribed by Proposition 4.21.

In the case of a non-exceptional $(M-1)$-curve, we still obtain that $(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0$ is at most two points. Since all *real* intersection points in $F \pitchfork \mathbf{C}G_0$ are positive, we therefore obtain that $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 \in \{m, m-2\}$.

If the degree m is even, then the following observations prove the dichotomy about $g_0 \in \mathbf{RP}^2_{\pm}$.

(1) If $\mathbf{R}F \cap \mathbf{R}G_0$ has m points, then the only possible way for s_F to have an isolated fixed point is if it comes from filling an oval on which σ acts like rotation by 180°. This means that there is exactly one oval of the curve that has g_0 in its interior, and in particular $g_0 \in \mathbf{RP}^2_+$.

(2) If $\mathbf{R}F \cap \mathbf{R}G_0$ is $m-2$ points, then there are exactly two (positive) points in $(F \smallsetminus \mathbf{R}F) \cap \mathbf{C}G_0$, which will give rise to an isolated fixed point for s_F. Therefore, there cannot be an oval having g_0 in its interior, and in particular $g_0 \in \mathbf{RP}^2_-$. ∎

Proposition 4.32. *Let F be a flexible symmetric $(M-1)$-curve of degree m which is not exceptional. Then $\widetilde{\mathbf{R}}F$ is a collection of ovals meeting $\widetilde{\mathbf{R}}G_0$ only, together with possibly a one-sided component if m is odd. All those ovals intersect $\widetilde{\mathbf{R}}G_0$.*

Proof. There is only one circle component in $\mathrm{Fix}(s_F)$, thus we must have that $\widetilde{\mathbf{R}}G_0$ contains no other ovals than those already forming a circle in $\mathrm{Fix}(s_F)$ with the ovals of $\mathbf{R}F$ that meet $\mathbf{R}G_0$. ∎

The previous claim can be re-formulated into saying that a non-exceptional symmetric $(M-1)$-curve has

$$\widetilde{r} = \widetilde{q} = 0.$$

This has the following consequence.

Proposition 4.33. *If F is a flexible symmetric $(M-1)$-curve which is not exceptional, then Γ'_F is connected.*

Proof. It suffices to show that $b_0(\Gamma'_F) \leqslant 1$, since we already have $b_0(\Gamma'_F) \geqslant 1$ because $j \neq 0$. This is a direct application of Proposition 4.26, by studying all four cases depending on the parity of m and on the cardinality of $\mathbf{R}F \pitchfork \mathbf{R}G_0$. ∎

We now propose the following conjecture.

Conjecture 4.34. *If F is a flexible symmetric $(M-1)$-curve which is not exceptional, then $\widetilde{\mathbf{R}}F$ is type II.*

Proposition 4.35. *Conjecture 4.34 is true is the following cases:*

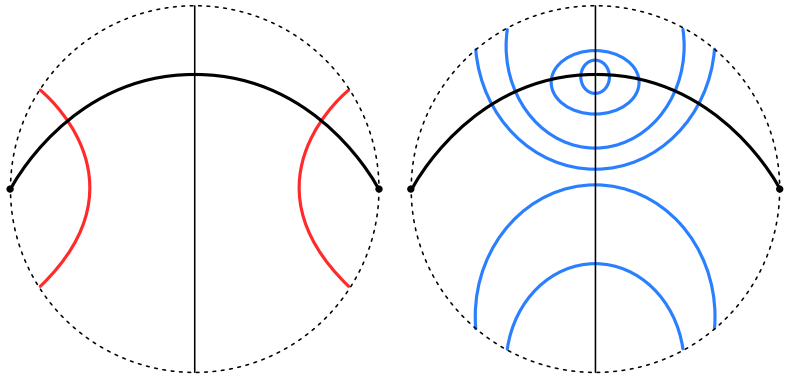
(1) if $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m-2$;

(2) if the curve F is algebraic;

(3) if the curve has odd degree.

Proof. Flexibly, assume that $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m-2$. Let $m = 2k+\varepsilon$, $\varepsilon \in \{0,1\}$. Then $b_0(\widetilde{\mathbf{R}}F) = k-1+\varepsilon$, which violates the Klein congruence Proposition 1.22(3).

In the algebraic case, consider the real pencil of symmetric lines that passes through the point g_0. This pencil is also real for the other conjugation $\widetilde{\text{conj}}$ (we may call its lines *bi-real lines*). To show the result, it suffices to treat the case where $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m$, since the other has been taken care of. We first treat the case where $m = 2k$ is even. In this case, we have $r = 1$ by Proposition 4.31. Assume by contradiction that $\widetilde{\mathbf{R}}F$ is type I. Then, by the same reasoning as in the proof of Theorem 4.27, we obtain that $\widetilde{\mathbf{R}}F$ is the hyperbolic scheme, which is a nesting of k ovals all meeting $\widetilde{\mathbf{R}}G_0$. Consider a point on $\widetilde{\mathbf{R}}G_0$ which is situated inside the inner-most oval of $\widetilde{\mathbf{R}}F$, and take the bi-real symmetric line Λ joining this point to g_0. We refer the reader to Figure 4.9.



(a) (b)

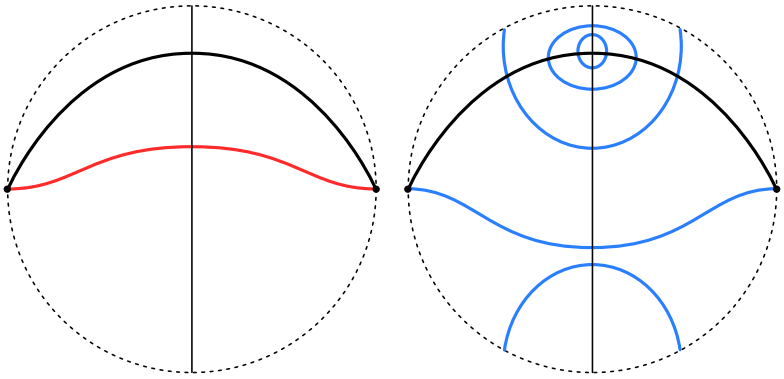
Figure 4.9. (a) The bi-real line inside $\mathbf{RP}^2$ intersects $\mathbf{R}F$ in at least 2 points. (b) In $\widetilde{\mathbf{RP}}^2$, the same line intersects $\widetilde{\mathbf{R}}F$ in at least $2k$ points.

If we count intersection points between the line $\mathbf{C}\Lambda$ with $\mathbf{C}F$, we count at least $2k+2$ points, which contradicts the Bézout theorem.

In the case where $m = 2k+1$ is odd this time, we cannot do the same procedure; we depict the situation in Figure 4.10.

This time, we only obtain $2k+1$ points of intersection. The last statement is simply a consequence of Proposition 4.20. Indeed, in this case, we have $b_0(\mathbf{R}F) = 2k^2-k$, and $j = k+1$. This means that there are no $(M-1)$-curves of odd degree with $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m$. ■

In order to attack Conjecture 4.34 for flexible curves of even degree, we could also proceed by contradiction. Indeed, if $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m$ and $\widetilde{\mathbf{R}}F$ is type I, we still have that $\widetilde{\mathbf{R}}F$ is the hyperbolic scheme by the Rokhlin–Mishachev oriention formula. The bi-real pencil of symmetric lines through g_0 can still be used, and all intersection points in $\mathbf{R}\Lambda \pitchfork \mathbf{R}F$ or $\widetilde{\mathbf{R}}\Lambda \pitchfork \widetilde{\mathbf{R}}F$ are positive. However, this does not give a contradiction in this case, since there *may* be intersection points in $(F \smallsetminus \mathbf{R}F \cup \widetilde{\mathbf{R}}F) \cap \mathbf{C}\Lambda$ which are negative, and this could be in agreement with $Q_{\mathbf{CP}^2}(F, \mathbf{C}\Lambda) = m$.



(a) (b)

Figure 4.10. The bi-real pencil in the case of an odd degree curve.

Recall the Gudkov–Krakhnov–Kharlamov congruence Theorem 1.32(3): if F is a flexible $(M-1)$-curve of even degree $m = 2k$, then $p - n \equiv k^2 \pm 1$ [8]. Just like Fiedler's congruence is a stronger version of the Gudkov–Rokhlin congruence for symmetric M-curves, it is a natural question to ask whether one can find a stronger congruence for symmetric $(M-1)$-curves too. We propose the following method that could possibly lead to finding such an improved congruence.

(1) Describe the mirror scheme $\widetilde{\mathbf{R}}F$.
(2) Consider the Arnold surface of the second kind $\mathcal{B}(F)$.
(3) Apply the Guillou–Marin congruence.

Question 4.36. *In the Gudkov–Krakhnov–Kharlamov congruence, we have the two possible alternatives $k^2 \pm 1$ for $p - n$ mod 8. Can we distinguish them simply from the cardinality of $\mathbf{R}F \pitchfork \mathbf{R}G_0$?*

We now switch to $(M-2)$-cures. This time, the surface $\Sigma(F)$ can be either a torus or a Klein bottle.

Proposition 4.37. *Let $F \subset \mathbf{CP}^2$ be an exceptional flexible $(M-2)$-curve of degree m. Then $m \in \{4,8\}$ and $\widetilde{\mathbf{R}}F = \varnothing$. Moreover:*

(1) if $m = 4$, then either F is type I, in which case it is the hyperbolic scheme and g_0 is situated inside the inner-most oval, or F is type II, in which case g_0 is the exterior of $\mathbf{R}F$;
(2) if $m = 8$, then $\mathbf{R}F$ is type I.

Proof. Necessarily, m is even and $m \geqslant 4$ since there are no $(M-2)$-conics. In this case, we have:

$$\frac{m}{2} = |Q_{\mathbf{CP}^2}(F, \mathbf{C}G_0)| \leqslant \frac{\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0}{2} \leqslant r,$$

with r the number of isolated fixed points for $s_F : \Sigma(F) \to \Sigma(F)$.

If F is type I, then $\Sigma(F)$ is a torus, and $\mathrm{Fix}(s_F)$ *does* contained isolated fixed points. In this case, by Corollary 4.6, we have that $\mathrm{Fix}(s_F)$ is at most 4 points, and thus $m \leqslant 8$. Moreover, $\mathrm{Fix}(s_F)$ does not contain circle components, thus $\widetilde{\mathbf{R}}F = \varnothing$. The real scheme of $\mathbf{R}F$ when $m = 4$ is immediate.

If F is type II, the surface $\Sigma(F)$ is a Klein bottle. By Proposition 4.7, we obtain that $m \leqslant 4$ and thus $m = 4$. This implies that $\mathbf{R}F$ is two ovals outside one another. Moreover, $\mathrm{Fix}(s_F)$ contains at most one circle, which necessarily comes from an oval of $\widetilde{\mathbf{R}}F$. Therefore, either $\widetilde{\mathbf{R}}F = \varnothing$, or $\widetilde{\mathbf{R}}F$ is one oval, which is necessarily enveloping the point g_0. Assume by contradiction that $\widetilde{\mathbf{R}}F$ is one oval. Consider the surface $\widetilde{\Sigma}(F)$ and its involution $\widetilde{s}_F$. They are constructed in the same way as $\Sigma(F)$ and s_F, but by filling the involution $\widetilde{s} : F/\widetilde{\mathrm{conj}} \to F/\widetilde{\mathrm{conj}}$ instead of $s : F/\mathrm{conj} \to F/\mathrm{conj}$ (note that $\mathbf{CP}^2/\widetilde{\mathrm{conj}}$ is diffeomorphic to $\mathbf{CP}^2/\mathrm{conj}$, but is not *equal* to it). In this case, $\widetilde{\Sigma}(F) \cong 3\mathbf{RP}^2$. By Proposition 4.7, we obtain that $\mathrm{Fix}(s_F)$ must contain at least one circle. This is excluded, since the only way would be if $\widetilde{\mathbf{R}}F \pitchfork \widetilde{\mathbf{R}}G_0 \neq \varnothing$, whereas we assumed the contrary.

We can construct explicit realizations of both exceptional $(M-2)$-quartics as follows; consider the following union of two conics (we give the affine equations):

$$F_I(x,y) = (x^2+y^2+1)(x^2+y^2+2) \text{ and } F_{II}(x,y) = ((x-1)^2+y^2-2)((x+1)^2+y^2-2).$$

Those are singular quartics, but all singular points lie in $\mathbf{CP}^2 \smallsetminus \mathbf{RP}^2 \cup \widetilde{\mathbf{RP}}^2$. Therefore, up to a symmetric perturbation, those are constructions of the two exceptional $(M-2)$-quartics. ■

Proposition 4.38. *Let F be a flexible symmetric $(M-2)$-curve of degree m which is not exceptional. Then $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 \in \{m, m-2, m-4\}$ and $b_0(\Gamma'_F) \in \{1,2\}$. Moreover:*

(1) if $b_0(\Gamma'_F) = 2$, then $\widetilde{\mathbf{R}}F$ only consists of components that meet $\widetilde{\mathbf{R}}G_0$;

(2) if $b_0(\Gamma'_F) = 1$, then $\widetilde{\mathbf{R}}F$ contains those components and possibly at most two more ovals, with either one oval having g_0 in its interior, or a pair of ovals not enveloping g_0 and swapped by σ (i.e. either $\widetilde{q} = 1$ or $\widetilde{r} = 1$).

In the case where $m = 2k$ is even, we have the following.

(1) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m$ and $(F, \mathbf{R}F)$ is type I, then g_0 is in the exterior of $\mathbf{R}F$, and $(F, \widetilde{\mathbf{R}}F)$ is type I if and only if $b_0(\Gamma'_F) = 2$ or $b_0(\Gamma'_F) = 1$ and $\widetilde{q} = 1$.

(2) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m$ and $(F, \mathbf{R}F)$ is type II, then either g_0 is exterior to $\mathbf{R}F$, or is situated inside exactly two ovals of $\mathbf{R}F$. If g_0 is inside two ovals, then $b_0(\Gamma'_F) = 1$.

(3) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m-2$, then $(F, \mathbf{R}F)$ is type II, g_0 is inside exactly one oval of $\mathbf{R}F$, $b_0(\Gamma'_F) = 1$ and $\widetilde{q} = \widetilde{r} = 0$.

(4) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m-4$, then $(F, \mathbf{R}F)$ is type II, g_0 is exterior to $\mathbf{R}F$, $b_0(\Gamma'_F) = 1$ and $\widetilde{q} = \widetilde{r} = 0$.

Proof. First, if $(F, \mathbf{R}F)$ is type I, then Lemma 4.15 gives that $\mathbf{R}F \pitchfork \mathbf{R}G_0$ is m points. If it is type II and $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 \neq m$, then $\Sigma(F)$ is a Klein bottle and $\mathrm{Fix}(s_F)$ contains at most two points (and if it does contain points, there are exactly two), which necessarily come either from a pair of conjugate points in $(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0$ or from an oval of $\mathbf{R}F$ which envelops g_0. This means that $\#(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0 \leqslant 4$, and thus $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 \in \{m-2, m-4\}$.

We now prove the statements regarding the even degree case.

(1) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m$ and $\mathbf{R}F$ is type I, then $\Sigma(F)$ is a torus. By Proposition 4.19, g_0 is exterior to $\mathbf{R}F$. By a drawing analogous to that in Figure 4.8 (simply add one "genus hole" to each half), we obtain that $(F, \widetilde{\mathbf{R}}F)$ is type I if and only if there are two blue components, *i.e.* two circles in $\mathrm{Fix}(s_F)$. This occurs only in the prescribes scenarii.

(2) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m$ and $\mathbf{R}F$ is type II, then $\Sigma(F)$ is a Klein bottle. From Proposition 4.7, either $\mathrm{Fix}(s_F)$ contains no isolated points, or it contains two. They each respectively correspond to the cases where g_0 is exterior to $\mathbf{R}F$ or situated inside two ovals of $\mathbf{R}F$. In this second case, we also have that $\mathrm{Fix}(s_F)$ contains at most (and thus exactly) one circle component, giving that $b_0(\Gamma'_F) = 1$ and $\widetilde{q} = \widetilde{r} = 0$.

(3) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m-2$, then $(F \smallsetminus \mathbf{R}F) \pitchfork \mathbf{C}G_0 \neq \varnothing$, and thus $\mathrm{Fix}(s_F)$ contains at least one isolated point, coming from such a pair of complex conjugate intersection points. Since $\mathrm{Fix}(s_F)$ also contains at least one circle component, Corollary 4.6 ensures that $\Sigma(F)$ cannot be orientable, and thus $(F, \mathbf{R}F)$ is type II. Moreover, Proposition 4.7 yields that there must be exactly two isolated points and at most one circle component. This ensures that $b_0(\Gamma'_F) = 1$, $\widetilde{q} = \widetilde{r} = 0$ and that g_0 is situated inside one oval of $\mathbf{R}F$ (this gives the additional isolated point).

(4) If $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = m-4$, the reasoning is very similar to that of the previous case. ∎

Again, deeper knowledge about the mirror curve might produce a refining of the Theorem 1.32(4) for symmetric $(M-2)$-curves. We refer the interesed reader to Trille's work [Tri03] for a refinement of the Kharlamov–Marin congruence, where the condition ensuring that F is type I is a congruence mod 4.

In view of the case study of exceptional $(M-i)$-curves for $i \in \{0,1,2\}$, one might be tempted to think that an exceptional curve always has an empty mirror curve. This does not hold. Indeed, in degree 8 for instance, one may consider a union of two circles in $\mathbf{RP}^2$ swapped by σ, together with the union of two more circles in $\widetilde{\mathbf{RP}}^2$ swapped by σ. The curve we obtain has $\mathbf{R}F$ containing two ovals outside one another, and $\widetilde{\mathbf{R}}F$ is also two ovals outside one another.

4.3.4 The Symmetric Classification in Low Degrees

We wish to answer Question 4.13 in low degrees for flexible curves. That is: given a flexible symmetric curve F of degree $m \leqslant 5$, what are the possible pairs of topological types for $(\mathbf{RP}^2, \mathbf{R}F \cup \mathbf{R}G_0 \cup \{g_0\})$ and $(\widetilde{\mathbf{RP}}^2, \widetilde{\mathbf{R}}F \cup \widetilde{\mathbf{R}}G_0 \cup \{g_0\})$? In fact, both those pairs are uniquely determined by the subset $\Gamma_F \subset \Omega$ defined in §4.3.1.

Recall that Ω is obtained by gluing the two discs $\pi \circ p(\mathbf{RP}^2)$ and $\pi \circ p(\widetilde{\mathbf{RP}}^2)$ along their common boundary $\pi \circ p(\mathbf{R}G_0)$ and an interior point $\pi \circ p(g_0)$. However, it is easier to represent Ω as a disc where its boundary points are identified in pairs when they share the same x-coordinate, and where the left-most and right-most boundary points are identified together. The subset $\pi \circ p(\mathbf{R}F)$ of Γ_F will be colored red, and $\pi \circ p(\widetilde{\mathbf{R}}F)$ will be colored blue; see Figure 4.11 for an example.

In order to recover $\mathbf{R}F$ and $\widetilde{\mathbf{R}}F$ from Γ_F, it suffices to lift through $\pi \circ p$ the red and blue parts accordingly. Equivalently, one can simply cut the disc used to represent Ω in half, take two copies of the corresponding half, and glue them together along $\pi \circ p(\mathbf{R}G_0)$ to obtain the disc representation of $\mathbf{RP}^2$ or $\widetilde{\mathbf{RP}}^2$ and of $\mathbf{R}F$ or $\widetilde{\mathbf{R}}F$ inside.

We first prove the following.

Theorem 4.39. *The classification of symmetric curves of degrees $m \leqslant 3$ is given by Figure 4.12.*

Proof. The case of degree one is immediate. In degree two, there are only two possibilities for $\mathbf{R}F$. It

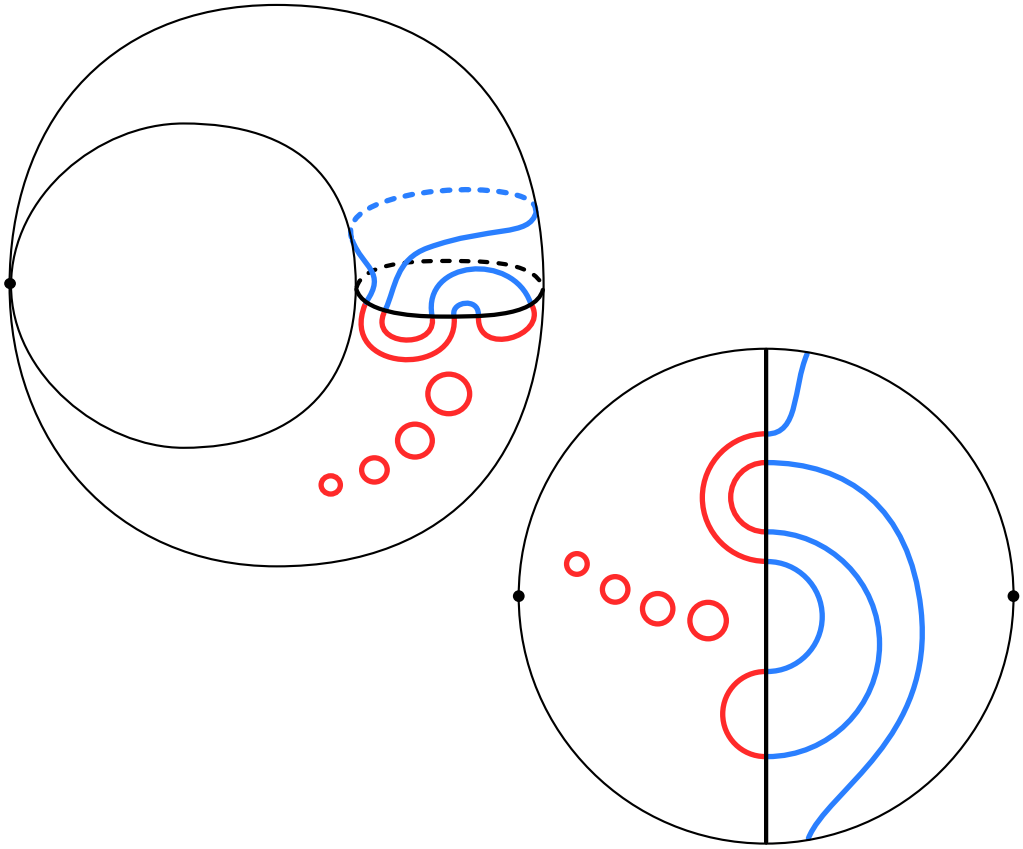


Figure 4.11. A representation of the subset Γ_F of Ω where F is the sextic depicted in Figure 4.5.

can be empty, in which case Proposition 4.25 ensures that $\widetilde{\mathbf{R}}F$ is not empty, and g_0 is located in $\widetilde{\mathbf{RP}}^2_+$, by Proposition 4.21 applied to the M-curve $\widetilde{\mathbf{R}}F$. It can also be a single oval, in which case either it is exceptional, and thus $\widetilde{\mathbf{R}}F = \varnothing$ and g_0 is inside that oval, or it intersects $\mathbf{R}G_0$ in two points, which means that $\widetilde{\mathbf{R}}F \neq \varnothing$ is a single oval as well.

For the case of degree three, there are again only two possibilities for $\mathbf{R}F$. If it is only a pseudo-line and contains no oval, then it is an $(M-1)$-curve, and Proposition 4.32 ensures that $\widetilde{\mathbf{R}}F$ also consists only of a pseudo-line without ovals. If, however, $\mathbf{R}F$ does contain an oval, then since it is a non-exceptional M-curve, Theorem 4.27 guarantees that $\widetilde{\mathbf{R}}F$ is also a pseudo-line with one oval. Lastly, the pseudo-line of $\widetilde{\mathbf{R}}F$ must intersect an oval of $\mathbf{R}F$, otherwise this would violate Proposition 4.28. ■

In degree four, the situation is already slightly more complicated. We can study the case of M-quartics first, then $(M-1)$-curves, *etc.*

There is an exceptional M-quartic, and there is only one possibility for a non-exceptional M-quartic.

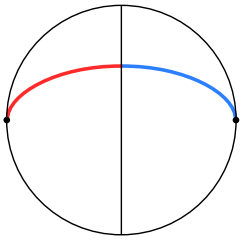
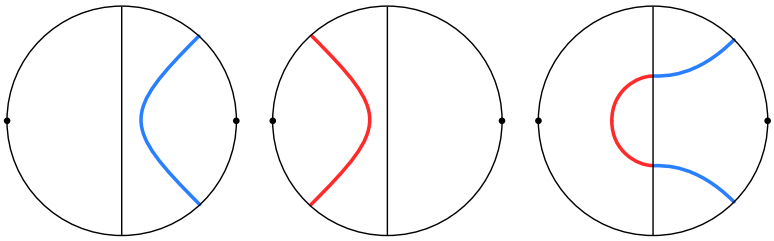
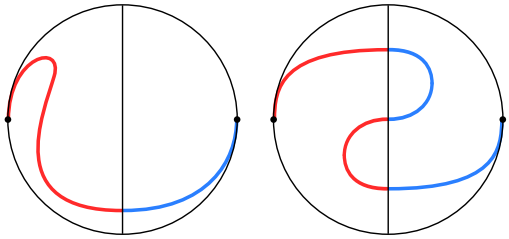
Degree	List of pairs of schemes
1	
2	
3	

Figure 4.12. The symmetric classification of flexible symmetric curves of degree less than 3.

In the first case, the mirror scheme is empty, and in the second, it is the hyperbolic scheme.

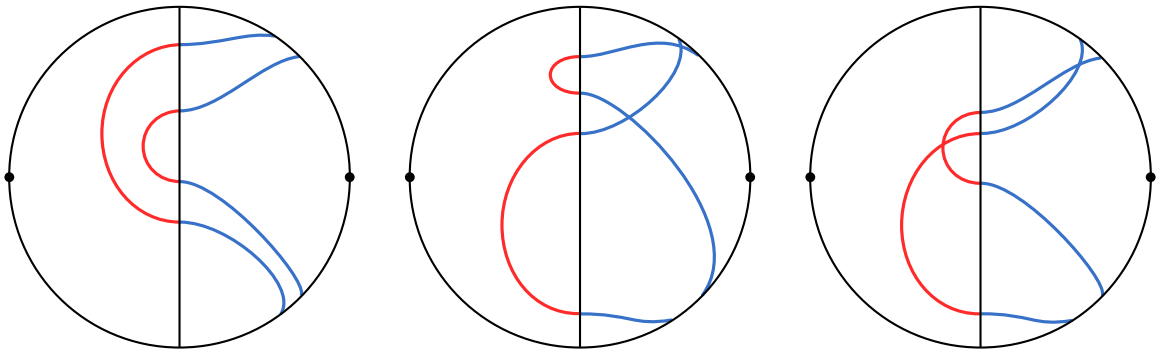
For $(M-1)$-quartics, there are no exceptional curves. There are two types of non-exceptional ones, depending on the cardinality of $\mathbf{R}F \pitchfork \mathbf{R}G_0$.

(1) If $\mathbf{R}F \pitchfork \mathbf{R}G_0$ is four points, then $\widetilde{\mathbf{R}}F$ is two ovals by Proposition 4.32. Moreover, g_0 is situated inside the only oval of $\mathbf{R}F$ which does not intersect $\mathbf{R}G_0$. In the algebraic case, we can also ensure that $\widetilde{\mathbf{R}}F$ is type II, by Proposition 4.35. This means that $\widetilde{\mathbf{R}}F$ is two ovals. Lastly, since $b_0(\Gamma'_F) = 1$ by Proposition 4.33, there is only one possibility for Γ_F, since g_0 must be situated in the exterior of the curve by Proposition 4.38 applied to the $(M-2)$-quartic $\widetilde{\mathbf{R}}F$.

(2) If $\mathbf{R}F \pitchfork \mathbf{R}G_0$ consists of two points only, then $\widetilde{\mathbf{R}}F$ is a single oval, by Proposition 4.32, and g_0 is necessarily in the exterior of that oval.

Both scenarii can occur algebraically. Indeed, we have the following observations.

(1) If L is a real line which is transverse to $\mathbf{R}G_0$ and such that $g_0 \notin \mathbf{R}L$, then the union $F = L \times \sigma(L)$ of L and $\sigma(L)$ is a singular conic, $\mathbf{R}F$ has a single double point located on $\mathbf{R}G_0$ and $\widetilde{\mathbf{R}}F$ is an acnode located on $\widetilde{\mathbf{R}}G_0$ (in fact, those singularities are the same point, which is a non-degenerate



(a) (b) (c)

Figure 4.13. (a) The union of conics where one is inside the other. (b) The union where they are outside one another. (c) The union where they meet transversely in two points.

critical point of F). Taking a small perturbation of F resolves the singularity of $\mathbf{R}L$, and either kills the acnode or makes it evolve into an oval.

(2) A non-exceptional M-conic is self-dual, in the sense that the mirror scheme is also a non-exceptional M-conic.

(3) The union of two non-exceptional conics can occur in a handful of possible ways, three of which are interesting for us and are depicted in Figure 4.13. This is a singular quartic with double point singularities only. Those can be perturbed in the class of symmetric curves, and Brusotti's theorem[†] still applies, meaning that we can choose in what way the smoothing occurs.

This can be used to construct *symmetric* curves by perturbing singular quartics as unions of conics and lines.

Now, for $(M-2)$-quartics, we study all possibilities. First, there are two exceptional curves, described in Proposition 4.37. Next, we assume that $\mathbf{R}F$ is the hyperbolic scheme and that $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = 4$. There are three options.

(1) $\widetilde{\mathbf{R}}F$ contains only two ovals, which are meeting $\widetilde{\mathbf{R}}G_0$. Then $\widetilde{\mathbf{R}}F$ is either the hyperbolic scheme or the non-dividing $(M-2)$-quartic. Both cases are constructed by an appropriate perturation of a singular quartic made up of the transverse union of conics.

(2) $\widetilde{\mathbf{R}}F$ contains one additional oval enveloping g_0 together with the two meeting $\widetilde{\mathbf{R}}G_0$. This means that $\widetilde{\mathbf{R}}F$ is an $(M-1)$-curve and its mirror curve $\mathbf{R}F$ is type I. This is prevented by Proposition 4.35, and thus this configuration cannot exist.

(3) $\widetilde{\mathbf{R}}F$ contains one additional pair of ovals swapped by σ. This means that $\widetilde{\mathbf{R}}F$ is an M-quartic, and this exists (this is the dual scenario of that when $\mathbf{R}F$ is the M-quartic.

[†] Although in this small degree case, one needs not use this fact since actual curves used to perturb are not too difficult to find explicitly.

Now, assume that $\mathbf{R}F$ is the non-dividing $(M-2)$-quartic, still with $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = 4$. Again, we test all possible cases.

(1) If $\widetilde{\mathbf{R}}F$ contains no other ovals than those two, this means that $\widetilde{\mathbf{R}}F$ is either the hyperbolic scheme or the other $(M-2)$-quartic. Again, both cases occur as perturbations of a union of conics.
(2) If $\widetilde{\mathbf{R}}F$ contains an additional oval enveloping g_0, this means that $\widetilde{\mathbf{R}}F$ is an $(M-1)$-quartic. This situation occurs as the dual of that when it is $\mathbf{R}F$ which is the $(M-1)$-quartic.
(3) If $\widetilde{\mathbf{R}}F$ contains two more ovals swapped by σ, this implies that $\widetilde{\mathbf{R}}F$ is an M-quartic with its mirror curve type II. This is prevented by Theorem 4.27, meaning that this situation does not occur.

If this time we have $\#\mathbf{R}F \pitchfork \mathbf{R}G_0 = 2$, then $\mathbf{R}F$ cannot be the hyperbolic scheme. This means that $\mathbf{R}F$ is composed of one oval meeting $\mathbf{R}G_0$ and one other oval enveloping g_0. Then we have that $\widetilde{\mathbf{R}}F$ only consists of one oval meeting $\widetilde{\mathbf{R}}G_0$, by Proposition 4.38.

The cases of $b_0(\mathbf{R}F) \in \{0,1\}$ have almost all been treated. Indeed, if $\mathbf{R}F$ is a single oval meeting $\mathbf{R}G_0$, then:

(1) the case $b_0(\widetilde{\mathbf{R}}F) = 4$ does not occur, since a non-exceptional M-quartic must have 4 intersection points with $\mathbf{R}G_0$;
(2) the case $b_0(\widetilde{\mathbf{R}}F) = 3$ means that this is the dual situation to one seen before with $\mathbf{R}F$ being an $(M-1)$-quartic;
(3) similarly, the case $b_0(\widetilde{\mathbf{R}}F) = 2$ is dual to one seen previously;
(4) the case where $b_0(\widetilde{\mathbf{R}}F) = 1$ is new, and it occurs as the curve $x^4 + y^4 - 1 = 0$.

Now, if $\mathbf{R}F = \varnothing$, then either $\widetilde{\mathbf{R}}F \neq \varnothing$ in which case we have already treated that case (or rather its dual), or $\widetilde{\mathbf{R}}F = \varnothing$. This occurs with the Fermat curve $x^4 + y^4 + 1 = 0$.

We therefore have completely proved the following.

Theorem 4.40. *The list of all possible degree 4 symmetric curves is shown in Figures 4.14, 4.15 and 4.16.*

4.3.5 Ragsdale's Conjecture for Symmetric M-Curves

Define $R = \frac{3k(k-1)}{2} + 1$. Ragsdale formulated the following conjecture[†].

Conjecture 4.41 (Ragsdale)**.** *Let F be an algebraic curve of even degree $m = 2k$. Then:*

$$p \leqslant R \text{ and } n \leqslant R.$$

Conjecture 4.41 was disproved by Itenberg in [Ite93] (see also [IV96]), but the examples he gave were not M-curves. In paricular, the Ragsdale conjecture is still open for M-curves.

We investigate this conjecture for symmetric curves. Consider the negative Arnold surface $\mathcal{A}_-(F) = F/\mathrm{conj} \cup p(\mathbf{RP}^2_-) \subset \mathbf{S}^4$. The involution $s : \mathbf{S}^4 \to \S^4$ restricts to $\mathcal{A}_-(F)$, and the quotient $\mathcal{A}_-(F)/s$ has boundary diffeomorphic to $\mathcal{A}_-(F) \cap \mathrm{Fix}(s)$. As in Definition 4.30, denote as $\Omega = \widetilde{\mathcal{R}}_-/s$ where $\widetilde{\mathcal{R}}_-$ intersects $p(\mathbf{RP}^2_-)$ in arcs.

[†] She actually conjectured that $n \leqslant R-1$, but this was later disproved by Viro with counter-examples with $n = R$.

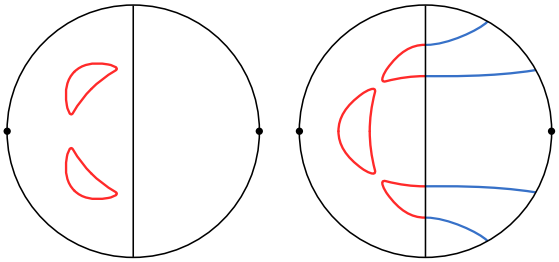
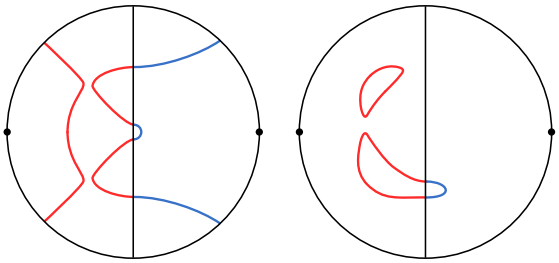
$b_0(\mathbf{R}F)$	**List of pairs of schemes**
4	
3	

Figure 4.14. The symmetric classification of all possible algebraic symmetric quartics (1/3).

Definition 4.42. *The* ***negative Arnold surface of the second kind*** *is the surface* $\mathcal{B}_-(F) = \mathcal{A}_-(F)/s\cup\Omega$.

We now assume until the end that F is a flexible symmetric M-curve of degree $m = 2k$. A computation gives:

$$\begin{cases} \chi(\mathcal{B}_-(F)) = -k^2+1+\frac{3k-(p-n)}{2}+\chi(\Omega)-\chi(\mathcal{A}_-(F)\cap\widetilde{\mathcal{R}}_-), \\ e(S,\mathcal{B}_-(F)) = k^2-1+(p-n)+e(S,\Omega). \end{cases}$$

We check that Ω is a collection of $\left\lfloor\frac{k}{2}\right\rfloor+1$ discs, and that $e(\Omega)=-1$. Moreover, $\mathcal{A}_-(F)\cap\Omega$ is either a collection of $\left\lfloor\frac{k}{2}\right\rfloor+1$ circles (mainly $\partial\Omega$), or it is that collection of circles together with the point g_0. This point is only present when $g_0\in\Omega$, that is when k is even. In particular, the surface $\mathcal{B}_-(F)$ is nodally immersed with a single double point when k is even. In the odd degree case, set $\mathcal{X}(F)=\mathcal{B}_-(F)$. In the even degree case, we can smooth the singularity by gluing a Hopf band to obtain an embedded surface $\mathcal{X}(F)$. We obtain:

$$\begin{cases} \chi(\mathcal{X}(F)) = -k^2+\left\lfloor\frac{k}{2}\right\rfloor+2+\frac{3k-(p-n)}{2}-[1+(-1)^k] \\ e(\mathcal{X}(F)) = k^2-2+(p-n)\pm[1+(-1)^k], \end{cases}$$

and we have the choice in the $\pm$. We pick the $-$ sign (to minimize the b_2^- later). This therefore yields:

$$\begin{cases} \chi(\mathcal{X}(F)) = -k^2+\left\lfloor\frac{k}{2}\right\rfloor+1-(-1)^k+\frac{3k-(p-n)}{2} \\ e(\mathcal{X}(F)) = k^2-3-(-1)^k+(p-n). \end{cases}$$

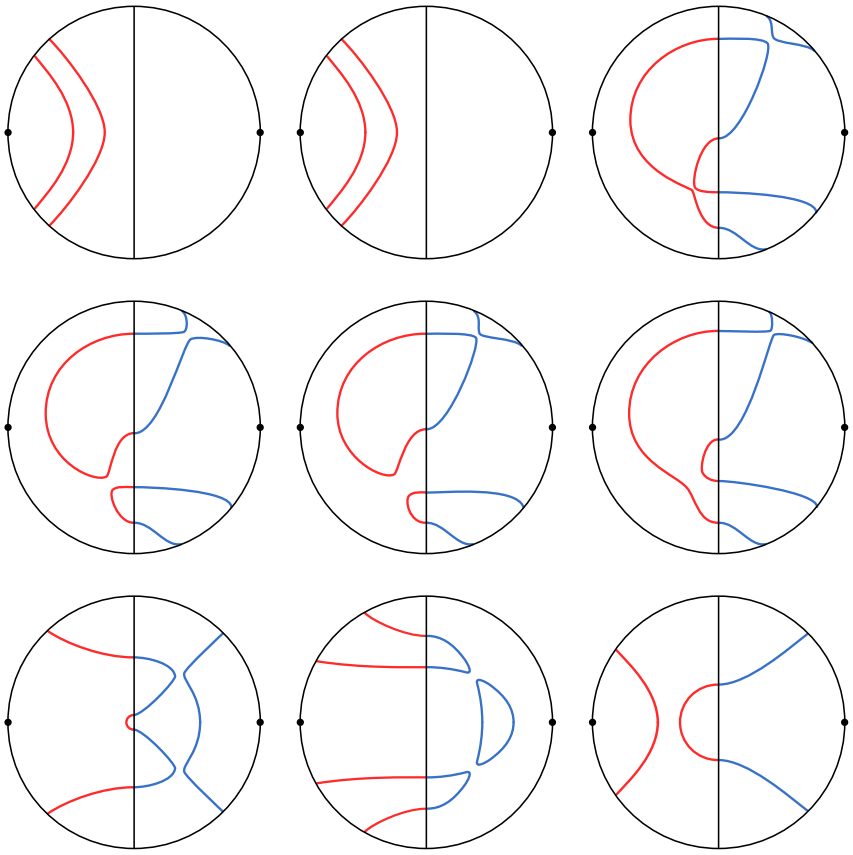
$b_0(\mathbf{R}F)$	**List of pairs of schemes**
2	

Figure 4.15. The symmetric classification of all possible algebraic symmetric quartics (2/3).

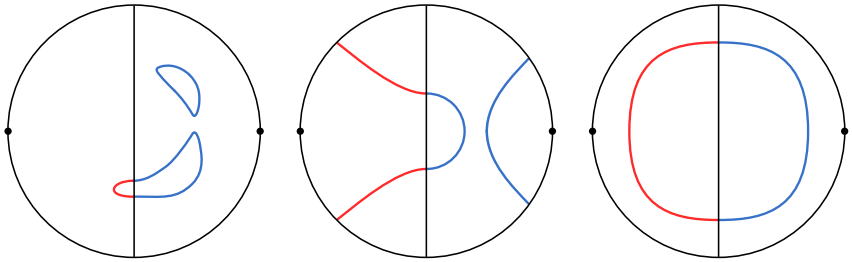
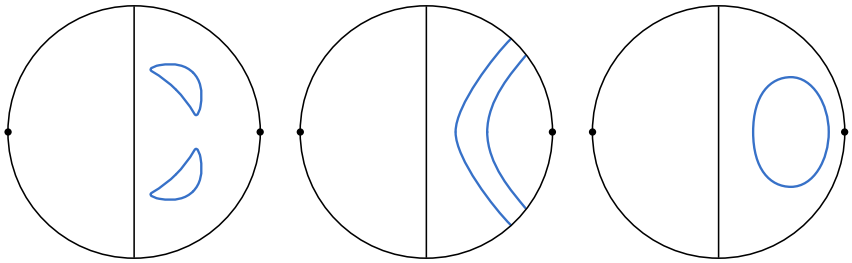
$b_0(\mathbf{R}F)$	**List of pairs of schemes**
1	
0	

Figure 4.16. The symmetric classification of all possible algebraic symmetric quartics (3/3). There is also the empty curve, not depicted.

Consider Y the 4-manifold which is the double branched cover of S ramified along $\mathcal{X}(F)$. By reasoning analogous to previous computations, we obtain:

$$b_2(Y) = k^2 - \left\lfloor \frac{k}{2} \right\rfloor + 1 + (-1)^k + \frac{(p-n) - 3k}{2} \text{ and } \sigma(Y) = \frac{-k^2 + 3 + (-1)^k - (p-n)}{2}.$$

This allows to compute that

$$2b_2^-(Y) = \frac{3k(k-1)}{2} - \left\lfloor \frac{k}{2} \right\rfloor + \frac{(-1)^k - 1}{2} + (p-n).$$

Using a membrane coming partially from $\widetilde{\mathbf{RP}}^2_{\pm}$ (the one opposite to that whose image is $\widetilde{\mathcal{R}}_-$), we can derive that $b_2^-(Y) \geqslant 1$. This yields the strengthened Petrovskii inequality

$$n \leqslant p + \frac{3k(k-1)}{2} - \left\lfloor \frac{k}{2} \right\rfloor + \frac{(-1)^k - 1}{2} - 2.$$

Using the fact that we must have $p + n = \frac{(m-1)(m-2)}{2} + 1$ and $p - n \equiv k^2$ [16], this restricts the values for p and n. We have, for instance, the following table.

$m = \cdots$	$n \in \cdots$	$\frac{3k(k-1)}{2} + 1 = \cdots$
8	$\{19, 11, 3\}$	19
10	$\{30, 22, 14, 6\}$	31
12	$\{50, 42, 34, 26, 18, 10, 2\}$	46
14	$\{79, 71, 63, 55, 47, 39, 31, 23, 15, 7\}$	64
16	$\{101, 93, 85, 77, 69, 61, 53, 45, 37, 29, 21, 13, 5\}$	85
18	$\{132, 124, 116, 108, 100, 92, 84, 76, 68, 60, 52, 44, \cdots\}$	109

In particular, we see that $n \leqslant R$ is immediately true in degrees 8 and 10 for symmetric M-curves. In degree 12, to prove that $n \leqslant R$, it only suffices to show that $n = 50$ is impossible (or equivalently, that $p = 6$ is impossible). If there existed a curve of degree 12 with $p = 6$, the strengthened Petrovskii inequality gives:

$$n \leqslant 46,$$

a contradiction.

In the case of cures of degee 14, we need to rule out the case $(n, p) = (71, 8)$, since there are no curves with $n = 79$ (this implies that $p = 0$ because of $p + n = 79$). Again, if there was such a curve with $p = 8$, then the strengthened Petrovskii bound gives $n \leqslant 65$, a contradiction.

This means that the inequality $n \leqslant R$ is satisfied for symmetric M-curves of even degree $m \leqslant 14$.

Appendix: Plotting Algebraic Curves

The general idea is to use the following parametrization of the real projective plane:

$$\begin{array}{rccc} \Phi & : & \overline{\mathbf{D}}^2 & \longrightarrow & \mathbf{RP}^2 \\ & & (x,y) & \longmapsto & \left[x : y : \sqrt{1-x^2-y^2}\right]. \end{array}$$

This means that the image of the closed 2-disc $\overline{\mathbf{D}}^2$ under Φ is $\mathbf{RP}^2$, and that the line at infinity L_∞ is the image of the boundary circle. Note that the map Φ is $1:1$ from the interior $\mathbf{D}^2$ to the affine part $\mathbf{RP}^2 \setminus L_\infty$, and that it is $2:1$ from the circle $\partial\overline{\mathbf{D}}^2$ to L_∞.

Finally, recall that if

$$A(x,y) = \sum_{i+j\leqslant m} \lambda_{i,j} x^i y^j$$

is an affine curve, the associated projective curve has the equation

$$A_h(x_0,x_1,x_2) = \sum_{i+j\leqslant m} \lambda_{i,j} x_0^i x_1^j x_2^{m-i-j}.$$

Alternatively, this can be obtained by the following procedure :

$$A_h(x_0,x_1,x_2) = x_2^m A(x_0/x_2, x_1/x_2).$$

We will describe the procedure we used to plot projective curves out of their affine equations.

A.1 In Plain Python

This was done using the `Python 3.10.6` kernel. First, import the requires libraries.

```
# Imports
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp
```

Then, define the formal variables used to input the polynomial equations.

```
# Polynomial indeterminates
X = sp.Symbol("X")
Y = sp.Symbol("Y")
Z = sp.Symbol("Z")
```

Define the affine equation now. We take the example of a symmetric M-quartic obtained from the union of two ellipses

$$A(x,y) = (x^2+4y^2-1)(4x^2+y^2-1)$$

and by perturbing them into

$$A'(x,y) = A(x,y)+0.1.$$

```
# The equation
p = 10*(X**2+4*Y**2-1)*(4*X**2+Y**2-1)+1
```

Next, we need to homogenize the affine equation. This can be done by substitutions as follows.

```
# Homogenization
pp = sp.poly(p).total_degree()
q = p.subs([(X,X/Z),(Y,Y/Z)])
q = sp.simplify(Z**d*q)
```

We now need to create the function

$$f : \overline{\mathbf{D}}^2 \to \mathbf{R}$$

of which we want to plot the contour.

```
# Function to plot the zero-level of
P = sp.lambdify((X,Y,Z),q)
f = lambda x,y: P(x,y,np.sqrt(1-x**2-y**2))
```

We are now ready to plot. First, some initialization.

```
# Init
fig, ax = plt.subplots()

# The range of the plot
ax.set_aspect("equal")
plt.axis("off")
plt.xlim([-1.05,1.05])
plt.ylim([-1.05,1.05])

# The size of the actual image
fig.set_size_inches(4.15,4.15)
plt.tight_layout()
```

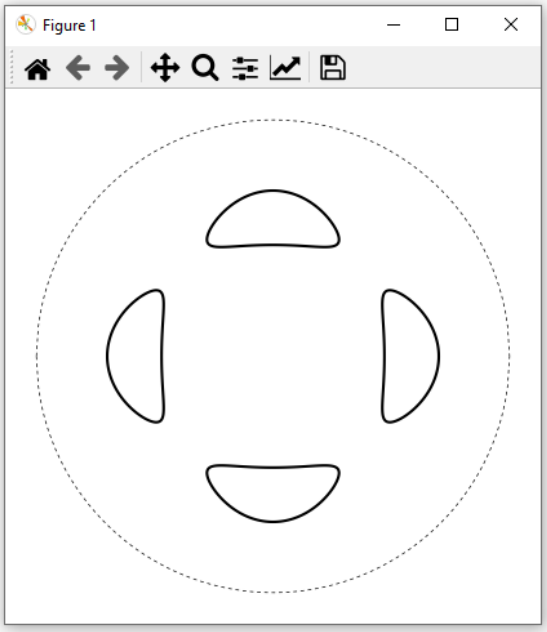
Now, we draw the dashed circle which represents $\mathbf{RP}^2$.

```
# We will use two semi-circles
cx = np.linspace(-1,1,num=250,endpoint=True)
cy = np.sqrt(1-cx**2)

# Style dictionary
cd = {"color":"k","linestyle":"--","linewidth":.5,"dashes":(4,4.1)}

# Plotting
plt.plot(*(cx,cy),**cd)
plt.plot(*(cx,-cy),**cd)
```

We can finally overlay the contour of f to plot the curve.

```
# The range
xr = np.linspace(-1,1,num=1000,endpoint=True)
yr = np.linspace(-1,1,num=1000,endpoint=True)

# The numpy object
x,y = np.meshgrid(xr, yr)
eq = f(x,y)

# Plotting
plt.contour(x,y,eq,[0],colors=["k"])

# Done!
plt.show()
```

With this example, we obtain the following plot.

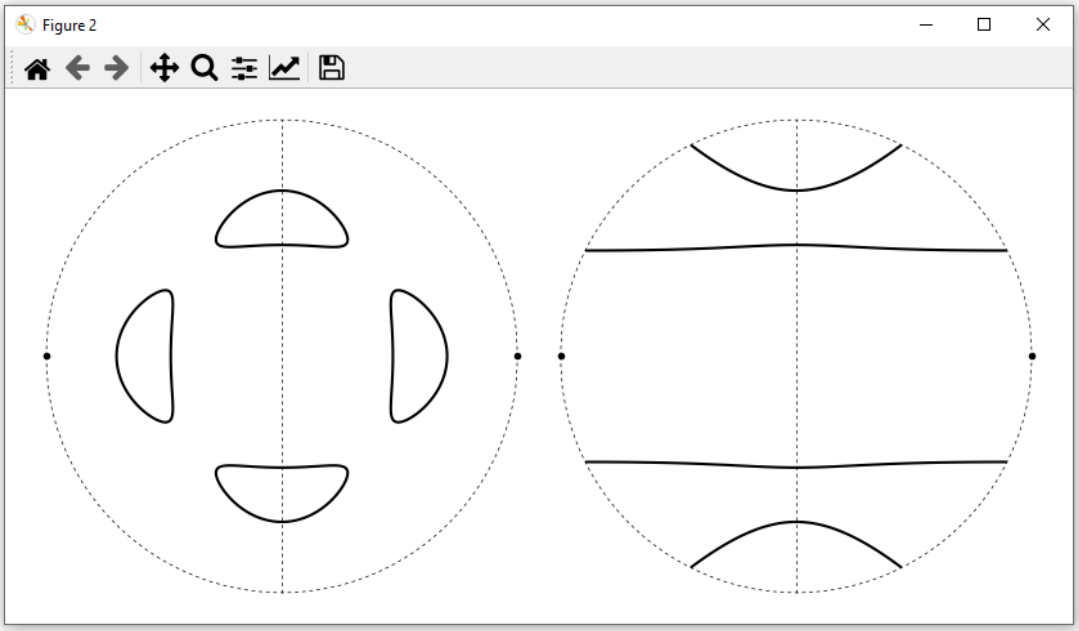


If we want to plot a *symmetric* curve, we can adapt the previous code into the following one.

```
# Imports
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp

# Polynomial indeterminates
X = sp.Symbol("X")
Y = sp.Symbol("Y")
Z = sp.Symbol("Z")

# The equation
p = 10*(X**2+4*Y**2-1)*(4*X**2+Y**2-1)+1

# Homogenization
d = sp.poly(p).total_degree()
q = p.subs([(X,X/Z),(Y,Y/Z)])
q = sp.simplify(Z**d*q)

# Functions to plot the zero-level of
P = sp.lambdify((X,Y,Z),q)
f = lambda x,y: P(x,y,np.sqrt(1-x**2-y**2))
g = lambda x,y: P(1j*x,y,np.sqrt(1-x**2-y**2))

# Init
fig, (ax1,ax2) = plt.subplots(ncols=2)

# Two semi-circles
cx = np.linspace(-1,1,num=250,endpoint=True)
cy = np.sqrt(1-cx**2)

# Semi-circles style dictionary
cd = {"color":"k","linestyle":"--","linewidth":.5,"dashes":(4,4.1)}

# The size of the actual image
fig.set_size_inches(2*4.15,4.15)
plt.tight_layout()

for ax in [ax1,ax2]:
    # The range of the plot
    ax.set_aspect("equal")
    ax.axis("off")
    ax.set_xlim([-1.05,1.05])
    ax.set_ylim([-1.05,1.05])

    # Dashed circle
    ax.plot(*(cx,cy),**cd)
    ax.plot(*(cx,-cy),**cd)

    # Elements of symmetry
    ax.plot([0,0],[-1,1],**cd)
    ax.plot([-1],[0],color="k",marker=".",markersize=6,label="a")
    ax.plot([1],[0],color="k",marker=".",markersize=6,label="a")
```

```
# The range
xr = np.linspace(-1,1,num=1000,endpoint=True)
yr = np.linspace(-1,1,num=1000,endpoint=True)

# The numpy objects
x,y = np.meshgrid(xr, yr)
eq1 = f(x,y)
eq2 = g(x,y)

# Plotting
ax1.contour(x,y,eq1,[0],colors=["k"])
ax2.contour(x,y,eq2,[0],colors=["k"])

# Done!
plt.show()
```

The result is the following plot this time. On the left, we see the curve $\mathbf{R}F$, and on the right, its mirror curve $\widetilde{\mathbf{R}}F$.



A.2 In Sage

We use the `SageMath 10.2` kernel. Note that this will return an error in lots of cases with the `SageMath <=9` kernel.

We will describe how to plot a symmetric curve, since that also covers the general case. Again, we start with some preparation.

In [1]:
```
# The polynomial ring
R.<x,y> = PolynomialRing(ZZ)

# The indeterminates
X = var("X")
Y = var("Y")
Z = var("Z")

# Intederminates for the parametrizations
u = var("u")
v = var("v")
```

In [2]:
```
# Colors
red = (1, 0, 0)
blue = (0, 0, 1)
yellow = (1, .75, 0)
black = (0, 0, 0)

# Style dictionaries
range_dict = ((u, -1, 1), (v, -1, 1))
style_dict = {"linewidth": 1, "frame": False}
curve_dict = {"borderwidth": 1, "bordercol": black}

# The unit disc
disc = u**2 + v**2 <= 1
```

Next, we input the curve. We will pick the same one as before.

In [3]:
```
P = 10*(x^2+4*y^2-1)*(4*x^2+y^2-1)+1
```

We homogenize, as before, and we compute the functions which we need the contour of.

In [4]:
```
# Homogenization
hP = P.homogenize()(x = X, y = Y, h = Z)

# Function to plot the contour of
f = hP(X = u, Y = v, Z = sqrt(1 - u**2 - v**2))
g = hP(X = sqrt(-1) * u, Y = v, Z = sqrt(1 - u**2 - v**2))
```

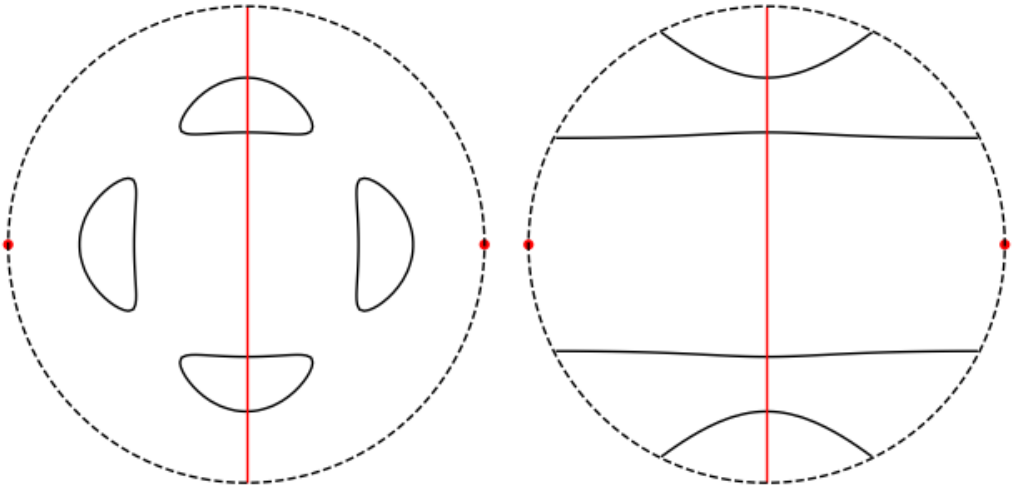
Finally, we can plot everything.

In order to simply display any curve, symmetric or not, without taking care of the mirror curve, one simply needs to remove all the corresponding lines of code, and to adapt the last command to simply

display one graphics unit instead of two.

In [5]:
```
# The curve itself
G1 = Graphics()
G1 += region_plot([f == 0, disc], *range_dict, **curve_dict)
G1 += implicit_plot(u, *range_dict, **style_dict, color = red)
G1 += point((-1, 0), color = red, size = 20)
G1 += point((1, 0), color = red, size = 20)
G1 += implicit_plot(u**2 + v**2 - 1, *range_dict, **style_dict, color = black,
    linestyle = "--")

# The mirror curve
G2 = Graphics()
G2 += region_plot([g == 0, disc], *range_dict, **curve_dict)
G2 += implicit_plot(u, *range_dict, **style_dict, color = red)
G2 += point((-1, 0), color = red, size = 20)
G2 += point((1, 0), color = red, size = 20)
G2 += implicit_plot(u**2 + v**2 - 1, *range_dict, **style_dict, color = black,
    linestyle = "--")

# Done!
graphics_array((G1, G2))
```

Out [5]:



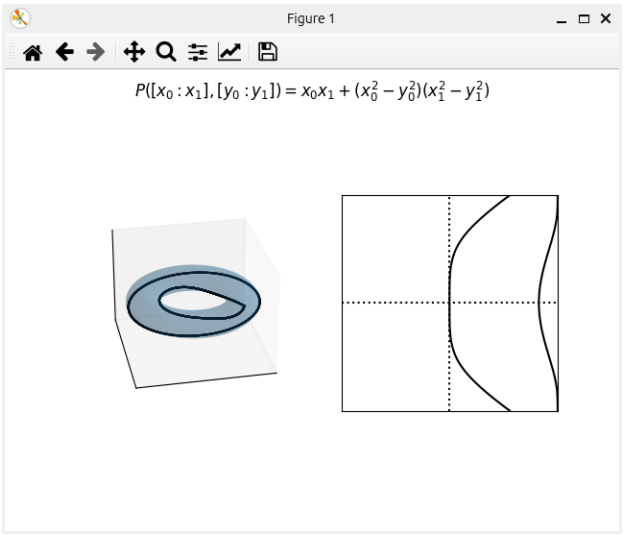
A.3 Curves on the Hyperboloid

We can adapt the previous method for plane curves to allow for plots of curves on the hyperboloid quadric.

```
# Imports
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
import sympy as sp

# Polynomial indeterminates
x0 = sp.Symbol("x_0")
x1 = sp.Symbol("x_1")
y0 = sp.Symbol("y_0")
y1 = sp.Symbol("y_1")

# The curve
P = (x0**2-y0**2)*(x1**2-y1**2)+x0*x1

# Parametrization of the torus
pp = lambda t: np.pi*(t+1)
T = lambda u,v: (((np.cos(pp(u)))+3)*np.cos(pp(v)),(np.cos(pp(u))+3)*np.sin(pp(v)),
    np.sin(pp(u)))
T2 = lambda u,v: (((1.1*np.cos(pp(u)))+3)*np.cos(pp(v)),(1.1*np.cos(pp(u))+3)*np.
    sin(pp(v)),1.1*np.sin(pp(u)))

# Lambda expression for the curve
Q = sp.lambdify((x0,x1,y0,y1),P)
f = lambda u,v: Q(u,np.sqrt(1-u**2),v,np.sqrt(1-v**2))

# The torus
angles = np.linspace(-1,1,500)
U, V = np.meshgrid(angles,angles)
X, Y, Z = T(U,V)

# Subplots shenanigans
fig = plt.figure()
ax1 = fig.add_subplot(121,projection="3d")
ax2 = fig.add_subplot(122)

# Limits for the plots
ax1.set_xlim3d(-4,4)
ax1.set_ylim3d(-4,4)
ax1.set_zlim3d(-4,4)
ax1.set_xticks([])
ax1.set_yticks([])
ax1.set_zticks([])

ax2.set_xlim(-1,1)
ax2.set_ylim(-1,1)
ax2.set_aspect("equal")
ax2.set_xticks([])
```

```
ax2.set_yticks([])

# Plot's title
tit = str(P).replace("**","^").replace("^1","").replace("*","")
tit = "$P([x_0:x_1],[y_0:y_1]!)="+tit+"$"
fig.suptitle(tit)

# The torus
ax1.plot_surface(X,Y,Z,alpha=.25)

# The contour level 0
cn = ax2.contour(U,V,eq,[0],colors=["k"])
V = cn.allsegs[0]

# Grabbing the paths and plotting them on the torus
for v in V:
    x, y = v[:,0], v[:,1]
    for i in range(len(x)-1):
        xx0, yy0, zz0 = T(x[i],y[i])
        xx1, yy1, zz1 = T(x[i+1],y[i+1])
        ax1.plot([xx0,xx1],[yy0,yy1],[zz0,zz1],"k-")

ax2.plot([-1,1],[0,0],"k:")
ax2.plot([0,0],[-1,1],"k:")

# Done
plt.show()
```

We obtain the following output plot.



A.4 Distance Functions on $\mathbf{CP}^2$

Recall from §3.1.1 that given any point $z \in \mathbf{CP}^2 \setminus (\mathbf{RP}^2 \cup Q)$, there exists a unique point $x \in \mathbf{RP}^2$ such that

$$\operatorname{dist}(z, \mathbf{RP}^2) = \operatorname{dist}(z, x).$$

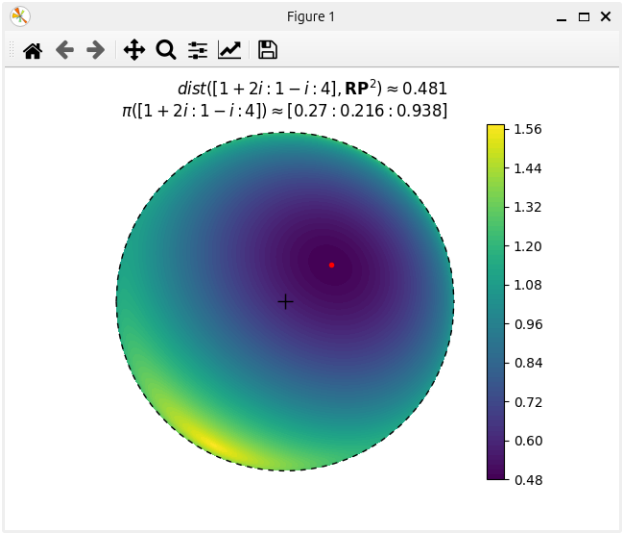
We denote as $\pi(z)$ this point which gives a map $\pi : \mathbf{CP}^2 \setminus (\mathbf{RP}^2 \cup Q) \to \mathbf{RP}^2$. Of course, this map can be extended to $\mathbf{CP}^2 \setminus Q$ by setting $\pi|_{\mathbf{RP}^2} = \mathrm{id}_{\mathbf{RP}^2}$. We are interested in visualizing this map.

Recall that the Fubini–Study distance between two points in $\mathbf{CP}^2$ is induced by the metric on the sphere. To compute the distance between a point and $\mathbf{RP}^2$, and to find the projection, this is simply a matter of optimizing a 2-variable function.

```
# Imports
import matplotlib.pyplot as plt
import numpy as np
import scipy.optimize as opt

# The complex point of interest
Z0 = [1+2j,1-1j,4]

# Distance functions
def dist(z1,z2):
    a, b, c = z1
    d, e, f = z2
    aa, bb, cc = a.conjugate(), b.conjugate(), c.conjugate()
    dd, ee, ff = d.conjugate(), e.conjugate(), f.conjugate()
    n, d = (a*dd+b*ee+c*ff)*(aa*d+bb*e+cc*f), (a*aa+b*bb+c*cc)*(d*dd+e*ee+f*ff)
    return np.arccos(np.sqrt(n/d)).real

def fun(x,y):
    return dist(Z0,[x,y,np.sqrt(1-x**2-y**2)])

mini = opt.fmin(lambda v: fun(v[0],v[1]) if v[0]**2+v[1]**2<=1 else np.inf,[0,0],
    disp=False)

# The contour
X, Y = np.meshgrid(np.linspace(-1,1,500), np.linspace(-1,1,500))
Z = fun(X,Y)

# Plotting
fig = plt.figure()
ax = fig.add_subplot()

# Plot range
ax.set_aspect("equal")
plt.axis("off")
plt.xlim([-1.05,1.05])
plt.ylim([-1.05,1.05])

# The contour
c = ax.contourf(X,Y,Z,100)
fig.colorbar(c)
```

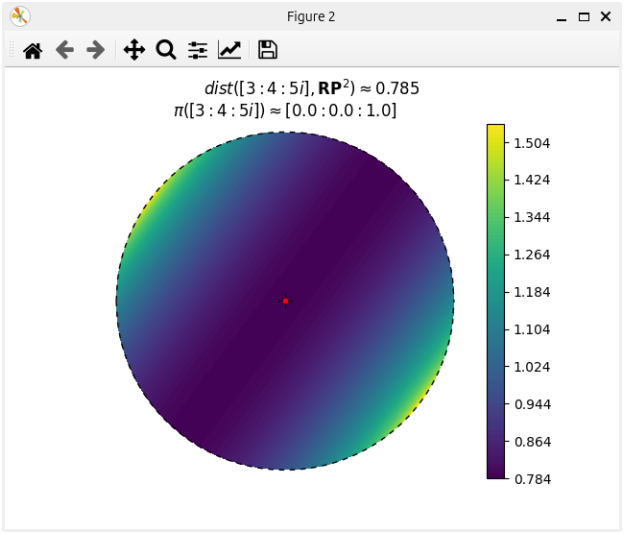
```
180
181 # Two semi-circles
182 T = np.linspace(-1,1,250)
183 C = np.sqrt(1-T**2)
184
185 style_dict = {"color":"k","linewidth":1,"linestyle":"--","dashes":(4,4.1)}
186
187 plt.plot(T,C,**style_dict)
188 plt.plot(T,-C,**style_dict)
189
190 # Middle cross
191 plt.plot([0],[0],color="k",marker="+",markersize=12,label="a")
192
193 # Projection point
194 plt.plot([mini[0]],[mini[1]],color="r",marker=".",markersize=6)
195
196 # Title
197 s0, s1, s2 = str(Z0[0]), str(Z0[1]), str(Z0[2])
198 s0, s1, s2 = s0.replace("(","").replace("!)",""), s1.replace("(","").replace("!)","
    "), s2.replace("(","").replace("!)","")
199 s0, s1, s2 = s0.replace("j","i").replace("1i","i"), s1.replace("j","i").replace("1i
    ","i"), s2.replace("j","i").replace("1i","i")
200
201 s3, s4, s5 = str(int(1000*mini[0])/1000), str(int(1000*mini[1])/1000), str(int(1000
    *np.sqrt(1-mini[0]**2-mini[1]**2))/1000)
202
203 plt.suptitle("$dist(["+s0+":"+s1+":"+s2+"],{\\bf RP}^2!)\\approx"+str(int(1000*fun(
    *mini))/1000)+"$")
204 plt.title("$\\pi(["+s0+":"+s1+":"+s2+"]!)\\approx["+s3+":"+s4+":"+s5+"]$")
205
206 # Done!
207 plt.show()
```

We get the following output.



It is interpreted in the following way. Colors in the disc indicate the distance between the corresponding point in $\mathbf{RP}^2$ and the fixed imaginary point z. The minimum is attained at the red dot, which corresponds to $\pi(z) \in \mathbf{RP}^2$.

We can look at what happens if we consider a point $z \in Q$. For instance, if $z = [2:3:5\mathbf{i}]$, then we get the following plot.



We notice that, as expected, there isn't uniqueness in a point $x \in \mathbf{RP}^2$ such that $\operatorname{dist}(z, \mathbf{RP}^2) = \operatorname{dist}(x, z)$ (and thus, the red dot is irrelevant here). In fact, there is a whole real line worth of such points, which can be seen on the plot as the "crease line" colored in dark blue. Moreover, we do see that $\operatorname{dist}(z, \mathbf{RP}^2) = \pi/4$.

We can use the previous plotting algorithm to do even more. If $F \subset \mathbf{CP}^2$ is a non-singular algebraic curve, then, letting

$$S(r) = \partial N_r = \left\{ z \in \mathbf{CP}^2 \;\middle|\; \operatorname{dist}(z, \mathbf{RP}^2) = r \right\}$$

for any $0 < r < \pi/4$, the intersection $\mathbf{C}F \cap S(r)$ is a link in a lens space $L(4,1)$. Moreover, we have the projection map

$$\pi : S(r) \cap \mathbf{C}F \to \mathbf{RP}$$

that brings this link to a collection of immersed circles in $\mathbf{RP}^2$. For all r but finitely many, this collection will be nodally immersed. We would like to depict this, as well as the evolution as r goes from being close to 0 (where we expect to see $\mathbf{R}F$) to approaching $\pi/4$ (and this should be a generic arrangement of m lines where m is the degree of the curve).

The idea is that we have a parametrization of ∂N_r by:

(1) points $x \in \mathbf{RP}^2$, and
(2) normal vectors $\mathbf{v} \in \nu_x \mathbf{RP}^2$ such that $\|\mathbf{v}\| = r$.

We can consider the following function:

$$\begin{array}{rccc} w & : & \mathbf{RP}^2 & \longrightarrow & \mathbf{R} \\ & & x & \longmapsto & \min\limits_{\substack{\mathbf{v}\in v_x\mathbf{RP}^2 \\ \|\mathbf{v}\|=r}} |f(z_{x,\mathbf{v}})|, \end{array}$$

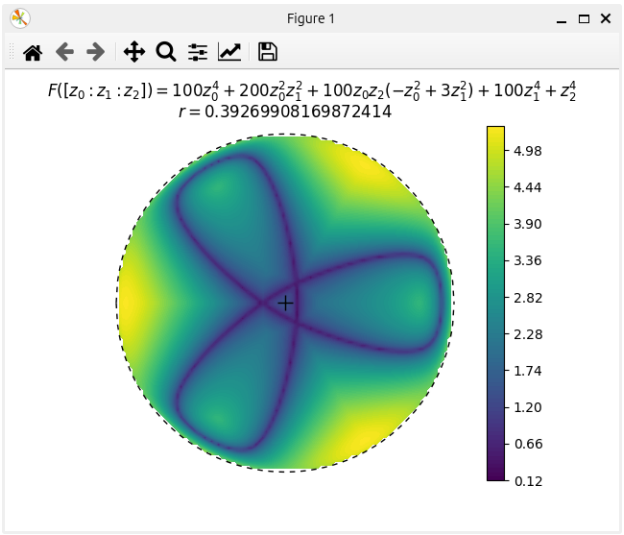
where $z_{x,\mathbf{v}}$ is the unique point in ∂N_r determined by $(x,\mathbf{v})$. We are therefore interested in plotting the zero-level set of w. We will in fact plot the filled contour of w.

```
# Imports
import matplotlib.pyplot as plt
import numpy as np
import sympy as sp
import scipy.optimize as opt
import scipy.linalg as ln

# Indeterminates
z0 = sp.Symbol("z0")
z1 = sp.Symbol("z1")
z2 = sp.Symbol("z2")

# The curve
F = 100*z0**4+200*z0**2*z1**2+100*z0*z2*(-z0**2+3*z1**2)+100*z1**4+z2**4

ff = sp.lambdify((z0,z1,z2),F)

# The level of the slice
r = np.pi/8

# Precision
prec = 100
hprec = 5

# Distance function
def dist(z1,z2):
    a, b, c = z1
    d, e, f = z2
    aa, bb, cc = a.conjugate(), b.conjugate(), c.conjugate()
    dd, ee, ff = d.conjugate(), e.conjugate(), f.conjugate()
    n, d = (a*dd+b*ee+c*ff)*(aa*d+bb*e+cc*f), (a*aa+b*bb+c*cc)*(d*dd+e*ee+f*ff)
    return np.arccos(np.sqrt(n/d)).real

# Normal geodesics to RP(2!)
def geo(x,y,s):
    a, b, c = x
    d, e, f = y
    C, S = np.cos(s), 1j*np.sin(s)
    return [C*a+S*d,C*b+S*e,C*c+S*f]

# Init
fig = plt.figure()
ax = fig.add_subplot()

```

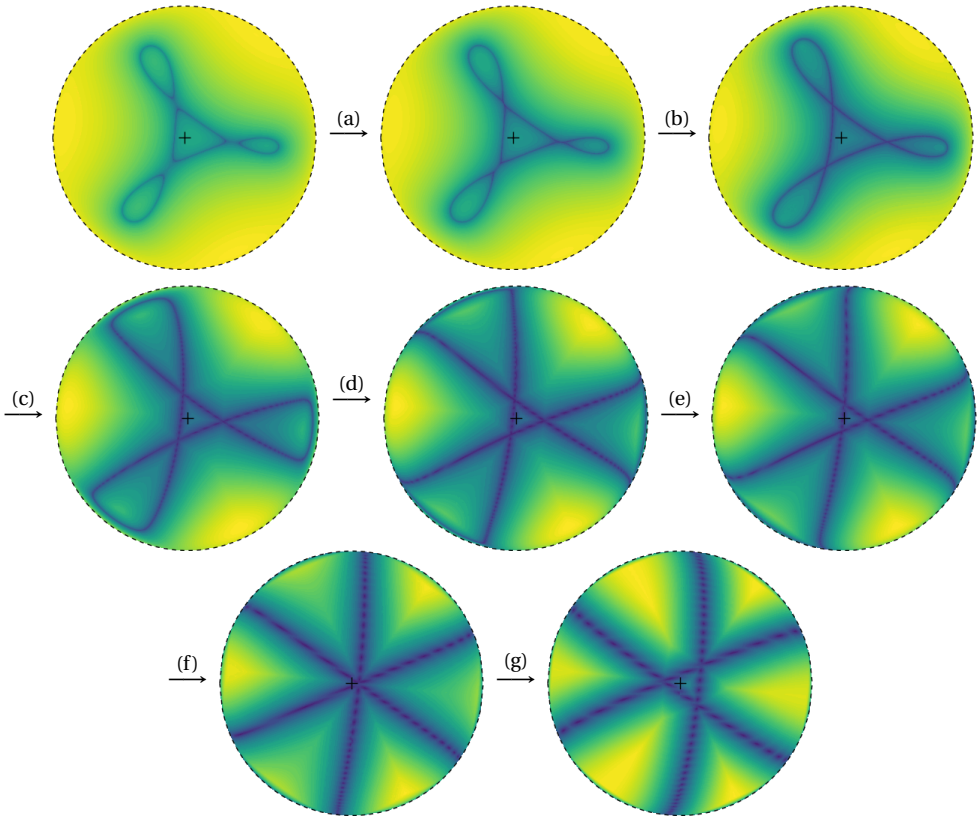
```
# Plot range
ax.set_aspect("equal")
plt.axis("off")
plt.xlim([-1.05,1.05])
plt.ylim([-1.05,1.05])

# The smooth indicator function of the link projection
def w(x,y):
    if x**2+y**2<=1:
        z = np.sqrt(1-x**2-y**2)
        v0, v1 = ln.null_space(np.array([[x,y,z]])).transpose()
        h = lambda t: np.abs(ff(*geo([x,y,z],np.cos(t)*v0+np.sin(t)*v1,r)))**r
        t0 = 0
        for ci in np.linspace(0,np.pi,num=hprec):
            t = opt.fmin(h,[ci],disp=False)[0]
            if h(t)<=h(t0): t0 = t
        return h(t0)
    else: return np.nan

# Computing the contour (the level [0] is the link projection!)
X, Y = np.meshgrid(np.linspace(-1,1,prec),np.linspace(-1,1,prec))
m, n = X.shape
Z = np.zeros((m,n))
for i in range(m):
    for j in range(n):
        Z[i,j] = w(X[i,j],Y[i,j])

# Plotting the filled contour
c = ax.contourf(X,Y,Z,100)
fig.colorbar(c)
ax.contour(X,Y,Z,[.1],colors="r")

# Two semi-circles
T = np.linspace(-1,1,250)
C = np.sqrt(1-T**2)

style_dict = {"color":"k","linewidth":1,"linestyle":"--","dashes":(4,4.1)}

plt.plot(T,C,**style_dict)
plt.plot(T,-C,**style_dict)

# Middle cross
plt.plot([0],[0],color="k",marker="+",markersize=12,label="a")

# Title
Fstring = str(F).replace("**","^").replace("*","").replace("z0","z_0").replace("z1"
    ,"z_1").replace("z2","z_2").replace(" ","")
plt.suptitle("$F([z_0:z_1:z_2]!)="+Fstring+"$")
plt.title("$r="+str(r)+"$")

# Done!
plt.show()
```

The curve we inputted was the $(M-1)$-quartic depicted in Figure 1.5 (where the x and y variables are

swapped). The output is the following plot, where the projection is the level set corresponding to the darkest blue color.



We now depict the evolution as r approaches $\pi/4$, this time with an M-quartic (slightly perturbed to avoid symmetries).



We describe what happens at each step.

(a) The curve undergoes a step where it is immerged but the point where it is not injective is not a double point (rather a tangency point between two branches).

(b) The previous operation happens two additional times, and the curve becomes the "clover leaf" with three double points.

(c) Three branches "escape to infinity".

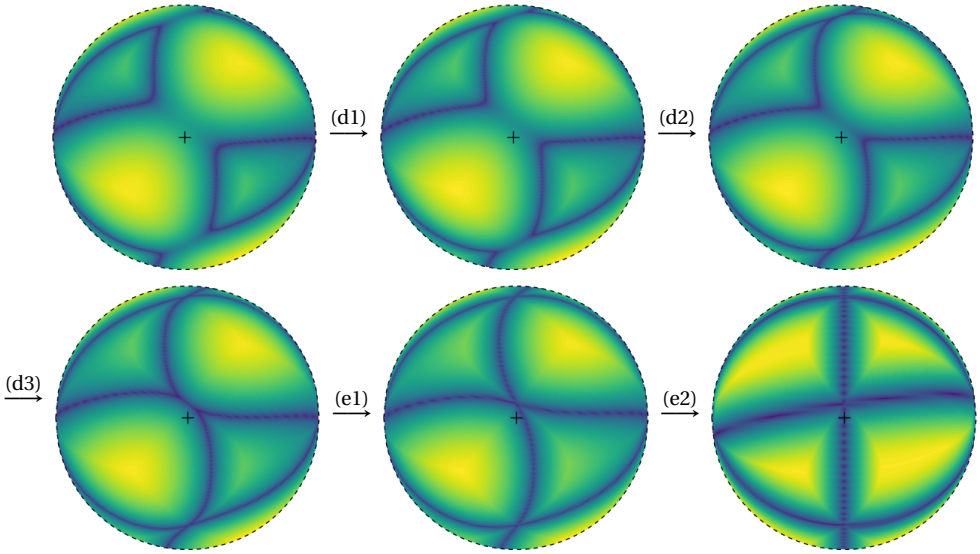
(d) Those branches start merging in a neighborhood of the line at infinity, and the curve undergoes a time where two branches meet tangentially.

(e) This happens two more times.

(f) The curve undergoes a "Reidemeister III type move", where it has three branches meeting at one point.

(g) The curve has become a generic arrangement of four lines (one line is the line at infinity).

Transformations (d) and (e) are slightly difficult to see, so we depict them after performing a projectiv transformation that brings the line at infinity through the origin.



This might be useful for the following project concerning the Ragsdale conjecture for M-curves.

(1) Show that the Ragsdale conjecture holds for a generic oriented line arrangement in $\mathbf{RP}^2$ whose double points are all resolved accordingly to orientations.

(2) Study all possible times $0 < r < \pi/4$ at which the projection of $\mathbf{C}F \cap S(r)$ is not a nodal immersion (that is, singular times such as having a tangency point or a triple point).

(3) Show that the Ragsdale conjecture is preserved through those singular events.

This means that one could derive the Ragsdale conjectue for an M-curve. Indeed, if r is close to $\pi/4$, the projection is a generic oriented line arrangement, and if r is close to 0, then the projection is embedded and is isotopic to $\mathbf{R}F$. Some elements of answer regarding the singular times can be found in [Gil92].

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Involutions and Real Flexible Curves on Complex Surfaces

The first part of Hilbert's sixteenth problem deals with the topology of non-singular real plane algebraic curves in the projective plane. As well-known, many topological properties of such curves are shared with the wider class of flexible curves, introduced by O. Viro in 1984.

The goal of this thesis is to further investigate the topological origins of the restrictions on real curves in connection with Hilbert's sixteenth problem. We add a natural condition to the definition of flexible curves, namely that they shall intersect an empty real conic Q like algebraic curves do, i.e. all intersections are positive. We see $\mathbf{CP}^2$ as a cylinder over a lens space $L(4,1) \times \mathbf{R}$ which is compactified by adding $\mathbf{RP}^2$ and Q respectively to the ends, and we use the induced decomposition of $\mathbf{S}^4 = \mathbf{CP}^2/\mathrm{conj}$.

It is a standard fact that Arnold's surface plays an essential role in the study of curves of even degree. We introduce an analogue of this surface for curves of odd degree. We generalize the notion of flexible curves further to include non-orientable surfaces as well. We say that a flexible curve is of degree m if its self-intersection is m^2 and it intersects the conic Q transversely in exactly $2m$ points. Our main result states that for a not necessarily orientable curve of odd degree $2k+1$, its number of non-empty ovals is no larger than $-\chi(F)/2 - k^2 + k + 1$, where $\chi(F)$ is the Euler characteristic of F. This upper bound simplifies to k^2 in the case of a usual flexible curve. We also generalize our result for flexible curves on quadrics, which provides a new restriction, even for algebraic curves.

In the introductory chapters, a thorough survey of the classical theory of real plane curves is outlined, both from the real and the complex points of view. Some results regarding the theory of knotted surfaces in 4-manifolds are laid down. More specifically, we review statements involving the Euler class of normal bundles of embedded surfaces. This eventually leads us to consider the non-orientable genus function of a 4-manifold. This forms a non-orientable counterpart of the Thom conjecture, proved by Kronheimer and Mrowka in 1994 in the orientable case. We almost entirely compute this function in the case of $\mathbf{CP}^2$, and we investigate that function on other 4-manifolds.

Finally, we digress around the new notion of non-orientable flexible curves, where we survey which known results still hold in that setting. We also focus on algebraic and flexible curves invariant under a holomorphic involution of $\mathbf{CP}^2$, a smaller class of curves introduced by T. Fiedler and called symmetric curves. We give a state of the art, and we formulate a collection of small results results regarding the position of a symmetric plane curve with respect to the elements of symmetry. We also propose a possible approach to generalize Fiedler's congruence $p - n \equiv k^2$ [16], holding for symmetric M-curves of even degree $2k$, into one for symmetric $(M-1)$-curves of even degree.

Keywords: Algebraic curves, flexible curves, Hilbert's 16th problem, double branched covers, nonorientable surfaces.

Involutions et courbes flexibles réelles sur des surfaces complexes

La première partie du seizième problème de Hilbert traite de la topologie des courbes algébriques réelles régulières dans le plan projectif. Il est bien connu que bon nombre des propriétés topologiques satisfaites par de telles courbes sont également vraies pour la classe plus large des courbes flexibles, introduites par O. Viro en 1984.

Le but de cette thèse est d'approfondir les origines topologiques des restrictions sur les courbes réelles, en lien avec le seizième problème de Hilbert. Nous ajoutons une condition naturelle à la définition de courbe flexible, à savoir qu'elles doivent intersecter une conique réelle vide Q comme une courbe algébrique, c'est-à-dire en des points positifs uniquement. Nous voyons $\mathbf{CP}^2$ comme un cylindre sur un espace lenticulaire $L(4,1) \times \mathbf{R}$, que l'on compactifie en ajoutant $\mathbf{RP}^2$ et Q aux bords, et nous utilisons la décomposition induite sur $\mathbf{S}^4 = \mathbf{CP}^2/\mathrm{conj}$.

C'est un fait standard que la surface d'Arnold joue un rôle essentiel dans l'étude des courbes de degré pair. Nous introduisons un analogue de cette surface pour des courbes de degré impair. Nous généralisons également la notion de courbe flexible pour inclure des surfaces non orientables. Nous considérons qu'une courbe flexible est de degré m si son auto-intersection est m^2 et si elle intersecte la conique Q de manière transverse en exactement $2m$ points. Notre résultat principal affirme que pour une telle courbe flexible (non nécessairement orientable) de degré impair $m = 2k+1$ ne peut pas posséder plus de $-\chi(F)/2 - k^2 + k + 1$, où $\chi(F)$ est la caractéristique d'Euler de F. Cette borne supérieure se simplifie en k^2 dans le cas d'une courbe flexible au sens usuel. Nous généralisons également notre résultat pour des courbes flexibles sur des quadriques, ce qui produit une nouvelle restriction, même pour des courbes algébriques.

Dans les chapitres introductifs, un aperçu détaillé de la théorie classique des courbes réelles planes est fait, en s'appuyant aussi bien sur le point de vue réel que complexe. Certains résultats à propos de la théorie des surfaces nouées dans les 4-variétés sont énoncés. Plus précisément, il est question de faits concernant la classe d'Euler du fibré normal d'une surface plongée. Cela nous amène ensuite à consider la fonction de genre non-orientable d'une 4-variété. Cela constitue un analogue de la conjecture de Thom (résolue par Kronheimer et Mrowka en 1994) pour des surfaces non orientables. Nous calculons presque totalement cette fonction pour $\mathbf{CP}^2$, et nous étudions cette fonction sur d'autres 4-variétés.

Enfin, nous digressons autour de la nouvelle notion de courbes flexibles non orientables, où nous dressons une liste de résultats connus qui restent vrai dans ce cadre. Nous nous concentrons aussi sur la classe des courbes algébriques et flexibles qui sont invariantes sous l'action d'une involution holomorphe de $\mathbf{CP}^2$, une notion introduite par T. Fiedler et appelées courbes symétriques. Nous donnons un état de l'art, et nous formulons une succession de petits résultats à propos de la disposition d'une courbe symétrique par rapport aux éléments de symétrie. Nous proposons également une approche pour tenter de généraliser la congruence de Fiedler $p - n \equiv k^2$ [16], valable pour des M-courbes symétriques de degré $2k$, à des $(M-1)$-courbes symétriques de degré $2k$.

Mots-clefs : Courbes algébriques, courbes flexibles, 16ème problème de Hilbert, revêtements doubles ramifiés, surfaces non orientables.

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Résumé : La première partie du seizième problème de Hilbert traite de la topologie des courbes algébriques réelles régulières dans le plan projectif. Il est bien connu que bon nombre des propriétés topologiques satisfaites par de telles courbes sont également vraies pour la classe plus large des courbes flexibles, introduites par O. Viro en 1984.
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