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THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES

ÉCOLE DOCTORALE N° 601

*Mathématiques, Télécommunications, Informatique, Signal, Systèmes,
Électronique*

Spécialité : *Mathématiques et leurs Interactions*

Par

Mahieddine ADIM

**Modèles continument stratifiés et systèmes multi-couches pour
les écoulements géophysiques**

Thèse présentée et soutenue à Rennes , le 4 juillet 2024

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FRENCH INTRODUCTION

1.1 Contexte général

Les océans, en tant que composants essentiels de notre planète, jouent un rôle central dans la régulation du climat, le cycle de l'eau, et la biodiversité marine. La compréhension des phénomènes océanographiques complexes qui animent ces vastes étendues d'eau est cruciale pour appréhender les mécanismes sous-jacents et les interactions qui façonnent notre environnement. Au cœur de cette compréhension se trouve la modélisation des écoulements fluides, dont la dynamique est régie par des équations fondamentales. Dans ce contexte, nous introduisons ces équations fondamentales qui décrivent l'évolution des écoulements incompressibles hétérogènes sous l'influence de la gravité

$$\begin{aligned}
& \partial_t \rho + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}} \rho + (w + w_*) \partial_z \rho = 0, \\
& \rho (\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}}) \mathbf{u} + (w + w_*) \partial_z \mathbf{u}) + \nabla_{\mathbf{x}} P + f \mathbf{k} \times \mathbf{u} = 0, \\
& \rho (\partial_t w + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}} w + (w + w_*) \partial_z w) + \partial_z P + g \rho = 0, \\
& \nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w = 0, \\
& P|_{z=\zeta} - P_{\text{atm}} = 0, \\
& \partial_t \zeta + (\mathbf{u} + \mathbf{u}_*)|_{z=\zeta} \cdot \nabla_{\mathbf{x}} \zeta - (w + w_*)|_{z=\zeta} = 0, \\
& w|_{z=-H} = 0.
\end{aligned} \tag{1.1.1}$$

Ici, t et (\mathbf{x}, z) représentent respectivement le temps et les variables d'espace horizontal-vertical, et nous utilisons $\nabla_{\mathbf{x}}$, $\nabla_{\mathbf{x}} \cdot$, $\Delta_{\mathbf{x}}$, pour désigner le gradient, la divergence et le laplacien par rapport à \mathbf{x} . Le champ vectoriel $(\mathbf{u}, w) \in \mathbb{R}^d \times \mathbb{R}$ représente la vitesse (horizontale et verticale), $\rho > 0$ est la densité, $P \in \mathbb{R}$ est la pression incompressible, tous étant définis dans le domaine spatial.

$$\Omega_t = \{(\mathbf{x}, z) : \mathbf{x} \in \mathbb{R}^d, -H < z < \zeta(t, \mathbf{x})\}$$

où $\zeta(t, \mathbf{x})$ décrit la position d'une surface libre, et H est la profondeur de la couche au repos. Le champ gravitationnel est supposé constant et vertical, et $g > 0$ est la constante d'accélération gravitationnelle, la constante $f \in \mathbb{R}$ et le vecteur constant \mathbf{k} représentent respectivement le paramètre de Coriolis et le vecteur unitaire dans la direction verticale. Enfin, les termes d'advection associés à la 'vitesse bolus' $(\mathbf{u}_*, w_*) \in \mathbb{R}^d \times \mathbb{R}$ ont été proposés par Gent et McWilliams [35]. Ces termes sont introduits pour prendre en compte la contribution effective des corrélations tourbillonnaires géostrophiques dans les modèles à grande échelle ne résolvant pas les tourbillons. Leurs formes spécifiques dans le cas le plus simple avec le paramètre de diffusivité constant κ se lisent comme suit:

$$\mathbf{u}_* = \kappa \partial_z \left(\frac{\nabla_x \rho}{\partial_z \rho} \right), \quad w_* = -\kappa \nabla_x \cdot \left(\frac{\nabla_x \rho}{\partial_z \rho} \right), \quad \kappa > 0 \quad (1.1.2)$$

En négligeant les termes effectifs d'advection (c'est-à-dire en fixant $\kappa = 0$), on retrouve les équations d'Euler pour des fluides incompressibles hétérogènes sous l'influence des forces gravitationnelles verticales ainsi que la force de Coriolis, où les deux dernières lignes de (1.1.1) modélisent respectivement l'équation cinématique à la surface libre et la condition d'imperméabilité du fond rigide. Dans l'équation (1.1.1), la pression P peut être obtenue à partir de sa valeur (atmosphérique) à la surface, P_{atm} , en résolvant le problème aux limites elliptique induit par la contrainte d'incompressibilité des champs de vitesse à divergence nulle. Cependant, dans le régime des eaux peu profondes, où l'échelle horizontale de la perturbation est grande par rapport à la profondeur de la couche H , les calculs formels (voir ci-dessous) suggèrent que les accélérations verticales peuvent être négligées et que la pression P satisfait approximativement la loi d'équilibre hydrostatique, c'est-à-dire

$$\partial_z P + g\rho = 0. \quad (1.1.3)$$

En remplaçant l'équation de la vitesse verticale dans (1.1.1) par l'identité dans (1.1.3) cela conduit aux équations hydrostatiques sous l'influence de la force de Coriolis:

$$\begin{aligned}
 \partial_t \rho + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_x \rho + (w + w_*) \partial_z \rho &= 0 \\
 \rho (\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_*) \cdot \nabla_x) \mathbf{u} + (w + w_*) \partial_z \mathbf{u}) + \nabla_x P + f \mathbf{k} \times \mathbf{u} &= 0 \\
 \partial_t \zeta + (\mathbf{u} + \mathbf{u}_*)|_{z=\zeta} \cdot \nabla_x \zeta - (w + w_*)|_{z=\zeta} &= 0 \\
 P &= P_{\text{atm}} + g \int_z^\zeta \rho(z', \cdot) dz', \\
 w &= - \int_{-H}^z \nabla_x \cdot \mathbf{u}(z', \cdot) dz'.
 \end{aligned} \tag{1.1.4}$$

Il convient de noter que le système précédent représente une simplification des équations primitives, lesquelles mettent en jeu des aspects tels que la diffusivité, la viscosité, les équations relatives aux traceurs (salinité S et température θ), ainsi qu'une relation d'état. Cette dernière est une équation exprimant la densité ρ en fonction de S et θ . Ces détails sont explicités dans l'article de Titi et Korn [51], de même que dans l'ouvrage de géophysique de Vallis [75][section 2.2 et section 10.7].

Les implications de la force de Coriolis deviennent significatives à grande échelle océanique. Cependant, malgré cela, dans la suite de l'analyse, nous choisirons de négliger les effets de la rotation de la Terre pour plusieurs raisons. D'une part, pour simplifier les systèmes traités dans ce manuscrit. D'autre part, nous nous intéresserons au caractère bien posé local de nos systèmes d'équations. Nous nous situons dans un régime où la rotation de la Terre n'est pas rapide, ce qui signifie que prendre en compte cette force n'apporte ni avantages significatifs ni difficultés supplémentaires à notre analyse. De plus, à travers nos résultats de caractère bien posé, nous prouvons l'absence d'instabilités à hautes fréquences qui se développent rapidement. Il est à noter que, dans ce cas, la présence de la force de Coriolis n'a pas d'impact conséquent.

Nous soulignons que notre étude est de nature théorique et ne vise pas à modéliser l'océan. Elle se concentre sur des modèles qui ne sont pas précis en termes de représentation physique des océans. Néanmoins, à travers ces modèles, nous abordons des enjeux pertinents, notamment l'hypothèse hydrostatique et la variation de densité.

La raison physique sous-jacente à l'introduction des vitesses de transport supplémentaires, notées \mathbf{u}_* et w_* et définies dans l'équation (1.1.2), au sein des systèmes d'équations (1.1.1) et (1.1.4), est que ces vitesses "bolus" agissent comme des ajustements aux équations d'Euler incompressibles standard qui s'appliquent indépendamment de l'inclusion ou non de l'hypothèse hydrostatique et sont dérivés des principes fondamentaux de la mécanique des fluides. Notons que dans ce manuscrit, notre intérêt portera sur ces vitesses de transport supplémentaires qui vont induire une diffusion, comme nous pourrons le voir dans les équations (1.2.11), ci-dessous. La modélisation déterministe de cette diffusivité effective résultant de la corrélation tourbillonnaire, que nous mettons en œuvre dans cette étude, remonte aux années 1990 et est créditée à Gent & McWilliams [35], comme référencé également dans [39, 40]. Cette approche trouve son origine dans la nécessité de modéliser les effets dissipatifs moyennés des tourbillons méso-échellonnés, qui étaient alors inabordables sur le plan computationnel à cette époque, sur l'écoulement à grande échelle au niveau macroscopique. Ils considèrent les inconnues (ρ, \mathbf{u}, w) comme les composantes à grande échelle des champs de densité et de vitesse, respectivement. L'approche implique l'introduction de correcteurs appropriés dans l'équation de conservation de la masse et dans les équations des traceurs (comme la salinité et la température). Plus précisément, ils suggèrent d'incorporer des champs de vitesse bolus \mathbf{u}_* et w_* , définis dans l'équation (1.1.2). L'ajout de la contribution de vitesse bolus dans l'équation de conservation de la quantité de mouvement a été suggéré dans [39, 36]. L'un des principaux éléments conduisant à la forme spécifique des champs de vitesse bolus est que l'effet dissipatif moyen des tourbillons méso-échellonnés devrait agir principalement le long des surfaces isopycnales (c'est-à-dire le long, et non pas à travers, des surfaces de densités égales). Il convient de mentionner que l'advection induite par les tourbillons selon Gent & McWilliams ne constitue qu'une partie de la paramétrisation sous-maille des tourbillons méso-échellonnés utilisée dans les modèles généraux de circulation océanique mondiale, qui incluent également la diffusion induite par les tourbillons de Redi [69]. Plus précisément, les équations pour les traceurs, comprenant la paramétrisation des tourbillons de Gent–McWilliams–Redi, se présentent comme suit (voir [51] et les références citées)

$$\partial_t C + (\mathbf{U}_3 \cdot \nabla_3) C = \nabla_3 \cdot (K_R \nabla_3 C) + \nabla_3 \cdot (K_{GM} \nabla_3 C), \quad (1.1.5)$$

où C est un traceur (typiquement la température θ et la salinité S), $\mathbf{U}_3 := (\mathbf{u}, w)$ représente le champ de vitesse tridimensionnel, et ∇_3 est le gradient tridimensionnel.

Ci-dessus, nous définissons K_R et K_{GM} par

$$K_R := \frac{K_I}{1 + |\mathbf{L}|^2} \begin{pmatrix} 1 + L_y^2 & -L_x L_y & L_x \\ -L_x L_y & 1 + L_x^2 & L_y \\ L_x & L_y & |\mathbf{L}|^2 \end{pmatrix} + \frac{K_D}{1 + |\mathbf{L}|^2} \begin{pmatrix} L_x^2 & L_x L_y & -L_x \\ L_x L_y & L_y^2 & -L_y \\ -L_x & -L_y & 1 \end{pmatrix} \quad (1.1.6)$$

$$K_{GM} := \kappa \begin{pmatrix} 0 & 0 & -L_x \\ 0 & 0 & -L_y \\ L_x & L_y & 0 \end{pmatrix}$$

où nous avons utilisé la notation $L := (L_x, L_y) = \frac{-\nabla_{\mathbf{x}} \rho}{\partial_z \rho}$, et les constantes strictement positifs K_I et K_D sont les coefficients de diffusivité isoneutre (tangente aux surfaces de densité neutre) et dianeutre (normale aux surfaces de densité neutre). On remarque que le tenseur K_R est symétrique, tandis que le tenseur K_{GM} est antisymétrique, de sorte que l'effet dissipatif dû à l'advection tourbillonnaire de Gent & McWilliams n'est pas évident. En fait, il y a eu un débat sur la nature adiabatique de la paramétrisation de Gent & McWilliams [38]. Comme le montre clairement la formulation des équations en coordonnées isopycnales (voir équations (1.2.11) ci-dessous), les termes d'advection tourbillonnaire induits par Gent & McWilliams introduisent une contribution diffusive à la variable d'épaisseur. C'est pourquoi le paramètre κ est souvent appelé le paramètre de diffusivité d'épaisseur.

Comme on a pu le constater dans l'équation (1.1.5), les termes de Redi-Gent & McWilliams ont été ajoutés aux équations des traceurs (température θ et salinité S), afin de prendre en compte l'effet de ces termes sur les équations d'Euler (1.1.1) et d'Euler hydrostatique (1.1.4), il est naturel de s'interroger sur l'impact de ces termes additionnels sur la variable de densité ρ . Il s'avère que, lorsqu'il est appliqué à la variable de densité, la contribution isoneutre de la diffusion induite par les tourbillons de Redi disparaît exactement, tandis que la contribution dianeutre devient un laplacien isotrope tridimensionnel, et la contribution de Gent & McWilliams peut être interprétée comme une advection.

$$\nabla_3 \cdot (K_R \nabla_3 \rho) = K_D \Delta_3 \rho, \quad \nabla_3 \cdot (K_{GM} \nabla_3 \rho) = (\mathbf{U}_3^* \cdot \nabla_3) \rho.$$

Où nous notons $\mathbf{U}_3^* := (\mathbf{u}_*, w_*)$ défini dans (1.1.2). Par conséquent, dans les situations où l'équation d'état $\rho = \rho(\theta, S)$ est une combinaison linéaire des traceurs (avec des

paramètres de diffusivité égaux pour tous les traceurs), les équations (1.1.5) conduisent à

$$\partial_t \rho + (\mathbf{U}_3 \cdot \nabla_3) \rho + (\mathbf{U}_3^* \cdot \nabla_3) \rho = K_D \Delta_3 \rho.$$

Étant donné que le coefficient de diffusivité dianeutre est généralement choisi beaucoup plus petit que le coefficient de diffusivité isoneutre, $0 \leq K_D \ll K_I$, il est raisonnable de négliger le côté droit de l'équation ci-dessus. Cela conduit aux équations de conservation de la masse dans (1.1.1) et (1.1.4). Il est essentiel de souligner que le choix de la variable ρ en tant que combinaison linéaire des traceurs θ et S dans le paragraphe précédent constitue une approche simple et élémentaire du point de vue de la modélisation, visant à simplifier le cadre d'étude des équations primitives. Cependant, il est possible d'étendre certains résultats de stabilité des équations primitives avec les contributions des termes de Redi-Gent & McWilliams (1.1.6), dans un contexte plus général, plus précisément dans le cas où ρ est une fonction régulière des traceurs mentionnés précédemment. Nous précisons particulièrement les résultats d'existence de solutions faibles, ainsi qu'un résultat de caractère bien posé global obtenu par Titi et Korn [51] dans ce contexte général.

1.2 Motivation de la thèse

1.2.1 Équations en jeu

Dans cette section, nous mettons en œuvre les systèmes d'équations qui constitueront le centre d'intérêt de ce manuscrit.

La justification rigoureuse de l'hypothèse hydrostatique dans le régime des eaux peu profondes, c'est-à-dire lorsque la longueur d'onde horizontale typique de l'écoulement est grande par rapport à la profondeur verticale de la couche, a été analysée de manière approfondie, que ce soit dans des situations de densité homogène [7, 57, 34, 58], avec des distributions de densité lisses [68, 67], ou dans le cadre bi-couches [14, 31, 28]. L'enjeu de cette thèse consiste à établir des ponts entre le cadre à stratification continue (la densité est continue) et le cadre à stratification nette (la densité est constante sur chacune des couches du fluide, plus particulièrement nous traiterons le cas bi-couches et multi-couches), dans le contexte de l'hypothèse hydrostatique.

Dans ce manuscrit, le système hydrostatique continument stratifié avec les termes additionnels de Gent & McWilliams occupera une place centrale. Cependant, avant de

le présenter explicitement, nous éclairons dans les paragraphes qui suivent les hypothèses ainsi que la méthodologie qui nous permet d'extraire ce système à partir des équations d'Euler (1.1.1).

Considérons maintenant des solutions régulières à l'équation (1.1.1) définies sur un intervalle de temps I_t . En supposant que l'écoulement est stablement stratifié, c'est-à-dire

$$\inf(-\partial_z \rho) > 0, \quad (1.2.1)$$

la densité $\rho : z \mapsto \rho(\cdot, \cdot, z)$ est une fonction inversible de z . Nous désignons son inverse par $\eta : \varrho \mapsto \eta(\cdot, \cdot, \varrho)$, afin de définir le changement de variable isopycnal suivant

$$\rho(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)) = \varrho, \quad \eta(t, \mathbf{x}, \rho(t, \mathbf{x}, z)) = z. \quad (1.2.2)$$

Nous supposons également que $\rho(t, \mathbf{x}, -H) = \rho_1$, $\rho(t, \mathbf{x}, \zeta(t, \mathbf{x})) = \rho_0$ pour $(t, \mathbf{x}) \in I_t \times \mathbb{R}^d$ où $\rho_0 < \rho_1$ sont deux densités de référence constantes, fixes et positives. Ensuite, nous avons

$$\eta : I_t \times \Omega \longmapsto \mathbb{R} \text{ avec } \Omega := \mathbb{R}^d \times (\rho_0, \rho_1) \text{ et } h := -\partial_\varrho \eta > 0, \quad (1.2.3)$$

l'inégalité précédente tenant compte de l'hypothèse de stratification stable. Nous introduisons maintenant

$$\check{\mathbf{u}}(t, \mathbf{x}, \varrho) = \mathbf{u}(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)), \quad \check{w}(t, \mathbf{x}, \varrho) = w(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)), \quad \check{P}(t, \mathbf{x}, \varrho) = P(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)).$$

À partir de la règle de la chaîne, nous déduisons que le système (1.1.1) en coordonnées isopycnal s'écrit

$$\begin{aligned} \partial_t \eta + \check{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \eta - \check{w} &= \kappa \Delta_{\mathbf{x}} \eta, \\ \varrho \left(\partial_t \check{\mathbf{u}} + \left(\left(\check{\mathbf{u}} - \kappa \frac{\nabla_{\mathbf{x}} h}{h} \right) \cdot \nabla_{\mathbf{x}} \right) \check{\mathbf{u}} \right) + \nabla_{\mathbf{x}} \check{P} + \frac{\nabla_{\mathbf{x}} \eta}{h} \partial_\varrho \check{P} &= 0, \\ \varrho \left(\partial_t \check{w} + \left(\check{\mathbf{u}} - \kappa \frac{\nabla_{\mathbf{x}} h}{h} \right) \cdot \nabla_{\mathbf{x}} \check{w} \right) - \frac{\partial_\varrho \check{P}}{h} + g \varrho &= 0, \\ -h \nabla_{\mathbf{x}} \cdot \check{\mathbf{u}} - (\nabla_{\mathbf{x}} \eta) \cdot (\partial_\varrho \check{\mathbf{u}}) + \partial_\varrho \check{w} &= 0, \\ \check{P} \Big|_{\varrho=\rho_0} &= P_{\text{atm}}, \quad \check{w} \Big|_{\varrho=\rho_1} = 0. \end{aligned} \quad (1.2.4)$$

Remarquez qu'en différenciant par rapport à ϱ la première équation et en utilisant la

quatrième équation (découlant de la contrainte d'incompressibilité), la conservation de la masse se lit

$$\partial_t h + \nabla_{\mathbf{x}} \cdot (h \check{\mathbf{u}}) = \kappa \Delta_{\mathbf{x}} h, \quad (1.2.5)$$

d'où l'apparition de la diffusion mentionné dans la section précédente.

À présent, nous pouvons présenter une version adimensionnée du système précédent. Nous nous intéressons aux déviations par rapport aux solutions stationnaires des équations d'Euler incompressibles avec densité variable :

$$(h_{\text{eq}}, \mathbf{u}_{\text{eq}}, w_{\text{eq}}, P_{\text{eq}}) = (h(\varrho), \mathbf{u}(\varrho), 0, \underline{P}(\varrho)),$$

qui satisfont la condition d'équilibre

$$\partial_{\varrho} \underline{P}(\varrho) = g \varrho h(\varrho).$$

Par conséquent, nous considérons des fluctuations (non nécessairement petites) de cette solution stationnaire, de sorte que nos inconnues admettent la décomposition suivante :

$$\begin{aligned} h(t, \mathbf{x}, \varrho) &= \underline{h}(\varrho) + h_{\text{pert}}(t, \mathbf{x}, \varrho), & \check{\mathbf{u}}(t, \mathbf{x}, \varrho) &= \mathbf{u}(\varrho) + \mathbf{u}_{\text{pert}}(t, \mathbf{x}, \varrho), \\ \check{w}(t, \mathbf{x}, \varrho) &= 0 + w_{\text{pert}}(t, \mathbf{x}, \varrho), & \check{P}(t, \mathbf{x}, \varrho) &= \underline{P}(\varrho) + P_{\text{pert}}(t, \mathbf{x}, \varrho). \end{aligned}$$

De plus, nous adimensionnons les équations avec les variables adimensionnées suivantes : nous définissons

$$\frac{h(t, \mathbf{x}, \varrho)}{H} = \tilde{h}(\varrho) + \tilde{h}(\tilde{t}, \tilde{\mathbf{x}}, \varrho) \quad \text{et} \quad \frac{\check{\mathbf{u}}(t, \mathbf{x}, \varrho)}{\sqrt{gH}} = \tilde{\mathbf{u}}(\varrho) + \tilde{\mathbf{u}}(\tilde{t}, \tilde{\mathbf{x}}, \varrho),$$

et

$$\frac{\lambda}{H} \frac{\check{w}(t, \mathbf{x}, \varrho)}{\sqrt{gH}} = \tilde{w}(\tilde{t}, \tilde{\mathbf{x}}, \varrho), \quad \frac{\check{P}(t, \mathbf{x}, \varrho)}{gH} = \frac{P_{\text{atm}}}{gH} + \int_{\rho_0}^{\varrho} \varrho' \tilde{h}(\varrho') d\varrho' + \tilde{P}(\tilde{t}, \tilde{\mathbf{x}}, \varrho),$$

où l'on rappelle que H représente la profondeur de la couche au repos. Pour λ une longueur horizontale de référence, nous utilisons les coordonnées adimensionnées suivantes

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{\lambda} \quad \text{et} \quad \tilde{t} = \frac{\sqrt{gH}}{\lambda} t. \quad (1.2.6)$$

Introduisant le paramètre adimensionné de diffusion (épaisseur), $\tilde{\kappa}$, et le paramètre de

faible profondeur, μ , à travers

$$\tilde{\kappa} = \frac{\kappa}{\lambda\sqrt{gH}} \quad \text{et} \quad \mu = \frac{H^2}{\lambda^2}, \quad (1.2.7)$$

en substituant les coordonnées/variables adimensionnées dans le système (1.2.4) et l'équation qui en découle, et en omettant les tildes et pert pour des raisons de lisibilité, on obtient

$$\begin{aligned} \partial_t h + \nabla_x \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) &= \kappa \Delta_x h, \\ \varrho \left(\partial_t \mathbf{u} + \left(\left(\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h} \right) \cdot \nabla_x \right) \mathbf{u} \right) + \nabla_x P + \frac{\nabla_x \eta}{\underline{h} + h} (\varrho \underline{h} + \partial_\varrho P) &= 0, \\ \mu \varrho \left(\partial_t w + \left(\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h} \right) \cdot \nabla_x w \right) - \frac{\partial_\varrho P}{\underline{h} + h} + \frac{\varrho h}{\underline{h} + h} &= 0, \\ -(\underline{h} + h) \nabla_x \cdot \mathbf{u} - \nabla_x \eta \cdot (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}) + \partial_\varrho w &= 0, \quad (\text{div.-free cond.}) \\ \eta(\cdot, \varrho) = \int_\varrho^{\rho_1} h(\cdot, \varrho') d\varrho', \quad P|_{\varrho=\rho_0} = 0, \quad w|_{\varrho=\rho_1} = 0. & \quad (\text{bound. cond.}) \end{aligned} \quad (1.2.8)$$

Le système hydrostatique est obtenu en fixant $\mu = 0$ dans (1.2.8). Plus précisément, en introduisant l'équilibre hydrostatique,

$$\frac{\partial_\varrho P}{\underline{h} + h} = \frac{\varrho h}{\underline{h} + h} \quad \text{et} \quad P|_{\varrho=\rho_0} = 0,$$

dans la deuxième équation de (1.2.8), cela implique le système hydrostatique continument stratifié avec les termes additionels de Gent & McWilliams qui nous intéresse dans ce manuscrit.

$$\begin{aligned} \partial_t h + \nabla_x \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) &= \kappa \Delta_x h, \\ \varrho \left(\partial_t \mathbf{u} + \left(\left(\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h} \right) \cdot \nabla_x \right) \mathbf{u} \right) + \nabla_x \psi &= 0, \end{aligned} \quad (1.2.9)$$

avec

$$\begin{aligned} \psi(t, \mathbf{x}, \varrho) &= \int_{\rho_0}^{\varrho} \varrho' h(t, \mathbf{x}, \varrho') d\varrho' + \varrho \int_{\varrho}^{\rho_1} h(t, \mathbf{x}, \varrho') d\varrho' \\ &= \rho_0 \int_{\rho_0}^{\rho_1} h(t, \mathbf{x}, \varrho') d\varrho' + \int_{\rho_0}^{\varrho} \int_{\varrho'}^{\rho_1} h(t, \mathbf{x}, \varrho'') d\varrho'' d\varrho'. \end{aligned} \quad (1.2.10)$$

Plus particulièrement, nous nous placerons dans le cadre d'une dimension spatiale ($\mathbf{x} = x \in \mathbb{R}$). Donc

$$\begin{cases} \partial_t h + \partial_x((\underline{h} + h)(\underline{u} + u)) = \kappa \partial_x^2 h, \\ \partial_t u + \left(\underline{u} + u - \kappa \frac{\partial_x h}{\underline{h} + h} \right) \partial_x u + \frac{1}{\varrho} \partial_x \psi = 0, \end{cases} \quad (1.2.11)$$

où

$$\begin{aligned} \psi(t, x, \varrho) &= \int_{\rho_0}^{\varrho} \varrho' h(t, x, \varrho') d\varrho' + \varrho \int_{\varrho}^{\rho_1} h(t, x, \varrho') d\varrho' \\ &= \rho_0 \int_{\rho_0}^{\rho_1} h(t, x, \varrho') d\varrho' + \int_{\rho_0}^{\varrho} \int_{\varrho'}^{\rho_1} h(t, x, \varrho'') d\varrho'' d\varrho'. \end{aligned}$$

Il est pertinent de mentionner que le système (1.2.8) (respectivement (1.2.9)) obtenu, n'est qu'une réécriture du système (1.1.1) (respectivement (1.1.4)) sous les hypothèses de stratification stable (1.2.1) et de continuité de densité.

Afin de faciliter la compréhension pour le lecteur, nous donnons le sens physique des variables considérées dans (1.2.11). De ce qui a été établi précédemment, quand la stratification est stable, c'est-à-dire lorsque la densité augmente avec la profondeur pour toutes les positions spatiales horizontales $x \in \mathbb{R}$, la reformulation avec des coordonnées isopycnals fait appel à l'inverse de la fonction de densité $z \mapsto \rho(\cdot, z)$, que nous notons $\varrho \mapsto \eta(\cdot, \varrho)$ pour $\varrho \in (\rho_0, \rho_1)$, où ρ_0 est la densité (constante) à la surface libre et ρ_1 est la densité (constante) au fond plat rigide. Étant donné que le graphe de $\eta(\cdot, \varrho)$ représente en coordonnées eulériennes la feuille isopycnale de densité ϱ , la fonction définie comme $\underline{h} + h = -\partial_{\varrho} \eta$ est la profondeur infinitésimale de cette feuille isopycnale. Ensuite, $\underline{u} + u$ est la composante horizontale de la vitesse des particules du fluide au niveau de la feuille isopycnale. Nous décomposons la profondeur et vitesses horizontales en $\underline{h} + h$ et $\underline{u} + u$ où $(\underline{h}, \underline{u})$ représentent l'écoulement de cisaillement de référence et sont des fonctions données dépendant uniquement de la variable de densité $\varrho \in (\rho_0, \rho_1)$, et (h, u) sont les inconnues représentant les déviations de l'équilibre et dépendent du temps t , de l'espace horizontal $x \in \mathbb{R}$ et de la variable de densité $\varrho \in (\rho_0, \rho_1)$. Enfin, ψ est le potentiel de Montgomery (tel que mentionné dans l'article de Holm et Long [44]), responsable de l'interaction entre les feuilles isopycnales.

Il est important de noter que pour obtenir le système (1.2.11), l'hypothèse de strati-

fication stable et continue (1.2.1) était nécessaire pour définir le changement de variable isopycnales, qui implique que des feuilles de densités égales réalisent un feuilletage du domaine du fluide ; voir, par exemple, [44, 13]. Mentionnons également que l'on peut utiliser le changement de variable isopycnales dans un cadre plus général que celui précédemment établi (la densité stablement stratifiée et continue), plus généralement, on peut considérer $\varrho = \underline{\rho}(r)$ avec $r \mapsto \underline{\rho}(r)$, pas forcément bijective ni même continue. Nous aurons un aperçu de cela dans le Chapitre 6 où nous choisirons un cas particulier de cette dernière fonction, qui s'adaptera à notre étude pour traiter le système de Saint-Venant bi-couches. Par ailleurs, la justification rigoureuse de la limite hydrostatique ($\mu \rightarrow 0$ dans (1.2.8)) se trouve dans le résultat de Duchêne et Bianchini [13].

En analysant attentivement la structure des équations du système continu stratifié avec les termes additionnels de Gent & McWilliams (1.2.11), nous observons une certaine similitude avec le système de Saint-Venant multi-couches avec les termes de Gent & McWilliams ajoutés d'une manière analogue (voir l'équation (1.2.12)). Dans ce manuscrit, nous expliquons précisément cette similitude, en d'autres termes, nous fournissons le dictionnaire nécessaire qui permet au lecteur de considérer le système à multi-couches comme une approximation du système continu stratifié (voir chapitre 5), et au contraire le système continu stratifié comme une approximation du système bi-couches (voir chapitre 6).

Le système multi-couches que l'on considère ici est le suivant

$$\begin{cases} \partial_t H_i + \partial_x((\underline{H}_i + H_i)(\underline{U}_i + U_i)) = \kappa \partial_x^2 H_i \\ \partial_t U_i + \left(\underline{U}_i + U_i - \kappa \frac{\partial_x H_i}{\underline{H}_i + H_i} \right) \partial_x U_i + g \sum_{j=1}^N \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j = 0, \end{cases} \quad \forall i \in \{1, \dots, N\}$$

(1.2.12)

Ce système correspond à une situation de N couches de fluides immiscibles avec des densités constantes ρ_i , $i \in \{1, \dots, N\}$. Quand $\kappa = 0$, en appliquant l'approximation hydrostatique et l'hypothèse d'écoulement en colonne, nous arrivons (voir par exemple [56, 9]) au système de Saint-Venant multi-couches (1.2.12) où $\underline{H}_i + H_i$ représente la profondeur de la $i^{\text{ème}}$ couche (\underline{H}_i étant la profondeur au repos et H_i la déviation) et $\underline{U}_i + U_i$ étant la vitesse horizontale moyenne de la couche (\underline{U}_i étant la vitesse de référence et U_i la déviation), et $g = 1$ sera la constante d'accélération de gravité après remise à échelle.

Comme précédemment mentionné, dans (1.2.12) (respectivement (1.2.11)), les termes proportionnels à κ ont été introduits par Gent et McWilliams [35] afin de représenter la contribution à grande échelle des tourbillons non résolus. Ils apparaissent sous la forme

de vitesses effectives supplémentaires $U_i^* := -\kappa \frac{\partial_x H_i}{H_i + H_i}$ (respectivement $u^* := -\kappa \frac{\partial_x h}{h+h}$) et agissent comme des contributions de diffusivité dans les équations de conservation de la masse. De manière intéressante, des termes similaires ont été introduits dans le travail de Duran, Vila, et Baraille [32] afin de contrôler l'énergie discrète d'un schéma numérique semi-implicite pour le système multi-couches (1.2.12). Du point de vue mathématique, comme établi dans ce manuscrit, l'effet régularisant des contributions de diffusivité est essentiel pour notre analyse car il fournit des estimations de stabilité appropriées, et nous supposons toujours que $\kappa > 0$.

Notons que cette approche par des modèles multi-couches pour approximer les équations d'Euler et de Navier-Stokes a déjà été abordée par plusieurs auteurs, parmi lesquels on trouve le résultat d'Audusse [4] qui a proposé un modèle de Saint-Venant multi-couches en discrétisant en profondeur, ce qui donne un ensemble de systèmes couplés de Saint-Venant (classique) pour approximer les équations de Navier-Stokes hydrostatique. Contrairement au modèle de Saint-Venant classique, où la vitesse horizontale ne dépend pas de la variable verticale et implique donc une perte d'informations (car le modèle ne décrit pas correctement le profil vertical de cette vitesse), le modèle multi-couches proposé permet d'étendre la plage de validité et fournit une description précise du profil vertical. Pour justifier (numériquement) ce dernier, l'auteur utilise une approche couche par couche, en adaptant des résultats bien connus sur le solveur de volume fini du système classique de Saint-Venant au cas multi-couche (voir également [6]). L'approche précédente diffère de celle établie dans ce manuscrit en termes de modélisation. En effet, la construction du modèle multi-couches proposé par l'auteur est réalisée de manière purement artificielle en discrétisant les hauteurs. En d'autres termes, l'interface est déterminée uniquement par la variation de la surface, ce qui implique un échange de particules de fluide entre ses différentes couches au fil du temps (voir également [5, 33]). Contrairement au modèle multi-couches traité dans ce manuscrit (1.2.12), qui est obtenu en suivant le profil de densité, permettant ainsi de discrétiser le domaine du fluide en une série de couches à densité constante. Dans ce modèle, la couche la plus dense se trouve au fond, et aucun échange de masses entre les couches n'est possible.

D'un autre point de vue, Camassa et Tiron [22] ont proposé un modèle bi-couches pour approximer le système d'Euler incompressible avec stratification continue. La construction de ce modèle ainsi que sa justification reposent sur le fait que, de manière similaire au système continument stratifié, celui-ci doit également capturer les ondes de vitesse d'amplitude infinitésimale et de longueur d'onde infinie en ajoutant la conservation de

certaines quantités physiques (qui caractérisent une configuration stratifiée), notamment la masse et l'énergie potentielle dans chacune des couches. Cette justification représente une approche physique du problème d'approximation, qui diffère de l'approche théorique que l'on considère dans ce manuscrit. Notre approche se base sur la démonstration du caractère bien posé et explicite la vitesse de convergence (qui dépend du nombre de couches considérées) pour justifier la validité de notre modèle multi-couches.

Comme mentionné précédemment, nous examinons l'approximation du système hydrostatique continu stratifié par le système multi-couches (la densité dans ce dernier étant constante sur chaque couche). Cela nous conduit naturellement à étudier le problème "inverse", qui consiste à démontrer que dans la limite où la densité (initialement continue) devient continue par morceaux, le système hydrostatique continu stratifié devient une approximation du système multi-couches (nous nous concentrons ici sur le cas à deux couches).

Cela nous ramène naturellement au système de Saint-Venant bi-couches (correspond à $N = 2$ dans (1.2.12)) qui est un modèle standard utilisé pour décrire les ondes internes dans les écoulements à densité stratifié, particulièrement dans des situations où la distribution de densité permet une représentation approximative du fluide en tant que deux couches avec des densités presque constantes séparées par une mince pycnocline ; voir par exemple [42, Chap. 6]. En plus de cette hypothèse de *stratification nette*, la dérivation formelle du système Saint-Venant bi-couches repose sur deux conditions supplémentaires. On suppose que la pression interne est *hydrostatique*, c'est-à-dire que la force de gradient de pression équilibre la force externe due à la gravité (voir (1.1.3)). On suppose également que la vitesse d'écoulement est *colonnaire*, ce qui signifie que la vitesse horizontale des particules fluides reste constante par rapport à la variable verticale à l'intérieur de chaque couche.

Naturellement, la validité du système Saint-Venant bi-couches dépend de l'attente que ces trois hypothèses restent précises sur une échelle de temps pertinente à mesure que l'écoulement évolue.

Dans le cadre bi-couches et en supposant que la pression est hydrostatique, l'hypothèse d'écoulement en colonne est propagée exactement par le flot. Cela signifie que le système de Saint-Venant bi-couches produit des solutions exactes aux équations d'Euler hydrostatiques (incompressibles) avec des distributions de densité et de vitesse horizontale qui sont constantes par morceaux par rapport à la variable verticale.

Dans ce manuscrit, nous examinons les solutions aux équations d'Euler hydrostatiques

à proximité de telles solutions, c'est-à-dire en relaxant les hypothèses de stratification nette ainsi que celle du mouvement en colonne. Nous démontrons que, pour des données initiales suffisamment proches du cadre bi-couches, les solutions émergentes des équations d'Euler hydrostatiques restent proches de la solution prédite par le système de Saint-Venant bi-couches sur une échelle de temps pertinente.

Plus précisément, les équations d'Euler hydrostatiques que nous considérons ont la forme :

$$\begin{aligned} \partial_t h + \partial_x((1+h)(\underline{u} + u)) &= \kappa \partial_x^2 h, \\ \partial_t u + \left(\underline{u} + u - \kappa \frac{\partial_x h}{1+h} \right) \partial_x u + \frac{1}{\underline{\rho}} \partial_x \Psi &= 0, \end{aligned} \quad (1.2.13)$$

où le potentiel de Montgomery Ψ est donné par

$$\Psi(t, x, r) = \underline{\rho}(r) \int_{-1}^r h(\cdot, r') dr' + \int_r^0 \underline{\rho}(r') h(\cdot, r') dr'. \quad (1.2.14)$$

Le système (1.2.14) n'est que la réécriture du système (1.2.11) à travers un changement de variables, plus précisément prendre $\varrho = \underline{\rho}(r)$ avec $r \mapsto \underline{\rho}(r)$ où la variable $r \in (-1, 0)$ réfère à la profondeur de la ligne isopycnale (remise à échelle). Cela permet de fixer la profondeur infinitésimale de référence des isopycnes à la valeur $\underline{h}(r) = 1$. Par ailleurs la profondeur totale du domaine du fluide au repos est fixée à $\int_{-1}^0 \underline{h}(r) dr = 1$ par le biais d'une mise à l'échelle appropriée. Il est à noter que le système (1.2.13) peut être dérivé du système à coordonnées eulériennes (1.1.1), même dans le cas où la stratification $\underline{\rho}$ n'est pas bijective, en particulier dans le cadre multi-couches. De plus, le système (1.2.13) est moins singulier dans la limite de stratification nette.

Enfin, après avoir motivé le lecteur sur le passage du système discret (Saint-Venant multi-couches) au système hydrostatique continuellement stratifié, et inversement, du système hydrostatique continuellement stratifié au système discret (Saint-Venant bi-couches), nous nous intéresserons également au système de Saint-Venant à une couche, incluant les termes additionnels de Gent & McWilliams (correspond à $N = 1$ dans (1.2.12) et dans le

cas d'une dimension spatiale d).

$$\begin{cases} \partial_t h + \nabla \cdot ((h + \underline{h})(\mathbf{u} + \underline{\mathbf{u}} + \mathbf{u}^*)) = 0, \\ \partial_t \mathbf{u} + ((\mathbf{u} + \underline{\mathbf{u}} + \mathbf{u}^*) \cdot \nabla) \mathbf{u} + \nabla h = 0, \\ h|_{t=0} = h_0, \inf h_0 > 0, \\ u|_{t=0} = u_0, \end{cases} \quad (1.2.15)$$

où $t > 0$ représente la variable temporelle, d la dimension spatiale horizontale, $\mathbf{x} = (x_1, \dots, x_d)$ les coordonnées spatiales horizontales, $(\underline{h}, \underline{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^d$ est un équilibre stable, $h = h(t, \mathbf{x})$ la déviation de \underline{h} , $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ la déviation de $\underline{\mathbf{u}}$. Physiquement $h + \underline{h}$ représente la hauteur de la couche et $\mathbf{u} + \underline{\mathbf{u}}$ représente le champ de vitesse horizontal moyen de la couche, et $\mathbf{u}^* = -\kappa \frac{\nabla h}{h + \underline{h}}$.

Nous attirons l'attention du lecteur sur le fait qu'en dehors des résultats concernant ce système, et ce pour des raisons de simplification et de lisibilité, nous considérerons la dimension spatiale $d = 1$.

Comme indiqué précédemment, lorsque $\kappa = 0$, nous retrouvons le système de Saint-Venant classique, ce dernier est une approximation d'ordre μ du système des water waves (système de deux équations d'évolution scalaires obtenues à partir du système d'Euler fondamental (1.1.1), en supposant que le fluide est homogène et que le champ de vitesse \mathbf{U} est irrotationnel), cette approximation est obtenue en utilisant des propriétés de l'opérateur de Dirichlet-to-Neumann (cet opérateur est défini à travers le potentiel de vitesse, qui découle de la propriété d'irrotationnalité et qui satisfait un certain problème de Laplace), où μ représente le paramètre de faible profondeur, défini précédemment (voir (1.2.7)). On suppose que μ est suffisamment petit pour se situer dans le régime des eaux peu profondes. Des informations détaillées sur la justification rigoureuse ainsi que d'autres propriétés de ce modèle peuvent être trouvées dans les références suivantes [28, 55]. Insistons également sur le fait que dans ce manuscrit, les résultats obtenus proposent une approche différente pour la justification de ce modèle, en faisant tout d'abord l'hypothèse hydrostatique (qui est valide dans le régime d'eau peu profonde), puis en regardant la limite de densité constante et d'écoulement en colonne.

Le lecteur a vraisemblablement noté notre intérêt particulier pour les systèmes de Saint-Venant à une couche et à deux couches (correspondant à (1.2.12) pour $N \in \{1, 2\}$). Il est à souligner que, pour ces deux systèmes, nous obtenons des résultats améliorés par

rapport au cas général. Ceci est attribuable au fait que les conditions d'hyperbolicité du problème sans diffusivité d'épaisseur ($\kappa = 0$) sont mieux comprises pour ces cas spécifiques.

1.2.2 Aspects de stabilité

Rappelons au lecteur quelques résultats classiques de stabilité connus par la communauté.

En ce qui concerne le système de Saint-Venant à une couche, il est établi que la présence de l'hypothèse de non-cavitation ($\underline{h} + h > 0$) implique la stricte hyperbolicité. De plus, ce système admet un symétriseur de Friedrich explicite, qui revient essentiellement à multiplier l'équation sur la vitesse horizontale par $\underline{h} + h$. L'étude de ce système peut être retrouvée dans [28], mais également dans un cadre plus général qui traite les systèmes quasi-linéaires du premier ordre, dont fait partie le système de Saint-Venant à une couche [12, 60]. Dans le cas particulier du toit rigide pour le système de Saint-Venant bi-couches des conditions d'hyperbolicité explicites ont été données par Guyenne, Lannes et Saut [43] et améliorées par Bresch et Renardy [20]. Contrairement au système de Saint-Venant à une couche et au cas particulier précédemment énoncé, la détermination explicite du domaine d'hyperbolicité du système de Saint-Venant bi-couches (à surface libre) n'est pas possible. Il est à noter que, dans ce cas, même l'hypothèse de non-cavitation ($\underline{h}_1 + h_1 > 0$ et $\underline{h}_2 + h_2 > 0$) ne garantit pas l'hyperbolicité de celui-ci. Néanmoins, certaines approches existent pour clarifier cette hyperbolicité, dont l'approche géométrique établie par Ovsjannikov [65] mais également celle de Barros et Choi [10] ainsi que Viríssimo et Milewski [76], que nous détaillerons d'une manière claire et concise dans le Chapitre 6.

Comme mentionné précédemment pour le cas du Saint-Venant bi-couches, nous ne disposons pas également d'une formule explicite pour le domaine d'hyperbolicité du Saint-Venant multi-couches ($\kappa = 0$ dans le système (1.2.12) avec $N \geq 2$) et il n'existe pas d'approche géométrique qui permet d'avoir une meilleure compréhension de celui-ci. Cette difficulté découle de la présence de plusieurs variables dans ce système, rendant l'analyse compliquée, voire même impossible. Néanmoins, dans [27, 62], les auteurs fournissent des conditions suffisantes pour une stricte hyperbolicité lorsque le fluide est stablement stratifié (c'est-à-dire lorsque $\rho_1 < \rho_2 < \dots < \rho_N$), mais ces conditions sont obtenues à l'aide d'arguments perturbatifs par rapport à la situation sans vitesses de cisaillement et

elles dégènèrent lorsque $N \rightarrow \infty$ vers

$$\{(\underline{H}_i + H_i, \underline{U}_i + U_i) \in \mathbb{R}^{2N} : \underline{H}_i + H_i > 0, \underline{U}_1 + U_1 = \underline{U}_2 + U_2 = \dots = \underline{U}_N + U_N\},$$

empêchant toute étude au niveau non linéaire ou incluant des vitesses de cisaillement. Nous laissons le lecteur se référer à la discussion intéressante dans [70, §5] concernant les résultats de stabilité pour les systèmes multi-couches lors de l'augmentation du nombre de couches, en relation avec les résultats de stabilité pour le système continu stratifié. On y rappelle le célèbre critère de stabilité de Miles [61] et Howard [45], empêchant l'instabilité spectrale des écoulements de cisaillement continu stratifiés si le nombre de Richardson local est partout supérieur à $1/4$. Il est également mentionné explicitement que cette notion de stabilité (ainsi que d'autres résultats, obtenus par Holm et Long [44] et Arbanel et al. [1]) est en général trop faible pour impliquer le contrôle des déviations par rapport aux équilibres, en particulier au niveau non linéaire.

Selon notre compréhension actuelle, le caractère bien posé du problème à valeur initiale dans des espaces de régularité finie pour le système continu stratifié (1.2.11), en l'absence de diffusivité d'épaisseur ($\kappa = 0$), avec des données initiales satisfaisant le critère de Miles et Howard, reste un problème ouvert. Nous invitons le lecteur à se référer au travail de Kukavica et al. [52] pour l'existence et l'unicité d'une solution dans des espaces de fonctions analytiques (dans le cadre du toit rigide), et à Cao, Li & Titi [23] parmi de nombreux autres travaux (voir [58] pour un compte rendu récent) pour la situation avec des contributions de viscosité horizontale et de diffusivité. Alors que les travaux mentionnés précédemment traitent des équations d'Euler hydrostatiques écrites en coordonnées eulériennes et ne reposent pas sur l'hypothèse de stable stratification, le travail de Bianchini et Duchêne [13] est plus proche de notre cadre car il traite spécifiquement le système en coordonnées isopycnales avec les termes supplémentaires de Gent et McWilliams, c'est-à-dire (1.2.11). Ils démontrent que des données initiales suffisamment régulières satisfaisant l'hypothèse de non-cavitation $(\underline{h} + h)|_{t=0} \geq h_* > 0$ conduisent à une unique solution sur un intervalle de temps $[0, T]$ avec $T^{-1} \lesssim 1 + \kappa^{-1}(|\underline{u}'|_{L^2((\rho_{\text{surf}}, \rho_{\text{bott}}))}^2 + M_0^2)$ où M_0 est la taille des déviations initiales par rapport à l'équilibre.

Enfin pour cloturer cette section, il est pertinent de mentionner que dans le cas où $\kappa = 0$ et en la présence de la force de Coriolis le calcul explicite du domaine d'hyperbolicité n'est pas possible/connu pour le système multi-couches (1.2.12). Néanmoins le résultat de Stewart et Dellar [71], démontre numériquement que le seuil pour les vitesses de ci-

saillement, en dessous duquel les équations à trois couches restent hyperboliques, n'est pas modifié par l'inclusion de la force de Coriolis complète. Les équations standards d'eaux peu profondes ne perdent leur hyperbolicité qu'en raison de cisaillements parallèles à la direction de propagation des vagues, mais la force de Coriolis complète introduit un autre mécanisme de perte d'hyperbolicité en raison de cisaillements dans la direction perpendiculaire.

1.2.3 Résultats principaux

Notre premier résultat concerne le système de Saint-Venant (shallow water) à une couche avec les termes additionnels de Gent & McWilliams (3.1.1). Nous présentons un résultat de caractère bien posé dans des espaces de régularité finie et en temps court pour ce système. La référence à un temps "court" provient du fait que le temps d'existence et d'unicité obtenu dépend du paramètre κ de ces termes additionnels. Ce résultat est obtenu en se basant fortement sur la diffusion induite par ces termes en κ , ce qui nous permet d'utiliser les théorèmes généraux bien établis pour le caractère bien posé des équations de transport et de transport-diffusion. De manière analogue et en employant la même stratégie que précédemment sur le système à multi-couches (1.2.12), nous obtenons un caractère bien posé de celui-ci, avec un temps d'existence qui dépend du nombre de couches N et du paramètre κ . En un second temps, nous changeons de stratégie en utilisant la méthode d'énergie. Nous démontrons que l'on peut améliorer le temps d'existence et d'unicité de la solution du système à une couche, en un temps qui est indépendant du paramètre κ . Toujours dans le même esprit et avec la même stratégie, nous améliorons ce résultat sur le système à multi-couches, en montrant que sous des hypothèses naturelles et en particulier l'hypothèse de stratification stable (en fait, nous supposons pour simplifier que les densités sont équidistribuées, c'est-à-dire $(\rho_{i+1} - \rho_i = \frac{\rho_{\text{bott}} - \rho_{\text{surf}}}{N})$) et l'hypothèse de non-cavitation $(\underline{H}_i + H_i)|_{t=0} \geq h^* > 0$, les solutions du système multi-couches (1.2.12) sont uniforme, uniques et existent sur un intervalle de temps analogue à celui du système continument stratifié (obtenu dans [13]), qui est en particulier *uniforme par rapport* à N et avec une meilleure dépendance en κ , nous y ferons référence par caractères bien posés en temps long (*ie Large time well-posedness*) (contrairement au caractère bien posé en temps court (*ie small time well-posedness*) quand le temps d'existence a priori dépend de N). Ceci repose sur l'exploitation des propriétés de symétrie du système qui proviennent du dernier terme de l'équation (1.2.12)₂, que l'on peut voir comme le terme permettant de transmettre les informations entre les couches du fluide. Grâce à ce terme, on obtient

des compensations naturelles qui apparaissent au cours de notre raisonnement avec la méthode de l'énergie. Enfin et pour cette première partie nous montrons qu'en présence des termes de Gent & McWilliams, l'énergie des deux systèmes (3.1.1) et (1.2.12) est décroissante.

Notre deuxième résultat principal énonce que pour toute solution suffisamment régulière du système continument stratifié (1.2.11) satisfaisant l'hypothèse de non-cavitation et des bornes appropriées, les solutions des systèmes multi-couches (1.2.12) avec des densités convenablement choisies ρ_i , des profondeurs de référence H_i , des vitesses de fond U_i et une déviation initiale $(H_i, U_i)|_{t=0}$ sont à une distance $\mathcal{O}(1/N^2)$ de la solution continument stratifiée.

Afin d'atteindre ces objectifs, nous nous appuyons principalement sur deux éléments :

1. un résultat de *cohérence*, qui affirme que les solutions suffisamment régulières du système continument stratifié peuvent être projetées en des fonctions à valeurs N -dimensionnelles satisfaisant le système de Saint-Venant multi-couches à un terme de reste de taille $\mathcal{O}(1/N^2)$
2. une estimation de *stabilité* appropriée sur le système linéarisé du Saint-Venant multi-couches, qui est uniforme par rapport à N .

Le résultat de cohérence revient à contrôler la différence entre la projection N -dimensionnelle de la contribution du potentiel de Montgomery, $\frac{1}{\varrho} \partial_x \psi$ dans (1.2.11), et la contribution correspondante dans (1.2.12) lorsque $(H_i)_{i \in \{1, \dots, N\}}$ est la projection N -dimensionnelle de h . L'estimation $\mathcal{O}(1/N^2)$ est obtenue en exploitant des développements de Taylor à des valeurs appropriées de densités ρ_i dans l'esprit de la règle du point milieu dans les quadratures numériques.

L'estimation de stabilité sur (1.2.12) repose sur la méthode d'énergie, pour une fonctionnelle d'énergie soigneusement construite. Afin d'obtenir des bornes appropriées sur la bonne échelle de temps, nous utilisons une structure partiellement symétrique du système hyperbolique (lorsque $\kappa = 0$) et nous nous appuyons sur l'effet de régularisation de la diffusivité uniquement lorsque c'est nécessaire. Cette structure partiellement symétrique et la construction de la fonctionnelle d'énergie associée reposent sur une décomposition de l'opérateur linéaire

$$\Gamma_N : (H_i)_{i \in \{1, \dots, N\}} \mapsto \left(\sum_{j=1}^N \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} H_j \right)_{i \in \{1, \dots, N\}}$$

qui fait écho directement à une décomposition analogue de l'opérateur continu

$$\mathcal{M} : h \mapsto \frac{1}{\varrho} \left(\rho_{\text{surf}} \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} h(\varrho') d\varrho' + \int_{\rho_{\text{surf}}}^{\varrho} \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} h(\varrho'') d\varrho'' d\varrho' \right)$$

et qui est un ingrédient clé dans l'étude de Bianchini et Duchêne. Plus généralement, nos estimations d'énergie reflètent celles de [13] après avoir introduit un dictionnaire entre les opérateurs continus et les opérateurs discrets analogues, avec plusieurs adaptations pour que cela corresponde à notre cadre.

En ce qui concerne le problème "inverse" (approximation du système bi-couches par le système continument stratifié), nous remarquons que les solutions du système de Saint-Venant bi-couches (6.2.6) (ou (1.2.12) avec $N = 2$) fournissent des solutions exactes à (1.2.13). Plus précisément, si nous notons

$$\begin{aligned} \underline{\rho}_{\text{bl}}(r) &= \rho_s \mathbf{1}_{(-\underline{H}_s, 0)}(r) + \rho_b \mathbf{1}_{(-1, -\underline{H}_s)}(r), \\ \underline{u}_{\text{bl}}(r) &= \underline{U}_s \mathbf{1}_{(-\underline{H}_s, 0)}(r) + \underline{U}_b \mathbf{1}_{(-1, -\underline{H}_s)}(r), \\ u_{\text{bl}}(\cdot, r) &= U_s \mathbf{1}_{(-\underline{H}_s, 0)}(r) + U_b \mathbf{1}_{(-1, -\underline{H}_s)}(r), \\ h_{\text{bl}}(\cdot, r) &= \frac{H_s}{\underline{H}_s} \mathbf{1}_{(-\underline{H}_s, 0)}(r) + \frac{H_b}{\underline{H}_b} \mathbf{1}_{(-1, \underline{H}_s)}(r), \end{aligned} \tag{1.2.16}$$

où H_s, U_s, H_b, U_b sont des solutions du Saint-Venant bi-couches avec les termes additionnels de Gent & McWilliams (6.2.6) (ou (1.2.12) avec $N = 2$), alors $(\underline{\rho}_{\text{bl}}, \underline{u}_{\text{bl}}, h_{\text{bl}}, u_{\text{bl}})$ est une solution de (1.2.13)-(1.2.14). Dans ce travail, nous comparerons ces solutions avec celles provenant de données satisfaisant (1.2.16) seulement de manière approximative. Plus précisément, nos résultats sont de deux types.

1. Nous démontrons que des solutions fortes du système Saint-Venant bi-couches (6.2.6) émergent à partir de données initiales suffisamment régulières satisfaisant certaines conditions d'hyperbolicité.
2. Nous démontrons que des solutions fortes aux équations d'Euler hydrostatiques (1.2.13) émergent à partir de données proches des profils constants par morceaux donnés par (1.2.16), et que ces solutions restent proches des solutions à deux couches.

Notez que pour atteindre notre objectif de comparer des solutions avec une distribution de densité constante par morceaux avec une solution ayant une distribution de densité continue, nous souhaitons considérer des déviations qui peuvent être importantes dans

de petites régions autour de la pycnocline. Cela exige de sélectionner soigneusement la topologie mesurant la taille des déviations dans (2). De plus, notre stratégie repose sur la construction d'une solution intermédiaire approximative, qui est démontrée comme étant proche de la solution correspondante aux équations d'Euler hydrostatiques dans une topologie forte, et proche de la solution du bi-couches dans une topologie plus faible. Ainsi, cette solution intermédiaire approximative améliore la description de la solution exacte dans la pycnocline.

Une deuxième remarque importante est que bien que la contribution de la diffusion d'épaisseur soit essentielle pour nos estimations de stabilité, nous souhaitons contrôler et comparer les solutions sur un intervalle de temps qui est uniforme par rapport au paramètre de diffusion d'épaisseur, $0 < \kappa \leq 1$. La dépendance par rapport au paramètre d'épaisseur n'apparaîtra que comme une restriction sur la taille des déviations admissibles. En ce qui concerne (1), cela est rendu possible par le fait bien connu que le système de Saint-Venant bi-couches est bien posé (sous certaines conditions d'hyperbolicité) lorsque $\kappa = 0$ [65], cependant, obtenir le résultat correspondant pour $\kappa > 0$ n'est pas direct et nécessite une utilisation fine de la structure des paramètres de diffusion d'épaisseur. En ce qui concerne (2), nous utilisons l'existence de la solution du bi-couches et par conséquent l'existence de la solution approximante améliorée pour utiliser un argument de bootstrap sur le contrôle de solutions suffisamment proches des équations d'Euler hydrostatiques sur l'échelle de temps pertinente. À cet égard, l'effet régularisant des contributions de diffusion d'épaisseur est fortement utilisé.

Notons que le deuxième résultat (2) s'applique à *n'importe quelle* solution donnée (suffisamment régulière) des équations d'Euler hydrostatiques. Ainsi, notre travail fournit les mêmes estimations de stabilité autour d'autres solutions, construites par exemple dans le cadre multi-couches ou d'ondes simples.

1.2.4 Travaux connexes

À notre connaissance, c'est la première fois que la convergence entre les solutions des systèmes multi-couches et les équations continument stratifiées est rigoureusement démontrée (alors que des liens formels sont discutés, par exemple, dans les travaux pionniers de Benton [11], Su [72], Killworth [50]), à l'exception notable du travail de Chen et Walsh [24]. Dans ce dernier travail, les auteurs examinent le cadre des ondes progressives périodiques sans l'approximation hydrostatique (et sans contributions de diffusivité). Ils démontrent que pour des ondes progressives périodiques suffisamment petites et solutions

aux équations continument stratifiées satisfaisant des hypothèses naturelles (en particulier que la stratification est stable), une régularité suffisante de la densité de ligne de courant et l'absence de points de stagnation), il existe des solutions d'ondes progressives correspondantes associées à n'importe quelle fonction de densité de ligne de courant lisse par morceau (d'où, en particulier, des solutions aux systèmes multi-couches qui correspondent à des fonctions de densité de ligne de courant constantes par morceaux, le champ de vitesse étant irrotationnel à l'intérieur de chaque couche), dans un voisinage L^∞ , et que l'application des fonctions de densité de ligne de courant aux ondes progressives correspondantes est Lipschitz continue dans des espaces fonctionnels appropriés, la distance entre deux densités de ligne de courant étant mesurée à travers la norme L^∞ .

Ce résultat, qui est assez robuste et polyvalent, ne se compare pas directement au nôtre. Plus important encore, le cadre des ondes progressives est bien sûr très différent de notre cadre permettant des dynamiques non triviales. En résumé, le problème d'ondes progressives peut être reformulé en un problème de nature elliptique, alors que nos systèmes d'équations sont hyperboliques/paraboliques. De manière cohérente, les outils utilisés dans notre travail, à l'exception d'un changement de coordonnées semi-lagrangien pour fixer le domaine du fluide (appelé transformation Dubreil-Jacotin dans le contexte des ondes progressives, équivalent au changement de coordonnées isopycnales), sont très différents des outils utilisés dans [24]. De plus, l'approximation hydrostatique modifie également la nature du problème, car elle élimine les effets dispersifs. En tant que tels, les ondes non linéaires progressives sont inexistantes dans le cadre de l'approximation hydrostatique, et pourraient être remplacées—si l'on souhaite étudier des dynamiques simplifiées— par des ondes simples [25, 64].

Enfin, comme discuté dans [24], la limite dite "inverse", consistant à approcher une situation à bi-couches (ou multi-couches) avec des problèmes en continument stratifié, a été étudiée (dans le cadre des ondes progressives) en particulier par James [47], après Turner et Amick [74, 3].

1.3 Plan de la thèse

Dans la première partie du Chapitre 3 nous nous intéressons en premier lieu au système de Saint-Venant (shallow water) à une couche avec les termes additionnels de Gent & McWilliams précédemment annoncés (voir l'équation (1.1.2)). Nous présentons un résultat de caractère bien posé dans des espaces de régularité finie et en temps court (le

temps dépend du paramètre κ de ces termes additionnels) voir Théorème 3.2.4. Nous montrons dans la Proposition 3.2.7 que ce système à une énergie décroissante. Ensuite on améliore ce temps d'existence et d'unicité de la solution de ce système en un temps qui est indépendant du paramètre κ (voir Proposition 3.2.9).

Dans la seconde partie de ce chapitre, et à partir de tout ce qui va suivre, nous considérerons une dimension d'espace ($d = 1$) afin de ne pas alourdir la lecture et pour obtenir de manière plus simple les résultats du chapitre 7 qui généralisent les résultats connus sur les espaces de régularité finie. Dans cette partie, nous examinons le système de Saint-Venant (shallow water) multi-couches pour la propagation d'ondes de gravité dans les écoulements à densité stratifié, avec les termes additionnels de Gent & McWilliams (1.2.12). Dans le même esprit que précédemment, nous établissons un caractère bien posé de ce système dans les espaces de régularité finie avec un temps non uniforme par rapport aux paramètres κ et le nombre de couches N (voir Théorème 3.3.1). Enfin nous clôturons ce chapitre en établissant que l'énergie associée à ce système est également décroissante (voir Proposition 3.3.2).

Dans le chapitre 4, nous améliorons le temps d'existence et d'unicité pour le système de Saint-Venant multi-couches obtenu précédemment en recourant à la méthode d'énergie (voir Théorème 4.5.1). La mise en œuvre de cette méthode requiert une étape cruciale de quasilinearisation, permettant d'obtenir les estimations de stabilité nécessaires pour conclure l'argument de continuité utilisée. L'amélioration de ce temps d'existence et d'unicité vise à obtenir un temps présentant une meilleure dépendance en κ et qui est uniforme par rapport au nombre de couches N .

Dans le chapitre 5, nous abordons la question de la convergence du système multi-couches (1.2.12) vers le système continu stratifié (1.2.11) lorsque le nombre de couches N tend vers l'infini et nous fournissons un taux de convergence explicite qui est d'ordre $1/N^2$ (voir Theorem 5.4.1).

Dans le chapitre 6, dans la Section 6.2, nous étudions le système de Saint-Venant bi-couches. Nous rappelons d'abord certains résultats connus sans diffusion d'épaisseur, puis nous examinons le système avec des contributions de diffusion d'épaisseur. Le résultat principal est la Proposition 6.2.8, qui donne, pour des données initiales suffisamment régulières satisfaisant un critère d'hyperbolicité, l'existence et le contrôle de solutions fortes au système de Saint-Venant bi-couches sur un intervalle de temps uniforme par rapport à $0 < \kappa \leq 1$, tandis que la Proposition 6.2.9 énonce la convergence forte attendue

quand $\kappa \rightarrow 0$. Dans la Section 6.3, nous étudions les équations d'Euler hydrostatiques. Nous fournissons d'abord des estimations de stabilité par rapport aux perturbations des équations et des données en utilisant des distances appropriées. En deuxième étape, nous introduisons des solutions approximatives associées à une solution exacte de référence donnée et à des données voisines. En nous appuyant sur ces solutions approximatives, nous démontrons la Proposition 6.3.8, qui contrôle la différence entre la solution de référence et les solutions exactes des équations d'Euler hydrostatiques émergeant de données voisines. Ensemble, la Proposition 6.2.8 et la Proposition 6.3.8 fournissent la justification rigoureuse annoncée selon laquelle, pour des données initiales suffisamment proches du cadre à deux couches et satisfaisant un critère d'hyperbolicité limitant notamment la taille admissible des vitesses de cisaillement, les solutions émergentes des équations d'Euler hydrostatiques restent proches de la solution prédite par le système de Saint-Venant bi-couches sur une échelle de temps pertinente. Le résultat rigoureux est présenté dans le Théorème 6.3.1.

Le chapitre 7 représente une boîte à outils pour les chapitres 4 et 5, car il regroupe la généralisation au cadre des fonctions à valeurs dans \mathbb{R}^N tous les lemmes et propositions classiques (de produits, composition, injections...) que l'on retrouve pour les fonctions numérique dans les espaces de régularité finie, notamment les espaces de Sobolev.

ENGLISH INTRODUCTION

2.1 General context

The oceans, as essential components of our planet, play a central role in regulating climate, the water cycle, and marine biodiversity. Understanding the complex oceanographic phenomena that animate these vast expanses of water is crucial for grasping the underlying mechanisms and interactions that shape our environment. At the heart of this understanding lies the modeling of fluid flows, whose dynamics are governed by fundamental equations. In this context, we introduce these fundamental equations that describe the evolution of heterogeneous, incompressible flows under the influence of gravity

$$\begin{aligned}
& \partial_t \rho + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}} \rho + (w + w_*) \partial_z \rho = 0, \\
& \rho (\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}}) \mathbf{u} + (w + w_*) \partial_z \mathbf{u}) + \nabla_{\mathbf{x}} P + f \mathbf{k} \times \mathbf{u} = 0, \\
& \rho (\partial_t w + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}} w + (w + w_*) \partial_z w) + \partial_z P + g \rho = 0, \\
& \nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w = 0, \\
& P|_{z=\zeta} - P_{\text{atm}} = 0, \\
& \partial_t \zeta + (\mathbf{u} + \mathbf{u}_*)|_{z=\zeta} \cdot \nabla_{\mathbf{x}} \zeta - (w + w_*)|_{z=\zeta} = 0, \\
& w|_{z=-H} = 0.
\end{aligned} \tag{2.1.1}$$

Here, t and (\mathbf{x}, z) represent respectively time and the horizontal-vertical spatial variables, and we use $\nabla_{\mathbf{x}}$, $\nabla_{\mathbf{x}} \cdot$, and $\Delta_{\mathbf{x}}$ to denote the gradient, divergence, and Laplacian with respect to \mathbf{x} . The vector field $(\mathbf{u}, w) \in \mathbb{R}^d \times \mathbb{R}$ represents the velocity (horizontal and vertical), $\rho > 0$ is the density, $P \in \mathbb{R}$ is the incompressible pressure, all defined in the spatial domain.

$$\Omega_t = \{(\mathbf{x}, z) : \mathbf{x} \in \mathbb{R}^d, -H < z < \zeta(t, \mathbf{x})\}$$

where $\zeta(t, \mathbf{x})$ describes the position of a free surface, and H is the depth of the resting layer. The gravitational field is assumed to be constant and vertical, with $g > 0$ as the

gravitational acceleration constant, $f \in \mathbb{R}$ as the Coriolis parameter, and the constant vector \mathbf{k} representing the unit vector in the vertical direction. Finally, the advection terms associated with the 'bolus velocity' $(\mathbf{u}_*, w_*) \in \mathbb{R}^d \times \mathbb{R}$ were proposed by Gent and McWilliams [35]. These terms are introduced to account for the contribution of geostrophic eddy correlations to the effective transport velocities in non-eddy-resolving (large-scale) models. Their specific forms in the simplest case of constant diffusivity parameter κ read as follows:

$$\mathbf{u}_* = \kappa \partial_z \left(\frac{\nabla_x \rho}{\partial_z \rho} \right), \quad w_* = -\kappa \nabla_x \cdot \left(\frac{\nabla_x \rho}{\partial_z \rho} \right), \quad \kappa > 0 \quad (2.1.2)$$

Neglecting the effective advection terms (i.e., setting $\kappa = 0$), we recover the Euler equations for heterogeneous, incompressible fluids under the influence of vertical gravitational forces and the Coriolis force, where the last two lines of (2.1.1) model respectively the kinematic equation at the free surface and the impermeability condition of the rigid bottom. In equation (2.1.1), the pressure P can be obtained from its (atmospheric) value at the surface, P_{atm} , by solving the elliptic boundary-value problem induced by the incompressibility constraint of divergence free velocity fields. Yet, in the shallow water regime, where the horizontal scale of the perturbation is large compared with the depth of the layer H , formal computations (see below) suggest that vertical accelerations can be neglected and that pressure P approximately satisfies the hydrostatic balance law, that is

$$\partial_z P + g\rho = 0. \quad (2.1.3)$$

By replacing the equation for the vertical velocity in (2.1.1) with the identity in (2.1.3), this leads to the hydrostatic equations under the influence of the Coriolis force:

$$\begin{aligned} \partial_t \rho + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_x \rho + (w + w_*) \partial_z \rho &= 0 \\ \rho (\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_*) \cdot \nabla_x) \mathbf{u} + (w + w_*) \partial_z \mathbf{u}) + \nabla_x P + f \mathbf{k} \times \mathbf{u} &= 0 \\ \partial_t \zeta + (\mathbf{u} + \mathbf{u}_*)|_{z=\zeta} \cdot \nabla_x \zeta - (w + w_*)|_{z=\zeta} &= 0 \\ P &= P_{\text{atm}} + g \int_z^\zeta \rho(z', \cdot) dz', \\ w &= - \int_{-H}^z \nabla_x \cdot \mathbf{u}(z', \cdot) dz'. \end{aligned} \quad (2.1.4)$$

It is worth noting that the previous system represents a simplification of the primitive

equations, which involve aspects such as diffusivity, viscosity, tracer equations (salinity S and temperature θ), as well as an equation of state. This latter is an equation expressing density ρ as a function of S and θ . These details are elaborated in the paper by Titi and Korn [51], as well as in Vallis's geophysics textbook [75][section 2.2 et section 10.7].

The implications of the Coriolis force become significant at large oceanic scales. However, despite this, in our analysis, we choose to neglect the effects of Earth's rotation for several reasons. On one hand, to simplify the systems treated in this manuscript. On the other hand, we focus on the local well-posedness of our systems of equations. We operate in a regime where Earth's rotation is not rapid, meaning that accounting for this force brings neither significant advantages nor additional difficulties to our analysis. Furthermore, through our well-posedness results, we demonstrate the absence of rapidly growing high-frequency instabilities. It is noteworthy that in this scenario, the presence of the Coriolis force has no substantial impact.

We emphasize that our study is theoretical in nature and does not aim to model the ocean. It focuses on models that are not precise in terms of the physical representation of oceans. Nevertheless, through these models, we address relevant issues, notably the hydrostatic assumption and density variation.

The underlying physical reason for introducing additional transport velocities, denoted by \mathbf{u}_* and w_* and defined in equation (2.1.2), within the systems of equations (2.1.1) and (2.1.4), is that these "bolus" velocities act as adjustments to the standard incompressible Euler equations that apply independently of whether the hydrostatic assumption is included or not, and are derived from the fundamental principles of fluid mechanics. Note that in this manuscript, our interest will focus on these additional transport velocities which induce diffusion, as we will see in equations (2.2.11) below. The deterministic modeling of this effective diffusivity induced by eddy correlation, which we implement in this study, dates back to the 1990s and is credited to Gent & McWilliams [35], as referenced also in [39, 40]. This approach originates from the necessity to model the averaged dissipative effects of mesoscale eddies, which were computationally infeasible at that time, on the large-scale flow at the macroscopic level. They consider the unknowns (ρ, \mathbf{u}, w) as the large-scale components of the density and velocity fields, respectively. The approach involves introducing suitable correctors to the mass conservation equation and equations

for tracers (such as salinity and temperature). More precisely, they suggest incorporating bolus velocity fields \mathbf{u}_* and w_* , defined in equation (2.1.2). The addition of the bolus velocity contribution in the momentum conservation equation was suggested in [39, 36]. One of the main ingredient leading to the specific form of the bolus velocity fields is that the averaged dissipative effect of mesoscale eddies should act predominantly along isopycnals sheets (that is along and not across surfaces of equal densities). It should be mentioned that the Gent & McWilliams eddy-induced advection is only part of the subgridscale parametrization of mesoscale eddies used in global ocean general circulation models, which also include Redi's eddy-induced diffusion [69]. Specifically, the equations for tracers including the so-called Gent–McWilliams–Redi eddy parametrization read (see [51] and references therein)

$$\partial_t C + (\mathbf{U}_3 \cdot \nabla_3) C = \nabla_3 \cdot (K_R \nabla_3 C) + \nabla_3 \cdot (K_{GM} \nabla_3 C), \quad (2.1.5)$$

where C is a tracer (typically the temperature θ and the salinity S), $\mathbf{U}_3 := (\mathbf{u}, w)$ denotes the three-dimensional velocity field, and ∇_3 the three-dimensional gradient. Above, we define K_R and K_{GM} by

$$K_R := \frac{K_I}{1 + |\mathbf{L}|^2} \begin{pmatrix} 1 + L_y^2 & -L_x L_y & L_x \\ -L_x L_y & 1 + L_x^2 & L_y \\ L_x & L_y & |\mathbf{L}|^2 \end{pmatrix} + \frac{K_D}{1 + |\mathbf{L}|^2} \begin{pmatrix} L_x^2 & L_x L_y & -L_x \\ L_x L_y & L_y^2 & -L_y \\ -L_x & -L_y & 1 \end{pmatrix} \quad (2.1.6)$$

$$K_{GM} := \kappa \begin{pmatrix} 0 & 0 & -L_x \\ 0 & 0 & -L_y \\ L_x & L_y & 0 \end{pmatrix}$$

where we used the notation $L := (L_x, L_y) = \frac{-\nabla_{\mathbf{x}} \rho}{\partial_z \rho}$, and the non-negative constants K_I and K_D are the isoneutral (tangent to neutral density surfaces) and dianeutral (normal to neutral density surfaces) diffusivity coefficients. Notice that the tensor K_R is symmetric, while the tensor K_{GM} is skew-symmetric, so that the dissipative effect due to Gent & McWilliams eddy-advection is not obvious. In fact, there has been some debate about the adiabatic nature of the Gent & McWilliams parametrization [38]. As it is evident from the formulation of the equations in isopycnal coordinates (see equation (2.2.11) below), the Gent & McWilliams eddy-induced advection terms introduce a diffusive contribution to the thickness variable. This is why the parameter κ is often referred to as the thickness diffusivity parameter.

As observed in equation (2.1.5), the Redi-Gent & McWilliams terms have been added to the tracer equations (temperature θ and salinity S) to account for their effect on the Euler equations (2.1.1) and hydrostatic Euler equations (2.1.4). It is natural to question the impact of these additional terms on the density variable ρ . It turns out that, when applied to the density variable, the isoneutral contribution of Redi's eddy-induced diffusion vanishes exactly, while the dianeutral contribution becomes an isotropic three-dimensional Laplacian, and the Gent & McWilliams contribution can be interpreted as advection.

$$\nabla_3 \cdot (K_R \nabla_3 \rho) = K_D \Delta_3 \rho, \quad \nabla_3 \cdot (K_{GM} \nabla_3 \rho) = (\mathbf{U}_3^* \cdot \nabla_3) \rho,$$

where we denote $\mathbf{U}_3^* := (\mathbf{u}_*, w_*)$ defined in (2.1.2). Consequently, in situations where the equation of state $\rho = \rho(\theta, S)$ is a linear combination of the tracers (with equal diffusivity parameters for all tracers), the equations (2.1.5) yield

$$\partial_t \rho + (\mathbf{U}_3 \cdot \nabla_3) \rho + (\mathbf{U}_3^* \cdot \nabla_3) \rho = K_D \Delta_3 \rho.$$

Since the dianeutral diffusivity coefficient is typically set much smaller than the isoneutral diffusivity coefficient, $0 \leq K_D \ll K_I$, it makes sense to neglect the right-hand side of the above equation. This results in the mass conservation equations in (2.1.1) and (2.1.4). It is essential to emphasize that the choice of the variable ρ as a linear combination of the tracers θ and S in the preceding paragraph constitutes a simple and elementary approach from a modeling perspective, aiming to simplify the framework of the primitive equations. However, it is possible to extend certain stability results of the primitive equations with the contributions of the Redi-Gent & McWilliams terms (2.1.6), in a more general context, specifically in the case where ρ is a smooth function of the aforementioned tracers. We particularly specify the results of existence of weak solutions, as well as a global well-posedness result obtained by Titi and Korn [51] in this general context.

2.2 Motivation of the thesis

2.2.1 Equations at play

In this section, we implement the systems of equations that will be the focus of this manuscript.

The rigorous justification of the hydrostatic assumption in the shallow-water regime, i.e, when the typical horizontal wavelength of the flow is large compared to the vertical depth of the layer, has been thoroughly analyzed in situations of homogeneous density [7, 57, 34, 58], with smooth density distributions [68, 67], or in the bi-layer framework [14, 31, 28]. The challenge of this thesis is to establish bridges between the continuous stratification framework (where density is continuous) and the sharp stratification framework (where density is constant on each layer of the fluid, particularly focusing on the bi-layer and multi-layer cases), in the context of the hydrostatic assumption.

In this manuscript, the continuously stratified hydrostatic system with the additional Gent & McWilliams terms will occupy a central place. However, before explicitly presenting it, we clarify in the following paragraphs the assumptions and methodology that allow us to extract this system from the Euler equations (2.1.1).

Let us consider smooth solutions to equation (2.1.1) defined over a time interval I_t . Assuming that the flow is stably stratified, i.e.

$$\inf(-\partial_z \rho) > 0, \tag{2.2.1}$$

the density $\rho : z \mapsto \rho(\cdot, \cdot, z)$ is an invertible function of z . We denote its inverse by $\eta : \varrho \mapsto \eta(\cdot, \cdot, \varrho)$, in order to define the following isopycnal change of variable:

$$\rho(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)) = \varrho, \quad \eta(t, \mathbf{x}, \rho(t, \mathbf{x}, z)) = z. \tag{2.2.2}$$

We also assume that $\rho(t, \mathbf{x}, -H) = \rho_1$, $\rho(t, \mathbf{x}, \zeta(t, \mathbf{x})) = \rho_0$ for $(t, \mathbf{x}) \in I_t \times \mathbb{R}^d$ where $\rho_0 < \rho_1$ are two fixed and positive constant reference densities. Then we have

$$\eta : I_t \times \Omega \mapsto \mathbb{R} \text{ with } \Omega := \mathbb{R}^d \times (\rho_0, \rho_1) \text{ and } h := -\partial_\varrho \eta > 0, \tag{2.2.3}$$

the latter inequality accounting for the stable stratification assumption. We now introduce

$$\check{\mathbf{u}}(t, \mathbf{x}, \varrho) = \mathbf{u}(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)), \quad \check{w}(t, \mathbf{x}, \varrho) = w(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)), \quad \check{P}(t, \mathbf{x}, \varrho) = P(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)).$$

From the chain rule, we infer that system (2.1.1) in isopycnal coordinates reads

$$\begin{aligned} \partial_t \eta + \check{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \eta - \check{w} &= \kappa \Delta_{\mathbf{x}} \eta, \\ \varrho \left(\partial_t \check{\mathbf{u}} + \left(\left(\check{\mathbf{u}} - \kappa \frac{\nabla_{\mathbf{x}} h}{h} \right) \cdot \nabla_{\mathbf{x}} \right) \check{\mathbf{u}} \right) + \nabla_{\mathbf{x}} \check{P} + \frac{\nabla_{\mathbf{x}} \eta}{h} \partial_{\varrho} \check{P} &= 0, \\ \varrho \left(\partial_t \check{w} + \left(\check{\mathbf{u}} - \kappa \frac{\nabla_{\mathbf{x}} h}{h} \right) \cdot \nabla_{\mathbf{x}} \check{w} \right) - \frac{\partial_{\varrho} \check{P}}{h} + g \varrho &= 0, \\ -h \nabla_{\mathbf{x}} \cdot \check{\mathbf{u}} - (\nabla_{\mathbf{x}} \eta) \cdot (\partial_{\varrho} \check{\mathbf{u}}) + \partial_{\varrho} \check{w} &= 0, \\ \check{P} \Big|_{\varrho=\rho_0} &= P_{\text{atm}}, \quad \check{w} \Big|_{\varrho=\rho_1} = 0. \end{aligned} \tag{2.2.4}$$

Notice that differentiating with respect to ϱ the first equation and using the fourth equation (stemming from the incompressibility constraint), the mass conservation reads

$$\partial_t h + \nabla_{\mathbf{x}} \cdot (h \check{\mathbf{u}}) = \kappa \Delta_{\mathbf{x}} h, \tag{2.2.5}$$

hence the emergence of the diffusion mentioned in the preceding section.

At this point, we are ready to introduce a dimensionless version of the previous system. We are interested in deviations from steady solutions to the incompressible Euler equations with variable density:

$$(h_{\text{eq}}, \mathbf{u}_{\text{eq}}, w_{\text{eq}}, P_{\text{eq}}) = (\underline{h}(\varrho), \underline{\mathbf{u}}(\varrho), 0, \underline{P}(\varrho)),$$

which satisfy the equilibrium condition

$$\partial_{\varrho} \underline{P}(\varrho) = g \varrho \underline{h}(\varrho).$$

Therefore, we consider (non-necessarily small) fluctuations of that steady solution, so that our unknowns admit the following decomposition:

$$\begin{aligned} h(t, \mathbf{x}, \varrho) &= \underline{h}(\varrho) + h_{\text{pert}}(t, \mathbf{x}, \varrho), & \check{\mathbf{u}}(t, \mathbf{x}, \varrho) &= \underline{\mathbf{u}}(\varrho) + \mathbf{u}_{\text{pert}}(t, \mathbf{x}, \varrho), \\ \check{w}(t, \mathbf{x}, \varrho) &= 0 + w_{\text{pert}}(t, \mathbf{x}, \varrho), & \check{P}(t, \mathbf{x}, \varrho) &= \underline{P}(\varrho) + P_{\text{pert}}(t, \mathbf{x}, \varrho). \end{aligned}$$

Furthermore, we non-dimensionalize the equations through the following scaled variables: we set

$$\frac{h(t, \mathbf{x}, \varrho)}{H} = \tilde{h}(\varrho) + \tilde{h}(\tilde{t}, \tilde{\mathbf{x}}, \varrho) \quad \text{and} \quad \frac{\check{\mathbf{u}}(t, \mathbf{x}, \varrho)}{\sqrt{gH}} = \tilde{\mathbf{u}}(\varrho) + \tilde{\mathbf{u}}(\tilde{t}, \tilde{\mathbf{x}}, \varrho),$$

and

$$\frac{\lambda}{H} \frac{\check{w}(t, \mathbf{x}, \varrho)}{\sqrt{gH}} = \tilde{w}(\tilde{t}, \tilde{\mathbf{x}}, \varrho), \quad \frac{\check{P}(t, \mathbf{x}, \varrho)}{gH} = \frac{P_{\text{atm}}}{gH} + \int_{\rho_0}^{\varrho} \varrho' \tilde{h}(\varrho') d\varrho' + \tilde{P}(\tilde{t}, \tilde{\mathbf{x}}, \varrho),$$

where we recall that H represents the depth of the resting layer. For λ a reference horizontal length, we use the following dimensionless coordinates

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{\lambda} \quad \text{and} \quad \tilde{t} = \frac{\sqrt{gH}}{\lambda} t. \quad (2.2.6)$$

Introducing the dimensionless (thickness) diffusion parameter, $\tilde{\kappa}$ and the shallowness parameter, μ , through

$$\tilde{\kappa} = \frac{\kappa}{\lambda \sqrt{gH}} \quad \text{and} \quad \mu = \frac{H^2}{\lambda^2}, \quad (2.2.7)$$

substituting the scaled coordinates/variables in system (2.2.4) and the subsequent equation and dropping the tildes for the sake of readability yields

$$\begin{aligned} \partial_t h + \nabla_x \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) &= \kappa \Delta_x h, \\ \varrho \left(\partial_t \mathbf{u} + \left(\left(\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h} \right) \cdot \nabla_x \right) \mathbf{u} \right) + \nabla_x P + \frac{\nabla_x \eta}{\underline{h} + h} (\varrho \underline{h} + \partial_\varrho P) &= 0, \\ \mu \varrho \left(\partial_t w + \left(\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h} \right) \cdot \nabla_x w \right) - \frac{\partial_\varrho P}{\underline{h} + h} + \frac{\varrho h}{\underline{h} + h} &= 0, \\ -(\underline{h} + h) \nabla_x \cdot \mathbf{u} - \nabla_x \eta \cdot (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}) + \partial_\varrho w &= 0, \quad (\text{div.-free cond.}) \\ \eta(\cdot, \varrho) = \int_{\varrho}^{\rho_1} h(\cdot, \varrho') d\varrho', \quad P|_{\varrho=\rho_0} = 0, \quad w|_{\varrho=\rho_1} = 0. & \quad (\text{bound. cond.}) \end{aligned} \quad (2.2.8)$$

The hydrostatic system is obtained by setting $\mu = 0$ in (2.2.8). Specifically, plugging the hydrostatic balance

$$\frac{\partial_\varrho P}{\underline{h} + h} = \frac{\varrho h}{\underline{h} + h} \quad \text{and} \quad P|_{\varrho=\rho_0} = 0,$$

into the second equation of (2.2.8), this yields the continuously stratified hydrostatic sys-

tem with the additional Gent & McWilliams terms, which is of interest in this manuscript.

$$\begin{aligned} \partial_t h + \nabla_x \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) &= \kappa \Delta_x h, \\ \varrho \left(\partial_t \mathbf{u} + \left(\left(\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h} \right) \cdot \nabla_x \right) \mathbf{u} \right) + \nabla_x \psi &= 0, \end{aligned} \quad (2.2.9)$$

with

$$\begin{aligned} \psi(t, \mathbf{x}, \varrho) &= \int_{\rho_0}^{\varrho} \varrho' h(t, \mathbf{x}, \varrho') d\varrho' + \varrho \int_{\varrho}^{\rho_1} h(t, \mathbf{x}, \varrho') d\varrho' \\ &= \rho_0 \int_{\rho_0}^{\rho_1} h(t, \mathbf{x}, \varrho') d\varrho' + \int_{\rho_0}^{\varrho} \int_{\varrho'}^{\rho_1} h(t, \mathbf{x}, \varrho'') d\varrho'' d\varrho'. \end{aligned} \quad (2.2.10)$$

More specifically, we will consider the framework of one spatial dimension ($\mathbf{x} = x \in \mathbb{R}$). Thus,

$$\begin{cases} \partial_t h + \partial_x((\underline{h} + h)(\underline{u} + u)) = \kappa \partial_x^2 h, \\ \partial_t u + \left(\underline{u} + u - \kappa \frac{\partial_x h}{\underline{h} + h} \right) \partial_x u + \frac{1}{\varrho} \partial_x \psi = 0, \end{cases} \quad (2.2.11)$$

where

$$\begin{aligned} \psi(t, x, \varrho) &= \int_{\rho_0}^{\varrho} \varrho' h(t, x, \varrho') d\varrho' + \varrho \int_{\varrho}^{\rho_1} h(t, x, \varrho') d\varrho' \\ &= \rho_0 \int_{\rho_0}^{\rho_1} h(t, x, \varrho') d\varrho' + \int_{\rho_0}^{\varrho} \int_{\varrho'}^{\rho_1} h(t, x, \varrho'') d\varrho'' d\varrho'. \end{aligned}$$

It is pertinent to mention that the system (2.2.8) (respectively (2.2.9)) obtained is merely a rewriting of the system (2.1.1) (respectively (2.1.4)) under the assumptions of stable stratification (2.2.1) and density continuity.

In order to facilitate understanding for the reader, we provide the physical interpretation of the variables considered in (2.2.11). As established previously, when stratification is stable, i.e., when density increases with depth for all horizontal spatial positions $x \in \mathbb{R}$, the reformulation with isopycnal coordinates involves the inverse of the density function $z \mapsto \rho(\cdot, z)$, which we denote as $\varrho \mapsto \eta(\cdot, \varrho)$ for $\varrho \in (\rho_0, \rho_1)$, where ρ_0 is the (constant) density at the free surface and ρ_1 is the (constant) density at the rigid flat bottom. Given that the graph of $\eta(\cdot, \varrho)$ represents, in Eulerian coordinates, the isopycnal sheet of density ϱ , the function defined as $\underline{h} + h = -\partial_\varrho \eta$ represents the infinitesimal depth of this isopycnal

sheet. Then, $\underline{u} + u$ represents the horizontal component of the fluid particle velocity at the isopycnal sheet. We decompose the depth and horizontal velocities into $\underline{h} + h$ and $\underline{u} + u$ where $(\underline{h}, \underline{u})$ represent the reference shear flow and are given functions depending only on the density variable $\varrho \in (\rho_0, \rho_1)$, and (h, u) are the unknowns representing deviations from equilibrium and depend on time t , horizontal space $x \in \mathbb{R}$, and the density variable $\varrho \in (\rho_0, \rho_1)$. Finally, ψ is the Montgomery potential (as mentioned in the paper by Holm and Long [44]), responsible for the interaction between isopycnal sheets.

It is important to note that to obtain the system (2.2.11), the assumption of stable and continuous stratification (2.2.1) was necessary to define the isopycnal change of variable, which implies that sheets of equal density form a foliation of the fluid domain; see, for example, [44, 13]. It is also worth mentioning that the use of the isopycnal change of variable can be applied in a more general framework than the one previously established (stable and continuous density stratification). More generally, we can consider $\varrho = \underline{\rho}(r)$ with $r \mapsto \underline{\rho}(r)$, not necessarily bijective or even continuous. We will have an overview of this in Chapter 6 where we will choose a particular case of this latter function, which will suit our study to deal with the bi-layer Saint-Venant system. Furthermore, the rigorous justification of the hydrostatic limit ($\mu \rightarrow 0$ in (2.2.8)) is found in the result by Duchêne and Bianchini [13].

By carefully analyzing the structure of the equations in the continuously stratified system with the additional Gent & McWilliams terms (2.2.11), we observe a certain similarity with the multi-layer shallow water system with Gent & McWilliams terms added in a similar manner (see equation (2.2.12)). In this manuscript, we precisely explain this similarity, in other words, we provide the necessary dictionary that allows the reader to consider the multi-layer system as an approximation of the continuously stratified system (see Chapter 5), and conversely, the continuously stratified system as an approximation of the bi-layer system (see Chapter 6).

The multi-layer system considered here is as follows

$$\begin{cases} \partial_t H_i + \partial_x((\underline{H}_i + H_i)(\underline{U}_i + U_i)) = \kappa \partial_x^2 H_i \\ \partial_t U_i + \left(\underline{U}_i + U_i - \kappa \frac{\partial_x H_i}{\underline{H}_i + H_i} \right) \partial_x U_i + g \sum_{j=1}^N \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j = 0, \end{cases} \quad \forall i \in \{1, \dots, N\} \quad (2.2.12)$$

This system corresponds to a situation of N layers of immiscible fluids with constant densities ρ_i , $i \in \{1, \dots, N\}$. When $\kappa = 0$, by applying the hydrostatic approximation and

the columnar motion assumption, we arrive (see for example [56, 9]) at the multi-layer shallow water system (2.2.12) where $\underline{H}_i + H_i$ represents the depth of the i^{th} layer (\underline{H}_i being the resting depth and H_i the deviation) and $\underline{U}_i + U_i$ is the average horizontal velocity of the layer (\underline{U}_i being the reference velocity and U_i the deviation), and $g = 1$ will be the gravitational acceleration constant after rescaling.

As previously mentioned, in (2.2.12) (respectively (2.2.11)), the terms proportional to κ were introduced by Gent and McWilliams [35] to represent the large-scale contribution of unresolved eddies. They appear in the form of additional effective velocities $U_i^* := -\kappa \frac{\partial_x H_i}{\underline{H}_i + H_i}$ (respectively $u^* := -\kappa \frac{\partial_x h}{h+h}$) and act as diffusivity contributions in the mass conservation equations. Interestingly, similar terms were introduced in the work of Duran, Vila, and Baraille [32] to control the discrete energy of a semi-implicit numerical scheme for the multi-layer system (2.2.12). From a mathematical perspective, as established in this manuscript, the regularization effect of diffusivity contributions is essential for our analysis as it provides appropriate stability estimates, and we will always assume that $\kappa > 0$.

Note that the approach of using multi-layer models to approximate the Euler and Navier-Stokes equations has been addressed by several authors. Among them, Audusse [4] proposed a multi-layer shallow water model by discretizing in depth, which leads to a set of coupled classical shallow water systems to approximate the hydrostatic Navier-Stokes equations. Unlike the classical shallow water model, where the horizontal velocity does not depend on the vertical variable and thus implies a loss of information (as the model does not accurately describe the vertical profile of this velocity), the proposed multi-layer model extends the validity range and provides an accurate description of the vertical profile. To justify this numerically, the author uses a layer-by-layer approach, adapting well-known results on the finite volume solver of the classical shallow water system to the multi-layer case (see also [6]). The previous approach differs from the one established in this manuscript in terms of modeling. Indeed, the construction of the multi-layer model proposed by the author is done purely artificially by discretizing the heights. In other words, the interface is determined solely by the variation of the surface, which implies an exchange of fluid particles between its different layers over time (see also [5, 33]). In contrast to the multi-layer model treated in this manuscript (2.2.12), which is obtained by following the density profile, thereby discretizing the fluid domain into a series of layers with constant density. In this model, the densest layer is at the bottom, and no exchange of mass between the layers is possible.

From another perspective, Camassa and Tiron [22] proposed a bi-layer model to approximate the incompressible Euler system with continuous stratification. The construction of this model, as well as its justification, relies on the fact that, similar to the continuously stratified system, it must also capture infinitesimal amplitude velocity waves with infinite wavelength in addition to conserving certain physical quantities (which characterize a stratified configuration), including mass and potential energy in each of the layers. This justification represents a physical approach to the approximation problem, which differs from the theoretical approach considered in this manuscript. Our approach is based on demonstrating well-posedness and explicitly stating the convergence rate (which depends on the number of layers considered) to justify the validity of our multi-layer model.

As mentioned earlier, we are examining the approximation of the continuously stratified hydrostatic system by the multi-layer system (with density constant on each layer). This naturally leads us to study the "inverse" problem, which is to demonstrate that in the limit where the density (initially continuous) becomes piecewise constant, the continuously stratified hydrostatic system becomes an approximation of the multi-layer system (we focus here on the two-layer case).

This naturally brings us back to the bi-layer shallow water system (corresponding to $N = 2$ in (2.2.12)), which is a standard model used to describe internal waves in density-stratified flows, particularly in situations where the density distribution allows an approximate representation of the fluid as two layers with nearly constant densities separated by a thin pycnocline; see, for example, [42, Chap. 6]. In addition to this *sharp stratification* assumption, the formal derivation of the bi-layer shallow water system relies on two additional conditions. It is assumed that the internal pressure is hydrostatic, meaning that the pressure gradient force balances the external force due to gravity (see (2.1.3)). It is also assumed that the flow velocity is *columnar*, which means that the horizontal velocity of fluid particles remains constant with respect to the vertical variable within each layer.

Naturally, the validity of the bi-layer shallow water system depends on the expectation that these three assumptions remain accurate over a relevant timescale as the flow evolves.

In the bi-layer framework and assuming hydrostatic pressure, the columnar assumption is exactly propagated by the flow. This means that the bi-layer shallow water system produces exact solutions to the hydrostatic (incompressible) Euler equations with density and horizontal velocity distributions that are piecewise constant with respect to the

vertical variable.

In this manuscript, we examine solutions to the hydrostatic Euler equations near such solutions, i.e., by relaxing the assumptions of sharp stratification and columnar motion. We demonstrate that for initial data sufficiently close to the bi-layer framework, the emerging solutions of the hydrostatic Euler equations remain close to the solution predicted by the bi-layer shallow water system over a relevant timescale.

More precisely, the hydrostatic Euler equations we consider take the form:

$$\begin{aligned} \partial_t h + \partial_x((1+h)(\underline{u} + u)) &= \kappa \partial_x^2 h, \\ \partial_t u + \left(\underline{u} + u - \kappa \frac{\partial_x h}{1+h} \right) \partial_x u + \frac{1}{\underline{\rho}} \partial_x \Psi &= 0, \end{aligned} \tag{2.2.13}$$

where the Montgomery potential Ψ is given by:

$$\Psi(t, x, r) = \underline{\rho}(r) \int_{-1}^r h(\cdot, r') dr' + \int_r^0 \underline{\rho}(r') h(\cdot, r') dr'. \tag{2.2.14}$$

The system (2.2.14) is merely a rewriting of the system (2.2.11) through a change of variables, specifically by taking $\varrho = \underline{\rho}(r)$ with $r \mapsto \underline{\rho}(r)$ where the variable $r \in (-1, 0)$ refers to the depth of the isopycnal line (scaled). This allows fixing the infinitesimal reference depth of the isopycnals to the value $\underline{h}(r) = 1$. Additionally, the total depth of the fluid domain is set to $\int_{-1}^0 \underline{h}(r) dr = 1$ through appropriate scaling. It is noteworthy that the system (2.2.13) can be derived from the system in Eulerian coordinates (2.1.1), even in cases where the stratification $\underline{\rho}$ is not bijective, particularly in the multi-layered framework. Moreover, the system (2.2.13) is less singular in the limit of sharp stratification.

Finally, after motivating the reader regarding the transition from the discrete system (multi-layered shallow water) to the continuously stratified hydrostatic system, and vice versa, from the continuously stratified hydrostatic system to the discrete system (bi-layered shallow water), we will also investigate the single-layer shallow water system, including the additional terms of Gent & McWilliams (corresponding to $N = 1$ in (2.2.12)

and in the case of spatial dimension d).

$$\begin{cases} \partial_t h + \nabla \cdot ((h + \underline{h})(\mathbf{u} + \underline{\mathbf{u}} + \mathbf{u}^*)) = 0, \\ \partial_t \mathbf{u} + ((\mathbf{u} + \underline{\mathbf{u}} + \mathbf{u}^*) \cdot \nabla) \mathbf{u} + \nabla h = 0, \\ h|_{t=0} = h_0, \inf h_0 > 0, \\ u|_{t=0} = u_0, \end{cases} \quad (2.2.15)$$

where $t > 0$ represents the time variable, d is the horizontal spatial dimension, $\mathbf{x} = (x_1, \dots, x_d)$ the horizontal spatial coordinates, $(\underline{h}, \underline{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^d$ is a stable equilibrium, $h = h(t, x)$ is the deviation from \underline{h} , $\mathbf{u} = \mathbf{u}(t, x)$ is deviation from $\underline{\mathbf{u}}$. Physically $h + \underline{h}$ represents the layer height and $\mathbf{u} + \underline{\mathbf{u}}$ represents the average horizontal velocity field of the layer, and $\mathbf{u}^* = -\kappa \frac{\nabla h}{h + \underline{h}}$.

We draw the reader's attention to the fact that, apart from the results concerning this system, for the sake of simplification and readability, we will consider the spatial dimension $d = 1$.

As previously mentioned, when $\kappa = 0$, we recover the classical shallow water system. The latter is an approximation of order μ of the water waves system (a system of two scalar evolution equations derived from the fundamental Euler system (2.1.1), assuming the fluid is homogeneous and the velocity field \mathbf{U} is irrotational), this approximation is achieved by exploiting properties of the Dirichlet-to-Neumann operator (this operator is defined through the velocity potential resulting from irrotationality and satisfying a specific Laplace problem). Here μ represents the shallowness parameter defined earlier (see (2.2.7)). We assume that μ is sufficiently small to be in the shallow water regime. Detailed information on the rigorous justification as well as other properties of this model can be found in the following references [28, 55]. It's also worth emphasizing that in this manuscript, the obtained results propose a different approach for justifying this model. We first make the hydrostatic assumption (which is valid in the shallow water regime), and then look at the limit of constant density and columnar motion.

The reader likely noticed our particular interest in the single-layer and two-layer Saint-Venant systems (corresponding to (2.2.12) for $N \in \{1, 2\}$). It is worth noting that for these two systems, we obtain improved results compared to the general case. This is attributable to the better understanding of the hyperbolicity conditions of the problem without thickness diffusivity ($\kappa = 0$) for these specific cases.

2.2.2 Stability aspects

Let's remind the reader of some classical stability results known to the community.

Regarding the single-layer shallow water system, it is established that the presence of the non-cavitation assumption ($\underline{h} + h > 0$) implies strict hyperbolicity. Furthermore, this system admits an explicit Friedrich symmetrizer, which essentially consists in multiplying the equation for horizontal velocity by $\underline{h} + h$. The study of this system can be found in [28], but also in a more general framework that deals with quasi-linear first-order systems, including the single-layer shallow water system [12, 60]. In the particular case of the rigid lid for the bi-layer shallow water system, explicit hyperbolicity conditions have been provided by Guyenne, Lannes, and Saut [43] and improved by Bresch and Renardy [20]. Unlike the single-layer shallow water system and the previously stated particular case, determining the explicit hyperbolicity domain of the bi-layer shallow water system (with a free surface) is not feasible. It is worth noting that, in this case, even the non-cavitation assumption ($\underline{h}_1 + h_1 > 0$ and $\underline{h}_2 + h_2 > 0$) does not guarantee its hyperbolicity. Nonetheless, several approaches exist to clarify this hyperbolicity, including the geometric approach established by Ovsjannikov [65], as well as those of Barros and Choi [10] and Vir'issimo and Milewski [76], which we will detail in a clear and concise manner in Chapter 6.

As mentioned earlier for the case of the bi-layer shallow water system, we also lack an explicit formula for the hyperbolicity domain of the multi-layer shallow water system ($\kappa = 0$ in the system (2.2.12) with $N \geq 2$), and there is no geometric approach that provides a better understanding of it. This difficulty arises from the presence of multiple variables in this system, making the analysis complicated, if not impossible. However, in [27, 62], the authors provide sufficient conditions for strict hyperbolicity when the fluid is stably stratified (i.e., when $\rho_1 < \rho_2 < \dots < \rho_N$), but these conditions are obtained using perturbative arguments with respect to the situation without shear velocities, and they degenerate as $N \rightarrow \infty$ towards

$$\{(\underline{H}_i + H_i, \underline{U}_i + U_i) \in \mathbb{R}^{2N} : \underline{H}_i + H_i > 0, \underline{U}_1 + U_1 = \underline{U}_2 + U_2 = \dots = \underline{U}_N + U_N\},$$

preventing any study at the nonlinear level or including shear velocities. We refer the reader to the insightful discussion in [70, §5] regarding stability results for multilayer systems as the number of layers increases, in relation to stability results for continuously

stratified systems. The famous stability criterion of Miles [61] and Howard [45] is recalled there, preventing spectral instability of continuously stratified shear flows if the local Richardson number is everywhere greater than $1/4$. It is also explicitly mentioned that this notion of stability (along with other results, obtained by Holm and Long [44] and Arbanel et al. [1]) is generally too weak to imply control of deviations from equilibrium, especially at the nonlinear level.

According to our current understanding, the well-posedness of the initial value problem in finite regularity spaces for the continuously stratified system (2.2.11), in the absence of thickness diffusivity ($\kappa = 0$), with initial data satisfying the Miles and Howard criterion, remains an open problem. We invite the reader to refer to the work of Kukavica et al. [52] for the existence and uniqueness of a solution in spaces of analytic functions (within the rigid lid framework), and to Cao, Li & Titi [23] among many other works (see [58] for a recent overview) for the situation with contributions from horizontal viscosity and diffusivity. While the previously mentioned works deal with the hydrostatic Euler equations written in Eulerian coordinates and do not rely on the assumption of stable stratification, the work of Bianchini and Duchêne [13] is closer to our framework as it specifically addresses the system in isopycnal coordinates with the additional terms of Gent and McWilliams, i.e., (2.2.11). They demonstrate that sufficiently regular initial data satisfying the no-cavitation assumption $(\underline{h} + h)|_{t=0} \geq h_* > 0$ lead to a unique solution over a time interval $[0, T]$ with $T^{-1} \lesssim 1 + \kappa^{-1}(|\underline{u}'|_{L^2((\rho_{\text{surf}}, \rho_{\text{bott}}))}^2 + M_0^2)$ where M_0 is the size of the initial deviations from equilibrium.

Finally, to conclude this section, it is pertinent to mention that in the case where $\kappa = 0$ and in the presence of the Coriolis force, the explicit calculation of the hyperbolicity domain is not possible/known for the multi-layer system (2.2.12). However, the result by Stewart and Dellar [71] numerically demonstrates that the threshold for shear velocities, below which the three-layer equations remain hyperbolic, is not altered by the inclusion of the full Coriolis force. The standard shallow water equations only lose hyperbolicity due to shears parallel to the wave propagation direction, but the full Coriolis force introduces another mechanism of hyperbolicity loss due to shears in the perpendicular direction.

2.2.3 Main results

Our first result concerns the single-layer shallow water system with the additional Gent & McWilliams terms (3.1.1). We present a well-posedness result in finite regularity spaces and small time for this system. The reference to "small" time comes from the fact

that the existence and uniqueness time obtained depends on the parameter κ of these additional terms. This result is achieved by relying heavily on the diffusion induced by these terms in κ , which allows us to use well-established general theorems for the well-posedness of transport and transport-diffusion equations. Similarly, employing the same strategy as before for the multi-layer system (2.2.12), we obtain a well-posedness result for it, with an existence time depending on the number of layers N and the parameter κ . Secondly, we switch strategies and utilize the energy method. We demonstrate that we can improve the existence and uniqueness time of the solution for the single-layer system, to a time that is independent of the parameter κ . Continuing in the same spirit and with the same strategy, we enhance this result for the multi-layer system, showing that under natural assumptions and particularly the assumption of stable stratification (in fact, we assume for simplicity that the densities are evenly distributed, i.e. $(\rho_{i+1} - \rho_i = \frac{\rho_{\text{bott}} - \rho_{\text{surf}}}{N})$) and the non-cavitation assumption $(\underline{H}_i + H_i)|_{t=0} \geq h^* > 0$, the solutions of the multi-layer system (2.2.12) are uniform, unique, and exist over a time interval analogous to that of the continuously stratified system (obtained in [13]), which is particularly *uniform with respect to N* and with a better dependence on κ . We will refer to it as *Large time* well-posedness (as opposed to *small time* well-posedness where the a priori existence time depends on N). This relies on exploiting the symmetry properties of the system stemming from the last term of equation (2.2.12)₂, which can be seen as the term facilitating information exchange between the fluid layers. Through this term, natural compensations emerge during our reasoning with the energy method. Finally, in this first part, we demonstrate that in the presence of the Gent & McWilliams terms, the energy of both systems (3.1.1) and (2.2.12) is decreasing.

Our second main result states that for any sufficiently regular solution of the continuously stratified system (2.2.11) satisfying the non-cavitation assumption and appropriate bounds, the solutions of the multi-layer system (2.2.12) with suitably chosen densities ρ_i , reference depths \underline{H}_i , background velocities \underline{U}_i and an initial deviation $(H_i, U_i)|_{t=0}$ are at a distance $\mathcal{O}(1/N^2)$ to the continuously stratified solution.

In order to achieve these goals, we rely mostly on two ingredients:

1. a *consistency* result, stating that sufficiently regular solutions to the continuously stratified system may be projected into N -dimensional valued functions satisfying the multi-layer shallow water systems up to a remainder term of size $\mathcal{O}(1/N^2)$;
2. a suitable *stability* estimate on the linearized multi-layer shallow water systems, which is uniform with respect to N .

The consistency result amounts to controlling the difference between the N -dimensional projection of the contribution of the Montgomery potential, $\frac{1}{\varrho} \partial_x \psi$ in (2.2.11), and the corresponding contribution in (2.2.12) when $(H_i)_{i \in \{1, \dots, N\}}$ is the N -dimensional projection of h . The $\mathcal{O}(1/N^2)$ estimate is obtained by exploiting Taylor expansions at suitable values of densities ρ_i in the spirit of the midpoint rule in numerical quadratures.

The stability estimate on (2.2.12) relies on the energy method, for a carefully constructed energy functional. In order to obtain suitable bounds on the correct time scale, we make use of a partial symmetric structure of the hyperbolic system (when $\kappa = 0$) and rely on the regularization effect of diffusivity only when necessary. This partial symmetric structure and the construction of the associated energy functional relies on a decomposition of the linear operator

$$\Gamma_N : (H_i)_{i \in \{1, \dots, N\}} \mapsto \left(\sum_{j=1}^N \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} H_j \right)_{i \in \{1, \dots, N\}}$$

which echoes directly an analogous decomposition of continuous operator

$$\mathcal{M} : h \mapsto \frac{1}{\varrho} \left(\rho_{\text{surf}} \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} h(\varrho') d\varrho' + \int_{\rho_{\text{surf}}}^{\varrho} \int_{\varrho'}^{\rho_{\text{bott}}} h(\varrho'') d\varrho'' d\varrho' \right)$$

and which is a key ingredient in the study of Bianchini and Duchêne. More generally, our energy estimates mirrors the ones in [13] after introducing a dictionary between continuous operators and analogous discrete operators, with several adaptations to fit our framework.

Regarding the "inverse" problem (approximation of the bi-layer system by the continuously stratified system), we note that the solutions of the bi-layer shallow water system (6.2.6) (or (2.2.12) with $N = 2$) provide exact solutions to (2.2.13). More precisely, if we denote

$$\begin{aligned} \underline{\rho}_{\text{bl}}(r) &= \rho_s \mathbf{1}_{(-\underline{H}_s, 0)}(r) + \rho_b \mathbf{1}_{(-1, -\underline{H}_s)}(r), \\ \underline{u}_{\text{bl}}(r) &= \underline{U}_s \mathbf{1}_{(-\underline{H}_s, 0)}(r) + \underline{U}_b \mathbf{1}_{(-1, -\underline{H}_s)}(r), \\ u_{\text{bl}}(\cdot, r) &= U_s \mathbf{1}_{(-\underline{H}_s, 0)}(r) + U_b \mathbf{1}_{(-1, -\underline{H}_s)}(r), \\ h_{\text{bl}}(\cdot, r) &= \frac{H_s}{\underline{H}_s} \mathbf{1}_{(-\underline{H}_s, 0)}(r) + \frac{H_b}{\underline{H}_b} \mathbf{1}_{(-1, \underline{H}_s)}(r), \end{aligned} \tag{2.2.16}$$

where H_s, U_s, H_b, U_b are solutions of the bi-layer shallow water system with the additional

Gent & McWilliams terms (6.2.6) (or (2.2.12) with $N = 2$), then $(\rho_{\text{bl}}, \underline{u}_{\text{bl}}, h_{\text{bl}}, u_{\text{bl}})$ is a solution to (2.2.13)-(2.2.14). In this work we shall compare these solutions with the ones emerging from data satisfying (2.2.16) only approximately. Specifically, our results are twofold.

1. We prove that strong solutions to the bi-layer shallow water system (6.2.6) emerge from sufficiently regular initial data satisfying some hyperbolicity conditions.
2. We prove that strong solutions to the hydrostatic Euler equations (2.2.13) emerge from data close to the piecewise constant profiles given by (2.2.16), and that these solutions remain close to the bi-layer solutions.

Notice that to accommodate our aim of comparing solutions with a piecewise constant density distribution with a solution with a continuous density distribution, we wish to consider deviations that can be large in some small regions around the pycnocline. This demands to carefully select the topology measuring the size of deviations in (2). Additionally, our strategy relies on the construction of an intermediate approximate solution, which is proved to be close to the corresponding solution to the hydrostatic Euler equation in a strong topology, and close to the bi-layer solution in a weaker topology. Hence this intermediate approximate solution improves the description of the exact solution in the pycnocline.

A second important remark is that while the contribution of thickness diffusivity is essential to our stability estimates, we wish to control and compare solutions on a time interval which is uniform with respect to the thickness diffusivity parameter $0 < \kappa \leq 1$. The dependency on the thickness parameter will appear only as a restriction on the size of admissible deviations. Concerning (1), this is made possible by the well-known fact that the bi-layer shallow water system is well-posed (under some hyperbolicity conditions) when $\kappa = 0$ [65], yet obtaining the corresponding result for $\kappa > 0$ is not straightforward and demands to use finely the structure of the thickness diffusivity parameters. Concerning (2), we use the existence of the bilayer solution and consequently the existence of the improved approximate solution to bootstrap the control of sufficiently close solutions to the hydrostatic Euler equations on the relevant timescale. For that matter the regularizing effect of thickness diffusivity contributions is strongly used.

Let us point out that the second result (2) applies to *any given* (sufficiently regular) solution to the hydrostatic Euler equations. Hence our work provides the same stability estimates around other solutions, constructed for instance in the framework of multiple layers or simple waves.

2.2.4 Connected results

To the best of our knowledge, this is the first time that the convergence between solutions to multi-layer systems and continuously stratified equations is rigorously proved (while formal connections are discussed for instance in the pioneering works of Benton [11], Su [72], Killworth [50]), with the notable exception of the work of Chen and Walsh [24]. In the latter work, the authors consider the framework of periodic traveling waves without the hydrostatic approximation (and without diffusivity contributions). They prove that for sufficiently small periodic traveling wave solutions to the continuously stratified equations satisfying natural assumptions (in particular that the stratification is stable, sufficient regularity of the streamline density, and the absence of stagnation points), there exist corresponding traveling wave solutions associated with any piecewise smooth streamline density function (hence in particular solutions to multi-layer systems which correspond to piecewise constant streamline density functions, the velocity field being irrotational within each layer), in an L^∞ neighborhood, and that the mapping from streamline density functions to the corresponding traveling wave is Lipschitz continuous in suitable functional spaces, the distance between two streamline densities being measured through the L^∞ norm.

This result, which is quite strong and versatile, does not directly compare with ours. Most importantly, the framework of traveling waves is of course very different from our framework allowing non-trivial dynamics. Roughly speaking, the traveling wave problem can be recast into a problem of elliptic nature, while our systems of equations are hyperbolic/parabolic. Consistently the tools used in our work, with the exception of a semi-Lagrangian change of coordinates to fix the fluid domain (called the Dubreil-Jacotin transformation in the context of traveling waves, which is equivalent to the isopycnal change of coordinates) are very different from the tools used in [24]. Moreover, the hydrostatic approximation also modifies the nature of the problem, as it discards dispersive effects. As such, nonlinear traveling waves are inexistent within the hydrostatic approximation framework, and could be replaced —if one desires to study simplified dynamics— with simple waves [25, 64].

Finally, as discussed in [24], the "reverse" limit consisting in approaching a bi-layer (or multi-layer) situation with continuously stratified problems has been studied (in the framework of traveling waves) in particular by James [47], after Turner and Amick [74, 3].

2.3 Plan of the thesis

In the first part of Chapter 3, we first focus on the single-layer shallow water system, with the additional Gent & McWilliams terms as previously introduced (see Equation (2.1.2)). We present a small-time well-posedness result in finite regularity spaces (the time depends on the parameter κ of these additional terms), see Theorem 3.2.4. We demonstrate in Proposition 3.2.7 that this system has decreasing energy. Then, we improve the existence and uniqueness time of the solution to this system to a time independent of the parameter κ , as shown in Proposition 3.2.9.

In the second part of this chapter, and throughout the following, we consider the spacial dimension to be one-dimensional i.e. ($d = 1$) for the sake of readability and to more straightforwardly derive the results of Chapter 7, which generalize known results in finite regularity spaces. In this part, we examine the multi-layer shallow water system for gravity wave propagation in stratified density flows, with the additional Gent & McWilliams terms (2.2.12). Following a similar approach as before, we establish a well-posedness of this system in finite regularity spaces with a nonuniform time with respect to the parameters κ and the number of layers N (see Theorem 3.3.1). Finally, we conclude this chapter by demonstrating that the energy associated with this system also decreases (see Proposition 3.3.2).

In Chapter 4, we improve the existence and uniqueness time for the multi-layer shallow water system obtained previously by employing the energy method (see Theorem 4.5.1). The implementation of this method requires a crucial step of quasilinearization, which allows obtaining the stability estimates necessary to conclude the continuity argument used. Improving this existence and uniqueness time aims to achieve a time that has a better dependence on κ and that is uniform with respect to the number of layers N .

In Chapter 5, we address the convergence of the multi-layer system (2.2.12) towards the continuously stratified system (2.2.11) as the number of layers N tends to infinity, providing an explicit convergence rate of order $1/N^2$ (see Theorem 5.4.1).

In Chapter 6, in Section 6.2, we study the bi-layer shallow water systems. We first recall some known results without thickness diffusivity, and then consider the system with thickness diffusivity contributions. The main result is Proposition 6.2.8, which provides for any sufficiently regular initial data satisfying some hyperbolicity criterion the existence and control of strong solutions to the bi-layer shallow water system on a time interval

which is uniform with respect to $0 < \kappa \leq 1$, while Proposition 6.2.9 states the expected strong convergence as $\kappa \rightarrow 0$. In Section 6.3, we study the hydrostatic Euler equations. We provide first some stability estimates with respect to perturbations of the equations and of the data using suitable distances. As a second step we introduce approximate solutions associated with some given reference exact solution and close-by data. Building upon these approximate solutions, we prove Proposition 6.3.8, which controls the difference between the reference solution and exact solutions to the hydrostatic Euler equations emerging from close-by data. Together, Proposition 6.2.8 and Proposition 6.3.8 provide the announced rigorous justification that for initial data suitably close to the bi-layer framework and satisfying some hyperbolicity criterion limiting in particular the admissible size of shear velocities, the emerging solutions to the hydrostatic Euler equations remain close to the solution predicted by the bilayer shallow water system on a relevant timescale. The rigorous statement is displayed in Theorem 6.3.1.

Chapter 7 serves as a toolbox for Chapters 4 and 5, as it generalizes classic lemmas and propositions (such as those regarding products, compositions, injections, etc.) found for scalar functions in spaces of finite regularity, notably Sobolev spaces, to the framework of functions with values in \mathbb{R}^N .

THE REGULARIZED SHALLOW WATER SYSTEM

3.1 Introduction

In this chapter, we examine a model describing wave motion in oceanography, which is motivated by the work of Peter R. Gent and James C. McWilliams. The model is represented by the following system:

$$\begin{cases} \partial_t h + \nabla \cdot ((h + \underline{h})(\mathbf{u} + \underline{\mathbf{u}} + \mathbf{u}^*)) = 0, \\ \partial_t \mathbf{u} + ((\mathbf{u} + \underline{\mathbf{u}} + \mathbf{u}^*) \cdot \nabla) \mathbf{u} + \nabla h = 0, \\ h|_{t=0} = h_0, \inf h_0 > 0, \\ u|_{t=0} = u_0, \end{cases} \quad (3.1.1)$$

where $t > 0$ is the time variable, d the horizontal spacial dimension, $x = (x_1, \dots, x_d)$ the horizontal spatial coordinates, $(\underline{h}, \underline{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^d$ is an equilibrium, $h = h(t, x)$ is the deviation from \underline{h} , $\mathbf{u} = \mathbf{u}(t, x)$ is the deviation from $\underline{\mathbf{u}}$. Physically $h + \underline{h}$ represents the layer height and $\mathbf{u} + \underline{\mathbf{u}}$ represents the average horizontal velocity field of the layer. We notice that if $\mathbf{u}^* = 0$ the system corresponds to the shallow water (Saint-Venant) system which is a hyperbolic symmetrizable system and is well known and studied for example we give the book of Sylvie Benzoni-Gavage and Denis Serre [12] as a reference.

The main focus of this chapter is the choice of $\mathbf{u}^* = -\kappa \frac{\nabla h}{h + \underline{h}}$ where $\kappa > 0$ is the thickness diffusivity. This choice is motivated by the work of Peter R. Gent and James C. McWilliams on isopycnal mixing and eddy viscosity (see [35] equation (15)). It is a natural and suitable choice for them for many reasons for example in the case where κ is a positive constant a Laplacian will appear in (3.1.1)₁ which will have a smoothing effect on h .

We show that the initial value problem is in fact well-posed locally in time in regular

Sobolev spaces for a single layered fluid and we establish that its corresponding system has decreasing energy. Moreover, in a natural way we extend the previous well-posedness result to the case of a fluid with finite number of layers. Finally, relying on energy methods found in the book of Sylvie Benzoni-Gavage and Denis Serre [12] we improve the well-posedness result for the case of a single layered fluid by extending the time existence compered with the parameters considered in this problem.

3.2 The single layered shallow-water system

3.2.1 Small-time well-posedness

In this sub-section we begin by giving classical results on transport equations and transport diffusion equations which will be the ingredient to prove the small-time well-posedness of the single-layered shallow water system (3.2.1).

The following theorem results from combining [8, Theorem 3.19] and [8, Theorem 3.14].

Theorem 3.2.1. *Let $T > 0$, $d \in \mathbb{N}^*$, $\sigma > 1 + \frac{d}{2}$. Let $f_0 \in H^\sigma(\mathbb{R}^d)$, $g \in L^1(0, T; H^\sigma(\mathbb{R}^d))$, and \mathbf{v} be the time-dependent vector field such that $\mathbf{v} \in L^\rho(0, T; H^s(\mathbb{R}^d))$ for some $\rho > 1$, and $\nabla \mathbf{v} \in L^1(0, T; H^{\sigma-1}(\mathbb{R}^d))$. The transport equation*

$$\begin{cases} \partial_t f + (\mathbf{v} \cdot \nabla) f = g \\ f|_{t=0} = f_0 \end{cases}$$

has a unique solution $f \in C([0, T]; H^\sigma(\mathbb{R}^d))$. Moreover there exists $C > 0$ such that

$$\|f\|_{L_T^\infty(H^\sigma)} \leq e^{CV(T)} (\|f_0\|_{H^\sigma} + \|g\|_{L_T^1(H^\sigma)}),$$

where $V'(t) := \|\nabla \mathbf{v}(t)\|_{H^{\sigma-1}}$.

Theorem 3.2.2. *Let $\kappa > 0$, $d \in \mathbb{N}^*$, $T > 0$, $2 \leq \rho \leq \infty$, $\sigma > 1 + \frac{d}{2}$, and \mathbf{v} and V be defined as in Theorem 3.2.1. If $g \in L^2([0, T]; H^{\sigma-1}(\mathbb{R}^d))$ and $V(T)$ is finite with $f_0 \in H^\sigma(\mathbb{R}^d)$ there exists $\gamma > 0$ (which depends only on d and s) and a unique $f \in L^\rho([0, T]; H^{\sigma+\frac{2}{\rho}}(\mathbb{R}^d)) \cap C([0, T]; H^\sigma(\mathbb{R}^d))$ such that f is solution to the transport diffusion equation*

$$\partial_t f + \mathbf{v} \cdot \nabla f - \kappa \Delta f = g,$$

Moreover it satisfies :

$$\kappa^{\frac{1}{\rho}} \|f\|_{L_T^\rho(H^{\sigma+\frac{2}{\rho}})} \leq \gamma e^{\gamma(1+\kappa T)^{\frac{1}{\rho}} V(T)} \left((1+\kappa T)^{\frac{1}{\rho}} \|f_0\|_{H^\sigma} + (1+\kappa T)^{\frac{1}{\rho}+\frac{1}{2}} \kappa^{-\frac{1}{2}} \|g\|_{L_T^2(H^{\sigma-1})} \right).$$

Remark 3.2.3. The estimates in the previous theorem are found in [8, Theorem 3.38], the existence part follows using properties of the heat equation's semi-group and Duhamel's formula.

Theorem 3.2.4. Let $d \in \mathbb{N}^*$, $s > 1 + \frac{d}{2}$, $h_* > 0$, $M, \underline{M} > 0$, and $0 < \kappa \leq 1$. There exists $T = \frac{\kappa}{C(s, h^*, M, \underline{M}, d)}$ (with $C(s, h^*, M, \underline{M}, d) > 1$ and non-decreasing function of M), such that for all $(\underline{h}, \underline{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^d$ with

$$\underline{h} \geq h_*; \quad |\underline{h}| + |\underline{\mathbf{u}}| \leq \underline{M},$$

for any $(h_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d)^{1+d}$ with

$$\begin{aligned} M_0 &:= \|h_0\|_{H^s} + \|\mathbf{u}_0\|_{(H^s)^d} \leq M, \\ h_0 + \underline{h} &\geq 2h^*, \end{aligned}$$

there exists a unique (h, \mathbf{u}) such that

$$\begin{aligned} h &\in L^2([0, T]; H^{s+1}(\mathbb{R}^d)) \cap C([0, T]; H^s(\mathbb{R}^d)), \\ \mathbf{u} &\in C([0, T]; H^s(\mathbb{R}^d)^d) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^d)^d), \end{aligned}$$

and (h, \mathbf{u}) is solution to

$$\begin{cases} \partial_t h + (\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla h + (h + \underline{h}) \nabla \cdot \mathbf{u} = \kappa \Delta h, \\ \partial_t \mathbf{u} + \left((\mathbf{u} + \underline{\mathbf{u}} - \kappa \frac{\nabla h}{h + \underline{h}}) \cdot \nabla \right) \mathbf{u} + \nabla h = 0, \\ h|_{t=0} = h_0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \quad (3.2.1)$$

moreover

$$\|h\|_{L^\infty(0, T_M; H^s)} \leq 2\gamma M_0, \quad \kappa^{\frac{1}{2}} \|h\|_{L^2(0, T_M; H^{s+1})} \leq 2\gamma M_0, \quad \|\mathbf{u}\|_{L^\infty(0, T_M; (H^s)^d)} \leq 2M_0, \quad (3.2.2)$$

and $h + \underline{h} \geq h^*$ on $[0, T] \times \mathbb{R}^d$, where γ is a universal constant.

Proof. For $T_0 > 0$, we define the following set

$$A_{T_0,s} \times B_{T_0,s} \stackrel{def}{=} \left(L^2([0, T_0]; H^{s+1}(\mathbb{R}^d)) \cap C([0, T_0]; H^s(\mathbb{R}^d)) \right) \times C([0, T_0]; H^s(\mathbb{R}^d)^d).$$

The system is viewed as a system of a transport-diffusion equation and a transport equation, coupled only through order-zero source terms. We put $h^{-1} = 0$, $\mathbf{u}^{-1} = 0$ and we define the sequence $(h^n, \mathbf{u}^n)_{n \in \mathbb{N}}$ by the following Picard iterative scheme:

$$\begin{cases} \partial_t h^{n+1} + (\mathbf{u}^n + \underline{\mathbf{u}}) \cdot \nabla h^{n+1} + (h^n + \underline{h}) \nabla \cdot \mathbf{u}^n = \kappa \Delta h^{n+1}, \\ \partial_t \mathbf{u}^{n+1} + ((\mathbf{u}^n + \underline{\mathbf{u}} - \kappa \frac{\nabla h^n}{h^n + \underline{h}}) \cdot \nabla) \mathbf{u}^{n+1} + \nabla h^n = 0, \\ h_{|t=0}^{n+1} = h_0, \quad \inf_{[0, T_n] \times \mathbb{R}^d} h^{n+1} + \underline{h} \geq h_*, \\ \mathbf{u}_{|t=0}^{n+1} = \mathbf{u}_0. \end{cases} \quad (3.2.3)$$

Step 1: We begin to show inductively that for any $n \in \mathbb{N}$, there exists $T_n > 0$, such that (h^n, \mathbf{u}^n) is in $A_{T_n,s} \times B_{T_n,s}$ and that $\inf_{[0, T_n] \times \mathbb{R}^d} (h^n + \underline{h}) \geq h_*$.

It is immediate that $(h^{-1}, \mathbf{u}^{-1}) \in A_{T,s} \times B_{T,s}$, and that $\inf_{[0, T] \times \mathbb{R}^d} (h^{-1} + \underline{h}) = \underline{h} \geq h_*$, $\forall T > 0$.

We suppose that $\exists T_n > 0$ such that $(h^n, \mathbf{u}^n) \in A_{T_n,s} \times B_{T_n,s}$, and $\inf_{[0, T_n] \times \mathbb{R}^d} h^n + \underline{h} \geq h_*$ for a certain $n \in \mathbb{N}$. Let $f_0 = \mathbf{u}_0$; $\mathbf{v} = \mathbf{u}^n + \underline{\mathbf{u}} - \kappa \frac{\nabla h^n}{h^n + \underline{h}}$; $g = -\nabla h^n$.

We define

$$F_{h_*}(x) = \begin{cases} \frac{1}{x+\underline{h}} - \frac{1}{\underline{h}} & \text{when } x + \underline{h} \geq h_* \\ w(x) & \text{elsewhere} \end{cases},$$

where w can be chosen such that $F_{h_*} \in C_b^\infty(\mathbb{R})$.

We notice that $F_{h_*}(0) = 0$ hence by [8, Theorem 2.89] $F_{h_*}(h^n) \in H^s(\mathbb{R}^d)$ and

$$\|F_{h_*}(h^n)\|_{H^s} \leq C(s, h_*, \underline{M}, \|\nabla h^n\|_{H^{s-1}}) \|h^n\|_{H^s},$$

where $C(s, h_*, \underline{M}, \|\nabla h^n\|_{H^{s-1}})$ depends non-decreasingly on $\|\nabla h^n\|_{H^{s-1}}^{-1}$. Hence

$$\|\nabla \mathbf{v}(t)\|_{H^{s-1}} \lesssim \|\mathbf{u}^n(t)\|_{H^s} + \kappa \|h^n(t)\|_{H^{s+1}} C(s, h_*, \underline{M}, \|h^n\|_{L_{T_n}^\infty(H^s)}) \|h^n\|_{L_{T_n}^\infty(H^s)} + \kappa \|h^n(t)\|_{H^{s+1}}.$$

Since \mathbf{u}^{n+1} satisfies the following transport equation

$$\partial_t \mathbf{u}^{n+1} + \mathbf{v} \cdot \nabla \mathbf{u}^{n+1} = g,$$

and moreover by what proceeds one can easily check that f_0, \mathbf{v}, g satisfy the hypothesis of Theorem 3.2.1, then there exists a unique $\mathbf{u}^{n+1} \in B_{T_n, s}$ which satisfies the equation (3.2.3)₂ in a distributional sense, and the following estimate:

$$\|\mathbf{u}^{n+1}\|_{L_{T_n}^\infty(H^s)} \leq \left(\|\mathbf{u}_0\|_{H^s} + \|\nabla h^n\|_{L_{T_n}^1(H^s)} \right) \exp \left(C \int_0^{T_n} \|\nabla \left(\mathbf{u}^n + \underline{\mathbf{u}} - \kappa \frac{\nabla h^n}{h^n + \underline{h}} \right) (t)\|_{H^{s-1}} dt \right).$$

In the same way we put $f_0 = h_0, \mathbf{v} = \mathbf{u}^n + \underline{\mathbf{u}}, g = -(h^n + \underline{h})\nabla \cdot \mathbf{u}^n$ then h^{n+1} satisfies the following transport diffusion equation:

$$\partial_t h^{n+1} + (\mathbf{v} \cdot \nabla) h^{n+1} = \kappa \Delta h^{n+1} + g,$$

and moreover one can easily check that f_0, \mathbf{v}, g satisfy the hypothesis of Theorem 3.2.2 then there exists a unique $h^{n+1} \in A_{T_n, s}$ such that it satisfies the equation (3.2.3)₁ in a distributional sense and the following estimate:

$$\begin{aligned} \kappa^{\frac{1}{2}} \|h^{n+1}\|_{L_{T_n}^2(H^{s+1})} &\leq \gamma \exp \left(\gamma (1 + \kappa T_n)^{\frac{1}{2}} \int_0^{T_n} \|\nabla \mathbf{u}^n(t)\|_{H^{s-1}} dt \right) \\ &\times \left((1 + \kappa T_n) \kappa^{-\frac{1}{2}} \|(h^n + \underline{h})\nabla \cdot \mathbf{u}^n\|_{L_{T_n}^2(H^{s-1})} + (1 + \kappa T_n)^{\frac{1}{2}} \|h_0\|_{H^s} \right). \end{aligned}$$

It remains to show that $\inf_{[0, T_{n+1}] \times \mathbb{R}^d} h^{n+1} + \underline{h} \geq h_*$ for a convenient T_{n+1} .

Let us assume that $T_{n+1} \leq T_n$. We know that

$$\begin{aligned} \left\| (\mathbf{u}^n \cdot \nabla h^{n+1} - \kappa \Delta h^{n+1} + (h^n + \underline{h})\nabla \cdot \mathbf{u}^n)(t) \right\|_{H^{s-1}} &\lesssim \|\mathbf{u}^n\|_{L_{T_n}^\infty(H^{s-1})} \|h^{n+1}\|_{L_{T_n}^\infty(H^s)} \\ &+ \|h^{n+1}\|_{L_{T_n}^\infty(H^{s+1})} + \underline{M} \|\mathbf{u}^n\|_{L_{T_n}^\infty(H^s)} \\ &+ \|h^n\|_{L_{T_n}^\infty(H^s)} \|\mathbf{u}^n\|_{L_{T_n}^\infty(H^s)}, \end{aligned}$$

which implies that $\partial_t h^{n+1}$ (in a distributional sense) is in $L^2(0, T_n; H^{s-1}(\mathbb{R}^d))$ therefore $h^{n+1} \in H^1([0, T_n]; H^{s-1}(\mathbb{R}^d)) \hookrightarrow C([0, T_n]; H^{s-1}(\mathbb{R}^d))$, consequently it satisfies the funda-

mental theorem of calculus:

$$\forall t \in [0, T_{n+1}] \quad h^{n+1}(t, \cdot) + \underline{h} = h_0(\cdot) + \underline{h} + \int_0^t \partial_t h^{n+1}(\tau, \cdot) d\tau,$$

and by Sobolev injection there exists $c = c(s, \underline{M}) > 0$ (which may increase if necessary throughout the proof), such that

$$\int_0^t \partial_t h^{n+1}(\tau, \cdot) d\tau \leq c\sqrt{T_{n+1}} \|\partial_t h^{n+1}\|_{L^2_{T_{n+1}}(H^{s-1})}.$$

Choosing T_{n+1} such that $c\sqrt{T_{n+1}} \|\partial_t h^{n+1}\|_{L^2_{T_{n+1}}(H^{s-1})} \leq h_*$ we obtain:

$$\forall t \in [0, T_{n+1}] \quad h^{n+1}(t, \cdot) + \underline{h} \geq h_*,$$

therefore $(h^{n+1}, \mathbf{u}^{n+1}) \in A_{T_{n+1}, s} \times B_{T_{n+1}, s}$ and $h^{n+1} + \underline{h} \geq h_*$ on $[0, T_{n+1}] \times \mathbb{R}^d$.

The iterative scheme is well defined at least in small time which at this point depends on n . To overcome this dependency using the fact that the initial data is bounded we are going to bound the sequence $(h^n, \mathbf{u}^n)_{n \in \mathbb{N}}$ in a certain space of its regularity.

Step 2: We are going to find an appropriate time $T = T(s, d, \kappa, h_*, M, \underline{M})$, such that $\forall n \in \mathbb{N}$, $(h^n, \mathbf{u}^n) \in B_T(0)$ and $h^n + \underline{h} \geq h_*$ on $[0, T] \times \mathbb{R}^d$ where

$$B_T(0) \stackrel{\text{def}}{=} \{(h, \mathbf{u}) \in A_{T, s} \times B_{T, s}; \|h\|_{L^\infty_T(H^s)} \leq 2e\gamma M_0, \kappa^{\frac{1}{2}} \|h\|_{L^2_T(H^{s+1})} \leq 2e\gamma M_0, \|\mathbf{u}\|_{L^\infty_T(H^s)} \leq 2eM_0\}.$$

We suppose that for $n \in \mathbb{N}$, $(h^n, \mathbf{u}^n) \in B_T(0)$ and $h^n(t, \cdot) + \underline{h} \geq h_*$, for all $t \in [0, T]$. Using the estimates of the previous step of the proof we have:

$$\begin{aligned} \|\mathbf{u}^{n+1}\|_{L^\infty_T(H^s)} &\leq \exp\left(C(T \times 2eM + \kappa^{\frac{1}{2}}(2e\gamma M)^2 C(s, h_*, \underline{M}, M)T^{\frac{1}{2}} + \kappa^{\frac{1}{2}}2e\gamma MT^{\frac{1}{2}})\right) \\ &\quad \times \left(M_0 + 2e\gamma\kappa^{-\frac{1}{2}}T^{\frac{1}{2}}M_0\right). \end{aligned}$$

Moreover

$$\|h^{n+1}\|_{L^\infty_T(H^s)} \leq \gamma \exp(\gamma T \times 2eM) \left((1 + (\kappa T)^{\frac{1}{2}}) \kappa^{-\frac{1}{2}} (2e\gamma M d + d) 2eM_0 T^{\frac{1}{2}}_M + M_0 \right),$$

and,

$$\begin{aligned} \kappa^{\frac{1}{2}} \|h^{n+1}\|_{L_T^2(H^{s+1})} &\leq \gamma e^{\gamma(1+(\kappa T)^{\frac{1}{2}})2TeM} \\ &\times \left((1 + \kappa T) \kappa^{-\frac{1}{2}} (2ce\gamma d + d) 2eM_0 T^{\frac{1}{2}} + M_0 + (\kappa T)^{\frac{1}{2}} M_0 \right). \end{aligned}$$

On the other hand $\forall t \in [0, T]$

$$\int_0^t \partial_t h^{n+1}(\tau, \cdot) d\tau \leq c \left(((2eM)^2\gamma + 2eMd + (2eM)^2\gamma d)T + \kappa^{\frac{1}{2}} 2eM\gamma T^{\frac{1}{2}} \right).$$

Hence there exists $C(s, h_*, M, \underline{M}, d) > 1$ such that for the choice of $0 < T \leq \frac{\kappa}{C(s, h_*, M, \underline{M}, d)}$ and by using the previous estimates and by induction we have $(h^n, \mathbf{u}^n) \in B_T(0)$ and $h^n + \underline{h} \geq h_*$, in $[0, T] \times \mathbb{R}^d$.

Step 3: We now prove that $(h^n, \mathbf{u}^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $A_{T,0} \times B_{T,0}$.

We put $S_{n+1,m} = \mathbf{u}^{n+m+1} - \mathbf{u}^{n+1}$, $f_0 = 0$, $\mathbf{v} = \mathbf{u}^{n+m} + \underline{\mathbf{u}} - \kappa \frac{\nabla h^{n+m}}{h^{n+m} + \underline{h}}$,
 $-g = \left((\mathbf{u}^{n+m} - \mathbf{u}^n - \kappa \left(\frac{\nabla h^{n+m}}{h^{n+m} + \underline{h}} - \frac{\nabla h^n}{h^n + \underline{h}} \right)) \cdot \nabla \right) \mathbf{u}^{n+1} + \nabla(h^{n+m} - h^n)$.
 Hence $S_{n+1,m}$ satisfies the following transport equation

$$\partial_t S_{n+1,m} + \mathbf{v} \cdot \nabla S_{n+1,m} = g,$$

since $S_{n+1,m} \in C([0, T]; L^2(\mathbb{R}^d)^d)$ and satisfies the transport equation in a distributional sense, by Theorem 3.2.1 the solution $S_{n+1,m}$ is unique and it satisfies the corresponding a priori estimate.

We put $W_{n+1,m} = h^{n+m+1} - h^{n+1}$, $f_0 = 0$, $\mathbf{v} = \mathbf{u}^{n+m} + \underline{\mathbf{u}}$,
 $-g = (\mathbf{u}^{n+m} - \mathbf{u}^n) \cdot \nabla h^{n+1} + h^{n+m} \nabla \cdot (\mathbf{u}^{n+m} - \mathbf{u}^n) + (h^{n+m} - h^n) \nabla \cdot \mathbf{u}^n + \nabla \cdot (\mathbf{u}^{n+m} - \mathbf{u}^n)$.
 Hence $W_{n+1,m}$ satisfies the following transport diffusion equation

$$\partial_t W_{n+1,m} + \mathbf{v} \cdot \nabla W_{n+1,m} = \kappa \Delta W_{n+1,m} + g,$$

since $W_{n+1,m} \in L^2(0, T; H^1(\mathbb{R}^d)) \cap C([0, T]; L^2(\mathbb{R}^d))$ and it satisfies the transport diffusion equation by uniqueness $W_{n+1,m}$ satisfies the corresponding a priori estimate.

Consequently using the function F_{h_*} defined in **Step 1** and [8, Theorem.2.89] and the product estimates in Sobolev spaces to bound \mathbf{v}, g (in both cases of transport and transport-diffusion equations) we can deduce from the a priori estimates of both $W_{n+1,m}$ and $S_{n+1,m}$ that there exists, $\beta = \beta(s, d, h_*, M, \underline{M}) > 0$ such that :

$$\begin{aligned} \|W_{n+1,m}\|_{L_T^\infty(L^2)} + \kappa^{\frac{1}{2}} \|W_{n+1,m}\|_{L_T^2(H^1)} + \|S_{n+1,m}\|_{L_T^\infty(L^2)} &\leq \beta \kappa^{-\frac{1}{2}} \left(\|W_{n,m}\|_{L_T^\infty(L^2)} \right. \\ &\quad \left. + \kappa^{\frac{1}{2}} \|W_{n,m}\|_{L_T^2(H^1)} + \|S_{n,m}\|_{L_T^\infty(L^2)} \right) T^{\frac{1}{2}}. \end{aligned}$$

Finally with T as in **Step 2** by augmenting $C(s, h^*, d, M)$ such that:

$$\begin{aligned} \|W_{n+1,m}\|_{L_T^\infty(L^2)} + \kappa^{\frac{1}{2}} \|W_{n+1,m}\|_{L_T^2(H^1)} + \|S_{n+1,m}\|_{L_T^\infty(L^2)} &\leq \frac{1}{2} \left(\|W_{n,m}\|_{L_T^\infty(L^2)} \right. \\ &\quad \left. + \kappa^{\frac{1}{2}} \|W_{n,m}\|_{L_T^2(H^1)} + \|S_{n,m}\|_{L_T^\infty(L^2)} \right). \end{aligned}$$

Implies that $(h^n, \mathbf{u}^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $A_{T,0} \times B_{T,0}$ endowed with the norm $\|h\|_{L_T^\infty(L^2)} + \kappa^{\frac{1}{2}} \|h\|_{L_T^2(H^1)} + \|u\|_{L_T^\infty(L^2)}$, therefore there exists $(h, \mathbf{u}) \in A_{T,0} \times B_{T,0}$ such that $(h^n, \mathbf{u}^n) \longrightarrow (h, \mathbf{u})$.

Step 4: Passing to the limit.

Using the classical argument of weak convergence of a subsequence in Sobolev spaces and from what proceeds by using an interpolation argument we can pass to the limit in the equations (one can see for example [8] as a reference of this argument) and consequently (h, u) satisfies (3.2.2),(3.2.1).

Step 5: Regularity of the solution.

Since $h \in L^2(0, T_M; H^{s+1}(\mathbb{R}^d)) \cap L^\infty(0, T_M; H^s(\mathbb{R}^d))$ is solution to the transport diffusion equation (3.2.1)₁, by Theorem 3.2.2 it is the unique solution hence $h \in L^2(0, T; H^{s+1}(\mathbb{R}^d)) \cap C([0, T]; H^s(\mathbb{R}^d))$.

Using the fact that $h^n \rightarrow h$ in $L^\infty([0, T] \times \mathbb{R}^d)$, it follows that

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d \quad h^n(t, x) + \underline{h} \geq h_* \implies \forall (t, x) \in [0, T] \times \mathbb{R}^d \quad h(t, x) + \underline{h} \geq h_*.$$

Since $\mathbf{u} \in L^\infty(0, T_M; H^s(\mathbb{R}^d)^d)$ and satisfies the transport equation (3.2.1)₂, by Theorem 3.2.1 \mathbf{u} is the unique solution and $\mathbf{u} \in C([0, T]; H^s(\mathbb{R}^d)^d)$, again using the equation we can confirm that $\mathbf{u} \in C^1([0, T]; H^{s-1}(\mathbb{R}^d)^d)$. \square

Definition 3.2.5. Let $d \in \mathbb{N}^*$, $h_* > 0$, $s > 1 + \frac{d}{2}$, $(h_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d)^{1+d}$ and $(\underline{h}, \underline{\mathbf{u}})$ as in Theorem 3.2.4. Denote for $T > 0$

$$E_{T,s} := \left(L^2([0, T]; H^{s+1}(\mathbb{R}^d)) \cap C([0, T]; H^s(\mathbb{R}^d)) \right) \times \left(C([0, T]; H^s(\mathbb{R}^d)^d) \times C^1([0, T]; H^{s-1}(\mathbb{R}^d)^d) \right),$$

and define

$$T_{max} := \sup \{ T > 0; \text{there exists } (h, \mathbf{u}) \in E_{T,s}, \text{ with } (h, \mathbf{u}) \text{ solution to (3.2.1)} \\ \text{and } \inf_{[0, T] \times \mathbb{R}^d} h + \underline{h} \geq h_* \}.$$

We say that (h, \mathbf{u}) is a maximal solution of the system (3.2.1) if

$$h \in L^2_{loc}([0, T_{max}); H^{s+1}(\mathbb{R}^d)) \times C([0, T_{max}); H^s(\mathbb{R}^d)), \\ \mathbf{u} \in C([0, T_{max}); H^s(\mathbb{R}^d)^d) \times C^1([0, T_{max}); H^{s-1}(\mathbb{R}^d)^d)$$

$$\text{and } \inf_{[0, T_{max}) \times \mathbb{R}^d} h + \underline{h} \geq h_*.$$

From the previous theorem it follows that the system (3.2.1) has a unique maximal solution. Consequently we state the following theorem.

Theorem 3.2.6. Let $0 < \kappa \leq 1$, $d \in \mathbb{N}^*$, $h_* > 0$, $s > 1 + \frac{d}{2}$, $(h_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d)^{1+d}$ and $(\underline{h}, \underline{\mathbf{u}})$ as in Theorem 3.2.4 such that $\inf_{\mathbb{R}^d} h_0 + \underline{h} > 0$, there exists a unique $T_{max} > 0$ and a unique (h, \mathbf{u}) with

$$h \in L^2_{loc}([0, T_{max}); H^{s+1}(\mathbb{R}^d)) \cap C([0, T_{max}); H^s(\mathbb{R}^d)), \\ \mathbf{u} \in C([0, T_{max}); H^s(\mathbb{R}^d)^d) \cap C^1([0, T_{max}); H^{s-1}(\mathbb{R}^d)^d),$$

such that (h, \mathbf{u}) is a maximal solution of the system (3.2.1) and it satisfies the non cavi-

tation assumption, for $(t, x) \in [0, T_{max}) \times \mathbb{R}^d$

$$h(t, x) + \underline{h} \geq h_*.$$

Moreover we have

- If moreover $(h_0, \mathbf{u}_0) \in H^{s'}(\mathbb{R}^d)^{1+d}$ with $s' > s$, then $(h, \mathbf{u}) \in C([0, T_{max}); H^{s'}(\mathbb{R}^d)^{1+d})$ (propagation of regularity).
- $(h_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d)^{1+d} \mapsto (h, \mathbf{u}) \in C([0, T_{max}); H^s(\mathbb{R}^d)^{1+d})$ is continuous (flow map continuity).

Proof. The first part of the theorem derives naturally from Theorem 3.2.4, as for the continuity of the flow map and the propagation of regularity this is classical and well known in literature one can check [8] for these type of results. \square

3.2.2 Decreasing total energy

Applying the scalar product of (3.2.1)₁ with h and the scalar product of (3.2.1)₂ with $(h + \underline{h})\mathbf{u}$, integrating by parts and combining the two equations we naturally define the energy of the system (3.2.1) by the following functional

$$\forall t \in [0, T], \quad \mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^d} h(t, x)^2 + (h + \underline{h})|\mathbf{u}|^2(t, x) dx. \quad (3.2.4)$$

In the following proposition we will prove that in fact this energy is decreasing.

Proposition 3.2.7. *Let $d \in \mathbb{N}^*$, $s > 1 + \frac{d}{2}$, $h_* > 0$, $T > 0$, $(\underline{h}, \underline{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^d$ and (h, \mathbf{u}) solution of the system (3.2.1) defined on $[0, T] \times \mathbb{R}^d$, where $h + \underline{h} \geq h_*$ on $[0, T] \times \mathbb{R}^d$, such that h, \mathbf{u} are smooth enough. Then the energy \mathcal{E} defined in (3.2.4) is decreasing.*

Proof. Omitting to write the variable (t, x) in what follows to lighten the text, and differentiating with respect to the time t the energy functional we consequently have

$$\frac{d}{dt} \mathcal{E}(t) = \int_{\mathbb{R}^d} (\partial_t h) h dx + \frac{1}{2} \int_{\mathbb{R}^d} \partial_t h |\mathbf{u}|^2 dx + \int_{\mathbb{R}^d} (h + \underline{h}) \partial_t \mathbf{u} \cdot \mathbf{u} dx.$$

Integrating by parts we obtain the following identities.

$$\begin{aligned} \int_{\mathbb{R}^d} (\partial_t h) h dx &= - \int_{\mathbb{R}^d} ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla h) h dx - \int_{\mathbb{R}^d} h(h + \underline{h})(\nabla \cdot \mathbf{u}) dx + \kappa \int_{\mathbb{R}^d} (\Delta h) h dx \\ &= \int_{\mathbb{R}^d} (h + \underline{h}) \nabla h \cdot (\mathbf{u} + \underline{\mathbf{u}}) dx - \kappa \int_{\mathbb{R}^d} |\nabla h|^2 dx. \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^d} \partial_t h |\mathbf{u}|^2 dx &= - \frac{1}{2} \int_{\mathbb{R}^d} ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla (h + \underline{h}) + (h + \underline{h}) \nabla \cdot (\mathbf{u} + \underline{\mathbf{u}}) - \kappa \Delta h) |\mathbf{u}|^2 dx \\ &= - \frac{1}{2} \int_{\mathbb{R}^d} ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla h) |\mathbf{u}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^d} (h + \underline{h})(\nabla \cdot \mathbf{u}) |\mathbf{u}|^2 dx + \frac{\kappa}{2} \int_{\mathbb{R}^d} (\Delta h) |\mathbf{u}|^2 dx, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^d} (h + \underline{h}) \partial_t \mathbf{u} \cdot \mathbf{u} dx &= - \int_{\mathbb{R}^d} ((h + \underline{h})(\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla) \mathbf{u} \cdot \mathbf{u} dx + \kappa \int_{\mathbb{R}^d} (\nabla h \cdot \nabla) \mathbf{u} \cdot \mathbf{u} dx \\ &\quad - \int_{\mathbb{R}^d} (h + \underline{h}) \nabla h \cdot \mathbf{u} dx, \end{aligned}$$

where

$$- \int_{\mathbb{R}^d} ((h + \underline{h})(\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla) \mathbf{u} \cdot \mathbf{u} dx = \frac{1}{2} \int_{\mathbb{R}^d} (\nabla h \cdot (\mathbf{u} + \underline{\mathbf{u}})) |\mathbf{u}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} (h + \underline{h})(\nabla \cdot \mathbf{u}) |\mathbf{u}|^2 dx$$

and

$$\kappa \int_{\mathbb{R}^d} (\nabla h \cdot \nabla) \mathbf{u} \cdot \mathbf{u} dx = - \frac{\kappa}{2} \int_{\mathbb{R}^d} (\Delta h) |\mathbf{u}|^2 dx.$$

Gathering the above equalities we obtain

$$\frac{d}{dt} \mathcal{E}(t) = \int_{\mathbb{R}^d} (h + \underline{h}) \nabla h \cdot \underline{\mathbf{u}} dx - \kappa \int_{\mathbb{R}^d} |\nabla h|^2 dx,$$

using the fact that

$$\int_{\mathbb{R}^d} (h + \underline{h}) \nabla h \cdot \underline{\mathbf{u}} dx = 0,$$

we have

$$\frac{d}{dt} \mathcal{E}(t) = -\kappa \int_{\mathbb{R}^d} |\nabla h|^2 dx,$$

therefore the energy \mathcal{E} of system (3.2.1) is decreasing. \square

3.2.3 Large-time well-posedness

In this sub-section we improve the result obtained in Theorem 3.2.4 and this is done by using the energy method. We start by obtaining energy estimates on the linearization of system (3.2.1), using these estimates we give the large-time wellposedness of the single-layered shallow water system (3.2.1).

We consider the system

$$\begin{cases} \partial_t \dot{h} + (\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla \dot{h} + (h + \underline{h}) \nabla \cdot \dot{\mathbf{u}} = \mathbf{R} + \kappa \Delta \dot{h}, \\ \partial_t \dot{\mathbf{u}} + \left((\mathbf{u} + \underline{\mathbf{u}} - \kappa \frac{\nabla h}{h + \underline{h}}) \cdot \nabla \right) \dot{\mathbf{u}} + \nabla \dot{h} = \mathbf{R}, \\ \dot{h}|_{t=0} = \dot{h}_0, \\ \dot{\mathbf{u}}|_{t=0} = \dot{\mathbf{u}}_0. \end{cases} \quad (3.2.5)$$

Lemma 3.2.8. *Let $h_*, M, \underline{M} > 0$. There exists $C = C(M, \underline{M}, h_*) > 0$ such that for all $0 < \kappa \leq 1$, for any $\underline{h}, \underline{\mathbf{u}} \in \mathbb{R}$ such that*

$$|\underline{h}| \leq \underline{M},$$

and for any $T > 0$ such that (h, \mathbf{u}) is solution to the system (3.2.1) on $[0, T]$ with $(h, u) \in C^1([0, T]; H^s(\mathbb{R}^d)^{d+1}) \cap C([0, T]; H^{s+1}(\mathbb{R}^d)^{d+1})$, satisfying for almost every $t \in [0, T]$ the upper bound

$$\|\nabla h(t, \cdot)\|_{L_x^\infty} + \|\nabla \cdot \mathbf{u}(t, \cdot)\|_{L_x^\infty} + \|h(t, \cdot)\|_{L_x^\infty} \leq M,$$

and the lower bound

$$\inf_{x \in \mathbb{R}} h(t, x) + \underline{h} \geq h_*, \quad (3.2.6)$$

and for any $(\dot{h}, \dot{\mathbf{u}}) \in C([0, T]; (L^2(\mathbb{R}))^{1+d})$ with $\dot{h} \in L^2(0, T; H^1(\mathbb{R}))$ satisfying (3.2.5) with

$\mathbf{R}, \mathbf{R} \in L^2(0, T; (L^2(\mathbb{R}))^{1+d})$, the following estimate holds

$$\mathcal{E}^{\frac{1}{2}}(\dot{h}, \dot{\mathbf{u}})(t) + \frac{\kappa^{\frac{1}{2}}}{2} \|\nabla \dot{h}\|_{L^2(0,t;L_x^2)} \leq \left(\mathcal{E}(\dot{h}, \dot{\mathbf{u}})^{\frac{1}{2}}(0) + C \int_0^t \|\mathbf{R}(\tau, \cdot)\|_{L_x^2} + \|\mathbf{R}(\tau, \cdot)\|_{L_x^2} d\tau \right) \times \exp(C(h_*^{-1}, M)t),$$

where we denote

$$\forall t \in [0, T], \quad \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) := \frac{1}{2} \int_{\mathbb{R}^d} |\dot{h}|^2(t, x) + (h(t, x) + \underline{h}) |\dot{\mathbf{u}}|^2(t, x) dx. \quad (3.2.7)$$

Proof. Omitting to write the variable (t, x) in what follows to lighten the text, and differentiating with respect to the time t the energy functional we consequently have

$$\frac{d}{dt} \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) = \int_{\mathbb{R}^d} (\partial_t \dot{h}) \dot{h} dx + \frac{1}{2} \int_{\mathbb{R}^d} \partial_t h |\dot{\mathbf{u}}|^2 dx + \int_{\mathbb{R}^d} (h + \underline{h}) \partial_t \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dx.$$

Integrating by parts we have

$$\int_{\mathbb{R}^d} (\partial_t \dot{h}) \dot{h} dx = - \int_{\mathbb{R}^d} ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla \dot{h}) \dot{h} dx - \int_{\mathbb{R}^d} (h + \underline{h}) (\nabla \cdot \dot{\mathbf{u}}) \dot{h} dx - \kappa \int_{\mathbb{R}^d} |\nabla \dot{h}|^2 dx + \int_{\mathbb{R}^d} \mathbf{R} \dot{h} dx,$$

and

$$- \int_{\mathbb{R}^d} ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla \dot{h}) \dot{h} dx = \frac{1}{2} \int_{\mathbb{R}^d} (\nabla \cdot \mathbf{u}) |\dot{h}|^2 dx,$$

$$- \int_{\mathbb{R}^d} (h + \underline{h}) (\nabla \cdot \dot{\mathbf{u}}) \dot{h} dx = \int_{\mathbb{R}^d} \dot{h} (\nabla h \cdot \dot{\mathbf{u}}) dx + \int_{\mathbb{R}^d} (h + \underline{h}) \nabla \dot{h} \cdot \dot{\mathbf{u}} dx,$$

hence

$$\begin{aligned} \int_{\mathbb{R}^d} (\partial_t \dot{h}) \dot{h} dx &= \frac{1}{2} \int_{\mathbb{R}^d} (\nabla \cdot \mathbf{u}) |\dot{h}|^2 dx + \int_{\mathbb{R}^d} \dot{h} (\nabla h \cdot \dot{\mathbf{u}}) dx + \int_{\mathbb{R}^d} (h + \underline{h}) \nabla \dot{h} \cdot \dot{\mathbf{u}} dx \\ &\quad - \kappa \int_{\mathbb{R}^d} |\nabla \dot{h}|^2 dx + \int_{\mathbb{R}^d} \mathbf{R} \dot{h} dx \end{aligned} \quad (3.2.8)$$

$$\begin{aligned} \int_{\mathbb{R}^d} (h + \underline{h}) \partial_t \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dx &= - \int_{\mathbb{R}^d} (h + \underline{h}) ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla) \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dx + \kappa \int_{\mathbb{R}^d} (\nabla h \cdot \nabla) \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dx \\ &\quad - \int_{\mathbb{R}^d} (h + \underline{h}) \nabla \dot{h} \cdot \dot{\mathbf{u}} dx + \int_{\mathbb{R}^d} (h + \underline{h}) \mathbf{R} \cdot \dot{\mathbf{u}} dx, \end{aligned}$$

moreover

$$- \int_{\mathbb{R}^d} (h + \underline{h}) ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla) \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dx = \frac{1}{2} \int_{\mathbb{R}^d} (\nabla h \cdot (\mathbf{u} + \underline{\mathbf{u}})) |\dot{\mathbf{u}}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} (h + \underline{h}) (\nabla \cdot \mathbf{u}) |\dot{\mathbf{u}}|^2 dx$$

$$\kappa \int_{\mathbb{R}^d} (\nabla h \cdot \nabla) \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dx = -\frac{\kappa}{2} \int_{\mathbb{R}^d} (\Delta h) |\dot{\mathbf{u}}|^2 dx,$$

consequently

$$\begin{aligned} \int_{\mathbb{R}^d} (h + \underline{h}) \partial_t \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dx &= \frac{1}{2} \int_{\mathbb{R}^d} (\nabla h \cdot (\mathbf{u} + \underline{\mathbf{u}})) |\dot{\mathbf{u}}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} (h + \underline{h}) (\nabla \cdot \mathbf{u}) |\dot{\mathbf{u}}|^2 dx - \frac{\kappa}{2} \int_{\mathbb{R}^d} (\Delta h) |\dot{\mathbf{u}}|^2 dx \\ &\quad - \int_{\mathbb{R}^d} (h + \underline{h}) \nabla \dot{h} \cdot \dot{\mathbf{u}} dx + \int_{\mathbb{R}^d} (h + \underline{h}) \mathbf{R} \cdot \dot{\mathbf{u}} dx, \end{aligned} \tag{3.2.9}$$

$$\frac{1}{2} \int_{\mathbb{R}^d} \partial_t h |\dot{\mathbf{u}}|^2 dx = -\frac{1}{2} \int_{\mathbb{R}^d} ((\mathbf{u} + \underline{\mathbf{u}}) \cdot \nabla h + (h + \underline{h}) \nabla \cdot \mathbf{u} - \kappa \Delta h) |\dot{\mathbf{u}}|^2 dx. \tag{3.2.10}$$

Gathering (3.2.8), (3.2.9) and (3.2.10) we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) &= \frac{1}{2} \int_{\mathbb{R}^d} (\nabla \cdot \mathbf{u}) |\dot{h}|^2 dx + \int_{\mathbb{R}^d} \dot{h} (\nabla h \cdot \dot{\mathbf{u}}) dx - \kappa \int_{\mathbb{R}^d} |\nabla \dot{h}|^2 dx + \int_{\mathbb{R}^d} \mathbf{R} \dot{h} dx \\ &\quad + \int_{\mathbb{R}^d} (h + \underline{h}) \mathbf{R} \cdot \dot{\mathbf{u}} dx, \end{aligned} \tag{3.2.11}$$

consequently

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) + \kappa \int_{\mathbb{R}^d} |\nabla \dot{h}|^2(t) dx &\leq C(h_*^{-1}) \left(\|\nabla h(t, \cdot)\|_{L_x^\infty} + \|\nabla \cdot \mathbf{u}(t, \cdot)\|_{L_x^\infty} \right) \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) \\ &\quad + C(h_*^{-1}) \left(\|\mathbf{R}(t)\|_{L_x^2} + \left(\|h(t, \cdot)\|_{L_x^\infty} + |\underline{h}| \right) \|\mathbf{R}(t)\|_{L_x^2} \right) \mathcal{E}^{\frac{1}{2}}(\dot{h}, \dot{\mathbf{u}})(t). \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) + \kappa \int_{\mathbb{R}^d} |\nabla \dot{h}|^2(t) dx &\leq C(h_*^{-1}, M) \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) \\ &\quad + C(h_*^{-1}, M, \underline{M}) (\|\mathbf{R}(t)\|_{L_x^2} + \|\mathbf{R}(t)\|_{L_x^2}) \mathcal{E}^{\frac{1}{2}}(\dot{h}, \dot{\mathbf{u}})(t). \end{aligned}$$

The proof is then a consequence of Gronwall lemma. \square

Proposition 3.2.9. *Let $s > 1 + \frac{d}{2}$, $h_* > 0$, $d \in \mathbb{N}^*$, $M > 0$, $0 < \kappa \leq 1$, $(h_0, \mathbf{u}_0) \in H^s(\mathbb{R}^d)^{1+d}$ and $(\underline{h}, \underline{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^d$ such that*

$$\begin{aligned} M_0 &:= \mathcal{E}(\Lambda^s h, \Lambda^s \mathbf{u})(0) \leq M, \\ \inf_{\mathbb{R}^d} h_0 + \underline{h} &\geq 2h_*, \\ |\underline{h}| &\leq \underline{M}, \end{aligned}$$

then there exists $C(h_*^{-1}, M, \underline{M}) > 0$, $0 < T^* < \infty$ such that for T_{max} , h , \mathbf{u} as in Theorem 3.2.6 with (h, \mathbf{u}) the maximal solution to (3.2.1) with $(h, \mathbf{u})|_{t=0} = (h_0, \mathbf{u}_0)$. Then T_{max} is as follows

$$T_{max} \geq T^* \geq \frac{1}{C(h_*^{-1}, M, \underline{M}) M^{\frac{1}{2}}}.$$

Moreover for all $(t, x) \in [0, T^*] \times \mathbb{R}^d$

$$\begin{aligned} \mathcal{E}(\Lambda^s h, \Lambda^s \mathbf{u})(t) &\leq M_0, \\ h(t, x) + \underline{h} &\geq h_*. \end{aligned}$$

Where \mathcal{E} is defined as in Lemma 3.2.8, and $C(M, \underline{M}, h_*)$ is a non-decreasing function of M .

Proof. We suppose (h, \mathbf{u}) sufficiently regular to justify the computation done in proof of Lemma 3.2.8. Let $(\dot{h}, \dot{\mathbf{u}}) := (\Lambda^s h, \Lambda^s \mathbf{u})$, and applying the operator Λ^s to system (3.2.1),

we obtain system (3.2.5) with the corresponding rest terms

$$\begin{aligned} \mathbf{R} &= -[\Lambda^s, \mathbf{u}] \nabla h - [\Lambda^s, h] \nabla \cdot \mathbf{u}, \\ \mathbf{R} &= -[\Lambda^s, \mathbf{v}^*] \nabla \mathbf{u} = -[\Lambda^s, \mathbf{u}] \nabla \mathbf{u} + \frac{\kappa}{h + \underline{h}} [\Lambda^s, \nabla h] \nabla \mathbf{u} + \kappa \left[\Lambda^s, \frac{1}{h + \underline{h}} \right] (\nabla h)(\nabla \mathbf{u}), \end{aligned}$$

where $\mathbf{v}^* = \mathbf{u} + \underline{\mathbf{u}} - \kappa \frac{\nabla h}{h + \underline{h}}$.

In order to obtain a differential inequality on $\mathcal{E}(\dot{h}, \dot{\mathbf{u}})$ it remains to estimate the rest terms \mathbf{R}, \mathbf{R} in L^2 spacial norm.

Since $s > 1 + \frac{d}{2}$, by using commutator estimates in Sobolev spaces (see [53]) there exists $C_1 > 0$ such that

$$\|\mathbf{R}(t)\|_{L_x^2} \leq C_1 \|\dot{\mathbf{u}}(t)\|_{L_x^2} \|\dot{h}(t)\|_{L_x^2}.$$

Moreover one has for any $s_0 > \frac{1}{2}, \sigma \geq 0$, using classical product estimates and composition estimates in Sobolev spaces (see [53]) we have

$$\begin{aligned} \left| \nabla \frac{1}{h + \underline{h}} \right|_{H_x^\sigma} &= \left| \frac{\nabla h}{(h + \underline{h})^2} \right|_{H_x^\sigma} \leq \left| \frac{\nabla h}{\underline{h}^2} \right|_{H_x^\sigma} + \left| \left(\frac{1}{\underline{h}^2} - \frac{1}{(h + \underline{h})^2} \right) \nabla h \right|_{H_x^\sigma} \\ &\lesssim (h_*^{-1})^2 |\nabla h|_{H_x^\sigma} + \left| \left(\frac{1}{\underline{h}^2} - \frac{1}{(h + \underline{h})^2} \right) \right|_{H_x^{s_0}} |\nabla h|_{H_x^\sigma} \\ &\quad + \left\langle \left| \left(\frac{1}{\underline{h}^2} - \frac{1}{(h + \underline{h})^2} \right) \right|_{H_x^\sigma} |\nabla h|_{H_x^{s_0}} \right\rangle_{\sigma > s_0} \\ &\leq C (h_*^{-1}, \|h\|_{H_x^{s_0}}) \|\nabla h\|_{H_x^\sigma}, \end{aligned}$$

Hence since $s > 1 + \frac{d}{2}$, using commutator estimates in Sobolev spaces there exists $C(h_*^{-1}, \|\dot{h}\|_{L_x^2}) > 0$ such that

$$\|\mathbf{R}(t)\|_{L_x^2} \leq \|\dot{\mathbf{u}}(t)\|_{L_x^2}^2 + \kappa C(h_*, \|\dot{h}\|_{L_x^2}) \|\dot{\mathbf{u}}(t)\|_{L_x^2} \left(\|\nabla \dot{h}(t)\|_{L_x^2} + \|\dot{h}(t)\|_{L_x^2}^2 \right). \quad (3.2.12)$$

Furthermore we have the following estimates:

$$\begin{aligned} \|\dot{\mathbf{u}}(t, \cdot)\|_{L_x^2}^2 &\leq \frac{2}{h_*} \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t), \\ \|\dot{h}(t, \cdot)\|_{L_x^2} &\leq \sqrt{2} \mathcal{E}^{\frac{1}{2}}(\dot{h}, \dot{\mathbf{u}})(t). \end{aligned}$$

Hence, by using Cauchy Schwarz inequality and the previous estimates in addition to

the continuous injection $H^{s-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ equation (3.2.11) and by augmenting $C(h_*^{-1}, \|\dot{h}\|_{L_x^2})$, we have using Lemma 3.2.8 the following estimate

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) + \frac{\kappa}{4} \int_{\mathbb{R}^d} |\nabla \dot{h}|^2(t) dx &\leq C(h_*^{-1}) \left(1 + \|h(t, \cdot)\|_{L_x^\infty} + |\underline{h}|\right) \mathcal{E}^{\frac{3}{2}}(\dot{h}, \dot{\mathbf{u}})(t) \\ &+ \kappa C(h_*^{-1}, \|\dot{h}\|_{L_x^2}) \left(\|h(t, \cdot)\|_{L_x^\infty} + |\underline{h}|\right)^2 \mathcal{E}^2(\dot{h}, \dot{\mathbf{u}})(t) \\ &+ \kappa C(h_*^{-1}, \|\dot{h}\|_{L_x^2}) \left(\|h(t, \cdot)\|_{L_x^\infty} + |\underline{h}|\right) \mathcal{E}^2(\dot{h}, \dot{\mathbf{u}})(t). \end{aligned} \quad (3.2.13)$$

Once we established inequality (3.2.13) the goal now is to obtain a control on the energy $\mathcal{E}(\dot{h}, \dot{\mathbf{u}})$. Let

$$T^* := \sup \left\{ 0 \leq T < T_{max}; \forall t \in (0, T], \quad (\mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) \leq 2M_0, \text{ and } \inf_{\mathbb{R}^d} h(t, \cdot) + \underline{h} \geq h_*) \right\}.$$

By continuity in time of the solution we deduce that $T^* > 0$. Let $t \in (0, T^*)$, using inequality (3.2.13) we have

$$\frac{d}{dt}\mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) \leq C(h_*^{-1}, M_0, \underline{M}) M_0^{\frac{1}{2}} \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) + \kappa C(h_*^{-1}, M_0, \underline{M}) M_0 \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t),$$

thus

$$\frac{d}{dt}\mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) \leq C(h_*^{-1}, M_0, \underline{M}) (M_0^{\frac{1}{2}} + \kappa M_0) \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t),$$

where $C(s, h^*, M_0, \underline{M}) > 0$ is a non-decreasing function of M_0 . By using Gronwall's inequality, we have

$$\mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) \leq \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(0) \exp \left(C(h_*^{-1}, M_0, \underline{M}) (M_0^{\frac{1}{2}} + \kappa M_0) t \right), \quad (3.2.14)$$

using the fact that $0 < \kappa < 1$ and augmenting $C(h_*^{-1}, M_0, \underline{M})$ we find that there exists $C_1 \geq 1$ depending only on \underline{M}, M, h_* such that

$$C(h_*^{-1}, M_0, \underline{M}) M_0^{\frac{1}{2}} t \leq C_1^{-1} \implies \mathcal{E}(\dot{h}, \dot{\mathbf{u}})(t) \leq \frac{3}{2} M_0.$$

Since $(\underline{h}, h, \underline{\mathbf{u}}, \mathbf{u})$ satisfy the following heat equation

$$\partial_t h + \underline{\mathbf{u}} \cdot \nabla h = \kappa \Delta h + g,$$

where $g := -(\mathbf{u} \cdot \nabla h + (h + \underline{h}) \nabla \cdot \mathbf{u})$, by positivity of the heat kernel and since $\underline{\mathbf{u}}$ is constant we have

$$\inf_{x \in \mathbb{R}^d} h(t, x) \geq \inf_{x \in \mathbb{R}^d} h(0, x) - \|g\|_{L^1(0, t; L_x^\infty)}.$$

Using product estimates in Sobolev spaces (see [53]) and the fact that $s > 1 + \frac{d}{2}$ we have

$$\begin{aligned} \|g(t)\|_{L_x^\infty} &\lesssim |\underline{h}| \|\mathbf{u}(t, \cdot)\|_{L_x^2} + \|\mathbf{u}(t, \cdot)\|_{H_x^s} \|h(t, \cdot)\|_{H_x^s} \\ &\leq C(h_*^{-1}, M_0, \underline{M}) M_0^{\frac{1}{2}}. \end{aligned}$$

Hence augmenting C_1 if necessary we find that

$$C(h_*^{-1}, M_0, \underline{M}) M_0^{\frac{1}{2}} t \leq C_1^{-1} \implies \forall x \in \mathbb{R}^d, \quad h(t, x) + \underline{h} \geq \frac{3}{2} h_*.$$

Hence by continuity in time of the solution we find that for all $t \in (0, T_*)$ satisfying $C(h_*^{-1}, M_0, \underline{M}) M_0^{\frac{1}{2}} t \leq C_1^{-1}$, there exists $\delta > 0$ such that $[t - \delta, t + \delta] \subset (0, T_*)$. By a continuity argument we consequently have

$$T_{max} \geq T^* \geq \frac{1}{C(h_*^{-1}, M_0, \underline{M}) M_0^{\frac{1}{2}}}.$$

This estimate on T_{max} is obtained for (h, \mathbf{u}) at least in $C^1([0, T]; H^s(\mathbb{R}^d)^{d+1}) \cap C([0, T]; H^{s+1}(\mathbb{R}^d)^{d+1})$, we deduce the same estimate on T_{max} if the initial data (h_0, \mathbf{u}_0) were only in $H^s(\mathbb{R}^d)^{1+d}$ (consequently (h, \mathbf{u}) will have the same regularity as given in Theorem 3.2.6) and it is obtained by regularizing these initial data (h_0, \mathbf{u}_0) and using the continuity of the flow map and the propagation of regularity found in Theorem 3.2.6, the proof is then concluded. \square

Remark 3.2.10. *The previous result is an improvement of the result found in Theorem 3.2.4 since there is no more dependency on κ for the lower bound on the time existence T_{max} .*

3.3 The multi-layer shallow-water system

Let us remind the reader that, as stated in the introduction, besides the results concerning the previous chapter on the one-layer shallow water system, and for simplicity and readability reasons, the dimension will be set as $d = 1$.

3.3.1 The nonuniform well-posedness

As it has been seen in sub-section 3.2.1, the shallow-water system with the additional Gent & McWilliams terms is well-posed with a time of existence that isn't uniform with respect to κ (see Theorem 3.2.4). We can easily adapt the proof of Theorem 3.2.4 (with some minor modifications) to the case of a finite number of layer of the fluid where the time T in addition to the dependency on κ will also depend on the number of layers N and hence we state following theorem.

Theorem 3.3.1. *Let $s > 1 + \frac{1}{2}$, $N \in \mathbb{N}^*$, $h_* > 0$, $M, \underline{M} > 0$, $\rho_i > 0$ for all $i \in \{1, \dots, N\}$, $0 < \kappa \leq 1$, there exists $T = \frac{\kappa}{C(s, h_*, M, \underline{M})}$ (where $C(s, h_*, M, \underline{M}) > 1$ and non-decreasing function of M and \underline{M}), such that for all $(\underline{H}, \underline{U}) := ((\underline{H}_1, \dots, \underline{H}_N), (\underline{U}_1, \dots, \underline{U}_N)) \in \mathbb{R}^{2N}$ with*

$$\inf_{i \in \{1, \dots, N\}} \underline{H}_i \geq h_*; \quad \sup_{i \in \{1, \dots, N\}} |\underline{H}_i| + |\underline{U}_i| \leq \underline{M}, \quad (3.3.1)$$

for any $(H_0, U_0) := ((H_{1,0}, \dots, H_{N,0}), (U_{1,0}, \dots, U_{N,0})) \in H^s(\mathbb{R})^{2N}$ with

$$M_{0,N} := \sup_{i \in \{1, \dots, N\}} \|H_{i,0}\|_{H^s} + \|U_{i,0}\|_{H^s} \leq M, \\ H_{i,0} + \underline{H}_i \geq 2h_*, \quad \forall i \in \{1, \dots, N\}.$$

There exists a unique (H, U) with

$$H = (H_1, \dots, H_N) \in \left(L^2([0, T]; H^{s+1}(\mathbb{R})) \cap C([0, T]; H^s(\mathbb{R})) \right)^N \text{ and} \\ U = (U_1, \dots, U_N) \in \left(C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R})) \right)^N,$$

such that (H, U) is solution to

$$\begin{cases} \partial_t H_i + \partial_x((\underline{H}_i + H_i)(\underline{U}_i + U_i)) = \kappa \partial_x^2 H_i \\ \partial_t U_i + \left(\underline{U}_i + U_i - \kappa \frac{\partial_x H_i}{\underline{H}_i + H_i} \right) \partial_x U_i + \sum_{j=1}^N \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j = 0, \end{cases} \quad \forall i \in \{1, \dots, N\}, \quad (3.3.2)$$

with initial data $(H, U)|_{t=0} = (H_0, U_0)$.

Moreover

$$\|H_i\|_{L^\infty(0, T; H^s)} \leq 2e\gamma M_{0, N}, \quad \kappa^{\frac{1}{2}} \|H_i\|_{L^2(0, T; H^{s+1})} \leq 2e\gamma M_{0, N}, \quad \|U_i\|_{L^\infty(0, T; H^s)} \leq 2e M_{0, N},$$

and $H_i + \underline{H}_i \geq h_*$ on $[0, T] \times \mathbb{R}$, for all $i \in \{1 \dots N\}$, where γ is the same as in Theorem 3.2.2.

Proof. We will give a sketch of the proof since it follows the same steps as in Theorem 3.2.4. The main idea is to adapt the proof of Theorem 3.2.4 for each layer. Let $i \in \{1, \dots, N\}$ fixed, system (3.3.2) is viewed as a system of a transport-diffusion equation and a transport equation, coupled only through order-zero source terms. We put $H_i^{-1} = 0$, $U_i^{-1} = 0$ and we define the sequence $(H_i^n, U_i^n)_{n \in \mathbb{N}}$ by the following Picard iterative scheme:

$$\begin{cases} \partial_t H_i^{n+1} + (\underline{U}_i + U_i^n) \partial_x H_i^{n+1} + (\underline{H}_i + H_i^n) \partial_x U_i^n = \kappa \partial_x^2 H_i^{n+1} \\ \partial_t U_i^{n+1} + \left(\underline{U}_i + U_i^n - \kappa \frac{\partial_x H_i^n}{\underline{H}_i + H_i^n} \right) \partial_x U_i^{n+1} + \sum_{j=1}^N \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j^n = 0, \\ H_i^{n+1}|_{t=0} = H_{i,0}, \quad H_i^{n+1} + \underline{H}_i \geq h_* \\ U_i^{n+1}|_{t=0} = U_{i,0}. \end{cases} \quad (3.3.3)$$

For **Step 1** by applying the same procedure and relying on the estimates 3.3.1 we easily show inductively that for any $i \in \{1, \dots, N\}$ fixed, $n \in \mathbb{N}$, there exists $T_n > 0$ (which can depend on i), such that (H_i^n, U_i^n) is in $A_{T_n, s} \times B_{T_n, s}$ and that $\inf_{[0, T_n] \times \mathbb{R}^d} (H_i^n + \underline{H}_i) \geq h_*$. The major change is the estimate obtained on U_i^{n+1} .

$$\begin{aligned} \|U_i^{n+1}\|_{L_{T_n}^\infty(H^s)} &\leq \left(\|U_{i,0}\|_{H^s} + \sum_{j=1}^N \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \|\partial_x H_j^n\|_{L_{T_n}^1(H^s)} \right) \\ &\quad \times \exp \left(C \int_0^{T_n} \left\| \partial_x \left(U_i^n + \underline{U}_i - \kappa \frac{\partial_x H_i^n}{H_i^n + \underline{H}_i} \right) (t) \right\|_{H^{s-1}} dt \right). \end{aligned} \quad (3.3.4)$$

For **Step 2** using the induction argument in addition to estimates on H_i^{n+1} and relying on (3.3.4) we easily deduce that there exists $C(s, N, h_*, M, \underline{M}, d) > 1$ such that for the choice of $0 < T \leq \frac{\kappa}{C(s, h_*, M, \underline{M}, N, d)}$ we have $(H_i^n, U_i^n) \in B_T(0)$ and $H_i^n + \underline{H}_i \geq h_*$ in $[0, T] \times \mathbb{R}$ and that is for all fixed $i \in \{1, \dots, N\}$.

For **Step 3** we show that $((H_1^n, U_1^n), \dots, (H_N^n, U_N^n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $(A_{T,0} \times B_{T,0})^N$ endowed with its natural norm defined as follows,

$$\text{for } (H, U) = ((H_1, \dots, H_N), (U_1, \dots, U_N))$$

$$\|(H, U)\|_{(A_{T,0} \times B_{T,0})^N} := \sum_{i=1}^N \left(\|H_i\|_{L_T^\infty(L_x^2)} + \kappa^{\frac{1}{2}} \|H_i\|_{L_T^2(H_x^1)} + \|U_i\|_{L_T^\infty(L_x^2)} \right),$$

hence defining for $i \in \{1, \dots, N\}$

$$\begin{aligned} S_{n+1,m,i} &= U_i^{n+m+1} - U_i^{n+1}, \\ W_{n+1,m,i} &= H_i^{n+m+1} - H_i^{n+1}, \end{aligned}$$

and using the exact same estimates found in **Step 3** of the proof of Theorem 3.2.4 with the corresponding f_0, \mathbf{v}, g and by following the same procedure we easily show that by augmenting $C(s, N, h_*, M, \underline{M}, d)$, $((H_1^n, U_1^n), \dots, (H_N^n, U_N^n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $(A_{T,0} \times B_{T,0})^N$.

Finally **Step 4** and **Step 5** are easily deduced by adapting the last two steps of the proof of Theorem 3.2.4. \square

3.3.2 Decreasing total energy

Based on (3.2.4) we define naturally the energy functional of system (3.3.2) for $N \in \mathbb{N}^*$ and $t \in [0, T]$ by

$$\begin{aligned} \mathcal{E}(t) = & \frac{\rho_1}{2N} \int_{\mathbb{R}} \left(\sum_{j=1}^N H_j(t, x) \right)^2 dx + \frac{1}{2N} \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N H_j(t, x) \right)^2 dx \quad (3.3.5) \\ & + \frac{1}{2} \int_{\mathbb{R}} \sum_{j=1}^N \rho_j (\underline{H}_j + H_j(t, x)) |U_j|^2(t, x) dx + \int_{\mathbb{R}} \sum_{j=1}^N \rho_j \underline{U}_j (\underline{H}_j + H_j(t, x)) U_j(t, x) dx. \end{aligned}$$

In the following proposition we will prove that in fact this energy is decreasing.

Proposition 3.3.2. *Let $N \in \mathbb{N}^*$, $s > 1 + \frac{1}{2}$, $h_* > 0$, $T > 0$,*

$(\underline{H}, \underline{U}) := ((\underline{H}_1, \dots, \underline{H}_N), (\underline{U}_1, \dots, \underline{U}_N)) \in \mathbb{R}^{2N}$, and for $H = (H_1, \dots, H_N)$, $U = (U_1, \dots, U_N)$ smooth enough such that (H, U) is solution to system (3.3.2) defined on $[0, T] \times \mathbb{R}$, where

$\inf_{i \in \{1, \dots, N\} \times \mathbb{R}} \underline{H}_i + H_i(t, \cdot) \geq h_$ for $t \in [0, T]$, moreover supposing that $\rho_N > \dots > \rho_1 > 0$.*

Then the energy \mathcal{E} defined in (3.3.5) is decreasing.

Proof. Omitting to write the variable (t, x) in what follows to lighten the text, and differentiating with respect to the time t the energy functional we consequently have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) = & \frac{\rho_1}{N} \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_t H_j \right) \left(\sum_{j=1}^N H_j \right) dx + \frac{1}{N} \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N \partial_t H_j \right) \left(\sum_{j=k+1}^N H_j \right) dx \\ & + \frac{1}{2} \int_{\mathbb{R}} \sum_{j=1}^N \rho_j (\partial_t H_j) |U_j|^2 dx + \int_{\mathbb{R}} \sum_{j=1}^N \rho_j (\underline{H}_j + H_j) (\partial_t U_j) U_j dx \\ & + \int_{\mathbb{R}} \sum_{j=1}^N \rho_j \underline{U}_j (\underline{H}_j + H_j) \partial_t U_j dx + \int_{\mathbb{R}} \sum_{j=1}^N \rho_j \underline{U}_j \partial_t H_j U_j dx. \end{aligned}$$

Integrating by parts we obtain the following identities.

$$\begin{aligned}
 \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_t H_j \right) \left(\sum_{j=1}^N H_j \right) dx &= - \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_x ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) \right) \left(\sum_{j=1}^N H_j \right) dx \\
 &\quad + \kappa \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_x^2 H_j \right) \left(\sum_{j=1}^N H_j \right) dx \\
 &= \int_{\mathbb{R}} \left(\sum_{j=1}^N ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) \right) \left(\sum_{j=1}^N \partial_x H_j \right) dx - \kappa \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_x H_j \right)^2 dx,
 \end{aligned}$$

we automatically have

$$\begin{aligned}
 \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N \partial_t H_j \right) \left(\sum_{j=k+1}^N H_j \right) dx \\
 = \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) \right) \left(\sum_{j=k+1}^N \partial_x H_j \right) dx \\
 - \kappa \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \int_{\mathbb{R}} \left(\sum_{j=k+1}^N \partial_x H_j \right)^2 dx,
 \end{aligned}$$

moreover

$$\frac{1}{2} \int_{\mathbb{R}} \sum_{j=1}^N \partial_t H_j |U_j|^2 \rho_j dx = -\frac{1}{2} \int_{\mathbb{R}} \sum_{j=1}^N \partial_x ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) |U_j|^2 \rho_j dx + \frac{\kappa}{2} \int_{\mathbb{R}} \sum_{j=1}^N \partial_x^2 H_j |U_j|^2 \rho_j dx,$$

furthermore

$$\begin{aligned}
 \int_{\mathbb{R}} \sum_{j=1}^N \rho_j (\underline{H}_j + H_j) (\partial_t U_j) U_j dx &= - \int_{\mathbb{R}} \sum_{j=1}^N ((\underline{H}_j + H_j)(\underline{U}_j + U_j) \partial_x U_j) U_j \rho_j dx \\
 &\quad + \kappa \int_{\mathbb{R}} \sum_{j=1}^N (\partial_x H_j \partial_x U_j) U_j \rho_j dx \\
 &\quad - \frac{1}{N} \int_{\mathbb{R}} \sum_{j=1}^N (\underline{H}_j + H_j) U_j \left(\sum_{i=1}^N \min(\rho_i, \rho_j) (\partial_x H_i) \right) dx \\
 &= \frac{1}{2} \int_{\mathbb{R}} \sum_{j=1}^N \partial_x ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) |U_j|^2 \rho_j dx \\
 &\quad - \frac{\kappa}{2} \int_{\mathbb{R}} \sum_{j=1}^N (\partial_x^2 H_j) |U_j|^2 \rho_j dx \\
 &\quad - \frac{1}{N} \int_{\mathbb{R}} \sum_{j=1}^N (\underline{H}_j + H_j) U_j \left(\sum_{i=1}^N \min(\rho_i, \rho_j) (\partial_x H_i) \right) dx,
 \end{aligned}$$

in the same way

$$\begin{aligned}
 \int_{\mathbb{R}} \sum_{j=1}^N \rho_j \underline{U}_j (\underline{H}_j + H_j) \partial_t U_j dx &= \int_{\mathbb{R}} \sum_{j=1}^N \partial_x ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) U_j \underline{U}_j \rho_j dx \\
 &\quad - \kappa \int_{\mathbb{R}} \sum_{j=1}^N (\partial_x^2 H_j) U_j \underline{U}_j \rho_j dx \\
 &\quad - \frac{1}{N} \int_{\mathbb{R}} \sum_{j=1}^N (\underline{H}_j + H_j) \underline{U}_j \left(\sum_{i=1}^N \min(\rho_i, \rho_j) (\partial_x H_i) \right) dx,
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{\mathbb{R}} \sum_{j=1}^N \rho_j \underline{U}_j \partial_t H_j U_j dx &= - \int_{\mathbb{R}} \sum_{j=1}^N \partial_x ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) U_j \underline{U}_j \rho_j dx \\
 &\quad + \kappa \int_{\mathbb{R}} \sum_{j=1}^N (\partial_x^2 H_j) U_j \underline{U}_j \rho_j dx.
 \end{aligned}$$

Gathering the above equalities we obtain

$$\begin{aligned}
 \frac{d}{dt} \mathcal{E}(t) &= \frac{\rho_1}{N} \int_{\mathbb{R}} \left(\sum_{j=1}^N ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) \right) \left(\sum_{j=1}^N \partial_x H_j \right) dx \\
 &+ \frac{1}{N} \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) \right) \left(\sum_{j=k+1}^N \partial_x H_j \right) dx \\
 &- \frac{1}{N} \int_{\mathbb{R}} \sum_{j=1}^N (\underline{H}_j + H_j) (\underline{U}_j + U_j) \left(\sum_{i=1}^N \min(\rho_i, \rho_j) (\partial_x H_i) \right) dx \\
 &- \frac{\kappa \rho_1}{N} \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_x H_j \right)^2 dx - \frac{\kappa}{N} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \int_{\mathbb{R}} \left(\sum_{j=k+1}^N \partial_x H_j \right)^2 dx \\
 &= \frac{1}{N} \mathcal{S}_1 + \frac{\kappa}{N} \mathcal{S}_2,
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{S}_1 &= \rho_1 \int_{\mathbb{R}} \left(\sum_{j=1}^N ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) \right) \left(\sum_{j=1}^N \partial_x H_j \right) dx \\
 &+ \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N ((\underline{H}_j + H_j)(\underline{U}_j + U_j)) \right) \left(\sum_{j=k+1}^N \partial_x H_j \right) dx \\
 &- \int_{\mathbb{R}} \sum_{j=1}^N (\underline{H}_j + H_j) (\underline{U}_j + U_j) \left(\sum_{i=1}^N \min(\rho_i, \rho_j) (\partial_x H_i) \right) dx. \\
 \mathcal{S}_2 &= -\rho_1 \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_x H_j \right)^2 dx - \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \int_{\mathbb{R}} \left(\sum_{j=k+1}^N \partial_x H_j \right)^2 dx.
 \end{aligned}$$

For the sake of readability and in order to deal with \mathcal{S}_1 , we introduce the following notations

$$\begin{aligned}
 a_j &:= \underline{H}_j + H_j, \\
 b_j &:= \underline{U}_j + U_j, \\
 c_j &:= \partial_x H_j,
 \end{aligned}$$

hence using the fact that $\rho_N > \dots > \rho_1 > 0$, we have

$$\begin{aligned}
 \mathcal{S}_1 &= \rho_1 \int_{\mathbb{R}} \left(\sum_{j=1}^N (a_j b_j) \right) \left(\sum_{j=1}^N c_j \right) dx + \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N (a_j b_j) \right) \left(\sum_{j=k+1}^N c_j \right) dx \\
 &\quad - \int_{\mathbb{R}} \sum_{j=1}^N a_j b_j \left(\sum_{i=1}^N \min(\rho_i, \rho_j) c_i \right) dx \\
 &= \rho_1 \int_{\mathbb{R}} \left(\sum_{j=1}^N (a_j b_j) \right) \left(\sum_{j=1}^N c_j \right) dx + \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N (a_j b_j) \right) \left(\sum_{j=k+1}^N c_j \right) dx \\
 &\quad - \int_{\mathbb{R}} \sum_{j=1}^N a_j b_j \left(\sum_{i=1}^N \left(\rho_1 + \sum_{k=1}^{\min(i,j)-1} (\rho_{k+1} - \rho_k) \right) c_i \right) dx \\
 &= \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N (a_j b_j) \right) \left(\sum_{j=k+1}^N c_j \right) dx - \int_{\mathbb{R}} \sum_{j=1}^N \sum_{i=1}^N \sum_{k=1}^{\min(i,j)-1} (\rho_{k+1} - \rho_k) a_j b_j c_i dx,
 \end{aligned}$$

by exchanging the order of summation in the second term we have

$$\begin{aligned}
 \mathcal{S}_1 &= \int_{\mathbb{R}} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \left(\sum_{j=k+1}^N (a_j b_j) \right) \left(\sum_{j=k+1}^N c_j \right) dx - \int_{\mathbb{R}} \sum_{k=1}^{N-1} \sum_{j=k+1}^N \sum_{i=k+1}^N (\rho_{k+1} - \rho_k) a_j b_j c_i dx \\
 &= 0.
 \end{aligned}$$

Consequently

$$\begin{aligned}
 \frac{d}{dt} \mathcal{E}(t) &= \frac{\kappa}{N} \mathcal{S}_2 \\
 &= -\frac{\kappa \rho_1}{N} \int_{\mathbb{R}} \left(\sum_{j=1}^N \partial_x H_j \right)^2 dx - \frac{\kappa}{N} \sum_{k=1}^{N-1} (\rho_{k+1} - \rho_k) \int_{\mathbb{R}} \left(\sum_{j=k+1}^N \partial_x H_j \right)^2 dx,
 \end{aligned}$$

since $\rho_N > \dots > \rho_1 > 0$, the energy \mathcal{E} of system (3.3.2) is decreasing. \square

3.4 Conclusion

In this chapter the small-time well-posedness of the single layered shallow-water system with additional Gent & McWilliams terms (3.1.1) is obtained, its large-time well-posedness (when the time of existence and uniqueness is independent of the parameter κ) is then

obtained by the energy method. Moreover we proved that the former system had a decreasing energy.

We showed the well-posedness of the multi-layer system (3.3.2) with a lower bound on the time of existence that depends N and κ , this result is obtained relying strongly on the results of transport and transport-diffusion equations found in [8]. In the strategy of the proof we neglected the structure of the equations and the fact there is a correspondence between the layers of the fluid, this correspondence can be seen through the last term of (3.3.2)₂, which morally represents an exchange of information between the N layers of the fluid. Finally we showed that this system has a decreasing energy. Note that in the small-time well-posedness result Theorem 3.3.1, we didn't assume any hypotheses on stable stratification nor impose any regularity on the density profile as opposed to the result of the following chapter, mainly Theorem 4.5.1. Here, we impose regularity on the density profile in addition to stable stratification that can be seen in equation (4.1.1). In addition to these assumptions and exploiting the symmetric structure of the multi-layer system, we obtain a time of existence that is independent of N and that has a better dependency on the parameter κ .

LARGE-TIME WELL-POSEDNESS OF THE MULTI-LAYER SHALLOW WATER SYSTEM

4.1 Notations and conventions

In what follows, we use as convention that $g = 1$ and $\rho_{\text{bott}} - \rho_{\text{surf}} = 1$. This choice can be enforced without loss of generality through a suitable rescaling of the variables and unknowns. We also set $(\rho_i)_{i \in \{1, \dots, N\}}$ as

$$\forall i \in \{1, \dots, N\}, \quad \rho_i = \rho_{\text{surf}} + (i - \frac{1}{2}) \frac{\rho_{\text{bott}} - \rho_{\text{surf}}}{N}. \quad (4.1.1)$$

This choice is motivated by the following chapter, where we approach a continuous distribution in an equidistributed manner.

We shall use matrix and vector formulations when dealing with the multi-layer system (2.2.12). We then use capital letters without indices for N -dimensional vectors, sans serif fonts for N -by- N matrices, and indices to denote each components.

For $F = (F_1, \dots, F_N)^t$, $G = (G_1, \dots, G_N)^t$ two vectors, we define the product $FG := (F_1 G_1, \dots, F_N G_N)^t$, and for a function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, $\varphi(F) := (\varphi(F_1), \dots, \varphi(F_N))^t$.

With these conventions, (2.2.12) reads

$$\begin{cases} \partial_t H + (\underline{U} + U) \partial_x H + (\underline{H} + H) \partial_x U = \kappa \partial_x^2 H, \\ \partial_t U + \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) \partial_x U + \Gamma \partial_x H = 0, \end{cases} \quad (4.1.2)$$

with $\underline{H} = (\underline{H}_1, \dots, \underline{H}_N)^t$, $\underline{U} = (\underline{U}_1, \dots, \underline{U}_N)^t$, $H = (H_1, H_2, \dots, H_N)^t$, $U = (U_1, U_2, \dots, U_N)^t$, and $\Gamma_{i,j} := \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i}$, for all $1 \leq i, j \leq N$.

We now introduce matrix operations which will be used in our proofs.

— Denote $\text{Id} \in M_N(\mathbb{R})$ the identity matrix, $\text{P} := \text{Diag}(1, 0, \dots, 0) \in M_N(\mathbb{R})$ the projection onto the first component, and $\text{C} = \text{Id} - \text{P}$.

- We define the operator $\mathbb{T} := \sqrt{N}\mathbb{P} \in M_N(\mathbb{R})$ which can be interpreted as the discrete analogue of the trace operator on the surface (acting on the density variable in the continuously stratified case).
- Let $\mathbb{S} \in M_N(\mathbb{R})$ be the discrete integration operator defined as

$$\mathbb{S} := \frac{1}{N} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

We also define $\mathbb{S}_0 \in M_{N,N-1}(\mathbb{R})$ to be the matrix \mathbb{S} without the last column.

- Let $\mathbb{D}_\rho \in M_{N-1,N}(\mathbb{R})$ be the discrete differentiation operator defined as

$$\mathbb{D}_\rho := N \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

With an abuse of notation we will write $\mathbb{D}_\rho^2 = \tilde{\mathbb{D}}_\rho \mathbb{D}_\rho \in M_{N-2,N}(\mathbb{R})$ the second order discrete differentiation operator, where $\tilde{\mathbb{D}}_\rho$ is the matrix \mathbb{D}_ρ without the last line and last column. This matrix is used in order to define $\|\cdot\|_{H^{s,k}}$ (see below). Moreover we will use the convention $\mathbb{D}_\rho^0 = \text{Id}$.

- Let $\mathbb{M} \in M_{N-1,N}(\mathbb{R})$ be defined as

$$\mathbb{M} := \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

With an abuse of notation we will write $\mathbb{M}^2 = \tilde{\mathbb{M}}\mathbb{M} \in M_{N-2,N}(\mathbb{R})$, where $\tilde{\mathbb{M}}$ is the matrix \mathbb{M} without the last line and last column, hence it is in the matrix space $M_{N-2,N-1}(\mathbb{R})$. Moreover we will use the convention $\mathbb{M}^0 = \text{Id}$. The operator \mathbb{M} appears naturally in the Leibniz formula that is described in Appendix 7, and will be particularly useful to obtain sharp estimates in that chapter.

- We denote $R_u, R_d \in M_{N-1, N}(\mathbb{R})$ the upwards and downwards reduction operators defined for $F = (F_1, \dots, F_N)^t \in \mathbb{R}^N$, by

$$R_u F = (F_2, \dots, F_N)^t \quad \text{and} \quad R_d F = (F_1, \dots, F_{N-1})^t.$$

These operators are used in the Appendix 7, they appear naturally due to the definition of the norm $\|\cdot\|_{H^{s,k}}$ down below.

- For any $q \geq 1$ we define endow \mathbb{R}^N with the following normalized l^q norms:

$$|F|_{l^q} := \frac{\left(\sum_{i=1}^N |F_i|^q \right)^{\frac{1}{q}}}{N^{\frac{1}{q}}} \quad \text{and} \quad |F|_{l^\infty} := \sup_{i \in \{1, \dots, N\}} |F_i|$$

Moreover for $p, q \geq 1$, we introduce the following functional spaces

$$L_x^p(l^q) = \{F = (F_1, \dots, F_N) \in (L^p(\mathbb{R}^d))^N; \| |F|_{l^q} \|_{L_x^p} < \infty\}$$

$$l^q(L_x^p) = \{F = (F_1, \dots, F_N) \in (L^p(\mathbb{R}^d))^N; |(\|F_i\|_{L_x^p})_{1 \leq i \leq N}|_{l^q} < \infty\}$$

endowed with the following norms

$$\begin{aligned} \|F\|_{L_x^p(l^q)} &:= \| |F|_{l^q} \|_{L_x^p}, \\ \|F\|_{l^q(L_x^p)} &:= |(\|F_i\|_{L_x^p})_{1 \leq i \leq N}|_{l^q}. \end{aligned}$$

Notice that $\|F\|_{L_x^2(l^2)} = \|F\|_{l^2(L_x^2)}$.

- Let $\langle \cdot, \cdot \rangle_{L_x^2 l^2}$ be the scalar product of the Hilbert space $l^2(L_x^2(\mathbb{R}))$ such that for $F, G \in l^2(L_x^2(\mathbb{R}))$ we have

$$\langle F, G \rangle_{L_x^2 l^2} := \int_{\mathbb{R}} \langle F(x), G(x) \rangle_{l^2} dx = \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}} F_i(x) G_i(x) dx.$$

- For $F \in \mathbb{R}^N$ and for $k \in \{0, 1, 2\}$ we define the following norm

$$|F|_{w^{k, \infty}} := \sum_{l=0}^k |D_\rho^l F|_{l^\infty}.$$

— Let $s \in \mathbb{R}$, $k \in \{0, 1, 2\}$, where $k \leq s$, we define the functional space

$$H^{s,k}(\mathbb{R}) = \{F : \mathbb{R} \longrightarrow \mathbb{R}^N; \forall j \in \mathbb{N}, j \leq k, D_\rho^j F \in H^{s-j}(\mathbb{R})^N\},$$

endowed with the topology of the norm

$$\|F\|_{H^{s,k}}^2 = \sum_{j=0}^k \|D_\rho^j F\|_{L^2(H_x^{s-j})}^2.$$

Note that the functional space $H^{s,k}(\mathbb{R})$ coincides with $H^s(\mathbb{R})^N$ and their corresponding norms are equivalent (although not uniformly with respect to N). Moreover in this article we chose to work with this norm in order to obtain estimates uniform with respect to N (see section 4.3). We recall that each power of D_ρ when applied to a vector reduces its size.

— We use standard notations for functions depending on time. For instance, for $k \in \mathbb{N}$ and X a Banach space as above, $C^k([0, T]; X)$ is the space of functions with values in X which are continuously differentiable up to order k , and $L^p(0, T; X)$ the p -integrable X valued functions. All these spaces are endowed with their natural norms.

4.2 Introduction

This chapter is dedicated to the proof of Theorem 4.5.1, concerning the well-posedness of the initial-value problem for the multilayer system (2.2.12). Let us for convenience rewrite the system here, using conventions and notations introduced in Section 4.1.

$$\begin{cases} \partial_t H + (\underline{U} + U)\partial_x H + (\underline{H} + H)\partial_x U = \kappa \partial_x^2 H, \\ \partial_t U + \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H}\right) \partial_x U + \Gamma \partial_x H = 0, \end{cases} \quad (4.2.1)$$

where we recall that for all $i \in \{1, \dots, N-1\}$, $\rho_{i+1} - \rho_i = \frac{1}{N}$ and $\Gamma_{i,j} = \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i}$.

The goal in this section is to improve the nonuniform well-posedness result of the multi-layer shallow water system (see Theorem 3.3.1) by obtaining a time of existence uniform with respect to N . In order to achieve this goal we shall rely on the energy method. We extract the quasilinear structure of the equations in Section 4.3 and provide useful estimates on the extracted linearized equations in Section 4.4, while the completion

of the proof of Theorem 4.5.1 is achieved in Section 4.5.

A crucial ingredient in our energy estimates, dictating our choice of the energy functional, is the following decomposition

$$\rho\Gamma = \rho_1(\text{TS})^t\text{TS} + \text{S}^t\text{CS} \quad (4.2.2)$$

(recall Section 4.1 for the definition of the matrices C, S and T). This decomposition mimics an analogous one of the operator

$$\varrho\mathcal{M} : h \mapsto \rho_{\text{surf}} \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} h(\varrho') d\varrho' + \int_{\rho_{\text{surf}}}^{\varrho} \int_{\varrho'}^{\rho_{\text{bott}}} h(\varrho'') d\varrho'' d\varrho',$$

used by Bianchini and Duchêne in [13] to obtain the analogous well-posedness result for the continuously stratified system (2.2.11).

4.3 Quasilinearization

In this subsection we focus on linearizing system (4.2.1), this is done by applying the operators S, T, ∂_x , D_ρ to the equations so as to obtain linear equations satisfied by the derivatives of our unknowns H, U .

Once we obtain these equations, in the following Lemma (Lemma 4.3.1) we estimate the remainder terms in the $l^2(L_x^2)$ norm. The main difficulty is to obtain an upper bound of the remainder terms which has a regularity that doesn't exceed the regularity of the unknowns H, U found in (4.5.2).

Note that in point 1 of this Lemma the term $(\underline{H} + H)\partial_x\partial_x^\alpha U$ (respectively $\Gamma\partial_x\partial_x^\alpha H$) in equation (4.3.1) (respectively equation (4.3.2)) is not included in the remainder $R_{\alpha,0}$ (respectively $\mathbf{R}_{\alpha,0}$) this is due to the fact that we will use the structure of the system and take advantage of the appearing symmetry which will allow us to have some important compensations of these terms in the energy method (see Lemma 4.4.2), where we will control the variable $\mathbf{S}H$ in $l^2(L_x^2)$ norm. As for the other points of the Lemma where the operator D_ρ^j intervenes and in order to prepare the control of the variable H and higher derivatives of the variable $\mathbf{S}H$ (see energy (4.5.2)) we will view our system as a coupling of transport and transport-diffusion equations and where we will rely heavily on the effects of the κ diffusion when estimating as in the previous chapter. Finally we emphasize the

fact that the strategy of point **1** which is relying on the structure of the system and less on the κ diffusion will enable us to obtain a better time dependency (with respect to the constant κ) in the large time well-posedness result (see Theorem **4.5.1**).

Lemma 4.3.1. *Let $s \in \mathbb{N}$ such that $s > 2 + \frac{1}{2}$, and $M, \underline{M}, h_* > 0$. There exists $C = C(s, M, \underline{M}, h_*) > 0$, such that for any $N \in \mathbb{N}^*$, $\kappa > 0$ and any $(\underline{H}, \underline{U}) \in w^{2,\infty}$ such that*

$$|\boldsymbol{\rho}|_{L^\infty} + |\boldsymbol{\rho}^{-1}|_{L^\infty} + |\underline{H}|_{w^{2,\infty}} + |\mathbf{D}_\rho \underline{U}|_{w^{1,\infty}} \leq \underline{M},$$

and any $(H, U) \in C([0, T]; H^{s,2}(\mathbb{R}))$ solution to (4.2.1) with some $T > 0$ and satisfying

$$\|H(t, \cdot)\|_{H^{s-1,1}} + \|\mathbf{S}H(t, \cdot)\|_{H^{s,1}} + \|\mathbf{T}SH(t, \cdot)\|_{H^{s,0}} + \|U(t, \cdot)\|_{H^{s,2}} + \kappa^{\frac{1}{2}} \|H(t, \cdot)\|_{H^{s,2}} \leq M$$

for all $t \in [0, T]$ and

$$\inf_{(t,x,i) \in (0,T) \times \mathbb{R} \times \{1, \dots, N\}} \frac{H_i}{H_i} + H_i(t, x) \geq h_*, \quad \text{the following holds.}$$

1. For all $\alpha \in \mathbb{N}$ with $0 \leq \alpha \leq s$, we have

$$\partial_t \partial_x^\alpha H + (\underline{U} + U) \partial_x \partial_x^\alpha H + (\underline{H} + H) \partial_x \partial_x^\alpha U = \kappa \partial_x^2 \partial_x^\alpha H + \mathbf{R}_{\alpha,0} \quad (4.3.1)$$

$$\partial_t \partial_x^\alpha U + \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) \partial_x \partial_x^\alpha U + \Gamma \partial_x \partial_x^\alpha H = \mathbf{R}_{\alpha,0} \quad (4.3.2)$$

where for every $t \in [0, T]$, $(\mathbf{S}R_{\alpha,0}(t, \cdot), \mathbf{T}SR_{\alpha,0}(t, \cdot), \mathbf{R}_{\alpha,0}(t, \cdot)) \in l^2(L^2(\mathbb{R}))^2 \times l^2(L^2(\mathbb{R}))$ and

$$\|\mathbf{S}R_{\alpha,0}(t, \cdot)\|_{l^2(L^2_x)} + \|\mathbf{T}SR_{\alpha,0}(t, \cdot)\|_{l^2(L^2_x)} + \|\mathbf{R}_{\alpha,0}(t, \cdot)\|_{l^2(L^2_x)} \leq CM(1 + \kappa \|\partial_x H\|_{H^{s,2}}).$$

2. For all $\alpha \in \mathbb{N}$, $j \in \{0, 1\}$ with $0 \leq \alpha \leq s - 1 - j$, we have

$$\partial_t \partial_x^\alpha \mathbf{D}_\rho^j H + (\mathbf{M}^j(\underline{U} + U)) \partial_x \partial_x^\alpha \mathbf{D}_\rho^j H = \kappa \partial_x^2 \partial_x^\alpha \mathbf{D}_\rho^j H + \mathcal{R}_{\alpha,j} \quad (4.3.3)$$

where for every $t \in [0, T]$, $\mathcal{R}_{\alpha,j}(t, \cdot) \in l^2(L^2(\mathbb{R}))$ and

$$\|\mathcal{R}_{\alpha,j}(t, \cdot)\|_{l^2(L^2_x)} \leq CM.$$

3. For any $\alpha \in \mathbb{N}$, $j \in \{1, 2\}$, such that $0 \leq \alpha \leq s - j$, it holds

$$\partial_t \mathbf{D}_\rho^j \mathbf{S} \partial_x^\alpha H + \mathbf{R}_d(\mathbf{M}^{j-1}(\underline{U} + U)) \partial_x (\mathbf{D}_\rho^j \mathbf{S} \partial_x^\alpha H) = \kappa \partial_x^2 (\mathbf{D}_\rho^j \mathbf{S} \partial_x^\alpha H) + R_{\alpha,j} \quad (4.3.4)$$

$$\partial_t \mathbf{D}_\rho^j \partial_x^\alpha U + \left(\mathbf{M}^j \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) \right) \partial_x (\mathbf{D}_\rho^j \partial_x^\alpha U) = \mathbf{R}_{\alpha,j} \quad (4.3.5)$$

where for every $t \in [0, T]$, $(R_{\alpha,j}(t, \cdot), \mathbf{R}_{\alpha,j}(t, \cdot)) \in l^2(L^2(\mathbb{R})) \times l^2(L^2(\mathbb{R}))$ and

$$\|R_{\alpha,j}(t, \cdot)\|_{l^2(L_x^2)} + \|\mathbf{R}_{\alpha,j}(t, \cdot)\|_{l^2(L_x^2)} \leq CM(1 + \kappa \|\partial_x H\|_{H^{s,2}}).$$

4. For any $\alpha \in \mathbb{N}$, $j \in \{0, 1, 2\}$, $0 \leq \alpha \leq s - j$, it holds

$$\partial_t \mathbf{D}_\rho^j \partial_x^\alpha H + (\mathbf{M}^j(\underline{U} + U)) \partial_x (\mathbf{D}_\rho^j \partial_x^\alpha H) = \kappa \partial_x^2 (\mathbf{D}_\rho^j \partial_x^\alpha H) + r_{\alpha,j} + \partial_x \mathbf{r}_{\alpha,j} \quad (4.3.6)$$

where for every $t \in [0, T]$, $(r_{\alpha,j}(t, \cdot), \mathbf{r}_{\alpha,j}(t, \cdot)) \in l^2(L^2(\mathbb{R})) \times l^2(L^2(\mathbb{R}))$ and

$$\kappa^{\frac{1}{2}} \|r_{\alpha,j}(t, \cdot)\|_{l^2(L_x^2)} + \|\mathbf{r}_{\alpha,j}(t, \cdot)\|_{l^2(L_x^2)} \leq CM.$$

Proof.

In this proof we remind the reader that we intensively and without mentioning use the definition of the commutators found in Section 7.1 as well as other notations for our estimates found in this section as well.

Estimates for $\mathbf{S}R_{\alpha,0}$, $\mathbf{T}SR_{\alpha,0}$ with $0 \leq \alpha \leq s$.

$$\text{We have } R_{\alpha,0} = -[\partial_x^\alpha, U] \partial_x H - [\partial_x^\alpha, H] \partial_x U \text{ and } \mathbf{R}_{\alpha,0} = -\left[\partial_x^\alpha, \underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right] \partial_x U.$$

We notice that

$$\mathbf{S}R_{\alpha,0} = -\mathbf{S}([\partial_x^\alpha; U, \partial_x H] + (\partial_x^\alpha U)(\partial_x H)) - \mathbf{S}([\partial_x^\alpha; H, \partial_x U] + (\partial_x^\alpha H)(\partial_x U)).$$

On the other hand using Lemma 7.2.1 and since \mathbf{S}_0 , \mathbf{S} , \mathbf{R}_u , \mathbf{D}_ρ commute with ∂_x we have

$$\begin{aligned} -\mathbf{S}[\partial_x^\alpha; U, \partial_x H] &= -\left(\mathbf{S} \partial_x^\alpha (U \partial_x H) - \mathbf{S}(U \partial_x \partial_x^\alpha H) - \mathbf{S}((\partial_x^\alpha U)(\partial_x H)) \right) \\ &= -\left(\partial_x^\alpha (U \partial_x \mathbf{S}H) - \partial_x^\alpha (\mathbf{S}_0((\mathbf{D}_\rho(U))(\mathbf{R}_u \partial_x \mathbf{S}H))) - U \partial_x \mathbf{S} \partial_x^\alpha H \right. \\ &\quad \left. + \mathbf{S}_0((\mathbf{D}_\rho(U))(\mathbf{R}_u \partial_x \mathbf{S} \partial_x^\alpha H)) - (\partial_x^\alpha U)(\partial_x \mathbf{S}H) + \mathbf{S}_0((\mathbf{D}_\rho(\partial_x^\alpha U))(\mathbf{R}_u \partial_x \mathbf{S}H)) \right) \\ &= -[\partial_x^\alpha; U, \partial_x \mathbf{S}H] + \mathbf{S}_0[\partial_x^\alpha; \mathbf{D}_\rho(U), \mathbf{R}_u \partial_x \mathbf{S}H] \end{aligned}$$

and

$$-\mathbf{S}((\partial_x^\alpha H)(\partial_x U)) = -\left((\partial_x U)(\mathbf{S}\partial_x^\alpha H) - \mathbf{S}_0((\mathbf{D}_\rho(\partial_x U))(\mathbf{R}_u(\mathbf{S}\partial_x^\alpha H))) \right),$$

and finally we find

$$\begin{aligned} \mathbf{S}R_{\alpha,0} &= -[\partial_x^\alpha; U, \partial_x \mathbf{S}H] - (\partial_x U)(\mathbf{S}\partial_x^\alpha H) + \mathbf{S}_0\left([\partial_x^\alpha; \mathbf{D}_\rho(U), \mathbf{R}_u \partial_x \mathbf{S}H] + (\mathbf{D}_\rho(\partial_x U))(\mathbf{R}_u \mathbf{S}\partial_x^\alpha H)\right) \\ &\quad - \mathbf{S}\left((\partial_x^\alpha U)(\partial_x H) + [\partial_x^\alpha; H, \partial_x U]\right). \end{aligned}$$

Since $\sqrt{N}\mathbf{P}[\partial_x^\alpha; U, \partial_x \mathbf{S}H] = [\partial_x^\alpha; U, \partial_x \mathbf{T}SH]$, we immediately recover a similar equation for the trace case

$$\begin{aligned} \mathbf{T}SR_{\alpha,0} &= -[\partial_x^\alpha; U, \partial_x \mathbf{T}SH] - (\partial_x U)(\mathbf{T}\mathbf{S}\partial_x^\alpha H) + \mathbf{T}\mathbf{S}_0\left([\partial_x^\alpha; \mathbf{D}_\rho(U), \mathbf{R}_u \partial_x \mathbf{S}H] + (\mathbf{D}_\rho(\partial_x U))(\mathbf{R}_u \mathbf{S}\partial_x^\alpha H)\right) \\ &\quad - \mathbf{T}\mathbf{S}\left((\partial_x^\alpha U)(\partial_x H) + [\partial_x^\alpha; H, \partial_x U]\right). \end{aligned}$$

Using Lemma 7.2.3 and since $s > \frac{1}{2} + \frac{3}{2}$, we have

$$\|(\partial_x U)(\mathbf{S}\partial_x^\alpha H)\|_{l^2(L_x^2)} \leq \|\partial_x U\|_{l^\infty(L_x^\infty)} \|\mathbf{S}\Lambda^s H\|_{l^2(L_x^2)} \lesssim \|\mathbf{S}H\|_{H^{s,0}} \|U\|_{H^{s,1}},$$

and

$$\|(\partial_x U)(\mathbf{T}\mathbf{S}\partial_x^\alpha H)\|_{l^2(L_x^2)} \leq \|\partial_x U\|_{l^\infty(L_x^\infty)} \|\mathbf{T}\mathbf{S}\Lambda^s H\|_{l^2(L_x^2)} \lesssim \|\mathbf{T}\mathbf{S}H\|_{H^{s,0}} \|U\|_{H^{s,1}}.$$

Using Lemma 7.2.5 (6) and since $s - 2 > \frac{1}{2}$ we obtain

$$\begin{aligned} \|[\partial_x^\alpha; U, \partial_x \mathbf{S}H]\|_{l^2(L_x^2)} &\lesssim \|U\|_{H^{\max(s-2+\frac{3}{2}, s-\frac{1}{2}), 1}} \|\partial_x \mathbf{S}H\|_{H^{\max(s-1, s-2+1), 0}} \\ &\lesssim \|U\|_{H^{s,1}} \|\mathbf{S}H\|_{H^{s,0}}, \end{aligned}$$

and

$$\begin{aligned} \|[\partial_x^\alpha; U, \partial_x \mathbf{T}SH]\|_{l^2(L_x^2)} &\lesssim \|U\|_{H^{\max(s-2+\frac{3}{2}, s-\frac{1}{2}), 1}} \|\partial_x \mathbf{T}SH\|_{H^{\max(s-1, s-2+1), 0}} \\ &\lesssim \|U\|_{H^{s,1}} \|\mathbf{T}SH\|_{H^{s,0}}. \end{aligned}$$

Moreover using Lemma 7.2.2, and estimate (7.2.5) and since $s - 1 > \frac{1}{2} + 1$ we have

$$\begin{aligned} \left\| \left[\partial_x^\alpha; \mathbf{D}_\rho(U), \mathbf{R}_u \partial_x \mathbf{S}H \right] \right\|_{l^1} \left\|_{L_x^2} \right\| &\lesssim \left\| \Lambda^{s-1}(\mathbf{D}_\rho(U)) \right\|_{l^2(L_x^2)} \left\| \Lambda^{s-2+1}(\partial_x \mathbf{S}H) \right\|_{l^2(L_x^2)} \\ &\quad + \left\| \Lambda^{s-2+1}(\mathbf{D}_\rho(U)) \right\|_{l^2(L_x^2)} \left\| \Lambda^{s-1}(\partial_x \mathbf{S}H) \right\|_{l^2(L_x^2)} \\ &\lesssim \|U\|_{H^{s,1}} \|\mathbf{S}H\|_{H^{s,0}}, \end{aligned}$$

and

$$\begin{aligned} \left\| \left[\partial_x^\alpha; H, \partial_x U \right] \right\|_{l^1} \left\|_{L_x^2} \right\| &\lesssim \left\| \Lambda^{s-1} H \right\|_{l^2(L_x^2)} \left\| \Lambda^{s-2+1}(\partial_x U) \right\|_{l^2(L_x^2)} \\ &\quad + \left\| \Lambda^{s-1}(\partial_x U) \right\|_{l^2(L_x^2)} \left\| \Lambda^{s-2+1} H \right\|_{l^2(L_x^2)} \\ &\lesssim \|H\|_{H^{s-1,0}} \|U\|_{H^{s,0}}. \end{aligned}$$

Using Lemma 7.2.2 and classical product estimates in Sobolev spaces and the fact that and since $s - 1 > \frac{1}{2} + 1$ we have

$$\begin{aligned} \left\| (\partial_x^\alpha U)(\partial_x H) \right\|_{l^1} \left\|_{L_x^2} \right\| &\lesssim \left\| \Lambda^s(U) \right\|_{l^2(L_x^2)} \left\| \Lambda^{s-2}(\partial_x H) \right\|_{l^2(L_x^2)} \\ &\lesssim \|U\|_{H^{s,0}} \|H\|_{H^{s-1,0}}, \end{aligned}$$

and

$$\begin{aligned} \left\| (\mathbf{D}_\rho(\partial_x U))(\mathbf{R}_u \mathbf{S} \partial_x^\alpha H) \right\|_{l^1} \left\|_{L_x^2} \right\| &\lesssim \left\| \Lambda^{s-2+1}(\mathbf{D}_\rho U) \right\|_{l^2(L_x^2)} \left\| \Lambda^s \mathbf{S}H \right\|_{l^2(L_x^2)} \\ &\lesssim \|U\|_{H^{s,1}} \|\mathbf{S}H\|_{H^{s,0}}. \end{aligned}$$

Using the previously established estimates and the fact that $\|\mathbf{S}_0\|_{l^1 \rightarrow l^2} \leq 1$, $\|\mathbf{TS}_0\|_{l^1 \rightarrow l^2} \leq 1$, $\|\mathbf{S}\|_{l^1 \rightarrow l^2} \leq 1$ (Lemma 7.2.2), we have

$$\left\| \mathbf{S}R_{\alpha,0} \right\|_{l^2(L_x^2)} + \left\| \mathbf{TS}R_{\alpha,0} \right\|_{l^2(L_x^2)} \leq (\|H\|_{H^{s-1,0}} + \|\mathbf{S}H\|_{H^{s,0}} + \|\mathbf{TS}H\|_{H^{s,0}}) \|U\|_{H^{s,1}}. \quad (4.3.7)$$

Estimate for $\mathbf{R}_{\alpha,0}$ with $0 \leq \alpha \leq s$.

$$\mathbf{R}_{\alpha,0} = -[\partial_x^\alpha, U] \partial_x U + \kappa \frac{1}{\underline{H} + H} [\partial_x^\alpha, \partial_x H] \partial_x U + \kappa \left[\partial_x^\alpha, \frac{1}{\underline{H} + H} \right] (\partial_x H)(\partial_x U).$$

Using Lemma 7.2.5 (4) and since $s - \frac{3}{2} > \frac{1}{2}$, we have

$$\begin{aligned} \|[\partial_x^\alpha, U]\partial_x U\|_{l^2(L_x^2)} &\lesssim \|\partial_x U\|_{H^{s-1,1}} \|\partial_x U\|_{H^{s-1,1}} \\ &\lesssim \|U\|_{H^{s,1}}^2. \end{aligned}$$

Using Lemma 7.2.5 (4) and since $s - \frac{3}{2} > \frac{1}{2}$, we have

$$\begin{aligned} \left\| \kappa \left[\partial_x^\alpha, \frac{1}{\underline{H} + H} \right] (\partial_x H)(\partial_x U) \right\|_{l^2(L_x^2)} &\lesssim \kappa \left(\left\| \Lambda^{s-\frac{3}{2}} \left(\partial_x \frac{1}{\underline{H} + H} \right) \right\|_{l^\infty(L_x^2)} \|\Lambda^{s-1}((\partial_x H)(\partial_x U))\|_{l^2(L_x^2)} \right. \\ &\quad \left. + \|\Lambda^{s-\frac{3}{2}}((\partial_x H)(\partial_x U))\|_{l^\infty(L_x^2)} \left\| \Lambda^{s-1} \left(\partial_x \frac{1}{\underline{H} + H} \right) \right\|_{l^2(L_x^2)} \right). \end{aligned}$$

Moreover one has for any $s_0 > \frac{1}{2}$, $\sigma \geq 0$ and any $i \in \{1, \dots, N\}$, using classical product estimates and composition estimates in Sobolev spaces, and recalling the notations found in Section 7.1, we have

$$\begin{aligned} \left| \partial_x \frac{1}{\underline{H}_i + H_i} \right|_{H_x^\sigma} &= \left| \frac{\partial_x H_i}{(\underline{H}_i + H_i)^2} \right|_{H_x^\sigma} \leq \left| \frac{\partial_x H_i}{\underline{H}_i^2} \right|_{H_x^\sigma} + \left| \left(\frac{1}{\underline{H}_i^2} - \frac{1}{(\underline{H}_i + H_i)^2} \right) \partial_x H_i \right|_{H_x^\sigma} \\ &\lesssim (h_*^{-1})^2 |\partial_x H_i|_{H_x^\sigma} + \left| \left(\frac{1}{\underline{H}_i^2} - \frac{1}{(\underline{H}_i + H_i)^2} \right) \right|_{H_x^{s_0}} |\partial_x H_i|_{H_x^\sigma} \\ &\quad + \left\langle \left| \left(\frac{1}{\underline{H}_i^2} - \frac{1}{(\underline{H}_i + H_i)^2} \right) \right|_{H_x^\sigma} |\partial_x H_i|_{H_x^{s_0}} \right\rangle_{\sigma > s_0} \\ &\leq C (h_*^{-1}, \|H_i\|_{H_x^{s_0}}) \|\partial_x H_i\|_{H_x^\sigma}, \end{aligned}$$

and therefore

$$\left\| \Lambda^{s-\frac{3}{2}} \left(\partial_x \frac{1}{\underline{H} + H} \right) \right\|_{l^\infty(L_x^2)} \lesssim C \left(h_*^{-1}, \|\Lambda^{s-\frac{3}{2}} H\|_{l^\infty(L_x^2)} \right) \|\Lambda^{s-\frac{3}{2}} \partial_x H\|_{l^\infty(L_x^2)},$$

and

$$\left\| \Lambda^{s-1} \left(\partial_x \frac{1}{\underline{H} + H} \right) \right\|_{l^2(L_x^2)} \lesssim C \left(h_*^{-1}, \|\Lambda^{s-\frac{3}{2}} H\|_{l^\infty(L_x^2)} \right) \|\Lambda^{s-1} \partial_x H\|_{l^2(L_x^2)}.$$

Using classical product estimates in Sobolev spaces we have

$$\|\Lambda^{s-\frac{3}{2}}((\partial_x H)(\partial_x U))\|_{l^\infty(L_x^2)} \lesssim \|\partial_x \Lambda^{s-\frac{3}{2}} H\|_{l^\infty(L_x^2)} \|\Lambda^{s-\frac{3}{2}} \partial_x U\|_{l^\infty(L_x^2)} \lesssim \|H\|_{H^{s,1}} \|U\|_{H^{s,1}}.$$

By Lemma 7.2.5 ((2) and (4)) and since $s - 1 > \frac{1}{2} + \frac{1}{2}$

$$\|\Lambda^{s-1}((\partial_x H)(\partial_x U))\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s,1}} \|U\|_{H^{s,1}}.$$

and

$$\begin{aligned} \left\| \kappa \frac{1}{\underline{H} + H} [\partial_x^\alpha, \partial_x H] \partial_x U \right\|_{l^2(L_x^2)} &\lesssim \kappa h_*^{-1} (\|\partial_x^2 H\|_{H^{s-1,1}} \|\partial_x U\|_{H^{s-1,1}}) \\ &\lesssim \kappa h_*^{-1} (\|\partial_x H\|_{H^{s,1}} \|U\|_{H^{s,1}}). \end{aligned}$$

Consequently, using Lemma 7.2.3,

$$\|\mathbf{R}_{\alpha,0}\|_{l^2(L_x^2)} \lesssim \|U\|_{H^{s,1}}^2 + \kappa C(h_*^{-1}, \|H\|_{H^{s-1,1}}) (\|H\|_{H^{s,1}}^2 + \|\partial_x H\|_{H^{s,1}}) \|U\|_{H^{s,1}}. \quad (4.3.8)$$

Estimates for $\mathcal{R}_{\alpha,0} = -[\partial_x^\alpha, \underline{U} + U] \partial_x H - \partial_x^\alpha ((\underline{H} + H) \partial_x U)$, with $0 \leq \alpha \leq s - 1$.

Since $s - 1 > \frac{1}{2} + \frac{1}{2}$, and using Lemma 7.2.5 ((2) and (3)) and Lemma 7.2.3 we have

$$\|\partial_x^\alpha (H \partial_x U)\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s-1,0}} \|U\|_{H^{s,1}} + \|H\|_{H^{s-1,1}} \|U\|_{H^{s,0}},$$

and

$$\|[\partial_x^\alpha, U] \partial_x H\|_{L^2(L_x^2)} \lesssim \|U\|_{H^{s,1}} \|H\|_{H^{s-1,0}},$$

moreover

$$\|\partial_x^\alpha (\underline{H} \partial_x U)\|_{l^2(L_x^2)} \lesssim |\underline{H}|_{l^\infty} \|U\|_{H^{s,0}}.$$

Hence

$$\|\mathcal{R}_{\alpha,0}\|_{l^2(L_x^2)} \lesssim (|\underline{H}|_{l^\infty} + \|H\|_{H^{s-1,1}}) \|U\|_{H^{s,1}}. \quad (4.3.9)$$

Estimates for $\mathcal{R}_{\alpha,1} = -[\partial_x^\alpha \mathbf{D}_\rho, \underline{U} + U] \partial_x H - \partial_x^\alpha \mathbf{D}_\rho ((\underline{H} + H) \partial_x U)$, with $0 \leq \alpha \leq s - 2$.

By Lemma 7.2.1 we have

$$\begin{aligned} \partial_x^\alpha \mathbf{D}_\rho((\underline{H} + H)\partial_x U) &= \partial_x^\alpha((\mathbf{D}_\rho \underline{H})(\mathbf{M}\partial_x U)) + \partial_x^\alpha((\mathbf{D}_\rho H)(\mathbf{M}\partial_x U)) + \partial_x^\alpha((\mathbf{M}\underline{H})(\mathbf{D}_\rho \partial_x U)) \\ &\quad + \partial_x^\alpha((\mathbf{M}H)(\mathbf{D}_\rho \partial_x U)). \end{aligned}$$

Using Lemma 7.2.5 (1) and since $s - 2 > \frac{1}{2}$ we have

$$\begin{aligned} \|\partial_x^\alpha((\mathbf{D}_\rho H)(\mathbf{M}\partial_x U))\|_{l^2(L_x^2)} &\lesssim \|\mathbf{D}_\rho H\|_{H^{s-2,0}} \|\mathbf{M}\partial_x U\|_{H^{s-\frac{3}{2},1}} \\ &\lesssim \|H\|_{H^{s-1,1}} \|U\|_{H^{s,1}}, \end{aligned}$$

and

$$\begin{aligned} \|\partial_x^\alpha((\mathbf{M}H)(\mathbf{D}_\rho \partial_x U))\|_{l^2(L_x^2)} &\lesssim \|\mathbf{M}H\|_{H^{s-\frac{3}{2},1}} \|\mathbf{D}_\rho \partial_x U\|_{H^{s-2,0}} \\ &\lesssim \|H\|_{H^{s-1,1}} \|U\|_{H^{s,1}}. \end{aligned}$$

Using Lemma 7.2.7 (2) and since $s - 2 > \frac{1}{2}$ we have

$$\|[\partial_x^\alpha \mathbf{D}_\rho, U]\partial_x H\|_{l^2(L_x^2)} \lesssim \|U\|_{H^{s,2}} \|H\|_{H^{s-1,1}},$$

once again with Lemma 7.2.1

$$\|[\partial_x^\alpha \mathbf{D}_\rho, \underline{U}]\partial_x H\|_{l^2(L_x^2)} = \|\partial_x^\alpha((\mathbf{D}_\rho \underline{U})(\mathbf{M}\partial_x H))\|_{l^2(L_x^2)} \leq |\mathbf{D}_\rho \underline{U}|_{l^\infty} \|H\|_{H^{s-1,0}}.$$

We immediately have

$$\|\partial_x^\alpha((\mathbf{D}_\rho \underline{H})(\mathbf{M}\partial_x U))\|_{l^2(L_x^2)} \leq |\mathbf{D}_\rho \underline{H}|_{l^\infty} \|U\|_{H^{s-1,0}},$$

and

$$\|\partial_x^\alpha((\mathbf{M}\underline{H})(\mathbf{D}_\rho \partial_x U))\|_{l^2(L_x^2)} \leq |\underline{H}|_{l^\infty} \|U\|_{H^{s,1}}.$$

Hence

$$\|\mathcal{R}_{\alpha,1}\|_{l^2(L_x^2)} \lesssim (|\underline{H}|_{w^{1,\infty}} + \|H\|_{H^{s-1,1}}) \|U\|_{H^{s,2}} + |\mathbf{D}_\rho \underline{U}|_{l^\infty} \|H\|_{H^{s-1,0}}. \quad (4.3.10)$$

Estimates for $R_{\alpha,j}$ for $j \in \{1, 2\}$, with $0 \leq \alpha \leq s - j$.

For $j = 1$ using the fact that $D_\rho S = R_d$, we have

$$R_{\alpha,1} = R_d \left(-[\partial_x^{\alpha+1}, U]H - \partial_x^\alpha ((\underline{H})(\partial_x U)) \right).$$

Using Lemma 7.2.5 (4) and since $s - 1 > \frac{1}{2} + 1$ we have

$$\begin{aligned} \|[\partial_x^{\alpha+1}, U]H\|_{l^2(L_x^2)} &\lesssim \|\partial_x U\|_{H^{s-1,1}} \|H\|_{H^{s-1,1}} \\ &\lesssim \|U\|_{H^{s,1}} \|H\|_{H^{s-1,1}}. \end{aligned}$$

and

$$\|\partial_x^\alpha ((\underline{H})(\partial_x U))\|_{l^2(L_x^2)} \leq \|\underline{H}\|_{l^\infty} \|U\|_{H^{s,0}}.$$

For $j = 2$, using Lemma 7.2.1 and the fact that $D_\rho^2 S = R_d D_\rho$, we have

$$R_{\alpha,2} = R_d \left(-\llbracket \partial_x^\alpha D_\rho, U + U \rrbracket \partial_x H - \partial_x^\alpha D_\rho ((\underline{H} + H)\partial_x U) \right).$$

Using Lemma 7.2.7 (2), Lemma 7.2.6 (1) and since $s - 2 > \frac{1}{2}$, we have

$$\|\llbracket \partial_x^\alpha D_\rho, U \rrbracket \partial_x H\|_{l^2(L_x^2)} \lesssim \|U\|_{H^{s,2}} \|H\|_{H^{s-1,1}},$$

and

$$\|\partial_x^\alpha D_\rho ((\underline{H})(\partial_x U))\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s-1,1}} \|U\|_{H^{s,1}}.$$

Moreover we have

$$\|\llbracket \partial_x^\alpha D_\rho, \underline{U} \rrbracket \partial_x H\|_{l^2(L_x^2)} \leq \|D_\rho \underline{U}\|_{l^\infty} \|H\|_{H^{s-1,0}},$$

and

$$\|\partial_x^\alpha D_\rho ((\underline{H})(\partial_x U))\|_{l^2(L_x^2)} \lesssim \|\underline{H}\|_{w^{1,\infty}} \|U\|_{H^{s,1}}.$$

Hence

$$\| R_{\alpha,j} \|_{l^2(L_x^2)} \lesssim (\| H \|_{H^{s-1,1}} + | \underline{H} |_{w^{1,\infty}} + | D_\rho \underline{U} |_{l^\infty}) (\| U \|_{H^{s,2}} + \| H \|_{H^{s-1,0}}). \quad (4.3.11)$$

Estimates for $\mathbf{R}_{\alpha,j} = - \left[\partial_x^\alpha D_\rho^j \underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right] \partial_x U - \partial_x^\alpha (D_\rho^j (\Gamma \partial_x H))$, with $j \in \{1, 2\}$ and $0 \leq \alpha \leq s - j$.

By means of Lemma 7.2.1 and Lemma 7.2.4, we easily infer

$$\left\| \frac{\partial_x H}{\underline{H}} \right\|_{H^{s,2}} \leq C(h_*^{-1}, | D_\rho \underline{H} |_{l^\infty}, | D_\rho^2 \underline{H} |_{l^\infty}) \| \partial_x H \|_{H^{s,2}}.$$

Since $s > \frac{3}{2} + \frac{1}{2}$, by Lemma 7.2.7 we have

$$\| [\partial_x^\alpha D_\rho^j, U] \partial_x U \|_{l^2(L_x^2)} \lesssim \| U \|_{H^{s,2}} \| \partial_x U \|_{H^{s-1,2}}.$$

By Lemma 7.2.1, we have

$$\| [\partial_x^\alpha D_\rho^j, U] \partial_x U \|_{l^2(L_x^2)} \lesssim (| D_\rho \underline{U} |_{l^\infty} + | D_\rho^2 \underline{U} |_{l^\infty}) \| U \|_{H^{s,1}}.$$

Moreover using Lemma 7.2.7, Lemma 7.2.6 (2) and Lemma 7.2.8 it follows that

$$\begin{aligned} \kappa \left\| \left[\partial_x^\alpha D_\rho^j, \frac{\partial_x H}{\underline{H} + H} \right] \partial_x U \right\|_{l^2(L_x^2)} &\lesssim \kappa \left\| \frac{\partial_x H}{\underline{H} + H} \right\|_{H^{s,2}} \| \partial_x U \|_{H^{s-1,2}} \\ &\lesssim \kappa \left\| \frac{\partial_x H}{\underline{H}} \right\|_{H^{s,2}} \| U \|_{H^{s,2}} + \kappa \| U \|_{H^{s,2}} \| \partial_x H \|_{H^{s,2}} \left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s-1,1}} \\ &\quad + \kappa \| U \|_{H^{s,2}} \| \partial_x H \|_{H^{s-1,1}} \left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s,2}} \\ &\lesssim \kappa C(h_*^{-1}, | D_\rho \underline{H} |_{w^{1,\infty}}) \| \partial_x H \|_{H^{s,2}} \| U \|_{H^{s,2}} \\ &\quad + \kappa \| U \|_{H^{s,2}} \| \partial_x H \|_{H^{s,2}} C(h_*^{-1}, \underline{M}, M) \| H \|_{H^{s-1,1}} \\ &\quad + \kappa \| U \|_{H^{s,2}} \| H \|_{H^{s,2}} C(h_*^{-1}, \underline{M}, M) \| H \|_{H^{s,2}}. \end{aligned}$$

Using (4.2.2) we have

$$\partial_x^\alpha (D_\rho^j (\Gamma \partial_x H)) = \partial_x^\alpha (D_\rho^j (\boldsymbol{\rho}_1 \boldsymbol{\rho}^{-1} (\text{TS})^t \partial_x \text{TSH})) + \partial_x^\alpha (D_\rho^j (\boldsymbol{\rho}^{-1} \text{S}^t \text{C} \partial_x \text{SH})),$$

consequently using Lemma 7.2.1 and the fact that $D_\rho \boldsymbol{\rho} = -(1 \cdots, 1)^t$, $D_\rho^2 \boldsymbol{\rho} = (0, \cdots, 0)^t$, $D_\rho(\text{TS})^t(\text{TS}) = 0$ and $D_\rho \mathbf{S} = \mathbf{R}_d$, $D_\rho \mathbf{S}^t = -\mathbf{R}_u$, one has

$$\begin{aligned} \|\partial_x^\alpha (D_\rho^j (\boldsymbol{\rho}_1 \boldsymbol{\rho}^{-1} (\text{TS})^t \partial_x \text{TS} H))\|_{l^2(L_x^2)} &\lesssim \boldsymbol{\rho}_1 C (|\boldsymbol{\rho}^{-1}|_{l^\infty}) \|\partial_x^{\alpha+1} \text{TS} H\|_{l^2(L_x^2)} \\ &\lesssim \boldsymbol{\rho}_1 C (|\boldsymbol{\rho}^{-1}|_{l^\infty}) \|\text{TS} H\|_{H^{s,0}} \end{aligned}$$

and

$$\begin{aligned} \|\partial_x^\alpha (D_\rho^j (\boldsymbol{\rho}^{-1} \mathbf{S}^t \mathbf{C} \partial_x \mathbf{S} H))\|_{l^2(L_x^2)} &\lesssim C (|\boldsymbol{\rho}^{-1}|_{l^\infty}) \|\partial_x^{\alpha+1} \mathbf{S} H\|_{l^2(L_x^2)} \\ &\lesssim C (|\boldsymbol{\rho}^{-1}|_{l^\infty}) \|\mathbf{S} H\|_{H^{s,1}}. \end{aligned}$$

Hence

$$\begin{aligned} \|\mathbf{R}_{\alpha,j}\|_{l^2(L_x^2)} &\lesssim C (\boldsymbol{\rho}_1, |\boldsymbol{\rho}^{-1}|_{l^\infty}) (\|\text{TS} H\|_{H^{s,0}} + \|\mathbf{S} H\|_{H^{s,0}}) + (|\mathbf{D}_\rho \underline{U}|_{w^{1,\infty}} + \|U\|_{H^{s,2}}) \|U\|_{H^{s,2}} \\ &\quad + \kappa C (h_*^{-1}, |\underline{H}|_{w^{2,\infty}}, \|H\|_{H^{s-1,1}}) (\|H\|_{H^{s,2}}^2 + \|\partial_x H\|_{H^{s,2}}) \|U\|_{H^{s,2}}. \end{aligned} \tag{4.3.12}$$

Estimates for $r_{\alpha,0} = -[\partial_x^{\alpha+1}; U, H] + (\partial_x^\alpha U) (\partial_x H)$, $\mathbf{r}_{\alpha,0} = -(\partial_x^\alpha U)(H) - \partial_x^\alpha (\underline{H}U)$, with $0 \leq \alpha \leq s$.

Using Lemma 7.2.5 with Lemma 7.2.3 and since $s - \frac{3}{2} > \frac{1}{2}$ we have

$$\|[\partial_x^{\alpha+1}; U, H]\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s,1}} \|U\|_{H^{s,1}},$$

and

$$\|(\partial_x^\alpha U) (\partial_x H)\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s,1}} \|U\|_{H^{s,0}},$$

and

$$\|(\partial_x^\alpha U)(H)\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s-1,1}} \|U\|_{H^{s,0}}.$$

Finally,

$$\|\partial_x^\alpha (\underline{H}U)\|_{l^2(L_x^2)} \lesssim |\underline{H}|_{l^\infty} \|U\|_{H^{s,0}}.$$

Estimates for $r_{\alpha,j}$, $\mathbf{r}_{\alpha,j}$ for $j \in \{1, 2\}$ such that $0 \leq \alpha \leq s - j$.

We have

$$\begin{aligned} r_{\alpha,j} &= -\llbracket \partial_x^{\alpha+1} \mathbf{D}_\rho^j; U, H \rrbracket + (\partial_x^\alpha \mathbf{D}_\rho^j U)(\mathbf{M}^j \partial_x H), \\ \mathbf{r}_{\alpha,j} &= -\llbracket \mathbf{D}_\rho^j, \underline{U} \rrbracket \partial_x^\alpha H - \partial_x^\alpha (\mathbf{D}_\rho^j (\underline{H}U)) - (\partial_x^\alpha \mathbf{D}_\rho^j U)(\mathbf{M}^j H). \end{aligned}$$

Using Lemma 7.2.1, we have

$$\|\llbracket \mathbf{D}_\rho^j, \underline{U} \rrbracket \partial_x^\alpha H\|_{l^2(L_x^2)} \lesssim |\mathbf{D}_\rho \underline{U}|_{l^\infty} \|H\|_{H^{s-1,1}} + |\mathbf{D}_\rho^2 \underline{U}|_{l^\infty} \|H\|_{H^{s-2,0}},$$

and

$$\|\partial_x^\alpha (\mathbf{D}_\rho^j (\underline{H}U))\|_{l^2(L_x^2)} \lesssim |\underline{H}|_{l^\infty} \|U\|_{H^{s,2}} + |\mathbf{D}_\rho \underline{H}|_{l^\infty} \|U\|_{H^{s-1,1}} + |\mathbf{D}_\rho^2 \underline{H}|_{l^\infty} \|U\|_{H^{s-2,0}}.$$

Moreover using Lemma 7.2.3 and since $s > \frac{3}{2} + \frac{1}{2}$ we have

$$\|(\partial_x^\alpha \mathbf{D}_\rho^j U)(\mathbf{M}^j H)\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s-1,1}} \|U\|_{H^{s,2}},$$

and

$$\|(\partial_x^\alpha \mathbf{D}_\rho^j U)(\mathbf{M}^j \partial_x H)\|_{l^2(L_x^2)} \lesssim \|H\|_{H^{s,1}} \|U\|_{H^{s,2}}.$$

Since $\alpha + 1 \leq s + 1 - j$ and $s + 1 \geq 2$, $s + 1 > \frac{5}{2} + \frac{1}{2}$ then by Lemma 7.2.7 (3)

$$\|\llbracket \partial_x^{\alpha+1} \mathbf{D}_\rho^j; U, H \rrbracket\|_{l^2(L_x^2)} \lesssim \|U\|_{H^{s,2}} \|H\|_{H^{s,2}}.$$

Hence for $j \in \{0, 1, 2\}$ such that $0 \leq \alpha \leq s - j$

$$\begin{aligned} \|r_{\alpha,j}\|_{l^2(L_x^2)} &\lesssim \|H\|_{H^{s,2}} \|U\|_{H^{s,2}}, \\ \|\mathbf{r}_{\alpha,j}\|_{l^2(L_x^2)} &\lesssim (|\underline{H}|_{w^{2,\infty}} + |\mathbf{D}_\rho \underline{U}|_{w^{1,\infty}} + \|H\|_{H^{s-1,1}})(\|U\|_{H^{s,2}} + \|H\|_{H^{s-1,1}}). \end{aligned} \tag{4.3.13}$$

Collecting estimates (4.3.9) – (4.3.13) the lemma is proved. \square

4.4 Energy estimates

In this section we give some energy estimates of linear equations arising in Lemma 4.3.1. We first recall some energy estimates of transport and transport diffusion equations:

$$\partial_t \dot{H} + \dot{U} \partial_x \dot{H} = \kappa \partial_x^2 \dot{H} + R + \partial_x \mathbf{R}. \quad (4.4.1)$$

$$\partial_t \dot{H} + \dot{U} \partial_x \dot{H} = R. \quad (4.4.2)$$

Lemma 4.4.1.

1. *There exists a universal constant $C_0 > 0$ such that for any $N \in \mathbb{N}^*$, $\kappa > 0$ and $T > 0$, for any $\dot{U} \in L^\infty(0, T; l^\infty(L_x^\infty(\mathbb{R})))$ with $\partial_x \dot{U} \in L^1(0, T; l^\infty(L_x^\infty(\mathbb{R})))$, for any $(R, \mathbf{R}) \in L^2(0, T; l^2(L_x^2(\mathbb{R})))$ and for any $\dot{H} \in C^0([0, T]; l^2(L_x^2(\mathbb{R})))$ with $\partial_x \dot{H} \in L^2(0, T; l^2(L_x^2(\mathbb{R})))$ such that (4.4.1) holds in $L^2(0, T; (H^{1,0}(\mathbb{R}))')$ we have*

$$\begin{aligned} & \|\dot{H}\|_{L^\infty(0, T; l^2(L_x^2))} + \kappa^{\frac{1}{2}} \|\partial_x \dot{H}\|_{L^2(0, T; l^2(L_x^2))} \\ & \leq C_0 \left(\|\dot{H}|_{t=0}\|_{l^2(L_x^2)} + \|R\|_{L^1(0, T; l^2(L_x^2))} + \kappa^{-\frac{1}{2}} \|\mathbf{R}\|_{L^2(0, T; l^2(L_x^2))} \right) \\ & \quad \exp \left(C_0 \|\partial_x \dot{U}\|_{L^1(0, T; l^\infty(L_x^\infty))} \right). \end{aligned} \quad (4.4.3)$$

2. *There exists a universal constant $C_0 > 0$ such that for any $N \in \mathbb{N}^*$, $T > 0$, and any $\dot{H} \in L^\infty(0, T; l^2(L_x^2(\mathbb{R})))$ solution of (4.4.2) with initial data $\dot{H}|_{t=0} \in l^2(L_x^2)$, with $R \in L^1(0, T; l^2(L_x^2(\mathbb{R})))$, we have*

$$\|\dot{H}\|_{L^\infty(0, T; l^2(L_x^2))} \leq \left(\|\dot{H}|_{t=0}\|_{l^2(L_x^2)} + \|R\|_{L^1(0, T; l^2(L_x^2))} \right) \exp \left(C_0 \|\partial_x \dot{U}\|_{L^1(0, T; l^\infty(L_x^\infty))} \right). \quad (4.4.4)$$

Proof. Let $i \in \{1, \dots, N\}$ fixed, then \dot{H}_i satisfies the following equation

$$\partial_t \dot{H}_i + \dot{U}_i \partial_x \dot{H}_i = \kappa \partial_x^2 \dot{H}_i + R_i + \partial_x \mathbf{R}_i.$$

It is standard (see Chapter 3 in [8]) that we have the estimate

$$\begin{aligned} & \|\dot{H}_i\|_{L^\infty(0,T;L_x^2)} + \kappa^{\frac{1}{2}} \|\partial_x \dot{H}_i\|_{L^2(0,T;L_x^2)} \\ & \leq C_0 \left(\|\dot{H}_i|_{t=0}\|_{L_x^2} + \|R_i\|_{L^1(0,T;L_x^2)} + \kappa^{-\frac{1}{2}} \|\mathbf{R}_i\|_{L^2(0,T;L_x^2)} \right) \\ & \quad \exp \left(C_0 \|\partial_x \dot{U}\|_{L^1(0,T;l^\infty(L_x^\infty))} \right). \end{aligned}$$

Taking the l^2 -norm of this estimate and using triangular inequality we infer (4.4.3).

In the same way (4.4.4) is obtained using standard energy estimates for the transport equation (see Th. 3.14 in [8]). □

We now consider system

$$\begin{aligned} \partial_t \dot{H} + (\underline{U} + U) \partial_x \dot{H} + (\underline{H} + H) \partial_x \dot{U} &= \kappa \partial_x^2 \dot{H} + R, \\ \partial_t \dot{U} + \Gamma \partial_x \dot{H} + \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) \partial_x \dot{U} &= \mathbf{R}, \end{aligned} \quad (4.4.5)$$

where we recall that Γ satisfies the decomposition (4.2.2).

The following Lemma provides an energy estimate for the previous system, the main challenge in this Lemma is to rely as minimum as possible on the diffusion (κ term) in order to obtain sharp estimates with respect to κ . We overcome this difficulty by relying on the structure of the system and using a thoughtfully picked symmetrizer which will enable us to take advantage of natural compensation that will then appear.

Lemma 4.4.2. *Let $h_*, M, \rho_1 > 0$. There exists $C = C(M, \rho_1, h_*) > 0$ (independent of N) such that for all $\kappa > 0$, $N \in \mathbb{N}^*$ and any $\underline{H}, \underline{U} \in \mathbb{R}^N$ and any $T > 0$ and (H, U) solution to the system (4.2.1) on $[0, T]$ with $(H, U) \in C([0, T]; H^{s,2}(\mathbb{R}^2))$, $H \in L^2(0, T; H^{s+1,2}(\mathbb{R}))$, satisfying for almost every $t \in [0, T]$ the upper bound*

$$\|\partial_x H(t, \cdot)\|_{L_x^\infty(L^2)} + \|\partial_x U(t, \cdot)\|_{l^\infty(L_x^\infty)} \leq M,$$

and the lower bound

$$\inf_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}} \underline{H}_i + H_i(t, x) \geq h_*, \quad (4.4.6)$$

and for any $(\dot{H}, \dot{U}) \in C([0, T]; l^2(L^2(\mathbb{R}^2)))$ with $\dot{H} \in L^2(0, T; l^2(H^1(\mathbb{R})))$ satisfying (4.4.5)

with $R, \mathbf{R} \in L^2(0, T; l^2(L^2(\mathbb{R})))$, the following estimate holds.

$$\begin{aligned} \mathcal{E}^{\frac{1}{2}}(\dot{H}, \dot{U})(t) + \frac{\kappa^{\frac{1}{2}}}{2} \|\partial_x \mathbf{CS}\dot{H}\|_{L^2(0,t;l^2(L_x^2))} + (\rho_1 \kappa)^{\frac{1}{2}} \|\partial_x \mathbf{TS}\dot{H}\|_{L^2(0,t;l^2(L_x^2))} \\ \leq \left(\mathcal{E}^{\frac{1}{2}}(\dot{H}, \dot{U})(0) + C \int_0^t \mathcal{E}^{\frac{1}{2}}(R, \mathbf{R})(\tau) d\tau \right) \\ \times \exp \left(C \int_0^t \left(1 + \kappa^{-1} \|\mathbf{D}_\rho(\underline{U} + U)(\tau, \cdot)\|_{L_x^\infty(l^2)}^2 \right) d\tau \right), \end{aligned}$$

where we denote

$$\mathcal{E}(\dot{H}, \dot{U})(t) := \frac{1}{2} \|\mathbf{CS}\dot{H}\|_{l^2(L_x^2)}^2 + \frac{\rho_1}{2} \|\mathbf{TS}\dot{H}\|_{l^2(L_x^2)}^2 + \frac{1}{2} \int_{\mathbb{R}} \langle \dot{U}, \rho(\underline{H} + H)\dot{U} \rangle_{l^2} dx.$$

Proof. In this proof we consider (\dot{H}, \dot{U}) to be a sufficiently regular solution to the system (4.4.5) so that the following computations and integration by parts hold true. The case of (\dot{H}, \dot{U}) with the regularity mentioned in the Lemma is then done by the classical argument of regularization and passing to the limit.

By applying CS to (4.4.5)₁ and then testing it against $\mathbf{CS}\dot{H}$, and using Abel's summation of Lemma 7.2.1 we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{CS}\dot{H}\|_{l^2(L_x^2)}^2 + \langle (\underline{U} + U) \partial_x \mathbf{CS}\dot{H}, \mathbf{CS}\dot{H} \rangle_{L_x^2 l^2} - \langle \mathbf{CS}_0 \left((\mathbf{D}_\rho(\underline{U} + U)) (\mathbf{R}_u(\partial_x \mathbf{S}\dot{H})) \right), \mathbf{CS}\dot{H} \rangle_{L_x^2 l^2} \\ + \langle \mathbf{CS} \left((\underline{H} + H) \partial_x \dot{U} \right), \mathbf{CS}\dot{H} \rangle_{L_x^2 l^2} = -\kappa \|\partial_x \mathbf{CS}\dot{H}\|_{l^2(L_x^2)}^2 + \langle \mathbf{CSR}, \mathbf{CS}\dot{H} \rangle_{L_x^2 l^2}. \end{aligned}$$

In the same way by applying TS to (4.4.5)₁ and then testing it against $\rho_1 \mathbf{TS}\dot{H}$ and using Abel's summation of Lemma 7.2.1 we obtain

$$\begin{aligned} \frac{\rho_1}{2} \frac{d}{dt} \|\mathbf{TS}\dot{H}\|_{l^2(L_x^2)}^2 + \rho_1 \kappa \|\partial_x \mathbf{TS}\dot{H}\|_{l^2(L_x^2)}^2 = -\rho_1 \langle \mathbf{T} \left((\underline{U} + U) \partial_x \mathbf{S}\dot{H} \right), \mathbf{TS}\dot{H} \rangle_{L_x^2 l^2} + \rho_1 \langle \mathbf{TSR}, \mathbf{TS}\dot{H} \rangle_{L_x^2 l^2} \\ + \rho_1 \langle \mathbf{TS}_0 \left((\mathbf{D}_\rho(\underline{U} + U)) (\mathbf{R}_u \partial_x \mathbf{S}\dot{H}) \right), \mathbf{TS}\dot{H} \rangle_{L_x^2 l^2} - \rho_1 \langle \mathbf{TS} \left((\underline{H} + H) \partial_x \dot{U} \right), \mathbf{TS}\dot{H} \rangle_{L_x^2 l^2}. \end{aligned}$$

By testing (4.4.5)₂ against $\rho(\underline{H} + H)\dot{U}$ and using the identity (4.2.2) and the fact that

(H, U) satisfies (4.2.1) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\langle \dot{U}, \rho(\underline{H} + H)\dot{U} \rangle_{L_x^2 l^2} \right) + \rho_1 \langle \text{TS} \partial_x \dot{H}, \text{TS} \left((\underline{H} + H)\dot{U} \right) \rangle_{L_x^2 l^2} + \langle \text{CS} \partial_x \dot{H}, \text{CS} \left((\underline{H} + H)\dot{U} \right) \rangle_{L_x^2 l^2} \\ = \langle \mathbf{R}, \rho(\underline{H} + H)\dot{U} \rangle_{L_x^2 l^2}. \end{aligned}$$

Collecting the above, $\mathcal{E}(\dot{H}, \dot{U})$ satisfies the following differential equation:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{H}, \dot{U})(t) + \kappa \|\partial_x \text{CS} \dot{H}\|_{l^2(L_x^2)}^2 + \rho_1 \kappa \|\partial_x \text{TS} \dot{H}\|_{l^2(L_x^2)}^2 \\ = - \langle (\underline{U} + U) \partial_x \text{CS} \dot{H}, \text{CS} \dot{H} \rangle_{L_x^2 l^2} - \rho_1 \langle \mathbf{T} \left((\underline{U} + U) \partial_x \text{S} \dot{H} \right), \text{TS} \dot{H} \rangle_{L_x^2 l^2} \end{aligned} \quad (4.4.7)$$

$$+ \langle \text{CS}_0 \left((\text{D}_\rho(\underline{U} + U))(\text{R}_u(\partial_x \text{S} \dot{H})) \right), \text{CS} \dot{H} \rangle_{L_x^2 l^2} + \rho_1 \langle \text{TS}_0 \left((\text{D}_\rho(\underline{U} + U))(\text{R}_u \partial_x \text{S} \dot{H}) \right), \text{TS} \dot{H} \rangle_{L_x^2 l^2} \quad (4.4.8)$$

$$- \rho_1 \langle \text{TS} \partial_x \dot{H}, \text{TS} \left((\underline{H} + H)\dot{U} \right) \rangle_{L_x^2 l^2} - \langle \text{CS} \partial_x \dot{H}, \text{CS} \left((\underline{H} + H)\dot{U} \right) \rangle_{L_x^2 l^2} \quad (4.4.9)$$

$$- \rho_1 \langle \text{TS} \left((\underline{H} + H) \partial_x \dot{U} \right), \text{TS} \dot{H} \rangle_{L_x^2 l^2} - \langle \text{CS} \left((\underline{H} + H) \partial_x \dot{U} \right), \text{CS} \dot{H} \rangle_{L_x^2 l^2} \quad (4.4.10)$$

$$+ \langle \text{CSR}, \text{CS} \dot{H} \rangle_{L^2 x l^2} + \langle \mathbf{R}, \rho(\underline{H} + H)\dot{U} \rangle_{L_x^2 l^2} + \rho_1 \langle \text{TSR}, \text{TS} \dot{H} \rangle_{L_x^2 l^2}. \quad (4.4.11)$$

Before estimating all the terms in the previous equality we notice that

$$\frac{1}{2} \|\text{CS} \dot{H}\|_{l^2(L_x^2)}^2 + \frac{\rho_1}{2} \|\text{TS} \dot{H}\|_{l^2(L_x^2)}^2 + \frac{\rho_1 h_*}{2} \|\dot{U}\|_{l^2(L_x^2)}^2 \leq \mathcal{E}(\dot{H}, \dot{U})(t).$$

Hence we can now begin the estimating process to establish a differential inequality.

We estimate the terms of (4.4.8) using Lemma 7.2.2 and Cauchy Schwarz inequality

$$\begin{aligned} \left| \langle \text{CS}_0 \left((\text{D}_\rho(\underline{U} + U))(\text{R}_u(\partial_x \text{S} \dot{H})) \right), \text{CS} \dot{H} \rangle_{L_x^2 l^2} \right| &\leq \int_{\mathbb{R}} |(\text{D}_\rho(\underline{U} + U))(\text{R}_u(\partial_x \text{S} \dot{H}))|_{l^1} |\text{CS} \dot{H}|_{l^2} dx \\ &\leq \|\text{D}_\rho(\underline{U} + U)\|_{L_x^\infty(l^2)} \|\partial_x \text{CS} \dot{H}\|_{l^2(L_x^2)} \|\text{CS} \dot{H}\|_{l^2(L_x^2)} \\ &\leq \frac{\kappa}{2} \|\partial_x \text{CS} \dot{H}\|_{l^2(L_x^2)}^2 + \kappa^{-1} \|\text{D}_\rho(\underline{U} + U)\|_{L_x^\infty(l^2)}^2 \mathcal{E}(\dot{H}, \dot{U})(t). \end{aligned}$$

In the same way we have

$$\begin{aligned} \left| \rho_1 \langle \text{TS}_0 \left((\text{D}_\rho(\underline{U} + U))(\text{R}_u \partial_x \text{S} \dot{H}) \right), \text{TS} \dot{H} \rangle_{L_x^2 l^2} \right| &\leq \frac{\kappa}{4} \|\partial_x \text{CS} \dot{H}\|_{l^2(L_x^2)}^2 \\ &\quad + 2\kappa^{-1} \rho_1 \|\text{D}_\rho(\underline{U} + U)\|_{L_x^\infty(l^2)}^2 \mathcal{E}(\dot{H}, \dot{U})(t). \end{aligned}$$

We estimate now the terms in (4.4.7). Using integration by parts, we have

$$-\langle (\underline{U} + U)\partial_x \text{CS}\dot{H}, \text{CS}\dot{H} \rangle_{L_x^2 l^2} = \frac{1}{2} \langle (\partial_x U)(\text{CS}\dot{H}), \text{CS}\dot{H} \rangle_{L_x^2 l^2},$$

and consequently

$$\left| -\langle (\underline{U} + U)\partial_x \text{CS}\dot{H}, \text{CS}\dot{H} \rangle_{L_x^2 l^2} \right| \leq \frac{1}{2} \|\partial_x U\|_{l^\infty(L_x^\infty)} \|\text{CS}\dot{H}\|_{l^2(L_x^2)}^2 \leq \|\partial_x U\|_{l^\infty(L_x^\infty)} \mathcal{E}(\dot{H}, \dot{U})(t).$$

In the same way

$$\left| -\rho_1 \langle \mathbb{T} \left((\underline{U} + U)\partial_x \text{S}\dot{H} \right), \text{TS}\dot{H} \rangle_{L_x^2 l^2} \right| \leq \frac{\rho_1}{2} \|\partial_x U\|_{l^\infty(L_x^\infty)} \|\text{TS}\dot{H}\|_{l^2(L_x^2)}^2 \leq \|\partial_x U\|_{l^\infty(L_x^\infty)} \mathcal{E}(\dot{H}, \dot{U})(t)$$

Let us now consider the terms (4.4.9) and (4.4.10). Integrating by parts it follows that

$$-\langle \text{CS} \left((\underline{H} + H)\partial_x \dot{U} \right), \text{CS}\dot{H} \rangle_{L_x^2 l^2} = \langle \text{CS} \left((\partial_x H)(\dot{U}) \right), \text{CS}\dot{H} \rangle_{L_x^2 l^2} + \langle \text{CS} \left((\underline{H} + H)\dot{U} \right), \partial_x \text{CS}\dot{H} \rangle_{L_x^2 l^2}$$

and

$$\begin{aligned} -\rho_1 \langle \text{TS} \left((\underline{H} + H)\partial_x \dot{U} \right), \text{TS}\dot{H} \rangle_{L_x^2 l^2} &= \rho_1 \langle \text{TS} \left((\partial_x H)(\dot{U}) \right), \text{TS}\dot{H} \rangle_{L_x^2 l^2} \\ &\quad + \rho_1 \langle \text{TS} \left((\underline{H} + H)\dot{U} \right), \partial_x \text{TS}\dot{H} \rangle_{L_x^2 l^2}. \end{aligned}$$

Hence,

$$(4.4.9) + (4.4.10) = \langle \text{CS} \left((\partial_x H)(\dot{U}) \right), \text{CS}\dot{H} \rangle_{L_x^2 l^2} + \rho_1 \langle \text{TS} \left((\partial_x H)(\dot{U}) \right), \text{TS}\dot{H} \rangle_{L_x^2 l^2}.$$

Using Lemma 7.2.2 it follows that

$$\begin{aligned} \left| \langle \text{CS} \left((\partial_x H)(\dot{U}) \right), \text{CS}\dot{H} \rangle_{L_x^2 l^2} \right| &\leq \int_{\mathbb{R}} |\partial_x H|_{l^2} |\dot{U}|_{l^2} |\text{CS}\dot{H}|_{l^2} \\ &\leq \|\partial_x H\|_{L_x^\infty(l^2)} \|\dot{U}\|_{l^2(L_x^2)} \|\text{CS}\dot{H}\|_{l^2(L_x^2)} dx \\ &\leq \frac{1}{\sqrt{h_* \rho_1}} \|\partial_x H\|_{L_x^\infty(l^2)} \mathcal{E}(\dot{H}, \dot{U})(t), \end{aligned}$$

and similarly,

$$\begin{aligned} \left| \rho_1 \langle \mathbf{TS} \left((\partial_x H)(\dot{U}) \right), \mathbf{TS}\dot{H} \rangle_{L_x^2 l^2} \right| &\leq \rho_1 \int_{\mathbb{R}} |\partial_x H|_{l^2} |\dot{U}|_{l^2} |\mathbf{TS}\dot{H}|_{l^2} dx \\ &\leq \frac{1}{\sqrt{h_*}} \|\partial_x H\|_{L_x^\infty(l^2)} \mathcal{E}(\dot{H}, \dot{U})(t). \end{aligned}$$

The terms of (4.4.11) are easily controlled using Cauchy-Schwarz inequality.

$$\begin{aligned} \langle \mathbf{CSR}, \mathbf{CS}\dot{H} \rangle_{L_x^2 l^2} + \langle \mathbf{R}, \rho(\underline{H} + H)\dot{U} \rangle_{L_x^2 l^2} + \rho_1 \langle \mathbf{TSR}, \mathbf{TS}\dot{H} \rangle_{L_x^2 l^2} \\ \leq \mathcal{E}(\dot{H}, \dot{U})^{\frac{1}{2}}(t) \mathcal{E}(R, \mathbf{R})^{\frac{1}{2}}(t). \end{aligned}$$

Gathering all the above estimates we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{H}, \dot{U})(t) + \frac{\kappa}{2} \|\partial_x \mathbf{CS}\dot{H}\|_{l^2(L_x^2)}^2 + \rho_1 \kappa \|\partial_x \mathbf{TS}\dot{H}\|_{l^2(L_x^2)}^2 \\ \leq \left(2(1 + \rho_1) \kappa^{-1} \|\mathbf{D}_\rho(\underline{U} + U)\|_{L_x^\infty(l^2)}^2 + 2 \|\partial_x U\|_{l^\infty(L_x^\infty)} + \left(\frac{1}{\sqrt{h_*}} + \frac{1}{h_* \rho_1} \right) \|\partial_x H\|_{L_x^\infty(l^2)} \right) \mathcal{E}(\dot{H}, \dot{U})(t) \\ + \mathcal{E}(\dot{H}, \dot{U})^{\frac{1}{2}}(t) \mathcal{E}(R, \mathbf{R})^{\frac{1}{2}}(t), \end{aligned}$$

and hence there exists $C = C(h_*^{-1}, \rho_1, M)$ such that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{H}, \dot{U})(t) + \frac{\kappa}{2} \|\partial_x \mathbf{CS}\dot{H}\|_{l^2(L_x^2)}^2 + \rho_1 \kappa \|\partial_x \mathbf{TS}\dot{H}\|_{l^2(L_x^2)}^2 \\ \leq C \left(1 + \kappa^{-1} \|\mathbf{D}_\rho(\underline{U} + U)\|_{L_x^\infty(l^2)}^2 \right) \mathcal{E}(\dot{H}, \dot{U})(t) + C \mathcal{E}(\dot{H}, \dot{U})^{\frac{1}{2}}(t) \mathcal{E}(R, \mathbf{R})^{\frac{1}{2}}(t). \end{aligned}$$

We conclude by using Gronwall's inequality. \square

4.5 Large time well-posedness of the multi-layer system

In this subsection we state and prove the main result of this chapter, that concerns the well-posedness of system (4.2.1) with a time of existence uniform with respect to the number of layers N .

Theorem 4.5.1. *Let $s \in \mathbb{N}$ be such that $s > 2 + \frac{1}{2}$, and $\underline{M}, M^*, h_*, h^* > 0$. Then, there exists $C > 0$ such that for any $N \in \mathbb{N}^*$ and any $\kappa > 0$,*

— for any $(\underline{\rho}, \underline{H}, \underline{U}) \in \mathbb{R}^{3N}$ such that

$$|\underline{\rho}|_{l^\infty} + |\underline{\rho}^{-1}|_{l^\infty} + |\underline{H}|_{w^{2,\infty}} + |\underline{U}|_{w^{2,\infty}} \leq \underline{M},$$

— for any initial data $(H_0, U_0) \in H^{s,2}(\mathbb{R})^2$ with

$$M_0 := \|H_0\|_{H^{s-1,1}} + \|SH_0\|_{H^{s,2}} + \|\text{TSH}_0\|_{H^{s,0}} + \|U_0\|_{H^{s,2}} + \kappa^{\frac{1}{2}} \|H_0\|_{H^{s,2}} \leq M^*$$

and

$$\inf_{(x,i) \in \mathbb{R} \times \{1, \dots, N\}} \underline{H}_i + (H_0)_i(x) \geq h_*, \quad \sup_{(x,i) \in \mathbb{R} \times \{1, \dots, N\}} \underline{H}_i + (H_0)_i(x) \leq h^*,$$

the following holds. Denoting

$$T^{-1} = C \left(1 + \kappa^{-1} (\|D_\rho \underline{U}\|_{l^2}^2 + M_0^2) \right), \quad (4.5.1)$$

there exists a unique strong solution $(H, U) \in C([0, T]; H^{s,2}(\mathbb{R})^2)$ to (4.2.1) with initial data $(H, U)|_{t=0} = (H_0, U_0)$. Moreover, $H \in L^2(0, T; H^{s+1,2}(\mathbb{R}))$ and one has, for any $t \in [0, T]$, the lower and upper bounds

$$\inf_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}^d} \underline{H}_i + H_i(t, x) \geq \frac{h_*}{2}, \quad \sup_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}^d} \underline{H}_i + H_i(t, x) \leq 2h^*,$$

and the estimate

$$\|(H, U)\|_s(t) \leq CM_0,$$

where we define

$$\|(H, U)\|_s(t) := \|H(t, \cdot)\|_{H^{s-1,1}} + \|SH(t, \cdot)\|_{H^{s,2}} + \|\text{TSH}(t, \cdot)\|_{H^{s,0}} + \|U(t, \cdot)\|_{H^{s,2}} + \kappa^{\frac{1}{2}} \|H(t, \cdot)\|_{H^{s,2}} \quad (4.5.2)$$

$$+ \kappa^{\frac{1}{2}} \|\partial_x H\|_{L^2(0,t;H^{s-1,1})} + \kappa^{\frac{1}{2}} \|\partial_x SH\|_{L^2(0,t;H^{s,2})} + \kappa^{\frac{1}{2}} \|\partial_x \text{TSH}\|_{L^2(0,t;H^{s,0})} + \kappa \|\partial_x H\|_{L^2(0,t;H^{s,2})}. \quad (4.5.3)$$

Proof. Let us denote $T^* \in (0, +\infty]$ the maximal time of existence and uniqueness of $(H, U) \in C([0, T^*]; H^{s,2}(\mathbb{R})^2)$, $H \in L^2(0, T^*; H^{s+1,2}(\mathbb{R}))$ as provided by Theorem 3.3.1, and

$$T_* = \sup \left\{ 0 < T < T^* : \forall t \in (0, T], \quad \frac{h_*}{2} \leq \underline{H}_i + H_i(t, x) \leq 2h^* \text{ and } \|(H, U)\|_s(t) \leq c_0 M_0 \right\},$$

where $c_0 > 1$ will be determined later on. By continuity in time of the solution we deduce that $T_* > 0$. Let $t \in (0, T_*)$. By Lemma 4.3.1 and Lemma 4.4.2 and using the fact that $|\mathbf{S}|_{l^2 \rightarrow l^2} \leq \frac{1}{\sqrt{N}} |\mathbf{TS}|_{l^2 \rightarrow l^2} + |\mathbf{CS}|_{l^2 \rightarrow l^2}$, we find that there exists $c > 1$ depending on ρ_1, ρ_N, h_*, h^* and $C > 0$ depending additionally on $\underline{M}, c_0 M_0$ such that

$$\begin{aligned} & \|\Lambda^s \mathbf{S}H(t, \cdot)\|_{l^2(L_x^2)} + \|\Lambda^s \mathbf{TSH}(t, \cdot)\|_{l^2(L_x^2)} + \|\Lambda^s U(t, \cdot)\|_{l^2(L_x^2)} \\ & + \kappa^{\frac{1}{2}} \|\partial_x \mathbf{S} \Lambda^s H\|_{L^2(0, t; l^2(L_x^2))} + \kappa^{\frac{1}{2}} \|\partial_x \Lambda^s \mathbf{TSH}\|_{L^2(0, t; l^2(L_x^2))} \\ & \leq c \left(\|\Lambda^s \mathbf{S}H_0\|_{l^2(L_x^2)} + \|\Lambda^s U_0\|_{l^2(L_x^2)} + \|\Lambda^s \mathbf{TSH}_0\|_{l^2(L_x^2)} + C c_0 M_0 (t + \sqrt{t}) \right) \\ & \quad \times \exp \left(C \int_0^t \left(1 + \kappa^{-1} \|\mathbf{D}_\rho(\underline{U} + U)(\tau, \cdot)\|_{L_x^\infty(l^2)}^2 \right) d\tau \right). \end{aligned} \tag{4.5.4}$$

For all $j \in \{1, 2\}$ and applying Lemma 4.3.1 and Lemma 4.4.1 (1) it follows that

$$\begin{aligned} & \|\mathbf{D}_\rho^j \mathbf{S}H(t, \cdot)\|_{H^{s-j, 0}} + \kappa^{\frac{1}{2}} \|\partial_x \mathbf{D}_\rho^j \mathbf{S}H\|_{L^2(0, t; H^{s-j, 0})} \\ & \leq C_0 \left(\|\mathbf{D}_\rho^j \mathbf{S}H_0\|_{H^{s-j, 0}} + C c_0 M_0 (t + \sqrt{t}) \right) \exp \left(C_0 \int_0^t \|\partial_x U(\tau, \cdot)\|_{l^\infty(L_x^\infty)} d\tau \right), \end{aligned} \tag{4.5.5}$$

and moreover using Lemma 4.4.1 (2) we have

$$\begin{aligned} & \|\mathbf{D}_\rho^j U(t, \cdot)\|_{H^{s-j, 0}} \leq \left(\|\mathbf{D}_\rho^j U_0\|_{H^{s-j, 0}} + C c_0 M_0 (t + \sqrt{t}) \right) \\ & \quad \times \exp \left(C_0 \int_0^t \left\| \partial_x \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) (\tau, \cdot) \right\|_{l^\infty(L_x^\infty)} d\tau \right). \end{aligned} \tag{4.5.6}$$

For $j \in \{0, 1\}$, and using Lemma 4.3.1 and Lemma 4.4.1 (1) we have

$$\begin{aligned} & \| \mathbf{D}_\rho^j H(t, \cdot) \|_{H^{s-1-j,0}} + \kappa^{\frac{1}{2}} \| \partial_x \mathbf{D}_\rho^j H \|_{L^2(0,t;H^{s-1-j,0})} \\ & \leq C_0 \left(\| \mathbf{D}_\rho^j H_0 \|_{H^{s-1-j,0}} + C c_0 M_0 t \right) \exp \left(C_0 \int_0^t \| \partial_x U(\tau, \cdot) \|_{l^\infty(L_x^\infty)} d\tau \right). \end{aligned}$$

Finally, for any $j \in \{0, 1, 2\}$ using Lemma 4.3.1 and Lemma 4.4.1 (1) we have

$$\begin{aligned} & \kappa^{\frac{1}{2}} \| \mathbf{D}_\rho^j H(t, \cdot) \|_{H^{s-j,0}} + \kappa \| \partial_x \mathbf{D}_\rho^j H \|_{L^2(0,t;H^{s-j,0})} \\ & \leq C_0 \left(\kappa^{\frac{1}{2}} \| \mathbf{D}_\rho^j H_0 \|_{H^{s-j,0}} + C c_0 M_0 (t + \sqrt{t}) \right) \exp \left(C_0 \int_0^t \| \partial_x U(\tau, \cdot) \|_{l^\infty(L_x^\infty)} d\tau \right). \end{aligned} \tag{4.5.7}$$

Since $s > 2 + \frac{1}{2}$, by using Lemma 7.2.3, and the continuous injection $\| \cdot \|_{l^\infty(L_x^\infty)} \leq \| \cdot \|_{H^{s-\frac{3}{2},1}}$ and the fact that

$$\| \partial_x U(\tau, \cdot) \|_{l^\infty(L_x^\infty)} \leq \| U(\tau, \cdot) \|_{H^{s-\frac{1}{2},1}},$$

and

$$\begin{aligned} \left\| \partial_x \left(\frac{\partial_x H}{\underline{H} + H} \right) (\tau, \cdot) \right\|_{l^\infty(L_x^\infty)} & \leq \left\| \left(\frac{\partial_x^2 H}{\underline{H} + H} \right) (\tau, \cdot) \right\|_{l^\infty(L_x^\infty)} + \left\| \left(\frac{\partial_x H}{\underline{H} + H} \right)^2 (\tau, \cdot) \right\|_{l^\infty(L_x^\infty)} \\ & \leq C (h_*^{-1}) \left(\| \partial_x H(\tau, \cdot) \|_{H^{s-\frac{1}{2},1}} + \| \partial_x H(\tau, \cdot) \|_{H^{s-\frac{3}{2},1}}^2 \right), \end{aligned}$$

we have then

$$\left\| \partial_x \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} (\tau, \cdot) \right) \right\|_{l^\infty(L_x^\infty)} \leq C (h_*^{-1}) \left(\| U(\tau, \cdot) \|_{H^{s,2}} + \kappa \| \partial_x H(\tau, \cdot) \|_{H^{s,2}} + \kappa \| H(\tau, \cdot) \|_{H^{s,2}}^2 \right)$$

and

$$\| \mathbf{D}_\rho(\underline{U} + U) \|_{L_x^\infty(l^2)}^2 \lesssim \| \mathbf{D}_\rho \underline{U} \|_{l^2}^2 + \| U \|_{H^{s,2}}^2 \leq \| \mathbf{D}_\rho \underline{U} \|_{l^2}^2 + (c_0 M_0)^2.$$

Hence gathering estimates (4.5.4) – (4.5.7) we find that

$$\| (H, U) \|_s(t) \leq c \left(M_0 + C c_0 M_0 (t + \sqrt{t}) \right) \exp \left(C \left(t + \sqrt{t} + \int_0^t (1 + \kappa^{-1} (\| \mathbf{D}_\rho \underline{U} \|_{l^2}^2 + (c_0 M_0)^2)) d\tau \right) \right),$$

where we recall that $c > 1$ depends on h_* and h^* , and $C > 0$ depends on $\underline{M}, h_*, h^*, c_0 M_0$. Hence choosing $c_0 = 2c$, we find that there exists $C_1 \geq 1$ depending only on $\underline{M}, M^*, h_*, h^*$ such that

$$t(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + M_0^2)) \leq C_1^{-1} \implies \|(H, U)\|_s(t) \leq \frac{3}{4}c_0 M_0.$$

Moreover we notice that since for all $i \in \{1, \dots, N\}$

$$\partial_t H_i = \kappa \partial_x^2 H_i + g_i, \quad \text{with} \quad g_i = \partial_x (H_i U_i + \underline{H}_i U_i + \underline{U}_i H_i),$$

by the positivity of the heat kernel we have

$$\begin{aligned} \inf_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}^d} H_i(t, x) &\geq \inf_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}^d} H_i(0, x) - \|g\|_{L^1(0,t; l^\infty(L^\infty))}, \\ \sup_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}^d} H_i(t, x) &\leq \sup_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}^d} H_i(0, x) + \|g\|_{L^1(0,t; l^\infty(L^\infty))}. \end{aligned}$$

By Lemma 7.2.3 and by Lemma 7.2.6

$$\begin{aligned} \|g\|_{l^\infty(L_x^\infty)} &\lesssim |\underline{H}|_{w^{1,\infty}} \|U\|_{H^{s,1}} + |\underline{U}|_{w^{1,\infty}} \|H\|_{H^{s,1}} + \|U\|_{H^{s,1}} \|H\|_{H^{s,1}} \\ &\leq C(\underline{M})(1 + \kappa^{-1} M_0^2). \end{aligned}$$

Hence augmenting C_1 if necessary we find that

$$t(1 + \kappa^{-1} M_0^2) \leq C_1^{-1} \implies \forall (i, x) \in \{1, \dots, N\} \times \mathbb{R}^d, \quad \frac{2}{3}h_* \leq \underline{H}_i + H_i(t, x) \leq \frac{3}{2}h^*.$$

Hence by continuity in time of the solution we find that for all $t \in (0, T_*)$ satisfying that $t(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + M_0^2)) \leq C_1^{-1}$, there exists $\delta > 0$ such that $[t - \delta, t + \delta] \subset (0, T_*)$. By a continuity argument we deduce that $T_* > (C_1(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + M_0^2)))^{-1}$, which completes the proof. \square

4.6 Conclusion

In this chapter, we showed the Large time well-posedness of the multilayer shallow water system. We have overcome the dependency on the number of layers N of the time of existence and uniqueness obtained in the previous chapter (see Theorem 3.3.1) by fully exploiting the structure of the equations, and the assumption of stable stratification. This

was achieved by relying less on the diffusion induced by $\kappa > 0$ and by taking advantage of the compensations resulting from the symmetry properties of system (3.3.2). Using the energy method, we obtained a time of existence and uniqueness uniform with respect to N with a better dependency on the parameter κ . Furthermore, this time is the discrete analogue of the one obtained in [13], within the continuous stratification framework.

APPROXIMATION OF THE CONTINUOUSLY STRATIFIED HYDROSTATIC SYSTEM BY THE MULTI-LAYER SYSTEM

This chapter is dedicated to the proof of our second main result, Theorem 5.4.1. We prove that considering a sufficiently regular solution of the hydrostatic continuously stratified system (5.2.1) satisfying the non-cavitation assumption and appropriate bounds, the solutions to the multi-layer systems (2.2.12) with suitably chosen densities ρ_i , reference depths H_i , background velocities U_i and initial data $(H_i, U_i)|_{t=0}$ are at a distance $\mathcal{O}(1/N^2)$ to the continuously stratified solution.

This convergence result is deduced from a consistency result obtained in Section 5.2, and on stability estimates derived in Section 5.3. The proof of Theorem 5.4.1 is then completed in Section 5.4.

5.1 Introduction

As seen in the previous chapter, through the following figure 5.1 we represent the choice we made to discretize a continuous streamline density with a piecewise constant function with equidistributed values (*ie throughout the chapter ρ_i satisfies (4.1.1)*), it is relevant to mention that this choice does not convey a restriction on admissible density profiles.

We shall map functions defined on $[\rho_{\text{surf}}, \rho_{\text{bott}}]$ (associated with the continuously stratified framework) to N -dimensional vectors (associated with the multi-layer framework) with the following operator:

$$P_N : \begin{array}{l} \mathcal{C}([\rho_{\text{surf}}, \rho_{\text{bott}}]) \\ f \end{array} \begin{array}{l} \rightarrow \\ \mapsto \end{array} \begin{array}{l} \mathbb{R}^N \\ (f(\rho_i))_{1 \leq i \leq N} \end{array}$$

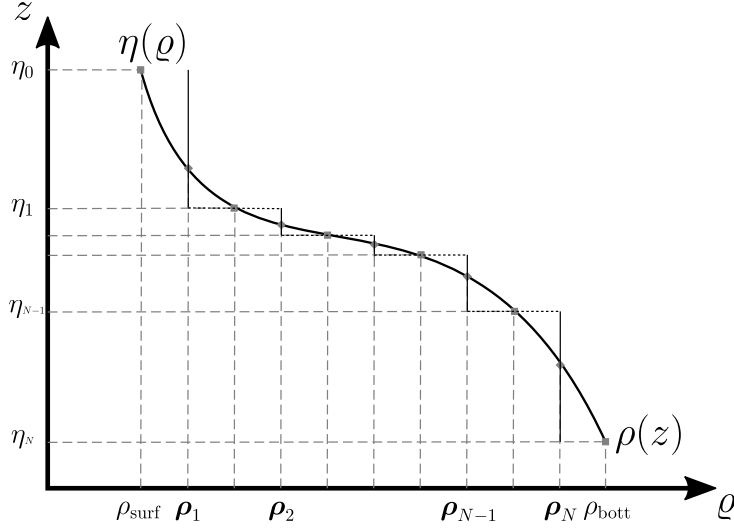


Figure 5.1 – A density profile and its piecewise constant approximation

Let us now describe some functional spaces we use in this chapter.

- In order to describe tensoried functional spaces for functions with variables in the strip $\Omega := \mathbb{R} \times (\rho_{\text{surf}}, \rho_{\text{bott}})$, we use the equivalent notations $H^s(\mathbb{R}) = H_x^s$ (the usual L^2 -based Sobolev space on \mathbb{R}) and $W^{k,\infty}(\mathbb{R}) = W_x^{k,\infty}$ (the L^∞ -based Sobolev space on \mathbb{R}), and similarly $L^2((\rho_{\text{surf}}, \rho_{\text{bott}})) = L_\rho^2$ and $W^{k,\infty}((\rho_{\text{surf}}, \rho_{\text{bott}})) = W_\rho^{k,\infty}$. We denote for instance

$$L_\rho^2 L_x^\infty = L^2(\rho_{\text{surf}}, \rho_{\text{bott}}; L^\infty(\mathbb{R})) = \{f \in L_{loc}^1(\Omega; \mathbb{R}) : \text{ess sup}_{x \in \mathbb{R}} |f(\cdot, x)| \in L^2((\rho_{\text{surf}}, \rho_{\text{bott}}))\}.$$

- Let $s \in \mathbb{R}$, $k \in \mathbb{N}$, where $k \leq s$, we define the functional space

$$X^{\infty,s,k}(\Omega) = X_{\rho,x}^{\infty,s,k} = \{f \in L_{loc}^1(\Omega; \mathbb{R}); \forall j \in \mathbb{N}, j \leq k, \partial_\rho^j f \in L_\rho^\infty(H_x^{s-j}(\mathbb{R}))\},$$

endowed with the topology of the norm

$$\|f\|_{\infty,s,k}^2 = \sum_{j=0}^k \|\partial_\rho^j f\|_{L_\rho^\infty(H_x^{s-j})}^2.$$

5.2 Consistency

Let $\rho_{\text{bott}} > \rho_{\text{surf}} > 0$ and $\kappa > 0$. Let $(\underline{h}, \underline{u})$ and (h, u) be a sufficiently smooth solution to the continuously stratified system

$$\begin{cases} \partial_t h + \partial_x((\underline{h} + h)(\underline{u} + u)) = \kappa \partial_x^2 h, \\ \partial_t u + \left(\underline{u} + u - \kappa \frac{\partial_x h}{\underline{h} + h} \right) \partial_x u + \frac{1}{\varrho} \mathcal{M} \partial_x h = 0, \\ h|_{t=0} = h_0, \\ u|_{t=0} = u_0, \end{cases} \quad (5.2.1)$$

where $(\mathcal{M}\eta)(\cdot, \varrho) = \rho_{\text{surf}} \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} \eta(\cdot, \varrho') d\varrho' + \int_{\rho_{\text{surf}}}^{\varrho} \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} \eta(\cdot, \varrho'') d\varrho'' d\varrho'$.

Let $N \in \mathbb{N}^*$ and recall (see Section 4.1) the definition of the linear operator P_N

$$P_N : \begin{array}{l} \mathcal{C}([\rho_{\text{surf}}, \rho_{\text{bott}}]) \\ f \end{array} \rightarrow \begin{array}{l} \mathbb{R}^N \\ \mapsto (f(\boldsymbol{\rho}_i))_{1 \leq i \leq N} \end{array}$$

where $\boldsymbol{\rho}_i = \rho_{\text{surf}} + (i - \frac{1}{2}) \frac{\rho_{\text{bott}} - \rho_{\text{surf}}}{N}$.

Notice that $P_N(fg) = P_N(f)P_N(g)$. Hence applying P_N to (5.2.1), we see that $(P_N h, P_N u)$ satisfy

$$\begin{cases} \partial_t P_N h + \partial_x(P_N(\underline{h} + h)P_N(\underline{u} + u)) = \kappa \partial_x^2 P_N h, \\ \partial_t P_N u + P_N \left(\underline{u} + u - \kappa \frac{\partial_x h}{\underline{h} + h} \right) \partial_x P_N u + \Gamma \partial_x P_N h = \mathbf{R}_N, \\ P_N h|_{t=0} = P_N(h_0), \\ P_N u|_{t=0} = P_N(u_0), \end{cases} \quad (5.2.2)$$

with $\Gamma_{i,j} = \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i}$ and

$$\mathbf{R}_N = \Gamma \partial_x P_N h - P_N \left(\frac{1}{\varrho} \mathcal{M} \partial_x h \right). \quad (5.2.3)$$

The following Lemma determines the size of \mathbf{R}_N previously defined, by providing an upper bound that depends on the variable h and vanishes when the number of layers N tends to infinity. The main challenge is to estimate carefully such that the latter property remains true, moreover imposing as minimal as possible regularity on h throughout the process of estimating.

Lemma 5.2.1. *For any $\rho_{\text{bott}} > \rho_{\text{surf}} > 0$, there exists $C > 0$ such that for any $N \in \mathbb{N}^*$,*

and any sufficiently regular function $h : \mathbb{R} \times (\rho_{\text{surf}}, \rho_{\text{bott}}) \rightarrow \mathbb{R}$, \mathbf{R}_N defined by (5.2.3) satisfies

$$\|\mathbf{R}_N\|_{H^{s,2}} \leq \frac{C}{N^2} \left(\|\partial_\varrho \partial_x \Lambda^s h\|_{L^\infty(L_x^2)} + \|\partial_\varrho^2 \partial_x \Lambda^s h\|_{L^\infty(L_x^2)} \right).$$

Proof.

We recall that the densities ρ_i are defined for all $i \in \{1, 2, \dots, N\}$ as $\rho_i = \rho_{\text{surf}} + \frac{i-1}{N}(\rho_{\text{bott}} - \rho_{\text{surf}})$, and in this proof we extend this definition to $i \in \{\frac{1}{2}, \frac{3}{2}, \dots, N + \frac{1}{2}\}$. In this proof, we denote by $C(\rho_{\text{surf}}, \rho_{\text{bott}}) > 0$ a constant that depends only on ρ_{surf} and ρ_{bott} and which will increase if necessary throughout the proof. For $i \in \{1, \dots, N\}$ fixed and for almost every $(t, x) \in [0, T] \times \mathbb{R}$ with T the time of existence and uniqueness of the solution (h, u) , integration by parts yields

$$\begin{aligned} (\mathcal{M}\partial_x h)(x, \rho_i) &= \rho_{\text{surf}} \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} \partial_x h(x, \varrho) d\varrho + \int_{\rho_{\text{surf}}}^{\rho_i} \int_{\varrho}^{\rho_{\text{bott}}} \partial_x h(x, \varrho') d\varrho' d\varrho \\ &= \int_{\rho_{\text{surf}}}^{\rho_i} \varrho \partial_x h(x, \varrho) d\varrho + \rho_i \int_{\rho_i}^{\rho_{\text{bott}}} \partial_x h(x, \varrho) d\varrho \\ &= \int_{\rho_{\text{surf}}}^{\rho_{\text{bott}}} \min(\varrho, \rho_i) \partial_x h(x, \varrho) d\varrho. \end{aligned}$$

This yields the following identities.

Firstly, one has

$$\begin{aligned} (\rho \Lambda^s \mathbf{R}_N)_i &= - \sum_{j=1}^N \int_{\rho_{j-\frac{1}{2}}}^{\rho_{j+\frac{1}{2}}} \min(\varrho, \rho_i) \partial_x \Lambda^s h(\cdot, \varrho) - \min(\rho_i, \rho_j) \partial_x \Lambda^s h(\cdot, \rho_j) d\varrho \\ &= - \sum_{j=1}^{i-1} \int_{\rho_{j-\frac{1}{2}}}^{\rho_{j+\frac{1}{2}}} (\varrho \partial_x \Lambda^s h(\cdot, \varrho) - \rho_j \partial_x \Lambda^s h(\cdot, \rho_j)) d\varrho - \rho_i \sum_{j=i+1}^N \int_{\rho_{j-\frac{1}{2}}}^{\rho_{j+\frac{1}{2}}} (\partial_x \Lambda^s h(\cdot, \varrho) - \partial_x \Lambda^s h(\cdot, \rho_j)) d\varrho \\ &\quad - \int_{\rho_{i-\frac{1}{2}}}^{\rho_i} (\varrho \partial_x \Lambda^s h(\cdot, \varrho) - \rho_i \partial_x \Lambda^s h(\cdot, \rho_i)) d\varrho - \rho_i \int_{\rho_i}^{\rho_{i+\frac{1}{2}}} (\partial_x \Lambda^s h(\cdot, \varrho) - \partial_x \Lambda^s h(\cdot, \rho_i)) d\varrho. \end{aligned}$$

Hence by numerical integration midpoint rule (applied to the first two terms) and the

rectangle rule (applied to the last two terms) we infer

$$\|(\Lambda^s \mathbf{R}_N)_i\|_{L^2(\mathbb{R}^d)} \leq \frac{C(\rho_{\text{surf}}, \rho_{\text{bott}})}{N^2} \left(\|\partial_\varrho \partial_x \Lambda^s h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} + \|\partial_\varrho^2 \partial_x \Lambda^s h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} \right). \quad (5.2.4)$$

Secondly, one has

$$\begin{aligned} \left(D_\rho(\rho \Lambda^{s-1} \mathbf{R}_N) \right)_i &= N \left((\rho \Lambda^{s-1} \mathbf{R}_N)_i - (\rho \Lambda^{s-1} \mathbf{R}_N)_{i+1} \right) \\ &= -N \sum_{j=1}^N \int_{\rho_{j-\frac{1}{2}}}^{\rho_{j+\frac{1}{2}}} q_i(\varrho) \partial_x \Lambda^{s-1} h(\cdot, \varrho) - q_i(\rho_j) \partial_x \Lambda^{s-1} h(\cdot, \rho_j) d\varrho \end{aligned}$$

where $q_i : \varrho \mapsto \min(\varrho, \rho_i) - \min(\varrho, \rho_{i+1})$. After simple computations we infer

$$\begin{aligned} \left(D_\rho(\rho \Lambda^{s-1} \mathbf{R}_N) \right)_i &= -N \left(\int_{\rho_i}^{\rho_{i+1}} (\rho_i - \varrho) \partial_x \Lambda^{s-1} h(\cdot, \varrho) - 2N(\rho_i - \rho_{i+1}) \partial_x \Lambda^{s-1} h(\cdot, \rho_{i+1}) d\varrho \right) \\ &\quad - N(\rho_i - \rho_{i+1}) \left(\int_{\rho_{i+1}}^{\rho_{i+\frac{3}{2}}} \partial_x \Lambda^{s-1} h(\cdot, \varrho) - \partial_x \Lambda^{s-1} h(\cdot, \rho_{i+1}) d\varrho \right) \\ &\quad - \sum_{j=i+2}^N \left(\int_{\rho_{j-\frac{1}{2}}}^{\rho_{j+\frac{1}{2}}} \partial_x \Lambda^{s-1} h(\cdot, \varrho) - \partial_x \Lambda^{s-1} h(\cdot, \rho_j) d\varrho \right). \end{aligned}$$

Consequently, using the Taylor expansion of $\varrho \mapsto \partial_x \Lambda^{s-1} h(\cdot, \varrho)$ up to the first order in the neighborhood of ρ_{i+1} and up to the second order in the neighborhood of ρ_j for $j \in \{i+2, \dots, N\}$, we obtain

$$\| (D_\rho(\rho \Lambda^{s-1} \mathbf{R}_N))_i \|_{L_x^2} \leq \frac{C(\rho_{\text{bott}})}{N^2} \left(\|\partial_\varrho \partial_x \Lambda^{s-1} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} + \|\partial_\varrho^2 \partial_x \Lambda^{s-1} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} \right).$$

Using Lemma 7.2.1 (2) and the fact that $D_\rho \rho = -(1, \dots, 1)^t \in \mathbb{R}^{N-1}$ in addition to the previous estimate (5.2.4), we infer

$$\| (D_\rho \Lambda^{s-1} \mathbf{R}_N)_i \|_{L_x^2} \leq \frac{C(\rho_{\text{surf}}, \rho_{\text{bott}})}{N^2} \left(\|\partial_\varrho \partial_x \Lambda^{s-1} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} + \|\partial_\varrho^2 \partial_x \Lambda^{s-1} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} \right). \quad (5.2.5)$$

Lastly, one has

$$\begin{aligned} \left(D_\rho^2(\boldsymbol{\rho}\Lambda^{s-2}\mathbf{R}_N) \right)_i &= N \left((D_\rho(\boldsymbol{\rho}\Lambda^{s-2}\mathbf{R}_N))_i - (D_\rho(\boldsymbol{\rho}\Lambda^{s-2}\mathbf{R}_N))_{i+1} \right) \\ &= -N^2 \sum_{j=1}^N \left(\int_{\rho_{j-\frac{1}{2}}}^{\rho_{j+\frac{1}{2}}} k_i(\varrho) \partial_x \Lambda^{s-2} h(\cdot, \varrho) - k_i(\rho_j) \partial_x \Lambda^{s-2} h(\cdot, \rho_j) d\varrho \right), \end{aligned}$$

where $k_i : \varrho \mapsto \min(\varrho, \rho_i) - 2 \min(\varrho, \rho_{i+1}) + \min(\varrho, \rho_{i+2})$. After simple computations we infer

$$\begin{aligned} \left(D_\rho^2(\boldsymbol{\rho}\Lambda^{s-2}\mathbf{R}_N) \right)_i &= -N^2 \left(\int_{\rho_i}^{\rho_{i+1}} (\rho_i - \varrho) \partial_x \Lambda^{s-2} h(\cdot, \varrho) d\varrho + \int_{\rho_{i+1}}^{\rho_{i+2}} (\rho_i - 2\rho_{i+1} + \varrho) \partial_x \Lambda^{s-2} h(\cdot, \varrho) d\varrho \right. \\ &\quad \left. - \frac{1}{N} (\rho_i - \rho_{i+1}) \partial_x \Lambda^{s-2} h(\cdot, \rho_{i+1}) \right). \end{aligned}$$

Consequently, using the Taylor expansion of $\varrho \mapsto \partial_x \Lambda^{s-2} h(\cdot, \varrho)$ up to the second order in the neighborhood of ρ_{i+1} , we obtain

$$\|D_\rho^2(\boldsymbol{\rho}\Lambda^{s-2}\mathbf{R}_N)\|_{L_x^2} \leq \frac{C(\rho_{\text{bott}})}{N^2} \left(\|\partial_\varrho \partial_x \Lambda^{s-2} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} + \|\partial_\varrho^2 \partial_x \Lambda^{s-2} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} \right).$$

Using Lemma 7.2.1 (3) and the fact that $D_\rho^2 \boldsymbol{\rho} = (0, \dots, 0)^t \in \mathbb{R}^{N-2}$ in addition to the previous estimates (5.2.4), (5.2.5), we infer

$$\|D_\rho^2(\Lambda^{s-2}\mathbf{R}_N)\|_{L_x^2} \leq \frac{C(\rho_{\text{surf}}, \rho_{\text{bott}})}{N^2} \left(\|\partial_\varrho \partial_x \Lambda^{s-2} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} + \|\partial_\varrho^2 \partial_x \Lambda^{s-2} h(t, \cdot, \cdot)\|_{L_\varrho^\infty(L_x^2)} \right).$$

The proof is complete. □

5.3 Stability

In this subsection we will provide the key ingredients towards stability estimates on the difference between the solutions to the continuously stratified system (5.2.1) (after applying the projection operator P_N) and the corresponding solutions to (4.2.1). These stability estimates are obtained by considering the linearized system satisfied by the difference and their derivatives, carefully estimating the remainders that result from this linearization.

In the following Lemma we estimate these remainder terms.

Lemma 5.3.1. *Let $s \in \mathbb{N}$, $s > 2 + \frac{1}{2}$, there exists $C > 0$, such that for any $N \in \mathbb{N}^*$, $\kappa > 0$, and any (\underline{H}, H, U) and (\underline{h}, h, u) sufficiently smooth respectively on \mathbb{R} and $\Omega := (\rho_{\text{surf}}, \rho_{\text{bott}}) \times \mathbb{R}$, setting*

$$\begin{aligned} R' &= -(U - P_N u) \partial_x P_N h - (H - P_N h) \partial_x P_N u, \\ \mathbf{R}' &= - \left((U - P_N u) - \kappa \left(\frac{\partial_x H}{\underline{H} + H} - P_N \left(\frac{\partial_x h}{\underline{h} + h} \right) \right) \right) \partial_x P_N u, \end{aligned}$$

the following estimates hold.

1.

$$\|R'\|_{H^{s-1,1}} \leq C \|U - P_N u\|_{H^{s-1,1}} \|\partial_x h\|_{\infty, s-1,1} + C \|H - P_N h\|_{H^{s-1,1}} \|\partial_x u\|_{\infty, s-1,1}.$$

2.

$$\begin{aligned} \|SR'\|_{H^{s,0}} + \|TSR'\|_{H^{s,0}} &\leq C \|U - P_N u\|_{H^{s,0}} \|\partial_x h\|_{\infty, s,0} + C \|S(H - P_N h)\|_{H^{s,0}} \|\partial_x u\|_{\infty, s-1,1} \\ &\quad + C \|TS(H - P_N h)\|_{H^{s,0}} \|\partial_x u\|_{\infty, s-1,1} + C \|H - P_N h\|_{H^{s-1,0}} \|\partial_x u\|_{\infty, s,0}. \end{aligned}$$

3.

$$\|\mathbf{R}'\|_{H^{s,0}} \leq C \|U - P_N u\|_{H^{s,0}} \|\partial_x u\|_{\infty, s,0} + C \kappa \left\| \frac{\partial_x H}{\underline{H} + H} - P_N \left(\frac{\partial_x h}{\underline{h} + h} \right) \right\|_{H^{s,0}} \|\partial_x u\|_{\infty, s,0}.$$

4.

$$\|\mathbf{R}'\|_{H^{s,2}} \leq C \|U - P_N u\|_{H^{s,2}} \|\partial_x u\|_{\infty, s,2} + C \kappa \left\| \frac{\partial_x H}{\underline{H} + H} - P_N \left(\frac{\partial_x h}{\underline{h} + h} \right) \right\|_{H^{s,2}} \|\partial_x u\|_{\infty, s,2}.$$

Proof. The proof of 1., 3., and 4. follows immediately from Lemma 7.2.1, Lemma 7.2.5 (2) and Lemma 7.2.9. For 2. we use Lemma 7.2.1 (1) to infer the identity valid for $0 \leq \alpha \leq s$

$$\begin{aligned} \partial_x^\alpha SR' &= S \partial_x^\alpha \left((U - P_N u) (\partial_x P_N h) \right) - (\partial_x P_N u) (S \partial_x^\alpha (H - P_N h)) \\ &\quad + S_0 \left((D_\rho (\partial_x P_N u)) (R_u S \partial_x^\alpha (H - P_N h)) \right) - S \left([\partial_x^\alpha, \partial_x P_N u] (H - P_N h) \right). \end{aligned}$$

The estimate on $\|SR'\|_{H^{s,0}}$ follows from Lemma 7.2.2, Lemma 7.2.5 (2), and Lemma 7.2.9. The estimate for $\|TSR'\|_{H^{s,0}}$ follows in the same way from the analogous identity. \square

The following Lemma is the follow-up of the previous one. In order to close the bootstrap argument (see Theorem 5.4.1) we estimate the term $\left\| \frac{\partial_x H}{\underline{H} + H} - P_N \left(\frac{\partial_x h}{\underline{h} + h} \right) \right\|_{H^{s,2}}$. The main challenge is to provide careful estimates such that the upper bound has at most the size of $\|(H - P_N h, U - P_N u)\|_s$ where $\|\cdot\|_s$ is defined in (4.5.2).

Lemma 5.3.2. *Let $h_*, \underline{M}, M_c, M > 0$, $s \in \mathbb{N}$ such that $s > 2 + \frac{1}{2}$. There exists $C(h_*^{-1}, \underline{M}, M_c, M) > 0$ such that for any $N \in \mathbb{N}^*$, and any $\underline{h} \in W^{2,\infty}((\rho_{\text{surf}}, \rho_{\text{bott}}))$, $\underline{H} = P_N(\underline{h})$, and any h and H sufficiently smooth respectively on \mathbb{R} and $\Omega := (\rho_{\text{surf}}, \rho_{\text{bott}}) \times \mathbb{R}$, satisfying*

$$\begin{aligned} \|\underline{H}\|_{W^{2,\infty}} &\leq \underline{M}, \quad \|H\|_{H^{s-1,1}} \leq M, \quad \|h\|_{\infty, s-1, 1} \leq M_c, \\ \inf_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}} \underline{H}_i + H_i(x) &\geq h_*, \quad \inf_{(x,\varrho) \in \mathbb{R} \times (\rho_{\text{surf}}, \rho_{\text{bott}})} \underline{h}(\varrho) + h(x, \varrho) \geq h_*, \end{aligned}$$

one has

$$\begin{aligned} \left\| \frac{\partial_x H}{\underline{H} + H} - P_N \left(\frac{\partial_x h}{\underline{h} + h} \right) \right\|_{H^{s,2}} &\leq C(h_*^{-1}, \underline{M}, M_c, M) \times \\ &\left(\|\partial_x(H - P_N h)\|_{H^{s,2}} + \|H - P_N h\|_{H^{s,2}} (\|H\|_{H^{s,2}} + \|\partial_x h\|_{\infty, s-1, 1}) \right. \\ &\left. + \|H - P_N h\|_{H^{s-1,1}} \|\partial_x h\|_{\infty, s, 2} + \|H - P_N h\|_{H^{s-1,1}} \|\partial_x h\|_{\infty, s-1, 1} (\|H\|_{H^{s,2}} + \|h\|_{\infty, s, 2}) \right). \end{aligned}$$

Proof. We notice that

$$\begin{aligned} \frac{\partial_x H}{\underline{H} + H} - P_N \left(\frac{\partial_x h}{\underline{h} + h} \right) &= \partial_x(H - P_N h) \left(\frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right) + \frac{\partial_x(H - P_N h)}{\underline{H}} \\ &+ \frac{\partial_x P_N h}{\underline{H}^2} (P_N h - H) + \partial_x P_N h \left(\frac{1}{(\underline{H} + H)(\underline{H} + P_N h)} - \frac{1}{\underline{H}^2} \right) (P_N h - H), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{(\underline{H} + H)(\underline{H} + P_N h)} - \frac{1}{\underline{H}^2} &= \left(\frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right) \left(\frac{1}{\underline{H} + P_N h} - \frac{1}{\underline{H}} \right) \\ &+ \frac{1}{\underline{H}} \left[\left(\frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right) + \left(\frac{1}{\underline{H} + P_N h} - \frac{1}{\underline{H}} \right) \right]. \end{aligned}$$

Since $s > 2 + \frac{1}{2}$, using Lemma 7.2.6 (2), Lemma 7.2.8 and Lemma 7.2.9 we obtain the desired estimate. \square

We can now collect all estimates on remainders which will be used in the proof of

our second main result, namely Theorem 5.4.1. The following Lemma is the analogue of Lemma 4.3.1 except this time instead of focusing on the variables H, U , the focus will be on the variables $H - P_N h, U - P_N u$. By means of Lemma 5.3.1 and Lemma 5.3.2, we obtain remainder terms which are controlled by $\|(H - P_N h, U - P_N u)\|_s$ where $\|\cdot\|_s$ is defined in (4.5.2).

Lemma 5.3.3. *Let $\rho_{\text{bott}} > \rho_{\text{surf}} > 0$, $s \in \mathbb{N}$, $s > 2 + \frac{1}{2}$, $h_*, \underline{M}, M_c, M > 0$. Then there exists $C > 0$ such that for any $N \in \mathbb{N}^*$, $\kappa \in (0, 1]$, and any (h, u) solution to (5.2.1) with $\underline{h}, \underline{u} \in W^{2,\infty}((\rho_{\text{surf}}, \rho_{\text{bott}}))$ and (H, U) solution to (4.2.1) with $\underline{H} = P_N \underline{h}, \underline{U} = P_N \underline{u}$, assuming that these solutions both exist on a time interval $[0, T)$ with $T > 0$, and that the following estimates hold for any $t \in [0, T)$*

$$\begin{aligned} & |\boldsymbol{\rho}|_{l^\infty} + |\boldsymbol{\rho}^{-1}|_{l^\infty} + |\underline{H}|_{w^{2,\infty}} + |\underline{U}|_{w^{2,\infty}} + |\underline{h}|_{W^{2,\infty}} + |\underline{u}|_{W^{2,\infty}} \leq \underline{M}, \\ & \|H(t, \cdot)\|_{H^{s-1,1}} + \|U(t, \cdot)\|_{H^{s,2}} + \kappa^{\frac{1}{2}} \|H(t, \cdot)\|_{H^{s,2}} \leq M, \\ & \|h(t, \cdot)\|_{\infty, s+1, 2} + \|\partial_x u(t, \cdot)\|_{\infty, s, 2} \leq M_c, \end{aligned}$$

$$\inf_{(i,x) \in \{1, \dots, N\} \times \mathbb{R}} \underline{H}_i + H_i(t, x) \geq h_*, \quad \inf_{(x,\varrho) \in \mathbb{R} \times [\rho_{\text{surf}}, \rho_{\text{bott}}]} \underline{h}(\varrho) + h(t, x, \varrho) \geq h_*,$$

then the following holds.

1. For all $\alpha \in \mathbb{N}$ with $0 \leq \alpha \leq s$, we have

$$\begin{aligned} \partial_t \partial_x^\alpha (H - P_N h) + (\underline{U} + U) \partial_x \partial_x^\alpha (H - P_N h) + (\underline{H} + H) \partial_x \partial_x^\alpha (U - P_N u) \\ = \kappa \partial_x^2 \partial_x^\alpha (H - P_N h) + \mathbf{R}'_{\alpha,0}, \end{aligned} \tag{5.3.1}$$

$$\partial_t \partial_x^\alpha (U - P_N u) + \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) \partial_x \partial_x^\alpha (U - P_N u) + \Gamma \partial_x \partial_x^\alpha (H - P_N h) = \mathbf{R}'_{\alpha,0}, \tag{5.3.2}$$

where for every $t \in [0, T]$, $(\mathbf{S}\mathbf{R}'_{\alpha,0}(t), \mathbf{T}\mathbf{S}\mathbf{R}'_{\alpha,0}(t), \mathbf{R}'_{\alpha,0}(t)) \in l^2(L^2(\mathbb{R}))^3$ and

$$\begin{aligned} \|\mathbf{S}\mathbf{R}'_{\alpha,0}(t, \cdot)\|_{l^2(L^2_x)} + \|\mathbf{T}\mathbf{S}\mathbf{R}'_{\alpha,0}(t, \cdot)\|_{l^2(L^2_x)} \leq \\ C(\|U - P_N u\|_{H^{s,1}} + \|\mathbf{S}(H - P_N h)\|_{H^{s,0}} + \|\mathbf{T}\mathbf{S}(H - P_N h)\|_{H^{s,0}} + \|H - P_N h\|_{H^{s-1,0}}). \end{aligned}$$

$$\begin{aligned} \|\mathbf{R}'_{\alpha,0}\|_{l^2(L^2_x)} \leq \|\mathbf{R}_N\|_{H^{s,2}} + C(1 + \kappa \|\partial_x H\|_{H^{s,1}}) \|U - P_N u\|_{H^{s,1}} \\ + C(\|H - P_N h\|_{H^{s-1,1}} + \kappa^{\frac{1}{2}} \|H - P_N h\|_{H^{s,2}} + \kappa \|\partial_x (H - P_N h)\|_{H^{s,2}}). \end{aligned}$$

2. For all $\alpha \in \mathbb{N}$, $j \in \{0, 1\}$ with $0 \leq \alpha \leq s - 1 - j$, we have

$$\partial_t \partial_x^\alpha \mathbf{D}_\rho^j (H - P_N h) + (\mathbf{M}^j(\underline{U} + U)) \partial_x \partial_x^\alpha \mathbf{D}_\rho^j (H - P_N h) = \kappa \partial_x^2 \partial_x^\alpha \mathbf{D}_\rho^j (H - P_N h) + \mathcal{R}'_{\alpha,j} \quad (5.3.3)$$

where for every $t \in [0, T]$, $\mathcal{R}'_{\alpha,j}(t) \in l^2(L^2(\mathbb{R}))$ and

$$\|\mathcal{R}'_{\alpha,j}(t, \cdot)\|_{l^2(L^2_x)} \leq C(\|U - P_N u\|_{H^{s,2}} + \|H - P_N h\|_{H^{s-1,1}}).$$

3. For any $\alpha \in \mathbb{N}$, $j \in \{1, 2\}$ such that $0 \leq \alpha \leq s - j$, it holds

$$\begin{aligned} \partial_t \mathbf{D}_\rho^j \mathbf{S} \partial_x^\alpha (H - P_N h) + \mathbf{R}_d(\mathbf{M}^{j-1}(\underline{U} + U)) \mathbf{D}_\rho^j \mathbf{S}(\partial_x^\alpha \partial_x (H - P_N h)) \\ = \kappa \partial_x^2 (\partial_x^\alpha \mathbf{D}_\rho^j \mathbf{S}(H - P_N h)) + \mathbf{R}'_{\alpha,j} \end{aligned} \quad (5.3.4)$$

$$\partial_t \mathbf{D}_\rho^j \partial_x^\alpha (U - P_N u) + \mathbf{M}^j \left(\underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) \partial_x (\mathbf{D}_\rho^j \partial_x^\alpha (U - P_N u)) = \mathbf{R}'_{\alpha,j} \quad (5.3.5)$$

where for every $t \in [0, T]$, $(\mathbf{R}'_{\alpha,j}(t), \mathbf{R}'_{\alpha,j}(t)) \in l^2(L^2(\mathbb{R})) \times l^2(L^2(\mathbb{R}))$ and

$$\|\mathbf{R}'_{\alpha,j}(t, \cdot)\|_{l^2(L^2_x)} \leq C(\|U - P_N u\|_{H^{s,2}} + \|H - P_N h\|_{H^{s-1,1}}).$$

$$\begin{aligned} \|\mathbf{R}'_{\alpha,j}\|_{l^2(L^2_x)} &\leq \|\mathbf{R}_N\|_{H^{s,2}} + C(1 + \kappa \|\partial_x H\|_{H^{s,2}}) \|U - P_N u\|_{H^{s,2}} \\ &\quad + C(\|\mathbf{TS}(H - P_N h)\|_{H^{s,0}} + \|\mathbf{S}(H - P_N h)\|_{H^{s,1}}) \\ &\quad + C(\|H - P_N h\|_{H^{s-1,1}} + \kappa^{\frac{1}{2}} \|H - P_N h\|_{H^{s,2}} + \kappa \|\partial_x (H - P_N h)\|_{H^{s,2}}). \end{aligned}$$

4. For any $\alpha \in \mathbb{N}$, $j \in \{0, 1, 2\}$, $0 \leq \alpha \leq s - j$, it holds

$$\partial_t \mathbf{D}_\rho^j \partial_x^\alpha (H - P_N h) + \mathbf{M}^j(\underline{U} + U) \partial_x (\mathbf{D}_\rho^j \partial_x^\alpha (H - P_N h)) = \kappa \partial_x^2 (\partial_x^\alpha \mathbf{D}_\rho^j (H - P_N h)) + r'_{\alpha,j} + \partial_x \mathbf{r}'_{\alpha,j} \quad (5.3.6)$$

where for every $t \in [0, T]$, $(r'_{\alpha,j}(t), \mathbf{r}'_{\alpha,j}(t)) \in l^2(L^2(\mathbb{R})) \times l^2(L^2(\mathbb{R}))$ and

$$\begin{aligned} \|r'_{\alpha,j}\|_{l^2(L^2_x)} &\leq C \|H - P_N h\|_{H^{s,2}}, \\ \|\mathbf{r}'_{\alpha,j}\| &\leq C(\|U - P_N u\|_{H^{s,2}} + \|H - P_N h\|_{H^{s-1,1}}). \end{aligned}$$

Above, we denote by \mathbf{R}_N the term defined in (5.2.3).

Proof. Under the hypothesis of the Lemma, $(P_N h, P_N u)$ is a solution to (5.2.2), consequently $(H - P_N h, U - P_N u)$ satisfy the following system

$$\begin{cases} \partial_t(H - P_N h) + (\underline{U} + U)\partial_x(H - P_N h) + (\underline{H} + H)\partial_x(U - P_N u) = \kappa\partial_x^2(H - P_N h) + R', \\ \partial_t(U - P_N u) + \left(\underline{U} + U - \kappa\frac{\partial_x H}{\underline{H} + H}\right)\partial_x(U - P_N u) + \Gamma\partial_x(H - P_N h) = -\mathbf{R}_N + \mathbf{R}', \end{cases} \quad (5.3.7)$$

where

$$\begin{aligned} \mathbf{R}_N &= \Gamma\partial_x P_N h - P_N \left(\frac{1}{\varrho}\partial_x \psi\right), \\ R' &= -(U - P_N u)\partial_x P_N h - (H - P_N h)\partial_x P_N u, \\ \mathbf{R}' &= -\left((U - P_N u) - \kappa\left(\frac{\partial_x H}{\underline{H} + H} - P_N\left(\frac{\partial_x h}{\underline{h} + h}\right)\right)\right)\partial_x P_N u. \end{aligned}$$

The contribution from \mathbf{R}_N is trivial in this proof. We have estimated R' and \mathbf{R}' in Lemma 5.3.1 and Lemma 5.3.2. The remaining contributions are estimated following the same steps as in Lemma 4.3.1.

Estimates for $\mathcal{R}'_{\alpha,j}$, for all $\alpha \in \mathbb{N}$, $j \in \{0, 1\}$ with $0 \leq \alpha \leq s - 1 - j$, we have

$$\mathcal{R}'_{\alpha,j} = \partial_x^\alpha \mathbf{D}_\rho^j \mathbf{R}' - \llbracket \partial_x^\alpha \mathbf{D}_\rho^j, \underline{U} + U \rrbracket \partial_x(H - P_N h) - \partial_x^\alpha \mathbf{D}_\rho^j ((\underline{H} + H)\partial_x(U - P_N u)),$$

we obtain an estimate of the above by using the same estimates as $\mathcal{R}_{\alpha,j}$ (adapted to our case) in Lemma 4.3.1 and then applying Lemma 5.3.1 and Lemma 7.2.9, consequently we have

$$\begin{aligned} \|\mathcal{R}'_{\alpha,j}(t, \cdot)\|_{l^2(L_x^2)} &\lesssim (\|\partial_x h\|_{\infty, s-1, 1} + |\underline{H}|_{w^1, \infty} + \|H\|_{H^{s-1, 1}}) \|U - P_N u\|_{H^{s, 2}} \\ &\quad + (\|\partial_x u\|_{\infty, s-1, 1} + |\mathbf{D}_\rho \underline{U}|_{l^\infty} + \|U\|_{H^{s, 2}}) \|H - P_N h\|_{H^{s-1, 1}}. \end{aligned}$$

Estimates for $S\mathcal{R}'_{\alpha,0}$, $\mathbf{T}S\mathcal{R}'_{\alpha,0}$ and $\mathbf{R}'_{\alpha,0}$ with $0 \leq \alpha \leq s$.

For all $\alpha \in \mathbb{N}$ with $0 \leq \alpha \leq s$, using Abel's summation Lemma 7.2.1 we have

$$\begin{aligned} SR'_{\alpha,0} &= \partial_x^\alpha SR' - [\partial_x^\alpha, U, \partial_x S(H - P_N h)] - (\partial_x(U - P_N u))(\partial_x^\alpha SH) \\ &\quad + S_0 \left([\partial_x^\alpha; D_\rho U, R_u \partial_x S(H - P_N h)] + (D_\rho(\partial_x(U - P_N u)))(R_u S \partial_x^\alpha H) \right) \\ &\quad - S \left((\partial_x^\alpha U)(\partial_x(H - P_N h)) + [\partial_x^\alpha; H, \partial_x(U - P_N u)] \right), \end{aligned}$$

$$\begin{aligned} TSR'_{\alpha,0} &= \partial_x^\alpha TSR' - [\partial_x^\alpha, U, \partial_x TS(H - P_N h)] - (\partial_x(U - P_N u))(\partial_x^\alpha TSH) \\ &\quad + TS_0 \left([\partial_x^\alpha; D_\rho U, R_u \partial_x S(H - P_N h)] + (D_\rho(\partial_x(U - P_N u)))(R_u S \partial_x^\alpha H) \right) \\ &\quad - TS \left((\partial_x^\alpha U)(\partial_x(H - P_N h)) + [\partial_x^\alpha; H, \partial_x(U - P_N u)] \right), \end{aligned}$$

and

$$\begin{aligned} \mathbf{R}'_{\alpha,0} &= -\partial_x^\alpha \mathbf{R}_N + \partial_x^\alpha \mathbf{R}' - [\partial_x^\alpha, U] \partial_x(U - P_N u) + \frac{\kappa}{\underline{H} + H} [\partial_x^\alpha, \partial_x H] \partial_x(U - P_N u) \\ &\quad + \kappa \left[\partial_x^\alpha, \frac{1}{\underline{H} + H} \right] ((\partial_x H)(\partial_x(U - P_N u))). \end{aligned}$$

We obtain an estimate using the previous identities by using the same estimates of $SR_{\alpha,0}$, $TSR_{\alpha,0}$ and $\mathbf{R}_{\alpha,0}$ (adapted to our case) in Lemma 4.3.1. Moreover, applying Lemma 5.3.1 we find

$$\begin{aligned} &\| SR'_{\alpha,0}(t, \cdot) \|_{l^2(L_x^2)} + \| TSR'_{\alpha,0}(t, \cdot) \|_{l^2(L_x^2)} \\ &\lesssim (\| \partial_x h \|_{\infty, s, 0} + \| TSH \|_{H^{s,0}} + \| SH \|_{H^{s,0}} + \| H \|_{H^{s-1,0}}) \| U - P_N u \|_{H^{s,1}} \\ &\quad + (\| \partial_x u \|_{\infty, s-1, 1} + \| U \|_{H^{s,1}}) (\| S(H - P_N h) \|_{H^{s,0}} + \| TS(H - P_N h) \|_{H^{s,0}}) \\ &\quad + (\| \partial_x u \|_{\infty, s, 0} + \| U \|_{H^{s,0}}) \| H - P_N h \|_{H^{s-1,0}}. \end{aligned}$$

In the same way, and using additionally Lemma 5.3.2, we find

$$\begin{aligned} & \| \mathbf{R}'_{\alpha,0} \|_{l^2(L_x^2)} \lesssim \| \mathbf{R}_N \|_{H^{s,2}} \\ & + C(h_*^{-1}, \| H \|_{H^{s-1,1}}) (\| U \|_{H^{s,1}} + \| \partial_x u \|_{\infty,s,0} + \kappa (\| H \|_{H^{s,1}}^2 + \| \partial_x H \|_{H^{s,1}})) \times \| U - P_N u \|_{H^{s,1}} \\ & + C(h_*^{-1}, \underline{M}, M_c, M) \kappa \| \partial_x u \|_{\infty,s,0} \\ & \times \left(\| \partial_x (H - P_N h) \|_{H^{s,2}} + \| H - P_N h \|_{H^{s,2}} (\| H \|_{H^{s,2}} + \| \partial_x h \|_{\infty,s-1,1}) \right. \\ & \left. + \| H - P_N h \|_{H^{s-1,1}} \| \partial_x h \|_{\infty,s,2} + \| H - P_N h \|_{H^{s-1,1}} \| \partial_x h \|_{\infty,s-1,1} (\| H \|_{H^{s,2}} + \| h \|_{\infty,s,2}) \right). \end{aligned}$$

Estimates for $R'_{\alpha,j}$, $j \in \{1, 2\}$, for any $\alpha \in \mathbb{N}$, such that $0 \leq \alpha \leq s - j$.

Recalling that $D_\rho \mathbf{S} = \mathbf{R}_d$ and $D_\rho^2 \mathbf{S} = \mathbf{R}_d D_\rho$ we have

$$\begin{aligned} R'_{\alpha,1} &= \mathbf{R}_d \partial_x^\alpha R' + \mathbf{R}_d \left(-[\partial_x^{\alpha+1}, \underline{U} + U] \partial_x^\alpha (H - P_N h) - \partial_x^\alpha ((\underline{H} + H) \partial_x (U - P_N u)) \right) \\ &= \mathbf{R}_d \partial_x^\alpha R' + \mathbf{R}_d \left(\partial_x^\alpha ((\partial_x U) (H - P_N h)) - \partial_x^\alpha ((H) (\partial_x (U - P_N u))) \right) \\ &\quad + \mathbf{R}_d \left(-[\partial_x^{\alpha+1}, U] (H - P_N h) - \partial_x^\alpha ((\underline{H}) (\partial_x (U - P_N u))) \right) \end{aligned}$$

and

$$R'_{\alpha,2} = \mathbf{R}_d D_\rho \partial_x^\alpha R' + \mathbf{R}_d \left(-[\partial_x^\alpha D_\rho, \underline{U} + U] \partial_x (H - P_N h) - \partial_x^\alpha D_\rho ((\underline{H} + H) (\partial_x (U - P_N u))) \right).$$

Using Lemma 7.2.5 (1) to estimate the second and third term of $R'_{\alpha,1}$, and Lemma 5.3.1 to estimate the contributions of R' and the same estimates as $R_{\alpha,1}$, $R_{\alpha,2}$ (adapted to our case) in Lemma 4.3.1 for the remaining contributions, we find

$$\begin{aligned} \| R'_{\alpha,j}(t, \cdot) \|_{l^2(L_x^2)} &\lesssim (\| \partial_x h \|_{\infty,s-1,1} + \| H \|_{H^{s-1,1}} + | \underline{H} |_{w^{1,\infty}}) \| U - P_N u \|_{H^{s,2}} \\ &\quad + (\| \partial_x u \|_{\infty,s-1,1} + \| U \|_{H^{s,2}} + | D_\rho \underline{U} |_{l^\infty}) \| H - P_N h \|_{H^{s-1,1}}. \end{aligned}$$

Estimates for $\mathbf{R}'_{\alpha,j}$, for any $\alpha \in \mathbb{N}$, $j \in \{1, 2\}$ such that $0 \leq \alpha \leq s - j$, we have

$$\begin{aligned} \mathbf{R}'_{\alpha,j} &= -\partial_x^\alpha D_\rho^j \mathbf{R}_N + \partial_x^\alpha D_\rho^j \mathbf{R}' - \left[\partial_x^\alpha D_\rho^j, \underline{U} + U - \kappa \frac{\partial_x H}{\underline{H} + H} \right] \partial_x (U - P_N u) \\ &\quad - \partial_x^\alpha D_\rho^j (\Gamma \partial_x (H - P_N h)). \end{aligned}$$

Where Γ is as in (4.2.2). Again we use the corresponding estimates for $\mathbf{R}_{\alpha,j}$ (adapted to

our case) in Lemma 4.3.1, Lemma 5.3.1 and Lemma 5.3.2 for the contribution of \mathbf{R}' , and deduce

$$\begin{aligned}
\| \mathbf{R}'_{\alpha,j} \|_{l^2(L_x^2)} &\lesssim \| \mathbf{R}_N \|_{H^{s,2}} + (\| \partial_x u \|_{\infty,s,2} + \| U \|_{H^{s,2}} + | \mathbf{D}_\rho \underline{U} |_{w^{1,\infty}}) \times \| U - P_N u \|_{H^{s,2}} \\
&+ C(\boldsymbol{\rho}_1, | \boldsymbol{\rho}^{-1} |_{l^\infty}) (\| \mathbf{TS}(H - P_N h) \|_{H^{s,0}} + \| \mathbf{S}(H - P_N h) \|_{H^{s,0}}) \\
&+ C(h_*^{-1}, | \underline{H} |_{w^{2,\infty}}, \| H \|_{H^{s-1,1}}) \kappa (\| H \|_{H^{s,2}}^2 + \| \partial_x H \|_{H^{s,2}}) \times \| U - P_N u \|_{H^{s,2}} \\
&+ C(h_*^{-1}, \underline{M}, M_c, M) \kappa \| \partial_x u \|_{\infty,s,2} \\
&\times \left(\| \partial_x (H - P_N h) \|_{H^{s,2}} + \| H - P_N h \|_{H^{s,2}} (\| H \|_{H^{s,2}} + \| \partial_x h \|_{\infty,s-1,1}) \right. \\
&\quad \left. + \| H - P_N h \|_{H^{s-1,1}} \| \partial_x h \|_{\infty,s,2} + \| H - P_N h \|_{H^{s-1,1}} \| \partial_x h \|_{\infty,s-1,1} (\| H \|_{H^{s,2}} + \| h \|_{\infty,s,2}) \right).
\end{aligned}$$

Estimates for $r'_{\alpha,j}$ and $\mathbf{r}'_{\alpha,j}$, for any $\alpha \in \mathbb{N}$, $j \in \{0, 1, 2\}$, $0 \leq \alpha \leq s - j$, we have

$$\begin{aligned}
r'_{\alpha,j} &= -[\partial^{\alpha+1} \mathbf{D}_\rho^j; U, H - P_N h] + (\partial_x^\alpha \mathbf{D}_\rho^j U)(\mathbf{M}^j \partial_x (H - P_N h)) \\
\mathbf{r}'_{\alpha,j} &= -[\mathbf{D}_\rho^j, \underline{U}] \partial_x^\alpha (H - P_N h) - \partial_x^\alpha \mathbf{D}_\rho^j (\underline{H}(U - P_N u)) - (\partial_x^\alpha \mathbf{D}_\rho^j U)(\mathbf{M}^j (H - P_N h)) \\
&\quad - \partial_x^\alpha \mathbf{D}_\rho^j ((U - P_N u) P_N h).
\end{aligned}$$

The estimate of the last term of $\mathbf{r}'_{\alpha,j}$ is obtained using Lemma 7.2.1 and Lemma 7.2.9. The remaining contributions are obtained using the estimates of $r_{\alpha,j}$ and $\mathbf{r}_{\alpha,j}$ (adapted to our case) in Lemma 4.3.1, and we obtain

$$\begin{aligned}
\| r'_{\alpha,j} \|_{l^2(L_x^2)} &\lesssim \| U \|_{H^{s,2}} \| H - P_N h \|_{H^{s,2}} \\
\| \mathbf{r}'_{\alpha,j} \| &\lesssim (| \underline{H} |_{w^{2,\infty}} + \| h \|_{\infty,s,2}) \| U - P_N u \|_{H^{s,2}} \\
&\quad + (| \mathbf{D}_\rho \underline{U} |_{w^{1,\infty}} + \| U \|_{H^{s,2}}) \| H - P_N h \|_{H^{s-1,1}}.
\end{aligned}$$

This concludes the proof. □

5.4 Convergence

In this subsection we state and prove our second main result. Specifically, we show that considering any sufficiently regular solutions to the continuously stratified system (5.2.1) that is bounded and satisfies the non-cavitation assumptions on a given time interval, we can construct corresponding solutions (H, U) to the multi-layer system (4.2.1) on the same

time interval provided that the number of layers N is sufficiently large, and we quantify the convergence rate between the continuously stratified and multi-layer solutions as N goes to infinity.

Theorem 5.4.1. *Let $\rho_{\text{bott}} > \rho_{\text{surf}} > 0$, $s \in \mathbb{N}$ such that $s > 2 + \frac{1}{2}$, and \underline{M} , M_c , h_* , h^* , $\kappa > 0$. Moreover, consider \underline{h} , $\underline{u} \in W^{2,\infty}((\rho_{\text{surf}}, \rho_{\text{bott}}))$ such that*

$$|\underline{h}|_{W^{2,\infty}} + |\underline{u}|_{W^{2,\infty}} \leq \underline{M},$$

and $(h, u) \in C([0, T]; X_{\varrho, x}^{\infty, s+1, 2})$ solution to (5.2.1) on a time interval $[0, T]$ with $T > 0$ such that for all $t \in [0, T]$

$$\inf_{(x, \varrho) \in \mathbb{R} \times (\rho_{\text{surf}}, \rho_{\text{bott}})} \underline{h}(\varrho) + h(t, x, \varrho) \geq h_*, \quad \sup_{(x, \varrho) \in \mathbb{R} \times (\rho_{\text{surf}}, \rho_{\text{bott}})} \underline{h}(\varrho) + h(t, x, \varrho) \leq h^* \quad (5.4.1)$$

and

$$\|h(t, \cdot)\|_{\infty, s+1, 2} + \|u(t, \cdot)\|_{\infty, s+1, 2} + \|\partial_{\varrho} \partial_x \Lambda^s h(t, \cdot)\|_{L_{\varrho}^{\infty}(L_x^2)} + \|\partial_{\varrho}^2 \partial_x \Lambda^s h(t, \cdot)\|_{L_{\varrho}^{\infty}(L_x^2)} \leq M_c. \quad (5.4.2)$$

Then there exists $c > 0$ and $N_0 \in \mathbb{N}^*$ such that for all $N \geq N_0$ and any initial data $(H_0, U_0) \in H^s(\mathbb{R})^{2N}$ satisfying

$$\|(H_0 - P_N h(0, \cdot), U_0 - P_N u(0, \cdot))\|_s \leq c M_c,$$

the solution to (4.2.1) with $\underline{H} = P_N \underline{h}$, $\underline{U} = P_N \underline{u}$ and satisfying $(H, U)_{t=0} = (H_0, U_0)$ defined in Theorem 4.5.1 is well-defined on the time interval $[0, T]$ and satisfies for any $t \in [0, T]$

$$\inf_{(x, i) \in \mathbb{R} \times \{1, \dots, N\}} \underline{H}_i + (H)_i(t, x) \geq \frac{h_*}{2}, \quad \sup_{(x, i) \in \mathbb{R} \times \{1, \dots, N\}} \underline{H}_i + (H)_i(t, x) \leq 2h^*, \quad (5.4.3)$$

and

$$\|(H, U)\|_s(t) \leq \alpha(1 + \sqrt{\kappa t}) M_c, \quad (5.4.4)$$

with α a universal constant, and the difference between the two solutions satisfies

$$\begin{aligned} \|(H - P_N h, U - P_N u)\|_s(t) &\leq C \|(H - P_N h, U - P_N u)\|_s(0) \exp\left(\frac{C}{2}(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + (CM_c)^2)t\right) \\ &\quad + \frac{C}{2N^2} \int_0^t \exp\left(\frac{C}{2}(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + (CM_c)^2)(t - \tau)\right) d\tau, \end{aligned} \quad (5.4.5)$$

where $\|\cdot\|_s$ is defined as in Theorem 4.5.1, and C depends only on $\rho_{\text{bott}}, \rho_{\text{surf}}, \mathcal{S}, \underline{M}, M_c, h_*, h^*$ (while N_0 and c depend also on T and κ).

Proof. From Theorem 4.5.1 (or Theorem 3.3.1) we set $T_N > 0$ the maximal time of existence and uniqueness of the solution $(H, U) \in C([0, T_N]; H^s(\mathbb{R})^{2N})$ with $H \in L^2(0, T_N; H^{s+1,2}(\mathbb{R}))$ to the system (4.2.1) with $(H, U)_{t=0} = (H_0, U_0)$. We set $T_* \in [0, T_N]$ the larger value such that for all $t \in [0, T_*)$ we have (5.4.3) and

$$\|(H, U)\|_s(t) \leq c_0 M_c,$$

where $c_0 > 0$ will be determined below. We can choose $c > 0$ sufficiently small such that, by continuity of the energy functional we have that $T_* > 0$, and we consider below $t \in (0, T_*)$ such that $t \leq T$.

Notice that the control of the difference $(H - P_N h, U - P_N u)(t)$ induces a control of $(H, U)(t)$ by triangular inequality: using Lemma 7.2.9, there exists $\alpha > 0$ a universal constant such that

$$\begin{aligned} \|(H, U)\|_s(t) &\leq \|(H - P_N h, U - P_N u)\|_s(t) + \|(P_N h, P_N u)\|_s(t) \\ &\leq \|(H - P_N h, U - P_N u)\|_s(t) + \alpha(1 + \sqrt{\kappa t})M_c. \end{aligned}$$

In the following, we prove the estimate (5.4.5) for $t \in (0, T_*)$, that is assuming (5.4.3) and $\|(H, U)\|_s(t) \leq c_0 M_c$. From (5.4.5) we infer that under the assumptions

$$\|(H - P_N h, U - P_N u)\|_s(0) \leq \epsilon \frac{M_c}{2C} \exp\left(-\frac{C}{2}(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + (c_0 M_c)^2)T)\right), \quad (5.4.6)$$

$$\frac{1}{N^2} \frac{\exp\left(\frac{C}{2}(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + (c_0 M_c)^2)T)\right) - 1}{(1 + \kappa^{-1}(\|\mathbf{D}_\rho \underline{U}\|_{l^2}^2 + (c_0 M_c)^2))} \leq \epsilon \frac{M_c}{2}, \quad (5.4.7)$$

where $\epsilon > 0$ is arbitrary, we have

$$\|(H, U)\|_s(t) \leq (\epsilon + \alpha(1 + \sqrt{\kappa t}))M_c.$$

Similarly, choosing ϵ sufficiently small, we can infer (5.4.3) with $h_*/2$ (respectively $2h^*$) replaced with $3h_*/4$ (respectively $3h^*/2$) from (5.4.5)-(5.4.6)-(5.4.7). Hence setting $c_0 = 2(\epsilon + \alpha(1 + \sqrt{T}))$, the usual continuity argument implies that $T_N \geq T_* \geq T$, and the conclusions of the Theorem hold.

Let us now establish (5.4.5), assuming (5.4.3) and $\|(H, U)\|_s(t) \leq c_0M_c$. To this aim we follow the proof of Theorem 4.5.1, using the stability estimates analogous to Lemma 4.4.1 and Lemma 4.4.2 together with the estimates on remainders of Lemma 5.2.1 and Lemma 5.3.3. We use below the results and notations of Lemma 5.3.3.

Following the proof of Lemma 4.4.2 and integrating in time the differential inequality instead of using Gronwall's lemma (and using the fact that $\|\mathbf{S}\|_{l^2 \rightarrow l^2} \leq \frac{1}{\sqrt{N}} \|\mathbf{TS}\|_{l^2 \rightarrow l^2} + \|\mathbf{CS}\|_{l^2 \rightarrow l^2}$) we obtain

$$\begin{aligned} & \|\mathbf{S}(H - P_N h)(t, \cdot)\|_{H^{s,0}}^2 + \|\mathbf{TS}(H - P_N h)(t, \cdot)\|_{H^{s,0}}^2 + \|(U - P_N u)(t, \cdot)\|_{H^{s,0}}^2 \\ & + \kappa \|\partial_x \mathbf{S}(H - P_N h)\|_{L^2(0,t;H^{s,0})}^2 + \kappa \|\partial_x \mathbf{TS}(H - P_N h)\|_{L^2(0,t;H^{s,0})}^2 \\ & \leq c \left(\|\mathbf{S}(H - P_N h)(0, \cdot)\|_{H^{s,0}}^2 + \|(U - P_N u)(0, \cdot)\|_{H^{s,0}}^2 + \|\mathbf{TS}(H - P_N h)(0, \cdot)\|_{H^{s,0}}^2 \right) \\ & + c \int_0^t C(\rho_{\text{surf}}, h_*, c_0 M_c) (1 + \kappa^{-1} \|\mathbf{D}_\rho(\underline{U} + U)(\tau, \cdot)\|_{L_x^\infty(l^2)}) \left(\|(H - P_N h, U - P_N u)\|_s(\tau) \right)^2 d\tau \\ & + c \int_0^t \left(\sum_{0 \leq \alpha \leq s} (\|\mathbf{SR}'_{\alpha,0}(\tau, \cdot)\|_{l^2(L_x^2)} + \|\mathbf{TSR}'_{\alpha,0}(\tau, \cdot)\|_{l^2(L_x^2)} + \|\mathbf{R}'_{\alpha,0}(\tau, \cdot)\|_{l^2(L_x^2)}) \right) \\ & \quad \times \left(\|(H - P_N h, U - P_N u)\|_s(\tau) \right) d\tau, \end{aligned} \tag{5.4.8}$$

where $c > 1$ depends on h_* , h^* , ρ_{surf} and ρ_{bott} .

Similarly, integrating in time the differential inequalities that yield Lemma 4.4.1 and recalling that, by definition,

$$\begin{aligned} & \|(H - P_N h)(t, \cdot)\|_{H^{s-1,1}} + \|(U - P_N u)(t, \cdot)\|_{H^{s,2}} + \kappa^{\frac{1}{2}} \|(H - P_N h)(t, \cdot)\|_{H^{s,2}} \\ & \leq \|(H - P_N h, U - P_N u)\|_s(t), \end{aligned}$$

we find that for $j \in \{1, 2\}$,

$$\begin{aligned} & \| \mathbf{D}_\rho^j \mathbf{S}(H - P_N h)(t, \cdot) \|_{H^{s-j,0}}^2 + \kappa \| \partial_x \mathbf{D}_\rho^j \mathbf{S}(H - P_N h) \|_{L^2(0,t;H^{s-j,0})}^2 \\ & \lesssim \| \mathbf{D}_\rho^j \mathbf{S}(H - P_N h)(0) \|_{H^{s-j,0}}^2 + \int_0^t \| U(\tau, \cdot) \|_{H^{s,2}} \left(\| (H - P_N h, U - P_N u) \|_s(\tau) \right)^2 d\tau \\ & \quad + \int_0^t \left(\sum_{0 \leq \alpha \leq s-j} \| \mathbf{R}'_{\alpha,j}(\tau, \cdot) \|_{l^2(L_x^2)} \right) (\tau) \| (H - P_N h, U - P_N u) \|_s(\tau) d\tau, \end{aligned} \quad (5.4.9)$$

for $j \in \{1, 2\}$,

$$\begin{aligned} & \| \mathbf{D}_\rho^j (U - P_N u)(t, \cdot) \|_{H^{s-j,0}}^2 \lesssim \| \mathbf{D}_\rho^j (U - P_N u)(0) \|_{H^{s-j,0}}^2 \\ & \quad + \int_0^t \left\| \partial_x \left(U - \kappa \frac{\partial_x H}{\underline{H} + H} \right) (\tau, \cdot) \right\|_{l^\infty(L_x^\infty)} \left(\| (H - P_N h, U - P_N u) \|_s(\tau) \right)^2 d\tau \\ & \quad + \int_0^t \left(\sum_{0 \leq \alpha \leq s-j} \| \mathbf{R}'_{\alpha,j}(\tau, \cdot) \|_{l^2(L_x^2)} \right) \left(\| (H - P_N h, U - P_N u) \|_s(\tau) \right) d\tau, \end{aligned} \quad (5.4.10)$$

for $j \in \{0, 1\}$,

$$\begin{aligned} & \| \mathbf{D}_\rho^j (H - P_N h)(t, \cdot) \|_{H^{s-j-1,0}}^2 + \kappa \| \partial_x \mathbf{D}_\rho^j (H - P_N h) \|_{L^2(0,t;H^{s-j-1,0})}^2 \\ & \lesssim \| \mathbf{D}_\rho^j (H - P_N h)(0) \|_{H^{s-j-1,0}}^2 + \int_0^t \| U(\tau, \cdot) \|_{H^{s,2}} \left(\| (H - P_N h, U - P_N u) \|_s(\tau) \right)^2 d\tau \\ & \quad + \int_0^t \left(\sum_{0 \leq \alpha \leq s-j-1} \| \mathcal{R}'_{\alpha,j}(\tau, \cdot) \|_{l^2(L_x^2)} \right) \left(\| (H - P_N h, U - P_N u) \|_s(\tau) \right) d\tau, \end{aligned} \quad (5.4.11)$$

for $j \in \{0, 1, 2\}$,

$$\begin{aligned} & \kappa \| \mathbf{D}_\rho^j (H - P_N h)(t, \cdot) \|_{H^{s-j,0}}^2 + \frac{\kappa^2}{2} \| \partial_x \mathbf{D}_\rho^j (H - P_N h) \|_{L^2(0,t;H^{s-j,0})}^2 \\ & \lesssim \kappa \| \mathbf{D}_\rho^j (H - P_N h)(0) \|_{H^{s-j,0}}^2 + \int_0^t \| U(\tau, \cdot) \|_{H^{s,2}} \left(\| (H - P_N h, U - P_N u) \|_s(\tau) \right)^2 d\tau \\ & \quad + \kappa^{\frac{1}{2}} \int_0^t \left(\sum_{0 \leq \alpha \leq s-j} \| r'_{\alpha,j}(\tau, \cdot) \|_{l^2(L_x^2)} \right) \| (H - P_N h, U - P_N u) \|_s(\tau) d\tau \\ & \quad + \int_0^t \left(\sum_{0 \leq \alpha \leq s-j} \| \mathbf{r}'_{\alpha,j}(\tau, \cdot) \|_{l^2(L_x^2)} \right)^2 d\tau, \end{aligned} \quad (5.4.12)$$

where we used Cauchy-Schwarz and Young inequalities for the contribution of $\mathbf{r}'_{\alpha,j}$.

Moreover as seen in the proof of Theorem 4.5.1 we have

$$\left\| \partial_x \left(U - \kappa \frac{\partial_x H}{\underline{H} + H}(\tau, \cdot) \right) \right\|_{l^\infty(L_x^\infty)} \leq C(\|U(\tau, \cdot)\|_{H^{s,2}} + \kappa \|\partial_x H(\tau, \cdot)\|_{H^{s,2}} + \kappa \|H(\tau, \cdot)\|_{H^{s,2}}^2), \quad (5.4.13)$$

with $C = C(h_*^{-1}, c_0 M_c)$ and

$$\|D_\rho(\underline{U} + U)\|_{L_x^\infty(l^2)}^2 \lesssim \|D_\rho \underline{U}\|_{l^2}^2 + \|U\|_{H^{s,2}}^2 \leq \|D_\rho \underline{U}\|_{l^2}^2 + (c_0 M_c)^2. \quad (5.4.14)$$

Hence gathering estimates (5.4.8)-(5.4.12) with (5.4.13) and (5.4.14), together with the remainder estimates obtained in Lemma 5.2.1 and Lemma 5.3.3, and using the control of (h, u) in (5.4.2) and that for all $\tau \in [0, t]$, $\|(H, U)\|_s(\tau) \leq c_0 M_c$, we obtain

$$\begin{aligned} \left(\|(H - P_N h, U - P_N u)\|_s(t) \right)^2 &\leq C^2 \left(\|(H - P_N h, U - P_N u)\|_s(0) \right)^2 \\ &+ C \int_0^t (1 + \kappa^{-1} (\|D_\rho \underline{U}\|_{l^2}^2 + (c_0 M_c)^2)) \left(\|(H - P_N h, U - P_N u)\|_s(\tau) \right)^2 d\tau \\ &+ \frac{C}{N^2} \int_0^t \|(H - P_N h, U - P_N u)\|_s(\tau) d\tau \\ &+ C \sup_{\tau \in [0, t]} \left(\|(H - P_N h, U - P_N u)\|_s(\tau) \right) \left(\int_0^t \left(\|(H - P_N h, U - P_N u)\|_s(\tau) \right)^2 d\tau \right)^{1/2}, \end{aligned}$$

where $C := C(\rho_{\text{bott}}, \rho_{\text{surf}}^{-1}, h^*, h_*^{-1}, \underline{M}, c_0 M_c) > 1$ and the last term stems from the use of Cauchy-Schwarz inequality in contributions involving either $\|\partial_x(H - P_N h)\|_{H^{s,2}}$ or $\|\partial_x H\|_{H^{s,2}}$. Using Young inequality and augmenting C , this contribution can be absorbed in terms of the other ones, and we have simply

$$\begin{aligned} \sup_{\tau \in [0, t]} \left(\|(H - P_N h, U - P_N u)\|_s(\tau) \right)^2 &\leq C^2 \left(\|(H - P_N h, U - P_N u)\|_s(0) \right)^2 \\ &+ C \int_0^t (1 + \kappa^{-1} (\|D_\rho \underline{U}\|_{l^2}^2 + (c_0 M_c)^2)) \sup_{\tau' \in [0, \tau]} \left(\|(H - P_N h, U - P_N u)\|_s(\tau') \right)^2 d\tau \\ &+ \frac{C}{N^2} \int_0^t \sup_{\tau' \in [0, \tau]} \|(H - P_N h, U - P_N u)\|_s(\tau') d\tau. \end{aligned}$$

By Gronwall's lemma we infer that $\sup_{\tau \in [0, t]} \|(H - P_N h, U - P_N u)\|_s(\tau) \leq Z(t)$ for all

$t \in [0, T_*)$, where Z is the solution to

$$\begin{cases} Z'(\tau) = C(1 + \kappa^{-1}(\|D_\rho \underline{U}\|_{l^2}^2 + (c_0 M_c)^2)Z(\tau) + \frac{C}{N^2}Z^{\frac{1}{2}}(\tau) & \tau \in [0, T_*), \\ Z(0) = C^2\|(H - P_N h, U - P_N u)\|_s(0)^2, \end{cases}$$

whose explicit solution is

$$\begin{aligned} Z(t) = & \left(C\|(H - P_N h, U - P_N u)\|_s(0) \exp\left(\frac{C}{2}(1 + \kappa^{-1}(\|D_\rho \underline{U}\|_{l^2}^2 + (C M_c)^2)t\right) \right. \\ & \left. + \frac{C}{2N^2} \int_0^t \exp\left(\frac{C}{2}(1 + \kappa^{-1}(\|D_\rho \underline{U}\|_{l^2}^2 + (C M_c)^2)(t - \tau)\right) d\tau \right)^2. \end{aligned}$$

Consequently we obtain (5.4.5), and the proof is complete. \square

Remark 5.4.2. — The existence of $T > 0$ and (h, u) sufficiently smooth solutions to the system (5.2.1) over the time interval $[0, T]$ satisfying (5.4.1) and (5.4.2), as considered in the previous theorem, results from Theorem 1.1 in [13].

— The dependency of c and N_0 on T and κ is expressed in (5.4.6) and (5.4.7).

Remark 5.4.3. We used the operator

$$P_N : \begin{array}{ll} \mathcal{C}([\rho_{\text{surf}}, \rho_{\text{bott}}]) & \rightarrow \mathbb{R}^N \\ f & \mapsto (f(\rho_i))_{1 \leq i \leq N} \end{array}$$

to map functions defined on $[\rho_{\text{surf}}, \rho_{\text{bott}}]$ to N -dimensional vectors because it enjoys the property $P_N(fg) = P_N(f)P_N(g)$. However we can adapt the proof of Theorem 5.4.1 so as to replace P_N with

$$\overline{P}_N : \begin{array}{ll} \mathcal{C}([\rho_{\text{surf}}, \rho_{\text{bott}}]) & \rightarrow \mathbb{R}^N \\ f & \mapsto \left(\frac{1}{\rho_{i+1/2} - \rho_{i-1/2}} \int_{\rho_{i-1/2}}^{\rho_{i+1/2}} f(\varrho) d\varrho \right)_{1 \leq i \leq N} \end{array}$$

In this case the physical meaning of the discretized objects is more explicit. For instance, $(\overline{P}_N(\underline{u} + u))_i$ is the i -th layer-averaged horizontal velocity, whereas $(\overline{P}_N(\underline{h} + h))_i = N(\eta_{i-1} - \eta_i)$ is the rescaled layer depth; see Figure 5.1.

5.5 Conclusion

In this chapter we showed that for a sufficiently regular solution (h, u) to the continuously stratified system that exists and is controlled over a time interval $[0, T]$, where $T > 0$, under the assumption that the density is stably stratified $\rho_{\text{bott}} > \rho_{\text{surf}} > 0$, the solution (H, U) to the multi-layer system exists on the same time interval, with a size of $\mathcal{O}(1 + \sqrt{\kappa T})$. Moreover, the size of the error between the two solutions, for N large enough, is of size $\mathcal{O}(1/N^2)$ as long as the initial data remain close (in the sense that the difference is also of size $\mathcal{O}(1/N^2)$). Note that in the latter result, the multiplicative constant of $1/N^2$ depends on the parameter κ . As in the previous chapter, the stable stratification of density plays an important role, this enabled us to close and conclude our energy method. To establish the comparison between the solutions, we used the projection operator P_N (see definition in the beginning of the introduction 5.1) to project the continuous solution (h, u) into an N -dimensional valued function, but other choices could be made; see Remark 5.4.3. Our results rigorously justify the multi-layer strategy to approximate solutions to the continuously stratified system, at least in the stably stratified situation.

SHARP STRATIFICATION

6.1 Introduction

This chapter is the natural follow-up to the previous chapters. Our main interest is to rigorously justify the "inverse" limit, by which we mean the convergence of the hydrostatic Euler equations (Equations (2.2.13) – (2.2.14)), towards the bi-layer shallow water system (6.2.6) in the sharp stratification limit.

We begin this chapter by announcing well-known results on the bi-layer shallow water system (in both the presence and absence of thickness diffusivity $0 < \kappa \leq 1$). Then, we provide a rigorous justification of the limit of the former system when κ vanishes. This result was made possible since we showed that for sufficiently regular initial data satisfying certain hyperbolicity conditions, the time interval of existence obtained is uniform with respect to the thickness diffusivity κ . This enables us to obtain stability estimates uniform with respect to κ and conclude the convergence result.

In the second part of the chapter, and in order to prove the sharp stratification limit, we begin by establishing sharp stability estimates for the hydrostatic Euler equations. Then, we construct an approximating solution to the latter system that is close in a strong topology (L_r^∞), such that it remains also close to the reference solution in a weak topology (L_r^1). This then enables us to conclude the argument and justify the limit (see Proposition 6.3.3).

6.2 The bi-layer shallow water system

6.2.1 The system without thickness diffusivity

We consider the system

$$\begin{cases} \partial_t H_s + \partial_x((\underline{H}_s + H_s)(\underline{U}_s + U_s)) = 0, \\ \partial_t H_b + \partial_x((\underline{H}_b + H_b)(\underline{U}_b + U_b)) = 0, \\ \partial_t U_s + (\underline{U}_s + U_s)\partial_x U_s + \partial_x H_s + \partial_x H_b = 0, \\ \partial_t U_b + (\underline{U}_b + U_b)\partial_x U_b + \frac{\rho_s}{\rho_b}\partial_x H_s + \partial_x H_b = 0. \end{cases} \quad (6.2.1)$$

We shall also always assume $\rho_s \geq 0$ and $\rho_b > 0$. Through rescaling and Galilean invariance we can assume without loss of generality that $\underline{H}_s + \underline{H}_b = 1$ and $\underline{U}_s + \underline{U}_b = 0$.

In compact form, the system reads

$$\partial_t \mathbf{U} + \mathbf{A}(\underline{\mathbf{U}} + \mathbf{U})\partial_x \mathbf{U} = 0$$

with $\mathbf{U} := (H_s, H_b, U_s, U_b)$, $\underline{\mathbf{U}} := (\underline{H}_s, \underline{H}_b, \underline{U}_s, \underline{U}_b)$ and where we introduce the matrix-valued function

$$\mathbf{A} : (H_s, H_b, U_s, U_b) \in \mathbb{R}^4 \mapsto \begin{pmatrix} U_s & 0 & H_s & 0 \\ 0 & U_b & 0 & H_b \\ 1 & 1 & U_s & 0 \\ \frac{\rho_s}{\rho_b} & 1 & 0 & U_b \end{pmatrix}. \quad (6.2.2)$$

The following Lemma concerning the hyperbolicity domain of the bilayer shallow water system is proved in [65, 10, 76].

Lemma 6.2.1. *Let $0 < \rho_s < \rho_b$ and $\mathbf{U} := (H_s, H_b, U_s, U_b) \in \mathbb{R}^4$ be such that that $H_s, H_b > 0$. There exist two values $0 < \text{Fr}_- < \text{Fr}_+$ such that the following holds:*

1. *If $|U_b - U_s| < \sqrt{H_b} \text{Fr}_-$, then there exist four distinct real eigenvalues of the matrix $\mathbf{A}(\mathbf{U})$.*
2. *If $\sqrt{H_b} \text{Fr}_- < |U_b - U_s| < \sqrt{H_b} \text{Fr}_+$, then there exist two distinct real eigenvalues of the matrix $\mathbf{A}(\mathbf{U})$ and two distinct complex conjugate eigenvalues.*
3. *If $|U_b - U_s| > \sqrt{H_b} \text{Fr}_+$, then there exist four distinct real eigenvalues of the matrix $\mathbf{A}(\mathbf{U})$.*

Moreover, Fr_- and Fr_+ depend only and smoothly on $H_s/H_b \in (0, +\infty)$ and $\rho_s/\rho_b \in (0, 1)$.

Remark 6.2.2. *Ovsjannikov [65]—revisited by Barros and Choi [10] and then by Virísimo and Milewski [76]— provided a nice geometrical approach to the critical values Fr_+ and Fr_- . The characteristic polynomial associated to $\mathbf{A}(\mathbf{U})$ is*

$$P(\lambda) = ((U_b - \lambda)^2 - H_b)((U_s - \lambda)^2 - H_s) - \frac{\rho_s}{\rho_b} H_s H_b.$$

Notice that $\lambda \in \mathbb{R}$ is a real root of P if and only if $(p_s, p_b) := (\frac{U_s - \lambda}{\sqrt{H_s}}, \frac{U_b - \lambda}{\sqrt{H_b}})$ satisfies the following identities:

$$(p_s^2 - 1)(p_b^2 - 1) = \frac{\rho_s}{\rho_b}, \quad p_s \sqrt{H_s} - U_s = p_b \sqrt{H_b} - U_b. \quad (6.2.3)$$

The first equality describes a fourth-order curve parametrized by ρ_s/ρ_b having four axes of symmetry and consisting of an inner closed curve and four hyperbolic branches and the second equality describes the straight line with slope $\sqrt{H_s/H_b}$ and intercept $(U_b - U_s)/\sqrt{H_b}$. In this geometrical approach, Fr_- and Fr_+ (and their opposite) are the intercepts of the tangents to the fourth-order curves with slope $\sqrt{H_s/H_b}$.

Figure 6.1 reproduces the aforementioned curves and straight lines for several parameter values.

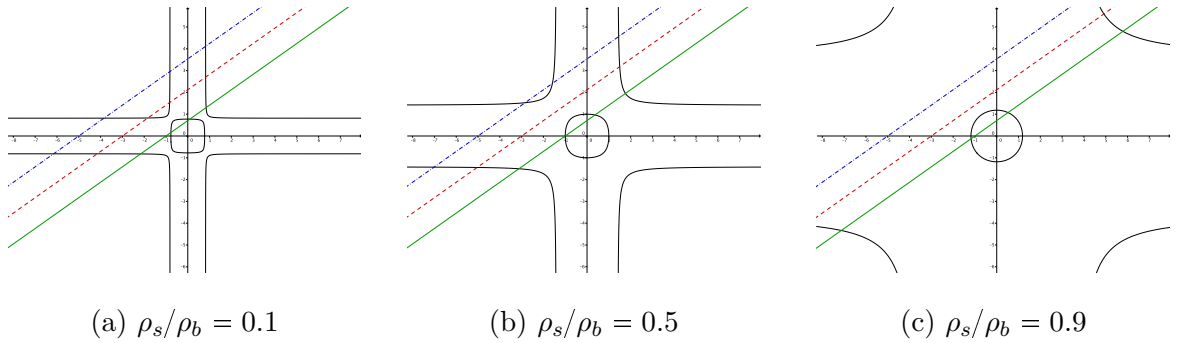


Figure 6.1 – Solutions to (6.2.3) with $H_s = 1/3$, $H_b = 2/3$, and different values for ρ_s/ρ_b . Solutions to the quartic equation are in black (plain). Solutions to the linear equation with $(U_b - U_s)/\sqrt{H_b} = 1/2$ (green, plain), $(U_b - U_s)/\sqrt{H_b} = 3/2$ (red, dashed) and $(U_b - U_s)/\sqrt{H_b} = 5/2$ (blue, dot-dashed).

In this work we restrict our analysis to the hyperbolic domain described by Lemma 6.2.1(1). While standard theory for strictly hyperbolic systems guarantees the existence of a symmetrizer to (6.2.1) by using spectral projections, the following Lemma provides an (almost) explicit expression for such a symmetrizer.

Lemma 6.2.3. *Let $0 < \rho_s < \rho_b$ and $\mathbf{U} := (H_s, H_b, U_s, U_b) \in \mathbb{R}^4$ be such that that $H_s, H_b > 0$ and*

$$|U_s - U_b| < \sqrt{H_b} \text{Fr}_- \quad (6.2.4)$$

where $\text{Fr}_- = \text{Fr}_-(H_s/H_b, \rho_s/\rho_b) > 0$ has been defined in Lemma 6.2.1.

There exists $\lambda \in [\min(\{U_s, U_b\}), \max(\{U_s, U_b\})]$ such that, denoting $U_\ell^\lambda := U_\ell - \lambda \in [-|U_b - U_s|, |U_b - U_s|]$ for $\ell \in \{s, b\}$, the matrix

$$\mathbf{S}^\lambda(\mathbf{U}) := \begin{pmatrix} \frac{\rho_s}{\rho_b} & \frac{\rho_s}{\rho_b} & \frac{\rho_s}{\rho_b} U_s^\lambda & 0 \\ \frac{\rho_s}{\rho_b} & 1 & 0 & U_b^\lambda \\ \frac{\rho_s}{\rho_b} U_s^\lambda & 0 & \frac{\rho_s}{\rho_b} H_s & 0 \\ 0 & U_b^\lambda & 0 & H_b \end{pmatrix}$$

satisfies (i) $\mathbf{S}^\lambda A$ is symmetric; and (ii) \mathbf{S}^λ is symmetric, definite positive.

Moreover, λ can be chosen so that $\frac{\lambda - U_\ell}{\sqrt{H_\ell}}$ (for $\ell \in \{s, b\}$) depends only and smoothly on $H_s/H_b > 0$, $\rho_s/\rho_b \in (0, 1)$, and $(U_b - U_s)/\sqrt{H_b} \in (-\text{Fr}_-, \text{Fr}_-)$.

Proof. It is straightforward to check that \mathbf{S}^λ and $\mathbf{S}^\lambda A$ are symmetric (and real-valued) for any value of $\lambda \in \mathbb{R}$. In order to prove that \mathbf{S}^λ is definite positive for a suitable choice of λ , we rely on Sylvester's criterion. We obtain the requirements

$$H_b > 0, \quad \frac{\rho_s}{\rho_b} H_s H_b > 0, \quad \frac{\rho_s}{\rho_b} H_s H_b - \frac{\rho_s}{\rho_b} H_s (U_b^\lambda)^2 > 0$$

and

$$\left(\frac{\rho_s}{\rho_b}\right)^2 \left((U_b^\lambda)^2 - H_b \right) \left((U_s^\lambda)^2 - H_s \right) - \frac{\rho_s}{\rho_b} H_s H_b > 0.$$

The last inequality is equivalent to $P(\lambda) > 0$ where P is the aforementioned characteristic polynomial. By Lemma 6.2.1, under the condition (6.2.4) there are four distinct real roots to P , which we can denote $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ and $P(\lambda) > 0$ for any $\lambda \in (\lambda_2, \lambda_3)$. Moreover for all $\lambda \in (\lambda_2, \lambda_3)$, $(p_s^\lambda, p_b^\lambda) := \left(\frac{U_s - \lambda}{\sqrt{H_s}}, \frac{U_b - \lambda}{\sqrt{H_b}}\right)$ belongs to the domain delimited by the inner closed curve, and in particular we have $(p_\ell^\lambda)^2 < 1$. Hence we find that for all $\lambda \in (\lambda_2, \lambda_3)$ all principal minors are positive, and hence \mathbf{S}^λ is definite positive.

By the standard perturbation theory [49], p_ℓ^λ for $\ell \in \{s, b\}$ and $\lambda \in \{\lambda_2, \lambda_3\}$ depend smoothly on $H_s/H_b > 0$, $\rho_s/\rho_b \in (0, 1)$ and $(U_b - U_s)/\sqrt{H_b} \in (-\text{Fr}_-, \text{Fr}_-)$. What is more, we can always choose (smoothly) $\lambda \in (\lambda_2, \lambda_3)$ so that $p_s^\lambda > 0$ and $p_b^\lambda < 0$ when $U_b < U_s$, or $p_s^\lambda < 0$ and $p_b^\lambda > 0$ when $U_b > U_s$, which corresponds to enforcing $\lambda \in [\min(\{U_s, U_b\}), \max(\{U_s, U_b\})]$. This concludes the proof. \square

The following proposition follows from the standard theory on strictly hyperbolic systems (see e.g. [12]). For convenience, we define for $\varsigma \in (0, 1)$ a compact subset of the domain of strict hyperbolicity as

$$\mathfrak{p}^\varsigma := \left\{ (\rho_s, \rho_b, H_s, H_b, U_s, U_b) \in \mathbb{R}^6 : \right. \\ \left. \varsigma/2 \leq \rho_s/\rho_b \leq 1-\varsigma/2, \quad \varsigma \leq H_s/H_b \leq \varsigma^{-1}, \quad H_s+H_b \geq \varsigma, \quad \text{Fr}_- - \frac{|U_b-U_s|}{\sqrt{H_b}} \geq \varsigma \right\} \quad (6.2.5)$$

where $\text{Fr}_- = \text{Fr}_-(\rho_s/\rho_b, H_s/H_b)$ is defined in Lemma 6.2.1.

Proposition 6.2.4 (Well-posedness). *Let $s \geq s_0 > 3/2$, $\varsigma > 0$ and $M_0 > 0$. There exist $C > 0$ and $T > 0$ such that the following holds.*

For all $(\rho_s, \rho_b, \underline{H}_s, \underline{H}_b, \underline{U}_s, \underline{U}_b) \in \mathbb{R}^6$ such that $\underline{H}_s + \underline{H}_b = 1$ and $\underline{U}_s + \underline{U}_b = 0$ and $(H_s^0, H_b^0, U_s^0, U_b^0) \in H^s(\mathbb{R})^4$ such that

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s^0(x), \underline{H}_b + H_b^0(x), \underline{U}_s + U_s^0(x), \underline{U}_b + U_b^0(x)) \in \mathfrak{p}^\varsigma$$

and

$$\|(H_s^0, H_b^0, U_s^0, U_b^0)\|_{H^{s_0}} \leq M_0$$

there exists a unique $(H_s, H_b, U_s, U_b) \in C([0, T^]; H^s(\mathbb{R})^4) \cap C^1([0, T^*]; H^{s-1}(\mathbb{R})^4)$ maximal-in-time (classical) solution to (6.2.1) emerging from the initial data $(H_s, H_b, U_s, U_b)|_{t=0} = (H_s^0, H_b^0, U_s^0, U_b^0)$.*

Moreover, one has $T^ > T/M_0$ and for any $t \in [0, T/M_0]$ one has*

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s(t, x), \underline{H}_b + H_b(t, x), \underline{U}_s + U_s(t, x), \underline{U}_b + U_b(t, x)) \in \mathfrak{p}^{\varsigma/2}$$

and

$$\|(H_s(t, \cdot), H_b(t, \cdot), U_s(t, \cdot), U_b(t, \cdot))\|_{H^s} \leq C \|(H_s^0, H_b^0, U_s^0, U_b^0)\|_{H^s}.$$

Moreover, the maximal existence time (resp. the emerging solution in $C([0, T^]; H^s(\mathbb{R})^4)$) is a lower semi-continuous (resp. continuous) function of the initial data in $H^s(\mathbb{R})^4$ and if $T^* < \infty$ then*

$$\|(H_s(t, \cdot), H_b(t, \cdot), U_s(t, \cdot), U_b(t, \cdot))\|_{H^{s_0}} \rightarrow \infty \text{ as } t \rightarrow T^*.$$

6.2.2 The system with diffusivity

We now consider the system

$$\begin{cases} \partial_t H_s + \partial_x((\underline{H}_s + H_s)(\underline{U}_s + U_s)) = \kappa \partial_x^2 H_s, \\ \partial_t H_b + \partial_x((\underline{H}_b + H_b)(\underline{U}_b + U_b)) = \kappa \partial_x^2 H_b, \\ \partial_t U_s + (\underline{U}_s + U_s - \kappa \frac{\partial_x H_s}{\underline{H}_s + H_s}) \partial_x U_s + \partial_x H_s + \partial_x H_b = 0, \\ \partial_t U_b + (\underline{U}_b + U_b - \kappa \frac{\partial_x H_b}{\underline{H}_b + H_b}) \partial_x U_b + \frac{\rho_s}{\rho_b} \partial_x H_s + \partial_x H_b = 0. \end{cases} \quad (6.2.6)$$

Proposition 6.2.5 (Small time well-posedness). *Let $s \geq s_0 > 3/2$, $\varsigma \in (0, 1)$, $M_0 > 0$ and $c > 1$. There exists $T > 0$ such that the following holds.*

For all $\kappa > 0$, for all $(\rho_s, \rho_b, \underline{H}_s, \underline{H}_b, \underline{U}_s, \underline{U}_b) \in \mathbb{R}^6$ such that $\underline{H}_s + \underline{H}_b = 1$ and $\underline{U}_s + \underline{U}_b = 0$ and for all $(H_s^0, H_b^0, U_s^0, U_b^0) \in H^s(\mathbb{R})^4$ such that

$$0 \leq \rho_s / \rho_b \leq \varsigma^{-1} \quad \text{and} \quad \forall x \in \mathbb{R}, \quad \underline{H}_s + H_s^0(x) \geq \varsigma, \quad \underline{H}_b + H_b^0(x) \geq \varsigma$$

and

$$\|(H_s^0, H_b^0, U_s^0, U_b^0)\|_{H^{s_0}} \leq M_0$$

there exists a unique $(H_s, H_b, U_s, U_b) \in C([0, T^]; H^s(\mathbb{R})^4)$ maximal-in-time strong solution to (6.2.6) emerging from the initial data $(H_s, H_b, U_s, U_b)|_{t=0} = (H_s^0, H_b^0, U_s^0, U_b^0)$.*

Moreover, $T^ > \kappa T$ and for any $t \in [0, \kappa T]$ one has*

$$\forall x \in \mathbb{R}, \quad \underline{H}_s + H_s(t, x) \geq \varsigma/c, \quad \underline{H}_b + H_b(t, x) \geq \varsigma/c$$

and

$$\max(\{\|(H_s, H_b, U_s, U_b)\|_{L^\infty(0, t; H^s)}, \kappa^{1/2} \|(\partial_x H_s, \partial_x H_b)\|_{L^2(0, t; H^s)}\}) \leq c \|(H_s^0, H_b^0, U_s^0, U_b^0)\|_{H^s}.$$

Moreover, the maximal existence time (resp. the emerging solution in $C([0, T^]; H^s(\mathbb{R})^4)$) is a lower semi-continuous (resp. continuous) function of the initial data in $H^s(\mathbb{R})^4$ and if $T^* < \infty$ then*

$$\|(H_s(t, \cdot), H_b(t, \cdot), U_s(t, \cdot), U_b(t, \cdot))\|_{H^{s_0}} \rightarrow \infty \text{ as } t \rightarrow T^*.$$

Proof. The proof has been given in Theorem 3.2.4, but we sketch it here for convenience.

We view (6.2.6) as a system of two transport-diffusion equations and two transport equations, coupled only through order-zero source terms:

$$\begin{cases} \partial_t H_s + (\underline{U}_s + U_s) \partial_x H_s - \kappa \partial_x^2 H_s = -(\underline{H}_s + H_s) \partial_x U_s, \\ \partial_t H_b + (\underline{U}_b + U_b) \partial_x H_b - \kappa \partial_x^2 H_b = -(\underline{H}_b + H_b) \partial_x U_b, \\ \partial_t U_s + (\underline{U}_s + U_s - \kappa \frac{\partial_x H_s}{\underline{H}_s + H_s}) \partial_x U_s = -\partial_x H_s - \partial_x H_b, \\ \partial_t U_b + (\underline{U}_b + U_b - \kappa \frac{\partial_x H_b}{\underline{H}_b + H_b}) \partial_x U_b = -\frac{\rho_s}{\rho_b} \partial_x H_s - \partial_x H_b. \end{cases}$$

The standard theory on transport and transport-diffusion equations (see [8]) allows to bootstrap the standard fixed-point strategy through Picard iterates

$$\begin{cases} \partial_t H_s^{n+1} + (\underline{U}_s + U_s^n) \partial_x H_s^{n+1} - \kappa \partial_x^2 H_s^{n+1} = -(\underline{H}_s + H_s^n) \partial_x U_s^n, \\ \partial_t H_b^{n+1} + (\underline{U}_b + U_b^n) \partial_x H_b^{n+1} - \kappa \partial_x^2 H_b^{n+1} = -(\underline{H}_b + H_b^n) \partial_x U_b^n, \\ \partial_t U_s^{n+1} + (\underline{U}_s + U_s^n - \kappa \frac{\partial_x H_s^n}{\underline{H}_s + H_s^n}) \partial_x U_s^{n+1} = -\partial_x H_s^n - \partial_x H_b^n, \\ \partial_t U_b^{n+1} + (\underline{U}_b + U_b^n - \kappa \frac{\partial_x H_b^n}{\underline{H}_b + H_b^n}) \partial_x U_b^{n+1} = -\frac{\rho_s}{\rho_b} \partial_x H_s^n - \partial_x H_b^n, \end{cases}$$

which defines a sequence satisfying the following estimates (where c_0 is a non-essential constant depending on s)

$$\begin{aligned} & \max(\{ \| (H_s^{n+1}, H_b^{n+1}, U_s^{n+1}, U_b^{n+1}) \|_{L^\infty(0,t;H^s)}, \kappa^{1/2} \| (\partial_x H_s^{n+1}, \partial_x H_b^{n+1}) \|_{L^2(0,t;H^s)} \}) \\ & \leq \left(\| (H_s^0, H_b^0) \|_{H^s} + \kappa^{-1/2} (\| \underline{H}_s + H_s^n \|_{L^\infty(0,t;H^s)}) \| U_s^n \|_{L^2(0,t;H^s)} + (\| \underline{H}_b + H_b^n \|_{L^\infty(0,t;H^s)}) \| U_b^n \|_{L^2(0,t;H^s)} \right) \\ & \quad + \| (U_s^0, U_b^0) \|_{H^s} + \left(1 + \frac{\rho_s}{\rho_b} \right) \| (\partial_x H_s^n, \partial_x H_b^n) \|_{L^1(0,t;H^s)} \\ & \quad \times \exp \left(c_0 \| (U_s^n, U_b^n) \|_{L^1(0,t;H^s)} + c_0 \kappa \| \left(\frac{\partial_x H_s^n}{\underline{H}_s + H_s^n}, \frac{\partial_x H_b^n}{\underline{H}_b + H_b^n} \right) \|_{L^1(0,t;H^s)} \right) \end{aligned}$$

and converging in $C(0, t; H^s)$ provided $t \in (0, \kappa T]$ where T is chosen sufficiently small.

The proof of the continuity of the flow map can be obtained along the same lines, using the continuity with respect to the initial data and Lipschitz-continuity with respect to source terms of the transport-diffusion and transport equations. \square

Remark 6.2.6. *The terminology ‘‘Small time well-posedness’’ in Proposition 6.2.5 refers to the fact that the time of existence and control of solutions of the above result is limited to $T^* \gtrsim \kappa$, and in particular may vanish as $\kappa \searrow 0$. Notice that, differently from the statement of Proposition 6.2.4, we do not assume that the flow is stably stratified, namely*

$\rho_s < \rho_b$. Assuming additionally that the flow is stably stratified, we recall that we improved this result in situations with small shear velocities and small deviations from the shear equilibrium, and obtained in Theorem 4.5.1 the existence and uniform control of solution up to times $T^* \gtrsim (1 + \kappa^{-1}(|\underline{U}_b - \underline{U}_s|^2 + M_0^2))^{-1}$. In the following results, we complete the picture by showing that, in the situation where the shear velocity is small enough to guarantee that the flow is in the hyperbolic domain of the non-diffusive equation, then the time of existence is uniform with respect to $\kappa \in (0, 1]$. In fact we shall prove the expected property that solutions to the diffusive system (6.2.6) converge as $\kappa \searrow 0$ towards corresponding solutions to the non-diffusive system (6.2.1), as long as the non-diffusive solution is bounded.

In order to obtain stability estimates that are uniform with respect to $\kappa \in (0, 1]$, we rely on two main ideas. Firstly, we shall use energy estimates using the explicit symmetrizer adapted to the non-diffusive system introduced in Lemma 6.2.3 (while the strategy in Chapter 4 precisely in Lemma 4.4.2 and Theorem 4.5.1 we used only its block-diagonal component). Because the non-diagonal components of the symmetrizer behave poorly with respect to the diffusive contributions, we need another ingredient. Specifically, we notice that the total velocities $V_\ell := U_\ell - \kappa \frac{\partial_x H_\ell}{\underline{H}_\ell + H_\ell}$ ($\ell \in \{s, b\}$) associated to solutions to (6.2.6) satisfy the system

$$\begin{cases} \partial_t H_s + \partial_x((\underline{H}_s + H_s)(\underline{U}_s + V_s)) = 0, \\ \partial_t H_b + \partial_x((\underline{H}_b + H_b)(\underline{U}_s + V_b)) = 0, \\ \partial_t V_s + (\underline{U}_s + V_s - \kappa \frac{\partial_x H_s}{\underline{H}_s + H_s}) \partial_x V_s + \partial_x H_s + \partial_x H_b = \kappa \partial_x^2 V_s, \\ \partial_t V_b + (\underline{U}_b + V_b - \kappa \frac{\partial_x H_b}{\underline{H}_b + H_b}) \partial_x V_b + \frac{\rho_s}{\rho_b} \partial_x H_s + \partial_x H_b = \kappa \partial_x^2 V_b. \end{cases} \quad (6.2.7)$$

We observe that diffusive terms act as effective viscosity contributions on the total velocities. The last two equations read equivalently

$$\begin{cases} \partial_t V_s + (\underline{U}_s + V_s) \partial_x V_s + \partial_x H_s + \partial_x H_b = \frac{\kappa}{\underline{H}_s + H_s} \partial_x((\underline{H}_s + H_s) \partial_x V_s), \\ \partial_t V_b + (\underline{U}_b + V_b) \partial_x V_b + \frac{\rho_s}{\rho_b} \partial_x H_s + \partial_x H_b = \frac{\kappa}{\underline{H}_b + H_b} \partial_x((\underline{H}_b + H_b) \partial_x V_b), \end{cases}$$

and we recognize the shallow-water equations with degenerate viscosity contributions which were advocated by Gent in [37] and derived from the Navier–Stokes equations in [41, 19]. In their analysis of such systems (and generalizations thereof), Bresch and Desjardins introduced the so-called BD entropy in [17, 15, 16] (see also [18] for a refined analysis),

which is based precisely in the reformulation of (6.2.7) as (6.2.6) (in dimension $d = 1$).

In the same spirit, we combine the regularizing effects of the effective diffusivity and viscosity terms with aforementioned energy estimates, which allows us to obtain suitable stability estimates presented in Lemma 6.2.7, below.

Applying Lemma 6.2.7 to (the derivatives of) solutions to (6.2.6)-(6.2.7), we find a time of existence which is uniform with respect to κ . We state the result in forthcoming Proposition 6.2.8.

Applying Lemma 6.2.7 to (the derivatives of) the difference between solutions to (6.2.6)-(6.2.7) and corresponding solutions to the non-diffusive system (6.2.1) ($\kappa = 0$) yields the aforementioned convergence of the former towards the latter as $\kappa \searrow 0$, on a time interval defined by the solutions without diffusivity. We state the result in forthcoming Proposition 6.2.9.

Lemma 6.2.7 (Stability). *Let $\varsigma \in (0, 1)$ and $M > 0$. There exists $c > 0$ depending only on ς and $C > 0$ depending also on M such that the following holds.*

Let $\kappa \in (0, 1]$, $0 < \rho_s < \rho_b$ and $\mathbf{U} := (H_s, H_b, U_s, U_b)$, $\mathbf{V} := (H_s, H_b, V_s, V_b) \in C([0, T]; W^{1, \infty}(\mathbb{R}^4)) \cap C^1([0, T]; L^\infty(\mathbb{R}^4))$ be such that for all $t \in [0, T]$, the hyperbolicity condition holds:

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, H_s(t, x), H_b(t, x), U_s(t, x), U_b(t, x)) \in \mathfrak{p}^\varsigma,$$

where \mathfrak{p}^ς is defined in (6.2.5), and

$$\|\mathbf{U}(t, \cdot)\|_{W^{1, \infty}} + \|\mathbf{V}(t, \cdot)\|_{W^{1, \infty}} + \kappa \|(\partial_x^2 H_s(t, \cdot), \partial_x^2 H_b(t, \cdot))\|_{L^\infty} + \kappa^{-1} \|(\mathbf{U} - \mathbf{V})(t, \cdot)\|_{L^\infty} + \|\partial_t \mathbf{U}(t, \cdot)\|_{L^\infty} \leq M.$$

Let $\dot{\mathbf{U}} := (\dot{H}_s, \dot{H}_b, \dot{U}_s, \dot{U}_b)$ and $\dot{\mathbf{V}} := (\dot{H}_s, \dot{H}_b, \dot{V}_s, \dot{V}_b)$ be sufficiently regular solutions to the linearized equations with remainders

$$\begin{aligned} \partial_t \dot{\mathbf{U}} + \mathbf{A}^\kappa(\mathbf{U}) \partial_x \dot{\mathbf{U}} &= \kappa \mathbf{D}_1 \partial_x^2 \dot{\mathbf{U}} + \mathbf{R}_U, \\ \partial_t \dot{\mathbf{V}} + \mathbf{A}^\kappa(\mathbf{V}) \partial_x \dot{\mathbf{V}} &= \kappa \mathbf{D}_2 \partial_x^2 \dot{\mathbf{V}} + \mathbf{R}_V, \end{aligned}$$

where we denote

$$\mathbf{A}^\kappa : (H_s, H_b, U_s, U_b) \mapsto \begin{pmatrix} U_s & 0 & H_s & 0 \\ 0 & U_b & 0 & H_b \\ 1 & 1 & U_s - \kappa \frac{\partial_x H_s}{H_s} & 0 \\ \frac{\rho_s}{\rho_b} & 1 & 0 & U_b - \kappa \frac{\partial_x H_b}{H_b} \end{pmatrix},$$

and

$$\mathbf{D}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{D}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, denote $R := (R_s, R_b)$ such that

$$\dot{V}_s = \dot{U}_s - \kappa \frac{\partial_x \dot{H}_s}{H_s} + R_s, \quad \dot{V}_b = \dot{U}_b - \kappa \frac{\partial_x \dot{H}_b}{H_b} + R_b. \quad (6.2.8)$$

Then, for any $t \in [0, T]$, one has the estimate

$$\begin{aligned} & \|\dot{U}(t, \cdot)\|_{L^2} + \|\dot{V}(t, \cdot)\|_{L^2} + c\kappa^{1/2} \|\partial_x \dot{V}(t, \cdot)\|_{L^2(0,t;L^2)} \leq c^{-1} (\|\dot{U}(t=0, \cdot)\|_{L^2} + \|\dot{V}(t=0, \cdot)\|_{L^2}) \exp(C M t) \\ & + C \int_0^t (\|\mathbf{R}_U(t', \cdot)\|_{L^2} + \|\mathbf{R}_V(t', \cdot)\|_{L^2} + M \|R(t', \cdot)\|_{H^1}) \exp(C M (t - t')) dt. \end{aligned}$$

Proof. Denote

$$\mathbf{S}^\lambda : (H_s, H_b, U_s, U_b) \mapsto \begin{pmatrix} \frac{\rho_s}{\rho_b} & \frac{\rho_s}{\rho_b} & \frac{\rho_s}{\rho_b} U_s^\lambda & 0 \\ \frac{\rho_s}{\rho_b} & 1 & 0 & U_b^\lambda \\ \frac{\rho_s}{\rho_b} U_s^\lambda & 0 & \frac{\rho_s}{\rho_b} H_s & 0 \\ 0 & U_b^\lambda & 0 & H_b \end{pmatrix}$$

where $U_\ell^\lambda := U_\ell - \lambda$ (for $\ell \in \{s, b\}$) with λ provided in Lemma 6.2.3. Using that $\mathbf{S}^\lambda(\cdot)$ and

$S^\lambda(\cdot)A^0(\cdot)$ are symmetric, and integration by parts, we have the energy identities

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left((S^\lambda(\mathbf{U})\dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + (S^\lambda(\mathbf{U})\dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \right) \\
 &= (S^\lambda(\mathbf{U})\partial_t \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + (S^\lambda(\mathbf{U})\partial_t \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} + \frac{1}{2} ([\partial_t, S^\lambda(\mathbf{U})]\dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \frac{1}{2} ([\partial_t, S^\lambda(\mathbf{V})]\dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \\
 &= -(S^\lambda(\mathbf{U})A^\kappa(\mathbf{U})\partial_x \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} - (S^\lambda(\mathbf{U})A^\kappa(\mathbf{V})\partial_x \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} + \kappa(S^\lambda(\mathbf{U})D_1\partial_x^2 \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} \\
 &\quad + \kappa(S^\lambda(\mathbf{U})D_2\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} + (S^\lambda(\mathbf{U})\mathbf{R}_U, \dot{\mathbf{U}})_{L^2} + (S^\lambda(\mathbf{U})\mathbf{R}_V, \dot{\mathbf{V}})_{L^2} \\
 &\quad + \frac{1}{2} ([\partial_t, S^\lambda(\mathbf{U})]\dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \frac{1}{2} ([\partial_t, S^\lambda(\mathbf{U})]\dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \\
 &= \frac{1}{2} ([\partial_x, S^\lambda(\mathbf{U})A^0(\mathbf{U})]\dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \frac{1}{2} ([\partial_x, S^\lambda(\mathbf{V})A^0(\mathbf{V})]\dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \\
 &\quad - (S^\lambda(\mathbf{U})(A^\kappa(\mathbf{U}) - A^0(\mathbf{U}))\partial_x \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} - (S^\lambda(\mathbf{V})(A^\kappa(\mathbf{V}) - A^0(\mathbf{V}))\partial_x \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \\
 &\quad - ((S^\lambda(\mathbf{U}) - S^\lambda(\mathbf{V}))A^\kappa(\mathbf{V})\partial_x \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} + \frac{1}{2} ([\partial_t, S^\lambda(\mathbf{U})]\dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \frac{1}{2} ([\partial_t, S^\lambda(\mathbf{U})]\dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \\
 &\quad + (S^\lambda(\mathbf{U})\mathbf{R}_U, \dot{\mathbf{U}})_{L^2} + (S^\lambda(\mathbf{U})\mathbf{R}_V, \dot{\mathbf{V}})_{L^2} + \kappa(S^\lambda(\mathbf{U})D_1\partial_x^2 \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \kappa(S^\lambda(\mathbf{U})D_2\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \\
 &=: A - (S^\lambda(\mathbf{U})(A^\kappa(\mathbf{U}) - A^0(\mathbf{U}))\partial_x \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \kappa(S^\lambda(\mathbf{U})D_1\partial_x^2 \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \kappa(S^\lambda(\mathbf{U})D_2\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2}.
 \end{aligned}$$

By means of Cauchy–Schwarz inequality we find that

$$|A| \leq C (\|\dot{\mathbf{U}}\|_{L^2} + \|\dot{\mathbf{V}}\|_{L^2}) \times (M\|\dot{\mathbf{U}}\|_{L^2} + M\|\dot{\mathbf{V}}\|_{L^2} + \kappa M\|\partial_x \dot{\mathbf{V}}\|_{L^2} + \|\mathbf{R}_U\|_{L^2} + \|\mathbf{R}_V\|_{L^2}),$$

where C denotes a multiplicative constant depending only on ς and M , and which may change from line to line. We now focus on the remaining terms. We first notice that defects of symmetry in $S^\lambda(\mathbf{U})(A^\kappa(\mathbf{U}) - A^0(\mathbf{U}))$ arise only in the first two rows, and that the first two components of $\dot{\mathbf{U}}$ equal the first two components of $\dot{\mathbf{V}}$. Hence using integration by parts and Cauchy–Schwarz inequality we infer

$$|(S^\lambda(\mathbf{U})(A^\kappa(\mathbf{U}) - A^0(\mathbf{U}))\partial_x \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2}| \leq \kappa C M \|\dot{\mathbf{U}}\|_{L^2} \times (\|\dot{\mathbf{U}}\|_{L^2} + \|\partial_x \dot{\mathbf{V}}\|_{L^2}).$$

Then, again making use of the identity $D_1\partial_x^2 \dot{\mathbf{U}} = (\partial_x^2 \dot{H}_1, \partial_x^2 \dot{H}_2, 0, 0) = D_1\partial_x^2 \dot{\mathbf{V}}$ we infer that

$$\begin{aligned}
 \kappa(S^\lambda(\mathbf{U})D_1\partial_x^2 \dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + \kappa(S^\lambda(\mathbf{U})D_2\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} &= \kappa(S^\lambda(\mathbf{U})(D_1 + D_2)\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \\
 &\quad + \kappa(S^\lambda(\mathbf{U})D_1\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{U}} - \dot{\mathbf{V}})_{L^2}.
 \end{aligned}$$

After integration by parts, and since $D_1 + D_2 = \text{Id}$, we find that

$$\kappa(\mathbf{S}^\lambda(\mathbf{U})(D_1 + D_2)\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2} \leq -\kappa(\mathbf{S}^\lambda(\mathbf{U})\partial_x \dot{\mathbf{V}}, \partial_x \dot{\mathbf{V}})_{L^2} + \kappa C M \|\partial_x \dot{\mathbf{V}}\|_{L^2} \|\dot{\mathbf{V}}\|_{L^2}.$$

Then, using from (6.2.8) that $\kappa\partial_x \dot{H}_\ell = H_\ell(\dot{U}_\ell - \dot{V}_\ell) + R_\ell$ (where $\ell \in \{s, b\}$), we obtain the identities

$$\begin{aligned} \kappa(\mathbf{S}^\lambda(\mathbf{U})D_1\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{U}} - \dot{\mathbf{V}})_{L^2} &= \kappa \sum_{\ell \in \{s, b\}} (\rho_\ell U_\ell^\lambda \partial_x^2 \dot{H}_\ell, \dot{U}_\ell - \dot{V}_\ell)_{L^2} \\ &= \sum_{\ell \in \{s, b\}} (\rho_\ell U_\ell^\lambda \partial_x (H_\ell(\dot{U}_\ell - \dot{V}_\ell) + R_\ell), \dot{U}_\ell - \dot{V}_\ell)_{L^2} \\ &= \sum_{\ell \in \{s, b\}} \frac{\rho_\ell}{2} ((U_\ell^\lambda \partial_x H_\ell - H_\ell \partial_x U_\ell^\lambda)(\dot{U}_\ell - \dot{V}_\ell), \dot{U}_\ell - \dot{V}_\ell)_{L^2} \\ &\quad + \kappa \sum_{\ell \in \{s, b\}} \rho_\ell (U_\ell^\lambda \partial_x (H_\ell R_\ell), \dot{U}_\ell - \dot{V}_\ell)_{L^2}, \end{aligned}$$

where we used integration by parts in the last line. We infer

$$\kappa(\mathbf{S}^\lambda(\mathbf{U})D_1\partial_x^2 \dot{\mathbf{V}}, \dot{\mathbf{U}} - \dot{\mathbf{V}})_{L^2} \leq C M (\|\dot{\mathbf{U}}\|_{L^2} + \|\dot{\mathbf{V}}\|_{L^2})^2 + C M (\|\dot{\mathbf{U}}\|_{L^2} + \|\dot{\mathbf{V}}\|_{L^2}) \|R\|_{H^1}.$$

Combining all these estimate and denoting

$$\mathcal{E} := (\mathbf{S}^\lambda(\mathbf{U})\dot{\mathbf{U}}, \dot{\mathbf{U}})_{L^2} + (\mathbf{S}^\lambda(\mathbf{U})\dot{\mathbf{V}}, \dot{\mathbf{V}})_{L^2},$$

one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathcal{E} + \kappa(\mathbf{S}^\lambda(\mathbf{U})\partial_x \dot{\mathbf{V}}, \partial_x \dot{\mathbf{V}})_{L^2} &\leq C M (\|\dot{\mathbf{U}}\|_{L^2} + \|\dot{\mathbf{V}}\|_{L^2} + \kappa\|\partial_x \dot{\mathbf{V}}\|_{L^2}) (\|\dot{\mathbf{U}}\|_{L^2} + \|\dot{\mathbf{V}}\|_{L^2}) \\ &\quad + C(\|\mathbf{R}_U\|_{L^2} + \|\mathbf{R}_V\|_{L^2} + M\|R\|_{H^1}) (\|\dot{\mathbf{U}}\|_{L^2} + \|\dot{\mathbf{V}}\|_{L^2}). \end{aligned} \quad (6.2.9)$$

By using that $\mathbf{S}^\lambda(\mathbf{U})$ is definite positive, we find that there exists $c > 0$ depending only on ς such that

$$\mathcal{E} \geq c^2 \|\dot{\mathbf{U}}\|_{L^2}^2 + c^2 \|\dot{\mathbf{V}}\|_{L^2}^2, \quad (\mathbf{S}^\lambda(\mathbf{U})\partial_x \dot{\mathbf{V}}, \partial_x \dot{\mathbf{V}})_{L^2} \geq c^2 \|\partial_x \dot{\mathbf{V}}\|_{L^2}^2.$$

Hence we find (using the Peter Paul inequality and augmenting C) that

$$\frac{1}{2} \frac{d}{dt} \mathcal{E} + \frac{1}{2} c^2 \kappa \|\partial_x \dot{\mathbf{V}}\|_{L^2}^2 \leq C M \mathcal{E} + C(\|\mathbf{R}_U\|_{L^2} + \|\mathbf{R}_V\|_{L^2} + M\|R\|_{H^1}) \mathcal{E}^{1/2},$$

and the result follows by Gronwall's Lemma. \square

Proposition 6.2.8 (Large-time well-posedness). *Let $s \geq s_0 > 3/2$, $\varsigma \in (0, 1)$ and $M_0 > 0$. There exists $C > 0$ and $T > 0$ such that the following holds.*

Let $\kappa \in (0, 1]$, $(\rho_s, \rho_b, \underline{H}_s, \underline{H}_b, \underline{U}_s, \underline{U}_b) \in \mathbb{R}^6$ such that $\underline{H}_s + \underline{H}_b = 1$ and $\underline{U}_s + \underline{U}_b = 0$, and let $(H_s^0, H_b^0, U_s^0, U_b^0) \in H^{s+1}(\mathbb{R})^2 \times H^s(\mathbb{R})^2$ such that the hyperbolicity condition holds:

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s^0(x), \underline{H}_b + H_b^0(x), \underline{U}_s + U_s^0(x), \underline{U}_b + U_b^0(x)) \in \mathfrak{p}^\varsigma,$$

where \mathfrak{p}^ς is defined in (6.2.5), and

$$\|(H_s^0, H_b^0, U_s^0, U_b^0, \kappa \partial_x H_s^0, \kappa \partial_x H_b^0)\|_{H^{s_0}} \leq M_0.$$

Denote $(H_s, H_b, U_s, U_b) \in C([0, T^*]; H^s(\mathbb{R})^4)$ the maximal-in-time solution to (6.2.6) emerging from the initial data $(H_s, H_b, U_s, U_b)|_{t=0} = (H_s^0, H_b^0, U_s^0, U_b^0)$ as defined in Proposition 6.2.5.

One has $T^* > T/M_0$ and for any $t \in [0, T/M_0]$,

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s(t, x), \underline{H}_b + H_b(t, x), \underline{U}_s + U_s(t, x), \underline{U}_b + U_b(t, x)) \in \mathfrak{p}^{\varsigma/2},$$

and

$$\begin{aligned} & \| (H_s, H_b, U_s, U_b, \kappa \partial_x H_s, \kappa \partial_x H_b)(t, \cdot) \|_{H^s} + \| (\partial_t H_s, \partial_t H_b, \partial_t U_s, \partial_t U_b)(t, \cdot) \|_{H^{s-1}} \\ & \leq C \| (H_s^0, H_b^0, U_s^0, U_b^0, \kappa \partial_x H_s^0, \kappa \partial_x H_b^0) \|_{H^s}. \end{aligned}$$

Proof. We assume that the initial data is smooth, so that $(H_s, H_b, U_s, U_b, V_s, V_b)$ are smooth on their domain of existence. The general case is obtained by regularizing the initial data and passing to the limit, thanks to the persistence of regularity and continuity of the flow map stated in Proposition 6.2.5.

Denote $V_\ell := U_\ell - \kappa \frac{\partial_x H_\ell}{\underline{H}_\ell + H_\ell}$ ($\ell \in \{s, b\}$), $\Lambda^s := (\text{Id} - \partial_x^2)^{s/2}$ and

$$(\dot{H}_s, \dot{H}_b, \dot{U}_s, \dot{U}_b, \dot{V}_s, \dot{V}_b) := (\Lambda^s H_s, \Lambda^s H_b, \Lambda^s U_s, \Lambda^s U_b, \Lambda^s V_s, \Lambda^s V_b).$$

Applying the operator Λ^s to (6.2.6) and (6.2.7), we obtain

$$\begin{cases} \partial_t \dot{H}_s + (\underline{H}_s + H_s) \partial_x \dot{U}_s + (\underline{U}_s + U_s) \partial_x \dot{H}_s = \kappa \partial_x^2 \dot{H}_s + R_H(H_s, U_s), \\ \partial_t \dot{H}_b + (\underline{H}_b + H_b) \partial_x \dot{U}_b + (\underline{U}_b + U_b) \partial_x \dot{H}_b = \kappa \partial_x^2 \dot{H}_b + R_H(H_b, U_b), \\ \partial_t \dot{U}_s + (\underline{U}_s + U_s - \kappa \frac{\partial_x H_s}{\underline{H}_s + H_s}) \partial_x \dot{U}_s + \partial_x \dot{H}_s + \partial_x \dot{H}_b = R_U(H_s, U_s), \\ \partial_t \dot{U}_b + (\underline{U}_b + U_b - \kappa \frac{\partial_x H_b}{\underline{H}_b + H_b}) \partial_x \dot{U}_b + \frac{\rho_s}{\rho_b} \partial_x \dot{H}_s + \partial_x \dot{H}_b = R_U(H_b, U_b), \end{cases}$$

and

$$\begin{cases} \partial_t \dot{H}_s + (\underline{H}_s + H_s) \partial_x \dot{V}_s + (\underline{U}_s + V_s) \partial_x \dot{H}_s = R_H(H_s, V_s), \\ \partial_t \dot{H}_b + (\underline{H}_b + H_b) \partial_x \dot{V}_b + (\underline{U}_b + V_b) \partial_x \dot{H}_b = R_H(H_b, V_b), \\ \partial_t \dot{V}_s + (\underline{U}_s + V_s - \kappa \frac{\partial_x H_s}{\underline{H}_s + H_s}) \partial_x \dot{V}_s + \partial_x \dot{H}_s + \partial_x \dot{H}_b = \kappa \partial_x^2 \dot{V}_s + R_U(H_s, V_s), \\ \partial_t \dot{V}_b + (\underline{U}_b + V_b - \kappa \frac{\partial_x H_b}{\underline{H}_b + H_b}) \partial_x \dot{V}_b + \frac{\rho_s}{\rho_b} \partial_x \dot{H}_s + \partial_x \dot{H}_b = \kappa \partial_x^2 \dot{V}_s + R_U(H_b, V_b), \end{cases}$$

with remainders R_H and R_V defined as

$$R_H(H, V) := -[\Lambda^s, H] \partial_x V - [\Lambda^s, V] \partial_x H \quad \text{and} \quad R_U(H, V) := -[\Lambda^s, V - \kappa \frac{\partial_x H}{\underline{H} + H}] \partial_x V.$$

Moreover, applying the operator Λ^s to the identity $V_\ell = U_\ell - \kappa \frac{\partial_x H_\ell}{\underline{H}_\ell + H_\ell}$ ($\ell \in \{s, b\}$) yields

$$\dot{V}_\ell = \dot{U}_\ell - \kappa \frac{\partial_x \dot{H}_\ell}{\underline{H}_\ell + H_\ell} + R(H_\ell),$$

with

$$R(H) = -\kappa [\Lambda^s, \frac{1}{\underline{H} + H}] \partial_x H.$$

Standard commutator estimates and composition estimates in Sobolev spaces; see *e.g.* [54, Appendix B] yield

$$\begin{aligned} \|R_H(H, V)\|_{L^2} &\leq C (\|H\|_{H^{s_0}} + \|V\|_{H^{s_0}}) (\|H\|_{H^s} + \|V\|_{H^s}), \\ \|R_U(H, V)\|_{L^2} &\leq C (\kappa \|\partial_x H\|_{H^{s_0}} + \|V\|_{H^{s_0}}) (\kappa \|\partial_x H\|_{H^s} + \|V\|_{H^s}), \\ \|R(H)\|_{H^1} &\leq C (\|H\|_{H^{s_0}} + \kappa \|\partial_x H\|_{H^{s_0}}) (\|H\|_{H^s} + \kappa \|\partial_x H\|_{H^s}), \end{aligned}$$

where C is a positive constant depending only on $s, s_0, \|H\|_{H^{s_0}}$ and $\inf_{\mathbb{R}}(\underline{H} + H) > 0$.

Moreover, notice that by using the equations (6.2.6), (6.2.7) and the identity

$\kappa \partial_x H_\ell = (\underline{H}_\ell + H_\ell)(U_\ell - V_\ell)$ for $\ell \in \{s, b\}$, we have

$$\begin{aligned} & \|(\partial_t H_s, \partial_t H_b, \partial_t U_s, \partial_t U_b)\|_{H^{s-1}} + \kappa \|(\partial_x H_s, \partial_x H_b)\|_{H^s} + \kappa^{-1} \|(U_s - V_s, U_b - V_b)\|_{H^{s-1}} \\ & \leq C \|(H_s, H_b, U_s, U_b, V_s, V_b)\|_{H^s}, \end{aligned}$$

where the multiplicative constant C depends on $\|(H_s, H_b, U_s, U_b, V_s, V_b)\|_{H^{s_0}}$ and $\inf_{\mathbb{R}}(\underline{H}_\ell + H_\ell) > 0$ (for $\ell \in \{s, b\}$).

We may thus apply Lemma 6.2.7, and infer that we can set C depending on s_0, M_0, ς , and c depending only on ς , so that as long as

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s(t, x), \underline{H}_b + H_b(t, x), \underline{U}_s + U_s(t, x), \underline{U}_b + U_b(t, x)) \in \mathfrak{p}^{\varsigma/2}, \quad (6.2.10)$$

and

$$\|(H_s, H_b, U_s, U_b, V_s, V_b)(t, \cdot)\|_{H^{s_0}} \leq 2cM_0 \quad (6.2.11)$$

one has (using the standard $H^{s_0} \subset W^{1, \infty}$ continuous embedding and the fact that $0 \leq \underline{H}_s, \underline{H}_b \leq 1$ and $\underline{U}_s + \underline{U}_b = 0$)

$$\begin{aligned} & \|(H_s, H_b, U_s, U_b, V_s, V_b)(t, \cdot)\|_{H^{s_0}} \leq cM_0 \exp(CM_0 t) \\ & + CM_0 \int_0^t \|(H_s, H_b, U_s, U_b, V_s, V_b)(t', \cdot)\|_{H^{s_0}} \exp(CM_0(t-t')) dt. \end{aligned}$$

Applying Gronwall's Lemma, we find that if $CM_0 t$ is smaller than a universal constant, then

$$\|(H_s, H_b, U_s, U_b, V_s, V_b)(t, \cdot)\|_{H^s} \leq \frac{3c}{2} M_0.$$

What is more we have (augmenting C if necessary)

$$|(H_s(t, \cdot) - H_s^0, H_b(t, \cdot) - H_b^0, U_s(t, \cdot) - U_s^0, U_b(t, \cdot) - U_b^0)| \leq \int_0^t \|(\partial_t H_s, \partial_t H_b, \partial_t U_s, \partial_t U_b)\|_{L^\infty} \leq CM_0 t.$$

Hence lowering further $CM_0 t$, we infer that $(\rho_s, \rho_b, H_s(t, x), H_b(t, x), U_s(t, x), U_b(t, x)) \in \mathfrak{p}^{\varsigma/4}$ for all $x \in \mathbb{R}$. By the usual continuity argument we infer that the assumptions (6.2.10) and (6.2.11) do hold for $t \in [0, T/M_0]$ with T depending on s, s_0, M_0, ς . This yields the lower bound on the maximal time of existence and the claimed upper bound on the solution follows from the above estimates replacing s_0 with s . \square

Combined together, Proposition 6.2.5 and Proposition 6.2.8 yield a time of existence

for solutions to (6.2.6) emerging from sufficiently regular initial data which is independent of $\kappa \in (0, 1]$. The following result describes the behavior of these solutions as $\kappa \searrow 0$.

Proposition 6.2.9 (Convergence). *Let $s \geq s_0 > 3/2$, $\varsigma \in (0, 1)$ and $M_0 > 0$.*

Let $\kappa \in (0, 1]$, $(\rho_s, \rho_b, \underline{H}_s, \underline{H}_b, \underline{U}_s, \underline{U}_b) \in \mathbb{R}^6$ such that $\underline{H}_s + \underline{H}_b = 1$ and $\underline{U}_s + \underline{U}_b = 0$, and $(H_s^0, H_b^0, U_s^0, U_b^0) \in H^{s+2}(\mathbb{R})^4$ be such that the hyperbolicity condition holds

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s^0(x), \underline{H}_b + H_b^0(x), \underline{U}_s + U_s^0(x), \underline{U}_b + U_b^0(x)) \in \mathfrak{p}^\varsigma,$$

where \mathfrak{p}^ς is defined in (6.2.5), and

$$\|(H_s^0, H_b^0, U_s^0, U_b^0, \kappa \partial_x H_s^0, \kappa \partial_x H_b^0)\|_{H^{s_0}} \leq M_0.$$

Denote

- (H_s, H_b, U_s, U_b) the maximal solution to the non-diffusive system (6.2.1) emerging from the initial data

$$(H_s, H_b, U_s, U_b)|_{t=0} = (H_s^0, H_b^0, U_s^0, U_b^0)$$

as defined in Proposition 6.2.4;

- for all $\kappa > 0$, $(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa)$ the maximal solution to system (6.2.6) emerging from the initial data

$$(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa)|_{t=0} = (H_s^0, H_b^0, U_s^0, U_b^0)$$

as defined in Proposition 6.2.5;

- $T_0 > 0$ and $c_0 > 1$ such that for all $t \in [0, T_0]$,

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s(t, x), \underline{H}_b + H_b(t, x), \underline{U}_s + U_s(t, x), \underline{U}_b + U_b(t, x)) \in \mathfrak{p}^{s/c_0}$$

and

$$\|(H_s, H_b, U_s, U_b)(t, \cdot)\|_{H^{s+2}} \leq c_0 M_0.$$

Then there exists $\kappa_0 > 0$ and $C > 0$, both depending only on $s, s_0, \varsigma, M_0, T_0, c_0$, such that for all $\kappa \in (0, \kappa_0]$, $(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa)(t, \cdot)$ is well-defined for all $t \in [0, T_0]$ and satisfies the

hyperbolicity condition

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s^\kappa(t, x), \underline{H}_b + H_b^\kappa(t, x), \underline{U}_s + U_s^\kappa(t, x), \underline{U}_b + U_b^\kappa(t, x)) \in \mathfrak{p}^{s/(2c_0)}$$

and the upper bound

$$\|(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa)(t, \cdot)\|_{H^s} \leq 2c_0 M_0.$$

Moreover, one has for all $t \in [0, T_0]$

$$\|(H_s^\kappa - H_s, H_b^\kappa - H_b, U_s^\kappa - U_s, U_b^\kappa - U_b)(t, \cdot)\|_{H^s} \leq \kappa C M_0.$$

Proof. Denote $V_\ell := U_\ell$ and $V_\ell^\kappa := U_\ell^\kappa - \kappa \frac{\partial_x H_\ell^\kappa}{\underline{H}_\ell + H_\ell^\kappa}$ ($\ell \in \{s, b\}$), $\Lambda^s := (\text{Id} - \partial_x^2)^{s/2}$, and

$$(\dot{H}_s, \dot{H}_b, \dot{U}_s, \dot{U}_b, \dot{V}_s, \dot{V}_b) := (\Lambda^s(H_s^\kappa - H_s), \Lambda^s(H_b^\kappa - H_b), \Lambda^s(U_s^\kappa - U_s), \Lambda^s(U_b^\kappa - U_b), \Lambda^s(V_s^\kappa - V_s), \Lambda^s(V_b^\kappa - V_b))$$

Substracting (6.2.6), (6.2.7) and (6.2.1) we obtain

$$\begin{cases} \partial_t \dot{H}_s + (\underline{H}_s + H_s^\kappa) \partial_x \dot{U}_s + (\underline{U}_s + U_s^\kappa) \partial_x \dot{H}_s = \kappa \partial_x^2 \dot{H}_s + \kappa \Lambda^s \partial_x^2 H_s + R_H(H_s^\kappa, U_s^\kappa, H_s, U_s), \\ \partial_t \dot{H}_b + (\underline{H}_b + H_b^\kappa) \partial_x \dot{U}_b + (\underline{U}_b + U_b^\kappa) \partial_x \dot{H}_b = \kappa \partial_x^2 \dot{H}_b + \kappa \Lambda^s \partial_x^2 H_b + R_H(H_b^\kappa, U_b^\kappa, H_b, U_b), \\ \partial_t \dot{U}_s + (\underline{U}_s + U_s^\kappa - \kappa \frac{\partial_x H_s^\kappa}{\underline{H}_s + H_s^\kappa}) \partial_x \dot{U}_s + \partial_x \dot{H}_s + \partial_x \dot{H}_b = \kappa \Lambda^s \left(\frac{\partial_x H_s^\kappa}{\underline{H}_s + H_s^\kappa} \partial_x U_s \right) + R_U(H_s^\kappa, U_s^\kappa, H_s, U_s), \\ \partial_t \dot{U}_b + (\underline{U}_b + U_b^\kappa - \kappa \frac{\partial_x H_b^\kappa}{\underline{H}_b + H_b^\kappa}) \partial_x \dot{U}_b + \frac{\rho_s}{\rho_b} \partial_x \dot{H}_s + \partial_x \dot{H}_b = \kappa \Lambda^s \left(\frac{\partial_x H_b^\kappa}{\underline{H}_b + H_b^\kappa} \partial_x U_b \right) + R_U(H_b^\kappa, U_b^\kappa, H_b, U_b), \end{cases}$$

and

$$\begin{cases} \partial_t \dot{H}_s + (\underline{H}_s + H_s^\kappa) H_s^\kappa \partial_x \dot{V}_s + (\underline{U}_s + V_s^\kappa) \partial_x \dot{H}_s = R_H(H_s^\kappa, V_s^\kappa, H_s, V_s), \\ \partial_t \dot{H}_b + (\underline{H}_b + H_b^\kappa) H_b^\kappa \partial_x \dot{V}_b + (\underline{U}_b + V_b^\kappa) \partial_x \dot{H}_b = R_H(H_b^\kappa, V_b^\kappa, H_b, V_b), \\ \partial_t \dot{V}_s + (\underline{U}_s + V_s^\kappa - \kappa \frac{\partial_x H_s^\kappa}{\underline{H}_s + H_s^\kappa}) \partial_x \dot{V}_s + \partial_x \dot{H}_s + \partial_x \dot{H}_b \\ \quad \quad \quad = \kappa \partial_x^2 \dot{V}_s + \kappa \Lambda^s \partial_x^2 V_s + \kappa \Lambda^s \left(\frac{\partial_x H_s^\kappa}{\underline{H}_s + H_s^\kappa} \partial_x V_s \right) + R_U(H_s^\kappa, V_s^\kappa, H_s, V_s), \\ \partial_t \dot{V}_b + (\underline{U}_b + V_b^\kappa - \kappa \frac{\partial_x H_b^\kappa}{\underline{H}_b + H_b^\kappa}) \partial_x \dot{V}_b + \frac{\rho_s}{\rho_b} \partial_x \dot{H}_s + \partial_x \dot{H}_b \\ \quad \quad \quad = \kappa \partial_x^2 \dot{V}_b + \kappa \Lambda^s \partial_x^2 V_b + \kappa \Lambda^s \left(\frac{\partial_x H_b^\kappa}{\underline{H}_b + H_b^\kappa} \partial_x V_b \right) + R_U(H_b^\kappa, V_b^\kappa, H_b, V_b), \end{cases}$$

where, for any $\ell \in \{s, b\}$, we denote

$$\begin{aligned} R_H(H_\ell^\kappa, U_\ell^\kappa, H_\ell, U_\ell) &= -\Lambda^s((H_\ell^\kappa - H_\ell)\partial_x U_\ell + (U_\ell^\kappa - U_\ell)\partial_x H_\ell) \\ &\quad - [\Lambda^s, H_\ell^\kappa](\partial_x U_\ell^\kappa - \partial_x U_\ell) - [\Lambda^s, U_\ell^\kappa](\partial_x H_\ell^\kappa - \partial_x H_\ell), \\ R_U(H_\ell^\kappa, U_\ell^\kappa, H_\ell, U_\ell) &= -\Lambda^s((U_\ell^\kappa - U_\ell)\partial_x U_\ell) - [\Lambda^s, U_\ell^\kappa - \kappa \frac{\partial_x H_\ell^\kappa}{\underline{H}_\ell + H_\ell^\kappa}](\partial_x U_\ell^\kappa - \partial_x U_\ell). \end{aligned}$$

Moreover, one has by definition

$$\dot{V}_\ell = \dot{U}_\ell - \kappa \frac{\partial_x \dot{H}_\ell}{\underline{H}_\ell + H_\ell^\kappa} + R(H_\ell^\kappa, H_\ell)$$

where

$$R(H_\ell^\kappa, H_\ell) = -\kappa [\Lambda^s, \frac{1}{\underline{H}_\ell + H_\ell^\kappa}] \partial_x H_\ell^\kappa - \kappa \frac{\partial_x \Lambda^s H_\ell}{\underline{H}_\ell + H_\ell^\kappa}.$$

Standard commutator estimates and composition estimates in Sobolev spaces; see *e.g.* [54, Appendix B] yield

$$\begin{aligned} \|R_H(H_\ell^\kappa, V_\ell^\kappa, H_\ell, V_\ell)\|_{L^2} &\leq C(\|\partial_x H_\ell\|_{H^s} + \|\partial_x V_\ell\|_{H^s} + \|\partial_x H_\ell^\kappa\|_{H^{s-1}} + \|\partial_x V_\ell^\kappa\|_{H^{s-1}})(\|\dot{H}_\ell\|_{L^2} + \|\dot{V}_\ell\|_{L^2}), \\ \|R_U(H_\ell^\kappa, V_\ell^\kappa, H_\ell, V_\ell)\|_{L^2} &\leq C(\|\partial_x V_\ell\|_{H^s} + \kappa \|\partial_x H_\ell^\kappa\|_{H^s} + \|\partial_x V_\ell^\kappa\|_{H^{s-1}})\|\dot{V}_\ell\|_{L^2}, \\ \|R(H_\ell^\kappa, H_\ell)\|_{H^1} &\leq C\kappa(\|\partial_x H_\ell^\kappa\|_{H^s} + \|\partial_x H_\ell\|_{H^{s+1}}), \end{aligned}$$

where C is a positive constant depending only on $s, s_0, \|H_\ell^\kappa\|_{H^{s_0}}$ and $\inf_{\mathbb{R}} \underline{H}_\ell + H_\ell^\kappa > 0$ for any $\ell \in \{s, b\}$.

Moreover notice that by the equations (6.2.6) and (6.2.7) and using the identity $\kappa \partial_x H_\ell^\kappa = (\underline{H}_\ell + H_\ell^\kappa)(U_\ell^\kappa - V_\ell^\kappa)$ we have

$$\begin{aligned} \|(\partial_t H_s^\kappa, \partial_t H_b^\kappa, \partial_t U_s^\kappa, \partial_t U_b^\kappa)\|_{H^{s-1}} + \kappa \|(\partial_x H_s^\kappa, \partial_x H_b^\kappa)\|_{H^s} + \kappa^{-1} \|(U_s^\kappa - V_s^\kappa, U_b^\kappa - V_b^\kappa)\|_{H^{s-1}} \\ \leq C \|(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa, V_s^\kappa, V_b^\kappa)\|_{H^s}, \end{aligned}$$

where the multiplicative constant C depends on $\|(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa, V_s^\kappa, V_b^\kappa)\|_{H^{s_0}}$ and $\inf_{\mathbb{R}} \underline{H}_\ell + H_\ell^\kappa > 0$ for $\ell \in \{s, b\}$.

We may thus apply Lemma 6.2.7, and infer that we can set C depending on s, s_0, M_0, ς , and c depending only on ς , so that as long as

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s^\kappa(t, x), \underline{H}_b + H_b^\kappa(t, x), \underline{U}_s + U_s^\kappa(t, x), \underline{U}_b + U_b^\kappa(t, x)) \in \mathfrak{p}^{\varsigma/(2c_0)}, \quad (6.2.12)$$

and

$$\|(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa, V_s^\kappa, V_b^\kappa)(t, \cdot)\|_{H^s} \leq 2c_0 M_0 \quad (6.2.13)$$

one has, using that $(\dot{H}_s, \dot{H}_b, \dot{U}_s, \dot{U}_b, \dot{V}_s, \dot{V}_b)|_{t=0} = (0, 0, 0, 0, -\kappa\Lambda^s(\frac{\partial_x H_s^\kappa}{H_s + H_s^\kappa})|_{t=0}, -\kappa\Lambda^s(\frac{\partial_x H_b^\kappa}{H_b + H_b^\kappa})|_{t=0})$,

$$\begin{aligned} & \|(\dot{H}_s, \dot{H}_b, \dot{U}_s, \dot{U}_b, \dot{V}_s, \dot{V}_b)(t, \cdot)\|_{L^2} + c\kappa^{1/2} \|(\partial_x \dot{H}_s, \partial_x \dot{H}_b, \partial_x \dot{V}_s, \partial_x \dot{V}_b)\|_{L^2(0,t;L^2)} \leq \kappa C M_0 \exp(C c_0 M_0 t) \\ & + C \int_0^t \left(c_0 M_0 \|(\dot{H}_s, \dot{H}_b, \dot{U}_s, \dot{U}_b, \dot{V}_s, \dot{V}_b, \kappa \partial_x \dot{H}_s, \kappa \partial_x \dot{H}_b)(t', \cdot)\|_{L^2} \right. \\ & \left. + \kappa \|(\partial_x H_s^\kappa, \partial_x H_b^\kappa, \partial_x^2 H_s, \partial_x^2 H_b, \partial_x^2 V_s, \partial_x^2 V_b, \partial_x U_s, \partial_x U_b, \partial_x V_s, \partial_x V_b)(t', \cdot)\|_{H^s} \right) \exp(C c_0 M_0 (t-t')) dt. \end{aligned}$$

Now, we use in the integrand the triangle inequality

$$\|(\partial_x H_s^\kappa, \partial_x H_b^\kappa)(t', \cdot)\|_{H^s} \leq \|(\partial_x \dot{H}_s, \partial_x \dot{H}_b)(t', \cdot)\|_{L^2} + \|(\partial_x H_s, \partial_x H_b)(t', \cdot)\|_{H^s}.$$

The first contribution may be absorbed by the left-hand side if κ is sufficiently small (depending on $c, C, c_0 M_0, T_0$), and the second contribution is estimated, as other terms, using the assumption

$$\sup \left\{ \|(H_s, H_b, U_s, U_b)(t, \cdot)\|_{H^{s+2}} : t \in [0, T_0] \right\} \leq c_0 M_0.$$

Applying then Gronwall's Lemma, we find that

$$\|(\dot{H}_s, \dot{H}_b, \dot{U}_s, \dot{U}_b, \dot{V}_s, \dot{V}_b)(t, \cdot)\|_{L^2} \leq \kappa C c_0 M_0 K \quad (6.2.14)$$

where K depends only on $C c_0 M_0 T_0$.

By using again the triangle inequality, we can lower further κ (depending on c_0) so that (6.2.14) implies

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, H_s^\kappa(t, x), H_b^\kappa(t, x), U_s^\kappa(t, x), U_b^\kappa(t, x)) \in \mathfrak{p}^{2\zeta/(3c_0)}$$

and

$$\|(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa, V_s^\kappa, V_b^\kappa)(t, \cdot)\|_{H^s} \leq \frac{3}{2} c_0 M_0.$$

Hence by the usual continuity argument we infer that $(H_s^\kappa, H_b^\kappa, U_s^\kappa, U_b^\kappa)(t, \cdot)$ is well-defined for all $t \in [0, T_0]$, and (6.2.12)-(6.2.13)-(6.2.14) hold. This concludes the proof. \square

6.3 The hydrostatic Euler equations

In this section we study the stability of the hydrostatic Euler equations for stratified flows:

$$\begin{aligned} \partial_t h + \partial_x((1+h)(\underline{u} + u)) &= \kappa \partial_x^2 h, \\ \partial_t u + \left(\underline{u} + u - \kappa \frac{\partial_x h}{1+h} \right) \partial_x u + \frac{1}{\underline{\rho}} \partial_x \Psi &= 0, \end{aligned} \quad (6.3.1)$$

where the Montgomery potential Ψ is given by

$$\Psi(\cdot, r) = \underline{\rho}(r) \int_{-1}^r h(\cdot, r') dr' + \int_r^0 \underline{\rho}(r') h(\cdot, r') dr' =: (\mathbf{M}[\underline{\rho}]h)(r). \quad (6.3.2)$$

We recall that our stability results must accommodate with solutions generated by the bilayer system that are piecewise constant and allow the comparison with continuously stratified flows. Hence we must allow for deviations that can be large pointwise, while smallness stems from integration (with respect to the r -variable). In practice we shall manipulate simultaneously the pointwise as well as the L_r^1 topologies depending on the need. This is the case for instance in the following Lemma, measuring the Lipschitz continuity with respect to the density variable of the Montgomery operator $\mathbf{M}[\underline{\rho}]$ defined (6.3.2).

Lemma 6.3.1. *Let $\underline{M} > 0$. There exists $C > 0$ such that for any $\underline{\rho}_\ell$ ($\ell \in \{1, 2\}$) such that*

$$\|(\underline{\rho}_\ell, \frac{1}{\underline{\rho}_\ell})\|_{L_r^1 \times L_r^\infty} \leq \underline{M},$$

and for any $h \in L_r^\infty$, one has for almost any $r \in (-1, 0)$,

$$\left| \left(\frac{1}{\underline{\rho}_1} \mathbf{M}[\underline{\rho}_1]h - \frac{1}{\underline{\rho}_2} \mathbf{M}[\underline{\rho}_2]h \right)(r) \right| \leq \left(\underline{M}^3 |\underline{\rho}_1 - \underline{\rho}_2|(r) + \underline{M} \|\underline{\rho}_1 - \underline{\rho}_2\|_{L_r^1} \right) \|h\|_{L_r^\infty}$$

Proof.

$$\left(\frac{1}{\underline{\rho}_1} \mathbf{M}[\underline{\rho}_1]h - \frac{1}{\underline{\rho}_2} \mathbf{M}[\underline{\rho}_2]h \right)(r) = \left(\frac{1}{\underline{\rho}_1(r)} - \frac{1}{\underline{\rho}_2(r)} \right) \int_r^0 \underline{\rho}_1(r') h(r') dr' + \frac{1}{\underline{\rho}_2(r)} \int_r^0 (\underline{\rho}_1(r') - \underline{\rho}_2(r')) h(r') dr'.$$

□

Note also that we are seeking stability estimates for the system (6.3.1)-(6.3.2) with respect to perturbations of the equations —in particular through $\underline{\rho} \approx \underline{\rho}_{\text{bl}}$ and $\underline{u} \approx \underline{u}_{\text{bl}}$ — and with respect to perturbations of the initial data.

In Section 6.3.1, we first state a local well-posedness result associated with the initial-value problem for the system (6.3.1)-(6.3.2), and then provide some stability estimates. These two results by themselves are not sufficient to bootstrap in a standard manner the strong convergence of solutions as the size of deviations shrink, because the topology involved in the first result, namely L_r^∞ , is stronger than the topology used in the second result, which is roughly speaking L_r^1 . For that matter we introduce and study in Section 6.3.2 a refined approximate solution which, compared with the original reference solution, improves the description of solutions associated with nearby profiles. Specifically, the refined approximate solution satisfies the following three properties:

- i. it is well-defined and controlled on a time interval which is uniform with respect to $\kappa \in (0, 1]$;
- ii. the difference with respect to the nearby solution is controlled for the strong norm associated with L_r^∞ ;
- iii. the difference with respect to the reference solution is controlled for the weak norm associated with L_r^1 .

The resulting convergence result, Proposition 6.3.8, is stated and proved in Section 6.3.3.

6.3.1 Stability estimates

Proposition 6.3.2 (Well-posedness). *Let $s \geq s_0 > 3/2$, $\varsigma \in (0, 1)$, $\underline{M}, M_0 > 0$ and $c > 1$. There exists $T > 0$ such that the following holds.*

For all $\kappa \in (0, 1]$, all $(\underline{\rho}, \underline{u}) \in L^\infty((-1, 0))$ such that

$$\|(\underline{u}, \underline{\rho}, \frac{1}{\underline{\rho}})\|_{L_r^\infty} \leq \underline{M},$$

and all $(h^0, u^0) \in L^\infty((-1, 0); H^s(\mathbb{R})^2)$ such that for almost all $r \in (-1, 0)$,

$$\forall x \in \mathbb{R}, \quad 1 + h^0 \geq \varsigma,$$

and

$$\|(h^0, u^0)\|_{L_r^\infty H_x^{s_0}} \leq M_0$$

there exists a unique $(h, u) \in C([0, T^]; L^\infty((-1, 0); H^s(\mathbb{R})^2))$ maximal-in-time strong solution to (6.3.1)-(6.3.2) emerging from the initial data $(h, u)|_{t=0} = (h^0, u^0)$.*

Moreover, $T^* > \kappa T$ and for any $t \in [0, \kappa T]$ and almost all $r \in (-1, 0)$ one has

$$\forall x \in \mathbb{R}, \quad 1 + h(t, x, r) \geq \varsigma/c$$

and

$$\max(\{\|(h, u)\|_{L^\infty(0,t;L_r^\infty H_x^s)}, \kappa^{1/2} \|\partial_x h\|_{L^2(0,t;L_r^\infty H_x^s)}\}) \leq c \|(h^0, u^0)\|_{L_r^\infty H_x^s}.$$

Moreover, the maximal existence time (resp. the emerging solution in $C([0, T^*]; L^\infty((-1, 0); H^s(\mathbb{R}^2)))$) is a lower semi-continuous (resp. continuous) function of the initial data in $L^\infty((-1, 0); H^s(\mathbb{R}^2))$ and if $T^* < \infty$ then

$$\|(h(t, \cdot), u(t, \cdot))\|_{L_r^\infty H_x^{s_0}} \rightarrow \infty \text{ as } t \rightarrow T^*.$$

Proof. The proof is very similar to the proof of Proposition 6.2.5, using estimates for transport and transport-diffusion equations pointwisely with respect to the variable $r \in (-1, 0)$. The essential arguments are that $L^\infty((-1, 0))$ is a Banach algebra and that differentiation with respect to the space variable ∂_x as well as the Fourier multiplier $\Lambda^s = (\text{Id} - \partial_x^2)^{s/2}$ commute with the operator $\frac{1}{\rho} \mathbf{M}[\rho]$, and that the linear operator $\frac{1}{\rho} \mathbf{M}[\rho] : L_r^\infty L_x^2 \rightarrow L_r^\infty L_x^2$ is bounded for any $\rho \in L^\infty((-1, 0))$. \square

Proposition 6.3.3 (Stability). *Let $s > 3/2$, $\varsigma \in (0, 1)$ and $\underline{M}, M_1, M_2 > 0$. There exists $C > 0$ such that the following holds.*

Let $\kappa \in (0, 1]$ and $T > 0$ be such that

$$CT \leq \kappa.$$

Let $\underline{\rho}_\ell, \underline{u}_\ell$ (for $\ell \in \{1, 2\}$) be such that

$$\|(\underline{\rho}_\ell, \frac{1}{\underline{\rho}_\ell})\|_{L_r^\infty} \leq \underline{M}.$$

Let (h_ℓ, u_ℓ) be solutions to (6.3.1)-(6.3.2) (with $\underline{\rho} = \underline{\rho}_\ell$ and $\underline{u} = \underline{u}_\ell$) defined on the interval $[0, T]$ and satisfying

$$\|(\partial_x h_1, \partial_x u_1)\|_{L_T^\infty L_r^\infty H_x^s} \leq M_1, \quad \|(h_2, u_2)\|_{L_T^\infty L_r^\infty H_x^s} + \kappa^{1/2} \|\partial_x h_2\|_{L_r^\infty L_T^2 H_x^s} \leq M_2,$$

and

$$\text{ess inf}_{(t,x,r) \in [0,T] \times \mathbb{R} \times (-1,0)} 1 + h_1(t, x, r) \geq \varsigma, \quad \text{ess inf}_{(t,x,r) \in [0,T] \times \mathbb{R} \times (-1,0)} 1 + h_2(t, x, r) \geq \varsigma.$$

Then one has

$$\begin{aligned} & \max(\{\|(h_1 - h_2, u_1 - u_2)\|_{L_T^\infty L_r^1 H_x^s}, \kappa^{1/2} \|\partial_x(h_1 - h_2)\|_{L_T^2 L_r^1 H_x^s}\}) \\ & \leq 2\|(h_1^0 - h_2^0, u_1^0 - u_2^0)\|_{L_r^1 H_x^s} + CT \|(\underline{\rho}_1 - \underline{\rho}_2, \underline{u}_1 - \underline{u}_2)\|_{L_r^1} \end{aligned} \quad (6.3.3)$$

and for almost any $r \in (-1, 0)$,

$$\begin{aligned} & \max(\{\|(h_1 - h_2, u_1 - u_2)(\cdot, r)\|_{L_T^\infty H_x^s}, \kappa^{1/2} \|\partial_x(h_1 - h_2)(\cdot, r)\|_{L_T^2 H_x^s}\}) \\ & \leq 2(\|(h_1^0 - h_2^0, u_1^0 - u_2^0)(r)\|_{H_x^s} + \|(h_1^0 - h_2^0, u_1^0 - u_2^0)\|_{L_r^1 H_x^s}) \\ & \quad + CT (|(\underline{\rho}_1 - \underline{\rho}_2, \underline{u}_1 - \underline{u}_2)(r)| + \|(\underline{\rho}_1 - \underline{\rho}_2, \underline{u}_1 - \underline{u}_2)\|_{L_r^1}). \end{aligned} \quad (6.3.4)$$

Proof. Let us denote $\dot{h} := h_1 - h_2$, $\dot{u} := u_1 - u_2$, $\underline{\dot{u}} := \underline{u}_1 - \underline{u}_2$ and $\dot{\rho} = \rho_1 - \rho_2$. We have on the time interval $I := [0, T]$

$$\begin{cases} \partial_t \dot{h} + (\underline{u}_2 + u_2) \partial_x \dot{h} - \kappa \partial_x^2 \dot{h} = r_1 + r_2, \\ \partial_t \dot{u} + (\underline{u}_2 + u_2 + u_2^*) \partial_x \dot{u} = r_3, \end{cases}$$

where we denote $u_2^* := -\kappa \frac{\partial_x h_2}{1+h_2}$,

$$r_1 := -(\underline{\dot{u}} + \dot{u}) \partial_x h_1 - \dot{h} \partial_x u_1, \quad r_2 := -(1 + h_2) \partial_x \dot{u}$$

and

$$r_3 := -\left(\underline{\dot{u}} + \dot{u} - \kappa \frac{\partial_x \dot{h}}{1+h_2} + \kappa \dot{h} \frac{\partial_x h_1}{(1+h_1)(1+h_2)}\right) \partial_x u_1 - \left(\frac{1}{\rho_1} \mathbf{M}[\underline{\rho}_1] - \frac{1}{\rho_2} \mathbf{M}[\underline{\rho}_2]\right) \partial_x h_1 - \frac{1}{\rho_2} \mathbf{M}[\underline{\rho}_2] \partial_x \dot{h}.$$

We can now use standard estimates and transport-diffusion and transport equations ([8]) to infer that there exists c_0 depending only on s such that for almost any $r \in (-1, 0)$, one has

$$\begin{aligned} \max(\{\|\dot{h}(\cdot, r)\|_{L_T^\infty H_x^s}, \kappa^{1/2} \|\partial_x \dot{h}(\cdot, r)\|_{L_T^2 H_x^s}\}) & \leq \left(\|\dot{h}(t=0, \cdot, r)\|_{H_x^s} + \|r_1(\cdot, r)\|_{L_T^1 H_x^s} + \kappa^{-\frac{1}{2}} \|r_2(\cdot, r)\|_{L_T^2 H_x^{s-1}}\right) \\ & \quad \times \exp(c_0 \|\partial_x u_2(\cdot, r)\|_{L_T^1 H^{s-1}}) \end{aligned}$$

and

$$\|\dot{u}(\cdot, r)\|_{L_T^\infty H_x^s} \leq \left(\|\dot{u}(t=0, \cdot, r)\|_{H_x^s} + \|r_3(\cdot, r)\|_{L_T^1 H_x^s}\right) \times \exp(c_0 \|\partial_x u_2(\cdot, r) + \partial_x u_2^*(\cdot, r)\|_{L_T^1 H_x^{s-1}}).$$

Using that $H^{s'}(\mathbb{R})$ is a Banach algebra for all $s' > 1/2$, we find that for any $t \in [0, T]$

$$\|r_1(t, \cdot, r)\|_{H_x^s} \lesssim M_1 \times (|\dot{u}(r)| + \|\dot{u}(t, \cdot, r)\|_{H_x^s} + \|\dot{h}(t, \cdot, r)\|_{H_x^s}), \quad \|r_2(t, \cdot, r)\|_{H_x^{s-1}} \lesssim (1 + M_2) \|\dot{u}(t, \cdot, r)\|_{H_x^s},$$

and, making additionally use of standard composition estimates in Sobolev spaces ([54, Appendix B]) and Lemma 6.3.1,

$$\begin{aligned} \|r_3(t, \cdot, r)\|_{H_x^s} &\leq C(s, \underline{M}, M_1, M_2, \varsigma) \left(|\dot{u}(r)| + \|\dot{u}(t, \cdot, r)\|_{H_x^s} + \kappa \|\dot{h}(t, \cdot, r)\|_{H_x^{s+1}} + |\dot{\rho}(r)| + \|\dot{\rho}\|_{L_r^1} \right) M_1 \\ &\quad + \underline{M}^2 \|\partial_x \dot{h}(t, \cdot)\|_{L_r^1 H_x^s}. \end{aligned}$$

Finally, using product and composition estimates on $u_2^* = -\kappa \partial_x h_2 + \kappa \frac{h_2}{1+h_2} \partial_x h_2$, we have

$$\|\partial_x u_2^*(t, \cdot, r)\|_{H_x^{s-1}} \leq C(M_2, \varsigma^{-1}) \kappa \|\partial_x h_2(t, \cdot, r)\|_{H_x^s}.$$

Collecting these estimates we infer that there exists $C > 0$ depending only on $s, \underline{M}, M_1, M_2, \varsigma$ such that

$$\begin{aligned} &\max(\{\|\dot{h}(\cdot, r), \dot{u}(\cdot, r)\|_{L_T^\infty H_x^s}, \kappa^{1/2} \|\partial_x \dot{h}(\cdot, r)\|_{L_T^2 H_x^s}\}) \leq \exp(CM_2(T + \kappa^{1/2}T^{1/2})) \\ &\quad \times \left(\|\dot{h}(t=0, \cdot, r), \dot{u}(t=0, \cdot, r)\|_{H_x^s} + C\kappa^{-1/2}T^{1/2} (\|\dot{u}(\cdot, r)\|_{L_T^\infty H_x^s} + \kappa^{1/2} \|\partial_x \dot{h}\|_{L_T^2 L_r^1 H_x^s}) \right. \\ &\quad \left. + CM_1T \times (|\dot{u}(r)| + |\dot{\rho}(r)| + \|\dot{\rho}\|_{L_r^1} + \|\dot{h}(\cdot, r)\|_{L_T^\infty H_x^s} + \|\dot{u}(\cdot, r)\|_{L_T^\infty H_x^s} + \kappa T^{-1/2} \|\partial_x \dot{h}(\cdot, r)\|_{L_T^2 H_x^s}) \right). \end{aligned}$$

Hence there exists $C > 0$, depending only on $s, \underline{M}, M_1, M_2, \varsigma$ such that for any T sufficiently small so that one has $M_1T \leq (36C)^{-1}$, $(\kappa^{-1/2} + \kappa^{1/2}M_1)T^{1/2} \leq (18C)^{-1}$ and $M_2T + M_2\kappa^{1/2}T^{1/2} \leq C^{-1} \ln(3/2)$ one has

$$\begin{aligned} &\frac{5}{6} \max(\{\|\dot{h}(\cdot, r), \dot{u}(\cdot, r)\|_{L_T^\infty H_x^s}, \kappa^{1/2} \|\partial_x \dot{h}(\cdot, r)\|_{L_T^2 H_x^s}\}) \\ &\leq \frac{3}{2} \left(\|\dot{h}(t=0, \cdot, r), \dot{u}(t=0, \cdot, r)\|_{H_x^s} + \frac{1}{18} \kappa^{1/2} \|\partial_x \dot{h}\|_{L_T^2 L_r^1 H_x^s} + CM_1T (|\dot{u}(r)| + |\dot{\rho}(r)| + \|\dot{\rho}\|_{L_r^1}) \right). \end{aligned} \tag{6.3.5}$$

Integrating this inequality with respect to the variable r and using Minkowski's inequality, we infer the first stability estimate, (6.3.3). Plugging (6.3.3) in the right-hand side of (6.3.5), the second stability estimate, (6.3.4), follows immediately. \square

Remark 6.3.4. *The restriction $T \lesssim \kappa$ in Propositions 6.3.2 and 6.3.3 is quite stringent.*

In [13], some improved stability estimates concerning the system (6.3.1)-(6.3.2) were derived by the authors. The latter estimates exploit a partial symmetric structure of the equations, and demand some extra regularity with respect to the variable r . Because we cannot afford such regularity since our stability estimates will be used with piecewise constant functions, we use in the proof of Proposition 6.3.3 in a stronger way the parabolic regularization of thickness diffusivity.

In order to obtain a final result on a timescale which is independent of the parameter $\kappa \in (0, 1]$ we shall exploit some a priori control on a reference solution and restrict to initial data as well as shear velocity and density distributions that are close to the reference data.

6.3.2 Refined approximation

In this section we consider a given reference solution to the hydrostatic Euler equation (6.3.1)-(6.3.2) and build from it a refined approximate solution associated with nearby profiles. Specifically, let $\underline{\rho}_{\text{ref}}, \underline{u}_{\text{ref}}, h_{\text{ref}}, u_{\text{ref}}$ be a solution to

$$\begin{aligned} \partial_t h_{\text{ref}} + \partial_x((1 + h_{\text{ref}})(\underline{u}_{\text{ref}} + u_{\text{ref}})) &= \kappa \partial_x^2 h_{\text{ref}}, \\ \partial_t u_{\text{ref}} + \left(\underline{u}_{\text{ref}} + u_{\text{ref}} - \kappa \frac{\partial_x h_{\text{ref}}}{1 + h_{\text{ref}}} \right) \partial_x u_{\text{ref}} + \frac{1}{\underline{\rho}_{\text{ref}}} \mathbf{M}[\underline{\rho}_{\text{ref}}] \partial_x h_{\text{ref}} &= 0, \end{aligned} \quad (6.3.6)$$

where we recall that the operator \mathbf{M} is defined in (6.3.2). Considering profiles $(\underline{\rho}, \underline{u})$ which are in some sense close to $(\underline{\rho}_{\text{ref}}, \underline{u}_{\text{ref}})$ we construct approximate solutions $(h_{\text{app}}, u_{\text{app}})$ as the solutions to

$$\begin{aligned} \partial_t h_{\text{app}} + \partial_x((1 + h_{\text{app}})(\underline{u} + u_{\text{app}})) &= \kappa \partial_x^2 h_{\text{app}}, \\ \partial_t u_{\text{app}} + \left(\underline{u} + u_{\text{app}} - \kappa \frac{\partial_x h_{\text{app}}}{1 + h_{\text{app}}} \right) \partial_x u_{\text{app}} &= -\frac{1}{\underline{\rho}} \mathbf{M}[\underline{\rho}] \partial_x h_{\text{ref}}. \end{aligned} \quad (6.3.7)$$

Notice first that $(h_{\text{app}}, u_{\text{app}})$ satisfies approximately the hydrostatic Euler equations associated with profiles $(\underline{\rho}, \underline{u})$.

Proposition 6.3.5. *For any $s \geq 0$ the refined approximate solution $(h^{\text{app}}, u^{\text{app}})$ satisfies*

$$\begin{aligned} \partial_t h_{\text{app}} + \partial_x((1 + h_{\text{app}})(\underline{u} + u_{\text{app}})) &= \kappa \partial_x^2 h_{\text{app}}, \\ \partial_t u_{\text{app}} + \left(\underline{u} + u_{\text{app}} - \kappa \frac{\partial_x h_{\text{app}}}{1 + h_{\text{app}}} \right) \partial_x u_{\text{app}} + \frac{1}{\underline{\rho}} \mathbf{M}[\underline{\rho}] \partial_x h_{\text{app}} &= r_{\text{rem}} \end{aligned} \quad (6.3.8)$$

with

$$\|r_{\text{rem}}(t, \cdot)\|_{L_r^\infty H_x^s} \leq \|\underline{\rho}\|_{L_r^\infty} \|\frac{1}{\underline{\rho}}\|_{L_r^\infty} \|(h_{\text{ref}} - h_{\text{app}})(t, \cdot)\|_{L_r^1 H_x^{s+1}}$$

Proof. We have

$$\begin{aligned} r_{\text{rem}}(\cdot, r) &= \left(\frac{1}{\underline{\rho}} \mathbf{M}[\underline{\rho}] (\partial_x h_{\text{app}} - \partial_x h_{\text{ref}}) \right) (\cdot, r) \\ &= \int_{-1}^r (\partial_x h_{\text{app}} - \partial_x h_{\text{ref}})(\cdot, r') \, dr' + \frac{1}{\underline{\rho}(r)} \int_r^0 \underline{\rho}(r') (\partial_x h_{\text{app}} - \partial_x h_{\text{ref}})(\cdot, r') \, dr', \end{aligned}$$

and the result follows since $1 \leq \sup(\{\underline{\rho}(r')/\underline{\rho}(r) : (r, r') \in (-1, 0)^2\}) \leq \|\underline{\rho}\|_{L_r^\infty} \|\frac{1}{\underline{\rho}}\|_{L_r^\infty}$. \square

That the above remainder term r_{app} is small is a consequence of the subsequent Proposition 6.3.7. We first prove that for any initial data $(h_{\text{app}}, u_{\text{app}})|_{t=0} = (h^0, u^0)$, the emerging solution $(h_{\text{app}}, u_{\text{app}})$ is well-defined and controlled on a time interval uniform with respect to $\kappa \in (0, 1]$.

Proposition 6.3.6. *Let $s > 3/2$, $\varsigma \in (0, 1)$, $\underline{M}, M_0, M_{\text{ref}} > 0$ and $c > 1$. There exists $C > 0$ and $T > 0$ such that the following holds.*

Let $h_{\text{ref}} \in C([0, T_{\text{ref}}]; L^1((0, 1); H_x^{s+1}(\mathbb{R})))$ be such that

$$\|\partial_x h_{\text{ref}}\|_{L_{T_{\text{ref}}}^1 L_r^1 H_x^s} \leq M_{\text{ref}}.$$

For all $\kappa \in (0, 1]$, all $(\underline{\rho}, \underline{u}) \in L^\infty((-1, 0))$ such that

$$\|(\underline{\rho}, \frac{1}{\underline{\rho}})\|_{L_r^\infty} \leq \underline{M},$$

and all $(h^0, u^0) \in L^\infty((0, 1); H^s(\mathbb{R})^2)$ such that for almost all $r \in (-1, 0)$

$$\forall x \in \mathbb{R}, \quad 1 + h^0 \geq \varsigma$$

and

$$\max(\{\|h^0\|_{L_r^\infty H_x^{s-1}}, \kappa^{1/2} \|h^0\|_{L_r^\infty H_x^s}, \|u^0\|_{L_r^\infty H_x^s}\}) \leq M_0,$$

there exists a unique $(h_{\text{app}}, u_{\text{app}}) \in C([0, T^]; L^\infty((0, 1); H_x^s(\mathbb{R})^2))$ maximal solution to (6.3.7) emerging from the initial data $(h_{\text{app}}, u_{\text{app}})|_{t=0} = (h^0, u^0)$. Moreover one has $T^* \geq T_{\text{app}} := \min(\{T_{\text{ref}}, T\})$ and for any $t \in [0, T_{\text{app}}]$ and almost any $r \in (-1, 0)$ one has the upper*

bound

$$\begin{aligned} & \max(\{\|h_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^\infty H_x^{s-1}}, \kappa^{1/2}\|h_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^\infty H_x^s}, \kappa\|\partial_x h_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^2 H_x^s}\}) + \|u_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^\infty H_x^s} \\ & \leq c \max(\{\|h^0(\cdot, r)\|_{H_x^{s-1}}, \kappa^{1/2}\|h^0(t=0, \cdot, r)\|_{H_x^s}, \|u^0(\cdot, r)\|_{H_x^s}\}) + CM_{\text{ref}}. \end{aligned}$$

Proof. The existence and uniqueness of $(h_{\text{app}}, u_{\text{app}}) \in C([0, T^*]; L^\infty((0, 1); H_x^s(\mathbb{R}^2)))$ maximal solution to (6.3.7) is obtained as in Proposition 6.3.2. We set $T \in [0, T^*)$. By standard estimates on transport and transport-diffusion equations ([8]) applied to (6.3.7) we have

$$\begin{aligned} \max(\{\|h_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^{s-1}}, \kappa^{1/2}\|\partial_x h_{\text{app}}(\cdot, r)\|_{L_T^2 H_x^{s-1}}\}) & \leq \left(\|h^0(\cdot, r)\|_{H_x^{s-1}} + \|r_{\text{app}}(\cdot, r)\|_{L_T^1 H_x^{s-1}} \right) \\ & \quad \times \exp(c_0 \|\partial_x u_{\text{app}}(\cdot, r)\|_{L_T^1 H^{s-1}}), \end{aligned}$$

$$\begin{aligned} \max(\{\|h_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^s}, \kappa^{1/2}\|\partial_x h_{\text{app}}(\cdot, r)\|_{L_T^2 H_x^s}\}) & \leq \left(\|h^0(\cdot, r)\|_{H_x^s} + \kappa^{-1/2}\|r_{\text{app}}(\cdot, r)\|_{L_T^2 H_x^{s-1}} \right) \\ & \quad \times \exp(c_0 \|\partial_x u_{\text{app}}(\cdot, r)\|_{L_T^1 H^{s-1}}), \end{aligned}$$

and

$$\|u_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^s} \leq \left(\|u^0(\cdot, r)\|_{H_x^s} + \|r_{\text{ref}}(\cdot, r)\|_{L_T^1 H_x^s} \right) \times \exp(c_0 \|\partial_x u_{\text{app}}(\cdot, r) + \partial_x u_{\text{app}}^*(\cdot, r)\|_{L_T^1 H_x^{s-1}}),$$

where we denote $u_{\text{app}}^* := -\kappa \frac{\partial_x h_{\text{app}}}{1+h_{\text{app}}}$, $r_{\text{app}} := -(1+h_{\text{app}})\partial_x u_{\text{app}}$, $r_{\text{ref}} = -\frac{1}{\underline{\rho}} \mathbf{M}[\underline{\rho}] \partial_x h_{\text{ref}}$, and the constant c_0 depends only on s .

Now we notice that for almost any $r \in (-1, 0)$,

$$\begin{aligned} \|\partial_x u_{\text{app}}^*(t, \cdot, r)\|_{H_x^{s-1}} & \leq C(\|h_{\text{app}}(t, \cdot, r)\|_{H_x^{s-1}}, \varsigma^{-1}) \kappa \|\partial_x h_{\text{app}}(t, \cdot, r)\|_{H_x^s}, \\ \|r_{\text{app}}(t, \cdot, r)\|_{H_x^{s-1}} & \leq C(\|h_{\text{app}}(t, \cdot, r)\|_{H_x^{s-1}}) \|u_{\text{app}}(t, \cdot, r)\|_{H_x^s}, \\ \|r_{\text{ref}}(t, \cdot, r)\|_{H_x^s} & \leq C(\|\underline{\rho}\|_{L_r^\infty} \|\frac{1}{\underline{\rho}}\|_{L_r^\infty}) \|\partial_x h_{\text{ref}}(t, \cdot)\|_{L_T^1 H_x^s}. \end{aligned}$$

From this we infer that there exists C , depending only on s , $\|h_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^{s-1}}, \varsigma^{-1}, \|\underline{\rho}\|_{L_r^\infty} \|\frac{1}{\underline{\rho}}\|_{L_r^\infty}$

such that

$$\begin{aligned}
 & \max(\{\|h_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^{s-1}}, \kappa^{1/2}\|h_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^s}, \kappa\|\partial_x h_{\text{app}}(\cdot, r)\|_{L_T^2 H_x^s}, \|u_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^s}\}) \\
 & \leq \left(\max(\{\|h^0(\cdot, r)\|_{H_x^{s-1}}, \kappa^{1/2}\|h^0(t=0, \cdot, r)\|_{H_x^s}, \|u^0(\cdot, r)\|_{H_x^s}\}) \right. \\
 & \quad \left. + C(T + T^{1/2})\|u_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^s} + C\|\partial_x h_{\text{ref}}\|_{L_T^1 L_r^1 H_x^s} \right) \\
 & \quad \times \exp(CT\|u_{\text{app}}(\cdot, r)\|_{L_T^\infty H_x^s} + CT^{1/2}\kappa\|\partial_x h_{\text{app}}(\cdot, r)\|_{L_T^2 H_x^s}).
 \end{aligned}$$

By the standard continuity argument, we find that

$$\begin{aligned}
 & \max(\{\|h_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^\infty H_x^{s-1}}, \kappa^{1/2}\|h_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^\infty H_x^s}, \kappa\|\partial_x h_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^2 H_x^s}, \|u_{\text{app}}(\cdot, r)\|_{L_{T_{\text{app}}}^\infty H_x^s}\}) \\
 & \leq c \max(\{\|h^0(\cdot, r)\|_{H_x^{s-1}}, \kappa^{1/2}\|h^0(t=0, \cdot, r)\|_{H_x^s}, \|u^0(\cdot, r)\|_{H_x^s}\}) + CM_{\text{ref}}
 \end{aligned}$$

for almost all $r \in (0, 1)$ and for all $T_{\text{app}} \in [0, T]$ such that

$$(T_{\text{app}} + T_{\text{app}}^{1/2})M_0 \leq C^{-1}$$

where C depends only on $s, \varsigma, \underline{M}, cM_0$. This concludes the proof. \square

We conclude this section by investigating the difference between the reference solution and the refined approximate solution.

Proposition 6.3.7. *Let $s > 3/2$, $\varsigma \in (0, 1)$ and $\underline{M}, M_{\text{ref}}, M_{\text{app}} > 0$. There exists $C > 0$ such that the following holds.*

Let $\kappa \in (0, 1]$, $T > 0$ and let $\underline{\rho}_{\text{ref}}, \underline{u}_{\text{ref}}, \underline{\rho}, \underline{u} \in L_r^\infty$ be such that

$$\|(\underline{\rho}_{\text{ref}}, \frac{1}{\underline{\rho}_{\text{ref}}}, \underline{\rho}, \frac{1}{\underline{\rho}})\|_{L_r^\infty} \leq \underline{M}.$$

Let $(h_{\text{ref}}, u_{\text{ref}}) \in C([0, T]; L^\infty((-1, 0); H^{s+1}(\mathbb{R})^2))$ be a solution to (6.3.6) (that is (6.3.1)-(6.3.2) with $(\underline{\rho}, \underline{u}) = (\underline{\rho}_{\text{ref}}, \underline{u}_{\text{ref}})$) defined on the time interval $[0, T]$ and satisfying

$$\|(h_{\text{ref}}, u_{\text{ref}})\|_{L_T^\infty L_r^\infty H_x^{s+1}} \leq M_{\text{ref}}.$$

Let $(h_{\text{app}}, u_{\text{app}}) \in C([0, T]; L^\infty((-1, 0); H^{s+1}(\mathbb{R})^2))$ be solution to (6.3.7) defined on the time interval $[0, T]$ and satisfying

$$\|(h_{\text{app}}, u_{\text{app}})\|_{L_T^\infty L_r^\infty H_x^s} + \kappa^{1/2}\|\partial_x h_{\text{app}}\|_{L_r^\infty L_T^2 H_x^s} \leq M_{\text{app}},$$

and such that for all $t \in [0, T]$ and almost all $r \in (-1, 0)$

$$\inf_{x \in \mathbb{R}} 1 + h_{\text{ref}}(t, x, r) \geq \varsigma, \quad \inf_{x \in \mathbb{R}} 1 + h_{\text{app}}(t, x, r) \geq \varsigma.$$

Then one has

$$\begin{aligned} \|(h_{\text{ref}} - h_{\text{app}}, u_{\text{ref}} - u_{\text{app}})(\cdot, r)\|_{L_T^\infty H_x^s} &\leq \left(\|(h_{\text{ref}} - h_{\text{app}}, u_{\text{ref}} - u_{\text{app}})(t = 0, \cdot, r)\|_{H_x^s} \right. \\ &\quad \left. + C\kappa \left(|(\underline{\rho}_{\text{ref}} - \underline{\rho}, \underline{u}_{\text{ref}} - \underline{u})(r)| + \|(\underline{\rho}_{\text{ref}} - \underline{\rho})\|_{L_r^1} \right) \right) \exp(CT/\kappa). \end{aligned} \quad (6.3.9)$$

Proof. Let us denote $\dot{h} := h_{\text{ref}} - h_{\text{app}}$, $\dot{u} := u_{\text{ref}} - u_{\text{app}}$, $\dot{\underline{u}} := \underline{u}_{\text{ref}} - \underline{u}$ and $\dot{\underline{\rho}} = \underline{\rho}_{\text{ref}} - \underline{\rho}$. We have on the time interval $I := [0, T]$ for which both function are well-defined

$$\begin{cases} \partial_t \dot{h} + (\underline{u} + u_{\text{app}}) \partial_x \dot{h} - \kappa \partial_x^2 \dot{h} = r_1 + r_2, \\ \partial_t \dot{u} + (\underline{u} + u_{\text{app}} + u_{\text{app}}^*) \partial_x \dot{u} = r_3, \end{cases}$$

where $u_{\text{app}}^* := -\kappa \frac{\partial_x h_{\text{app}}}{1 + h_{\text{app}}}$,

$$r_1 := -(\dot{\underline{u}} + \dot{u}) \partial_x h_{\text{ref}} - \dot{h} \partial_x u_{\text{ref}}, \quad r_2 := -(1 + h_{\text{app}}) \partial_x \dot{u}$$

and

$$r_3 := -\left(\dot{\underline{u}} + \dot{u} - \kappa \frac{\partial_x \dot{h}}{1 + h_{\text{app}}} + \kappa \dot{h} \frac{\partial_x h_{\text{ref}}}{(1 + h_{\text{ref}})(1 + h_{\text{app}})} \right) \partial_x u_{\text{ref}} - \left(\frac{1}{\underline{\rho}_{\text{ref}}} \mathbf{M}[\underline{\rho}_{\text{ref}}] - \frac{1}{\underline{\rho}} \mathbf{M}[\underline{\rho}] \right) \partial_x h_{\text{ref}}.$$

We can proceed as in the proof of Proposition 6.3.3 with some straightforward adjustments as for the contributions of r_3 since the contribution $-\frac{1}{\underline{\rho}} \mathbf{M}[\underline{\rho}] \partial_x \dot{h}$ is nonexistent.

We infer that there exists C depending only on $s, \varsigma, \underline{M}, M_{\text{app}}, M_{\text{ref}}$ such that for all $\kappa \in (0, 1]$ and $T_0 \in (0, T]$ such that $CT_0 \leq \kappa$, one has for almost any $r \in (-1, 0)$,

$$\max(\{ \|(\dot{h}, \dot{u})(\cdot, r)\|_{L_{T_0}^\infty H_x^s}, \kappa^{1/2} \|\partial_x \dot{h}(\cdot, r)\|_{L_{T_0}^2 H_x^s} \}) \leq 2 \|(\dot{h}, \dot{u})(t = 0, \cdot, r)\|_{H_x^s} + C^2 T_0 (|\dot{\underline{\rho}}(r)| + |\dot{\underline{u}}(r)| + \|\dot{\underline{\rho}}\|_{L_r^1}).$$

Iterating this control on $T_n := \min(\{n\kappa/C, T\})$, we find that

$$\|(\dot{h}, \dot{u})(\cdot, r)\|_{L_{T_n}^\infty H_x^s} \leq 2^{n+1} \|(\dot{h}, \dot{u})(t = 0, \cdot, r)\|_{H_x^s} + C 2^n \kappa (|\dot{\underline{\rho}}(r)| + |\dot{\underline{u}}(r)| + \|\dot{\underline{\rho}}\|_{L_r^1}),$$

which yields the claimed estimate. \square

6.3.3 Convergence

We now conclude our analysis with the following stability result for solutions to the hydrostatic Euler equations.

Proposition 6.3.8 (Convergence). *Let $s > 3/2$, $\underline{M}, M_{\text{ref}}, M_0 > 0$, and $\kappa \in (0, 1]$. Then there exists C, T independent of κ and $\delta_0 > 0$ (depending on κ) such that the following holds.*

Let $(\underline{\rho}_{\text{ref}}, \underline{u}_{\text{ref}}) \in L^\infty((-1, 0))^2$ and $(h_{\text{ref}}, u_{\text{ref}}) \in C([0, T]; L^\infty((-1, 0); H^{s+2}(\mathbb{R}^2)))$ be a solution to (6.3.6) (that is (6.3.1)-(6.3.2) with $(\underline{\rho}, \underline{u}) = (\underline{\rho}_{\text{ref}}, \underline{u}_{\text{ref}})$) defined on the time interval $[0, T]$ such that

$$\|(\underline{\rho}_{\text{ref}}, \frac{1}{\underline{\rho}_{\text{ref}}})\|_{L_r^\infty} \leq \underline{M}, \quad \|(h_{\text{ref}}, u_{\text{ref}})\|_{L_T^\infty L_r^\infty H_x^{s+2}} + \|\partial_x h_{\text{ref}}\|_{L_T^1 L_r^1 H_x^{s+2}} \leq M_{\text{ref}}.$$

Let $(\underline{\rho}, \underline{u}) \in L^\infty((-1, 0))^2$ and $(h^0, u^0) \in L^\infty((-1, 0); H^{s+2}(\mathbb{R}^2))$ be such that

$$\|(\underline{\rho}, \frac{1}{\underline{\rho}})\|_{L_r^\infty} \leq \underline{M}, \quad \|(h^0, \kappa^{1/2} \partial_x h^0)\|_{L_r^\infty H_x^{s+1}} + \|u^0\|_{L_r^\infty H_x^{s+2}} \leq M_0$$

and

$$\|(\underline{\rho}_{\text{ref}} - \underline{\rho}, \underline{u}_{\text{ref}} - \underline{u})\|_{L_r^1} + \|(h_{\text{ref}}|_{t=0} - h^0, u_{\text{ref}}|_{t=0} - u^0)\|_{L_r^1 H_x^{s+1}} \leq \delta_0.$$

Then $(h, u) \in C([0, T^*]; L^\infty((-1, 0); H^{s+2}(\mathbb{R}^2)))$ the maximal-in-time solution to (6.3.1)-(6.3.2) emerging from initial data $(h, u)|_{t=0} = (h^0, u^0)$ is defined on the time interval $[0, T]$ and we have for almost all $r \in (-1, 0)$,

$$\begin{aligned} \|(h_{\text{ref}} - h, u_{\text{ref}} - u)(\cdot, r)\|_{L_T^\infty H_x^s} &\leq \left(\|(h_{\text{ref}} - h, u_{\text{ref}} - u)(t=0, \cdot, r)\|_{H_x^s} + \|(h_{\text{ref}} - h, u_{\text{ref}} - u)(t=0, \cdot)\|_{L_r^1 H_x^{s+1}} \right. \\ &\quad \left. + |(\underline{\rho}_{\text{ref}} - \underline{\rho}, \underline{u}_{\text{ref}} - \underline{u})(r)| + \|(\underline{\rho}_{\text{ref}} - \underline{\rho}, \underline{u}_{\text{ref}} - \underline{u})\|_{L_r^1} \right) \times C \exp(CT/\kappa). \end{aligned} \quad (6.3.10)$$

Proof. We first recall that Proposition 6.3.7 provides an estimate on the difference between the reference solution $(h_{\text{ref}}, u_{\text{ref}}) \in C([0, T]; L^\infty((-1, 0); H^{s+2}(\mathbb{R}^2)))$ and the corresponding approximate solution defined as the solution to the system (6.3.7) with initial data $(h_{\text{app}}, u_{\text{app}})|_{t=0} = (h^0, u^0)$, $(h_{\text{app}}, u_{\text{app}}) \in C([0, T]; L^\infty((-1, 0); H_x^{s+2}(\mathbb{R}^2)))$, whose existence and control on the time interval $[0, T]$ (lessening T if necessary) is provided by

Proposition 6.3.6. Specifically we have for almost any $r \in (-1, 0)$

$$\begin{aligned} \|(h_{\text{ref}} - h_{\text{app}}, u_{\text{ref}} - u_{\text{app}})(\cdot, r)\|_{L_T^\infty H_x^{s+1}} &\leq \left(\|(h_{\text{ref}} - h_{\text{app}}, u_{\text{ref}} - u_{\text{app}})(t=0, \cdot, r)\|_{H_x^{s+1}} \right. \\ &\quad \left. + C\kappa \left(|(\underline{\rho}_{\text{ref}} - \underline{\rho}, \underline{u}_{\text{ref}} - \underline{u})(r)| + \|(\underline{\rho}_{\text{ref}} - \underline{\rho})\|_{L_r^1} \right) \right) \exp(CT/\kappa), \end{aligned}$$

where C depends only on $s, \varsigma, \underline{M}, M_{\text{ref}}$ and M_0 .

We then consider the difference between the exact solution $(h, u) \in C([0, T^*]; L^\infty((-1, 0); H_x^{s+2}(\mathbb{R}^2)))$ —whose existence is provided by Proposition 6.3.2— and the approximate solution. By means of the consistency result, Proposition 6.3.5, we can adapt the proof of Proposition 6.3.3 and we find that under the assumptions that

$$\|(h, u)\|_{L_T^\infty L_r^\infty H_x^s} + \kappa^{1/2} \|\partial_x h\|_{L_r^\infty L_T^2 H_x^s} \leq M \quad \text{and} \quad \operatorname{ess\,inf}_{(t,x,r) \in [0,T] \times \mathbb{R} \times (-1,0)} 1 + h(t, x, r) \geq \varsigma \quad (6.3.11)$$

there exists $C > 0$ depending only on $s, \varsigma, \underline{M}, M_{\text{ref}}, M$ such that for all $\kappa \in (0, 1]$ and $t \in (0, \min(\{T, T^*\}))$ such that $Ct \leq \kappa$ and for almost any $r \in (-1, 0)$ one has

$$\begin{aligned} &\max(\{\|(h - h_{\text{app}}, u - u_{\text{app}})(\cdot, r)\|_{L_t^\infty H_x^s}, \kappa^{1/2} \|\partial_x(h - h_{\text{app}})(\cdot, r)\|_{L_t^2 H_x^s}\}) \\ &\leq 2\|(h - h_{\text{app}}, u - u_{\text{app}})(t=0, \cdot, r)\|_{L_r^\infty H_x^s} + Ct \|(h_{\text{ref}} - h_{\text{app}})(t, \cdot)\|_{L_r^1 H_x^{s+1}}. \end{aligned}$$

Iterating the stability estimate on $T_n := \min(\{nT_0, T\})$ we deduce that (augmenting C if necessary)

$$\begin{aligned} &\max(\{\|(h - h_{\text{app}}, u - u_{\text{app}})\|_{L_t^\infty L_r^\infty H_x^s}, \kappa^{1/2} \|\partial_x(h - h_{\text{app}})\|_{L_r^\infty L_t^2 H_x^s}\}) \\ &\leq \exp(CT/\kappa) \|(h - h_{\text{app}}, u - u_{\text{app}})(t=0, \cdot)\|_{L_r^\infty H_x^s} + C\kappa \exp(CT/\kappa) \|h_{\text{ref}} - h_{\text{app}}\|_{L_T^\infty L_r^1 H_x^{s+1}}. \end{aligned}$$

Since by construction $(h_{\text{app}}, u_{\text{app}})|_{t=0} = (h, u)|_{t=0}$ and using the above control on $h_{\text{ref}} - h_{\text{app}}$ we infer (augmenting C if necessary)

$$\begin{aligned} &\max(\{\|(h - h_{\text{app}}, u - u_{\text{app}})\|_{L_t^\infty L_r^\infty H_x^s}, \kappa^{1/2} \|\partial_x(h - h_{\text{app}})\|_{L_r^\infty L_t^2 H_x^s}\}) \\ &\leq C\kappa \exp(CT/\kappa) \|(h_{\text{ref}} - h, u_{\text{ref}} - u)(t=0, \cdot)\|_{L_r^1 H_x^{s+1}} + C(\kappa \exp(CT/\kappa))^2 \|(\underline{\rho}_{\text{ref}} - \underline{\rho}, \underline{u}_{\text{ref}} - \underline{u})\|_{L_r^1}. \end{aligned}$$

Notice that this estimate implies by triangle inequality an upper bound on $\|(h, u)\|_{L_t^\infty L_r^\infty H_x^s} + \kappa^{1/2} \|\partial_x h\|_{L_r^\infty L_T^2 H_x^s}$ which (choosing M sufficiently large and δ_0 sufficiently small) enforces strictly the condition (6.3.11), so that by the standard continuity argument the above

holds without restriction on $t \in (0, \min(\{T, T^*\}))$. By the persistence of regularity stated in Proposition 6.3.2 we infer that $T^* > T$ and the above holds for any $t \in (0, T]$. The estimate (6.3.10) then immediately follows from the triangle inequality. \square

Thanks to Proposition 6.2.8 (concerning the well-posedness and control of solutions to the bilayer system) on one hand and Proposition 6.3.8 (concerning the control of the deviations of nearby solutions to some given reference solutions) on the other hand, one infers immediately the announced rigorous justification of the propagation in time of the columnar motion and sharp stratification assumptions in near-bilayer situations (within the hydrostatic framework). Specifically, we have the following result.

Theorem 6.3.1. *Let $s \geq s_0 > 3/2$, $\underline{M} > 0$, $M_0 > 0$, $\varsigma \in (0, 1)$, and $\kappa \in (0, 1]$. Then there exist $C > 0, T > 0$ independent of κ and $\delta_0 > 0$ (depending on κ) such that the following holds.*

Let $(\underline{\rho}, \underline{u}) \in L^\infty((-1, 0))^2$ and $(h^0, u^0) \in L^\infty((-1, 0); H^{s+2}(\mathbb{R})^2)$ be such that

$$\|(\underline{\rho}, \frac{1}{\underline{\rho}})\|_{L_r^\infty} \leq \underline{M}, \quad \|(h^0, \kappa^{1/2} \partial_x h^0)\|_{L_r^\infty H_x^{s+1}} + \|u^0\|_{L_r^\infty H_x^{s+2}} \leq M_0$$

and there exists $(\rho_s, \rho_b, \underline{H}_s, \underline{H}_b, \underline{U}_s, \underline{U}_b) \in \mathbb{R}^6$ such that $\underline{H}_s + \underline{H}_b = 1$ and $\underline{U}_s + \underline{U}_b = 0$ as well as $(H_s^0, H_b^0, U_s^0, U_b^0) \in H^{s+4}(\mathbb{R})^2 \times H^{s+3}(\mathbb{R})^2$ such that the hyperbolicity condition holds:

$$\forall x \in \mathbb{R}, \quad (\rho_s, \rho_b, \underline{H}_s + H_s^0(x), \underline{H}_b + H_b^0(x), \underline{U}_s + U_s^0(x), \underline{U}_b + U_b^0(x)) \in \mathfrak{p}^\varsigma,$$

where \mathfrak{p}^ς is defined in (6.2.5), and

$$\|(H_s^0, H_b^0, U_s^0, U_b^0, \kappa \partial_x H_s^0, \kappa \partial_x H_b^0)\|_{H^{s+3}} \leq M_0$$

and such that denoting $(\underline{\rho}_{\text{bl}}^0, \underline{u}_{\text{bl}}^0, h_{\text{bl}}^0, u_{\text{bl}}^0)$ through (2.2.16) we have

$$\|(h_{\text{bl}}^0 - h^0, u_{\text{bl}}^0 - u^0)\|_{L_r^1 H_x^{s+1}} + \|(\underline{\rho}_{\text{bl}} - \underline{\rho}, \underline{u}_{\text{bl}} - \underline{u})\|_{L_r^1} \leq \delta_0,$$

then

1. *there exists $(H_s, H_b, U_s, U_b) \in C([0, T]; H^{s+3}(\mathbb{R})^4)$ solution to (??) emerging from the initial data*

$$(H_s, H_b, U_s, U_b)|_{t=0} = (H_s^0, H_b^0, U_s^0, U_b^0);$$

2. there exists $(h, u) \in C([0, T]; L^\infty((-1, 0); H^{s+2}(\mathbb{R})^2))$ solution to (2.2.13) emerging from initial data

$$(h, u)|_{t=0} = (h^0, u^0);$$

3. denoting $(\underline{\rho}_{\text{bl}}, \underline{u}_{\text{bl}}, h_{\text{bl}}, u_{\text{bl}})$ through (2.2.16) we have for almost all $r \in (-1, 0)$,

$$\begin{aligned} \|(h_{\text{bl}}-h, u_{\text{bl}}-u)(\cdot, r)\|_{L_T^\infty H_x^s} &\leq \left(\|(h_{\text{bl}}^0-h^0, u_{\text{bl}}^0-u^0)(\cdot, r)\|_{H_x^s} + \|(h_{\text{bl}}^0-h^0, u_{\text{bl}}^0-u^0)\|_{L_r^1 H_x^{s+1}} \right. \\ &\quad \left. + |(\underline{\rho}_{\text{bl}} - \underline{\rho}, \underline{u}_{\text{bl}} - \underline{u})(r)| + \|(\underline{\rho}_{\text{bl}} - \underline{\rho}, \underline{u}_{\text{bl}} - \underline{u})\|_{L_r^1} \right) \times C \exp(Ct/\kappa). \end{aligned}$$

Remark 6.3.9. Proposition 6.3.8 is not limited to the bilayer framework and applies to any suitably regular reference solution. Hence it can be combined with results analogous to Proposition 6.2.8 to provide results analogous to Theorem 6.3.1 in the one-layer and multi-layer frameworks.

We recall that the result analogous to Proposition 6.2.8 in the one-layer framework (that is associated with the standard shallow water equation with thickness diffusivity that was discussed for instance in [37]) is stated and proved in Proposition 3.2.9. Notice it requires neither the discussion on the hyperbolicity domain of the non-diffusive equations (since the standard non-cavitation assumption guarantees hyperbolicity) nor the discussion on the parabolic regularization of the total velocity (since the natural symmetrizer behaves well with diffusivity contributions).

A result analogous to Proposition 6.2.8 in the multi-layer framework follows from combining the result of [27] with the analysis of Section 6.2.2. In the former, it is proved that assuming sufficiently small shear velocities is a sufficient condition for the (strict) hyperbolicity of the multi-layer system in the stably stratified situation. Notice however that this smallness condition is implicit, and not uniform with respect to the number of layers.

6.4 Conclusion

In this chapter we recalled some known results of the bi-layer shallow water system with and without thickness diffusivity (κ). We showed that for any sufficiently regular initial data satisfying some hyperbolicity criterion, strong solutions to the bilayer shallow water system (6.2.6) exist and are controlled on a time interval which is uniform with respect to $0 < \kappa \leq 1$, moreover we rigorously proved the expected strong convergence as $\kappa \rightarrow 0$ (see

Proposition 6.2.9). Furthermore we studied the hydrostatic Euler equations. We provided some stability estimates with respect to perturbations of the equations and of the data using suitable distances. As a second step we introduced approximate solutions associated with some given reference exact solution and close-by data. Building upon these approximate solutions, we proved Proposition 6.3.8 which controls the difference between the reference solution and exact solutions to the hydrostatic Euler equations emerging from close-by data. Together, Proposition 6.2.8 and Proposition 6.3.8 provide the announced rigorous justification that for initial data suitably close to the bilayer framework and satisfying some hyperbolicity criterion limiting in particular the admissible size of shear velocities, the emerging solutions to the hydrostatic Euler equations remain close to the solution predicted by the bilayer shallow water system on a relevant timescale (independent of κ)(see Theorem 6.3.1).

TECHNICAL TOOLS

In this section we provide several preliminary results that will be used throughout this paper. In particular, we provide embeddings, product and commutator estimates adapted to our functional framework, and which mostly follow from standard estimates in Sobolev spaces.

Before stating these results we introduce notations.

7.1 Notations

- For any $a, b \in \mathbb{R}$, we use the notation $a \lesssim b$ if there exists $C > 0$, independent of relevant parameters, such that $a \leq Cb$.
- We generically denote by $C(\dots)$ some positive function that has a non decreasing dependence on its arguments.
- We set

$$\langle B_a \rangle_{a>b} = \begin{cases} 0 & \text{if } a \leq b, \\ B_a & \text{otherwise,} \end{cases}$$

- We denote $\Lambda = (\text{Id} - \partial_x^2)^{1/2}$, so that for all $s \geq 0$, $H^s(\mathbb{R}) = \{f \in L^2(\mathbb{R}); \Lambda^s f \in L^2(\mathbb{R})\}$.
- Let I be an interval and X a Banach space equipped with the norm $\|\cdot\|_X$ then we define the following spaces
 - For $1 \leq p < \infty$, the space $L^p(I; X)$ consists of all strongly measurable functions $f : I \rightarrow X$, such that

$$\int_I \|f\|_X^p < \infty,$$

equipped with the norm

$$\|f\|_{L^p(I; X)} = \left(\int_I \|f\|_X^p \right)^{\frac{1}{p}}.$$

When $p = \infty$, the space $L^\infty(I; X)$ consists of all strongly measurable functions $f : I \rightarrow X$ such that

$$\|f\|_{L^\infty(I;X)} = \sup_{x \in I} \|f(x)\|_X.$$

where \sup denotes the essential supremum.

- The space $C(I; X)$ consists of all functions $f : I \rightarrow X$, that are continuous at every point of I .

Where f is said to be continuous at $x_0 \in I$, when $f(x) \rightarrow f(x_0)$ in X as $x \rightarrow x_0$.

- When useful, we provide insights on the variables at stake in aforementioned functional spaces by means of subscripts. For instance for $f : (x, r) \in \mathbb{R} \times (-1, 0) \mapsto f(x, r) \in \mathbb{R}$ we may denote

$$\|f\|_{L_r^\infty H_x^s} = \text{ess sup} \left(\{ \|f(\cdot, r)\|_{H^s}, r \in (-1, 0) \} \right).$$

- We sometimes also use subscripts to provide information on the interval at stake in functional spaces. For instance for $T > 0$ and $f : (t, x) \in [0, T] \times \mathbb{R} \mapsto f(t, x) \in \mathbb{R}$ we may denote

$$\|f\|_{L_T^\infty H_x^s} = \text{ess sup} \left(\{ \|f(t, \cdot)\|_{H^s}, t \in [0, T] \} \right).$$

- Let $\mathbf{A} = (A_{ij})_{1 \leq i, j \leq N}$ and $U = (U_1, \dots, U_N)^t$ where A_{ij}, U_i are in a suitable functional spaces such that the linear operator P is well defined on that space (most of the time $P = \Lambda^s$ with $s \geq 0$ or $P = \partial_x^\alpha$ with $\alpha \in \mathbb{N}$) we define the commutators

$$[P, \mathbf{A}]U := P(\mathbf{A}U) - \mathbf{A}P(U)$$

$$[P; \mathbf{A}, U] := P(\mathbf{A}U) - \mathbf{A}P(U) - P(\mathbf{A})U.$$

For the definition of the operators $(D_\rho^j, M^j \dots)$ and other definitions used in this manuscript one can return to section 4.1 and section 5.1.

We conclude this section by introducing a new type of commutator adapted to the Leibnitz formula displayed in Lemma 7.2.1 below.

Definition 7.1.1. Let $A, B \in \mathbb{R}^N$, $j \in \{0, 1, 2\}$, we define, the following commutators:

$$\begin{aligned} \llbracket \partial_x^\alpha D_\rho^j, A \rrbracket B &:= \partial_x^\alpha (D_\rho^j(AB)) - (M^j A)(\partial_x^\alpha D_\rho^j B), \\ \llbracket \partial_x^\alpha D_\rho^j; A, B \rrbracket &:= \partial_x^\alpha (D_\rho^j(AB)) - (M^j A)(\partial_x^\alpha D_\rho^j B) - (M^j B)(\partial_x^\alpha D_\rho^j A). \end{aligned}$$

7.2 Adapted classical results

Recalling the convention that $M^0 = \text{Id}$, we have

$$\begin{aligned} \llbracket \partial_x^\alpha D_\rho^0, A \rrbracket B &= [\partial_x^\alpha, A]B, \\ \llbracket \partial_x^\alpha D_\rho^0; A, B \rrbracket &= [\partial_x^\alpha; A, B]. \end{aligned}$$

Lemma 7.2.1. Let $N \in \mathbb{N}^*$, $F, G \in \mathbb{R}^N$, we have the following identities:

1. (Abel's summation)

$$S(FG) = F(SG) - S_0((D_\rho F)(R_u SG)).$$

2. (first order Leibnitz formula)

$$D_\rho(FG) = (D_\rho F)(MG) + (MF)(D_\rho G).$$

3. (second order Leibnitz formula)

$$D_\rho^2(FG) = (D_\rho^2 F)(M^2 G) + (M^2 F)(D_\rho^2 G) + 2(MD_\rho F)(MD_\rho G).$$

Proof. 1. Let $F = (F_1, \dots, F_N)^t \in \mathbb{R}^N$, $G = (G_1, \dots, G_N)^t \in \mathbb{R}^N$, the identity results from the fact that

$$\sum_{j=i}^N F_j G_j - F_i \sum_{k=i}^N G_k = - \sum_{j=i}^{N-1} (F_j - F_{j+1}) \sum_{k=j+1}^N G_k, \quad \forall i \in \{1, \dots, N-1\}.$$

2. It results from the fact that

$$(FG)_{i+1} - (FG)_i = \left(\frac{F_i + F_{i+1}}{2} \right) (G_{i+1} - G_i) + \left(\frac{G_i + G_{i+1}}{2} \right) (F_{i+1} - F_i).$$

3. It results directly from multiplying the first order Leibnitz formula by \tilde{D}_ρ and reap-

plying this formula a second time using the fact that $\tilde{D}_\rho M = MD_\rho$.

□

Lemma 7.2.2. *Let $N \in \mathbb{N}^*$, $F \in \mathbb{R}^N$. For any $p, q \in [1, \infty]$ the following estimates hold true:*

$$|F|_{l^p} \leq |F|_{l^q}, \text{ when } p \leq q.$$

$$\|T\|_{l^\infty \rightarrow l^2} \leq 1; \quad \|S\|_{l^p \rightarrow l^q} \leq 1.$$

The same continuity estimates hold for S_0 instead of S .

The following Lemma provides a continuous embedding for the space $H^{s+1/2,1}(\mathbb{R})$ with a continuity constant that is uniform with respect to the dimension parameter N . This estimate will be very useful and frequently used throughout this paper in order to obtain sharp estimates when using the energy method. In the proof of this Lemma we mimic the proof of Lemma A.1 in [13] which provides the analogous result in the (infinite-dimensional) continuous setting. Moreover the proof relies on the use of the operator $S_0 D_\rho$ which has an almost Identity like behavior, in addition to an auxiliary function ϕ carefully chosen such that it can receive the unwanted discrete derivative and enables us to have a sharper estimate, and finally the proof is achieved by the use of Parseval equality.

Lemma 7.2.3. *Let $N \in \mathbb{N}^*$, $s \in \mathbb{R}$, $F \in H^{s+\frac{1}{2},1}(\mathbb{R})$. There exists $C > 0$ independent of N such that*

$$\|\Lambda^s F\|_{l^\infty(L_x^2)} \leq C \|F\|_{H^{s+\frac{1}{2},1}}.$$

Proof. Let us assume that N is even, and define $\phi := \left(\overbrace{1, \dots, 1}^{\frac{N}{2}}, \frac{N-2}{N}, \dots, \frac{4}{N}, \frac{2}{N}, 0 \right)^t \in \mathbb{R}^N$,

hence $D_\rho(\phi) = N \left(\overbrace{0, \dots, 0}^{\frac{N}{2}-1}, \frac{2}{N}, \dots, \frac{2}{N} \right)^t$. Furthermore we notice that

$$|\phi|_{l^\infty} \leq 1, \quad |D_\rho(\phi)|_{l^\infty} \leq 2.$$

Let $i \in \{1, \dots, \frac{N}{2}\}$. Using Lemma 7.2.2 with Parseval equality we have

$$\begin{aligned}
 \|\Lambda^s F_i\|_{L_x^2}^2 &= \int_{\mathbb{R}} |\Lambda^s F_i|^2 \\
 &= \int_{\mathbb{R}} \left(S_0 \left(D_\rho(\phi(\Lambda^s F)^2) \right) \right)_i \\
 &= \int_{\mathbb{R}} \left(S_0 \left((M\phi)(D_\rho((\Lambda^s F)^2)) \right) \right)_i + \int_{\mathbb{R}} \left(S_0 \left((M(\Lambda^s F)^2)(D_\rho(\phi)) \right) \right)_i \\
 &= 2 \int_{\mathbb{R}} \left(S_0 \left((M\phi)(\Lambda^{s-\frac{1}{2}} D_\rho(F))(M\Lambda^{s+\frac{1}{2}} F) \right) \right)_i + \int_{\mathbb{R}} \left(S_0 \left((M(\Lambda^s F)^2)(D_\rho(\phi)) \right) \right)_i \\
 &\leq 2 \|\phi\|_{l^\infty} \int_{\mathbb{R}} |(\Lambda^{s-\frac{1}{2}}(D_\rho(F)))(\Lambda^{s+\frac{1}{2}} F)|_{l^1} + \int_{\mathbb{R}} |M(\Lambda^s F)^2|_{l^1} \|D_\rho(\phi)\|_{l^\infty} \\
 &\lesssim \|F\|_{H^{s+\frac{1}{2},1}}^2.
 \end{aligned}$$

Let $\tilde{F}_i := F_{N-i+1}$ for $i \in \{1, \dots, N\}$ and from what proceeds and for $i \in \{1, \dots, \frac{N}{2}\}$ we have

$$\|\Lambda^s \tilde{F}_i\|_{L_x^2}^2 \lesssim \|\Lambda^{s+\frac{1}{2}} \tilde{F}\|_{l^2(L_x^2)}^2 + \|\Lambda^{s-\frac{1}{2}}(D_\rho \tilde{F})\|_{l^2(L_x^2)}^2.$$

Hence the result follows immediately since $\|\Lambda^{s-\frac{1}{2}}(D_\rho \tilde{F})\|_{l^2(L_x^2)} = \|\Lambda^{s-\frac{1}{2}}(D_\rho F)\|_{l^2(L_x^2)}$ and $\|\Lambda^{s+\frac{1}{2}} \tilde{F}\|_{l^2(L_x^2)} = \|\Lambda^{s+\frac{1}{2}} F\|_{l^2(L_x^2)}$. We can easily adapt this proof when N is odd. \square

We first collect some useful facts concerning the discrete derivation operator.

Lemma 7.2.4. *Let $N \in \mathbb{N}^*$, $F \in \mathbb{R}^N$, $h_* > 0$ and $\underline{H} \in \mathbb{R}^N$ such that $\inf_{i \in \{1, \dots, N\}} \underline{H}_i \geq h_*$.*

Then there exists $C(h_^{-1})$ independent of N such that*

$$1. \quad |D_\rho \frac{1}{\underline{H}}|_{l^\infty} \leq C(h_*^{-1}) |D_\rho \underline{H}|_{l^\infty} \quad \text{and} \quad |D_\rho^2 \frac{1}{\underline{H}}|_{l^\infty} \leq C(h_*^{-1}) (|D_\rho^2 \underline{H}|_{l^\infty} + |D_\rho \underline{H}|_{l^\infty}^2).$$

Proof. 1. We have for $i \in \{1, \dots, N-2\}$

$$\begin{aligned}
 \left(D_\rho^2 \frac{1}{\underline{H}} \right)_i &= N^2 \left(\frac{1}{\underline{H}_i} - \frac{2}{\underline{H}_{i+1}} + \frac{1}{\underline{H}_{i+2}} \right) \\
 &= N^2 \left(\frac{2(\underline{H}_{i+2} - \underline{H}_{i+1})(\underline{H}_{i+1} - \underline{H}_i) - \underline{H}_{i+1}(\underline{H}_{i+2} - 2\underline{H}_{i+1} + \underline{H}_i)}{\underline{H}_i \underline{H}_{i+1} \underline{H}_{i+2}} \right)
 \end{aligned}$$

and the result follows immediately. \square

The upcoming Lemma 7.2.5 and Lemma 7.2.6 are generalization of the classical product

and commutator estimates in Sobolev spaces adapted in the case of the Sobolev space $H^{s,k}$ for N dimensional vector fields.

Lemma 7.2.5. *Let $N \in \mathbb{N}^*$, $t_0 > \frac{1}{2}$.*

1. *For any $s, s_1, s_2 \in \mathbb{R}$ such that $s_1 \geq s$, $s_2 \geq s$, $s_1 + s_2 \geq 0$ and $s_1 + s_2 \geq s + t_0$, there exists $C > 0$ independent of N such that for any $(p, q) \in \{(2, \infty), (\infty, 2)\}$ and any $F = (F_1, \dots, F_N)^t \in (H^{s_1}(\mathbb{R}^d))^N$ and $G = (G_1, \dots, G_N)^t \in (H^{s_2}(\mathbb{R}^d))^N$, $FG \in l^2(H^s(\mathbb{R}^d))$ and*

$$\|FG\|_{l^2(H^s)} \leq C \|F\|_{l^p(H_x^{s_1})} \|G\|_{l^q(H_x^{s_2})} \leq \begin{cases} C \|F\|_{H^{s_1+\frac{1}{2},1}} \|G\|_{H^{s_2,0}} \\ or \\ C \|F\|_{H^{s_1,0}} \|G\|_{H^{s_2+\frac{1}{2},1}} \end{cases}$$

2. *For any $s \geq -t_0$, there exists $C > 0$ independent of N such that for any $(p, q), (\tilde{p}, \tilde{q}) \in \{(2, \infty), (\infty, 2)\}$ and any $F = (F_1, \dots, F_N)^t \in (H^s(\mathbb{R}^d))^N$ and $G = (G_1, \dots, G_N)^t \in (H^s(\mathbb{R}^d))^N \cap (H^{t_0}(\mathbb{R}^d))^N$, $FG \in l^2(H^s(\mathbb{R}^d))$ and*

$$\begin{aligned} \|FG\|_{l^2(H_x^s)} &\leq C \|F\|_{l^p(H_x^{t_0})} \|G\|_{l^q(H_x^s)} + C \left\langle \|F\|_{l^{\tilde{p}}(H_x^s)} \|G\|_{l^{\tilde{q}}(H_x^{t_0})} \right\rangle_{s>t_0} \\ &\leq C \begin{cases} \|F\|_{H^{\max(t_0,s)+\frac{1}{2},1}} \|G\|_{H^{s,0}} \\ or \\ \|F\|_{H^{\max(t_0+\frac{1}{2},s),1}} \|G\|_{H^{\max(t_0+\frac{1}{2},s),1}} \end{cases} \end{aligned}$$

3. *For any $s \in \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$ such that $s_1 \geq s$, $s_2 \geq s - 1$ and $s_1 + s_2 \geq s + t_0$, there exists $C > 0$ independent of N such that for any $(p, q) \in \{(2, \infty), (\infty, 2)\}$ and any $F = (F_1, \dots, F_N)^t \in (H^{s_1}(\mathbb{R}^d))^N$ and $G = (G_1, \dots, G_N)^t \in (H^{s_2}(\mathbb{R}^d))^N$, $[\Lambda^s, F]G \in (L^2(\mathbb{R}^d))^N$ and*

$$\|[\Lambda^s, F]G\|_{l^2(L_x^2)} \leq C \|\Lambda^{s_1} F\|_{l^p(L_x^2)} \|\Lambda^{s_2} G\|_{l^q(L_x^2)} \leq \begin{cases} C \|F\|_{H^{s_1+\frac{1}{2},1}} \|G\|_{H^{s_2,0}} \\ or \\ C \|F\|_{H^{s_1,0}} \|G\|_{H^{s_2+\frac{1}{2},1}} \end{cases}$$

4. *For any $s \geq 0$, there exists $C > 0$ independent of N such that for any $(p, q), (\tilde{p}, \tilde{q}) \in \{(2, \infty), (\infty, 2)\}$ and any $G = (G_1, \dots, G_N)^t \in (H^{s-1}(\mathbb{R}^d))^N$ and any $F = (F_1, \dots, F_N)^t \in$*

$(L^\infty(\mathbb{R}^d))^N$ with $\partial_x F_i \in H^{s-1}(\mathbb{R}^d) \cap H^{t_0}(\mathbb{R}^d)$, one has $[\Lambda^s, F]G \in (L^2(\mathbb{R}^d))^N$, and

$$\begin{aligned} \| [\Lambda^s, F]G \|_{l^2 L_x^2} &\leq C \| \Lambda^{t_0}(\partial_x F) \|_{l^p(L_x^2)} \| \Lambda^{s-1}G \|_{l^q(L_x^2)} \\ &\quad + C \left\langle \| \Lambda^{s-1}(\partial_x F) \|_{l^{\tilde{p}}(L_x^2)} \| \Lambda^{t_0}G \|_{l^{\tilde{q}}(L_x^2)} \right\rangle_{s > t_0+1} \\ &\leq C \begin{cases} \| \partial_x F \|_{H^{\max(t_0+\frac{1}{2}, s-\frac{1}{2}), 1}} \| G \|_{H^{s-1, 0}} \\ \text{or} \\ \| \partial_x F \|_{H^{\max(t_0+\frac{1}{2}, s-1), 1}} \| G \|_{H^{\max(s-1, t_0+\frac{1}{2}), 1}} \end{cases} \end{aligned}$$

5. For any $s \geq 0$, there exists $C > 0$ independent of N such that for any $(p, q), (\tilde{p}, \tilde{q}) \in \{(2, \infty), (\infty, 2)\}$ and for any $F = (F_1, \dots, F_N)^t$, $G = (G_1, \dots, G_N)^t$ with $F_i, G_i \in H^s(\mathbb{R}^d) \cap H^{t_0+1}(\mathbb{R}^d)$ for all $i \in \{1, \dots, N\}$, one has $[\Lambda^s; F, G] \in (L^2(\mathbb{R}^d))^N$ and

$$\begin{aligned} \| [\Lambda^s; F, G] \|_{l^2 L_x^2} &\leq C \| \Lambda^{t_0+1}F \|_{l^p(L_x^2)} \| \Lambda^{s-1}G \|_{l^q(L_x^2)} + \| \Lambda^{s-1}F \|_{l^{\tilde{p}}(L_x^2)} \| \Lambda^{t_0+1}G \|_{l^{\tilde{q}}(L_x^2)} \\ &\leq C \begin{cases} \| F \|_{H^{\max(t_0+\frac{3}{2}, s-\frac{1}{2}), 1}} \| G \|_{H^{\max(s-1, t_0+1), 0}} \\ \text{or} \\ \| F \|_{H^{\max(t_0+\frac{3}{2}, s-1), 1}} \| G \|_{H^{\max(t_0+\frac{3}{2}, s-1), 1}} \end{cases} \end{aligned}$$

The result holds true if we replace Λ^s by ∂_x^α for $0 \leq \alpha \leq s$.

Proof. 1. The proof results immediately from the following classical product estimate in Sobolev spaces for scalar function f, g

$$\| fg \|_{H_x^s} \leq C \| f \|_{H_x^{s_1}} \| g \|_{H_x^{s_2}}, \quad (7.2.1)$$

then conclude with Lemma 7.2.3.

2. The proof results immediately from the following classical tame estimate for products in Sobolev spaces and for scalar function f, g

$$\| fg \|_{H_x^s} \leq C \| f \|_{H_x^{t_0}} \| g \|_{H_x^s} + C \left\langle \| f \|_{H_x^s} \| g \|_{H_x^{t_0}} \right\rangle_{s > t_0}, \quad (7.2.2)$$

then conclude with Lemma 7.2.3.

3. We fix $(p, q) \in \{(2, \infty), (\infty, 2)\}$ and we have $[\Lambda^s; F]G = ([\Lambda^s, F_i]G_i)_i$, hence the proof results immediately using the following classical commutator estimate in Sobolev

spaces for scalar functions f, g

$$\|[\Lambda^s, f]g\|_{L_x^2} \lesssim \|\Lambda^{s_1} f\|_{L_x^2} \|\Lambda^{s_2} g\|_{L_x^2}, \quad (7.2.3)$$

then conclude with Lemma 7.2.3.

4. The proof results immediately from using the following classical tame estimate for commutators in Sobolev spaces for scalar functions f, g

$$\|[\Lambda^s, f]g\|_{L_x^2} \leq C \|\Lambda^{t_0}(\partial_x f)\|_{L_x^2} \|\Lambda^{s-1} g\|_{L_x^2} + C \left\langle \|\Lambda^{s-1}(\partial_x f)\|_{L_x^2} \|\Lambda^{t_0} g\|_{L_x^2} \right\rangle_{s > t_0 + 1}, \quad (7.2.4)$$

then conclude with Lemma 7.2.3.

5. We have $[\Lambda^s; F, G] = (\Lambda^s(F_i G_i) - \Lambda^s(F_i)G_i - F_i \Lambda^s(G_i))_i$, the result follows immediately by using the following classical estimate for the symmetric commutator and for scalar functions f, g , where $[\Lambda^s; f, g] := \Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f \in L^2(\mathbb{R})$

$$\|[\Lambda^s; f, g]\|_{L_x^2} \leq C \|f\|_{H_x^{t_0+1}} \|g\|_{H_x^{s-1}} + C \|f\|_{H_x^{s-1}} \|g\|_{H_x^{t_0+1}}, \quad (7.2.5)$$

then conclude by Lemma 7.2.3. □

Lemma 7.2.6. *Let $N \in \mathbb{N}^*$, $t_0 > \frac{1}{2}$ then we have the following*

1. *Let $s \in \mathbb{N}$ such that $s \geq t_0 + \frac{1}{2}$, then there exists C independent of N such that for any $F, G \in H^{s,k}$, $k \in \{1, 2\}$ we have*

$$\|FG\|_{H^{s,k}} \leq C \|F\|_{H^{s,k}} \|G\|_{H^{s,k}}.$$

2. *Let $s \in \mathbb{N}$ such that $s \geq t_0 + \frac{3}{2}$, then there exists C independent of N such that for any $F, G \in H^{s,2}$ it holds*

$$\|FG\|_{H^{s,2}} \leq C (\|F\|_{H^{s,2}} \|G\|_{H^{s-1,1}} + \|F\|_{H^{s-1,1}} \|G\|_{H^{s,2}}).$$

Proof. 1. Let $F, G \in H^{s,1}$ then by Lemma 7.2.1

$$\begin{aligned} \|FG\|_{H^{s,1}}^2 &= \|\Lambda^s(FG)\|_{l^2(L_x^2)}^2 + \|\Lambda^{s-1}(\mathcal{D}_\rho(FG))\|_{l^2(L_x^2)}^2 \\ &\lesssim \|\Lambda^s(FG)\|_{l^2(L_x^2)}^2 + \|\Lambda^{s-1}((MF)(\mathcal{D}_\rho G))\|_{l^2(L_x^2)}^2 + \|\Lambda^{s-1}((MG)(\mathcal{D}_\rho F))\|_{l^2(L_x^2)}^2. \end{aligned}$$

Since $s \geq t_0 + \frac{1}{2}$, and using Lemma 7.2.5 (2) we have

$$\|\Lambda^s(FG)\|_{l^2(L_x^2)} \lesssim \|F\|_{H^{s,1}} \|G\|_{H^{s,1}},$$

and

$$\|\Lambda^{s-1}((MF)D_\rho G)\|_{l^2(L_x^2)} \lesssim \|F\|_{H^{s,1}} \|G\|_{H^{s,1}}.$$

Symmetrically $\|\Lambda^{s-1}((MG)(D_\rho F))\|_{l^2(L_x^2)} \lesssim \|F\|_{H^{s,1}} \|G\|_{H^{s,1}}$. Consequently $H^{s,1}$ is a Banach algebra.

Let $F, G \in H^{s,2}$ then $\|FG\|_{H^{s,2}}^2 = \|FG\|_{H^{s,1}}^2 + \|\Lambda^{s-2}(D_\rho^2(FG))\|_{l^2(L^2)}^2$. Using the fact that $H^{s,1}$ is a Banach algebra we have

$$\|FG\|_{H^{s,1}} \lesssim \|F\|_{H^{s,2}} \|G\|_{H^{s,2}}.$$

By Lemma 7.2.1

$$D_\rho^2(FG) = (M^2F)(D_\rho^2G) + (M^2G)(D_\rho^2F) + 2(MD_\rho F)(MD_\rho G).$$

Then by Lemma 7.2.5 (2)

$$\|\Lambda^{s-2}((M^2F)(D_\rho^2G))\|_{l^2(L_x^2)} \lesssim \|F\|_{H^{s,1}} \|G\|_{H^{s,2}}.$$

Symmetrically we deduce that $\|\Lambda^{s-2}((M^2G)(D_\rho^2F))\|_{l^2(L_x^2)} \lesssim \|F\|_{H^{s,2}} \|G\|_{H^{s,1}}$. Moreover by Lemma 7.2.5 (1) we have

$$\|\Lambda^{s-2}((MD_\rho G)(MD_\rho F))\|_{l^2(L_x^2)} \lesssim \|F\|_{H^{s,2}} \|G\|_{H^{s,2}}.$$

2. The proof follows the steps seen previously in 1 with this time using Lemma 7.2.3 and Lemma 7.2.5 and choosing the convenient values of $p, q, \tilde{p}, \tilde{q}$, in addition relying on the fact that $s \geq t_0 + \frac{3}{2}$ when estimating.

□

The following Lemma is an adaptation of the classical commutator estimates in Sobolev spaces to the case of N dimensional vector fields, contrary to the previous Lemmas the

derivation operator includes spacial derivatives represented by ∂_x^α and discrete derivative represented by D_ρ^j .

Lemma 7.2.7. *Let $N \in \mathbb{N}^*$, $t_0 > \frac{1}{2}$, then the following holds*

1. *Let $s \in \mathbb{N}$ with $s \geq t_0 + \frac{3}{2}$, then there exists $C > 0$ independent of N such that*
 - *For any $\alpha \in \mathbb{N}$, with $0 \leq \alpha \leq s$*

$$\|[\partial_x^\alpha, F]G\|_{l^2(L_x^2)} \leq C \|F\|_{H^{s,2}} \|G\|_{H^{s-1,1}}.$$

The result holds true if we replace ∂_x^α by Λ^s .

- *For any $j \in \{1, 2\}$ and $\alpha \in \mathbb{N}$ such that $0 \leq \alpha \leq s - j$*

$$\|[\partial_x^\alpha D_\rho^j, F]G\|_{l^2(L_x^2)} \leq C \|F\|_{H^{s,2}} \|G\|_{H^{s-1,j}}.$$

The result holds true if we replace ∂_x^α by Λ^{s-j} .

2. *Let $s \in \mathbb{N}$ with $s > 2 + \frac{1}{2}$, then there exists $C > 0$ independent of N such that for $\alpha \in \mathbb{N}$ with $0 \leq \alpha \leq s - 2$ we have*

$$\|[\partial_x^\alpha D_\rho, F]G\|_{l^2(L_x^2)} \leq C \|F\|_{H^{s,2}} \|G\|_{H^{s-2,1}}.$$

The result holds true if we replace ∂_x^α by Λ^{s-2} .

3. *Let $s \in \mathbb{N}$ with $s \geq t_0 + \frac{5}{2}$, then there exists $C > 0$ independent of N such that for any $j \in \{1, 2\}$ and $\alpha \in \mathbb{N}$ such that $0 \leq \alpha \leq s - j$ then*

$$\|[\partial_x^\alpha D_\rho^j, F, G]\|_{l^2(L_x^2)} \leq C \|F\|_{H^{s-1,2}} \|G\|_{H^{s-1,2}}.$$

The result holds true if we replace ∂_x^α by Λ^{s-j} .

Proof.

1. Using Lemma 7.2.5 (4) we have

$$\|[\partial_x^\alpha, F]G\|_{l^2(L_x^2)} \leq C \left(\|\partial_x F\|_{H^{t_0+\frac{1}{2},1}} \|G\|_{H^{s-1,0}} + \|\partial_x F\|_{H^{s-1,0}} \|G\|_{H^{t_0+\frac{1}{2},1}} \right).$$

For $j = 1$ and $0 \leq \alpha \leq s - 1$, we have by Lemma 7.2.1 and Lemma 7.2.5 ((2) and

(4)

$$\begin{aligned}
 \llbracket \partial_x^\alpha \mathbf{D}_\rho, F \rrbracket G &= \partial_x^\alpha (\mathbf{D}_\rho (FG)) - (\mathbf{M}F)(\partial_x^\alpha (\mathbf{D}_\rho G)) \\
 &= \partial_x^\alpha ((\mathbf{M}F)(\mathbf{D}_\rho G) + (\mathbf{M}G)(\mathbf{D}_\rho F)) - (\mathbf{M}F)(\partial_x^\alpha (\mathbf{D}_\rho G)) \\
 &= [\partial_x^\alpha, \mathbf{M}F] \mathbf{D}_\rho G + \partial_x^\alpha ((\mathbf{M}G)(\mathbf{D}_\rho F)) \\
 &\lesssim \|\partial_x \mathbf{M}F\|_{H^{\max(t_0 + \frac{1}{2}, s - \frac{3}{2}), 1}} \|\mathbf{D}_\rho G\|_{H^{s-2, 0}} + \|\mathbf{D}_\rho F\|_{H^{\max(t_0 + \frac{1}{2}, s-1), 1}} \|\mathbf{M}G\|_{H^{\max(t_0 + \frac{1}{2}, s-1), 1}}.
 \end{aligned}$$

For $j = 2$ and $0 \leq \alpha \leq s - 2$, we have by Lemma 7.2.1 we have by Lemma 7.2.1 and Lemma 7.2.5 ((2) and (4))

$$\begin{aligned}
 \llbracket \partial_x^\alpha \mathbf{D}_\rho^2, F \rrbracket G &= \partial_x^\alpha (\mathbf{D}_\rho^2 (FG)) - (\mathbf{M}^2 F)(\partial_x^\alpha (\mathbf{D}_\rho^2 G)) \\
 &= [\partial_x^\alpha, \mathbf{M}^2 F] \mathbf{D}_\rho^2 G + 2\partial_x^\alpha ((\mathbf{M} \mathbf{D}_\rho G)(\mathbf{M} \mathbf{D}_\rho F)) + \partial_x^\alpha ((\mathbf{D}_\rho^2 F)(\mathbf{M}^2 G)) \\
 &\lesssim \|\partial_x \mathbf{M}^2 F\|_{H^{\max(t_0 + \frac{1}{2}, s - \frac{5}{2}), 1}} \|\mathbf{D}_\rho^2 G\|_{H^{s-3, 0}} + \|\mathbf{M} \mathbf{D}_\rho F\|_{H^{\max(t_0, s-2) + \frac{1}{2}, 1}} \|\mathbf{M} \mathbf{D}_\rho G\|_{H^{s-2, 0}} \\
 &\quad + \|\mathbf{D}_\rho^2 F\|_{H^{s-2, 0}} \|\mathbf{M}^2 G\|_{H^{\max(t_0, s-2) + \frac{1}{2}, 1}}.
 \end{aligned}$$

2. The result follows from the fact that

$$\llbracket \partial_x^\alpha \mathbf{D}_\rho, F \rrbracket G = [\partial_x^\alpha, \mathbf{M}F] \mathbf{D}_\rho G + \partial_x^\alpha ((\mathbf{M}G)(\mathbf{D}_\rho F)),$$

and using Lemma 7.2.5 (2) and (4) and choosing the corresponding $(p, q) \in \{2, \infty\}$ with $t_0 = s - \frac{3}{2}$, then using Lemma 7.2.3.

3. For $j = 1$ and $0 \leq \alpha \leq s - 1$, we have by Lemma 7.2.1 and Lemma 7.2.5 (4)

$$\begin{aligned}
 \llbracket \partial_x^\alpha \mathbf{D}_\rho; F, G \rrbracket &= \partial_x^\alpha (\mathbf{D}_\rho (FG)) - (\mathbf{M}F)(\partial_x^\alpha \mathbf{D}_\rho G) - (\mathbf{M}G)(\partial_x^\alpha \mathbf{D}_\rho F) \\
 &= \partial_x^\alpha ((\mathbf{M}F)(\mathbf{D}_\rho G) + (\mathbf{M}G)(\mathbf{D}_\rho F)) - (\mathbf{M}F)(\partial_x^\alpha \mathbf{D}_\rho G) - (\mathbf{M}G)(\partial_x^\alpha \mathbf{D}_\rho F) \\
 &= [\partial_x^\alpha, \mathbf{M}F] \mathbf{D}_\rho G + [\partial_x^\alpha, \mathbf{M}G] \mathbf{D}_\rho F \\
 &\lesssim \|\partial_x \mathbf{M}F\|_{H^{\max(t_0 + \frac{1}{2}, s-2), 1}} \|\mathbf{D}_\rho G\|_{H^{\max(t_0 + \frac{1}{2}, s-2), 1}} \\
 &\quad + \|\partial_x \mathbf{M}G\|_{H^{\max(t_0 + \frac{1}{2}, s-2), 1}} \|\mathbf{D}_\rho F\|_{H^{\max(t_0 + \frac{1}{2}, s-2), 1}}.
 \end{aligned}$$

For $j = 2$ and $0 \leq \alpha \leq s - 2$, we have by Lemma 7.2.1 and Lemma 7.2.5 ((2) and

(4)

$$\begin{aligned}
 \llbracket \partial_x^\alpha D_\rho^2; F, G \rrbracket &= \partial_x^\alpha (D_\rho^2(FG)) - (M^2 F)(\partial_x^\alpha D_\rho^2 G) - (M^2 G)(\partial_x^\alpha D_\rho^2 F) \\
 &= \partial_x^\alpha ((D_\rho^2 F)(M^2 G) + (D_\rho^2 G)(M^2 F) + 2(MD_\rho F)(MD_\rho G)) \\
 &\quad - (M^2 F)(\partial_x^\alpha D_\rho^2 G) - (M^2 G)(\partial_x^\alpha D_\rho^2 F) \\
 &= [\partial_x^\alpha, M^2 G] D_\rho^2 F + [\partial_x^\alpha, M^2 F] D_\rho^2 G + 2\partial_x^\alpha ((MD_\rho F)(MD_\rho G)) \\
 &\lesssim \|\partial_x M^2 G\|_{H^{\max(t_0 + \frac{1}{2}, s-3), 1}} \|D_\rho^2 F\|_{H^{s-3, 0}} + \|\partial_x M^2 F\|_{H^{\max(t_0 + \frac{1}{2}, s-3), 1}} \|D_\rho^2 G\|_{H^{s-3, 0}} \\
 &\quad + \|MD_\rho F\|_{H^{\max(t_0 + \frac{1}{2}, s-2), 1}} \|MD_\rho G\|_{H^{\max(t_0 + \frac{1}{2}, s-2), 1}}.
 \end{aligned}$$

This concludes the proof. \square

Difficulties arise when estimating nonlinear and non-polynomial quantities in the space $H^{s,k}$. In the following Lemma we tackle the problem specifically for the quantity $\frac{1}{\underline{H}+H} - \frac{1}{\underline{H}}$ which arises naturally in several places.

Lemma 7.2.8. *Let $h_*, \underline{M}, M^* > 0$, $t_0 > \frac{1}{2}$ and $s \in \mathbb{N}$ such that $s \geq t_0 + \frac{3}{2}$, there exists $C(h_*^{-1}, \underline{M}, M^*) > 0$, such that for any $N \in \mathbb{N}^*$ and any $\underline{H} \in w^{2,\infty}$, $H \in H^{s,2}(\mathbb{R})$, satisfying*

$$|\underline{H}|_{w^{2,\infty}} \leq \underline{M}; \quad \|H\|_{H^{s-1,1}} \leq M^*,$$

$$\inf_{(x,i) \in \mathbb{R} \times \{1, \dots, N\}} \underline{H}_i + H_i(x) \geq h_*,$$

the following holds

$$\left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s,2}} \leq C(h_*^{-1}, \underline{M}, M^*) \|H\|_{H^{s,2}}. \quad (7.2.6)$$

$$\left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s-1,1}} \leq C(h_*^{-1}, \underline{M}, M^*) \|H\|_{H^{s-1,1}}. \quad (7.2.7)$$

Proof. For $i \in \{1, \dots, N\}$ fixed, since s is sufficiently large we notice by Sobolev injection that H_i vanishes at infinity as a consequence we have $\underline{H}_i \geq h_*$. Let $\left(\frac{1}{\underline{H}+H} - \frac{1}{\underline{H}}\right)_i = \frac{1}{\underline{H}_i+H_i} - \frac{1}{\underline{H}_i} = \varphi_i(H_i)$, where $\varphi_i \in C^\infty(\mathbb{R}; \mathbb{R})$ such that $\varphi_i(x) = \frac{1}{\underline{H}_i+x} - \frac{1}{\underline{H}_i}$ when $\underline{H}_i + x \geq h_*$, moreover we notice that there exists $C(\alpha, h_*^{-1})$ (independent of i) such that $\|\varphi_i^{(\alpha)}\|_{L^\infty([h_* - \underline{H}_i, +\infty])} \leq C(\alpha, h_*^{-1})$, using the fact that $\underline{H} \in w^{2,\infty}$ we can choose φ_i in such a way that there exists $C(\alpha, h_*^{-1})$ (independent of i) such that $\|\varphi_i^{(\alpha)}\|_{L^\infty(]-\infty, h_* - \underline{H}_i])} \leq C(\alpha, h_*^{-1})$. Hence using the composition Lemma in Sobolev spaces (see Lemma A.4 in

[13]) and Lemma 7.2.3 we have

$$\left\| \Lambda^s \left(\frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right) \right\|_{l^2(L_x^2)} \leq C(h_*^{-1}, \|H\|_{l^\infty(H_x^{t_0})}) \|H\|_{H^{s,0}}. \quad (7.2.8)$$

$$\left\| \Lambda^{s-1} \left(\frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right) \right\|_{l^2(L_x^2)} \leq C(h_*^{-1}, \|H\|_{l^\infty(H_x^{t_0})}) \|H\|_{H^{s-1,0}}. \quad (7.2.9)$$

By Lemma 7.2.1 and Lemma 7.2.4 we find that

$$\begin{aligned} \left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s,2}} &\leq C(h_*^{-1}, |\underline{H}|_{w^{2,\infty}}) \left\| \frac{H}{\underline{H} + H} \right\|_{H^{s,2}} \\ \left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s-1,1}} &\leq C(h_*^{-1}, |\underline{H}|_{w^{1,\infty}}) \left\| \frac{H}{\underline{H} + H} \right\|_{H^{s-1,1}}. \end{aligned}$$

We have for $i \in \{1, \dots, N-1\}$

$$\begin{aligned} \left(D_\rho \left(\frac{H}{\underline{H} + H} \right) \right)_i &= \left[-N(\underline{H}_i - \underline{H}_{i+1}) \left(\frac{H_i + H_{i+1}}{2} \right) + N(H_i - H_{i+1}) \left(\frac{H_i + H_{i+1}}{2} \right) \right] \\ &\quad \times \frac{1}{(\underline{H}_i + H_i)(\underline{H}_{i+1} + H_{i+1})}. \end{aligned}$$

then

$$D_\rho \left(\frac{H}{\underline{H} + H} \right) = [-(D_\rho \underline{H})(MH) + (D_\rho H)(M\underline{H})] R_d \left(\frac{1}{\underline{H} + H} \right) R_u \left(\frac{1}{\underline{H} + H} \right).$$

We have for $i \in \{1, \dots, N-2\}$

$$\left(D_\rho \left(R_d \left(\frac{1}{\underline{H} + H} \right) R_u \left(\frac{1}{\underline{H} + H} \right) \right) \right)_i = \frac{-2(MD_\rho(\underline{H} + H))_i}{(\underline{H}_i + H_i)(\underline{H}_{i+1} + H_{i+1})(\underline{H}_{i+2} + H_{i+2})}.$$

and hence

$$\begin{aligned} D_\rho^2 \left(\frac{H}{\underline{H} + H} \right) &= \left[-(D_\rho^2 \underline{H})(M^2 H) + (D_\rho^2 H)(M^2 \underline{H}) \right] M \left(R_d \left(\frac{1}{\underline{H} + H} \right) R_u \left(\frac{1}{\underline{H} + H} \right) \right) \\ &\quad - 2[-M((D_\rho \underline{H})(MH)) + M((D_\rho H)(M\underline{H}))] (MD_\rho(H + \underline{H})) \\ &\quad \times R_d R_d \left(\frac{1}{\underline{H} + H} \right) R_d R_u \left(\frac{1}{\underline{H} + H} \right) R_u R_u \left(\frac{1}{\underline{H} + H} \right). \end{aligned}$$

Using the previous equalities and the following arguments

- The spatial derivatives commute with the operators $R_d, R_u, M, D_\rho, D_\rho^2$.
- The operators R_d, R_u, M are bounded in l^2 and l^∞ .
- Lemma 7.2.6 and estimates (7.2.8) and (7.2.9).

We obtain

$$\begin{aligned} \left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s,2}} &\leq C(h_*^{-1}, |\underline{H}|_{w^{2,\infty}}, \|H\|_{H^{s-1,1}}) \|H\|_{H^{s,2}}. \\ \left\| \frac{1}{\underline{H} + H} - \frac{1}{\underline{H}} \right\|_{H^{s-1,1}} &\leq C(h_*^{-1}, |\underline{H}|_{w^{1,\infty}}, \|H\|_{H^{s-1,1}}) \|H\|_{H^{s-1,1}}. \end{aligned}$$

Hence the proof is then completed since the constant C is increasing in its arguments. \square

We conclude this section by collecting continuity estimates on the operator P_N defined by

$$P_N : \begin{array}{ll} \mathcal{C}([\rho_{\text{surf}}, \rho_{\text{bott}}]) & \rightarrow \mathbb{R}^N \\ f & \mapsto (f(\boldsymbol{\rho}_i))_{1 \leq i \leq N} \end{array}$$

Lemma 7.2.9. *Let $s \in \mathbb{N}$, $s \geq 2$ there exists $C > 0$ such that for any $N \in \mathbb{N}^*$ and any $f \in X^{\infty, s, 2}(\Omega)$ the following holds*

$$\|P_N f\|_{H^{s,0}} \leq C \|f\|_{\infty, s, 0}; \quad \|P_N f\|_{H^{s,1}} \leq C \|f\|_{\infty, s, 1} \quad \|P_N f\|_{H^{s,2}} \leq C \|f\|_{\infty, s, 2}.$$

Proof. It is immediate that

$$\|P_N f\|_{H^{s,0}}^2 \leq \|f\|_{L_\varrho^\infty(H_x^s)}^2.$$

For $i \in \{1, \dots, N\}$, using the mean value theorem

$$\int_{\mathbb{R}} |N(\Lambda^{s-1} f(x, \boldsymbol{\rho}_i) - \Lambda^{s-1} f(x, \boldsymbol{\rho}_{i+1}))|^2 dx \leq \|\partial_\varrho f\|_{L_\varrho^\infty(H_x^{s-1})}^2,$$

hence $\|\Lambda^{s-1} D_\rho P_N f\|_{l^2(L_x^2)} \lesssim \|\partial_\varrho f\|_{L_\varrho^\infty(H_x^{s-1})}$, and $\|P_N f\|_{H^{s,1}} \lesssim (\|f\|_{L_\varrho^\infty(H_x^s)} + \|\partial_\varrho f\|_{L_\varrho^\infty(H_x^{s-1})})$.

Using the Taylor expansion for the map $\varrho \mapsto \Lambda^{s-2} f(\cdot, \varrho)$ we have

$$\int_{\mathbb{R}} |N^2(\Lambda^{s-2} f(x, \boldsymbol{\rho}_i) - 2\Lambda^{s-2} f(x, \boldsymbol{\rho}_{i+1}) + \Lambda^{s-2} f(x, \boldsymbol{\rho}_{i+2}))|^2 dx \leq 2 \|\partial_\varrho^2 \Lambda^{s-2} f\|_{L_\varrho^\infty(L_x^2)}^2,$$

consequently,

$$\|\Lambda^{s-2} D_\rho^2 P_N f\|_{l^2(L_x^2)} \leq 2 \|\partial_\varrho^2 \Lambda^{s-2} f\|_{L_\varrho^\infty(L_x^2)}.$$

This concludes the proof.

□

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Titre : Modèles continument stratifiés et systèmes multi-couches pour les écoulements géophysiques

Mot clés : *EDP non-linéaires, océanographie, stratification de densité, modèles hydrostatiques, systèmes de Saint-Venant, paramétrisation de Gent & McWilliams*

Résumé : *Dans cette thèse, nous établissons rigoureusement des ponts entre les écoulements continument stratifiés et les écoulements multi-couches. Dans une première partie, nous considérons le système de Saint-Venant multi-couches avec un terme supplémentaire diffusif qui a un effet régularisant, dont la motivation provient des travaux des océanographes Gent & McWilliams sur le mélange isopycnal et la diffusivité des tourbillons, et qui pourrait être interprété comme un terme de turbulence. En exploitant la structure de ce système, nous obtenons un dictionnaire qui nous permet d'interpréter ce système multi-couches comme une discrétisation de la formulation en coordonnées isopycnales*

du système hydrostatique continument stratifié avec le terme diffusif de Gent & McWilliams ajouté de manière similaire. Nous montrons la convergence de la solution discrète vers la solution continue à mesure que le nombre de couches tend vers l'infini, et nous fournissons un taux de convergence explicite. Dans une deuxième partie, dans cette thèse, nous abordons la limite "inverse", nous montrons rigoureusement que, sous certaines conditions d'hyperbolicité et dans un cadre topologique bien choisi, la solution du système continument stratifié converge vers le système de Saint-Venant bi-couches dans la limite de stratification nette.

Title: Continuously stratified models and multi-layer systems for geophysical flows

Keywords: *Non-linear PDEs, oceanography, density stratification, Hydrostatic models, shallow water systems, Gent & McWilliams parameterization*

Abstract: *In this thesis, we rigorously establish bridges between continuously stratified flows and multi-layer flows. In the first part, we consider the multi-layer shallow water system with an additional diffusive term that has a regularizing effect, motivated by the work of oceanographers Gent & McWilliams on isopycnal mixing and eddy diffusivity, which can be interpreted as a turbulence term. By exploiting the structure of this system, we derive a dictionary that allows us to interpret this multi-layer system as a discretization of the formulation in isopycnal coordinates of the continuously*

stratified hydrostatic system with the Gent & McWilliams diffusive term added in a similar manner. We demonstrate the convergence of the discrete solution to the continuous solution as the number of layers tends to infinity, and we provide an explicit convergence rate. In the second part of this thesis, we address the "inverse" limit. We rigorously show that, under certain hyperbolicity conditions and within a well-chosen topological framework, the solution of the continuously stratified system converges to the bi-layer shallow water system in the limit of sharp stratification.