Asymptotic analysis of multiple integrals : beyond the classical beta-ensembles

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“Laisse moi
croire qu’un au revoir ne
nous sépare jamais.”

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Abstract

This thesis develops techniques to extract the large-N behavior of N-fold multiple integrals. There are two main motivations for developing such techniques. First, mulitple integrals can arise as integral representations of special functions. Similar to the integral representation for the Euler's Γ function, having powerful tools to extract information from these expressions, such as the regularity of the function, asymptotic behaviors, or explicit expression for specific parameter values, is crucial. Second, multiple integrals appear in statistical mechanics, such as partition functions of multi-particle systems in the classical picture or as building blocks of correlation functions in a quantum framework.

Techniques from large deviations theory, Riemann-Hilbert problems and the transport- based methods form the basic arsenal to access their asymptotic behavior. Currently, one of the rare model in the existing literature that can be dealt with in a large generality, is the partition function of the β-ensembles [APS01, EM03, BI05b, BG13a, BG13b, CGM15] which can be seen as the law of the spectrum of certain random matrix models. The understanding of this model follows from more than fifty years of extensive study and surprising links with various fields such as the enumeration of maps [BIPZ78], random tilings [Joh02], integrable systems [GM22], statistics [PA14], or quantum mechanics [GMGW98] were discovered. However, there is interest in exploring generalization of this model, and this is the main subject of this thesis.

This thesis focuses on two models that go beyond the classical β-ensembles. The first one concerns the β-ensembles in the high temperature, *viz* when β scales with N as $\beta = P/N$ with P constant. The major difference compared to the constant β-case is that one has to account for entropic effects due to the high temperature. These effects are due to the contributions of the Lebesgue integration measure which is here of the same order in N as the energy. The second model of interest is the so-called *sinh-model*, introduced in [BGK16] to study the type of integrals introduced in [Luk01] which are conjectured to be linked with some form factor in quantum field theory. This model involves a hyperbolic Vandermonde determinant as well as an unscaled potential term, which makes certain nice properties of the usual β-ensembles inaccessible. Additionally, for interesting applications in physics, one has to deal with potentials which depend in N in a more involved way, resulting in a more sophisticated analysis.

This manuscript is divided into five parts::

- The first part of this thesis is an introduction to the field of multiple integrals analysis. It describes links to special functions, classical and quantum integrable models and random matrix theory. It also provides a brief overview of the various existing methods in the literature as well as a presentation of the models that we consider in this thesis. A rec of each chapter can be found at the end of this introduction.
- Based on **[DGM23]**[1], this chapter establishes a central limit theorem for the linear statistics of a β-ensemble in the high temperature regime. The result goes beyond the setting of quadratic potentials considered earlier in the litterature [NT18]. The proof is based on

[1]The bold notation is used for the contributions of the author.

the inversion of the so-called master operator, which is done by establishing a link with a Schrödinger operator. This result is a generalization of existing techniques developped for the circular case [HL21] to a non-compact setting where more care is needed.

- Based on **[DG24]**, this chapter establishes the existence of an asymptotic expansion for the partition function of the model studied in Chapter 1 for potentials given by the sum of a quadratic potential and a bounded smooth function. The proof is based on the analysis of the loop equations in the setting where the equilibrium measure is supported on the whole real line. Our proof includes continuity results for the master operator, from which the large-N behavior of the linear-satistics is deduced. To conclude about the partition function, a continuity result for the equilibrium measure with respect to the potential dependance is also shown.
- Based on **[DGK24]**, this chapter deals with the sinh-model. Working with a N-dependent potential, the explicit form of the leading asymptotic for Lukyanov's conjecture for the vacuum expectation value of the exponential of the field operator of the quantum sinh-Gordon model [Luk01] is argued. The conjecture is valid under the very natural hypothesis that a certain remainder term is negligible. This chapter heavily relies on the explicit expression for the equilibrium measure found in [BGK16], allowing for explicit computations.
- The last part is a conclusion summarizing the content of this thesis. We also address some natural questions following our results.
- The appendices contain general facts about unbounded operators, large deviation principles, and Riemann-Hilbert problems.

Contents

Introduction

"The problem with identity is that it wrecks your ability to think clearly. Identifying with a belief makes you feel like you have to be ready to defend it, which motivates you to focus your attention on collecting evidence in its favor. Identity makes you reflexively reject arguments that feel like attacks on you or the status of your group." Julia Galef

Contents

I.1 Integral representations

I.1.1 The Birth of Integration theory

In the XVI-*th* and XVII-*th* century, significant work about integrals was independently achieved by Leibniz and Newton, respectively, in their seminal books *Nova Methodus pro Maximis et Minimis* and *Philosophiæ Naturalis Principia Mathematica*, see the excellent reviews [Edw12, Ham16]. Since the mathematical rigor was not yet developped and the logical revolution of the XIX-XX-*th* centuries had not yet occurred, the manipulations were very speculative and more philosophical in nature. It is commonly accepted that, more or less at the same period, these two mathematicians created *the theory of calculus* independently of each other. The advances of the XVII-*th* century followed millennia of investigation into "*quadrature*[2] and tangent" problems by Greek, Chinese and Arabic mathematicians.

Often in science, big discoveries and paradigms shifts arise from joint efforts. In the early XVII-*th* century, Fermat manipulated quantities like derivatives, although he didn't present it as an important tool nor introduced a notation for it. Moreover, mathematicians like Torricelli and Barrow, Newton's teacher, understood the inverse relationship between tangents and *quadrature*, or in modern terms, between differentiation and integration. Barrow, in his *Geometrical Lectures* published in 1670, even explicitly stated a geometric theorem about this inverse relationship. It was noticed at this time, from the works of Galileo, for example, that representing the velocity versus time of a moving object on a graph allowed one to deduce the distance traveled by computing the area under the curve. The contributions of Newton and Leibniz was not so much in their discovery of the fundamental theorem of calculus, namely that:

$$f(x) - f(a) = \int_a^x f'(t)dt \tag{1}$$

whatever these symbols mean, but rather by stressing the importance of these concepts and exploring the consequences of these formulas.

The techniques and discoveries of the XVII-*th* century were formalized by Riemann in 1854 in his *Habilitationsschrift*, where he posed *"What is one to understand by* $\int_a^b f(x)dx$ *?"*. His formalization employed the rigorous notion of limit and the so called ε-δ *techniques* developed by Bolzano, Weierstrass and Cauchy in the early XIX-*th* century. Riemann's approach involved approximating

[2]The problem of the quadrature of a surface has to be understood as the attempt of computing its area. The intuition behind it was almost always to divide the surface into small pieces for which the corresponding area could be computed, with the hope that the total area could then be determined.

the area under the curve using small rectangles. By successively choosing smaller and smaller rectangles (thus taking the limit), he could determine the exact value of the area under the curve.

With the development, among many other fields, of thermodynamics with physicists like Gibbs and Boltzmann and probability theory with the works by Gauss and Laplace, the integral became a central object in both mathematics and physics. Riemann's approach was quite successful in providing a solid foundation for calculus, the use of Fourier series and more generally the ability to integrate discontinuous functions.

In 1902, Lebesgue introduced a new theory of integration in his doctoral thesis. In his framework, equation (1) held for a broader[3] class of functions compared to Riemann's framework. Although Lebesgue's theory was later extended to provide meaning to more integrals of functions, such as the Henstock–Kurzweil integral and Pfeffer's integral [SK11, Pfe92], it became the standard method for integrating functions. One reason for that is the convenience of the Lebesgue spaces and the fact that most of the theorems can be shown easily in this framework. Its construction involves a measure, which is the central object of measure theory, providing a solid foundation for probability theory.

I.1.2 Integral representations

In Lebesgue's theory, theorems which were already available in Riemann's framework could be demonstrated more easily. One example is the Arzela's dominated convergence theorem [Lux71]. Some of these theorems allow for the handling of functions in the form of parameter-dependent integrals, namely,

$$g : x \mapsto \int_{a(x)}^{b(x)} f(x,t)dt. \tag{2}$$

These theorems established that, under certain natural hypotheses on f, g is a continuous/differentiable function. The fundamental theorems of Lebesgue's theory imply that when solving a differential equation, one could, in principle, be satisfied with finding a solution of the form (2). This is because, assuming that f satisfies certain hypotheses, one could recover all the information obtained from an explicit expression. Therefore, an expression can be considered 'explicit' if one can extract all the desired information from it. This information can include regularity properties, exact values at certain parameter values, and asymptotic behavior.

Together with other tools, such as the Laplace's method or its generalization the saddle-point method [AF03, Chapter 6], there is a sufficiently strong arsenal that one can use in many cases to deal with one-dimensional integrals of the form

$$z \mapsto \int_{\mathscr{C}} f(z,\zeta)d\zeta$$

where $\mathscr{C}$ is a curve in the complex plane. For example, one is able to deduce the regularity of this function assuming f is regular enough or its limit when $z \to z_0$, assuming the knowledge on the same limit about f. Many of the so-called *special functions* admit expressions of such a form.

An example of a special function is the celebrated Euler's *Gamma* function Γ, defined by

$$\Gamma(z) \overset{(\text{def})}{=} \int_0^{+\infty} e^{-t}t^z dt \qquad \text{for } \Re(z) > 0.$$

This defines a holomorphic function in the right half-plane. From this expression, one is able to find limits, demonstrate smoothness properties and algebraic relations as well as compute Γ's

[3] Although, Riemann-integrability on bounded intervals imply Lebesgue integrability, this is not the case for the improper Riemann integral. However, one can argue that functions which are only Lebesgue integrable have a much more general form than the improper Riemann integrable ones.

values at specific points. The Gamma function belongs to a broad class of functions called *higher transcendental functions* which generalizes the *transcendental functions*, such as exp or cos. The latter are themselves generalizations of the simplest possible functions, namely the polynomials. They constitute generalizations in the sense that to move from one class to a higher one, one needs to take a limit (from polynomial functions to transcendental functions) or integrate (from transcendental to higher transcendental functions) which also ammounts to take a limit. With an extensive literature on the subject, the higher transcendental class is considered now very well understood. The next natural step is then to try to understand the next class of special functions *i.e.* the generalization of the previous class. A class of functions that generalizes the higher transcendental functions is the so-called *Painlevé class of functions*.

In the early XX-*th*-century, Painlevé and his school studied second order nonlinear rational differential equations. Specifically, they focused on a subclass of these equations, namely those for which the only singularities of the solutions depending on the initial data are poles. Painlevé and Gambier showed that there exists only fifty equations of such a type. Furthermore, they demonstrated that fourty-four of these equations could be reduced to the remaining six (or to trivial ones), yielding the famous Painlevé equations I-VI. Their solutions, known as *Painlevé transcendents* play a significant role in mathematical physics and many other areas today. The corresponding equations can be seen as reductions of the so-called *integrable* partial differential equations (PDE) such as the *Korteweg-de Vries* equation, the *non-linear Schrödinger* equation or the *sine-Gordon* equation [FIKN23]. Their solutions also appear in random matrix theory in the so-called *Gaussian Unitary Ensemble*, *i.e.* the probability distribution on Hermitian matrices proportional to:

$$\exp\left(-\mathrm{Tr}(H^2)\right) dH \tag{3}$$

where $dH \overset{(\mathrm{def})}{=} \prod_{i=1}^{N} dH_{ii} \prod_{1\leq i<j\leq N} d\mathrm{Re}(H_{ij}) d\mathrm{Im}(H_{ij})$ stands for the Lebesgue measure on $\mathbb{R}^N \times \mathbb{C}^{N(N-1)/2}$, the space of Hermitian matrices. For a random matrix sampled from this distribution, the probability of finding no eigenvalues in an interval of the type $[0, x]$ for $x > 0$, as the size N of the matrix goes to infinity, is given by an expression of the form [PS97]:

$$\exp\left(\int_0^{\pi x} \frac{\sigma(t)}{t} dt\right)$$

where σ is a solution of Painlevé V. This limiting probability can be represented by a *Fredholm determinant*, *i.e.* the determinant of the operator $\mathsf{id} - \mathsf{K}_{\mathrm{sine},x}$ acting on $L^2([0, x])$ where $\mathsf{K}_{\mathrm{sine},x}$ is the so-called *Sine-kernel operator* [BGK16, (1.1.3)]. One may also obtain an expression for the asymptotic distribution of the law of the fluctuations of the largest eigenvalue, known as the Tracy-Widom law, of the cumulative distribution function F [TW94]. $F(x)$ is given by the Fredholm determinant of $\mathsf{id} - \mathsf{K}_{\mathrm{Airy},x}$ acting on $L^2(]x, +\infty[)$ where $\mathsf{K}_{\mathrm{Airy},x}$ is the so-called *Airy-kernel operator* [BGK16, (1.1.4)]. This determinant can also be expressed in terms of a Painlevé transcendent. Namely, we have,

$$\det\left(\mathsf{id} - \mathsf{K}_{\mathrm{Airy},x}\right) = \exp\left(\int_x^{+\infty} (x-t) u(t)^2 dt\right) \tag{4}$$

where u is a solution of Painlevé II. Painlevé transcendents also appear in other *probabilistic integrable models* such as in the computation of the correlation functions in the 2D-Ising model [Bot18] or in random growth processes [For03].

These functions allow for obtaining "explicit" expressions for sophisticated objects. However, these quantities do not admit finite-order integral representations. They are generally expressed as series whose n-*th* summand is given by a n-fold integral of a simple combination of simple transcendental functions. To turn them into powerful mathematical tools, one must be able to extract all the knowledge out of such representations.

Although maps like $x \mapsto \det(\mathsf{id} - \mathsf{K}_x)$ for a general x-dependent operator K_x are in general hard to study, there is a subclass of operators called *integrable integral operators* [IIKS90] for which it is possible to extract a lot of information. These operators are integral operators whose kernels are of the following form:

$$\mathsf{k}_x(\xi,\eta) = \frac{\sum_{a=1}^N f_a(\xi)g_a(\eta)}{\xi-\eta}, \qquad \text{with} \quad \sum_{a=1}^N f_a(\xi)g_a(\xi) = 0.$$

For this class of operators, a strong link with the so-called *Riemann-Hilbert theory* exists, the basic theory is described in Appendix C or in the reference [Dei19] where the link is described. This connection has been explored in the literature in [Sak68, JMMS80, IIKS90].

Riemann-Hilbert theory includes powerful method and effective tools to investigate such operators, such as the Deift-Zhou nonlinear steepest-descent method [DZ92, DZ93] or the concept of local parametrix [Its81]. The Riemann-Hilbert theory is even able to handle the layer of Painlevé transcendents. Indeed, the latter can be expressed as some function of a matrix entry of the solution of a 2×2 matrix Riemann-Hilbert problem (RHP). With the present tools in this theory, one can extract all the information about the Painlevé transcendents [FIKN23].

We have argued that there exist tools to study the various mentionned special functions. However, a natural question arises *"What is the next layer of special functions ?"* Such a layer should be a class of functions such that their asymptotic behavior is described in terms of the previous layer of functions. For example, the asymptotic behavior the Painlevé II transcendent u appearing in (4), is given by $u(x) \sim \mathrm{Ai}(x)$ as $x \to +\infty$ [BBD08] where Ai stands for the *Airy-function* given on the real line by:

$$\mathrm{Ai}(x) \overset{(\mathrm{def})}{=} \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{t^3}{3} + xt\right) dt.$$

To construct the next layers of special functions, one might be interested in studying representations of sophisticated algebraic structures. Indeed, it was first noticed by Bargmann in the 40's [Bar47] that classical special functions can arise as matrix elements of representations of elements of certain Lie groups [KV95]. A simple example of such a link is given by specific representations of the Euclidean group of the plane $E(2)$, which consists of all the transformations that preserve the Euclidean distance, namely translations and rotations. The elements of this group can be parametrized by three real numbers r, θ, φ in the following manner:

$$g(r,\theta,\varphi) \overset{(\mathrm{def})}{=} \begin{pmatrix} \cos\theta & -\sin\theta & r\cos\varphi \\ \sin\theta & \cos\theta & r\sin\varphi \\ 0 & 0 & 1 \end{pmatrix} \in E(2).$$

Moreover, there exists a specific representation T_ρ, depending on a real parameter ρ, of $E(2)$ on $L^2(\mathbb{S})$, the space of square integrable functions on the unit sphere of the plane, that allows one to express the *Bessel function of the first kind* J_n as a matrix element. It arises as a solution of a second order linear differential equation and can be defined for all integer n by

$$J_n(z) \overset{(\mathrm{def})}{=} \int_{-\pi}^{\pi} e^{\mathrm{i}z\sin\zeta - \mathrm{i}n\zeta}\frac{d\zeta}{2\pi}.$$

By taking the orthonormal basis of $L^2(\mathbb{S})$ composed of functions $e_n : z \mapsto \frac{1}{\sqrt{2\pi}}e^{\mathrm{i}nz}$, one can prove that

$$J_{n-m}(\rho r) = e^{\mathrm{i}n\theta + \mathrm{i}(m-n)\left(\varphi+\frac{\pi}{2}\right)} \left\langle \mathsf{T}_\rho[g(r,\theta,\varphi)]e_n, e_m\right\rangle_{L^2(\mathbb{S})}.$$

This picture also includes, the hypergeometric functions ${}_2F_1(a,b,c,z)$ (respectively the *Gegenbauer polynomials* $C_n^{(\alpha)}(z)$) which arises as matrix elements of a representation of the *special linear group*

$SL_2(\mathbb{C})$ (resp. the *special orthogonal group* $SO_N(\mathbb{R})$) [KV95, Chapter 7 & 9]. Additionally, the relationships between special functions, such as summation identities or addition formulas, stem from this algebraic structure. Such formulas can arise from the fact that the representation verifies the homorphism property or by writing a change of basis formula in the representation space. The parameters labelling a special function often arise as the parameters labelling the representation or a certain group element.

To consider a higher layer composed of more sophisticated functions, one can consider matrix elements of algebraic structures more subtle than groups, namely the *quantum groups* [Dri86], [Isa23]. Such structures appeared in the 80's, in the study of quantum integrable systems and more precisely in the context of the *quantum inverse scattering method* [Yan67, Bax72, STF79]. They are not groups, but they can still be considered as encoding the symmetries of a physical system.

I.1.3 Large-N behavior of multiple N-fold integrals

This thesis aims at developing techniques to capture more information about certain special functions that are given in terms of N-fold multiple integrals. More precisely, our goal in this thesis is to extend the techniques allowing one to capture the large-N behaviour of integrals of the generic form:

$$Z_N = \int_{\mathbb{R}^N} \prod_{i,j=1}^{N} k_N(x_i, x_j). \prod_{i=1}^{N} w_N(x_i) dx_i \tag{5}$$

where k_N is a N-dependent kernel and w_N a weight making the integral convergent which may also depend on N. These integrals arise in quantum integrable models as in Lukyanov's conjecture [Luk01] about the quantum sinh-Gordon model, see Section I.7 for more details. Indeed, by applying the classical separation of variables method, it was conjectured that the vacuum expectation value of the exponential of the field operator was given in terms of a ratio of integrals given under the general form:

$$Z_N^{\mathrm{Luk}} = \int_{\mathbb{R}^N} \prod_{1\le i<j\le N} \frac{\sinh\left[\omega_1|x_i - x_j|\right]}{\omega_1} \frac{\sinh\left[\omega_2|x_i - x_j|\right]}{\omega_2}. \prod_{i=1}^{N} e^{-V(x_i)} dx_i, \qquad \omega_1, \omega_2 \ge 0.$$

Above V is a function growing sufficiently fast at infinity. They can also be given as building blocks of volumes appearing in asymptotic geometric analysis. In [KMP98, GP07], the authors showed that these integrals arise as building blocks of the volume of the unit ball of self-adjoint matrices for the p-Schatten norm, $p > 3$. These integrals take the form:

$$Z_N^{\mathrm{Scha}} = \int_{\mathbb{R}^N} \prod_{1\le i<j\le N} |x_i^a - x_j^a|^b. \prod_{i=1}^{N} |x_i|^c e^{-abN|x_i|^p} dx_i, \qquad a, b, c \ge 0.$$

Such integrals also arise as normalizing constant in probabilistic models also known as *partition functions*. This terminology is due to the statistical mechanics approach to study these models. An example of such a model is the partition function of the law of the spectrum of certain random matrices. The spectrum of a random matrix sampled from the *Gaussian Unitary Ensemble*, introduced in (3), is given by the following p.d.f.

$$\frac{\prod_{1\le i<j\le N} |x_i - x_j|^2. \prod_{i=1}^{N} e^{-|x_i|^2} dx_i}{Z_N^{\mathrm{GUE}}}, \qquad Z_N^{\mathrm{GUE}} = \int_{\mathbb{R}^N} \prod_{1\le i<j\le N} |x_i - x_j|^2. \prod_{i=1}^{N} e^{-|x_i|^2} dx_i.$$

Other examples of random matrix models give rise to similar expressions with different domains (the real half-line, or a bounded interval) or potentials. Surprisingly, the double product can also

contain hyperbolic sines such as in (5). In the work [CW14], the authors showed that the spectrum of a Hermitian matrix sampled from the distribution proportional to

$$\exp\left(-N\mathrm{Tr}\left(V(M)-AM\right)\right)dM, \qquad \text{with} \quad A \overset{(\mathrm{def})}{=} \mathrm{diag}\left(1/N,\dots,(N-1)/N\right),$$

admits the following partition function:

$$Z_N^{\mathrm{C,W}} = \int_{\mathbb{R}^N} \prod_{1\leq i<j\leq N} |x_i-x_j| \sinh|x_i-x_j| . \prod_{i=1}^N e^{-NV(x_i)} dx_i.$$

Finally, these integrals also appear in statistical mechanics, such as in the *Coulomb gas* setting [Ser18]. This model represents confined electrically charged particles interacting with one another. In this setting, the partition function is very important as it's possible to recover the average of observables from it and detect the presence of phases transitions.

There already exist techniques to extract the large N-behaviour designed for certain integral [EM03, BG13a, BG13b, BGK15, CGM15, BGK16, CKM23, Ser23] . Nonetheless, the general case remains largely open. In order to make these integral representations really explicit, one needs to design techniques to extract all the information from them. Before expliciting the considered type of integrals in this thesis, we review some of the cases where one is able to grasp this information. This review will allow us to give an overview of the existing methods such as large deviation principles, Riemann-Hilbert techniques or the loop equations analysis.

I.2 A simple case using large deviation techniques

A simple case where one is able to capture the asymptotic behaviour of a N-fold integral is the following case:

$$\mathcal{J}_N[V,W] \overset{(\mathrm{def})}{=} \int_{\mathbb{R}^N} \prod_{i,j=1}^N e^{W(\lambda_i,\lambda_j)} . \prod_{i=1}^N e^{-NV(\lambda_i)} d\lambda_i. \tag{6}$$

with W a smooth bounded function and V a function growing sufficiently fast at infinity so that the integral converges. This integral can be seen as the partition function of a classical statistical mechanics system with N particles on the real line, confined by a potential V and interacting through a two-body interaction W. Unless V and W are chosen very specifically, one cannot expect to compute explicitly the integral $\mathcal{J}_N[V,W]$. However, one can ask the following question: "Does there exist coefficients $F_2[V,W]$, $F_1[V,W]$... such that

$$\log \mathcal{J}_N[V,W] = N^2 F_2[V,W] + N F_1[V,W] + ..."$$

To show such an asymptotic expansion (AE), a probilistic point of view on $\mathcal{J}_N[V,W]$ is useful and for this purpose, we define the following probability measure:

$$d\mathbb{P}_N^V(\lambda_1,\dots,\lambda_N) \overset{(\mathrm{def})}{=} \frac{\prod_{i=1}^N e^{-NV(\lambda_i)}}{\mathcal{I}_N[V]} d\lambda_i, \qquad \text{with} \quad \mathcal{I}_N[V] \overset{(\mathrm{def})}{=} \int_{\mathbb{R}^N} \prod_{a=1}^N e^{-NV(\lambda_a)} d^N\lambda.$$

In the following, it will be useful to consider the *empirical measure*, namely, the random measure given by

$$\widehat{\mu}_N \overset{(\mathrm{def})}{=} \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}.$$

In this picture, the particles λ_i are considered random variables whose joint law is $\mathbb{P}_N^V$. The empirical measure is an important quantity as it describes the density of particles for a given configuration $(\lambda_1, \dots, \lambda_N)$. It is also useful as it can be seen that the following equation holds:

$$\mathcal{J}_N[V,W] = \mathcal{I}_N[V]\mathbb{E}_N^V\left[e^{N^2 F_W[\widehat{\mu}_N]}\right], \qquad \text{where } F_W[\mu] \overset{(\text{def})}{=} \iint_{\mathbb{R}^2} W(s,t)d\mu(s)d\mu(t). \tag{7}$$

The first term in the RHS, $\mathcal{I}_N[V]$, can be easily managed, as by Fubini's theorem, it can be recast as

$$\mathcal{I}_N[V] = \left(\int_{\mathbb{R}} e^{-NV(\lambda)}d\lambda\right)^N.$$

This one-dimensional integral is of Laplace type hence the eponymous method allows one to capture its leading asymptotic behaviour. Indeed, assuming that the function V is strictly convex and hence attains its minimizer at a unique point x_0, an application of the Laplace method demonstrates that:

$$\frac{1}{N^2}\log \mathcal{I}_N[V] \underset{N\to\infty}{\longrightarrow} -\min V = -V(x_0). \tag{8}$$

The second term can be dealt with the theory of *large deviation principles* (LDP) which we recall briefly here and in Appendix B. A sequence of probability measures (ν_N) on a Polish space $\mathcal{S}$ satisfies a LDP if there exists a function I, called the *rate function* and a positive sequence v_N going to infinity, called the *speed*, such that:

$$\begin{aligned} &\forall F \text{ closed set of } \mathcal{S}, && \liminf_N \frac{1}{v_N}\log \nu_N(F) \geq -\inf_{x\in F} I(x), \\ &\forall O \text{ open set of } \mathcal{S}, && \limsup_N \frac{1}{v_N}\log \nu_N(O) \leq -\inf_{x\in O} I(x). \end{aligned}$$

We stress, that the function I should be lower semi-continuous and positive. This definition can be non-rigorously summarized as:

$$\nu_N(A) \approx e^{-v_N \inf_A I} \qquad \text{as} \quad N\to\infty,$$

which asserts the exponential decay of the probability of rare events (or large deviations from the expected behavior at large N). We say in that case that a LDP(v_N, I) holds. The advantage of this theory is that it admits theorems allowing one to capture the large-N behaviour of terms like $\mathbb{E}_N^V\left[e^{N^2 F_W[\widehat{\mu}_N]}\right]$ when F_W satisfies certain properties.

One can show, by the boundedness and continuity of W, that F_W is bounded and continuous on $\mathcal{M}_1(\mathbb{R})$, the space of probability measures on $\mathbb{R}$, equipped with its weak topology. Furthermore, it can be proven without too much effort that the sequence of probability measures $\mathcal{P}_N$ on $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}))$, defined as the law of $\widehat{\mu}_N$ seen as a probability measure-valued random variable under $\mathbb{P}_N^V$, satisfies LDP(N^2, I_V) for a certain rate function I_V. Therefore one is able, by applying Varadhan's lemma 3.0.5, to deduce the following asymptotic behaviour

$$\frac{1}{N^2}\log \mathbb{E}_N^V\left[e^{N^2 F_W[\widehat{\mu}_N]}\right] \underset{N\to\infty}{\longrightarrow} \sup_{\mu\in\mathcal{M}_1(\mathbb{R})}\{F_W(\mu) - I_V(\mu)\}. \tag{9}$$

To rigorously show the following asymptotic behaviour, *via* (7), (8) and (9):

$$\frac{1}{N^2}\log \mathcal{J}_N[V,W] \underset{N\to\infty}{\longrightarrow} \sup_{\mu\in\mathcal{M}_1(\mathbb{R})}\{F_W(\mu) - I_V(\mu)\} - V(x_0),$$

it is necessary to prove that the law of $\widehat{\mu}_N$ indeed satisfies a LDP(N^2, I_V). It might be tempting to use the famous Sanov's theorem for the law of the empirical measure of iid random variables but

the N-dependence of each law prevents one from applying this theorem. This can be however done by hand and it can be shown that for every $\mu \in \mathcal{M}_1(\mathbb{R})$:

$$I_V(\mu) \overset{(\text{def})}{=} \int_{\mathbb{R}} V(x) d\mu(x).$$

Computing, the limit of $N^{-2} \log \mathcal{J}_N[V, W]$ amounts to minimize the following quantity

$$\int_{\mathbb{R}} V(x) d\mu(x) - \iint_{\mathbb{R}^2} W(x, y) d\mu(x) d\mu(y).$$

Without specifying the functions of V and W, this functional is non-trivial to minimize with respect to μ hence doesn't provide an easy computation of the limit $\lim_{N\to\infty} N^{-2} \log \mathcal{J}_N[V, W]$. However, this allows at least to show the existence of the limit under reasonable assumptions on V and W.

This machinery works for all integrals involving a two-body interaction which is bounded and continuous. This class of integrals is well-understood and there exists techniques to deal with them. However, one can also consider showing this type of result for more involved interactions such as the singular ones, *i.e.* unbounded or discontinuous interaction W. These integrals are way more difficult to handle as it requires a lot of effort to adapt the techniques. An instance of such an integral is the partition function of the so-called *β-ensembles*.

I.3 The β-ensembles

The partition function of the so-called *β-ensembles* given by:

$$\mathcal{Z}_N^{(\beta)}[V] \overset{(\text{def})}{=} \int_{\mathbb{R}^N} p_N^{V,\beta}(\lambda_1, \dots, \lambda_N) d^N\lambda, \qquad \text{with } p_N^{V,\beta}(\lambda_1, \dots, \lambda_N) \overset{(\text{def})}{=} \prod_{i<j}^{N} |\lambda_i - \lambda_j|^\beta . \prod_{i=1}^{N} e^{-NV(\lambda_i)}. \tag{10}$$

Above V is a confining potential growing fast enough at infinity, *i.e.* faster than logarithmically, making this integral convergent.

The integral $\mathcal{Z}_N^{(\beta)}[V]$ is equal to $\mathcal{J}_N[V, W]$ with $W(x, y) \overset{(\text{def})}{=} \beta \log|x - y| \mathbf{1}_{x<y}$. This function is not bounded and thus the function F_W, introduced in (7) is not bounded either. Indeed, one can see it by showing that the sequence $\left(F_W\left(\mathcal{U}([0, N^{-1}])\right)\right)_N$, where $\mathcal{U}([0, N^{-1}])$ is the uniform measure on $[0, N^{-1}]$, is not bounded. Indeed,

$$-\beta \iint_{x<y} \log|x - y| d\mathcal{U}([0, N^{-1}])(x) d\mathcal{U}([0, N^{-1}])(y) = \frac{3}{4}\beta + \frac{\beta}{2} \log N \underset{N\to\infty}{\longrightarrow} +\infty.$$

This argument shows that Varadhan's lemma cannot be applied. One must then use new tools in order to capture the asymptotic behavior of $\mathcal{Z}_N^{(\beta)}[V]$.

I.3.1 Definition and links to random matrix theory

The probability measure associated to this partition function is the so-called *β-ensembles*, given by:

$$d\mathbb{P}_N^{V,\beta}(\lambda_1, \dots, \lambda_N) \overset{(\text{def})}{=} p_N^{V,\beta}(\lambda_1, \dots, \lambda_N) \frac{d^N\lambda}{\mathcal{Z}_N^{(\beta)}[V]}. \tag{11}$$

Excellent reviews about this probabilistic model are available in [Meh04, DG09, AGZ10a, For10, PS11, BGK16, FGRV16, Gui19].The probability measure $\mathbb{P}_N^{V,\beta}$ arose historically as the joint-law of the eigenvalues of certain $N \times N$ random matrices. Indeed for $\beta = 1, 2, 4$ and $V(x) = \beta x^2/4$, $\mathbb{P}_N^{V,\beta}$,

called the *Gaussian β-ensemble* (GβE) arises as the joint-law of the eigenvalues of the following random matrix

$$M = \frac{A + {}^{\dagger}A}{\sqrt{2}}, \qquad \text{with} \qquad (A_{i,j})_{1\leq i,j\leq N} \overset{\text{law}}{=} \bigotimes_{i,j=1}^{N} \mathcal{N}_{\mathbb{F}}(0,1),\ \forall i,j \in [\![1,N]\!].$$

Above, $\beta = 1$ (resp. 2, resp. 4), † denotes the real transpose (resp. complex conjugate transpose, resp. quaternionic conjugate transpose), $\mathbb{F}$ denotes the field $\mathbb{R}$ (resp. $\mathbb{C}$, resp. $\mathbb{H}$). These matrix ensembles are called respectively the *Gaussian Orthogonal Ensemble* (GOE), *Gaussian Unitary Ensemble* (GUE) and *Gaussian Symplectic Ensemble* (GSE). The names of these ensembles refer to the fact that the law of a random matrix sampled from this ensemble is invariant under deterministic conjugaison by an element of this group. Namely, for the GOE it reads: $M \overset{\text{law}}{=} O^{\top}MO$, for all $O \in \mathcal{O}_N(\mathbb{R})$.

To establish the explicit form for the law of the spectrum, the idea, see [AGZ10a, Theorem 2.5.2], is to check that such a random matrix M decomposes into $^{\dagger}UDU$ where U is orthogonal (resp. hermitian, resp. symplectic) when $\beta = 1$ (resp. 2, resp. 4) and D real diagonal. Heuristically, a random matrix in this ensemble can thus be identified with its preimage with respect to the "bijective"[4] map $(U, D) \mapsto M$. By a change of variables, since U can be, at least heuristically, parametrized by $\beta N(N-1)/2$ parameters, one can show that the Jacobian must a polynomial in the eigenvalues. Furthermore, by degree considerations and the fact that this map fails to be a bijection as soon as $D_{ii} = D_{jj}$ for $i \neq j$, the Jacobian must be proportionnal to $|\Delta(\lambda_1, \ldots, \lambda_N)|^{\beta}$, where $\Delta(\lambda_1, \ldots, \lambda_N)$ is the Vandermonde determinant associated with the numbers $(\lambda_1, \ldots, \lambda_N)$. Integrating with respect to U yields the form of $\mathbb{P}_N^{V,\beta}$.

These three ensembles correspond, when interpreting real symmetric (resp. complex hermitian, resp. quaternionic self-dual) matrices as hamiltonian operators, to three types of quantum systems. In 1962, Dyson [Dys62] in a very important paper established a link between the GOE (resp. GUE, resp. GSE) and systems presenting time and spin-reversal symmetry (resp. with no time reversal symmetry, resp. with time-reversal symmetry) [Haa01]. There exists nine others type of random matrices modeling quantum systems with symmetries [Iva02]. In this picture, β seems to play the role of the dimension over the field of the reals and it has been an important question whether or not the β-ensemble arises as the law of the spectrum of certain interesting random matrix models for general $\beta \geq 0$ and more general V, see [Ede10].

In an important work [DE02a], Dumitriu and Edelman managed to find a tridiagonal matrix ensemble with independent coefficients (up to symmetry). Namely, by exploiting the fact that for $\beta = 1, 2, 4$, the law of the matrix was invariant under conjugation by certain group elements, they managed to exhibit an explicit random matrix M_β, with independent entries up to symmetry, whose spectrum follows the distribution of the β-ensembles. The law of M_β is given by:

$$M_\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{\beta(N-1)} & 0 & \cdots \\ \chi_{\beta(N-1)} & \mathcal{N}(0,2) & \chi_{\beta(N-2)} & 0 \\ 0 & \chi_{\beta(N-2)} & \ddots & \ddots \\ & & \ddots & \chi_\beta \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix} \tag{12}$$

with the χ_α distribution defined by the following p.d.f.

$$f_{\chi_\alpha}(x) \overset{(\text{def})}{=} \frac{2^{1-a/2}}{\Gamma(a/2)} x^{a-1} e^{-x^2/2}$$

[4] This map is clearly not bijective since M can have eigenvalues with multiplicities and U is not unique. However, this map can be made bijective by throwing away hermitian matrices forming a set of null Lebesgue measure, *i.e.* by restricting to matrices with distinct eigenvalues.

and where all the coefficients are independent up to the symmetry. This representation makes sense for all $\beta \geq 0$ and the spectrum of M_β follows the GβE. This tridiagonal representation was later generalized to polynomial potentials, as discussed in [KRV16]. Although the coefficients are no longer independent, the matrix does exhibit a kind of Markov field property.

The β-ensemble for $\beta = 2$ and quadratic potential can also be seen as the law of the ground state of a system composed of N free fermions confined in a quadratic potential [DLDMS15]. There also exist some discrete version of the β-ensemble which arise in the context of lozenge tilings of hexagons [Gui19].

Finally, explicit expressions exist for the law of the spectrum of other random matrix models, such as the Laguerre-ensemble or the Jacobi ensemble. These expressions are quite similar in the sense that they also contain a Vandermonde and a confining term. The main difference lies in the domain of integration, which must be changed to $[0, +\infty[$ for the Laguerre-ensemble and $[0, 1]$ for the Jacobi-ensemble. In the first case, one usually chooses a potential which present a singularity at 0 such as $V(x) = -\alpha \log|x| + |x|$ for $\alpha > 0$.

Having introduced the β-ensembles and shown several links it has with many other models, we now review the existing techniques to obtain the asymptotic behavior of its partition function $\mathcal{Z}_N^{(\beta)}[V]$.

I.3.2 Large deviation principle and leading order for $\mathcal{Z}_N^{(\beta)}[V]$

The β-ensembles can be viewed as an interacting particle system, or more precisely a *log-gas* with N particles subject to a confining potential V, which prevents them from escaping to infinity. It also involves a 2-body repulsive (singular) interaction that makes more likely for these particles to be well-spaced. With this point of view, $\mathbb{P}_N^{V,\beta}$ introduced in (11), can be interpreted as a Gibbs measure at inverse temperature $\beta > 0$:

$$p_N^{V,\beta}(\lambda_1, \dots, \lambda_N) = e^{-\beta N^2 \mathcal{H}_N(\lambda_1, \dots, \lambda_N)}$$

with Hamiltonian $\mathcal{H}$ given by, when the λ_i are distinct,

$$\mathcal{H}_N^V(\lambda_1, \dots, \lambda_N) \overset{(\text{def})}{=} \int_{\mathbb{R}} V(x) d\widehat{\mu}_N(x) - \iint_{x \neq y} \log|x-y| d\widehat{\mu}_N(x) d\widehat{\mu}_N(y)^5.$$

Here, the empirical measure is defined as $\widehat{\mu}_N = N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$. This object encodes the macroscopic behavior of the system and provides important information about $p_N^{V,\beta}$ and $\mathcal{Z}_N^{(\beta)}[V]$ since $\mathcal{H}_N^V$ solely depends on $\widehat{\mu}_N$. It is known since Wigner [Wig55] that, when V is quadratic, $\widehat{\mu}_N$ converges to the semi-circle law μ_{sc} given by $d\mu_{\text{sc}}(x) \overset{(\text{def})}{=} \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)dx$. Moreover, the convergence of the empirical measure of the β-ensembles for general V has been obtained *via* the LDP theory, see [AGZ10a, Theorem 2.6.1] for a recent proof.

Theorem I.3.1 ([BAG97]) *The family of random measures $(\widehat{\mu}_N)_{N\geq 1}$ satisfies, in $\mathcal{M}_1(\mathbb{R})$ equipped with the weak topology, a* LDP(N^2, I_V^β) *with strictly convex, good rate function I_V^β given by*

$$I_V^\beta(\mu) \overset{(\text{def})}{=} \mathcal{E}_{V,\beta}(\mu) - \inf_{\nu \in \mathcal{M}_1(\mathbb{R})} \mathcal{E}_{V,\beta}(\nu).$$

The functional $\mathcal{E}_{V,\beta}$ is given for all $\mu \in \mathcal{M}_1(\mathbb{R})$ by

$$\mathcal{E}_{V,\beta}(\mu) \overset{(\text{def})}{=} \iint_{\mathbb{R}^2} \left(\frac{V(x)}{2} + \frac{V(y)}{2} - \frac{\beta}{2} \log|x-y| \right) d\mu(x) d\mu(y). \tag{13}$$

[5] This double integral can be seen as the one fold-integral of the log of the characteristic polynomial of the "associated" random matrix against $\widehat{\mu}_N$.

The unique minimizer is attained at a Lebesgue absolutely continuous probability measure $d\mu_{V,\beta}(x) \overset{\text{(def)}}{=} \rho_{V,\beta}(x)dx$. *It is called the equilibrium measure and is characterized by the following Euler-Lagrange equations:*

$$V(x) - \beta \int_{\mathbb{R}} \log|x-y| d\mu_{V,\beta}(y) \begin{cases} \geq C_{V,\beta} & \text{outside supp } \mu_{V,\beta} \\ = C_{V,\beta} & \mu_{V,\beta} - as \end{cases}. \tag{14}$$

$\mu_{V,\beta}$ *is compactly supported. Finally, under* $\mathbb{P}_N^{V,\beta}$, $\widehat{\mu}_N$ *converges weakly towards* $\mu_{V,\beta}$ *almost surely and the free energy reads*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathcal{Z}_N^{(\beta)}[V] = -\int_{\mathbb{R}} V(x) d\mu_{V,\beta}(x) + \beta \iint_{\mathbb{R}^2} \log|x-y| d\mu_{V,\beta}(x) d\mu_{V,\beta}(y). \tag{15}$$

Several comments need to be made about this theorem. First, for all $\beta \geq 0$ and V quadratic, one recovers the semi-circle distribution

$$d\mu_{V,\beta}(x) = \sqrt{\beta - 4x^2} \mathbf{1}_{[-\frac{\sqrt{\beta}}{2}, \frac{\sqrt{\beta}}{2}]}(x) \frac{dx}{\beta\pi}.$$

Though, only in few other cases, is the equilibrium measure explicitly known such as the potential $V(x) = x^{2n}$ and $\beta = 2$ [Dei99, Section 6] in which the equilibrium measure reads:

$$\rho_{V,2}(x) = \frac{n}{\pi} \sqrt{a^2 - x^2} . \left(x^{2n-2} + \sum_{j=1}^{n-1} x^{2n-2-2j} a^{2j} \prod_{l=1}^{j} \frac{2l-1}{2l} \right) \mathbf{1}_{[-a,a]}(x) dx, \tag{16}$$

with explicit endpoints given by:

$$a \overset{\text{(def)}}{=} \left(n \prod_{l=1}^{n} \frac{2l-1}{2l} \right)^{-\frac{1}{2n}}. \tag{17}$$

Another case where the equilibrium density is explicit is for the potential $V(x) = \frac{x^4}{20} - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x$ and $\beta = 1$, [CKI10, Example 1.2] where $\rho_{V,1}$ is given by

$$\rho_{V,1}(x) = \frac{1}{10\pi} (x+2)^{1/2} (2-x)^{5/2} \mathbf{1}_{[-2,2]}(x). \tag{18}$$

In this example, one can see that the density $\rho_{V,1}$ doesn't vanish like a square-root at the endpoint 2. In general, one usually resorts a one-dimensional integral representation for $\mu_{V,\beta}$ which is obtained by solving a scalar RHP [Dei99, Section 6] which we will detail below. Also, note that $\mu_{V,\beta}$ plays a significant role in the analysis of orthogonal polynomials by Riemann-Hilbert and the non-linear steepest descent method, see the previous reference for a review and Subsection I.3.4. The properties have been extensively studied [Lan72, DKM98, ST13].

Secondly, from the Euler-Lagrange characterization (14), one can deduce that, if V is analytic, $\mu_{V,\beta}$ is supported on a finite union of segments $\bigcup_{l=1}^{m} [a_l, b_l]$ with $m \geq 1$ and $-\infty < a_1 < b_1 < a_2 < \cdots < b_m < +\infty$ [DKM98, Theorem 1.38]. The case $m = 1$ is called the *one-cut* regime while the case $m > 1$ is called the *multi-cut* regime. Strict convexity and smoothness hypotheses on V ensures that $m = 1$ [Joh98, Prop 3.1], but there is no clear characterization of all potentials leading to the one-cut regime. Furthermore, it might happen that for a point $x_0 \in \text{supp } \mu_{V,\beta}$, the p.d.f. of $\mu_{V,\beta}$ might vanish like $(x - x_0)^{2k}$ for an integer k. When this situation occurs, we say that we are in the *critical case*. An instance of this class is $\mu_{V,\beta}$ when $V = \frac{x^4}{4} - x^2$. Indeed, the equilibrium measure admits the following expression, see [CK06]

$$d\mu_{V,\beta}(x) = \frac{1}{2\pi} \sqrt{|x-2|.|x+2|} x^2 \mathbf{1}_{[-2,2]}(x) dx.$$

For general V, one may obtain an integral representation or the equilibrium density $\rho_{V,\beta}$ by solving a scalar RHP. Indeed, assuming the one-cut regime, one differentiates the Euler-Lagrange equation (14) to get:

$$\fint_a^b \frac{\rho_{V,\beta}(y)dy}{x-y} = V'(x) \ \ \forall x \in [a,b], \qquad F(z) \overset{\text{(def)}}{=} \frac{1}{(z-a)^{1/2}(z-b)^{1/2}} \int_a^b \frac{\rho_{V,\beta}(y)dy}{z-y}. \quad (19)$$

where $\fint$ denotes the Cauchy principal value with the choice of the principal branch of the logarithm. It follows from a direct computation that, for all $x \in [a,b]$:

$$F_+(x) + F_-(x) = -2\pi \frac{\rho_{V,\beta}(x)}{\sqrt{(b-x)(x-a)}} \qquad \text{and} \qquad F_+(x) - F_-(x) = \frac{-2\mathrm{i}V'(x)}{\sqrt{(b-x)(x-a)}}.$$

Above $F_\pm$ denotes the boundary value of F from the $\pm$ side and is defined by:

$$F_\pm(x) \overset{\text{(def)}}{=} \lim_{\substack{z \to x \\ z \in \pm \text{ side non}-\text{tangentially}}} F(z),$$

where the $\pm$ side is defined as the . Since F decays sufficiently fast at infinity, $F(z)$ verifies for all $z \in \mathbb{C} \setminus [a,b]$,

$$F(z) = \int_a^b \frac{dy}{2\mathrm{i}\pi} \frac{2V'(y)}{(z-y)\sqrt{(b-y)(y-a)}}.$$

Thus, one finally obtains the following expression for $\rho_{V,\beta}$:

$$\rho_{V,\beta}(x) = \fint_a^b \frac{\sqrt{(b-x)(x-a)}}{\sqrt{(b-y)(y-a)}} \frac{V'(y)}{y-s} \frac{dy}{2\pi^2}. \quad (20)$$

Thirdly, it seems that the expression obtained in (13) for the rate function is very similar to the one associated with the integral $\mathcal{J}_N[V,W]$ for a bounded continuous 2-body interaction W. The fact that the logarithmic interaction is singular, and more precisely that

$$\mu \in \mathcal{M}_1(\mathbb{R}) \mapsto \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y)$$

fails to be a bounded continuous function, implies that Varadhan's lemma doesn't apply. The main achievement of the proof was to overcome this singularity.

LDP techniques allow one to capture the leading behavior of $\log \mathcal{Z}_N^{(\beta)}[V]$ when N goes to infinity, but to go beyond this leading order, one must use more involved techniques such as the study of the fluctuations of the eigenvalues.

I.3.3 Fluctuations and subleading order for $\mathcal{Z}_N^{(\beta)}[V]$

Once one knows about the limiting behavior of $\widehat{\mu}_N$, one is interested in the fluctuations around this limit, namely whether a *central limit theorem* (CLT) for *linear statistics* holds. More precisely, a result of this type has been proven first, for $\beta = 2$ and polynomial V (of even degree and positive leading coefficient) in [Joh98],

$$N \int_{\mathbb{R}} f(x) d\left(\widehat{\mu}_N - \mu_{V,\beta}\right)(x) \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\Big(m(f), \sigma^2(f)\Big) \quad (21)$$

for $m(f), \sigma(f) \in \mathbb{R}$ and f smooth such that f and f' don't grow too fast at infinity. It was later generalized to all β and a large class of V in [BG13a] and to more general settings in [BG13b, Shc13, BLS18, LLW19]. It was proven that the size of these fluctuations does not change when changing β and V. However, the size of fluctuations has been shown to be very dependent of the

regularity of the test-function. Indeed for indicator functions $f = \mathbf{1}_{[a,b]}$, which are very singular, [Kil07, BMP22] show that the fluctuations are again Gaussian but of size $\sqrt{\log N}/N$. Mesoscopic CLT's *i.e.* results of the type:

$$N \int_{\mathbb{R}} f\left(\frac{x - x_0}{v_N}\right) d\left(\widehat{\mu}_N - \mu_{V,\beta}\right)(x) \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathcal{N}\Big(m(f), \sigma^2(f)\Big); \qquad v_N \underset{N\to+\infty}{\longrightarrow} 0,\ N v_N \underset{N\to+\infty}{\longrightarrow} +\infty$$

where f is a compactly supported function have also been shown in [BL18, Lam21a, Pei24]. One can choose $v_N = N^{-\alpha}$ where $\alpha \in (0,1)$ for example. CLT's for individual particles at an intermediate distance from the edge, *i.e.* for the k-th particle λ_k with $k = k(N) \to +\infty$ as $N \to +\infty$, were shown in [Gus05, BEY14a, BMP22]. These results show that the fluctuations of this k-th particle around the k-th quantile γ_k of $\mu_{V,\beta}$, defined by:

$$\int_{-\infty}^{\gamma_k} d\mu_{V,\beta}(x) = \frac{k}{N},$$

are of typical size $\sqrt{\log N}/N$.

These results can be seen as a second order asymptotic expansion (AE) for the partition function in the sense that one usually shows (21) by proving the convergence of Laplace transforms. In the following, we denote by $H^s(O)$, the s-th Sobolev subspace of functions supported on an open set $O \subset \mathbb{R}^m$, defined by which belongs to:

$$H^s(R^m) \overset{(\text{def})}{=} \left\{u \in L^2(\mathbb{R}^m), \|u\|_{H^s(\mathbb{R}^m)} < +\infty\right\},$$

where

$$\|u\|^2_{H^s(\mathbb{R}^m)} \overset{(\text{def})}{=} \int_{\mathbb{R}^m} (1 + \|t\|_2)^{2s} \left|\mathcal{F}[u](t_1, \ldots, t_m)\right|^2 d^m\underline{t}.$$

Above, $\|.\|_2$ denotes the Euclidean norm on $\mathbb{R}^m$. Then we are able to state the following celebrated theorem in random matrix theory:

Theorem I.3.2 ([Joh98]) *For $h \in H^{2+\alpha}(O)$ where $O \subset \mathbb{R}$ is some open neighborhood of* $\operatorname{supp}\mu_{V,2}$*, $\alpha > 0$, such that $|h(x)| = O(V(x))$ at infinity and $|h'(x)| \leq q(x)$ for some polynomial q, there exists a quadratic form σ^2 such that the following holds:*

$$\log \mathbb{E}_N^{V,\beta}\left[e^{\sum_{i=1}^N h(\lambda_i)}\right] \underset{N\to\infty}{=} N\int_{\mathbb{R}} h(x) d\mu_{V,\beta}(x) + \left(\frac{2}{\beta} - 1\right)\int_{\mathbb{R}} h(x) d\nu_{V,\beta}(x) + \sigma^2(h) + o(1). \quad (22)$$

The quadratic form $\sigma^2(h)$ can be written as:

$$\sigma^2(h) \overset{(\text{def})}{=} \frac{-1}{2\pi^2\beta}\int_a^b dt h(t) \int_a^b \frac{\sqrt{(b-s)(s-a)}}{\sqrt{(t-a)(b-t)}} \frac{h'(s)}{s-t} ds$$

and admits a form of $1/2$-Sobolev norm type given by:

$$\sigma^2(h) = \frac{1}{4\beta}\sum_{k=1}^{+\infty} |k^{1/2} a_k(h)|^2, \qquad \text{with} \quad a_k(h) \overset{(\text{def})}{=} \frac{2}{\pi}\int_0^\pi h\left(\frac{a+b}{2} + \frac{b-a}{2}\cos\theta\right)\cos k\theta d\theta. \quad (23)$$

The signed measure $\nu_{V,\beta}$ is given by for $\beta = 2$ and quadratic V by

$$d\nu_{x^2,2}(x) \overset{(\text{def})}{=} \frac{1}{2}\left(\delta_{-2} + \delta_2\right) - \frac{\mathbf{1}_{[-2,2]}}{\pi\sqrt{4-x^2}} dx. \quad (24)$$

Furthermore, there exists established expansions for the partition function associated to the quadratic potential $V_G(x) \overset{(\text{def})}{=} x^2/2$, $\log \mathcal{Z}_N^{(\beta)}[V_G]$, namely the Mehta's formula [Meh04, Formula 17.6.7] which expresses $\mathcal{Z}_N^{(\beta)}[V_G]$ as a Selberg integral. It reads

$$\mathcal{Z}_N^{(\beta)}[V_G] = (2\pi)^{N/2} N^{-\frac{\beta}{4}N^2 + N(\frac{\beta}{4}-\frac{1}{2})} \frac{\prod_{a=1}^{N} \Gamma\left(1+\frac{a\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)^N}. \tag{25}$$

From there, one can obtain the following asymptotic expansion:

$$\log \mathcal{Z}_N^{(\beta)}[V_G] \underset{N\to\infty}{=} \left\{\frac{\beta}{4}\log\left(\frac{\beta}{2}\right) - \frac{3\beta}{8}\right\} N^2 + \frac{\beta}{2} N \log N + \left\{\left(\frac{1}{2}+\frac{\beta}{4}\right)\log\left(\frac{\beta}{2e}\right) + \log(2\pi) - \log\Gamma\left(1+\frac{\beta}{2}\right)\right\} N$$
$$+ \frac{1}{12}\left\{3 + \frac{\beta}{2} + \frac{2}{\beta}\right\}\log N + \partial_1\chi\left(0, \frac{2}{\beta}, 1\right) + \frac{\log(2\pi)}{2} + o(1). \tag{26}$$

Above, the function $\chi(s, z_1, z_2)$ is a special case of the general Barnes ζ function which is the meromorphic continuation (with respect to s) of the function defined for $\mathrm{Re}(s) > 2$ by the formula

$$\chi(s, z_1, z_2) \overset{(\text{def})}{=} \sum_{(n_1,n_2)\in\mathbb{N}^2\setminus\{(0,0)\}} \frac{1}{(n_1 z_1 + n_2 z_2)^s}. \tag{27}$$

From (22) and (26), one can deduce the AE of the partition function for modified potentials $V_h(x) = x^2/2 + N^{-1}h(x)$:

$$\log \mathcal{Z}_N^{(\beta)}[V_h] \underset{N\to\infty}{=} \log \mathcal{Z}_N^{(\beta)}[V_G] + N\int_{\mathbb{R}} h(x) d\mu_{V_h,\beta}(x) + \left(\frac{2}{\beta} - 1\right)\int_{\mathbb{R}} h(x) d\mu_{V_h,\beta}(x) + \sigma^2(h) + o(1).$$

While the picture in the one-cut case is simple, it is much more subtle in the multi-cut case. Indeed, it is possible in that case that the fluctuations are not Gaussian as it was noticed in [Pas06, Shc11, Shc13, BG13b]. Indeed in the multi-cut case, one should add a bounded oscillatory term in N to the above equation.

CLTs allow one to go beyond the leading order behavior of the partition function. One can wonder if it possible to access its AE even beyond that ? We review some of the existing methods. We first start with the $\beta = 2$ case and the orthogonal polynomials method.

I.3.4 The asymptotic expansion of the partition function at $\beta = 2$

The case $\beta = 2$ for the partition function $\mathcal{Z}_N^{(\beta)}[V]$ enjoys a particular expression. Indeed, in this case, the corresponding β-ensemble enjoys a so-called *determinantal property*, *i.e.* its n-th correlation function $\rho(x_1, \ldots, x_n)$ is given by a $n \times n$ determinant $\det[K(x_i, x_j)]$. Similarly, the partition function $\mathcal{Z}_N^{(2)}[V]$ can be recast in terms of the following monic orthogonal polynomials $(\pi_k^N)_k$ defined as follows:

$$\pi_k^N(x) = x^k + \ldots \qquad \text{and} \qquad \forall k, l, \left\langle \pi_k^N, \pi_l^N \right\rangle \overset{(\text{def})}{=} \int_{\mathbb{R}} \pi_k^N(x)\pi_l^N(x) e^{-NV(x)} dx = \frac{\delta_{k,l}}{\gamma_k^{(N)}}.$$

With these definitions, the partition function $\mathcal{Z}_N^{(2)}[V]$ admits the following expression:

$$\mathcal{Z}_N^{(2)}[V] = N! \det_{1\le i,j\le N} \int_{\mathbb{R}} x^{i+j-1} w_N(x) dx = N! \prod_{i=0}^{N-1} \left(\gamma_i^{(N)}\right)^{-1}, \qquad w_N(x) \overset{(\text{def})}{=} e^{-NV(x)}. \tag{28}$$

In principle, this relation just transports the difficulty of extracting the asymptotics of $\mathcal{Z}_N^{(2)}[V]$ towards the one of computing the $\gamma_i^{(N)}$'s. The idea is that these orthogonal polynomials are integrable, meaning that, in principle, every question about their asymptotics can be answered. The existence of an asymptotic expansion for potentials leading to one-cut equilibrium measures have been obtained in [EM03] and [BI05a] as well as for the two-cut case [CGM15]. In a recent work [CKM23], this AE has been obtained for singular potentials, *i.e.*, for $\log \mathcal{Z}_N^{(2)}[|x|^\gamma]$, with $\gamma > 0$, up to $o(1)$. This analysis relies on the following RHP [FIK92].

Theorem I.3.3 *The following* 2×2 *RHP:*

- $\mathcal{Y} = \mathcal{Y}^{(N)} \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$.
- $\mathcal{Y}$ *admits continuous* $\pm$ *boundary-value* $\mathcal{Y}_\pm$ *on* $\mathbb{R}$.
- *For all* $x \in \mathbb{R}$, $\mathcal{Y}_+(x) = \mathcal{Y}_-(x) \begin{pmatrix} 1 & w_N(x) \\ 0 & 1 \end{pmatrix}$
- $\mathcal{Y}(z) = \left(I_2 + O(z^{-1})\right) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}$

admits a unique solution $\mathcal{Y}$ *which is given by*

$$\forall z \in \mathbb{C},\ \mathcal{Y}^{(N)}(z) \overset{\text{(def)}}{=} \begin{pmatrix} \pi_N^N(z) & \mathsf{C}_{\mathbb{R}}\left[\pi_N^N w_N\right](z) \\ -2\mathrm{i}\pi\gamma_{N-1}^{(N)}\pi_{N-1}^N(z) & -2\mathrm{i}\pi\gamma_{N-1}^{(N)}\mathsf{C}_{\mathbb{R}}\left[\pi_{N-1}^N w_N\right](z) \end{pmatrix}.$$

We recall that the boundary values of a function haave been defined in (C.1) and we refer to Appendix C for basic definitions about RHPs.

This very important result allows one to use the vast literature of RHPs to extract the large N behavior of the coefficients $\gamma_i^{(N)}$. The main tool for this procedure is the so-called *Deift-Zhou non-linear steepest descent method* [DZ92, DZ93], see [DLW96, Dei99] for reviews. It consists of performing invertible transformations from the unique solution $\mathcal{Y}$ of the above RHP

$$\mathcal{Y} \leftrightsquigarrow \mathcal{T} \leftrightsquigarrow \mathcal{S} \leftrightsquigarrow \mathcal{R}$$

until the RHP for $\mathcal{R}$ is such that the jump matrix is close to the identity. This implies that its solution is asymptotically close to the identity. Before explaining the different transformations occurring in this method, we make the assumption that V is an analytic potential leading to the one-cut regime. This implies that the equilibrium density $\rho_{V,2}$, supported on $[a, b]$ can be continued analytically in a complex neighborhood of its support.

The first transformation is explicitly given by

$$\mathcal{T}(z) \overset{\text{(def)}}{=} e^{N\frac{C_{V,2}}{2}\sigma_3}\mathcal{Y}(z)e^{-Ng(z)\sigma_3}e^{-N\frac{C_{V,2}}{2}\sigma_3}, \qquad g(z) \overset{\text{(def)}}{=} \int_{\mathbb{R}} \log(z-t)d\mu_{V,2}(t),\ \ \mathrm{Im}(z) \neq 0.$$

with

$$\sigma_3 \overset{\text{(def)}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The function $\mathcal{T}$ solves the following RHP

- $\mathcal{T} \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$.
- $\mathcal{T}$ admits continuous $\pm$ boundary-value $\mathcal{T}_\pm$ on $\mathbb{R}$.
- For all $x \in \mathbb{R}$, $\mathcal{T}_+(x) = \mathcal{T}_-(x)G_{\mathcal{T}}(x)$ with

$$G_{\mathcal{T}}(x) \overset{\text{(def)}}{=} \begin{pmatrix} e^{N(g_-(x)-g_+(x))} & e^{N\left(g_-(x)+g_+(x)-V(x)+C_{2,V}\right)} \\ 0 & e^{-N(g_-(x)-g_+(x))} \end{pmatrix}. \tag{29}$$

- $\mathcal{T}(z) = I_2 + O(z^{-1})$ as $z \to \infty$.

This transformation has the effect of normalizing the first RHP. Indeed, one gets $\mathcal{T}(z) = I_2 + O(z^{-1})$ at infinity as an asymptotic condition for $\mathcal{T}$. Furthermore, one can notice that $G_{\mathcal{T}}$ admits the following factorization:

$$G_{\mathcal{T}}(x) = e^{-N\varphi_-(x)\sigma_3} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} e^{N\varphi_+(x)\sigma_3} \qquad \text{with} \quad \varphi(z) \overset{\text{(def)}}{=} \mathrm{i}\pi \int_a^z d\mu_{V,2}(\zeta).$$

One can show that $g_+(x) - g_-(x) = 2\varphi(x)$ for all $x \in (a, b)$. This leads to the second step. One uses the unique solution $\mathcal{T}$ to construct the unique solution $\mathcal{S}$ of a new RHP with a modified contour $\Sigma_{\mathcal{S}}$ given in Figure I.1:

- $\mathcal{S} \in \mathcal{O}(\mathbb{C} \setminus \Sigma_{\mathcal{S}})$.
- $\mathcal{S}$ admits continuous $\pm$ boundary-value $\mathcal{S}_\pm$ on $\Sigma_{\mathcal{S}}$ (the + (resp. $-$) side is always on the left (resp. right) side as one traverses the contour).
- $\mathcal{S}_+(z) = \mathcal{S}_-(z) G_{\mathcal{S}}(z)$ for all $z \in \Sigma_{\mathcal{S}}$ where $G_{\mathcal{S}}$ is given in Figure I.1.
- $\mathcal{S}(z) = I_2 + O(z^{-1})$ as $z \to \infty$.

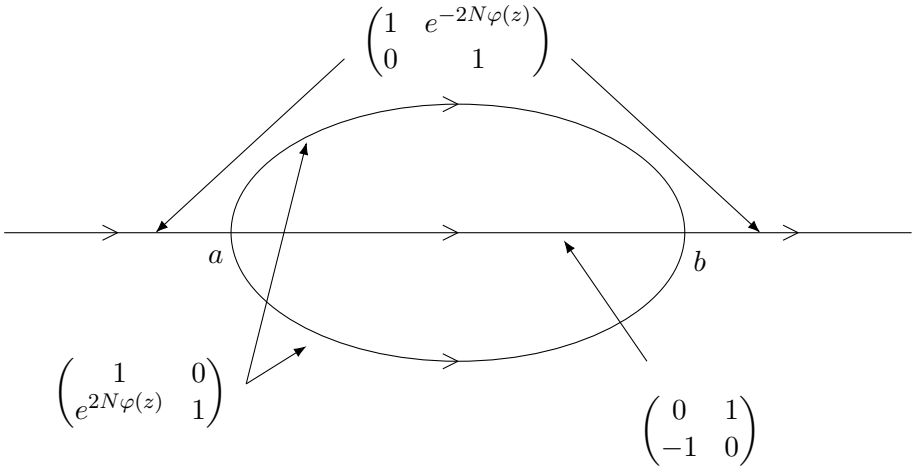

Figure I.1: Piecewise definition of the jump matrix $G_{\mathcal{S}}$ on the contour $\Sigma_{\mathcal{S}}$. Note that the specific shape of $\Sigma_{\mathcal{S}}$ is not important.

Without entering into too many details, one can show that the function

$$x \in \mathbb{C} \mapsto g_+(x) - g_-(x),$$

appearing in (29), is purely imaginary on $[a, b]$ and of real part positive above (a, b) and negative below. Thus $G_{\mathcal{T}}$ contains rapidly oscillating terms on $[a, b]$. The transformation $\mathcal{T} \rightsquigarrow \mathcal{S}$ transforms these oscillatory terms into exponentially fast decaying terms. This allows one to deal with a RHP whose jump matrix satisfies

$$G_{\mathcal{S}}(z) \underset{N\to\infty}{\longrightarrow} \begin{cases} G_{\mathcal{S}}^{\infty}(z) \overset{\text{(def)}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \ z \in [a, b], \\ I_2 & \text{otherwise.} \end{cases}$$

Thus when N goes to infinity, one can expect that the solution $\mathcal{S}$ converges to the unique solution $\mathcal{S}^\infty$ of the below RHP:

- $\mathcal{S}^\infty \in \mathcal{O}(\mathbb{C} \setminus [a, b])$,

- $\mathcal{S}_+^\infty(z) = \mathcal{S}_-^\infty(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for all $z \in [a,b]$,
- $\mathcal{S}^\infty(z) = I_2 + O(z^{-1})$ as $z \to \infty$.

Indeed, since the jump matrix of $\mathcal{S}$, $G_{\mathcal{S}}$ converges pointwise towards the identity oustide of $[a,b]$, we expect that the other jumps will disappear. This RHP can be explicitly solved (it is done by diagonalizing $G_{\mathcal{S}}^\infty$) but proving that $\mathcal{S} \underset{N\to\infty}{\longrightarrow} \mathcal{S}^\infty$ (for some appropriate norms) is delicate. When two jump matrices G_1, G_2, corresponding to two RHP on some contour of the complex plane Σ, are such that $\|G_1 - G_2\|_{L^2(\Sigma)\cap L^\infty(\Sigma)} \underset{N\to\infty}{\longrightarrow} 0$, then the corresponding solutions X_1, X_2 also satisfy $\|X_1 - X_2\|_{L^2(\Sigma)} \underset{N\to\infty}{\longrightarrow} 0$ [Dei99, Corollary 7.108]. The problem arises near the endpoints a and b, where $G_{\mathcal{S}}$ may not converge uniformly to $G_{\mathcal{S}}^\infty$.

To overcome this issue, one has to perform a last transformation $\mathcal{S} \rightsquigarrow \mathcal{R}$ by constructing a so-called *local parametrix* $\mathcal{S}_{\text{par}}$ which is a the solution of a RHP on a contour $\Sigma_{\mathcal{S}_{\text{par}}}$. The construction of this 2×2 matrix-valued function is quite sophisticated so we do not recall it here. Nonetheless, its role is to solve explicitly the previous RHP in the neighborhoods of the endpoints a and b such that after setting $\mathcal{R} \overset{(\text{def})}{=} \mathcal{S}\mathcal{S}_{\text{par}}^{-1}$, one can show that $\mathcal{U}$ is a holomorphic function on $\Sigma_{\mathcal{S}_{\text{par}}}$ with jumps given by a matrix $G_{\mathcal{R}}$. One can finally show that $\|G_{\mathcal{R}} - I_2\|_{L^\infty(\Sigma_{\mathcal{S}_{\text{par}}})} \underset{N\to\infty}{\longrightarrow} 0$. This allows one to solve uniquely the RHP for $\mathcal{R}$ and by performing the inverse transformations one gets [Dei99, Theorem 7.171]:

Theorem I.3.4 *It holds that $\mathcal{T}(z) = (I_2 + o(1))\mathcal{S}_\infty(z)$ uniformly for $|\mathrm{Im}(z)| \geq \varepsilon$.*

This theorem is sufficient to determine the asymptotic behavior of γ_i's and concludes about the AE of $\mathcal{Z}_N^{(2)}[V]$. Applying this method, the authors of [EM03] obtained the following AE for polynomials potentials of the form $V(.,t_1,\dots,t_n) : \lambda \mapsto \lambda^2/2 + \sum_{i=1}^n t_i\lambda^i$:

$$N^{-2}\log\frac{\mathcal{Z}_N^{(2)}[V(.,t_1,\dots,t_n)]}{\mathcal{Z}_N^{(2)}[V(.,0,\dots,0)]} = \sum_{k=0}^{M} e_k(t_1,\dots,t_n)N^{-2k} + O(N^{-2M-2}). \tag{30}$$

This expansion holds for all $M > 0$. The numbers $e_k(t_1,\dots,t_n)$ are analytic functions of $\underline{t} \overset{(\text{def})}{=} (t_1,\dots,t_n)$ and possess a combinatorial interpretation. There exists a connection between asymptotics of multiple integrals, such as the partition function $\mathcal{Z}_N^{(2)}[V]$ and the enumeration of maps. As argued in [BIPZ78, Bou11, E$^+$16], there is a link between the so-called *formal matrix integrals* and generating functions of maps. This statement can be seen at the level of equation (30) by the following result:

Theorem I.3.5 *The numbers $e_k(\underline{t})$ appearing in* (30) *admit the following expression:*

$$e_k(\underline{t}) = \sum_{(m_1,\dots,m_n)\in(\mathbb{N}^*)^n} \kappa_k(m_1,\dots,m_n)\prod_{i=1}^n \frac{(-t_i)^{m_i}}{m_i!}.$$

The coefficient $\kappa_k(m_1,\dots,m_n)$ is the number of k-maps with m_j j-valent vertices for $j \in [\![1,n]\!]$.

A *map* is a combinatorial object that can be defined as a graph embedded in a Riemann surface so that: *(i)* the images of the edges do not intersect, and *(ii)* the connected components of the embedding are simply connected. A *k-map* is then a map embedded in a surface of genus k.

This connection was first observed by the seminal works [BIPZ78, tH93]. The previous theorem can be seen as a motivation for computing the AE of certain integrals, as one may recover some combinatorial species in the coefficients.

After having addressed the special case $\beta = 2$, we conclude this section by summarizing several results about the β-ensembles. This will lead us to consider generalizations of this model.

I.3.5 Universality

The concept of *universality* refers to the phenomenon observed experimentally that there are features of many physical systems that do not depend on the details of such systems. An example of universality is that a large class of systems exhibit the same so-called *critical exponents* at phase transitions points. This observation also holds mathematically in many random models, where they tend to exhibit the same limiting behavior as their complexity (often represented by a size parameter N) tends to infinity. Remarkably, this limiting behavior appears to depend very little on the initial parameters of the model [Dei06]. Classic examples of this phenomenon include the famous law of large numbers and the central limit theorem for independent variables. The emergence of the Tracy-Widom law in systems with high correlation, such as the six-vertex model, the β-ensembles, or the random growth models, can also be interpreted as instances of universality [BG16].

The renormalization group theory *à la* Wilson provides a mathematical explanation for this phenomenon [LL75, CCEF10, Ott23]. In this framework, the limiting behavior arises as an attractive fixed point of the renormalization operator. This operator *coarse-grains* the system as its complexity increases, bringing the system closer to this attractor.

In the realm of random matrix theory, universality results are well-established. While Wigner's theorem serves as the most evident illustration, universal features also emerge when investigating the local behavior of eigenvalues.

It has been proven that the local statistics within the bulk of β-ensembles converge towards the so-called Sineβ process, initially demonstrated for Gaussian β-ensembles [VV09]. Analogous results can be found in the literature concerning the two-dimensional Coulomb case, such as the *Ginibre point process* [BS09], and the *Riesz-β process* in Riesz gases [Bou22].

For β-ensembles, this result is considered universal as it ultimately does not depend on the choice of the potential, as shown in [BEY12, BEY14b]. Other universality results have been shown using orthogonal polynomials in [PS97, Wid99, BI99, DKM$^+$99a, DKM$^+$99b, DGKV07, DG07b, PS08, Shc11]. Instead, it solely depends on the inverse temperature β. This universality class even encompasses general interactions that exhibit logarithmic local behavior, as demonstrated in [Ven13]. Thus, one may wonder what happens when modifying the model more substantially, specifically the N-dependence of the temperature, the interaction, or scaling of the potential. These issues will be addressed in Sections I.5 and I.7.

In Section I.5, we will remove the scaling of the potential and make β N-dependent such that $N\beta$ is constant. In Section I.7, we will use a different interaction and eliminate the scaling by N of the potential. This involves relinquishing certain convenient properties inherent in the β-ensembles that enable the analysis. Despite the significant implications of these changes, we expect that most of the results and heuristics in the β-ensembles will remain valid. However, proving these results will require addressing a substantial technical gap arising from these modifications.

Having addressed the special case $\beta = 2$ for the AE of $\log \mathcal{Z}_N^{(\beta)}[V]$, we now give details about a general method for deriving the complete AE (up to any finite order N^{-k}) for $\mathcal{Z}_N^{(\beta)}[V]$. This method, called the *loop equations analysis*, will play a significant role in this thesis.

I.4 The asympotic expansion of $\mathcal{Z}_N^{(\beta)}[V]$ by the loop equations analysis

I.4.1 Overview of the method

For the general case, let V be a general potential, V_{ref} be a reference potential such that the large-N behavior of $\mathcal{Z}_N^{(\beta)}[V_{\text{ref}}]$ is known (the quadratic potential is a good example because of Mehta's

formula (25)), and define

$$V_t \overset{(\mathrm{def})}{=} tV + (1-t)V_{\mathrm{ref}}, \qquad \forall t \in [0,1], \tag{31}$$

then one has

$$\log\left(\frac{\mathcal{Z}_N^{(\beta)}[V]}{\mathcal{Z}_N^{(\beta)}[V_{\mathrm{ref}}]}\right) = \int_0^1 \partial_t \log \mathcal{Z}_N^{(\beta)}[V_t]dt = -N^2 \int_0^1 \mathbb{E}_N^{V_t}\left[\int_{\mathbb{R}} (V(x) - V_{\mathrm{ref}}(x))\, d\widehat{\mu}_N(x)\right] dt. \tag{32}$$

This equation links the partition function with the *linear statistics i.e.* expressions given, for an integer $n \geq 1$ and a smooth function f_n by

$$\langle f_n \rangle^W_{\overset{n}{\otimes}\widehat{\mu}_N} \overset{(\mathrm{def})}{=} \mathbb{E}_N^W\left[\int_{\mathbb{R}} f_n(x_1,\dots,x_n) d^{\otimes^n}\widehat{\mu}_N(x_1,\dots,x_n)\right]. \tag{33}$$

The computation (32) allows one, in principle, to deduce the large-N behavior of the partition function from an integration of the asymptotics linear statistics. Suppose one has:

$$\begin{aligned}
\langle V - V_{\mathrm{ref}} \rangle^{V_t}_{\widehat{\mu}_N} &= \sum_{a=0}^{K} \frac{c_a(t)}{N^a} + O\left(N^{-(K+1)}\right), \\
\log \mathcal{Z}_N^{(\beta)}[V_{\mathrm{ref}}] &= aN^2 + bN\log N + cN + d\log N + \sum_{a=0}^{K} \frac{d_a}{N^a} + O\left(N^{-(K+1)}\right),
\end{aligned}$$

one might hope to obtain the N^{-1} expansion for $N^{-2}\log \mathcal{Z}_N^{(\beta)}[V]$ in the form:

$$\begin{aligned}
\log \mathcal{Z}_N^{(\beta)}[V] = \left(a - \int_0^1 c_0(t)dt\right) N^2 + bN\log N + \left(c - \int_0^1 c_1(t)dt\right) N + d\log N \\
+ \sum_{a=0}^{K} \frac{d_a - \int_0^1 c_{a+2}(t)dt}{N^a} + O\left(N^{-(K+1)}\right).
\end{aligned} \tag{34}$$

Since the asymptotics of $\log \mathcal{Z}_N^{(\beta)}[V_{\mathrm{ref}}]$ can be obtained by dealing only with properties of the Γ function by (25), the main step in this procedure is to obtain the asymptotics of $\langle V - V_{\mathrm{ref}} \rangle^{V_t}_{\widehat{\mu}_N}$ and make the integration step rigorous.

To obtain the AE of the linear-satistic $\langle V - V_{\mathrm{ref}} \rangle^{V_t}_{\widehat{\mu}_N}$, one relies on a technique called the analysis of the *loop* or *Schwinger-Dyson equations*. It consists of a tower of equations indexed by $n \in \mathbb{N}^*$ which makes it possible to extract the large-N behavior of the linear statistics. These equations can be proven by integration by parts or by using the fact that $\mathcal{Z}_N^{(\beta)}[V]$ is invariant under change of variables. The first one takes the following form:

$$\frac{\beta}{2}\langle \mathcal{D}[f] \rangle_{\widehat{\mu}_N \otimes \widehat{\mu}_N} + \frac{\beta - 2}{N}\langle f' \rangle_{\widehat{\mu}_N} - \langle V' f \rangle_{\widehat{\mu}_N} = 0, \qquad \text{with } \mathcal{D}[f] \overset{(\mathrm{def})}{=} \frac{f(x) - f(y)}{x - y}. \tag{35}$$

This first loop equation links the 2-linear statistic $\langle \mathcal{D}[f] \rangle_{\widehat{\mu}_N \otimes \widehat{\mu}_N}$ with 1-linear statistics. Taking the formal limit of (35) as $N \to \infty$, one obtains the following equation:

$$\frac{\beta}{2}\int_{\mathbb{R}^2} \mathcal{D}[f](x,y) d\mu_{V,\beta}(x) d\mu_{V,\beta}(y) = \int_{\mathbb{R}} V'(x) f(x) d\mu_{V,\beta}(x). \tag{36}$$

This equation can be derived rigorously, by differentiating the Euler-Lagrange equation (14) and then integrating against $f(x)d\mu_{V,\beta}(x)$. To explain the mechanism behind extracting the large-N

asymptotic of the linear statistics out of the loop equations, we need to recenter (35) using (36). Explicitly, by using the recentred measure $\mathcal{L}_N \overset{(\text{def})}{=} \widehat{\mu}_N - \mu_{V,\beta}$ one obtains:

$$\langle \mathcal{K}[f] \rangle_{\mathcal{L}_N} + \frac{\beta}{2} \langle \mathcal{D}[f] \rangle_{\mathcal{L}_N \otimes \mathcal{L}_N} + \frac{1-\beta/2}{N} \langle f' \rangle_{\mathcal{L}_N} = -\frac{1-\beta/2}{N} \langle f' \rangle_{\mu_{V,\beta}} . \tag{37}$$

Above, the operator $\mathcal{K}$ is called the *master-operator* and is defined by:

$$\mathcal{K}[f](x) \overset{(\text{def})}{=} \beta \int_{\mathbb{R}} \frac{f(x)-f(y)}{x-y} d\mu_{V,\beta}(y) - V'(x) f(x). \tag{38}$$

This operator is well-known to appear in this context, see [BG13a, BG13b, Shc13, BLS18]. Inverting this operator and obtaining sufficiently good controls on it with respect to appropriate norms are the technical requirements for analyzing the loop equations. This was achieved in [BFG15] relying on techniques developed in [Tri85] and we will present the idea to demonstrate its inversion. Once it is inverted, equation (37) can be recast as:

$$\langle f \rangle_{\mathcal{L}_N} = -\frac{\beta-2}{N} \langle \mathcal{K}^{-1}[f]' \rangle_{\mu_{V,\beta}} - \frac{\beta-2}{N} \langle \mathcal{K}^{-1}[f]' \rangle_{\mathcal{L}_N} - \frac{\beta}{2} \left\langle \mathcal{D} \circ \mathcal{K}^{-1}[f] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N} . \tag{39}$$

For the following, it is also useful to state the n-th loop equation. For that, we need several notations, let f_n a function of n variables then we set:

$$\begin{aligned} \mathcal{K}_1^{-1}[f_n](x_1,\dots,x_n) &\overset{(\text{def})}{=} \mathcal{K}^{-1}\left[f_n(.,x_2,\dots,x_n)\right](x_1), \\ \mathcal{D}_1[f_n](x_1,\dots,x_{n+1}) &\overset{(\text{def})}{=} \mathcal{D}\left[f_n(.,x_3,\dots,x_{n+1})\right](x_1,x_2) \\ \Theta^{(a)}[f_n](x_1,\dots,x_{n-1}) &\overset{(\text{def})}{=} f_n(x_1,\dots,x_{a-1},x_1,x_a,\dots,x_{n-1}) \end{aligned} \tag{40}$$

With the operators defined previously, we can state the n-th loop equation for a bounded smooth function f_n of $n>1$ variables:

$$\begin{aligned} \langle f_n \rangle_{\overset{n}{\otimes}\mathcal{L}_N} = \frac{1}{N^2} \sum_{a=2}^{n} \langle \Theta^{(a)} \circ \mathcal{K}_1^{-1}[\partial_a f_n] \rangle_{\overset{n-1}{\otimes}\mathcal{L}_N} + \frac{\beta}{2} \langle \mathcal{D}_1 \circ \mathcal{K}_1^{-1}[f_n] \rangle_{\overset{n+1}{\otimes}\mathcal{L}_N} + \frac{1-\beta}{N} \langle \partial_1 \mathcal{K}_1^{-1}[f_n] \rangle_{\mu_{V,\beta}\overset{n-1}{\otimes}\mathcal{L}_N} \\ + \frac{1}{N^2} \sum_{a=2}^{n} \langle \Theta^{(a)} \circ \mathcal{K}_1^{-1}[\partial_a f_n] \rangle_{\mu_{V,\beta}\overset{n-2}{\otimes}\mathcal{L}_N} + \frac{1-\beta}{N} \langle \partial_1 \mathcal{K}_1^{-1}[f_n] \rangle_{\overset{n}{\otimes}\mathcal{L}_N} . \end{aligned} \tag{41}$$

One of the main feature of the above equation is that it relates n-linear statistics with $n+1$, $n-1$ and $n-2$ linear statistics. An other point is that the inverse of the master operator appears in every linear statistics of the RHS. But, the method of the loop equations analysis consists in estimating every term of the RHS. To identify if a term is, with respect to a fixed order $O(N^{-K})$, either a remainder or a contribution, one needs a bound on the linear-statistics called an *a priori bound*.

I.4.2 The a priori bound

This bound takes the following form:

$$| \langle f_n \rangle_{\overset{n}{\otimes}\mathcal{L}_N} | \leq \frac{C}{N^{\frac{n}{2}-\varepsilon}} \| \mathcal{R}_\kappa[f_n] \|_{W_n^\infty(\mathbb{R}^n)}, \tag{42}$$

where $C>0$ is a positive constant indedpendent of f_n and where the W_n^∞-norm and the regularization $\mathcal{R}_\kappa$, $\kappa>0$, are defined by:

$$\| g \|_{W_n^\infty(\mathbb{R}^n)} \overset{(\text{def})}{=} \max_{0\leq i\leq n} \| g^{(i)} \|_{L^\infty(\mathbb{R}^n)}, \qquad \mathcal{R}_\kappa[g](x_1,\dots,x_n) \overset{(\text{def})}{=} \prod_{a=1}^{n} e^{-\kappa V(x_a)} . g(x_1,\dots,x_n). \tag{43}$$

This bound can be proven by *concentration of measure* and was first shown in [BdMPS95] and is legit for all function f_n such that $\mathcal{R}_\kappa[f_n] \in W_n^\infty(\mathbb{R}^n)$.

Under convenient assumptions on the growth of the potential, it is possible to bound the p.d.f. $\mathcal{Z}_N^{(\beta)}[V]^{-1}p_N^{V,\beta}$, introduced (10), as follows:

$$\forall(\lambda_1,\dots,\lambda_N)\in\mathbb{R}^N,\ \mathcal{Z}_N^{(\beta)}[V]^{-1}p_N^{V,\beta}(\lambda_1,\dots,\lambda_N)\leq \exp\left\{-N^2D^2[\widehat{\mu}_{N,\mathrm{reg}},\mu_{V,\beta}]+O(N\log N)\right\} \quad (44)$$

where the pseudo-distance D^2 (possibly infinite) is defined for all $\mu,\nu\in\mathcal{M}_1(\mathbb{R})$ by:

$$D^2(\mu,\nu)\overset{(\mathrm{def})}{=}-\iint_{\mathbb{R}^2}\log|x-y|d(\mu-\nu)(x)d(\mu-\nu)(y)=\int_0^{+\infty}\frac{\left|\mathcal{F}[\mu-\nu](t)\right|^2}{t}dt, \quad (45)$$

where $\mathcal{F}$ denotes the Fourier transform of a measure. One can notice that for atomic measures, this quantity is infinite. Therefore, if one wants to prove that $\mathcal{Z}_N^{(\beta)}[V]^{-1}p_N^{V,\beta}(\lambda_1,\dots,\lambda_N)$ is small as soon as $\widehat{\mu}_N$ is far from $\mu_{V,\beta}$ with respect to D, it is convenient to regularize $\widehat{\mu}_N$ into $\widehat{\mu}_{N,\mathrm{reg}}$ following a procedure developed in [MMS14]. The regularized empirical measure $\widehat{\mu}_{N,\mathrm{reg}}$ consists of a convolution of $\widehat{\mu}_N$ with the uniform measure on very small intervals, $[0,e^{-N}]$ for example. Thus, since $\widehat{\mu}_N$ is close to $\widehat{\mu}_{N,\mathrm{reg}}$, $\mathcal{Z}_N^{(\beta)}[V]^{-1}p_N^{V,\beta}(\lambda_1,\dots,\lambda_N)$ is small as soon as $\widehat{\mu}_N$ is far from $\mu_{V,\beta}$. To start, one can show that for a general bounded continuous function f_n,

$$\langle f_n\rangle_{\overset{n}{\otimes}\mathcal{L}_N}=\langle f_{n|c}\rangle_{\overset{n}{\otimes}\mathcal{L}_N}+\text{exponential errors}$$

where $f_{n|c}\overset{(\mathrm{def})}{=}f_n.\phi_c$ with ϕ_c a compactly supported smooth function whose support is sufficiently large. We can thus suppose in the rest of this section that f_n is a compactly supported function. To conclude on the *a priori* bound, one can use the fact that $\mathcal{L}_N=(\widehat{\mu}_N-\widehat{\mu}_{N,\mathrm{reg}})+(\widehat{\mu}_{N,\mathrm{reg}}-\mu_{V,\beta})$. By choosing a sufficiently close regularization $\widehat{\mu}_{N,\mathrm{reg}}$, one can see that for $f_n\in W_n^\infty(\mathbb{R}^n)$:

$$\langle f_n\rangle_{\overset{n}{\otimes}\mathcal{L}_N}=\langle f_n\rangle_{\overset{n}{\otimes}\widehat{\mu}_{N,\mathrm{reg}}-\mu_{V,\beta}}+\text{ exponential errors.}$$

Furthermore, for the linear statistic on the RHS, particles in the event:

$$\Omega\overset{(\mathrm{def})}{=}\{D^2(\mu_{N,\mathrm{reg}},\mu_{V,\beta})>M/N^{1-\varepsilon}\},\qquad\qquad \varepsilon>0$$

(such that the empirical measure is far from the equilibrium measure) will only yield exponential errors $O(e^{-cMN^{1+\varepsilon}})$ for a constant $c>0$ because of (44).

$$\langle f_n\rangle_{\overset{n}{\otimes}\widehat{\mu}_{N,\mathrm{reg}}}=\left|\mathbb{E}_N^{V,P}\left[\mathbf{1}_{\Omega^c}\int_{\mathbb{R}^n}f_n(x_1,\dots,x_n)\prod_{i=1}^n d(\widehat{\mu}_{N,\mathrm{reg}}-\mu_{V,\beta})(x_i)\right]\right|+\text{ exponential errors.}$$

By Plancherel formula and Cauchy-Schwarz inequality, one gets the following inequalities:

$$\begin{aligned}&\left|\mathbb{E}_N^{V,P}\left[\mathbf{1}_{\Omega^c}\int_{\mathbb{R}^n}f_n(x_1,\dots,x_n)\prod_{i=1}^n d(\widehat{\mu}_{N,\mathrm{reg}}-\mu_{V,\beta})(x_i)\right]\right|\\&\qquad\leq\left(\int_{\mathbb{R}^n}\left|\mathcal{F}\left[f_n\right](\varphi_1,\dots,\varphi_n)\right|^2\prod_{a=1}^n|\varphi_a|\frac{d^n\varphi}{(2\pi)^n}\right)^{1/2}.\mathbb{E}_N^{V,P}\left[\mathbf{1}_{\Omega_N^c}2^{\frac{n}{2}}D^n(\widehat{\mu}_{N,\mathrm{reg}},\mu_{V,\beta})\right]\\&\leq\frac{(2M)^{\frac{n}{2}}}{N^{\frac{n}{2}(1-\varepsilon)}}\left(\int_{\mathbb{R}^n}\left|\mathcal{F}\left[f_n\right](\varphi_1,\dots,\varphi_n)\right|^2\left\{1+\left(\sum_{a=1}^n|\varphi_a|^2\right)^{1/2}\right\}^n\frac{d^n\varphi}{(2\pi)^n}\right)^{1/2}\leq(2M)^{\frac{n}{2}}\frac{\|f_n\|_{H^{n/2}(\mathbb{R}^n)}}{N^{\frac{n}{2}(1-\varepsilon)}}.\end{aligned}$$

Finally, by using the fact that f_n is a compactly supported function, we can bound the $H^{n/2}$-norm of f_n by $C\|\mathcal{R}_\kappa[f_n]\|_{W_n^\infty(\mathbb{R})}$ with $C>0$ independent of f_n. Note that, in order to control the exponential errors, one had to rely on the fact that $\|\mathcal{R}_\kappa[f_n]\|_{W_1^\infty(\mathbb{R}^n)}$ is finite. Since $\varepsilon>0$ can be taken as small as needed, this establishes (42).

I.4.3 The inversion the of the master operator

To estimate each term in (39) and (41) *via* the *a priori* bound, it is necessary to verify that it is applicable *i.e.* that the test-function f_n of every linear statistics involved in these equations are such that $\mathcal{R}_\kappa[f_n] \in W_n^\infty$-norm for n sufficiently large. In order to prove that, it is necessary to obtain controls on the operators involved in (41). One then needs to show the continuity of the operators $\mathcal{K}^{-1}$, $\mathcal{D}\circ\mathcal{K}^{-1}$ and $\Theta^{(a)}\circ\mathcal{K}^{-1}$ with respect to the norm $\|\mathcal{R}_\kappa[.]\|_{W_1^\infty(\mathbb{R}^n)}$. Since the controls on the other operators can be proven quite directly, the only non-trivial thing to prove is the continuity of $\mathcal{K}^{-1}$. It follows from the integral representation of $\mathcal{K}^{-1}[f]$ that one can show when inverting $\mathcal{K}$, see [BFG15, Lemma 3.2]. Indeed, once an integral representation for $\mathcal{K}^{-1}[f]$ is found, the standard tools of functional analysis allow one to obtain the desired continuity results.

It can be derived as follows, let g be a smooth function such that $\mathcal{K}[f]=g$. One first notes that it is possible to recast $\mathcal{K}[f](x)$ by

$$\mathcal{K}[f](x) = -V'_{\text{eff}}(x)f(x) - \beta \fint_{\mathbb{R}} \frac{f(y)}{x-y} d\mu_{V,\beta}(y), \qquad V_{\text{eff}}(x) \overset{\text{(def)}}{=} V(x) - \beta\int_{\mathbb{R}} \log|x-y| d\mu_{V,\beta}(y) - C_{V,\beta}.$$

The effective potential appearing here is such that, by the Euler-Lagrange equations (14), one has $V_{\text{eff}}(x)=0$ on $\text{supp}\mu_{V,\beta}$ and is positive elsewhere. In the following, we assume that the potential V leads to the one-cut regime and set supp $\mu_{V,\beta}=[a,b]$. Let $x\in]a,b[$, one gets the following singular integral equation:

$$-\beta \fint_{\mathbb{R}} \frac{f(y)}{x-y} d\mu_{V,\beta}(y) = g(x). \tag{46}$$

By setting:

$$\mathsf{F}(z) \overset{\text{(def)}}{=} \frac{-\beta}{(z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}} \int_a^b \frac{f(y)}{x-y} d\mu_{V,\beta}(y),$$

where we use the principal determination of the logarithm, one can verify from (46) that F is the unique solution of the following scalar RHP:

- $\mathsf{F}\in\mathcal{O}(\mathbb{C}\backslash]a,b[)$ and admits continuous $\pm$ boundary values on $]a,b[$.
- For all $s\in]a,b[$, $\mathsf{F}_+(s)-\mathsf{F}_-(s) = \dfrac{-2\mathrm{i}g(s)}{\sqrt{(b-s)(s-a)}}$.
- $\mathsf{F}(z)=O(z^{-2})$ as $z\to\infty$.

Provided that:

$$\int_a^b \frac{g(u)}{\sqrt{(b-u)(u-a)}} du = 0, \tag{47}$$

the unique solution of the above RHP is known. This leads to:

$$\mathsf{F}(z) = \int_a^b \frac{du}{2\mathrm{i}\pi} \frac{-2\mathrm{i}g(u)}{(u-z)\sqrt{(b-u)(u-a)}}.$$

If constraint (47) is not satisfied, $\mathsf{F}(z)\neq O(z^{-2})$ as $z\to\infty$. Finally, from the definition of F, it can be seen that:

$$\mathsf{F}_+(z)+\mathsf{F}_-(z) = \frac{2\pi\beta f(s)\rho_{V,\beta}(s)}{\sqrt{(b-s)(s-a)}}.$$

This leads to the following expression for $f(s)$ with $s\in[a,b]$:

$$f(s) = -\frac{\beta}{\rho_{V,\beta}(s)} \fint_a^b \frac{\sqrt{(b-s)(s-a)}}{\sqrt{(b-u)(u-a)}} \frac{g(u)}{u-s} \frac{du}{2\pi^2} \overset{\text{(def)}}{=} -\frac{\beta}{\rho_{V,\beta}(s)} \mathcal{V}[g](s). \tag{48}$$

For $s \notin [a,b]$, one can find that:

$$f(s) = -\frac{g(s) - 2\pi\mathcal{V}[g](s)}{V'(s) - 2\pi\mathcal{V}[V'](s)}. \tag{49}$$

From these integral representations, it is not hard to read the regularity as well as obtaining a control of the form $\|\mathcal{R}_\kappa \mathcal{K}^{-1}[g]\|_{W^\infty_n(\mathbb{R})} \leq C\|\mathcal{R}_{\kappa'}[g]\|_{W^\infty_{n+2}(\mathbb{R})}$ for $\kappa' > \kappa > 0$ which is exactly what is needed to conduct the loop equation analysis.

I.4.4 The loop equations analysis

In order to identify the first leading order term, which is of the order N^{-1} for $\langle f \rangle_{\mathcal{L}_N}$, we start by examining (39). The first term of the RHS of (39) clearly contributes to this order, while the second is evidently a $o(N^{-1})$. For the third one, the *a priori* bound is only able to say that:

$$\left\langle \mathcal{D} \circ \mathcal{K}^{-1}[f] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N} = O(N^{-1+2\varepsilon}).$$

This is not precise enough so we need to investigate the loop equation at level 2. Setting $f_2 \overset{\text{(def)}}{=} \mathcal{D} \circ \mathcal{K}^{-1}[f]$, it reads:

$$\begin{aligned}\langle f_2 \rangle_{\overset{2}{\otimes}\mathcal{L}_N} = \frac{1}{N^2}\left\langle \Theta^{(2)} \circ \mathcal{K}_1^{-1}[\partial_2 f_2] \right\rangle_{\mathcal{L}_N} + \frac{\beta}{2}\left\langle \mathcal{D}_1 \circ \mathcal{K}_1^{-1}[f_2] \right\rangle_{\overset{3}{\otimes}\mathcal{L}_N} + \frac{1-\beta}{N}\left\langle \partial_1 \mathcal{K}_1^{-1}[f_2] \right\rangle_{\mu_{V,\beta}\otimes\mathcal{L}_N} \\ + \frac{1}{N^2}\left\langle \Theta^{(2)} \circ \mathcal{K}_1^{-1}[\partial_2 f_2] \right\rangle_{\mu_{V,\beta}} + \frac{1-\beta}{N}\left\langle \partial_1 \mathcal{K}_1^{-1}[f_2] \right\rangle_{\overset{2}{\otimes}\mathcal{L}_N}.\end{aligned} \tag{50}$$

By applying the *a priori* bound (42) to the first term of the RHS of (50), we get the following estimate:

$$\frac{1}{N^2}\left\langle \Theta^{(2)} \circ \mathcal{K}_1^{-1}[\partial_2 f_2] \right\rangle_{\mathcal{L}_N} = O\left(N^{-\frac{5}{2}+\varepsilon}\right). \tag{51}$$

Hence by picking $\varepsilon > 0$ small enough, we can conclude that it is negligible and doesn't contribute to the expansion up to $o(N^{-1})$. The same application of the *a priori* bound for the second, third and fifth term of (50) predicts the respective estimates $O\left(N^{-\frac{3}{2}+3\varepsilon}\right)$, $O\left(N^{-\frac{3}{2}+\varepsilon}\right)$ and $O\left(N^{-2+2\varepsilon}\right)$. The fourth one is a deterministic term so we can finally conclude that:

$$\langle f_2 \rangle_{\mathcal{L}_N \otimes \mathcal{L}_N} = O\left(N^{-\frac{3}{2}+2\varepsilon}\right), \tag{52}$$

and hence that:

$$\langle f \rangle_{\mathcal{L}_N} = -\frac{\beta-2}{N}\left\langle \mathcal{K}^{-1}[f]' \right\rangle_{\mu_{V,\beta}} + O\left(N^{-\frac{3}{2}+\varepsilon}\right). \tag{53}$$

This allows us to improve the *a priori* bound into an *a posteriori* bound for the one linear statistics. The 1-linear statistic are indeed more precisely a $O(N^{-1})$ rather than an imprecise $O(N^{-\frac{1}{2}+\varepsilon})$.

We now explain how to get the subleading order of $\langle f \rangle_{\mathcal{L}_N}$ namely its AE up to $o(N^{-2})$. The first loop equation (39) contains the 1-linear statistic $-\frac{\beta-2}{N}\left\langle \mathcal{K}^{-1}[f]' \right\rangle_{\mathcal{L}_N}$. By (53), we know its AE up to $o(N^{-2})$, namely:

$$-\frac{\beta-2}{N}\left\langle \mathcal{K}^{-1}[f]' \right\rangle_{\mathcal{L}_N} = \frac{(\beta-2)^2}{N^2}\left\langle \mathcal{K}^{-1}\left[\mathcal{K}^{-1}[f]'\right]' \right\rangle_{\mathcal{L}_N} + o(N^{-2}). \tag{54}$$

It thus remains to obtain the AE of $\left\langle \mathcal{D} \circ \mathcal{K}^{-1}[f] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}$ up to $o(N^{-2})$. The first term in (50) is clearly a $o(N^{-2})$. So is the fifth term by (52). The third term is a 1-linear statistic preceded by a factor N^{-1} so by (53), we know its AE up to $o(N^{-2})$. The fourth one is also clearly a contribution of the right order so we collect it. The application of the *a priori* bound for the 3-linear statistic

$\frac{\beta}{2}\langle \mathcal{D}_1 \circ \mathcal{K}_1^{-1}[f_2]\rangle_{\overset{3}{\otimes}\mathcal{L}_N}$ yields an imprecise $O(N^{-\frac{3}{2}+3\varepsilon})$. To capture more precisely its N-behavior, it is necessary to investigate the third loop equation.

Skipping some details, we will repeat the same analysis in this equation. The *a posteriori* bound on the 1-linear statistic and the *a priori* bound on the 2 and 3-linear statistics will be enough to show that all the terms will be $o(N^{-2})$ except for the 4-th linear statistic:

$$\frac{\beta}{2}\langle \mathcal{D}_1 \circ \mathcal{K}_1^{-1} \circ \mathcal{D}_1 \circ \mathcal{K}_1^{-1}[f_2]\rangle_{\overset{4}{\otimes}\mathcal{L}_N}.$$

The latter is estimated by the *a priori* bound as a $O(N^{-2+4\varepsilon})$ which is too imprecise. It will thus be necessary to investigate the fourth loop equation to improve this estimate and get the right order. In this fourth equation, the estimates we currently have on the k-linear statistics for $k \in [\![1,5]\!]$ wil be precise enough to show that all these terms are of order $O(N^{-\frac{5}{2}+5\varepsilon})$. This is then enough to conclude to conclude about the AE of $\langle f\rangle_{\mathcal{L}_N}$ up to $o(N^{-2})$ and get the following improved *a posteriori* bounds:

$$\langle f_4\rangle_{\overset{4}{\otimes}\mathcal{L}_N} = O\left(N^{-\frac{5}{2}+5\varepsilon}\right), \qquad \langle f_3\rangle_{\overset{3}{\otimes}\mathcal{L}_N} = O\left(N^{-\frac{5}{2}+5\varepsilon}\right), \qquad \langle f_2\rangle_{\overset{2}{\otimes}\mathcal{L}_N} = \frac{c}{N^2} + O\left(N^{-\frac{5}{2}+5\varepsilon}\right),$$

where c comes from collecting the contributions coming from the third and fourth terms in (50). This allows us to find the AE of $\langle f\rangle_{\mathcal{L}_N}$ up to $O(N^{-\frac{5}{2}+5\varepsilon})$. This machinery allows one to push the AE of k-linear statistics up to any order by investigating enough loop equations and at each step improving the *a priori* bounds into *a posteriori* bounds. This is enough to conclude about the AE in powers of N^{-1} of k-linear statistics up to any order and this for any $k \geq 1$. In particular, the loop equations analysis provides an AE for the 1-linear-statistic involved in (32), which takes the following form for all $M > 0$,

$$\langle V - V_{\mathrm{ref}}\rangle_{\mathcal{L}_N}^{V_t} = \sum_{k=1}^{M} \frac{c_k(V - V_{\mathrm{ref}}, V_t)}{N^k} + O\left(N^{-(M+1)}\right). \tag{55}$$

I.4.5 Integration of the asymptotic expansion of linear statistics

The last step of this method consists of integrating this AE as in (32) in order to deduce the AE for $\mathcal{Z}_N^{(\beta)}[V]$. To achieve this, one needs to be able to make the integrals of each coefficient well-defined. Indeed, using the notations of the beginning of this Subsection, one gets:

$$\int_0^1 \langle V - V_{\mathrm{ref}}\rangle_{\widehat{\mu}_N}^{V_t}\, dt = \int_0^1 \langle V - V_{\mathrm{ref}}\rangle_{\mu_{V_t,\beta}}\, dt + \sum_{k=1}^{M} \frac{\int_0^1 c_k(V - V_{\mathrm{ref}}, V_t)dt}{N^k} + \int_0^1 O(N^{-(M+1)})dt. \tag{56}$$

These integrals might not be well-defined and could even be infinite if the corresponding integrands are not integrable. However, it can be shown that each coefficient $c_k(V - V_{\mathrm{ref}}, V_t)$ is a finite sum of linear statistics with respect to products of $\mu_{V_t,\beta}$. The kind of result that is needed here is a continuity property for the map $t \in [0,1] \mapsto \mu_{V_t,\beta}$. Following [BG13a, Lemma 5.1], one can show that starting from a potential V_{ref} such that the corresponding equilibrium measure $\mu_{V_{\mathrm{ref}},\beta}$ has the same support as $\mu_{V,\beta}$, the equilibrium measure associated with the interpolating potential V_t is given by:

$$\mu_{V_t,\beta} = t\mu_{V,\beta} + (1-t)\mu_{V_{\mathrm{ref}},\beta}. \tag{57}$$

This can be proven by using that the characterization of the equilibrium measure (14), which is affine in $\mu_{V,\beta}$ and V. With this result, one can integrate the coefficients $t \mapsto c_k(V - V_{\mathrm{ref}}, V_t)$ as well as the remainder appearing in (56). The latter is composed of all the linear statistics that we were able to neglect and can be controled by:

$$|O(N^{-(M+1)})| \leq C_t \frac{\|\mathcal{R}_\kappa[V - V_{\mathrm{ref}}]\|_{W^\infty_{c(M)}(\mathbb{R})}}{N^{M+1}},$$

where $\mathcal{R}_\kappa$ is the regularization appearing in (42), C_t is a t-dependent constant composed of the continuity constants of the different operators involved in the loop equations and $c(M)$ is a large enough integer. By the interpolating form of $\mu_{V_t,\beta}$, it is straightforward to show that these constants are continuous with respect to t, yielding a finite integral. This results in the fact that the integral of the remainder $\int_0^1 O(N^{-(M+1)})dt$ is itself a remainder $O(N^{-(M+1)})$, allowing us to conclude about (34). This constitutes the method of the analysis of the loop equations.

We now introduce the model that will be the main focus of Chapters 1 and 2, namely the β-ensembles at high temperature.

I.5 The β-ensembles at high temperature

I.5.1 The model

The *β-ensembles at high temperature* is described by the following probability measure

$$d\mathbb{P}_N^{V,P}(\lambda_1,\dots,\lambda_N) \overset{(\text{def})}{=} p_N^{V,P}(\lambda_1,\dots,\lambda_N)\frac{d^N\lambda}{\mathcal{Z}_N[V]} \tag{58}$$

with

$$p_N^{V,P}(\lambda_1,\dots,\lambda_N) \overset{(\text{def})}{=} \prod_{i<j}^N |\lambda_i-\lambda_j|^{\frac{2P}{N}}.\prod_{i=1}^N e^{-V(\lambda_i)}, \qquad \mathcal{Z}_N[V] \overset{(\text{def})}{=} \int_{\mathbb{R}^N} p_N^{V,P}(\lambda_1,\dots,\lambda_N)d^N\lambda. \tag{59}$$

The primary difference from the classical β-ensembles is the replacement of a constant $\beta \geq 0$ with an N-dependent vanishing coefficient of $2P/N$. In a statistical mechanics approach, this coefficient is interpreted as the inverse temperature. Therefore, when N is large, it corresponds to a high-temperature regime. It can also be viewed as a parameter that tunes the strength of the two-body interaction. Consequently, in this regime, the particles are less correlated compared to the β-constant regime. This statement will be formalized later by presenting results that assert the presence of disorder in this regime.

Another difference is that NV is replaced by V, which places the confining contribution on the same scale with respect to N as the repulsive interaction. Explicitly, $p_N^{V,P}$ can be rewritten as:

$$p_N^{V,P}(\lambda_1,\dots,\lambda_N) = e^{-N\mathcal{H}_N^P(\widehat{\mu}_N)} \quad \text{with} \quad \mathcal{H}_N^P(\mu) \overset{(\text{def})}{=} \int_{\mathbb{R}} V(x)d\mu(x) - 2P\iint_{x\neq y}\log|x-u|d\mu(x)d\mu(y).$$

We know, by Sanov's theorem 3.0.4 and the AE of the partition function of the classical β-ensemble, that the entropy[6], regardless of the potential, is of magnitude order N. In the classical regime, the Hamiltonian was scaled by N^2, making the entropy negligible in the description of the macroscopic behaviour of the system. Indeed in the classical regime, we have seen in Theorem I.3.1 that the empirical measure $\widehat{\mu}_N$, converges to an equilibrium measure $\mu_{V,\beta}$ which is the minimizer of a certain functional $\mathcal{E}_{V,\beta}$. This functional only depends on the potential V and the logarithmic interaction between the particles. However, since the leading scale in the high temperature regime is N, we will observe the entropy coming into play, which is one reason why this regime is particularly interesting.

I.5.2 Global behavior and fluctuations

As in the β-constant case, one is first interested in the global behavior of the particles, which can be understood by the study of the asymptotics of $\widehat{\mu}_N$. Similar to Theorem I.3.1, LDP provide tools to access this information. The following theorem holds:

[6]Indeed in[Shc13, Formula (1.20)] one can see the entropy appearing as a subleading term.

Theorem I.5.1 *The family of random measures $(\widehat{\mu}_N)_{N\geq 1}$ satisfies, in $\mathcal{M}_1(\mathbb{R})$ equipped with the weak topology, a $\mathrm{LDP}(N, I_V^P)$ with strictly convex, good rate function I_V^P given by:*

$$I_V^P(\mu) \overset{(\mathrm{def})}{=} \mathcal{E}_{V,P}(\mu) - \inf_{\nu\in\mathcal{M}_1(\mathbb{R})} \mathcal{E}_{V,P}(\nu). \tag{60}$$

The functional $\mathcal{E}_{V,P}$ is defined for all $\mu \in \mathcal{M}_1(\mathbb{R})$ by

$$\mathcal{E}_{V,P}(\mu) \overset{(\mathrm{def})}{=} \begin{cases} \displaystyle\int_{\mathbb{R}} \left(V(x) + \log \frac{d\mu(x)}{dx}\right) d\mu(x) - P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y) & \text{if } d\mu \ll dx, \\ +\infty & \text{otherwise.} \end{cases} \tag{61}$$

The unique minimizer is attained at a Lebesgue absolutely continuous probability measure $\mu_{V,P}$ which is called the equilibrium measure. Its p.d.f. is denoted by $\rho_{V,P}$ and $\mu_{V,P}$ is characterized by the following Euler-Lagrange equations:

$$V(x) - 2P\int_{\mathbb{R}} \log|x-y| d\mu_{V,P}(y) + \log \frac{d\mu_{V,P}(x)}{dx} \begin{cases} > C_{V,P} & \text{outside supp } \mu_{V,P} \\ = C_{V,P} & \mu_{V,P} - as \end{cases}. \tag{62}$$

Moreover, $\mu_{V,P}$ is supported on the whole real line. Finally, under $\mathbb{P}_N^{V,P}$, $\widehat{\mu}_N$ converges weakly towards $\mu_{V,P}$ almost surely and the free energy reads:

$$\lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}_N[V] = -\int_{\mathbb{R}} \left(V(x) + \log \frac{d\mu_{V,P}(x)}{dx}\right) d\mu_{V,P}(x) + P \iint_{\mathbb{R}^2} \log|x-y| d\mu_{V,P}(x) d\mu_{V,P}(y).$$

The LDP was established in [GZ19], see [CGZ14, 1.2] for a discussion about this regime. It was also proven in [BGP15], that for the Gaussian β-ensembles as long as $\beta \gg N^{-1}$, so for smaller temperatures than the regime of our interest, the structure of the LDP in Theorem I.3.1 remains the same, and entropy doesn't play any role. If the logarithmic interaction was not singular, this result would follow from a successive application of Sanov's theorem 3.0.4 and Varadhan's lemma 3.0.5. Overcoming the singularity of the logarithm is the main difficulty in this proof. The second part of the theorem, namely the study of the equilibrium measure $\mu_{V,P}$, was adressed in [GM22]. The authors established, in the continuation of [Spo20], a connection between $\mu_{V,P}$ and the *equilibrium measure* of the Toda chain $\nu_{V,P}^{\mathrm{Toda}}$, see Section I.6. The authors also showed a continuity result for $\mu_{V,P}$ with respect to the P-dependence.

Several comments have to be made about the previous theorem. First, the LDP holds at a lower speed compared to the classical regime, since the scaling of the Hamiltonian has been decreased from N^2 to N.

Secondly, the form of the rate function I_V^P given in (60), has also changed, and entropic effects start to kick in. These effects automatically make the equilibrium measure, absolutely continuous with respect to Lebesgue and supported on the whole real line. The latter is the main difference with the classical regime, where $\mu_{V,\beta}$ is compactly supported. Since the mathematical entropy is equal, by convention, to minus the physical entropy, minimizing I_V^P amounts to compromise between minimizing the energy with respect to μ

$$\int_{\mathbb{R}} V(x) d\mu(x) - P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y)$$

while at the same time, maximizing its physical entropy. Hence, $\mu_{V,P}$ is the result of a simultaneous competition between energy minimization and entropy maximization.

Finally, due to the presence of the entropy, the Euler-Lagrange equation has a different form than the one found in Theorem I.3.1. While the same equation in the classical β-ensemble was affine in

μ, it is non-linear in the high temperature regime. This has the effect of making way more difficult the understanding of the dependence of $\rho_{V,P}$ with respect to V. As explained in Subsection I.4.5, to perform the method of the loop equations analysis, one has to obtain a continuity result for $t \mapsto \rho_{V_t,P}$ where V_t is the interpolating potential defined in (31). This can be proven for $t \mapsto \rho_{V_t,\beta}$ in the classical regime without too much effort but the same proof doesn't work at all in the high temperature regime. Indeed, one cannot rely anymore on the linearity of the Euler-Lagrange equation (14) with respect to μ. We were however able to prove this result in this case, see Theorem I.8.6 relying on Banach fixed-point theorem. We will explain the basic idea of this proof in Subsection I.8.2.

Equation (62) can solved explicitly in the case of a quadratic potential [ABG12] and reads:

$$\rho_{V_G,P}(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \frac{1}{|\hat{f}_\alpha(x)|^2}, \qquad \hat{f}_\alpha(x) \overset{\text{(def)}}{=} \sqrt{\frac{P}{\Gamma(P)}} \int_0^{+\infty} t^{P-1} e^{-\frac{t^2}{2} + \mathrm{i}xt} dt.$$

It was also shown that this density interpolates between the case $P \to 0$, where one recovers the Gaussian distribution and $P \to +\infty$ where one recovers the semi-circle law. This phenomenon, discovered in [ABG12], is sometimes called the *Gauss-Wigner crossover* and was generalized to more general V [NT20a, Remark 2.1]. This density is also explicit in other models, namely the Jacobi and Laguerre ensembles, involving hypergeometric functions, see [Maz22]. This measure was also studied in the framework of Coulomb gases, it is called the *thermal equilibrium measure* in [AS22]. A general explicit form for $\rho_{V,P}$ can be obtained by taking the exponential of (62). This leads to the following epxression:

$$\rho_{V,P}(x) = \exp\left(-V(x) + 2P \int_{\mathbb{R}} \log|x-y| d\mu_{V,P}(y) + C_{V,P}\right) \qquad x - ae. \tag{63}$$

Furthermore, it can be shown that the logarithmic potential, as expressed above, exhibits the following asymptotic behavior as $|x| \to \infty$:

$$\int_{\mathbb{R}} \log|x-y| d\mu_{V,P}(y) = \log|x| + O(1/x).$$

This leads to the following asymptotic behavior for $\rho_{V,P}$:

$$\rho_{V,P}(x) \underset{|x|\to\infty}{\sim} |x|^{2P} \exp\left(-V(x) + C_{V,P}\right). \tag{64}$$

We thus observe that $\rho_{V,P}$ decays exponentially fast at infinity.

I.5.3 Fluctuations in the high temperature regime

Once the law of large numbers is established *i.e.* the convergence of the empirical measure, one becomes interested in the fluctuations around the limit, namely a CLT. For $N^{-1} \ll \beta \ll \log N$, it was shown in [BGP15] that for quadratic potentials $V_G(x) = x^2$ and smooth test functions h:

$$N\beta \int_{\mathbb{R}} h(x) d(\widehat{\mu}_N - \mu_{\mathrm{sc}})(x) \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \delta_{\int_{\mathbb{R}} h(t) d\nu_{x^2,2}(t)}$$

The measure $\nu_{x^2,2}$ is the one appearing in (24). This result motivates the understanding of the form of the fluctuations when $\beta = 2P/N$ for $P > 0$, namely the high temperature regime. In this regime, several results exist to describe the fluctuations of linear statistics. All of them involve a rescaling by $\sqrt{N}$ in order to observe the Gaussian fluctuations:

- for quadratic potentials and test-functions with polynomial growth was first proven in [NT18]. This result is a natural extension of [BGP15] as it allowed to deal with a higher temperature.

- In the circular case, for smooth enough test-functions and general smooth enough potential V, a CLT was proven in [HL21]. The authors successfully inverted the master operator and provided an explicit description of the limiting variance in terms of this inverse. This result is very general and allows for non-quadratic potentials. Furthermore, they demonstrated that the Gauss-Wigner crossover is valid at the level of the limiting variance.Specifically, they showed that as the parameter P, on which the limiting variance depends, approaches 0 (respectively, $+\infty$), the variance converges to that of the CLT for independent and iid particles (respectively, for the constant β case).
- On the real line with general potential V smooth enough and growing fast enough at infinity for bounded smooth enough functions, a CLT was shown in **[DGM23]**. This result, which is a contribution of the author of this manuscript and their co-author, is discussed in Chapter 1. It extends the techniques used in the circular case to a non-compact setting, specifically the real line. Additionally, we successfully inverted the master operator Ξ in this context and obtained an explicit description of the limiting variance $\phi \mapsto \sigma^2_{\beta-\mathrm{ens,HT}}(V,P,\phi)$.
- A CLT on the real line, polynomial potentials and test-functions with polynomial growth followed was established in [MM24]. This result, however, is limited to polynomial potentials and does not extend to other types of potentials. Nevertheless, it encompasses a broader class of test functions beyond merely sufficiently smooth and bounded functions.

To highlight the contribution of this thesis, we present our result explicitly:

Theorem I.5.2 *Assume that V satisfies Assumptions 1.1.1 and Assumption 1.1.2. Then for all $\phi \in \mathcal{C}^2(\mathbb{R})$ with ϕ, ϕ' and ϕ'' bounded such that $\int_{\mathbb{R}} \phi(x) d\mu_{V,P}(x) = 0$, we have the following convergence:*

$$\sqrt{N}\int_{\mathbb{R}} \phi(x) d(\widehat{\mu}_N - \mu_{V,P})(x) \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\left(0, \sigma^2_{\beta-\mathrm{ens,HT}}(V,P,\phi)\right) \tag{65}$$

where the limiting variance $\sigma^2_{\beta-\mathrm{ens,HT}}(V,P,\phi)$ is given by

$$\begin{aligned}\sigma^2_{\beta-\mathrm{ens,HT}}(V,P,\phi) \overset{(\mathrm{def})}{=} \langle \phi', \Xi^{-1}[\phi]\rangle_{L^2(\mu_{V,P})} &= \int_{\mathbb{R}} \left(\Xi^{-1}[\phi]'(x)^2 + V''(x)\Xi^{-1}[\phi](x)^2\right) d\mu_{V,P}(x) \\ &+ P\iint_{\mathbb{R}^2} \left(\frac{\Xi^{-1}[\phi](x) - \Xi^{-1}[\phi](y)}{x-y}\right)^2 d\mu_{V,P}(x) d\mu_{V,P}(y)\,. \end{aligned} \tag{66}$$

We will give more details about the proof in Subsection I.8.1. A common approach to proving this type of result, developped in [Joh98] and used in [BLS18], is based on the following convergence of Laplace transforms:

$$\mathbb{E}_N^{V,P}\left[\exp\left(\sqrt{N}t\int_{\mathbb{R}} \phi(x) d(\widehat{\mu}_N - \mu_{V,P})(x)\right)\right] \xrightarrow[N\to\infty]{} \exp\left(-\frac{t^2}{2}\sigma^2_{\beta-\mathrm{ens,HT}}(V,P,\phi)\right).$$

I.5.4 Local behavior

The local behavior of the particles have also been understood in this model, namely the convergence of local statistics. It was proven in [BGP15] that the local statistics $\sum_{i=1}^N \delta_{N(\lambda_i - E)}$ for any $E \in \mathbb{R}$ and when the λ_i are distributed according to the Gaussian β-ensemble at high temperature, converge to a Poisson point process with density $\rho_{V,P}(E) > 0$. An alternative proof was also presented later in [NT18]. This result was later generalized to general potential V in [NT20a]. Finally, this result was extended to more general geometric settings and interactions in [Lam21b]. These results are consistant with the fact that in the constant β-case, the bulk local statistics converges toward a Sine-β process. Indeed, it was proven in [AD14a] that this process converges toward a Poisson point process as β goes to 0.

Regarding the edge, the question of the local behavior was first raised in [AD14b] where it was shown that the Tracy Widom law TW_β, *i.e.* the properly rescaled limiting law of the fluctuations of the largest eigenvalue of the β-ensembles, converges to a Gumbel distribution as $\beta \to 0$. The latter describes the behavior of the maximum of multiple identically independently distributed particles. It was later proven in [Pak18] that for quadratic potentials and $\beta \ll \frac{1}{\log N}$ the edge of the spectrum *i.e.* the law of $\max_{1\leq i\leq N} \lambda_i$, when properly rescaled, converges to a Gumbel law. The convergence of the local statistics at the edge was also demonstrated in this paper. At the level of stochastic operators, this convergence was also shown to hold [DL22]. This result was finally generalized to $\beta N \to$ cste and for general V in [Lam21b]. The result goes as follows:

Theorem I.5.3 *For $V(x) = |x|^\alpha$, there exists $\eta_N > 0$ which behaves like $(\log N)^{\frac{1}{\alpha}}$ such that the following convergence holds:*

$$\alpha \eta_N^{\alpha-1} \left(\max_{i=1,\dots,N} x_i - \eta_N \right) \xrightarrow[N\to\infty]{\text{law}} \text{Gumbel}(0,1).$$

The precise result for $V(x) = |x|^\alpha$ yields the following formula for the speed η_N:

$$\eta_N = (\log N)^{1/\alpha} \left(1 + \frac{2P - \alpha + 1}{\alpha^2} \frac{\log\log N}{\log N} - \frac{\log(\alpha) - C_{V,P}}{\alpha \log N} \right)$$

with $C_{V,P}$ given as (62). In Chapter 1, we will use the localization of the spectrum for general V [Lam21b][7], in order to say that for ϕ a smooth enough test-function and with high probability:

$$\int_{\mathbb{R}} \phi(x) d\widehat{\mu}_N(x) \approx \int_{\mathbb{R}} \phi \mathbf{1}_{[-E_N, E_N]}(x) d\widehat{\mu}_N(x)$$

with $E_N \underset{N\to\infty}{\longrightarrow} +\infty$. This truncation argument will be crucial in showing the CLT for β-ensemble at high temperature **[DGM23]**.

I.5.5 The master operator and the differences with the classical regime

Due to the change in leading scale in the high temperature regime compared to the constant β regime (N *versus* N^2), the master operator $\mathcal{K}$, introduced in (38), governing the fluctuations of the empirical measure around the equilibrium measure $\mu_{V,P}$, also changes. The main difference, except that one has to replace $\mu_{V,\beta}$ by $\mu_{V,P}$ and add a recentring term, is that one has to add a differential term. It is defined as:

$$\Xi[\phi](x) \overset{\text{(def)}}{=} 2P \int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{x - y} d\mu_{V,P}(y) - V'(x)\phi(x) + \phi'(x) - 2P \int_{\mathbb{R}} \mathcal{H}[\phi \rho_{V,P}](y) d\mu_{V,P}(y)\,. \quad (67)$$

Above $\mathcal{H}$ denotes the Hilbert transform defined by:

$$\mathcal{H}[f](x) \overset{\text{(def)}}{=} \fint_{\mathbb{R}} \frac{f(y)}{y - x} dy$$

and satisfies

$$\left(\int_{\mathbb{R}} \log|. - y| d\mu_{V,P}(y) \right)'(x) = \mathcal{H}[\rho_{V,P}](x).$$

The first two terms, upon replacing $\mu_{V,P} \to \mu_{V,\beta}$ constitutes the master operator $\mathcal{K}$ introduced in (38). The fourth term is a constant recentring term such that $\int_{\mathbb{R}} \Xi[\phi](x) d\mu_{V,P}(x) = 0$, while the third term is the differential term in question. The effect of that term makes Ξ an unbounded operator which must be inverted on an appropriate domain, requiring some technicalities.

[7] The result for general V, although not presented as a result, can be directly derived from the arguments presented in this paper.

As explained in Subsection I.4.3, the main task when analyzing the loop equations, as will be the case in Chapter 2, is to invert this master operator Ξ and obtain continuity results with respect to appropriate norms. In practice, it is much easier to invert this operator $\mathcal{K}$ in the classical regime than Ξ in the high temperature regime. Another setup where the inversion of the master operator is more involved than the classical regime of β-ensembles is [BGK15].

In the high temperature regime, the inversion of the master operator, see Subsection I.4.3 for details, turns out to be different. Indeed, using the Euler-Lagrange equations (62), one can see that the analogue of the derivative of the effective potential, namely:

$$V_{\text{eff}}^{\text{HT}\prime}(x) = V'(x) + 2P\int_{\mathbb{R}} \frac{d\mu_{V,P}(y)}{y-x},$$

is not equal to 0 on the support of $\mu_{V,P}$ but rather equal to $-\dfrac{\rho'_{V,P}(x)}{\rho_{V,P}(x)}$ which doesn't vanish on an interval. This allows one to recast the operator Ξ as:

$$\Xi[\phi](x) = \phi'(x) + \frac{\rho'_{V,P}(x)}{\rho_{V,P}(x)}\phi(x) + 2P\mathcal{H}[\phi\rho_{V,P}](x) - 2P\int_{\mathbb{R}} \mathcal{H}[\phi\rho_{V,P}](y)d\mu_{V,P}(y)\,. \tag{68}$$

As it will be explained in Subsection I.8.1 and Chapter 1 Section 1.6, this operator can be inverted on some subspace H of the weighted Sobolev space [CL20, Zhi98]:

$$H^1(\mu_{V,P}) \overset{(\text{def})}{=} \Big\{u \in L^2(\mu_{V,P}), u' \in L^2(\mu_{V,P})\Big\}.$$

We were able to prove its inversion in **[DGM23]**:

Theorem I.5.4 (Inversion of the master operator) $\Xi : \mathfrak{D}(\Xi) \longrightarrow \mathsf{H}$ *is invertible. Furthermore there exists* $C > 0$*, such that for all* $f \in \mathsf{H}$*,*

$$\big\|\Xi^{-1}[f]\big\|_{L^2(\mu_{V,P})} \leq C\|f'\|_{L^2(\mu_{V,P})}. \tag{69}$$

Above $\mathfrak{D}(\Xi)$ denotes the *domain* of Ξ, a functional space which we don't explicit here. The idea of the proof of such a result is explained in Subsection I.8.1.

The advantage of such an invertible differential operator is that, in principle, the inverse is regularizing. This means that while $\Xi[\phi]$ is defined only for smooth enough function due to the differential term, which is encoded in the definition of $\mathfrak{D}(\Xi)$, the inverse $\Xi^{-1}[\psi]$ will be smooth and admit an integral representation. This integral representation is very useful to derive the continuity results that we need in order to conduct the loop equation analysis. We were able to prove these in **[DG24]**:

Theorem I.5.5 *Let* $n \geq 1$*, there exists a* $C > 0$ *such that for all* $f \in H^{n+1}(\mathbb{R})$*:*

$$\|\Xi^{-1}[f]\|_{H^n(\mathbb{R})} \leq C\|f\|_{H^{n+1}(\mathbb{R})}.$$

Moreover, there exists $C' > 0$*, such that for all* $g \in W^{\infty}_{n+1}(\mathbb{R})$*,*

$$\|\Xi^{-1}[g]\|_{W^{\infty}_{n}(\mathbb{R})} \leq C'\|g\|_{W^{\infty}_{n+1}(\mathbb{R})}.$$

We present the idea of the proof in Subsection I.8.2.

Finally, the differences between the constant β case and the high temperature regime are summarized in Figure I.2.

Besides being an interesting generalization of the classical-case, the high-temperature regime has connections with physics through the so-called *integrable systems*. One instance of such a physical system is the *Toda chain* and the next section is dedicated to it.

β constant	$\beta \sim N^{-1}$
LDP speed N^2 -energy	LDP speed N - energy +entropy
supp $\mu_{\text{eq}} = [a, b]$	supp $\mu_{\text{eq}} = \mathbb{R}$; exponential decay at ∞
CLT speed N	CLT speed $\sqrt{N}$
Sine-β/Airy β local statistics	Poisson statistics
Fluct$(\max x_i) \sim F_\beta$ (Tracy Widom)	Fluct$(\max x_i) \sim$ Gumbel
Master operator $\phi \to \mathcal{K}[\phi]$	$\phi \to \mathcal{K}[\phi] + \phi'$

Figure I.2: Summary of the differences between the classical regime and the high temperature regime for the β-ensembles.

I.6 The Toda chain

This section introduces the so-called *Toda chain*. This physical model is integrable and admits a link with the previous β-ensembles as it was discovered in [Spo20]. In that work, the author aims at establishing a so-called *hydrodynamic equation* for the Toda chain. The main result of Chapter 1 has a link with this research line. We will briefly describe the model and its main features, as well as its link with the result of Chapter 1.

I.6.1 Definition of the system

First introduced in 1967 in [Tod67], the *Toda chain* or *Toda lattice* is a one-dimensional many-body system composed of N particles at positions $\mathbf{q} \overset{\text{(def)}}{=} (q_i)_{1 \le i \le N}$ and momenta $\mathbf{p} \overset{\text{(def)}}{=} (p_i)_{1 \le i \le N}$. By imposing periodic boundary conditions, namely $q_{N+j} = q_j + \ell$ with $\ell \in \mathbb{R}$, these particles interact through a potential $\mathcal{U}(q_1, \dots, q_N) \overset{\text{(def)}}{=} \sum_{i=1}^N e^{q_i - q_{i+1}}$, so that the Hamiltonian of the system is given by:

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) \overset{\text{(def)}}{=} \frac{1}{2} \sum_{i=1}^N p_i^2 + e^{-r_i}, \qquad r_j \overset{\text{(def)}}{=} q_{j+1} - q_j.$$

An important feature of this Hamiltonian is the form of the two-body interaction. Particle j interacts only with particles $j-1$ and $j+1$, in a sense it is a "*nearest label interaction*". In this setup, if $\{.,.\}$ denotes the *Poisson bracket* defined for two functions $f(\mathbf{p}, \mathbf{q}, t)$, $g(\mathbf{p}, \mathbf{q}, t)$, by:

$$\{f, g\} \overset{\text{(def)}}{=} \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

the equations of motion read, with the notation $\dot{x}(t) \overset{\text{(def)}}{=} \frac{d}{dt} x(t)$:

- $\{q_k, p_j\} = \delta_{j,k}, \qquad \{q_k, q_j\} = \{p_k, p_j\} = 0 \qquad$ for all $j, k \in [\![1, N]\!]$,
- $\dot{q}_k = p_k, \qquad \dot{p}_k = e^{-r_{k-1}} - e^{-r_k} \qquad$ for all $k \in [\![1, N]\!]$.

Since

$$\frac{d}{dt} \sum_{i=1}^N p_i = \sum_{i=1}^N e^{-r_{j-1}} - e^{-r_j} = e^{-r_0} - e^{-r_N} = 0,$$

the total momentum is a conserved quantity together with the Hamiltonian. Flaschka [Fla74] and Manakov [Man75] independently showed, after it was conjectured in the literature following numerical simulations, that the system is integrable. Indeed, they showed that the system exhibits N integrals of motion, namely N local conserved quantities (other than the total momentum and energy).

The Toda chain can be thought either as the discretization of a continuous fluid $(q(x,t), p(x,t))$ or as a chain with a fixed number of particles.

I.6.2 Integrability and the Lax pair

We now introduce a new set of dependent variables, called the *Flaschka variables*, to express these conserved quantities:

$$a_k = e^{\frac{q_k - q_{k+1}}{2}}, \quad 1 \leq k \leq N.$$

With the $2N$ variables $\mathbf{a} \overset{\text{(def)}}{=} (a_k)_{k=1}^N$ and $\mathbf{p}$, the evolution equations given previously can be recast as:

$$\dot{a}_k = -(p_{k+1} - p_k)a_k, \quad 1 \leq k \leq N. \tag{70}$$

The integrability of the Toda chain can be seen *via* the existence of a *Lax pair* (L_N, B_N). The very existence of this pair of matrices is sufficient to exhibit N independent integrals of motion, as we will see below. We set:

$$L_N \overset{\text{(def)}}{=} \frac{1}{2}\begin{pmatrix} -p_1 & a_1 & 0 & \dots & & a_N \\ a_1 & -p_2 & a_2 & 0 & \dots & 0 \\ 0 & a_2 & -p_3 & a_3 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & a_{N-2} & -p_{N-1} & a_{N-1} \\ a_N & \dots & & 0 & a_{N-1} & -p_N \end{pmatrix}, \tag{71}$$

$$B_N \overset{\text{(def)}}{=} \begin{pmatrix} 0 & -a_1 & 0 & \dots & & a_N \\ a_1 & 0 & -a_2 & 0 & \dots & 0 \\ 0 & a_2 & 0 & -a_3 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & a_{N-2} & 0 & -a_{N-1} \\ -a_N & \dots & & 0 & a_{N-1} & 0 \end{pmatrix}.$$

Again, the dynamics of the system can be recast as $\dot{L}_N(t) = [B_N(t), L_N(t)]$. A Lax pair is simply a pair of matrices (or operators in some contexts) that satisfies this relationship.

The fact that such a pair provides the appropriate number of integrals of motion can be seen from the fact that $L_N(t)$ is *isospectral*, *i.e.* its eigenvalues do not evolve in time. Indeed, let $t, t_0 \geq 0$ and consider $U_N(t, t_0)$ as the solution of the Cauchy problem

$$\forall t \geq 0, \ \frac{d}{dt}U_N(t, t_0) = B_N(t)U_N(t, t_0), \qquad U_N(t_0, t_0) = I_N,$$

it can be seen that $M_N(t) \overset{\text{(def)}}{=} L_N(t)U_N(t, t_0)$ and $O_N(t) \overset{\text{(def)}}{=} U_N(t, t_0)L(t_0)$ verify the same Cauchy problem

$$\forall t \geq 0, \ \frac{d}{dt}P_N(t) = B(t)P_N(t), \qquad P_N(t_0) = L(t_0).$$

From this, we conclude that $L_N(t) = U_N(t, t_0)L_N(t_0)U_N(t, t_0)^{-1}$, thus the characteristic polynomial of L_N is constant over time and so do its roots, *i.e.* the eigenvalues $\lambda^N \overset{\text{(def)}}{=} (\lambda_i^N)_{i=1}^N$ of L_N.

In general, there is no hope to expect that λ_i^N of L_N are local quantities (in the sense that it would depend only on a susbset of the variables $\mathbf{a}$ and $\mathbf{b}$). Physically, what is more relevant is defining a set of *local conserved quantities* as follows:

$$Q^{[n],N} \overset{\text{(def)}}{=} \sum_{i=1}^N \left(\lambda_i^N\right)^n = \sum_{i=1}^N \left(L_N^n\right)_{i,i} = \sum_{i=1}^N Q_i^{[n],N}.$$

The first equality shows that $Q^{[n],N}$ is indeed a conserved quantity while the third one allows to show that its corresponding density is local:

$$Q_i^{[n],N} \overset{\text{(def)}}{=} \sum_{j_1=1}^N \dots \sum_{j_{n-1}=1}^N \left(L_N\right)_{i,j_1} \left(L_N\right)_{j_1,j_2} \dots \left(L_N\right)_{j_{n-1},i}. \tag{72}$$

Indeed, one can observe from (71) that the above equation involves only variables a_{i+k}, p_{i+k} for $k \in [\![0, n-1]\!]$ modulo N.

The three first local conserved quantities are interpreted as stretches, momenta and the energy density:

$$Q_i^{[0]} \overset{\text{(def)}}{=} r_i, \quad Q_i^{[1]} = p_i, \quad Q_i^{[2]} = \frac{1}{2}(p_i^2 + a_i^2 + a_{i-1}^2)$$

As we will see, these local conserved quantities can be used to describe the state of the system at long times.

I.6.3 Statistical description of the system and GGE

Most books on statistical mechanics begin with the assumption that the physical system of interest *thermalizes* in the sense that it reaches a thermal equilibrium state (a state of maximal entropy and equirepartied energy). The *Gibbs ensemble* or *canonical ensemble* (GE) is a description of a physical system at thermal equilibrium. It is represented by a probability distribution over the system's microstates. The probability of a microstate having an energy E_i is given by:

$$\mathbb{P}_{\text{GE}}(E_i) \overset{\text{(def)}}{=} \frac{e^{-\beta E_i}}{\sum_j e^{-\beta E_j}}$$

where $\beta \geq 0$ is the inverse temperature and acts as a Lagrange multiplier ensuring that this distribution maximizes entropy under fixed constraints [RAS15]. This distribution is invariant over the system's dynamics, since energy is conserved over time.

The long-time behavior of integrable systems can be modelled using the so-called *Generalized Gibbs Ensemble* (GGE), introduced in [Jay57]. Subsequent experiments [RDYO07, VR16] have shown that this model fits well with certain experiments. Like the GE, the GGE is a probabilistic description given by:

$$\mathbb{P}_{\text{GGE}}(Q_1, Q_2, \dots) \overset{\text{(def)}}{=} \frac{e^{-\sum_n \mu_n Q_n}}{Z_{\text{GGE}}}$$

where the Q_n's represent the *local conserved charges* of the system (including energy and many others). The Lagrange multipliers μ_n are, just like β, tuning parameters. They are often referred as *chemical potentials*. This measure is also invariant under the dynamic.

For the Toda chain, the GGE depends on a potential V, which grow faster than x^2 at infinity, and a parameter $P > 0$, and is given by:

$$d\mathbb{T}_N^{V,P}(p,a) \overset{\text{(def)}}{=} \frac{e^{-\text{Tr}(V(L_N))} \prod_{i=1}^N a_i^{2P-1} dp_i da_i}{Z_{N,\text{Toda}}^P[V]}. \tag{73}$$

Choosing a polynomial $V = \sum_{k=0}^n \mu_k X^k$, we find that $\text{Tr}\,(V(L_N)) = \sum_{k=0}^n \mu_k Q^{[k],N}$. Here, P is a parameter called the pressure which can be seen as the Lagrange multiplier corresponding to the constraint that $\sum_{j=0}^n r_j$ is conserved. For the choice of $V_G = X^2/2$, we obtain:

$$\text{Tr}\,(V_G(L_N)) = \frac{1}{2}\sum_{k=1}^N p_k^2 + \sum_{k=1}^N a_k^2,$$

and thus

$$d\mathbb{T}_N^{V_G,P}(a,p) = \frac{\prod_{k=1}^N e^{-p_k^2/2} dp_k \prod_{k=1}^N a_k^{2P-1} e^{-a_k^2} da_k}{Z_{N,\text{Toda}}^P[V_G]}.$$

This measure is the joint law of N standard Gaussian random variables and N χ variable with parameter $2P$, all independent of each other. Under this distribution, L_N is a random Jacobi matrix with extra coefficients in the top-right and bottom-left corners, which takes the following form:

$$L_N \overset{\text{law}}{=} \begin{pmatrix} \mathcal{N}(0,1) & \frac{\chi_{2P}}{\sqrt{2}} & 0 & \dots & & \frac{\chi_{2P}}{\sqrt{2}} \\ \frac{\chi_{2P}}{\sqrt{2}} & \mathcal{N}(0,1) & \frac{\chi_{2P}}{\sqrt{2}} & 0 & \dots & 0 \\ 0 & \frac{\chi_{2P}}{\sqrt{2}} & \mathcal{N}(0,1) & \frac{\chi_{2P}}{\sqrt{2}} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & \frac{\chi_{2P}}{\sqrt{2}} & \mathcal{N}(0,1) & \frac{\chi_{2P}}{\sqrt{2}} \\ \frac{\chi_{2P}}{\sqrt{2}} & \dots & & 0 & \frac{\chi_{2P}}{\sqrt{2}} & \mathcal{N}(0,1) \end{pmatrix}.$$

This matrix looks like the Jacobi matrix given by Dumitriu and Edelamn [DE02a] and forms the basis for comparing it with the β-ensembles.

Following partial proofs of [Spo20], the authors of [GM22] managed to establish rigorously the comparison between the GGE of the Toda chain and the β-ensembles at high temperature. They proved that the empirical measure of the eigenvalues of L_N satisfies, under $\mathbb{T}_N^{V+x^2/2,P}$ defined in (73), a LDP with good rate function admitting a unique minimizer $\nu_{V,P}^{\text{Toda}}$. As a consequence, they showed that this unique minimizer is then the weak-limit of this empirical measure. The meausre $\nu_{V,P}^{\text{Toda}}$ can be linked to the equilibrium measure of the β-ensembles at high temperature by the following theorem, which was first conjectured in [Spo20]:

Theorem I.6.1 *[GM22] For any bounded continuous function f, it holds that:*

$$\int_{\mathbb{R}} f(x) d\nu_{V,P}^{\text{Toda}}(x) = \partial_P \left(P \int_{\mathbb{R}} f(x) d\mu_{V,P}(x) \right). \tag{74}$$

Understanding the limiting behavior of the eigenvalues of L_N under the GGE allows in principle to justify a coarse-grained description for the Toda chain, namely the *hydrodynamic description.*

I.6.4 Hydrodynamic equation: the computation of the currents

For the Toda chain, it is believed that a hydrodynamic equation holds. A hydrodynamic description corresponds to a coarser description of the system which is supposed to hold at intermediate large time scales. For integrable systems, this description is called the *generalized hydrodynamic* description, see the review [Doy20, BBDV22, Ess22]. More explicitly, after many collisions between the particles which leads to mixing, the description of the system *via* Boltzmann equations becomes accurate. This description leads to the relaxation of the system and entropy maximization. This maximization holds at large spatial scales but also at intermediate mesoscopic scales. This scale corresponds to a partition of the space into smaller parts named *fluid cells* which are considered big thermodynamically speaking. Each cell is supposed to have reach equilibrium at long enough times. Indeed in each fluid cell, after a certain relaxation time, the local state is supposed to approach a GGE distribution.

In one-dimensional systems, the hydrodynamic equations have the following form:

$$\begin{cases} \partial_t \rho(x,t) + \partial_x.(v\rho)(x,t) = 0 \\ \partial_t v(x,t) + v(x,t)\partial_x v(x,t) = \dfrac{-1}{\rho(x,t)} \partial_x \mathsf{P}(\rho(x,t)) \end{cases} \tag{75}$$

where ρ is the local density of the fluid, v is its velocity and $\mathsf{P}(\rho)$ is the pressure of the fluid.

The basic assumption for a hydrodynamic description to hold is the *local entropy maximization* also called *local thermodynamic equilibrium* property. This means that at mesoscopic scale *i.e.* in fluid

cells, the distribution of the system maximizes the entropy. Formally, it states that the average of a local observable $o(x,t)$ at position x and time t can be, after long enough time, well approximated by the GGE-average $\langle o(0,0)\rangle_{\underline{\beta}(x,t)}$. The subscript $\underline{\beta}(x,t)$ denotes the Lagrange multiplier(s), which depends on x and t, which determines the local (Generalized) Gibbs distribution of maximum entropy. In the case of the Toda chain, let $o\left((a_{i+k},p_{i+k})_{-K\leq i\leq K}\right)$ be an observable depending only on finite subset of particles whose indices are in $[\![i-K,i+K]\!]\subset[\![1,N]\!]$ for some $i\in[\![1,N]\!]$. Let $x\in[0,1]$, $t\in\mathbb{R}$, such that $i/N \underset{N\to\infty}{\longrightarrow} x$, the local entropy maximization assumption reads explicitly:

$$\frac{1}{Nt}\int_0^{Nt} o\left(\left(a_{i+k}(s),p_{i+k}(s)\right)_{-K\leq i\leq K}\right)ds \underset{N\to\infty}{\longrightarrow} \mathbb{E}_N^{V_{x,t},P_{x,t}}\left[o\left(\left(a_{i+k},p_{i+k}\right)_{-K\leq i\leq K}\right)\right].$$

The conservation laws and the computations of the limit of the averages of the conserved charges and its corresponding currents represent the basis for deriving a generalized hydrodynamic description. Since the charges $Q_i^{[n],N}$, defined in (72), are local and Q_i^N is conserved, there must exist *local currents* $J_i^{[n],N}$, which satisfy the following conservation law:

$$\frac{d}{dt}Q_i^{[n]}(t)+\partial_i J_i^{[n]}(t)=0. \tag{76}$$

where we have set $\partial_i f(i)\overset{(\text{def})}{=} f(i+1)-f(i)$. By differentiating $Q_i^{[n]}$ with respect to t, we get:

$$\frac{d}{dt}Q_i^{[n],N}(t)=\left(B_N(t)L_N^n(t)-L_N^n(t)B_N(t)\right)_{i,i}=a_{i-1}(t)\left(L_N^n(t)\right)_{i,i-1}-a_i(t)\left(L_N^n(t)\right)_{i+1,i}.$$

The conservation law (76), corresponds to setting $J_i^{[n],N}(t)\overset{(\text{def})}{=} a_{i-1}(t)\left(L_N^n(t)\right)_{i,i-1}$. To derive a hydrodynamic equation of type (75), it is necessary to compute the limits of the average of the density of the conserved charges at time 0, $\lim_{N\to\infty}\mathbb{E}_N^{V,P}[Q_i^{[n],N}(0)]$, and of the average current at time 0 $\lim_{N\to\infty}\mathbb{E}_N^{V,P}[J_i^{[n],N}(0)]$ for generic parameters P and V. Indeed, to write (75), one has to make sense of the local density of particles ρ and its local velocity v. While the GGE-averages of the local conserved charge densities provide a definition for ρ, the GGE-averages of the currents are enough to define v.

To derive these limits one uses the links with the β-ensembles at high temperature. For the conserved charges, this can be done *via*:

$$\begin{aligned}\mathbb{E}_N^{V,P}[Q_i^{[n],N}(0)]=\mathbb{E}_N^{V,P}[\mathrm{Tr}L_N(0)^n]=\mathbb{E}_N^{V,P}\left[\int_{\mathbb{R}}x^n d\widehat{\mu}_N(x)\right]&\underset{N\to\infty}{\longrightarrow}\int_{\mathbb{R}}x^n d\nu_{V,P}^{\mathrm{Toda}}(x)\\&=\partial_P\left(P\int_{\mathbb{R}}x^n d\mu_{V,P}(x)\right),\end{aligned} \tag{77}$$

where $\mu_{V,P}$ is the equilibrium measure of the β-ensembles at high temperature introduced in Theorem I.5.1. The measure $\nu_{V,P}^{\mathrm{Toda}}$ is the weak-limit of the empirical measure of the spectrum of the Lax matrix L of the Toda chain when distributed by the GGE $\mathbb{T}_N^{V,P}$ and the last equality comes from Theorem I.6.1.

Furthermore, to deal with the currents, we define the matrix $L_N^{\downarrow}$ by

$$(L_N^{\downarrow})_{i,j}=(\delta_{j<i}+\delta_{i=1,j=N})(L_N)_{i,j}.$$

With this definition, the currents can be recast as $J_i^{[n],N}(0)=\mathrm{Tr}(L_N(0)L_N^{\downarrow}(0))$. The knowledge of the limiting behavior of the eigenvalues of L_N is not sufficient anymore to compute $\mathbb{E}_N^{V,P}\left[J_i^{[n],N}(0)\right]$.

However, it was argued in [Spo21, (6.10)] that the limit of the average of the currents was linked to the fluctuations of the spectrum of L_N by

$$\partial_P \lim_{N\to\infty} \mathbb{E}_N^{V,P}[J_i^{[n],N}(0)] = \lim_{N\to\infty} \frac{1}{N} \mathrm{Cov}_N^{V,P}\left(\mathrm{Tr}L_N(0), \mathrm{Tr}L_N^n(0)\right) \tag{78}$$

with covariance taken with respect to $\mathbb{T}_N^{V,P}$. Furthermore, by replacing the traces by integrals against the empirical measure, equation (78) can be rewritten as:

$$\partial_P \lim_{N\to\infty} \mathbb{E}_N^{V,P}[J_i^{[n],N}(0)] = \lim_{N\to\infty} \mathrm{Cov}_N^{V,P}\left(\sqrt{N}\int_{\mathbb{R}} x d\widehat{\mu}_N(x), \sqrt{N}\int_{\mathbb{R}} x^n d\widehat{\mu}_N(x)\right). \tag{79}$$

It was shown in [MM24], that a CLT holds for the linear statistics $\displaystyle\int_{\mathbb{R}} x^k d\widehat{\mu}_N(x)$, namely:

$$\int_{\mathbb{R}} x^k d(\widehat{\mu}_N - \nu_{V,P}^{\mathrm{Toda}})(x) \xrightarrow[N\to\infty]{\mathrm{law}} \mathcal{N}\left(0, \sigma^2_{\mathrm{Toda}}(V,P,x^k)\right).$$

Extracting the variance $\sigma^2_{\mathrm{Toda}}(V,P,x^k)$ allows one to compute the derivative of the limits of the currents. The authors showed that the limiting variances of the CLT's for the Toda chain and the high temperature regime of β-ensembles are linked through:

$$\sigma^2_{\mathrm{Toda}}(V,P,x^k) = \partial_P\left(P\sigma^2_{\beta-\mathrm{ens,HT}}(V,P,x^k)\right).$$

Unfortunately, they didn't manage to find an explicit form for any of these variances. As explained before, in **[DGM23]** we showed a central limit theorem for the linear statistics with bounded test functions in the β-ensembles at high temperature but with an explicit[8] form for the limiting variance $\sigma^2_{\beta-\mathrm{ens,HT}}(V,P,f)$ but our expression is only valid for bounded smooth enough functions. Nonetheless, we expect that this expression remains valid when $f(x) = x^k$. Our expectation is based on the fact that the CLT, at least for polynomial potentials V, holds for polynomial test-functions by [MM24]. The previous reasoning establishes a conjecture for the computation of $\sigma^2_{\mathrm{Toda}}(V,P,x^k)$ and thus for the averaged currents $\lim_{N\to\infty} \mathbb{E}_N^{V,P}[J_i^{[n],N}(0)]$.

Having reviewed the theory of β-ensembles at high temperature and presented one motivation for studying this regime, we now present another model which is more involved than the classical β-ensembles: the so-called *sinh*-model.

I.7 The sinh-model

I.7.1 Motivations from quantum integrable systems

A quantum integrable system is a class of quantum systems such that, in principle, one is able to compute most quantities of interest. There exists four types of such systems:

- quantum spin chains: 1+1 dimensional systems such as the XXX Heisenberg spin-chain
- quantum many-body systems: the Calogero-Sutherland model an example of this class.
- quantum field theories:for instance the sinh-Gordon in 1+1 dimensions. We are going to present it in the following.
- 2D lattice systems: the six-vertex model represents one of them.

[8] The reader might have understood by now that this word is very fuzzy.

It is given by a Hilbert space $\mathfrak{h}$, the state space and a Hamiltonian $\mathcal{H}$ which is, in this setting, a self-adjoint operator. In this quantum picture, observables are also realized as self-adjoint operators. on $\mathfrak{h}$.

The main quantities of interest in a quantum integrable system are the finite temperature correlation functions, namely thermal expectation values of products of local operators $\mathcal{O}_i(x_i, t_i)$ at inverse temperature $\beta \geq 0$ given by:

$$\langle \mathcal{O}_1(x_1,t_1)\dots\mathcal{O}_n(x_n,t_n)\rangle_\beta \overset{(\text{def})}{=} \frac{\operatorname{Tr}\left[\mathcal{O}_1(x_1,t_1)\dots\mathcal{O}_n(x_n,t_n)e^{-\beta\mathcal{H}}\right]}{\operatorname{Tr}\left[e^{-\beta\mathcal{H}}\right]}.$$

The main reason for the interest in these quantities is that they can be directly observed in experiments. Assuming that the Hamiltonian $\mathcal{H}$ possesses a unique *ground state* $|\Psi_{\text{gs}}\rangle$[9], namely an eigenstate for the lowest eigenvalue of $\mathcal{H}$, then the *zero temperature* (*i.e.* $\beta = \infty$) correlation functions are the simplest case among all $\beta \geq 0$. Indeed, in that case,

$$\langle \mathcal{O}_1(x_1,t_1)\dots\mathcal{O}_n(x_n,t_n)\rangle_\infty = \langle \Psi_{\text{gs}} \,|\, \mathcal{O}_1(x_1,t_1)\dots\mathcal{O}_n(x_n,t_n) \,|\, \Psi_{\text{gs}}\rangle\,.$$

This fact is analogous as in the classical setting of Gibbs ensembles where in the limit $\beta \to +\infty$, this measure simplifies in a uniform law over the ground states.

By picking, an orthonormal basis of $\mathfrak{h}$, $(|\Psi_i\rangle)_{i\geq 0}$, one is able to define the *form factors* for a local operator $\mathcal{O}$, which is a great tool to compute the correlation functions. They are defined as matrix elements for general operators $\mathcal{O}$ by:

$$\mathcal{F}_{\mathcal{O}}(\Psi_k,\Psi_l) \overset{(\text{def})}{=} \langle \Psi_k \,|\, \mathcal{O} \,|\, \Psi_l\rangle\,.$$

Under certain conditions, such as the translational invariance of the model, one is able to relate the n-point correlation functions at zero temperature to the form factors by a so-called form-factors expansion, namely for space-time points $(x_i,t_i)_{1\leq i\leq n}$:

$$\langle \mathcal{O}_1(x_1,t_1)\dots\mathcal{O}_n(x_n,t_n)\rangle_\infty = \sum_{i_1,\dots i_{n-1}\geq 0}\prod_{a=2}^{n}\left\{\mathcal{F}_{\mathcal{O}_a}(\Psi_{i_{a-1}},\Psi_{i_a})\right\}.\prod_{j=1}^{n-1} e^{\mathrm{i}(x_{a+1}-x_a)\widehat{P}_{i_a}-\mathrm{i}(t_{a+1}-t_a)\widehat{E}_{i_a}}.$$

The correlations functions of a model are often computed through a procedure called the *algebraic Bethe ansatz* or the *quantum inverse scattering method* (QISM), see [Yan67, Bax72, STF79]. However there exists a class of model such that this machinery is not feasible namely the models which don't possess a particular state called the *vacuum state*. The quantum Toda chain is one instance of these models. To overcome the failure of this method, a generalization was developed by Sklyanin in [Skl89, Skl95]. It allows one to determine the spectrum, eigenvectors and correlation functions in quantum integrable models and can be thought as a way to map multi-parameter multidimensional spectral problems into multi-parameter one-dimensional spectral problem which is in principle considerably easier. This method is called the *quantum separation of variables* method.

The method also allows one to obtain expressions for the form factors associated to generic operators $\mathcal{O}$ of the form:

$$\langle \Psi \,|\, \mathcal{O} \,|\, \Phi\rangle = \int_{\mathbb{R}^N} F_{\mathcal{O}}(y_1,\dots,y_N).\prod_{a=1}^{N}\overline{q_\Psi(y_a)}q_\Psi(y_a)d\mu_N(\mathbf{y}_N). \tag{80}$$

[9] In the sense that its lowest eigenvalue is non-degenerate.

where the q-function are solutions of some finite difference one dimensionnal equations and the measure μ_N admits the following general form:

$$d\mu_N(x_1,\dots,x_N) \overset{\text{(def)}}{=} \prod_{i<j}^{N} \frac{\sinh[\omega_1(x_i-x_j)]}{\omega_2}\frac{\sinh[\omega_2(x_i-x_j)]}{\omega_2}.\prod_{i=1}^{N} dx_i.$$

Writing equation (80) explicitly and supposing that $\overline{q_\Psi(y_a)}q_\Psi(y_a)$ can be written as $\exp(-W_{\Phi,\Psi}(y_a))$, one obtain an expression close to the partition function of the β-ensembles $\mathcal{Z}_N^{(\beta)}[W_{\Phi,\Psi}]$:

$$\mathfrak{Z}_N[W_{\Phi,\Psi},F_{\mathcal{O}}] \overset{\text{(def)}}{=} \int_{\mathbb{R}^N} F_{\mathcal{O}}(y_1,\dots,y_N)\prod_{i<j}^{N}\frac{\sinh[\omega_1(x_i-x_j)]}{\omega_2}\frac{\sinh[\omega_2(x_i-x_j)]}{\omega_2}\prod_{a=1}^{N} e^{-W_{\Phi,\Psi}(y_a)}d^Ny. \tag{81}$$

This method is very general and was applied to many models such as the sinh-Gordon quantum field theory in 1+1 dimensions. In the next Subsection, we briefly describe the model and state a conjecture about the form factors in this model.

I.7.2 An integrable quantum field theory: the sinh-Gordon model

Developed during the XX-*th* century by physicists like Feynman, Tomonaga, Dirac, Schwinger and Dyson among others, quantum field theory is the quantum analogue for classical field theory in classical physics. It is an approach to model the evolution of particles during interactions, and it is the foundation of the so-called *standard-model*, one of the experimentally most accurate theory of physics ever written. This model is incredibly efficient in predicting most of the phenomenons surrounding us and led to many Nobel Prizes. It has been given a rigorous construction in some cases, namely in the free-case also known as the Klein-Gordon equation or in the so-called ϕ^4 model. Nevertheless, in the full generality, most of the questions remain open, and it is fair to say that this theory lacks a rigorous mathematical foundation when dealing with non-trivial interactions. It is therefore a challenge to make these theories well-defined and one could choose to tackle first the simplest ones (even though incredibly hard), namely the one admitting an integrable structure. An instance of this class is the *sinh-Gordon model*.

The classical sinh-Gordon model describes the evolution of a scalar field $\phi(x,t)$ by the following partial differential equation:

$$\big(\partial_t^2-\partial_x^2\big)\phi(x,t)+\frac{m^2}{g}\sinh\big(\phi(x,t)\big)=0$$

where m is the mass of the field and g is a coupling constant. This equation is associated with the extremalization of the action $\mathcal{S}_{\text{sinh}}$ given by density the $\mathcal{L}(\phi,\partial_\mu\phi)$:

$$\mathcal{S}_{\text{sinh}}[\phi] \overset{\text{(def)}}{=} \int_{\mathbb{R}^2}\mathcal{L}_{\text{sinh}}(\phi,\partial_\mu\phi)dxdt, \qquad \mathcal{L}_{\text{sinh}}(\phi,\partial_\mu\phi) \overset{\text{(def)}}{=} \frac{1}{2}\partial_\mu\partial^\mu\phi(x,t)-\frac{m^2}{g^2}\cosh\big(\phi(x,t)\big). \tag{82}$$

In the classical picture, it can be shown that this equation is integrable in the sense that it admits a Lax pair [DF19, Section 3.2.1]. It is well known that the quantum analogue of this theory is integrable using the fact that the scattering matrix is scalar [ZZ79]. We refer to [KLM21] for a review on the quantum sinh-Gordon model. In the finite volume case (namely on a torus of radius $R>0$), by applying the QSVM, Lukyanov [Luk01] conjectured an expression for the form factors associated to the exponential of the field operator, namely:

$$\forall a\in\mathbb{R},\quad \Big\langle \Psi_{\text{gs}}\Big|\, e^{a\phi}\,\Big|\,\Psi_{\text{gs}}\Big\rangle \overset{\text{(def)}}{=} \lim_{N\infty}\left[\left(\frac{N}{mR}\right)^{\theta}\frac{\mathcal{I}_N(R,a)}{\mathcal{I}_N(R,0)}\right], \qquad \text{with } \theta \overset{\text{(def)}}{=} \frac{a^2}{2(1+b^2)(1+b^{-2})}, \tag{83}$$

where $b \overset{(\text{def})}{=} g^2/(16\pi + 2g^2)$ and

$$\mathcal{I}_N(R,a) \overset{(\text{def})}{=} \int_{\mathbb{R}^N} \prod_{i<j}^{N} \sinh\left[(1+b^{-2})(\gamma_i-\gamma_j)\right] \sinh\left[\frac{\gamma_i-\gamma_j}{1+b^2}\right] \prod_{i=1}^{N} e^{-W_a(\gamma_i)} d^N\gamma. \tag{84}$$

In this definition, the function W_a is defined by:

$$W_a(\gamma) \overset{(\text{def})}{=} \mathfrak{r}\cosh(\gamma) - \alpha\gamma - \int_{\mathbb{R}} \frac{\mathfrak{g}(\gamma')}{\cosh(\gamma-\gamma')} \frac{d\gamma'}{2\pi} \qquad \text{with } \mathbf{g}(\lambda) \overset{(\text{def})}{=} \log\left(1+e^{-\epsilon(\gamma')}\right) \tag{85}$$

Above, the function ε verifies the following equation:

$$\varepsilon(\gamma) + \int_{\mathbb{R}} \frac{d\gamma'}{2\pi} K(\gamma-\gamma') \log\left(1+e^{-\epsilon(\gamma')}\right) = 2\pi m R\cosh(\gamma) \tag{86}$$

where

$$K(\lambda) \overset{(\text{def})}{=} \frac{4\cosh(\lambda)\sin\left(\frac{\pi}{1+\mathfrak{b}^2}\right)}{\cosh(2\lambda) - \cos\left(\frac{2\pi}{1+\mathfrak{b}^2}\right)} \qquad \text{and} \qquad \tau \overset{(\text{def})}{=} \frac{\pi(b^2-b^{-2})}{4+2b^2+2b^{-2}}.$$

It was shown in [FKS99] that (86) admits a unique soluution in $L^\infty(\mathbb{R})$. Several years later, a lattice discretization of the sinh-Gordon model in finite volume was proposed in [BT06]. The discretized field is defined over N sites, and it was shown that when $\hbar \to 0^+$ and $N \to \infty$, one recovers the classical sinh-Gordon model. In this sense, one can consider that the limit $N \to \infty$ of the discretization provides a definition of a quantum sinh-Gordon model in finite volume. This discretization was also shown to be integrable, in the sense that N integrals of motion were exhibited. Recently, in [GGV24], a construction based on the Gaussian free field was also proposed.

In Chapter 3, we will try to follow the path developed in [BGK16] to make progress by studying the limit (83), see Subsection I.8.3 for a brief description.

We now introduce the *sinh-model* which is a probabilistic model which arises when tackling the study of the multiple integrals defined in (81) with $F_{\mathcal{O}}(y_1,\dots,y_N) = \prod_{a=1}^{N} e^{-\alpha y_a}$. We will introduce this model and discuss some of its properties before presenting the results we have obtained.

I.7.3 Definition of the model and comparison with β-ensembles

The *sinh-model* is the probabilistic model introduced in [BGK16], $\omega_1, \omega_2 > 0$ and a potential V growing sufficiently fast at infinity, by

$$d\mathbf{P}^V_{N,\sinh}(\lambda_1,\dots,\lambda_N) \overset{(\text{def})}{=} \mathfrak{z}_{N,\sinh}[V]^{-1} P^V_{N,\sinh}(\lambda_1,\dots,\lambda_N) d^N\lambda \tag{87}$$

with

$$P^V_{N,\sinh}(\lambda_1,\dots,\lambda_N) \overset{(\text{def})}{=} \prod_{i<j}^{N} \Big[\sinh\left(\pi\omega_1|\lambda_i-\lambda_j|\right)\sinh\left(\pi\omega_2|\lambda_i-\lambda_j|\right)\Big] . \prod_{i=1}^{N} e^{-V(\lambda_i)}.$$

This model is interesting because by applying the loop equations method (Subsection I.4) in $\mathbf{P}^V_{N,\sinh}$, one can access the AE of the corresponding partition function. The latter represents the general form of the multiple integral obtained by applying the QSVM to certain quantum integrable models (81). Additionally, it can be seen as a generalization of the β-ensembles.

Indeed, these two systems share several commonalities. They both represent an interacting system of N particles on the real line subjected to a confining potential V with a repulsive two-body interaction. However, there are some important differences between these two models. The most obvious difference is the form of the interaction. Although they exhibit the same singular behavior

when particles are close, the hyperbolic sine function arising in (87) makes the model less computationally tractable in principle. An illustration of this fact is the loss of homogeneity when the interaction is changed:

$$|x| \rightsquigarrow \sinh(\pi\omega_1|\lambda_i-\lambda_j|)\sinh(\pi\omega_2|\lambda_i-\lambda_j|).$$

This makes rescaling changes of variables of the type $\lambda_i = \tau_N\mu_i$, less efficient when studying the distribution $\mathbf{P}^{V,\beta}_{N,\sinh}$ compared to the β-ensemble case where one can just discard the scaling τ_N. This will lead to major complications in the analysis of $\mathfrak{z}^{\beta}_{N,\sinh}[V]$.

A second important difference is the absence of scaling by N of the potential, this type of interacting particle systems is said to have *non-varying weights* (see [BGK16, Section 1.4] for a discussion). It is worth reminding the reader that the presence of such a scaling resulted in the confining and repulsive terms having the same magnitude in terms of N. Consequently, at large N, the system exhibited a global behavior where particles remained, with overwhelming probability, within a bounded N-independent region $[a,b]$. Without this scaling, and with a repulsive interaction (of strength N^2) stronger than the confinement (of strength N), it is expected that the particles will primarily experience repulsion until the effect of the potential becomes significant. As a result, when N is large, the particles are more likely to localize within an N-dependent region $[a_N,b_N]$, where a_N and b_N tend to infinity. Introducing a change of variables $\lambda_i = \tau_N\mu_i$, with $\tau_N \to \infty$ as $N\to\infty$, allows us to rebalance these contributions in terms of N and ensures that the particles remain within an interval $[a,b]$ with high probability. The cost of this procedure is that one must retain this scaling throughout the entire analysis due to the loss of homogeneity of the sinh interaction. One must then proceed with caution due to the increased complexity resulting from the additional N-dependence and multiple scales in the problem, namely N and τ_N. Indeed, by performing an integration by parts, it can be observed that the first loop equation of the model [BGK16, Proposition 3.2.3] involves N-dependent operators, significantly increasing the technicalities involved in analyzing the loop equations.

To determine the appropriate scaling, assuming the configuration $(\mu_1,\dots,\mu_N)$ is sufficiently typical, we find:

$$\sum_{i=1}^{N} V(\tau_N\mu_i) \underset{N\to\infty}{\sim} NV(\tau_N)$$

and

$$\sum_{i<j}^{N} \log\Big[\sinh(\pi\omega_1\tau_N|\mu_i-\mu_j|)\sinh(\pi\omega_2\tau_N|\mu_i-\mu_j|)\Big] \underset{N\to\infty}{\sim} CN^2\tau_N.$$

Thus, to get $NV(\tau_N) \underset{N\to\infty}{\sim} CN^2\tau_N$, we need, for V growing like $|x|^q$ as $x\to\infty$, $\tau_N = N^{\frac{1}{q-1}}$ and for V growing like $e^{|x|}$ at infinity, one finds the following hyperbolic equation $e^{\tau_N} = N\tau_N$ which at the first order leads to $\tau_N = \log N$. We therefore consider the following particle system:

$$d\mathbb{P}^{V}_{N,\sinh}(\lambda_1,\dots,\lambda_N) \overset{(\text{def})}{=} \mathcal{Z}_{N,\sinh}[V]^{-1} p^{V}_{N,\sinh}(\lambda_1,\dots,\lambda_N)d^N\lambda \tag{88}$$

with

$$p^{V}_{N,\sinh}(\lambda_1,\dots,\lambda_N) \overset{(\text{def})}{=} \prod_{i<j}^{N} \sinh(\pi\omega_1\tau_N|\lambda_i-\lambda_j|)\sinh(\pi\omega_2\tau_N|\lambda_i-\lambda_j|)\cdot\prod_{i=1}^{N} e^{-N\tau_N V_N(\lambda_i)} \tag{89}$$

and τ_N adapted to V. In the above expression, we have set $V_N(x) \overset{(\text{def})}{=} (N\tau_N)^{-1}V(\tau_N x)$. In [BGK16], the authors considered a general, strictly convex, smooth, N-independent potential $V_N = V$, with $\tau_N = N^{\alpha}$ and managed to show *via* the loop equations analysis, a large-N AE

for $\log \mathcal{Z}^{\beta}_{N,\sinh}[V]$ up to $o(1)$. The appropriate framework for the study of Lukyanov's conjecture, is to establish such a development for $\log \mathcal{Z}^{1}_{N,\sinh}[V]$ up to $o(1)$ with V as in (85). In the following, we will consider such V. Since the potential V grows exponentially fast at infinity, we choose the scaling $\tau_N = \log N$ in the rest of this section. We now explain some features of the model and describe the global behavior of the particles as N tends to infinity.

I.7.4 Global behavior of the particles

Just as for the β-ensembles, the strategy for understanding the global behavior of the particles is to analyze the limiting behavior for large N of the empirical measure $\widehat{\mu}_N = N^{-1}\sum_{i=1}^{N}\delta_{\lambda_i}$. This is summarized by the following result.

Theorem I.7.1 *The family of random measures $(\widehat{\mu}_N)_{N\geq 1}$ satisfies, in $\mathcal{M}_1(\mathbb{R})$ equipped with the weak topology, a $\mathrm{LDP}(N^2\log N, I_{V,\sinh})$ with strictly convex, good rate function:*

$$I_{V,\sinh} \overset{(\mathrm{def})}{=} \mathcal{E}_{V,\sinh} - \inf_{\nu\in\mathcal{M}_1(\mathbb{R})}\mathcal{E}_{V,\sinh}(\nu).$$

The functional $\mathcal{E}_{V,\sinh}$ is defined for all $\mu\in\mathcal{M}_1(\mathbb{R})$ by:

$$\mathcal{E}_{V,\sinh}(\mu) \overset{(\mathrm{def})}{=} \begin{cases} -\dfrac{\pi(\omega_1+\omega_2)}{2}\displaystyle\iint_{\mathbb{R}^2}|x-y|d\mu(x)d\mu(y) & \text{if } \mu\left([-1,1]^c\right)=0,\\ +\infty & \text{if } \mu\left([-1,1]^c\right)>0. \end{cases} \tag{90}$$

The unique minimizer is attained at the Rademacher probability measure, i.e. at $\mathsf{Rad} \overset{(\mathrm{def})}{=} \frac{1}{2}(\delta_{-1}+\delta_1)$. Finally, under $\mathbb{P}^{V}_{N,\sinh}$, $\widehat{\mu}_N$ converges weakly towards Rad almost surely and the free energy reads

$$\lim_{N\to\infty}\frac{1}{N^2\log N}\log\mathcal{Z}_{N,\sinh}[V] = \frac{\pi(\omega_1+\omega_2)}{2}\iint_{\mathbb{R}^2}|x-y|d\mathsf{Rad}(x)d\mathsf{Rad}(y) = \frac{\pi(\omega_1+\omega_2)}{2}.$$

This theorem can be proved following a similar approach to [BGK16, Appendix B]. The rate function $I_{V,\sinh}$ in this LDP is a bit different to the one obtained in [BGK16, Theorem 2.1.1] which is valid for potentials growing like $a|x|^q$ at infinity. In our case, choosing the stronger confining potential V growing exponentially fast at infinity corresponds, heuristically, to taking $q\to+\infty$. As a result, $I_{V,\sinh}$ does not depend on the parameters defining (85). One reason for this is that, at large N, it is extremely unlikely to have a substantial number of particles outside of $[-1,1]$ due to the strong confining effect of V. Moreover, within $[-1,1]$, the particles only "feel" the confinement as a correction to the repulsion. Concretely, the contribution of the potential within $[-1,1]$ is equal to $N^2\tau_N V_N(\lambda)\sim N^{1+|\lambda|}$ for $|\lambda|\leq 1$, while the repulsive effect is proportional to $N^2\log N$.

In our case, it is convenient to work with a N-dependent equilibrium measure that approaches the Rademacher law as N grows large, while still enjoying some useful properties. Another instance where one uses a N-dependent equilibrium measure is [BF06, Section 5]. The idea is that $\mu^{(N)}_{\mathrm{eq}}$ and $\mathcal{E}^{(N)}_{V,\sinh}$ absorb much more the N dependence of $\mathcal{Z}^{\beta}_{N,\sinh}[V]$ than Rad and $\mathcal{E}_{V,\sinh}$.

A more suitable choice to capture the large-N behavior of the particles is the minimizer of the following functional:

$$\mathcal{E}^{(N)}_{V,\sinh}[\mu] \overset{(\mathrm{def})}{=} \int_{\mathbb{R}} V_N(x)d\mu(x) - \frac{1}{2\tau_N}\iint_{\mathbb{R}^2}\log\prod_{a=1}^{2}\sinh\left[\pi\omega_a\tau_N(x-y)\right]d\mu(x)d\mu(y).$$

It satisfies the following theorem.

Theorem I.7.2 *$\mathcal{E}^{(N)}_{V,\sinh}$ is strictly convex. Its unique minimizer denoted $\mu^{(N)}_{\mathrm{eq}}$ and called the equilibrium measure, is absolutely continuous with respect to the Lebesgue measure with p.d.f. $\rho^{(N)}_{\mathrm{eq}}$, and it is characterized by the following Euler-Lagrange equations:*

$$V_N(x) - \frac{1}{2\tau_N}\int_{\mathbb{R}} \log \prod_{a=1}^{2} \sinh\left[\pi\omega_a \tau_N(x-y)\right] d\mu^{(N)}_{\mathrm{eq}}(y) \begin{cases} > C^{\sinh}_{V_N} & \text{outside supp } \mu^{(N)}_{\mathrm{eq}}, \\ = C^{\sinh}_{V_N} & \mu^{(N)}_{\mathrm{eq}} - as. \end{cases} \tag{91}$$

Finally, it is compactly supported on $[a_N, b_N]$ and has a vanishing square-root behavior at the endpoints.

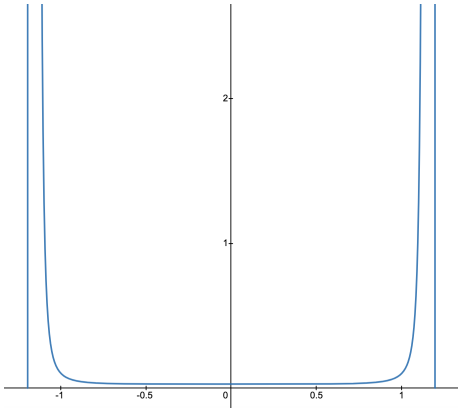

Figure I.3: The equilibrium density $\rho^{(N)}_{\mathrm{eq}}$.

This measure whose general form is represented in Figure I.3, is very similar to $\mu_{V,\beta}$ introduced in Theorem I.3.1. Indeed, it is Lebesgue-continuous, compactly supported and also exhibits the vanishing square-root behavior at the endpoints. Furthermore, to capture more properties of $\rho^{(N)}_{\mathrm{eq}}$, by differentiating equality (91), one finds that it satisfies the following singular integral equation:

$$\forall x \in [a_N, b_N],\ \mathcal{S}_N[\rho^{(N)}_{\mathrm{eq}}](x) = V_N'(x), \qquad \mathcal{S}_N[\phi](x) \overset{(\mathrm{def})}{=} \fint_{a_N}^{b_N} S\left(\tau_N(x-y)\right)\phi(y)dy \tag{92}$$

where $S(x) \overset{(\mathrm{def})}{=} \sum_{a=1}^{2} \frac{\pi\omega_a}{2}\coth\left(\pi\omega_a x\right)$. We stress that in the singular equation (92), which is analogous to (19), the unknown is the triplet $(\rho^{(N)}_{\mathrm{eq}}, a_N, b_N)$. As previously explained, one would like to solve this equation in order to have a manageable expression for the density and access the knowledge of linear statistics and, hence the partition function $\mathcal{Z}_{N,\sinh}[V]$.

While in the classical β-ensemble case, this equation can be easily solved by means of a scalar RHP, the situation here is much more involved. The operator $\mathcal{S}_N$ belongs to the class of so-called *truncated Wiener-Hopf operators*. When defined on sufficiently irregular Sobolev spaces, this operator can be inverted by using a Wiener-Hopf factorization involving the solution χ of an auxiliary 2×2 RHP. This method is greatly inspired by the technique developed in [Kre62, Nov80].

More explicitly, to solve equation $\mathcal{S}_N[\varphi](\xi) = H(\xi)$, it is convenient to first solve the *regularized equation*

$$\forall x \in [a_N, b_N],\ \mathcal{S}_{N,\gamma}[\phi](x) = H(x), \qquad \mathcal{S}_N[\phi](x) \overset{(\mathrm{def})}{=} \fint_{a_N}^{b_N} S_\gamma\left(\tau_N(x-y)\right)\phi(y)dy \tag{93}$$

where $S_\gamma(x) \overset{(\mathrm{def})}{=} S(x)\mathbf{1}_{[-\gamma\overline{x}_N, \gamma\overline{x}_N]}$ and $\overline{x}_N \overset{(\mathrm{def})}{=} \tau_N(b_N - a_N)$. To obtain the final answer, one can send the parameter $\gamma \to +\infty$ once that all the computations are made. Furthermore, equation (93)

can be recasted as

$$\mathscr{S}_{N,\gamma}[\varphi](\xi) \overset{(\text{def})}{=} \int_0^{\overline{x}_N} S_\gamma(\xi-\eta)\varphi(\eta)\frac{d\eta}{2\mathrm{i}\pi} = h(\xi), \qquad \text{with} \quad \begin{cases} \varphi(\eta) \overset{(\text{def})}{=} \phi\left(\dfrac{\eta + a_N\tau_N}{\tau_N}\right), \\ h(\xi) \overset{(\text{def})}{=} \dfrac{\tau_N}{2\mathrm{i}\pi} H\left(\dfrac{\eta + a_N\tau_N}{\tau_N}\right). \end{cases} \tag{94}$$

The solutions of $\mathscr{S}_{N,\gamma}[\varphi] = h$ can be mapped to a solution Φ of a 2×2 RHP.

Lemma I.7.3 *[BGK16, Lem 4.1.1] Let $h \in H^s([0,\overline{x}_N])$, $s<0$. For any solution $\varphi \in H^s([0,\overline{x}_N])$, of $\mathscr{S}_{N,\gamma}[\varphi] = h$. There exists a two-dimensional vector function $\Phi \in \mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ such that $\varphi = \mathcal{F}^{-1}[(\Phi_1)_+]$ and Φ is a solution to the boundary value problem:*

- $(\Phi_a)_\pm \in \mathcal{F}[H^s(\mathbb{R}^\pm)]$ *for $a\in\{1,2\}$, and there exists $C>0$ such that:*

$$\forall \mu>0,\ \ \forall a\in\{1,2\}, \quad \int_{\mathbb{R}} |\Phi_a(\lambda \pm \mathrm{i}\mu)|^2.(1+|\lambda|+|\mu|)^{2s}\,d\lambda < C,$$

- *We have the jump equation for $\Phi_+(\lambda) = G_\chi(\lambda).\Phi_-(\lambda) + \mathbf{H}(\lambda)$ for $\lambda\in\mathbb{R}$, with:*

$$G_\chi(\lambda) \overset{(\text{def})}{=} \begin{pmatrix} e^{\mathrm{i}\lambda\overline{x}_N} & 0 \\ \dfrac{1}{2\mathrm{i}\pi\beta}\mathcal{F}[S_\gamma](\lambda) & -e^{-\mathrm{i}\lambda\overline{x}_N} \end{pmatrix} \text{ and } \mathbf{H}(\lambda) \overset{(\text{def})}{=} \begin{pmatrix} 0 \\ -e^{-\mathrm{i}\lambda\overline{x}_N}\mathcal{F}[h_e](\lambda) \end{pmatrix}. \tag{95}$$

It turns out that this boundary value problem can be solved by constructing a factorization of G_χ. This leads to the formulation of the following 2×2, RHP for χ which can be solved:

Theorem I.7.4 *[BGK16, Prop 4.2.1] For N sufficiently large, there exists a unique solution to the 2×2 RHP:*

- $\chi \in \mathcal{M}_2(\mathcal{O}(\mathbb{C}\setminus\mathbb{R}))$,
- *χ admits continuous $\pm$-boundary values on $\mathbb{R}$,*
- $\chi(\lambda) = \mathsf{A}(\lambda)\left(I_2 + \dfrac{\mathsf{B}(\lambda)}{\lambda} + O(\lambda^{-2})\right)$ *as λ goes to infinity.*
- *For all $\xi\in\mathbb{R}$, $\chi_+(\xi) = G_\chi(\xi)\chi_-(\xi)$, where G_χ has been introduced in* (95).

In the above theorem, we do not provide the exact expressions of A and B as they are not needed for the purpose of this discussion. However, they can be found in the corresponding reference. The RHP above can be solved by using the Deift-Zhou non-linear steepest descend method. This allows us to invert the operator $\mathscr{S}_{N,\gamma}$ on $H^s([0,\overline{x}_N])$ for s big enough and obtain an integral representation in terms of χ for the inverse. After letting $\gamma\to+\infty$, we obtain the following result:

Theorem I.7.5 *[BGK16, Prop 4.3.8] Let $0<s<1/2$, the operator $\mathcal{S}_N : H^s([a_N,b_N]) \to \mathfrak{X}_s(\mathbb{R})$ is continuous and invertible where*

$$\mathfrak{X}_s(\mathbb{R}) \overset{(\text{def})}{=} \left\{ h\in H^s(\mathbb{R}),\ \int_{\mathbb{R}+\mathrm{i}\varepsilon} \chi_{11}(\mu)\mathcal{F}[h](\tau_N\mu)e^{-\mathrm{i}\tau_N\mu b_N}d\mu = 0 \right\}.$$

The inverse $\mathcal{W}_N$ is given for all $\phi\in\mathfrak{X}_s(\mathbb{R})$ by

$$\mathcal{W}_N[\phi](\xi) \overset{(\text{def})}{=} \frac{(\tau_N)^2}{2\pi}\int_{\mathbb{R}+2\mathrm{i}\varepsilon}\frac{d\lambda}{2\mathrm{i}\pi}e^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}\int_{\mathbb{R}+\mathrm{i}\varepsilon}\frac{d\mu}{2\mathrm{i}\pi}\frac{K_N(\lambda,\mu)e^{-\mathrm{i}\mu\tau_N b_N}}{\mu-\lambda}\mathcal{F}\left[\phi\mathbf{1}_{[a_N,b_N]}\right](\tau_N\mu) \tag{96}$$

for a N-dependent kernel K_N given in terms of χ and a fixed $\varepsilon>0$.

In [BGK16, Section 4], the authors were able to deduce many properties about the large-N behavior of χ, which translates into knowledge of the operator $\mathcal{W}_N$ for large N. This enables us to find the

solution of (92) for choices of $a_N < b_N$ such that $b_N - a_N > \eta$ uniformly in N for some $\eta > 0$. Imposing the following constraints on a_N and b_N,

$$\int_{a_N}^{b_N} \rho_{\mathrm{eq}}^{(N)}(\xi)d\xi = 1, \qquad \int_{\mathbb{R}+\mathrm{i}\varepsilon} \chi_{11}(\mu)\mathcal{F}[V_N'\mathbf{1}_{[a_N,b_N]}](\tau_N\mu)e^{-\mathrm{i}\tau_N\mu b_N}d\mu = 0 \quad (97)$$

allows one to uniquely determine a_N and b_N such that the following holds:

$$\rho_{\mathrm{eq}}^{(N)} = \mathcal{W}_N[V_N']. \quad (98)$$

The first equation guarantees that $\mu_{\mathrm{eq}}^{(N)}$ has mass 1, while the second ensures, using the linear form in the definition of $\mathfrak{X}_s(\mathbb{R})$, that it vanishes as a square root at the endpoints.

Compared to the derivation of an explicit expression for the equilibrium measure in the classical β-ensemble case which leads to (20), it is fair to say that the above procedure is considerably more sophisticated. Indeed, in the first case, by solving a scalar RHP, one is rewarded with an explicit expression, in the parameters (except for the endpoints) for the equilibrium density. This provides a lot of knowledge on this function such as: its regularity, its vanishing behavior at the endpoints or its dependence on additional parameters introduced in the model. In the sinh-model, it requires to solve a 2×2 RHP which is, in general, considerably harder than dealing with a scalar RHP. Furthermore, while the formula found in (98) which involves the operator $\mathcal{W}_N$, introduced in (96), is very satisfying on many aspects as it allows one to understand the regularity, the N-dependence or again the behavior at the endpoints, it is way more involved than (20). Indeed, the expression is now a triple integral and it depends on the entries of χ which are non-explicit functions. These facts imply that it requires way more efforts to extract the information out of this expression. Indeed, it is necessary to extract the large-N behavior of $\chi(\lambda)$, which requires a lot of effort in order to undestand the behavior in N and ξ of $\rho_{\mathrm{eq}}^{(N)}(\xi)$.

We will now explain the strategies used to obtain the asymptotics of $\mathcal{Z}_{N,\sinh}[V]$.

I.7.5 The partition function

Several strategies exist to obtain an AE for $\log \mathcal{Z}_{N,\sinh}[V]$ up to $o(1)$. The first strategy is the loop equations analysis, described for the classical β-ensembles in Subsection I.4. By using this method for a N-independent potential V, the authors of [BGK16], managed to obtain an explicit expression for the coefficients of the AE of $\log \mathcal{Z}_{N,\sinh}[V]$ up to $o\Big(\frac{(\log N)^\alpha}{N^\gamma}\Big)$ for $\alpha, \gamma > 0$. To conduct the loop equations analysis, the authors identified the master operator $\mathcal{U}_N$ (which is then N-dependent) which has the following expression:

$$\mathcal{U}_N[\phi] \overset{(\mathrm{def})}{=} \phi\left\{V' - \mathcal{S}_N\left[\rho_{\mathrm{eq}}^{(N)}\right]\right\} + \mathcal{S}_N\left[\phi\rho_{\mathrm{eq}}^{(N)}\right]. \quad (99)$$

By exploiting the inversion of the singular integral transform $\mathcal{S}_N$ from Theorem I.7.5, the authors managed to invert $\mathcal{U}_N$ and obtain an integral representation for $\mathcal{U}_N^{-1}$ [BGK16, Prop 5.2.1]. This expression is similar to the one found for the classical β-ensembles (48) and (49). On $[a_N, b_N]$, since the first term of (99) vanishes by (91), this representation reads, for a smooth ϕ in $\mathfrak{X}_s(\mathbb{R})$ with $0 < s < 1/2$, and $x \in [a_N, b_N]$:

$$\mathcal{U}_N^{-1}[\phi](x) = \frac{\mathcal{W}_N[\phi](x)}{\rho_{\mathrm{eq}}^{(N)}(x)} \quad (100)$$

Since $\rho_{\mathrm{eq}}^{(N)}$ vanishes like a square-root at the endpoints, and $\mathcal{W}_N$ is the inverse of the singular integral transform $\mathcal{S}_N$, (100) is very similar to (48). Outside $[a_N, b_N]$, it reads $\mathcal{U}_N^{-1}[\phi](x) = \ell_{\sinh}(x)^{-1}k_{\sinh}(x)$ with

$$\ell_{\sinh}(x) \overset{(\mathrm{def})}{=} \mathcal{S}_N \circ \mathcal{W}_N[V'](x) - V'(x), \qquad k_{\sinh}(x) \overset{(\mathrm{def})}{=} \mathcal{S}_N \circ \mathcal{W}_N[\phi](x) - \phi(x),$$

which is again very similar to (49). One also needs to obtain controls with respect to the W_n^∞-norm to apply the a priori bound and conduct the loop equations analysis. However, this representation is not completely satisfying for extracting this control, as one has to take care of a vanishing denominator when N goes to infinity in (100). Indeed, in the case of a N-dependent potential V_N introduced in (85), we know that $\rho_{\text{eq}}^{(N)}$ approaches the Rademacher measure. Thus, we expect that $\rho_{\text{eq}}^{(N)}(\pm 1) \underset{N\to\infty}{\longrightarrow} +\infty$ and that for $x \neq \pm 1$, $\rho_{\text{eq}}^{(N)}(x) \underset{N\to\infty}{\longrightarrow} 0$.

Moreover, from the form of the density found by computing $\mathcal{W}_N[V_N']$, it is expected that the convergence to 0 far from the endpoints is polynomially fast. However, for sufficiently smooth functions ϕ one can expect that the numerator will also vanish exponentially fast in the bulk. Using the explicit form of $\mathcal{W}_N$ (96), along with the knowledge on the asymptotics of χ and the residue theorem, it is possible to obtain a sufficiently explicit representation of that term. From this expression, the exponential decay in the bulk becomes apparent. One can deduce from it that a linear statistic like $\langle \mathcal{U}_N^{-1}[\phi] \rangle_{\mu_{\text{eq}}^{(N)}}$ is of order 1, and its explicit limit can be obtained.

The second approach can be pursued *via* the asymptotics of the so-called *bi-orthogonal polynomials*. A good reference for the theory of such polynomials is [Bor98]. Inspired by the orthogonal polynomials approach for obtaining the partition function of the β-ensembles for $\beta = 2$, as explained in Subsection I.3.4, there exists a generalization for other models whose interaction can be written as a product of two different Vandermonde determinants, as is the case here, namely:

$$\prod_{i<j}^{N} \prod_{p=1}^{2} \sinh\left[\pi \omega_p \tau_N (\lambda_i - \lambda_j)\right] = \prod_{i=1}^{N} \frac{e^{-\pi(\omega_1+\omega_2)\tau_N(N-1)\lambda_i}}{2^{N-1}} \cdot \prod_{p=1}^{2} \det_{1,\leq i,j\leq N} \left(e^{2\pi\omega_p \lambda_i (j-1)} \right). \tag{101}$$

Thus, similarly as in (28), it is possible to recast the partition function as:

$$\mathcal{Z}_{N,\sinh}[V] = \prod_{i=1}^{N} \gamma_i^{-1} \tag{102}$$

where the γ_i's correspond to the normalization with respect to orthogonality conditions for a system of polynomials $(P_i, Q_i)_{i\geq 0}$ called the *bi-orthogonal polynomials*. In [CR13], the authors managed to prove that such polynomials admita RH formulation in the same spirit as in Theorem I.3.3. From this RHP, one can expect to extract, again using the Deift-Zhou non-linear steepest descent method, the large-N behavior of the γ_i's and hence the AE of $\mathcal{Z}_{N,\sinh}[V]$.

However, currently, the literature only contains the formalization of this approach in the case of varying-weights. The generalization to the non-varying-weights case implies rescaling the integral. As we have argued before, this would introduce another scale τ_N into the large-N analysis of this RHP. Extending the proof to this setup would require a significant effort.

We now state the results obtained in this thesis as well as summarize Chapters 1, 2 and 3.

I.8 Summary of research results

I.8.1 Chapter 1: CLT for real β-ensembles at high temperature

This result is derived from joint work with Ronan Memin **[DGM23]**. We establish for the β-ensembles at high temperature, under very general conditions on the potential V, a central limit theorem for linear statistics with bounded smooth enough functions.

We consider the probability measure $\mathbb{P}_N^{V,P}$ on $\mathbb{R}^N$, as introduced in (58), where V is quite general, *i.e.*, it grows sufficiently fast at infinity and is smooth enough. Examples of such potentials include

polynomials of even degree with positive leading coefficient or the hyperbolic cosine function. The goal of this article is to study the fluctuations of the linear statistics, rescaled by $\sqrt{N}$, namely:

$$\sqrt{N}\int_{\mathbb{R}}\phi(x)d(\widehat{\mu}_N-\mu_{V,P})(x), \qquad \phi \text{ smooth and bounded}$$

where $\widehat{\mu}_N$ is the empirical measure and $\mu_{V,P}$ is the equilibrium measure introduced in Theorem I.5.1. More precisely, we demonstrate that as N tends to infinity, this quantity converges in distribution to a normal distribution $\mathcal{N}(0,\sigma^2(\phi))$ for some $\sigma^2(\phi)>0$.

A common approach to proving this type of result, first developped in [Joh98] and used in [BLS18], is based on the following convergence of Laplace transforms:

$$\mathbb{E}_N^{V,P}\left[\exp\left(\sqrt{N}t\int_{\mathbb{R}}\phi(x)d(\widehat{\mu}_N-\mu_{V,P})(x)\right)\right]\underset{N\to\infty}{\longrightarrow}\exp\left(-\frac{t^2}{2}\sigma^2(\phi)\right).$$

Our strategy is based on a well-known change of variables in the partition function $\mathcal{Z}_N[V]$ defined in (59), see [Joh98, Shc14b, BFG15, BLS18]. Take a compactly supported smooth function $\phi:\mathbb{R}\to\mathbb{R}$ and perform the following change of variables in $\mathcal{Z}_N[V]$,

$$x_i\to y_i+\frac{t}{\sqrt{N}}\phi(y_i) \qquad 1\le i\le N.$$

This yields:

$$\mathcal{Z}_N[V]=\int_{\mathbb{R}^N}\prod_{i<j}\left|y_i-y_j+\frac{t}{\sqrt{N}}(\phi(y_i)-\phi(y_j))\right|^{\frac{2P}{N}}.e^{-\sum_{i=1}^N V\left(y_i+\frac{t}{\sqrt{N}}\phi(y_i)\right)}.\prod_{i=1}^N\left(1+\frac{t}{\sqrt{N}}\phi'(y_i)\right)d^Ny\,.$$

Expanding the different terms in this integral, we obtain:

$$\mathcal{Z}_N[V]=\int_{\mathbb{R}^N}p_N^{V,P}(y_1,\dots,y_N).\exp\left\{\frac{t}{\sqrt{N}}\left[\frac{2P}{N}\sum_{i<j}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}+\sum_{i=1}^N\Big(\phi'(y_i)-V'(y_i)\phi(y_i)\Big)\right]\right.$$
$$\left.-\frac{t^2}{2}\sigma_N^2(\phi)+O\Big(N^{-\frac{3}{2}}t^3\|\phi\|_{W_1^\infty(\mathbb{R})}\Big)\right\}d^Ny, \quad (103)$$

where $\sigma_N^2(\phi)$ is a quadratic form in ϕ converging to a limiting variance $\sigma^2(\phi)$. Recalling the definition of the master operator Ξ defined in (67), one can observe that in (103), the term proportional to t is almost $\sqrt{N}\int_{\mathbb{R}}\Xi[\phi]d\widehat{\mu}_N(x)$. Therefore, we can set the error term as follows:

$$\text{error term}\overset{\text{(def)}}{=}-t\sqrt{N}\left[\int_{\mathbb{R}}\Xi[\phi](x)d\widehat{\mu}_N(x)-\frac{2P}{N}\sum_{i<j}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}-\sum_{i=1}^N\Big(\phi'(y_i)-V'(y_i)\phi(y_i)\Big)\right]$$
$$-\sigma_N^2(\phi)+\sigma_N^2(\phi)+O(t^3N^{-\frac{3}{2}})$$

After dividing both parts of the equation by $\mathcal{Z}_N[V]$, and because of the identity $\int_{\mathbb{R}}\Xi[\phi](x)d\mu_{V,P}(x)=0$ (coming from (62)), we deduce from (103) the following convergence:

$$\mathbb{E}_N^{V,P}\left[\exp\left[t\sqrt{N}\int_{\mathbb{R}}\Xi[\phi](x)d(\widehat{\mu}_N-\mu_{V,P})(x)+\text{error term}\right]\right]\underset{N\to\infty}{\longrightarrow}\exp\Big(-\frac{t^2}{2}\sigma^2(\phi)\Big). \quad (104)$$

The error term can be managed by using large deviation principle and concentration of measure type techniques to argue that $\widehat{\mu}_N$ is close to $\mu_{V,P}$ as $N\to\infty$ with high probability. The proof proceeds as follows: define the following distance d by

$$d(\mu,\mu')\overset{\text{(def)}}{=}\sup_{\substack{\|f\|_{\text{Lip}}\le 1\\ \|f\|_{1/2}\le 1}}\left\{\left|\int f(x)d\mu(x)-\int f(x)d\mu'(x)\right|\right\},$$

where $\|f\|_{\mathrm{Lip}}$ denotes the Lipschitz constant of f, and $\|f\|_{1/2}^2 \overset{(\mathrm{def})}{=} \int_{\mathbb{R}} |t|\,|\mathcal{F}[f](t)|^2\,dt$, with $\mathcal{F}$ being the Fourier transform on $L^2(\mathbb{R})$. By using a regularization of the empirical measure $\widehat{\mu}_N$ developped in [MMS14], we obtain the following concentration result:

Proposition I.8.1 *There exists $K \in \mathbb{R}$ (depending on P and on V), such that for any $N \geq 1$ and $r > 0$,*

$$\mathbb{P}_N^{V,P}\left(d(\hat{\mu}_N, \mu_{V,P}) > r\right) \leq e^{-Nr^2\frac{P\pi^2}{2}+5P\log N+K}. \tag{105}$$

Since some building blocks of the error term arising in equation (104) also appear in the classical β-ensembles, it is well-known that this type of concentration result allows one to neglect this term (see [Gui19, Cor. 4.16]).

Once this term is handled and shown to be negligible, equation (104) demonstrates the central limit theorem for test functions of the form $\Xi[\phi]$, where ϕ is compactly supported and satisfies certain regularity conditions. To extend this result to functions ϕ supported on the entire real line, more precise information about the size of the spectrum[10] is required.

In the constant β-ensembles, particles will localize in a bounded interval $[a, b]$, namely the smallest interval containing the support of $\mu_{V,\beta}$. Therefore, one can expect the contribution of ϕ outside of a compact set to be negligible. Specifically, it is expected that the following probability

$$\mathbb{P}_N^{V,\beta}\left(\left|\int_{\mathbb{R}} \phi(x)d\widehat{\mu}_N(x) - \int_{\mathbb{R}} \phi(x)\mathbf{1}_{[a,b]}(x)d\widehat{\mu}_N(x)\right| > t\right)$$

is really small.

However, in the high-temperature regime, since the support of $\mu_{V,P}$ is the entire real line, more precise estimates on the size of the spectrum are required. In the works [Pak18, Lam21b], the authors managed to localize the spectrum in an interval of the form $[-E_N, E_N]$, where $E_N \underset{N\to\infty}{\sim} V^{-1}(\log N)$. This allows for the proof of the central limit theorem of the form $\Xi[\phi]$, where ϕ is smooth enough and decays fast enough at infinity, *i.e.*:

$$\sqrt{N}\int_{\mathbb{R}} \Xi[\phi](x)d(\widehat{\mu}_N - \mu_{V,P})(x) \xrightarrow[N\to\infty]{\mathrm{law}} \mathcal{N}\left(0, \sigma^2(\phi)\right)$$

for $\sigma^2(\phi) \geq 0$. To complete the proof and obtain a central limit theorem for the linear statistics, and this for an explicit and general class of functions ψ, one must invert the operator Ξ and repeat the previous procedure with $\phi \overset{(\mathrm{def})}{=} \Xi^{-1}[\psi]$. The analog of this operator was inverted in [HL21], where the particles lie on the unit circle. However, due to the fact that, in the circular case, this operator then acts on compactly supported functions, the techniques do not generalize easily. Inverting this unbounded operator, which acts on functions defined on an unbounded domain, is therefore one of the main obstacles in the proof, and more involved techniques are required. We now explain how we managed to overcome this difference from the circular case.

To invert Ξ, it is enough to invert the operator $\mathcal{L}$, defined as $\mathcal{L}[\phi] \overset{(\mathrm{def})}{=} \Xi[\phi']$, since $\Xi^{-1}[\phi] = (\mathcal{L}^{-1}[\phi])'$. The heuristic is that the operator $\mathcal{L}$ is a second-order differential operator that is positive on a certain Hilbert space. Following [HL21], this operator can be analyzed using Hilbert space techniques. To use this set of tools, we must restrict ourselves to the class of potentials V such that the associated equilibrium measure $\mu_{V,P}$ satisfies the Poincaré inequality:

$$\exists C_{\mathrm{Poin}}^{V,P} > 0, \forall f \in \mathcal{C}_c^1(\mathbb{R}), \qquad \mathrm{Var}_{\mu_{V,P}}(f) \leq C_{\mathrm{Poin}}^{V,P}\int_{\mathbb{R}} |f'(x)|^2 d\mu_{V,P}(x).$$

[10] The set of integration variables $(y_1, \ldots, y_N)$ is heuristically thought as the spectrum of a random matrix

Polynomial potentials of even degree with positive leading coefficient or the hyperbolic cosine are example of such potentials. This is verified for any smooth potential that is strictly convex outside of a compact set. With this inequality, one can show that the pair $(\mathsf{H}, \langle ., . \rangle_{\mathsf{H}})$, where

$$\mathsf{H} \overset{(\text{def})}{=} \left\{ u \in L^2(\mu_{V,P}) \;\middle|\; u' \in L^2(\mu_{V,P}), \int_{\mathbb{R}} u(x) d\mu_{V,P}(x) = 0 \right\}, \qquad \langle u, v \rangle_{\mathsf{H}} \overset{(\text{def})}{=} \langle u', v' \rangle_{L^2(\mu_{V,P})} ,$$

defines a Hilbert space. In particular, the operator $\mathcal{L}$, seen as an unbounded operator on the previous weighted Sobolev space can be decomposed as

$$-\mathcal{L} = \mathcal{A} + 2P\mathcal{W} , \tag{106}$$

where $\mathcal{A}$ and $\mathcal{W}$ are given for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ by:

$$\mathcal{A}[\phi] \overset{(\text{def})}{=} -\frac{(\phi' \rho_{V,P})'}{\rho_{V,P}}, \qquad \mathcal{W}[\phi] \overset{(\text{def})}{=} -\mathcal{H}[\phi' \rho_{V,P}] + \int_{\mathbb{R}} \mathcal{H}[\phi' \rho_{V,P}](y) d\mu_{V,P}(y) \tag{107}$$

where $\mathcal{H}$ denotes the Hilbert transform. It is possible to show that $\mathcal{A}$ and $\mathcal{W}$ are symmetric, positive semi-definite operators on this space and that their respective kernels are given by the one-dimensionnal vector space of constant functions. Furthermore, by exploiting that $\mathcal{A}$ is a Sturm-Liouville operator *i.e.* an oeprator of the form $\dfrac{d}{dx}\left(p\dfrac{d}{dx}\right) + q$, one can show that $\mathcal{A}$ is conugated to a Schrödinger operator of the form $-\Delta + w_V$ *via*:

$$\frac{1}{\sqrt{\rho_{V,P}}} \mathcal{A} \sqrt{\rho_{V,P}} = -\Delta + w_V ,$$

where w_V is a potential such that $w_V(x) \to +\infty$ as $|x| \to \infty$. It is a standard fact from the theory of such operators that such a Schrödinger operator is diagonalizable, and thus, so is $\mathcal{A}$. From this, one can deduce that $\mathcal{A}$ is positive on a domain $\mathcal{D}(\mathcal{A})$, *i.e.* there exists $c > 0$ such that for all $\phi \in \mathcal{D}(\mathcal{A})$,

$$\langle \mathcal{A}[\phi], \phi \rangle_{\mathsf{H}} > c \|\phi\|_{\mathsf{H}}^2 .$$

Above, c can be taken equal to $\lambda_1(\mathcal{A})$, the smallest eigenvalue of $\mathcal{A}$. Thus, by (106), $-\mathcal{L}$ is also positive. This fact allows us to extend $\mathcal{L}$ to a domain $\mathcal{D}(\mathcal{L})$, known as the Friedrichs extension see Appendix A, such that it is invertible on this space.

Theorem I.8.2 (Inversion of the master operator) *$-\mathcal{L} : \mathcal{D}(\mathcal{L}) \longrightarrow \mathsf{H}$ is bijective. Furthermore, $(-\mathcal{L})^{-1}$ is positive, continuous from $(\mathsf{H}, \|.\|_{\mathsf{H}})$ to $(\mathcal{D}(\mathcal{L}), \|.\|_{\mathsf{H}})$. More precisely, for all $f \in \mathsf{H}$,*

$$\|\mathcal{L}^{-1}[f]\|_{\mathsf{H}} \leq \lambda_1(\mathcal{A})^{-1/2} \|f\|_{\mathsf{H}} . \tag{108}$$

By construction, it remains obvious that its inverse $\mathcal{L}^{-1}$ satisfies (108). This continuity result will be crucial for Chapter 2 to get controls on Ξ^{-1} with respect to L^2 and L^∞ norms.

Once this operator is inverted, it is necessary to show that there exists a class of functions ϕ such that $\Xi^{-1}[\phi]$ satisfies the regularity and decay conditions required to consider non-compactly supported functions. We showed that the class of functions, denoted by $\mathcal{T}$, defined by:

$$\mathcal{T} \overset{(\text{def})}{=} \left\{ f \in \mathcal{C}^2(\mathbb{R}) \,|\, f, f', f'' \text{ are bounded}, \int_{\mathbb{R}} f(x) d\mu_{V,P}(x) = 0 \right\}. \tag{109}$$

is such that for every $\phi \in \mathcal{T}$, $\Xi^{-1}[\phi]$ is smooth and decay fast enough to make rigorous the heuristic change of variables in $\mathcal{Z}_N[V]$ detailed above. This can be shown by using the regularity conditions on the domain $\mathcal{D}(\mathcal{L})$ and the integral representation for Ξ^{-1}

$$\Xi^{-1}[\phi](x) = \frac{1}{\rho_{V,P}(x)} \int_x^{+\infty} d\mu_{V,P}(t) \left[\phi(t) + 2P \Big\{ \mathcal{H}[\Xi^{-1}[\phi]\rho_{V,P}](t) - \int_{\mathbb{R}} \mathcal{H}[\Xi^{-1}[\phi]\rho_{V,P}](y) d\mu_{V,P}(y) \Big\} \right] . \tag{110}$$

which comes from the following resolvent formula:

$$\mathcal{L}^{-1}[\phi] = -\mathcal{A}^{-1}\left[\phi + 2P\mathcal{W} \circ \mathcal{L}^{-1}[\phi]\right].$$

This integral representation allows one to capture the growth of $\Xi^{-1}[\phi]$ and its derivatives and verify that if $\phi \in \mathcal{T}$, then $\Xi^{-1}[\phi]$ is smooth enough and decays rapidly enough at infinity so that the CLT is valid for ϕ. All these ingredients allowed us to conclude about Theorem I.5.2.

I.8.2 Chapter 2: Asymptotics of the partition function for β-ensembles at high temperature

This result is based on the following article **[DG24]**. We establish the existence of an all-order AE of the partition function $\mathcal{Z}_N[V_{G,\phi}]$ defined in (59) and $V_{G,\phi}(x) = x^2/2 + \phi(x)$ where ϕ is a bounded smooth function.

Theorem I.8.3 (AE of the partition function) *Let $\phi \in L^2(\mathbb{R})$, such that $\phi^{(k)} \in L^2(\mathbb{R})$ for all $k \geq 0$. There exists a unique sequence $(c_i)_{i\geq 0} \in \mathbb{R}^{\mathbb{N}}$ depending on ϕ and P, such that for all $K \geq 0$,*

$$\frac{1}{N}\log \mathcal{Z}_N[V_{G,\phi}] = \sum_{i=0}^{K} \frac{c_i}{N^i} + O\left(N^{-(K+1)}\right).$$

The strategy is based on the analysis of the loop equations, see Subsection I.4. In order to make the previous scheme work, one needs to be able to obtain the following results:

(i) An *a priori* bound on the linear statistics, namely

$$|\langle f_n \rangle_{\overset{n}{\otimes}\mathcal{L}_N}| \leq \frac{C}{N^{c(n)}}\mathcal{N}_n^{(m_n)}(f_n)$$

where $(c(n)), (m_n)$ are increasing sequences going to ∞ as $n \to \infty$, $C > 0$ a positive constant and $\mathcal{N}_n^{(m)}$ a norm on functions of n variables which admit derivatives of order m.

(ii) Continuity inequalities for the inverse of the master operator on the functional spaces suggested by $\mathcal{N}_n^{(m)}$ namely:

$$\mathcal{N}_n^{(m)}\left(\Xi_1^{-1}[f]\right) \leq C\mathcal{N}_n^{(m+1)}(f)$$

(iii) A control on the dependence in t of integrals against $\mu_{V_{G,\phi,t},P}$ where $V_{G,\phi,t}(x) = x^2/2 + t\phi(x)$.

These results are the main requirements that allow one to perform the loop equations analysis, and therefore to first deduce an AE for the linear statistics.

For the point *(i)*, we follow the scheme used in [BGK16, Corollary 3.1.10]. **When the particles concentrate on a compact set**, *i.e.* when the limiting measure is compactly supported (such as $\mu_{V,\beta}$ for the classical β-ensembles, or $\mu_{\mathrm{eq}}^{(N)}$ for the sinh-model), one can show that

$$\langle f \rangle_{\overset{n}{\otimes}\mathcal{L}_N} = \langle f_{|\mathrm{c}} \rangle_{\overset{n}{\otimes}\mathcal{L}_N} + \text{ exponential errors} \tag{111}$$

where $f_{|\mathrm{c}}(\xi_1, \ldots, \xi_n) \overset{(\mathrm{def})}{=} f(\xi_1, \ldots, \xi_n)\prod_{i=1}^n \chi(\xi_i)$ where χ is a compactly supported function equal to 1 on a set containing the support of the equilibrium measure. Thus, in this framework, following the scheme explained in Subsection I.4, deriving an *a priori* bound on $\langle f \rangle_{\overset{n}{\otimes}\mathcal{L}_N}$ for general f amounts to show it for the truncated function f_{c} as our estimates are much more imprecise than exponential errors. By the procedure detailled in Subsection I.4, one obtains a control in terms of $\|f_{\mathrm{c}}\|_{H^{n/2}(\mathbb{R}^n)}$. Since this function is compactly supported, it is possible to transform the $H^{n/2}$-norms in terms of a W_n^∞-norm, namely:

$$\|f_{\mathrm{c}}\|_{H^{n/2}(\mathbb{R}^n)} \leq \|f_{\mathrm{c}}\|_{W_n^\infty(\mathbb{R}^n)} \leq \|f\|_{W_n^\infty(\mathbb{R}^n)}.$$

These bounds allow to only require a finite W_n^∞-norm property on the functions for which the *a priori* bound is admissible. We were however not able to reproduce this procedure in the high temperature regime.

Indeed, showing that $\langle f\rangle_{\otimes\mathcal{L}_N}^n - \langle f_{|\mathrm{c}}\rangle_{\otimes\mathcal{L}_N}^n$ goes to zero exponentially fast as N goes to infinity, even if ones takes a N-dependent truncation, namely chooses a smooth function χ_N such that, it is equal to 1 on $[-E_N, E_N]$ and 0 on $[-E_N-1, E_N+1]^c$ where $E_N \underset{N\to\infty}{\longrightarrow} +\infty$ is difficult here. The unability to construct a convenient truncation which would produce (111) leads to the requirement to deal with several norms: ($H^{n/2}$ and W_1^∞) at the same time. The *a priori* bound is supposed to be applied to the linear statistics involved in the loop equations, *i.e.* linear statistics with test-functions of the form $\mathsf{A}\circ\Xi^{-1}[f]$, where Ξ^{-1} is the inverse of the master operator and $\mathsf{A}\in\{\mathcal{D}, \Theta^{(a)}\}$ which were defined in (35) and (40). In order to apply it for this type of linear-statistics, one must then obtain the continuity with respect t o two norms, for the different operators involved and especially the master operator. This requirement increases considerably the complexity and technicality of the proof. Point *(i)* is still completed.

For the point *(ii)*, the main difficulty is to obtain continuity results with respect to W_n^∞ and $H^{n/2}$ norms, for the master operator Ξ^{-1}. We obtained them namely:
Theorem I.8.4 (Controls on the master operator) *Let $n\geq 1$, there exists a $C>0$ such that for all $f\in H^{n+1}(\mathbb{R})$:*

$$\|\Xi^{-1}[f]\|_{H^n(\mathbb{R})} \leq C\|f\|_{H^{n+1}(\mathbb{R})}.$$

Moreover, there exists $C'>0$, such that for all $g\in W_{n+1}^\infty(\mathbb{R})$,

$$\|\Xi^{-1}[g]\|_{W_n^\infty(\mathbb{R})} \leq C'\|g\|_{W_{n+1}^\infty(\mathbb{R})}.$$

They follow by obtaining a convenient integral representation for the inverse $\Xi^{-1}[\phi]$. Our strategy is based on the resolvent formula. By recalling (68), we get:

$$f(x)\rho_{V,P}(x) = -(\Xi^{-1}[f]\rho_{V,P})'(x) - 2P\rho_{V,P}(x)\mathcal{H}[\Xi^{-1}[f]\rho_{V,P}](x) + 2P\rho_{V,P}(x)\int_{\mathbb{R}}\mathcal{H}[\Xi^{-1}[f]\rho_{V,P}](y)d\mu_{V,P}(y). \quad (112)$$

and thus by integrating on $[x,+\infty[$, we get:

$$\Xi^{-1}[f](x) = \frac{1}{\rho_{V,P}(x)}\int_x^{+\infty} d\mu_{V,P}(t)\left[f(t) + 2P\Big\{\mathcal{H}[\Xi^{-1}[f]\rho_{V,P}](t) - \int_{\mathbb{R}}\mathcal{H}[\Xi^{-1}[f]\rho_{V,P}](y)d\mu_{V,P}(y)\Big\}\right]. \quad (113)$$

This integrand is composed of two terms: the term in $f(t)$, which can be handled by standard techniques, and the term proportional to $2P$. The latter involves the master operator Ξ^{-1} which makes this formula circular. However, because of the L^2-isometry property of the Hilbert transform and, crucially, the continuity result (108), one can show the continuity of this expression in f with respect to L^2 and L^∞ norms. Furthermore, to show the continuity with stronger norms namely H^n or W_n^∞, one must differentiate and manage the diverging parts appearing. This can be done by using integration by parts. This can be seen at the level of the operator $\mathcal{A}$ which is a building block of $\mathcal{L}$ (107), we have:

$$\mathcal{A}^{-1}[f]'(x) = \frac{1}{\rho_{V,P}(x)}\int_x^{+\infty} f(t)d\mu_{V,P}(t) \quad (114)$$

which implies by differentiating and integrating by parts:

$$\begin{aligned}\mathcal{A}^{-1}[f]''(x) &= \frac{-\rho'_{V,P}(x)}{\rho_{V,P}(x)^2}\int_x^{+\infty} f(t)d\mu_{V,P}(t) - f(x)\\ &= \frac{-\rho'_{V,P}(x)}{\rho_{V,P}(x)^2}\left[f(t)\frac{\rho_{V,P}(t)}{\rho'_{V,P}(t)}\rho_{V,P}(t)\right]_x^{+\infty} + \frac{\rho'_{V,P}(x)}{\rho_{V,P}(x)^2}\int_x^{+\infty}\left(f\frac{\rho_{V,P}}{\rho'_{V,P}}\right)'(t)d\mu_{V,P}(t) - \phi(x)\\ &= \frac{\rho'_{V,P}(x)}{\rho_{V,P}(x)^2}\int_x^{+\infty}\left(f\frac{\rho_{V,P}}{\rho'_{V,P}}\right)'(t)d\mu_{V,P}(t). \quad (115)\end{aligned}$$

Since, in the last integral, $\rho'_{V,P}$, which vanishes at least once, appears in the denominator, this formula can only hold in a neighborhood of $+\infty$ (the same can be done near $-\infty$). Even though, the prefactor of the integral diverges strongly, the order of this prefactor being roughly $V'(x)e^{2V(x)}$, the integrand decay at the inverse same speed and by integrating, a factor $1/V'(x)$ must be added which allow obtaining an integrable decaying behavior at infinity. This procedure can be applied to deal with every derivative of $\mathcal{A}^{-1}[f]'$ and all the derivatives of $\Xi^{-1}[f]$. To deal with the behavior of $\mathcal{A}^{-1}[f]''$ in a compact set, the first equality in (115) is convenient and doesn't exhibit any singularity. These formulations allow obtaining sufficiently good controls with respect to the two norms involved in the problem. This completes point *(ii)*.

Points *(i)* and *(ii)*, make the loop equations analysis possible. This leads to the existence of the AE for the linear statistics for general potential V:

Theorem I.8.5 (AE of linear statistics) *Under assumptions 2.1.1 on the potential V, for all $\phi \in L^2(\mathbb{R}^k)$, such that $\phi^{(j)} \in L^2(\mathbb{R}^k)$ for all $j \geq 0$, there exists a unique sequence $(b_i)_{i\geq\lceil k/2\rceil}$ depending on V, ϕ and P such that forall $K > 0$:*

$$\langle\phi\rangle^V_{\otimes^k\mathcal{L}_N} = \sum_{i=\lceil k/2\rceil}^{K}\frac{b_i}{N^i} + O\left(N^{-(K+1)}\right).$$

Once the AE is obtained up to any order for the linear statistics and this, for smooth bounded functions, one can try to integrate this AE into one for the partition function *via* the following formula:

$$\begin{aligned}\log\mathcal{Z}_N[V_{G,\phi}] &= \log\mathcal{Z}_N[V_G] - N\int_0^1\langle\phi\rangle^{V_{G,\phi,t}}_{\widehat{\mu}_N}\,dt\\ &\sim \sum_{k=0}^{K}\frac{c_{G,k}}{N^k} + O(N^{-(K+1)}) + \sum_{k=0}^{K}\frac{\int_0^1 c_k(\phi,t)dt}{N^k} + \int_0^1 O(N^{-(K+1)})dt \quad (116)\end{aligned}$$

Since the AE of $\log\mathcal{Z}_N[V_G]$ can be obtained up to any order in N^{-1} as in (26), in order to make this integration-step rigorous, one needs to integrate each term $c_k(\phi,t)$ arising in the AE of the linear statistics as well as the remainder term which also depends on t. One can show that proving that these integrals are well-defined boils down to realize point *(iii)*.

Let ϕ smooth and bounded, V a general potential and define $V_{\phi,t} : x \mapsto V(x)+t\phi(x)$ where $t \in [0,1]$. One can show that every quantity that we need to integrate depend on t only *via* $\rho_{V_{\phi,t}}$. Showing that the map $t \mapsto \rho_{V_{\phi,t}} \in W^\infty_n(\mathbb{R})$ is continuous for all $n \in \mathbb{N}$ is enough to integrate all of these terms depending on t and conclude about the AE for $\log\mathcal{Z}_N[V_{G,\phi}]$.

For the classical β-ensemble, this continuity result is easy to show. We argued in Subsection I.4.5, following [BG13a], that if V and W are such that $\mu_{V,\beta}$ and $\mu_{W,\beta}$ have the same support $[a,b]$, then the equilibrium measure associated to the convex combination of potentials $tV + (1-t)W$ is the convex combination of the equilibrium measure $t\mu_{V,\beta} + (1-t)\mu_{W,\beta}$, (57). **For the high temperature regime**, this step turns out to be way more involved.

First it is easy to see, taking two potentials V and W, that

$$\rho_{tV+(1-t)W,P} \neq t\rho_{V,P} + (1-t)\rho_{W,P}. \tag{117}$$

Indeed, the above equation isn't compatible with the asymptotic behavior at infinity found in (64). To show that $\rho_{V_{\phi,t},P}$ converges uniformly to $\rho_{V_{\phi,t_0},P}$ when $t \to t_0$, we set the following definition:

$$u_t \overset{(\text{def})}{=} \frac{\rho_{V_{\phi,t},P} - \rho_{V_{\phi,t_0},P}}{\delta t} \frac{1}{\rho_{V_{\phi,t_0},P}}, \qquad \delta t \overset{(\text{def})}{=} t - t_0. \tag{118}$$

By using (63), one can show that u_t solves the following equation:

$$1 + \delta t u_t(x) = \exp\left\{ \delta t \Big(-\phi(x) + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0},P}(y) + 2P \int_{\mathbb{R}} \log|x-y| u_t(y) d\mu_{V_{\phi,t_0},P}(y) \right.$$
$$\left. -2P \iint_{\mathbb{R}^2} \log|y-z| u_t(z) d\mu_{V_{\phi,t_0},P}(z) d\mu_{V_{\phi,t_0},P}(y) + \frac{1}{\delta t} \int_{\mathbb{R}} \left[\log\left(1+\delta t u_t(y)\right) - \delta t u_t(y)\right] d\mu_{V_{\phi,t_0},P}(y) \Big) \right\}.$$

By linearizing the exponential in the above equation, one can show that there exists a continuous invertible operator $\mathcal{T} \overset{(\text{def})}{=} -\mathcal{L} \circ \mathcal{A}^{-1}$, and some t-continuous operators $\mathcal{U}_t$, $\mathcal{V}_t$ such that u_t is a fixed point of the following equation:

$$u_t = \mathcal{T}^{-1} \circ \mathcal{V}_t[u_t], \qquad \text{with} \quad \mathcal{V}_t[u] \overset{(\text{def})}{=} -\phi + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) + \delta t \mathcal{U}_t[u]. \tag{119}$$

One can show that the operator $\mathcal{T}^{-1}$ is continuous and that $\mathcal{V}_t$ is Lipschitz-continuous with respect to L^∞ norms, with Lipschitz constant proportional to δt. Thus, by picking t close enough to t_0, which amounts to take δt small enough, an application of Banach fixed-point theorem leads to the fact that u_t is the unique fixed point of the t-continuous operator $\mathcal{T}^{-1} \circ \mathcal{V}_t$. It is then a standard fact that the unique fixed point of a contractive operator continuous in t is itself continuous in t. This reasoning allows us prove the following theorem:

Theorem I.8.6 *Under assumptions 2.1.1 on the potential V, for all $i \in \mathbb{N}$ and for all $\phi \in L^2(\mathbb{R})$, such that $\phi^{(k)} \in L^2(\mathbb{R})$ for all $k \geq 0$,*

$$\|\rho_{V_{\phi,t}} - \rho_{V_{\phi,t'}}\|_{W_i^\infty(\mathbb{R})} \underset{t \to t'}{\to} 0$$

where $V_{\phi,t} : x \mapsto V(x) + t\phi(x)$ where $t \in [0,1]$. The $W_i^\infty(\mathbb{R})$-norm is defined as $\|f\|_{W_n^\infty(\mathbb{R})} \overset{(\text{def})}{=} \max_{k \in [\![0,n]\!]} \|f^{(k)}\|_{L^\infty(\mathbb{R})}$. Furthermore, for all $x \in \mathbb{R}$, $t \mapsto \rho_{V_{\phi,t}}(x) \in \mathcal{C}^\infty(\mathbb{R})$ and satisfies the following integro-differential equation for all $t \geq 0$ and $x \in \mathbb{R}$:

$$\partial_t \rho_{V_{\phi,t}}(x) = \left(-\phi(x) + \int_{\mathbb{R}} \phi(s) \rho_{V_{\phi,t}}(s) ds \right) \rho_{V_{\phi,t}}(x).$$

I.8.3 Chapter 3: On the equilibrium measure for the Lukyanov integral

This article is based on the preprint **[DGK24]** which gives an explicit form for the leading order term in the AE of $\log \frac{\mathcal{Z}_N^{\sinh}[V_\alpha]}{\mathcal{Z}_N^{\sinh}[V_0]}$, in terms of the parameters α, ω_1 and ω_2. This result allows justifying that the limit in Lukyanov's conjecture about the vacuum expectation value of the exponential of the field operator, introduced in (83), exists.

The starting point is the interpolation equation, namely

$$\log \frac{\mathcal{Z}_N^{\sinh}[V_{\mathfrak{b}}]}{\mathcal{Z}_N^{\sinh}[V_0]} = -N\log N \int_0^{\mathfrak{b}} \partial_t \log \mathcal{Z}_N^{\sinh}[V_t]dt = N\log N \int_0^{\mathfrak{b}} \langle \mathrm{id}\rangle_{\widehat{\mu}_N}^{V_t}\, dt$$
$$\underset{N\to\infty}{=} N\log N \int_0^{\mathfrak{b}} \langle \mathrm{id}\rangle_{\mu_{\mathrm{eq},t}^{(N)}}\, dt + \mathfrak{R}_N. \quad (120)$$

While we are only able to show that the remainder term is of order $O(\sqrt{N}\log^2 N)$, we make the following assumption:

Assumptions I.8.7

$$\mathfrak{R}_N \underset{N\to\infty}{=} O(1). \quad (121)$$

We know that the limiting measure of $\widehat{\mu}_N$ is the singular measure $\frac{1}{2}(\delta_{-1}+\delta_1)$. Since this measure doesn't capture enough of the N-dependence of $\widehat{\mu}_N$, it is more convenient to use the N-dependent equilibrium measure $\mu_{\mathrm{eq}}^{(N)}$. The first step that one needs to take care is the construction of this measure. We rely on the equation $\mathcal{S}_N[\rho_{\mathrm{eq}}^{(N)}] = V_N'$, where $\mathcal{S}_N$ and V_N have been introduced in (92) and (85), and on the inversion of this operator, see Theorem I.7.5. The potential is explicit, and we recall its coarse form:

$$V_N(\xi) = \frac{\mathfrak{r}}{N\tau_N}\cosh(\xi\tau_N) - \frac{\alpha}{N}\xi + \text{negligible errors.}$$

The exponential errors come from the integral term in (85) which decreases like $O(e^{-2|x|})$ at infinity. With the rescaling in τ_N, it can be seen that this term decreases in $\frac{1}{N^3\tau_N}$. This term will only contributes to the remainders and for this heuristic, we can neglect its effects.

By the explicit form of V_N, using the residue theorem and growth conditions at infinity of the Riemann-Hilbert solution χ, one is able to obtain the following explicit expression for $\rho_{\mathrm{eq}}^{(N)}$ and this for general $a_N < b_N$:

$$\rho_{\mathrm{eq}}^{(N)}(\xi) = \frac{\mathrm{i}\tau_N}{2\pi N}\int_{\mathbb{R}+2\mathrm{i}\varepsilon} \frac{d\lambda e^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}}{2\mathrm{i}\pi}\left(\frac{\mathfrak{r}}{2}\Big[\frac{K(\lambda,\mathrm{i})e^{\tau_N b_N}}{\lambda-\mathrm{i}} - \frac{K(\lambda,-\mathrm{i})e^{-\tau_N a_N}}{\lambda+\mathrm{i}}\Big] - \frac{\mathfrak{b}\chi_{11}(\lambda)\chi_{12,+}(0)}{\lambda}\right)$$
$$+ \text{negligible errors,} \quad (122)$$

where the kernel K is defined as follows:

$$K(\lambda,\mu) \overset{(\mathrm{def})}{=} \chi_{11}(\lambda)\chi_{12}(\mu) - \frac{\mu}{\lambda}\chi_{11}(\mu)\chi_{12}(\lambda).$$

As explained in Subsection I.7.4, to construct $\rho_{\mathrm{eq}}^{(N)}$, one needs to find the endpoints a_N and b_N. It is done by exploiting the constraints. The first constraint considered is the one ensuring the square-root like vanishing behavior of $\rho_{\mathrm{eq}}^{(N)}$, namely $\mathfrak{C}_{1,N} = 0$ with

$$\mathfrak{C}_{1,N} \overset{(\mathrm{def})}{=} \int_{\mathbb{R}+\mathrm{i}\varepsilon} \frac{d\mu}{2\mathrm{i}\pi} e^{-\mathrm{i}\tau_N b_N\mu}\chi_{11}(\mu)\mathcal{F}\big[V_N'\mathbf{1}_{[a_N,b_N]}\big](\tau_N\mu). \quad (123)$$

By the residue theorem and the knowledge on the asymptotics of χ, we get:

$$\mathfrak{C}_{1,N} = \frac{\mathrm{i}}{N\tau_N}\left(\frac{\mathfrak{r}}{2}\left(-e^{\tau_N b_N}\chi_{11}(\mathrm{i}) + e^{-\tau_N a_N}\chi_{11}(-\mathrm{i})\right) + \mathfrak{b}\chi_{11,+}(0)\right) + \text{error term.} \quad (124)$$

The second constraint on the mass $\mathfrak{C}_{2,N} = 1$ where

$$\mathfrak{C}_{2,N} \overset{(\mathrm{def})}{=} \int_{a_N}^{b_N} \rho_{\mathrm{eq}}^{(N)}(\xi)d\xi \quad (125)$$

can also be computed by using Fubini's theorem and the form found in (122) to get:

$$2\pi N\mathfrak{C}_{2,N} = \mathfrak{b}\chi'_{11,+}(0)\chi_{12,+}(0) + \frac{\mathfrak{r}}{2}\Big\{\big(e^{\tau_N b_N} + e^{-\tau_N a_N}\big)\Big[\mathrm{i}\chi_{11,+}(0)\chi_{12}(\mathrm{i}) - \mathrm{i}\chi_{11}(\mathrm{i})\chi_{12,+}(0)\Big] \\ + \chi_{11}(\mathrm{i})\chi_{11,+}(0)e^{\tau_N b_N} - \chi_{11}(\mathrm{i})\chi'_{12,+}(0)\big(e^{\tau_N b_N} - e^{-\tau_N a_N}\big)\Big\}.$$

Using the local inversion theorem, one is able to find a unique couple (a_N, b_N) solving these equations and then conclude on the construction of $\rho_{\mathrm{eq}}^{(N)}$. Before trying to compute $\int_{a_N}^{b_N} \xi\rho_{\mathrm{eq}}^{(N)}(\xi)d\xi$, one needs to obtain the asymptotic behavior of a_N and b_N, this can be done by solving explicitly the two constraints which leads to

$$b_N = 1 + \frac{1}{\log N}\log\Big(\frac{2\pi\beta R_\uparrow(\mathrm{i})\sqrt{\omega_1+\omega_2}}{\mathfrak{r}}\Big) + \text{error}$$
$$a_N = -1 - \frac{1}{\log N}\log\Big(\frac{2\pi\beta R_\uparrow(\mathrm{i})\sqrt{\omega_1+\omega_2}}{\mathfrak{r}}\Big) + \text{error}$$

where $R_\uparrow$ is an explicit special function depending on the parameter ω_1, ω_2. Finally, using Fubini and all the asymptotics that we computed, this is enough to obtain the following leading order term for the expectation value of $\mu_{\mathrm{eq}}^{(N)}$:

$$\langle \mathrm{id}\rangle_{\mu_{\mathrm{eq},t}^{(N)}} \underset{N\to\infty}{=} \frac{t}{\pi(\omega_1+\omega_2)N} + O\left(\frac{1}{N\log N}\right). \tag{126}$$

This leads to the following conjecture:

Conjecture I.8.8 *Under the assumption* (121)

$$\log\frac{\mathcal{Z}_N^{\sinh}[V_{\mathfrak{b}}]}{\mathcal{Z}_N^{\sinh}[V_0]} \underset{N\to\infty}{=} \frac{\mathfrak{b}^2}{2\pi(\omega_1+\omega_2)}\log N + O(1). \tag{127}$$

Chapter 1

CLT for real β-ensembles at high temperature

"When it is asked, What is the foundation of all conclusions from experience? we can give no satisfactory answer but must return to the same point, and confess that the empirical evidence of cause and effect is ultimately based on habit or custom, not on logical reasoning or any form of necessary connection." David Hume

Contents

1.1 Introduction and main result

The β-ensemble of dimension $N \geq 1$ with parameter $\beta > 0$ and potential V is the probability measure on $\mathbb{R}^N$ given by

$$d\mathbf{P}_N^{\beta,V}(x_1,\dots,x_N) = \frac{1}{Z_N(V,\beta)} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,. \tag{1.1}$$

The potential V has to be chosen so that the partition function

$$Z_N(V,\beta) = \int_{\mathbb{R}^N} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N$$

is finite. This is the case for example if for some $\beta' > \max(1,\beta)$,

$$\liminf_{|x|\to\infty} \frac{V(x)}{N\beta' \log|x|} > 1 \,, \tag{1.2}$$

see [AGZ10b, equation (2.6.2)]. The parameter β, which is allowed to depend on N, is the so-called inverse temperature.

Under the special choice of $V_G(x) = \frac{x^2}{2}$, the measure (1.1) can be seen as the joint law of the (unordered) eigenvalues of certain matrix models:

- For $\beta = 1$ (resp. $\beta = 2$), it is the law of the eigenvalues of the Gaussian Orthogonal Ensemble (resp. Gaussian Unitary Ensemble), [AGZ10b, Theorem 2.5.2].
- For general $\beta > 0$, potentially depending on N, it is the law of the spectrum of certain tri-diagonal random matrices as shown by Dumitriu and Edelman in [DE02b].

We consider here the *high temperature regime* where β scales as $1/N$, and write $\beta = \frac{2P}{N}$ for some $P > 0$. The corresponding measure is therefore

$$d\mathbb{P}_N^{V,P}(x_1, \dots, x_N) = \frac{1}{\mathcal{Z}_N^P[V]} \prod_{i<j} |x_i - x_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,, \tag{1.3}$$

with partition function

$$\mathcal{Z}_N^P[V] = \int_{\mathbb{R}^N} \prod_{i<j} |x_i - x_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,. \tag{1.4}$$

It was shown in [GZ19] that under $\mathbb{P}_N^{V,P}$, the sequence of empirical measures

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$$

satisfies a large deviation principle at speed N with strictly convex, good rate function. As a consequence, $\hat{\mu}_N$ converges almost surely in distribution towards a deterministic measure μ_V as N goes to infinity, meaning that almost surely, for every bounded continuous $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f(x) d\hat{\mu}_N(x) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(x) d\mu_V(x) \,.$$

The limiting measure μ_V can be seen to have a density ρ_V which satisfies for almost every $x \in \mathbb{R}$

$$V(x) - 2P \int_{\mathbb{R}} \log|x - y| \rho_V(y) dy + \log \rho_V(x) = C_{V,P} \,, \tag{1.5}$$

where $C_{V,P}$ is constant (see [GM22, Lemma 3.2] for example).

The β-ensemble in the regime $\beta N \underset{N\to\infty}{\longrightarrow} 2P > 0$ has drawn a lot of attention from the random matrix and statistical physics communities lately. This regime was first considered by [CL97] with the study of Dyson Brownian motion with vanishing repulsive coefficient scaled like $\frac{1}{N}$. Gases of vortices were also studied with temperature proportional to N in [BG99]. The limiting density was then described in the case of the quadratic potential in [ABG12], as a crossover between the Wigner semicircle law (fixed $\beta > 0$ case) and the Gaussian density (case $\beta = 0$). The fluctuations of the eigenvalues in the bulk and at the edge of a configuration were studied for example in [BGP15, NT18, NT20b, Pak18, Lam21b]. These fluctuations were shown to be described by Poisson statistics in this regime. Recently, Spohn uncovered in [Spo20] a link between the study of the Classical Toda chain and the β-ensemble in the high temperature regime, showing that the limiting density of states of the classical Toda chain, distributed according to the generalized Gibbs ensemble with polynomial potential, can be computed by means of the limiting empirical measure of the β-ensemble at high temperature. In [Maz22], the author established this relation using the matrix

representation of the β-ensemble and a moment method, and in [GM22] the authors proved a large deviation principle for the empirical measure of the Toda chain, establishing the previous result for potentials with polynomial growth. See also [Spo22, GM23, MM22] for a similar link between the Ablowitz-Ladik lattice and the circular β-ensemble at high temperature. This relation can be further pushed to compute the limiting *currents* of the Toda chain through the central limit theorem for the empirical measure in the β ensemble. The computation of these currents is a crucial step to the derivation of a hydrodynamic equation for the Toda chain, and to the analysis of the correlations of the locally conserved quantities at equilibrium through *linearized hydodynamics*, see [Spo21]. The derivation of the currents *via* the central limit theorem is justified in the recent paper [MM24].

The Central Limit Theorem for the fluctuations of the linear statistics of β-ensembles was first established by [Joh98] for $\beta = 2$ and polynomial potential, then generalized and further developed in the regime where β is fixed in [Shc13, BG13a, BG13b, BLS18, LLW19]. Also an optimal local law was found in this regime in [BMP22]. The CLT was obtained in the high-temperature regime $\beta N \to 2P > 0$ by Nakano and Trinh in [NT18, Theorem 4.9] for quadratic V, relying on the tridiagonal representation for the β-ensemble with quadratic potential in [DE02b]. In [HL21], the authors prove the CLT in the case of the *circular* β-ensemble at high temperature with general potential, using a normal approximation method involving the spectral analysis of an operator associated to the limiting covariance structure. Their method allowed them to derive a Berry-Esseen bound, *i.e.* a speed of convergence of the fluctuations towards a Gaussian variable.

In this paper, we adapt part of the arguments of [HL21] to our setup. More precisely, we show that for a class of regular, convex potentials V satisfying a growth condition of the type

$$\lim_{|x|\to\infty} \frac{V''(x)}{V'(x)^2} = 0\,,$$

denoting $\nu_N = \hat{\mu}_N - \mu_V$ and considering test functions f belonging to the range of a certain integro-differential operator, the scaled fluctuations of $\hat{\mu}_N$, defined by

$$\sqrt{N}\nu_N(f) \overset{(\mathrm{def})}{=} \sqrt{N}\left(\int_{\mathbb{R}} f(x)d\mu_N(x) - \int_{\mathbb{R}} f(x)d\mu_V(x)\right)\,,$$

converge in law towards centered Gaussian law with variance depending on f.

When considering the fixed temperature regime, *i.e.* β fixed, one has to renormalize the x_i's by $\sqrt{N}$. It is shown in [AGZ10b, Theorem 2.6.1] that the measure

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i/\sqrt{N}}$$

satisfies a large deviation principle, and the limiting measure is characterized in [AGZ10b, Lemma 2.6.2] by an equation similar to (1.5). In fact, the term $\log \rho_V$ in the left-hand side of (1.5) is the only difference in the equation characterizing the limiting measure in the fixed β case. We point out the very similar characterization of the equilibrium measure corresponding to the minimization problem arising in [BGK16]. There again, the limiting measure is compactly supported. The term $\log \rho_V$ is of prime importance because its presence implies that the support of ρ_V is the whole real line. It leads to technicalities to deal with the behavior at infinity of most of the associated objects, namely dealing with weighted Lebesgue spaces $L^2(\mu_V)$ and the corresponding Sobolev spaces $H^k(\mu_V)$.

Our strategy is based on a change of variables in the partition function $\mathcal{Z}_N^P[V]$ (1.4), used for the β-ensemble at fixed temperature introduced in [Joh98, BFG15, Shc14b], and used in [Gui19, BGK16] to derive the loop equations and in [BLS18] to derive a CLT in the β-ensemble with β fixed. The

outline of the argument goes as follows: Take $\phi : \mathbb{R} \to \mathbb{R}$ smooth, vanishing fast enough at infinity, and do the change of variables in $\mathcal{Z}_N^P[V]$, $x_i = y_i + tN^{-1/2}\phi(y_i)$, $1 \leq i \leq N$, to get

$$\mathcal{Z}_N^P[V] = \int_{\mathbb{R}^N} \prod_{i<j} \left| y_i - y_j + \frac{t}{\sqrt{N}}\Big(\phi(y_i) - \phi(y_j)\Big)\right|^{\frac{2P}{N}} . \exp\left(- \sum_{i=1}^N V\left(y_i + \frac{t}{\sqrt{N}}\phi(y_i)\right)\right) \times \prod_{i=1}^N \left(1 + \frac{t}{\sqrt{N}}\phi'(y_i)\right) d^N y\,.$$

Expanding the different terms in this integral, one gets:

$$\mathcal{Z}_N^P[V] = \int_{\mathbb{R}^N} \prod_{i<j} |y_i - y_j|^{\frac{2P}{N}} . e^{-\sum_{i=1}^N V(y_i)} . \exp\left\{ \frac{t}{\sqrt{N}}\left[\frac{2P}{N}\sum_{i<j}\frac{\phi(y_i) - \phi(y_j)}{y_i - y_j} + \sum_{i=1}^N \Big(\phi'(y_i) - V'(y_i)\phi(y_i)\Big)\right]\right\} . \exp\Big(- \frac{t^2}{2}\sigma_N^2(\phi)\Big) d^N y\,,$$

where the term $\sigma_N^2(\phi)$ converges towards a limiting variance $\sigma^2(\phi)$ depending on ϕ, P and V. After dividing both parts of the equation by $\mathcal{Z}_N^P[V]$, and because of equation (1.5) characterizing μ_V, one can deduce from the last equation the convergence of the Laplace transform

$$\mathbb{E}\left[e^{t\sqrt{N}(\nu_N(\Xi[\phi]) + \text{error term})}\right] \underset{N\to\infty}{\longrightarrow} \exp\Big(\frac{t^2}{2}\sigma^2(\phi)\Big)\,, \tag{1.6}$$

where Ξ is a linear operator acting on test functions and defined by

$$\Xi[\phi](x) \overset{\text{(def)}}{=} 2P\int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{x - y} d\mu_V(y) + \phi'(x) - V'(x)\phi(x) - 2P\int_{\mathbb{R}} \mathcal{H}[\phi\rho_V](y) d\mu_V(y)\,. \tag{1.7}$$

Once the error term is taken care of, (1.6) shows the central limit theorem for test functions of the form $\Xi[\phi]$. Following [HL21], the operator $\mathcal{L}$ given by

$$\mathcal{L}[\phi] \overset{\text{(def)}}{=} \Xi[\phi'] \tag{1.8}$$

can be analyzed using Hilbert space techniques. In particular, the operator $\mathcal{L}$, seen as an unbounded operator of the Hilbert space

$$\mathsf{H} = \left\{ u \in L^2(\mu_V) \;\middle|\; u' \in L^2(\mu_V), \int_{\mathbb{R}} u(x) d\mu_V(x) = 0 \right\}, \qquad \langle u, v\rangle_{\mathsf{H}} = \left\langle u', v'\right\rangle_{L^2(\mu_V)}\,,$$

can be decomposed as

$$-\mathcal{L} = \mathcal{A} + 2P\mathcal{W}\,,$$

where $\mathcal{A}$ is a positive Sturm-Liouville operator and $\mathcal{W}$ is positive and self-adjoint. Such a writing allows us to show that $-\mathcal{L}$ is invertible, see Theorem 1.6.7.

We now state the assumptions we make on the potential V. Recall that a probability measure μ supported on $\mathbb{R}$ satisfies the Poincaré inequality if there exists $C > 0$ such that for all $f \in \mathcal{C}^1(\mathbb{R})$ with compact support:

$$\mathrm{Var}_\mu(f) \overset{\text{(def)}}{=} \int_{\mathbb{R}} \left(f(x) - \int_{\mathbb{R}} f(y) d\mu(y)\right)^2 d\mu(x) \leq C\int_{\mathbb{R}} f'(x)^2 d\mu(x)\,. \tag{1.9}$$

Assumptions 1.1.1 *The potential V satisfies:*

i) $V \in \mathcal{C}^3(\mathbb{R})$, $V(x) \underset{|x|\to+\infty}{\longrightarrow} +\infty$, $|V'(x)| \underset{|x|\to+\infty}{\longrightarrow} +\infty$ *and is such that* μ_V *satisfies the Poincaré inequality* (1.9).

ii) For all polynomial $Q \in \mathbb{R}[X]$ *and* $\alpha > 0$, $Q\left(V'(x)\right) e^{-V(x)} = \underset{|x|\to\infty}{o}(x^{-\alpha})$.

iii) Furthermore, for any sequence $(x_N)_N$ *such that* $|x_N|$ *goes to infinity, and for all real* $a < b$, *we have, as* N *goes to infinity,*

$$\frac{1}{V'(x_N)^2} \sup_{a\le x\le b} |V''(x_N + x)| \underset{N\to\infty}{\longrightarrow} 0\,.$$

iv) The function $\dfrac{1}{V'^2}$ *is integrable at infinity,* $\dfrac{V''(x)}{V'(x)} = \underset{|x|\to\infty}{O}(1)$ *and* $\dfrac{V^{(3)}(x)}{V'(x)} = \underset{|x|\to\infty}{O}(1)$.

Since we need another assumption to state our result, we postpone to Remark 1.1.3 explicit examples of potentials admissible for our main result. We now discuss the previous assumptions.

- Because *i)* implies that V goes to infinity faster than linearly, we will see that it ensures exponential decay at infinity of ρ_V. Recalling the sufficient condition for $\mathbb{P}_N^{V,P}$ of equation (1.2) to be defined, this first assumption implies that there exists $\alpha > 0$ such that $\liminf_{|x|\to\infty} \frac{V(x)}{|x|} > \alpha$. This guarantees in particular that the β-ensemble (1.3) is well-defined for all $N \ge 1$ and $P \ge 0$. We will use the fact that μ_V satisfies the Poincaré inequality to ensure that H endowed with $\langle\cdot,\cdot\rangle_{\mathsf{H}}$ is a Hilbert space.
- The second assumption ensures that any power of V' (and of V'' by *iv)*) is in $L^2(\mu_V)$ and that ρ_V, which behaves like e^{-V} up to a sub-exponential factor, belongs to the Sobolev space $H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$. Indeed, for $k \le 2$, using *iv)*, ${\rho_V}^{(k)}$ behaves at infinity like $(V')^k\rho_V$ as shown in Lemma 1.2.2 which is in $L^2(\mathbb{R})$ by assumption *ii)*.
- Assumption *iii)* will be used to localize the minimum/maximum point of a typical configuration $(x_1, \dots, x_N)$ following the law $\mathbb{P}_N^{V,P}$: this will be done in Corollary 1.4.2, which comes as a consequence of [Lam21b, Theorem 3.4]. More precisely, Corollary 1.4.2 establishes that for some sequences $(\alpha_N^+)_N, (\alpha_N^-)_N$ and $(E_N^-)_N, (E_N^+)_N$, all going to infinity, the random variables

$$\alpha_N^+\left(\max_{1\le j\le N} x_j - E_N^+\right) \qquad \text{and} \qquad \alpha_N^-\left(\max_{1\le j\le N} x_j - E_N^-\right)$$

converge in distribution. For large N, the scalars E_N^+ and E_N^- can thus be seen as the edges of a typical configuration. Furthermore,

$$V(E_N^\pm) \sim \log N\,. \tag{1.10}$$

We refer to Section 1.4 for detailed statements, which we use to lift the result of Proposition 1.5.1 from compactly supported functions to more general functions.

- We use Assumption *iv)* to control integral remainders in the proof of Theorem 1.7.1, ensuring that $\mathcal{L}^{-1}$ is regular enough *i.e.* that for sufficiently smooth functions f, $\mathcal{L}^{-1}[f]' \in H^2(\mathbb{R})$.

We will need another technical assumption to ensure that Taylor remainders arising in the proof of Theorem 1.5.2 are negligible.

Assumptions 1.1.2 *With the notations of Theorem 1.4.1, we have*

$$\sup_{d(x,I_N)\le 1} \left|V^{(3)}(x)\right| = o(N^{1/2})\,,$$

where $I_N = \left[E_N^- - 2; E_N^+ + 2\right]$.

Remark 1.1.3 Taking $V = V_{\text{conv}} + \phi$ with V_{conv}, $\phi \in \mathcal{C}^3(\mathbb{R})$ such that $\phi^{(k)}$ is bounded for $k = 0, \dots, 3$, V_{conv} is convex with $|V'_{\text{conv}}| \to +\infty$ at infinity, satisfying hypotheses *ii)*, *iii)*, *iv)* and Assumption *1.1.2*, such that there exists $\varepsilon > 0$ such that $V_{\text{conv}} - 2Pf_\varepsilon$ is convex (see Lemma 1.2.4), then V satisfies Assumptions *1.1.1* and *1.1.2*. The main point that needs to be checked is that the measure μ_V satisfies the Poincaré inequality, this will be done in Proposition 1.2.6. The function f_ε is introduced as a function that behaves like $\log|x|$ at infinity, but has second derivative which is as small as desired.

The type of functions V_{conv} that one can consider is typically the convex polynomials or $\cosh(\alpha x)$. On the other hand a *scaled* potential like e^{x^2} has a faster growing derivative at infinity and doesn't satisfy assumptions *iii)* and *iv)*.

We are now able to state the main result, *i.e.* the central limit theorem for functions belonging to the image of the operator $\mathcal{L}$ introduced in (1.8).

Theorem 1.1.4 *Assume that V satisfies Assumptions 1.1.1 and Assumption 1.1.2. Then for all $\phi \in \mathcal{C}^2(\mathbb{R})$ with ϕ, ϕ' and ϕ'' bounded such that $\int_{\mathbb{R}} \phi(x) d\mu_V(x) = 0$, we have the convergence:*

$$\sqrt{N}\nu_N(\phi) \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\left(0, (\sigma_P^V)^2(\phi)\right) \tag{1.11}$$

where the limiting variance $(\sigma_P^V)^2(\phi)$ is given by

$$(\sigma_P^V)^2(\phi) = \langle \phi, \mathcal{L}^{-1}[\phi] \rangle_{\mathsf{H}} = \int_{\mathbb{R}} \left(\mathcal{L}^{-1}[\phi]''(x)^2 + V''(x)\mathcal{L}^{-1}[\phi]'(x)^2 \right) d\mu_V(x) + P \iint_{\mathbb{R}^2} \left(\frac{\mathcal{L}^{-1}[\phi]'(x) - \mathcal{L}^{-1}[\phi]'(y)}{x - y} \right)^2 d\mu_V(x) d\mu_V(y)\,. \tag{1.12}$$

Remark 1.1.5 In particular, let $\phi_1, \dots, \phi_d \in \mathcal{C}^1(\mathbb{R})$, all satisfying the hypotheses of Theorem 1.1.4. Considering for $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$ the function $\phi = \sum_{j=1}^d t_j \phi_j$, ϕ also satisfies the hypotheses. Then we have

$$\mathbb{E}\left[e^{i\sqrt{N}(t_1\nu_N(\phi_1) + \dots + t_d\nu_N(\phi_d))} \right] = \mathbb{E}\left[e^{i\sqrt{N}\nu_N(\phi)} \right],$$

and the last expectation converges towards the characteristic function of a centered Gaussian variable. Thus, we see that the vector $\sqrt{N}(\nu_N(\phi_1), \dots, \nu_N(\phi_d))$ converges towards a centered Gaussian vector whose covariance matrix is given by $\left(\langle \phi_i, \mathcal{L}^{-1}[\phi_j] \rangle_{\mathsf{H}}\right)_{1 \le i,j \le d}$.

Remark 1.1.6 In Theorem 1.1.4, it's possible to consider functions such that $\int_{\mathbb{R}} \phi(x) d\mu_V(x) \neq 0$ if one replaces ϕ by $\phi - \int_{\mathbb{R}} \phi(x) d\mu_V(x)$. Also, the assumption 1.1.1 *iv)* is restrictive. In the setting of a polynomial potential V, by a more careful analysis, one might improve the result, by considering less regular test-functions ϕ.

As a tool to deal with the error term of equation (1.6), we establish a concentration inequality for the empirical measure. This inequality is stated in terms of the following distance over the set of probability distributions $\mathcal{P}(\mathbb{R})$.

For $\mu, \mu' \in \mathcal{P}(\mathbb{R})$ we define the distance

$$d(\mu, \mu') = \sup_{\substack{\|f\|_{\text{Lip}} \le 1 \\ \|f\|_{1/2} \le 1}} \left\{ \left| \int f(x) d\mu(x) - \int f(x) d\mu'(x) \right| \right\}, \tag{1.13}$$

where $\|f\|_{\text{Lip}}$ denotes the Lipschitz constant of f, and $\|f\|_{1/2}^2 = \int_{\mathbb{R}} |t| \, |\mathcal{F}[f](t)|^2 \, dt$, where $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R})$ which takes the following expression $\mathcal{F}[f](t) = \int_{\mathbb{R}} f(x) e^{-\mathrm{i}tx} dx$ for f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

We then have:

Theorem 1.1.7 *There exists $K \in \mathbb{R}$ (depending on P and on V), such that for any $N \geq 1$ and $r > 0$,*

$$\mathbb{P}_N^{V,P}\left(d(\hat{\mu}_N, \mu_V) > r\right) \leq e^{-Nr^2\frac{P\pi^2}{2}+5P\log N+K}. \tag{1.14}$$

This result is the analog of [HL21, Theorem 1.4].

The paper is organized as follows. In Section 1.2 we discuss the regularity of the equilibrium density ρ_V under Assumption 1.1.1. In Section 1.3 we prove Theorem 1.1.7. Section 1.4 is dedicated to the localization of the edge of a typical configuration, mentioned in the discussion preceding the statement of Assumption *1.1.2*. We next prove in Section 1.5 the convergence of the Laplace transform of $\sqrt{N}\nu_N(\mathcal{L}\phi)$ for general functions ϕ which establishes Theorem 1.1.4 for functions of the form $\mathcal{L}\phi$. Section 1.6 is dedicated to the diagonalization and inversion of $\mathcal{L}$ given by (1.8). In Section 1.7, we show regularity properties of $\mathcal{L}^{-1}$ to establish Theorem 1.1.4. We detail in Appendix 1.8 elements of proof for the spectral theory of Schrödinger operators, used in Section 1.6.

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1.2 Regularity of the equilibrium measure and Hilbert transform

In this section, we discuss the regularity properties of the equilibrium density ρ_V, namely its decay at infinity and its smoothness, and give formulas for its two first derivatives.

The Hilbert transform, whose definition we recall, plays a central role in the analysis of the equilibrium measure. It is first defined on the Schwartz class through $\forall \phi \in \mathcal{S}(\mathbb{R})$, $\forall x \in \mathbb{R}$,

$$\mathcal{H}[\phi](x) \overset{(\text{def})}{=} \fint_{\mathbb{R}} \frac{\phi(t)}{t-x}dt = \lim_{\varepsilon \downarrow 0} \int_{|t-x|>\varepsilon} \frac{\phi(t)}{t-x}dt = \int_0^{+\infty} \frac{\phi(x+t)-\phi(x-t)}{t}dt, \tag{1.15}$$

where $\fint$ denotes the Cauchy principal value integral, and then extended to $L^2(\mathbb{R})$ thanks to property *ii)* of Lemma 1.2.1: $\|f\|_{L^2(dx)} = \frac{1}{\pi}\|\mathcal{H}[f]\|_{L^2(dx)}$. The last expression in (1.15) is a definition where the integral converges in the classical sense. We also recall the definition of the logarithmic potential U^f of a density of probability $f : \mathbb{R} \to \mathbb{R}$, given for $x \in \mathbb{R}$ by

$$U^f(x) \overset{(\text{def})}{=} -\int_{\mathbb{R}} \log|x-y| f(y)dy\,. \tag{1.16}$$

Because we assume $f \in L^1(\mathbb{R})$ to be nonnegative, U^f takes values in $[-\infty, +\infty)$. If f integrates the function log, *i.e* $\int_{\mathbb{R}} \log|x| f(x)dx < +\infty$, then U^f takes real values. Additionally, one can check that the logarithmic potential and the Hilbert transform of f are linked through the distributional identity $\left(U^f\right)' = \mathcal{H}[f]$.

We recall in the next lemma some properties of the Hilbert transform that we will use in the rest of the paper.

Lemma 1.2.1 (Properties of the Hilbert transform)

i) Fourier transform: For all $\phi \in L^2(\mathbb{R})$, $\mathcal{F}\Big[\mathcal{H}[\phi]\Big](\omega) = \mathrm{i}\pi sgn(\omega)\mathcal{F}[\phi](\omega)$ for all $\omega \in \mathbb{R}$.

ii) As a consequence, $\frac{1}{\pi}\mathcal{H}$ *is an isometry of* $L^2(\mathbb{R})$*, and* $\mathcal{H}$ *satisfies on* $L^2(\mathbb{R})$ *the identity* $\mathcal{H}^2 = -\pi^2 I$.

iii) Derivative: For any $f \in H^1(\mathbb{R})$, $\mathcal{H}[f]$ *is also* $H^1(\mathbb{R})$ *and* $\mathcal{H}[f]' = \mathcal{H}[f']$.

iv) For all $p > 1$*, the Hilbert transform can be extended as a bounded operator* $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$.

v) Skew-self adjointness: For any $f, g \in L^2(\mathbb{R})$, $\langle \mathcal{H}[f], g\rangle_{L^2(\mathbb{R})} = -\langle f, \mathcal{H}[g]\rangle_{L^2(\mathbb{R})}$.

Proof We refer to [Kin09] for the proofs of these properties. □

As a consequence of [GZ19], $\hat{\mu}_N$ converges almost surely under $\mathbb{P}_N^{V,P}$ towards the unique minimizer of the energy-functional $\mathcal{E}_{V,P}$, defined for $\mu \in \mathcal{P}(\mathbb{R})$ by

$$\mathcal{E}_{V,P}(\mu) \overset{(\text{def})}{=} \begin{cases} \displaystyle\int_{\mathbb{R}} \Big[V(x) + \log\Big(\frac{d\mu(x)}{dx}\Big)\Big] d\mu(x) - P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y) \text{ if } \mu \ll dx \\ +\infty \text{ otherwise} \end{cases} . \tag{1.17}$$

Here we wrote $\mu \ll dx$ for "μ is absolutely continuous with respect to Lebesgue measure".

Consequently, following [GM22, Lemma 3.2], the density ρ_V of μ_V satisfies equation (1.5), which we rewrite here for convenience.

$$V(x) - 2P \int_{\mathbb{R}} \log|x-y| \rho_V(y) dy + \log \rho_V(x) = C_{V,P}\,, \tag{1.18}$$

where $C_{V,P}$ is a constant (depending on V and P). Using this equation, we will show in the next lemma that ρ_V decays exponentially and is twice continuously differentiable *via* the representation:

$$\forall x \in \mathbb{R},\ \rho_V(x) = \exp\Big(- V(x) - 2PU^{\rho_V}(x) - C_{V,P}\Big).$$

In the Gaussian potential case *i.e.* $V_G(x) = \dfrac{x^2}{2}$, an explicit formula has been found first in [ABG12], and in [TS15] *via* a different method:

$$\rho_P^{V_G}(x) = \frac{\Gamma(P)}{P\sqrt{2\pi}} \frac{\exp\Big(-\dfrac{x^2}{2}\Big)}{\displaystyle\int_0^{+\infty} t^{P-1} e^{-\frac{t^2}{2} + \mathrm{i}xt} dt}.$$

It has been established in [BGP15] that $\sqrt{P+1}\rho_P^{V_G}(\sqrt{P+1}x)$ converges to the Gaussian distribution when P goes to zero and the semi-circle law when P goes to infinity. So in the Gaussian case, μ_V can be seen as an interpolation between the Gaussian distribution and the semi-circular one. In fact, this interpolation result holds true for general potentials, see [NT20b, Remark 2.1].

In the next lemma, we prove that ρ_V has the same regularity as V.

Lemma 1.2.2 *Under Assumption 1.1.1,*

- *The support of* μ_V *is* $\mathbb{R}$ *and there exists a constant* C_P^V *such that for all* $x \in \mathbb{R}$,

$$\rho_V(x) \le C_P^V (1+|x|)^{2P} e^{-V(x)}\,.$$

- *The density* ρ_V *is in* $\mathcal{C}^3(\mathbb{R})$ *and we have*

$$\rho_V' = -\Big(V' + 2P\mathcal{H}[\rho_V]\Big)\rho_V \tag{1.19}$$

and

$$\rho_V'' = \Big(- 2P\mathcal{H}[\rho_V]' - V'' + V'^2 + 4P^2\mathcal{H}[\rho_V]^2 + 4PV'\mathcal{H}[\rho_V]\Big)\rho_V\,. \tag{1.20}$$

Proof For the first point, [GM22, Lemma 3.2] establishes that the support of μ_V is the whole real axis, and that under the first condition of Assumptions 1.1.1, we have the bound, valid for all $x \in \mathbb{R}$

$$\rho_V(x) \leq \frac{K_P^V}{(1+|x|)^2}\,, \tag{1.21}$$

with K_P^V a positive constant. Using (1.18) and the fact that

$$\log|x-y| \leq \log\big(1+|x|\big) + \log\big(1+|y|\big)\,,$$

we see that for all $x \in \mathbb{R}$,

$$\rho_V(x) \leq C_P^V \exp\Big(-V(x) + 2P\log(1+|x|)\Big)\,, \tag{1.22}$$

with

$$C_P^V = \exp\Big(2P\int_{\mathbb{R}} \log(1+|y|)\rho_V(y)dy + C_{V,P}\Big)$$

which is indeed finite by (1.21).

For the second point, we use that $\big(U^{\rho_V}\big)' = \mathcal{H}[\rho_V]$ weakly and equation (1.18) to conclude on the distributional identity

$$\rho_V' = \Big(-V' - 2P\mathcal{H}[\rho_V]\Big)\rho_V\,.$$

By the second point of Assumption 1.1.1, $V'(x)e^{-V(x)+2P\log(1+|x|)} = o(x^{-1})$ as $|x| \to \infty$, thus by (1.22), $V'\rho_V \in L^2(\mathbb{R})$. Also since ρ_V is $L^2(\mathbb{R})$ and bounded, we deduce, by using that $\mathcal{H}\big[L^2(\mathbb{R})\big] = L^2(\mathbb{R})$, that $\mathcal{H}[\rho_V]\rho_V \in L^2(\mathbb{R})$. Adding up these terms we get $\rho_V \in H^1(\mathbb{R})$. Because $\mathcal{H}[\rho_V]' = \mathcal{H}[\rho_V']$ in a weak sense by Lemma 1.2.1, $\mathcal{H}[\rho_V] \in H^1(\mathbb{R})$. By the classical fact that $H^1(\mathbb{R})$ is contained in the set of 1/2-Hölder functions $\mathcal{C}^{1/2}(\mathbb{R})$, we have $\mathcal{H}[\rho_V] \in \mathcal{C}^{1/2}(\mathbb{R})$ and so $U^{\rho_V} \in \mathcal{C}^{1,1/2}(\mathbb{R})$, the set of functions in $\mathcal{C}^1(\mathbb{R})$ with derivative of class 1/2-Hölder.
Using the fact that V is continuously differentiable, the previous equation for the weak derivative of ρ_V then ensures that $\rho_V \in \mathcal{C}^1(\mathbb{R})$ and equation (1.19) holds in the strong sense.

Differentiating (in a weak sense) equation (1.19) we obtain

$$\rho_V'' = \Big(-2P\mathcal{H}[\rho_V]' - V'' + V'^2 + 4P^2\mathcal{H}[\rho_V]^2 + 4PV'\mathcal{H}[\rho_V]\Big)\rho_V\,.$$

The three first terms belong to $L^2(\mathbb{R})$ for the same reasons as before. Since $\rho_V \in H^1(\mathbb{R})$n by Lemma 1.2.1*iii)* so is $\mathcal{H}[\rho_V] \in H^1(\mathbb{R})$, it is then bounded over $\mathbb{R}$ hence the two last term are in $L^2(\mathbb{R})$ when multiplied by ρ_V. Finally, we can conclude that $\rho_V \in H^2(\mathbb{R})$ and so that $\mathcal{H}[\rho_V] \in H^2(\mathbb{R})$ with $\mathcal{H}[\rho_V]'' = \mathcal{H}[\rho_V'']$ (in a weak sense). As before, we conclude that $\rho_V \in \mathcal{C}^2(\mathbb{R})$ and that equation (1.20) holds in a strong sense. By the exact same method, we can show that $\rho_V \in \mathcal{C}^3(\mathbb{R})$. □

We next show that the Hilbert transform of ρ_V is continuous and decays at infinity.

Lemma 1.2.3 *Let $u \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} u(t)dt \overset{\text{(def)}}{=} \lim_{A\to\infty}\int_{-A}^{A} u(t)dt$ exists and $f : t \mapsto tu(t) \in H^1(\mathbb{R})$ then*

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} u(t)dt}{x}.$$

Moreover if $\displaystyle\int_{\mathbb{R}} u(t)dt = 0$, $\int_{\mathbb{R}} f(t)dt$ exists and $g : t \mapsto t^2u(t) \in H^1(\mathbb{R})$, then

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} tu(t)dt}{x^2}.$$

As a consequence, we obtain that $\mathcal{H}[\rho_V](x) \underset{|x|\to\infty}{\sim} -x^{-1}$ and the logarithmic potential U^{ρ_V} is Lipschitz bounded, with bounded derivative $\mathcal{H}[\rho_V]$.

Proof Let $u \in L^2(\mathbb{R})$, such that $\int_{\mathbb{R}} u(t)dt$ exists and $f : t \mapsto tu(t) \in H^1(\mathbb{R})$. Then

$$x\mathcal{H}[u](x) + \int_{\mathbb{R}} u(t)dt = \int_{\mathbb{R}} \Big[\frac{xu(x+t) - xu(x-t)}{2t} + \frac{u(x+t)}{2} + \frac{u(x-t)}{2}\Big]dt = \mathcal{H}[f](x).$$

Since $f \in H^1(\mathbb{R})$, so is $\mathcal{H}[f]$, proving that it goes to zero at infinity. Hence

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} u(t)dt}{x}$$

Moreover if $\int_{\mathbb{R}} u(t)dt = 0$, $\int_{\mathbb{R}} f(t)dt$ exists and $g : t \mapsto t^2u(t) \in H^1(\mathbb{R})$, then by the same argument:

$$x^2\mathcal{H}[u](x) = x\mathcal{H}[f](x) = \mathcal{H}[g](x) - \int_{\mathbb{R}} f(t)dt$$

where $g(t) = t^2u(t)$. We deduce that $\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} tu(t)dt}{x^2}$ since $\mathcal{H}[g]$ goes to zero at infinity.□

The following Lemmas will be useful to show that if $V = V_{\text{conv}} + \phi$ with V_{conv} such that there exists $\alpha > 0$ such that for $|x|$ big enough, $V''_{\text{conv}}(x) > \alpha$ and ϕ bounded then V satisfies Poincaré inequality.

Lemma 1.2.4 (Asymptotic of the logarithmic potential) *We have the following asymptotic expansion at infinity* $U^{\rho_V}(x) = -\log|x| + \underset{|x|\to\infty}{O}(x^{-1})$.

Proof Since $\mathcal{H}[\rho_V](x) = -x^{-1} + \underset{|x|\to\infty}{O}(x^{-2})$, and recalling that U^{ρ_V} (defined by (1.16)) satisfies $(U^{\rho_V})'(x) = \mathcal{H}[\rho_V](x)$, we deduce the result by integrating $t \mapsto \mathcal{H}[\rho_V](t) + 1/t$ in a neighborhood of infinity. □

We conclude this section by stating the Poincaré inequality for the measure μ_V under the assumption that V is a bounded perturbation of a strictly convex potential V_{conv}.

Lemma 1.2.5 *Let $\varepsilon > 0$, there exists a function $f \in \mathcal{C}^2(\mathbb{R})$ such that $f_\varepsilon(x) + \log|x| = \underset{|x|\to\infty}{O}(1)$, and $\|f''_\varepsilon\|_\infty \le \varepsilon$.*

Proof Indeed, for fixed $\varepsilon > 0$, let

$$f_\varepsilon(x) \overset{(\text{def})}{=} -\log\left(\sqrt{\varepsilon^{-1} + x^2}\right).$$

It is straightforward to verify that for all $x \in \mathbb{R}$,

$$|f''_\varepsilon(x)| = \left|\frac{\varepsilon^{-1} - x^2}{(\varepsilon^{-1} + x^2)^2}\right| \le \varepsilon\,. \qquad \square$$

Proposition 1.2.6 *Assume that $V = V_{\text{conv}} + \phi$, where $V_{\text{conv}} \in \mathcal{C}^3(\mathbb{R})$ with V_{conv} convex and ϕ bounded. Assume that there exists $\varepsilon > 0$ such that $V_{\text{conv}} - 2Pf_\varepsilon$ is convex (f_ε being given by Lemma 1.2.5). Then, the measure μ_V satisfies the Poincaré inequality: there exists a constant $C > 0$ such that for all $f \in \mathcal{C}^1_c(\mathbb{R})$,*

$$\mathrm{Var}_{\mu_V}(f) = \int_{\mathbb{R}} \left(f(x) - \int_{\mathbb{R}} f(y)d\mu_V(y)\right)^2 d\mu_V(x) \le C\int_{\mathbb{R}} |f'(x)|^2 d\mu_V(x)\,. \tag{1.23}$$

Proof We use the fact that if μ_1, μ_2 are two absolutely continuous probability measures supported on $\mathbb{R}$ such that $\frac{1}{C} \le \frac{d\mu_1}{d\mu_2} \le C$ for some $C > 0$ and μ_1 satisfies Poincaré inequality with constant C_1 then so does μ_2 for some other constant. Indeed, in that case let $f \in \mathcal{C}^1_c(\mathbb{R})$, we have:

$$\mathrm{Var}_{\mu_2}(f) = \inf_a \int_{\mathbb{R}} (f(x) - a)^2\, d\mu_2(x) \le C\,\mathrm{Var}_{\mu_1}(f) \le C^2C_1\int_{\mathbb{R}} |f'(x)|^2 d\mu_2(x).$$

Here we take $d\mu_2(x) \overset{(\text{def})}{=} \rho_V(x)dx$ and we want to compare it to a measure μ_1 supported on $\mathbb{R}$ defined by $d\mu_1(x) = \frac{1}{Z}\exp\big(-W(x)\big)dx$ for some convex function W. The measure μ_1 then clearly verifies the Poincaré inequality: this fact comes as a direct consequence of [BBCG08, Corollary 1.9], which states that if a probability measure μ has a log-concave density on $\mathbb{R}$, then it satisfies (1.23). With the definition $W \overset{(\text{def})}{=} V_{\text{conv}} - 2Pf_\varepsilon$ with $\varepsilon > 0$ such that $V_{\text{conv}} - 2Pf_\varepsilon$ is convex, $W - V - 2PU^{\rho_V}$ is bounded on $\mathbb{R}$. It is then not hard to see that $\frac{1}{C} \leq \frac{d\mu_1}{d\mu_V} \leq C$ for some $C > 0$ which allows to conclude that μ_V satisfies the Poincaré inequality. □

Remark 1.2.7 The previous proof is easily extended to potentials $V = V_{\text{conv}} + \phi$, with V_{conv} satisfying: there exists $\alpha > 0$ such that for $|x|$ big enough, $V''_{\text{conv}}(x) > \alpha$; such as $V(x) = x^{2n} + \phi(x)$ with ϕ bounded and $n \geq 1$. This potential indeed fails to satisfy the condition $V'' - 2Pf''_\varepsilon \geq 0$, and we need another trick.

On the other side, one can notice that $x \mapsto V(x) - 2Pf_\varepsilon(x + x_0)$ for x_0 big enough will be convex. Indeed, the "lack of convexity" of $-f_\varepsilon$ will be compensated since $f''_\varepsilon(x + x_0) > 0$ will occur in the region where $V''_{\text{conv}}(x) > \alpha$. Therefore we can use the same argument as the one of Proposition 1.2.6 to conclude that μ_V satisfies Poincaré inequality.

Remark 1.2.8 We will apply later inequality (1.23) to more general functions than $\mathcal{C}^1_c(\mathbb{R})$, namely functions of the weighted Sobolev space $H^1(\mu_V)$, defined in Section 1.6; which can be seen as the completion of $\mathcal{C}^\infty_c(\mathbb{R})$ with respect to the norm $\|u\|_{L^2(\mu_V)} + \|u'\|_{L^2(\mu_V)}$.

1.3 Concentration inequality, proof of Theorem 1.1.7

We prove in this section the concentration Theorem 1.1.7. Its proof is a direct adaptation of Theorem 1.4 of [HL21], which shows the analogous estimate in the circular setup. It is inspired by [MMS14] and based on a comparison between a configuration $\mathbf{x}_N = (x_1, \dots, x_N)$ sampled with $\mathbb{P}_N^{V,P}$ and a regularized version $\mathbf{y}_N = (y_1, \dots, y_N)$, which we describe here.

Definition 1.3.1 *$y_1 \overset{(\text{def})}{=} x_1$, and for $0 \leq k \leq N-1$, $y_{k+1} \overset{(\text{def})}{=} y_k + \max\{x_{k+1} - x_k, N^{-3}\}$.*

Note that the configuration $\mathbf{y}_N$ given by the previous definition satisfies $y_{k+1} - y_k \geq N^{-3}$, and $\mathbf{y}_N$ is close to $\mathbf{x}_N$ in the sense that

$$\sum_{k=1}^{N} |x_k - y_k| \leq \frac{1}{2N}\,. \tag{1.24}$$

Indeed, by construction we have $|x_k - y_k| = y_k - x_k \leq (k-1)N^{-3}$, and we get the bound by summing these inequalities.

The key point of the proof of Theorem 1.1.7 is comparing the empirical measure $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, where $\mathbf{x}_N$ follows $\mathbb{P}_N^{V,P}$, to the regularized measure

$$\widetilde{\mu}_N \overset{(\text{def})}{=} \lambda_{N^{-5}} * \frac{1}{N}\sum_{i=1}^{N} \delta_{y_i}, \tag{1.25}$$

i.e. the convolution of $\lambda_{N^{-5}}$ and the empirical measure, where $\lambda_{N^{-5}}$ is the uniform measure on $[0, N^{-5}]$. The interest of introducing the measure $\widetilde{\mu}_N$ is that it is close to $\hat{\mu}_N$, while having a finite energy $\mathcal{E}_{V,P}(\widetilde{\mu}_N)$, given by (1.17). Finally, notice that the empirical measure doesn't change when reordering $x_1, \dots, x_N$, and thus we do not lose in generality for our purposes in assuming that $x_1 \leq \ldots \leq x_N$ in Definition 1.3.1.

We now introduce a distance on $\mathcal{P}(\mathbb{R})$ which is well-suited to our context.

Definition 1.3.2 *For $\mu, \mu' \in \mathcal{P}(\mathbb{R})$ we define the distance (possibly infinite) $D(\mu, \mu')$ by*

$$D(\mu,\mu') \overset{(\text{def})}{=} \left(-\int \log|x-y| d(\mu-\mu')(x) d(\mu-\mu')(y) \right)^{1/2} \tag{1.26}$$
$$= \left(\int_0^{+\infty} \frac{1}{t} \left| \mathcal{F}[\mu-\mu'](t) \right|^2 dt \right)^{1/2},$$

where the Fourier transform of a signed measure ν is defined by $\mathcal{F}[\nu](x) \overset{(\text{def})}{=} \int_{\mathbb{R}} e^{-\mathrm{i}tx} d(\mu-\mu')(x)$.

Let $f : \mathbb{R} \to \mathbb{R}$ with finite 1/2 norm $\|f\|_{1/2} \overset{(\text{def})}{=} \left(\int_{\mathbb{R}} |t| \, |\mathcal{F}[f](t)|^2 \, dt \right)^{1/2}$. By Plancherel theorem and Hölder inequality, for any $\mu, \mu' \in \mathcal{P}(\mathbb{R})$, setting $\nu = \mu - \mu'$,

$$\left| \int_{\mathbb{R}} f(x) d\mu(x) - \int_{\mathbb{R}} f(x) d\mu'(x) \right|^2 = \left| \frac{1}{2\pi} \int_{\mathbb{R}} |t|^{1/2} \mathcal{F}[f](t) \frac{\overline{\mathcal{F}[\nu](t)}}{|t|^{1/2}} dt \right|^2 \leq \frac{1}{2\pi^2} \|f\|_{1/2}^2 D^2(\mu,\mu').$$

Therefore the metric d defined in (1.13) is dominated by D:

$$d(\mu,\mu') \leq \frac{1}{\sqrt{2}\pi} D(\mu,\mu'). \tag{1.27}$$

The following lemma shows how the distance D is related to the energy-functional $\mathcal{E}_{V,P}$ defined in (1.17), we will write $\mathcal{E}$ for simplicity.

Lemma 1.3.3 *We have for any absolutely continuous $\mu \in \mathcal{P}(\mathbb{R})$ with finite energy $\mathcal{E}(\mu)$,*

$$\mathcal{E}(\mu) - \mathcal{E}(\mu_V) = P D^2(\mu, \mu_V) + \int_{\mathbb{R}} \log \frac{d\mu(x)}{d\mu_V(x)} d\mu(x) \,. \tag{1.28}$$

Proof (of Lemma 1.3.3) Subtracting $\mathcal{E}(\mu) - \mathcal{E}(\mu_V)$ we find

$$\mathcal{E}(\mu) - \mathcal{E}(\mu_V) = \int_{\mathbb{R}} V(x) d(\mu-\mu_V)(x) + \int_{\mathbb{R}} \log \frac{d\mu(x)}{dx} d\mu(x) - \int_{\mathbb{R}} \log \rho_V(x) d\mu_V(x)$$
$$- P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y) + P \iint_{\mathbb{R}^2} \log|x-y| d\mu_V(x) d\mu_V(y) \,. \tag{1.29}$$

Now, if ν is a signed measure of mass zero, integrating (1.18) we get

$$\int_{\mathbb{R}} V(x) d\nu(x) - 2P \iint_{\mathbb{R}^2} \log|x-y| d\nu(x) d\mu_V(y) + \int_{\mathbb{R}} \log \rho_V(x) d\nu(x) = 0 \,.$$

We take $\nu = \mu - \mu_V$, and get

$$\int_{\mathbb{R}} V(x) d(\mu-\mu_V)(x) = 2P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu_V(y) - 2P \iint_{\mathbb{R}^2} \log|x-y| d\mu_V(x) d\mu_V(y)$$
$$- \int_{\mathbb{R}} \log \rho_V(x) d\mu(x) + \int_{\mathbb{R}} \log \rho_V(x) d\mu_V(x) \,.$$

Plugging this last identity in (1.29), we find

$$\mathcal{E}(\mu) - \mathcal{E}(\mu_V) = -P \iint_{\mathbb{R}^2} \log|x-y| d\nu(x) d\nu(y) + \int_{\mathbb{R}} \log \frac{d\mu(x)}{d\mu_V(x)} d\mu(x)$$

which establishes the result. □

Proof (Proof of Theorem 1.1.7) We first give a lower bound for the partition function $\mathcal{Z}_N^P[V]$ (1.4) of $\mathbb{P}_N^{V,P}$. We rewrite it as

$$\mathcal{Z}_N^P[V] = \int_{\mathbb{R}^N} \exp\left(\frac{2P}{N}\sum_{i<j}\log|x_i-x_j| - \sum_{i=1}^N\Big[V(x_i)+\log\rho_V(x_i)\Big]\right) d\mu_V(x_1)\dots d\mu_V(x_N)\,,$$

and apply Jensen inequality to obtain:

$$\begin{aligned}\log\mathcal{Z}_N^P[V] &\geq \int_{\mathbb{R}^N}\left(\frac{2P}{N}\sum_{i<j}\log|x_i-x_j| - \sum_{i=1}^N\Big[V(x_i)+\log\rho_V(x_i)\Big]\right) d\mu_V(x_1)\dots d\mu_V(x_N)\\ &\geq P(N-1)\iint_{\mathbb{R}^2}\log|x-y|d\mu_V(x)d\mu_V(y) - N\int_{\mathbb{R}}\Big[V(x)+\log\rho_V(x)\Big]d\mu_V(x)\\ &\geq -N\mathcal{E}\big(\mu_V\big) - P\iint_{\mathbb{R}^2}\log|x-y|d\mu_V(x)d\mu_V(y).\end{aligned}$$

Using this estimate and the fact that for $1\leq i,j\leq N$ we have $|x_i-x_j|\leq|y_i-y_j|$, with $\mathbf{y}_N = (y_1,\dots,y_N)$ of definition 1.3.1, we deduce the bound on the density of probability

$$\begin{aligned}\frac{d\mathbb{P}_N^{V,P}}{d^Nx}(x_1,\dots,x_N) \leq \exp\Big\{N\mathcal{E}(\mu_V) + P\iint_{\mathbb{R}^2}\log|x-y|d\mu_V(x)d\mu_V(y)\\ + \frac{P}{N}\sum_{i\neq j}\log|y_i-y_j| - \sum_{i=1}^N V(x_i)\Big\}\,. \quad (1.30)\end{aligned}$$

Recalling (1.25), we now show the following estimate:

$$\sum_{i\neq j}\log|y_i-y_j| \leq 2 + N^2\iint_{\mathbb{R}^2}\log|x-y|d\widetilde{\mu}_N(x)d\widetilde{\mu}_N(y) + 5N\log N + \frac{3}{2}N\,. \quad (1.31)$$

Let $i\neq j$ and $u,v\in[0,N^{-5}]$. Since for $x\neq 0$ and $|h|\leq\frac{|x|}{2}$, we have $\big|\log|x+h|-\log|x|\big|\leq\frac{2|h|}{|x|}$, we deduce that:

$$\big|\log|y_i-y_j+u-v| - \log|y_i-y_j|\big| \leq \frac{2|u-v|}{|y_i-y_j|} \leq \frac{2N^{-5}}{N^{-3}} = \frac{2}{N^2}.$$

Thus, summing over $i\neq j$ and integrating with respect to u and v, we get:

$$\begin{aligned}\sum_{i\neq j}\log|y_i-y_j| &\leq 2 + \sum_{i\neq j}\iint_{\mathbb{R}^2}\log|y_i-y_j+u-v|d\lambda_{N^{-5}}(u)d\lambda_{N^{-5}}(v)\\ &= 2 + N^2\iint_{\mathbb{R}^2}\log|x-y|d\widetilde{\mu}_N(x)d\widetilde{\mu}_N(y) - N\iint_{\mathbb{R}^2}\log|u-v|d\lambda_{N^{-5}}(u)d\lambda_{N^{-5}}(v)\,.\end{aligned}$$

The last integral is equal to $-\frac{3}{2}-5\log N$, so we deduce (1.31). We now combine (1.30) and (1.31). Recall (1.17) and set

$$c_N \overset{(\text{def})}{=} P\left(\iint_{\mathbb{R}^2}\log|x-y|d\mu_V(x)d\mu_V(y) + 3/2 + 2/N\right)\,.$$

Then, we get:

$$\begin{aligned}\frac{d\mathbb{P}_N^{V,P}}{d^N x}(x_1,\dots,x_N) &\le e^{c_N} N^{5P} \exp\Big\{ \big[N \Big\{ \mathcal{E}(\mu_V) - \mathcal{E}(\widetilde{\mu}_N) + \int_{\mathbb{R}} \left(V(x) + \log \frac{d\widetilde{\mu}_N(x)}{dx} \right) d\widetilde{\mu}_N(x) \Big\} \\ &\quad - \sum_{i=1}^N V(x_i) \big] \Big\} \\ &= e^{c_N} N^{5P} \exp\Big\{ - NPD^2(\widetilde{\mu}_N, \mu_V) + N \int_{\mathbb{R}} \left(V(x) + \log \rho_V(x) \right) d\widetilde{\mu}_N(x) \\ &\quad - \sum_{i=1}^N V(x_i) \Big\},\end{aligned}$$

where we used equation (1.28) in the last equality. Using again equation (1.18) we then see that the density $\frac{d\mathbb{P}_N^{V,P}}{d^N x}(x_1,\dots,x_N)$ is bounded by

$$e^{c_N} N^{5P} \exp\left[-NPD^2(\widetilde{\mu}_N, \mu_V) + 2PN \iint_{\mathbb{R}^2} \log|x-y| d(\widetilde{\mu}_N - \hat{\mu}_N)(x) d\mu_V(y) \right] \prod_{i=1}^N \rho_V(x_i)\,.$$

Recalling (1.16), we used that $\iint_{\mathbb{R}^2} \log|x-y| d(\widetilde{\mu}_N - \hat{\mu}_N)(x) d\mu_V(y) = -\int_{\mathbb{R}} U^{\rho_V}(x) d(\widetilde{\mu}_N - \hat{\mu}_N)(x)$.
As a consequence of the bound on the density $\frac{d\mathbb{P}_N^{V,P}}{d^N x}(x_1,\dots,x_N)$ we established, we have for all $r > 0$

$$\mathbb{P}_N^{V,P}\left(D^2(\widetilde{\mu}_N, \mu_V) > r \right) \le e^{-NPr + c_N + 5P\log N} \int_{\mathbb{R}^N} \exp\left\{ -2PN \int_{\mathbb{R}} U^{\rho_V} d(\widetilde{\mu}_N - \hat{\mu}_N) \right\} \prod_{i=1}^N \rho_V(x_i) dx_i\,. \tag{1.32}$$

Next, we show that $-N\int_{\mathbb{R}} U^{\rho_V} d(\widetilde{\mu}_N - \hat{\mu}_N)$ is bounded. By Lemma 1.2.3, U^{ρ_V} is differentiable with bounded derivative $\mathcal{H}[\rho_V]$ on $\mathbb{R}$. As a consequence,

$$\begin{aligned}\left| N \int_{\mathbb{R}} U^{\rho_V} d(\widetilde{\mu}_N - \hat{\mu}_N) \right| &\le \sum_{i=1}^N \int_{\mathbb{R}} |U^{\rho_V}(y_i + u) - U^{\rho_V}(x_i)|\, d\lambda_{N^{-5}}(u) \\ &\le \|\mathcal{H}[\rho_V]\|_\infty \left(\sum_{i=1}^N |y_i - x_i| + N \int_{\mathbb{R}} u d\lambda_{N^{-5}}(u) \right) \\ &\le \|\mathcal{H}[\rho_V]\|_\infty \Big(\frac{1}{2N} + N^{-4}/2 \Big),\end{aligned}$$

where we used (1.24) in the last inequality. Therefore, we deduce from (1.32)

$$\mathbb{P}_N^{V,P}\left(D^2(\widetilde{\mu}_N, \mu_V) > r \right) \le e^{-NPr + c_N + 5P\log N + \frac{2P}{N}\|\mathcal{H}[\rho_V]\|_\infty} = e^{-NPr + 5P\log N + K_N} \tag{1.33}$$

with $K_N \overset{(\text{def})}{=} c_N + \frac{2P}{N}\|\mathcal{H}[\rho_V]\|_\infty$. Since $(c_N)_N$ is bounded, so is $(K_N)_N$.
Finally, let f be a Lipschitz bounded function with $\|f\|_{\text{Lip}} \le 1$, then, we have (as we did for U^{ρ_V})

$$\left| \int_{\mathbb{R}} f(x) d\hat{\mu}_N(x) - \int_{\mathbb{R}} f(x) d\widetilde{\mu}_N(x) \right| \le N^{-2}\,.$$

Thus by (1.27)

$$d(\hat{\mu}_N, \mu_V) \le d(\hat{\mu}_N, \widetilde{\mu}_N) + d(\widetilde{\mu}_N, \mu_V) \le N^{-2} + \frac{1}{\sqrt{2}\pi} D(\widetilde{\mu}_N, \mu_V)\,,$$

and for any N such that $r - N^{-2} \geq r/2$ (in particular $r - N^{-2} > 0$) we get

$$\mathbb{P}_N^{V,P}\left(d(\hat{\mu}_N, \mu_V) > r\right) \leq \mathbb{P}_N^{V,P}\left(\frac{1}{2\pi^2} D^2(\widetilde{\mu}_N, \mu_V) > (r - N^{-2})^2\right) \leq \mathbb{P}_N^{V,P}\left(\frac{1}{2\pi^2} D^2(\widetilde{\mu}_N, \mu_V) > r^2/4\right),$$

and the last term is bounded by $e^{-Nr^2\frac{P\pi^2}{2}+5P\log N+K}$ for some K large enough, which concludes the proof. □

As a consequence of Theorem 1.1.7, we are able to control the quantities

$$\zeta_N(\phi) \overset{\text{(def)}}{=} \iint_{\mathbb{R}^2} \frac{\phi(x) - \phi(y)}{x - y} d(\hat{\mu}_N - \mu_V)(x) d(\hat{\mu}_N - \mu_V)(y) \tag{1.34}$$

for a certain class of test functions ϕ.

Corollary 1.3.4 *There exists $C, K > 0$ such that for all $\phi \in \mathcal{C}^2(\mathbb{R}) \cap H^2(\mathbb{R})$ with bounded second derivative, we have for $\varepsilon > 0$ and N large enough,*

$$\mathbb{P}_N^{V,P}\left(\sqrt{N}|\zeta_N(\phi)| \leq N^{-1/2+\varepsilon}\right) \geq 1 - \exp\left\{-\frac{PN^{\varepsilon}}{2C\|\psi\|_{H^2(\mathbb{R})}} + 5P\log N + K\right\}$$

with $N_2(\phi) = \|\phi'\|_{L^2(dx)} + \|\phi''\|_{L^2(dx)}$.

Proof We follow the proof given in [Gui19, Cor. 4.16] and adapt it to our setting. Let us denote by $\widetilde{\zeta_N}(\phi)$ the quantity

$$\iint_{\mathbb{R}^2} \frac{\phi(x) - \phi(y)}{x - y} d(\widetilde{\mu}_N - \mu_V)(x) d(\widetilde{\mu}_N - \mu_V)(y)\,.$$

We have the almost sure inequality, by a Taylor estimate

$$|\zeta_N(\phi) - \widetilde{\zeta_N}(\phi)| \leq 2N^{-2}\|\phi''\|_{\infty}\,. \tag{1.35}$$

Thus, for any $\delta > 0$,

$$\begin{aligned}\mathbb{P}_N^{V,P}\left(|\zeta_N(\phi)| > \delta\right) &\leq \mathbb{P}_N^{V,P}\left(|\zeta_N(\phi) - \widetilde{\zeta_N}(\phi)| > \delta/2\right) + \mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \delta/2\right) \\ &\leq \mathbb{P}_N^{V,P}\left(2N^{-2}\|\phi''\|_{\infty} > \delta/2\right) + \mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \delta/2\right),\end{aligned}$$

where the first term of the right-hand side is either 0 or 1. With $\delta = N^{-1+\varepsilon}$, $\varepsilon > 0$, it is zero for N large enough. For such a choice of δ, and for N large enough,

$$\mathbb{P}_N^{V,P}\left(|\zeta_N(\phi)| > N^{-1+\varepsilon}\right) \leq \mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \frac{1}{2}N^{-1+\varepsilon}\right).$$

We next show that, for some $C > 0$ independent of ϕ, we have

$$|\widetilde{\zeta_N}(\phi)| \leq CD^2(\widetilde{\mu}_N, \mu_V)\|\phi\|_{H^2(\mathbb{R})}\,. \tag{1.36}$$

We begin by showing this inequality for $\psi \in \mathcal{S}(\mathbb{R})$. By using the inverse Fourier transform we have

$$\begin{aligned}\widetilde{\zeta}_N(\psi) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} \frac{\int_{\mathbb{R}} dt\mathcal{F}[\psi](t)e^{\mathrm{i}tx} - \int_{\mathbb{R}} dt\mathcal{F}[\psi](t)e^{\mathrm{i}ty}}{x - y} d\big(\widetilde{\mu}_N - \mu_V\big)(x) d\big(\widetilde{\mu}_N - \mu_V\big)(y) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} dt\mathrm{i}t\mathcal{F}[\psi](t)\iint_{\mathbb{R}^2} e^{\mathrm{i}ty}\frac{e^{\mathrm{i}t(x-y)} - 1}{\mathrm{i}t(x-y)} d\big(\widetilde{\mu}_N - \mu_V\big)(x) d\big(\widetilde{\mu}_N - \mu_V\big)(y) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} dt\mathrm{i}t\mathcal{F}[\psi](t)\iint_{\mathbb{R}^2} e^{\mathrm{i}ty}\int_0^1 d\alpha e^{\mathrm{i}\alpha t(x-y)} d\big(\widetilde{\mu}_N - \mu_V\big)(x) d\big(\widetilde{\mu}_N - \mu_V\big)(y) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} dt\mathrm{i}t\mathcal{F}[\psi](t)\int_0^1 d\alpha \int_{\mathbb{R}} e^{\mathrm{i}\alpha tx} d\big(\widetilde{\mu}_N - \mu_V\big)(x)\int_{\mathbb{R}} e^{\mathrm{i}(1-\alpha)ty} d\big(\widetilde{\mu}_N - \mu_V\big)(y)\end{aligned}$$

We then apply in order the triangular inequality, Cauchy-Schwarz inequality, a change of variable and the fact that $|\mathcal{F}[\widetilde{\mu}_N - \mu_V]|^2$ is an even function.

$$\begin{aligned}
|\widetilde{\zeta}_N(\psi)| &\leq \frac{1}{2\pi}\int_{\mathbb{R}} dt\,|t\mathcal{F}[\psi](t)| \int_0^1 d\alpha\,|\mathcal{F}[\widetilde{\mu}_N - \mu_V](\alpha t)|\,.\,\left|\mathcal{F}[\widetilde{\mu}_N - \mu_V]\big((1-\alpha)t\big)\right| \\
&\leq \frac{1}{2\pi}\int_{\mathbb{R}} dt\,|t\mathcal{F}[\psi](t)| \Big(\int_0^1 d\alpha\,|\mathcal{F}[\widetilde{\mu}_N - \mu_V](\alpha t)|^2\Big)^{\frac{1}{2}} \Big(\int_0^1 d\alpha\,\left|\mathcal{F}[\widetilde{\mu}_N - \mu_V]\big((1-\alpha)t\big)\right|^2\Big)^{\frac{1}{2}} \\
&\leq \frac{1}{2\pi}\int_{\mathbb{R}} dt\,|t\mathcal{F}[\psi](t)| \int_0^1 d\alpha\,|\mathcal{F}[\widetilde{\mu}_N - \mu_V](\alpha t)|^2 \\
&\leq \frac{1}{2\pi}\int_0^{+\infty} dt\,|t\mathcal{F}[\psi](t)| \int_0^1 \frac{td\alpha}{t\alpha}\,|\mathcal{F}[\widetilde{\mu}_N - \mu_V](\alpha t)|^2 \\
&+ \frac{1}{2\pi}\int_{-\infty}^0 dt\,|t\mathcal{F}[\phi](t)| \int_0^1 \frac{-td\alpha}{-t\alpha}\,|\mathcal{F}[\widetilde{\mu}_N - \mu_V](\alpha t)|^2 \\
&\leq \frac{1}{2\pi}\int_{\mathbb{R}} dt\,|t\mathcal{F}[\psi](t)|\,D^2(\widetilde{\mu}_N, \mu_V) \\
&\leq \frac{1}{2\pi}\Big(\int_{\mathbb{R}} dt\,|t\mathcal{F}[\psi](t)|^2\,(1+t^2)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \frac{dt}{1+t^2}\Big)^{\frac{1}{2}} D^2(\widetilde{\mu}_N, \mu_V) \\
&\leq \frac{1}{2\sqrt{\pi}} D^2(\widetilde{\mu}_N, \mu_V) N_2(\psi) \\
&\leq \frac{1}{2\sqrt{\pi}} D^2(\widetilde{\mu}_N, \mu_V) \|\psi\|_{H^2(\mathbb{R})}
\end{aligned}$$

By density of $\mathcal{S}(\mathbb{R})$ in $H^2(\mathbb{R})$, and since $\widetilde{\zeta}_N : \Big(H^2(\mathbb{R}), \|\cdot\|_{H^2(\mathbb{R})}\Big) \to \mathbb{R}$ is continuous, the inequality still holds for ϕ. Thus, using equation (1.33),

$$\begin{aligned}
\mathbb{P}_N^{V,P}\left(|\widetilde{\zeta_N}(\phi)| > \frac{1}{2}N^{-1+\varepsilon}\right) &\leq \mathbb{P}_N^{V,P}\left(D^2(\widetilde{\mu}_N, \mu_V) > \frac{N^{-1+\varepsilon}}{2C\|\phi\|_{H^2(\mathbb{R})}}\right) \\
&\leq \exp\left\{-P\frac{N^{\varepsilon}}{2C\|\phi\|_{H^2(\mathbb{R})}} + 5P\log N + K\right\},
\end{aligned}$$

which concludes the proof. □

1.4 Localization of the edge of a configuration

In [Lam21b, Theorem 1.8, Theorem 3.4], Lambert was able to control the edge (*i.e* the minimum and the maximum) of a typical configuration $(x_1, \dots, x_N)$ distributed according to $\mathbb{P}_N^{V,P}$, by showing that the random measure

$$\Xi_N \overset{(\text{def})}{=} \sum_{j=1}^N \delta_{\varphi_N^{-1}(x_j)}$$

converges in distribution towards a Poisson point process for a function φ_N which takes the form

$$\varphi_N(x) \overset{(\text{def})}{=} E_N + \alpha_N^{-1}x\,.$$

Before being more precise on the construction of $(E_N)_N$ and $(\alpha_N)_N$, we explain, following [Lam21b], how one can use this convergence to localize the edge of a typical configuration $(x_1, \dots, x_N)$. Let us assume for a moment that Ξ_N converges towards a Poisson point process with intensity $\theta(x) = e^{-x}$, with $E_N \to +\infty$. In particular, the random variable

$$\Xi_N(t, +\infty)$$

converges in distribution towards a Poisson random variable with mean $\int_t^{+\infty} e^{-x}dx$. Combined with the equalities

$$\mathbb{P}_N^{V,P}\Big(\Xi_N(t,+\infty)=0\Big) = \mathbb{P}_N^{V,P}\Big(\forall\ 1\le j\le N,\ \varphi_N^{-1}(x_j)=\alpha_N(x_j-E_N)\le t\Big)$$
$$= \mathbb{P}_N^{V,P}\left(\alpha_N\left(\max_{1\le j\le N} x_j - E_N\right)\le t\right),$$

we deduce that for all $t\in\mathbb{R}$

$$\mathbb{P}_N^{V,P}\left(\alpha_N\left(\max_{1\le j\le N} x_j - E_N\right)\le t\right) \underset{N\to\infty}{\longrightarrow} \exp(-e^{-t})\,.$$

Therefore, the random variable

$$\alpha_N\left(\max_{1\le j\le N} x_j - E_N\right)$$

converges in distribution to the Gumbel law, showing that the maximum of a configuration is of order E_N. Furthermore, as will be clear from the construction of α_N and E_N, α_N is positive, and goes to infinity as N goes to infinity.

Replacing in the previous analysis $\theta(x)=e^x$ and $E_N\to-\infty$, we would have deduced in the same fashion that

$$\alpha_N\left(\min_{1\le j\le N} x_j - E_N\right)$$

converges in law.

With the above notations, we can apply [Lam21b, Theorem 3.4] to our context.

Theorem 1.4.1 *Let $v=\pm$. There exists $(E_N^v)_N$, $(\alpha_N^v)_N$ sequences of real numbers with $|E_N^v|\to+\infty$, $\alpha_N^v>0$ for large enough N, satisfying $V'(E_N^v)=\alpha_N^v v$, such that:*

a) $\dfrac{Ne^{-V(E_N^v)+2P\log|E_N^v|+\lambda_V^P}}{\alpha_N^v} \underset{N\to\infty}{\longrightarrow} 1$ *(recall $C_{V,P}$ is defined through equation (1.5)),*

b) $\frac{\log(\alpha_N^v)}{N}\underset{N\to\infty}{\longrightarrow}0$ *and* $\alpha_N^v|E_N^v|\underset{N\to\infty}{\longrightarrow}+\infty$ *,*

c) For all compact $K\subset\mathbb{R}$,

$$(\alpha_N^v)^{-2}\sup_{x\in K}\left|V''(\varphi_N(x))\right|\underset{N\to\infty}{\longrightarrow}0\,.$$

As a consequence, the random measure Ξ_N converges in distribution as $N\to\infty$ to a Poisson point process with intensity $\theta(x)=e^{-vx}$.

Proof We prove it in the case $v=+$, the case where $v=-$ being similar. We show that there exists a sequence $(E_N^+)_N$ going to $+\infty$ satisfying $f(E_N^+)=-\log N$, where we defined the function f by

$$f(x) = -V(x)+2P\log|x|+C_{V,P}-\log|V'(x)|\,.$$

Recalling Assumption 1.1.1 *i)*, $|V'|$ goes to infinity at infinity, thus $\alpha_N^+=V'(E_N^+)\to+\infty$ (in the case $v=-1$ we would have looked for a sequence $(E_N^-)_N$ going to $-\infty$ and $\alpha_N^-=-V'(E_N^-)$).
As a consequence of Assumptions 1.1.1,*ii)*, one shows that $\log|V'|$ is negligible with respect to V at infinity. Therefore, because $\dfrac{\log|x|}{V(x)}\underset{|x|\to\infty}{\longrightarrow}0$,

$$f(x) = -V(x)+\underset{x\to+\infty}{o}(V(x))\,. \tag{1.37}$$

Because $f(x) \underset{x\to+\infty}{\longrightarrow} -\infty$ there exists $(E_N^+)_N$ going to infinity such that for all $N \geq 1$, $f(E_N^+) = -\log N$. Setting $x = E_N^+$ in (1.37), we obtain that $-V(E_N^+) \sim f(E_N^+) = -\log N$. Property c) follows from Assumptions 1.1.1, point *iii)*, along with the fact that α_N^{-1} stays bounded.

It remains to show that $\dfrac{\log(\alpha_N^+)}{N} = \dfrac{\log|V'(E_N^+)|}{N} \underset{N\to\infty}{\longrightarrow} 0$. By construction, we have

$$\frac{\log|V'(E_N^+)|}{N} = \frac{\log\left(Ne^{-V(E_N^+)+2P\log E_N^+ + \lambda_P}\right)}{N} = -\frac{V(E_N^+)}{N} + o(1)\,.$$

Using that $V(E_N^+) \sim \log N$, we can conclude that $\log|V'(E_N^+)| = o(N)$ which concludes the proof.□

By the discussion preceding Theorem 1.4.1, we deduce the following corollary.

Corollary 1.4.2 (Edge of a configuration) *Let $E_N^\pm$, $\alpha_N^\pm \overset{(\text{def})}{=} |V'(E_N^\pm)|$ be the sequences of Theorem 1.4.1 associated with $v = \pm 1$. Then, both random variables*

$$\alpha_N^+\left(\max_{1\leq j\leq N} x_j - E_N^+\right)$$

and

$$\alpha_N^-\left(\min_{1\leq j\leq N} x_j - E_N^-\right)$$

converge to a Gumbel law, whose distribution function is given for $t \geq 0$ by $\mathcal{G}([0,t]) = \exp(e^{-t})$. Furthermore, $V(E_N^\pm) \sim \log N$ and $\alpha_N^\pm \underset{N\to\infty}{\longrightarrow} \pm\infty$.

Remark 1.4.3 Note that[Lam21b, Theorem 3.4] applies for V of class $\mathcal{C}^2$ outside of a compact set, allowing to take $V(x) = |x|^a$ for $a > 1$. In this case, we find $E_N^\pm \sim \pm(\log N)^{1/a}$. If $V(x) = \cosh(x)$, we find $E_N^+ \sim -E_N^- \sim \operatorname{arg\,cosh}(\log N) \sim \log\log N$.

The next lemma will be convenient in the proof of Theorem 1.5.2 when dealing with error terms.

Lemma 1.4.4 *With the notations of Corollary 1.4.2, we have*

$$\mu_V([E_N^-, E_N^+]^c) = o(N^{-1/2})\,.$$

Proof Let $0 < \delta < 1$, to be specified later. We have

$$\int_{E_N^+}^{+\infty} d\mu_V(x) = \int_{E_N^+}^{+\infty} \rho_V(x)^\delta \rho_V(x)^{1-\delta} dx \leq \int_{\mathbb{R}} \rho_V(x)^\delta dx \sup_{[E_N^+,+\infty[} \rho_V^{1-\delta}\,.$$

By the first inequality of Lemma 1.2.2, the integral is finite. Also from the same inequality, we have for some constant C' and x big enough $\rho_V(x) \leq C'e^{-\frac{3}{4}V(x)}$. Because V is increasing in a neighborhood of $+\infty$, we get for N large enough

$$\sup_{[E_N^+,+\infty[} \rho_V^{1-\delta} \leq C'^{1-\delta} e^{-(1-\delta)\frac{3}{4}V(E_N^+)}\,.$$

Taking $\delta > 0$ such that $\frac{1}{2} - (1-\delta)\frac{3}{4} \overset{(\text{def})}{=} -\gamma < 0$ and using that $V(E_N^+) = \log N + o(\log N)$ (established in the proof of Theorem 1.4.1),

$$\sqrt{N}\int_{E_N^+}^{+\infty} d\mu_V(x) \leq Ke^{-\gamma\log N + (1-\delta)\frac{3}{4}o(\log N)}\,,$$

and the right-hand side goes to zero as N goes to infinity. The integral $\displaystyle\int_{-\infty}^{E_N^-} d\mu_V(x)$ can be handled in the same way. □

Remark 1.4.5 We could improve the proof to show that $\mu_V([E_N^-, E_N^+]^c) \sim N^{-1+\varepsilon}$ for any $\varepsilon > 0$, but showing that it is $o(N^{\frac{1}{2}})$ is sufficient for what we need and requires less carefulness.

1.5 Laplace transform for smooth test functions, proof of Theorem 1.1.4

Section 1.3 allows us to justify in Proposition 1.5.1 the heuristics we gave in equation (1.6) for ϕ having compact support. We will then extend in Theorem 1.5.2 this result to a more general set of functions, by an approximation by compactly supported functions, using Corollary 1.4.2.

Proposition 1.5.1 *For $\phi \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ with compact support, we have for any real t, as N goes to infinity,*

$$\mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\nu_N(\Xi[\phi])\right)\right] \underset{N\to\infty}{\longrightarrow} \exp\left\{\frac{t^2}{2}q_P(\phi)\right\}, \tag{1.38}$$

where $\Xi[\phi]$ is given by equation (1.7)*, and $q_P(\phi)$ is given by*

$$q_P(\phi) \overset{(\mathrm{def})}{=} \int_{\mathbb{R}}\Big(\phi'(x)^2 + V''(x)\phi(x)^2\Big)d\mu_V(x) + P\iint_{\mathbb{R}^2}\Big(\frac{\phi(x)-\phi(y)}{x-y}\Big)^2 d\mu_V(x)d\mu_V(y)\,. \tag{1.39}$$

Proof Let $\phi \in \mathcal{C}_c^1(\mathbb{R},\mathbb{R})$, and let $t \in \mathbb{R}$. We perform in equation (1.4) the change of variables $x_i = y_i + \frac{t}{\sqrt{N}}\phi(y_i)$, $1 \le i \le N$, which is a diffeomorphism for N big enough. We thus have

$$\mathcal{Z}_N^P[V] = \int_{\mathbb{R}^N}\prod_{1\le i<j\le N}\Big|y_i - y_j + \frac{t}{\sqrt{N}}\big(\phi(y_i)-\phi(y_j)\big)\Big|^{\frac{2P}{N}}.\exp\Big[-\sum_{i=1}^N V\Big(y_i + \frac{t}{\sqrt{N}}\phi(y_i)\Big)\Big]$$
$$.\prod_{i=1}^N\left(1+\frac{t}{\sqrt{N}}\phi'(y_i)\right)d^N y, \tag{1.40}$$

and we develop separately the different terms of this integral. The first term can be written as:

$$\prod_{i<j}|y_i-y_j|^{2P/N}\prod_{i<j}\left|1+\frac{t}{\sqrt{N}}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}\right|^{2P/N},$$

The second product above, setting $\Delta\phi_{i,j} \overset{(\mathrm{def})}{=} \dfrac{\phi(y_i)-\phi(y_j)}{y_i-y_j}$ and using Taylor-Lagrange theorem, equals:

$$\exp\left(\frac{2P}{N}\sum_{i<j}\log\left|1+\frac{t}{\sqrt{N}}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}\right|\right)$$
$$= \exp\left(\frac{2P}{N}\sum_{i<j}\left(\frac{t}{\sqrt{N}}\Delta\phi_{i,j} - \frac{t^2}{2N}(\Delta\phi_{i,j})^2 + R_{N,1}(i,j)\right)\right), \tag{1.41}$$

where we noticed that $1+\dfrac{t}{\sqrt{N}}\Delta\phi_{i,j} \ge 1 - \dfrac{t}{\sqrt{N}}\|\phi'\|_\infty > 0$ if N is big enough, and where

$$|R_{N,1}(i,j)| \le \frac{|t|^3}{3N^{3/2}}\|\phi'\|_\infty^3.$$

Again by Taylor-Lagrange theorem, the second term in (1.40) equals

$$\exp\Big(-\sum_{i=1}^N\Big(V(y_i) + \frac{t}{\sqrt{N}}V'(y_i)\phi(y_i) + \frac{t^2}{2N}V''(y_i)\phi(y_i)^2 + R_{N,2}(i)\Big)\Big),$$

where $R_{N,2}(i) = \dfrac{t^3}{6N^{3/2}} V^{(3)}\left(y_i + \dfrac{t\theta_i}{\sqrt{N}}\phi(y_i)\right)\phi(y_i)^3$ for some $\theta_i \in [0,1]$. This remainder thus verifies, for N large enough:

$$|R_{N,2}(i)| \leq \frac{|t|^3}{6N^{3/2}} \|\phi\|_\infty^3 \sup_{d(x,\text{supp }\phi)\leq 1} |V^{(3)}(x)|.$$

The last term reads

$$\prod_{i=1}^N \left(1 + \frac{t}{\sqrt{N}}\phi'(y_i)\right) = \exp\left(\sum_{i=1}^N \left(\frac{t}{\sqrt{N}}\phi'(y_i) - \frac{t^2}{2N}\phi'(y_i)^2 + R_{N,3}(i)\right)\right),$$

with $|R_{N,3}(i)| \leq \dfrac{t^3}{3N^{3/2}}\|\phi'\|_\infty^3$. Dividing both sides of equation (1.40) by $\mathcal{Z}_N^P[V]$, we get:

$$\begin{aligned}\mathbb{E}_N^{V,P}\Bigg[\exp\left\{t\sqrt{N}\left(P\iint_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\hat{\mu}_N(x)d\hat{\mu}_N(y) + \int_{\mathbb{R}}(\phi'(x) - V'(x)\phi(x))d\hat{\mu}_N(x)\right)\right\} \\ \times \exp\left\{\frac{t^2}{2}\left(-P\iint_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\hat{\mu}_N(x)d\hat{\mu}_N(y) - \int_{\mathbb{R}}(V''(x)\phi^2(x) + \phi'(x)^2)d\hat{\mu}_N\right)\right\} \\ \times \exp\{K_N(t,\phi)\}\Bigg] = 1,\end{aligned}$$

with $|K_N(t,\phi)| \leq \dfrac{c(t,\phi)}{\sqrt{N}}$ where $c(t,\phi) \geq 0$ is independent of N. This bound implies that, taking the limit $N \to \infty$, we can get rid of K_N:

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}_N^{V,P}\Bigg[\exp\left\{t\sqrt{N}\left(P\iint_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\hat{\mu}_N(x)d\hat{\mu}_N(y) + \int_{\mathbb{R}}\big[\phi'(x) - V'(x)\phi(x)\big]d\hat{\mu}_N(x)\right)\right\} \\ \times\exp\left\{\frac{t^2}{2}\left(-P\iint_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\hat{\mu}_N(x)d\hat{\mu}_N(y) - \int_{\mathbb{R}}(V''(x)\phi^2(x) + \phi'(x)^2)d\hat{\mu}_N(x)\right)\right\}\Bigg] = 1.\end{aligned}$$

Using Fubini's theorem (the function $(x,y) \mapsto \dfrac{\phi(x)-\phi(y)}{x-y}$ being bounded continuous on $\mathbb{R}^2$), the first line in the expectation value can be rewritten as $e^{t\sqrt{N}\Lambda_N}$ with:

$$\Lambda_N \overset{\text{(def)}}{=} 2P\iint_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\mu_V(x)d(\hat{\mu}_N-\mu_V)(y) + \int_{\mathbb{R}}\big[\phi'(x) - V'(x)\phi(x)\big]d(\hat{\mu}_N-\mu_V)(x) + P\zeta_N(\phi) \tag{1.42}$$

where $\zeta_N(\phi)$ is given by (1.34), and where we differentiated Equation (1.5) — every term involved being differentiable by the results of Section 1.2— to deduce

$$\int_{\mathbb{R}}\big[V'(x)\phi(x) - \phi'(x)\big]d\mu_V(x) - P\iint_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\mu_V(x)d\mu_V(y) = 0\,.$$

Let $F : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$F(\mu) \overset{\text{(def)}}{=} -P\iint_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\mu(x)d\mu(y) - \int_{\mathbb{R}}\big[V''(x)\phi(x)^2 + \phi'(x)^2\big]d\mu(x)\,. \tag{1.43}$$

It is continuous for the topology of weak convergence since all the functions in the integrals are bounded continuous. So far we have established that:

$$\lim_{N\to\infty}\mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)\right)\right] = 1, \tag{1.44}$$

with Λ_N given by (1.42). We now replace in the latter equation the term $F(\hat{\mu}_N)$ by its limiting expression, $F(\mu_V)$. Fix a metric that is compatible with the weak convergence of probability measures on $\mathbb{R}$. For example,

$$d_{\mathrm{Lip}}(\mu,\nu) \overset{(\mathrm{def})}{=} \sup \left| \int_{\mathbb{R}} f(x) d\mu(x) - \int_{\mathbb{R}} f(x) d\nu(x) \right| , \tag{1.45}$$

where the supremum runs over $f : \mathbb{R} \to \mathbb{R}$ bounded and Lipschitz with $\|f\|_\infty \leq 1$ and Lipschitz constant $|f|_{\mathrm{Lip}} \leq 1$. By the large deviations principle for $(\hat{\mu}_N)_N$ under the probability (1.3) established by [GZ19, Theorem 1.1], for all $\delta > 0$ the event $A_N(\delta) \overset{(\mathrm{def})}{=} \{d_{\mathrm{Lip}}(\hat{\mu}_N, \mu_V) > \delta\}$ has (for N big enough) probability smaller than e^{-Nc_δ} where $c_\delta > 0$. Hence, it is convenient to decompose the expectation of equation (1.44) into:

$$\mathbb{E}_N^{V,P}\left[\mathbf{1}_{A_N(\delta)} \exp\left(t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)\right)\right] + \mathbb{E}_N^{V,P}\left[\mathbf{1}_{A_N(\delta)^c} \exp\left(t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)\right)\right] \overset{(\mathrm{def})}{=} \mathcal{I}^{(1)} + \mathcal{I}^{(2)}.$$

We can bound the second term by using Cauchy-Schwarz inequality:

$$0 \leq \mathcal{I}^{(2)} \leq \mathbb{P}(A_N(\delta)^c)^{\frac{1}{2}} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)\right)\right]^{1/2} = e^{-Nc_\delta/2} \underset{N\to+\infty}{O}\left(e^{\sqrt{N}C(\phi,V,t)}\right) ;$$

where we used the rough bound

$$|\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\mu_N)| \leq \sqrt{N}c_1(\phi,V) + \frac{t^2}{2}c_2(V,\phi)$$

for some $c_1(\phi,V)$, $c_2(\phi,V) > 0$ independent of N. By taking the limit $N \to +\infty$, we conclude that:

$$\begin{aligned} 1 = \lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)\right)\right] \\ = \lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_V)\leq\delta\}} \exp\left(t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)\right)\right]. \end{aligned}$$

By continuity of F there is some $\varepsilon(\delta)$ which goes to 0 as $\delta \to 0$ such that, for $d_{\mathrm{Lip}}(\nu,\mu_V) \leq \delta$, we have $|F(\nu) - F(\mu_V)| \leq \varepsilon(\delta)$. Taking the (decreasing) limit as δ goes to zero, we deduce that:

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N + \frac{t^2}{2}F(\hat{\mu}_N)\right)\right] = \lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_V)\leq\delta\}} e^{t\sqrt{N}\Lambda_N}\right] e^{\frac{t^2}{2}F(\mu_V)}.$$

But the same large deviations argument shows that:

$$\lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_V)\leq\delta\}} \exp\left(t\sqrt{N}\Lambda_N\right)\right] = \lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N\right)\right].$$

Thus, we have shown that:

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}_N^{V,P}\Bigg[\exp\Bigg\{t\sqrt{N}\Big(2P\iint_{\mathbb{R}^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_V(x) d(\hat{\mu}_N - \mu_V)(y) + P\zeta_N(\phi) \\ + \int_{\mathbb{R}} (\phi'(x) - V'(x)\phi(x)) d(\hat{\mu}_N - \mu_V)(x)\Big)\Bigg\}\Bigg] = \exp\Big(-\frac{t^2}{2}F(\mu_V)\Big), \end{aligned} \tag{1.46}$$

which establishes that $\sqrt{N}\Lambda_N = \sqrt{N}\Big(\nu_N(\Xi[\phi]) + P\zeta_N(\phi)\Big)$ converges in law towards a centered Gaussian random variable with announced variance. We finally get rid of the remaining term $\zeta_N(\phi)$, using Corollary 1.3.4: taking $\varepsilon = 1/4$ for example, we see in particular that $\sqrt{N}\zeta_N(\phi)$ converges in probability towards zero. The conclusion follows from Slutsky's lemma. □

We now extend the result of Proposition 1.5.1 to a more general set of functions. With the notations of Proposition 1.5.1, the following theorem holds:

Theorem 1.5.2 *Let $\phi \in H^2(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ such that ϕ'' is bounded. Additionally, suppose that $V^{(3)}\phi^2$, $V''\phi\phi'$, $V''\phi^2$ and $V'\phi$ are bounded. Then, recalling* (1.39) *we have the convergence as N goes to infinity*

$$\sqrt{N}\nu_N(\Xi[\phi]) \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\left(0, q_P(\phi)\right) .$$

Proof For $N \geq 1$, let E_N^-, E_N^+ be given by Corollary 1.4.2. Let $\chi_N : \mathbb{R} \to [0,1]$ be $\mathcal{C}^2$ with compact support such that:

$$\chi_N(x) = 1 \text{ for } x \in [E_N^- - 1, E_N^+ + 1] \text{ and } \chi_N(x) = 0 \text{ for } x \in [E_N^- - 2, E_N^+ + 2]^c$$

and such that, denoting $\phi_N = \phi\chi_N$, $\sup_N \|\phi_N^{(k)}\|_\infty + \|\phi_N^{(k)}\|_{L^2(\mathbb{R})} < +\infty$ for $k = 0, 1, 2$ (we assumed $\phi \in H^2(\mathbb{R})$ and $\|\phi''\|_\infty < +\infty$, in particular ϕ, ϕ' and ϕ'' are bounded and such a χ_N exists). The point of cutting ϕ outside the set $[E_N^- - 1, E_N^+ + 1]$ is that with high probability, the empirical measure $\hat{\mu}_N$ doesn't see the difference between ϕ and ϕ_N.

The support of ϕ_N is then contained in $[E_N^- - 2, E_N^+ + 2]$, and we now argue that the proof of Proposition 1.5.1 can be adapted so that:

$$\sqrt{N}\nu_N(\Xi[\phi_N]) \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}(0, q_P(\phi)) . \tag{1.47}$$

Similarly as in Proposition 1.5.1, we perform, in $\mathcal{Z}_N^P[V]$, the change of variables $x_i = y_i + \frac{t}{\sqrt{N}}\phi_N(y_i)$, $1 \leq i \leq N$, which is the same as before, but with ϕ replaced by ϕ_N. First, with $I_N \overset{\text{(def)}}{=} [E_N^- - 2, E_N^+ + 2]$, the error term

$$K_N(t, \phi_N) \leq 2\frac{t^3}{3N^{1/2}}\|\phi_N'\|_\infty^3 + \frac{t^3}{6N^{1/2}}\|\phi_N\|_\infty \sup_{d(x, I_N)\leq 1} |V^{(3)}(x)|$$

of the proof of Proposition 1.5.1 is still going to zero, because of our choice of χ_N and Assumption 1.1.2. As previously, we then have:

$$\lim_{N\to\infty} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N(\phi_N) + \frac{t^2}{2}F_N(\hat{\mu}_N)\right)\right] = 1 \tag{1.48}$$

with

$$\Lambda_N(\phi_N) \overset{\text{(def)}}{=} 2P \iint_{\mathbb{R}^2} \frac{\phi_N(x) - \phi_N(y)}{x - y} d\mu_V(x) d(\hat{\mu}_N - \mu_V)(y) + P\zeta_N(\phi_N) + \int_{\mathbb{R}} \left[\phi_N'(x) - V'(x)\phi_N(x)\right] d(\hat{\mu}_N - \mu_V)(x) ,$$

where ζ_N is given by (1.34), and

$$F_N(\hat{\mu}_N) \overset{\text{(def)}}{=} -P \iint_{\mathbb{R}^2} \left(\frac{\phi_N(x) - \phi_N(y)}{x - y}\right)^2 d\hat{\mu}_N(x) d\hat{\mu}_N(y) - \int_{\mathbb{R}} \left[V''(x)\phi_N(x)^2 + \phi_N'(x)^2\right] d\hat{\mu}_N(x) .$$

Taking again the distance d_{Lip} defined in (1.45), one can check that for μ, ν probability measures over $\mathbb{R}$:

$$|F_N(\mu) - F_N(\nu)| \leq C_N d_{\text{Lip}}(\mu, \nu) ,$$

where C_N is a term depending on the norms $\|\phi_N'\|_\infty, \|\phi_N''\|_\infty$, $\|V''\phi_N^2\|_\infty$ and $\|(V''\phi_N^2)'\|_\infty$. The choice of χ_N and the fact that ϕ is chosen so that $V^{(3)}\phi^2$ and $V''\phi\phi'$ are bounded, guarantee that

$\|(V''\phi_N^2)'\|_\infty$ is bounded in N. The other norms are easily bounded by hypothesis. Therefore C_N can be seen to be uniformly bounded in N, and we find some $C \geq 0$ independent of N such that

$$|F_N(\mu) - F_N(\nu)| \leq C d_{\mathrm{Lip}}(\mu, \nu)\,.$$

As in proposition 1.5.1, we use the large deviation principle for $(\hat{\mu}_N)$ to deduce that:

$$\lim_{N\to+\infty} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N(\phi_N) + \frac{t^2}{2}F_N(\hat{\mu}_N)\right)\right] = \lim_{N\to+\infty} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N(\phi_N)\right)\right] e^{\frac{t^2}{2}F_N(\mu_V)}\,.$$

By dominated convergence, $F_N(\mu_V)$ converges to $F(\mu_V)$, the function F being given by (1.43). This shows the convergence as N goes to infinity

$$\lim_{N\to+\infty} \mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\Lambda_N(\phi_N)\right)\right] = \exp\Big(-\frac{t^2}{2}F(\mu_V)\Big)\,,$$

and we deduce that $\sqrt{N}\Big(\nu_N(\Xi[\phi_N]) + P\zeta_N(\phi_N)\Big)$ converges towards a centered Gaussian variable with variance $-F(\mu_V) = q_P(\phi)$. Because $\sup_N \|\phi_N\|_{H^2(\mathbb{R})}$ is finite, we can again apply Corollary 1.3.4 to deduce the convergence in law (1.47). We now have the ingredients to conclude, by showing that the characteristic function

$$\mathbb{E}_N^{V,P}\left[\exp\left(\mathrm{i}t\sqrt{N}\nu_N(\Xi[\phi])\right)\right]$$

converges to the characteristic function of a Gaussian variable with appropriate variance. Denoting

$$\widetilde{\Xi}[\phi](x) \overset{(\mathrm{def})}{=} 2P\int_{\mathbb{R}} \frac{\phi(x)-\phi(y)}{x-y} d\mu_V(y) + \phi'(x) - V'(x)\phi(x) = \Xi[\phi](x) + 2P\int_{\mathbb{R}} \mathcal{H}[\phi\rho_V](y) d\mu_V(y)\,,$$

we have:

$$\nu_N(\Xi[\phi]) = \nu_N(\widetilde{\Xi}[\phi]).$$

We aim at establishing that:

$$\mathbb{E}_N^{V,P}\left[\exp\left(\mathrm{i}t\sqrt{N}\nu_N(\widetilde{\Xi}[\phi])\right)\right] = \mathbb{E}_N^{V,P}\left[\exp\left(\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi](x)d\hat{\mu}_N(x)\right)\right] \times \exp\Big(-\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi](x)d\mu_V(x)\Big)$$

converges towards the appropriate characteristic function.

By Corollary 1.4.2, the probability under $\mathbb{P}_N^{V,P}$ of the event $\mathcal{E}_N = \Big\{x_1,\dots,x_N \in [E_N^- - 1, E_N^+ + 1]\Big\}$ converges to 1. Along with the convergence (1.47), we deduce that:

$$\begin{aligned}\exp\Big(-\frac{t^2}{2}q_P(\phi)\Big) &= \lim_N \mathbb{E}_N^{V,P}\left[\exp\left(\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi_N](x)d\hat{\mu}_N(x)\right)\right]\exp\left(-\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi_N](x)d\mu_V(x)\right)\\ &= \lim_N \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\mathcal{E}_N}\exp\left(\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi_N](x)d\hat{\mu}_N(x)\right)\right]\exp\Big(-\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi_N](x)d\mu_V(x)\Big)\,,\end{aligned}$$

where we used that:

$$\left|\mathbb{E}_N^{V,P}\left[\mathbf{1}_{\mathcal{E}_N^c}\exp\left(\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi_N](x)d\hat{\mu}_N(x)\right)\right]\exp\Big(-\mathrm{i}t\sqrt{N}\int_{\mathbb{R}}\widetilde{\Xi}[\phi_N](x)d\mu_V(x)\Big)\right| \leq \mathbb{P}_N^{V,P}(\mathcal{E}_N^c) \underset{N\to+\infty}{\longrightarrow} 0\,.$$

Using that $\phi_N = \phi$ on $J_N = [E_N^- - 1, E_N^+ + 1]$,

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{\Xi}[\phi_N](x)d\mu_V(x) &= 2P \iint_{\mathbb{R}^2} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_V(x)d\mu_V(y) + \int \big[\phi_N'(x) - V'(x)\phi_N(x)\big]d\mu_V(x) \\ &= 2P \iint_{J_N^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_V(x)d\mu_V(y) + 2P \iint_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_V(x)d\mu_V(y) \\ &+ \int_{J_N} \big[\phi'(x) - V'(x)\phi(x)\big]d\mu_V(x) + \int_{J_N^c} \big[\phi(x)\chi_N'(x) + \phi'(x)\chi_N(x) - V'(x)\phi(x)\chi_N(x)\big]d\mu_V(x)\,.\end{aligned}$$

By boundedness of $(\|\phi_N'\|_\infty)_N$, the second term is bounded by

$$C_P \iint_{(J_N^2)^c} d\mu_V(x)d\mu_V(y) \le 2C_P\mu_V(J_N^c) = o(N^{-1/2})\,,$$

where we used the union bound and Lemma 1.4.4. By the same estimate and the fact that χ_N can be chosen so that $(\|\chi_N'\|_\infty)_N$ is bounded, and because ϕ', $V'\phi$ are bounded, the last integral term is also $o(N^{-1/2})$. By the previous arguments, we also conclude that

$$2P \iint_{(J_N^2)^c} \frac{\phi(x)-\phi(y)}{x-y} d\mu_V(x)d\mu_V(y) + \int_{J_N^c} \big[\phi'(x) - V'(x)\phi(x)\big]d\mu_V(x) = o(N^{-1/2})\,,$$

thus

$$\int_{\mathbb{R}} \widetilde{\Xi}[\phi_N](x)d\mu_V(x) = \int_{\mathbb{R}} \widetilde{\Xi}[\phi]d\mu_V(x) + o(N^{-1/2})\,,$$

and so far we have :

$$\exp\Big(-\frac{t^2}{2}q_P(\phi)\Big) = \lim_N \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\mathcal{E}_N} \exp\Big(\mathrm{i}t\sqrt{N}\int_{\mathbb{R}} \widetilde{\Xi}[\phi_N](x)d\hat{\mu}_N(x)\Big)\right] \exp\Big(-\mathrm{i}t\sqrt{N}\int_{\mathbb{R}} \widetilde{\Xi}[\phi](x)d\mu_V(x)\Big)\,.$$

Finally, using that on $\mathcal{E}_N$ the measure $\hat{\mu}_N$ is supported on J_N (because each $x_i \in J_N$), using $\phi_N = \phi$ on J_N, we get:

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{\Xi}[\phi_N](x)d\hat{\mu}_N(x) &= 2P \iint_{J_N^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_V(x)d\hat{\mu}_N(y) + 2P \iint_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_V(x)d\hat{\mu}_N(y) \\ &+ \int_{J_N} \big[\phi'(x) - V'(x)\phi(x)\big]d\hat{\mu}_N(x) \\ &= 2P \iint_{J_N^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_V(x)d\hat{\mu}_N(y) + \int_{\mathbb{R}} \big[\phi'(x) - V'(x)\phi(x)\big]d\hat{\mu}_N(x) + o(N^{-1/2})\,,\end{aligned}$$

Where in the second line we used, using Lemma 1.4.4 again, that

$$\iint_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_V(x)d\hat{\mu}_N(y) = \iint_{J_N\times J_N^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_V(x)d\hat{\mu}_N(y) = o(N^{-1/2})\,,$$

and the same estimate holds for ϕ_N replaced by ϕ. Therefore,

$$\log \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\mathcal{E}_N} \exp\Big(\mathrm{i}t\sqrt{N}\int_{\mathbb{R}} \widetilde{\Xi}[\phi](x)d\hat{\mu}_N(x)\Big)\right] \exp\Big(-\mathrm{i}t\sqrt{N}\int_{\mathbb{R}} \widetilde{\Xi}[\phi](x)d\mu_V(x)\Big) \underset{N\to\infty}{\longrightarrow} -\frac{t^2}{2}q_P(\phi)\,.$$

This establishes that:

$$\lim_N \mathbb{E}_N^{V,P}\left[\exp\Big(\mathrm{i}t\sqrt{N}\int_{\mathbb{R}} \Xi[\phi](x)d\hat{\mu}_N(x)\Big)\right] = \exp\Big(-\frac{t^2}{2}q_P(\phi)\Big)\,,$$

which concludes the proof. □

Remark 1.5.3 Taking ϕ such that ϕ' satisfies the conditions of Theorem 1.5.2, we have:

$$\mathbb{E}_N^{V,P}\left[\exp\left(t\sqrt{N}\nu_N(\mathcal{L}[\phi])\right)\right] \underset{N\to\infty}{\longrightarrow} \exp\left\{\frac{t^2}{2}q_P(\phi')\right\}, \tag{1.49}$$

where the operator $\mathcal{L}$ is defined as $\mathcal{L}[\phi] \overset{(\text{def})}{=} \Xi[\phi']$, *i.e.*

$$\mathcal{L}[\phi](x) = 2P\int_{\mathbb{R}} \frac{\phi'(x)-\phi'(y)}{x-y}d\mu_V(y) + \phi''(x) - V'(x)\phi'(x) - 2P\int_{\mathbb{R}} \mathcal{H}[\phi'\rho_V](y)d\mu_V(y)\,. \tag{1.50}$$

Note that $q_P^V(\phi') = \left(\sigma_P^V\right)^2(\mathcal{L}[\phi])$ where σ_P^V is defined in (1.12). By Theorem 1.7.1, the class of functions in $\mathcal{L}^{-1}[\mathcal{T}]$ where

$$\mathcal{T} \overset{(\text{def})}{=} \left\{ f \in \mathcal{C}^2(\mathbb{R}) \,|\, f, f', f'' \text{ are bounded}, \int_{\mathbb{R}} f(x)d\mu_V(x) = 0 \right\} \tag{1.51}$$

satisfies (1.49). This proves Theorem 1.1.4.

1.6 Inversion of $\mathcal{L}$

This section is dedicated to the definition of $\mathcal{L}$ given by (1.8) and its domain and then we focus on its inversion. We rely heavily on results of Appendix 1.8: the diagonalization of the operator $\mathcal{A}$ by the use of the theory of Schrödinger operators.

Let $P > 0$ be fixed. We introduce the operators $\mathcal{A}$ and $\mathcal{W}$, acting on sufficiently smooth functions of $L^2(\mu_V)$, by

$$\mathcal{A}[\phi] \overset{(\text{def})}{=} -\frac{\left(\phi'\rho_V\right)'}{\rho_V} = -\left(\phi'' + \frac{\rho_V'}{\rho_V}\phi'\right) \quad \text{and} \quad \mathcal{W}[\phi] \overset{(\text{def})}{=} -\mathcal{H}\big[\phi'\rho_V\big] + \int_{\mathbb{R}} \mathcal{H}\big[\phi'\rho_V\big](y)d\mu_V(y)\,. \tag{1.52}$$

One can show that the operator $\mathcal{A}$ corresponds to the operator verifying:

$$\langle\phi,\psi\rangle_{\mathsf{H}} = \int_{\mathbb{R}} \phi'(x)\psi'(x)d\mu_V(x) = \int_{\mathbb{R}} \phi(x)\mathcal{A}[\psi](x)d\mu_V(x) = \langle\phi,\mathcal{A}[\psi]\rangle_{L^2(\mu_V)}$$

We first show the following decomposition of $\mathcal{L}$.

Lemma 1.6.1 *For ϕ twice differentiable we have the following pointwise identity*

$$-\mathcal{L}[\phi] = \mathcal{A}[\phi] + 2P\mathcal{W}[\phi]\,. \tag{1.53}$$

Proof We write for $x \in \mathbb{R}$

$$2P\int_{\mathbb{R}} \frac{\phi'(x)-\phi'(y)}{x-y}\rho_V(y)dy = -2P\phi'(x)\mathcal{H}[\rho_V](x) + 2P\mathcal{H}[\phi'\rho_V](x)\,. \tag{1.54}$$

Then,

$$\mathcal{L}[\phi] = \phi'' - V'\phi' - 2P\phi'\mathcal{H}[\rho_V] + 2P\mathcal{H}\big[\phi'\rho_V\big] - 2P\int_{\mathbb{R}} \mathcal{H}[\phi'\rho_V](y)d\mu_V(y)\,.$$

By (1.19) we have $-V' - 2P\mathcal{H}[\rho_V] = \dfrac{\rho_V'}{\rho_V}$, which concludes the proof. □

In order to state the next theorem, whose proof we detail in the Appendix, we introduce the following Sobolev-type spaces. Let

$$H^1_{V'}(\mathbb{R}) \overset{(\text{def})}{=} \left\{ u \in H^1(\mathbb{R}),\, uV' \in L^2(\mathbb{R}) \right\}.$$

As explained in the Appendix, $\mathcal{A}$, defined in (1.52), is linked to a Schrödinger operator $\mathcal{S} \overset{(\text{def})}{=} -\Delta + w_V$ *via* $\mathcal{A}[\phi] = \rho_V^{-1/2}\mathcal{S}[\rho_V^{1/2}\phi]$. Here w_V is defined in (1.70), continuous bounded by below and grows at infinity like $\frac{V'^2}{4}$. This operator $\mathcal{S}$ is defined on the following space:

$$\mathcal{D}(\mathcal{S}) = \left\{ u \in H^1_{V'}(\mathbb{R}), \sqrt{\rho_V}\mathcal{A}\left[\frac{1}{\sqrt{\rho_V}}u\right] \in L^2(\mathbb{R}) \right\}.$$

Now, let

$$\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) \overset{(\text{def})}{=} \rho_V^{-1/2}\mathcal{D}(\mathcal{S})$$

and its homogeneous counterpart

$$\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \overset{(\text{def})}{=} \left\{ u \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}),\ \int_{\mathbb{R}} u(x)d\mu_V(x) = 0 \right\}.$$

Finally, we let $L^2_0(\mu_V)$ be the subset of $L^2(\mu_V)$ of zero mean functions with respect to ρ_V.

We detail the proof of the following theorem in Appendix 1.8 which is based on Schrödinger operators theory.

Theorem 1.6.2 (Diagonalization of $\mathcal{A}$ in $L^2_0(\mu_V)$) *There exists a sequence $0 < \lambda_1 < \lambda_2 < \dots$ going to infinity, and a complete orthonormal set $(\phi_n)_{n\geq 1}$ of $L^2_0(\mu_V)$ of associated eigenfunctions for $\mathcal{A}$, meaning that*

- $\text{span}\{\phi_n,\ n \geq 1\}$ *is dense in* $L^2_0(\mu_V)$,
- *For all* i, j, $\langle \phi_i, \phi_j \rangle_{L^2(\mu_V)} = \delta_{i,j}$,
- *For all* $n \geq 1$, $\mathcal{A}[\phi_n] = \lambda_n \phi_n$.

Furthermore, each ϕ_n is in $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \to L^2_0(\mu_V)$ is bijective, and we have the writing, for $u \in L^2_0(\mu_V)$

$$\mathcal{A}^{-1}[u] = \sum_{n\geq 1} \lambda_n^{-1} \langle u, \phi_n \rangle_{L^2(\mu_V)} \phi_n \,.$$

We see the operators $\mathcal{A}$ and $\mathcal{W}$ as unbounded operators on the space

$$\mathsf{H} = \left\{ u \in H^1(\mu_V) \,|\, \int_{\mathbb{R}} u(x)d\mu_V(x) = 0 \right\}$$

endowed with the inner product $\langle u, v \rangle_{\mathsf{H}} = \langle u', v' \rangle_{L^2(\mu_V)}$. This defines an inner product on H and makes it a complete space: it can be seen that $H^1(\mu_V)$ is the completion of $\mathcal{C}^\infty_c(\mathbb{R})$ with respect to the inner product $\langle u, v \rangle_{L^2(\mu_V)} + \langle u', v' \rangle_{L^2(\mu_V)}$, see [Zhi98]. The space H is then the kernel of the bounded (with respect to $\|\cdot\|_{\mathsf{H}}$) linear form, $\langle \widetilde{1}, \cdot \rangle_{L^2(\mu_V)}$ on $H^1(\mu_V)$, and both inner products are equivalent on H because of the Poincaré inequality, Proposition 1.2.6. The use of H is motivated by the fact that both $\mathcal{A}$ and $\mathcal{W}$ are self-adjoint positive on this space as we show in Lemma 1.6.4.

In the next proposition, we deduce from Theorem 1.6.2 the diagonalization of $\mathcal{A}$ in H.

Proposition 1.6.3 (Diagonalization of $\mathcal{A}$ in H) *With the same eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ as in Theorem 1.6.2, there exists a complete orthonormal set $(\psi_n)_{n\geq 1}$ of H formed by eigenfunctions of $\mathcal{A}$.*

Proof With $(\phi_n)_{n\geq 1}$ of Theorem 1.6.2,

$$\begin{aligned}
\delta_{i,j} = \langle \phi_i, \phi_j \rangle_{L^2(\mu_V)} &= \frac{1}{\lambda_j} \langle \phi_i, \mathcal{A}[\phi_j] \rangle_{L^2(\mu_V)} \\
&= \frac{1}{\lambda_j} \langle \phi_i', \phi_j' \rangle_{L^2(\mu_V)} \\
&= \frac{1}{\lambda_j} \langle \phi_i, \phi_j \rangle_{\mathsf{H}}.
\end{aligned}$$

With $\psi_n = \frac{1}{\sqrt{\lambda_n}}\phi_n$, $(\psi_n)_{n\geq 1}$ is then orthonormal with respect to the inner product of H. To show that $\mathrm{span}\{\psi_n, n \geq 1\}$ is dense in H, let $u \in \mathsf{H}$ be such that for all $j \geq 1$, $\langle u, \phi_j\rangle_{\mathsf{H}} = 0$. In the last series of equalities, replace ϕ_i by u: we see that u is orthogonal to each ϕ_j in $L^2(\mu_V)$, thus u is a constant as shown in the proof of Lemma 1.8.10, and because $u \in \mathsf{H}$ it has zero mean against μ_V. This shows that $u = 0$. □

We set for what follows $\mathcal{D}(\mathcal{A}) \overset{(\mathrm{def})}{=} \{u \in \mathsf{H} \,|\, \mathcal{A}[u] \in \mathsf{H}\}$ and $\mathcal{D}(\mathcal{W}) \overset{(\mathrm{def})}{=} \{u \in \mathsf{H} \,|\, \mathcal{W}[u] \in \mathsf{H}\}$.

Lemma 1.6.4 *The following properties hold:*

- *The operator $\mathcal{W} : \mathcal{D}(\mathcal{W}) \to \mathsf{H}$ is positive: for all $\phi \in \mathcal{D}(\mathcal{W})$,*

$$\langle \mathcal{W}[\phi], \phi\rangle_{\mathsf{H}} = \frac{1}{2}\|\phi'\rho_V\|_{1/2}^2 \geq 0\,,$$

  with equality only for $\phi = 0$, where the $1/2$-norm of u is given by

$$\|u\|_{1/2}^2 \overset{(\mathrm{def})}{=} \int_{\mathbb{R}} |x|.\,|\mathcal{F}[u](x)|^2\,dx\,.$$

- *Both $\mathcal{A}$ and $\mathcal{W}$ are self-adjoint for the inner product of H.*

Proof To prove the first point, let $\phi \in \mathcal{D}(\mathcal{W})$. Then,

$$\begin{aligned} 2\pi\,\langle \mathcal{W}[\phi], \phi\rangle_{\mathsf{H}} = -2\pi\,\big\langle \mathcal{H}[\phi'\rho_V]', \phi'\rho_V\big\rangle_{L^2(dx)} &= -\big\langle ix\mathcal{F}\big[\mathcal{H}[\phi'\rho_V]\big], \mathcal{F}[\phi'\rho_V]\big\rangle_{L^2(dx)} \\ &= \pi\,\big\langle\, |\, x\,|\,\mathcal{F}[\phi'\rho_V], \mathcal{F}[\phi'\rho_V]\big\rangle_{L^2(dx)} = \pi\|\phi'\rho_V\|_{1/2}^2 \geq 0\,, \end{aligned}$$

and because ϕ is in H, this last quantity is zero if and only if ϕ vanishes.
For the second point, note that both $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{W})$ contain $\mathcal{C}_c^\infty(\mathbb{R}) \cap \mathsf{H}$, so $\mathcal{A}$ and $\mathcal{W}$ are densely defined. Let $\psi_i, \psi_j \in \mathrm{span}(\psi_n, n \geq 1)$, then by orthonormality

$$\langle \mathcal{A}[\psi_i], \psi_j\rangle_{\mathsf{H}} = \delta_{i,j}\lambda_i = \langle \psi_i, \mathcal{A}[\psi_j]\rangle_{\mathsf{H}}$$

hence $\mathcal{A}$ is symmetric. Furthermore, we notice that we have

$$\mathcal{D}(\mathcal{A}) = \{u \in \mathsf{H}, \sum_{n\geq 1} \lambda_n^2 |\,\langle u, \psi_n\rangle_{\mathsf{H}}\,|^2 < +\infty\}.$$

Indeed, we extend $\mathcal{A}$ on the space of the RHS by for all $v \in \{u \in \mathsf{H}, \sum_{n\geq 1} \lambda_n^2 |\,\langle u, \psi_n\rangle_{\mathsf{H}}\,|^2 < +\infty\}$,

$$\mathcal{A}[v] \overset{(\mathrm{def})}{=} \sum_{n\geq 1} \lambda_n\,\langle v, \psi_n\rangle_{\mathsf{H}}\,\psi_n.$$

It is then clear that $\mathcal{A}[v] \in \mathsf{H}$. The converse is straightforward. Moreover, it is standard that $\mathcal{D}(\mathcal{A})$ is therefore the completion of span $(\psi_n, n \geq 1)$ under the graph norm $\|u\|_{\mathcal{A}}^2 \overset{(\mathrm{def})}{=} \|u\|_{\mathsf{H}}^2 + \|\mathcal{A}[u]\|_{\mathsf{H}}^2$ which makes $\big(\mathcal{A}, \mathcal{D}(\mathcal{A})\big)$ a closed operator. Moreover, $\big(\mathcal{A}, \mathcal{D}(\mathcal{A})\big)$ is self-adjoint since for $v \in \mathcal{D}(\mathcal{A}^*)$, there exists $f_v \in \mathsf{H}$, such that for all $u \in \mathcal{C}_c^\infty(\mathbb{R}) \cap \mathsf{H}$,

$$\langle (\mathcal{A}[u])', \rho_V v'\rangle_{L^2(\mathbb{R})} = \langle u', \rho_V f_v'\rangle_{L^2(\mathbb{R})}\,.$$

This implies that, in the sense of distributions, $(\mathcal{A}[v])' = f_V' \in L^2(\mu_V)$ and therefore $\mathcal{A}[v] = f_V + C$ for a constant C. Finally, since we necessarily have, $\int_{\mathbb{R}} \mathcal{A}[v](x)d\mu_V(x) = 0$, $\mathcal{A}[v] \in \mathsf{H}$ *i.e.* $v \in \mathcal{D}(\mathcal{A})$.

Now let $u, v \in \mathcal{D}(\mathcal{W})$, using Plancherel's isometry and *i)* of Lemma 1.2.1, we see that

$$\langle \mathcal{W}[u], v\rangle_{\mathsf{H}} = \big\langle (\mathcal{W}[u])', v'\rho_V\big\rangle_{L^2(dx)} = \frac{1}{2}\,\big\langle\, |\, x\,|\,\mathcal{F}[u'\rho_V], \mathcal{F}[v'\rho_V]\big\rangle_{L^2(dx)}\,,$$

and this last expression is symmetric in (u,v), so $\mathcal{W}$ is symmetric. To see that $\mathcal{W}$ is self-adjoint, one must show that $\mathcal{D}(\mathcal{W}^*) \subset \mathcal{D}(\mathcal{W})$. Let $v \in \mathcal{D}(\mathcal{W}^*)$, then the map $u \in \mathcal{D}(\mathcal{W}) \mapsto \langle \mathcal{W}[u], v\rangle_{\mathsf{H}}$ can be extended to a continuous linear functional on H, so there exists $f_v \in \mathsf{H}$ such that for all $u \in \mathcal{C}_c^\infty(\mathbb{R}) \cap \mathsf{H}$,

$$\langle -\mathcal{H}[u'\rho_V]', v'\rho_V\rangle_{L^2(\mathbb{R})} = \langle u'\rho_V, f_v'\rangle_{L^2(\mathbb{R})}\,.$$

Therefore, in the sense of distributions $-\mathcal{H}[v'\rho_V]' = f_v' \in L^2(\mu_V)$. Since for all $f \in \mathsf{H}$, $\mathcal{W}[f] \in L_0^2(\mu_V)$, we just showed that $\mathcal{W}[v] \in \mathsf{H}$ ie $v \in \mathcal{D}(\mathcal{W})$. This concludes the proof. □

Definition 1.6.5 (Quadratic form associated to $-\mathcal{L}$) *We define for all $u, v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$ the quadratic form associated to $-\mathcal{L}$ by*

$$q_{-\mathcal{L}}(u,v) = \langle \mathcal{A}[u], \mathcal{A}[v]\rangle_{L^2(\mu_V)} + P\,\langle \mathcal{F}[u'\rho_V], \mathcal{F}[v'\rho_V]\rangle_{L^2(|x|dx)}\,.$$

We also define the form domain of $\mathcal{L}$ by

$$Q(\mathcal{L}) = \Big\{ u \in \mathsf{H}, q_{-\mathcal{L}}(u,u) < +\infty \Big\}\,.$$

Note that for all $u, v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$, $q_{-\mathcal{L}}(u,v) = \langle -\mathcal{L}[u], v\rangle_{\mathsf{H}}$ and that whenever $u \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{W})$,

$$q_{-\mathcal{L}}(u,u) = \langle \mathcal{A}[u], u\rangle_{\mathsf{H}} + 2P\,\langle \mathcal{W}[u], u\rangle_{\mathsf{H}} \geq \lambda_1(\mathcal{A})\|u\|_{\mathsf{H}}^2 \tag{1.55}$$

by Proposition 1.6.3 and Lemma 1.6.4. We can now extend $q_{-\mathcal{L}}$ to its form domain $Q(\mathcal{L})$ which is equal to $\Big\{ u \in \mathsf{H}, \mathcal{A}[u] \in L^2(\mu_V),\, \mathcal{F}[u'\rho_V] \in L^2(|x|dx) \Big\} = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$. The equality comes from the fact that $\mathcal{A}^{-1}\Big(L_0^2(\mu_V)\Big) = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, that $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \subset \mathsf{H}$ and that $\mathcal{F}[u'\rho_V] \in L^2(|x|dx)$ whenever $u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, indeed $u'\rho_V \in H^1(\mathbb{R})$ because $(u'\rho_V)' = -\rho_V \mathcal{A}[u] \in L^2(\mathbb{R})$. We now define $\mathcal{D}(\mathcal{L})$ the domain of definition of $-\mathcal{L}$ by:

$$\mathcal{D}(\mathcal{L}) \overset{(\text{def})}{=} \Big\{ u \in Q(\mathcal{L}), v \mapsto q_{-\mathcal{L}}(u,v) \text{ can be extended to a continuous linear functional on } \mathsf{H} \Big\}.$$

Proposition 1.6.6 $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A})$.

Proof Let $u \in \mathcal{D}(\mathcal{L})$, by Riesz's theorem there exists $f_u \in \mathsf{H}$, such that $q_{-\mathcal{L}}(u,v) = \langle f_u, v\rangle_{\mathsf{H}}$ for all $v \in Q(\mathcal{L})$, we set $-\mathcal{L}u \overset{(\text{def})}{=} f_u$, it is called the Friedrichs extension of $-\mathcal{L}$. Then for all $v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$, by integration by part and anti selft-adjointness of $\mathcal{H}$, we get:

$$\langle -\mathcal{L}[u], v\rangle_{\mathsf{H}} = q_{-\mathcal{L}}(u,v) = \langle u, \mathcal{A}[v]\rangle_{\mathsf{H}} + 2P\,\langle u, \mathcal{W}[v]\rangle_{\mathsf{H}}\,,$$

hence we deduce the distributional identity $-\mathcal{L}[u] = \mathcal{A}[u] + 2P\mathcal{W}[u]$. Since $u \in \mathsf{H}$, $u'\rho_V \in L^2(\mathbb{R})$, hence $\mathcal{H}[u'\rho_V] \in L^2(\mathbb{R})$ so $\mathcal{W}[u] \in L^2(\mathbb{R})$. Furthermore $u \in \mathcal{D}(\mathcal{L}) \subset Q(\mathcal{L}) = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, so $(\mathcal{W}[u])' = \mathcal{H}\big[\rho_V \mathcal{A}[u]\big] \in L^2(\mathbb{R})$ which implies that $\mathcal{W}[u] \in \mathsf{H}$. Finally, since $\mathcal{L}[u] \in \mathsf{H}$, $\mathcal{A}[u] \in \mathsf{H}$ so $u \in \mathcal{D}(\mathcal{A})$.

Similarly, if $u \in \mathcal{D}(\mathcal{A})$, for the same reasons as before $\mathcal{W}[u] \in \mathsf{H}$ thus $\mathcal{L}[u] = -\mathcal{A}[u] - 2P\mathcal{W}[u] \in \mathsf{H}$ *ie* $u \in \mathcal{D}(\mathcal{L})$. □

We are now ready to state the main theorem of this section, that is the inversion of $\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$.

Theorem 1.6.7 (Inversion of $\mathcal{L}$) *$-\mathcal{L} : \mathcal{D}(\mathcal{L}) \longrightarrow \mathsf{H}$ is bijective. Furthermore, $(-\mathcal{L})^{-1}$ is positive, continuous from $(\mathsf{H}, \|.\|_{\mathsf{H}})$ to $(\mathcal{D}(\mathcal{L}), \|.\|_{\mathsf{H}})$. More precisely, for all $f \in \mathsf{H}$,*

$$\|\mathcal{L}^{-1}[f]\|_{\mathsf{H}} \leq \lambda_1(\mathcal{A})^{-1/2}\|f\|_{\mathsf{H}}.$$

Proof Let $f \in \mathsf{H}$, since $\langle f, .\rangle_{\mathsf{H}}$ is a linear form on $Q(\mathcal{L}) = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ which is, by (1.55), continuous with respect to $q_{-\mathcal{L}}$, one can applies Riesz's theorem so there exists a unique $u_f \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, such that for all $v \in Q(\mathcal{L})$, $\langle f, v\rangle_{\mathsf{H}} = q_{-\mathcal{L}}(u_f, v)$. Since, u_f is clearly in $\mathcal{D}(\mathcal{L})$ by definition of the Friedrichs extension of $-\mathcal{L}$, we have $-\mathcal{L}[u_f] = f$. Finally, the continuity of $\mathcal{L}^{-1}$ with respect to $\|.\|_{\mathsf{H}}$ follows from (1.55). □

Remark 1.6.8 We can diagonalize $\mathcal{L}$ by the same argument we used in Appendix 1.8 to diagonalize $\mathcal{A}$ in $L^2_0(\mu_V)$.

We now prove, following Remark 1.5.3, a more compact formula for the variance such as the one appearing in [HL21]. With $\mathcal{T}$ defined in (1.51), we have

Lemma 1.6.9 *The following equality holds for all $\phi \in \mathcal{T}$*

$$\langle \mathcal{L}^{-1}[\phi], \phi \rangle_{\mathsf{H}} = \left(\sigma_P^V\right)^2(\phi) \overset{(\text{def})}{=} \int_{\mathbb{R}} \left(\mathcal{L}^{-1}[\phi]''(x)^2 + V''(x)\mathcal{L}^{-1}[\phi]'(x)^2 \right) d\mu_V(x) + P \iint_{\mathbb{R}^2} \left(\frac{\mathcal{L}^{-1}[\phi]'(x) - \mathcal{L}^{-1}[\phi]'(y)}{x-y} \right)^2 d\mu_V(x) d\mu_V(y) \quad (1.56)$$

Proof It suffices to show that $\left(\sigma_P^V\right)^2(\mathcal{L}[\phi]) = \langle \mathcal{L}[\phi], \phi \rangle_{\mathsf{H}}$ for all $\phi \in \mathcal{D}(\mathcal{L})$.

$$\langle \mathcal{L}[\phi], \phi \rangle_{\mathsf{H}} = -\int_{\mathbb{R}} \Big(\frac{(\phi' \rho_V)'}{\rho_V} \Big)'(x) \phi'(x) d\mu_V(x) - 2P \int_{\mathbb{R}} \mathcal{H}[\phi' \rho_V]'(x) \phi' d\mu_V(x).$$

Integration by parts in the first integral leads to:

$$-\int_{\mathbb{R}} \Big(\frac{(\phi' \rho_V)'}{\rho_V} \Big)'(x) \phi'(x) d\mu_V(x) = \int_{\mathbb{R}} \Big(\frac{(\phi' \rho_V)'}{\rho_V} \Big)(x)^2 d\mu_V(x) = \int_{\mathbb{R}} \left\{ \phi''^2 - \phi'(x)^2 \frac{\rho_V''}{\rho_V}(x) + \phi'(x)^2 \left(\frac{\rho_V'}{\rho_V} \right)^2 (x) \right\} d\mu_V(x).$$

Since

$$\frac{\rho_V''}{\rho_V} = \Big(-V'' - 2P\mathcal{H}[\rho_V]' + V'^2 + 4P^2 \mathcal{H}[\rho_V]^2 + 4PV'\mathcal{H}[\rho_V] \Big) = -V'' - 2P\mathcal{H}[\rho_V]' + \Big(\frac{\rho_V'}{\rho_V} \Big)^2,$$

we obtain

$$\langle \mathcal{L}[\phi], \phi \rangle_{\mathsf{H}} = \int_{\mathbb{R}} \big(\phi''(x)^2 + V''(x)\phi'(x)^2 \big) d\mu_V(x) - 2P \int_{\mathbb{R}} \mathcal{H}[\phi' \rho_V]'(x) \phi'(x) d\mu_V(x) + 2P \int_{\mathbb{R}} \mathcal{H}[\rho_V]'(x) \phi'(x)^2 d\mu_V(x).$$

We conclude using the fact that for all $\phi \in \mathcal{D}(\mathcal{L})$, we have:

$$\iint_{\mathbb{R}^2} \Big(\frac{\phi'(x) - \phi'(y)}{x-y} \Big)^2 d\mu_V(x) d\mu_V(y) = 2 \int_{\mathbb{R}} \big(\mathcal{H}[\rho_V]'(x) \phi'(x)^2 - \mathcal{H}[\phi' \rho_V]'(x) \phi'(x) \big) d\mu_V(x).$$

We now state a result that could allow one to obtain an explicit formulation for $\mathcal{L}^{-1}$ by the help of Fredholm determinant theory for Hilbert-Schmidt operators. The reader can refer to [GGK12] for more details.

Definition 1.6.10 (Fredholm determinant) *Let $\mathcal{K}$ be a kernel Hilbert-Schmidt operator on $L^2(\mu_V)$, we define the Fredholm 2-determinant of $\mathsf{id} + \mathcal{K}$ by*

$$\det_2 (\mathsf{id} + \mathcal{K}) \overset{(\text{def})}{=} 1 + \sum_{n \geq 1} \frac{1}{n!} \int_{\mathbb{R}^n} \begin{vmatrix} 0 & k(t_1,t_2) & \dots & k(t_1,t_n) \\ k(t_2,t_1) & 0 & \dots & k(t_2,t_n) \\ \vdots & & & \vdots \\ k(t_n,t_1) & k(t_n,t_2) & \dots & 0 \end{vmatrix} \prod_{i=1}^n d\mu_V(t_i).$$

where $|M|$ is the usual determinant for a matrix M.

Theorem 1.6.11 (Explicit expression for $\mathcal{L}^{-1}$) *For all $u \in \mathsf{H}$, such that*

$$x \mapsto \frac{1}{\rho_V(x)} \int_x^{+\infty} u(t)\rho_V(t)dt$$

is integrable at $+\infty$, we have:

$$-\mathcal{L}^{-1}[u] = \mathcal{A}^{-1}[u] - \mathcal{A}^{-1} \circ \mathcal{R}\big[u\big].$$

In this expression $\mathcal{A}^{-1}[u]$ is given by

$$\mathcal{A}^{-1}[u](x) = -\int_x^{+\infty} \frac{ds}{\rho_V(s)} \int_s^{+\infty} u(t)\rho_V(t)dt + C,$$

where $C \overset{(\text{def})}{=} \displaystyle\int_{\mathbb{R}} \rho_V(y)dy \int_y^{+\infty} \frac{ds}{\rho_V(s)} \int_s^{+\infty} u(t)\rho_V(t)dt$. The operator $\mathcal{R}$ is the kernel operator defined for all $v \in L^2(\mu_V)$ by:

$$\mathcal{R}[v](x) \overset{(\text{def})}{=} \int_{\mathbb{R}} r(x,y)v(y)d\mu_V(y),$$

where

$$r(x,y) \overset{(\text{def})}{=} \frac{1}{\det_2(\mathsf{id} - \mathcal{K})} \sum_{n\geq 1} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \begin{vmatrix} k(x,y) & k(x,t_1) & \dots & k(x,t_n) \\ k(t_1,y) & 0 & \dots & k(t_1,t_n) \\ \vdots & & & \vdots \\ k(t_n,y) & k(t_n,t_1) & \dots & 0 \end{vmatrix} \prod_{i=1}^n d\mu_V(t_i), \tag{1.57}$$

where $\mathcal{K}$ is the kernel operator defined for all $w \in L^2(\mu_V)$ by:

$$\mathcal{K}[w](x) \overset{(\text{def})}{=} \int_{\mathbb{R}} k(x,y)w(y)dy, \tag{1.58}$$

with

$$k(x,y) \overset{(\text{def})}{=} 2P\log|x-y| - 2P\int_{\mathbb{R}} \log|z-y|d\mu_V(z). \tag{1.59}$$

Finally, $2P\mathcal{W} \circ \mathcal{A}^{-1} = -\mathcal{K}$.

Proof Let $f \in \mathsf{H}$, there exists a unique $u \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}[u] = f$. Since $(u'\rho_V)' = -\rho_V\mathcal{A}[u] \in L^2(\mathbb{R})$, we get $u'\rho_V \in H^1(\mathbb{R})$, so $u'(x)\rho_V(x) \underset{|x|\to+\infty}{\longrightarrow} 0$. By definition, $-\dfrac{(u'\rho_V)'}{\rho_V} = f$ hence

$$\mathcal{A}^{-1}[f]'(x)\rho_V(x) = u'(x)\rho_V(x) = \int_x^{+\infty} f(t)\rho_V(t)dt. \tag{1.60}$$

Using the fact that $\displaystyle\int_{\mathbb{R}} u(x)\rho_V(x)dx = 0$, integrating again we get:

$$u(x) = -\int_x^{+\infty} \frac{ds}{\rho_V(s)} \int_s^{+\infty} f(t)\rho_V(t)dt + C$$

where $C = \displaystyle\int_{\mathbb{R}} \rho_V(x)dx \int_x^{+\infty} \frac{ds}{\rho_V(s)} \int_s^{+\infty} f(t)\rho_V(t)dt$. Now using that that for all $g \in \mathsf{H}$

$$-\mathcal{L}^{-1} = \Big(\mathcal{A} + 2P\mathcal{W}\Big)^{-1} = \mathcal{A}^{-1} \circ \Big(\mathsf{id} + 2P\mathcal{W} \circ \mathcal{A}^{-1}\Big)^{-1}$$

We only need to compute the expression of $\left(\mathsf{id}+2P\mathcal{W}\circ\mathcal{A}^{-1}\right)^{-1}$. Using (1.60), let $v\in\mathcal{A}\big[\mathcal{D}(\mathcal{L})\big]=\mathsf{H}$. We obtain:

$$\mathcal{W}\circ\mathcal{A}^{-1}[v](x)=-\fint_{\mathbb{R}}\frac{ds}{s-x}\int_s^{+\infty}dtv(t)\rho_V(t)+\int_{\mathbb{R}}d\mu_V(y)\fint_{\mathbb{R}}\frac{ds}{s-y}\int_s^{+\infty}dtv(t)\rho_V(t).$$

By Sokhotski-Plejmel formula, we have:

$$\fint_{\mathbb{R}}\frac{ds}{s-x}\int_s^{+\infty}dtv(t)\rho_V(t)=\lim_{\varepsilon\to0^+}\lim_{M\to+\infty}\int_{-M}^{M}\frac{ds}{2}\Big\{\frac{1}{s-x+\mathrm{i}\varepsilon}+\frac{1}{s-x-\mathrm{i}\varepsilon}\Big\}\int_s^{+\infty}dtv(t)\rho_V(t).$$

We then proceed to an integration by part:

$$\fint_{\mathbb{R}}\frac{ds}{s-x}\int_s^{+\infty}dtv(t)\rho_V(t)=\lim_{\varepsilon\to0^+}\lim_{M\to+\infty}\Big[\frac{\log\left((s-x)^2+\varepsilon^2\right)}{2}\int_s^{+\infty}dtv(t)\rho_V(t)\Big]_{-M}^{M}+\int_{\mathbb{R}}ds\log|x-s|v(s)\rho_V(s).$$

To conclude that:

$$\mathcal{W}\circ\mathcal{A}^{-1}[v](x)=-\int_{\mathbb{R}}ds\log|x-s|v(s)\rho_V(s)+\int_{\mathbb{R}}d\mu_V(y)\int_{\mathbb{R}}ds\log|y-s|v(s)\rho_V(s),$$

we just need to show that

$$\log|x|\int_x^{+\infty}dtv(t)\rho_V(t)\underset{|x|\to\infty}{\longrightarrow}0.$$

The latter can be seen by Cauchy-Schwarz inequality:

$$\Big|\log(x)\int_x^{+\infty}dtv(t)\rho_V(t)\Big|\leq|\log(x)|\|v\|_{L^2(\mu_V)}.\rho_V(x)^{1/4}\Big(\int_{\mathbb{R}}\rho_V(t)^{1/2}dt\Big)^{1/2}.$$

In this inequality, we used that ρ_V is decreasing exponentially fast in a neighborhood of $+\infty$, hence

$$\log(x)\int_x^{+\infty}dtv(t)\rho_V(t)\underset{x\to+\infty}{\longrightarrow}0.$$

Using that $\displaystyle\int_{\mathbb{R}}v(t)\rho_V(t)dt=0$, we have $\displaystyle\int_x^{+\infty}v(t)\rho_V(t)dt=-\int_{-\infty}^{x}v(t)\rho_V(t)dt$, therefore the exact same argument allows us to conclude when x goes to $-\infty$. We obtain the following equality:

$$(\mathsf{id}+2P\mathcal{W}\circ\mathcal{A}^{-1})[v](x)=v(x)-2P\int_{\mathbb{R}}ds\log|x-s|v(s)\rho_V(s)+2P\int_{\mathbb{R}}d\mu_V(y)\int_{\mathbb{R}}ds\log|y-s|v(s)\rho_V(s).$$

With the operator $\mathcal{K}$ given in (1.58), by [GGK12, Section XII.2], we can conclude that the inverse of the kernel operator $\mathsf{id}-\mathcal{K}$ is $\mathsf{id}-\mathcal{R}$ where $\mathcal{R}$ is given in (1.57). This concludes the proof. □

1.7 Regularity of the inverse of $\mathcal{L}$ and completion of the proof of Theorem 1.1.4

Since we have proven the central limit theorem for functions of the type $\mathcal{L}\phi$ with ϕ regular enough and satisfying vanishing asymptotic conditions at infinity, we exhibit a class of functions f for which $\mathcal{L}^{-1}f$ is regular enough to satisfy conditions of Theorem 1.5.2. Define the subset $\mathcal{T}$ of H by

$$\mathcal{T}\overset{(\mathrm{def})}{=}\left\{f\in\mathcal{C}^2(\mathbb{R})\,|\,f,f',f''\text{ are bounded},\int_{\mathbb{R}}fd\mu_V=0\right\}.$$

Theorem 1.7.1 *For $f \in \mathcal{T}$, there exists a unique $u \in \mathcal{C}^3(\mathbb{R})$ such that $u' \in H^2(\mathbb{R})$ with $u^{(3)}$ bounded which verifies:*

- $u'(x) = \underset{|x|\to\infty}{O}\big(V'(x)^{-1}\big)$
- $u''(x) = \underset{|x|\to\infty}{O}\big(V'(x)^{-1}\big)$
- $u^{(3)}(x) = \underset{|x|\to\infty}{O}\big(V'(x)^{-1}\big)$

and such that $f = \mathcal{L}[u]$.

Proof Let $f \in \mathcal{T} \subset \mathsf{H}$, then since $-\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathsf{H}$ is bijective, there exists a unique $u \in \mathcal{D}(\mathcal{L})$ such that $-\mathcal{L}[u] = f$ *i.e.*:

$$-u'' - \frac{\rho_V'}{\rho_V}u' - 2P\mathcal{H}[u'\rho_V] + 2P\int_{\mathbb{R}} \mathcal{H}[u'\rho_V](t)d\mu_V(t) = f. \tag{1.61}$$

Hence we have

$$-(u'\rho_V)' = \rho_V\Big(f + 2P\mathcal{H}[u'\rho_V] - 2P\int_{\mathbb{R}} \mathcal{H}[u'\rho_V](t)d\mu_V(t)\Big). \tag{1.62}$$

Since $u \in \mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A}) = \{u \in \mathsf{H} \mid \mathcal{A}u \in \mathsf{H}\}$, the functions $u'\rho_V$ and its distributional derivatives $(u'\rho_V)' = -\rho_V\mathcal{A}u$ and $(u'\rho_V)'' = -\frac{\rho_V'}{\rho_V}\rho_V^{1/2}.\Big(\rho_V^{1/2}\mathcal{A}u\Big) - \rho_V\Big(\mathcal{A}u\Big)'$ are in $L^2(\mathbb{R})$. In particular $u'\rho_V$ goes to zero at infinity, and $\mathcal{H}[u'\rho_V] \in H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$. So we can integrate (1.62) on $[x, +\infty[$, since by Lemma 1.2.3, the right-hand side is a $\underset{|x|\to\infty}{O}(\rho_V(x))$, to get the following expression for x large enough:

$$u'(x)\rho_V(x) = \int_x^{+\infty} \frac{\rho_V(t)}{\rho_V'(t)}\Big(f + 2P\mathcal{H}[u'\rho_V] - 2P\int_{\mathbb{R}} \mathcal{H}[u'\rho_V](s)d\mu_V(s)\Big).\rho_V'(t)dt. \tag{1.63}$$

From this expression, we see that $u' \in \mathcal{C}^2(\mathbb{R})$. We now check the integrability condition at infinity and the claimed boundedness properties of $u', u'', u^{(3)}$. After proceeding to an integration by parts, which is permitted by the previous arguments, we obtain:

$$u'(x) = -\frac{\rho_V(x)}{\rho_V'(x)}\Big(f(x) + 2P\mathcal{H}[u'\rho_V](x) - 2P\int_{\mathbb{R}} \mathcal{H}[u'\rho_V](t)d\mu_V(t)\Big) - R_1(x), \tag{1.64}$$

where we set $R_1(x) \overset{(\text{def})}{=} \frac{1}{\rho_V(x)}\int_x^{+\infty}\left[\frac{\rho_V(t)}{\rho_V'(t)}\Big(f + 2P\mathcal{H}[u'\rho_V] - 2P\int_{\mathbb{R}} \mathcal{H}[u'\rho_V](s)d\mu_V(s)\Big)\right]'\rho_V(t)dt.$
We will need to show that R_1 is a remainder of order $\underset{x\to+\infty}{O}\big(V'(x)^{-2}\big)$ at infinity. In this case, we will have $u'(x) = \underset{x\to+\infty}{O}\big(V'(x)^{-1}\big)$ which will be useful for the following. If we reinject (1.64) in (1.61), we find:

$$\begin{aligned} u'' = &-\Big(f + 2P\mathcal{H}[u'\rho_V] - 2P\int_{\mathbb{R}} \mathcal{H}[u'\rho_V](t)d\mu_V(t)\Big) \\ &- \frac{\rho_V'}{\rho_V}\left[-\frac{\rho_V}{\rho_V'}\Big(f + 2P\mathcal{H}[u'\rho_V] - 2P\int_{\mathbb{R}} \mathcal{H}[u'\rho_V](t)d\mu_V(t)\Big) - R_1\right] = \frac{\rho_V'}{\rho_V}R_1. \end{aligned} \tag{1.65}$$

Hence

$$u''(x) = \frac{\rho_V'}{\rho_V^2}(x)\int_x^{+\infty}\rho_V(t)dt\Bigg\{\underbrace{\Big(\frac{\rho_V}{\rho_V'}\Big)'(t)}_{=\underset{t\to+\infty}{O}\left(\frac{V''(t)}{V'(t)^2}\right)}\underbrace{\Big[f+2P\mathcal{H}[u'\rho_V]-2P\int_{\mathbb{R}}\mathcal{H}[u'\rho_V](s)d\mu_V(s)\Big](t)}_{=\underset{t\to+\infty}{O}(1)}$$
$$+\underbrace{\frac{\rho_V}{\rho_V'}(t)}_{=\underset{t\to+\infty}{O}\left(V'(t)^{-1}\right)}\underbrace{\Big(f'-2P\mathcal{H}[\rho_V\mathcal{A}u]\Big)(t)}_{=\underset{t\to+\infty}{O}(1)}\Bigg\}.$$

We recall that because of Assumptions 1.1.1 *iv)*, $\underset{t\to+\infty}{O}\Big(\frac{V''(t)}{V'(t)^2}\Big) = \underset{t\to+\infty}{O}\big(V'(t)^{-1}\big)$. The fact that $\mathcal{H}[\rho_V\mathcal{A}u]$ is bounded comes again from lemma 1.2.3. Differentiating yields

$$u^{(3)}(x) = \Big(\frac{\rho_V'}{\rho_V}\Big)'(x)R_1(x) - \Big(\frac{\rho_V'}{\rho_V}(x)\Big)^2R_1(x) - \frac{\rho_V'}{\rho_V}(x)g(x),$$

where we have set $g \overset{(\text{def})}{=} \left[\frac{\rho_V}{\rho_V'}\Big(f+2P\mathcal{H}[u'\rho_V]-2P\int_{\mathbb{R}}\mathcal{H}[u'\rho_V](t)d\mu_V(t)\Big)\right]'$. Now, by the same integration by parts argument as in (1.65), we obtain:

$$-\Big(\frac{\rho_V'(x)}{\rho_V(x)}\Big)^2R_1(x) - \frac{\rho_V'(x)}{\rho_V(x)}g(x)$$
$$= -\frac{\rho_V'(x)^2}{\rho_V(x)^3}\Big[g(t)\frac{\rho_V(t)}{\rho_V'(t)}\rho_V(t)\Big]_x^{+\infty} + \frac{\rho_V'(x)^2}{\rho_V(x)^3}\int_{\mathbb{R}}\Big(g\frac{\rho_V}{\rho_V'}\Big)'(t)\rho_V(t)dt - \frac{\rho_V'(x)}{\rho_V(x)}g(x)$$
$$= \Big(\frac{\rho_V'(x)}{\rho_V(x)}\Big)^2R_2(x),$$

where

$$R_2(x) \overset{(\text{def})}{=} \frac{1}{\rho_V(x)}\int_x^{+\infty}\left\{\left[\Big(f+2P\mathcal{H}[u'\rho_V]-2P\int_{\mathbb{R}}\mathcal{H}[u'\rho_V](t)d\mu_V(t)\Big)\frac{\rho_V}{\rho_V'}\right]'\frac{\rho_V}{\rho_V'}\right\}'(t)\rho_V(t)dt. \quad (1.66)$$

The above integration by part is justified by the fact that g goes indeed to zero at $+\infty$. Hence

$$u^{(3)}(x) = \Big(\frac{\rho_V'}{\rho_V}\Big)'(x)R_1(x) + \Big(\frac{\rho_V'}{\rho_V}(x)\Big)^2R_2(x).$$

Since $\left(\frac{\rho_V'}{\rho_V}\right)'(x) = \underset{x\to+\infty}{O}\Big(V'(x)\Big)$, using Lemma 1.2.2 and Assumptions 1.1.1 *iv)*, and $\frac{\rho_V'}{\rho_V}(x) = \underset{x\to+\infty}{O}\Big(V'(x)\Big)$, it just remains to check that

$$R_1(x) = \underset{x\to+\infty}{O}\Big(V'(x)^{-2}\Big) \qquad \text{and that} \qquad R_2(x) = \underset{x\to+\infty}{O}\Big(V'(x)^{-3}\Big).$$

Since the integrand in R_1 is a $\underset{t\to+\infty}{O}\Big(V'(t)^{-1}\Big)$, by a comparison argument, we get:

$$R_1(x) = \underset{x\to+\infty}{O}\big(I_1(x)\big) \quad \text{where} \quad I_1(x) \overset{(\text{def})}{=} \frac{1}{\rho_V(x)}\int_x^{+\infty}\frac{\rho_V(t)}{V'(t)}dt.$$

Furthermore, by integration by parts:

$$I_1(x) \overset{(\text{def})}{=} -\frac{\rho_V(x)}{V'(x)\rho_V'(x)} - \frac{1}{\rho_V(x)}\int_x^{+\infty} \rho_V(t)\Big(\frac{\rho_V}{V'\rho_V'}\Big)'(t)dt$$
$$= \underset{x\to+\infty}{O}\Big(V'(x)^{-2}\Big) - \frac{1}{\rho_V(x)}\int_x^{+\infty}\rho_V(t)dt\Big\{ -\frac{V''(t)}{V'(t)^2}\frac{\rho_V}{\rho_V'} + \frac{1}{V'(t)}\Big(\frac{\rho_V}{\rho_V'}\Big)'(t)\Big\}. \quad (1.67)$$

By the same argument as before, the last integral is of the form $\frac{1}{\rho_V(x)}\int_x^{+\infty} \underset{t\to+\infty}{O}\Big(\frac{\rho_V(t)}{V'(t)^2}\Big)dt$ so we can conclude that:

$$\frac{1}{\rho_V(x)}\int_x^{+\infty}\rho_V(t)dt\Big\{ -\frac{V''(t)}{V'(t)^2}\frac{\rho_V}{\rho_V'} + \frac{1}{V'(t)}\Big(\frac{\rho_V}{\rho_V'}\Big)'(t)\Big\} = \underset{x\to+\infty}{O}\Big(\frac{1}{\rho_V(x)V'(x)^2}\int_x^{+\infty}\rho_V(t)dt\Big),$$

where we used that $t\mapsto V'(t)^{-2}$ is decreasing on $[x,+\infty[$ for x big enough. In order to conclude that $R_1(x) = \underset{x\to+\infty}{O}\Big(V'(x)^{-2}\Big)$, one just needs to check that $\frac{1}{\rho_V(x)}\int_x^{+\infty}\rho_V(t)dt = \underset{x\to+\infty}{o}(1)$. By two integration by parts, we obtain:

$$\frac{1}{\rho_V(x)}\int_x^{+\infty}\rho_V(t)dt = -\frac{\rho_V(x)}{\rho_V'(x)} + \Big(\frac{\rho_V}{\rho_V'}\Big)'(x)\frac{\rho_V(x)}{\rho_V'(x)} + \frac{1}{\rho_V(x)}\int_x^{+\infty}\rho_V(t)\Big[\Big(\frac{\rho_V}{\rho_V'}\Big)'\frac{\rho_V}{\rho_V'}\Big]'(t)dt$$
$$= \underset{x\to+\infty}{o}(1) + \frac{1}{\rho_V(x)}\int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_V(t)}{V'(t)^2}\Big)dt$$
$$= \underset{x\to+\infty}{o}(1) + \underset{x\to+\infty}{O}\Big(\frac{1}{\rho_V(x)}\int_x^{+\infty}\frac{\rho_V(t)}{V'(t)^2}dt\Big) = \underset{x\to+\infty}{o}(1)$$

since $t\mapsto\rho_V(t)$ is decreasing on $[x,+\infty[$ for x big enough and that $t\mapsto V'(t)^{-2}$ is integrable at infinity by assumption *iv)*. This allows to conclude indeed that $R_1(x) = \underset{x\to+\infty}{O}\Big(V'(x)^{-2}\Big)$.

Now, we have to check that $R_2(x) = \underset{x\to+\infty}{O}\Big(V'(x)^{-3}\Big)$. By differentiating (1.66), by the same arguments as before,

$$R_2(x) \overset{(\text{def})}{=} \underset{x\to+\infty}{O}\big(I_2(x)\big) \quad \text{where} \quad I_2(x) \overset{(\text{def})}{=} \frac{1}{\rho_V(x)}\int_x^{+\infty}\frac{\rho_V(t)}{V'(t)^2}dt.$$

Moreover, by integration by part, we get just as before:

$$I_2(x) \overset{(\text{def})}{=} -\frac{\rho_V(x)}{V'(x)^2\rho_V'(x)} - \frac{1}{\rho_V(x)}\int_x^{+\infty}\rho_V(t)\Big(\frac{\rho_V}{V'^2\rho_V'}\Big)'(t)dt$$
$$= \underset{x\to+\infty}{O}\Big(V'(x)^{-3}\Big) - \frac{1}{\rho_V(x)}\int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_V(t)}{V'(t)^3}\Big)dt\Big\}$$
$$= \underset{x\to+\infty}{O}\Big(V'(x)^{-3}\Big) - \underset{x\to+\infty}{O}\Big(\frac{1}{\rho_V(x)V'(x)^3}\int_x^{+\infty}\rho_V(t)dt\Big) = \underset{x\to+\infty}{O}\Big(V'(x)^{-3}\Big). \quad (1.68)$$

Finally, we can conclude that, by doing the same thing near $-\infty$,

$$u'(x) = \underset{|x|\to+\infty}{O}\Big(V'(x)^{-1}\Big), \qquad u''(x) = \underset{|x|\to+\infty}{O}\Big(V'(x)^{-1}\Big) \quad \text{and} \quad u^{(3)}(x) = \underset{|x|\to+\infty}{O}\Big(V'(x)^{-1}\Big), \quad (1.69)$$

which establishes that these functions are in L^2 in a neighborhood of ∞, again by assumption *iv)*. Since we already showed that $u\in\mathcal{C}^3(\mathbb{R})\subset H^3_{\text{loc}}(\mathbb{R})$, it establishes that $u\in H^3(\mathbb{R})\cap\mathcal{C}^3(\mathbb{R})$ with $u^{(3)}$ bounded. To complete the proof we just have to show that $(u')^2V^{(3)}$, $u'u''V''$, $(u')^2V''$ and $u'V'$ are bounded which is easily checked by (1.69) and assumption 1.1.1 *iv)*. □

1.8 Appendix: proof of Theorem 1.6.2

In order to analyze $\mathcal{A}$, we let, for $u \in L^2(\mathbb{R})$,

$$\mathcal{S}[u] \overset{(\text{def})}{=} \rho_V^{1/2}\mathcal{A}[\rho_V^{-1/2}u]\,.$$

Note that $u \in \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big) \mapsto \rho_V^{-1/2}u \in \big(L^2(\mu_V), \|.\|_{L^2(\mu_V)}\big)$ is an isometry. It turns out that it will be easier to study first the operator $\mathcal{S}$ in order to get the spectral properties of $\mathcal{A}$.

Proposition 1.8.1 *The operator $\mathcal{S}$ is a Schrödinger operator: it admits the following expression for all $u \in C_c^2(\mathbb{R})$: $\mathcal{S}[u] = -u'' + w_V u$ with*

$$w_V \overset{(\text{def})}{=} \frac{1}{2}\left(\frac{1}{2}V'^2 - V'' + 2PV'\mathcal{H}[\rho_V] - 2P\mathcal{H}[\rho_V'] + 2P^2\mathcal{H}[\rho_V]^2\right) = \frac{1}{2}\Big[(\log\rho_V)'' + \frac{1}{2}(\log\rho_V)'^2\Big]\,. \tag{1.70}$$

Furthermore, w_V is continuous and we have $w_V(x) \underset{\infty}{\sim} \dfrac{V'(x)^2}{4} \underset{|x|\to\infty}{\longrightarrow} +\infty$.

Proof We compute directly

$$\begin{aligned}
\frac{\Big(\rho_V\big(\rho_V^{-1/2}u\big)'\Big)'}{\rho_V} &= \big(\rho_V^{-1/2}u\big)'' + \frac{\rho_V'}{\rho_V}\big(\rho_V^{-1/2}u\big)' \\
&= \big(\rho_V^{-1/2}u' - \frac{1}{2}\rho_V^{-3/2}\rho_V'u\big)' + \rho_V'\rho_V^{-3/2}u' - \frac{1}{2}\rho_V^{-5/2}\big(\rho_V'\big)^2u \\
&= \rho_V^{-1/2}u'' + \frac{1}{4}\rho_V^{-5/2}\big(\rho_V'\big)^2u - \frac{1}{2}\rho_V^{-3/2}\rho_V''u \\
&= \rho_V^{-1/2}\Big[u'' + \frac{1}{4}\rho_V^{-2}\big(\rho_V'\big)^2u - \frac{1}{2}\rho_V^{-1}\rho_V''u\Big] \\
&= \rho_V^{-1/2}\left(u'' - \frac{1}{2}\Big[\big(\frac{\rho_V''}{\rho_V}\big) - \frac{1}{2}\big(\frac{\rho_V'}{\rho_V}\big)^2\Big]u\right) \\
&= \rho_V^{-1/2}\left(u'' - \frac{1}{2}\Big[(\log\rho_V)'' + \frac{1}{2}(\log\rho_V)'^2\Big]u\right) = \rho_V^{-1/2}\Big(u'' - w_Vu\Big)\,.
\end{aligned}$$

Now, using Lemma 1.2.2, we have

$$w_V = \frac{1}{2}\left(\frac{1}{2}V'^2 - V'' + 2PV'\mathcal{H}[\rho_V] - 2P\mathcal{H}[\rho_V'] + 2P^2\mathcal{H}[\rho_V]^2\right)\,.$$

Notice that $\mathcal{H}[\rho_V']$ and $\mathcal{H}[\rho_V]$ are bounded since they belong to $H^1(\mathbb{R})$, as we showed in Lemma 1.2.2 that ρ_V is $H^2(\mathbb{R})$. Along with Assumption 1.1.1 *iii)* and Lemma 1.2.3, we deduce $w_V(x) \underset{\infty}{\sim} \frac{1}{4}V'^2(x)$. □

Remark 1.8.2 Note that the function w_V need not be positive on $\mathbb{R}$. In fact, neglecting the terms involving the Hilbert transforms of ρ_V and ρ_V', w_V would only be positive outside of a compact set. However, using the positivity of $\mathcal{A}$, which will be shown further in the article, we can show that the operator $-u'' + w_Vu$ is itself positive on $L^2(\mathbb{R})$. It can also be checked that, by integration by parts, $\mathcal{S}$ is self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R})$ with the inner product of $L^2(\mathbb{R})$.

We now introduce an extension of $\mathcal{S}$ by defining its associated bilinear form.

Definition 1.8.3 (Quadratic form associated to $\mathcal{S}$)
Let $\alpha \geq 0$ such that $w_V + \alpha \geq 1$. We define the quadratic form associated to $\mathcal{S} + \alpha\mathsf{id}$, defined for all $u \in \mathcal{C}_c^\infty(\mathbb{R})$ by

$$q_\alpha(u,u) \overset{(\text{def})}{=} \int_{\mathbb{R}} u'(x)^2dx + \int_{\mathbb{R}} u^2(x)(w_V(x)+\alpha)dx$$

This quadratic form can be extended to a larger domain denoted by $Q(\mathcal{S}+\alpha\mathsf{id})$, called the *form domain* of the operator $\mathcal{S}+\alpha\mathsf{id}$. By the theory of Schrödinger operators, it is well-known (see [Dav96, Theorem 8.2.1]) that such a domain is given by

$$Q(\mathcal{S}+\alpha\mathsf{id}) = \Big\{u \in H^1(\mathbb{R}), u(w_V+\alpha)^{1/2} \in L^2(\mathbb{R})\Big\} = \big\{u \in H^1(\mathbb{R}), uV' \in L^2(\mathbb{R})\big\} \overset{(\text{def})}{=} H^1_{V'}(\mathbb{R})\,.$$

The space $H^1_{V'}(\mathbb{R})$ can be seen to be the completion of $\mathcal{C}^\infty_c(\mathbb{R})$ under the norm q_α. Now that the quadratic form associated to $\mathcal{S}+\alpha\mathsf{id}$ has been extended to its form domain, it is possible to go back to the operator and extend it by its Friedrichs extension.

Theorem 1.8.4 (Friedrichs extension of $\mathcal{S}+\alpha\mathsf{id}$)
There exists an extension $(S+\alpha\mathsf{id})_F$ of the operator $\mathcal{S}+\alpha\mathsf{id}$, called the Friedrichs extension of $\mathcal{S}+\alpha\mathsf{id}$ defined on $\mathcal{D}\Big((S+\alpha\mathsf{id})_F\Big) = \Big\{u \in H^1_{V'}(\mathbb{R}), -u''+(w_V+\alpha)u \in L^2(\mathbb{R})\Big\}$.

Proof We denote

$$\mathcal{D}\Big((S+\alpha\mathsf{id})_F\Big) \overset{(\text{def})}{=} \Big\{v \in H^1_{V'}(\mathbb{R}), u \in H^1_{V'}(\mathbb{R}) \mapsto q_\alpha(u,v) \\ \text{can be extended to a continuous linear functional on } L^2(\mathbb{R})\Big\}$$

If $v \in \mathcal{D}\Big((S+\alpha\mathsf{id})_F\Big)$, by Riesz's theorem there exists a unique $f_v \in L^2(\mathbb{R})$ such that $q_\alpha(u,v) = \langle u, f_v\rangle_{L^2(dx)}$ holds for all $u \in L^2(\mathbb{R})$ and we can set $(S+\alpha\mathsf{id})_F[v] \overset{(\text{def})}{=} f_v$. Note that it is indeed a way of extending $\mathcal{S}+\alpha\mathsf{id}$ since for all $u,v \in \mathcal{C}^\infty_c(\mathbb{R})$, $q_\alpha(u,v) = \langle u, (\mathcal{S}+\alpha\mathsf{id})[v]\rangle_{L^2(dx)}$.

We want to show that $\mathcal{D}\Big((S+\alpha\mathsf{id})_F\Big) = \Big\{u \in H^1_{V'}(\mathbb{R}), -u''+(w_V+\alpha)u \in L^2(\mathbb{R})\Big\}$. Let $f \in \mathcal{D}\Big((S+\alpha\mathsf{id})_F\Big)$ and $g \overset{(\text{def})}{=} (S+\alpha\mathsf{id})_F[f] \in L^2(\mathbb{R})$. By definition of q_α, for all $u \in \mathcal{C}^\infty_c(\mathbb{R})$:

$$\int_{\mathbb{R}} gu dx = \int_{\mathbb{R}} f'(x)u'(x)dx + \int_{\mathbb{R}} (w_V(x)+\alpha)f(x)u(x)dx = -\int_{\mathbb{R}} f(x)u''(x)dx + \int_{\mathbb{R}} (w_V(x)+\alpha)f(x)u(x)dx$$

Therefore in the sense of distributions, we get $-f''+(w_V+\alpha) = g$ which is a function in $L^2(\mathbb{R})$, hence $f \in \Big\{u \in H^1_{V'}(\mathbb{R}), -u''+(w_V+\alpha)u \in L^2(\mathbb{R})\Big\}$. Conversely, if $f \in H^1_{V'}(\mathbb{R})$ such that $-f''+(w_V+\alpha)f \in L^2(\mathbb{R})$, it is possible to extend $u \mapsto q_\alpha(f,u)$ to a continuous linear form on $L^2(\mathbb{R})$ by

$$u \mapsto \int_{\mathbb{R}} u\Big(-f''(x)+f(x)(w_V(x)+\alpha)\Big)dx$$

which concludes the fact that $\mathcal{D}\Big((S+\alpha\mathsf{id})_F\Big) = \Big\{u \in H^1_{V'}(\mathbb{R}), -u''+(w_V+\alpha)u \in L^2(\mathbb{R})\Big\}$. □

In the following, we will only deal with $(\mathcal{S}+\alpha\mathsf{id})_F : \mathcal{D}\Big((S+\alpha\mathsf{id})_F\Big) \longrightarrow L^2(\mathbb{R})$ and denote it $\mathcal{S}+\alpha\mathsf{id} : \mathcal{D}(\mathcal{S}+\alpha\mathsf{id}) \longrightarrow L^2(\mathbb{R})$.

Remark 1.8.5 Note that in the previous proof, the application of Riesz's theorem doesn't allow to say that $(\mathcal{S}+\alpha\mathsf{id}) : v \in \Big(\mathcal{D}(S+\alpha\mathsf{id}), \|.\|_{q_\alpha}\Big) \mapsto f_v \in \Big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\Big)$, where $\|.\|_{q_\alpha}$ stands for the norm associated to the bilinear positive definite form q_α, is continuous. It can be seen by the fact that
$v \in \Big(\mathcal{D}(S+\alpha\mathsf{id}), \|.\|_{q_\alpha}\Big) \mapsto q(.,v) \in \Big(L^2(\mathbb{R})', \|.\|_{L^2(dx)'}\Big)$, where $L^2(\mathbb{R})'$ stands for the topological dual of $L^2(\mathbb{R})$ equipped with its usual norm, is not continuous. Indeed the $\|.\|_{q_\alpha}$ norm doesn't control the second derivative of v and hence doesn't provide any module of continuity for the $L^2(\mathbb{R})$-extended linear form $q(.,v)$.

Also note that, even though it would be convenient that $\mathcal{D}\Big((S+\alpha\mathsf{id}\Big) = L^2(\mathbb{R},(w_V+\alpha)^2dx) \cap H^2(\mathbb{R})$ it is not true without more properties on w_V. Such a result holds, for example when w_V belongs to B_2, the class of reverse Hölder weights, see [ABA07, Theorem 1.1].

Theorem 1.8.6 (Inversion of $\mathcal{S}+\alpha\mathsf{id}$)
For every $f \in L^2(\mathbb{R})$, there exists a unique $u \in \mathcal{D}\Big(S+\alpha\mathsf{id}\Big)$ such that $(\mathcal{S}+\alpha\mathsf{id})[u] = f$. Furthermore, the map $(S+\alpha\mathsf{id})^{-1}$ is continuous from $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ to $\big(\mathcal{D}(S+\alpha\mathsf{id}), \|.\|_{q_\alpha}\big)$.
Proof Let $f \in L^2(\mathbb{R})$, the map $u \longmapsto \langle u, f\rangle_{L^2(dx)}$ is continuous on $\big(H^1_{V'}(\mathbb{R}), \|.\|_{q_\alpha}\big)$ which is a Hilbert space. Therefore by Riesz's theorem, there exists a unique $v_f \in H^1_{V'}(\mathbb{R})$ such that for all $u \in H^1_{V'}(\mathbb{R})$, $\langle f, u\rangle_{L^2(dx)} = q_\alpha(v_f, u)$ from which we deduce that, in the sense of distributions, $f = -v_f'' + (w_V+\alpha)v_f$ which implies that $v_f \in \mathcal{D}(S+\alpha\mathsf{id})$. Since $v_f \in \mathcal{D}(S+\alpha\mathsf{id})$, we have then for all $u \in L^2(\mathbb{R})$, $\langle f, u\rangle_{L^2(dx)} = q_\alpha(v_f, u) = \langle(\mathcal{S}+\alpha)[v_f], u\rangle_{L^2(dx)}$, hence $(\mathcal{S}+\alpha\mathsf{id})[v_f] = f$. Finally, by Riesz's theorem, $f \in L^2(\mathbb{R}) \mapsto v_f \in H^1_{V'}(\mathbb{R})$ is continuous hence so is $(\mathcal{S}+\alpha\mathsf{id})^{-1}$. □
Remark 1.8.7 It would be tempting to use Banach's isomorphism theorem to say that since $(\mathcal{S}+\alpha\mathsf{id})^{-1}$ is bijective and continuous, so must be $\mathcal{S}+\alpha\mathsf{id}$. But since $\big(\mathcal{D}(S+\alpha\mathsf{id}), \|.\|_{q_\alpha}\big)$ is not a Banach space (it's not closed in $H^1_{V'}(\mathbb{R})$) we can't apply it.

We are now able to diagonalize the resolvent of $\mathcal{S}$.
Theorem 1.8.8 (Diagonalization of $(\mathcal{S}+\alpha\mathsf{id})^{-1}$)
There exists a complete orthonormal set $(\psi_n)_{n\geq 0}$ of $L^2(\mathbb{R})$ (meaning that

$$\overline{\mathrm{span}\{\psi_n,\ n\geq 0\}}^{\|.\|_{L^2(dx)}} = L^2(\mathbb{R})$$

and $\langle\psi_i, \psi_j\rangle_{L^2(dx)} = \delta_{i,j}$), where each $\psi_n \in \mathcal{D}(S+\alpha\mathsf{id})$ and $\big(\mu_n(\alpha)\big)_{n\geq 0} \in [0,1]^{\mathbb{N}}$ with $\mu_n(\alpha) \underset{N\to\infty}{\longrightarrow} 0$ such that $(\mathcal{S}+\alpha\mathsf{id})^{-1}[\psi_n] = \mu_n(\alpha)\psi_n$ for all $n \geq 0$. We also have

$$\Big|\Big|\Big|\big(\mathcal{S}+\alpha\mathsf{id}\big)^{-1}\Big|\Big|\Big|_{L^2(dx)} \overset{(\mathrm{def})}{=} \sup_{\|f\|_{L^2(dx)}=1} \|\big(\mathcal{S}+\alpha\mathsf{id}\big)^{-1} f\|_{L^2(dx)} \leq 1.$$

Proof By Proposition 1.8.1, $w_V+\alpha$ is continuous and goes to infinity at infinity. By Rellich criterion [RS78, Theorem XIII.65], the unit ball of $\mathcal{D}(S+\alpha\mathsf{id})$, *i.e.* the set

$$\Big\{u \in \mathcal{D}(S+\alpha\mathsf{id}),\ \int_{\mathbb{R}} u'(x)^2dx + \int_{\mathbb{R}}(w_V(x)+\alpha)u^2(x)dx \leq 1\Big\}$$

considered as a subset of $L^2(\mathbb{R})$ is relatively compact in $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$. Hence, we can conclude that the injection $\iota : \big(\mathcal{D}(S+\alpha\mathsf{id}), \|.\|_{q_\alpha}\big) \longrightarrow \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ is a compact operator. Since $(S+\alpha\mathsf{id})^{-1} : \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big) \longrightarrow \big(\mathcal{D}(S+\alpha\mathsf{id}), \|.\|_{q_\alpha}\big)$ is continuous then $(\mathcal{S}+\alpha\mathsf{id})^{-1}$ is compact from $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ to itself. The fact that $(\mathcal{S}+\alpha\mathsf{id})^{-1}$ is self-adjoint and positive allows us to apply the spectral theorem to obtain $\big(\mu_n(\alpha)\big)_{n\geq 0}$ positive eigenvalues verifying $\mu_n(\alpha) \underset{N\to\infty}{\longrightarrow} 0$ by compactness and a Hilbertian basis $(\psi_n)_{n\geq 0} \in L^2(\mathbb{R})^{\mathbb{N}}$, such that for all $n \geq 0$, $(\mathcal{S}+\alpha\mathsf{id})^{-1}[\psi_n] = \mu_n(\alpha)\psi_n$. It is then easy to see that for all n, $\psi_n \in \mathcal{D}(S+\alpha\mathsf{id})$ since they belong to the range of $(\mathcal{S}+\alpha\mathsf{id})^{-1}$. Finally, since for all $\phi \in L^2(\mathbb{R})$, $\langle(\mathcal{S}+\alpha\mathsf{id})[\phi], \phi\rangle_{L^2(dx)} \geq \|\phi\|^2_{L^2(dx)}$, the spectrum of $(\mathcal{S}+\alpha\mathsf{id})^{-1}$ is contained in $[0,1]$. It allows us to conclude that $\Big|\Big|\Big|(\mathcal{S}+\alpha\mathsf{id})^{-1}\Big|\Big|\Big|_{L^2(dx)} \leq 1$. □

Since for all $u \in H^1_{V'}(\mathbb{R})$, $(S+\alpha)[u] \in L^2(\mathbb{R})$ *iff* $\mathcal{S}[u] \in L^2(\mathbb{R})$, if we define $\mathcal{D}(\mathcal{S})$ in the same manner that we did before, $\mathcal{D}(\mathcal{S}) = \mathcal{D}(S+\alpha\mathsf{id})$. It is now straightforward to see how to extend $\mathcal{A} = \rho_V^{-1/2}\mathcal{S}\rho_V^{1/2}$ on $\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) \overset{(\mathrm{def})}{=} \rho_V^{-1/2}\mathcal{D}(\mathcal{S})$ equipped with the norm $\|.\|_{q_\alpha,\rho_V}$ to $\big(L^2(\mu_V), \|.\|_{L^2(\mu_V)}\big)$. The norm $\|.\|_{q_\alpha,\rho_V}$ is defined for all $u \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ by

$$\|u\|_{q_\alpha,\rho_V} = \int_{\mathbb{R}} u'(x)^2 d\mu_V(x) + \int_{\mathbb{R}} u^2(x)(w_V(x)+\alpha)d\mu_V(x)\,.$$

It is easy to see that $(\mathcal{A}+\alpha\mathsf{id})^{-1}$ is continuous.

Remark 1.8.9 The kernel of $\mathcal{A}$ is generated by the function $\widetilde{1}$. Indeed if $\phi \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ is in the kernel of $\mathcal{A}$ then

$$0 = -\frac{\left(\phi'\rho_V\right)'}{\rho_V} \Rightarrow \exists c \in \mathbb{R},\ \phi' = \frac{c}{\rho_V}$$

But since ϕ' is in $L^2(\mu_V)$ then $c = 0$ which implies that ϕ is constant. We must restrict $\mathcal{A}$ to the orthogonal of Ker$\mathcal{A}$ with respect to the inner product of $L^2(\mu_V)$, *i.e.*

$$\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \overset{(\text{def})}{=} \left\{ u \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) \mid \int_{\mathbb{R}} u(x) d\mu_V(x) = 0 \right\}.$$

Doing so makes $\mathcal{A}$ injective.

Before inverting $\mathcal{A}$, we need the following lemma:

Lemma 1.8.10 *The following equality holds*

$$(\mathcal{A} + \alpha \mathsf{id})\Big(\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})\Big) = L^2_0(\mu_V) \overset{(\text{def})}{=} \Big\{ u \in L^2(\mu_V),\ \int_{\mathbb{R}} u(x) d\mu_V(x) = 0 \Big\}.$$

Proof Let $\phi = \widetilde{c}$ for $c \in \mathbb{R}$, $(\mathcal{A} + \alpha \mathsf{id})[\phi] = \widetilde{\alpha c}$ then $(\mathcal{A} + \alpha \mathsf{id})[\mathbb{R}.\widetilde{1}] = \mathbb{R}\widetilde{1}$. Hence since $\mathcal{A} + \alpha \mathsf{id}$ is self-adjoint with respect to the inner product of $L^2(\mu_V)$ and that $\mathbb{R}\widetilde{1}$ is stable by $\mathcal{A} + \alpha \mathsf{id}$, then $(\mathcal{A}+\alpha \mathsf{id})\Big[(\mathbb{R}.\widetilde{1})^\perp \cap \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})\Big] \subset (\mathbb{R}.\widetilde{1})^\perp$. For the converse, let $u \in (\mathbb{R}.\widetilde{1})^\perp$, since $\mathcal{A}+\alpha \mathsf{id}$ is bijective, there exists $v \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ such that $u = (\mathcal{A} + \alpha \mathsf{id})[v]$. For all $w \in \mathbb{R}.\widetilde{1}$,

$$0 = \langle u, w \rangle_{L^2(\mu_V)} = \langle (\mathcal{A} + \alpha \mathsf{id})[v], w \rangle_{L^2(\mu_V)} = \langle v, (\mathcal{A} + \alpha \mathsf{id})[w] \rangle_{L^2(\mu_V)}$$

Hence $v \in \big((\mathcal{A} + \alpha \mathsf{id})[\mathbb{R}\widetilde{1}]\big)^\perp = \mathbb{R}\widetilde{1}^\perp$ and so $(\mathbb{R}.\widetilde{1})^\perp \subset (\mathcal{A} + \alpha \mathsf{id})\Big[(\mathbb{R}.\widetilde{1})^\perp\Big]$. □

It is easy to see that $L^2_0(\rho_V)$ is a closed subset of $L^2(\mu_V)$ as it is the kernel of the linear form $\phi \in L^2(\mu_V) \mapsto \Big\langle \phi, \widetilde{1} \Big\rangle_{L^2(\mu_V)}$, making it a Hilbert space.

Proposition 1.8.11 (Diagonalization and invertibility of $\mathcal{A}$) *There exists a complete orthonormal set of* $\Big(L^2_0(\rho_V), \langle ., . \rangle_{L^2(\mu_V)}\Big)$, $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})^{\mathbb{N}}$ *such that* $\mathcal{A}[\phi_n] = \lambda_n \phi_n$ *(meaning that*

$$\overline{\text{span}\{\phi_n,\ n \geq 0\}}^{\|.\|_{L^2(\mu_V)}} = L^2_0(\mu_V)$$

and $\langle \phi_i, \phi_j \rangle_{L^2(\mu_V)} = \delta_{i,j}$*). Furthermore,* $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \longrightarrow L^2_0(\mu_V) \overset{(\text{def})}{=} \Big\{ u \in L^2(\mu_V),\ \int_{\mathbb{R}} u(x) d\mu_V(x) = 0 \Big\}$ *is bijective,* $\mathcal{A}^{-1}$ *is continuous when considered as an operator of* $L^2_0(\mu_V)$*.*

Proof Since $(\mathcal{S} + \alpha \mathsf{id})^{-1}$ considered as an operator of $L^2(\mathbb{R})$, is compact so is $(\mathcal{A} + \alpha \mathsf{id})^{-1}$ on $L^2(\mu_V)$ and since $\mathcal{A}$ is self-adjoint, by the spectral theorem, $(\mathcal{A} + \alpha \mathsf{id})^{-1}$ is diagonalizable. With the notations of Theorem 1.8.8, $(\mathcal{A} + \alpha \mathsf{id})^{-1}$ has eigenvalues $\big(\mu_n(\alpha)\big)_{n\geq 0}$ and corresponding eigenfunctions $\phi_n = \rho_V^{-1/2}\psi_n \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$. Hence for all $n \in \mathbb{N}$, $\mathcal{A}[\phi_n] = \lambda_n \phi_n$ with $\lambda_n \overset{(\text{def})}{=} \big(\frac{1}{\mu_n(\alpha)} - \alpha\big)$. Now,

$$\lambda_n \|\phi_n\|^2_{L^2(\mu_V)} = \int_{\mathbb{R}} \mathcal{A}[\phi_n](x)\phi_n(x) d\mu_V(x) = -\int_{\mathbb{R}} (\rho_V \phi_n')'(x)\phi_n(x) dx = \int_{\mathbb{R}} \phi_n'(x)^2 d\mu_V(x) \geq 0\,.$$

Furthermore, the kernel of $\mathcal{A}$ is $\mathbb{R}.\widetilde{1}$, thus the spectrum of $\mathcal{A}$ restricted to $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ is positive. But since $(\mathcal{A}+\alpha \mathsf{id})^{-1}$ is a compact operator of $L^2(\mu_V)$ and that $(\mathcal{A}+\alpha \mathsf{id})$ maps $\mathbb{R}.\widetilde{1}^\perp$ to $\mathbb{R}.\widetilde{1}^\perp$ with respect to the inner product of $L^2(\mu_V)$ (see lemma 1.8.10), then $\big(\mathcal{A} + \alpha \mathsf{id}\big)^{-1}$ is compact as an operator from $L^2_0(\mu_V)$ to itself. By Fredholm alternative, for every $\lambda \in \mathbb{R}\ \lambda \neq 0$, either $(\mathcal{A} + \alpha \mathsf{id})^{-1} - \lambda \mathsf{id}$ is

bijective either $\lambda \in \mathrm{Sp}\big((\mathcal{A}+\alpha\mathsf{id})^{-1}\big)$. These conditions are equivalent to: either $\mathcal{A}+(\alpha-\frac{1}{\lambda})\mathsf{id}$ is bijective as an operator from $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ to $L^2_0(\mu_V)$, either $-\alpha+\frac{1}{\lambda}\in \mathrm{Sp}\big(\mathcal{A}\big)$. If we set $\lambda=\frac{1}{\alpha}$ then either $\mathcal{A}$ is bijective either $0\in\mathrm{Sp}(\mathcal{A})$, since the latter is wrong then $\mathcal{A}:\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})\to L^2_0(\mu_V)$ is bijective. The spectrum of $\mathcal{A}$ is $\left(\frac{1}{\mu_n(\alpha)}-\alpha\right)_{n\geq 0}\subset(\lambda_1,+\infty)\subset(0,+\infty)$, where λ_1 is the smallest eigenvalue, hence we deduce that $\left|\left|\left|\mathcal{A}^{-1}\right|\right|\right|_{L^2(\mu_V)}\leq\lambda_1^{-1}$. □

Chapter 2

Asymptotics of the partition function for β-ensembles at high temperature

"In the end, "what counts as science" is less important than how science gets done. Science is about what is true. Any proposal about how the world truly is should be able to be judged by the methods of science, even if the answer is "we don't know and never will." Excluding a possible way the world could be on the basis of a philosophical predisposition is contrary to the spirit of science. The real question is, how should scientific practice accommodate this possibility?" Sean M. Carroll

Contents

2.1 Introduction

2.1.1 Setting of the problem

Let $P > 0$ and V be a function growing sufficiently fast at infinity. The real β-ensemble at high temperature is the particle system on $\mathbb{R}$, $\{x_i\}_{i=1}^N$ with the following distribution:

$$d\mathbb{P}_N^V(\underline{x}) = p_N^V(\underline{x})dx_1 \dots dx_N \qquad \text{with} \qquad p_N^V(\underline{x}) \overset{(\text{def})}{=} \frac{1}{\mathcal{Z}_N[V]} \prod_{i<j}^N |x_i - x_j|^{2P/N} \prod_{i=1}^N e^{-V(x_i)} \tag{2.1}$$

where $\mathcal{Z}_N[V] > 0$ is the partition function that ensures that $\mathbb{P}_N^V$ is a probability measure on $\mathbb{R}^N$, namely

$$\mathcal{Z}_N[V] = \int_{\mathbb{R}^N} \prod_{i<j}^N |x_i - x_j|^{2P/N} \prod_{i=1}^N e^{-V(x_i)} dx_i. \tag{2.2}$$

Here, the factor 2 in the two-body interaction is irrelevant and just makes the equations "nicer". The main goal of this article is to establish the existence of the large N-asymptotic expansion of $\log \mathcal{Z}_N[V]$ under some assumptions on V using the technique first used in [ACM92, ArM90] and later developed in [BG13a, BG13b]. Note that when $2P/N$ is replaced by a N-independent $\beta \geq 0$ (that one can interpret as a coupling constant that measures the strength of the interaction), the distribution is known as the real β-*ensemble* in the fixed temperature regime and represents for polynomial potential the joint law of eigenvalues of the so-called Orthogonal (resp. Unitary, resp. Symplectic) ensemble for $\beta = 1$ (resp. $\beta = 2$, resp. $\beta = 4$) (see [AGZ10a]). For general $\beta \geq 0$ and quadratic V, the β-ensemble was expressed as the law of the spectrum of tridiagonal random matrices with independent entries [DE02a]. The result was then extended to general polynomial V in [KRV16].

When β is fixed, a great deal is known about this model because of twenty years of intensive study. First, central limit theorems were proven in [Joh98, BG13a, Shc14a, BLS18, Lam21a, LLW19], asymptotic of the partition function in [BG13a, BG13b, BGK15], local laws [BYY14, BEY14b, CFLW21, BMP22, Pei24] and universality results [PS97, DKM$^+$99c, DG07a, DGKV07]. For $\beta = 2$ constant and V polynomial, the asymptotic expansion of the partition function has the form

$$N^{-2} \log Z_N^{(2)}[V] \sim \sum_{g \geq 0} N^{-2g} c_g$$

where the previous equality has to be understood in the sense of an asymptotic expansion. The coefficients $(c_g)_{g\geq 0}$ of this expansion correspond to enumerations of maps and, more generally, the asymptotic expansion of $\log Z_N^\beta[V]$ gives information on the enumerations of graphs embedded in surfaces [Mar14, MTY05]. In [DFGZ23], the authors were able to establish the asymptotics for the moments of Hilbert-Schmidt norms of matrices uniformly distributed on unit balls. They were able to link these moments with the partition function of β-ensembles with singular potential, for which it is possible to obtain the leading asymptotics.

The study of the β-ensembles at high temperature has attracted a lot of attention recently since links were discovered with integrable systems, including the famous *classical Toda chain* [Tod67]. The integrable structure of this system, namely the existence of a sufficient number of conserved quantities, can be established by the existence of a so-called *Lax matrix*, whose spectrum is invariant under the dynamics. At long times, the model doesn't *thermalize*, *i.e.* it doesn't reach thermal equilibrium but is rather described by a more sophisticated probability measure called the *Generalized Gibbs Ensemble* (GGE) [Jay57]. This is due to the existence of a set of locally conserved quantities, which highly constrains the dynamics. In the context of the Toda chain, the GGE has been studied in [Spo20] and a link was established with the Gaussian β-ensembles. In the case of a Gaussian potential, it was shown that the distribution of the Lax matrix under the GGE was similar to the law of the tridiagonal representation of the Gaussian β-ensembles of Dumitriu and Edelman. This link was explored in [GM22] for more general potentials *via* large deviation techniques.

However, the high temperature regime had already been the subject of research with the pioneering works [CL97, BG99, ABG12]. More recently, large deviation principles (LDP) [Pak20, GZ19] and central limit theorems for β-ensembles at high temperature were shown in the circular case [HL21]. In the real case [NT18], such a result was obtained for quadratic potentials and polynomial test-functions, for general potentials with smooth bounded functions in **[DGM23]** and for polynomial potentials and polynomial test-functions in [MM24]. Convergence results of bulk local statistics to Poisson point-processes were established, first in [BGP15] and later in [NT18], for Gaussian potential, then generalized to general potentials in [NT20a] and finally to general interactions, potentials and geometric settings in [Lam21b]. The latter work, includes the convergence of the edge statistics and the asymptotic law of the fluctuations of the edge of the spectrum, namely a Gumbel distribution which was first discovered in the quadratic case by [Pak18]. This is in adequation with the fact that this law can be seen as the limit of the β-Tracy Widom law when $\beta \to 0$ [AD14b]. Finally, in [FM21], the authors analyzed the so-called loop equations to deduce the moments of the subdominant correction of the equilibrium measures corresponding to Gaussian, Laguerre and Jacobi ensembles. A natural extension of all these results is to obtain the large N asymptotic expansion of the $n-$linear statistics $\langle f\rangle^V_{\overset{n}{\otimes}\mathcal{L}_N}$ for general test-functions f where $L_N \overset{(\text{def})}{=} N^{-1}\sum_{a=1}^N \delta_{x_a}$ is the empirical distribution, $\mathcal{L}_N \overset{(\text{def})}{=} L_N - \mu_V$ and

$$\langle f\rangle^V_{\mu_1\otimes\ldots\otimes\mu_k} \overset{(\text{def})}{=} \mathbb{E}_N^V\left[\int_{\mathbb{R}} f(x_1,\ldots,x_k)\prod_{i=1}^k d\mu_i(x_i)\right]. \tag{2.3}$$

By [GZ19], the sequence of random probability measures $(L_N)_N$ satisfies a large deviation principle at speed N with a strictly convex, good rate function I_V^P. The latter is defined by $I_V^P \overset{(\text{def})}{=} \mathcal{E} - \inf_{\mu\in\mathcal{M}_1(\mathbb{R})}\mathcal{E}(\mu)$ where, for all absolutely continuous probability measures μ with respect to the Lebesgue measure:

$$\mathcal{E}(\mu) \overset{(\text{def})}{=} \int_{\mathbb{R}} V d\mu - 2P\iint_{\mathbb{R}^2}\log|x-y|d\mu(x)d\mu(y) + \int_{\mathbb{R}}\log\left(\frac{d\mu}{dx}\right)d\mu$$

and $+\infty$ for other probability measures on $\mathbb{R}$. In this functional, the last term is called the *entropy* of the measure μ, and because of conventions, it represents the negative physical entropy. This

term is negligible when the inverse temperature β is constant and the potential is scaled by N (it doesn't appear in the corresponding good rate function [AGZ10a]) but has the same order of magnitude as the energy when the correlation between particles is small/the temperature is high (see [Dea08, (34)] for a discussion). Minimizing the functional $\mathcal{E}$ amounts to minimizing the energy while maximizing the (physical) entropy.

We call the unique minimizer of $\mathcal{E}$ and denote it μ_V (we omit the P-dependence since this parameter is fixed throughout the entire article) the *equilibrium measure* which is Lebesgue continuous with p.d.f. ρ_V characterized by

$$V(x) - 2P\int_{\mathbb{R}} \log|x-y|\rho_V(y)dy + \log\rho_V(x) = \lambda_V \qquad x - ae \tag{2.4}$$

where λ_V is a constant (see [GM22, Lemma 3.2]). In this context, ρ_V is supported on $\mathbb{R}$ because of the presence of the entropy in the minimizing equation. This is a major difference with the β-constant case and N-scaled potential, where the equilibrium measure is compactly supported. This equation can be rewritten, assuming ρ_V is continuous (which is true as long as V is smooth; see [DGM23, Lemma 2.2]), as

$$\forall x\in\mathbb{R},\ \rho_V(x) = \exp\Big(-V(x) - 2PU^{\rho_V}(x) + \lambda_V\Big), \qquad U^{\rho_V}(x) \overset{(\text{def})}{=} -\int_{\mathbb{R}} \log|x-y|\rho_V(y)dy. \tag{2.5}$$

One observes that U^{ρ_V} diverges logarithmically to $-\infty$ at infinity [DGM23, Lemma 2.4]. Hence, assuming that V grows fast enough at infinity, instead of a compactly supported measure as in the constant β case, the equilibrium density is an exponentially fast decaying function at infinity in the case $\beta N = 2P$. Furthermore, this measure can be seen as an interpolation (with the appropriate scaling in P) called originally the Gauss-Wigner crossover [ABG12] between the equilibrium of the classical β-ensembles (when P goes to $+\infty$) and the measure $d\mu(x) = e^{-V(x)}dx/Z$ (when P goes to 0) [NT20a]. Note that under the choice of $V_G(x) = x^2/2$, the density has an explicit form [ABG12]:

$$\rho_{V_G}(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\frac{1}{|\hat{f_\alpha}(x)|^2}, \qquad \hat{f_\alpha}(x) \overset{(\text{def})}{=} \sqrt{\frac{P}{\Gamma(P)}}\int_0^{+\infty} t^{P-1}e^{-\frac{t^2}{2}+ixt}dt.$$

This density is also explicit in some other models, namely the Laguerre and Jacobi ensembles, involving hypergeometric functions [Maz22].

Our goal here is to follow a strategy introduced in [BG13a, BG13b] and used in [BGK15, BGK16] to establish the existence of the large N-asymptotic of $\log\mathcal{Z}_N[V]$ for a general potential of the form $V_{G,\phi}(x) \overset{(\text{def})}{=} x^2/2 + \phi$ where ϕ is a bounded smooth function. Namely we wish to show that for all $K \geq 0$, there exists $c_0, \dots, c_K \in \mathbb{R}$ depending on ϕ and P such that:

$$\frac{1}{N}\log\mathcal{Z}_N[V_{G,\phi}] = \sum_{i=0}^{K}\frac{c_i}{N^i} + O\left(N^{-(K+1)}\right).$$

Before stating the main results and explaining how to obtain them, we need to introduce some objects. An object that appears naturally when tackling this model is the so-called *master operator* Ξ defined, for sufficiently smooth ϕ, by:

$$\forall x\in\mathbb{R}, \quad \Xi[\phi](x) \overset{(\text{def})}{=} \phi'(x) + \big(\log\rho_V\big)'(x)\phi(x) + 2P\left(\mathcal{H}[\phi\rho_V](x) - \int_{\mathbb{R}}\mathcal{H}[\phi\rho_V](y)d\mu_V(y)\right) \tag{2.6}$$

where $\mathcal{H}$ denotes the *Hilbert transform*, which is defined by $\mathcal{H}[f](x) = \fint_{\mathbb{R}}\frac{f(y)}{y-x}dy$ and where the integral has to be understood as a Cauchy principal value. The main difference between Ξ and the master operator $\mathcal{K}$ that arises in the classical β-ensembles is the first derivative term of the

RHS in (2.6). Because of this term, Ξ is then an unbounded operator. While in the classical β-ensembles, $\mathcal{K}$ is easily invertible and controled, see [BFG15] dealing such an operator demands lot of technicalities in general. In particular, inverting and obtaining controls is much more difficult in this case.

One can understand the operator Ξ as controlling the fluctuations of the empirical measure L_N with respect to μ_V in the sense that when one writes, for a sufficiently smooth function ϕ,

$$\langle \phi \rangle_{L_N}^V = \langle \phi \rangle_{\mu_V} + R_N,$$

the remainder R_N is expressed as a sum of linear statistics of functions involving $\Xi^{-1}[\phi]$ and goes to zero when N is large. Above, the superscript for the expectation value with respect to μ_V can be omitted since the measure is deterministic. Furthermore, as we will show in the present work, if V_1 and V_2 are two potentials, Ξ^{-1} appears naturally when one wants to study the variation between the two corresponding equilibrium measures, namely $\mu_{V_1} - \mu_{V_2}$. Finally, by exploiting the fact that the partition function $\mathcal{Z}_N[V]$, introduced in (2.2), is invariant under the transformation $\lambda_i \to \lambda_i + tN^{-1/2}\phi(\lambda_i)$ for ϕ sufficiently smooth, the authors showed a central limit theorem in **[DGM23]** *i.e.* that

$$\sqrt{N} \int \Xi[\phi] d\left(L_N - \mu_V\right) \overset{\text{law}}{\Rightarrow} \mathcal{N}\left(0, \sigma^2(\phi)\right)$$

for σ^2 a positive quadratic form. It has been shown in this same article that this operator is invertible and that when ϕ is smooth, so is $\Xi^{-1}[\phi]$. This was done by inverting the operator $\mathcal{L}$, defined by $\mathcal{L}[\phi] \overset{\text{(def)}}{=} \Xi\left[\phi'\right]$, on the Hilbert space:

$$\mathsf{H} \overset{\text{(def)}}{=} \left\{ u \in L^2\left(\mu_V\right) \;\middle|\; u' \in L^2\left(\mu_V\right), \int_{\mathbb{R}} u d\mu_V = 0 \right\}, \qquad \langle u, v \rangle_{\mathsf{H}} \overset{\text{(def)}}{=} \left\langle u', v' \right\rangle_{L^2(\mu_V)}. \tag{2.7}$$

Once the operator $\mathcal{L}$ is inverted, this straightforwardly implies that Ξ is invertible with inverse $\Xi^{-1}[\psi] = \left(\mathcal{L}^{-1}[\psi]\right)'$.

The establishment of the large-N behavior of the 1-linear statistics is based on the so-called *loop equations* or *Schwinger-Dyson* equations introduced in [ACM92, ArM90]; see [BGK16] for a more precise state of the art.

2.1.2 Assumptions

For the rest of the paper, we use the following list of assumptions on the potential V:

Assumptions 2.1.1 *The potential V satisfies:*

(i) $V \in \mathcal{C}^\infty(\mathbb{R})$,

(ii) $V(x) \underset{|x|\to+\infty}{\longrightarrow} +\infty$ *and* $|V'(x)| \underset{|x|\to+\infty}{\longrightarrow} +\infty$,

(iii) The measure μ_V satisfies the Poincaré inequality i.e. there exists $C_{\text{Poinc}} > 0$ (depending on V and P) such that for all $f \in \mathcal{C}_c^1(\mathbb{R})$:

$$\text{Var}_{\mu_V}(f) \overset{\text{(def)}}{=} \int_{\mathbb{R}} \left(f(x) - \int_{\mathbb{R}} f(y) d\mu_V(y) \right)^2 d\mu_V(x) \leq C_{\text{Poinc}} \int_{\mathbb{R}} f'(x)^2 d\mu_V(x). \tag{2.8}$$

(iv) For all polynomial $Q \in \mathbb{R}[X]$ and $\alpha > 0$, all $p \geq 0$, $Q\left(V^{(p)}(x)\right) e^{-V(x)} = \underset{|x|\to\infty}{o}\left(x^{-\alpha}\right)$.

(v) The function $\dfrac{1}{V'^2}$ is integrable at infinity, and $\dfrac{V^{(k)}(x)}{V'(x)} = \underset{|x|\to\infty}{O}(1)$ for $k \geq 2$.

Assumption *(i)* is necessary to ensure that μ_V and $\Xi^{-1}[\phi]$ for ϕ smooth are smooth (its derivatives involve derivatives of V).

Assumption *(ii)* is sufficient condition for $\mathcal{Z}_N[V]$ to be well-defined. Indeed, the assumption on V' implies that V grows faster than linearly at infinity, which implies that the β-ensemble is well-defined. Another consequence is that ρ_V has exponential decay at infinity. The fact that V' goes to infinity is also necessary to ensure that $\Xi^{-1}[\phi]^{(k)}(x) \underset{|x|\to\infty}{\longrightarrow} 0$ for all $k \geq 0$ and for bounded smooth ϕ.

Assumption *(iii)* implies that Ξ defined in (2.6), is invertible; see [DGM23, Proposition 2.6]. The authors showed that for any potential of the form $V = V_{\text{conv}} + \phi$, where V_{conv} is a strictly convex potential outside of a compact set and ϕ a bounded function, μ_V satisfies the Poincaré inequality.

Assumption *(iv)* is necessary to ensure that the equilibrium density ρ_V is smooth.

Assumption *v)* allows one to prove that Ξ^{-1} is continuous with respect to the appropriate norms. Indeed, when differentiating $\Xi^{-1}[\phi]$, for ϕ a smooth function, quantities behaving at infinity like $V^{(k)}(x)V'(x)^{-1}$ will naturally arise. On the other hand, we will integrate some functions that behave like $V'(x)^{-2}$ at infinity.

These conditions are satisfied, for example, for every V in the following class

$$\left\{x \mapsto a_n x^{2n} + \phi(x),\ n > 0, a_n > 0, a_i \in \mathbb{R}, \phi^{(k)} \text{ bounded } \forall k \in \mathbb{N}\right\} \cup \left\{x \mapsto \frac{e^{\gamma x} + e^{-\gamma x}}{\alpha}, \alpha > 0, \gamma \in \mathbb{R}\right\}.$$

These potentials satisfy assumptions *(iii)*; see [DGM23, Proposition 2.6, Remark 2.7]for a discussion about it. On the other hand, potentials like $V(x) = e^{x^2}$ violate assumption *(v)*, therefore they do not fit in our analysis.

2.1.3 Main results

To state the next result, we recall that $\mathcal{L}_N \overset{(\text{def})}{=} L_N - \mu_V$.

Theorem 2.1.2 (Asymptotic expansion of linear statistics) *Under assumptions 2.1.1 on the potential V, for all smooth function $\phi \in L^2(\mathbb{R}^k)$, such that $\phi^{(j)} \in L^2(\mathbb{R}^k)$ for all $j \geq 0$, there exists a unique sequence $(b_i)_{i \geq \lceil k/2 \rceil}$ depending on V, ϕ and P such that forall $K > 0$:*

$$\langle \phi \rangle^V_{\otimes^k \mathcal{L}_N} = \sum_{i=\lceil k/2 \rceil}^{K} \frac{b_i}{N^i} + O\left(N^{-(K+1)}\right).$$

Our goal is to obtain the existence of an asymptotic expansion for $\mathcal{Z}_N[V_{G,\phi}]$ where $V_{G,\phi}(x) \overset{(\text{def})}{=} x^2/2 + \phi(x)$ and ϕ is a smooth function. As will be explained further, one would like to deduce the asymptotic expansion of $\log \mathcal{Z}_N[V_{G,\phi}]$ from an integration of the one for $t \in [0,1] \mapsto \langle \phi \rangle^{V_{G,\phi,t}}_{\mathcal{L}_N}$ where $V_{G,\phi,t}(x) \overset{(\text{def})}{=} x^2/2 + t\phi(x)$. In order to make this step rigorous, we need the following continuity result with respect to t.

Theorem 2.1.3 *Under assumptions 2.1.1 on the potential V, for all $i \in \mathbb{N}$ and for all smooth function $\phi \in L^2(\mathbb{R})$, such that $\phi^{(k)} \in L^2(\mathbb{R})$ for all $k \geq 0$,*

$$\|\rho_{V_{\phi,t}} - \rho_{V_{\phi,t'}}\|_{W_i^\infty(\mathbb{R})} \underset{t \to t'}{\to} 0$$

where $V_{\phi,t} : x \mapsto V(x) + t\phi(x)$ where $t \in [0,1]$. The $W_i^\infty(\mathbb{R})$-norm is defined as $\|f\|_{W_n^\infty(\mathbb{R})} \overset{(\text{def})}{=} \max_{k \in [\![0,n]\!]} \|f^{(k)}\|_{L^\infty(\mathbb{R})}$. Furthermore, for all $x \in \mathbb{R}$, $t \mapsto \rho_{V_{\phi,t}}(x) \in \mathcal{C}^\infty(\mathbb{R})$ and satisfies the following

integro-differential equation for all $t \geq 0$ and $x \in \mathbb{R}$:

$$\partial_t \rho_{V_{\phi,t}}(x) = \left(-\phi(x) + \int_{\mathbb{R}} \phi(s) \rho_{V_{\phi,t}}(s) ds \right) \rho_{V_{\phi,t}}(x).$$

Theorem 2.1.2 together with Theorem 2.1.3 allow us to deduce the following theorem.

Theorem 2.1.4 (Asymptotic expansion of the partition function) *Let $\phi \in L^2(\mathbb{R})$ be a smooth function, such that $\phi^{(k)} \in L^2(\mathbb{R})$ for all $k \geq 0$. There exists a unique sequence $(c_i)_{i \geq 0} \in \mathbb{R}^{\mathbb{N}}$ depending on ϕ and P, such that for all $K \geq 0$,*

$$\frac{1}{N} \log \mathcal{Z}_N \left[V_{G,\phi} \right] = \sum_{i=0}^{K} \frac{c_i}{N^i} + O\left(N^{-(K+1)} \right).$$

The leading term c_0 is given by:

$$c_0 \overset{\text{(def)}}{=} - \int_{\mathbb{R}} V_{G,\phi}(x) d\mu_{V_{G,\phi}}(x) + P \iint_{\mathbb{R}^2} \log|x-y| d\mu_{V_{G,\phi}}(x) d\mu_{V_{G,\phi}}(y) - \int_{\mathbb{R}} \log\left(\frac{d\mu_{V_{G,\phi}}(x)}{dx} \right) d\mu_{V_{G,\phi}}(x). \tag{2.9}$$

The first subleading term c_1 is given in terms of the Euler-Mascheroni constant γ and Ξ_t^{-1} the inverse of the master operator associated with the potential $V_{G,\phi,t}$. It can be written as

$$\begin{aligned} c_1 \overset{\text{(def)}}{=} \gamma \frac{P}{2} + \frac{\log(1+P)}{2} + \frac{1}{2} \sum_{j \geq 1} \left(\log\left(1 + \frac{P+1}{j} \right) - \log\left(1 + \frac{1}{j} \right) - \frac{P}{j} \right) \\ - P \int_0^1 \left[\left\langle \partial_1 \widetilde{\Xi_t^{-1}}[\phi] \right\rangle_{\mu_{V_{G,\phi,t}}} + \left\langle \Theta^{(2)} \circ \widetilde{\Xi_{t,1}^{-1}} \left[\partial_2 \mathcal{D} \circ \widetilde{\Xi_t^{-1}} \phi \right] \right\rangle_{\mu_{V_{G,\phi,t}}} \right] dt. \end{aligned} \tag{2.10}$$

Above, $\Theta^{(2)}$ and $\mathcal{D}$ are explicit operators given in Section 2.3 while

$$\Xi_t^{-1}[\phi](x) = \frac{1}{\rho_{V_{G,\phi,t}}(x)} \int_x^{+\infty} \mathcal{T}_t[\phi](y) d\mu_{V_{G,\phi,t}}(y)$$

where $\mathcal{T}_t$ is an explicit kernel operator given in (2.77). The symbol $\widetilde{\Xi_1^{-1}}$ is also defined in Sections 2.3 and 2.4. It is a standard fact (see for example [GZ19, Theorem 1.2]) that the leading term c_0 is the free energy of the model *i.e.*, $c_0 = \lim_{N \to \infty} N^{-1} \log \mathcal{Z}_N[V_{G,\phi}] = -\inf_{\mu} \mathcal{E}(\mu) = -\mathcal{E}(\mu_{V_{G,\phi}})$ where the infimum runs over probability measures on $\mathbb{R}$.

2.1.4 Outline of the proof

This strategy is based on the following *interpolation equation* of the form:

$$\log \mathcal{Z}_N[V_{G,\phi}] = \log \mathcal{Z}_N[V_G] - N \int_0^1 \langle \phi \rangle_{L_N}^{V_{G,\phi,t}} dt. \tag{2.11}$$

On the RHS, it is convenient to have $\log \mathcal{Z}_N[V_G]$ since, by Mehta's formula [Meh04, 17.6.7], one can extract its asymptotic expansion at large N. Once this identity is obtained, the derivation of the asymptotic expansion of the *free energy* of the model $\log \mathcal{Z}_N[V]$ follows from a similar expnasion for the 1-linear statistics as soon as one has sufficiently precise controls on the remainder's dependence on the data of the problem.

We now explain how to derive Theorem 2.1.2 for a general potential V satisfying assumptions 2.1.1. The proof is based on the analysis of the Schwinger-Dyson equations (SDE). It consists of a tower

of equations that link linear statistics of different orders together. The simplest equation is the one at level 1, which reads for any ϕ smooth enough:

$$\langle \phi \rangle_{\mathcal{L}_N}^V = \frac{P}{N} \left\langle \left(\Xi^{-1}[\phi]\right)' \right\rangle_{\mu_V} + \frac{P}{N} \left\langle \left(\Xi^{-1}[\phi]\right)' \right\rangle_{\mathcal{L}_N}^V - P \left\langle \mathcal{D} \circ \Xi^{-1}[\phi] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V \tag{2.12}$$

where $\mathcal{D}$ is the operator defined for all $x \neq y$ by $\mathcal{D}[\phi](x,y) \overset{\text{(def)}}{=} \dfrac{\phi(x) - \phi(y)}{x - y}$. This equation links the 1-linear statistic and the 2-linear statistic $\left\langle \mathcal{D} \circ \Xi^{-1}[\phi] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V$. The deduction of the asymptotic expansion for linear statistics from the SDE is based on a so-called *a priori bound*, which we will assume for now, of the following form:

$$| \langle \phi \rangle_{\overset{k}{\otimes} \mathcal{L}_N} | \leq C \frac{\|\phi\|}{N^{k/2}} \tag{2.13}$$

for a norm $\|.\|$ that we don't make precise here. Note however, that in the high temperature, one has a to deal with a more complex norm which requires integrability conditions on the functions we apply this bound to. Assuming we know that

$$\left\langle \mathcal{D} \circ \Xi^{-1}[\phi] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V = \frac{\alpha_1(\phi)}{N} + o(N^{-1}) \text{ and that } \langle \psi \rangle_{\mathcal{L}_N}^V = o(1),$$

for some $\alpha(\phi) \in \mathbb{R}$, then (2.12) allows one to obtain the leading order asymptotic for the 1-statistic:

$$\langle \phi \rangle_{\mathcal{L}_N}^V = N^{-1} \Big(P \left\langle \left(\Xi^{-1}[\phi]\right)' \right\rangle_{\mu_V} - \alpha_1(\phi) P + o(1) \Big) \overset{\text{(def)}}{=} \gamma_1(\phi) N^{-1} + o(N^{-1}).$$

Assuming now that for $n = 2$

$$\left\langle \mathcal{D} \circ \Xi^{-1}[\phi] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V = \sum_{i=1}^{n} \frac{\alpha_i(\phi)}{N^i} + o(N^{-n}), \tag{2.14}$$

it is not hard to see that one can iteratively derive the expansion of $\langle \phi \rangle_{\mathcal{L}_N}^V$ and get:

$$\langle \phi \rangle_{\mathcal{L}_N}^V = \sum_{i=1}^{n} \frac{\gamma_i(\phi)}{N^i} + o(N^{-n}), \qquad \gamma_2(\phi) = P\alpha_1 \left(\left(\Xi^{-1}[\phi]\right)' \right) - P\alpha_i(\phi).$$

By the same procedure, one can see that the extraction of the asymptotic expansion up to order $n > 2$ of the 1-linear statistics boils down to extracting the one for the 2-linear statistics.

To achieve that, one needs to investigate the loop equation at level 2, which has the following form for a smooth function ϕ_2 of 2 variables

$$\langle \phi_2 \rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V = \frac{1}{N} \langle \mathcal{U}[\phi_2] \rangle_{\mathcal{L}_N}^V + \langle \mathcal{V}[\phi_2] \rangle_{\overset{3}{\otimes} \mathcal{L}_N}^V + \frac{1}{N} \langle \mathcal{W}[\phi_2] \rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V + \frac{1}{N} \langle \mathcal{Y}[\phi_2] \rangle_{\mu_V}$$

with $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Y}$ some operators. From estimate (2.13) that we assumed at the beginning, we know that

$$\langle \mathcal{U}[\phi_2] \rangle_{\mathcal{L}_N}^V = O(N^{-1/2}), \qquad \langle \mathcal{V}[\phi_2] \rangle_{\overset{3}{\otimes} \mathcal{L}_N}^V = O(N^{-3/2}), \qquad \langle \mathcal{W}[\phi_2] \rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V = O(N^{-1}).$$

It is straightforward to see that only the term $N^{-1} \langle \mathcal{Y}[\phi_2] \rangle_{\mu_V}$ yields a non-negligible contribution to the expansion of $\langle \phi_2 \rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}^V$ at precision $o(N^{-1})$. In order to push it up to $o(N^{-2})$, one needs to obtain the asymptotic expansion for the 3-linear statistics and so on. Each additional order in the asymptotic expansion requires analysing a higher level SDE. Hopefully, each time only a finite number of equations need to be analyzed in order to get all the contributions, and the estimate

allows one to neglect all the other terms. Finally, in order to apply the estimate to neglect the remainders, one needs to show that the operators involved in the SDE preserve enough of the regularity of the function they act on, especially for the inverse of the master operator Ξ^{-1}. In this setting, one has to obtain way more subtle controls compared to the constant β-setting. This is due to the fact that (2.13) involves a more complex norm than just a L^∞-norm. Moreover, finding a manageable integral representation for $\Xi^{-1}[\phi]$ in order to extract controls out of it, is a highly non-trivial step. This makes the proof of the continuity of $\Xi^{-1}[\phi]$ with respect to ϕ quite technical. Finding such a form and proving continuity results for Ξ^{-1} is one of the main technical contribution of this article.

When integrating the asymptotic expansion of the 1-linear statistics $\langle\phi\rangle_{L_N}^{V_{G,\phi,t}}$, one needs to justify that the resulting integrals are finite, *i.e.* that the integrands are integrable. Justifying that $t \mapsto \rho_{V_{G,\phi,t}}$ is continuous with respect to the uniform convergence norm (of the function and all its derivatives) is enough to conclude this. A transport-based approximation of an analogue of $\rho_{V_{G,\phi,t}}$ was constructed in [LS17] in the case of the 2D Coulomb gas. While for the classical β-ensembles, it is not hard to show, under some hypotheses on the supports of V and W, that

$$\mu_{tV+(1-t)W} = t\mu_V + (1-t)\mu_W,$$

and deduce the continuity of $t \mapsto \mu_{tV+(1-t)W}$, to analog result in the high temperature regime is much more involved. This is due to the non-linearity of (2.4).

To show this result, our method is based on an application of the Banach fixed-point theorem to (2.5). In the model with fixed β the analogous step may be done easily because of the linearity, with respect to the equilibrium measure, of the characterizing equation (the analogue (2.4)). Due to the presence of entropy, ρ_V is the solution of a non-linear integral equation.

2.1.5 Notations and conventions

- Let X be a open set of $\mathbb{R}^p$, we denote by $\mathcal{C}^k(X)$ (resp. $L^p(X)$) the space of functions differentiable k-times for which the k-th derivative is continuous (resp. p^{th}-power integrable functions) on X. $\mathcal{C}_c^k(X)$ denotes the space of functions of class k on X with compact support. For $p \in [\![1, +\infty]\!]$, we denote by $L^p(X)$ the usual Lebesgue spaces on X and by $L^p(\mu)$ the Lebesgue spaces with respect to a borelian measure μ on $\mathbb{R}$. Furthermore, we define $L_0^2(\mu)$ by $\left\{u \in L^2(\mu),\ \int_{\mathbb{R}} u d\mu = 0\right\}$. For a function of several variables f, we denote the derivative operator with respect to its i-th variable by $\partial_i f$.
- The space of functions f such that $f^{(k)} \in L^\infty(\mathbb{R})$ for all $k = 0, \dots, n$ will be denoted $W_n^\infty(\mathbb{R})$. Its norm is classically $\|f\|_{W_n^\infty(\mathbb{R})} \overset{(\text{def})}{=} \max_{k \in [\![0,n]\!]} \|f^{(k)}\|_{L^\infty(\mathbb{R})}$.
- Let $f \in L^2(\mathbb{R})$, we denote by $\mathcal{H}[f]$ the Hilbert transform of f defined by

$$\mathcal{H}[f](x) \overset{(\text{def})}{=} \fint_{\mathbb{R}} \frac{f(y)}{y-x} dy$$

  where $\fint$ stands for the Cauchy principal value integral.
- We denote the Fourier transform of $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$\mathcal{F}[f](t) \overset{(\text{def})}{=} \int_{\mathbb{R}} f(x) e^{-\mathrm{i}tx} dx.$$

  When μ is a signed measure over $\mathbb{R}$, we shall denote its Fourier transform by the same symbol $\mathcal{F}[\mu]$.

- The 1/2-norm is defined for any function f which makes this quantity finite

$$\|f\|_{1/2}^2 \overset{(\text{def})}{=} \int_{\mathbb{R}} |t| \, |\mathcal{F}[f](t)|^2 \, dt.$$

- We denote by $\mathcal{M}_1(\mathbb{R})$ the set of probability measures over $\mathbb{R}$. For $\mu, \mu' \in \mathcal{M}_1(\mathbb{R})$ we define the distance (possibly infinite) D by

$$D(\mu, \mu') := \left(\int_0^{+\infty} \frac{1}{t} \left| \mathcal{F}[\mu - \mu'](t) \right|^2 dt \right)^{1/2}. \tag{2.15}$$

- We define the Sobolev spaces for all $m \geq 0$ by

$$H^m(\mathbb{R}^n) \overset{(\text{def})}{=} \left\{ u \in L^2(\mathbb{R}^n), \|u\|_{H^m(\mathbb{R}^n)} < +\infty \right\}$$

where

$$\|u\|_{H^m(\mathbb{R}^n)}^2 \overset{(\text{def})}{=} \int_{\mathbb{R}^n} (1 + \|t\|_2)^{2m} \, |\mathcal{F}[u](t_1, \ldots, t_n)|^2 \, d^n \underline{t}.$$

Above, $\|.\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$. If $\mu \in \mathcal{M}_1(\mathbb{R})$, we also define

$$H^k(\mu) \overset{(\text{def})}{=} \left\{ u \in L^2(\mu), \, u^{(k)} \in L^2(\mu) \right\}.$$

Outline of the paper. In Section 2.2, we establish an a priori bound on the n-linear statistics that will be crucial in order to analyze the loop equations. To prove this bound, we first prove a concentration inequality for the empirical measure. In Section 2.3, we establish controls on the operators that appear as building blocks of the loop equations. In Section 2.4, we prove controls on the so-called master operator. These will play a crucial role in the analysis of the loop equations. We then state the loop equations and establish the large N asymptotic expansion of the linear statistics in Section 2.5. In Section 2.6, we establish the continuity of the equilibrium density associated with the interpolation between the Gaussian potential and the potential considered in this paper, and this with respect to the interpolation parameter. Section 2.7 is dedicated to the expansion of the partition function and an explicit form for the free energy associated with the Gaussian potential, as well as the interpolation formula. We conclude with Theorem 2.1.4 thanks to the results shown in Section 2.6, including Theorem 2.1.3. This allows us to integrate the asymptotic expansion obtained in Theorem 2.1.2. We detail in Appendix 2.9 some results obtained in [**DGM23**] upon which this article largely relies. In Appendix 2.10, we prove the continuity and the integrability of the constants that appear in our problem.

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2.2 A priori bound on the linear statistics

As explained in the introduction, before analyzing the loop equations, one needs a bound that quantifies how small is a function integrated n times against the recentred empirical measure $\mathcal{L}_N \overset{(\text{def})}{=} L_N - \mu_V$. Before addressing this, let us recall certain properties enjoyed by μ_V and the concentration results established in [**DGM23**].

2.2.1 Equilibrium measure

We recall the definition of the logarithmic potential (or sometimes called Symm's operator) U^f of a function $f : \mathbb{R} \to \mathbb{R}$. When it is defined, the latter is given for all $x \in \mathbb{R}$ by

$$U^f(x) = -\int_{\mathbb{R}} \log|x-y| f(y) dy \,. \tag{2.16}$$

One can check that $\big(U^f\big)' = \mathcal{H}[f]$.

We now describe the regularity of the equilibrium density ρ_V characterized by (2.4).

Lemma 2.2.1 *[DGM23, Lemma 2.2]*

- *The support of μ_V is $\mathbb{R}$ and there exists a P-dependent constant C_V such that for all $x \in \mathbb{R}$,*
$$\rho_V(x) \le C_V(1+|x|)^{2P} e^{-V(x)} \,.$$
- *The density $\rho_V \in \mathcal{C}^\infty(\mathbb{R})$ and it holds*
$$\rho_V{}' = -\Big(V' + 2P\mathcal{H}[\rho_V]\Big)\rho_V, \tag{2.17}$$
as well as
$$\rho_V{}'' = \Big(-2P\mathcal{H}[\rho_V]' - V'' + V'^2 + 4P^2\mathcal{H}[\rho_V]^2 + 4PV'\mathcal{H}[\rho_V]\Big)\rho_V \,. \tag{2.18}$$

2.2.2 Concentration inequality

We now use an idea introduced by [MMS14] and based on a comparison between a configuration $\underline{x} = (x_1, \dots, x_N)$ sampled with $\mathbb{P}_N^{V,P}$ and a regularized version $\underline{y} = (y_1, \dots, y_N)$, which we describe here.

Definition 2.2.2 *Let $\underline{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and suppose (up to reordering) that $x_1 \le x_2 \le \dots \le x_N$. We define $\underline{y} \in \mathbb{R}^N$ by:*

$$y_1 \overset{\text{(def)}}{=} x_1 \qquad \text{and} \qquad \forall k \in [\![0, N-1]\!], \quad y_{k+1} \overset{\text{(def)}}{=} y_k + \max\Big\{x_{k+1} - x_k, e^{-(\log N)^2}\Big\} \,.$$

We denote by $L_N^{(y)} \overset{\text{(def)}}{=} \dfrac{1}{N}\displaystyle\sum_{a=1}^n \delta_{y_a}$ and also define $L_{N,u}^{(y)} \overset{\text{(def)}}{=} L_N^{(y)} * \mathcal{U}_N$ the convolution between $L_N^{(y)}$ and $\mathcal{U}_N$ the uniform measure on $\Big[0, N^{-2}e^{-(\log N)^2}\Big]$.

Note that the configuration $\underline{y}$ given by the previous definition satisfies $y_{k+1} - y_k \ge e^{-(\log N)^2}$, and $\underline{y}$ is close to $\underline{x}$ in the sense that

$$\sum_{k=1}^N |x_k - y_k| \le N^2 e^{-(\log N)^2} \,. \tag{2.19}$$

One can note that we have $|x_k - y_k| = y_k - x_k \le (k-1)e^{-(\log N)^2}$, and we get (2.19) by summing these inequalities. As in the proof of [DGM23, Theorem 1.5], we obtain a bound on the density:

Theorem 2.2.3 *For all $N \ge 1$ and $\underline{x} = \big(x_1, \dots, x_N\big) \in \mathbb{R}^N$,*

$$p_N^{V,P}\big(\underline{x}\big) \le \exp\Big(-NPD^2\Big(L_{N,u}^{(y)}, \mu_V\Big) + K_V + 2P(\log N)^2\Big) \prod_{i=1}^N \rho_V(x_i) \tag{2.20}$$

where $K_V \overset{\text{(def)}}{=} 2P\|\mathcal{H}[\rho_V]\|_\infty + C + P\Big|\displaystyle\iint_{\mathbb{R}^2} \log|x-y| d\mu_V(x) d\mu_V(y)\Big|$ for some fixed, V-independent constant C and with D as given in (2.15).

Note that we have to keep the dependance on V in all of the constants involved in our problem.

2.2.3 A priori bound on linear statistics

Thanks to the bound given in Theorem 2.2.3, we can prove the below *a priori* bound on the linear statistics. This bound is *a priori* in the sense that it is not optimal, namely, we will show later that for the n-linear statistics are $O(N^{-\lceil n/2 \rceil})$ versus $O(N^{-n(1-\varepsilon)/2})$ as predicted by the a priori bound. Anyway, this will allow us to neglect, in the loop equations, the terms that are integrated with respect to $\bigotimes^n \mathcal{L}_N$, where $\mathcal{L}_N \overset{(\text{def})}{=} L_N - \mu_V$. The following theorem and its proof are just adaptations of [BGK16, Corollary 3.1.10].

Theorem 2.2.4 (A priori bound on linear statistics) *Let $\varepsilon > 0$, there exits $C_{n,\varepsilon} > 0$ such that for all f in $W_1^\infty(\mathbb{R}^n) \cap H^{n/2}(\mathbb{R}^n)$, it holds*

$$\left| \langle f \rangle_{\bigotimes^n \mathcal{L}_N} \right| \leq \frac{C_{n,\varepsilon} e^{K_V}}{N^{\frac{n}{2}(1-\varepsilon)}} \left(\|f\|_{W_1^\infty(\mathbb{R}^n)} + \|f\|_{H^{n/2}(\mathbb{R}^n)} \right).$$

where K_V is defined in Theorem 2.2.3.

Proof We use the decomposition $\mathcal{L}_N = \left(L_N - L_{N,u}^{(y)}\right) + \mathcal{L}_{N,u}^{(y)}$ where $\mathcal{L}_{N,u}^{(y)} = L_{N,u}^{(y)} - \mu_V$ and obtain:

$$\langle f \rangle_{\otimes^n \mathcal{L}_N} = \sum_{l=0}^{n-1} \sum_{i_1 < \cdots < i_l}^{n} \mathbb{E}_N^{V,P} \left[\int_{\mathbb{R}^n} f(\xi_1, \ldots, \xi_n) \prod_{a=1}^{l} d\mathcal{L}_{N,u}^{(y)}(\xi_{i_a}) \prod_{\substack{a=1 \\ \neq i_1, \ldots, i_l}}^{n} d\left(L_N - L_{N,u}^{(y)}\right)(\xi_a) \right] + \langle f \rangle_{\otimes^n \mathcal{L}_{N,u}^{(y)}}. \quad (2.21)$$

Since the x_i's are not far from the y_i's, we have the following bound by the mean value theorem and the fact that all the involved measures are probability measures:

$$\mathbb{E}_N^{V,P} \left[\int_{\mathbb{R}^n} f(\xi_1, \ldots, \xi_n) \prod_{a=1}^{l} d\mathcal{L}_{N,u}^{(y)}(\xi_{i_a}) \prod_{\substack{a=1 \\ \neq i_1, \ldots, i_l}}^{n} d\left(L_N - L_{N,u}^{(y)}\right)(\xi_a) \right] \leq C_n \|f\|_{W_1^\infty(\mathbb{R}^n)} N e^{-(\log N)^2} \quad (2.22)$$

for some constant $C_n > 0$ only depending on n.

Let's focus now on $\langle f \rangle_{\otimes^n \mathcal{L}_{N,u}^{(y)}}$. We know by Theorem 2.2.3 that

$$\mathbb{P}_N^{V,P}(\Omega_N) = e^{K_V} O\left(e^{-cN^\varepsilon}\right) \qquad \text{where} \qquad \Omega_N \overset{(\text{def})}{=} \left\{ \lambda \in \mathbb{R}^N,\ D^2\left[L_{N,u}^{(y)}, \mu_V\right] > \frac{1}{N^{1-\varepsilon}} \right\}$$

for some $c > 0$ independent of V and for a remainder controlled V-independently. It ensures that:

$$\left| \langle f \rangle_{\otimes^n \mathcal{L}_{N,u}^{(y)}} \right| \leq C e^{K_V} e^{-cN^\varepsilon} \|f\|_{W_0^\infty(\mathbb{R}^n)} + \mathfrak{R}_N[f]$$

where

$$\mathfrak{R}_N[f] \overset{(\text{def})}{=} \mathbb{E}_N^{V,P} \left[\mathbf{1}_{\Omega_N^c} \int_{\mathbb{R}^n} f(\xi_1, \ldots, \xi_n) d\mathcal{L}_{N,u}^{(y)\,\otimes^n}(\xi_1, \ldots, \xi_n) \right].$$

By Plancherel formula and Cauchy-Schwarz inequality, one gets:

$$\begin{aligned}\mathfrak{R}_N[f] &= \mathbb{E}_N^{V,P}\left[\mathbf{1}_{\Omega_N^c}\int_{\mathbb{R}^n}\mathcal{F}[f](\varphi_1,\dots,\varphi_n)\prod_{a=1}^n\mathcal{F}\left[\mathcal{L}_{N,u}^{(y)}\right](-\varphi_a)\frac{d^n\varphi}{(2\pi)^n}\right] \\ &\leq \left(\int_{\mathbb{R}^n}|\mathcal{F}[f](\varphi_1,\dots,\varphi_n)|^2\prod_{a=1}^n|\varphi_a|\frac{d^n\varphi}{(2\pi)^n}\right)^{1/2}.\mathbb{E}_N^{V,P}\left[\mathbf{1}_{\Omega_N^c}2^{\frac{n}{2}}D^n\left[L_{N,u}^{(y)},\mu_V\right]\right] \\ &\leq \frac{2^{\frac{n}{2}}}{N^{\frac{n}{2}(1-\varepsilon)}}\left(\int_{\mathbb{R}^n}|\mathcal{F}[f](\varphi_1,\dots,\varphi_n)|^2\left\{1+\left(\sum_{a=1}^n|\varphi_a|^2\right)^{1/2}\right\}^n\frac{d^n\varphi}{(2\pi)^n}\right)^{1/2} \\ &\leq 2^{\frac{n}{2}}\frac{\|f\|_{H^{n/2}(\mathbb{R}^n)}}{N^{\frac{n}{2}(1-\varepsilon)}}\end{aligned} \tag{2.23}$$

which concludes the proof. □

2.3 Properties and control of the operators involved

In this section, we will set some definitions of operators which arise as building blocks of the loop equations. After defining them, we will prove their continuity on appropriate spaces. This will ultimately allow to apply the *a priori* bounds given in Theorem 2.2.4.

2.3.1 Definitions

The operators that will appear in the loop equations at level $n \geq 2$ will be constructed *via* the following extension procedure, allowing one to extend operators acting on l variables into operators acting on $n+l$ variables.

Definition 2.3.1 (Extension of operators) *Given an operator $\mathcal{O}$ that acts on functions of one variable and yields a function of l variables, ϕ a function of n variables, we define $\mathcal{O}_1$ by:*

$$\mathcal{O}_1[\phi](\xi_1,\dots,\xi_{n+l-1}) = \mathcal{O}\left[\phi(.,\xi_{l+1},\dots,\xi_{n+l-1})\right](\xi_1,\dots,\xi_l) \tag{2.24}$$

2.3.2 Control on the non-commutative derivative operator

A first example of an operator appearing in the loop equations is the non-commutative derivative (NCD) operator.

Definition 2.3.2 *Let $f \in \mathcal{C}^1(\mathbb{R})$, we define the NCD operator $\mathcal{D}[f]$ by:*

$$\forall x,y\in\mathbb{R},\ \mathcal{D}[f](x,y) = \begin{cases}\dfrac{f(x)-f(y)}{x-y} & \text{if } x\neq y\\ f'(x) & \text{if } x=y\end{cases}.$$

In the following, $p \geq 2$ is fixed.

Theorem 2.3.3 (Control for the NCD operator) *Let $n \geq 1$, there exists $C(n) > 0$ such that for all $f \in \mathcal{C}^n(\mathbb{R}^{p-1}) \cap H^{n+1}(\mathbb{R}^{p-1})$,*

$$\|\mathcal{D}_1[f]\|_{H^n(\mathbb{R}^p)} \leq C(n)\|f\|_{H^{n+1}(\mathbb{R}^{p-1})}.$$

Before showing this inequality, we need to show a general form of the derivatives of $\mathcal{D}_1[f]$.

Lemma 2.3.4 (General form for derivatives of $\mathcal{D}_1[f]$) *Let $\underline{m} = (m_1,\dots,m_p) \in \mathbb{N}^p$ satisfy $m_1 \geq m_2$ and $\sum_{i=1}^p m_i \leq n$. Let $x_1,\dots,x_p \in \mathbb{R}$ be such that $x_1 \neq x_2$, then one has:*

$$\partial^{\underline{m}}\mathcal{D}_1[f](x_1,\dots,x_n) = \sum_{j=0}^{m_2}C_{m_1,m_2,j}\frac{\left(g^{(j)}(x_2) - \sum_{k=0}^{m_1-j}\frac{g^{(k+j)}(x_1)}{k!}(x_2-x_1)^k\right)}{(x_2-x_1)^{m_1+m_2+1-j}} \tag{2.25}$$

with $C_{m_1,m_2,j} \overset{(\text{def})}{=} \binom{m_2}{j}(m_1+m_2-j)!(-1)^{m_2-j}$ *and* $g = \partial_2^{m_3}\dots\partial_{p-1}^{m_p} f(.,x_3,\dots,x_p)$.

Proof First, it is easy to verify that $\mathcal{D}_1[f] \in \mathcal{C}^n(\mathbb{R}^p)$ for $x_1 \neq x_2$. Secondly, when $n \geq 2$, by the Schwarz theorem, the order of the partial derivatives does not matter. It is only the derivatives with respect to x_1 and x_2 that are non-trivial to compute. Indeed, let $\underline{x} = (x_1,\dots,x_p) \in \mathbb{R}^p$ be such that $x_1 \neq x_2$, then

$$\partial_3^{m_3}\dots\partial_p^{m_p}\mathcal{D}_1[f](x_1,\dots,x_p) = \frac{g(x_1)-g(x_2)}{x_1-x_2}$$

with $g \overset{(\text{def})}{=} \partial_2^{m_3}\dots\partial_p^{m_p} f(.,x_3,\dots,x_p)$. By applying the Leibniz formula when differentiating m_1 times with respect to x_1, one gets:

$$\partial_{x_1}^{m_1}\partial_{x_3}^{m_3}\dots\partial_{x_p}^{m_p}\mathcal{D}_1[f](x_1,\dots,x_p) = \frac{m_1!}{(x_2-x_1)^{m_1+1}}\left(g(x_2) - \sum_{k=0}^{m_1}\frac{g^{(k)}(x_1)}{k!}(x_2-x_1)^k\right).$$

Again, we differentiate m_2 times with respect to x_2 and apply the Leibniz formula to get (2.25).□

We proved Lemma 2.3.4 for $m_1 \geq m_2$. Since $\mathcal{D}_1[f]$ is symmetric under the exchange of the two first variables, we can always assume that $m_1 \geq m_2$. The idea of the proof of Theorem 2.3.3 is to prove separately the L^2 control on $\partial^{\underline{m}}\mathcal{D}_1[f]$ close to the singularity (the diagonal) and far from it. To do so, we will use the Taylor formula with integral remainder to deal with the singularity and Lemma 2.3.4 when we are at a fixed distance from the diagonal.

Proof (of Theorem 2.3.3) Let $\underline{m} \overset{(\text{def})}{=} (m_1,\dots,m_p) \in \mathbb{N}^p$ be such that $m \overset{(\text{def})}{=} \sum_{i=1}^p m_i \leq n$. Without loss of generality, we can assume that $m_1 \geq m_2$. Let's show that

$$\left\|\partial_1^{m_1}\partial_2^{m_2}\dots\partial_p^{m_p}\mathcal{D}_1[f]\right\|_{L^2(\mathbb{R}^p)} \leq C\|f\|_{H^{m+1}(\mathbb{R}^{p-1})}$$

with $C>0$ independent of f. We first show this inequality on the subspace $\{\underline{x} \in \mathbb{R}^n, |x_1-x_2| \leq 1\}$. First note that

$$\mathcal{D}_1[f](x_1,x_2,\dots,x_p) = \int_0^1 \partial_1 f\left(x_1+t(x_2-x_1),x_3,\dots,x_p\right)dt$$

an so by differentiating under the integral sign and by Jensen's inequality, we obtain:

$$\left|\partial_1^{m_1}\partial_2^{m_2}\dots\partial_p^{m_p}\mathcal{D}_1[f](x_1,x_2,\dots,x_p)\right|^2 \leq \int_0^1 (1-t)^{2m_1}t^{2m_2}\partial_1^{m_1+m_2+1}\partial_2^{m_3}\dots\partial_{p-1}^{m_p} f\left(x_1+t(x_2-x_1),x_3,\dots,x_p\right)^2 dt.$$

Hence, by integrating with respect to $\underline{x}$, changing x_2-x_1 into $\widetilde{x_2}$, and using Fubini, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\partial_1^{m_1}\partial_2^{m_2}\dots\partial_p^{m_p}\mathcal{D}_1[f](x_1,x_2,\dots,x_p)\right|^2\mathbf{1}_{|x_2-x_1|<1}d^nx\\
&\leq \int_0^1 dt(1-t)^{2m_1}t^{2m_2}\int_{-1}^1 d\widetilde{x_2}\int_{\mathbb{R}}dx_1\int_{\mathbb{R}}dx_3\dots\int_{\mathbb{R}}dx_p\partial_1^{m_1+m_2+1}\partial_2^{m_3}\dots\partial_{p-1}^{m_p}f(x_1+t\widetilde{x_2},x_3,\dots,x_p)^2\\
&\leq C(m_1,m_2)\|\partial_1^{m_1+m_2+1}\partial_2^{m_3}\dots\partial_{p-1}^{m_p}f\|^2_{L^2(\mathbb{R}^{p-1})}\\
&\leq C(m_1,m_2)\|f\|^2_{H^{m+1}(\mathbb{R}^{p-1})}.
\end{aligned}$$

Now we deal with the subset $\{\underline{x} \in \mathbb{R}^n, |x_1-x_2| \geq 1\}$. By Jensen's inequality and Lemma 2.3.4, we

get:

$$\int_{\mathbb{R}} dx_1 \int_{|x_2-x_1|>1} dx_2 \left|\partial_1^{m_1}\partial_2^{m_2}\dots\partial_p^{m_p}\mathcal{D}_1[f](x_1,x_2,\dots,x_p)\right|^2 \leq (m_2+1)\sum_{j=0}^{m_2} C^2_{m_1,m_2,j}$$
$$\times \int_{\mathbb{R}} dx_1 \int_{|x_2-x_1|>1} dx_2 \frac{\left(g^{(j)}(x_2) - \sum_{k=0}^{m_1-j} \frac{g^{(k+j)}(x_1)}{k!}(x_2-x_1)^k\right)^2}{|x_2-x_1|^{2m_1+2m_2+2-2j}}$$

with $g = \partial_2^{m_3}\dots\partial_{p-1}^{m_p} f(.,x_3,\dots,x_p)$. Again by Jensen's inequality, we get:

$$\int_{\mathbb{R}} dx_1 \int_{|x_2-x_1|>1} dx_2 \left|\partial_1^{m_1}\partial_2^{m_2}\dots\partial_p^{m_p}\mathcal{D}_1[f](x_1,x_2,\dots,x_p)\right|^2 \leq (m_2+1)\sum_{j=0}^{m_2} C^2_{m_1,m_2,j}(m_1-j+1)$$
$$\times \int_{\mathbb{R}} dx_1 \int_{|x_2-x_1|>1} dx_2 \frac{g^{(j)}(x_2)^2 + \sum_{k=0}^{m_1-j} \frac{g^{(k+j)}(x_1)^2}{k!^2}(x_2-x_1)^{2k}}{|x_2-x_1|^{2m_1+2m_2+2-2j}}.$$

For all $j \in [\![0, m_2]\!]$, the double integral in the last line can be estimated with another constant $C(m_1,m_2)$ depending only on m_1 and m_2. For that, we use Fubini's theorem:

$$\int_{\mathbb{R}} dx_1 \int_{|x_2-x_1|>1} dx_2 \frac{g^{(j)}(x_2)^2 + \sum_{k=0}^{m_1-j} \frac{g^{(k+j)}(x_1)^2}{k!^2}(x_2-x_1)^{2k}}{|x_2-x_1|^{2m_1+2m_2+2-2j}}$$
$$= \int_{\mathbb{R}} dx_2 g^{(j)}(x_2)^2 \int_{|x_2-x_1|>1} \frac{dx_1}{|x_2-x_1|^{2m_1+2m_2+2-2j}}$$
$$+ \sum_{k=0}^{m_1-j} \frac{1}{k!^2}\int_{\mathbb{R}} dx_1 g^{(k+j)}(x_1)^2 \int_{|x_2-x_1|>1} \frac{dx_2}{|x_2-x_1|^{2m_1+2m_2+2-2(j+k)}} \leq C(m_1,m_2)\left\|g\right\|^2_{H^{m_1}(\mathbb{R})}.$$

Hence, after suming over j and changing the constant appropriately, we integrate over $x_3,\dots,x_p$ to obtain:

$$\int_{\mathbb{R}^{p-2}} dx_3\dots dx_p \int_{\mathbb{R}} dx_1 \int_{|x_2-x_1|>1} dx_2 \left|\partial_1^{m_1}\partial_2^{m_2}\dots\partial_p^{m_p}\mathcal{D}_1[f](x_1,x_2,\dots,x_p)\right|^2$$
$$\leq C(m_1,m_2)\sup_{l\in[\![1,m_1]\!]}\left\|\partial_1^l\dots\partial_{p-1}^{m_p} f\right\|^2_{L^2(\mathbb{R}^{p-1})} \leq C(m_1,m_2)\left\|f\right\|^2_{H^{m+1}(\mathbb{R}^{p-1})}.$$

This is enough to conclude. □

Since in Theorem 2.2.4, the bound on the linear statistic involves the $W_1^\infty(\mathbb{R}^p)$-norm, we state the following result.

Proposition 2.3.5 *There exists a $C>0$ such that for all $f \in \mathcal{C}^n(\mathbb{R}^{p-1}) \cap W_n^\infty(\mathbb{R}^{p-1})$,*

$$\|\mathcal{D}_1[f]\|_{W_n^\infty(\mathbb{R}^p)} \leq C(n)\|f\|_{W_{n+1}^\infty(\mathbb{R}^{p-1})}.$$

Proof This follows from Lemma 2.3.4 together with the Taylor formula with integral remainder.□

2.4 Control on the master operator Ξ

In this section, we study the so-called master operator which will play an essential role in the following. Indeed, proving continuity of this operator is a crucial step if one wants to analyze the loop equations.

2.4.1 Definition

We recall the definition of the operator $\mathcal{L}$.

Definition 2.4.1 *We define, for a sufficiently smooth function f, the operator*

$$\mathcal{L}[f] \overset{(\text{def})}{=} \Xi\big[f'\big] = -\mathcal{A}[f] - 2P\mathcal{W}[f]$$

where

$$\mathcal{A}[f] \overset{(\text{def})}{=} -\frac{(f'\rho_V)'}{\rho_V}, \qquad \mathcal{W}[f] \overset{(\text{def})}{=} -\mathcal{H}\left[f'\rho_V\right] + \int_{\mathbb{R}} \mathcal{H}\left[f'\rho_V\right](y) d\mu_V(y).$$

$\mathcal{L}$ is an unbounded operator on the space H defined in (2.7). This space is indeed a Hilbert-space by the fact that μ_V verifies the Poincaré inequality (see assumption *(iii)*). Its domain is defined by $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A}) \overset{(\text{def})}{=} \{u \in \mathsf{H},\, \mathcal{A}[u] \in \mathsf{H}\}$ by [DGM23, Theorem 6.7]. We show that the functions belonging to this set are smooth.

Lemma 2.4.2 (Regularity of the inverse of the derivative) *[DGM23, Theorem 7.1] Let $v \in \mathcal{D}(\mathcal{L})$, then $v' \in \mathcal{C}^1(\mathbb{R})$.*

For the next theorem, we recall that $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathsf{H}$ is a diagonalizable operator with positive countable spectrum. We denote by $\lambda_1(\mathcal{A}) > 0$ its smallest eigenvalue. This quantity has a role in our problem since for all $f \in \mathsf{H}$, $\|\mathcal{L}^{-1}[f]\|_{\mathsf{H}} \leq \lambda_1(\mathcal{A})^{-1/2}\|f\|_{\mathsf{H}}$ see [DGM23, Theorem 6.7].

Theorem 2.4.3 (Inversion of the master operator) $\Xi : \mathfrak{D}(\Xi) \longrightarrow \mathsf{H}$ *is invertible, of inverse defined for all $g \in \mathsf{H}$ by:*

$$\Xi^{-1}[g] \overset{(\text{def})}{=} \big(\mathcal{L}^{-1}[g]\big)'$$

where $\mathfrak{D}(\Xi) \overset{(\text{def})}{=} \big\{f \in \mathcal{C}^0(\mathbb{R}),\, \exists v \in \mathfrak{D}(\mathcal{L}),\, f = v'\big\}$. Furthermore for all $f \in \mathsf{H}$,

$$\big\|\Xi^{-1}[f]\big\|_{L^2(\mu_V)} \leq C_{\mathcal{L}}\|f'\|_{L^2(\mu_V)} \tag{2.26}$$

where $C_{\mathcal{L}} \overset{(\text{def})}{=} \lambda_1(\mathcal{A})^{-1/2}$.

Proof To prove that Ξ is invertible on $\mathfrak{D}(\Xi)$, the only thing to prove is that for all $v \in \mathfrak{D}(\mathcal{L})$, $v' \in \mathcal{C}^0(\mathbb{R})$ which is true by Lemma 2.4.2. The estimate comes from the fact that given $f \in \mathsf{H}$, one has $\Xi^{-1}[f] = \big(\mathcal{L}^{-1}[f]\big)'$. Then

$$\big\|\Xi^{-1}[f]\big\|_{L^2(\mu_V)} = \big\|\mathcal{L}^{-1}[f]\big\|_{\mathsf{H}} \leq C_{\mathcal{L}}\|f\|_{\mathsf{H}}. \qquad \square$$

The crucial step when one wants to analyze the loop equations, is to obtain controls on the master operator which we will show in this section. These bounds will allow us to apply the bound obtained in Theorem 2.2.4 to functions like $\Xi^{-1}[\phi]$.

2.4.2 Preliminaries

We define an operator $\mathcal{O}$ whose iterations will appear in the derivatives of the inverse of the master operator (which exists because of Lemma 2.4.8).

Definition 2.4.4 *Let $\mathcal{O}$ be the operator defined on smooth enough functions by:*

$$\mathcal{O}[f](x) \overset{(\text{def})}{=} \left(\frac{f\rho_V}{\rho_V'}\right)'(x) \qquad x - a\text{-}e \tag{2.27}$$

In order to give a more precise description of $\mathcal{O}^k$, which will allow us to analyse its asymptotics at infinity, we need the following definition.

Definition 2.4.5 (Differential degree) *Let f be a function of one variable defined on $\mathbb{R}$ differentiable n times, we define the differential degree denoted by d^f_∂ with respect to f by*

$$d^f_\partial\left(\prod_{k=0}^{n}\left(f^{(k)}\right)^{\alpha_k}\right) \overset{(\text{def})}{=} \sum_{k=0}^{n} k\alpha_k$$

For example the differential degree with respect to f of $(f')^2$ and $f''f$ is 2, while $d^f_\partial\left(\left(f^{(3)}\right)^2\right) = 6$.

Remark 2.4.6 For example, with $\alpha \overset{(\text{def})}{=} \frac{\rho_V}{\rho_V'}$, it holds that:

- $\mathcal{O}[f] = \alpha' f + \alpha f'$.
- $\mathcal{O}^2[f] = (\alpha\alpha')' f + 3\alpha\alpha' f' + \alpha^2 f''$.
- $\mathcal{O}^3[f] = \left(\alpha\left(\alpha\alpha'\right)'\right)' f + \left(4\alpha^2\alpha'' + 7\alpha\alpha'^2\right) f' + \left(6\alpha'\alpha^2\right) f'' + \alpha^3 f^{(3)}$.

Using the notion of differential degree, we are now able to state the next theorem.

Theorem 2.4.7 *Let $k \geq 1$, $f \in \mathcal{C}^k(\mathbb{R})$, there exists a family of polynomials $(P^k_a)_{0\leq a\leq k}$ such that*

$$\mathcal{O}^k[f] = \sum_{j=0}^{k} f^{(k-j)} P^k_j(\alpha, \ldots, \alpha^{(j)}), \qquad \text{with} \qquad \alpha \overset{(\text{def})}{=} \frac{\rho_V}{\rho_V'} \tag{2.28}$$

In fact, $P^k_j\left(\alpha, \ldots, \alpha^{(j)}\right)$, $j \in [\![0, k]\!]$, is the unique homogeneous polynomial in $j+1$ variables, with differential degree with respect to α equal to j, degree k and with coefficients independent of V satisfying the following reccurence relations:

- $P^{k+1}_0(\alpha) = \alpha P^k_0(\alpha) = \alpha^{k+1}$
- $\forall j \in [\![1, k]\!]$, $P^{k+1}_j\left(\alpha, \ldots, \alpha^{(j)}\right) = \left(\alpha P^k_{j-1}(\alpha, \ldots, \alpha^{(j-1)})\right)' + \alpha P^k_j\left(\alpha, \ldots, \alpha^{(j)}\right)$
- $P^{k+1}_{k+1}\left(\alpha, \ldots, \alpha^{(k+1)}\right) = \left(\alpha P^k_k\left(\alpha, \ldots, \alpha^{(k)}\right)\right)' = \left((\alpha'\alpha)' \ldots \alpha\right)'$

Proof Let's prove it by induction. For $k = 1$, $\mathcal{O}[f] = \alpha' f + \alpha f'$ and so by setting $P^1_0(\alpha) = \alpha$, which is homogeneous, of degree 1 and of differential degree 0, and $P^1_1(\alpha, \alpha') = \alpha'$ which is of degree 1 and differential degree 1, this proves the claim. Suppose that (2.28) holds at rank $k \in \mathbb{N}^*$, then:

$$\begin{aligned}
\mathcal{O}^{k+1}[f] = \left(\alpha\mathcal{O}^k[f]\right)' &= \sum_{j=0}^{k} f^{(k-j)}\left[\alpha P^k_j(\alpha, \ldots, \alpha^{(j)})\right]' + \sum_{j=0}^{k} f^{(k-j+1)} \alpha P^k_j\left(\alpha, \ldots, \alpha^{(j)}\right) \\
&= \left[\alpha P^k_k\left(\alpha, \ldots, \alpha^{(k)}\right)\right]' f + \sum_{j=0}^{k-1} f^{(k-j)}\left\{\left[\alpha P^k_j\left(\alpha, \ldots, \alpha^{(j)}\right)\right]' + \alpha P^k_{j+1}\left(\alpha, \ldots, \alpha^{(j+1)}\right)\right\} \\
&\qquad + \alpha P^k_0(\alpha) f^{(k+1)}
\end{aligned}$$

Hence by setting $P^{k+1}_0(\alpha) \overset{(\text{def})}{=} \left(\alpha P^k_0(\alpha)\right)'$, $P^{k+1}_{k+1}\left(\alpha, \ldots, \alpha^{(k+1)}\right) \overset{(\text{def})}{=} \left(\alpha P^k_k\left(\alpha, \ldots, \alpha^{(k)}\right)\right)'$ and for all $j \in [\![0, k-1]\!]$, $P^{k+1}_{j+1}\left(\alpha, \ldots, \alpha^{(j+1)}\right) \overset{(\text{def})}{=} \left(\alpha P^k_j(\alpha, \ldots, \alpha^{(j)})\right)' + \alpha P^k_j\left(\alpha, \ldots, \alpha^{(j)}\right)$, we obtain the desired form of (2.28) and the recurrence relations. It remains to check that the homogeneity and degree conditions hold at rank $k+1$. This follows from the recurrence relations for the P^k_j's. □

2.4.3 Closed form for Ξ^{-1}

Before showing a closed form for the derivatives of $\Xi_1^{-1}[f]$ and their L^2 properties, we first prove that, if f is sufficiently smooth, they indeed exist.

Lemma 2.4.8 (Regularity of the inverse) *Let $f \in \mathsf{H}$ such that $f\rho_V \in H^n(\mathbb{R})$ with $n \geq 2$, then $\rho_V \Xi^{-1}[f] \in H^{n+1}(\mathbb{R})$. Furthermore if $f \in \mathsf{H} \cap \mathcal{C}^n(\mathbb{R})$ is such that $f\rho_V \in H^n(\mathbb{R})$, then one also has $\Xi^{-1}[f] \in \mathcal{C}^{n+1}(\mathbb{R})$.*

Note that the last condition is verified whenever f and its derivatives are continuous and grow slower than e^{-V} at infinity. The proof uses the operators $\mathcal{L}$ and $\mathcal{A}$ introduced in Definition 2.4.1.

Proof We recall that $\Xi^{-1}[f] = \left(\mathcal{L}^{-1}[f]\right)'$. When $f \in \mathsf{H}$, we know that $\rho_V \left(\mathcal{L}^{-1}[f]\right)' \in H^2(\mathbb{R})$. This is because $\mathcal{L}^{-1}[f] \in \mathcal{D}(\mathcal{L}) \subset \{u \in \mathsf{H},\, \mathcal{A}[u] \in \mathsf{H}\}$ and $\left(\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right)' = \rho_V \mathcal{A}\left[\mathcal{L}^{-1}[f]\right] \in H^1(\mathbb{R})$. We want to show that $\rho_V \left(\mathcal{L}^{-1}[f]\right)'' \in H^n(\mathbb{R})$, let's show first that $\mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right] \in H^n(\mathbb{R})$. First observe that

$$\rho_V \mathcal{A}\mathcal{L}^{-1}[f] = -\rho_V f + 2P\rho_V \mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right] - 2P\rho_V \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right](y)dy$$

Hence, since $\rho_V \left(\mathcal{L}^{-1}[f]\right)' \in H^2(\mathbb{R})$, so is $\mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right]$. Moreover, the function

$2P\rho_V \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right](y)dy$ clearly belongs to $H^n(\mathbb{R})$ for all $n \in \mathbb{N}$ hence $\rho_V \mathcal{A} \circ \mathcal{L}^{-1}[f] = \left(\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right)' \in H^2(\mathbb{R})$ and hence $\rho_V \left(\mathcal{L}^{-1}[f]\right)' \in H^3(\mathbb{R})$. By induction, this shows that $\left(\mathcal{L}^{-1}[f]\right)' = \Xi^{-1}[f] \in \dfrac{1}{\rho_V} H^{n+1}(\mathbb{R}) \subset \mathcal{C}^n(\mathbb{R})$ by Sobolev-Hölder embedding theorem and hence we can conclude that $\mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right] \in H^{n+1}(\mathbb{R})$. Since

$$\left(\mathcal{L}^{-1}[f]\right)'' = f - \frac{\rho_V'}{\rho_V}\left(\mathcal{L}^{-1}[f]\right)' - 2P\left(\mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right] - \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \left(\mathcal{L}^{-1}[f]\right)'\right](y)d\mu_V(y)\right) \quad (2.29)$$

and that $\dfrac{\rho_V'}{\rho_V} \in \mathcal{C}^\infty(\mathbb{R})$, we can then conclude that, under the assumption that $f \in \mathcal{C}^n(\mathbb{R})$, $\left(\Xi^{-1}[f]\right)' = \mathcal{L}^{-1}[f] \in \mathcal{C}^n(\mathbb{R})$, hence $\Xi^{-1}[f] \in \mathcal{C}^{n+1}(\mathbb{R})$. □

Lemma 2.4.9 *There exists $M_V > 0$ such that $\forall |x| \geq M_V$, $\left|\dfrac{\rho_V'}{\rho_V}(x)\right| \geq 1$.*

Proof From Lemma 2.9.2, $\mathcal{H}[\rho_V]$ is bounded and by assumption *(ii)*, $V'(x)$ goes to infinity, the conclusion follows from the fact that $\dfrac{\rho_V'}{\rho_V}(x) = -V'(x) - 2P\mathcal{H}[\rho_V]$. □

We are now able to prove that a closed form holds for the derivatives of Ξ^{-1}. The idea is to use the resolvant formula which gives that for all $f \in \mathsf{H}$,

$$\mathcal{L}^{-1}[f] = -\mathcal{A}^{-1}\left[f + 2P\mathcal{W} \circ \mathcal{L}^{-1}[f]\right] \quad (2.30)$$

and for all $x \in \mathbb{R}$,

$$\mathcal{A}^{-1}[f](x) = \frac{1}{\rho_V(x)} \int_x^{\pm\infty} f(t)\rho_V(t)dt. \quad (2.31)$$

It doesn't matter if one chooses $+\infty$ or $-\infty$ in (2.31) since $\int_{\mathbb{R}} f(t)\rho_V(t)dt = 0$ but it will be convenient to make the choice $\mathrm{sgn}(x)\infty$ for reasons that will appear further. Before establishing the continuity for Ξ^{-1}, we need to introduce an operator $\mathcal{X}$ that takes a function in $\dfrac{1}{\rho_V} H^n(\mathbb{R})$ and produces one belonging to $\dfrac{1}{\rho_V} H^n(\mathbb{R}) \cap \mathsf{H}$ by means of a recentring.

Definition 2.4.10 *Let $\phi \in \dfrac{1}{\rho_V} H^n(\mathbb{R})$, we define the operator $\mathcal{X}$ by*

$$\mathcal{X}[\phi](\xi) = \phi(\xi) - \int_{\mathbb{R}} \phi(t)d\mu_V(t).$$

For any $\phi \in \frac{1}{\rho_V} H^n(\mathbb{R})$, it is clear that $\rho_V \mathcal{X}[\phi] \in H^n(\mathbb{R})$ *i.e.* $\mathcal{X}[\phi] \in \frac{1}{\rho_V} H^n(\mathbb{R})$. We denote by

$$\widetilde{\Xi_1} \overset{(\text{def})}{=} \Xi_1 \circ \mathcal{X}_1, \qquad \widetilde{\Xi_1^{-1}} \overset{(\text{def})}{=} \Xi_1 \circ \mathcal{X}_1$$

and, given a general operator $\mathcal{U}$, we adopt the notation $\widetilde{\mathcal{U}}$ for the operator $\mathcal{U} \circ \mathcal{X}$.

Theorem 2.4.11 *Let $f \in \mathcal{C}^n(\mathbb{R}) \cap \left(\frac{1}{\rho_V} H^n(\mathbb{R}) \right)$, for all $|x| > M_V$ with M_V given in Lemma 2.4.9, for all $k \in [\![1, n+1]\!]$ it holds that*

$$\left(\widetilde{\Xi^{-1}}[f] \right)^{(k)} = \sum_{a=0}^{k-1} Q_a^k \left(\theta, \dots, \theta^{(a)} \right) \beta_{k-a}, \qquad \text{where} \qquad \theta \overset{(\text{def})}{=} \frac{\rho_V'}{\rho_V} \tag{2.32}$$

with the convention that $\widetilde{\Xi^{-1}}[f] \overset{(\text{def})}{=} \beta_0$. The β_i's are defined, for all $|x| > M_V$, for all $i \in [\![1, k]\!]$, by:

$$\beta_i(x) \overset{(\text{def})}{=} \frac{-1}{\rho_V(x)} \int_x^{\text{sgn}(x)\infty} dt \rho_V(t) \mathcal{O}^i \left[\mathcal{X}[f] + 2P\mathcal{W} \circ \widetilde{\mathcal{L}^{-1}}[f] \right](t)$$

(see (2.27)). Above Q_a^k denotes the unique homogeneous polynomial in $a+1$ variables with degree $k-a$, with differential degree with respect to θ a and with coefficients independent of V satisfying the following induction relations:

$$Q_0^{k+1}(\theta) = \theta Q_0^k(\theta) = \theta^k \tag{2.33}$$

$$\forall a \in [\![1, k-1]\!], Q_a^{k+1} \left(\theta, \dots, \theta^{(a)} \right) = \theta Q_a^k \left(\theta, \dots, \theta^{(a)} \right) + Q_{a-1}^k \left(\theta, \dots, \theta^{(a)} \right)' \tag{2.34}$$

$$Q_k^{k+1} \left(\theta, \dots, \theta^{(k)} \right) = Q_{k-1}^k \left(\theta, \dots, \theta^{(k-1)} \right)' = \theta^{(k-1)} \tag{2.35}$$

Proof We prove this statement by induction. For $k = 1$, by (2.30) and (2.31), by setting

$$g \overset{(\text{def})}{=} -\mathcal{X}[f] - 2P\mathcal{W} \circ \widetilde{\mathcal{L}^{-1}}[f],$$

we get for all $x \in \mathbb{R}$,

$$\widetilde{\Xi^{-1}}[f](x) = \left(\widetilde{\mathcal{L}^{-1}}[f] \right)'(x) = \left(\mathcal{A}^{-1}[g] \right)'(x) = \frac{1}{\rho_V(x)} \int_x^{\text{sgn}(x)\infty} dt \rho_V(t) g(t) = \beta_0(x). \tag{2.36}$$

For $k = 2$, differentiating again, which is allowed by Lemma 2.4.8, we get for $|x|$ large enough:

$$(\widetilde{\mathcal{L}^{-1}}[f])''(x) = (\mathcal{A}^{-1}[g])''(x) = -g(x) - \frac{\rho_V'}{\rho_V}(x) \left(\mathcal{A}^{-1}[g] \right)'(x).$$

After performing an integration by parts in the last integral, we obtain

$$\left(\widetilde{\Xi^{-1}}[f] \right)'(x) = \left(\widetilde{\mathcal{L}^{-1}}[f] \right)''(x) = \frac{\rho_V'}{\rho_V^2}(x) \int_x^{\text{sgn}(x)\infty} dt \rho_V(t) \left(g \frac{\rho_V}{\rho_V'} \right)'(t) = (\theta \beta_1)(x).$$

By defining $Q_0^1(\theta) \overset{(\text{def})}{=} \theta$, it is readily seen that its degree is 1 and its differential degree with respect to θ is 0. Let $k \in [\![2, n]\!]$ and suppose that (2.32) is true at rank k, then by differentiating we get:

$$\left(\widetilde{\Xi^{-1}}[f] \right)^{(k+1)} = \sum_{a=0}^{k-1} Q_a^k \left(\theta, \dots, \theta^{(a)} \right) \beta_{k-a}' + Q_a^k \left(\theta, \dots, \theta^{(a)} \right)' \beta_{k-a}. \tag{2.37}$$

First, let $i \in [\![1,k]\!]$ and $|x|$ be large enough,

$$\beta_i'(x) = -\mathcal{O}^i[g](x) - \frac{\rho_V'}{\rho_V^2}(x) \int_x^{\mathrm{sgn}(x)\infty} dt \rho_V(t)\mathcal{O}^i[g](t) = \frac{\rho_V'}{\rho_V^2}(x) \int_x^{\mathrm{sgn}(x)\infty} dt \rho_V(t)\mathcal{O}^{i+1}[g](t) = (\theta\beta_{i+1})(x).$$

The second equality follows from an integration by parts and the fact that $\frac{\rho_V'}{\rho_V^2}\rho_V\frac{\rho_V}{\rho_V'}\mathcal{O}^i[g]$ goes to zero at infinity. Hence (2.37) becomes

$$\begin{aligned}\left(\widetilde{\Xi^{-1}}[f]\right)^{(k+1)} &= \sum_{a=0}^{k-1} \theta Q_a^k\left(\theta,\dots,\theta^{(a)}\right)\beta_{k+1-a} + Q_a^k\left(\theta,\dots,\theta^{(a)}\right)'\beta_{k-a} \\ &= \theta Q_0^k(\theta)\beta_{k+1} + \sum_{a=1}^{k-1}\left(\theta Q_a^k\left(\theta,\dots,\theta^{(a)}\right) + Q_{a-1}^k\left(\theta,\dots,\theta^{(a-1)}\right)'\right)\beta_{k+1-a} \\ &\qquad + Q_{k-1}^k\left(\theta,\dots,\theta^{(k-1)}\right)'\beta_1.\end{aligned}$$

By the definitions of $(Q_a^{k+1})_a$, it is clear that (2.32) is true at rank $k+1$. The fact that Q_a^{k+1} are homogeneous and have degree $k-a$ and differential degree a can be checked directly from the induction relations (2.33), (2.34), (2.35). □

Remark 2.4.12 When $V(x) = x^m$ with m even, it can be checked from (2.17) that for every $a \in [\![0,n-2]\!]$, $Q_a^n(\theta,\dots,\theta^{(a)})$ is of the form $c_{a,n}x^{m(n-1-a)-(n-1)} + T_a^n(x) + R_a^n\left(x,\mathcal{H}[\rho_V],\dots,\mathcal{H}\left[\rho_V^{(a)}\right]\right)(x)$ where $c_{a,n}$ is a real number, T_a^n is polynomial of degree strictly lower than $m(n-1-a)-(n-1)$ and R_a^n is also a polynomial of degree greater than 1. Since all these Hilbert transform vanish at infinity, such a polynomial expression goes to zero at infinity. This decomposition holds as long as the degree of the monomial is non-negative, otherwise it is zero. We give the first decompositions for $\left(\widetilde{\Xi^{-1}}[f]\right)^{(k)}$ for $k \in [\![0,3]\!]$:

$$\widetilde{\Xi^{-1}}[f] = \beta_0, \qquad \left(\widetilde{\Xi^{-1}}[f]\right)' = \theta\beta_1, \qquad \left(\widetilde{\Xi^{-1}}[f]\right)'' = \theta'\beta_1 + \theta^2\beta_2,$$

and

$$\left(\widetilde{\Xi^{-1}}[f]\right)^{(3)} = \theta''\beta_1 + \left(\theta\theta' + (\theta^2)'\right)\beta_2 + \theta^3\beta_3.$$

With the choice of potential $V(x) = x^m$ with m even, choosing a bounded function f with bounded derivatives at all orders and integrating by parts, it holds that for all$k \geq 0$, there exists $\gamma_0^{(k)},\dots,\gamma_k^{(k)} \in \mathbb{R}$,

$$|\beta_k(x)| \underset{|x|\to\infty}{\sim} \frac{\mathcal{O}^k[g](x)}{x^{m-1}} \underset{|x|\to\infty}{\sim} \frac{1}{x^{m-1}}\sum_{j=0}^{k} g^{(j)}(x)\left(\frac{\gamma_j^{(k)}}{x^{km-j}} + \underset{|x|\to\infty}{o}\left(\frac{1}{x^{km-j}}\right)\right).$$

When $V(x) = \cosh(\alpha x)$, by the same computation, we get for different $\gamma_j^{(k)}$

$$|\beta_k(x)| \underset{|x|\to\infty}{\sim} e^{-\alpha|x|}\sum_{j=0}^{k}\gamma_j^{(k)} g^{(j)}(x)\left(e^{-k\alpha|x|} + \underset{|x|\to\infty}{o}\left(e^{-k\alpha|x|}\right)\right).$$

2.4.4 Controls on the inverse of the master operator

Since we are going to use the polynomials, P_j^k and Q_j^k defined previously in Theorem 2.4.7 and 2.4.11, a lot in our estimates on $\widetilde{\Xi_1^{-1}}$, we first need the following lemma. With $\alpha = \frac{\rho_V}{\rho_V'}\theta^{-1}$, the following result holds.

Lemma 2.4.13 *For all $k \geq 1$, for all $j \in [\![0,k]\!]$*

(i) $P_j^k\left(\alpha,\dots,\alpha^{(j)}\right)(x) = \underset{|x|\to\infty}{O}\left(V'(x)^{-k}\right)$,

(ii) $P_j^k\left(\alpha,\dots,\alpha^{(j)}\right)'(x) = \underset{|x|\to\infty}{O}\left(V'(x)^{-k}\right)$,

(iii) $Q_j^k\left(\theta,\dots,\theta^{(j)}\right)(x) = \underset{|x|\to\infty}{O}\left(V'(x)^{k-j}\right)$.

Proof For *i)*, by the Faà di Bruno's formula, for all $n \geq 0$,

$$\alpha^{(n)} = \left(\frac{-1}{V' + 2P\mathcal{H}[\rho_V]}\right)^{(n)} = -\sum_{\lambda\vdash n} \frac{(-1)^{|\lambda|}|\lambda|!}{(V' + 2P\mathcal{H}[\rho_V])^{|\lambda|+1}} \prod_{i=1}^{n} \frac{\left(V^{(i+1)} + 2P\mathcal{H}[\rho_V]^{(i)}\right)^{\lambda_i}}{\lambda_i!(i!)^{\lambda_i}}.$$

where the sum is over $\lambda \overset{(\text{def})}{=} (\lambda_1,\dots,\lambda_n)$ such that $\sum_{i=1}^n i\lambda_i = n$. From assumption *v)* and Lemma 2.9.2, we see that

$$|\alpha^{(n)}(x)| \leq \sum_{\lambda\vdash n} C_\lambda \underset{|x|\to\infty}{O}\left(V'(x)^{-1}\right) = \underset{|x|\to\infty}{O}\left(V'(x)^{-1}\right).$$

Hence P_j^k, as a homogeneous polynomial in $\left(\alpha,\dots,\alpha^{(j)}\right)$ of degree k, is a $\underset{|x|\to\infty}{O}\left(V'(x)^{-k}\right)$.

For the point *ii)*, one has to notice that for each monomial

$$A_n \overset{(\text{def})}{=} \left[\prod_{i=1}^{n}\left(\alpha^{(i)}\right)^{l_i}\right]' = \sum_{j=1}^{n} l_j \alpha^{(j+1)}\left(\alpha^{(j)}\right)^{l_j-1}\prod_{i\neq j}^{n}\left(\alpha^{(i)}\right)^{l_i}.$$

But, we have proven that for all $i \in \mathbb{N}$, $\alpha^{(i)} = \underset{|x|\to\infty}{O}\left(V'(x)^{-1}\right)$, so by denoting $l \overset{(\text{def})}{=} \sum_{i=1}^n l_i = \deg(A_n)$, $A_n(x) = \underset{|x|\to\infty}{O}\left(V'(x)^{-l}\right)$. Therefore, any homogeneous polynomial of degree k such as P_j^k in the variables $\left(\alpha,\dots,\alpha^{(j)}\right)$ is a $\underset{|x|\to\infty}{O}\left(V'(x)^{-k}\right)$. Finally for the point *iii)*, it is clear that for all $j \geq 0$,

$$\theta^{(j)}(x) = V^{(j+1)}(x) - 2P\mathcal{H}\left[\rho_V\right]^{(j)}(x) = \underset{|x|\to\infty}{O}\left(V'(x)\right).$$

Thus $Q_j^{(k)}\left(\theta,\dots,\theta^{(j)}\right)$ as a homogeneous polynomial of degree $k-j$, is a $\underset{|x|\to\infty}{O}\left(V'(x)^{k-j}\right)$. □

Theorem 2.4.14 ($H^n(\mathbb{R})$-continuity of $\widetilde{\Xi_1^{-1}}$) *There exists a constant $\mathtt{C}(\widetilde{\Xi_1^{-1}}, H^n) > 0$ depending only on n and V such that for all $f \in H^{n+1}(\mathbb{R}^p)$,*

$$\left\|\widetilde{\Xi_1^{-1}}[f]\right\|_{H^n(\mathbb{R}^p)} \leq \mathtt{C}(\widetilde{\Xi_1^{-1}}, H^n)\|f\|_{H^{n+1}(\mathbb{R}^p)}.$$

Under the choice of potential $V_{\phi,t}$ defined in Theorem 2.1.3, for $\phi \in \cap_{k\geq 0}H^k(\mathbb{R})$ the map $t \in [0,1] \mapsto \mathtt{C}(\widetilde{\Xi_1^{-1}}, H^n)$ is continuous.

The idea of the proof is to use the closed form for the operator $\mathcal{O}^k$ defined in 2.4.4, found in Theorem 2.4.7 and inject it in the β_k which appear in the closed form for $\widetilde{\Xi^{-1}}^{(n)}$ in Theorem 2.4.11.

Proof Let $m \leq n$ and $(m_1,\dots,m_p) \in \mathbb{N}^p$ be such that $\sum_{i=1}^p m_i = m$. Let $x_2,\dots,x_p \in \mathbb{R}^{p-1}$ be fixed, we define $h : x_1 \mapsto \partial_2^{m_2}\dots\partial_p^{m_p} f(x_1,\dots,x_p)$ and $g = -\mathcal{X}[h] - 2P\mathcal{W} \circ \widetilde{\mathcal{L}^{-1}}[h]$. With these notations,

$$\partial^{\underline{m}}\widetilde{\Xi_1^{-1}}[f](x_1,\dots,x_p) = \widetilde{\Xi^{-1}}[h]^{(m_1)}(x_1).$$

We can then apply Theorem 2.4.11 and Theorem 2.4.7, so for $|x_1|$ large enough, we get

$$\begin{aligned}\widetilde{\Xi^{-1}}[h]^{(m_1)}(x_1) &= \sum_{a=0}^{m_1-1} Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)\beta_{m_1-a}(x_1)\\ &= \sum_{a=0}^{m_1-1} Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)\frac{1}{\rho_V(x_1)}\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt\rho_V(t)\mathcal{O}^{m_1-a}[g](t)\\ &= \sum_{a=0}^{m_1-1}\sum_{b=0}^{m_1-a} Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)\frac{1}{\rho_V(x_1)}\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt\rho_V(t)g^{(m_1-a-b)}(t)P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(t).\end{aligned} \tag{2.38}$$

Moreover an integration by parts yields:

$$\begin{aligned}\widetilde{\Xi^{-1}}[h]^{(m_1)}(x_1) = \sum_{a=0}^{m_1-1}\sum_{b=0}^{m_1-a} Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)\Bigg(&- g^{(m_1-a-b)}(x_1)\alpha(x_1)P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(x_1)\\ &+\frac{1}{\rho_V(x_1)}\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt\rho_V(t)\Bigg[g^{(m_1-a-b+1)}(t)\alpha(t)P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)(t)\\ &+ g^{(m_1-a-b)}(t)\left[\alpha P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t)\Bigg]\Bigg).\end{aligned} \tag{2.39}$$

We now use $M_V > 0$ from Lemma 2.4.9 so that $P_b^a\left(\alpha,\dots,\alpha^{(b)}\right)$ that appear above are well-defined on $[-M_V,M_V]^c$ *i.e.* don't have any singularity. Hence by integrating with respect to x_1, $\left(\Xi^{-1}[h]^{(m_1)}\right)^2$ on $[M_V,+\infty[$, we get, by Jensen's inequality, for a constant $C(n_1)>0$ depending only on m_1:

$$\begin{aligned}&\int_{M_V}^{+\infty} dx_1\left(\widetilde{\Xi^{-1}}[h]^{(m_1)}(x_1)\right)^2\\ &\le C(m_1)\sum_{a=0}^{m_1-1}\sum_{b=0}^{m_1-a}\int_M^{+\infty} dx_1 Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)^2\Bigg\{g^{(m_1-a-b)}(x_1)^2\alpha(x_1)^2P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(x_1)^2\\ &\quad+\frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt\rho_V(t)g^{(m_1-a-b+1)}(t)\alpha(t)P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)(t)\right]^2\\ &\quad+\frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt\rho_V(t)g^{(m_1-a-b)}(t)\left[\alpha P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t)\right]^2\Bigg\}.\end{aligned}$$

We want to bound this expression by $\|g\|^2_{H^{m_1+1}(\mathbb{R})}$, but since $g=-h+2P\mathcal{H}\left[\rho_V\Xi^{-1}[h]\right]+\mathfrak{c}$, where $\mathfrak{c}=\int_{\mathbb{R}}h(y)d\mu_V(y)-2P\int_{\mathbb{R}}\mathcal{H}\left[\rho_V\Xi^{-1}[h]\right](y)d\mu_V(y)$, the constant terms will fail to be in $L^2(\mathbb{R})$. We thus have to treat these terms separately. In the previous sum, g is differentiated everywhere except in the term $b=m_1-a$ so this is the only value of b where we have to deal with $\mathfrak{c}$. By

defining:

$$\mathfrak{f}^{(1),V}_{n_1,a,b} : x \mapsto Q^{n_1}_a\left(\theta,\dots,\theta^{(a)}\right)(x)\alpha(x)P^{n_1-a}_b(\alpha,\dots,\alpha^{(b)})(x) \tag{2.40}$$

$$\mathfrak{f}^{(2),V}_{n_1,a,b} : x \mapsto \frac{Q^{n_1}_a\left(\theta,\dots,\theta^{(a)}\right)(x)}{\rho_V(x)} \int_x^{\mathrm{sgn}(x)\infty} dt\rho_V(t)\left[\alpha P^{n_1-a}_b\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t), \tag{2.41}$$

by using Jensen's inequality and inequality (2.26) we get :

$$\begin{aligned}
\mathfrak{c}^2 C(m_1)\sum_{a=0}^{m_1-1}\int_{M_V}^{+\infty} dx_1\left[\mathfrak{f}^{(1),V}_{m_1,a,m_1-a}(x_1)^2+\mathfrak{f}^{(2),V}_{m_1,a,m_1-a}(x_1)^2\right]&\\
\le C(m_1)\sup_{0\le a<m_1}\left(\|\mathfrak{f}^{(1),V}_{m_1,a,m_1-a}\|^2_{L^2([-M_V,M_V]^c)}+\|\mathfrak{f}^{(2),V}_{m_1,a,m_1-a}\|^2_{L^2([-M_V,M_V]^c)}\right)&\\
\times\left(\|h\|^2_{L^2(\mu_V)}+\left\|\mathcal{H}\left[\rho_V\Xi^{-1}[h]\right]\right\|^2_{L^2(\mu_V)}\right)&\\
\le C(V,m_1)\|\rho_V\|_{L^\infty(\mathbb{R})}\left(\|h\|^2_{L^2(\mathbb{R})}+\|\rho_V\|_{L^\infty(\mathbb{R})}\pi^2\|\Xi^{-1}[h]\|^2_{L^2(\mu_V)}\right)&\\
\le C(V,m_1)\left(\|h\|^2_{L^2(\mathbb{R})}+\|\rho_V\|_{L^\infty(\mathbb{R})}\pi^2C^2_{\mathcal{L}}\|h'\|^2_{L^2(\mu_V)}\right)&\\
\le C_1(V,m_1)\|\partial_2^{m_2}\dots\partial_p^{m_p}f(.,x_2,\dots,x_p)\|^2_{H^1(\mathbb{R})}&
\end{aligned}$$

where at the end, the constant $C_1(V,m_1)$ is defined by:

$$\begin{aligned}
C_1(V,n_1)\overset{(\mathrm{def})}{=}C(n_1)\max_{0\le a<n_1}\left(\|\mathfrak{f}^{(1),V}_{n_1,a,n_1-a}\|^2_{L^2([-M_V,M_V]^c)}+\|\mathfrak{f}^{(2),V}_{n_1,a,n_1-a}\|^2_{L^2([-M_V,M_V]^c)}\right)&\\
\times\|\rho_V\|_{L^\infty(\mathbb{R})}\left(1+\|\rho_V\|^2_{L^\infty(\mathbb{R})}\pi^2C^2_{\mathcal{L}}\right).&
\end{aligned} \tag{2.42}$$

Above, the first integral that appears is well-defined, since by Lemma 2.4.13, one can check by assumption *(v)* that $\mathfrak{f}^{(1)}_{m_1,a,m_1-a}(x_1)$ and $\mathfrak{f}^{(2)}_{m_1,a,m_1-a}(x_1)$ behave like $\underset{|x_1|\to\infty}{O}\left(V'(x_1)^{-2}\right)$ which is integrable by assumption *(v)* again.

In the following, we set $\mathfrak{g}\overset{(\mathrm{def})}{=}g-\mathfrak{c}$. We can now replace g' by $\mathfrak{g}'$ since we handled all the terms involving $\mathfrak{c}$. By Cauchy-Schwarz inequality, with $C_2(V,m_1)$ defined by:

$$\begin{aligned}
C_2(V,n_1)=C(n_1)\max_{\substack{0\le a<n_1\\0\le b\le n_1-a}}\left(\left\|\mathfrak{f}^{(1),V}_{n_1,a,b}\right\|^2_{L^\infty([-M_V,M_V]^c)}&\right.\\
\left.+\|\mathfrak{f}^{(3),V}_{n_1,a,b}\|^2_{L^2([-M_V,M_V]^c)}+\left\|\mathfrak{f}^{(4),V}_{n_1,a,b}\right\|^2_{L^2([-M_V,M_V]^c)}\right),&
\end{aligned} \tag{2.43}$$

with

$$\mathfrak{f}^{(3),V}_{n_1,a,b} : x \mapsto \left|\frac{Q^{n_1}_a\left(\theta,\dots,\theta^{(a)}\right)(x)}{\rho_V(x)}\right| \int_x^{\mathrm{sgn}(x)\infty} dt\rho_V(t)^2\alpha(t)^2P^{n_1-a}_b\left(\alpha,\dots,\alpha^{(b)}\right)(t)^2\Bigg|^{1/2}, \tag{2.44}$$

$$\mathfrak{f}^{(4),V}_{n_1,a,b} : x \mapsto \left|\frac{Q^{n_1}_a\left(\theta,\dots,\theta^{(a)}\right)(x)}{\rho_V(x)}\right| \int_x^{\mathrm{sgn}(x)\infty} dt\rho_V(t)^2\left[\alpha P^{n_1-a}_b\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t)\Bigg|^{1/2}, \tag{2.45}$$

we get

$$\sum_{a=0}^{m_1-1}\sum_{b=0}^{m_1-a}\int_{M_V}^{+\infty} dx_1 Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)^2\Bigg\{\mathfrak{g}^{(m_1-a-b)}(x_1)^2\alpha(x_1)^2 P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(x_1)^2 + \frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt\rho_V(t)\mathfrak{g}^{(m_1-a-b+1)}(t)\alpha(t)P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)(t)\right]^2 + \frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt\rho_V(t)\mathfrak{g}^{(m_1-a-b)}(t)\left[\alpha P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t)\right]^2\Bigg\} \le C_2(V,m_1)\|\mathfrak{g}\|^2_{H^{m_1+1}(\mathbb{R})}.$$

Finally, by using that $\mathfrak{g} = -h - 2P\mathcal{H}\left[\rho_V\Xi^{-1}[h]\right]$, that $\pi^{-1}\mathcal{H}$ is an isometry in $L^2(\mathbb{R})$ and that for all $u \in H^1(\mathbb{R})$, $\mathcal{H}[u]' = \mathcal{H}[u']$, we obtain:

$$\left\|\Xi^{-1}[h]^{(m_1)}\right\|_{L^2([M_V,+\infty[)} \le 2\max_{i=1,2} C_i(V,m_1)^{1/2}\Big(\|h\|_{H^{m_1+1}(\mathbb{R})} + 2P\pi\|\rho_V\Xi^{-1}[h]\|_{H^{m_1+1}(\mathbb{R})}\Big).$$

We now use the form stated in Lemma 2.9.3, to conclude that

$$\left\|\Xi_1^{-1}[h]^{(m_1)}\right\|_{L^2([M_V,+\infty[)} \le 2\max_{i=1,2} C_i(V,m_1)^{1/2}\left(2P\pi C_3(V,m_1)+1\right)\|h\|_{H^{m_1+1}(\mathbb{R})}.$$

where $C_3(V,m_1)$ is explicitly given in Lemma 2.9.3. By the exact same bounds, on $]-\infty,-M_V]$, we finally obtain

$$\left\|\Xi^{-1}\left[h\right]^{(m_1)}\right\|_{L^2([-M_V,M_V]^c)} \le 4\max_{i=1,2} C_i(V,m_1)^{1/2}\left(2P\pi C_3(V,m_1)+1\right)\|h\|_{H^{m_1+1}(\mathbb{R})}.$$

Now relaxing the dependance on $x_2,\dots,x_p \in \mathbb{R}$ and integrating with respect to these variables, we get

$$\left\|\partial^{\underline{m}}\Xi_1^{-1}[f]\right\|_{L^2([-M_V,M_V]^c\times\mathbb{R}^{p-1})} \le 4\max_{i=1,2} C_i(V,m_1)^{1/2}\left(2P\pi C_3(V,m_1)+1\right) \times\left(\int_{\mathbb{R}^{p-1}}\|\partial_2^{m_2}\dots\partial_p^{m_p}f(.,x_2,\dots,x_p)\|^2_{H^{m_1+1}(\mathbb{R})}dx_2\dots dx_p\right)^{1/2}.$$

Thus we deduce that for a constant $C(n)>0$ only depending on n such that

$$\left\|\Xi_1^{-1}[f]\right\|_{H^n([-M_V,M_V]^c\times\mathbb{R}^{p-1})} \le C(n)\max_{m_1\le n}\max_{i=1,2} C_i(V,m_1)^{1/2}\left(2P\pi C_3(V,m_1)+1\right)\|f\|_{H^{n+1}(\mathbb{R}^p)}. \tag{2.46}$$

Now, we prove the control on $[-M_V,M_V]\times\mathbb{R}^{p-1}$, we fix $x_2,\dots,x_p \in \mathbb{R}$. By Cauchy-Schwarz inequality:

$$\int_{-M_V}^{M_V}\left|\Xi^{-1}\left[h\right]^{(m_1)}(x)\right|^2 dx \le \left\|\rho_V^{-1}\right\|^2_{L^\infty([-M_V,M_V])}\left\|\rho_V\Xi^{-1}\left[h\right]^{(m_1)}\right\|^2_{L^2(\mathbb{R})} \le C_3(V,m_1)^2\left\|\rho_V^{-1}\right\|^2_{L^\infty([-M_V,M_V])}\|h\|^2_{H^{m_1}(\mathbb{R})}$$

where the last inequality comes from Lemma 2.9.3. Again relaxing the dependance on $x_2,\dots,x_p \in \mathbb{R}$ and integrating with respect to these variables, we get for a constant $C(n)>0$ independent of n:

$$\left\|\Xi_1^{-1}[f]\right\|_{H^n([-M_V,M_V]\times\mathbb{R}^{p-1})} \le C(n)\max_{m_1\le n} C_3(V,m_1)\left\|\rho_V^{-1}\right\|_{L^\infty([-M_V,M_V])}\|f\|_{H^{n+1}(\mathbb{R}^p)}.$$

Collecting the last bound, using that $C_3(V,m)$ is increasing in m and (2.46) leads to the conclusion for

$$\mathtt{C}(\widetilde{\Xi_1^{-1}}, H^n) \overset{(\text{def})}{=} C(n)\left[C_3(V,n)\left\|\rho_V^{-1}\right\|_{L^\infty([-M_V,M_V])} + (2P\pi C_3(V,n)+1)\max_{n_1\leq n}\max_{i=1,2} C_i(V,n_1)^{\frac{1}{2}}\right] \tag{2.47}$$

The fact that, upon choosing the potential $V_{\phi,t}$ with $\phi \in \cap_{k\geq 0}H^k(\mathbb{R})$, $t \mapsto \mathtt{C}(\widetilde{\Xi_1^{-1}}, H^n)$ is continuous is shown in Proposition 2.10.8. □

Theorem 2.4.15 ($W_n^\infty(\mathbb{R})$-continuity of $\widetilde{\Xi_1^{-1}}$) *Let $n \geq 1$, for all $f \in W_{n+1}^\infty(\mathbb{R}^p)$,*

$$\left\|\widetilde{\Xi_1^{-1}}[f]\right\|_{W_n^\infty(\mathbb{R}^p)} \leq \mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)\|f\|_{W_{n+1}^\infty(\mathbb{R}^p)}.$$

Under the choice of potential $V_{\phi,t}$ defined in Theorem 2.1.3, for $\phi \in \cap_{k\geq 0}H^k(\mathbb{R})$ the map $t \in [0,1] \mapsto \mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)$ is continuous.

Proof Let $f \in W_{n+1}(\mathbb{R}^p)$, let $m \leq n$ and $(m_1,\dots,m_p) \in \mathbb{N}^p$ be such that $\sum_{i=1}^p m_i = m$, let $x_2,\dots,x_p \in \mathbb{R}^{p-1}$, we set $h : x_1 \mapsto \partial_2^{m_2}\dots\partial_p^{m_p} f(x_1,\dots,x_p)$ we know by theorem 2.4.11 that

$$\widetilde{\Xi^{-1}}[h](x_1) = \frac{1}{\rho_V(x_1)}\int_{x_1}^{\operatorname{sgn}(x_1)\infty} dt\rho_V(t)g(t),$$

where $g \overset{(\text{def})}{=} -\mathcal{X}[h] + 2P\mathcal{X}\circ\mathcal{H}\big[\rho_V\widetilde{\Xi^{-1}}[h]\big]$. For the following we define

$$\mathcal{I}_1^V : x \mapsto \frac{1}{\rho_V(x)}\left|\int_x^{\operatorname{sgn}(x)\infty}\rho_V(t)dt\right|, \qquad \mathcal{I}_2^V : x \mapsto \frac{1}{\rho_V(x)}\left|\int_x^{\operatorname{sgn}(x)\infty}\rho_V(t)^2dt\right|^{1/2}. \tag{2.48}$$

By integration by parts, one can see that $\mathcal{I}_1^V(x) = \underset{|x|\to\infty}{O}\left(V'(x)^{-1}\right)$ is bounded on $\mathbb{R}$. So for the first and third term, by direct bounds:

$$\left|\frac{1}{\rho_V(x_1)}\int_{x_1}^{\operatorname{sgn}(x_1)\infty} dt\rho_V(t)\left(-h(t)+\int_{\mathbb{R}}h(s)d\mu_V(s)\right)\right| \leq 2\|\mathcal{I}_1^V\|_\infty\|h\|_\infty \leq 2\|\mathcal{I}_1^V\|_\infty\|f\|_{W_n^\infty(\mathbb{R}^p)}.$$

For the two last terms, we want to use that $\pi^{-1}\mathcal{H}$ is an isometry on $L^2(\mathbb{R})$, so we use Cauchy-Schwarz inequality and the fact that $\mathcal{I}_2^V(x) = \underset{|x|\to\infty}{O}\left(V'(x)^{-1/2}\right)$ is bounded on $\mathbb{R}$ so that:

$$\sup_{x_1\in\mathbb{R}}\left|\frac{2P}{\rho_V(x_1)}\int_{x_1}^{\operatorname{sgn}(x_1)\infty} dt\rho_V(t)\mathcal{X}\circ\mathcal{H}\left[\rho_V\widetilde{\Xi^{-1}}[h]\right](t)\right| \leq C_4(V)\left\|h'\right\|_\infty \leq C_4(V)\|f\|_{W_{n+1}^\infty(\mathbb{R}^p)}.$$

with

$$C_4(V) \overset{(\text{def})}{=} 2P\left(\|\mathcal{I}_2^V\|_\infty + \|\mathcal{I}_1^V\|_\infty\|\rho_V\|_\infty^{1/2}\right)\pi\|\rho_V\|_\infty^{1/2}C_{\mathcal{L}}. \tag{2.49}$$

Thus, by taking the supremum of $x_2,\dots,x_p \in \mathbb{R}^{p-1}$, we conclude that for $m_1 = 0$,

$$\left\|\partial^{\underline{m}}\widetilde{\Xi_1^{-1}}[f]\right\|_{L^\infty(\mathbb{R}^p)} \leq \left(2\|\mathcal{I}_1^V\|_\infty + C_4(V)\right)\|f\|_{W_{n+1}^\infty(\mathbb{R}^p)}. \tag{2.50}$$

For $m_1 \neq 0$y (2.38), for all $|x| > M_V$, *cf* Lemma 2.4.9:

$$\widetilde{\Xi^{-1}}[h]^{(m_1)}(x_1) = \sum_{a=0}^{m_1-1}\sum_{b=0}^{m_1-a}\frac{Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)}{\rho_V(x_1)}\int_{x_1}^{\operatorname{sgn}(x_1)\infty} g^{(m_1-a-b)}(t)P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(t)d\mu_V(t)$$

where $g \overset{(\text{def})}{=} -\mathcal{X}[h] - 2P\mathcal{W} \circ \widetilde{\mathcal{L}^{-1}}[h]$. Furthermore, setting

$$\mathfrak{c} \overset{(\text{def})}{=} \int_{\mathbb{R}} h(t)d\mu_V(t) - 2P \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}[h]\right](t)d\mu_V(t)$$

and

$$\mathfrak{f}^{(5),V}_{l,a,b} : x \mapsto \frac{Q^l_a\left(\theta, \dots, \theta^{(a)}\right)(x)}{\rho_V(x)} \int_x^{\operatorname{sgn}(x)\infty} \left|P^{l-a}_b(\alpha, \dots, \alpha^{(b)})(t)\right| \rho_V(t)dt, \tag{2.51}$$

we can bound every term in the previous sum, involving $\mathfrak{c}$, namely taking $b = m_1 - a$:

$$|\mathfrak{c}\mathfrak{f}^{(5)}_{m_1,a,m_1-a}(x_1)| \leq (1 + 2P\pi\|\rho_V\|_\infty C_{\mathcal{L}})\|h\|_{W^\infty_1(\mathbb{R})}\|\mathfrak{f}^{(5)}_{m_1,a,m_1-a}\|_{L^\infty([-M_V,M_V]^c)}.$$

We directly bound $\mathfrak{c}$ in the LHS, while to bound $\mathfrak{f}^{(5)}_{m_1,a,m_1-a}(x_1)$, we successfully applied Jensen's inequality, used the isometry property of $\pi^{-1}\mathcal{H}$ on $L^2(\mathbb{R})$ and used the inequality (2.26). Furthermore, the fact that $\mathfrak{f}^{(5)}_{m_1,a,m_1-a}$ is bounded on $[-M_V, M_V]^c$ comes from Lemma 2.4.13. Finally, by setting $\mathfrak{g} \overset{(\text{def})}{=} g - \mathfrak{c}$, it only remains to establish the following bounds:

$$\begin{aligned}
&\Big| \sum_{a=0}^{m_1-1} \sum_{b=0}^{m_1-a} Q^{m_1}_a\left(\theta, \dots, \theta^{(a)}\right)(x_1) \frac{1}{\rho_V(x_1)} \int_{x_1}^{\operatorname{sgn}(x_1)\infty} \mathfrak{g}^{(m_1-a-b)}(t) P^{m_1-a}_b(\alpha, \dots, \alpha^{(b)})(t)\rho_V(t)dt\Big| \\
&\qquad \leq \sum_{a=0}^{m_1-1} \sum_{b=0}^{m_1-a} \left\{ \left\| \frac{\left|Q^{m_1}_a\left(\theta, \dots, \theta^{(a)}\right)\right|}{\rho_V} \int_{\cdot}^{\operatorname{sgn}(.)\infty} \left|\mathcal{H}\left[\left(\rho_V \widetilde{\Xi^{-1}}[h]\right)^{(m_1-a-b)}\right](t)\right| \right. \right. \\
&\left. \left. \cdot \left|P^{m_1-a}_b(\alpha, \dots, \alpha^{(b)})(t)\right| \rho_V(t)dt \right\|_{L^\infty([-M_V,M_V]^c)} + \|h\|_{W^\infty_{m_1}(\mathbb{R})}\|\mathfrak{f}^{(5)}_{m_1,a,b}\|_{L^\infty([-M_V,M_V]^c)} \right\} \\
&\qquad \leq \sum_{a=0}^{m_1-1} \sum_{b=0}^{m_1-a} \left\{ \|h\|_{W^\infty_{m_1}(\mathbb{R})}\|\mathfrak{f}^{(5),V}_{m_1,a,b}\|_{L^\infty([-M_V,M_V]^c)} \right. \\
&\qquad\qquad \left. + \|\mathfrak{f}^{(6),V}_{m_1,a,b}\|_{L^\infty([-M_V,M_V]^c)} \left\| \mathcal{H}\left[\left(\rho_V \widetilde{\Xi^{-1}}[h]\right)^{(m_1-a-b)}\right] \right\|_{L^2(\mathbb{R})} \right\}
\end{aligned}$$

with

$$\mathfrak{f}^{(6),V}_{l,a,b} : x \mapsto \left|Q^l_a\left(\theta, \dots, \theta^{(a)}\right)(x)\right| \frac{1}{\rho_V(x)} \sqrt{\int_x^{\operatorname{sgn}(x)\infty} \left|P^{l-a}_b(\alpha, \dots, \alpha^{(b)})(t)\right|^2 \rho_V(t)^2 dt}. \tag{2.52}$$

For each a, b, $\mathfrak{f}^{(6),V}_{m_1,a,b}$ is bounded on $[-M_V, M_V]^c$ because of Lemma 2.4.13 and Lemma 2.9.2. By Cauchy-Schwarz inequality and Lemma 2.9.3, we get:

$$\left\| \mathcal{H}\left[\left(\rho_V \widetilde{\Xi^{-1}}[h]\right)^{(m_1-a-b)}\right] \right\|_{L^2(\mathbb{R})} \leq \pi \left\| \left(\rho_V \widetilde{\Xi^{-1}}[h]\right)^{(m_1-a-b)} \right\|_{L^2(\mathbb{R})} \leq \pi C_6(V, m_1)\|h\|_{W^\infty_{m_1}(\mathbb{R})}.$$

Finally by the same reasonnings as before, we get

$$\begin{aligned}
&\Big| \sum_{a=0}^{m_1-1} \sum_{b=0}^{m_1-a} \frac{Q^{m_1}_a\left(\theta, \dots, \theta^{(a)}\right)(x_1)}{\rho_V(x_1)} \int_{x_1}^{\operatorname{sgn}(x_1)\infty} \mathfrak{g}^{(m_1-a-b)}(t) P^{m_1-a}_b(\alpha, \dots, \alpha^{(b)})(t)\rho_V(t)dt\Big| \\
&\qquad\qquad \leq C_7(V, m_1)\|h\|_{W^\infty_{m_1}(\mathbb{R})}
\end{aligned}$$

where $C_7(V,m_1)$ is defined by

$$C_7(V,m_1) \overset{(\text{def})}{=} \max_{1\le l\le m_1} l^2 \left(\max_{\substack{0\le a<l\\ 0\le b\le l-a}} \|\mathfrak{f}^{(5),V}_{l,a,b}\|_{L^\infty([-M_V,M_V]^c)} + \pi C_6(V,l) \max_{\substack{0\le a<l\\ 0\le b\le l-a}} \|\mathfrak{f}^{(6),V}_{l,a,b}\|_{L^\infty([-M_V,M_V]^c)} \right). \tag{2.53}$$

Thus, we deduce that

$$\begin{aligned}&\|\widetilde{\Xi^{-1}}[h]^{(m_1)}\|_{L^\infty([-M_V,M_V]^c)}\\ &\quad\le \left[m_1(1+2P\pi\|\rho_V\|_\infty C_{\mathcal{L}}) \max_{0\le a<m_1} \|\mathfrak{f}^{(5),V}_{m_1,a,m_1-a}\|_{L^\infty([-M_V,M_V]^c)} + C_7(V,m_1)\right]\|h\|_{W^\infty_{m_1}(\mathbb{R})}. \end{aligned}\tag{2.54}$$

Now let $x\in[-M_V,M_V]$, by differentiating m_1 times (2.36), the Leibniz formula ensures that there exists polynomials $R^{m_1}_{m_1-a}$ depending on $(\theta,\dots,\theta^{(m_1-1-a)})$ and a polynomial S^{m_1} of degree m_1-1 depending on $(\theta,\dots,\theta^{(m_1)})$, whose coefficients are independent of V such that

$$\begin{aligned}\widetilde{\Xi^{-1}}[h]^{(m_1)}(x) = \frac{S^{m_1}(\theta,\dots,\theta^{(m_1)})(x)}{\rho_V(x)} \int_x^{\operatorname{sgn}(x)\infty} g(t)\rho_V(t)dt\\ + \sum_{a=0}^{m_1-2} R^{m_1}_{m_1-a}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)(x) g^{(a)}(x) - g^{(m_1-1)}(x).\end{aligned}\tag{2.55}$$

We recall that the function g is defined by

$$g \overset{(\text{def})}{=} -h + \int_{\mathbb{R}} h(t)d\mu_V(t) + 2P\left(\mathcal{H}\left[\rho_V\widetilde{\Xi^{-1}}[h]\right] - \int_{\mathbb{R}} \mathcal{H}\left[\rho_V\widetilde{\Xi^{-1}}[h]\right](t)d\mu_V(t)\right).$$

Then, for all, $x\in[-M_V,M_V]$, by the same bounds as before with $C_8(V,m_1)$ defined by

$$\begin{aligned}C_8(V,m_1) \overset{(\text{def})}{=} \max_{1\le l\le m_1} \Big(&\|S^l(\theta,\dots,\theta^{(l)})\|_{L^\infty([-M_V,M_V])}\|\rho_V^{-1}\|_{L^\infty([-M_V,M_V])}\\ &+ l\max_{0\le a\le l-2}\|R^l_{l-a}(\theta,\dots,\theta^{(l-1-a)})\|_{L^\infty([-M_V,M_V])} + 1\Big),\end{aligned}\tag{2.56}$$

we obtain:

$$\begin{aligned}\Big|\frac{S^{m_1}(\theta,\dots,\theta^{(m_1)})(x_1)}{\rho_V(x_1)} \int_{x_1}^{\operatorname{sgn}(x_1)\infty} \mathcal{X}[h](t)\rho_V(t)dt + \sum_{a=0}^{m_1-2} R^{m_1}_{m_1-a}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)(x_1)\mathcal{X}[h]^{(a)}(x_1)\\ - \mathcal{X}[h]^{(m_1-1)}(x_1)\Big| \le 2C_8(V,m_1)\|h\|_{W^\infty_{m_1-1}(\mathbb{R})}\end{aligned}$$

and

$$\begin{aligned}2P\Big|\frac{S^{m_1}(\theta,\dots,\theta^{(m_1)})(x_1)}{\rho_V(x_1)} \int_{x_1}^{\operatorname{sgn}(x_1)\infty} \rho_V(t)dt + R^{m_1}_{m_1}\left(\theta,\dots,\theta^{(m_1-1)}\right)(x_1)\Big| \int_{\mathbb{R}} |\mathcal{H}\left[\rho_V\widetilde{\Xi^{-1}}[h]\right](t)|d\mu_V(t)\\ \le 2P\pi\|\rho_V\|_\infty^{1/2} C_{\mathcal{L}} C_8(V,m_1)\|h\|_{W^\infty_1(\mathbb{R})}.\end{aligned}$$

It remains to bound the terms involving the Hilbert-transform. For that, we use that for all $\phi\in H^{m_1}(\mathbb{R})$ and $a\in[\![0,m_1-1]\!]$,

$$|\mathcal{H}[\phi]^{(a)}(x)| = \sqrt{\mathcal{H}[\phi^{(a)}](x)^2} = \sqrt{\int_{+\infty}^x 2\mathcal{H}[\phi^{(a)}](t)\mathcal{H}[\phi^{(a+1)}](t)dt} \le \pi\sqrt{2\|\phi^{(a)}\|_{L^2(\mathbb{R})}\|\phi^{(a+1)}\|_{L^2(\mathbb{R})}}.$$

Applying those results for $\phi = \rho_V \widetilde{\Xi^{-1}}[h] \in H^{m_1}(\mathbb{R})$, Lemma 2.4.8 allows us to conclude that:

$$\||\mathcal{H}[\phi]\|_{W^\infty_{m_1-1}(\mathbb{R})} \leq \sqrt{2}\pi \|\phi\|_{H^{m_1}(\mathbb{R})}.$$

We conclude by Lemma 2.9.3 that:

$$\||\mathcal{H}[\phi]\|_{W^\infty_{m_1-1}(\mathbb{R})} \leq \sqrt{2}\pi C_6(V,m_1)\|h\|_{W^\infty_{m_1}(\mathbb{R})}$$

and thus, with $C_8(V,m_1)$ defined in (2.56) we get:

$$2P\Big|\frac{S^{m_1}(\theta,\dots,\theta^{(m_1)})(x_1)}{\rho_V(x_1)} \int_{x_1}^{\mathrm{sgn}(x_1)\infty} \mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}[h]\right](t)\rho_V(t)dt$$
$$+ \sum_{a=0}^{m_1-2} R^{m_1}_{m_1-a}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)(x_1)\mathcal{H}\left[\rho_V\widetilde{\Xi^{-1}}[h]\right]^{(a)}(x_1) - \mathcal{H}\left[\rho_V\widetilde{\Xi^{-1}}[h]\right]^{(m_1-1)}(x_1)\Big|$$
$$\leq 2P\sqrt{2}\pi C_6(V,m_1)C_8(V,m_1)\|h\|_{W^\infty_{m_1-1}(\mathbb{R})}.$$

All the previous bounds yield

$$\left\|\widetilde{\Xi^{-1}}[h]^{(m_1)}\right\|_{L^\infty([-M_V,M_V])} \leq \left(2 + 2P\pi\|\rho_V\|_\infty^{1/2}C_{\mathcal{L}} + 2P\sqrt{2}\pi C_6(V,m_1)\right)C_8(V,m_1)\|h\|_{W^\infty_{m_1}(\mathbb{R})}.$$

Upon taking the supremum over $x_2,\dots,x_p \in \mathbb{R}^{p-1}$ in (39), and over $m \leq n$ we conclude that

$$\left\|\widetilde{\Xi_1^{-1}}[f]\right\|_{W^\infty_n(\mathbb{R}^p)} \leq \mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)\|f\|_{W^\infty_{n+1}(\mathbb{R}^p)},$$

with $\mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)$ defined as

$$\mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty) = \max_{m_1\leq n}\left[m_1(1+2P\pi\|\rho_V\|_\infty C_{\mathcal{L}}) \max_{0\leq a<l}\|\mathfrak{f}^{(5),V}_{m_1,a,m_1-a}\|_{L^\infty([-M_V,M_V]^c)} + C_7(V,m_1)\right]$$
$$+ \max_{m_1\leq n}\left(2 + 2P\pi\|\rho_V\|_\infty^{1/2}C_{\mathcal{L}} + 2P\sqrt{2}\pi C_6(V,m_1)\right)C_8(V,m_1) + 2\|\mathcal{I}_1^V\|_\infty + C_4(V). \quad (2.57)$$

The fact that, upon choosing the potential $V_{\phi,t}$ with $\phi \in \cap_{k\geq 0}H^k(\mathbb{R})$, $t \mapsto \mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)$ is continuous is shown in Proposition 2.10.8. □

Finally, we define the variable insertion operators which will also be involved in the loop equations.

Definition 2.4.16 *If ϕ is a function in n variables, we define the n-th variable insertion operator $\Theta^{(p)}$ as*

$$\Theta^{(p)}[\phi](\xi_1,\dots,\xi_{n-1}) = \phi(\xi_1,\dots,\xi_{p-1},\xi_1,\xi_p,\dots,\xi_{n-1}) \quad (2.58)$$

Corollary 2.4.17 *Let $n,p \geq 1$, $a \in [\![2,p+1]\!]$, for all $f \in W^\infty_{n+1}(\mathbb{R}^{p+1})$,*

$$\left\|\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}[f]\right\|_{W^\infty_n(\mathbb{R}^p)} \leq 2\mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)\|f\|_{W^\infty_{n+1}(\mathbb{R}^{p+1})}$$

where the constant $\mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)$ was introduced in Theorem 2.4.15.

Proof Let $f \in W^\infty_{n+1}(\mathbb{R}^{p+1})$, $x_1,\dots,x_p \in \mathbb{R}$, $\underline{m} \overset{(\mathrm{def})}{=} (m_1,\dots,m_p) \in \mathbb{N}^p$ such that $m \overset{(\mathrm{def})}{=} \sum_{i=1}^p m_i \leq n$.

$$\partial^{\underline{m}}\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}[f](x_1,\dots,x_p) = \partial_1^{m_1}\partial_2^{m_2}\dots\partial_{a-1}^{m_{a-1}}\partial_{a+1}^{m_a}\dots\partial_{p+1}^{m_p}\widetilde{\Xi_1^{-1}}[f](x_1,\dots,x_{a-1},x_1,x_a\dots,x_p)$$
$$+ \partial_2^{m_2}\dots\partial_{a-1}^{m_{a-1}}\partial_a^{m_1}\partial_{a+1}^{m_a}\partial_{p+1}^{m_p}\widetilde{\Xi_1^{-1}}[f](x_1,\dots,x_{a-1},x_1,x_a\dots,x_p).$$

Thus $\left\|\partial^{\underline{m}}\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}[f]\right\|_{L^\infty(\mathbb{R}^p)} \leq 2\left\|\widetilde{\Xi_1^{-1}}[f]\right\|_{W^\infty_n(\mathbb{R}^{p+1})} \leq 2\mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty)\|f\|_{W^\infty_{n+1}(\mathbb{R}^{p+1})}$. □

The last control that we need is on $\Theta^{(a)} \circ \widetilde{\Xi_1}^{-1}$ in H^n-norm. $\Theta^{(a)}$ was defined in Definition 2.4.16.

Theorem 2.4.18 (H^n-continuity for $\Theta^{(a)} \circ \widetilde{\Xi_1}^{-1}$) *Let $n \in \mathbb{N}$, $p \in [\![1, +\infty[\![$, let $a \in [\![2, p+1]\!]$, there exists a constant $\mathtt{C}(\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}, H^n) > 0$, such that for all $f \in H^{n+1}(\mathbb{R}^{p+1})$,*

$$\|\Theta^{(a)} \circ \widetilde{\Xi_1}^{-1}[f]\|_{H^n(\mathbb{R}^p)} \leq \mathtt{C}(\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}, H^n)\|f\|_{H^{n+1}(\mathbb{R}^{p+1})}.$$

Under the choice of potential $V_{\phi,t}$ defined in Theorem 2.1.3, for $\phi \in \cap_{k\geq 0} H^k(\mathbb{R})$ the map $t \in [0,1] \mapsto \mathtt{C}(\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}, H^n)$ is continuous.

Proof Let $f \in H^{n+1}(\mathbb{R}^{p+1})$, Let $f \in W^{\infty}_{n+1}(\mathbb{R}^{p+1})$, $x_1, \dots, x_p \in \mathbb{R}$, $\underline{m} \overset{(\text{def})}{=} (m_1, \dots, m_p) \in \mathbb{N}^p$ such that $m \overset{(\text{def})}{=} \sum_{i=1}^p m_i \leq n$. We set

$$h : (x, y) \mapsto \partial_2^{m_2} \dots \partial_{a-1}^{m_{a-1}} \partial_{a+1}^{m_a} \dots \partial_{p+1}^{m_p} f(x, x_2, \dots, x_{a-1}, y, x_a, \dots, x_p)$$

and

$$g(x, y) \overset{(\text{def})}{=} -h(x, y) + 2P\mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}\left[h(., y)\right]\right](x) + \mathfrak{c}(y)$$

where we have set $\mathfrak{c}(y) = \displaystyle\int_{\mathbb{R}} h(s, y) d\mu_V(s) - 2P \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}\left[h(., y)\right]\right](s) d\mu_V(s)$. Let $x \in \mathbb{R}$,

$$\partial^{\underline{m}} \Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}\left[f\right](x_1, \dots, x_p) = \widetilde{\Xi^{-1}}[\partial_2^{m_1} h(., x_1)](x_1) + \widetilde{\Xi^{-1}}[h(., x_1)]^{(m_1)}(x_1). \tag{2.59}$$

The first term is easy to control by Theorem 2.4.11,

$$\widetilde{\Xi^{-1}}[\partial_a^{m_1} h(., x_1)](x_1) = \frac{1}{\rho_V(x_1)} \int_{x_1}^{\text{sgn}(x_1)\infty} dt \rho_V(t) \Bigg\{ - \partial_2^{m_1} h(t, x_1) + \int_{\mathbb{R}} \partial_2^{m_1} h(s, x_1) d\mu_V(s)$$
$$+ 2P\mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}\left[\partial_a^{m_1} h(., x_1)\right]\right](t) - 2P \int_{\mathbb{R}} 2P\mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}\left[\partial_a^{m_1} h(., x_1)\right]\right](s) d\mu_V(s) \Bigg\}.$$

From the standart arguments that we used before,

$$\int_{\mathbb{R}} \widetilde{\Xi^{-1}}[\partial_2^{m_1} h(., x_1)](x_1)^2 dx_1 \leq 4(1 + 4P^2\pi^2 \|\rho_V\|^2_{L^\infty(\mathbb{R})} C_{\mathcal{L}}^2) \Bigg\{ \|\mathcal{I}_2^V\|^2_{L^\infty(\mathbb{R})} \|h\|^2_{H^{m_1+1}(\mathbb{R}^2)}$$
$$+ \|\rho_V\|_{L^\infty(\mathbb{R})} \left\|\mathcal{I}_1^V\right\|^2_{L^\infty(\mathbb{R})} \|h\|^2_{H^{m_1+1}(\mathbb{R}^2)} \Bigg\} \tag{2.60}$$

where $\mathcal{I}_1^V$ and $\mathcal{I}_2^V$ have been defined in (2.48).

We now deal with the second term in (2.59). By (2.39), we have for a constant $C(m_1) > 0$ depending only on m_1:

$$\int_{M_V}^{+\infty} \Xi^{-1}[h(.,x_1)]^{(m_1)}(x_1)^2 dx_1 \leq C(m_1) \sum_{a=0}^{m_1-1} \sum_{b=0}^{m_1-a} \int_{M_V}^{+\infty} dx_1 Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)^2$$
$$\times \Bigg\{ [\partial_1^{m_1-a-b} g(x_1,x_1)]^2 \alpha(x_1)^2 P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(x_1)^2$$
$$+ \frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt \rho_V(t)\partial_1^{m_1-a-b+1} g(t,x_1)\alpha(t) P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)(t)\right]^2$$
$$+ \frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt \rho_V(t)\partial_1^{m_1-a-b} g(t,x_1)\left[\alpha P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t)\right]^2 \Bigg\}.x \quad (2.61)$$

We first deal with the presence of $\mathfrak{c}$ in the sum. This term only arises in the sum when $b = m_1 - a$. By using the functions $\mathfrak{f}_{m_1,a,b}^{(j),V}$ defined in (2.40), we can bound these terms, for all $a \in [\![0, m_1-1]\!]$, by

$$\int_{M_V}^{+\infty} dx_1 \mathfrak{c}(x_1)^2 \left(\mathfrak{f}_{m_1,a,m_1-a}^{(1)}(x_1)^2 + \mathfrak{f}_{m_1,a,m_1-a}^{(2),V}(x_1)^2\right) \leq 2\left(\sum_{j\in\{1,2\}} \|\mathfrak{f}_{m_1,a,m_1-a}^{(j),V}\|^2_{L^\infty([-M_V,M_V]^c)}\right)$$
$$\times \int_{\mathbb{R}} dx_1 \left(\int_{\mathbb{R}} h(t,x_1)^2 d\mu_V(t) + 4P^2 \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}\left[h(.,x_1)\right]\right](t)^2 d\mu_V(t)\right)$$
$$\leq 2\left(\sum_{j\in\{1,2\}} \|\mathfrak{f}_{m_1,a,m_1-a}^{(j),V}\|^2_{L^\infty([-M_V,M_V]^c)}\right)(\|\rho_V\|_{L^\infty(\mathbb{R})} + 4P^2\pi^2\|\rho_V\|^3_{L^\infty(\mathbb{R})} C_{\mathcal{L}}^2)\|h\|^2_{H^1(\mathbb{R}^2)}.$$

Since we handled all the therms involving $\mathfrak{c}$, it just remains to bound (2.61) with the substitution $g(x,y) \rightsquigarrow \mathfrak{g}(x,y) \overset{(\mathrm{def})}{=} g(x,y) - \mathfrak{c}(y)$, namely:

$$\int_{M_V}^{+\infty} dx_1 Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)^2 \Bigg\{ \partial_1^{m_1-a-b}\mathfrak{g}(x_1,x_1)^2 \alpha(x_1)^2 P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(x_1)^2$$
$$+ \frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt \rho_V(t)\partial_1^{m_1-a-b+1}\mathfrak{g}(t,x_1)\alpha(t) P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)(t)\right]^2$$
$$+ \frac{1}{\rho_V(x_1)^2}\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty} dt \rho_V(t)\partial_1^{m_1-a-b}\mathfrak{g}(t,x_1)\left[\alpha P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t)\right]^2 \Bigg\}.$$

For the first term, we use the fact that for fixed $x \in \mathbb{R}$, $t \mapsto \partial_1^{m_1-a-b}\mathfrak{g}(t,x)$ goes to zero at infinity as an element of $H^1(\mathbb{R})$ and by Cauchy-Schwarz inequality, we get:

$$|\partial_1^{m_1-a-b}\mathfrak{g}(x_1,x_1)| = \sqrt{\int_{+\infty}^{x_1} 2\partial_1^{m_1-a-b}\mathfrak{g}(t,x_1)\partial_1^{m_1-a-b+1}\mathfrak{g}(t,x_1)dt} \leq \sqrt{2}\|\mathfrak{g}(.,x_1)\|_{H^{m_1-a-b+1}(\mathbb{R})}.$$

Furthermore, for all $x_1 > M_V$, $\mathfrak{f}_{m_1,a,b}^{(1),V} : x \mapsto Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x)\alpha(x)P_b^{m_1-a}(\alpha,\dots,\alpha^{(b)})(x)$ is bounded since it is continuous and a $\underset{|x|\to\infty}{O}\left(V'(x)^{-2}\right)$ by Lemma 2.4.13. We conclude, by Lemma

2.9.3 that, with $\mathfrak{f}^{(1),V}_{m_1,a,b}$ being given in (2.40),

$$\sum_{a=0}^{m_1-1}\sum_{b=0}^{m_1-a}\int_{M_V}^{+\infty} dx_1^2\partial_1^{m_1-a-b}\mathfrak{g}(x_1,x_1)^2\mathfrak{f}^{(1),V}_{m_1,a,b}(x_1)^2$$
$$\leq C(m_1)\max_{\substack{0\leq a<m_1\\0\leq b\leq m_1-a}}\|\mathfrak{f}^{(1),V}_{m_1,a,b}\|^2_{L^\infty([-M_V,M_V]^c)}\int_{M_V}^{+\infty}\|\mathfrak{g}(.,x_1)\|^2_{H^{m_1-a-b+1}(\mathbb{R})}dx_1$$
$$\leq C(m_1)\max_{\substack{0\leq a<m_1\\0\leq b\leq m_1-a}}\|\mathfrak{f}^{(1),V}_{m_1,a,b}\|^2_{L^\infty([-M_V,M_V]^c)}\|\mathfrak{g}\|^2_{H^{m_1+1}(\mathbb{R}^2)}$$
$$\leq C(m_1)\left[1+4P^2\pi^2\|\rho_V\|_{L^\infty(\mathbb{R})}C_3(V,n)\right]\max_{\substack{0\leq a<m_1\\0\leq b\leq m_1-a}}\|\mathfrak{f}^{(1),V}_{m_1,a,b}\|^2_{L^\infty([-M_V,M_V]^c)}\|\partial_3^{m_3}f(.,z)\|^2_{H^{m_1+1}(\mathbb{R}^2)}.$$

It just remains to bound

$$\int_{M_V}^{+\infty}dx_1\frac{Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)^2}{\rho_V(x_1)^2}\left\{\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty}dt\rho_V(t)\partial_1^{m_1-a-b+1}\mathfrak{g}(t,x_1)\alpha(t)P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)(t)\right]^2\right.$$
$$\left.+\left[\int_{x_1}^{\mathrm{sgn}(x_1)\infty}dt\rho_V(t)\partial_1^{m_1-a-b}\mathfrak{g}(t,x_1)\left[\alpha P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)\right]'(t)\right]^2\right\}.$$

For the first term, we use Cauchy-Schwarz inequality, Lemma 2.4.13 and the function $\mathfrak{f}^{(3),V}_{m_1,a,b}$ defined in (2.44) to get

$$\frac{|Q_a^{m_1}\left(\theta,\dots,\theta^{(a)}\right)(x_1)|^2}{\rho_V(x_1)^2}\left|\int_{x_1}^{\mathrm{sgn}(x_1)\infty}dt\rho_V(t)\partial_1^{m_1-a-b+1}\mathfrak{g}(t,x_1)\alpha(t)P_b^{m_1-a}\left(\alpha,\dots,\alpha^{(b)}\right)(t)\right|^2$$
$$\leq\|\mathfrak{f}^{(3),V}_{m_1,a,b}\|^2_{L^\infty([-M_V,M_V]^c)}\|\mathfrak{g}(.,x_1)\|^2_{H^{n_1+1}(\mathbb{R})}$$
$$\leq 2\|\mathfrak{f}^{(3),V}_{m_1,a,b}\|^2_{L^\infty([-M_V,M_V]^c)}\left(1+4P^2\pi^2\|\rho_V\|_{L^\infty(\mathbb{R})}C_3(V,n)\right)$$
$$\times\|\partial_2^{m_2}\dots\partial_{a-1}^{m_{a-1}}\partial_{a+1}^{m_a}\dots\partial_{p+1}^{m_p}f(.,x_2,\dots,x_{a-1},x_1,x_a,\dots,x_p)\|^2_{H^{m_1+1}(\mathbb{R})}.$$

We proceed in the exact same way for the second term and do the same thing on $]-\infty,-M_V]$, for every term we dealt with. Finally, by integrating with respect to x_1, collecting all the terms and then integrating over $x_1\in[-M_V,M_V]^c$ and over $x_2,\dots,x_p\in\mathbb{R}$, we get

$$\left\|\Theta^{(a)}\left[\widetilde{\Xi^{-1}}[\partial_2^{m_2}\dots\partial_{a-1}^{m_{a-1}}\partial_{a+1}^{m_a}\dots\partial_{p+1}^{m_p}f]^{(m_1)}\right]\right\|_{L^2([-M_V,M_V]^c\times\mathbb{R}^{p-1})}\leq C_{10}(V,n)\|f\|_{H^{n+1}(\mathbb{R}^{p+1})}$$

with $C_{10}(V,n)$ defined by

$$C_{10}(V,n)^2\overset{(\mathrm{def})}{=}C(n)\max_{m_1\leq n}\left\{\left(\|\rho_V\|_{L^\infty(\mathbb{R})}+4P^2\pi^2\|\rho_V\|^3_{L^\infty(\mathbb{R})}C_{\mathcal{L}}^2\right)\max_{\substack{0\leq a<m_1\\j\in\{1,2\}}}\|\mathfrak{f}^{(j),V}_{m_1,a,m_1-a}\|^2_{L^\infty([-M_V,M_V]^c)}\right.$$
$$\left.+\left[1+4P^2\pi^2\|\rho_V\|_{L^\infty(\mathbb{R})}C_3(V,n)\right]\max_{\substack{0\leq a<m_1\\0\leq b\leq m_1-a\\j\in\{1,3\}}}\|\mathfrak{f}^{(j),V}_{m_1,a,b}\|^2_{L^\infty([-M_V,M_V]^c)}\right\}. \quad (2.62)$$

It just remains to bound $\int_{-M_V}^{M_V} \widetilde{\Xi^{-1}}[h(.,x_1)]^{(m_1)}(x_1)^2 dx_1$. For that, we use (2.55)

$$\int_{-M_V}^{M_V} \widetilde{\Xi^{-1}}[h(.,x_1)]^{(m_1)}(x_1)^2 dx_1$$
$$\leq C(m_1) \int_{-M_V}^{M_V} dx_1 \Bigg\{ \frac{S^{m_1}(\theta,\dots,\theta^{(m_1)})(x_1)^2}{\rho_V(x_1)^2} \left(\int_{x_1}^{\mathrm{sgn}(x_1)\infty} g(t,x_1)\rho_V(t)dt \right)^2$$
$$+ \sum_{a=0}^{m_1-2} R_{m_1-a}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)(x_1)^2 [\partial_1^a g(x_1,x_1)]^2 + [\partial_1^{m_1-1} g(x_1,x_1)]^2 \Bigg\}.$$

By the same procedure as before, we first deal with $\mathfrak{c}$ defined at the beginning of the proof, this yields, with $\mathcal{I}_a$ defined in Theorem 2.4.15

$$\int_{-M_V}^{M_V} \mathfrak{c}(x_1)^2 dx_1 \left[\frac{S^{m_1}(\theta,\dots,\theta^{(m_1)})(x_1)^2}{\rho_V(x_1)^2} \left(\int_{x_1}^{\mathrm{sgn}(x_1)\infty} \rho_V(t)dt \right)^2 + R_{m_1}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)(x_1)^2 \right]$$
$$\leq C(m_1)\|h\|_{H^1(\mathbb{R}^2)}^2 \|\rho_V\|_{L^\infty(\mathbb{R})} \left(1 + 4P^2\pi^2 \|\rho_V\|_{L^\infty(\mathbb{R})}^2 C_{\mathcal{L}}^2\right) \Bigg\{ \|\mathcal{I}_1^V\|_\infty^2 \left\| S^{m_1}(\theta,\dots,\theta^{(m_1)}) \right\|_{L^\infty([-M_V,M_V])}^2$$
$$+ \left\| R_{m_1}^{m_1}\left(\theta,\dots,\theta^{(m_1-1)}\right) \right\|_{L^\infty([-M_V,M_V])}^2 \Bigg\}.$$

Hence as before, we can replace g by $\mathfrak{g}$ and conclude with the last bounds:

$$\int_{-M_V}^{M_V} \frac{S^{m_1}(\theta,\dots,\theta^{(m_1)})(x_1)^2}{\rho_V(x_1)^2} \left(\int_{x_1}^{\mathrm{sgn}(x_1)\infty} \mathfrak{g}(t,x_1)\rho_V(t)dt \right)^2 dx_1$$
$$\leq \|\mathfrak{g}\|_{L^2(\mathbb{R}^2)}^2 \left\| \mathcal{I}_2^V S^{m_1}(\theta,\dots,\theta^{(m_1)}) \right\|_{L^\infty([-M_V,M_V])}^2$$

and by Cauchy-Schwarz inequality,

$$\int_{-M_V}^{M_V} dx_1 \Big(\sum_{a=0}^{m_1-2} R_{m_1-a}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)(x_1)^2 \partial_1^a \mathfrak{g}(x_1,x_1)^2 + \partial_1^{m_1-1}\mathfrak{g}(x_1,x_1)^2 \Big)$$
$$\leq \left\{ 1 + \max_{a\in[\![0,m_1-2]\!]} \left\| R_{m_1-a}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right) \right\|_{L^\infty([-M_V,M_V])}^2 \right\}$$
$$\times \sum_{a=0}^{m_1-1} \int_{-M_V}^{M_V} dx_1 \int_{+\infty}^{x_1} dt \partial_1^{a+1}\mathfrak{g}(t,x_1)\partial_1^a \mathfrak{g}(t,x_1)$$
$$\leq C(m_1) \left\{ 1 + \max_{a\in[\![0,m_1-2]\!]} \left\| R_{m_1-a}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right) \right\|_{L^\infty([-M_V,M_V])}^2 \right\} \|\mathfrak{g}\|_{H^{m_1}(\mathbb{R}^2)}^2.$$

Moreover, by Lemma 2.9.3

$$\|\mathfrak{g}\|_{H^{m_1}(\mathbb{R}^2)} \leq \|h\|_{H^{m_1}(\mathbb{R}^2)} + 2P\pi \left\| \rho_V \widetilde{\Xi_1^{-1}}[h] \right\|_{H^{m_1}(\mathbb{R}^2)} \leq [1 + 2P\pi C_3(V,n)] \|h\|_{H^{m_1}(\mathbb{R}^2)}.$$

We can then conclude that

$$\int_{-M_V}^{M_V} dx_1 \Big(\sum_{a=0}^{m_1-2} R_{m_1-a}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)(x_1)^2 \partial_1^a \mathfrak{g}(x_1,x_1)^2 + \partial_1^{m_1-1}\mathfrak{g}(x_1,x_1)^2\Big)$$

$$\leq C(m_1)\left[1+2P\pi C_3(V,n)\right]\left\{1+\max_{a\in[\![0,m_1-2]\!]}\left\|R_{m_1-a}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)\right\|^2_{L^\infty([-M_V,M_V])}\right\}\|h\|^2_{H^{m_1}(\mathbb{R}^2)}.$$

Thus by integrating with respect to $z\in\mathbb{R}$, we get,

$$\left\|\Theta^{(a)}\left[\widetilde{\Xi^{-1}}[\partial_2^{m_2}\dots\partial_{a-1}^{m_{a-1}}\partial_{a+1}^{m_a}\dots\partial_{p+1}^{m_p}f]^{(m_1)}\right]\right\|_{L^2([-M_V,M_V]\times\mathbb{R}^{p-1})}\leq C_{11}(V,n)\|\partial_3^{m_3}f\|_{H^{m_1+1}(\mathbb{R}^{p+1})}$$

with $C_{11}(V,n)>0$ defined by

$$\begin{aligned}C_{11}(V,n)^2 &\overset{(\mathrm{def})}{=} \max_{m_1\leq n} C(m_1)\Bigg\{\|\rho_V\|_{L^\infty(\mathbb{R})}\left(1+4P^2\pi^2\|\rho_V\|^2_{L^\infty(\mathbb{R})}C_{\mathcal{L}}^2\right)\\
&\times\Big\{\|\mathcal{I}_1^V\|_\infty^2\left\|S^{m_1}(\theta,\dots,\theta^{(m_1)})\right\|^2_{L^\infty([-M_V,M_V])}+\left\|R_{m_1}^{m_1}\left(\theta,\dots,\theta^{(m_1-1)}\right)\right\|^2_{L^\infty([-M_V,M_V])}\\
&+[1+2P\pi C_3(V,n)]^2\Bigg(\|\mathcal{I}_2^V\|_\infty^2\left\|S^{m_1}(\theta,\dots,\theta^{(m_1)})\right\|^2_{L^\infty([-M_V,M_V])}\\
&+1+\max_{a\in[\![0,m_1-2]\!]}\left\|R_{m_1-a}^{m_1}\left(\theta,\dots,\theta^{(m_1-1-a)}\right)\right\|^2_{L^\infty([-M_V,M_V])}\Bigg)\Big\}.\end{aligned}\tag{2.63}$$

Collecting the bounds on the L^2-norms of $\Theta^{(a)}\left[\widetilde{\Xi^{-1}}[\partial_2^{m_2}\dots\partial_{a-1}^{m_{a-1}}\partial_{a+1}^{m_a}\dots\partial_{p+1}^{m_p}f]^{(m_1)}\right]$ on $[-M_V,M_V]^c\times\mathbb{R}$ and $[-M_V,M_V]\times\mathbb{R}$, we obtain:

$$\left\|\Theta^{(a)}\left[\widetilde{\Xi^{-1}}[\partial_2^{m_2}\dots\partial_{a-1}^{m_{a-1}}\partial_{a+1}^{m_a}\dots\partial_{p+1}^{m_p}f]^{(m_1)}\right]\right\|_{L^2(\mathbb{R}^p)}\leq 2\max_{i\in\{10,11\}}C_i(V,n)\|h\|_{H^{m_1+1}(\mathbb{R}^{p+1})}.$$

By combining the above equation together with (2.60) and taking the supremum over $m\leq n$, we get

$$\left\|\Theta^{(a)}\circ\widetilde{\Xi_1^{-1}}[f]\right\|_{H^n(\mathbb{R}^p)}\leq \mathtt{C}(\Theta^{(a)}\circ\widetilde{\Xi_1^{-1}},H^n)\|f\|_{H^{n+1}(\mathbb{R}^{p+1})}$$

with, C_{10} and C_{11} being given in (2.62) and (2.63), $\mathcal{I}_a^V$ being given in (2.48);

$$\begin{aligned}\mathtt{C}(\Theta^{(a)}\circ\widetilde{\Xi_1^{-1}},H^n) &\overset{(\mathrm{def})}{=} 2\max_{i\in\{10,11\}}C_i(V,n)\\
&+2\sqrt{(1+4P^2\pi^2\|\rho_V\|^2_{L^\infty(\mathbb{R})}C_{\mathcal{L}}^2)\left(\|\mathcal{I}_2^V\|^2_{L^\infty(\mathbb{R})}+\|\rho_V\|_{L^\infty(\mathbb{R})}\left\|\mathcal{I}_1^V\right\|^2_{L^\infty(\mathbb{R})}\right)}.\end{aligned}\tag{2.64}$$

This yields the conclusion. The fact that, upon choosing the potential $V_{\phi,t}$ with $\phi\in\cap_{k\geq 0}H^k(\mathbb{R})$, $t\mapsto\mathtt{C}(\Theta^{(a)}\circ\widetilde{\Xi_1^{-1}},H^n)$ is continuous is shown in Proposition 2.10.8. □

2.5 Asymptotic expansion of the linear statistics

2.5.1 Loop equations for general functions

We are now ready to state the loop equations, we recall the definition of a linear statistic was defined in (2.3).

Theorem 2.5.1 (Dyson-Schwinger equations) *The level 1 Schwinger-Dyson equation holds for all $\psi_1 \in \cap_{k\geq 0} H^k(\mathbb{R})$ and takes the form:*

$$\langle \psi_1 \rangle_{\mathcal{L}_N} = \frac{P}{N}\left\langle \partial_1 \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mu_V} + \frac{P}{N}\left\langle \partial_1 \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mathcal{L}_N} - P\left\langle \mathcal{D} \circ \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}. \tag{2.65}$$

For all $\psi_n \in \cap_{k\geq 0} H^k(\mathbb{R}^n)$, the level $n > 1$ Schwinger-Dyson equations reads:

$$\begin{aligned}\langle \psi_n \rangle_{\bigotimes^n \mathcal{L}_N} = {} & \frac{P}{N}\left\langle \partial_1 \widetilde{\Xi_1^{-1}}[\psi_n] \right\rangle_{\mu_V \bigotimes^{n-1} \mathcal{L}_N} + \frac{P}{N}\left\langle \partial_1 \widetilde{\Xi_1^{-1}}[\psi_n] \right\rangle_{\bigotimes^n \mathcal{L}_N} - P\left\langle \mathcal{D}_1 \circ \widetilde{\Xi_1^{-1}}[\psi_n] \right\rangle_{\bigotimes^{n+1} \mathcal{L}_N} \\ & - \frac{1}{N}\sum_{a=2}^{n}\left(\left\langle \Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}\big[\partial_a \psi_n\big] \right\rangle_{\bigotimes^{n-1} \mathcal{L}_N} + \left\langle \Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}\big[\partial_a \psi_n\big] \right\rangle_{\mu_V \bigotimes^{n-2} \mathcal{L}_N} \right). \end{aligned} \tag{2.66}$$

Proof See for example [BGK16, Prop 3.2.3] done in a similar context. □

2.5.2 Asymptotic expansion of linear statistics

The a priori bound on the linear statistics of Theorem 2.2.4, provides a starting point for obtaining the existence of their large-N asymptotic expansion in powers of N^{-1} up to any order through an analysis of the loop equations.

Theorem 2.5.2 *Let $\psi_k \in \bigcap_{m\geq 0} H^m(\mathbb{R}^k)$, then for all integer K, there exists a sequence $(d_a^{(k),V})_{a\geq \lceil k/2 \rceil} \in \mathbb{R}^N$ such that*

$$\langle \psi_k \rangle_{\bigotimes^k \mathcal{L}_N} = \sum_{a=\lceil k/2 \rceil}^{K} \frac{d_a^{(k),V}(\psi_k)}{N^a} + O\left(N^{-(K+1)}\right)$$

with

$$d_1^{(1),V}[\psi_1] = P\left\langle \partial_1 \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mu_V} + P\left\langle \Theta^{(2)} \circ \widetilde{\Xi_1^{-1}}\left[\partial_2 \mathcal{D} \circ \widetilde{\Xi^{-1}}[\psi_1]\right] \right\rangle_{\mu_V}.$$

Furthermore, there exists a sequence $(m_{K,k}) > 0$, increasing in K, such that for all $k \geq 1$ and $K \geq 0$, all $\psi_k \in \bigcap_{m\geq 0} H^m(\mathbb{R}^k)$,

$$\left| \langle \psi_k \rangle_{\bigotimes^k \mathcal{L}_N} - \sum_{a=\lceil k/2 \rceil}^{K} \frac{d_a^{(k),V}(\psi_k)}{N^a} \right| \leq \frac{\mathtt{C}_{\mathrm{rem}}(V,K,k)}{N^{K+1}} \mathcal{N}_{m_{K,k}}^{(k)}(\psi_k). \tag{2.67}$$

Above $\mathcal{N}_m^{(n)}(\psi_m) \overset{(\mathrm{def})}{=} \max\left(\|\psi_n\|_{W_n^\infty(\mathbb{R}^m)}, \|\psi_n\|_{H^n(\mathbb{R}^m)}\right)$, while $\mathtt{C}_{\mathrm{rem}}(V,K,k) > 0$ is a constant depending on V, K and k. Finally, under the choice of potential $V_{\phi,t}$ defined in Theorem 2.1.3, for $\phi \in \cap_{k\geq 0} H^k(\mathbb{R})$ the map $t \in [0,1] \mapsto \mathtt{C}_{\mathrm{rem}}(V_{\phi,t},K,k)$ is continuous.

Proof Using the first loop equation given in Theorem 2.5.1, we get:

$$\langle \psi_1 \rangle_{\mathcal{L}_N} = \frac{P}{N}\left\langle \partial_1 \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mu_V} + \frac{P}{N}\left\langle \partial_1 \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mathcal{L}_N} - P\left\langle \mathcal{D} \circ \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mathcal{L}_N \otimes \mathcal{L}_N}. \tag{2.68}$$

where we recall that $\widetilde{\Xi^{-1}} = \Xi^{-1} \circ \mathcal{X}$ defined in 2.4.10. The idea is to verify the hypotheses of Theorems 2.2.4 for each function involved in the Dyson-Schwinger equations. By Proposition 2.3.5 and Theorem 2.3.3, 2.4.14, 2.4.15, 2.4.17 and 2.4.18 and the fact that $\psi_k \in \cap_{m\geq 0} H^m(\mathbb{R}^k)$, we're ensured that all the norms are finite and that a n-linear statistic will be a $O\left(N^{-\frac{n}{2}(1-\varepsilon)}\right)$ where $\varepsilon > 0$ is fixed but can be chosen arbitrarly small.

We show by induction on K that there exists an asymptotic expansion up to $o(N^{-K})$ for any function $\psi_k \in \cap_{m\geq 0} H^m(\mathbb{R}^k)$ for all $k \leq 2K$.

For $K=1$, since the first term in (2.68) clearly contributes to the asymptotic expansion of $\langle \psi_1 \rangle_{\mathcal{L}_N}$ up to $o(N^{-1})$ so we focus on the two other terms. In (2.68), the second term is clearly a $o(N^{-1})$ since by Theorem 2.2.4, Theorem 2.4.14 and Theorem 2.4.15

$$\begin{aligned}|\left\langle \partial_1 \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mathcal{L}_N}| &\leq C_{1,\varepsilon} e^{K_V} N^{-(1-\varepsilon)/2} \left(\|\partial_1 \widetilde{\Xi^{-1}}[\psi_1]\|_{H^{1/2}(\mathbb{R})} + \|\partial_1 \widetilde{\Xi^{-1}}[\psi_1]\|_{W_1^\infty(\mathbb{R})} \right) \\ &\leq C_{1,\varepsilon} e^{K_V} N^{-(1-\varepsilon)/2} \left(\mathtt{C}(\widetilde{\Xi_1^{-1}}, H^2) + \mathtt{C}(\widetilde{\Xi_1^{-1}}, W_2^\infty) \right) \mathcal{N}_3(\psi_1).\end{aligned}$$

To obtain the expansion of the 2-linear statistic up to $o(N^{-1})$, we will need to use the loop equation at level 2 with $\psi_2 \overset{(\mathrm{def})}{=} \mathcal{D} \circ \widetilde{\Xi^{-1}}[\psi]$. Let $\psi_2 \in \cap_{k \geq 0} H^k(\mathbb{R}^2)$ be arbitrary for now. The level 2 equation reads:

$$\begin{aligned}\langle \psi_2 \rangle_{\overset{2}{\bigotimes} \mathcal{L}_N} = \frac{P}{N} \left\langle \partial_1 \widetilde{\Xi_1^{-1}}[\psi_2] \right\rangle_{\mu_V \bigotimes \mathcal{L}_N} + \frac{P}{N} \left\langle \partial_1 \widetilde{\Xi_1^{-1}}[\psi_2] \right\rangle_{\overset{2}{\bigotimes} \mathcal{L}_N} - \frac{1}{N} \left\langle \Theta^{(2)} \circ \widetilde{\Xi_1^{-1}}\big[\partial_2 \psi_2\big] \right\rangle_{\mathcal{L}_N} \\ - P \left\langle \mathcal{D}_1 \circ \widetilde{\Xi_1^{-1}}[\psi_2] \right\rangle_{\overset{3}{\bigotimes} \mathcal{L}_N} - \frac{1}{N} \left\langle \Theta^{(2)} \circ \widetilde{\Xi_1^{-1}}\big[\partial_2 \psi_2\big] \right\rangle_{\mu_V}. \quad (2.69)\end{aligned}$$

The first term is a $o(N^{-1})$ as a 1-linear statistic $\langle \psi \rangle_{\mathcal{L}_N}$ where $\psi(x) \overset{(\mathrm{def})}{=} \int_{\mathbb{R}} \partial_1 \widetilde{\Xi_1^{-1}}[\psi_2](x,y) d\mu_V(y)$. This function is indeed in $H^1(\mathbb{R})$ because of Theorem 2.4.14

$$\|\psi\|^2_{H^1(\mathbb{R})} \leq \|\rho_V\|_\infty \|\partial_1 \widetilde{\Xi_1^{-1}}[\psi_2]\|^2_{H^1(\mathbb{R}^2)} \leq \|\rho_V\|_\infty \mathtt{C}(\widetilde{\Xi_1^{-1}}, H^2)^2 \|\psi_2\|^2_{H^3(\mathbb{R}^2)}$$

and in $W_1^\infty(\mathbb{R})$ by Theorem 2.4.15

$$\|\psi\|_{W_1^\infty(\mathbb{R})} \leq \|\partial_1 \widetilde{\Xi_1^{-1}}[\psi_2]\|_{W_1^\infty(\mathbb{R}^2)} \leq \mathtt{C}(\widetilde{\Xi_1^{-1}}, W_2^\infty) \|\psi_2\|_{W_3^\infty(\mathbb{R}^2)}.$$

Thus by the a priori bound Theorem 2.2.4, we get

$$\left| \frac{P}{N} \left\langle \partial_1 \widetilde{\Xi_1^{-1}}[\psi_2] \right\rangle_{\mu_V \bigotimes \mathcal{L}_N} \right| \leq \frac{P C_{1,\varepsilon} e^{K_V}}{N^{1+(1-\varepsilon)/2}} \left[\mathtt{C}(\widetilde{\Xi_1^{-1}}, W_2^\infty) + \|\rho_V\|_\infty^{1/2} \mathtt{C}(\widetilde{\Xi_1^{-1}}, H^2) \right] \mathcal{N}_3^{(2)}(\psi_2). \quad (2.70)$$

The following two terms in (2.69) are also a $o(N^{-1})$ by the same reasons as before. By Theorem 2.2.4, the 3-linear statistics is a $o(N^{-1})$ for $\varepsilon > 0$ small enough. Hence, we obtain the expansion:

$$\langle \psi \rangle_{\mathcal{L}_N} = \frac{d_1^{(1)}(\psi_1)}{N} + o(N^{-1}) \text{ and } \langle \psi_2 \rangle_{\mathcal{L}_N \otimes \mathcal{L}_N} = \frac{d_1^{(2)}(\psi_2)}{N} + o(N^{-1})$$

where

$$d_1^{(1)}(\psi_1) \overset{(\mathrm{def})}{=} P \left\langle \partial_1 \widetilde{\Xi^{-1}}[\psi_1] \right\rangle_{\mu_V} + P \left\langle \Theta^{(2)} \circ \widetilde{\Xi_1^{-1}} \left[\partial_2 \mathcal{D} \circ \widetilde{\Xi^{-1}}[\psi_1] \right] \right\rangle_{\mu_V}$$

and

$$d_1^{(2)}(\psi_2) \overset{(\mathrm{def})}{=} - \left\langle \Theta^{(2)} \circ \widetilde{\Xi_1^{-1}}\big[\partial_2 \psi_2\big] \right\rangle_{\mu_V}.$$

More generally, suppose the desired expansion for $\langle \psi_k \rangle_{\overset{k}{\bigotimes} \mathcal{L}_N}$ holds up to $o(N^{-n})$ for all $k \in [\![0, 2n]\!]$ and for any function $\psi_k \in \cap_{m \geq 0} H^m(\mathbb{R}^k)$. Additionnaly, suppose that (2.67) is true for all $k \in [\![1, K-2]\!]$. Then, taking a general function $\psi_{2n+2} \in \cap_{m \geq 0} H^m(\mathbb{R}^{2n+2})$, the $(2n+2)$-th equation involves the $2n+3$ linear statistic $\left\langle \mathcal{D}_1 \circ \Xi_1^{-1}[\psi_{2n+2}] \right\rangle_{\overset{2n+3}{\bigotimes} \mathcal{L}_N}$ (see (2.66)). By Theorem 2.2.4, it will be a $o\left(N^{-(n+1)}\right)$ for ε small enough. The other terms will be either, $2n+1$-linear statistics with a N^{-1} prefactor and therefore behave like $O(N^{-(n+1)})$ for ε small enough, or either be $2n$-linear

statistics with a prefactor N^{-1}. For the latter, by hypothesis, we know the asymptotic expansion up to $O(N^{-n})$, thus with the prefactor N^{-1}, we deduce the following expansion for $\langle \psi_{2n+2} \rangle_{\bigotimes^{2n+2} \mathcal{L}_N}$

$$\langle \psi_{2n+2} \rangle_{\bigotimes^{2n+2} \mathcal{L}_N} = \frac{d_{n+1}^{(2n+2)}(\psi_{2n+2})}{N^{n+1}} + o\left(N^{-(n+1)}\right)$$

We will deduce from that, the asymptotic expansion of $\langle \psi_{2n+1} \rangle_{\bigotimes^{2n+1} \mathcal{L}_N}$ for a general ψ_{2n+1} belonging to $\cap_{k\geq 0} H^k(\mathbb{R}^{2n+1})$. In the $(2n+1)$-*th* loop equation (2.66), the $2n+2$ linear statistic will yield a non-trivial term of order $N^{-(n+1)}$ *i.e.*

$$\left\langle \mathcal{D}_1 \circ \Xi_1^{-1}[\psi_{2n+1}] \right\rangle_{\bigotimes^{2n+2} \mathcal{L}_N} = \frac{d_{n+1}^{(2n+2)}\left(\mathcal{D}_1 \circ \widetilde{\Xi_1^{-1}}[\psi_{2n+1}]\right)}{N^{n+1}} + o\left(N^{-(n+1)}\right).$$

Again, the $(2n+1)$-linear statistics with a prefactor N^{-1} will be $o(N^{-(n+1)})$. Finally for the $2n$ and $2n-1$ linear statistics with the prefactor N^{-1} appearing in the $(2n+1)$-*th* loop equation, we know the asymptotic expansion up to $o(N^{-(n+1)})$ by hypothesis.

To conclude on the asymptotic expansion up to $o(N^{-(n+1)})$ for the $2n$-linear statistics, just notice that for each term appearing in the $2n$ equation each term will either be a $2n+1$-linear statistics for which we know the asymptotic expansion up to $o(N^{-(n+1)})$, or a linear statistic for which we know, by hypothesis, the asymptotic expansion up to $o(N^{-n})$ (of order $2n$,$2n-1$ or $2n-2$), preceded by a factor N^{-1}. We can therefore conclude on the existence of the asymptotic expansion up to $o(N^{-(n+1)})$for the $2n$-linear statistics. Then applying the same arguments for $2n-1$,... 1-linear statistics allows us to conclude that the induction step is established.

Finally, to conclude on (2.67), one just has to notice that for all $n \geq 1$ and $\psi_n \in \cap_{k\geq 0} H^k(\mathbb{R}^n)$, for all $K \geq \lceil n/2 \rceil$,

$$\langle \psi_n \rangle_{\bigotimes^n \mathcal{L}_N} - \sum_{a=\lceil n/2 \rceil}^{K} \frac{d_a^{(n)}(\psi_n)}{N^a} = \frac{d_{K+1}^{(n)}(\psi_n)}{N^{K+1}} + \mathfrak{R}_{K+1}^{(n)}(\psi_n) \qquad \text{and} \qquad \mathfrak{R}_{K+1}^{(n)}(\psi_n) = o(N^{-(K+1)}).$$

Above, the remainder $\mathfrak{R}_{K+1}^{(n)}(\psi_n)$ contains all the negligible (by the a priori bound) statistics involving the operators $\widetilde{\Xi_1^{-1}}$, $\mathcal{D}_1 \circ \widetilde{\Xi_1^{-1}}$ and $\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}$, thus just as in (2.70), by using continuity of the different operators involved in each of the statistics, there exists $m_{K,n}^{(1)} > 0$, a polynomial $\mathtt{Q}_{K,n}^{(1)}$ in e^{K_V}, $\|\rho_V\|_\infty^{1/2}$, $\left(\mathtt{C}(\widetilde{\Xi_1^{-1}}, H^i)\right)_{0\leq i\leq m_{K,n}^{(1)}}$ $\left(\mathtt{C}(\widetilde{\Xi_1^{-1}}, W_i^\infty)\right)_{0\leq i\leq m_{K,n}^{(1)}}$ and $\left(\mathtt{C}(\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}, H^n)\right)_{0\leq i\leq m_{K,n}^{(1)}}$ with coefficients independent of V and a constant $C(K,n) > 0$ such that

$$\left|\mathfrak{R}_{K+1}^{(n)}[\psi_n]\right| \leq \frac{C^{(1)}(K,n)}{N^{K+1}} \mathtt{Q}_{K,n}^{(1)} \mathcal{N}_{m_{K,n}^{(1)}}^{(n)}(\psi_n).$$

To bound, $d_{K+1}^{(n)}(\psi_n)$ and extract the V-dependance, one just notices that it is a sum of linear statistics, involving as before the previous operators. By contintuity of the operators, there exists a polynomial $\mathtt{Q}_{K,n}^{(2)}$ in the previous operator norms and $\|\rho_V\|_\infty^{1/2}$ with coefficients independent of V such that

$$\left|d_{K+1}^{(n)}(\psi_n)\right| \leq C^{(2)}(K,n) \mathtt{Q}_{K,n}^{(2)} \mathcal{N}_{m_{K,n}^{(2)}}^{(n)}(\psi_n).$$

Thus setting $\mathtt{C}_{\mathrm{rem}}(V,K,n) = C^{(1)}(K,n)\mathtt{Q}_{K,n}^{(1)} + C^{(2)}(K,n)\mathtt{Q}_{K,n}^{(2)}$ and $m_{K,n} \overset{(\mathrm{def})}{=} \max\left(m_{K,n}^{(1)}, m_{K,n}^{(2)}\right)$ allows us to conclude about 2.67. The fact that $t \in [0,1] \mapsto \mathtt{C}_{\mathrm{rem}}(V_{\phi,t}, K, n)$ is continuous follows from the fact it is a polynomial in building blocks which are continuous as it is shown in Appendix 2.10, Lemma 2.10.10, Proposition 2.10.8. □

2.6 Parameter continuity of the equilibrium measure

We want to conclude about the asymptotic expansion of $\log \mathcal{Z}_N[V_{G,\phi}]$ for a smooth ϕ, by inserting the asymptotic expansion of the linear statistics of Theorem 2.5.2 in Lemma 2.7.3. In order to make that step rigorous, it is necessary to prove that all the linear statistics integrated with respect to the probability measure $\mathbb{P}_N^{V_{G,\phi,t}}$ with $t \in [0,1]$, when integrating them with respect to t, yield a definite and finite integral. Since all the quantities depend on t through $\mu_{V_{G,\phi,t}}$, we first prove a continuity result for $t \mapsto \rho_{V_{G,\phi,t}}$. The result that we are going to prove does not depend on the specificity of the Gaussian potential so in the following, we set $V_{\phi,t} : x \mapsto V(x) + t\phi(x)$ with V satisfying the assumptions 2.1.1.

While from the measure point of view, it is easy to show that $t \mapsto \mu_{V_{\phi,t}}$ is continuous for the weak topology of measures, it is not sufficient to deduce the continuity of the quantities involved in our problem. Indeed, in the controls we showed in Section 2.4 quantities like L^∞-norm of derivatives of ρ_{V_t} and $C_{\mathcal{L}}$, it will be necessary to show that $\|\rho_{V_{\phi,t}} - \rho_{V_{\phi,t_0}}\|_{W_i^\infty(\mathbb{R})} \underset{t\to t_0}{\longrightarrow} 0$ for all $t_0 \in [0,1]$.

2.6.1 Setting for Banach fixed-point theorem

Let $\phi \in \mathcal{C}^\infty(\mathbb{R})$, such that for all $k \in \mathbb{N}$, $\phi^{(k)} \in L^2(\mathbb{R})$, let $t_0 \in [0,1]$, we define the function u_t by:

$$\rho_{V_{\phi,t}} = (1 + \delta t u_t)\rho_{V_{\phi,t_0}} \qquad ie \qquad u_t = \frac{\rho_{V_{\phi,t}} - \rho_{V_{\phi,t_0}}}{\delta t}\frac{1}{\rho_{V_{\phi,t_0}}} \tag{2.71}$$

where $t \neq t_0$ and $\delta t \overset{(\text{def})}{=} t - t_0$. We will show, by Banach fixed-point theorem, that $x \mapsto u_t(x) \in \mathcal{C}^\infty(\mathbb{R})$, by Lemma 2.2.1, is the unique fixed-point of a t continuous operator. This will allow us to deduce that $t \mapsto u_t$ is continuous for the $\bigcap_{k\in\mathbb{N}} W_k^\infty$-norm. The continuity of $t \mapsto \rho_{V_t}$ will then follow.

In order to construct the operator of interest, we start with the following lemma.

Lemma 2.6.1 *Let $t, t_0 \in [0,1]$,*

$$\lambda_{V_{\phi,t}} = \lambda_{V_{\phi,t_0}} + \delta t \int_{\mathbb{R}} \phi(x) d\mu_{V_{\phi,t_0}}(x) - 2P\delta t \iint_{\mathbb{R}^2} \log|x-y| u_t(x) d\mu_{V_{\phi,t_0}}(x) d\mu_{V_{\phi,t_0}}(y) + \int_{\mathbb{R}} \left[\log\left(1 + \delta t u_t(x)\right) - \delta t u_t(x)\right] d\mu_{V_{\phi,t_0}}(x). \tag{2.72}$$

Here $\lambda_{V_{\phi,t}}$ denotes the constant appearing in (2.4) *with potential $V_{\phi,t}$.*

Proof We integrate with respect to $\mu_{V_{\phi,t_0}}$ (2.4) to get

$$\lambda_{V_{\phi,t}} = \int_{\mathbb{R}} V_{\phi,t_0}(x) d\mu_{V_{\phi,t_0}}(x) + \delta t \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) - 2P \iint_{\mathbb{R}^2} \log|x-y| d\mu_{V_{\phi,t}}(x) d\mu_{V_{\phi,t_0}}(y) + \int_{\mathbb{R}} \log \rho_{V_{\phi,t}}(x) d\mu_{V_{\phi,t_0}}(x).$$

After using the fact that $\int_{\mathbb{R}} u_t d\mu_{V_{\phi,t_0}} = 0$, that $\rho_{V_{\phi,,t}} = (1 + \delta t u_t)\rho_{V_{\phi,t_0}}$ and the characterization (2.4) of $\mu_{V_{\phi,t_0}}$, this yields the result. □

To show that u_t is a fixed point of a t-continuous operator, we need to invert and control the operator $\mathcal{T} \overset{(\text{def})}{=} \mathcal{L} \circ \mathcal{A}^{-1}$ (these operators were inroduced in Definition 2.4.1)which will appear naturally when comparing $\rho_{V_{\phi,t}}$ to $\rho_{V_{\phi,t_0}}$.

Proposition 2.6.2 *We define the operator $\mathcal{T}$ by $\mathcal{T}[v] \overset{(\text{def})}{=} v - \mathcal{K}[v]$ for all $v \in L^2(\mu_{V_{\phi,t_0}})$, where*

$$\mathcal{K}[v](x) \overset{(\text{def})}{=} 2P \int_{\mathbb{R}} k(x,y) v(y) \rho_{V_{\phi,t_0}}(y) dy$$

and

$$k(x,y) \overset{(\text{def})}{=} \left(\log \frac{|x-y|}{1+|x|} - \int_{\mathbb{R}} \log \frac{|z-y|}{1+|z|} \rho_{V_{\phi,t_0}}(z) dz \right).$$

$\mathcal{T} : L^2(\mu_{V_{\phi,t_0}}) \to L^2(\mu_{V_{\phi,t_0}})$ *is bijective and for all* $n \geq 0$, $\mathcal{T}\left[W_n^\infty(\mathbb{R})\right] = W_n^\infty(\mathbb{R})$. *Finally, for all* $n \in \mathbb{N}$, *there exists* $C_{\mathcal{T},n} > 0$ *such that for any* $v \in W_n^\infty(\mathbb{R})$,

$$\left\| \mathcal{T}^{-1}[v] \right\|_{W_n^\infty(\mathbb{R})} \leq C_{\mathcal{T},n} \|v\|_{W_n^\infty(\mathbb{R})}. \tag{2.73}$$

Proof It was shown in **[DGM23]** that $\mathcal{T}[v] = -\mathcal{L} \circ \mathcal{A}^{-1}[v]$. $\mathcal{A}$ and $\mathcal{L}$ are unbounded operators on H, it was also shown there that:

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \overset{(\text{def})}{=} \{\phi \in \mathsf{H},\, \mathcal{A}[\phi] \in \mathsf{H}\} \to \mathsf{H} \quad \text{and} \quad \mathcal{L} : \mathcal{D}(\mathcal{A}) \to \mathsf{H}$$

are bijective, thus so is $\mathcal{T} : \mathsf{H} \to \mathsf{H}$.

The fact that k verifies $\|k(x,y)\|_{L^2(\mu_{V_{\phi,t_0}} \otimes \mu_{V_{\phi,t_0}})} < +\infty$ implies that $\mathcal{K} : L^2(\mu_{V_{\phi,t_0}}) \to L^2(\mu_{V_{\phi,t_0}})$ is an Hilbert-Schmidt operator thus compact and so $\mathcal{T}$ is a Fredholm operator. We now show that the kernel of $\mathcal{T}$ is trivial. Let $v \in L^2(\mu_{V_{\phi,t_0}})$ such that $\mathcal{T}[v] = 0$ so $v = \mathcal{K}[v]$. The RHS is in $H^1(\mu_{V_{\phi,t_0}})$, indeed we have:

$$\mathcal{K}[v]'(x) = -\mathcal{H}[v\rho_{V_{\phi,t_0}}](x) - \frac{\operatorname{sgn}(x)}{1+|x|} \int_{\mathbb{R}} v(y) d\mu_{V_{\phi,t_0}}(y) \in L^2(\mu_{V_{\phi,t_0}}).$$

Moreover since $\int_{\mathbb{R}} \mathcal{K}[v] d\mu_{V_{\phi,t_0}} = 0$, we conclude that $v \in \mathsf{H}$. We can now conclude that $v = 0$ by the bijectivity of $\mathcal{T}$ on H. Finally, by Fredhom alternative, $\mathcal{T}$ is invertible on $L^2(\mu_{V_{\phi,t_0}})$ since it is injective.

We now prove that for all $n \in \mathbb{N}$, $\mathcal{T}\left[W_n^\infty(\mathbb{R})\right] = W_n^\infty(\mathbb{R})$. We proceed by induction. For $n = 0$, let $f \in L^\infty(\mathbb{R}) \subset L^2(\mu_{V_{\phi,t_0}})$. There exists a unique $v \in L^2(\mu_{V_{\phi,t_0}})$ such that $\mathcal{T}[v] = f$ so $v = f + \mathcal{K}[v]$ but since f and $\mathcal{K}[v]$ are bounded, so is $v \in L^\infty(\mathbb{R})$. Reciprocally, if $v \in L^\infty(\mathbb{R})$ so is $\mathcal{T}[v]$, hence $\mathcal{T}\left[L^\infty(\mathbb{R})\right] = L^\infty(\mathbb{R})$. Finally let $v \in L^\infty(\mathbb{R})$,

$$\left\| \mathcal{T}[v] \right\|_{L^\infty(\mathbb{R})} \leq \left(1 + 2P \max_{x \in \mathbb{R}} \int_{\mathbb{R}} |k(x,y)| d\mu_{V_{\phi,t_0}}(y) \right) \|v\|_{L^\infty(\mathbb{R})}. \tag{2.74}$$

Now suppose $\mathcal{T}\left[W_n^\infty(\mathbb{R})\right] = W_n^\infty(\mathbb{R})$ is true and let's show it for $n+1$. Let $f \in W_{n+1}^\infty(\mathbb{R}) \subset W_n^\infty(\mathbb{R})$, so by hypothesis, there exists $v \in W_n^\infty(\mathbb{R})$ such that:

$$\begin{aligned} f^{(n)}(x) = v^{(n)}(x) + 2P \left(\log(1+|.|)\right)^{(n)}(x) \int_{\mathbb{R}} v(y) d\mu_{V_{\phi,t_0}}(y) - 2P \int_{\mathbb{R}} \log|x-y| (v\rho_{V_{\phi,t_0}})^{(n)}(y) dy \\ + 2P\delta_{n,0} \iint_{\mathbb{R}^2} \frac{\log|z-y|}{1+|z|} v(y) d\mu_{V_{\phi,t_0}}(y) d\mu_{V_{\phi,t_0}}(z). \end{aligned} \tag{2.75}$$

We deduce that $v^{(n)}$ is differentiable of derivative:

$$v^{(n+1)}(x) = f^{(n+1)}(x) - 2P \left(\log(1+|.|)\right)^{(n+1)}(x) \int_{\mathbb{R}} v(y) d\mu_{V_{\phi,t_0}}(y) - 2P\mathcal{H}\left[(v\rho_{V_{\phi,t_0}})^{(n)} \right](x), \tag{2.76}$$

where $\left(\log(1+|.|)\right)^{(n)}(x) = \dfrac{n!\operatorname{sgn}(x)}{(1+|x|)^n}$ if n is odd and $\dfrac{n!}{(1+|x|)^n}$ if n is even. Since the two first terms in the RHS of (2.76) are clearly bounded, we just have to show that $\mathcal{H}\left[(v\rho_{V_{\phi,t_0}})^{(n)}\right] \in H^1(\mathbb{R})$.

By boundedness of $f^{(n+1)}$ and $v^{(i)}$ for all $i \leq n$, we have:

$$\begin{aligned}(v\rho_{V_{\phi,t_0}})^{(n+1)} &= \rho_{V_{\phi,t_0}}\left(v^{(n+1)} + \sum_{k=0}^{n}\binom{n+1}{k}\frac{\rho_{V_{\phi,t_0}}^{(n+1-k)}}{\rho_{V_{\phi,t_0}}}v^{(k)}\right)\\ &= \rho_{V_{\phi,t_0}}\Bigg(f^{(n+1)} - 2P\left(\log(1+|.|)\right)^{(n+1)}\int_{\mathbb{R}} v(y)d\mu_{V_{\phi,t_0}}(y) - 2P\mathcal{H}\left[(v\rho_{V_{\phi,t_0}})^{(n)}\right]\\ &\qquad + \sum_{k=0}^{n}\binom{n+1}{k}\frac{\rho_{V_{\phi,t_0}}^{(n+1-k)}}{\rho_{V_{\phi,t_0}}}v^{(k)}\Bigg) \in L^2(\mathbb{R}).\end{aligned}$$

Thus, it holds that $\mathcal{H}\left[(v\rho_{V_{\phi,t_0}})^{(n)}\right] \in H^1(\mathbb{R})$ and that it is bounded. Hence it proves that $v^{(n+1)} \in L^\infty(\mathbb{R})$ and hence $W^\infty_{n+1}(\mathbb{R}) \subset \mathcal{T}\left[W^\infty_{n+1}(\mathbb{R})\right]$. Conversely, if $v \in W^\infty_{n+1}(\mathbb{R})$, then $f \in W^\infty_n(\mathbb{R})$ by hypothesis and just as before, we show that (2.76) holds. We conclude that $f^{(n+1)} \in L^\infty(\mathbb{R})$ again by showing that $\mathcal{H}\left[(v\rho_{V_{\phi,t_0}})^{(n)}\right]$ is bounded by the fact that $v \in W^\infty_{n+1}(\mathbb{R})$. This establishes that $W^\infty_{n+1}(\mathbb{R}) = \mathcal{T}\left[W^\infty_{n+1}(\mathbb{R})\right]$.

Thus for all $n \in \mathbb{N}$, $\mathcal{T} : W^\infty_n(\mathbb{R}) \to W^\infty_n(\mathbb{R})$ is a bijective operator. Furthermore, it is a bounded operator by the fact that for all $1 \leq i \leq n$, there exists $C > 0$ such that for all $v \in W^\infty_n(\mathbb{R})$, by Leibniz formula:

$$\begin{aligned}\left|\mathcal{T}[v]^{(i)}(x)\right| &\leq \|v^{(i)}\|_{L^\infty(\mathbb{R})} + 2P\left\|\left(\log(1+|.|)\right)^{(i)}\right\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\\ &\quad + 2P\|v\|_{W^\infty_i(\mathbb{R})}\sum_{k=0}^{i}\binom{i}{k}\sup_{z\in\mathbb{R}}\int_{\mathbb{R}}\left|\log\frac{|z-y|}{1+|z|}\right|.\left|\frac{\rho^{(k)}_{V_{\phi,t_0}}}{\rho_{V_{\phi,t_0}}}(y)\right|d\mu_{V_{\phi,t_0}}(y) \leq C\|v\|_{W^\infty_i(\mathbb{R})}.\end{aligned}$$

Above we used (2.75) and the fact that $\int_{\mathbb{R}}(v\rho_{V_{\phi,t_0}})^{(i)}(y)dy = 0$ so:

$$\begin{aligned}\int_{\mathbb{R}}\log|x-y|(v\rho_{V_{\phi,t_0}})^{(i)}(y)dy &= \int_{\mathbb{R}}\log|x-y|(v\rho_{V_{\phi,t_0}})^{(i)}(y)dy - \log(1+|x|)\int_{\mathbb{R}}(v\rho_{V_{\phi,t_0}})^{(i)}(y)dy\\ &= \int_{\mathbb{R}}\log\frac{|x-y|}{1+|x|}(v\rho_{V_{\phi,t_0}})^{(i)}(y)dy.\end{aligned}$$

Thus we conclude that $\sup_{i\in[\![1,n]\!]}\left\|\mathcal{T}[v]^{(i)}\right\|_{L^\infty(\mathbb{R})} \leq C\|v\|_{W^\infty_n(\mathbb{R})}$. The bound on $\left\|\mathcal{T}[v]\right\|_{L^\infty(\mathbb{R})}$ was shown in (2.74). We finally conclude that $\mathcal{T} : W^\infty_n(\mathbb{R}) \to W^\infty_n(\mathbb{R})$ is bounded bijective between Banach spaces and by Banach isomorphism theorem so is $\mathcal{T}^{-1}$, this establishes (2.73) and completes the proof. □

Remark 2.6.3 An explicit expression for $\mathcal{T}^{-1}$ is available Fredholm determinant theory for invertible Hilbert-Schmidt operators, see [GGK12, Section XII]. For all $v \in L^2_0(\mu_{V_{\phi,t_0}})$,

$$\begin{aligned}\mathcal{T}^{-1}[v](x) &\overset{(\text{def})}{=} v(x)\\ &+ \frac{1}{\det_2(I-\mathcal{K})}\sum_{n\geq 1}\frac{(-1)^n}{n!}\int_{\mathbb{R}^{n+1}}\begin{vmatrix} k(x,s) & k(x,t_1) & \dots & k(x,t_n)\\ k(t_1,s) & 0 & \dots & k(t_1,t_n)\\ \vdots & & & \vdots\\ k(t_n,s) & k(t_n,t_1) & \dots & 0\end{vmatrix}v(s)d\mu_{V_{\phi,t_0}}(s)\prod_{i=1}^{n}d\mu_{V_{\phi,t_0}}(t_i).\end{aligned} \tag{2.77}$$

Above $\det_2$ stands for the 2-determinant. This formula was established in [DGM23, Theorem 6.11].

We are now able to show that u_t is a fixed point of a certain operator. We recall that ϕ was introduced in the beginning of Subsection 2.6.1.

Proposition 2.6.4 *For all $t \in [0,1]$, u_t defined in (2.71) is the unique measurable function such that $\int_{\mathbb{R}} u_t(x) d\mu_{V_{\phi,t_0}}(x) = 0$ and which satisfies:*

$$u_t = \mathcal{T}^{-1} \circ \mathcal{V}_t[u_t]$$

where $\mathcal{V}_t[u] \overset{(\text{def})}{=} -\phi + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) + \delta t \mathcal{U}_t[u]$, and

$$\mathcal{U}_t[v](x) \overset{(\text{def})}{=} \left(-\phi(x) + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) + \mathcal{K}[v](x) + \int_{\mathbb{R}} \frac{\log\Big(1+\delta t v(y)\Big) - \delta t v(y)}{\delta t} d\mu_{V_{\phi,t_0}}(y) \right)^2$$
$$\times \int_0^1 (1-s) ds \exp\left\{ s\delta t \left(-\phi(x) + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) + \mathcal{K}[v](x) \right) \right.$$
$$\left. + \int_{\mathbb{R}} \frac{\log\Big(1+\delta t v(y)\Big) - \delta t v(y)}{\delta t} d\mu_{V_{\phi,t_0}}(y) \right\} + \int_{\mathbb{R}} \frac{\log\Big(1+\delta t v(y)\Big) - \delta t v(y)}{(\delta t)^2} d\mu_{V_{\phi,t_0}}(y).$$

Proof Lemma 2.6.1 allows one to substitute $\lambda_{V_{\phi,t}}$ in the representation for $\rho_{V_{\phi,t}}$ by (2.72) hence leading to

$$\rho_{V_{\phi,t}} = (1+\delta t u_t)\rho_{V_{\phi,t_0}} = \exp\left(-V_{\phi,t_0} - 2PU^{\rho_{V_{\phi,t_0}}} + \lambda_{V_{\phi,t_0}} - \delta t\phi + \delta t \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) - 2P\delta t U^{u_t \rho_{V_{\phi,t_0}}} \right.$$
$$\left. - 2P\delta t \iint_{\mathbb{R}^2} \log|y-z| u_t(z) d\mu_{V_{\phi,t_0}}(z) d\mu_{V_{\phi,t_0}}(y) + \int_{\mathbb{R}} \Big[\log\Big(1+\delta t u_t(y)\Big) - \delta t u_t(y) \Big] d\mu_{V_{\phi,t_0}}(y) \right).$$

Recognizing $\rho_{V_{\phi,t_0}}$ *via* the first three terms in the exponential, u_t has to satisfy the following relation for all $x \in \mathbb{R}$,

$$1 + \delta t u_t(x) = \exp\left\{ \delta t \Big(-\phi(x) + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) + 2P \int_{\mathbb{R}} \log\frac{|x-y|}{1+|x|} u_t(y) d\mu_{V_{\phi,t_0}}(y) \right.$$
$$\left. - 2P \iint_{\mathbb{R}^2} \log\frac{|y-z|}{1+|y|} u_t(z) d\mu_{V_{\phi,t_0}}(z) d\mu_{V_{\phi,t_0}}(y) + \frac{1}{\delta t} \int_{\mathbb{R}} \left[\log\left(1+\delta t u_t(y)\right) - \delta t u_t(y) \right] d\mu_{V_{\phi,t_0}}(y) \Big) \right\}.$$

Above, we have used that $-U^{u_t \rho_{V_{\phi,t_0}}}(x) = \displaystyle\int_{\mathbb{R}} \log\frac{|x-y|}{1+|x|} u_t(y) d\mu_{V_{\phi,t_0}}(y)$ which is justified by the fact that $\displaystyle\int_{\mathbb{R}} u(y) d\mu_{V_{\phi,t_0}}(y) = 0$. Conversely, any u such that $\int_{\mathbb{R}} u(y) d\mu_{V_{\phi,t_0}}(y) = 0$ and satisfying the previous relation, verifies for all $x \in \mathbb{R}$,

$$V_t(x) + 2PU^w(x) + \log w(x) = \int_{\mathbb{R}} \Big\{ V_t(y) + 2PU^w(y) + \log w(y) \Big\} d\mu_{V_{\phi,t_0}}(y)$$

where we have set $w \overset{(\text{def})}{=} (1+\delta t u)\rho_{V_{\phi,t_0}}$. Because of this equation, w can be written in exponential form as in (2.5), it is thus positive and of mass 1 which makes $d\mu(x) \overset{(\text{def})}{=} w(x)dx$ a probability measure which satisfies the equation characterizing $\mu_{V_{\phi,t}}$, hence, by unicity of the solution of (2.4), $\mu_{V_{\phi,t}} = \mu$ and thus $u = u_t$.

We now expand exp into its Taylor-integral series of order 2, *i.e.* $e^x = 1 + x + x^2 \int_0^1 (1-s)e^{sx} ds$. By using that

$$\mathcal{K}[v] = 2P \int_{\mathbb{R}} \log \frac{|x-y|}{1+|x|} u(y) d\mu_{V_{\phi,t_0}}(y) - 2P \iint_{\mathbb{R}^2} \log \frac{|y-z|}{1+|y|} v(z) d\mu_{V_{\phi,t_0}}(z) d\mu_{V_{\phi,t_0}}(y)$$

and $\mathcal{T}[v] = v - \mathcal{K}[v]$, we get:

$$\begin{aligned}
\mathcal{T}[u_t](x) &= -\phi(x) + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) \\
&+ \delta t \Bigg[\left(-\phi(x) + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) + \mathcal{K}[u_t](x) + \frac{1}{\delta t} \int_{\mathbb{R}} \left[\log\left(1 + \delta t u_t(y)\right) - \delta t u_t(y) \right] d\mu_{V_{\phi,t_0}}(y) \right)^2 \\
&\qquad \times \int_0^1 \exp\Big(s\delta t \big(-\phi(x) + \int_{\mathbb{R}} \phi(y) d\mu_{V_{\phi,t_0}}(y) + \mathcal{K}[u_t](x) \\
&+ \int_{\mathbb{R}} \frac{\log\left(1 + \delta t u_t(y)\right) - \delta t u_t(y)}{\delta t} d\mu_{V_{\phi,t_0}}(y) \big) \Big) (1-s) ds + \int_{\mathbb{R}} \frac{\log\left(1 + \delta t u_t(y)\right) - \delta t u_t(y)}{(\delta t)^2} d\mu_{V_{\phi,t_0}}(y) \Bigg].
\end{aligned}$$

We next use the invertibility of $\mathcal{T}$ to conclude. □

The next theorem shows that for each t sufficiently close to t_0, $\mathcal{T}^{-1} \circ \mathcal{V}_t$ is contractive on a ball of fixed radius. Let $n \geq 0$, denote for all $R > 0$, $\overline{\mathscr{B}_n(0,R)}$ the closed ball of radius R_n for the $W_n^\infty(\mathbb{R})$-norm.

Theorem 2.6.5 *For all $\varepsilon_n > 0$ small enough, there exists $R_n > 0$, such that for all $t \in]t_0 - \varepsilon_n, t_0 + \varepsilon_n[$, the operator*

$$\mathcal{T}^{-1} \circ \mathcal{V}_t : \left(\overline{\mathscr{B}_n(0,R_n)}, \|.\|_{W_n^\infty(\mathbb{R})} \right) \longrightarrow \left(\overline{\mathscr{B}_n(0,R_n)}, \|.\|_{W_n^\infty(\mathbb{R})} \right)$$

is well-defined and continuous. Furthermore it is contractive, i.e. there exists $k_n \in]0,1[$, such that for all $v, w \in \overline{\mathscr{B}_n(0,R_n)}$,

$$\left\| \mathcal{T}^{-1} \circ \mathcal{V}_t[v] - \mathcal{T}^{-1} \circ \mathcal{V}_t[w] \right\|_{W_n^\infty(\mathbb{R})} \leq k_n \|v - w\|_{W_n^\infty(\mathbb{R})}.$$

Moreover k_n is independent of t on $]t_0 - \varepsilon_n, t_0 + \varepsilon_n[$ for $\varepsilon_n > 0$ small enough.

Proof Let $\varepsilon > 0$, $t \in]t_0 - \varepsilon, t_0 + \varepsilon[$, $v \in \overline{\mathscr{B}_0(0,R)}$ for an arbitrary $0 < R < \dfrac{1}{\delta t}$ (because otherwise the term $\int_{\mathbb{R}} \log(1+\delta t v) \rho_{V_{\phi,t_0}}$ in $\mathcal{V}_t[v]$ might be ill-defined). We first show that $\|\mathcal{V}_t[v]\|_{L^\infty(\mathbb{R})} < +\infty$. First, by Taylor-Lagrange inequality, we deduce that

$$\left| \int_{\mathbb{R}} \frac{\log\left[1 + \delta t v(y)\right] - \delta t v(y)}{\delta t} d\mu_{V_{\phi,t_0}}(y) \right| \leq \frac{\|v\|_{L^\infty(\mathbb{R})}^2 \delta t}{2} \sup_{x \in [-\|v\|_\infty, \|v\|_\infty]} \frac{1}{(1+\delta t x)^2} \leq \frac{R^2 \delta t}{2(1-\delta t R)^2}.$$

By recalling the definition of $\mathcal{V}_t$ in Proposition 2.6.4 and using the convexity of $x \mapsto x^2$, we get:

$$\begin{aligned}
\|\mathcal{V}_t[v]\|_{L^\infty(\mathbb{R})} \leq \delta t \Bigg[3 \Bigg(4\|\phi\|_{L^\infty(\mathbb{R})}^2 + R^2 \left\| \int_{\mathbb{R}} \left| \log \frac{|.-y|}{1+|.|} \right| d\mu_{V_{\phi,t_0}}(y) dy \right\|_{L^\infty(\mathbb{R})}^2 + \delta t^2 \frac{R^4}{4(1-\delta t R)^4} \Bigg) \\
\times \exp\left\{ \delta t \left(2\|\phi\|_{L^\infty(\mathbb{R})} + R \left\| \int_{\mathbb{R}} \left| \log \frac{|.-y|}{1+|.|} \right| d\mu_{V_{\phi,t_0}}(y) dy \right\|_{L^\infty(\mathbb{R})} + \delta t \frac{R^2}{2(1-\delta t R)^2} \right) \right\} + \frac{R^2}{2(1-\delta t R)^2} \Bigg] \\
+ 2\|\phi\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

The RHS is of the form $2\|\phi\|_\infty + \delta t g_0(\delta t)$ where g_0 is a positive function and with these notations

$$\left\|\mathcal{T}^{-1}\circ\mathcal{V}_t[v]\right\|_{L^\infty(\mathbb{R})} \leq \mathcal{C}_{\mathcal{T},0}\left(2\|\phi\|_{L^\infty(\mathbb{R})} + \delta t g_0(\delta t)\right).$$

Therefore, by choosing t such that $|t-t_0|<\varepsilon_0$ for $\varepsilon_0>0$ small enough, there exists $R_0>0$ such that $2\|\phi\|_\infty C_{\mathcal{T},0} < R_0 < \dfrac{1}{\varepsilon_0}$ and

$$\mathcal{T}^{-1}\circ\mathcal{V}_t\left(\overline{\mathscr{B}_0(0,R_0)}\right) \subset \overline{\mathscr{B}_0(0,R_0)}.$$

This makes the operator $\mathcal{T}^{-1}\circ\mathcal{V}_t : \left(\overline{\mathscr{B}_n(0,R_n)}, \|.\|_{W_n^\infty(\mathbb{R})}\right) \longrightarrow \left(\overline{\mathscr{B}_n(0,R_n)}, \|.\|_{W_n^\infty(\mathbb{R})}\right)$ well-defined for all $|t-t_0|<\varepsilon_0$. For the contractivity, let $u,v\in\overline{\mathscr{B}(0,R_0)}$, we get by Proposition 2.6.2,

$$\left\|\mathcal{T}^{-1}\Big[\mathcal{V}_t[u]-\mathcal{V}_t[v]\Big]\right\|_{L^\infty(\mathbb{R})} \leq C_{\mathcal{T},0}\left\|\mathcal{U}_t[u]-\mathcal{U}_t[v]\right\|_{L^\infty(\mathbb{R})}.$$

where $\mathcal{U}_t$ was defined in Proposition 2.6.4. We now want to control $\left|\mathcal{U}_t[u](x)-\mathcal{U}_t[v](x)\right|$. We have, by decomposing the sum

$$\left|\mathcal{U}_t[u](x)-\mathcal{U}_t[v](x)\right| \leq \delta t\left(\Delta_1 + \Delta_2(x)\mathsf{E}[u](x) + \Delta_3(x)\mathfrak{h}[v](x)^2\right)$$

where

$$\begin{aligned}
\mathfrak{h}[w](x) &\overset{(\text{def})}{=} -\phi(x) + \int_{\mathbb{R}}\phi(y)d\mu_{V_{\phi,t_0}}(y) + \mathcal{K}[w](x) + \int_{\mathbb{R}}\frac{\log\left(1+\delta t w(y)\right)-\delta t w(y)}{\delta t}d\mu_{V_{\phi,t_0}}\\
\mathsf{E}[w](x) &\overset{(\text{def})}{=} \int_0^1 \exp\Big[s\delta t\mathfrak{h}[w](x)\Big](1-s)ds\\
\Delta_1 &\overset{(\text{def})}{=} \int_{\mathbb{R}}\left|\frac{\log(1+\delta t u(y))-\delta t u(y)-\log(1+\delta t v(y))+\delta t v(y)}{(\delta t)^2}\right|d\mu_{V_{\phi,t_0}}(y)\\
\Delta_2(x) &\overset{(\text{def})}{=} \left|\mathfrak{h}[u](x)^2-\mathfrak{h}[v](x)^2\right|\\
\Delta_3(x) &\overset{(\text{def})}{=} \left|\mathsf{E}[u](x)-\mathsf{E}[v](x)\right|.
\end{aligned}$$

First, Taylor Lagrange inequality leads to:

$$\Delta_1 \leq \sup_{y\in[-R_0,R_0]}\frac{|y|}{1+\delta t y}\|u-v\|_{L^\infty(\mathbb{R})} \leq \frac{R_0}{1-\delta t R_0}\|u-v\|_{L^\infty(\mathbb{R})}.$$

Furthermore, by using $a^2-b^2=(a+b)(a-b)$, that $\mathfrak{h}[u]$ and $\mathfrak{h}[v]$ are bounded

$$\begin{aligned}
\Delta_2 &\leq 2\max\left(\|\mathfrak{h}[u]\|_{L^\infty(\mathbb{R})},\|\mathfrak{h}[v]\|_{L^\infty(\mathbb{R})}\right)\left[\left|\mathcal{K}[u-v](x)\right| + \delta t\Delta_1\right]\\
&\leq \left(4\|\phi\|_{L^\infty(\mathbb{R})} + 8PR_0\left\|\int_{\mathbb{R}}\left|\log\frac{|.-y|}{1+|.|}\right|d\mu_{V_{\phi,t_0}}(y)\right\|_{L^\infty(\mathbb{R})} + \frac{\delta t R_0^2}{(1-\delta t R_0)^2}\right)\|u-v\|_{L^\infty(\mathbb{R})}\\
&\qquad\times\left(\frac{R_0\delta t}{1-\delta t R_0} + 4P\left\|\int_{\mathbb{R}}\left|\log\frac{|.-y|}{1+|.|}\right|.\rho_{V_{\phi,t_0}}(y)dy\right\|_{L^\infty(\mathbb{R})}\right) \leq C(R_0)\|u-v\|_{L^\infty(\mathbb{R})}
\end{aligned}$$

Similarly, there exists $C(R_0)>0$ such that:

$$\Delta_3\|\mathfrak{h}[v]\|_{L^\infty(\mathbb{R})} \leq C(R_0)\delta t\|u-v\|_{L^\infty(\mathbb{R})}$$

which finally leads to the existence of $C_0 > 0$ such that:

$$\left\|\mathcal{U}_t[u] - \mathcal{U}_t[v]\right\|_{L^\infty(\mathbb{R})} \leq C_0 \delta t \|u - v\|_{L^\infty(\mathbb{R})}.$$

We now choose ε_0 again small enough such that $k_0 \overset{(\text{def})}{=} \delta t C_0 C_{\mathcal{T},0} < 1$, this concludes the proof that $\mathcal{T}^{-1} \circ \mathcal{V}_t$ is contractive on $\overline{\mathscr{B}_0(0, R_0)}$ with contractivity constant k_0.

To get the contractivity property for $\mathcal{T}^{-1} \circ \mathcal{V}_t$ on $W_n^\infty(\mathbb{R})$, we adapt a similar strategy. Let $u \in \overline{\mathscr{B}_n(0, R)}$ with $\delta t R < 1$, by Proposition 2.6.2,

$$\|\mathcal{T}^{-1}[\mathcal{V}_t[u]]\|_{W_n^\infty(\mathbb{R})} \leq C_{\mathcal{T},n} \|\mathcal{V}_t[u]\|_{W_n^\infty(\mathbb{R})}.$$

Furthermore, it is clear that every term appearing in the definition of $\mathcal{V}_t[u]$ belongs to $W_n(\mathbb{R})$, thus by the same argument as before there exists a positive function g_n such that,

$$\|\mathcal{T}^{-1} \circ \mathcal{V}_t[u]\|_{W_n(\mathbb{R})} \leq C_{\mathcal{T},n}\Big(2\|\phi\|_{W_n^\infty(\mathbb{R})} + \delta t g_n(R)\Big).$$

We conclude just as before that by taking δt small enough, $\mathcal{T}^{-1} \circ \mathcal{V}_t : \overline{\mathscr{B}_n(0, R)} \to \overline{\mathscr{B}_n(0, R)}$ is well-defined. Finally, just as before since for all $u, v \in W_n^\infty(\mathbb{R})$

$$\begin{aligned}
\mathcal{U}_t[u]^{(n)}(x) &= \delta t \sum_{k=0}^{n} \binom{n}{k} \left(\mathfrak{h}[u]^2\right)^{(k)}(x) \mathsf{E}[u]^{(n-k)}(x) \\
&= \delta t \sum_{0 \leq i \leq k \leq n} \binom{n}{k}\binom{k}{i} \mathfrak{h}[u]^{(i)}(x) \mathfrak{h}[u]^{(k-i)}(x) \mathsf{E}[u]^{(n-k)}(x).
\end{aligned}$$

Moreover, by the same controls as before it is easy to derive that for all $0 \leq i \leq k \leq n$, for all $u, v \in \overline{\mathscr{B}_n(0, R_n)}$,

$$\|\mathfrak{h}[u]^{(i)}\mathfrak{h}[u]^{(k-i)}\mathsf{E}[u]^{(n-k)} - \mathfrak{h}[v]^{(i)}\mathfrak{h}[v]^{(k-i)}\mathsf{E}[v]^{(n-k)}\|_{L^\infty(\mathbb{R})} \leq C(R_n, n, i, k)\|u - v\|_{W_n^\infty(\mathbb{R})}.$$

This is enough to conclude that

$$\left\|\mathcal{T}^{-1} \circ \mathcal{V}_t[u] - \mathcal{T}^{-1} \circ \mathcal{V}_t[v]\right\|_{W_n^\infty(\mathbb{R})} \leq \delta t C_{\mathcal{T},n} C(R_n, n, i, k)\|u - v\|_{W_n^\infty(\mathbb{R})}.$$

Finally, by taking δt small enough, we conclude that $\mathcal{T}^{-1} \circ \mathcal{V}_t$ is contractive on $\overline{\mathscr{B}_n(0, R_n)}$ with contractivity constant $k_n \overset{(\text{def})}{=} \delta t C_{\mathcal{T},n} C(R_n, n, i, k) < 1$. □

Remark 2.6.6 Note that the definition of u_t as an element of $W_n^\infty(\mathbb{R})$ depends on t_0 and that we only proved the characterization of u_t as a fixed point for $t \in]t_0 - \varepsilon_n, t_0 + \varepsilon_n[\backslash\{t_0\}$ with $\varepsilon_n > 0$ small enough (we stress that we successively lowered δt when increasing i.) Furthermore, since for all $v \in L^\infty(\mathbb{R})$

$$\mathcal{T}^{-1} \circ \mathcal{V}_{t_0}[v] = -\phi + \int_{\mathbb{R}} \phi d\mu_{V_{\phi,t_0}},$$

then we can set $u_{t_0} \overset{(\text{def})}{=} -\phi + \int_{\mathbb{R}} \phi d\mu_{V_{\phi,t_0}}$ which is obviously the unique fixed point in $\bigcap_{n \in \mathbb{N}} W_n^\infty(\mathbb{R})$ of $\mathcal{T}^{-1} \circ \mathcal{V}_{t_0}$.

2.6.2 Regularity of the equilibrium measure

We now prove the continuity of $t \mapsto u_t \in W_n^\infty(\mathbb{R})$.

Lemma 2.6.7 *Let $n \in \mathbb{N}$, $\varepsilon_n > 0$ and $R_n > 0$ be as in Theorem 2.6.5. For all $v \in \overline{\mathscr{B}_n(0, R_n)}$, for all $(t_p)_p \in (]t_0 - \varepsilon_n, t_0 + \varepsilon_n[)^{\mathbb{N}}$ such that $t_p \underset{p\to\infty}{\longrightarrow} t \in]t_0 - \varepsilon_n, t_0 + \varepsilon_n[$ then*

$$\|\mathcal{T}^{-1} \circ \mathcal{V}_{t_p}[v] - \mathcal{T}^{-1} \circ \mathcal{V}_t[v]\|_{W_n^\infty(\mathbb{R})} \underset{p\to\infty}{\longrightarrow} 0.$$

Proof The proof is based on the fact that there exists a neighboorhood U_n of t_0 such that for all $v \in \overline{\mathscr{B}_n(0,R_n)}$, $t \in U_n \mapsto \mathcal{V}_t[v] \in W_n^\infty(\mathbb{R})$ is continuous. Since $\mathcal{T}^{-1}$ is also continuous in $W_n^\infty(\mathbb{R})$-norm, we can conclude. □

Corollary 2.6.8 (Continuity of the fixed-point) *Let $n \in \mathbb{N}$, for all $t, t' \in]t_0-\varepsilon_n, t_0+\varepsilon_n[$,*

$$\|u_t - u_{t'}\|_{W_n^\infty(\mathbb{R})} \underset{t\to t'}{\longrightarrow} 0.$$

Proof Let $(t_n)_n \in (]t_0-\varepsilon, t_0+\varepsilon[)^{\mathbb{N}}$ such that $t_n \underset{n\to\infty}{\longrightarrow} t \in]t_0-\varepsilon_n, t_0+\varepsilon_n[$. First

$$\left\|u_{t_n} - \mathcal{T}^{-1}\circ\mathcal{V}_{t_n}[u_t]\right\|_{W_n^\infty(\mathbb{R})} = \left\|\mathcal{T}^{-1}\circ\mathcal{V}_{t_n}[u_{t_n}] - \mathcal{T}^{-1}\circ\mathcal{V}_{t_n}[u_t]\right\|_{W_n^\infty(\mathbb{R})} \leq k_i \left\|u_{t_n} - u_t\right\|_{W_n^\infty(\mathbb{R})}$$

and by the triangle inequality, we obtain:

$$\left\|u_t - \mathcal{T}^{-1}\circ\mathcal{V}_{t_n}[u_t]\right\|_{W_n^\infty(\mathbb{R})} \geq \|u_{t_n} - u_t\|_\infty - \left\|u_{t_n} - \mathcal{T}^{-1}\circ\mathcal{V}_{t_n}[u_t]\right\|_{W_n^\infty(\mathbb{R})} \geq (1-k_n)\left\|u_{t_n} - u_t\right\|_{W_n^\infty(\mathbb{R})}.$$

Nevertheless by Lemma 2.6.7, $\left\|u_t - \mathcal{T}^{-1}\circ\mathcal{V}_{t_n}[u_t]\right\|_{W_n^\infty(\mathbb{R})} \underset{n\to\infty}{\longrightarrow} 0$ because u_t is the fixed point of $\mathcal{T}^{-1}\circ\mathcal{V}_t$ which is a continuous operator with respect to t hence $\|u_{t_n} - u_t\|_{W_n^\infty(\mathbb{R})} \underset{n\to\infty}{\longrightarrow} 0$. □

Theorem 2.6.9 *Let $t, t_0 \in [0,1]$,*

$$\left\|\rho_{V_t} - \rho_{V_{\phi,t_0}}\right\|_{W_n^\infty(\mathbb{R})} \underset{t\to t_0}{\longrightarrow} 0.$$

Furthermore, for all $x \in \mathbb{R}$, $k \in \mathbb{N}$, $t \mapsto \partial_x^k \rho_{V_t}(x) \in \mathcal{C}^\infty(\mathbb{R})$ and satisfies the following partial differential equation:

$$\partial_t \partial_x^k \rho_{V_t}(x) = \partial_x^k \left[\left(-\phi + \int_{\mathbb{R}} \phi(s)\rho_{V_t}(s)ds\right)\rho_{V_t}\right](x).$$

Proof By setting $u_t \overset{(\text{def})}{=} \dfrac{\rho_{V_t} - \rho_{V_{t'}}}{t-t'}\dfrac{1}{\rho_{V_{t'}}}$,

$$\|\rho_{V_t} - \rho_{V_{t'}}\|_{W_n^\infty(\mathbb{R})} = |t-t'|\left\|\rho_{V_{t'}}u_t\right\|_{W_n^\infty(\mathbb{R})} \leq 2^n|t-t'|\left\|\rho_{V_{t'}}\right\|_{W_n^\infty(\mathbb{R})}\cdot\|u_t\|_{W_n^\infty(\mathbb{R})}.$$

By Corollary 2.6.8, $\|u_t\|_{W_n^\infty(\mathbb{R})} \underset{t\to t'}{\longrightarrow} \|u_{t'}\|_{W_n^\infty(\mathbb{R})}$, thus the right-hand side goes to zero proving the claim.

For the second point, we notice that

$$\left\|\frac{\rho_{V_t} - \rho_{V_{\phi,t_0}}}{\delta t} - \rho_{V_{\phi,t_0}}u_{t_0}\right\|_{W_n^\infty(\mathbb{R})} = \left\|(u_t - u_{t_0})\rho_{V_{\phi,t_0}}\right\|_{W_n^\infty(\mathbb{R})} \leq 2^n\|u_t - u_{t_0}\|_{W_n^\infty(\mathbb{R})}\|\rho_{V_{\phi,t_0}}\|_{W_n^\infty(\mathbb{R})}.$$

Since the RHS goes to zero as $t \to t_0$, $u_{t_0} = -\phi + \displaystyle\int_{\mathbb{R}} \phi d\mu_{V_{\phi,t_0}}$ and n is arbitrary, we conclude that, $x \in \mathbb{R}$, $t \mapsto \rho_{V_t}(x)$ is differentiable at every $t \in [0,1]$ of derivative

$$\partial_t \rho_{V_t}^{(k)}(x) = -\left(\rho_{V_t}\phi\right)^{(k)}(x) + \rho_{V_t}^{(k)}(x)\int_{\mathbb{R}} \phi(y)d\mu_{V_t}(y).$$

Since the above expression is again differentiable in t (one deals with the integral by dominated convergence theorem with the domination $|\phi(x)\partial_t\rho_{V_t}(x)| \leq 2\|\phi\|_\infty^2(1+\max_{s\in[0,1]}\|u_s\|_\infty)\rho_{V_{\phi,t_0}}(x)$ for an arbitrary t_0), we conclude that for all $x \in \mathbb{R}$, $t \mapsto \rho_{V_t}(x) \in \mathcal{C}^\infty(\mathbb{R})$. □

Corollary 2.6.10 (Convergence of moments) *Let $h \in \mathbb{N}$, $t \in [0,1]$ by denoting*

$$m(t,h) \overset{(\text{def})}{=} \int_{\mathbb{R}} x^h d\mu_{V_t}(x),$$

we have for all $t_0 \in [0,1]$

$$m(t,h) \underset{t \to t_0}{\to} m(t_0,h).$$

Proof For all $t \in [0,1]$, for all $x \in \mathbb{R}$, $\rho_{V_t}(x) \leq (1+\max_{s\in[0,1]} \|u_s\|_\infty)\rho_{V_{\phi,t_0}}(x)$, hence by dominated convergence theorem $|m(t,h) - m(t_0,h)| \underset{t\to t'}{\to} 0$. □

2.7 Proof of Theorem 2.1.4

2.7.1 Asymptotic expansion of the partition function for the Gaussian potential

The asymptotic behaviour of $\mathcal{Z}_N[V_G]$ can be deduced from Mehta's formula [Meh04, 17.6.7]

$$\mathcal{Z}_N[V_G] = (2\pi)^{N/2} \prod_{a=1}^{N} \frac{\Gamma\left(1+\frac{aP}{N}\right)}{\Gamma\left(1+\frac{P}{N}\right)}. \tag{2.78}$$

This will allow us to use this formula in our interpolating integration formula to deduce the asymptotic expansion of $\log \mathcal{Z}_N[V_\phi]$. From the previous equation, we can deduce the asymptotic behaviour of $\log \mathcal{Z}_N[V_G]$. It is given by the following theorem:

Theorem 2.7.1 *There exists a sequence $(g_k)_{k\geq 0} \in \mathbb{R}^N$, such that for all $K \geq 0$,*

$$\frac{1}{N}\log \mathcal{Z}_N[V_G] = \sum_{k=0}^{K} \frac{g_k}{N^k} + O\left(N^{-(K+1)}\right) \tag{2.79}$$

with

$$g_1 \overset{(\text{def})}{=} \gamma\frac{P}{2} + \frac{\log(1+P)}{2} + \frac{1}{2}\sum_{j\geq 1}\left(\log\left(1+\frac{P+1}{j}\right) - \log\left(1+\frac{1}{j}\right) - \frac{P}{j}\right).$$

Above γ denotes the Euler-Mascheroni constant.

Proof We first use (2.78) to deduce

$$\log \mathcal{Z}_N[V_G] = \frac{N\log(2\pi)}{2} + \sum_{a=1}^{N} \Gamma\left(1+\frac{aP}{N}\right) - N\log\Gamma\left(1+\frac{P}{N}\right). \tag{2.80}$$

Let $K > 0$, using the Taylor series expansion of $\log\Gamma$ around 1 (see [GR14, 8.342]), one has

$$-N\log\Gamma\left(1+\frac{P}{N}\right) = \gamma P - \sum_{k=1}^{+\infty} \frac{\zeta(k+1)}{k+1}\frac{(-P)^{k+1}}{N^k}. \tag{2.81}$$

where ζ denotes the Riemann ζ function. The second term in (2.80) can be estimated by using the Weierstrass product formula for $\frac{1}{\Gamma}$:

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{j=1}^{+\infty} (1+\frac{z}{j})e^{-z/j} \tag{2.82}$$

which is valid for any $z \in \mathbb{C}$. Hence we deduce that:

$$\sum_{k=1}^{N} \log\Gamma\left(1+\frac{kP}{N}\right) = -\sum_{k=1}^{N}\left(\gamma\Big(1+\frac{kP}{N}\Big) + \log\left(1+\frac{kP}{N}\right) - S_N(k)\right)$$
$$= -\gamma N - \gamma\frac{(N+1)P}{2} - \sum_{k=1}^{N}\log\left(1+\frac{kP}{N}\right) + \sum_{k=1}^{N} S_N(k) \quad (2.83)$$

where $S_N(k) \overset{(\text{def})}{=} -\sum_{j=1}^{+\infty}\left[\log\left(1+\frac{1}{j}+\frac{kP}{Nj}\right) - \frac{1}{j} - \frac{kP}{Nj}\right]$. By the Euler-Maclaurin formula, we have the following identity for any $K > 0$:

$$\sum_{k=1}^{N}\log\left(1+\frac{kP}{N}\right) = \int_0^N f_N(t)dt + \sum_{k=1}^{K+2}\frac{B_k}{k!}\left(f_N^{(k-1)}(N) - f_N^{(k-1)}(0)\right) + R_{K+2}^{(N)} \quad (2.84)$$

where $f_N(x) \overset{(\text{def})}{=} \log\left(1+\frac{xP}{N}\right)$ and B_k is the k-th Bernoulli number. The remainder $R_{K+2}^{(N)}$ is defined by $R_{K+2}^{(N)} \overset{(\text{def})}{=} (-1)^{K+1}\int_0^N f_N^{(K+2)}(t)\frac{\widetilde{B}_{K+2}(t-\lfloor t\rfloor)}{(K+2)!}dt$, where $\widetilde{B}_{K+2}$ is the $(K+2)$-th Bernoulli polynomial. By using the following bound on Bernoulli polynomials,

$$\forall x \in [0,1], \forall k > 0, \qquad |\widetilde{B}_k(x)| \leq 2\frac{k!}{(2\pi)^k}\zeta(k)$$

where ζ is the Riemann zeta function, $R_{K+2}^{(N)}$ can be controlled by the following inequalities:

$$|R_{K+2}^{(N)}| \leq \frac{2\zeta(K+2)}{(2\pi)^{K+2}}\int_0^N |f_N^{(K+2)}(t)|dt = \frac{2\zeta(K+2)}{(2\pi)^{K+2}}\int_0^N \frac{P^{K+2}}{N^{K+2}}\frac{(K+1)!}{\left(1+\frac{P}{N}t\right)^{K+2}}dt$$
$$= \frac{2\zeta(K+2)(K+1)!}{(2\pi)^{K+2}}\frac{P^{K+1}}{N^{K+1}}\int_0^P \frac{du}{(1+u)^{K+2}} = O\Big(N^{-(K+1)}\Big).$$

Extracting the large N-behaviour in (2.84) leads to

$$\sum_{k=1}^{N}\log\left(1+\frac{kP}{N}\right) = \frac{N}{P}\int_0^P \log(1+t)dt + B_1\log(1+P)$$
$$+ \sum_{k=2}^{K+2}\frac{B_k}{k!}(-1)^k(k-2)!\frac{P^{k-1}}{N^{k-1}}\left(\frac{1}{(1+P)^{k-1}} - 1\right) + O\Big(N^{-(K+1)}\Big)$$
$$= Nc_{-1} + \sum_{k=0}^{K}\frac{c_k}{N^k} + O\Big(N^{-(K+1)}\Big) \quad (2.85)$$

where $c_{-1} \overset{(\text{def})}{=} (1+P^{-1})\log(1+P) - 1$, $c_0 \overset{(\text{def})}{=} \frac{\log(1+P)}{2}$ and for all $k \in [\![1, K]\!]$,

$$c_k = \frac{-B_{k+1}(-P)^k}{k(k+1)}\left(\frac{1}{(1+P)^k} - 1\right).$$

Also by Fubini's theorem, we get,

$$\sum_{k=1}^{N} S_N(k) = -\sum_{j=1}^{+\infty}\left\{\sum_{k=1}^{N}\log\left(1+\frac{1}{j}+\frac{kP}{Nj}\right)-\frac{1}{j}-\frac{kP}{Nj}\right\}$$
$$=\sum_{j=1}^{+\infty}\left\{\frac{N}{j}+\frac{(N+1)P}{2j}+\sum_{k=1}^{N} g_{N,j}(k)\right\}$$

where $g_{N,j}(x)=\log\left(1+\frac{1}{j}+\frac{P}{jN}x\right)$. The first equality clearly shows that the RHS is a serie of general term bounded by $O\left(j^{-2}\right)$, so it converges and justifies the application of Fubini's theorem. Again by Euler-Maclaurin formula, we get:

$$\sum_{k=1}^{N} g_{N,j}(k) = -\int_0^N \log\left(1+\frac{1}{j}+t\frac{P}{Nj}\right)dt-\frac{1}{2}\left[\log\left(1+\frac{P+1}{j}\right)-\log\left(1+\frac{1}{j}\right)\right]$$
$$-\sum_{k=2}^{K+2}\frac{(-1)^k B_k}{k(k-1)}\left\{\left(1+\frac{P+1}{j}\right)^{1-k}-\left(1+\frac{1}{j}\right)^{1-k}\right\}\left(\frac{P}{Nj}\right)^{k-1}+R_{K+2}^{(N)}(j) \quad (2.86)$$

where again the new remainder $R_{K+2}^{(N)}(j)$ can be controlled *via*

$$|R_{K+2}^{(N)}(j)| \le \frac{2\zeta(K+2)}{(2\pi)^{K+2}}\int_0^N \frac{(K+1)!\left(\frac{P}{Nj}\right)^{K+2}}{\left(1+\frac{1}{j}+t\frac{P}{jN}\right)^{K+2}}dt$$
$$=\frac{2\zeta(K+2)}{(2\pi)^{K+2}}\left(\frac{P}{Nj}\right)^{K+1}\left\{\left(1+\frac{P+1}{j}\right)^{-(K+1)}-\left(1+\frac{1}{j}\right)^{-(K+1)}\right\}=O\left(\frac{1}{(Nj)^{K+1}}\right)$$

where $O\left(\frac{1}{j^{K+1}}\right)$ depends on K and P but not on N. Hence we deduce that

$$\sum_{k=1}^{N} g_{N,j}(k) = -N\int_0^1 \log\left(1+\frac{1+sP}{j}\right)ds-\frac{1}{2}\left[\log\left(1+\frac{P+1}{j}\right)-\log\left(1+\frac{1}{j}\right)\right]$$
$$+\sum_{k=1}^{K}\frac{(-P)^k B_{k+1}}{k(k+1)}\left\{\left(1+\frac{P+1}{j}\right)^{-k}-\left(1+\frac{1}{j}\right)^{-k}\right\}\frac{1}{(Nj)^k}+O\left((jN)^{-(K+1)}\right).$$

This leads to:

$$\sum_{k=1}^{N} S_N(k)=\sum_{j=1}^{+\infty}\left[\frac{N}{j}+\frac{(N+1)P}{2j}+\sum_{k=1}^{N} g_{N,j}(k)\right]=\sum_{j=1}^{+\infty}\left(u_j^{(1)}N+\sum_{k=0}^{K}u_j^{(k)}N^{-k}\right)+O(N^{-(K+1)})$$
$$=d_1 N+\sum_{k=0}^{K} d_{-k}N^{-k}+O\Big(N^{-(K+1)}\Big) \quad (2.87)$$

where for all $k=-1,0\dots K$, $\left(u_j^{(k)}\right)_{j>0}\in\ell^1(\mathbb{N}^*)$ and $d_{-k}\in\mathbb{R}$. This establishes the existence of the asymptotic expansion of $\log\mathcal{Z}_N[V_G]$ up to $O\Big(N^{-(K+1)}\Big)$. Collecting the different terms, leads to the formula for g_1. □

2.7.2 Free energy of the model

Only, in this Subsection, since the parameter P varies, we include the P-dependance of $\mathcal{Z}_N[V]$ in the notation and write $\mathcal{Z}_N^P[V]$ instead.

Theorem 2.7.2 (Free energy formula for Gaussian Potential) *Let $P > 0$, the free energy associated with the Gaussian potential is*

$$F(P) \overset{(\text{def})}{=} \lim_{N\to\infty} N^{-1} \log \mathcal{Z}_N^P [2PV_G] = -\frac{1+P}{2}\log(2P) + \frac{\log(2\pi)}{2} + \int_0^1 \log\Gamma(1+Px)\,dx \quad (2.88)$$

As P goes to $+\infty$, we have:

$$F(P) = -P\left(\frac{3+\log 2}{2}\right) - \frac{1+\log 2}{2} + \log(2\pi) + \frac{\log P}{12P} + O\left(P^{-1}\right) \quad (2.89)$$

Proof By a change of variable, it holds that $\mathcal{Z}_N^P[2PV_G] = \left(\sqrt{2P}\right)^{-N-P(N-1)} \mathcal{Z}_N^P[V_G]$. Hence by Mehta's formula (2.78),

$$F(P) \overset{(\text{def})}{=} \lim_{N\to\infty} N^{-1} \log \mathcal{Z}_N^P [2PV_G] = -\frac{1+P}{2}\log(2P) + \frac{\log(2\pi)}{2} + \int_0^1 \log\Gamma(1+Px)\,dx.$$

We can replace the last term by its asymptotic expansion so that

$$\int_0^1 \log\Gamma(1+Px)\,dx = \frac{(P+1)}{2}\log P - \frac{3P}{2} + \frac{\log(2\pi)-1}{2} + \frac{1}{12P}\log P + O\left(P^{-1}\right).$$

We used the classic formula to conclude

$$\log\Gamma(1+Px) = (1+Px)\log(1+Px) - 1 - Px - \frac{\log(1+Px)-\log(2\pi)}{2} + \frac{1}{12(1+Px)} + O\left(\frac{1}{(1+Px)^3}\right).$$

2.7.3 Interpolation with general potential

We first establish the link between the 1-linear statistics and the partition function with general potential and the one with Gaussian potential.

Lemma 2.7.3 *Let $V_t(x) = tV(x) + (1-t)V_G(x)$ with $t \in [0,1]$. We have*

$$\log \frac{\mathcal{Z}_N[V]}{\mathcal{Z}_N[V_G]} = -N\int_0^1 \langle V - V_G\rangle_{L_N}^{V_t}\,dt \quad (2.90)$$

Proof By the fundamental theorem of calculus:

$$\log \frac{\mathcal{Z}_N[V]}{\mathcal{Z}_N[V_G]} = \int_0^1 \partial_t \log \mathcal{Z}_N[V_t]dt = -\int_0^1 dt \int_{\mathbb{R}^N} p_N^{V_t}(\underline{x}) \sum_{i=1}^N \partial_t V_t(x_i) d^N\underline{x}.$$

Since $\displaystyle\sum_{i=1}^N \partial_t V_t(x_i) = N\int_{\mathbb{R}} [V(x) - V_G(x)]\,dL_N(x)$, where L_N is the empricial measure associated to the external potential V_t, it concludes the proof. □

Theorem 2.7.4 *For all $\phi \in \cap_{k\geq 0} H^k(\mathbb{R})$, there exists a sequence $(c_i)_{i\geq 0} \in \mathbb{R}^{\mathbb{N}}$ depending on ϕ and P such that for all $K \geq 0$*

$$\frac{1}{N}\log \mathcal{Z}_N[V_{G,\phi}] = \sum_{i=0}^K \frac{c_i}{N^i} + O\left(N^{-(K+1)}\right).$$

The leading term c_0 is equal to the following expression:

$$\int_{\mathbb{R}} V_{G,\phi}(x) d\mu_{V_{G,\phi}}(x) - P \iint_{\mathbb{R}^2} \log|x-y| d\mu_{V_{G,\phi}}(x) d\mu_{V_{G,\phi}}(y) + \int_{\mathbb{R}} \log\left(\frac{d\mu_{V_{G,\phi}}(x)}{dx}\right) d\mu_{V_{G,\phi}}(x).$$

The subleading term c_1 can be written as

$$c_1 \overset{(\mathrm{def})}{=} \gamma\frac{P}{2} + \frac{\log(1+P)}{2} + \frac{1}{2}\sum_{j\geq 1}\left(\log\left(1+\frac{P+1}{j}\right) - \log\left(1+\frac{1}{j}\right) - \frac{P}{j}\right) - P\int_0^1 \left[\left\langle \partial_1 \widetilde{\Xi^{-1}}\phi \right\rangle_{\mu_{V_{G,\phi,t}}} + \left\langle \Theta^{(2)} \circ \widetilde{\Xi_1^{-1}}\left[\partial_2 \mathcal{D} \circ \widetilde{\Xi^{-1}}\phi\right]\right\rangle_{\mu_{V_{G,\phi,t}}}\right] dt. \quad (2.91)$$

Proof By Lemma 2.7.3 and Theorem 2.7.1, to establish the asymptotic expansion of $\mathcal{Z}_N[V_{G,\phi}]$, it suffices to obtain the one for $\int_0^1 \langle\phi\rangle_{L_N}^{V_{G,\phi,t}} dt$. By Theorem 2.5.2, we get

$$\int_0^1 \langle\phi\rangle_{L_N}^{V_{G,\phi,t}} dt = \int_0^1 \langle\phi\rangle_{\mu_{V_{G,\phi,t}}} dt + \sum_{a=1}^{K} \frac{\int_0^1 d_a^{(1),V_{G,\phi,t}}(\phi) dt}{N^a} + \int_0^1 \left(\langle\phi\rangle_{\mathcal{L}_N}^{V_{G,\phi,t}} - \sum_{a=1}^{K} \frac{d_a^{(1),V_{G,\phi,t}}(\phi)}{N^a}\right) dt.$$

Finally, we conclude that the last integral is a $O\left(N^{-(K+1)}\right)$ by (2.67) and the continuity on $t \mapsto \mathsf{P}_{K,k}^{V_t}$ obtained in Proposition 2.10.11. Furthermore, by collecting order 1 for $\log \mathcal{Z}_N[V_G]$ and $\int_0^1 d_a^{(1),V_{G,\phi,t}}(\phi) dt$ for $a=1$ in Theorem 2.5.2, we infer on the value ofc_1. □

2.8 Conclusion

This work adapted the analysis of the loop equations method to prove the existence of a N^{-1} asymptotic expansion for a general class of potential. This class include all potentials given by $x^2+\phi$ where ϕ is a smooth bounded function. An immediate continuation of this result would be to extend it to more general confining potentials like x^4 for example. Our method relied on new controls on the equilibrium measures resulting from an energy minimization and entropy maximization. A natural question would be to extend these ideas to more general interactions.

2.9 Appendix: Lemmas and technical results

Lemma 2.9.1 (Properties of the Hilbert transform)

i) As a consequence, $\pi^{-1}\mathcal{H}$ is an isometry of $L^2(\mathbb{R})$, and $\mathcal{H}$ satisfies on $L^2(\mathbb{R})$ the identity $\mathcal{H}^2 = -\pi^2 I$.

ii) Derivative: For any $f \in H^1(\mathbb{R})$, $\mathcal{H}[f]$ is also $H^1(\mathbb{R})$ and $\mathcal{H}[f]' = \mathcal{H}[f']$.

iii) For all $p>1$, the Hilbert transform can be extended as a bounded operator $\mathcal{H}: L^p(\mathbb{R}) \to L^p(\mathbb{R})$.

iv) Skew-self adjointness: For any $f, g \in L^2(\mathbb{R})$, $\langle \mathcal{H}[f], g\rangle_{L^2(\mathbb{R})} = -\langle f, \mathcal{H}[g]\rangle_{L^2(\mathbb{R})}$.

v) For all $\delta>0$, for all $f \in L^1(\mathbb{R})$ such that $f' \in L^\infty(\mathbb{R})$, $\|\mathcal{H}[f]\|_\infty \leq (\delta^{-1}\|f\|_1 + 2\delta\|f'\|_\infty)$

Proof We refer to [Kin09] for the proofs of properties *i)-iv)*. To prove *v)*, let f be such a function,

$$|\mathcal{H}[f](x)| \leq \lim_{\varepsilon\to 0}\Big|\int_{\varepsilon\leq|x-y|\leq\delta} \frac{f(y)dy}{y-x}\Big| + \lim_{\varepsilon\to 0}\Big|\int_{\delta\leq|x-y|\leq\varepsilon^{-1}} \frac{f(y)dy}{y-x}\Big|.$$

The second term in the RHS can be bounded by $\delta^{-1}\|f\|_{L^1(\mathbb{R})}$ while the first term verifies

$$\lim_{\varepsilon\to 0}\Big|\int_{\varepsilon\leq|x-y|\leq\delta}\frac{f(y)dy}{y-x}|\leq\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x-y|\leq\delta}\left|\frac{f(y)-f(x)}{y-x}\right|dy+\lim_{\varepsilon\to 0}\left|\int_{\varepsilon\leq|x-y|\leq\delta}\frac{dy}{y-x}\right|.$$

The first term in the RHS can be bounded by $2\delta\|f'\|_\infty$ while the second is equal to 0. This allows to conclude. □

We recall some results obtained in **[DGM23]**.

Lemma 2.9.2 *Let $u\in L^2(\mathbb{R})$ be such that $\int_{\mathbb{R}}u(t)dt$ exists and let $f:t\mapsto tu(t)\in H^1(\mathbb{R})$ then*

$$\mathcal{H}[u](x)\underset{|x|\to\infty}{\sim}\frac{-\int_{\mathbb{R}}u(t)dt}{x}.$$

Moreover, if $\int_{\mathbb{R}}u(t)dt=0$, $\int_{\mathbb{R}}f(t)dt$ exists and $g:t\mapsto t^2u(t)\in H^1(\mathbb{R})$, then:

$$\mathcal{H}[u](x)\underset{|x|\to\infty}{\sim}\frac{-\int_{\mathbb{R}}tu(t)dt}{x^2}.$$

As a consequence, we obtain that $\mathcal{H}[\rho_V](x)\underset{|x|\to\infty}{\sim}-x^{-1}$ and the logarithmic potential U^{ρ_V} is Lipschitz bounded, with bounded derivative $\mathcal{H}[\rho_V]$.

Lemma 2.9.3 *Let $n\geq 1$, and $h\in H^n(\mathbb{R})$,*

$$\|\rho_V\widetilde{\Xi^{-1}}[h]\|_{H^n(\mathbb{R})}\leq C_3(V,n)\|h\|_{H^n(\mathbb{R})}.$$

with a constant $C_3(V,n)$, only depending on V and n. For the choice of potentiel $V=V_{\phi,t}$, for $\phi\in\mathcal{C}^\infty(\mathbb{R})$ with $\phi^{(k)}\in L^2(\mathbb{R})$ for all $k\in\mathbb{N}$ and $t\in[0,1]$, $t\mapsto C_3(V_{\phi,t},n)$ is a continuous function. Moreover, for all $h\in H^n(\mathbb{R})\cap W_n^\infty(\mathbb{R})$,

$$\|\rho_V\widetilde{\Xi^{-1}}[h]\|_{H^n(\mathbb{R})}\leq C_5(V,n)\|h\|_{W_n^\infty(\mathbb{R})}$$

with a constant $C_5(V,n)$, only depending on V and n. The function $t\mapsto C_5(V_{\phi,t},n)$ is also continuous.

Proof We first prove that for all $k\geq 0$, $h\in\dfrac{1}{\rho_V}H^k(\mathbb{R})$, there exists finite sets of indices $\mathfrak{I}_{l,a}^k$, $\mathfrak{J}_l^k$ and $\mathfrak{K}_l^k$ independent of V and polynomials $p_{a,1,1}^k$,$p_{a,b,c,d}^k$, $q_{a,b,c}^k$ in $\theta,\dots,\theta^{(k-1)}$, with coefficients

independent of V and of degree at most k, such that for all $x \in \mathbb{R}$,

$$\begin{aligned}
\widetilde{\Xi^{-1}}[h]^{(k)} &= \sum_{a=0}^{k-1} p^k_{a,1,1} h^{(a)} + \sum_{a=0}^{k-1} \sum_{b \in \mathfrak{I}^k_{2,a}} p^k_{a,b,2,1} \mathcal{H}\left[\rho_V p^k_{a,b,2,2} h^{(a)}\right] + \dots \\
&\quad + \sum_{a=0}^{k-1} \sum_{b \in \mathfrak{I}^k_{k,a}} p^k_{a,b,k,1} \mathcal{H}\Bigg[\rho_V p^k_{a,b,k,2} \mathcal{H}\Big[\rho_V p^k_{a,b,k,3} \mathcal{H}\Big[\dots \mathcal{H}\Big[\rho_V p^k_{a,b,k,k} h^{(a)} \Big]\Big] \dots \Big]}_{k-1} + q^k_{1,1} \widetilde{\Xi^{-1}}[h] \\
&+ \sum_{b \in \mathfrak{I}^k_2} q^k_{b,2,1} \mathcal{H}\left[\rho_V q^k_{b,2,2} \widetilde{\Xi^{-1}}[h]\right] + \dots + \sum_{b \in \mathfrak{I}^k_{k+1}} q^k_{b,k+1,1} \mathcal{H}\Bigg[\rho_V q^k_{b,k+1,2} \mathcal{H}\Big[\dots \mathcal{H}\Big[\rho_V q^k_{b,k+1,k+1} \widetilde{\Xi^{-1}}[h] \Big] \dots \Big]}_{k} \\
&\quad + \Bigg(r^k_{1,1} + \sum_{b \in \mathfrak{K}^k_2} r^k_{b,2,1} \mathcal{H}\left[\rho_V r^k_{b,2,2}\right] + \dots + \sum_{b \in \mathfrak{K}^k_k} r^k_{b,k,1} \mathcal{H}\Bigg[\rho_V r^k_{b,k,2} \mathcal{H}\Big[\dots \mathcal{H}\Big[\rho_V r^k_{b,k,k} \Big] \dots \Big]}_{k-1} \Bigg) \\
&\qquad \times \left(2P \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}[h]\right] d\mu_V - \int_{\mathbb{R}} h d\mu_V \right). \quad (2.92)
\end{aligned}$$

We prove it by induction, where for $n = 1$ one just uses the definition of Ξ for the initial case *i.e.*

$$\left(\widetilde{\Xi^{-1}}[h]\right)' = h - \int_{\mathbb{R}} h d\mu_V - \frac{\rho_V'}{\rho_V} \left(\widetilde{\Xi^{-1}}[h]\right)' - 2P\mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}[h]\right] + 2P \int_{\mathbb{R}} \mathcal{H}\left[\rho_V \widetilde{\Xi^{-1}}[h]\right] d\mu_V.$$

For the induction step, use a bootstrap argument. Suppose (2.92) holds at rank k, then differentiate and replace $\left(\widetilde{\Xi^{-1}}[h]\right)'$ by the RHS of the above relation to show that (2.92) holds at rank $k+1$. Now, by the Leibniz formula, for all $k \in [\![0, n]\!]$, it holds that

$$\left(\rho_V \Xi^{-1}[h]\right)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} \rho_V^{(k-i)} \Xi^{-1}[h]^{(i)}.$$

Furthermore by (2.92), by using successively that $\pi^{-1}\mathcal{H}$ is an isometry of $L^2(\mathbb{R})$, inequality (2.26) and Jensen's inequality, we obtain:

$$\max_{0 \le k \le n} \| \left(\rho_V \Xi^{-1}[h]\right)^{(k)}(x) \|_{L^2(\mathbb{R})} \le C_3(V, n) \|h\|_{H^k(\mathbb{R})}$$

with $C_3(V,n)$ given by

$$
\begin{aligned}
&C_3(V,n) \overset{(\text{def})}{=} C(n) \\
&\times \max_{i\le k\le n}\Bigg\{ i \max_{0\le a<i}\left\|\rho_V^{(k-i)} p^i_{a,b,1,1}\right\|_\infty + i\pi \max_{0\le a<i}|\mathfrak{I}^i_{2,a}| \max_{b\in\mathfrak{I}^i_{2,a}}\left(\left\|\rho_V^{(k-i)} p^i_{a,b,2,1}\right\|_\infty \|\rho_V p^i_{a,b,2,2}\|_\infty\right) + \dots \\
&\quad + i\pi^{i-1}\max_{0\le a<i}|\mathfrak{I}^i_{i,a}| \max_{b\in\mathfrak{I}^i_{i,a}}\left(\left\|\rho_V^{(k-i)} p^i_{a,b,i,1}\right\|_\infty \cdot \prod_{l=2}^{i}\|\rho_V p^i_{a,b,i,l}\|_\infty\right) + C_{\mathcal{L}}\left\|\rho_V^{(k-i)} q^i_{1,1}\right\|_\infty \|\rho_V^{1/2}\|_\infty \\
&\qquad + C_{\mathcal{L}}\pi|\mathfrak{J}^i_2| \max_{b\in\mathfrak{J}^i_2}\left\|\rho_V^{(k-i)} q^i_{b,2,1}\right\|_\infty \|\rho_V^{1/2} q^i_{b,2,2}\|_\infty \|\rho_V^{1/2}\|_\infty + \dots \\
&\qquad + C_{\mathcal{L}}\pi^i|\mathfrak{J}^i_{i+1}| \max_{b\in\mathfrak{J}^i_{i+1}}\left\|\rho_V^{(k-i)} q^i_{b,i+1,1}\right\|_\infty \prod_{l=2}^{i}\|\rho_V q^i_{b,i+1,l}\|_\infty \|\rho_V^{1/2} q^i_{b,i+1,i+1}\|_\infty \|\rho_V^{1/2}\|_\infty \\
&\quad + 2\|\rho_V\|_\infty (1+2P\pi C_{\mathcal{L}})\Big[\|r^i_{1,1}\frac{\rho_V^{(k-i)}}{\rho_V}\sqrt{\rho_V}\|_\infty + \pi|\mathfrak{K}^i_2| \max_{b\in\mathfrak{K}^i_2}\|\rho_V^{(k-i)} r^i_{b,2,1}\|_\infty \|\sqrt{\rho_V} r^i_{b,2,1}\|_\infty + \dots \\
&\qquad + \pi^{i-1}|\mathfrak{K}^i_i| \max_{b\in\mathfrak{K}^i_i}\|\rho_V^{(k-i)} r^i_{b,i,1}\|_\infty \prod_{l=2}^{i-1}\|\rho_V r^i_{b,i,l}\|_\infty \|\sqrt{\rho_V} r^i_{b,i,i}\|_\infty\Big]\Bigg\} \qquad (2.93)
\end{aligned}
$$

For the second inequality, if $h \in H^n(\mathbb{R}) \cap W^\infty_n(\mathbb{R})$, we use the fact the same inequalities but we use the following integrals at the end $\int_{\mathbb{R}}\left(p^i_{a,b,i,l} h^{(a)}\rho_V(t)\right)^2 dt \le \|h\|^2_{W^\infty_a(\mathbb{R})}\int_{\mathbb{R}}(p^i_{a,b,i,l}\rho_{V_t})^2 dt$. This leads to

$$
\|\rho_V \widetilde{\Xi^{-1}}[h]\|_{H^n(\mathbb{R})} \le C_5(V,n)\|h\|_{W^\infty_n(\mathbb{R})}
$$

with $C_5(V,n)$ given by

$$
\begin{aligned}
&C_5(V,n) \overset{(\text{def})}{=} C(n) \\
&\times \max_{i\le k\le n}\Bigg\{ i \max_{0\le a<i}\left\|\frac{\rho_V^{(k-i)}}{\rho_V} p^i_{a,b,1,1}\sqrt{\rho_V}\right\|_\infty + i\pi \max_{0\le a<i}|\mathfrak{I}^i_{2,a}| \max_{b\in\mathfrak{I}^i_{2,a}}\left(\left\|\rho_V^{(k-i)} p^i_{a,b,2,1}\right\|_\infty \|\sqrt{\rho_V} p^i_{a,b,2,2}\|_\infty\right) + \dots \\
&\quad + i\pi^{i-1}\max_{0\le a<i}|\mathfrak{I}^i_{i,a}| \max_{b\in\mathfrak{I}^i_{i,a}}\left(\left\|\rho_V^{(k-i)} p^i_{a,i,1}\right\|_\infty \cdot \prod_{l=2}^{i-1}\|\rho_V p^i_{a,i,l}\|_\infty \|\sqrt{\rho_V} p^i_{a,i,i}\|_\infty\right) \\
&\qquad + C_{\mathcal{L}}\left\|\rho_V^{(k-i)} q^i_{1,1}\right\|_\infty \|\rho_V\|^{1/2}_\infty + C_{\mathcal{L}}\pi|\mathfrak{J}^i_2| \max_{b\in\mathfrak{J}^i_2}\left\|\rho_V^{(k-i)} q^i_{b,2,1}\right\|_\infty \|\rho_V^{1/2} q^i_{b,2,2}\|_\infty + \dots \\
&\qquad + C_{\mathcal{L}}\pi^i|\mathfrak{J}^i_{i+1}| \max_{b\in\mathfrak{J}^i_{i+1}}\left\|\rho_V^{(k-i)} q^i_{b,i+1,1}\right\|_\infty \prod_{l=2}^{i}\|\rho_V q^i_{b,i+1,l}\|_\infty \|\rho_V^{1/2} q^i_{b,i+1,i+1}\|_\infty \\
&\quad + 2\left(1+2P\pi\|\rho_V\|^{1/2}_\infty C_{\mathcal{L}}\right)\Big[\|r^i_{1,1}\frac{\rho_V^{(k-i)}}{\rho_V}\sqrt{\rho_V}\|_{L^2(\mathbb{R})} + \pi|\mathfrak{K}^i_2| \max_{b\in\mathfrak{K}^i_2}\|\rho_V^{(k-i)} r^i_{b,2,1}\|_\infty \|\sqrt{\rho_V} r^i_{b,2,1}\|_\infty + \dots \\
&\qquad + \pi^{i-1}|\mathfrak{K}^i_i| \max_{b\in\mathfrak{K}^i_i}\|\rho_V^{(k-i)} r^i_{b,i,1}\|_\infty \prod_{l=2}^{i-1}\|\rho_V r^i_{b,i,l}\|_\infty \|\sqrt{\rho_V} r^i_{b,i,i}\|_\infty\Big]\Bigg\}. \qquad (2.94)
\end{aligned}
$$

The fact that $t \mapsto C_i(V_{\phi,t},n)$ is shown in Appendix 2.10. □

Remark 2.9.4 With $\theta = \frac{\rho_V'}{\rho_V}$, $g \overset{(\text{def})}{=} \widetilde{\Xi^{-1}}[f]$ and $c \overset{(\text{def})}{=} \int_{\mathbb{R}} (2P\mathcal{H}[\rho_V g](y) - f(y))\, d\mu_V(y)$ we have

$$
\begin{aligned}
g' &= f - \theta g - 2P\mathcal{H}[\rho_V g] + c \\
g'' &= -\theta f + f' - 2P\mathcal{H}[\rho_V f] + \left(\theta^2 - \theta'\right) g + 2P\theta\mathcal{H}[\rho_V g] + 4P^2\mathcal{H}[\rho_V \mathcal{H}[\rho_V g]] \\
&\quad + (-\theta - 2P\mathcal{H}[\rho_V])\, c. \\
g^{(3)} &= \left(\theta^2 - 2\theta'\right) f - \theta f' + f'' + 2P\theta\mathcal{H}[\rho_V f] - 2P\mathcal{H}[\rho_V \theta f] - 2P\mathcal{H}\left[\rho_V f'\right] + 4P^2\mathcal{H}\Big[\rho_V \mathcal{H}[\rho_V f]\Big] \\
&\quad + \Big[\left(\theta^2 - \theta'\right)' - \left(\theta^3 - \theta\theta'\right)\Big] g + 2P\left(\theta^2 - 2\theta'\right)\mathcal{H}[\rho_V g] - 4P^2\theta\mathcal{H}[\rho_V \mathcal{H}[\rho_V g]] + 4P^2\mathcal{H}[\rho_V \theta\mathcal{H}[\rho_V g]] \\
&\quad + 4P^2\mathcal{H}[\rho_V \mathcal{H}[\rho_V \theta g]] - 8P^3\mathcal{H}\Big[\rho_V \mathcal{H}\big[\rho_V \mathcal{H}[\rho_V g]\big]\Big] \\
&\quad + \Big(\left(\theta^2 - 2\theta'\right) + 2P\theta\mathcal{H}[\rho_V] - 2P\mathcal{H}[\rho_V \theta] + 4P^2\mathcal{H}\Big[\rho_V \mathcal{H}[\rho_V]\Big]\Big) c
\end{aligned}
$$

2.10 Appendix: Integrability of the constants

2.10.1 Parameter continuity of norms of certain functions

In this appendix, we work with $V = V_{G,\phi,t}$, $t \in [0,1]$ and $\phi \in \cap_{k\geq 0} H^k(\mathbb{R})$. We will show, that the constant $C_i(V_{G,\phi,t}, n)$ appearing in our problem, see Theorem 2.4.14, Theorem 2.4.15 and 2.4.18 will be continuous in t hence integrable on $[0,1]$. In this section, we denote for all $t \in [0,1]$,

$$\alpha_t \overset{(\text{def})}{=} \frac{\rho_{V_{G,\phi,t}}}{\rho'_{V_{G,\phi,t}}} \qquad \textit{and} \qquad \theta_t \overset{(\text{def})}{=} \frac{\rho'_{V_{G,\phi,t}}}{\rho_{V_{G,\phi,t}}}.$$

First of all, by Theorem 2.6.9, the map $t \mapsto \|\rho_{V_{G,\phi,t}}\|_{W_n^\infty(\mathbb{R})}$ is continuous for all $n \in \mathbb{N}$. This allows to conclude that

Lemma 2.10.1 *Let $t, t_0 \in [0,1]$, for all $n \in \mathbb{N}$,*

$$\left\|\mathcal{H}[\rho_{V_{G,\phi,t}} - \rho_{V_{G,\phi,t_0}}]\right\|_{W_n^\infty(\mathbb{R})} \underset{t\to t_0}{\longrightarrow} 0.$$

Proof We prove it by induction and use Lemma 2.9.1 and Theorem 2.6.9. For $n = 0$, we know that there exists $C > 0$, such that:

$$\left\|\mathcal{H}[\rho_{V_{G,\phi,t}} - \rho_{V_{G,\phi,t_0}}]\right\|_\infty \leq C\left(\|\rho_{V_{G,\phi,t}} - \rho_{V_{G,\phi,t_0}}\|_{L^1(\mathbb{R})} + \|\rho'_{V_{G,\phi,t}} - \rho'_{V_{G,\phi,t_0}}\|_\infty\right).$$

By Scheffé's lemma, the L^1 norm goes to zero and by Theorem 2.6.9 goes also to zero as t goes to t_0. Now suppose that $\left\|\mathcal{H}[\rho_{V_{G,\phi,t}} - \rho_{V_{G,\phi,t_0}}]\right\|_{W_n^\infty(\mathbb{R})} \underset{t\to t_0}{\longrightarrow} 0$ for some $n \geq 0$. We have that

$$\left\|\mathcal{H}[\rho^{(n+1)}_{V_{G,\phi,t}} - \rho^{(n+1)}_{V_{G,\phi,t_0}}]\right\|_\infty \leq C\left(\|\rho^{(n+1)}_{V_{G,\phi,t}} - \rho^{(n+1)}_{V_{G,\phi,t_0}}\|_{L^1(\mathbb{R})} + \|\rho^{(n+2)}_{V_{G,\phi,t}} - \rho^{(n+2)}_{V_{G,\phi,t_0}}\|_\infty\right).$$

Since for all $x \in \mathbb{R}$, $\rho^{(n+1)}_{V_{G,\phi,t_0}}(x)$ goes to zero and we have the following domination by Leibniz formula

$$|\rho^{(n+1)}_{V_{G,\phi,t}}(x)| \leq (1 + \max_{s\in[0,1]} \|u_s\|_{W^\infty_{n+1}(\mathbb{R})}) \sum_{k=0}^{n+1} |\rho^{(k)}_{V_{G,\phi,t_0}}(x)|.$$

By dominated convergence theorem, the L^1-norm goes to zero and the last term also trivially goes to zero by theorem 2.6.9. □

Secondly, we can set $M_{V_{G,\phi,t}}$ (see Lemma 2.4.9) equal to any value M such that

$$M > \max\left(1 + \|\phi'\|_\infty + 2P \max_{t\in[0,1]} \|\mathcal{H}[\rho_{V_{G,\phi,t}}]\|_\infty, 2\left(\|\phi'\|_\infty + 2P \max_{t\in[0,1]} \|\mathcal{H}[\rho_{V_{G,\phi,t}}]\|_\infty\right)\right).$$

We choose such a M, it is well-defined because of Lemma 2.10.1.

Lemma 2.10.2 *For all $t \in [0,1]$, for all $i, j \in \mathbb{N}$, for all $|x| \geq M$,*

$$\left|\frac{\rho'_{V_{G,\phi,t}}}{\rho_{V_{G,\phi,t}}}(x)\right| \geq 1 \qquad \text{and} \qquad \left|\left(\frac{\rho_{V_{G,\phi,t}}}{\rho'_{V_{G,\phi,t}}}\right)^{(i)}(x)^j\right| \leq \delta_{i,0}\frac{C_{0,j}}{|x|^j} + \frac{C_{i,j}}{|x|^{2j}}$$

for constants $C_{i,j} > 0$ independent of t.

Proof Let $x \in \mathbb{R}$, $-\dfrac{\rho'_{V_{G,\phi,t}}}{\rho_{V_{G,\phi,t}}}(x) = x + t\phi'(x) + 2P\mathcal{H}[\rho_{V_{G,\phi,t}}](x)$. Thus if $|x| \geq M$,

$$\left|\frac{\rho'_V}{\rho_V}(x)\right| \geq 1 + \left(\|\phi'\|_\infty - t|\phi'(x)|\right) + 2P\left(\max_{s\in[0,1]} \|\mathcal{H}[\rho_{V_{G,\phi,s}}]\|_\infty - |\mathcal{H}[\rho_{V_{G,\phi,t}}](x)|\right) \geq 1.$$

For the second point, one notices by differentiation and (2.17) that there exists polynomials P_k with coefficients independent of t such that

$$\left(\frac{\rho_{V_{G,\phi,t}}}{\rho'_{V_{G,\phi,t}}}\right)^{(i)}(x) = \sum_{k=1}^{i} \frac{\mathsf{P}_k\left(t\phi'(x), \ldots, t\phi^{(i+1)}(x), \mathcal{H}[\rho_{V_{G,\phi,t}}](x), \ldots, \mathcal{H}\left[\rho^{(i)}_{V_{G,\phi,t}}\right](x)\right)}{\left(x + t\phi'(x) + 2P\mathcal{H}[\rho_{V_{G,\phi,t}}](x)\right)^{k+1}}.$$

Furthermore, since $|x| \geq 2\left(\|\phi'\|_\infty + 2P\max_{t\in[0,1]} \|\mathcal{H}[\rho_{V_{G,\phi,t}}]\|_\infty\right)$ we have

$$\left|x + t\phi'(x) + 2P\mathcal{H}[\rho_{V_{G,\phi,t}}](x)\right| \geq \frac{|x|}{2} + \left(\frac{|x|}{2} - \|\phi'\|_\infty - 2P\max_{s\in[0,1]} \|\mathcal{H}[\rho_{V_s}]\|_\infty\right) \geq \frac{|x|}{2}.$$

Finally, the whole dependence in t and x of the numerator are in the entries which are bounded uniformly in t and x, we can conclude that each numerator in the sum is bounded by a constant $C_k > 0$. We can conclude that $\left|\left(\dfrac{\rho_{V_{G,\phi,t}}}{\rho'_{V_{G,\phi,t}}}\right)^{(i)}(x)\right| \leq 4i \max_{k\leq i} C_k|x|^{-2}$, raising to the power j leads to the conclusion. □

Lemma 2.10.3 *The map $t \mapsto \|\rho^{-1}_{V_{G,\phi,t}}\|_{L^\infty([-M,M])}$ is continuous.*

Proof Let $x \in [-M, M]$,

$$|\rho_{V_{G,\phi,t}}(x)^{-1} - \rho_{V_{G,\phi,t_0}}(x)^{-1}| = \left|\frac{\delta t u_t(x)\rho_{V_{G,\phi,t_0}}(x)}{\rho_{V_{G,\phi,t_0}}(x)\rho_{V_{G,\phi,t}}(x)}\right| \leq |\delta t|\frac{\|u_t\|_\infty \|\rho^{-1}_{V_{G,\phi,t_0}}\|_{L^\infty([-M,M])}}{(1 - |\delta t|\|u_t\|_\infty)}.$$

Taking the supremum over $x \in [-M, M]$ and let t goes to t_0 establishes the result. □

Now it remains to bound the L^2 or L^∞ norms of the functions $\mathfrak{f}^{(i),V_{G,\phi,t}}$ and $\mathcal{I}_a^{V_{G,\phi,t}}$ appearing in Theorem 2.4.14, 2.4.15 and 2.4.18.

Lemma 2.10.4 *$t \mapsto \|\mathcal{I}_a^{V_{G,\phi,t}}\|_\infty$ for all $a \in \{1, 2\}$ is continuous where $\mathcal{I}_a^{V_{G,\phi,t}}$ is defined in* (2.52).

Proof Let $x > 0$, $t, t_0 \in [0, 1]$, by the mean-value theorem and with u_t defined in Section 2.6, we

get

$$|\mathcal{I}_1^{V_{G,\phi,t}}(x) - \mathcal{I}_1^{V_{G,\phi,t_0}}(x)| = \left| \frac{1}{\rho_{V_{G,\phi,t}}(x)} \int_x^{+\infty} \rho_{V_{G,\phi,t}}(s)ds - \frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \rho_{V_{G,\phi,t_0}}(s)ds \right|$$
$$\leq \frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \rho_{V_{G,\phi,t_0}}(s)ds \left| \frac{1+\delta t u_t(s)}{1+\delta t u_t(x)} - 1 \right|$$
$$= \frac{|\delta t|}{\rho_{V_{G,\phi,t_0}}(x)(1-|\delta t|\|u_t\|_\infty)} \int_x^{+\infty} \rho_{V_{G,\phi,t_0}}(s)ds \, |u_t(s) - u_t(x)|$$
$$\leq \frac{|\delta t|\|u_t'\|_\infty}{(1-|\delta t|\|u_t\|_\infty)} \frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} (s-x)\rho_{V_{G,\phi,t_0}}(s)ds. \quad (2.95)$$

One thus concludes by showing that $x \in]0,+\infty[\mapsto \frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} (s-x)\rho_{V_{G,\phi,t_0}}(s)ds$ is bounded, since, in (2.95), δt goes to zero and $t \mapsto \|u_t\|_{W_1^\infty(\mathbb{R})}$ is bounded.

This can be proven by integration by parts

$$\frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} (s-x)\rho_{V_{G,\phi,t_0}}(s)ds = \frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \left[\frac{\rho_{V_{G,\phi,t_0}}}{\rho'_{V_{G,\phi,t_0}}}(s)\rho_{V_{G,\phi,t_0}}(s)s \right]_x^{+\infty}$$
$$- \frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \left[\frac{\rho_{V_{G,\phi,t_0}}}{\rho'_{V_{G,\phi,t_0}}}(s) + \left(\frac{\rho_{V_{G,\phi,t_0}}}{\rho'_{V_{G,\phi,t_0}}} \right)'(s)s \right] \rho_{V_{G,\phi,t_0}}(s)ds.$$

The first term in the right hand side is bounded, while by assumption *v)*, the last term is a

$$\frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \underset{x\to+\infty}{O} \left(\frac{\rho_{V_{G,\phi,t_0}}(s)}{s} \right) ds = \underset{x\to+\infty}{O} \left(\frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \frac{\rho_{V_{G,\phi,t_0}}(s)}{s} ds \right).$$

Again by an integration by parts, the last integral in the remainder is equal to

$$\frac{\rho_{V_{G,\phi,t_0}}(x)}{\rho'_{V_{G,\phi,t_0}}(x)x} + \underset{x\to+\infty}{O} \left(\frac{1}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \frac{\rho_{V_{G,\phi,t_0}}(s)}{s^2} ds \right).$$

Since $\rho_{V_{G,\phi,t_0}}$ is decreasing in a neighborhood at infinity the last remainder is a $\underset{x\to+\infty}{o}(1)$ while the first is behaves like x^{-2} at infinity. Finally, $x \mapsto \frac{x}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \rho_{V_{G,\phi,t_0}}(s)ds$ is bounded by the same exact technique.

Doing the same thing over $]-\infty,0]$ establishes that $t \mapsto \|\mathcal{I}_1^{V_{G,\phi,t}}\|_\infty$ is continuous.

Just as before, we get by the mean-value theorem,

$$|\mathcal{I}_2^{V_{G,\phi,t}}(x)^2 - \mathcal{I}_2^{V_{G,\phi,t_0}}(x)^2| \leq \frac{1}{\rho^2_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \rho_{V^2_{G,\phi,t_0}}(s)ds \left| \frac{(1+\delta t u_t(s))^2}{(1+\delta t u_t(x))^2} - 1 \right|$$
$$\leq \frac{|\delta t|\|u_t'\|_\infty(2+\|u_t\|_\infty)}{(1-|\delta t|\|u_t\|_\infty)^2} \frac{1}{\rho_{V_{G,\phi,t_0}}(x)^2} \int_x^{+\infty} (s-x)\rho_{V_{G,\phi,t_0}}(s)^2 ds. \quad (2.96)$$

We conclude by showing that $x \in [0,+\infty[\mapsto \frac{1}{\rho_{V_{G,\phi,t_0}}(x)^2} \int_x^{+\infty} (s-x)\rho_{V_{G,\phi,t_0}}(s)^2 ds$ is bounded which can again be proven by the same integration by parts and by doing the exact same thing on $]-\infty,0]$. Therefore by the fact that

$$0 \leq \left| \|\mathcal{I}_2^{V_{G,\phi,t}}\|_\infty^2 - \|\mathcal{I}_2^{V_{G,\phi,t_0}}\|_\infty^2 \right| \leq \| \left(\mathcal{I}_2^{V_{G,\phi,t}}\right)^2 - \left(\mathcal{I}_2^{V_{G,\phi,t_0}}\right)^2 \|_\infty \underset{t\to t_0}{\longrightarrow} 0$$

we conclude that $t \mapsto \|\mathcal{I}_2^{V_{G,\phi,t}}\|_\infty$ is continuous. □

Next, we show that any polynomial in $\theta_t \overset{\text{(def)}}{=} \dfrac{\rho'_{V_{G,\phi,t}}}{\rho_{V_{G,\phi,t}}}$ and its derivatives yield a continuous dependance in t.

Lemma 2.10.5 *Let P a polynomial in $\theta_t, \dots, \theta_t^{(k)}$ for some $k \geq 0$ with coefficients independent of t, let $l \in \mathbb{N}$ then then the two following maps are continuous:*

(i) $t \mapsto \|\mathsf{P}\big(\theta_t, \dots, \theta_t^{(k)}\big)\|_{L^\infty([-M,M])}$

(ii) $t \mapsto \|\sqrt{\rho^{(l)}_{V_{G,\phi,t}}}\mathsf{P}\big(\theta_t, \dots, \theta_t^{(k)}\big)\|_{L^\infty(\mathbb{R})}$

(iii) $t \mapsto \|\rho^{(l)}_{V_{G,\phi,t}}\mathsf{P}\big(\theta_t, \dots, \theta_t^{(k)}\big)\|_{L^\infty(\mathbb{R})}$.

Proof Proving these continuity results for any monomial in those variables is enough. Furthermore, since by continuity $x \in [-M, M] \mapsto \theta_t^{(i)}$ is bounded for all $i \leq k$, thus this monomial in $(\theta_t^{(i)})_{0\leq i\leq k}$ converges uniformly to the monomial in $(\theta_{t_0}^{(i)})_{0\leq i\leq k}$ as t goes to t_0. The arguments are that the product of two bounded, uniformly converging sequences of functions converges to the product of the limits and that for all $i > 0$, $\theta_t^{(i)}(x) - \theta_{t_0}^{(i)}(x) = -\delta t\phi^{(i+1)}(x) - 2P\mathcal{H}\left[\rho^{(i)}_{V_{G,\phi,t}} - \rho^{(i)}_{V_{G,\phi,t_0}}\right]$. The latter, when taking the supremum over $x \in [-M, M]$, goes to zero by Lemma 2.10.1. This establishes *(i)*. Furthermore, notice that *(iii)* implies *(ii)* since $\|\sqrt{\rho^{(l)}_{V_{G,\phi,t}}}\mathsf{P}\left(\theta_t, \dots, \theta^{(k)}\right)\|_{L^\infty(\mathbb{R})} = \|\rho^{(l)}_{V_{G,\phi,t}}\mathsf{P}\left(\theta_t, \dots, \theta_t^{(k)}\right)^2\|^{1/2}_{L^\infty(\mathbb{R})}$ and P is arbitrary so we only prove *(iii)*. Moreover since by Faà di Bruno's formula $\rho^{(l)}_{V_{G,\phi,t}} = \exp(\log \rho_{V_{G,\phi,t}})^{(l)}$ can be written as $\mathsf{Q}\left(\theta, \dots, \theta^{(l)}\right)\rho_V$ where Q is a polynomial with coefficients independent of t, it suffices to prove the result for $l = 0$.

For all $i \in \mathbb{N}$, we have $\theta_t^{(i)}(x) = -\delta_{i,0}x - t\phi^{(i+1)}(x) - 2P\mathcal{H}\left[\rho^{(i)}_{V_{G,\phi,t}}\right]$. Noticing that by Leibniz formula and the mean value theorem that for all $j \in \mathbb{N}$, for all $0 < \alpha < 1$

$$\begin{aligned}\left|x^j\rho_{V_{G,\phi,t}}(x)^\alpha - x^j\rho_{V_{G,\phi,t_0}}(x)^\alpha\right| &\leq \left|x^j\rho_{V_{G,\phi,t_0}}(x)^\alpha\right|.\left|\left(1 + \delta t u_t(x)\right)^\alpha - 1\right| \\ &\leq \frac{\alpha|\delta t|\max_{s\in[0,1]}\|u_s\|_{W_l^\infty(\mathbb{R})}}{\left(1 - |\delta t|\max_{s\in[0,1]}\|u_s\|_{W_l^\infty(\mathbb{R})}\right)^{1-\alpha}}\left\|x \mapsto x^j\rho_{V_{G,\phi,t_0}}(x)^\alpha\right\|_\infty\end{aligned}$$

where the existence of the max is justified by Corollary 2.6.8. Taking the supremum over $x \in \mathbb{R}$ and let t goes to t_0 shows that $t \mapsto \left(x \mapsto x^j\rho_{V_{G,\phi,t}}(x)^\alpha\right) \in L^\infty(\mathbb{R})$ is continuous. By boundedness and continuity with respect to the t parameter of $t \mapsto \phi^{(i+1)}(x) + 2P\mathcal{H}\left[\rho^{(i)}_{V_{G,\phi,t}}\right]$, we deduce that for all $i \in \mathbb{N}$ and $\alpha > 0$,

$$\|\theta_t^{(i)}\rho_{V_{G,\phi,t}}(x)^\alpha - \theta_{t_0}^{(i)}\rho_{V_{G,\phi,t_0}}(x)^\alpha\|_\infty \underset{t\to t_0}{\longrightarrow} 0.$$

From this last uniform convergence result, we show that by taking a monomial $\prod_{i=0}^k\left(\theta_t^{(i)}\right)^{l_i}$ such that $\sum_{i=0}^k l_i = m$, we deduce that $\rho_{V_{G,\phi,t}}\prod_{i=0}^k\left(\theta_t^{(i)}\right)^{l_i} = \prod_{i=0}^k\left(\theta_t^{(i)}\sqrt[m]{\rho_{V_{G,\phi,t}}}\right)^{l_i}$, as a product of bounded, uniformly converging t-sequences of functions, it converges uniformly. This concludes the proof. □

Lemma 2.10.6 (Continuity of uniform norms) *For all $j \in \{1, 2, 3, 5, 6\}$, $t \mapsto \|\mathfrak{f}^{(j),V_{G,\phi,t}}_{n_1,a,b}\|_{L^\infty([-M,M]^c)}$*

is continuous where we recall that

$$\mathfrak{f}^{(1),V_{G,\phi,t}}_{l,a,b} : x \mapsto Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)\alpha_t(x)P^{l-a}_b(\alpha_t,\dots,\alpha_t^{(b)})(x),$$

$$\mathfrak{f}^{(2),V_{G,\phi,t}}_{l,a,b} : x \mapsto \frac{Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)}{\rho_{V_{G,\phi,t}}(x)} \int_x^{\mathrm{sgn}(x)\infty} \left[\alpha_t P^{l-a}_b\left(\alpha_t,\dots,\alpha_t^{(b)}\right)\right]'(y)\rho_{V_{G,\phi,t}}(y)dy,$$

$$\mathfrak{f}^{(3),V_{G,\phi,t}}_{l,a,b} : x \mapsto \frac{Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)}{\rho_{V_{G,\phi,t}}(x)}\left|\int_x^{\mathrm{sgn}(x)\infty} \alpha_t(y)^2 P^{l-a}_b\left(\alpha_t,\dots,\alpha_t^{(b)}\right)(y)^2\rho_{V_{G,\phi,t}}(y)^2dy\right|^{1/2},$$

$$\mathfrak{f}^{(5),V_{G,\phi,t}}_{l,a,b} : x \mapsto \frac{Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)}{\rho_{V_{G,\phi,t}}(x)} \int_x^{\mathrm{sgn}(x)\infty} \left|P^{l-a}_b(\alpha_t,\dots,\alpha_t^{(b)})(y)\right|\rho_{V_{G,\phi,t}}(y)dy,$$

$$\mathfrak{f}^{(6),V_{G,\phi,t}}_{l,a,b} : x \mapsto \frac{\left|Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)\right|}{\rho_{V_{G,\phi,t}}(x)}\left|\int_x^{\mathrm{sgn}(x)\infty} \left|P^{l-a}_b(\alpha_t,\dots,\alpha_t^{(b)})(y)\right|^2\rho_{V_{G,\phi,t}}(y)^2dy\right|^{1/2}.$$

Proof First, one can check that, from Lemma 2.4.13, $\mathfrak{f}^{(1),V_{G,\phi,t}}_{n_1,a,b}(x) = \underset{|x|\to\infty}{O}(x^{-1})$ for all $t \in [0,1]$. Noticing that since $\alpha(x) = (-x - t\phi'(x) - 2P\mathcal{H}[\rho_{V_{G,\phi,t}}](x))^{-1} = \theta(x)^{-1}$, there exists $n > 0$ and a polynomial expression P with coefficients independent of t such that

$$\mathfrak{f}^{(1),V_{G,\phi,t}}_{n_1,a,b}(x) = \frac{\mathsf{P}\left(x^{-1}, t\phi',\dots,t\phi^{(k)}, P\mathcal{H}[\rho_{V_{G,\phi,t}}],\dots,P\mathcal{H}[\rho^{(k-1)}_{V_{G,\phi,t}}]\right)}{(1 + t\phi'(x)x^{-1} + 2P\mathcal{H}[\rho_{V_{G,\phi,t}}](x)x^{-1})^n}.$$

In the above expression, the numerator must be a $\underset{|x|\to\infty}{O}(x^{-1})$. We conclude from this closed form, that $\mathfrak{f}^{(1),V_{G,\phi,t}}_{n_1,a,b}$ converges uniformly to $\mathfrak{f}^{(1),V_{G,\phi,t_0}}_{n_1,a,b}$ when t goes to t_0 on $[-M,M]^c$. Indeed, $\mathfrak{f}^{(1),V_{G,\phi,t}}_{n_1,a,b}$ is a bounded rational function such that the denominator is bounded from below uniformly in t (see Lemma 2.10.2) and such that both the numerator and denominator converges uniformly. Thus, $t \mapsto \|\mathfrak{f}^{(1),V_{G,\phi,t}}_{n_1,a,b}\|_{L^\infty([-M,M]^c)}$ is continuous.

We only prove the continuity of $t \mapsto \|\mathfrak{f}^{(j),V_{G,\phi,t}}_{n_1,a,b}\|_{L^\infty([-M,M]^c)}$ in the case $j = 5$, since the same arguments also prove the cases $j \in \{2,3,6\}$. Since $|\rho_{V_{G,\phi,t}}(x) - \rho_{V_{G,\phi,t_0}}(x)| \le |\delta t| \max_{s\in[0,1]} \|u_s\|_\infty |\rho_{V_{G,\phi,t_0}}(x)$ and that the following map is uniformly bounded in $t \in [0,1]$ and $x > M$

$$g_t : x \mapsto \frac{Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)}{\rho_{V_{G,\phi,t}}(x)} \int_x^{+\infty} \left|P^{l-a}_b(\alpha_t,\dots,\alpha_t^{(b)})(y)\right|\rho_{V_{G,\phi,t_0}}(y)dy,$$

we can just show that g_t converges uniformly to t_0 as t goes to t_0. Moreover since

$$|\rho_{V_{G,\phi,t}}(x)^{-1} - \rho_{V_{G,\phi,t_0}}(x)^{-1}| \le |\delta t| \frac{\max_{s\in[0,1]} \|u_s\|_\infty}{1 - |\delta t| \max_{s\in[0,1]} \|u_s\|_\infty} \rho_{V_{G,\phi,t_0}}(x)^{-1},$$

it is enough to show the uniform convergence for $\dfrac{\rho_{V_{G,\phi,t}}}{\rho_{V_{G,\phi,t_0}}} g_t$. One can also notice that, for constants $C_{l,a}$ and $C_{l,a,b}$ independents of t,

$$|Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)| \le C_{l,a}|x|^{l-a} \qquad \text{and} \qquad \left|P^{l-a}_b(\alpha_t,\dots,\alpha_t^{(b)})(y)\right| \le C_{l,a,b}|x|^{-(l-a)}.$$

Finally, by writing $|x^{-(l-a)}Q^l_a\left(\theta_t,\dots,\theta_t^{(a)}\right)(x)|$ as polynomial in x^{-1}, $t\phi^{(i+1)}(x)$ and $\mathcal{H}[\rho^{(i)}_{V_{G,\phi,t}}]$ for $i \ge 0$ and $x^{l-a}P^{l-a}_b(\alpha_t,\dots,\alpha_t^{(b)})(x)$ as a rational function in those same variables, we conclude that

these functions converge uniformly towards the same functions at t_0. Therefore

$$x \mapsto x^{-(l-a)} Q_a^l\left(\theta_t, \dots, \theta_t^{(a)}\right)(x) \frac{x^{l-a}}{\rho_{V_{G,\phi,t_0}}(x)} \int_x^{+\infty} \frac{\rho_{V_{G,\phi,t_0}}(y)dy}{y^{l-a}} \left| y^{l-a} P_b^{l-a}(\alpha_t, \dots, \alpha_t^{(b)})(y) \right|$$

converges uniformly to the same functions at t_0. This establishes the proposition. □

Lemma 2.10.7 (Continuity L^2-norms) *For all $j \in [\![1,4]\!]$, the maps $t \mapsto \|\mathfrak{f}_{n_1,a,b}^{(j),V_{G,\phi,t}}\|_{L^2([-M,M]^c)}$ are continuous where*

$$\mathfrak{f}_{n_1,a,b}^{(4),V_{G,\phi,t}} : x \mapsto \frac{Q_a^{n_1}\left(\theta_t, \dots, \theta_t^{(a)}\right)(x)}{\rho_{V_{G,\phi,t}}(x)} \left| \int_x^{\mathrm{sgn}(x)\infty} \left[\alpha_t P_b^{n_1-a}\left(\alpha_t, \dots, \alpha_t^{(b)}\right)\right]'(y) \rho_{V_{G,\phi,t}}(y)^2 dy \right|^{1/2}. \tag{2.97}$$

Proof For the continuity of $t \mapsto \|\mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t}}\|_{L^2([-M,M]^c)}$, we use dominated convergence theorem. Since, we showed uniform convergence and that $\|\mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t}}\|_{L^\infty([-M,M]^c)} < +\infty$, we conclude that for all $x \in [-M,M]^c$,

$$\mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t}}(x)^2 \underset{t\to t_0}{\longrightarrow} \mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t_0}}(x)^2.$$

The domination follows from the fact $\mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t}}(x) = \underset{|x|\to\infty}{O}(x^{-2})$ and that all the dependance in t is bounded, hence there exists a constant $C_{n_1,a,b} > 0$ independent of t such that, for all $x \in [-M,M]^c$ and all $t \in [0,1]$,

$$|\mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t}}(x)| \leq \frac{C}{x^2}.$$

This establishes that $\|\mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t}}\|_{L^2([-M,M]^c)} \underset{t\to t_0}{\longrightarrow} \|\mathfrak{f}_{n_1,a,b}^{(1),V_{G,\phi,t_0}}\|_{L^2([-M,M]^c)}$.

We now establish the continuity for $t \mapsto \|\mathfrak{f}_{n_1,a,b}^{(3),V_{G,\phi,t}}\|_{L^2([-M,M]^c)}$, the case $j \in \{2,4\}$ is done with the exact same arguments. We want to use dominated convergence theorem, for $\left(\mathfrak{f}_{n_1,a,b}^{(3),V_{G,\phi,t}}\right)^2$. The latter, when $t \to t_0 \in [0,1]$, also converges uniformly since it is uniformly bounded and that we proved that $\mathfrak{f}_{n_1,a,b}^{(3),V_{G,\phi,t}}$ converges uniformly. It just remains to verify the domination hypothesis. By Lemma 2.4.13, we know that $\alpha(y)^2 P_b^{n_1-a}\left(\alpha_t, \dots, \alpha_t^{(b)}\right)(y)^2 = \underset{|y|\to\infty}{O}(y^{-2(n_1-a+1)})$. We conclude by Lemma 2.10.2 that there exists a constant $C_{n_1,a,b} > 0$ such that for all $y > M$,

$$\alpha_t(y)^2 P_b^{n_1-a}\left(\alpha_t, \dots, \alpha_t^{(b)}\right)(y)^2 \leq \frac{C_{n_1,a,b}}{y^{2(n_1-a+1)}}.$$

Similarly $\left|Q_a^{n_1}\left(\theta_t, \dots, \theta_t^{(a)}\right)(y)^2\right| \leq C_{n_1,a}|x|^{2(n_1-a)}$ for all $y > M$ and for $C_{n_1,a} > 0$ a constant independent of t and y. Finally, we get the following domination for an arbitrary t_0 and all $y > M$

$$\begin{aligned}
&\frac{Q_a^{n_1}\left(\theta_t, \dots, \theta_t^{(a)}\right)(x)^2}{\rho_{V_{G,\phi,t}}(x)^2} \int_x^{+\infty} \alpha_t(y)^2 P_b^{n_1-a}\left(\alpha_t, \dots, \alpha_t^{(b)}\right)(y)^2 \rho_{V_{G,\phi,t}}(y)^2 dy \\
&\qquad \leq \frac{C_{n_1,a}|x|^{2(n_1-a)}}{\rho_{V_{G,\phi,t_0}}(x)(1 - \max_{s\in[0,1]}\|u_s\|_\infty)} \int_x^{+\infty} \frac{C_{n_1,a,b}(1 + \max_{s\in[0,1]}\|u_s\|_\infty)\rho_{V_{G,\phi,t_0}}(y)dy}{y^{2(n_1-a+1)}}.
\end{aligned}$$

The RHS is in $L^1([M,+\infty[)$ by integration by parts as it was done in the proof of Lemma 2.10.4. We conclude by doing the same on $]-\infty,-M]$. □

Proposition 2.10.8 *With the choice of potential $V_{\phi,t}$, the following map is continuous*

$$t \in [0,1] \mapsto \left(\mathtt{C}(\widetilde{\Xi_1^{-1}}, H^n), \mathtt{C}(\widetilde{\Xi_1^{-1}}, W_n^\infty), \mathtt{C}(\Theta^{(a)} \circ \widetilde{\Xi_1^{-1}}, H^n)\right)$$

Proof By recalling the expression of those constants in (2.47), (2.57), (2.64), since in this appendix, all the building blocks in these constants were shown to be continuous we can conclude. □

2.10.2 Parameter-continuity of $C_{\mathcal{L}_t}$ and $K_{V_{G,\phi,t}}$

In [DGM23, App. A], the authors showed that for a general potential V, the operator $\mathcal{A}$ considered as an unbounded operator on H has the same spectrum as the Schrödinger operator $\mathcal{S} : \mathcal{D}(\mathcal{S}) \to L^2(\mathbb{R})$,defined by

$$\mathcal{S} \overset{(\text{def})}{=} -\Delta + w_V \quad with \quad \mathcal{D}(\mathcal{S}) \overset{(\text{def})}{=} \left\{ u \in H^1(\mathbb{R}), uV' \in L^2(\mathbb{R}), -u'' + w_V u \in L^2(\mathbb{R}), \int_{\mathbb{R}} u(x)dx = 0 \right\}$$

and

$$w_V \overset{(\text{def})}{=} \frac{1}{2}\left(\frac{1}{2}V'^2 - V'' + 2PV'\mathcal{H}[\rho_V] - 2P\mathcal{H}[\rho_V'] + 2P^2\mathcal{H}[\rho_V]^2\right) = \frac{1}{2}\Big[(\log\rho_V)'' + \frac{1}{2}(\log\rho_V)'^2\Big]. \tag{2.98}$$

Since $C_{\mathcal{L}} = \lambda_1(\mathcal{A})^{-1/2} = \lambda_1(\mathcal{S})^{-1/2}$ by Theorem 2.4.3, we just have to show that when choosing the potential $V = V_{G,\phi,t}$, the t-dependent Schrödinger operator $\mathcal{S}_t$ with potential $w_{V_{G,\phi,t}}$ produces a continuous smallest eigenvalue $\lambda_1(\mathcal{S}_t)$. Before that, we recall the essential material for manipulating $\mathcal{S}_t$.

Proposition 2.10.9 *The map $t \mapsto C_{\mathcal{L}_t} = \lambda_1(\mathcal{S}_t)^{-1/2}$ is continuous.*

Proof First for all $t > 0$, $\lambda_1(\mathcal{S}_t) > 0$. Secondly, we have the following equalities:

$$E_t = \min_{\substack{u\in\mathcal{D}(\mathcal{S}_t)\\ \|u\|_2=1}} \langle u, \mathcal{S}_t[u]\rangle_{L^2(\mathbb{R})} = \inf_{\substack{u\in\mathcal{C}_c^\infty(\mathbb{R})\\ \|u\|_2=1}} \int_{\mathbb{R}} (u')^2(y)dy + \int_{\mathbb{R}} u^2(y) w_{V_{G,\phi,t}}(y)dy.$$

From the previous section $w_{V_{t'}}$ converges uniformly to $w_{V_{G,\phi,t}}$ when t' goes to t. Hence for all $t, t' \in [0,1]$, $u \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\|u\|_2 = 1$

$$\left|\int_{\mathbb{R}} (u')^2(y)dy + \int_{\mathbb{R}} u^2(y) w_{V_{G,\phi,t}}(y)dy - \int_{\mathbb{R}} (u')^2(y)dy + \int_{\mathbb{R}} u^2(y) w_{V_{t'}}(y)dy\right| \le \|w_{V_{G,\phi,t}} - w_{V_{t'}}\|_{L^\infty(\mathbb{R})}$$

hence $\sup_{\substack{u\in\mathcal{C}_c^\infty(\mathbb{R})\\ \|u\|_2=1}} \left|\int_{\mathbb{R}} (u')^2(y) + \int_{\mathbb{R}} u^2(y) w_{V_t}(y) - \int_{\mathbb{R}} (u')^2(y)dy + \int_{\mathbb{R}} u^2(y) w_{V_{t'}}(y)dy\right|$ goes to zero as t' goes to t. Since uniform convergence is enough to ensure convergence of infinimums we get the result. □

We know prove the continuity of the constant $K_{V_{G,\phi,t}}$ introduced in Theorem 2.2.3.

Lemma 2.10.10 *The following map is continuous*

$$t \mapsto K_{V_{G,\phi,t}} = 2P\|\mathcal{H}[\rho_{V_{G,\phi,t}}]\|_\infty + C + P\Big|\iint_{\mathbb{R}^2} \log|x-y| d\mu_{V_{G,\phi,t}}(x) d\mu_{V_{G,\phi,t}}(y)\Big|$$

for C some fixed constant.

Proof We already proved the continuity of the map $t \mapsto \|\mathcal{H}[\rho_{V_{G,\phi,t}}]\|_{L^\infty(\mathbb{R})}$ in Lemma 2.10.1 so it just remains to show that the double integral is continuous with respect to t. We prove this by dominated convergence theorem. The function $(x,y) \mapsto \log|x-y|\rho_{V_{G,\phi,t}}(x)\rho_{V_{G,\phi,t}}(y)$ converges almost everywhere to

$$(x,y) \mapsto \log|x-y|\rho_{V_{G,\phi,t_0}}(x)\rho_{V_{G,\phi,t_0}}(y)$$

as t goes to t_0. Furthermore we have the following domination (x,y)-almost everywhere

$$\Big|\log|x-y|\Big|\rho_{V_{G,\phi,t}}(x)\rho_{V_{G,\phi,t}}(y) \le \Big|\log|x-y|\Big|(1 + \max_{s\in[0,1]}\|u_s\|_\infty)^2 \rho_{V_{G,\phi,t_0}}(x)\rho_{V_{G,\phi,t_0}}(y).$$

This allows us to conclude on the continuity of $t \mapsto K_{V_{G,\phi,t}}$. □

Proposition 2.10.11 *The map* $t \mapsto \mathsf{P}_{K,k}^{V_{G,\phi,t}}$ *where* $\mathsf{P}_{K,k}^{V}$ *appears in* (2.67) *is integrable on* $[0,1]$.

Proof By the bounds on $\left|\left|\left|\widetilde{\Xi_1}^{-1}\right|\right|\right|_{H^i}$, $\left|\left|\left|\widetilde{\Xi_1}^{-1}\right|\right|\right|_{W_i^\infty}$ and $\left|\left|\left|\Theta^{(a)} \circ \widetilde{\Xi_1}^{-1}\right|\right|\right|_{W_i^\infty}$ for $0 \leq i \leq m_{K,k}$ in Theorem 2.4.14, 2.4.15 and 2.4.17 plus the continuity results of Appendix 2.10, we conclude on the finitness of $\int_0^1 \mathsf{P}_{K,k}^{V_{G,\phi,t}} dt$. □

Chapter 3

On the equilibrium measure for the Lukyanov integral

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."
Bertrand Russell

Contents

3.1 Introduction and statement of results

3.1.1 The separation of variable integral for the exponent of the field

The seminal works of Al. Zamolodchikov [Zam90], following the pioneering considerations of Yang-Yang [YY69], introduced the concept of Thermodynamic Bethe Ansatz (TBA) as a key tool allowing one to describe the ground state energies of integrable quantum field theories in finite volume. The construction, as an input, utilises the model's S-matrix and describes the per-volume ground state energy in terms of a solution to a non-linear integral equation. In the case of the Sinh-Gordon 1+1 dimensional quantum field theory, the TBA description was conjectured simultaneously and

independently by Al. Zamolodchikov [Zam06] and Lukyanov [Luk01]. In that case, there is a unique TBA equation which takes the form

$$\varepsilon(\lambda) \;=\; 2\mathfrak{r}\sin\Big[\tfrac{\pi}{1+\mathfrak{b}^2}\Big]\cosh(\lambda) + \int_{\mathbb{R}} K(\lambda-\mu)\log\Big[1+\mathrm{e}^{-\varepsilon(\mu)}\Big] \quad \text{with} \quad K(\lambda) \;=\; \frac{4\cosh(\lambda)\sin\Big[\tfrac{\pi}{1+\mathfrak{b}^2}\Big]}{\cosh(2\lambda) \;-\; \cos\Big[\tfrac{2\pi}{1+\mathfrak{b}^2}\Big]}. \tag{3.1}$$

This equation involves two parameters $\mathfrak{r}, \mathfrak{b} > 0$. $\mathfrak{r} = CmR$ with $C > 0$ some constant, R the model's volume and m the mass parameter. Finally, $\mathfrak{b} > 0$ measures the interaction strength.

It was rigorously shown in [FKS99] that, for any $\mathfrak{r} > 0$, the non-linear integral equation (3.1) admits a unique solution in $L^\infty(\mathbb{R})$. With the solution at hand, the per-volume ground state energy admits the integral representation

$$-m\int_{\mathbb{R}} \frac{\mathrm{d}\lambda}{2\pi}\cosh(\lambda)\,\log\Big[1 \;+\; \mathrm{e}^{-\varepsilon(\lambda)}\Big]\,. \tag{3.2}$$

These TBA-like considerations were backed up by Bethe Ansatz calculations carried out for the lattice discretisation of the finite-volume Sinh-Gordon quantum field theory in [BT06, Tes08].

By invoking an analogy with the classical method of separation of variables, Lukyanov [Luk01] conjectured that the ground state expectation value of the exponential of the quantum Sinh-Gordon field $\mathrm{e}^{\alpha\varphi}$ may be deduced from the data contained in the large-N behaviour of the below integral

$$\mathfrak{z}_N\big[V_\alpha\big] \;=\; \int_{\mathbb{R}^N} \mathrm{d}^N\lambda \prod_{k<\ell}^N \Big\{ \sinh\Big[(1+\mathfrak{b}^2)(\lambda_k-\lambda_\ell)\Big]\cdot\sinh\Big[(1+\mathfrak{b}^{-2})(\lambda_k-\lambda_\ell)\Big]\Big\} \prod_{k=1}^N \mathrm{e}^{-V_\alpha(\lambda_k)}\,. \tag{3.3}$$

The potential V_α appearing above was expressed in terms of the solution ε to the TBA equation for the Sinh-Gordon model (3.1) as

$$V_\alpha(\lambda) \;=\; \mathfrak{r}\cosh(\lambda) \;-\alpha\lambda\; - \int_{\mathbb{R}} \frac{\mathrm{d}\mu}{2\pi}\cdot\frac{\mathbf{g}(\mu)}{\cosh(\lambda-\mu)} \qquad \text{with} \qquad \mathbf{g}(\mu) \;=\; 2\log\Big[1 \;+\; \mathrm{e}^{-\varepsilon(\lambda)}\Big]\,. \tag{3.4}$$

It was conjectured in [Luk01] that, as $N \to +\infty$,

$$\frac{\mathfrak{z}_N\big[V_{\alpha(\mathfrak{b}+\mathfrak{b}^{-1})}\big]}{\mathfrak{z}_N\big[V_0\big]} \;=\; \Big(\frac{N}{\mathfrak{r}}\Big)^{\frac{\alpha^2}{2}}\cdot\big\langle \mathrm{e}^{\alpha\varphi}\big\rangle_{\mathfrak{r}}\cdot\Big(1+\mathrm{o}(1)\Big)\,. \tag{3.5}$$

The constant term in these asymptotics $\big\langle \mathrm{e}^{\alpha\varphi}\big\rangle_{\mathfrak{r}}$ was conjectured to coincide with the ground state expectation value of the exponent of the properly normalised Sinh-Gordon quantum field in the finite volume R theory. In this description, the fields were normalised so that, as $\mathfrak{r} \to +\infty$, the two-point functions of the fields have trivial CFT-like normalisation in the short space-like Minkowski-distance regime, see [Luk01] for more details.

In fact, Lukyanov's integral falls into a much large class of N-fold integrals describing so-called form factor -*i.e.* matrix elements- of local operators in numerous quantum integrable models solvable by the quantum separation of variables method, see *e.g.* [Bab03, DKM03, DKM19, DM14, GMN12, KKN13, Koz13, Koz15, Nic13]. In such a setting the form factors are expressed as a ratio $\mathscr{Z}[W_N]/\mathscr{Z}[V_N]$ for certain potentials V_N, W_N and where

$$\mathscr{Z}[V_N] \;=\; \int_{\mathscr{C}^N} \mathrm{d}^N\lambda \prod_{k<\ell}^N \Big\{ \sinh\Big[\pi\omega_1(\lambda_k-\lambda_\ell)\Big]\cdot\sinh\Big[\pi\omega_2(\lambda_k-\lambda_\ell)\Big]\Big\} \prod_{k=1}^N \mathrm{e}^{-V_N(\lambda_k)}\,. \tag{3.6}$$

Above ω_1, ω_2 are related to a given model's coupling constants, and $\mathscr{C}$ is a model dependent curve in $\mathbb{C}$. The curve may or may not be compact or closed. Also, the potential V_N may or may not depend on N. However, typically, it is *not* varying with N as $V_N(\lambda) = NU(\lambda)$ for some N independent function U. For such multiple integrals, one is usually interested in the $N \to +\infty$ regime which allows one either to reach the thermodynamic or the continuum limit of the model. Thus, on top of testing Lukyanov's conjecture, the possibility to study of the large-N behaviour of this class of integrals will have numerous applications in the field of quantum integrable models. We should mention that the above integral falls into the class of bi-orthogonal ensembles [Bor98].

It is clear that the class of quantum separation of variables integrals $\mathscr{Z}[V_N]$ bears a strong structural ressembles with the spectral partition function of a random Hermitian matrix M sampled from a measure $\propto \mathrm{e}^{-N\mathrm{Tr}[V(M)]}\mathrm{d}M$, with M being the Lebesgue measure on the statistically independent entries. Indeed, the latter takes the form

$$\mathscr{Z}_{N;\mathrm{Herm}}[V] \;=\; \int_{\mathbb{R}^N} \mathrm{d}^N\lambda \prod_{k<\ell}^{N} |\lambda_k - \lambda_\ell|^2 \prod_{k=1}^{N} \mathrm{e}^{-NV(\lambda_k)} \tag{3.7}$$

In both cases, there appears a one-body confining potential and a repulsive two-body interaction vanishing as the square of the spacing between the integration variables. In fact, the two-body interaction is given by Vandermonde determinants in both cases: the square of a usual Vandermonde in the random matrix case (3.7) and the product of two-hyperbolic Vandermondes in the quantum separation of variables case (3.6). One could thus hope that the techniques allowing one to deal with large-N behaviour of the random matrix ensemble will also be fit for tackling the large-N behaviour of quantum separation of variables issued integrals. Unfortunately, the situation is way more intricate and takes its origin in crucial differences between these two types of integrals. Genuinely, the potentials arising in the quantum separation of variables case are *not*[†] varying with N as $V_N(\lambda) = NU(\lambda)$ for some N independent function U. Thus, the two-body and the one-body interactions in $\mathscr{Z}[V_N]$ evolve on different scales and one needs to dilatate the integration variables, in an appropriate fashion, so that both rescaled interactions equilibrate. While in the random matrix case the two-body interaction was behaving trivially under rescalings, this is not anymore the case in the quantum separation of variables setting. This introduces several additional scales in N to the problem what makes numerous of the steps developed for the random matrix case very tricky, technically speaking, to set in. We would like to mention that, in fact, certain instances of integrals of the type (3.6) did in fact appear directly in the random matrix literature. More precisely, the spectral part of a random Hermitian matrix's sampled from a measure $\propto \mathrm{e}^{-N\mathrm{Tr}[V(M)-AM]}\mathrm{d}M$ with $A = \mathrm{diag}(a_1, \ldots, a_N)$, $a_k = (k-1)/N$, admits the integral representation [CW14]

$$\mathscr{Z}_{N;\mathrm{Source}}[V] \;=\; \int_{\mathbb{R}^N} \mathrm{d}^N\lambda \prod_{k<\ell}^{N} \Big\{ (\lambda_k - \lambda_\ell) \sinh\big(\lambda_k - \lambda_\ell\big) \Big\} \prod_{k=1}^{N} \mathrm{e}^{-NV(\lambda_k)} \,. \tag{3.8}$$

The asymptotic expansion of such integrals may be deal with by using the techniques developed in [BGK15]. The work [CW14] proposed a Riemann–Hilbert approach for bi-orthogonal polynomials that could, in principle, allow one to extract the large-N behaviour of (3.8) and, more generally, (3.6). However, in the case of the quantum separation of variables issued integral, the necessity for rescaling the integration variables would introduce numerous technical complications to the large-N analysis technique outlined in [CW14]. In particular, it would demand to have a highly detailed control on the N-dependent equilibrium measure that will be obtained in the present paper.

The first progress in this direction of achieving a large-N analysis of multiple integrals of the form (3.6) was achieved in [BGK16] where techniques allowing one to deal with N-dependent two-body

[†]Else, indeed, the study of the large-N behaviour of quantum separartion of variables issued multiple integrals would follow from the application of techniques developed [BGK15]

interactions were developed. The aim of this work is to push further the results obtained in [BGK16] and lay the ground for rigorously justifying the presence of the the power-law term in N in (3.5), sand in a second stage for obtaining rigorously the whole expansion up to o(1). In order to apply concentration of measure techniques which were first developed for β-ensembles in [BG97] and, later, extended so as to allow to deal with more complex integrals in [BG13a, BG13b, BGK16], one first needs to have a good grasp on the so-called equilibrium measure. As explained in [BGK16], in the case of the partition function (3.3), the latter corresponds to the unique minimiser of an N-dependent functional on $\mathcal{M}^1(\mathbb{R})$, the space of probability measures on $\mathbb{R}$. The construction of the equilibrium measure is the main achievement of this work. As mentioned, this may also pave the way to the Riemann–Hilbert analysis of the large-N behaviour of (3.3) by means of bi-orthogonal polynomials. Our result allows us to back up the prediction on the leading large-N behaviour given in (3.5), although the lack of certain estimates does not allow us to turn our findings into a rigorous proof.

The paper is organised as follows.

3.1.2 The main results

It is easy to see that the repulsive nature of the sinh two-body interaction and the confining nature of the potential are of the same order of magnitude in N on a scale $\log N$. Hence, so as to deal with finite quantities, it appears convenient to rescale the integration variables in (3.3) by $\log N$. Then, it holds $\mathfrak{z}_N\big[V_\alpha\big] \,=\, \big[\log N\big]^N \mathcal{Z}_N\big[V_{N;\alpha}\big]$ with

$$\mathcal{Z}_N\big[V_{N;\alpha}\big] \,=\, \int_{\mathbb{R}^N} \mathrm{d}^N\lambda \prod_{a<b}^N \Big\{ \sinh\Big[\frac{\overline{\omega}_1}{2}(\lambda_a-\lambda_b)\Big]\cdot \sinh\Big[\frac{\overline{\omega}_2}{2}(\lambda_a-\lambda_b)\Big]\Big\} \prod_{a=1}^N \mathrm{e}^{-N\tau_N V_{N;\alpha}(\lambda_a)} \,. \tag{3.9}$$

The two periods $\overline{\omega}_a$ grow with N as

$$\overline{\omega}_a \,=\, 2\pi\tau_N\omega_a \qquad \text{with} \qquad \tau_N \,=\, \log N \,. \tag{3.10}$$

The rescaled confining potential takes the form

$$V_{N;\alpha}(\lambda) \,=\, \frac{\mathfrak{r}}{N\tau_N}\cosh\big[\tau_N\lambda\big] \,-\, \frac{\alpha\lambda}{N} \,-\, \int_{\mathbb{R}} \frac{\mathrm{d}\mu}{2\pi N}\cdot \frac{\mathbf{g}(\tau_N\mu)}{\cosh\big[\tau_N(\lambda-\mu)\big]} \,. \tag{3.11}$$

The connection to the Lukyanov integral imposes the following from for the periods

$$\omega_1 \,=\, \frac{1+\mathfrak{b}^2}{\pi} \quad \text{and} \quad \omega_2 = \frac{1+\mathfrak{b}^{-2}}{\pi} \,. \tag{3.12}$$

Following the techniques of [BG97] and their adaptation to the N-dependent setting developed in [BGK16], one may show that

$$\mathcal{Z}_N\big[V_{N;\alpha}\big] \,=\, \exp\Big\{ -N^2\tau_N \inf_{\mu\in\mathcal{M}^1(\mathbb{R})} \mathcal{E}_N[\mu] \,+\, \mathrm{O}\big(N\tau_N^2\big)\Big\} \,, \tag{3.13}$$

in which the N-dependent functional on $\mathcal{M}^1(\mathbb{R})$ takes the form

$$\mathcal{E}_N[\mu] \,=\, \int_{\mathbb{R}} \mathrm{d}\mu(s) V_{N;\alpha}(s) \,-\, \frac{1}{2\tau_N}\int_{\mathbb{R}^2} \mathrm{d}\mu(s)\mathrm{d}\mu(t)\log\Big\{ \prod_{a=1}^2 \sinh\Big(\overline{\omega}_a\frac{s-t}{2}\Big)\Big\} \,. \tag{3.14}$$

$\mathcal{E}_N$ is strictly convex, lower-continuous and has compact level sets, see [BGK16] for more details. As such, it admits a unique minimiser on $\mathcal{M}^1(\mathbb{R})$ denoted by $\widehat{\mu}_{\mathrm{eq};\alpha}$. Our main result is gathered in the

Theorem 3.1.1 *The equilibrium measure is Lebesgue continuous with a density $\widehat{\varrho}_{\mathrm{eq};\alpha}$ given by the square root of an analytic function. There exists $\mathfrak{r}_0$ such that, for any N and $\mathfrak{r} \geq \mathfrak{r}_0$, it is supported on the segment $\sigma_{N;\alpha} = [a_{N;\alpha}\,;b_{N;\alpha}]$. Moreover, there exists N_0 such that, for any $N \geq N_0$ it is given by*

$$\widehat{\varrho}_{\mathrm{eq};\alpha} \,=\, \mathcal{W}_N\big[V'_{N;\alpha}\mathbf{1}_{\sigma_{N;\alpha}}\big]_{\vert\substack{a_N\hookrightarrow a_{N;\alpha}\\ b_N\hookrightarrow b_{N;\alpha}}}\,. \tag{3.15}$$

where $\mathcal{W}_N$ is an explicit integral transform introduced in (3.46) *in which one should specify the periods ω_a as in* (3.12).

The endpoints $a_{N;\alpha}, b_{N;\alpha}$ of the support admit the large-N expansion

$$\begin{aligned}\tau_N b_{N;\alpha} \,&=\, \log\Big(\frac{\mathfrak{d}_0 N}{2}\Big) \,+\, \frac{\alpha}{N(1+\mathfrak{b}^2)(1+\mathfrak{b}^{-2})}\\ &+\, \frac{1}{N^2}\bigg\{\mathfrak{d}_1\Big(1\,+\,\frac{2\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi\mathfrak{r}}\mathbf{1}_{1<\zeta}\Big(1+\tfrac{\alpha}{\pi}\Big)\Big)\,-\,\frac{\alpha^2}{2(1+\mathfrak{b}^2)^2(1+\mathfrak{b}^{-2})^2}\bigg\}\,+\,\mathrm{O}\Big(\frac{1}{N^{3-\eta}}\Big)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\tau_N a_{N;\alpha} \,&=\, -\log\Big(\frac{\mathfrak{d}_0 N}{2}\Big) \,+\, \frac{\alpha}{N(1+\mathfrak{b}^2)(1+\mathfrak{b}^{-2})}\\ &-\, \frac{1}{N^2}\bigg\{\mathfrak{d}_1\Big(1\,+\,\frac{2\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi\mathfrak{r}}\mathbf{1}_{1<\zeta}\Big(1+\tfrac{\alpha}{\pi}\Big)\Big)\,+\,\frac{\alpha^2}{2(1+\mathfrak{b}^2)^2(1+\mathfrak{b}^{-2})^2}\bigg\}\,+\,\mathrm{O}\Big(\frac{1}{N^{3-\eta}}\Big)\end{aligned} \tag{3.17}$$

where $\eta > 0$ is fixed but can be taken as small as need be. The two constants $\mathfrak{d}_0, \mathfrak{d}_1$ arising in this expansion take the form

$$\mathfrak{d}_0 \,=\, \frac{2}{\mathfrak{r}\sqrt{\pi}}\prod_{\upsilon=\pm}\Big\{\big(1+\mathfrak{b}^{2\upsilon}\big)^{\frac{-1}{2(1+\mathfrak{b}^{2\upsilon})}}\cdot\Gamma\Big(\frac{1}{2(1+\mathfrak{b}^{2\upsilon})}\Big)\Big\} \quad \textit{and} \quad \mathfrak{d}_1 \,=\, \frac{\mathfrak{r}^2}{\pi}\prod_{\upsilon=\pm}\Big\{\frac{\sin\Big[\frac{\pi}{2(1+\mathfrak{b}^{2\upsilon})}\Big]}{1+\mathfrak{b}^{2\upsilon}}\Big\}\,. \tag{3.18}$$

The hardest part of the theorem consists in proving the form of the endpoints, the rest of its content follows from the techniques already developed in [BGK16].

We now explain one way to obtain the large-N behaviour of the ratio: $\mathfrak{z}_N\big[V_\alpha\big]/\mathfrak{z}_N\big[V_0\big]$, *i.e.* $\mathcal{Z}_N\big[V_{N;\alpha}\big]/\mathcal{Z}_N\big[V_{N;0}\big]$. It is direct to see that

$$\partial_\alpha \log \mathcal{Z}_N\big[V_{N;\alpha}\big] \,=\, N\tau_N\mathbb{E}_{N;\alpha}\Big[\int_{\mathbb{R}}\xi\mathrm{d}\mathrm{L}_N^{(\boldsymbol{\lambda}_N)}(\xi)\Big] \qquad \text{with} \qquad \mathrm{L}_N^{(\boldsymbol{\lambda}_N)} \,=\, \frac{1}{N}\sum_{a=1}^N\delta_{\lambda_a} \tag{3.19}$$

being the empirical distribution of the integration variables, $\boldsymbol{\lambda}_N = \big(\lambda_1,\ldots,\lambda_N\big)$, and $\mathbb{E}_{N;\alpha}$ referring to the expectation in respect to the probability measure $\mathbb{P}_{N;\alpha}$ on $\mathbb{R}^N$ with density

$$\mathfrak{p}_{N;\alpha}\big(\boldsymbol{\lambda}_N\big) \,=\, \frac{1}{\mathcal{Z}_N\big[V_{N;\alpha}\big]}\prod_{a<b}^N\Big\{\sinh\Big[\frac{\overline{\omega}_1}{2}(\lambda_a-\lambda_b)\Big]\cdot\sinh\Big[\frac{\overline{\omega}_2}{2}(\lambda_a-\lambda_b)\Big]\Big\}\prod_{a=1}^N\mathrm{e}^{-N\tau_N V_{N;\alpha}(\lambda_a)}\,. \tag{3.20}$$

Using concentration of measure techniques, one may show that under $\mathbb{P}_{N;\alpha}$ the empirical measure concentrates around the equilibrium measure in the sense that for smooth functions growing at most polynomially at infinity it holds

$$\Big|\mathbb{E}_{N;\alpha}\Big[\int_{\mathbb{R}}\phi(\xi)\mathrm{d}\Big(\widehat{\mu}_{\mathrm{eq};\alpha}-\mathrm{L}_N^{(\boldsymbol{\lambda}_N)}\Big)(\xi)\Big]\Big| \,\leq\, C\frac{\tau_N}{\sqrt{N}}\,. \tag{3.21}$$

for some constant depending on ϕ. This reasoning then entails that

$$\partial_\alpha \log \mathcal{Z}_N\big[V_{N;\alpha}\big] \;=\; N\tau_N \int_{\mathbb{R}} \xi \mathrm{d}\widehat{\mu}_{\mathrm{eq};\alpha}(\xi) \;+\; \mathrm{O}\Big(\sqrt{N}\tau_N^2\Big)\,. \tag{3.22}$$

The remainder is uniform in α. However the bounds issuing from the concentration are only *a priori* bounds and these may be improved by using the machinery of loop equations which allow one to improve the rigidity of the fluctuations, see [BG13a] for an implementation of the method in the case of β-ensembles, and compute the subdominant corrections contained in the O remainder above. However, in the case of the present multiple, there arise several technical difficulties in the analysis of the associated system of loop equations which go beyond the scope of the present analysis. We would however like to point out that under the specialisation (3.12), it holds

$$N\tau_N \int_{\mathbb{R}} \xi \mathrm{d}\widehat{\mu}_{\mathrm{eq};\alpha}(\xi) \;=\; \frac{\alpha \log N}{(1+\mathfrak{b}^2)(1+\mathfrak{b}^{-2})} \;+\; \mathrm{O}\big(1\big)\,, \tag{3.23}$$

with a remainder that is uniform in α.

This leads to the suggestive results

$$\log\left(\frac{\mathcal{Z}_N\big[V_{N;\alpha(\mathfrak{b}+\mathfrak{b}^{-1})^2}\big]}{\mathcal{Z}_N\big[V_{N;0}\big]}\right) \;=\; \frac{\alpha^2}{2}\log N \;+\; \mathfrak{R}_N \tag{3.24}$$

with $\mathfrak{R}_N$ a remainder that we are able only to estimate as $\mathrm{O}\Big(\sqrt{N}\tau_N^2\Big)$. However, we expect that this is an overestimate and that eventually, the machinery of loop equations will allow us to prove that $\mathfrak{R}_N = \mathrm{O}(1)$. Thus, while not being a rigorous proof thereof, the above provides a strong check of Lukyanov's conjecture.

3.2 The N-dependent equilibrium measure

3.2.1 General properties of the equilibrium measure

Consider the multiple integral

$$\mathfrak{X}_M \;=\; \int_{\mathbb{R}^M} \mathrm{d}^M\lambda \prod_{a<b}^{M} \Big\{ \sinh\Big[\pi\omega_1\tau_N(\lambda_a-\lambda_b)\Big]\cdot \sinh\Big[\pi\omega_2\tau_N(\lambda_a-\lambda_b)\Big]\Big\}^{\frac{1}{\tau_N}} \prod_{a=1}^{N} \mathrm{e}^{-MV_{N;\alpha}(\lambda_a)}\,. \tag{3.25}$$

There N is fixed and to be considered as an outer parameter. This kind of integral has been studied in [BGK16]. It was shown there that

$$\lim_{M\to+\infty}\Big\{\frac{1}{M^2}\log\mathfrak{X}_M\Big\} \;=\; -\inf_{\mu\in\mathcal{M}^1(\mathbb{R})}\mathcal{E}_N[\mu] \tag{3.26}$$

with $\mathcal{E}_N$ as defined in (3.14). The minimum is attained at a unique measure $\widehat{\mu}_{\mathrm{eq};\alpha}$ that has compact support given by a finite union of segments and is Lebesgue continuous with a density $\widehat{\varrho}_{\mathrm{eq};\alpha}$ given by the square root of an function analytic in an open neighbourhood of the support[†]. In particular, the density is smooth in the interior of the support and vanishes at least as a square root at the edges of the support.

Lemma 3.2.1 *There exists $\mathfrak{r}_0 > 0$ such that, for any $\mathfrak{r} \geq \mathfrak{r}_0$, the equilibrium measure has connected support*

$$\sigma_{N;\alpha} \;=\; \mathrm{supp}\big[\,\widehat{\mu}_{\mathrm{eq};\alpha}\big] \;=\; [a_{N;\alpha}\,;b_{N;\alpha}]\,. \tag{3.27}$$

[†]In our N dependent setting, the size of this neighbourhood will naturally depend on N

Proof —

It follows from Appendix C of [BGK16] that the support of the equilibrium measure will be connected, *viz.* of the form (3.27), as soon as $V_{N;\alpha}$ is strictly convex.

A direct calculation starting from (3.11) yields

$$V''_{N;\alpha}(\lambda) \,=\, \frac{\mathfrak{r}\tau_N}{N}\cosh\big[\tau_N\lambda\big] - \int_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi N}\cdot\Bigg\{\frac{1}{\cosh\big[\tau_N(\lambda-\mu)\big]} - 2\frac{\sinh^2\big[\tau_N(\lambda-\mu)\big]}{\cosh^3\big[\tau_N(\lambda-\mu)\big]}\Bigg\}\cdot\mathbf{g}(\tau_N\mu)\;. \tag{3.28}$$

It is easy to infer from the form of the non-linear integral equation satisfied by ε, that there exist $\mathfrak{r}$-independent $c_\varepsilon, c'_\varepsilon > 0$ such that

$$\mathrm{e}^{-\varepsilon(\lambda)} \,\leq\, c_\varepsilon \mathrm{e}^{-\mathfrak{r}c'_\varepsilon\cosh(\lambda)}\;. \tag{3.29}$$

One thus gets the lower bound

$$\begin{aligned} V''_{N;\alpha}(\lambda) \,\geq\, \frac{\mathfrak{r}\tau_N}{N}\cosh\big[\tau_N\lambda\big] \,-\, c_\varepsilon\tau_N^2\mathrm{e}^{-\mathfrak{r}c'_\varepsilon}\int_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi N}\cdot\Bigg|\frac{1}{\cosh\big[\tau_N(\mu)\big]} \,-\, 2\frac{\sinh^2\big[\tau_N(\lambda-\mu)\big]}{\cosh^3\big[\tau_N(\mu)\big]}\Bigg| \\ \geq\, \frac{\tau_N}{N}\Big\{\mathfrak{r} \,-\, C\mathrm{e}^{-\mathfrak{r}c'_\varepsilon}\Big\} > 0 \end{aligned} \tag{3.30}$$

where the last bound follows provided that $\mathfrak{r} \geq \mathfrak{r}_0$ for some $\mathfrak{r}_0 > 0$.

Further, it is a standard fact, see *e.g.* [Dei99] that $\widehat{\mu}_{\mathrm{eq};\alpha}$ corresponds to the unique solution to the variational problem

$$V_{N;\alpha}(\lambda) \,-\, \frac{1}{\tau_N}\int_{\mathbb{R}}\mathrm{d}\widehat{\mu}_{\mathrm{eq};\alpha}(s)\log\Bigg[\prod_{a=1}^{2}\sinh\Big(\overline{\omega}_a\frac{\lambda-s}{2}\Big)\Bigg] \qquad \begin{cases} = C^{(N)}_{\mathrm{eq};\alpha} & \lambda\in\sigma_{N;\alpha} \\ > C^{(N)}_{\mathrm{eq};\alpha} & \lambda\in\mathbb{R}\setminus\sigma_{N;\alpha} \end{cases}\;. \tag{3.31}$$

First of all, due to the smoothness of the equilibrium's measure density and its square root vanishing at the edges, we could formulate this problem in the strong sense, *i.e.* pointwise, and not *a.e.*. Also, we stress that due to the strict convexity of $V_{N;\alpha}$ for $\mathfrak{r} \geq \mathfrak{r}_0$, the second condition is immediately satisfied, see Appendix C of [BGK16] for more details.

This variational characterisation of the equilibrium measure allows one to obtain upper/lower bounds on the endpoints $a_{N;\alpha}/b_{N;\alpha}$.

Lemma 3.2.2 *There exists $N_0 > 0$ and $\varsigma > 0$ such that, for any $N \geq N_0$,*

$$a_{N;\alpha} \,\leq\, -\varsigma \qquad \textit{and} \qquad b_{N;\alpha} \,\geq\, \varsigma\;. \tag{3.32}$$

Proof —

The proof goes by contradiction. Thus assume that for any $\varsigma > 0$ and N_0 there exists $N \geq N_0$ such that

$$a_{N;\alpha} \,\geq\, -\varsigma \qquad \text{or} \qquad b_{N;\alpha} \,\leq\, \varsigma\;. \tag{3.33}$$

One may take $\sigma < 1/4$ and extracting sub-sequences if need be, one thus has a sequence $N_k \to +\infty$ such that, for any k, $b_{N_k;\alpha} \leq \varsigma$ or, for any k, $a_{N_k;\alpha} \geq -\varsigma$. We discuss in detail the first case scenario, the second one can be excluded in the same fashion.

We start by introducing the so-called effective potential

$$V_{\mathrm{eff}}(\lambda) \,=\, V_{N;\alpha}(\lambda) \,+\, \int_{\mathbb{R}}\mathrm{d}\widehat{\mu}_{\mathrm{eq};\alpha}(s)\mathfrak{f}_N(\lambda,s) \qquad \text{with} \qquad \mathfrak{f}_N(\lambda,s) \,=\, \frac{-1}{\tau_N}\log\Bigg[\prod_{a=1}^{2}\sinh\Big(\overline{\omega}_a\tfrac{\lambda-s}{2}\Big)\Bigg]\;. \tag{3.34}$$

One has that

$$\partial_\lambda \mathfrak{f}_N(\lambda, s) \,=\, -\sum_{a=1}^{2} \pi\omega_a \coth\big[\pi\omega_a \tau_N(\lambda - s)\big] \,<\, \quad \text{for} \quad \lambda \geq s\,. \tag{3.35}$$

which entails that $\mathfrak{f}_N(\lambda, s) - \mathfrak{f}_N(\mu, s) < 0$ for any $\lambda, \mu > s$. One may thus decompose for any $1/2 > \lambda > 1/4$

$$\begin{aligned} V_{\text{eff}}(\lambda) \,-\, V_{\text{eff}}\big(b_{N_k;\alpha}\big) \,=\, V_{N_k;\alpha}(\lambda) \,-\, V_{N_k;\alpha}\big(b_{N_k;\alpha}\big) \,+\, \int_{a_N}^{b_N} \mathrm{d}\,\widehat{\mu}_{\text{eq};\alpha}(s)\Big(\mathfrak{f}_{N_k}\big(\lambda, s\big) \,-\, \mathfrak{f}_{N_k}\big(\tfrac{1}{4}, s\big)\Big) \\ +\, \int_{a_N}^{b_N} \mathrm{d}\,\widehat{\mu}_{\text{eq};\alpha}(s)\Big(\mathfrak{f}_{N_k}\big(\tfrac{1}{4}, s\big) \,-\, \mathfrak{f}_{N_k}\big(b_{N_k;\alpha}, s\big)\Big)\,. \end{aligned} \tag{3.36}$$

One may then bound each term as follows. Since $1/4 > \varsigma > b_{N_k;\alpha}$, the last line produces a purely negative contribution. Further, it is direct to estimate that $V_{N_k;\alpha}(\lambda) = \mathrm{o}(1)$ as $k \to +\infty$. Likewise, if $-3/4 \leq b_{N_k;\alpha} \leq \sigma$ then one has that $V_{N_k;\alpha}\big(b_{N_k;\alpha}\big) = \mathrm{o}(1)$, while for $b_{N_k;\alpha} \leq -3/4$ it holds for k large enough that $V_{N_k;\alpha}\big(b_{N_k;\alpha}\big) \geq V_{N_k;\alpha}(\lambda)$. Thus, whatever the regime, it holds

$$V_{N_k;\alpha}\big(b_{N_k;\alpha}\big) \,-\, V_{N_k;\alpha}(\lambda) \,\leq\, \mathrm{o}(1)\,. \tag{3.37}$$

Finally, one has, for $\lambda > s$ that

$$\mathfrak{f}_N(\lambda, s) \,=\, -\pi(\omega_1 + \omega_2)(\lambda - s) \,+\, \mathrm{O}\Big(\frac{1}{\tau_N}\sum_{a=1}^{2} \mathrm{e}^{-\overline{\omega}_a(\lambda - s)}\Big)\,. \tag{3.38}$$

Thus, since $\lambda, 1/4$ are uniformly away from s in the integral arising in the first line of (3.36), one gets that

$$V_{\text{eff}}(\lambda) \,-\, C^{(N)}_{\text{eq};\alpha} \,=\, V_{\text{eff}}(\lambda) \,-\, V_{\text{eff}}\big(b_{N_k;\alpha}\big) \,\leq\, \mathrm{o}(1) \,-\, \pi(\omega_1 + \omega_2)\big(\lambda - \tfrac{1}{4}\big) \,<\, 0\,. \tag{3.39}$$

Since $\lambda \notin \sigma_{N;\alpha}$, this contradicts the variational equation (3.31).

One knows from Lemma 3.2.1 that the equilibrium measure is supported on the segment $[a_{N;\alpha}\,; b_{N;\alpha}]$. Since $\widehat{\varrho}_{\text{eq};\alpha}$ is smooth on $]a_{N;\alpha}\,; b_{N;\alpha}[$ and admits at worst square root singularities at the edges $a_{N;\alpha}, b_{N;\alpha}$, one has that $\widehat{\varrho}_{\text{eq};\alpha} \in H_s([a_{N;\alpha}\,; b_{N;\alpha}])$ for any $0 < s < 1/2$. In particular, one may differentiate the first relation given in (3.31) what yields the singular-integral equation satisfied by the equilibrium measure's density

$$\sum_{a=1}^{2} \pi\omega_a \fint_{a_{N;\alpha}}^{b_{N;\alpha}} \mathrm{d}s\, \widehat{\varrho}_{\text{eq};\alpha}(s) \coth\Big[\overline{\omega}_a \tfrac{\lambda - s}{2}\Big] \,=\, V'_{N;\alpha}(\lambda)\,. \tag{3.40}$$

Now, Lemma (3.2.2) ensures that $b_{N;\alpha} - a_{N;\alpha} \geq 2\sigma > 0$, so that $\widehat{\varrho}_{\text{eq};\alpha}$ solves a truncated Wiener-Hopf equation in which the renormalised difference of boundaries satisfies

$$\tau_N b_{N;\alpha} \,-\, \tau_N a_{N;\alpha} \underset{N\to+\infty}{\longrightarrow} +\infty\,. \tag{3.41}$$

This allows one to invoke Riemann–Hilbert techniques so as to solve the equation in the large-N regime and will provide the starting starting point for characterising the measure and its support thoroughly in the large-N regime.

3.2.2 A truncated Wiener-Hopf based representation for the equilibrium measure

It was established in [BGK16] that, provided that $b_N - a_N > \eta$ and that N is large enough, the singular integral operator $\mathcal{S}_N : H_s([a_N\,;b_N]) \to H_s(\mathbb{R})$ with $0 < s < 1/2$ defined for sufficiently regular functions by

$$\mathcal{S}_N[h](\lambda) \;=\; \sum_{a=1}^{2} \pi\omega_a \fint_{\mathbb{R}} \mathrm{d}\mu\, h(\mu) \coth\Big[\overline{\omega}_a \tfrac{\lambda-\mu}{2}\Big] \tag{3.42}$$

is invertible on the co-dimension 1 subspace of $H_s(\mathbb{R})$

$$\mathfrak{X}_s(\mathbb{R}) \;=\; \Big\{ g \in H_s(\mathbb{R}) \,:\, \mathcal{J}\big[g\big] \;=\; 0 \Big\} \qquad \text{with} \qquad \mathcal{J}\big[g\big] \;= \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\, \chi_{11}(\mu) \mathrm{e}^{-\mathrm{i}\bar{b}_N\mu} \mathcal{F}\big[g\big](\tau_N\mu)\,. \tag{3.43}$$

Above and in the following we agree upon

$$x_N \;=\; b_N - a_N\,, \quad \overline{x}_N \;=\; \tau_N x_N\,, \quad \bar{b}_N \;=\; \tau_N b_N \quad \text{and} \quad \bar{a}_N \;=\; \tau_N a_N\,. \tag{3.44}$$

Moreover, without further notice we shall assume in this subsection that the lower bound holds $b_N - a_N > \eta$ and that N is large enough. Finally, the Fourier transform $\mathcal{F}$ is defined, for $g \in L^1(\mathbb{R})$, as

$$\mathcal{F}[g](\mu) \;= \int_{\mathbb{R}} \mathrm{d}\eta\, g(\eta) \mathrm{e}^{\mathrm{i}\mu\eta}\,. \tag{3.45}$$

The closed subspace $\mathfrak{X}_s(\mathbb{R})$ along with the inverse $\mathcal{W}_N$ are both described in terms of a piecewise holomorphic 2×2 matrix valued function χ that we shall discuss below. First, however, we provide the expression for the inverse $\mathcal{W}_N$. For any sufficiently regular $g \in \mathfrak{X}_s(\mathbb{R})$, the latter takes the form of an integral transform

$$\mathcal{W}_N\big[g\big](\xi) \;=\; \frac{\tau_N^2}{2\pi} \int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}}{\mu-\lambda} \Big\{ \chi_{11}(\lambda)\chi_{12}(\mu) \,-\, \frac{\mu}{\lambda}\chi_{11}(\mu)\chi_{12}(\lambda) \Big\} \mathrm{e}^{-\mathrm{i}\bar{b}_N\mu} \times \mathcal{F}\big[g\big](\tau_N\mu)\,. \tag{3.46}$$

in which $\epsilon' > 0$ can be taken as small as need be.

$\mathcal{W}_N$ and $\mathcal{J}$ are both expressed in terms of the unique solution χ to the 2×2 Riemann–Hilbert problem

- $\chi \in \mathcal{O}\big(\mathbb{C}\setminus\mathbb{R}\big)$ and admits continuous $\pm$ boundary values on $\mathbb{R}$;
- $\chi_+(\lambda) \;=\; G_\chi(\lambda)\,\chi_-(\lambda)$, with a jump matrix

$$G_\chi(\lambda) \;=\; \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} & 0 \\ R(\lambda) & -\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \end{pmatrix} \qquad \text{where} \qquad R(\lambda) \;=\; \frac{\sinh\Big[\frac{\lambda}{2}\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big)\Big]}{2\sinh\Big[\frac{\lambda}{2\omega_1}\Big]\sinh\Big[\frac{\lambda}{2\omega_2}\Big]} \tag{3.47}$$

- as $\lambda \to \infty$

$$\chi(\lambda) \;=\; \begin{cases} \begin{pmatrix} -\mathrm{sgn}\Big[\Re(\lambda)\Big]\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} & 1 \\ -1 & 0 \end{pmatrix} \cdot \big[-\mathrm{i}\lambda\big]^{-\frac{\sigma_3}{2}} \cdot \Big(I_2 + \frac{\chi_1}{\lambda} + \mathrm{O}\Big(\frac{1}{\lambda^2}\Big)\Big)\,, & \lambda \in \mathbb{H}^+ \\ \begin{pmatrix} -1 & \mathrm{sgn}\Big[\Re(\lambda)\Big]\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \\ 0 & 1 \end{pmatrix} \cdot \big[\mathrm{i}\lambda\big]^{-\frac{\sigma_3}{2}} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\sigma_3} \cdot \Big(I_2 + \frac{\chi_1}{\lambda} + \mathrm{O}\Big(\frac{1}{\lambda^2}\Big)\Big)\,, & \lambda \in \mathbb{H}^- \end{cases}. \tag{3.48}$$

It was established in [BGK16] that χ satisfies the variable reflection relation

$$\chi(-\lambda) \;=\; \begin{pmatrix} \chi_{11}(\lambda) & -\lambda\,\chi_{11}(\lambda)+\chi_{12}(\lambda) \\ -\chi_{21}(\lambda) & \lambda\,\chi_{21}(\lambda)-\chi_{22}(\lambda) \end{pmatrix} \tag{3.49}$$

the complex conjugation property

$$\Big(\chi(\lambda^*)\Big)^* \;=\; \begin{pmatrix} -\chi_{11}(-\lambda) & \chi_{12}(-\lambda) \\ \chi_{21}(-\lambda) & -\chi_{22}(-\lambda) \end{pmatrix} \tag{3.50}$$

and that it holds

$$\det\big[\chi(\lambda)\big] \;=\; \mathrm{sgn}\big[\Im\lambda\big]\,,\lambda\in\mathbb{C}\setminus\mathbb{R}\,. \tag{3.51}$$

Again, it follows from [BGK16] that χ admits the large-N asymptotic behaviour valid as soon as b_N-a_N is bounded away from zero uniformly in N. Below, we list the uniform large-N asymptotic expansions in the regions of $\mathbb{C}$ which are pertinent for our needs. These regions are delimited by the real axis and the curves $\Gamma_{\uparrow/\downarrow}$ as depicted in Fig. 3.1. First, however, we point out that R

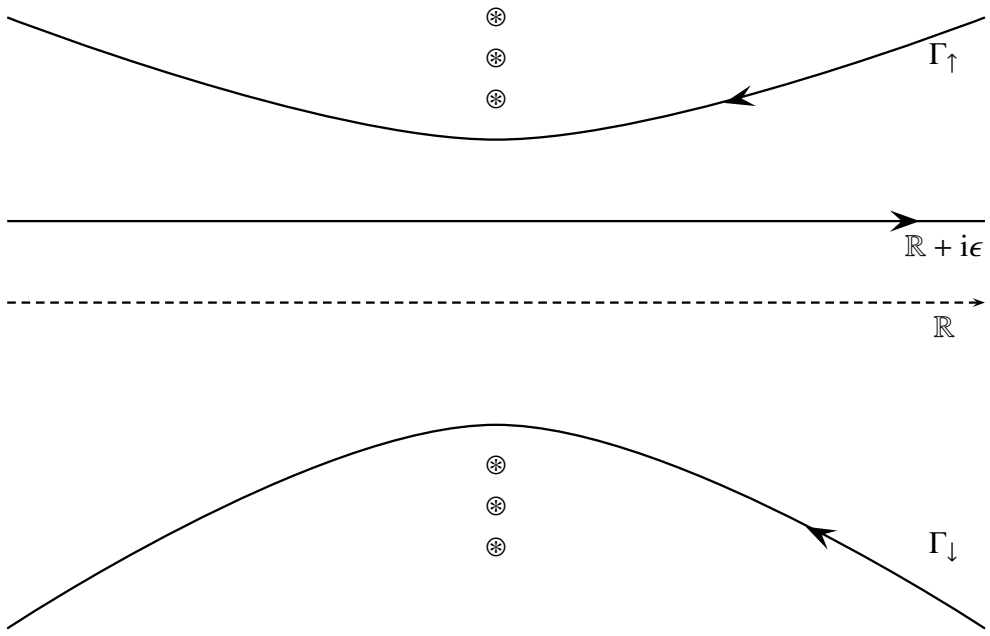

Figure 3.1: : Contour $\Gamma_{\uparrow/\downarrow}$ delimiting regions of uniform asymptotic expansion of χ.

admits a Wiener-Hopf-like factorisation

$$R(\lambda) \;=\; R_\uparrow(\lambda)\,R_\downarrow(\lambda) \tag{3.52}$$

where

$$R_\uparrow(\lambda)=\frac{\mathrm{i}}{\lambda}\cdot\sqrt{\omega_1+\omega_2}\cdot\left(\frac{\omega_2}{\omega_1+\omega_2}\right)^{\frac{\mathrm{i}\lambda}{2\pi\omega_1}}\cdot\left(\frac{\omega_1}{\omega_1+\omega_2}\right)^{\frac{\mathrm{i}\lambda}{2\pi\omega_2}}\cdot\frac{\prod_{p=1}^{2}\Gamma\Big(1-\dfrac{\mathrm{i}\lambda}{2\pi\omega_p}\Big)}{\Gamma\Big(1-\dfrac{\mathrm{i}\lambda(\omega_1+\omega_2)}{2\pi\omega_1\omega_2}\Big)} \tag{3.53}$$

and

$$R_{\downarrow}(\lambda) = \frac{\lambda}{2\pi\sqrt{\omega_1+\omega_2}} \cdot \left(\frac{\omega_2}{\omega_1+\omega_2}\right)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_1}} \cdot \left(\frac{\omega_1}{\omega_1+\omega_2}\right)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_2}} \cdot \frac{\prod_{p=1}^{2}\Gamma\left(\frac{\mathrm{i}\lambda}{2\pi\omega_p}\right)}{\Gamma\left(\frac{\mathrm{i}\lambda(\omega_1+\omega_2)}{2\pi\omega_1\omega_2}\right)} . \tag{3.54}$$

Note that

$$R_{\downarrow}(0) \;=\; -\mathrm{i}\sqrt{\omega_1+\omega_2} \qquad \text{and} \qquad \Big(\lambda R_{\uparrow}(\lambda)\Big)_{|\lambda=0} \;=\; \mathrm{i}\sqrt{\omega_1+\omega_2}\,. \tag{3.55}$$

Also, $R_{\uparrow}$ and $R_{\downarrow}$ satisfy to the relations

$$R_{\uparrow}(-\lambda) \;=\; \lambda^{-1}\cdot R_{\downarrow}(\lambda) \qquad \text{and} \qquad \Big(R_{\uparrow}(\lambda^*)\Big)^* \;=\; \lambda^{-1}\cdot R_{\downarrow}(\lambda)\,. \tag{3.56}$$

Furthermore, $R_{\uparrow/\downarrow}$ admit the asymptotic behaviour

$$R_{\uparrow}(\lambda) \;=\; \big(-\mathrm{i}\lambda\big)^{-\frac{1}{2}}\cdot\Big(1\;+\;\mathrm{O}\big(\lambda^{-1}\big)\Big) \qquad \text{for} \quad \lambda \underset{\lambda\in\mathbb{H}^+}{\longrightarrow} \infty \tag{3.57}$$

$$R_{\downarrow}(\lambda) \;=\; -\mathrm{i}\big(\mathrm{i}\lambda\big)^{\frac{1}{2}}\cdot\Big(1\;+\;\mathrm{O}\big(\lambda^{-1}\big)\Big) \qquad \text{for} \quad \lambda \underset{\lambda\in\mathbb{H}^-}{\longrightarrow} \infty\,. \tag{3.58}$$

The notation $\uparrow$ and $\downarrow$ indicates the direction, in respect to $\mathbb{R}+\mathrm{i}\epsilon$ in the complex plane where $R_{\uparrow/\downarrow}$ have no pole nor zeroes.

The mentioned uniform asymptotic expansions involves an auxiliary, piecewise analytic, matrix

$$\Pi(\lambda) \;=\; I_2 \;+\; \mathrm{O}\Big(\frac{\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}}{1+|\lambda|}\Big) \qquad \text{with} \qquad \zeta \;=\; \frac{2\pi\omega_1\omega_2}{\omega_1+\omega_2} \tag{3.59}$$

for any $\eta>0$ as small as need be and uniformly on $\mathbb{C}$. Moreover, the remainder is smooth in a_N and b_N with derivatives controlled as

$$\partial_{a_N}^{k}\partial_{b_N^{\ell}}\Pi(\lambda) \;=\; I_2\delta_{k;0}\delta_{\ell;0} \;+\; \mathrm{O}\Big(\tau_N^{k+\ell}\frac{\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}}{1+|\lambda|}\Big) \tag{3.60}$$

Finally, one has

$$\mathtt{P}_R(\lambda) \;=\; I_2 \;+\; \frac{\widetilde{\vartheta}_R}{\lambda}\Pi^{-1}(0)\sigma^{-}\Pi(0) \qquad \text{with} \qquad \widetilde{\vartheta}_R \;=\; \frac{1}{1\;-\;\big[\Pi'(0)\Pi^{-1}(0)\big]_{12}}\,. \tag{3.61}$$

The entries χ_{11},χ_{12} admit holomorphic continuations from $\mathbb{H}^{\mp}$ into some small tubular neighbourhood of $\mathbb{R}$ in $\mathbb{H}^{\pm}$. In particular, $\chi_{11;\pm}(\lambda),\chi_{12;\pm}(\lambda)$ are regular at $\lambda=0$. In their turn, χ_{11},χ_{12} admit meromorphic continuations from $\mathbb{H}^{\mp}$ into some small tubular neighbourhood of $\mathbb{R}$ in $\mathbb{H}^{\pm}$. They admit only one pole, which is simple at $\lambda=0$, and one has the behaviour

$$\chi_{21;-}(\lambda) \;=\; R_{\downarrow}(0)\widetilde{\vartheta}_R\Pi_{11}(0)\cdot\frac{1}{\lambda}+\mathrm{O}(1) \qquad \text{and} \qquad \chi_{22;-}(\lambda) \;=\; R_{\downarrow}(0)\widetilde{\vartheta}_R\Pi_{12}(0)\cdot\frac{1}{\lambda}+\mathrm{O}(1) \tag{3.62}$$

as $\lambda\to 0^+$.

• λ **between** $\mathbb{R}+\mathrm{i}\epsilon$ **and** $\Gamma_\uparrow$

In this region, it holds that

$$\chi(\lambda) \;=\; \chi_\infty(\lambda) \;+\; \delta\chi(\lambda) \tag{3.63}$$

where

$$\chi_\infty(\lambda) \;=\; \begin{pmatrix} 1/[\lambda R_\uparrow(\lambda)] \,-\, \mathrm{e}^{\mathrm{i}\lambda\overline{x}_{N;\alpha}}/R_\downarrow(\lambda) & 1/R_\uparrow(\lambda) \\ -R_\uparrow(\lambda) & 0 \end{pmatrix}, \tag{3.64}$$

while

$$\delta\chi(\lambda) \;=\; \begin{pmatrix} \Big\{\frac{1}{\lambda R_\uparrow(\lambda)} \,-\, \frac{\mathrm{e}^{\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{R_\downarrow(\lambda)}\Big\}\cdot\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{11} \,+\, \frac{[\delta(\Pi\mathtt{P}_R)(\lambda)]_{21}}{R_\uparrow(\lambda)} \\ -R_\uparrow(\lambda)\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{11} \end{pmatrix}$$
$$\begin{pmatrix} \Big\{\frac{1}{\lambda R_\uparrow(\lambda)} \,-\, \frac{\mathrm{e}^{\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{R_\downarrow(\lambda)}\Big\}\cdot\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{12} \,+\, \frac{[\delta(\Pi\mathtt{P}_R)(\lambda)]_{22}}{R_\uparrow(\lambda)} \\ -R_\uparrow(\lambda)\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{12} \end{pmatrix}. \tag{3.65}$$

The formulae for the remainder matrix involve

$$\delta(\Pi\mathtt{P}_R)(\lambda) \;=\; \begin{pmatrix} 1 & 0 \\ -1/\lambda & 1 \end{pmatrix}\cdot\left[\Pi\mathtt{P}_R(\lambda) \,-\, \begin{pmatrix} 1 & 0 \\ 1/\lambda & 1 \end{pmatrix}\right]. \tag{3.66}$$

Finally, one has the uniform estimate on the remainder

$$\delta\chi(\lambda) \;=\; \mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)\,. \tag{3.67}$$

• λ **between** $\Gamma_\downarrow$ **and** $\mathbb{R}$

In this region, it holds that

$$\chi(\lambda) \;=\; \chi_\infty(\lambda) \;+\; \delta\chi(\lambda) \tag{3.68}$$

where

$$\chi_\infty(\lambda) \;=\; \begin{pmatrix} \frac{-1}{R_\downarrow(\lambda)}\Big(1 \,-\, \frac{R_\downarrow(\lambda)}{\lambda R_\uparrow(\lambda)}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}\Big) & \frac{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{R_\uparrow(\lambda)} \\ \frac{R_\downarrow(\lambda)}{\lambda} & R_\downarrow(\lambda) \end{pmatrix}, \tag{3.69}$$

while

$$\delta\chi(\lambda) \;=\; \begin{pmatrix} \frac{-1}{R_\downarrow(\lambda)}\Big(1 \,-\, \frac{R_\downarrow(\lambda)}{\lambda R_\uparrow(\lambda)}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}\Big)\cdot\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{11} \,+\, \frac{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{R_\uparrow(\lambda)}\cdot[\delta(\Pi\mathtt{P}_R)(\lambda)]_{21} \\ \frac{R_\downarrow(\lambda)}{\lambda}\cdot\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{11} + R_\downarrow(\lambda)\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{21} \end{pmatrix}$$
$$\begin{pmatrix} \frac{-1}{R_\downarrow(\lambda)}\Big(1 \,-\, \frac{R_\downarrow(\lambda)}{\lambda R_\uparrow(\lambda)}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}\Big)\cdot\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{12} \,+\, \frac{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{R_\uparrow(\lambda)}\cdot[\delta(\Pi\mathtt{P}_R)(\lambda)]_{22} \\ \frac{R_\downarrow(\lambda)}{\lambda}\cdot\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{12} + R_\downarrow(\lambda)\big[\delta(\Pi\mathtt{P}_R)(\lambda)\big]_{22} \end{pmatrix}. \tag{3.70}$$

One has the uniform estimate on the remainder

$$\delta\chi(\lambda) \;=\; \mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)\,. \tag{3.71}$$

Moreover, one infers that

$$\begin{aligned} \chi_{21}(\lambda) \;&=\; \frac{R_\downarrow(0)\widetilde{\vartheta}_R\Pi_{11}(0)}{\lambda} \,+\, \mathrm{O}(1) \\ \chi_{22}(\lambda) \;&=\; \frac{R_\downarrow(0)\widetilde{\vartheta}_R\Pi_{12}(0)}{\lambda} \,+\, \mathrm{O}(1) \end{aligned} \qquad \text{as} \qquad \lambda\to 0 \qquad \text{with} \qquad \mathrm{Im}(\lambda)<0\,. \tag{3.72}$$

and that $\chi_{11}(\lambda)$ and $\chi_{12}(\lambda)$ admit bounded limits.

A remark is in order: while both the subspace constraint functional $\mathcal{J}$ and the inverse $\mathcal{W}_N$ involve the Fourier transform of g, their values *only* depend on the values of g on $[a_N\,;b_N]$. This follows from the jump conditions of χ.

Lemma 3.2.3 *Let $g_1, g_2 \in H_s(\mathbb{R})$, $0 < s < 1/2$ be such that $g_1 = g_2$ on $[a_N\,;b_N]$. Then,*

$$\mathcal{J}[g_1] \,=\, \mathcal{J}[g_2] \qquad \text{and} \qquad \mathcal{W}_N[g_1] \,=\, \mathcal{W}_N[g_2]\,. \tag{3.73}$$

Proof —

Since $g\mathbf{1}_{[a_N\,;b_N]} \in H_s(\mathbb{R})$, $0 < s < 1/2$ as soon as $g \in H_s(\mathbb{R})$, it is enough to show that $\mathcal{J}[g]$ and $\mathcal{W}_N[g]$ only depend on $g\mathbf{1}_{[a_N\,;b_N]}$. We discuss the proof only in the case of sufficiently regular g having fast decay at infinity although this can be done in full generality within the distributional setting of $H_s(\mathbb{R})$ functions.

One may decompose $\mathcal{J}$ in the form

$$\mathcal{J}[g] \,= \mathcal{J}_L[g] \,+\, \mathcal{J}_{\rm c}[g] \,+\, \mathcal{J}_R[g] \tag{3.74}$$

with

$$\mathcal{J}_L[g] \,=\!\int_{\mathbb{R}+{\rm i}\epsilon'}\!\frac{{\rm d}\mu}{2{\rm i}\pi}\,\chi_{11}(\mu)\int_{-\infty}^{a_N}{\rm d}\eta g(\eta){\rm e}^{{\rm i}\tau_N\mu(\eta-b_N)}\,,\quad \mathcal{J}_R[g] \,=\!\int_{\mathbb{R}+{\rm i}\epsilon'}\!\frac{{\rm d}\mu}{2{\rm i}\pi}\,\chi_{11}(\mu)\int_{b_N}^{+\infty}{\rm d}\eta g(\eta){\rm e}^{{\rm i}\tau_N\mu(\eta-b_N)} \tag{3.75}$$

and $\mathcal{J}_{\rm c}[g] \,=\, \mathcal{J}_{\rm c}\big[g\mathbf{1}_{[a_N\,;b_N]}\big]$.

We now show the vanishing of $\mathcal{J}_L[g]$ and $\mathcal{J}_R[g]$. Indeed, by using that χ_{11} is analytic in the upper half plane, that

$$\chi_{11}(\mu) \,\leq\, \frac{C}{|\mu|^{1/2}} \qquad \text{and} \qquad \int_{b_N}^{+\infty}{\rm d}\eta g(\eta){\rm e}^{{\rm i}\tau_N\mu(\eta-b_N)} \,=\, \frac{-g(b_N)}{{\rm i}\mu\tau_N} \,-\, \frac{1}{{\rm i}\mu\tau_N}\int_{b_N}^{+\infty}{\rm d}\eta g'(\eta){\rm e}^{{\rm i}\tau_N\mu(\eta-b_N)}\,, \tag{3.76}$$

one observes that one has the uniform bound on $\mathbb{H}^+$

$$\left|\chi_{11}(\mu)\int_{b_N}^{+\infty}{\rm d}\eta g(\eta){\rm e}^{{\rm i}\tau_N\mu(\eta-b_N)}\right| \,\leq\, \frac{C}{|\mu|^{3/2}} \tag{3.77}$$

and that the bounded functions are analytic in $\mathbb{H}^+$. This allows one to deform the integrations from $\mathbb{R}+{\rm i}\epsilon$ to $\mathbb{R}+{\rm i}M$ with $M>0$ as large as desired by virtue of Morera's theorem. Then, one has that

$$\begin{aligned}\big|\mathcal{J}_R\big[g\big]\big| = \left|\int_{\mathbb{R}+{\rm i}M}\!\frac{{\rm d}\mu}{2{\rm i}\pi}\,\chi_{11}(\mu)\int_{b_N}^{+\infty}{\rm d}\eta g(\eta){\rm e}^{{\rm i}\tau_N\mu(\eta-b_N)}\right| \,&\leq\, C\int_{\mathbb{R}}\frac{{\rm d}s}{\big[s^2+M^2\big]^{\frac{3}{2}}}\\ &=\, \frac{C}{\sqrt{M}}\int_{\mathbb{R}}\frac{{\rm d}s}{\big[s^2+1\big]^{\frac{3}{2}}} \underset{M\to+\infty}{\longrightarrow} 0\,.\end{aligned} \tag{3.78}$$

One carries out a similar reasoning regarding to $\mathcal{J}_L[g]$. The μ^{-1} decay rate of the Fourier transform at infinity and the existence of continuous + boundary values of χ_{11} on $\mathbb{R}$ ensure that

$$\begin{aligned}\mathcal{J}_L[g] \,=\!\int_{\mathbb{R}}\frac{{\rm d}\mu}{2{\rm i}\pi}\,\chi_{11;+}(\mu)\int_{-\infty}^{a_N}{\rm d}\eta g(\eta){\rm e}^{{\rm i}\tau_N\mu(\eta-b_N)} \,&=\!\int_{\mathbb{R}}\frac{{\rm d}\mu}{2{\rm i}\pi}\,\chi_{11;-}(\mu)\int_{-\infty}^{a_N}{\rm d}\eta g(\eta){\rm e}^{-{\rm i}\tau_N\mu(a_N-\eta)}\\ &=\int_{\mathbb{R}-{\rm i}M}\frac{{\rm d}\mu}{2{\rm i}\pi}\,\chi_{11}(\mu)\int_{-\infty}^{a_N}{\rm d}\eta g(\eta){\rm e}^{-{\rm i}\tau_N\mu(a_N-\eta)} \underset{M\to+\infty}{\longrightarrow} 0\,.\end{aligned} \tag{3.79}$$

This entails the claim relative to $\mathcal{J}[g]$. The results relative to $\mathcal{W}_N[g]$ are obtained in a very similar fashion.

3.2.3 A convenient representation for the inverse acting on $V'_{N;\alpha}$

In the general case, one may not expect to be able to simplify $\mathcal{W}_N[g]$ beyond its two-dimensional integral representation. However, since the potential of interest to us takes a very simple and specific form, such simplifications are possible in the case of $\mathcal{W}_N[V'_{N;\alpha}]$.

We first observe that the expression (3.63) for χ on the line $\mathbb{R}+\mathrm{i}\epsilon'$, $\epsilon' > \epsilon$ leads to the representation

$$\chi_{11}(\lambda) \,=\, \frac{Q_\uparrow(\lambda)}{\lambda R_\uparrow(\lambda)} \,-\, \mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\frac{Q_\downarrow(\lambda)}{R_\downarrow(\lambda)} \qquad \text{with} \qquad \left\{ \begin{array}{rcl} Q_\uparrow(\lambda) & = & 1+\delta\big(\Pi\mathtt{P}_R\big)_{11}(\lambda) \,+\, \lambda\delta\big(\Pi\mathtt{P}_R\big)_{21}(\lambda) \\ Q_\downarrow(\lambda) & = & 1+\delta\big(\Pi\mathtt{P}_R\big)_{11}(\lambda) \end{array} \right. \tag{3.80}$$

There, the functions $Q_{\uparrow/\downarrow}$ are holomorphic and bounded in the region enclosed by the curves $\Gamma_\uparrow$ and $\Gamma_\downarrow$. Similarly,

$$\chi_{12}(\lambda) \,=\, \frac{\widetilde{Q}_\uparrow(\lambda)}{\lambda R_\uparrow(\lambda)} \,-\, \mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\frac{\widetilde{Q}_\downarrow(\lambda)}{R_\downarrow(\lambda)} \qquad \text{with} \qquad \left\{ \begin{array}{rcl} \widetilde{Q}_\uparrow(\lambda) & = & \lambda+\delta\big(\Pi\mathtt{P}_R\big)_{12}(\lambda) \,+\, \lambda\delta\big(\Pi\mathtt{P}_R\big)_{22}(\lambda) \\ \widetilde{Q}_\downarrow(\lambda) & = & \delta\big(\Pi\mathtt{P}_R\big)_{12}(\lambda) \end{array} \right. \tag{3.81}$$

For further purpose, it is convenient to introduce the vectors

$$\boldsymbol{E}_L(\lambda) \,=\, \begin{pmatrix} \chi_{11}(\lambda) \\ -\dfrac{\chi_{12}(\lambda)}{\lambda} \end{pmatrix}, \quad \boldsymbol{E}_R(\lambda) \,=\, \begin{pmatrix} \chi_{12}(\lambda) \\ \lambda\,\chi_{11}(\lambda) \end{pmatrix} \tag{3.82}$$

so that

$$\Big(\boldsymbol{E}_L(\lambda),\boldsymbol{E}_R(\mu)\Big) \,=\, \chi_{11}(\lambda)\chi_{12}(\mu) \,-\, \frac{\mu}{\lambda}\chi_{11}(\mu)\chi_{12}(\lambda)\,. \tag{3.83}$$

The decomposition for χ_{11} and χ_{12} entail that $\boldsymbol{E}_R(\lambda) \,=\, \boldsymbol{E}_R^{(\uparrow)}(\lambda) \,-\, \mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\boldsymbol{E}_R^{(\downarrow)}(\lambda)$ with

$$\boldsymbol{E}_R^{(\uparrow)}(\lambda) \,=\, \frac{1}{\lambda R_\uparrow(\lambda)}\begin{pmatrix} \widetilde{Q}_\uparrow(\lambda) \\ \lambda Q_\uparrow(\lambda) \end{pmatrix}, \quad \boldsymbol{E}_R^{(\downarrow)}(\lambda) \,=\, \frac{1}{R_\downarrow(\lambda)}\begin{pmatrix} \widetilde{Q}_\downarrow(\lambda) \\ \lambda Q_\downarrow(\lambda) \end{pmatrix}, \tag{3.84}$$

and similarly $\boldsymbol{E}_L(\lambda) \,=\, \frac{1}{\lambda}\cdot\boldsymbol{E}_L^{(\uparrow)}(\lambda) \,-\, \frac{\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}}{\lambda}\cdot\boldsymbol{E}_L^{(\downarrow)}(\lambda)$

$$\boldsymbol{E}_L^{(\uparrow)}(\lambda) \,=\, \frac{1}{\lambda R_\uparrow(\lambda)}\begin{pmatrix} \lambda Q_\uparrow(\lambda) \\ -\widetilde{Q}_\uparrow(\lambda) \end{pmatrix}, \quad \boldsymbol{E}_L^{(\downarrow)}(\lambda) \,=\, \frac{1}{R_\downarrow(\lambda)}\begin{pmatrix} \lambda Q_\downarrow(\lambda) \\ -\widetilde{Q}_\downarrow(\lambda) \end{pmatrix}. \tag{3.85}$$

Finally, we also set

$$\mathscr{U}_{12}(\lambda) \;=\; \frac{Nu_N\lambda+\mathrm{i}v_N}{1+\lambda^2}\mathrm{i}\chi_{11}(\mathrm{i}) \tag{3.86}$$

$$\mathscr{U}_{11}(\lambda) \;=\; \frac{-\chi_{12}(\mathrm{i})}{1+\lambda^2}\cdot\Big(\mathrm{i}Nu_N+\lambda v_N\Big) \,-\, \frac{\mathrm{i}\chi_{11}(\mathrm{i})}{\mathrm{i}+\lambda}\frac{Nu_N-v_N}{2}\,, \tag{3.87}$$

in which we have introduced the shorthand notations

$$Nu_N \,=\, \mathrm{e}^{\bar{b}_N} \,+\, \mathrm{e}^{-\bar{a}_N} \qquad \text{and} \qquad v_N \,=\, \mathrm{e}^{\bar{b}_N} \,-\, \mathrm{e}^{-\bar{a}_N}\,, \tag{3.88}$$

Proposition 3.2.4 *One has the decomposition* $\mathcal{W}_N[V'_{N;\alpha}] \,=\, \sum_{a=1}^{3}\varpi_N^{(a)}$ *where*

$$\varpi_N^{(1)}(\xi) \,=\, \frac{\mathfrak{r}\tau_N}{4\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\mathrm{e}^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}\Big\{\frac{\chi_{12}(\lambda)}{\lambda}\mathscr{U}_{12}(\lambda) \,+\, \chi_{11}(\lambda)\mathscr{U}_{11}(\lambda)\Big\}\,, \tag{3.89}$$

and

$$\varpi_N^{(2)}(\xi) \,=\, \frac{\alpha\tau_N}{2\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\mathrm{e}^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}\frac{\chi_{11}(\lambda)\chi_{12;-}(0)}{\lambda}\,. \tag{3.90}$$

Finally, it holds that $\varpi_N^{(3)}(\xi) = \varpi_{N;\uparrow}^{(3)}(\xi) + \varpi_{N;\downarrow}^{(3)}(\xi) + \varpi_{N;0}^{(3)}(\xi)$*. The building blocks of this decomposition take the form, for* $\upsilon \in \{\uparrow,\downarrow\}$*,*

$$\varpi_{N;\upsilon}^{(3)}(\xi) = \frac{\epsilon_\upsilon\tau_N}{4\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\mathrm{e}^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}\Big(\boldsymbol{E}_L(\lambda),\boldsymbol{\mathcal{E}}^{(\upsilon)}(\lambda)\Big)\qquad \textit{with}\qquad \epsilon_\upsilon = \left\{\begin{array}{ll} 1 & \upsilon=\uparrow \\ -1 & \upsilon=\downarrow\end{array}\right. \tag{3.91}$$

Here, we have set

$$\boldsymbol{\mathcal{E}}^{(\uparrow)}(\lambda) \;=\; \mathrm{e}^{-\bar{b}_N}\frac{2\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi(\lambda+\mathrm{i})}\boldsymbol{E}_R^{(\uparrow)}(-\mathrm{i})\mathbf{1}_{1<\zeta} + \int_{\mathbb{R}-\mathrm{i}\varkappa_\eta}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\mu\mathcal{F}[\mathbf{g}](\mu)\,\mathrm{e}^{-\mathrm{i}\bar{b}_N\mu}}{(\mu-\lambda)\cosh\left[\frac{\pi\mu}{2}\right]}\boldsymbol{E}_R^{(\uparrow)}(\mu) \tag{3.92}$$

$$\boldsymbol{\mathcal{E}}^{(\downarrow)}(\lambda) \;=\; \mathrm{e}^{\bar{a}_N}\frac{2\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi(\mathrm{i}-\lambda)}\boldsymbol{E}_R^{(\downarrow)}(\mathrm{i})\mathbf{1}_{1<\zeta} + \int_{\mathbb{R}+\mathrm{i}\varkappa_\eta}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\mu\mathcal{F}[\mathbf{g}](\mu)\,\mathrm{e}^{-\mathrm{i}\bar{a}_N\mu}}{(\mu-\lambda)\cosh\left[\frac{\pi\mu}{2}\right]}\boldsymbol{E}_R^{(\downarrow)}(\mu) \tag{3.93}$$

while

$$\varpi_{N;0}^{(3)}(\xi) \,=\, \frac{-\tau_N}{4\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\mathrm{e}^{-\mathrm{i}\tau_N\lambda\xi}\Big(\boldsymbol{E}_L(\lambda),\boldsymbol{E}_R^{(\downarrow)}(\lambda)\Big)\frac{\lambda\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\left[\frac{\pi\lambda}{2}\right]}\,. \tag{3.94}$$

Above, we have introduced

$$\varkappa_\eta \,=\, (1-\eta)\min\big\{2,\zeta\big\}\qquad \textit{with}\qquad \zeta \,=\, 2\pi\frac{\omega_1\omega_2}{\omega_1+\omega_2}\,. \tag{3.95}$$

Proof —

Owing to Lemma 3.2.3, one may choose $V'_{N;\alpha}$ to take any values outside of $[a_N\,;b_N]$ so as to compute the Fourier transform occurring in the expression for $\mathcal{W}_N$, provided the function belongs to $H_s(\mathbb{R})$ with $0<s<1/2$. Thus, we choose to extend $V'_{N;\alpha}$ from $[a_N\,;b_N]$ to $\mathbb{R}$ as $V_{N;\alpha}(\lambda) = \mathfrak{v}_{N;\alpha}(\lambda)\mathbf{1}_{[a_N\,;b_N]}(\lambda) + \mathfrak{w}_N(\lambda)$, where

$$\mathfrak{v}_{N;\alpha}(\lambda) \,=\, \frac{\mathfrak{r}}{N\tau_N}\cosh\big[\tau_N\lambda\big] \,-\, \frac{\alpha\lambda}{N}\qquad\text{and}\qquad \mathfrak{w}_N(\lambda) \,=\, \int_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi N\tau_N}\cdot\frac{\mathbf{g}(\tau_N\mu)}{\cosh\big[\tau_N(\lambda-\mu)\big]}\,. \tag{3.96}$$

A direct calculation yields

$$\int_{a_N}^{b_N}\mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}\mu\tau_N(\eta-b_N)}\mathfrak{v}'_{N;\alpha}(\eta) \,=\, \frac{\mathfrak{r}}{2\tau_N N\mathrm{i}}\sum_{\sigma=\pm}\sigma\frac{\mathrm{e}^{\sigma\bar{b}_N} \,-\, \mathrm{e}^{-\mathrm{i}\mu\bar{x}_N}\mathrm{e}^{\sigma\bar{a}_N}}{\mu-\mathrm{i}\sigma} \,-\, \alpha\frac{1\,-\,\mathrm{e}^{-\mathrm{i}\mu\bar{x}_N}}{\mathrm{i}\mu N\tau_N}\,. \tag{3.97}$$

Further, one has that

$$\begin{aligned}\int_{\mathbb{R}}\mathrm{d}\eta\mathrm{e}^{\mathrm{i}\mu\tau_N\eta}\mathfrak{w}'_N(\eta) \;&=\; -\mathrm{i}\mu\tau_N\int_{\mathbb{R}}\mathrm{d}\eta\mathfrak{w}_N(\eta)\mathrm{e}^{\mathrm{i}\mu\tau_N\eta} \\ &=\; -\frac{\mathrm{i}\mu\tau_N}{2\pi N}\int_{\mathbb{R}}\mathrm{d}\eta\frac{\mathrm{e}^{\mathrm{i}\mu\tau_N\eta}}{\cosh(\tau_N\eta)}\cdot\int_{\mathbb{R}}\mathrm{d}\eta\mathrm{e}^{\mathrm{i}\mu\tau_N\eta}\mathbf{g}(\tau_N\eta) \;=\; -\frac{\mathrm{i}\mu\,\mathcal{F}[\mathbf{g}](\mu)}{2\tau_N N\cosh\left[\frac{\pi\mu}{2}\right]}\,.\end{aligned} \tag{3.98}$$

As a result, one obtains a decomposition $\mathcal{W}_N[V'_{N;\alpha}] \,=\, \sum_{a=1}^{3}\varpi_{N;\alpha}^{(a)}$ where

$$\begin{aligned}\varpi_N^{(1)}(\xi) = \frac{\tau_N\mathfrak{r}}{4\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'} -2mm\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\mathrm{e}^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}}{\mu-\lambda}\Big\{\chi_{11}(\lambda)\chi_{12}(\mu) \,-\, \frac{\mu}{\lambda}\chi_{11}(\mu)\chi_{12}(\lambda)\Big\} \\ \times\sum_{\sigma=\pm}\sigma\frac{\mathrm{e}^{\sigma\bar{b}_N} \,-\, \mathrm{e}^{-\mathrm{i}\mu\bar{x}_N}\mathrm{e}^{\sigma\bar{a}_N}}{\mu-\mathrm{i}\sigma}\,,\end{aligned}$$

$$\varpi_N^{(2)}(\xi) = -\frac{\tau_N \alpha}{2\mathrm{i}\pi N} \int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\tau_N \lambda(\xi - a_N)}}{\mu - \lambda} \Big\{ \chi_{11}(\lambda)\chi_{12}(\mu) \, - \, \frac{\mu}{\lambda} \chi_{11}(\mu)\chi_{12}(\lambda) \Big\} \times \frac{1 \, - \, \mathrm{e}^{-\mathrm{i}\mu \overline{x}_N}}{\mu} \, ,$$

and

$$\varpi_N^{(3)}(\xi) \; = \; -\frac{\mathrm{i}\tau_N}{4\pi N} \int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\tau_N \lambda(\xi - a_N)}}{\mu - \lambda} \Big\{ \chi_{11}(\lambda)\chi_{12}(\mu) \, - \, \frac{\mu}{\lambda} \chi_{11}(\mu)\chi_{12}(\lambda) \Big\} \times \mathrm{e}^{-\mathrm{i}\mu \bar{b}_N} \frac{\mu \, \mathcal{F}[\mathbf{g}](\mu)}{\cosh\Big[\dfrac{\pi\mu}{2}\Big]} \, .$$

We first compute the μ-integral arising in $\varpi_N^{(1)}$ and $\varpi_N^{(2)}$. For such a purpose, one starts by observing that χ_{1a} admits an analytic continuation from $\mathbb{H}^-$ to $\mathbb{H}^+$. Denoting this analytic continuation as $\chi_{1a;-}$, one has the relation

$$\chi_{1a;-}(\lambda) \; = \; \chi_{1a}(\lambda) \mathrm{e}^{-\mathrm{i}\lambda \overline{x}_N} \quad \text{with} \quad \Im[\lambda] > 0 \, . \tag{3.99}$$

Then, one may express $\varpi_N^{(1)}$ in the form

$$\begin{aligned} \varpi_N^{(1)}(\xi) \; = \; & \frac{\tau_N \mathfrak{r}}{4\mathrm{i}\pi N} \int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\tau_N \lambda(\xi - a_N)}}{\mu - \lambda} \Big\{ \chi_{11}(\lambda)\chi_{12}(\mu) - \frac{\mu}{\lambda} \chi_{11}(\mu)\chi_{12}(\lambda) \Big\} \sum_{\sigma = \pm} \frac{\sigma \mathrm{e}^{\sigma \bar{b}_N}}{\mu - \mathrm{i}\sigma} \\ & - \frac{\tau_N \mathfrak{r}}{4\mathrm{i}\pi N} \int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\tau_N \lambda(\xi - a_N)}}{\mu - \lambda} \Big\{ \chi_{11}(\lambda)\chi_{12;-}(\mu) \, - \, \frac{\mu}{\lambda} \chi_{11;-}(\mu)\chi_{12}(\lambda) \Big\} \sum_{\sigma = \pm} \frac{\sigma \mathrm{e}^{\sigma \bar{a}_N}}{\mu - \mathrm{i}\sigma} \end{aligned} \tag{3.100}$$

We could split the integral in two pieces since each integrand behaves at infinity as $\mathrm{O}\big(|\mu|^{-3/2}\big)$. Then, because of these bounds, one may take the first μ-integral by means of the residues of the poles located above the line $\mathbb{R} + \mathrm{i}\epsilon'$ and take the second μ-integral by means of the residues of the poles located below the line $\mathbb{R} + \mathrm{i}\epsilon'$. Note that there is no pole at $\mu - \lambda$ in the first integral so that only the pole at $\mu = \mathrm{i}$ contributes, while, in the second case, only the pole at $\mu = -\mathrm{i}$ does. This yields

$$\varpi_N^{(1)}(\xi) \; = \; \frac{\mathfrak{r}\tau_N}{4\mathrm{i}\pi N} \int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \mathrm{e}^{-\mathrm{i}\tau_N \lambda(\xi - a_N)} \sum_{\sigma = \pm} \mathrm{e}^{\bar{b}_N^{(\sigma)}} \frac{\chi_{11}(\lambda)\chi_{12}(\sigma\mathrm{i}) \, - \, \sigma\mathrm{i}\chi_{11}(\sigma\mathrm{i})\chi_{12}(\lambda)/\lambda}{\mathrm{i} - \sigma\lambda} \tag{3.101}$$

in which we have used the shorthand notations

$$\bar{b}_N^{(+)} \; = \; \bar{b}_N \qquad \text{and} \qquad \bar{b}_N^{(-)} \; = \; -\overline{a}_N \, . \tag{3.102}$$

Then, it is a matter of direct calculation to observe that owing to the inversion relation (3.49) one gets

$$\sum_{\sigma = \pm} \mathrm{e}^{\bar{b}_{N;\alpha}^{(\sigma)}} \frac{\chi_{11}(\lambda)\chi_{12}(\sigma\mathrm{i}) \, - \, \sigma\mathrm{i}\chi_{11}(\sigma\mathrm{i})\chi_{12}(\lambda)/\lambda}{\mathrm{i} - \sigma\lambda} \; = \; \frac{\chi_{12}(\lambda)}{\lambda} \mathscr{U}_{12}(\lambda) \, + \, \chi_{11}(\lambda) \mathscr{U}_{11}(\lambda) \tag{3.103}$$

with $\mathscr{U}_{1a}$ as given in (3.86)-(3.87). This was the last step before reaching (3.89).

The same reasoning yield (3.90): the part of the integral deformed up to $+\mathrm{i}\infty$ produces 0 while the part deformed to $-\mathrm{i}\infty$ picks a simple pole at $\mu = 0$.

We now turn on to rewriting $\varpi_N^{(3)}$, which can be recast as

$$\varpi_N^{(3)}(\xi) = \frac{\tau_N}{4\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\mathrm{e}^{-\mathrm{i}\tau_N\lambda(\xi-a_N)}}{\mu-\lambda}\Big\{\mathrm{e}^{-\mathrm{i}\mu\bar{b}_N}\Big(\boldsymbol{E}_L(\lambda),\boldsymbol{E}_R^{(\uparrow)}(\mu)\Big) - \mathrm{e}^{-\mathrm{i}\mu\bar{a}_N}\Big(\boldsymbol{E}_L(\lambda),\boldsymbol{E}_R^{(\downarrow)}(\mu)\Big)\Big\}\frac{\mu\,\mathcal{F}[\mathbf{g}](\mu)}{\cosh\Big[\frac{\pi\mu}{2}\Big]}\,. \quad (3.104)$$

One then splits the integral in two pieces, one containing $\boldsymbol{E}_R^{(\uparrow)}$ and the other one $\boldsymbol{E}_R^{(\downarrow)}$. Both integrals are well defined due to the $\mathrm{O}(\mu^{-k})$ behaviour at infinity of $\mathcal{F}[\mathbf{g}](\mu)$ ensured by the $\mathrm{O}\big(\mathrm{e}^{-\frac{\mathfrak{r}}{2}\cosh(\xi)}\big)$ control on $\mathbf{g}^{(k)}(\xi)$ for any $k\geq 0$ and the $\mathrm{O}\big(|\lambda|^{-3/2}\big)$ behaviour pointwise in μ. Then, in the integral involving $\boldsymbol{E}_R^{(\uparrow)}$, one moves the μ integration from $\mathbb{R}+\mathrm{i}\epsilon'$ to $\mathbb{R}-\mathrm{i}\varkappa_\eta$. There are four potential sources of poles in the integrand

$$\frac{1}{R_\uparrow(\lambda)}\ \text{poles at}\ \lambda=-\mathrm{i}\zeta n\,,\quad \frac{1}{R_\downarrow(\lambda)}\ \text{poles at}\ \lambda=\mathrm{i}\zeta n\,,\quad \frac{1}{\cosh\big[\frac{\pi\lambda}{2}\big]}\ \text{poles at}\ \lambda=\pm\mathrm{i}(1+2n) \quad (3.105)$$

with $n\in\mathbb{N}^*$, and $\lambda=\mu$. Thus, in deforming the integration contour, provided that $1<\zeta$, one picks up a pole at $\mu=-\mathrm{i}$. This then yields $\varpi_{N;\uparrow}^{(3)}(\xi)$

Similarly, in the second integral involving $\boldsymbol{E}_R^{(\downarrow)}$, one moves the μ-integration contour from $\mathbb{R}+\mathrm{i}\epsilon'$ to $\mathbb{R}+\mathrm{i}\varkappa_\eta$. This produces one contribution stemming from the pole at $\mu=\lambda$ and one contribution stemming from the pole at $\mu=\mathrm{i}$. The last contribution is only present if $1<\zeta$. The terms obtained in this way correspond to $\varpi_{N,\uparrow}^{(3)}(\xi)$ for the μ-integrals over $\mathbb{R}-\mathrm{i}\varkappa_\eta$ and $\varpi_{N,0}^{(3)}(\xi)$ for the residue at $\mu=\lambda$ part.

3.2.4 Support of the equilibrium measure

When constructing the equilibrium measure, on top of determining its density, one also needs to fix its support. Since the density belongs to $H_s(\mathbb{R})$, see [BGK16], with $0<s<1/2$, and satisfies the singular integral equation (3.40) throughout its support $[a_{N;\alpha}\,;b_{N;\alpha}]$ which satisfies owing to Lemma 3.2.2 the lower-bound $b_{N;\alpha}-a_{N;\alpha}\geq 2\varsigma>0$, one gets that, for any N large enough, one has the representation

$$\widehat{\varrho}_{\mathrm{eq};\alpha} \;=\; \mathcal{W}_N\big[V'_{N;\alpha}\big]_{|a_N,b_N\hookrightarrow a_{N;\alpha},b_{N;\alpha}}\,. \quad (3.106)$$

We stress that the inverse operator $\mathcal{W}_N$ given in (3.46) is now subordinate to the yet unknown pair of points $a_{N;\alpha}$, $b_{N;\alpha}$ delimiting the support. One then gets two additional constraints, one translating the fact that $V'_{N;\alpha}\in\mathcal{S}_N\big[H_s([a_{N;\alpha}\,;b_{N;\alpha}])\big]$ with $0<s<1/2$, and the second one expressing the unit mass property of the measure

$$0 \;=\; \mathcal{J}\big[V'_{N;\alpha}\big]_{|a_N,b_N\hookrightarrow a_{N;\alpha},b_{N;\alpha}} \qquad\text{and}\qquad 1 \;=\; \int_{a_{N;\alpha}}^{b_{N;\alpha}}\Big\{\mathcal{W}_N\big[V'_{N;\alpha}\big](\xi)\Big\}_{|a_N,b_N\hookrightarrow a_{N;\alpha},b_{N;\alpha}}\,. \quad (3.107)$$

In this Sub-Section, we shall establish that the constraints (3.107) admit a unique solution, for N large enough, provided that $a_{N;\alpha}\leq-\varsigma$ and $b_{N;\alpha}\geq\varsigma$, a property that is ensured by Lemma 3.2.2. This thus ensures that this solution does provide one with the support of the equilibrium measure. We close the subsection by establishing the explicit form of the first few terms in the large-N expansion of $a_{N;\alpha}$ and $b_{N;\alpha}$. This ends the proof of Theorem 3.1.1.

Proposition 3.2.5 *For given endpoints a_N,b_N satisfying $x_N>\eta$ for some $\eta>0$, the constraint functional $\mathcal{I}_1\big[V'_{N;\alpha}\big]$ defined in* (3.43) *admits the large-N asymptotic expansion*

$$\mathcal{J}\big[V'_{N;\alpha}\big] \;=\; \frac{\mathfrak{v}\chi_{11}(i)}{2\mathrm{i}N\tau_N}\big(\mathrm{e}^{\bar{b}_N}-\mathrm{e}^{-\bar{a}_N}\big)-\frac{\alpha\chi_{11;-}(0)}{\mathrm{i}N\tau_N}-\frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi N\tau_N R_\uparrow(-\mathrm{i})}\Big\{\mathrm{e}^{-\bar{b}_N}-\mathrm{e}^{\bar{a}_N}\Big\}\mathbf{1}_{1<\zeta}+\mathfrak{R}_{\mathcal{J}}(a_N,b_N) \quad (3.108)$$

with a remainder that is smooth in a_N, b_N and controlled as

$$\partial_{a_N}^k \partial_{b_N}^\ell \mathfrak{R}_{\mathcal{J}}(a_N, b_N) \;=\; \mathrm{O}\Big(\tau_N^{k+\ell} \frac{\mathrm{e}^{-\bar{b}_N \varkappa_\eta} + \mathrm{e}^{\bar{a}_N \varkappa_\eta}}{N\tau_N} \Big) \,. \tag{3.109}$$

Finally, $\varkappa_\eta$ is as introduced in (3.95) .

Proof —

Now one consecutively computes each of the contributions to $\mathcal{J}$. The one of $\mathfrak{v}_{N;\alpha}$ can be obtained in closed form. Indeed, one has

$$\begin{aligned} \mathcal{J}\big[\mathfrak{v}'_{N;\alpha}\big] \;=& \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi N\tau_N} \chi_{11}(\mu) \mathrm{e}^{-\mathrm{i}\mu\bar{b}_N} \Big\{ \sum_{\sigma=\pm} \frac{\mathfrak{r}\sigma \mathrm{e}^{\sigma\bar{b}_N}}{2\mathrm{i}(\mu - \mathrm{i}\sigma)} \;-\; \frac{\alpha}{\mathrm{i}\mu} \Big\} \\ & - \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi\tau_N N} \chi_{11}(\mu) \mathrm{e}^{-\mathrm{i}\mu\overline{x}_N} \mathrm{e}^{-\mathrm{i}\mu\bar{b}_N} \Big\{ \sum_{\sigma=\pm} \frac{\mathfrak{r}\sigma \mathrm{e}^{\sigma\bar{a}_N}}{2\mathrm{i}(\mu - \mathrm{i}\sigma)} \;-\; \frac{\alpha}{\mathrm{i}\mu} \Big\} \,. \end{aligned} \tag{3.110}$$

Note that, each integrand is a $\mathrm{O}\big(\mu^{-3/2}\big)$ at ∞. The first integral can be computed by taking the residues of the poles located above of $\mathbb{R} + \mathrm{i}\epsilon'$. There is a simple pole at $\mu = \mathrm{i}$. To compute the second integral, one observes that χ_{11} admits an analytic continuation from $\mathbb{H}^-$ to $\mathbb{H}^+$. Denoting this analytic continuation as $\chi_{11;-}$, it holds $\chi_{11;-}(\lambda) \;=\; \chi_{11}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}$ with $\Im[\lambda] > 0$. Thus, in the second integral, one replaces χ_{11} with $\chi_{11;-}$ and then takes the integral in terms of the residues at the poles located below of $\mathbb{R} + \mathrm{i}\epsilon'$. This yields

$$\begin{aligned} \mathcal{J}\big[\mathfrak{v}'_{N;\alpha}\big] \;&=\; \frac{\mathfrak{r}}{2\mathrm{i}\tau_N N} \Big(\chi_{11}(\mathrm{i}) \mathrm{e}^{-\bar{b}_N} \;-\; \chi_{11}(-\mathrm{i}) \mathrm{e}^{-\bar{a}_N} \Big) \;-\; \frac{\alpha \chi_{11;-}(0)}{\mathrm{i}\tau_N N} & (3.111) \\ &=\; \frac{\mathfrak{r}\chi_{11}(\mathrm{i})}{2\mathrm{i}\tau_N N} \Big(\mathrm{e}^{-\bar{b}_N} \;-\; \mathrm{e}^{-\bar{a}_N} \Big) \;-\; \frac{\alpha \chi_{11;-}(0)}{\mathrm{i}\tau_N N} \,. & (3.112) \end{aligned}$$

Here, we have simplified the expression owing to (3.50).

Further, observe that integrations by parts and the $\mathrm{O}\big(\mathrm{e}^{-\mathfrak{r}\cosh(\xi)}\big)$ decay of $\mathbf{g}^{(k)}(\xi)$ for any $k \geq 0$ ensure that $\mathcal{F}[\mathbf{g}](\mu) = \mathrm{O}\big(\mu^{-k}\big)$. Then, recalling the representation (3.80) and inserting into the expression for the constraint functional leads to

$$\begin{aligned} \mathcal{J}\big[\mathfrak{w}'_N\big] \;&=\; \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{4\mathrm{i}^2\pi N\tau_N} \frac{Q_\uparrow(\mu)\mathcal{F}[\mathbf{g}](\mu)}{R_\uparrow(\mu)\cosh\big[\frac{\pi\mu}{2}\big]} \mathrm{e}^{-\mathrm{i}\mu\bar{b}_N} \;-\; \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{4\mathrm{i}^2\pi N\tau_N} \frac{\mu Q_\downarrow(\mu)\mathcal{F}[\mathbf{g}](\mu)}{R_\downarrow(\mu)\cosh\big[\frac{\pi\mu}{2}\big]} \mathrm{e}^{-\mathrm{i}\mu\bar{a}_N} \\ &=\; \int_{\mathbb{R}-\mathrm{i}\varkappa_\eta} \frac{\mathrm{d}\mu}{4\mathrm{i}^2\pi N\tau_N} \frac{Q_\uparrow(\mu)\mathcal{F}[\mathbf{g}](\mu)}{R_\uparrow(\mu)\cosh\big[\frac{\pi\mu}{2}\big]} \mathrm{e}^{-\mathrm{i}\mu\bar{b}_N} \;-\; \int_{\mathbb{R}+\mathrm{i}\varkappa_\eta} \frac{\mathrm{d}\mu}{4\mathrm{i}^2\pi N\tau_N} \frac{\mu Q_\downarrow(\mu)\mathcal{F}[\mathbf{g}](\mu)}{R_\downarrow(\mu)\cosh\big[\frac{\pi\mu}{2}\big]} \mathrm{e}^{-\mathrm{i}\mu\bar{a}_N} \\ &\qquad - \frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi N\tau_N R_\uparrow(-\mathrm{i})} \Big\{ Q_\uparrow(-\mathrm{i}) \mathrm{e}^{-\bar{b}_N} \;-\; Q_\downarrow(\mathrm{i}) \mathrm{e}^{\bar{a}_N} \Big\} \mathbf{1}_{1<\zeta} \,. \end{aligned} \tag{3.113}$$

In the second line we have deformed the integration contour to $\mathbb{R} - \mathrm{i}\varkappa_\eta$ for the first integral and to $\mathbb{R} + \mathrm{i}\varkappa_\eta$. Note that, in the process, one only picks poles of $1/\cosh\big[\frac{\pi\mu}{2}\big]$ at $\pm\mathrm{i}$, and this provided that $1 < \zeta$. This generates the contribution of the last line.

Owing to the $\mathrm{O}(\mu^{-k})$ for any $k \geq 0$ decay of the integrand at infinity, one readily then estimates the first integral to be $\mathrm{O}\Big(\mathrm{e}^{-\varkappa_\eta \bar{b}_N}/N\tau_N\Big)$ and the second one to be $\mathrm{O}\Big(\mathrm{e}^{\varkappa_\eta \bar{a}_N}/N\tau_N\Big)$. Finally, one has $Q_{\uparrow/\downarrow}(\mu) \;=\; 1 \;+\; \mathrm{O}\big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\big)$ uniformly in $\mu \in \mathbb{R} \pm \mathrm{i}\varkappa_\eta$. Thus, all-in-all,

$$\mathcal{J}\big[\mathfrak{w}'_N\big] \;=\; -\frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi N\tau_N R_\uparrow(-\mathrm{i})} \Big\{ \mathrm{e}^{-\bar{b}_N} - \mathrm{e}^{\bar{a}_N} \Big\} \mathbf{1}_{1<\zeta} + \frac{1}{N\tau_N} \mathrm{O}\Big(\big[\mathrm{e}^{-\bar{b}_N} + \mathrm{e}^{\bar{a}_N} \big] \mathrm{e}^{-\zeta(1-\eta)\overline{x}_N} + \mathrm{e}^{-\varkappa_\eta \bar{b}_N} + \mathrm{e}^{\varkappa_\eta \bar{a}_N} \Big) \,. \tag{3.114}$$

It is clear from the previous handlings and the properties of $Q_{\uparrow/\downarrow}$ that the remainder is smooth in a_N, b_N and that each a_N or b_N derivative of the remainder worsens the control by a factor of τ_N.

The result then follows upon putting together all the estimates and exact expressions.

Proposition 3.2.6 *It holds*

$$\begin{aligned}\int_{a_N}^{b_N} \mathrm{d}\xi\, \mathcal{W}_N\big[V'_{N;\alpha}\big](\xi) \;=\; & \frac{-\alpha}{2\pi N}\Big[\chi'_{11;-}(0)\chi_{12;-}(0) \,-\, \chi'_{12;-}(0)\chi_{11;-}(0)\Big] \\ & -\, \frac{\mathfrak{r}}{4\mathrm{i}\pi N}\Big\{\big(\mathrm{e}^{\bar{b}_N} \,+\, \mathrm{e}^{-\bar{a}_N}\big)\big[\chi_{11;-}(0)\chi_{12}(\mathrm{i}) \,-\, \chi_{12;-}(0)\chi_{11}(\mathrm{i}) \,-\, \frac{\mathrm{i}}{2}\chi_{11;-}(0)\chi_{11}(\mathrm{i})\big] \\ & + \mathrm{i}\chi_{11;-}(0)\chi_{11}(\mathrm{i})\frac{\mathrm{e}^{\bar{b}_N} \,-\, \mathrm{e}^{-\bar{a}_N}}{2}\Big\} \,+\, \frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N R_\downarrow(\mathrm{i})R_\downarrow(0)}\big(\mathrm{e}^{-\bar{b}_N} + \mathrm{e}^{\bar{a}_N}\big)\mathbf{1}_{1<\zeta} \,+\, \mathrm{O}\Big(\frac{\mathrm{e}^{\varkappa_\eta \bar{a}_N} \,+\, \mathrm{e}^{-\varkappa_\eta \bar{b}_N}}{N}\Big)\,.\end{aligned} \tag{3.115}$$

Proof —

One starts from the partially integrated expression for $\mathcal{W}_N\big[V'_{N;\alpha}\big]$ obtained in Proposition 3.2.4. Then, with the notation of that proposition, one has

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \mathcal{W}_N\big[V'_{N;\alpha}\big](\xi) \;=\; \sum_{a=1}^{3}\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(a)}(\xi)\,. \tag{3.116}$$

Since,

$$\int_{a_N}^{b_N} \mathrm{d}\xi \mathrm{e}^{-\mathrm{i}\lambda\tau_N(\xi - a_N)} \;=\; \frac{1 \,-\, \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}}{\mathrm{i}\lambda\tau_N}\,, \tag{3.117}$$

one gets that

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(2)}(\xi) \;=\; -\frac{\alpha\chi_{12;-}(0)}{2\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\frac{\chi_{11}(\lambda) \,-\, \chi_{11;-}(\lambda)}{\lambda^2}\,. \tag{3.118}$$

There, we have used that χ_{1a} admits and analytic continuation from $\mathbb{H}^-$ to $\mathbb{H}^+$ denoted $\chi_{1a;-}(\lambda) = \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\chi_{1a}(\lambda)$ for $\Im(\lambda) > 0$. The integral is well defined in that the integrand behaves as $\mathrm{O}(|\lambda|^{-5/2}$ at ∞. Splitting it in two pieces and taking the integral involving χ_{11} by means of the residues at the poles located above $\mathbb{R} + 2\mathrm{i}\epsilon'$ -there are none- and the integral involving $\chi_{11;-}$ by means of the residues at the poles located below of $\mathbb{R} + 2\mathrm{i}\epsilon'$ -there is only one at $\lambda = 0$- one gets that

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(2)}(\xi) \;=\; -\frac{\alpha\chi_{12;-}(0)}{2\pi N}\partial_\lambda\chi_{11;-}(0)\,. \tag{3.119}$$

We next focus on the contribution involving $\varpi_N^{(1)}$. One gets

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(1)}(\xi) \;=\; -\frac{\mathfrak{r}}{4\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\Big\{\frac{\chi_{12}(\lambda) - \chi_{12;-}(\lambda)}{\lambda}\mathscr{U}_{12}(\lambda) \,+\, \big(\chi_{11}(\lambda) - \chi_{11;-}(\lambda)\big)\mathscr{U}_{11}(\lambda)\Big\}\,. \tag{3.120}$$

Splitting it in two pieces and taking the integral involving χ_{11} by means of the residues at the poles located above $\mathbb{R} + 2\mathrm{i}\epsilon'$ -there are none- and the integral involving $\chi_{11;-}$ by means of the residues at the poles located below of $\mathbb{R} + 2\mathrm{i}\epsilon'$ -there is only one at $\lambda = 0$- one gets that

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(1)}(\xi) \;=\; -\frac{\mathfrak{r}}{4\pi N}\Big\{\partial_\lambda\big(\chi_{12;-}\mathscr{U}_{12}\big)(0) \,+\, \chi_{11;-}(0)\mathscr{U}_{11}(0)\Big\}\,. \tag{3.121}$$

A direct calculation leads to

$$\mathscr{U}_{11}(0) \,=\, -\mathrm{i}Nu_N\chi_{12}(\mathrm{i}) - \chi_{11}(\mathrm{i})\frac{Nu_N - v_N}{2}\,, \quad \mathscr{U}_{12}(0) \,=\, -v_N\chi_{11}(\mathrm{i}) \quad \text{and} \quad \mathscr{U}'_{12}(0) \,=\, \mathrm{i}Nu_N\chi_{11}(\mathrm{i}). \tag{3.122}$$

Here, we remind that u_N and v_N have been introduced in (3.88). Upon inserting the above into the closed expression for the integral of $\varpi_N^{(1)}$, one eventually gets

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(1)}(\xi) \,=\, \frac{\mathfrak{r}}{4\pi N}\Big\{\mathrm{i}Nu_N\Big[\chi_{11;-}(0)\chi_{12}(\mathrm{i}) \,-\, \chi_{12;-}(0)\chi_{11}(\mathrm{i}) \,-\, \frac{\mathrm{i}}{2}\chi_{11;-}(0)\chi_{11}(\mathrm{i})\Big] \\ -\, v_N\Big[\frac{1}{2}\chi_{11;-}(0)\chi_{11}(\mathrm{i}) \,-\, \chi'_{12;-}(0)\chi_{11}(\mathrm{i})\Big]\Big\}\,. \tag{3.123}$$

Finally, we focus on estimating the contribution issuing from $\varpi_N^{(3)}$. Starting from the expression for the integrand provided in Proposition 3.2.4 along with (3.117), one gets that

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(3)}(\xi) \,=\, \mathcal{I}_{\downarrow}^{(3)} \,+\, \mathcal{I}_{\uparrow}^{(3)} \,+\, \mathcal{I}_{0}^{(3)}\,. \tag{3.124}$$

There, one has

$$\begin{aligned}\mathcal{I}_{\downarrow}^{(3)} \,&=\, \frac{1}{4\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^2}\Big(\big[\boldsymbol{E}_L^{(\uparrow)}(\lambda) \,+\, \big(1-\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\big)\boldsymbol{E}_L^{(\downarrow)}(\lambda)\big] \,-\, \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\lambda)\Big) \\ &=\, \frac{1}{4\pi N}\int_{\mathbb{R}+\mathrm{i}\varkappa_{\eta'}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^2}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda) \,+\, \big(1-\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\big)\boldsymbol{E}_L^{(\downarrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\lambda)\Big) \,+\, \mathbf{1}_{1<\zeta}\frac{\mathrm{e}^{\overline{a}_N}\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\mathrm{i}), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\mathrm{i})\Big) \\ &\quad +\, \frac{1}{4\pi N}\partial_\lambda\Big\{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\lambda)\Big)\Big\}_{|\lambda=0} \,-\, \frac{1}{4\pi N}\int_{\mathbb{R}-\mathrm{i}\varkappa_{\eta'}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^2}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\lambda)\Big)\,.\end{aligned} \tag{3.125}$$

Above, $\eta' > \eta$, and is taken small enough. Also, we have made use of the relation $\Big(\boldsymbol{E}_L^{(\downarrow)}(\mathrm{i}), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\mathrm{i})\Big) = 0$.

To estimate the various contributions more precisely, one needs the auxiliary estimates

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{E}_R^{(\downarrow)}(\mu)\Big) \,=\, \frac{-\mu}{R_\uparrow(\lambda)R_\downarrow(\mu)} \,+\, \mathrm{O}\Big(\sqrt{(1+|\lambda|)(1+|\mu|)}\cdot \mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big) \tag{3.126}$$

and

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{E}_R^{(\downarrow)}(\mu)\Big) \,=\, \mathrm{O}\Big(\sqrt{(1+|\lambda|)(1+|\mu|)}\cdot \mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)\,. \tag{3.127}$$

Note that the remainders are holomorphic in λ located between $\Gamma_\uparrow$ and $\Gamma_\downarrow$ and smooth in a_N, b_N, with each derivative adding a τ_N factor to the control. From that and the fact that $\mathcal{F}[\mathbf{g}] \in \mathcal{S}(\mathbb{R})$, one infers the bounds

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\lambda)\Big) \,=\, -\mathrm{e}^{\overline{a}_N}\frac{2\mathrm{i}\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi(\mathrm{i}-\lambda)R_\uparrow(\lambda)R_\downarrow(\mathrm{i})}\mathbf{1}_{1<\zeta} \,+\, \mathrm{O}\Big(\frac{\mathrm{e}^{\overline{a}_N\varkappa_\eta}}{\sqrt{1+|\lambda|}}\Big) \tag{3.128}$$

and

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\lambda)\Big) \,=\, \mathrm{O}\Big(\frac{\mathrm{e}^{\overline{a}_N\varkappa_\eta}}{\sqrt{1+|\lambda|}}\Big)\,. \tag{3.129}$$

There, the remainders enjoy the same properties as above. Inserting these bounds inside of the obtained representation for $\mathcal{I}_\uparrow^{(3)}$, one gets

$$\mathcal{I}_{\uparrow}^{(3)} \,=\, \mathrm{e}^{\overline{a}_N}\frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N R_\downarrow(\mathrm{i})}\mathbf{1}_{1<\zeta}\Big\{\frac{1}{\mathrm{i}R_\uparrow(\mathrm{i})} \,+\, \frac{1}{R_\downarrow(0)}\Big\} \,+\, \mathrm{O}\Big(\frac{\mathrm{e}^{\overline{a}_N\varkappa_\eta}}{N}\Big)\,. \tag{3.130}$$

The remainder is now only $\mathcal{C}^1$ in respect to a_N, b_N and partial a_N or b_N derivatives thereof enjoy the same control with a τ_N additional factor.

Similarly, one obtains

$$\begin{aligned}\mathcal{I}_{\uparrow}^{(3)} &= \frac{-1}{4\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^2}\Big(\big[\boldsymbol{E}_L^{(\uparrow)}(\lambda) + \big(1-\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\big)\boldsymbol{E}_L^{(\downarrow)}(\lambda)\big] - \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\uparrow)}(\lambda)\Big)\\
&= \frac{-1}{4\pi N}\int_{\mathbb{R}+\mathrm{i}\varkappa_{\eta'}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^2}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda)+\big(1-\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\big)\boldsymbol{E}_L^{(\downarrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\uparrow)}(\lambda)\Big) + \mathbf{1}_{1<\zeta}\frac{\mathrm{e}^{-\overline{b}_N-\overline{x}_N}\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\mathrm{i}), \boldsymbol{\mathcal{E}}^{(\downarrow)}(\mathrm{i})\Big)\\
&\quad - \frac{1}{4\pi N}\partial_\lambda\Big\{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\uparrow)}(\lambda)\Big)\Big\}_{|\lambda=0} - \frac{1}{4\pi N}\int_{\mathbb{R}-\mathrm{i}\varkappa_{\eta'}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^2}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\uparrow)}(\lambda)\Big)\,.\end{aligned} \tag{3.131}$$

As before, one gets the auxiliary estimates

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{E}_R^{(\uparrow)}(\mu)\Big) = \mathrm{O}\Big(\sqrt{(1+|\lambda|)(1+|\mu|)}\cdot\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big) \tag{3.132}$$

and

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{E}_L^{(\uparrow)}(\mu)\Big) = \frac{\lambda}{R_{\downarrow}(\lambda)R_{\uparrow}(\mu)} + \mathrm{O}\Big(\sqrt{(1+|\lambda|)(1+|\mu|)}\cdot\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)\,. \tag{3.133}$$

Again, the remainders are holomorphic in λ located between $\Gamma_{\uparrow}$ and $\Gamma_{\downarrow}$ and smooth in a_N, b_N, with each derivative adding a τ_N factor to the control. Thus,

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\uparrow)}(\lambda)\Big) = \mathrm{O}\Big(\frac{\mathrm{e}^{-\overline{b}_N-\zeta(1-\eta)\overline{x}_N}}{\sqrt{1+|\lambda|}}\Big) \tag{3.134}$$

and

$$\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{\mathcal{E}}^{(\uparrow)}(\lambda)\Big) = \frac{2\lambda\mathcal{F}[\mathbf{g}](\mathrm{i})\,\mathrm{e}^{-\overline{b}_N}}{\pi(\mathrm{i}+\lambda)R_{\downarrow}(\lambda)R_{\uparrow}(-\mathrm{i})}\mathbf{1}_{1<\zeta} + \mathrm{O}\Big(\frac{\mathrm{e}^{-\overline{b}_N\varkappa_\eta}}{\sqrt{1+|\lambda|}}\Big)\,. \tag{3.135}$$

There, the remainders enjoy the same properties as above. Hence, one gets that

$$\mathcal{I}_{\uparrow}^{(3)} = -\frac{\mathcal{F}[\mathbf{g}](\mathrm{i})\,\mathrm{e}^{-\overline{b}_N}}{2\pi^2 N R_{\uparrow}(-\mathrm{i})}\mathbf{1}_{1<\zeta}\int_{\mathbb{R}+\mathrm{i}\varkappa_\eta}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda R_{\downarrow}(\lambda)(\mathrm{i}+\lambda)} + \mathrm{O}\Big(\frac{\mathrm{e}^{-\varkappa_\eta\overline{b}_N}}{N}\Big)\,. \tag{3.136}$$

The remainder is $\mathcal{C}^1$ in respect to a_N, b_N and partial a_N or b_N derivatives thereof enjoy the same control with a τ_N additional factor. The remaining integral can be computed by means of taking the residues at $\lambda = 0$ and $\lambda = -\mathrm{i}$ located below of $\mathbb{R}+\mathrm{i}\varkappa_\eta$, leading eventually to

$$\mathcal{I}_{\uparrow}^{(3)} = \mathrm{e}^{-\overline{b}_N}\frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N R_{\downarrow}(\mathrm{i})}\mathbf{1}_{1<\zeta}\Big\{\frac{1}{\mathrm{i}R_{\uparrow}(\mathrm{i})} + \frac{1}{R_{\downarrow}(0)}\Big\} + \mathrm{O}\Big(\frac{\mathrm{e}^{-\varkappa_\eta\overline{b}_N}}{N}\Big)\,. \tag{3.137}$$

It remains to focus on $\mathcal{I}_0^{(3)}$ which takes the form

$$\begin{aligned}\mathcal{I}_0^{(3)} &= \frac{1}{4\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda}\Big(\mathrm{e}^{-\mathrm{i}\lambda\overline{a}_N}\big[\boldsymbol{E}_L^{(\uparrow)}(\lambda)+\big(1-\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\big)\boldsymbol{E}_L^{(\downarrow)}(\lambda)\big] - \mathrm{e}^{-\mathrm{i}\lambda\overline{b}_N}\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{E}_R^{(\downarrow)}(\lambda)\Big)\frac{\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\left[\frac{\pi\lambda}{2}\right]}\\
&= \frac{1}{4\pi N}\int_{\mathbb{R}+\mathrm{i}\varkappa_\eta}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda}\mathrm{e}^{-\mathrm{i}\lambda\overline{a}_N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{E}_R^{(\downarrow)}(\lambda)\Big)\frac{\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\left[\frac{\pi\lambda}{2}\right]} - \mathrm{e}^{\overline{a}_N}\frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\mathrm{i}), \boldsymbol{E}_R^{(\downarrow)}(\mathrm{i})\Big)\mathbf{1}_{1<\zeta}\\
&\quad + \frac{\mathcal{F}[\mathbf{g}](0)}{4\pi N}\Big(\boldsymbol{E}_L^{(\uparrow)}(0), \boldsymbol{E}_R^{(\downarrow)}(0)\Big) - \mathrm{e}^{-\overline{b}_N}\frac{\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N}\Big(\boldsymbol{E}_L^{(\uparrow)}(-\mathrm{i}), \boldsymbol{E}_R^{(\downarrow)}(-\mathrm{i})\Big)\mathbf{1}_{1<\zeta}\\
&\quad - \frac{1}{4\pi N}\int_{\mathbb{R}-\mathrm{i}\varkappa_\eta}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda}\mathrm{e}^{-\mathrm{i}\lambda\overline{b}_N}\Big(\boldsymbol{E}_L^{(\uparrow)}(\lambda), \boldsymbol{E}_R^{(\downarrow)}(\lambda)\Big)\frac{\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\left[\frac{\pi\lambda}{2}\right]}\,.\end{aligned} \tag{3.138}$$

It is then enough to invoke the previous auxiliary bounds to infer that $\Big(\boldsymbol{E}_L^{(\uparrow)}(0), \boldsymbol{E}_R^{(\downarrow)}(0)\Big) = \mathrm{O}\big(\mathcal{L}_N\big)$ while the two integral terms are a $N^{-1}\mathrm{O}\big(\mathrm{e}^{-\varkappa_\eta \bar{b}_N} + \mathrm{e}^{\varkappa_\eta \bar{a}_N}\big)$. Those auxiliary bounds also allow one to simplify the explicit contributions so that, up to subdominant corrections,

$$\mathcal{I}_0^{(3)} \,=\, \frac{\mathrm{i}\mathcal{F}[\mathbf{g}](\mathrm{i})}{2\pi^2 N R_\uparrow(\mathrm{i}) R_\downarrow(\mathrm{i})} \mathbf{1}_{1<\zeta}\Big(\mathrm{e}^{-\bar{b}_N} + \mathrm{e}^{\bar{a}_N}\Big) \,+\, \mathrm{O}\bigg(\frac{\mathrm{e}^{-\varkappa_\eta \bar{b}_N} + \mathrm{e}^{\varkappa_\eta \bar{a}_N}}{N}\bigg)\,. \tag{3.139}$$

The remainder is $\mathcal{C}^1$ in respect to a_N, b_N and partial a_N or b_N derivatives thereof enjoy the same control with a τ_N additional factor. By putting the three estimates together, we get

$$\int_{a_N}^{b_N} \mathrm{d}\xi\, \varpi_N^{(3)}(\xi) \,=\, \frac{\mathcal{F}[\mathbf{g}](\mathrm{i})\mathbf{1}_{1<\zeta}}{2\pi^2 N R_\downarrow(0) R_\downarrow(\mathrm{i})}\Big(\mathrm{e}^{-\bar{b}_N} + \mathrm{e}^{\bar{a}_N}\Big) \,+\, \mathrm{O}\bigg(\frac{\mathrm{e}^{-\varkappa_\eta \bar{b}_N} + \mathrm{e}^{\varkappa_\eta \bar{a}_N}}{N}\bigg)\,. \tag{3.140}$$

This entails the claim.

Below, we establish the unique solvability of the constraints (3.107) on the endpoints a_N, b_N under the hypothesis that $b_N - a_N \geq \eta$, for some fixed $\eta > 0$. Later, we will establish that these unique solutions do correspond to the endpoints of the support of the equilibrium measure.

Proposition 3.2.7 *Consider the subset of* $\mathbb{R}^2$

$$\mathcal{D}_\varsigma \,=\, [\varsigma\,; +\infty[\,\times\,]-\infty\,; -\varsigma]\,, \tag{3.141}$$

with $\varsigma > 0$ *and small. For any* $\eta > 0$ *there exists* N_0 *such that, for any* $N \geq N_0$*, there exists a unique solution* $\big(b_{N;\alpha}, a_{N;\alpha}\big) \in \mathcal{D}_\varsigma$ *to the constraint equations* (3.107)*. Moreover, it holds that*

$$b_{N;\alpha} \,=\, 1 \,+\, \mathrm{o}(1) \qquad \textit{and} \qquad a_{N;\alpha} \,=\, -1 \,+\, \mathrm{o}(1)\,. \tag{3.142}$$

Proof —

It follows from Propositions 3.2.5-3.2.6 and the expansion $\chi_{11}(i) = -\mathrm{i}/R_\uparrow(\mathrm{i}) + \mathrm{O}\big(\mathrm{e}^{-\overline{x}_N \varkappa_\eta}\big)$ with a differentiable remainder, that the constraints (3.107) are equivalent to the system of equations for a_N, b_N:

$$\mathrm{e}^{\bar{b}_N} \,-\, \mathrm{e}^{-\bar{a}_N} \,=\, 2\frac{\alpha \chi_{11;-}(0)}{\mathfrak{r}\chi_{11}(i)} \,+\, \mathrm{O}\bigg(\frac{\mathrm{e}^{-\bar{b}_N \widetilde{\varkappa}_\eta} + \mathrm{e}^{\bar{a}_N \widetilde{\varkappa}_\eta}}{N\tau_N}\bigg)\,, \tag{3.143}$$

and

$$\begin{aligned}\mathrm{e}^{\bar{b}_N} \,+\, \mathrm{e}^{-\bar{a}_N} \,=\, \frac{4\pi N}{\mathrm{i}\,\mathfrak{r}} \cdot \frac{1 \,+\, \dfrac{\alpha}{2\pi N}\Big[\chi'_{11;-}(0)\chi_{12;-}(0) \,-\, \chi'_{12;-}(0)\chi_{11;-}(0) \,+\, \dfrac{1}{2}\chi^2_{11;-}(0)\Big]}{\chi_{11;-}(0)\chi_{12}(\mathrm{i}) \,-\, \chi_{12;-}(0)\chi_{11}(\mathrm{i}) \,-\, \dfrac{\mathrm{i}}{2}\chi_{11;-}(0)\chi_{11}(\mathrm{i})} \\ +\, \mathrm{O}\Big(\overline{x}_N\Big(\mathrm{e}^{\widetilde{\varkappa}_\eta \bar{a}_N} \,+\, \mathrm{e}^{-\widetilde{\varkappa}_\eta \bar{b}_N}\Big)\Big)\,.\end{aligned} \tag{3.144}$$

There, we agree upon $\widetilde{\varkappa}_\eta = \min\big\{1, (1-\eta)\zeta\big\}$.

A direct calculation based on the expansions (3.63) and (3.68) leads to

$$\chi_{11;-}(0)\chi_{12}(\mathrm{i}) - \chi_{12;-}(0)\chi_{11}(\mathrm{i}) - \frac{\mathrm{i}}{2}\chi_{11;-}(0)\chi_{11}(\mathrm{i}) \,=\, \frac{-1}{R_\downarrow(0)R_\uparrow(\mathrm{i})}\bigg(1 + \mathrm{i}\mathrm{e}^{-\overline{x}_N}\frac{R_\uparrow(\mathrm{i})}{R_\downarrow(\mathrm{i})} + \mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)\bigg) \tag{3.145}$$

and

$$\chi'_{11;-}(0)\chi_{12;-}(0) \,-\, \chi'_{12;-}(0)\chi_{11;-}(0) \,+\, \frac{1}{2}\chi^2_{11;-}(0) \,=\, \mathrm{O}\Big(\overline{x}_N\,\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)\,. \tag{3.146}$$

This allows one to recast the second constraint in the form

$$\mathrm{e}^{\bar{b}_N} + \mathrm{e}^{-\bar{a}_N} = N\mathfrak{c}_0 \frac{1 + \overline{x}_N \mathrm{O}\Big(\mathrm{e}^{\widetilde{\varkappa}_\eta \bar{a}_N} + \mathrm{e}^{-\widetilde{\varkappa}_\eta \bar{b}_N} + \mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)}{1 + \mathrm{i}\mathrm{e}^{-\overline{x}_N} \dfrac{R_\uparrow(\mathrm{i})}{R_\downarrow(\mathrm{i})} + \mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)}$$
$$= N\mathfrak{c}_0 \cdot \Big(1 + \mathrm{O}\Big(\overline{x}_N \mathrm{e}^{\widetilde{\varkappa}_\eta \bar{a}_N} + \overline{x}_N \mathrm{e}^{-\widetilde{\varkappa}_\eta \bar{b}_N}\Big)\Big) . \quad (3.147)$$

There, we have set

$$\mathfrak{c}_0 = \frac{4\pi}{\mathfrak{r}} \sqrt{\omega_1 + \omega_2} R_\uparrow(\mathrm{i}) . \quad (3.148)$$

Finally, owing to

$$2\frac{\alpha \chi_{11;-}(0)}{\mathfrak{r}\chi_{11}(i)} = \frac{\alpha \mathfrak{c}_0}{\pi(\omega_1 + \omega_2)} \frac{1 + \mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)}{1 - \mathrm{i}\mathrm{e}^{-\overline{x}_N} \dfrac{R_\uparrow(\mathrm{i})}{R_\downarrow(\mathrm{i})} + \mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)} \quad (3.149)$$

one recasts the first constrain in the form

$$\mathrm{e}^{\bar{b}_N} - \mathrm{e}^{-\bar{a}_N} = \frac{\alpha \mathfrak{c}_0}{\pi(\omega_1 + \omega_2)} \frac{1 + \mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_N}\Big)}{1 - \mathrm{i}\mathrm{e}^{-\overline{x}_N} \dfrac{R_\uparrow(\mathrm{i})}{R_\downarrow(\mathrm{i})}} + \mathrm{O}\Big(\mathrm{e}^{-\bar{b}_N \widetilde{\varkappa}_\eta} + \mathrm{e}^{\bar{a}_N \widetilde{\varkappa}_\eta}\Big)$$
$$= \frac{\alpha \mathfrak{c}_0}{\pi(\omega_1 + \omega_2)} + \mathrm{O}\Big(\mathrm{e}^{\widetilde{\varkappa}_\eta \bar{a}_N} + \mathrm{e}^{-\widetilde{\varkappa}_\eta \bar{b}_N}\Big) . \quad (3.150)$$

In order to prove more efficiently the existence and uniqueness for N large enough of the system's solutions on the domain $\mathcal{D}_\varsigma$ introduced in (3.141), it is convenient to pass to the finite in N-variables $\big(u_N, v_N\big)$ defined through (3.88).

Note that upon defining

$$\Psi\big(x, y\big) = \Big(\frac{\mathrm{e}^{\tau_N x} + \mathrm{e}^{-\tau_N y}}{N}, \mathrm{e}^{\tau_N x} - \mathrm{e}^{-\tau_N y}\Big) , \quad (3.151)$$

one has that

$$\widehat{\mathcal{D}}_\varsigma = \Psi\big(\mathcal{D}_\varsigma\big) = \Big\{(u, v) \in \mathbb{R}^+ \times \mathbb{R} \,:\, u \geq \frac{1}{N^{1-\varsigma}} \text{ and } Nu - 2N^\varsigma \geq |v|\Big\} . \quad (3.152)$$

Then, one may recast the constraints in the form

$$u_N = \mathfrak{c}_0 + \delta\Phi_1\big(u_N, v_N\big) \qquad \text{and} \qquad v_N = \frac{\alpha \mathfrak{c}_0}{\pi(\omega_1 + \omega_2)} + \delta\Phi_2\big(u_N, v_N\big) . \quad (3.153)$$

The functions $\delta\Phi_a$ are smooth on $\widehat{\mathcal{D}}_\varsigma$ since the remainders in (3.147) and (3.150) are smooth on $\mathcal{D}_\varsigma$. Moreover, it is direct to estimate that throughout $\widehat{\mathcal{D}}_\varsigma$, it holds

$$\delta\Phi_a\big(u, v\big) = \mathrm{O}\Big(\Big(\tfrac{2}{Nu+v}\Big)^{\widetilde{\varkappa}_\eta(1-\eta')} + \Big(\tfrac{2}{Nu-v}\Big)^{\widetilde{\varkappa}_\eta(1-\eta')}\Big) = \mathrm{O}\Big(N^{-\varsigma\widetilde{\varkappa}_\eta(1-\eta')}\Big) , \quad (3.154)$$

with $\eta' > 0$ and small enough. Taken that the remainder's estimates also hold for the first derivatives up to additional τ_N factors, one gets that

$$\big|\partial_u \delta\Phi_a\big(u, v\big)\big| + \big|\partial_v \delta\Phi_a\big(u, v\big)\big| = \mathrm{O}\Big(N^{-\varsigma\widetilde{\varkappa}_\eta(1-\eta')}\Big) , \quad (3.155)$$

by slightly increasing η'. Thus, introducing the $\mathcal{C}^1$ diffeomorphism on $\widehat{\mathcal{D}}_\varsigma$

$$\mathbf{\Phi}(u,v) \;=\; \begin{pmatrix} \mathfrak{c}_0 - u + \delta\Phi_1\big(u,v\big) \\ \frac{\alpha\mathfrak{c}_0}{\pi(\omega_1+\omega_2)} - v + \delta\Phi_2\big(u,v\big) \end{pmatrix} \tag{3.156}$$

one has, uniformly throughout $\widehat{\mathcal{D}}_\varsigma$, that $\mathtt{D}\mathbf{\Phi} = -I_2 + \mathrm{O}\Big(N^{-\varsigma\widetilde{\varkappa}_\eta(1-\eta')}\Big)$. One is thus in the setting where one can invoke the local inversion theorem so as to ensure that $\mathbf{\Phi}$, as soon as N is large enough, is a local $\mathcal{C}^1$ diffeomorphism on $\widehat{\mathcal{D}}_\varsigma$ such that, for any $(u,v) \in \widehat{\mathcal{D}}_\varsigma$, there exist N-independent $s, s' > 0$ such that

$$\mathbf{\Phi} \;:\; B_{(u,v),s} \to \mathbf{\Phi}\Big(B_{(u,v),s}\Big) \supset B_{\mathbf{\Phi}(u,v),s'} \tag{3.157}$$

is a diffeomorphism.

This is enough so as to ensure that $\mathbf{\Phi}$ is a $\mathcal{C}^1$ diffeomorphism on $\widehat{\mathcal{D}}_\varsigma$. Indeed, assume that there exist $\big(u,v\big), \big(u',v'\big) \in \widehat{\mathcal{D}}_\varsigma$, $\big(u,v\big) \neq \big(u',v'\big)$ such that $\mathbf{\Phi}\big(u,v\big) = \mathbf{\Phi}\big(u',v'\big)$. Then, one has the relation

$$\left\{ \begin{array}{rcccl} u'-u &=& \delta\Phi_1\big(u,v\big) - \delta\Phi_1\big(u',v'\big) &=& \mathrm{O}\big(N^{-\varsigma\widetilde{\varkappa}_\eta(1-\eta')}\big) \\ v'-v &=& \delta\Phi_2\big(u,v\big) - \delta\Phi_2\big(u',v'\big) &=& \mathrm{O}\big(N^{-\varsigma\widetilde{\varkappa}_\eta(1-\eta')}\big) \end{array} \right. . \tag{3.158}$$

However, $\mathbf{\Phi}$ is injective on $B_{(u,v),s} \ni (u',v')$ what entails $\big(u,v\big) = \big(u',v'\big)$, a contradiction. This entail that $\mathbf{\Phi}$ is a diffeomorphism on $\widehat{\mathcal{D}}_\varsigma$.

Finally, by the estimates on $\delta\mathbf{\Phi}$, it holds that $\mathbf{\Phi}\Big(\mathfrak{c}_0\,,\,\frac{\alpha\mathfrak{c}_0}{\pi(\omega_1+\omega_2)}\Big) = \mathrm{O}\big(N^{-\varsigma\widetilde{\varkappa}_\eta(1-\eta')}\big)$. However, since there exists $s, s' > 0$ such that

$$\mathbf{\Phi}\Big(B_{\big(\mathfrak{c}_0\,,\,\frac{\alpha\mathfrak{c}_0}{\pi(\omega_1+\omega_2)}\big),s}\Big) \supset B_{\mathbf{\Phi}\big(\mathfrak{c}_0\,,\,\frac{\alpha\mathfrak{c}_0}{\pi(\omega_1+\omega_2)}\big),s'}\;, \tag{3.159}$$

it follows that $(0,0) \in \mathbf{\Phi}\big(\widehat{\mathcal{D}}_\varsigma\big)$, what ensures the existence and uniqueness of solutions to the system (3.107) on $\mathcal{D}_\varsigma$.

The form of the leading large-N behaviour for $a_{N;\alpha}, b_{N;\alpha}$ then follows readily.

Lemma 3.2.8 *The following large-N asymptotics hold*

$$\mathrm{e}^{\bar{b}_{N;\alpha}} \;=\; \frac{\mathfrak{c}_0 N}{2} + \frac{\alpha\mathfrak{c}_0}{2\pi(\omega_1+\omega_2)} - \frac{2\mathrm{i}R_\uparrow(\mathrm{i})}{N\mathfrak{c}_0 R_\downarrow(\mathrm{i})}\Big(1 + \frac{2\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi r}\mathbf{1}_{1<\zeta}\Big(1+\tfrac{\alpha}{\pi}\Big)\Big) + \mathrm{O}\Big(\frac{1}{N^{\varkappa_\eta}}\Big) \tag{3.160}$$

and

$$\mathrm{e}^{-\bar{a}_{N;\alpha}} \;=\; \frac{\mathfrak{c}_0 N}{2} - \frac{\alpha\mathfrak{c}_0}{2\pi(\omega_1+\omega_2)} - \frac{2\mathrm{i}R_\uparrow(\mathrm{i})}{N\mathfrak{c}_0 R_\downarrow(\mathrm{i})}\Big(1 + \frac{2\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi r}\mathbf{1}_{1<\zeta}\Big(1+\tfrac{\alpha}{\pi}\Big)\Big) + \mathrm{O}\Big(\frac{1}{N^{\varkappa_\eta}}\Big) \tag{3.161}$$

These asymptotic expansions involve the constant

$$\mathfrak{c}_0 \;=\; \frac{4\pi}{\mathfrak{r}}\sqrt{\omega_1+\omega_2}R_\uparrow(\mathrm{i}) \tag{3.162}$$

while $\varkappa_\eta$ is as introduced in (3.95)*.*

Moreover, it also holds that

$$\begin{aligned} \bar{b}_{N;\alpha} \;=\; & \log\Big(\frac{\mathfrak{c}_0 N}{2}\Big) + \frac{\alpha}{\pi N(\omega_1+\omega_2)} \\ & - \frac{1}{N^2}\Big\{\frac{4\mathrm{i}R_\uparrow(\mathrm{i})}{\mathfrak{c}_0^2 R_\downarrow(\mathrm{i})}\Big(1 + \frac{2\mathcal{F}[g](\mathrm{i})}{\pi\mathfrak{r}}\mathbf{1}_{1<\zeta}\Big(1+\tfrac{\alpha}{\pi}\Big)\Big) + \frac{\alpha^2}{2\pi^2N^2(\omega_1+\omega_2)^2}\Big\} + \mathrm{O}\Big(\frac{1}{N^{\varkappa_\eta+1}}\Big) \end{aligned} \tag{3.163}$$

and

$$\overline{a}_{N;\alpha} \,=\, -\log\left(\frac{\mathfrak{c}_0 N}{2}\right) \,+\, \frac{\alpha}{\pi N(\omega_1+\omega_2)} \\ +\, \frac{1}{N^2}\left\{\frac{4\mathrm{i}R_{\uparrow}(\mathrm{i})}{\mathfrak{c}_0^2 R_{\downarrow}(\mathrm{i})}\Big(1 \,+\, \frac{2\mathcal{F}[\mathbf{g}](\mathrm{i})}{\pi\mathfrak{r}}\mathbf{1}_{1<\zeta}\Big(1+\tfrac{\alpha}{\pi}\Big)\Big) \,-\, \frac{\alpha^2}{2\pi^2 N^2(\omega_1+\omega_2)^2}\right\} \,+\, \mathrm{O}\Big(\frac{1}{N^{\varkappa_\eta+1}}\Big) \quad (3.164)$$

Proof —

The result follows from direct calculations based on (3.69) and (3.64).

3.3 Large-N behaviour of the interpolating integral

Proposition 3.3.1 *It holds*

$$\int_{a_{N;\alpha}}^{b_{N;\alpha}} \mathrm{d}\xi\, \xi\, \widehat{\varrho}_{\mathrm{eq};\alpha}(\xi) \,=\, \frac{\alpha}{\pi(\omega_1+\omega_2)N\tau_N}\Big[\log\Big(\tfrac{N\mathfrak{c}_0}{2}\Big) \,+\, \mathrm{i}\log' R_{\downarrow}(0)\Big] \,+\, \mathrm{O}\Big(\frac{\tau_N}{N}\Big[N^{-1}\mathbf{1}_{1<\zeta} \,+\, N^{-\varkappa_\eta}\Big]\Big) \quad (3.165)$$

Proof —

One starts by observing that

$$\int_{a_{N;\alpha}}^{b_{N;\alpha}} \mathrm{d}\xi \mathrm{e}^{-\mathrm{i}\lambda\tau_N(\xi-a_{N;\alpha})} \,=\, \frac{a_{N;\alpha} \,-\, b_{N;\alpha}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{\mathrm{i}\lambda\tau_N} \,+\, \frac{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}} \,-\, 1}{(\lambda\tau_N)^2}\,. \quad (3.166)$$

Here, we agree that $\overline{x}_{N;\alpha} = \overline{b}_{N;\alpha} - \overline{a}_{N;\alpha}$. Thus, according to the partially integrated expression for $\mathcal{W}_N\big[V'_{N;\alpha}\big]$ obtained in Proposition 3.2.4, one may decompose the integral into three terms

$$\int_{a_{N;\alpha}}^{b_{N;\alpha}} \mathrm{d}\xi\, \xi\, \widehat{\varrho}_{\mathrm{eq};\alpha}(\xi) \,=\, \sum_{a=1}^{3} \mathcal{H}^{(a)} \quad (3.167)$$

where

$$\mathcal{H}^{(1)} = \frac{\mathfrak{r}\tau_N}{4\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\left\{\frac{\chi_{12}(\lambda)}{\lambda}\mathscr{U}_{12}(\lambda) + \chi_{11}(\lambda)\mathscr{U}_{11}(\lambda)\right\}\cdot\left\{\frac{a_{N;\alpha} \,-\, b_{N;\alpha}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{\mathrm{i}\lambda\tau_N} + \frac{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}} \,-\, 1}{(\lambda\tau_N)^2}\right\}, \quad (3.168)$$

$$\mathcal{H}^{(2)} \,=\, \frac{\alpha\tau_N}{2\mathrm{i}\pi N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\chi_{11}(\lambda)\chi_{12;-}(0)\cdot\left\{\frac{a_{N;\alpha} \,-\, b_{N;\alpha}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}}}{\mathrm{i}\lambda^2\tau_N} \,+\, \frac{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N;\alpha}} \,-\, 1}{\lambda^3\tau_N^2}\right\}, \quad (3.169)$$

and, by employing the notations introduced in Proposition 3.2.4

$$\mathcal{H}^{(3)} \,=\, \mathcal{H}^{(3)}_{\uparrow} \,+\, \mathcal{H}^{(3)}_{\downarrow} \,+\, \mathcal{H}^{(3)}_{0} \qquad \text{with} \qquad \mathcal{H}^{(3)}_{\upsilon} \,=\, \int_{a_{N;\alpha}}^{b_{N;\alpha}} \mathrm{d}\xi\, \xi\varpi^{(3)}_{N;\upsilon}(\xi)\,, \quad \upsilon\in\{\uparrow,\downarrow,0\}\,. \quad (3.170)$$

We stress that now, all the above quantities involve the solution χ subordinate to the choice of endpoints $a_{N;\alpha}$, $b_{N;\alpha}$.

The first two contributions can be computed in closed form. Indeed, by applying the previously introduced notations, one gets

$$\mathcal{H}^{(2)} \,=\, \frac{\alpha\tau_N}{2\mathrm{i}\pi N}\chi_{12;-}(0)\int_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\cdot\left\{\frac{a_{N;\alpha}\chi_{11}(\lambda) \,-\, b_{N;\alpha}\chi_{11;-}(\lambda)}{\mathrm{i}\lambda^2\tau_N} + \frac{\chi_{11;-}(\lambda) \,-\, \chi_{11}(\lambda)}{\lambda^3\tau_N^2}\right\}. \quad (3.171)$$

Due to the $\mathrm{O}\big(\lambda^{-5/2}\big)$ decay of the integrand at ∞

i) the contribution of the integrand involving χ_{11} can be evaluated by taking the residues of the integrand's poles located above $\mathbb{R}+2\mathrm{i}\epsilon'$. Since there are no poles, this part produces 0.

ii) The contribution of the integrand involving $\chi_{11;-}$ can be evaluated by taking the residues of the integrand's poles located below $\mathbb{R}+2\mathrm{i}\epsilon'$. The only poles present are the third and second order poles at 0.

All-in-all, one gets that

$$\mathcal{H}^{(2)} \;=\; -\frac{\alpha\tau_N}{2\mathrm{i}\pi N}\chi_{12;-}(0)\cdot\Big\{-\frac{b_{N;\alpha}}{\mathrm{i}\tau_N}\chi'_{11;-}(0)\;+\;\frac{1}{2\tau_N^2}\chi''_{11;-}(0)\Big\}\,. \tag{3.172}$$

The large-N behaviour of χ_{1a} given in (3.68), allows one to infer that

$$\chi_{12;-}(0)\;=\;\mathrm{O}\Big(\mathrm{e}^{-\zeta(1-\eta)\overline{x}_{N;\alpha}}\Big)\quad\text{and}\quad\chi^{(k)}_{11;-}(0)\;=\;\mathrm{O}\Big(\overline{x}^{k}_{N;\alpha}\Big)\quad\text{with}\quad k\in\mathbb{N}. \tag{3.173}$$

This entails that

$$\mathcal{H}^{(2)}\;=\;\mathrm{O}\bigg(\frac{|\alpha|\tau_N}{N}\cdot\mathrm{e}^{-\zeta(1-\eta)\overline{x}_{N;\alpha}}\bigg)\;=\;\mathrm{O}\bigg(\frac{|\alpha|\tau_N}{N^{1+2\zeta(1-\eta)}}\bigg)\,. \tag{3.174}$$

Similar handlings lead to

$$\begin{aligned}\mathcal{H}^{(1)}\;=\;-\frac{\mathfrak{r}\tau_N}{4\mathrm{i}\pi N}\Big\{&-\frac{b_{N;\alpha}}{\mathrm{i}\tau_N}\partial_\lambda\Big(\chi_{12;-}\mathscr{U}_{12}\Big)(0)\;+\;\frac{1}{2\tau_N^2}\partial_\lambda^2\Big(\chi_{12;-}\mathscr{U}_{12}\Big)(0)\\&-\frac{b_{N;\alpha}}{\mathrm{i}\tau_N}\Big(\chi_{11}\mathscr{U}_{11}\Big)(0)\;+\;\frac{1}{\tau_N^2}\partial_\lambda\Big(\chi_{11;-}\mathscr{U}_{11}\Big)(0)\Big\}\,.\end{aligned} \tag{3.175}$$

A long but straightforward calculation utilising the expansion (3.68) yields

$$\begin{aligned}\mathcal{H}^{(1)}\;=\;&\frac{\mathfrak{r}\tau_N}{4\mathrm{i}\pi NR_\downarrow(0)R_\uparrow(\mathrm{i})}\Big\{\overline{b}_{N;\alpha}\mathrm{e}^{\overline{b}_{N;\alpha}}\;+\;\overline{a}_{N;\alpha}\mathrm{e}^{-\overline{a}_{N;\alpha}}\;-\;\big(\mathrm{e}^{\overline{b}_{N;\alpha}}-\mathrm{e}^{-\overline{a}_{N;\alpha}}\big)\cdot\big(1-\mathrm{i}\log'R_\downarrow(0)\big)\\&+\;\mathrm{i}\frac{R_\uparrow(\mathrm{i})}{R_\downarrow(\mathrm{i})}\cdot\Big[\overline{b}_{N;\alpha}\mathrm{e}^{-\overline{b}_{N;\alpha}}\;+\;\overline{a}_{N;\alpha}\mathrm{e}^{\overline{a}_{N;\alpha}}\;-\;\mathrm{e}^{-\overline{x}_{N;\alpha}}\big(\mathrm{e}^{\overline{b}_{N;\alpha}}-\mathrm{e}^{-\overline{a}_{N;\alpha}}\big)\cdot\big(1+\mathrm{i}\log'R_\downarrow(0)\big)\Big]\Big\}\\&\times\Big(1+\mathrm{O}\big(\tau_N\mathrm{e}^{-\zeta(1-\eta)\overline{x}_{N;\alpha}}\big)\Big).\end{aligned} \tag{3.176}$$

Observe that, for large N, one has

$$\overline{b}_{N;\alpha}\mathrm{e}^{\overline{b}_{N;\alpha}}\;+\;\overline{a}_{N;\alpha}\mathrm{e}^{-\overline{a}_{N;\alpha}}\;=\;v_{N;\alpha}\log\Big(\frac{Nu_{N;\alpha}\mathrm{e}}{2}\Big)\;+\;\mathrm{O}\big(N^{-2}\big) \tag{3.177}$$

and

$$\overline{b}_{N;\alpha}\mathrm{e}^{-\overline{b}_{N;\alpha}}\;+\;\overline{a}_{N;\alpha}\mathrm{e}^{\overline{a}_{N;\alpha}}\;=\;\mathrm{O}\bigg(\frac{v_{N;\alpha}\tau_N}{(Nu_{N;\alpha})^2}\bigg)\,. \tag{3.178}$$

Finally, by inserting the large-N expansion of the endpoints obtained in Lemma 3.2.8, one gets

$$\mathcal{H}^{(1)}\;=\;\frac{\alpha}{\pi(\omega_1+\omega_2)N\tau_N}\bigg[\log\Big(\tfrac{N\mathfrak{c}_0}{2}\Big)+\mathrm{i}\log'R_\downarrow(0)\bigg]-\frac{2\alpha^2}{\pi^2(\omega_1+\omega_2)^2N^3\tau_N}+\mathrm{O}\bigg(\frac{1}{N^3}+\frac{\mathrm{e}^{-\zeta(1-\eta)\overline{x}_{N;\alpha}}}{N}\bigg)\,. \tag{3.179}$$

Hence, to conclude, it remains to estimate $\mathcal{H}^{(3)}$. We estimate separately each of the $\mathcal{H}^{(3)}_\upsilon$. For $\upsilon\in\{\uparrow,\downarrow\}$, by taking the expressions for $\varpi^{(3)}_{N;\uparrow/\downarrow}$ obtained in Proposition 3.2.4 one gets the representation

$$\mathcal{H}^{(3)}_\upsilon\;=\;\frac{\epsilon_\upsilon}{4\mathrm{i}\pi N\tau_N}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^3}\Big(\boldsymbol{F}_\uparrow(\lambda)\;+\;\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\boldsymbol{F}_\downarrow(\lambda),\boldsymbol{\mathcal{E}}^{(\upsilon)}(\lambda)\Big)\,, \tag{3.180}$$

where we have introduced

$$\boldsymbol{F}_{\downarrow}(\lambda) = \big(\mathrm{i}\bar{b}_{N;\alpha}\lambda + 1\big)\boldsymbol{E}_{L}^{(\uparrow)}(\lambda) \tag{3.181}$$

$$\boldsymbol{F}_{\uparrow}(\lambda) = -\big(\mathrm{i}\bar{a}_{N;\alpha}\lambda + 1\big)\boldsymbol{E}_{L}^{(\uparrow)}(\lambda) \,-\, \big(\mathrm{i}\bar{b}_{N;\alpha}\lambda + 1\big)\boldsymbol{E}_{L}^{(\downarrow)}(\lambda) \,+\, \mathrm{e}^{\mathrm{i}\lambda\overline{x}_{N;\alpha}}\big(\mathrm{i}\bar{a}_{N;\alpha}\lambda + 1\big)\boldsymbol{E}_{L}^{(\uparrow)}(\lambda)\,. \tag{3.182}$$

Thus, upon introducing $\widetilde{\varkappa}_{\eta} = (1-\eta)\min\{1,\zeta\}$ and deforming the integration contours, one gets

$$\mathcal{H}_{\upsilon}^{(3)} \,=\, \frac{\epsilon_{\upsilon}}{4\mathrm{i}\pi N\tau_{N}}\int_{\mathbb{R}+\mathrm{i}\widetilde{\varkappa}_{\eta}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^{3}}\Big(\boldsymbol{F}_{\uparrow}(\lambda),\boldsymbol{\mathcal{E}}^{(\upsilon)}(\lambda)\Big) \,-\, \frac{1}{2}\partial_{\lambda}^{2}\Big\{\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N}}\Big(\boldsymbol{F}_{\downarrow}(\lambda),\boldsymbol{\mathcal{E}}^{(\upsilon)}(\lambda)\Big)\Big\}_{|\lambda=0}$$
$$+\,\frac{\epsilon_{\upsilon}}{4\mathrm{i}\pi N\tau_{N}}\int_{\mathbb{R}-\mathrm{i}\widetilde{\varkappa}_{\eta}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^{3}}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_{N}}\Big(\boldsymbol{F}_{\downarrow}(\lambda),\boldsymbol{\mathcal{E}}^{(\upsilon)}(\lambda)\Big) \tag{3.183}$$

Then, using that within the band $|\Im(\lambda)| \leq \widetilde{\varkappa}_{\eta}$ one has the bounds

$$|\boldsymbol{\mathcal{E}}^{(\upsilon)}(\lambda)| \,\leq\, \frac{C}{1+|\lambda|}\Big(N^{-1}\mathbf{1}_{1<\zeta} \,+\, N^{-\varkappa_{\eta}}\Big)\,, \tag{3.184}$$

one readily gets that

$$\Big|\mathcal{H}_{\upsilon}^{(3)}\Big| \,\leq\, \frac{C\tau_{N}}{N}\Big(N^{-1}\mathbf{1}_{1<\zeta} \,+\, N^{-\varkappa_{\eta}}\Big)\,. \tag{3.185}$$

Finally, we focus on $\mathcal{H}_{0}^{(3)}$ which, upon using that $\Big(\boldsymbol{E}_{L}^{(\downarrow)}(\lambda),\boldsymbol{E}_{L}^{(\downarrow)}(\lambda)\Big) = 0$, may be recast as

$$\mathcal{H}_{0}^{(3)} \,=\, \frac{1}{4\mathrm{i}\pi N\tau_{N}}\int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^{2}}\frac{\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\Big[\frac{\pi\lambda}{2}\Big]}\Big(\boldsymbol{E}_{L}^{(\uparrow)}(\lambda),\boldsymbol{E}_{R}^{(\downarrow)}(\lambda)\Big)$$
$$\times\,\Big\{\big(\mathrm{i}\bar{a}_{N;\alpha}\lambda+1\big)\mathrm{e}^{-\mathrm{i}\lambda\bar{a}_{N;\alpha}} \,+\, \mathrm{e}^{-\mathrm{i}\lambda\bar{b}_{N;\alpha}}\big(\mathrm{i}\bar{b}_{N;\alpha}\lambda+1\big)\Big\}\,. \tag{3.186}$$

Then, it is enough to observe that throughout the strip $|\Im(\lambda)| \leq \widetilde{\varkappa}_{\eta}$

$$\Big(\boldsymbol{E}_{L}^{(\uparrow)}(\lambda),\boldsymbol{E}_{R}^{(\downarrow)}(\lambda)\Big) \,=\, -\frac{\lambda}{R(\lambda)} + \mathfrak{R}_{N}(\lambda) \qquad \text{with} \qquad \mathfrak{R}_{N}(\lambda) \,=\, \mathrm{O}\Big((1+|\lambda|)\mathrm{e}^{-\zeta(1-\eta)\overline{x}_{N;\alpha}}\Big)\,. \tag{3.187}$$

In particular, the leading term has a second order zero at $\lambda = 0$. This leads to

$$\mathcal{H}_{0}^{(3)} \,=\, \int_{\mathbb{R}+\mathrm{i}\widetilde{\varkappa}_{\eta}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^{2}}\frac{\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\Big[\frac{\pi\lambda}{2}\Big]}\Big(\boldsymbol{E}_{L}^{(\uparrow)}(\lambda),\boldsymbol{E}_{R}^{(\downarrow)}(\lambda)\Big)\frac{\mathrm{i}\bar{a}_{N;\alpha}\lambda+1}{4\mathrm{i}\pi N\tau_{N}}\mathrm{e}^{-\mathrm{i}\lambda\bar{a}_{N;\alpha}}$$
$$+\int_{\mathbb{R}-\mathrm{i}\widetilde{\varkappa}_{\eta}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda}\frac{\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\Big[\frac{\pi\lambda}{2}\Big]R(\lambda)}\frac{\mathrm{i}\bar{b}_{N;\alpha}\lambda+1}{4\mathrm{i}\pi N\tau_{N}}\mathrm{e}^{-\mathrm{i}\lambda\bar{b}_{N;\alpha}} + \int_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi\lambda^{2}}\frac{\mathcal{F}[\mathbf{g}](\lambda)}{\cosh\Big[\frac{\pi\lambda}{2}\Big]}\mathfrak{R}_{N}(\lambda)\frac{\mathrm{i}\bar{b}_{N;\alpha}\lambda+1}{4\mathrm{i}\pi N\tau_{N}}\mathrm{e}^{-\mathrm{i}\lambda\bar{b}_{N;\alpha}}\,. \tag{3.188}$$

Upon deforming further the integrals along $\mathbb{R}\pm\mathrm{i}\widetilde{\varkappa}_{\eta}$ and picking the residues of the simple pole at $\pm\mathrm{i}\zeta$ and then applying direct bounds, one eventually gets

$$\Big|\mathcal{H}_{0}^{(3)}\Big| \,\leq\, \frac{C}{N\tau_{N}}\Big\{\tau_{N}\Big[N^{-1}\mathbf{1}_{1<\zeta}+N^{-\varkappa_{\eta}}\Big] \,+\, \tau_{N}\mathrm{e}^{-\zeta(1-\eta)\overline{x}_{N;\alpha}+2\epsilon'\bar{b}_{N;\alpha}}\Big\} \,\leq\, \frac{C}{N}\Big[N^{-1}\mathbf{1}_{1<\zeta}+N^{-\varkappa_{\eta}}\Big]\,. \tag{3.189}$$

The claim then follows by putting the various estimates together.

Conclusion

In this work, we have provided a full characterisation of the equilibrium measure which governs the leading asymptotic expansion of the logarithm of the Lukyanov integral. This allowed us to check by means of explicit calculations the predictions relative to the leading term of the Lukyanov conjecture describing the large-N behaviour of a multiple-integral supposed to provide the lattice regularisation of the vacuum expectation value of the exponential of the field operator in the quantum Sinh-Gordon finite volume R field theory. Our calculations confirm this part of the conjecture. However, a lack of sharp bounds on the remainder, issuing from our incapacity to control, on sufficiently fine scales, the inverse of the master operator arising in the system of loop equations, does not allow us to prove that indeed the other corrections which could contribute to the asymptotics of the derivative $\partial_\alpha \log \mathcal{Z}_N\big[V_{N;\alpha}\big]$ will not do so on a stronger than $\log N$ scale. It would be extremely interesting to develop a much better understanding of the scaling regimes of the master operator appropriate for this setting.

We plan to address these questions, in full rigour, by alternative methods in further works.

Conclusion

"Space and time are the pure forms of sensible intuition; for they are represented, prior to all actual perception, as the conditions of the possibility of external phenomena." Immanuel Kant

Contents

We now give a conclusion and summarize some of the questions that arise from our work.

3.4 Central limit theorem

In Chapter 1, we managed to show a CLT for the real β-ensembles at high temperature. This was done by using some of the tools developped in [HL21] and extending them to a non-compact case. This required some technicalities and some new ideas. An example is the inversion of the master operator shown by using a link between the latter and a Schrödinger operator.

Question 1: Would it be possible to extend the class of test functions allowed by Theorem 1.1.4 ?

Since Theorem 1.1.4 only includes bounded functions while the other CLT's in the high temperature regime [NT18, HL21, MM24] allow for test-functions with polynomial growth, it seems likely that our approach could lead to the same type of conclusion. A possible way to achieve this task would be by specifying a potential (for example, a polynomial with a fixed even degree), but proving this extension with a general V could be challenging. Another way to improve the result could involve reproducing the change of variables in the partition function with finer estimates on the remainders, requiring less regularity on the test function ϕ of the change of variables $x_i \mapsto y_i + tN^{-1/2}\phi(y_i)$. Finally, when extending the set of functions for which the central limit theorem holds, the growth conditions we require on ϕ may not be optimal, and finer estimates could lead to less stringent conditions on the test functions.

Achieving such a task could lead to an explicit expression for the GGE-averages of the currents as it was argued in Subsection I.6.4. Indeed, currently we only know that this expression is valid for bounded functions but the averages of the currents requires the knowledge on the limiting variance $\sigma^2_{\beta-\mathrm{ens,HT}}(V, P, .)$ for polynomials test-functions.

As it is the case in the classical regime, the size of the fluctuations of linear statistics are expected to be different for singular test-function. It would be interesting to understand the fluctuations under the choice of indicator functions as test-functions.

Question 2: Could this analysis be reproduced for singular potentials V ?

It would be interesting to see if it would be possible to include some potentials V which present a singularity at some point. One could think of the potentials of the type $V : x \mapsto |x|^\gamma$ for $1 < \gamma < 2$. These potentials and their first derivative are continuous but their second derivative admit a singularity at the origin. Doing so would require to invert the master operator in that case which is not trivial.

Question 3: Would it be possible to understand the continuity of the variance with respect to the parameter P as in [HL21] ?

We recall that in this article, the authors obtained a CLT similar to our result in the circular case. Although their result allow for a broader class of test-functions, we obtained the same limiting variance $\sigma^2_{\beta-\text{ens,HT}}(V,P,.)$ **[DGM23]** as in their result. They managed to understand the asymptotic behaviour in P of the limiting variance. For all test-functions ψ regular and integrable, their result goes as follows:

$$\sigma^2_{\beta-\text{ens,HT}}(V,P,\psi) \underset{P\to 0}{\longrightarrow} \|\psi\|^2_{L^2(\mu_{V,0})} \overset{(\text{def})}{=} \int_{\mathbb{T}} \psi(x)^2 \frac{e^{-V(x)}}{\int_{\mathbb{T}} e^{-V(y)}dy} dx.$$

This variance corresponds to the limiting variance of the classical CLT for iid particles of law given by $\dfrac{e^{-V(x)}}{\int_{\mathbb{T}} e^{-V(y)}dy}dx$. They also managed to show that, under certain hypotheses on the equilibrium measure; $\mu_{V,P}$, that one recovers the 1/2-Sobolev norm when $P \to +\infty$, *i.e.*:

$$\sigma^2_{\beta-\text{ens,HT}}(V,P,\psi) \underset{P\to+\infty}{\longrightarrow} \|\psi\|^2_{\mathsf{H}^{1/2}} \overset{(\text{def})}{=} 2\sum_{k=1}^{+\infty} k|\widehat{\psi}_k|^2 \qquad \widehat{\psi}_k \overset{(\text{def})}{=} \int_{\mathbb{T}} \psi(x) e^{-\mathrm{i}kx}\frac{dx}{2\pi}.$$

As was shown in [Joh90], this corresponds to the limiting variance of the classical β-ensemble. This can be seen as a consequence of the Gauss-Wigner crossover as one expects that in the $P \longrightarrow 0$ limit, one recovers the equilibrium measure for iid particles, while in the $P \to +\infty$ limit, one should recover the equilibrium measure corresponding to the β-ensemble in the classical regime.

Showing this type of result in our case of the real line would require to have a control on the P-dependence of the equilibrium measure $\mu_{V,P}$. This dependence of $\mu_{V,P}$ has been understood in [GM22] and it would be interesting to see if this knowledge would be enough to adapt the arguments of [HL21, Section 8].

3.5 Asymptotic of the partition function

The inversion of the master operator in Chapter 1 paved the way for the applicabilty of the loop equations analysis method. In Chapter 2, to show the existence of a N^{-1} asymptotic expansion for the logarithm of the partition function for potentials of the form $x^2 + \phi$ with ϕ a bounded smooth function. Since the *a priori* bound shown there is different from the usual ones and includes a stronger norm, it was necessary to obtain more subtle continuity controls on the inverse of the master operator. This allowed me to derive the existence of an asymptotic expansion for the linear statistics. I was finally able to control the dependence with respect to the potential of the equilibrium measure which led to the final result.

Question 4: Would it be possible to adapt the proof of Chapter 2 in order to get more potentials than just the sum of a quadratic potential and a bounded smooth function ?

This is very likely that this AE holds for more general potentials. In many cases, the loop equations analysis was sufficient to include many more potentials like strictly convex polynomial potentials in the classical β-ensemble case [BG13a] or in the sinh-model [BGK16].

Extending this procedure to the class of strictly convex polynomial potentials, $V(x) = x^{2p} + x^2/2$ for example, would require to be able to obtain the AE of the following linear statistics

$$\mathbb{E}_N^{V,P}\left[\int_{\mathbb{R}} x^{2p} d\mathcal{L}_N(x)\right]. \tag{3.190}$$

While the a priori bound obtained in **[DG24]** allowed to deal only with bounded functions, it would be necessary to obtain a similar one for unbounded functions. To state things in a clearer way, we introduce the following exponential regularization, let ϕ_n a function in n real variables,

$$\mathcal{K}_\kappa[\phi](\xi_1,\dots,\xi_n) \overset{(\text{def})}{=} \prod_{a=1}^{n} e^{-\kappa V(\xi_a)}.\phi_n(\xi_1,\dots,\xi_n).$$

Obtaining an *a priori* bound of the form:

$$|\langle \phi_n \rangle_{\overset{n}{\otimes}\mathcal{L}_N}| \leq \frac{C}{N^{\frac{n}{2}-\varepsilon}} \left\{ \mathcal{N}_\kappa^{(n),\infty}[\phi_n] + \mathcal{N}_\kappa^{(n),2}[\phi_n] \right\}$$

where

$$\mathcal{N}_\kappa^{(n),\infty}[\phi_n] \overset{(\text{def})}{=} \|\mathcal{K}_\kappa[\phi_n]\|_{W_n^\infty(\mathbb{R}^n)}, \qquad \mathcal{N}_\kappa^{(n),\infty}[\phi_n] \overset{(\text{def})}{=} \|\mathcal{K}_\kappa[\phi_n]\|_{H^{n/2}(\mathbb{R}^n)},$$

would represent a major step towards proving the existence of an AE for linear statistics with unbounded functions. Even though it does seem reasonable to expect that such a bound hold, I was not able to prove this in **[DG24]**.

The inverse of the master operator should also be continuous with respect to these norms. Indeed, the method that I used to prove the controls with respect to the $H^{n/2}$ and W_n^∞ norms consisted on relying on an integral representation for the inverse Ξ^{-1}. Furthermore, by performing integration by parts and estimating the growth of every building blocks that appear in this description of the derivatives of the inverse, I managed to prove the desired expression. It seems that this procedure could be adapted to show continuity with respect to the norms $\mathcal{N}_\kappa^{(n),\infty}$ and $\mathcal{N}_\kappa^{(n),2}$.

The two steps would be enough to prove the existence of a large-N expansion for the linear statistic (3.190). To conclude about the large-N expansion of $\mathcal{Z}_N^P\left[x^{2p} + x^2/2\right]$, one must be able to prove that, setting $V_t(x) \overset{(\text{def})}{=} tx^{2p} + x^2/2$, the map $t \mapsto \rho_{V_t}$ is continuous with respect to the W_n^∞-norm. Our proof in the case where x^{2p} is replaced by a bounded smooth function ϕ, relied on the fact that $u_t \overset{(\text{def})}{=} \dfrac{\rho_{V_t} - \rho_{V_{t_0}}}{t - t_0} \dfrac{1}{\rho_{V_{t_0}}}$ is the fixed point of a contractive operator with respect to the W_n^∞-norm. The proof for the contractivity property heavily relied on the fact that ϕ was bounded and it would certainly not work anymore when ϕ is replaced by an unbounded function. Perhaps, it is still possible to prove that this operator is contractive but with some other norm, namely a weighted norm of the form $\mathsf{N}[f] \overset{(\text{def})}{=} \left\| \dfrac{f}{1+|V|} \right\|_{W_n^\infty(\mathbb{R})}$.

Solving these 3 problems would again allow one to prove the existence of an all-order AE for $\log \mathcal{Z}_N^P[V]$ for V in larger class that just a the sum of a quadratic potential and a bounded smooth function.

Question 5: Would it be possible to apply these techniques to grasp the asymptotic behaviour of the partition function of β-ensembles in the classical regime but for singular potentials ?

An obstacle for establishing the existence of an AE for $\log \mathcal{Z}_N^{(\beta)}[V]$ is often the lack of regularity of the potential. For example, when $V(x) = |x|^\gamma$ for $1 < \gamma < 2$, one can show that V'' will admit a singularity at zero. This will translate in the fact that the derivative of the equilibrium density $\rho'_{V,\beta}$ has a singularity at zero.

To see why it is an obstacle, one needs to write the first loop equation:

$$\langle \mathcal{D}[\phi] \rangle_{\widehat{\mu}_N \otimes \widehat{\mu}_N} - \langle V' \phi \rangle_{\widehat{\mu}_N} + \frac{\beta^{-1} - 1/2}{N} \langle h' \rangle_{\widehat{\mu}_N} = 0. \tag{3.191}$$

The obstruction in conducting the loop equations analysis comes from the presence of the derivative term, namely the third one. Indeed, when recentring around the usual equilibrium measure $\mu_{V,\beta}$, as explained in the introduction, the master operator $\mathcal{K}$, introduced in (38), will appear. After using $\phi = \mathcal{K}^{-1}[\psi]$ as the test-function in the recentred version (3.191), one will have to deal with a problematic 1-linear statistic given by $\langle \mathcal{K}^{-1}[\psi]' \rangle_{\mathcal{L}_N}$ where $\mathcal{L}_N \overset{(\text{def})}{=} \widehat{\mu}_N - \mu_{V,\beta}$ is the recentred empirical measure. The singularity of the equilibrium measure at zero wil imply a singularity for the inverse of the master operator acting on any function. This can be seen *via* (48). It will thus not be possible to apply the *a priori* bound for this function.

One way to overcome this difficulty is to recenter against a N-dependent measure $\mu_{V,\beta}^{(N)}$ with $\mathcal{C}^1$ p.d.f. which $\rho_{V,\beta}^{(N)}$ which satisfies for all h sufficiently regular:

$$0 = \frac{1}{2} \iint_{[a,b]^2} \frac{h(x) - h(y)}{x - y} d\mu_{V,\beta}^{(N)}(x) d\mu_{V,\beta}^{(N)}(y) - \int_a^b h(x) V'(x) d\mu_{V,\beta}^{(N)}(x) - \frac{\beta^{-1} - 1/2}{N} \int_a^b h'(x) d\mu_{V,\beta}^{(N)}(x).$$

This measure includes entropic effects just as the equilibrium measure of the β-ensembles in the high temperature regime but with a N^{-1} ponderation for the entropy. This recentring induces the following change on the loop equation at level 1:

$$0 = \frac{1}{2} \langle \mathcal{D}[h] \rangle_{\widetilde{\mathcal{L}}_N \otimes \widetilde{\mathcal{L}}_N} + \langle \mathcal{K}_N[\phi] \rangle_{\widetilde{\mathcal{L}}_N}, \qquad \widetilde{\mathcal{L}}_N \overset{(\text{def})}{=} \widehat{\mu}_N - \mu_{V,\beta}^{(N)}, \tag{3.192}$$

where the N-dependent master operator $\mathcal{K}_N$ is defined by:

$$\mathcal{K}_N[\phi](x) \overset{(\text{def})}{=} \int_{\mathbb{R}} \frac{h(x) - h(y)}{x - y} d\mu_{V,\beta}^{(N)}(y) - h(x) V'(x) + \frac{\beta^{-1} - 1/2}{N} h'(x). \tag{3.193}$$

This operator can be seen as the master operator in the high temperature Ξ, introduced in (67), but admitting an explicit dependence in N. It could in principle be inverted and controled using the same techniques developped in [**DGM23, DG24**].

Working with this more involved N-dependent equilibrium measure has the advantage of producing a first loop equation (3.192) which doesn't contain problematic terms anymore, *i.e.* a linear statistic involving a derivative.

It is an open question whether one would be able to conduct the loop equations analysis method for singular potentials with this new equilibrium measure and new master operator.

3.6 On the equilibrium measure for the Lukyanov integral

In Chapter 3, we were able to give a strong check of Lukyanov's conjecture. Indeed, relying on [BGK16], we obtain the main properties for the equilibrium measure relative to a specific N-dependent potential including an explicit asymptotic expansion for its endpoints a_N, b_N. This allowed us, under the natural hypothesis that a remainder termis sufficiently small, to see that the renormalization $N^{-\theta}$ for a certain θ in Lukyanov's conjecture is necessary in order to have a well-defined limit leading to an expression for the vacuum expectation value associated to the exponential of the field operator in the sinh-Gordon model in $1+1$ dimension and finite volume.

Question 6: Would it be possible to get rid of the assumption (121)?

We believe that this is doable by using the bi-orthogonal polynomials approach for the sinh-model. Following the RHP approach to bi-orthogonal polynomials developped in [CR13], it would be necessary to adapt the Deift-Zhou non-linear steepest descent in this case, namely with the hyperbolic Vandermonde as well the non-varying weight case. Having obtained several properties about the equilibrium measure, this would allow us to show in a future work that we can indeed get rid of this assumption.

Index of notations

Matrices, operators and transforms

- Tr refers to the usual trace of a matrix or of a trace-class operator.
- det refers to the usual determinant on matrices or to a Fredholm determinant according to context.
- id refers to the identity operator. I_N denotes the $N \times N$ indentity matrix.
- $\mathcal{M}_N(\mathbb{C})$ denotes the space of $N \times N$ matrices with complex entries.
- $O_N(\mathbb{R})$ denotes the orthogonal group of dimension N.
- $\mathcal{U}_N(\mathbb{C})$ denotes the unitary group of dimension N.
- $\mathcal{H}_N(\mathbb{C})$ denotes the space of Hermitian $N \times N$ matrices.
- $SL_N(\mathbb{C})$ denotes the special linear group of dimension N with complex coefficients.
- $SO_N(\mathbb{R})$ denotes the special orthogonal group of dimension N.
- $E(2)$ denotes the Euclidean group of dimension 2.
- $\mathcal{F}$ refers to the Fourier transform which is defined for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, by

$$\mathcal{F}[f](x) \overset{(\mathrm{def})}{=} \int_{\mathbb{R}} f(t) e^{-\mathrm{i}tx} dt.$$

  Note that the minus sign convention is changed into a + sign in Chapter 3.
- $\fint_I$ denotes the Cauchy principal-value on an interval I.
- $\mathcal{H}$ denotes the *Hilbert transform* and is defined for a function $f \in L^2(\mathbb{R})$, by

$$\mathcal{H}[f](x)_{\mathbb{R}} \overset{(\mathrm{def})}{=} \fint \frac{f}{y-x} dy.$$

- C_Σ denotes the Cauchy-transform on a contour $\Sigma \subset \mathbb{C}$ and is defined by

$$\mathsf{C}_\Sigma[f](z) \overset{(\mathrm{def})}{=} \int_\Sigma \frac{f(s)}{s-z} \frac{ds}{2\mathrm{i}\pi} \in \mathcal{O}\left(\mathbb{C} \setminus \Sigma\right).$$

Sets, functional spaces, norms and distances

- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{N}^*$ denotes the set of positive natural numbers.

- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{R}$ denotes the field of the real numbers.
- $\mathbb{C}$ denotes the field of the complex numbers.
- $\mathbb{H}^{\pm}$ denotes the set of complex numbers of imaginary part (strictly) positive/negative.
- $\mathbb{S}^1$ denotes the unit circle in the plane.
- $\mathcal{O}(U)$, where U is an open set of $\mathbb{C}$, denotes the space of holomorphic functions on U.
- $\mathcal{C}^k(U)$, where U is an open set of $\mathbb{R}^n$ and $k \in [\![0, +\infty]\!]$, denotes the space of functions differentiable k times whose k-th derivative is continuous.
- $\mathcal{M}_1(\mathbb{R})$ stands for the space of probability measures on $\mathbb{R}$.
- Let X be a Borelian set, we denote by $L^p(X)$ the space of p-th power integrable functions on X with its borelian σ-algebra and the Lebesgue measure. In case of another measure μ on $\mathbb{R}$, we write $L^p(\mu)$ space. The corresponding norms are defined by:

$$\|u\|_{L^p(X)} \overset{\text{(def)}}{=} \left(\int_X |u(x)|^p dx\right)^{1/p}, \qquad \|u\|_{L^p(\mu)} \overset{\text{(def)}}{=} \left(\int_{\mathbb{R}} |u(x)|^p d\mu(x)\right)^{1/p}$$

- $W_n^{\infty}(\mathbb{R}^p)$ denotes the space of functions f such that $f^{(k)} \in L^{\infty}(\mathbb{R}^p)$ for all $k = 0, \ldots, n$. Its norm is defined by $\|f\|_{W_n^{\infty}(\mathbb{R})} \overset{\text{(def)}}{=} \max_{k \in [\![0,n]\!]} \|f^{(k)}\|_{L^{\infty}(\mathbb{R})}$.
- We define the Sobolev spaces for all $m \geq 0$ by

$$H^m(\mathbb{R}^n) \overset{\text{(def)}}{=} \left\{u \in L^2(\mathbb{R}^n), \|u\|_{H^m(\mathbb{R}^n)} < +\infty\right\}$$

where

$$\|u\|^2_{H^m(\mathbb{R}^n)} \overset{\text{(def)}}{=} \int_{\mathbb{R}^n} (1 + \|t\|_2)^{2m} \left|\mathcal{F}[u](t_1, \ldots, t_n)\right|^2 d^n\underline{t}.$$

Above, $\|.\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$. If $\mu \in \mathcal{M}_1(\mathbb{R})$, we also define the weighted Sobolev space: with respect to μ by

$$H^k(\mu) \overset{\text{(def)}}{=} \left\{u \in L^2(\mu),\, u^{(k)} \in L^2(\mu)\right\}.$$

- D^2 denotes the "distance" on the space of measures defined by (45).

Special functions

- Γ denotes the Euler's Gamma function.
- ζ denotes the Zeta function.
- Ai denotes the Airy function.
- χ denotes a special case of the general Barnes ζ function in the Introduction defined in (27) and the solution of the Riemann-Hilbert problem relative to the sinh-model.
- B_n denotes the n-th Bernoulli polynomial.
- V_G denotes the Gaussian potential $V_G(x) = x^2/2$.

Constants

- γ denotes the Euler-Mascheroni constant.
- $C_{V,\beta}$ denotes the constant appearing in the characterization of $\mu_{V,\beta}$ (14).
- $C_{V,P}$ denotes the constant appearing in the characterization of $\mu_{V,P}$ (62).

Multiple integrals

- $\mathcal{Z}_N^{(\beta)}[V]$ denotes the partition function of the classical β-ensembles with at inverse temperature $\beta \geq 0$ and potential V defined in (10).
- $\mathcal{Z}_N[V]$ denotes the partition function of the β-ensembles in the high temperature regime with potential V defined in (59). The dependence in $P \geq 0$ is implicit here.

Measures

- $d^N\lambda$ denotes the Lebesgue measure on $\mathbb{R}^N$ with integration variables $(\lambda_1, \ldots, \lambda_N)$.
- $\mu_{V,\beta}$ denotes the equilibrium measure of the classical β-ensembles with parameter β and potential V defined in Theorem I.3.1. Its p.d.f. is denoted by $\rho_{V,\beta}$.
- $\mu_{V,P}$ denotes the equilibrium measure of β-ensembles in the large temperature regime with parameter P and potential V defined in Theorem I.5.1. Its p.d.f. is denoted by $\rho_{V,P}$. The measure is denoted by μ_V and its p.d.f. ρ_V in Chapter 1 and 2.
- μ_{sc} denotes the semi-circle distribution whose pdf is given in Subsection I.3.2.
- $\mu_{\mathrm{eq}}^{(N)}$ denotes the N-dependent equilibrium measure in the sinh-model.
- According to context, $\mathcal{L}_N$ denotes the recentred distribution $\widehat{\mu}_N - \mu_{V,\beta}$, $\widehat{\mu}_N - \mu_{V,P}$ or $\widehat{\mu}_N - \mu_{\mathrm{eq}}^{(N)}$.
- $\bigotimes_{i=1}^{n} \mu_i$ denotes the product measure of the μ_i's.
- $\mathcal{N}(m, \sigma^2)$ denotes the normal distribution with mean $m \in \mathbb{R}$ and variance $\sigma^2 > 0$.
- χ_α denotes the χ-distribution of parameter $\alpha > 0$.

Functionals

- $\mathcal{E}_{V,\beta}$ is the functional defined in (13) involved in the LDP for the classical β-ensembles.
- $\mathcal{E}_{V,P}$ is the functional defined in (61) involved in the LDP for the β-ensembles in the high temperature regime. It is sometimes simply denoted $\mathcal{E}$.

Appendix A

Functional analysis

This appendix is based on the following references [Ree12, RS03, RS78, Kat12, HS12].

3.1 Unbounded operators

3.1.1 First definitions

The theory of *unbounded operators* has been developed in the 1920-30's by Von Neumann and Stone to overcome functional analysis developed by Banach, Hahn, Helly etc... One reason for developing this theory comes from putting the ground for a mathematically rigorous framework for quantum mechanics.

Definition 3.1.1 *Let X and Y two Banach spaces, an **unbounded operator** is a linear $T : \mathcal{D}(T) \longrightarrow Y$ where $\mathcal{D}(T)$ is a vector subspace of X and is called the **domain** of T. If $\mathcal{D}(T)$ is dense in X, we say that T is **densely defined**.*

Typical unbounded operators are differential operator, for example $T \overset{(\text{def})}{=} \dfrac{d}{dx}$, which deteriorate the smoothness of the function it acts on.

A notion that is very useful when considering unbounded operators is the closed property. For that we define the ***graph*** $\Gamma(T)$ of an unbounded $T : X \longrightarrow Y$ as follows:

$$\Gamma(T) \overset{(\text{def})}{=} \{(x, T(x)) \in X \times Y, x \in \mathcal{D}(T)\}$$

Definition 3.1.2 *Let $T : X \longrightarrow Y$ an unbounded operator, we say that T is **closed** if its graph $\Gamma(T)$ is closed for the product norm $\|(x, y)\|_{X\times Y} \overset{(\text{def})}{=} \|x\|_X + \|y\|_Y$.*

Namely this tells us that for each $(x_n, y_n) \in X^{\mathbb{N}} \times Y^{\mathbb{N}}$, such that there $x_n \underset{n\to\infty}{\longrightarrow} x$ and $Tx_n \underset{n\to\infty}{\longrightarrow} y$, then $x \in \mathcal{D}(T)$ and $Tx = y$. It is also equivalent to $\mathcal{D}(T)$ being complete for the ***graph norm*** defined by $\|x\|_T \overset{(\text{def})}{=} \sqrt{\|x\|_X^2 + \|Tx\|_Y^2}$. Furthermore, if T is not closed but ***closable***, in the sense that $\overline{\Gamma(T)}$ is the graph a of an unbounded operator $\overline{T}$ (therefore automatically closed), we call $\overline{T}$ the ***closure*** of T.

It can be seen that for a closable operator T, the domain of its closure $\mathcal{D}\left(\overline{T}\right)$ is exactly the completion of $\Gamma(T)$ with respect to its graph norm $\|.\|_T$.

Example 3.1.3

- We take $X = Y = L^p(\mathbb{R}^d)$ for $p \geq 1$, $d \in \mathbb{N}^*$ and $f \in X$. We define the multiplication operator $M_f : g \in \mathcal{D}(M_f) \longrightarrow fg$ where

$$\mathcal{D}(M_f) \overset{(\text{def})}{=} \{g \in X, fg \in X\} = L^p\left(\mathbb{R}^d, 1 + |f|^p\right).$$

By the last equality, it is direct to conclude that $\mathcal{D}(M_f)$ with its graph norm is complete hence M_f is closed.

- The second example is about the Laplacian in $\mathbb{R}^d$. With the following notations, we take $X = Y = L^2(\mathbb{R}^d)$ and we define:

$$T_0 u \overset{(\text{def})}{=} \Delta u \text{ for all } u \in \mathcal{D}(T_0) \overset{(\text{def})}{=} \mathcal{C}_c^\infty(\mathbb{R})$$

$$T_1 u \overset{(\text{def})}{=} \Delta u \text{ for all } u \in \mathcal{D}(T_1) \overset{(\text{def})}{=} H^2(\mathbb{R}^d)$$

One can show that T_1 is the closure of T_0 using that $H^2(\mathbb{R}^d)$ can be seen as the completion of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with respect to the $H^2(\mathbb{R}^d)$-norm.

The following property is obvious.

Proposition 3.1.4 *Let $T : X \longrightarrow Y$ an unbounded closed operator, then*

$$T \text{ is a bounded operator} \qquad \text{iff} \qquad \mathcal{D}(T) = X.$$

3.1.2 Adjoint

Now, we would like to state an unbounded version of the spectral theorem. To do so, we need to put ourselves in an Hilbert space $\mathcal{H}$ with its inner product $\langle ., . \rangle$ and define a notion of adjointness for an operator that is not defined everywhere.

Definition 3.1.5 *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ an unbounded operator, we say the unbounded operator $T^* : \mathcal{H} \longrightarrow \mathcal{H}$ is the **adjoint** of T if for all $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$ it holds that*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle .$$

The domain of T^* is defined as follows

$$\mathcal{D}(T^*) \overset{(\text{def})}{=} \left\{ y \in \mathcal{H},\, x \in \mathcal{D}(T) \longmapsto \langle Tx, y \rangle \in \mathcal{H}' \right\}$$

where $\mathcal{H}'$ is the topological dual of $\mathcal{H}$.

We need to make several comments. First, one should notice that this is exactly the same definition as in the bounded case (*i.e.*, we define T^*y as the Riesz's representant of the linear form $x \mapsto \langle Tx, y \rangle$, except that in our case Riesz's theorem would fail to be applied for every $x \in \mathcal{H}$.) Secondly, in the definition of $\mathcal{D}(T)$, we wrote " $\in$ " for "‘can be extended into a continuous linear form on $\mathcal{H}$‘. Thirdly, one should notice that the adjoint T^* of a unbounded closed operator T exists if and only if T is densely defined since otherwise T^*y is not uniquely determined. Indeed, one could add to definition of T^*y, any $z \in \mathcal{D}(T)^\perp$. Thus in the following, we only consider densely defined unbounded operators T.

Proposition 3.1.6 *Let T a densely defined unbounded operator on $\mathcal{H}$, then T^* is a closed densely defined operator. Furthermore if T is closable then, T^* is densely defined.*

Now that we reviewed the basic construction of adjoints for unbounded operators, we can state the subtleties that arise when considering self-adjoint unbounded operators.

Definition 3.1.7 *Let T be a densely defined unbounded operator on $\mathcal{H}$, we say that T is **symmetric** if for all $u, v \in \mathcal{D}(T)$,*

$$\langle Tu, v \rangle = \langle u, Tv \rangle .$$

This definition is equivalent to T^* being an extension of T in the sense that $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and that on $\mathcal{D}(T)$, $T^* = T$. Furthermore, we say that T is ***self-adjoint*** if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $T = T^*$, then if T is self-adjoint, it is necessary closed. When it is only the closure of T, $\overline{T}$ that is self-adjoint then T is said to be ***essentially self-adjoint***.

Example 3.1.8
- The multiplication operator M_f on $L^2(\mathbb{R}^d)$ defined above is self-adjoint if and only if f takes real values almost everywhere.
- For the Laplacian on $\mathbb{R}^d$, it is direct to see that T_0 is symmetric and that T_1 is self-adjoint. T_1 being the closure of T_0 allows us to conclude that T_0 is only essentially self-adjoint.

Proposition 3.1.9 *Let T be a self-adjoint injective unbounded operator on $\mathcal{H}$, then $T^{-1} : T(\mathcal{D}(T)) \longrightarrow \mathcal{D}(T)$ is also self-adjoint.*

3.1.3 Spectral theorem for unbounded operators

In the following $\mathcal{H}$ denotes a Hilbert space.

Definition 3.1.10 *A projection-valued measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a map $\nu : \mathscr{B}(\mathbb{R}) \longrightarrow \mathcal{B}(\mathcal{H})$ such that*

i) *For all $A \in \mathscr{B}(\mathbb{R})$, $\nu(A)$ is a projection on $\mathcal{H}$.*

ii) *$\nu(\emptyset) = 0$, $\nu(\mathbb{R}) = Id_{\mathcal{H}}$.*

iii) *Let $(A_n)_{n\in\mathbb{N}}$ a sequence of disjoint sets in $\mathscr{B}(\mathbb{R})$, then*

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n).$$

The spectral theorem in the bounded compact case states that there exists an orthonormal basis $(\psi_j)_{j\in\mathbb{N}}$ composed of eigenfunctions for the operator T with corresponding real eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$T = \sum_{j\in\mathbb{N}} \lambda_j \langle ., \psi_j \rangle \psi_j.$$

This can be seen as an integral over the spectrum of T with respect to a projection-valued measure ν_T such that $\nu_T(\lambda_j) = \langle ., \psi_j \rangle \psi_j$ which is indeed the orthogonal projection on $\mathbb{C}.\psi_j$. We state the general version, when the spectrum might be continuous and hence the support of the projection-valued measure being non-discrete, below.

Theorem 3.1.11 *Let T be an unbounded self-adjoint operator on $\mathcal{H}$, then there exists projection-valued measure ν_T such that*

$$T = \int_{\mathbb{R}} x d\nu_T(x).$$

*The projection-valued measure ν_T is called the **spectral measure** of T.*

3.2 Form domains and Friedrichs extensions

This Section is dedicated to the description of a technique in operator theory called the *Friedrichs extension* and is based on a [Non21]. The technique can be heuristically explained as below.

If $(T, \mathcal{D}(T))$ is a symmetric, bounded by below operator on a Hilbert-space $\mathcal{H}$, then one can define an associated closable quadratic form $(q, \mathcal{D}(q))$ which is also bounded by below. By closable, we mean that there exist an extension, $(\overline{q}, \mathcal{D}(\overline{q}))$ of $(q, \mathcal{D}(q))$ which is closed. Finally, from this *closed* form one can, by Riesz's theorem, obtain an extension, $(\overline{T}, \mathcal{D}(\overline{T}))$, of $(T, \mathcal{D}(T))$, which is self-adjoint. This procedure is summarized as below:

$$(T, \mathcal{D}(T)) \longrightarrow (q, \mathcal{D}(q)) \longrightarrow (\overline{q}, \mathcal{D}(\overline{q})) \longrightarrow (\overline{T}, \mathcal{D}(\overline{T})).$$

3.2.1 Preliminaries

Definition 3.2.1 *A quadratic form $(q, \mathcal{D}(q))$ is said to be **closed** if the quadratic form is symmetric and bounded from below, i.e. there exists $C \in \mathbb{R}$, such that for all $u \in \mathcal{D}(q)$,*

$$q(u,u) \geq C\|u\|^2_{\mathcal{H}},$$

and such that $\mathcal{D}(q)$ equipped with the scalar product

$$\langle u, v\rangle_q \overset{(\text{def})}{=} q(u,v) + (C+1)\langle u, v\rangle_{\mathcal{H}}$$

*is a Hilbert-space. $(q, \mathcal{D}(q))$ is said to be **closable** if there exists a closed form $(\overline{q}, \mathcal{D}(\overline{q}))$ which extends it. In that case, we say that the latter is the closure of $(q, \mathcal{D}(q))$.*

Definition 3.2.2 *A quadratic form $(q, \mathcal{D}(q))$ is said to be **bounded** on $\mathcal{H}$ if $\mathcal{D}(q) = \mathcal{H}$ and such that there exists $C > 0$ such that for all $u, v \in \mathcal{H}$,*

$$|q(u,v)| \leq C.\|u\|_{\mathcal{H}}.\|v\|_{\mathcal{H}}.$$

*Furthermore, such a form is said to be **coercive** if it is bounded and there exists $\alpha > 0$, such that for all $u \in \mathcal{H}$,*

$$|q(u,u)| \geq \alpha.\|u\|^2_{\mathcal{H}}.$$

Furthermore, one can associate a quadratic form q_T associated to a general operator T

Proposition 3.2.3 *If $(T, \mathcal{D}(T))$ is a symmetric, bounded from below operator, then $(q_T, \mathcal{D}(T))$ its associated quadratic form defined for all $u, v \in \mathcal{D}(T)$, by $q_T(u,v) \overset{(\text{def})}{=} \langle T[u], v\rangle$, is bounded from below and closable.*

3.2.2 Extension of unbounded operators

In the present appendix, although the method is very general, one can think of T as the one-dimensional Schrödinger operator, *i.e.* for all $u \in \mathcal{C}^\infty_c(\mathbb{R})$, $T[u](x) = -u''(x) + (x^2+1)u(x)$. Suppose $\mathcal{H}$ is a dense Hilbert-space in a bigger Hilbert-space $\overline{\mathcal{H}}$. We define the following Hilbert-space $\overline{\mathcal{H}} = L^2(\mathbb{R})$.

Definition 3.2.4 *A symmetric operator $(T, \mathcal{D}(T))$ is said to be bounded by below if there exists $c \in \mathbb{R}$, such that for all $u \in \mathcal{D}(T)$,*

$$\langle T[u], u\rangle_{\overline{\mathcal{H}}} \geq c\|u\|^2_{\overline{\mathcal{H}}}.$$

One can verify that, our example is indeed bounded by below:

$$\langle T[u], u\rangle_{\overline{\mathcal{H}}} \overset{(\text{def})}{=} -\int_{\mathbb{R}} u''(x)u(x)dx + \int_{\mathbb{R}} x^2u(x)^2dx = \|u'\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}(x^2+1)u(x)^2dx \geq \|u\|^2_{L^2(\mathbb{R})}.$$

One thus considers its associated quadratic form defined for all $u \in D(q) \overset{(\text{def})}{=} \mathcal{C}^\infty_c(\mathbb{R})$ by

$$q(u,u) \overset{(\text{def})}{=} \langle T[u], u\rangle_{L^2(\mathbb{R})} = \|u'\|_{L^2(\mathbb{R})} + \int_{\mathbb{R}}(x^2+1)u(x)^2dx.$$

Furthermore, $(T, \mathcal{D}(T))$ is symmetric and bounded from below, hence $(q, \mathcal{D}(q))$ is closable and we can consider its closure $(\overline{q}, \mathcal{D}(\overline{q}))$ by Proposition 3.2.3. One can show that in our case, $\mathcal{D}(\overline{q}) \overset{(\text{def})}{=} \left\{u \in H^1(\mathbb{R}),\ |x|u \in L^2(\mathbb{R})\right\}$. In the following, we set $\mathcal{H} \overset{(\text{def})}{=} \mathcal{D}(\overline{q})$ equipped with its scalar product $\langle u, v\rangle_{\mathcal{H}} \overset{(\text{def})}{=} \overline{q}(u,v)$. $\overline{q}$ is indeed a scalar product it is coercive and symmetric.

We are now able to extend the initial operator $(T, \mathcal{D}(T))$ into a new operator $(\overline{T}, \mathcal{D}(\overline{T}))$.

Definition 3.2.5 *Let $(\overline{q}, \mathcal{D}(\overline{q}))$ be a quadratic form on $\mathcal{H}$. Then, there exists an operator $(T_q, \mathcal{D}(T_q))$ associated to q such that*

$$\mathcal{D}(T_q) \overset{(\text{def})}{=} \left\{ u \in \overline{\mathcal{H}},\, v \mapsto q(u,v) \text{ can be extended into a continuous linear form on } \overline{\mathcal{H}} \right\}$$

For all $u \in \mathcal{D}(T_q)$, $T_q[u]$ is defined as the Riesz's representant of the extension $v \mapsto q(u,v)$ in a continuous linear form.

Thus one is able to

Theorem 3.2.6 *Let $(T, \mathcal{D}(T))$ be a symmetric operator on $\overline{\mathcal{H}}$, consider $(\overline{q}, \mathcal{D}(\overline{q}))$ the closure of the quadratic form $(q_T, \mathcal{D}(q_T))$ associated to $(T, \mathcal{D}(T))$ as in Proposition 3.2.3. The domain $\mathcal{D}(q_T)$ is called the* ***form domain*** *of T and is denoted $\mathcal{Q}(T)$. Finally, the self-adjoint operator $(\overline{T}, \mathcal{D}(\overline{T}))$ associated to $(\overline{q}, \mathcal{D}(\overline{q}))$ is called the* ***Friedrichs extension*** *of T.*

Furthermore, by enoying the coercivity of $\overline{q}$, it is possible to state nice properties about its Friedrichs extension. It is summarized in the following theorem.

Theorem 3.2.7 *Suppose that $(\overline{q}, \mathcal{D}(\overline{q}))$ is a closed coercive quadratic form on $\mathcal{H}$ and that for all $u \in \mathcal{H}$, $\|u\|_{\mathcal{H}} \geq \|u\|_{\overline{\mathcal{H}}}$. Then, the operator $(T_q, \mathcal{D}(T_q))$ constructed in Definition 3.2.5 satisfies:*

- *$\mathcal{D}(T_q)$ is dense in $\overline{\mathcal{H}}$.*
- *$T_q : \mathcal{D}(T_q) \to \overline{\mathcal{H}}$ is bijective.*
- *T_q^{-1} is a bounded operator on $\overline{\mathcal{H}}$.*

Appendix B

Large deviation principles

One can find more details in the excellent books [RAS15, DZ98]. In this appendix, we fix $(\Omega, \mathcal{F}, \mathcal{P})$ a probability space.

Definition 3.0.1 *Let $\mathcal{X}$ be a topological space. A function $f : \mathcal{X} \to [-\infty, +\infty]$ is lower semi-continuous if for all $c \in \mathbb{R}$, $\{f \leq c\}$ is a closed susbet of $\mathcal{X}$.*

Definition 3.0.2 *Let $I : \mathcal{X} \to [0, +\infty]$ be a lower semi-continuous function and (r_N) be a positive sequence such that $r_N \underset{N\to\infty}{\longrightarrow} +\infty$. A sequence of probability measures $(\mu_N)_N$ in $\mathcal{M}_1(\mathcal{X})$ is said to satisfy a large deviation principle at speed r_N with rate function I if the following inequalities hold for all $A \subset \mathcal{X}$:*

$$-\inf_{x\in\dot{A}} I(x) \leq \liminf_{N\to\infty} \frac{1}{r_N} \log \mu_N(\dot{A}) \leq \limsup_{N\to\infty} \frac{1}{r_N} \log \mu_N(\overline{A}) \leq -\inf_{x\in\overline{A}} I(x). \tag{B.1}$$

We will just denote it by $\mathrm{LDP}(\mu_N, r_N, I)$ *in such a case.*

Definition 3.0.3 (Exponential tightness) *We say that a sequence of probability measures $(\mu_N)_N$ of $\mathcal{M}_1(\mathcal{X})$ is exponentially tight at scale u_N if for all $M > 0$, there exists a compact set K_M of $\mathcal{X}$ such that*

$$\lim_{N\to+\infty} \frac{1}{u_N} \log \mu_N\left(K_M^c\right) < -M. \tag{B.2}$$

The first result that we recall is the famous Sanov's theorem. The statement is a LDP for the law of the empirical distribution $\widehat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$ for a sequence of iid $\mathcal{S}$-valued random variables. We denote by ρ_N the law of the random element $\widehat{\mu}_N$, *i.e.* for all Borel subset $B \subset \mathcal{M}_1(\mathcal{S})$[1],

$$\rho_N(B) \overset{(\mathrm{def})}{=} \mathbb{P}\left(\widehat{\mu}_N \in B\right). \tag{B.3}$$

Theorem 3.0.4 (Sanov's theorem) *Let $\mathcal{S}$ be a Polish space, $\lambda \in \mathcal{M}_1(\mathcal{S})$ and a sequence of iid $\mathcal{S}$-valued random variables $(X_i)_{i\geq 1}$ of law λ. Then* $\mathrm{LDP}(\rho_N, N, H)$ *holds on the space $\mathcal{M}_1(\mathcal{S})$ with the convex rate function defined for all $\nu \in \mathcal{M}_1(\mathcal{S})$ by $H(\nu) \overset{(\mathrm{def})}{=} H(\nu|\lambda)$ where $\mathcal{H}(\nu|\lambda)$ is the relative enropy of ν with respect to λ defined by*

$$\mathcal{H}(\nu|\lambda) \overset{(\mathrm{def})}{=} \begin{cases} \displaystyle\int_{\mathcal{S}} \log \frac{d\nu}{d\lambda}(x) d\nu(x) & if \ \ \nu \ll \lambda, \\ +\infty & otherwise. \end{cases} \tag{B.4}$$

Theorem 3.0.5 (Varadhan's lemma) *Let (μ_N) be a sequence in $\mathcal{M}_1(\mathcal{X})$ such that* $\mathrm{LDP}(\mu_N, r_N, I)$ *for $r_N \underset{N\to\infty}{\longrightarrow} +\infty$ and I a rate function. Let $f : \mathcal{X} \to [-\infty, +\infty]$ be a bounded continuous function then,*

$$\lim_{N\to\infty} \frac{1}{r_N} \log \int_{\mathcal{X}} e^{r_N f(x)} d\mu_N(x) = \sup_{x, f(x)\wedge I(x) < +\infty} \Big(f(x) - I(x) \Big). \tag{B.5}$$

[1] This set is itself a Polish space when equipped the weak topology generated by $\mathcal{C}_b(\mathcal{S})$

The next result is about a sort of converse of Varadhan's lemma.

Theorem 3.0.6 (Bryc's inverse lemma) *Let $(\mu_N)_N$ be an exponentially tight at scale u_N sequence of probability measures of $\mathcal{M}_1(\mathcal{S})$ such that for all bounded continuous function f,*

$$\Lambda_f \overset{\text{(def)}}{=} \lim_{N\to\infty} \frac{1}{u_N} \log \int_{\mathcal{S}} e^{u_N f(x)} d\mu_N(x) \quad \text{exists.} \tag{B.6}$$

Then $\mathrm{LDP}(\mu_N, u_N, I)$ *holds with*

$$I(x) \overset{\text{(def)}}{=} \sup_f \{f(x) - \Lambda_f\} \tag{B.7}$$

where the supremum runs over all bounded continuous functions.

Appendix C

Riemann-Hilbert problems

Excellent reviews can be found in [Dei99, Chapter 1], [AF03, Chapter 7] and [Gak90]. We recall that given an oriented contour $\Sigma \subset \mathbb{C}$, we call the plus (respectively minus) side , the left (resp. right) side as one traverses the contour in the direction of the arrow. We define the *boundary values* of a function f defined in a neighborhood of Σ, when these objets are well-defined by,

$$\forall z \in \Sigma,\ f_{\pm}(z) \overset{(\text{def})}{=} \lim_{\substack{z' \to z \\ z' \in \pm \text{ side}}} f(z'). \tag{C.1}$$

Definition 3.0.1 (Riemann-Hilbert problem) *A scalar Riemann-Hilbert problem (RHP) requires in the data of an oriented contour $\Sigma \subset \mathbb{C}$ and a function g defined on Σ. This determines a RHP as follows: Seek for the set of solutions f satisfying the following conditions:*

- $f \in \mathcal{O}(\mathbb{C} \setminus \Sigma)$
- $f_+(z) = f_-(z)g(z)$ *for all* $z \in \Sigma$ *(Jump condition)*

The definition above is scalar but it is often the case that one seeks for matrix valued solutions. Also the jump condition we gave is multiplicative but one could also take an additive jump condition. One usually adds an asymptotic condition at infinity such as $f(z) = h(z) + o(z^{-1})$. This condition usually ensures the uniqueness of solution of the RHP.

The next definition concerns the so-called standard RHP, the ones that are easier to solve.
Definition 3.0.2 *A standard RHP is of the following form, given* (Σ, g)

- $f \in \mathcal{O}(\mathbb{C} \setminus \Sigma)$,
- $f_+(z) = f_-(z) + g(z)$ *for all* $z \in \Sigma$,
- $f(z) = O(z^{-1})$ *as* $z \to \infty$.

Definition 3.0.3 (Hilbert transform) *The Hilbert transform defined for all $f \in \mathcal{C}_c^\infty(\Sigma)$ by*

$$\forall x \in \mathbb{R},\ \mathcal{H}[f](x) \overset{(\text{def})}{=} \fint_{\Sigma} \frac{f(s)}{s-x} ds \tag{C.2}$$

and extended as a bounded operator on $L^2(\Sigma, ds)$.
Definition 3.0.4 (Cauchy operator,[Cal77]) *Let Σ be a self-intersecting Lipschitz curve in $\mathbb{C}$, we define the Cauchy operator C_Σ acting on $L^2(\Sigma, ds)$ by*

$$\forall f \in L^2(\Sigma, ds) \qquad \mathsf{C}_\Sigma[f](z) \overset{(\text{def})}{=} \int_{\Sigma} \frac{f(s)}{s-z} \frac{ds}{2\mathrm{i}\pi} \in \mathcal{O}(\mathbb{C} \setminus \Sigma) \tag{C.3}$$

For any $f \in L^2(\Sigma, ds)$, $\mathsf{C}_\Sigma[f]$ admits $L^2(\Sigma, ds)$ boundary values $\mathsf{C}_{\Sigma,\pm}[f]$. The operators $\mathsf{C}_{\Sigma,\pm}$ are bounded operators on $L^2(\Sigma, ds)$ which satisfies the Sokhotski–Plemelj formulas, namely

$$\mathsf{C}_{\Sigma,+}[f] - \mathsf{C}_{\Sigma,-}[f] = f, \qquad \mathsf{C}_{\Sigma,+}[f] + \mathsf{C}_{\Sigma,-}[f] = \frac{1}{\mathrm{i}\pi}\mathcal{H}[f]. \tag{C.4}$$

Proposition 3.0.5 *A standard RHP admits a unique solution given by $f \overset{(\mathrm{def})}{=} \mathsf{C}_\Sigma[g]$.*

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