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Julien Dardennes. Symplectic non-convexity of toric domains. Mathematics [math]. Université de Toulouse, 2024. English. NNT : 2024TLSES102 . tel-04760561

HAL Id: tel-04760561

<https://theses.hal.science/tel-04760561v1>

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Doctorat de l'Université de Toulouse

préparé à l'Université Toulouse III - Paul Sabatier

Non-convexité symplectique des domaines toriques

Thèse présentée et soutenue, le 10 juin 2024 par

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École doctorale

EDMITT - Ecole Doctorale Mathématiques, Informatique et Télécommunications de Toulouse

Spécialité

Mathématiques et Applications

Unité de recherche

IMT : Institut de Mathématiques de Toulouse

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Remerciements

Certains disent que je ne suis pas très doué pour remercier, dans ces quelques paragraphes je vais tâcher de faire de mon mieux. Tout d'abord, je voudrais remercier mon directeur de thèse Jean Gutt. Ton accompagnement tout au long de cette thèse a été d'un grand soutien et je te remercie de m'avoir fait découvrir les questions passionnantes sur lesquelles nous avons travaillé. Merci enfin pour ta bienveillance, ce fut une chance de t'avoir comme directeur de thèse.

Then, I would like to greatly thank Alberto Abbondandolo and Yaron Ostrover for their precious comments on the first version of this manuscript and for doing me the honor of being the rapporteurs of this thesis. I am also greatly thankful to Margherita Sandon, Felix Schlenk and Jean-François Barraud for accepting to be part of the jury.

I am grateful for the support that the Institut de Mathématiques de Toulouse provided me during this thesis and in particular for the PhD grant from MESRI. I acknowledge the support from the ANR CoSy which allowed me to attend many intellectually stimulating conferences.

Finally, I would like to thank warmly my collaborators. Vinicius, thank you for being so responsive and helpful. Jun, it was a real pleasure to work by your side, your advice and constant support were invaluable.

De retour en français, mes premières pensées vont vers Jean-François. Depuis la L3 où tu as accepté de superviser mon stage, jusqu'à la fin de ma thèse, tu as été d'un énorme soutien. Tu es la raison principale pour laquelle je me suis intéressé à la géométrie symplectique. Dès ce premier stage qui portait alors seulement sur l'homologie persistante, tu as su me transmettre ta passion pour le domaine et l'a entretenu tout au long des nombreuses conversations que nous avons eu depuis. Merci pour tout.

Mes pensées vont aussi vers mes anciens enseignants de mathématiques et de physique qui ont été nombreux à me pousser jusqu'ici. Tout particulièrement Anne Mounier à Castres qui est, et restera, un modèle d'enseignement pour moi.

Merci à mes mentors "presque" symplecticiens Simon Jubert et Florian Bertuol pour les nombreux conseils à propos du doctorat mais aussi à Hugo Martin. Merci

aux camarades doctorants: Robin, Nicolas, Romain, Jordan, Francesca, Athmane, Bruno, Benjamin et bien d'autres.

Merci aux amis de Rennes. Rosa et JB pour l'escalade et l'opéra. La coloc à l'avoine: Chemineau pour les séances de gainage et les rougails saucisse, mon second potto Mathias sans qui cette thèse aurait pu se finir plus tôt. Puis Hermès pour le lit gigogne et Julian toujours entre tradition et modernité dans le lit japonais.

Sans oublier Paolo, Thierry, Quentin, les chercheurs de plaques symétriques Antonin et Manu, Benoit pour les 2step. Mais aussi Mathieu, Raphael, Louison, Sandra, Clara, Vivien, Emmat, Liza, Alice, Anis, Anousone et tous ceux que j'oublie...

Finalement je tiens à remercier ma famille. Papi d'Aix, qui m'a encouragé à poursuivre ces études, j'en suis sûr que tu aurais été fier de me voir arriver jusqu'ici. Papi de Six fours, qui m'a transmis sa passion pour le savoir, les sciences et bien plus. Nos appels hebdomadaires sont inestimables. Merci aussi à mami, Arnaud, Aurélie (et Léon), Régis, Sylvie, Pauline, Quentin et Éric. Papa et maman, merci infiniment pour votre soutien ces 26 dernières années, sans vous rien n'aurait été possible, je vous en suis infiniment reconnaissant.

Et merci Capucine, t'es belle dans tes combats.

Résumé

La convexité joue un rôle particulier en géométrie symplectique, pourtant ce n'est pas une notion invariante par symplectomorphisme. Dans un article fondateur, Hofer, Wysocki et Zehnder ont montré que tout domaine fortement convexe est dynamiquement convexe, une notion, qui elle, est invariante par symplectomorphisme. Depuis plus de vingt ans, l'existence ou non de domaines dynamiquement convexes qui ne sont pas symplectomorphes à un convexe est restée une question ouverte. Récemment, Chaidez et Edtmair ont répondu à cette question en dimension 4. Ils ont établi un critère "quantitatif" de convexité symplectique puis ont construit des domaines dynamiquement convexes qui ne vérifient pas ce critère. Dans cette thèse, nous utilisons ce critère pour construire de nouveaux exemples de tels domaines en dimension 4, qui ont la propriété additionnelle d'être torique. De plus, nous estimons les constantes intervenant dans ce critère. Ce travail en collaboration avec Jean Gutt et Jun Zhang a été ensuite utilisé par Chaidez et Edtmair pour résoudre la question initiale en toute dimension. Dans un second temps, en collaboration avec Jean Gutt, Vinicius G.B.Ramos et Jun Zhang, nous étudions la distance des domaines dynamiquement convexes aux domaines symplectiquement convexes. Nous montrons qu'en dimension 4, celle-ci est arbitrairement grande aux yeux d'un analogue symplectique de la distance de Banach-Mazur. Au passage, nous reprovons de manière indépendante l'existence de domaines dynamiquement convexes non symplectiquement convexes en dimension 4.

Abstract

Convexity plays a special role in symplectic geometry, but it is not a notion that is invariant by symplectomorphism. In a seminal work, Hofer, Wysocki and Zehnder showed that any strongly convex domain is dynamically convex, a notion that is invariant by symplectomorphism. For more than twenty years, the existence or not of dynamically convex domains that are not symplectomorphic to a convex domain has remained an open question. Recently, Chaidez and Edtmair answered this question in dimension 4. They established a "quantitative" criterion of symplectic convexity and constructed dynamically convex domains that do not satisfy this criterion. In this thesis, we use this criterion to construct new examples of such domains in dimension 4, which have the additional property of being toric. Moreover, we estimate the constants involved in this criterion. This work in collaboration with Jean Gutt and Jun Zhang was later used by Chaidez and Edtmair to solve the initial question in all dimensions. Furthermore, in collaboration with Jean Gutt, Vinicius G.B.Ramos and Jun Zhang, we study the distance from dynamically convex domains to symplectically convex domains. We show that in dimension 4, this distance is arbitrarily large with respect to a symplectic analogue of the Banach-Mazur distance. Additionally, we independently reprove the existence of dynamically convex domains that are not symplectically convex in dimension 4.

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Introduction en français

Quand j'étais petit je voulais être joueur de tennis ou boulanger. Vingt ans plus tard, je me pose la question suivante:

Existe-t-il un analogue symplectique à la convexité?

Cette thèse est dédiée à l'étude de ce problème.

Géométrie symplectique

Historiquement, la géométrie symplectique tire ses prémices de l'étude de la mécanique céleste au XIXème siècle par Joseph Louis Lagrange et son étudiant Simon-Denis Poisson. Elle fournit le cadre géométrique adapté à l'espace des phases d'un système mécanique (théorème de Liouville), et s'est depuis développée en tant que sujet d'étude à part entière.

Formellement parlant, la *géométrie symplectique* est l'étude des *variétés symplectiques*, c'est à dire la donnée d'un couple (X, ω) où X est une variété différentielle lisse et ω est une 2-forme sur X qui est fermée et non-dégénérée. De part l'hypothèse de non-dégénérescence de la 2-forme, toute variété symplectique (X, ω) est de dimension paire et naturellement munie d'une forme volume ω^n . L'exemple prototypique de variété symplectique et qui sera au centre de l'étude de cette thèse est le suivant:

$$(\mathbb{R}^{2n}, x_1, \dots, x_n, y_1, \dots, y_n) \text{ muni de la 2-forme } \omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$$

En réalité, un théorème de Darboux stipule que toute variété symplectique est localement isomorphe à l'exemple ci-dessus. Dans la suite de cette thèse, on se restreindra aux *domaines* de \mathbb{R}^{2n} , c'est-à-dire la fermeture d'un ouvert connexe de \mathbb{R}^{2n} . Ainsi, l'intérieur de tout domaine hérite naturellement d'une structure symplectique fournie par la forme symplectique standard ω_0 . Pour comprendre cette géométrie, il est naturel de se demander dans quel cas peut on plonger un domaine dans un autre de façon symplectique. Un plongement lisse $\varphi : U \hookrightarrow V$ est dit *symplectique* si

$\varphi^*(\omega_0|_V) = \omega_0|_U$, ce que l'on dénote

$$U \xrightarrow{\text{Symp}} V$$

Dans le cas des domaines, la notion de volume symplectique coïncide avec le volume euclidien à une constante près:

$$\text{Vol}_{\mathbb{R}^{2n}}(A) = \frac{1}{n!} \int_A \omega_0^n = \frac{1}{n!} \text{Vol}_{\omega_0^n}(A)$$

Il s'en suit que toute transformation qui préserve la structure symplectique préserve nécessairement le volume et que l'existence d'un plongement $U \xrightarrow{\text{Symp}} V$ implique l'inégalité suivante

$$\text{Vol}(U) \leq \text{Vol}(V).$$

Cependant, la géométrie symplectique est plus rigide. Cette condition de volume ne suffit pas à garantir l'existence d'un tel plongement. En effet, un résultat de Gromov [Gro85] implique que la boule de rayon r se plonge dans le cylindre de rayon R si et seulement si $r \leq R$. Ce résultat a poussé Gromov [Gro85] ainsi que Ekeland et Hofer [EH89] à introduire la notion de *capacité symplectique*¹: une mesure 2-dimensionnelle symplectique qui saisit de l'information supplémentaire à celle du volume. Dix ans plus tard, Viterbo [Vit00] a énoncé la conjecture suivante à propos de ces capacités:

Conjecture (Viterbo). *Soit U un domaine convexe de \mathbb{R}^{2n} et c une capacité symplectique normalisée, alors:*

$$c(U) \leq (n! \text{Vol}(U))^{1/n}$$

De plus, le cas d'égalité est atteint si et seulement si U est symplectomorphe² à la boule.

Les capacités symplectiques et le volume sont invariants par symplectomorphismes², cependant, la convexité ne l'est pas. Cette remarque soulève naturellement la question suivante qui est le point de départ de cette thèse:

Question A. *Quels sont les domaines symplectomorphes à un convexe? Autrement dit, quels sont les domaines symplectiquement convexes?*

¹Cf Sous-section 1.1.2.

²Un symplectomorphisme est un difféomorphisme qui préserve la structure symplectique (cf Définition 0.1.7).

Géométrie de contact

Pour tenter de répondre partiellement à cette question nous aurons besoin des notions suivantes.

Une *structure de contact* sur une variété Y de dimension $2n - 1$ est la donnée d'un champ d'hyperplans maximale non intégrable ξ . Si, localement, $\xi = \ker(\alpha)$ où α est une 1-forme sur Y , cette condition se traduit par $\alpha \wedge (d\alpha)^{n-1} \neq 0$. Lorsque cette *forme de contact* est définie globalement, on dit que la structure est *coorientée*. Dans ce cas de figure, on associe naturellement à α un champ de vecteur R_α qui est transverse en tout point à la structure de contact. On appelle ce champ de vecteurs la *champ de Reeb* et l'étude de la dynamique de son flot est un sujet de recherche foisonnant en géométrie de contact.

En revenant aux domaines de \mathbb{R}^{2n} , un domaine à bord lisse est dit *étoilé*, si il contient l'origine et le champ de vecteurs radial est transverse au bord du domaine. Dans ce cas, le bord du domaine hérite naturellement d'une structure de contact coorientée et donc d'un champ de Reeb préféré.

État de l'art

Dans [HWZ98], Hofer, Wysocki et Zehnder ont montré que tout domaine fortement convexe³ de \mathbb{R}^4 possède soit 2, soit une infinité d'orbites de Reeb périodiques sur leur bord. Au passage, ils remarquent que tout domaine fortement convexe de \mathbb{R}^{2n} est *dynamiquement convexe*. La convexité dynamique est la propriété que l'*indice de Conley-Zehnder*⁴ de chaque orbite de Reeb périodique est plus grand que $n + 1$. De plus, la dynamique de Reeb est un invariant symplectique, c'est-à-dire invariant par symplectomorphismes. En particulier, la convexité dynamique est un invariant symplectique; une question naturelle est donc:

Question B. *Est-ce que tout domaine dynamiquement convexe est symplectomorphe à un convexe?*

Cette question est restée ouverte pendant 20 ans. Une réponse partielle a été fournie par Abbondandolo, Bramham, Hryniewicz et Salaomão [ABHSa18]. Ils ont construit des domaines dynamiquement convexes de \mathbb{R}^4 qui ne vérifient pas une version faible de la conjecture de Viterbo. Cela a conduit à l'alternative suivante en dimension 4: soit la Question B admet une réponse négative, soit la conjecture de Viterbo est fausse. En décembre 2020, Chaidez et Edtmair [CE22] ont scellé définitivement la Question B en construisant les premiers exemples de domaines de \mathbb{R}^4 dynamiquement

³i.e. un domaine convexe borné $U = \{x \in \mathbb{R}^{2n} \mid H(x) \leq 1\}$ avec H une fonction lisse telle qu'il existe $a > 0$ tel que $H''(x) \geq aId$ pour tout $x \in \mathbb{R}^{2n}$.

⁴Cf Appendice 4.4

convexes qui ne sont pas symplectiquement convexes. Leur résultat s'appuie sur un "critère quantitatif" satisfait par tout domaine symplectiquement convexe. Plus précisément, ce critère repose sur le produit de l'*invariant de Ruelle* et de la *systole*⁵:

Théorème. *Soit U un domaine symplectiquement convexe de \mathbb{R}^4 , alors*

$$c \leq \text{ru}(U) \cdot \text{sys}(U)^{\frac{1}{2}} \leq C \quad (1)$$

où c et C sont des constantes positives, indépendantes du domaine U .

Résultats principaux de cette thèse

En 2020, Gutt, Hutchings et Ramos [GHGBR22] ont montré que les domaines dynamiquement convexes de \mathbb{R}^4 , qui ont la propriété additionnelle d'être *toriques*, satisfont la version forte de la conjecture de Viterbo. Un domaine *torique* $X \subset \mathbb{C}^2 \simeq \mathbb{R}^4$ est un domaine invariant sous l'action du tore \mathbb{T}^2 i.e. si $(z_1, z_2) \in X$, $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2$ alors

$$(e^{i\theta_1}, e^{i\theta_2}).(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in X.$$

Cette symétrie permet d'associer de manière unique à chaque domaine torique une région dans le quadrant positif de \mathbb{R}^2 dite *image moment*. Une particularité de ces domaines est que leurs capacités symplectiques, leur convexité dynamique et leur invariant de Ruelle peuvent être plus facilement étudiés sur leur image moment.

De la même manière que pour la Question B, le résultat de [GHGBR22] suggère la question suivante.

Question B-bis. *Est-ce que tout domaine **torique** dynamiquement convexe est symplectomorphe à un convexe?*

Le premier résultat de cette thèse répond négativement à cette question. Nous développons une méthode pour créer de manière systématique de nouveaux exemples de domaines de \mathbb{R}^4 dynamiquement convexes mais non-symplectiquement convexes. En particulier, cette méthode s'applique aux domaines toriques.

Théorème 1 ([DGZ24]). *Soit X_Ω un domaine torique dynamiquement convexe de \mathbb{R}^4 , il existe une perturbation faible en terme de volume, telle que le domaine résultant $X_{\hat{\Omega}}$ est toujours dynamiquement convexe mais n'est pas symplectiquement convexe.*

De plus, nous estimons les constantes du critère numérique de Chaidez et Edtmair à l'aide des domaines toriques *strictement monotones*⁶.

⁵Cf Section 1.3.

⁶D'après [GHGBR22], cela revient à être dynamiquement convexe pour un domaine torique.

Théorème 2 ([DGZ24]). Soit \mathcal{SM}_4 l'ensemble des domaines toriques strictement monotones de \mathbb{R}^4 et \mathcal{CT}_4 l'ensemble des domaines toriques géométriquement convexes de \mathbb{R}^4 , alors

$$\inf_{X_\Omega \in \mathcal{SM}_4} \left(\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \right) = \frac{1}{2}.$$

et

$$\sup_{X_\Omega \in \mathcal{CT}_4} \left(\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \right) = 3.$$

En particulier, les constantes optimales c et C dans le critère de Chaidez et Edtmair satisfont $c \leq \frac{1}{2}$ and $C \geq 3$.

La force de cette construction est qu'elle s'étend naturellement aux dimensions supérieures. Ainsi, en généralisant leur critère numérique de convexité symplectique, Chaidez et Edtmair ont répondu à la Question B en toute dimension en s'appuyant sur notre construction. Ce travail en collaboration avec Jean Gutt et Jun Zhang a donné lieu à l'article suivant, **Symplectic non-convexity of toric domains**, publié dans *Communications in Contemporary Mathematics* (Vol. 26, No. 04, 2350010 (2024)).

Motivés par la distance de Banach-Mazur sur les convexes de \mathbb{R}^d , Ostrover et Polterovich ont introduit un analogue symplectique non-linéaire appelé la distance symplectique grossière d_c . Soit U et V des domaines de \mathbb{R}^4 :

$$d_c(U, V) := \inf \left\{ \log \lambda \geq 0 \mid \frac{1}{\lambda} U \xrightarrow{\text{Symp}} V \xrightarrow{\text{Symp}} \lambda U \right\}$$

Au vu des exemples trouvés précédemment par Chaidez-Edtmair et D.-Gutt-Zhang, on peut se demander:

Question C. À quel point les domaines dynamiquement convexes sont-ils éloignés des domaines symplectiquement convexes pour la distance symplectique grossière?

La seconde partie de cette thèse a consisté à répondre à cette question. D'après un résultat classique de John, tout domaine convexe de \mathbb{R}^4 est coincé entre un ellipsoïde et le dilaté de cet ellipsoïde par une constante qui ne dépend que de la dimension. Du point de vue de la distance symplectique grossière, cela se traduit par le fait que tout domaine symplectiquement convexe est à distance au plus $\log 2$ des ellipsoïdes. Cela fournit donc un deuxième critère de convexité symplectique.

Proposition ([DGRZ23]). Soit U un domaine de \mathbb{R}^4 symplectiquement convexe, alors

$$d_c(U, \mathcal{E}_4) := \inf_{E \in \mathcal{E}_4} d_c(U, E) \leq \log 2$$

où \mathcal{E}_4 désigne l'ensemble des ellipsoïdes de \mathbb{R}^4 .

À l'aide des capacités ECH⁷, nous avons trouvé une famille de domaines toriques dynamiquement convexes $(X_{\Omega_p})_{p \in]0,1]}$ telle que $d_c(X_{\Omega_p}, \mathcal{E}_4)$ tend vers l'infini quand p tend vers 0. En particulier, par inégalité triangulaire, lorsque p tend vers 0, la distance de X_{Ω_p} à l'ensemble des domaines symplectiquement convexes tend vers $+\infty$. Cela répond ainsi à la Question C: les domaines dynamiquement convexes sont arbitrairement éloignés des domaines symplectiquement convexes pour la distance symplectique grossière. Plus précisément, nous avons le théorème suivant:

Théorème 3 ([DGRZ23]). *Pour X_{Ω_p} tel que $p < \frac{1}{5}$, on a:*

$$d_c(X_{\Omega_p}, \mathcal{E}_4) \geq \frac{1}{8} \log \left(\frac{g(p)}{1 + \log 4 + \log g(p)} \right) \quad (2)$$

où $g(p) = 2^{\frac{2}{p}-2} \text{Vol}_{\mathbb{R}^4}(X_{\Omega_p})$ tend vers $+\infty$ quand $p \rightarrow 0$. En particulier, quand p satisfait la condition suivante

$$\frac{g(p)}{1 + \log 4 + \log g(p)} \geq 2^8, \quad (3)$$

alors X_{Ω_p} est dynamiquement convexe mais n'est pas symplectiquement convexe.

Ce travail en collaboration avec Jean Gutt, Vinicius G.B. Ramos et Jun Zhang a donné lieu à l'article suivant, **Coarse distance from dynamically convex to convex**, prépublication soumise (arXiv:2308.06604).

Structure du manuscrit

La structure du manuscrit est la suivante. Dans le **Chapitre préliminaire**, nous rappelons les définitions de base de la géométrie symplectique et de contact ainsi que certains liens entre ces deux dernières.

Dans le **Chapitre 1**, nous motivons la convexité symplectique en introduisant la conjecture de Viterbo puis nous présentons la convexité dynamique, quelques unes de ses applications ainsi que le résultat de Chaidez et Edtmair en dimension 4.

Dans le **Chapitre 2**, nous définissons les domaines toriques, étudions la dynamique sur leur bord et établissons une formule de leur invariant de Ruelle. Ensuite, nous calculons leurs capacités symplectiques dans de nombreux cas.

Le **Chapitre 3** est tiré en partie de l'article publié "*Symplectic non-convexity of toric domains*" en collaboration avec Jean Gutt et Jun Zhang. Il traite de la construc-

⁷Une famille de capacités symplectiques définies uniquement en dimension 4 (cf Sous-section 2.3.2).

tion de domaines toriques dynamiquement convexes qui ne sont pas symplectiquement convexes.

Le **Chapitre 4** est essentiellement composé de la prépublication "*Coarse distance from dynamically convex to convex*" en collaboration avec Jean Gutt, Vinicius G.B.Ramos et Jun Zhang. Dans celui-ci, nous montrons que les domaines dynamiquement convexes sont arbitrairement éloignés des domaines symplectiquement convexes pour la distance symplectique grossière. Au passage, nous répondons à la Question B de manière indépendante aux travaux de Chaidez et Edtmair.

Finalement, le **Chapitre Perspectives** traite de certaines directions de recherche motivées par les résultats de cette thèse.

Introduction

Symplectic Geometry

Historically, symplectic geometry finds its origins in the study of celestial mechanics in the 19th century by Joseph Louis Lagrange and his student Simon-Denis Poisson. It provides the appropriate geometric framework for the phase space of a mechanical system (Liouville's theorem) and has since developed as an independent subject of study.

Formally speaking, *symplectic geometry* is the study of *symplectic manifolds*, which is the data of a pair (X, ω) where X is a smooth differential manifold and ω is a 2-form on X that is closed and non-degenerate. Due to the non-degeneracy assumption of the 2-form, every symplectic manifold (X, ω) is of even dimension and naturally equipped with a volume form ω^n . The prototypical example of a symplectic manifold, which will be central to the study of this thesis, is:

$$(\mathbb{R}^{2n}, x_1, \dots, x_n, y_1, \dots, y_n) \text{ equipped with the 2-form } \omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$$

In fact, a theorem by Darboux states that every symplectic manifold is locally isomorphic to the above example. In what follows in this thesis, we will restrict ourselves to *domains* in \mathbb{R}^{2n} , i.e., the closure of a connected open set in \mathbb{R}^{2n} . Thus, the interior of any domain naturally inherits a symplectic structure provided by the standard symplectic form ω_0 . To understand this geometry, it is natural to ask in which cases one can embed one domain into another in a symplectic way. A smooth embedding $\varphi : U \hookrightarrow V$ is said to be *symplectic* if $\varphi^*(\omega_0|_V) = \omega_0|_U$. We denote this by

$$U \xrightarrow{\text{Symplectic}} V$$

where U and V are of the same dimension. In the case of domains, the notion of symplectic volume coincides with the Euclidean volume up to a constant:

$$\text{Vol}_{\mathbb{R}^{2n}}(A) = \frac{1}{n!} \int_A \omega_0^n = \frac{1}{n!} \text{Vol}_{\omega_0^n}(A)$$

It follows that any transformation preserving the symplectic structure necessarily preserves the volume, and the existence of an embedding $U \xrightarrow{\text{Symp}} V$ implies the following inequality

$$\text{Vol}(U) \leq \text{Vol}(V).$$

However, symplectic geometry is more rigid. This volume condition alone is not sufficient to guarantee the existence of such an embedding. Indeed, a result by Gromov [Gro85] is:

Theorem (Gromov).

$$\text{int} \left(B^{2n}(r) \right) \xrightarrow{\text{Symp}} Z^{2n}(R) \iff r \leq R$$

where $B^{2n}(r) := \{z \in \mathbb{C}^n \mid \pi \|z\|^2 \leq r\}$ and $Z^{2n}(R) := \{z \in \mathbb{C}^n \mid \pi |z_1|^2 \leq R\}$.

This result led Gromov [Gro85], as well as Ekeland and Hofer [EH89], to introduce the notion of *symplectic capacity*⁸: a 2-dimensional symplectic measure that captures additional information beyond volume. Ten years later, Viterbo [Vit00] stated the following conjecture regarding these capacities:

Conjecture (Viterbo). *Let U be a convex domain in \mathbb{R}^{2n} and c a normalized symplectic capacity, then:*

$$c(U) \leq (n! \text{Vol}(U))^{1/n}$$

Moreover, equality holds if and only if U is symplectomorphic⁹ to the ball.

Symplectic capacities and volume are invariant under symplectomorphisms⁹, however, convexity is not. This observation naturally raises the following question, which is the starting point of this thesis:

Question A. *Which domains are symplectomorphic to a convex domain? In other words, what are the symplectically convex domains?*

Contact Geometry

To address this question, we will need the following notions.

A *contact structure* on a manifold Y of dimension $2n - 1$ is a maximally non-integrable hyperplane field ξ on Y . If locally $\xi = \ker(\alpha)$, where α is a 1-form on Y , this condition translates to $\alpha \wedge (d\alpha)^{n-1} \neq 0$. When this *contact form* is defined globally, the structure is called *co-oriented*. In this case, we naturally associate to α a

⁸See Subsection 1.1.2.

⁹A symplectomorphism is a diffeomorphism that preserves the symplectic structure (cf Definition 0.1.7).

vector field R_α which is transverse to the contact structure at any point. This vector field is called the *Reeb vector field*, and the study of the dynamics of its flow is a rich subject of study in contact geometry.

Going back to domains in \mathbb{R}^{2n} , a domain with smooth boundary is *star-shaped* if it contains the origin and the radial vector field is transverse to the boundary of the domain. In this case, the boundary of the domain naturally inherits a co-oriented contact structure and thus a Reeb vector field.

State of the Art

In [HWZ98], Hofer, Wysocki, and Zehnder showed that every strongly convex¹⁰ domain in \mathbb{R}^4 has either 2 or infinitely many periodic Reeb orbits on its boundary. At the same time, they observed that every strongly convex domain in \mathbb{R}^{2n} is *dynamically convex*. Dynamical convexity is the property that the *Conley-Zehnder index*¹¹ of each periodic Reeb orbit is greater than $n + 1$. Moreover, the Reeb dynamics is a symplectic invariant, i.e., invariant under symplectomorphisms. In particular, dynamical convexity is a symplectic invariant; thus, a natural question arises:

Question B. *Is any dynamically convex domain symplectomorphic to a convex one?*

This question remained open for 20 years. A partial answer was provided by Abbondandolo, Bramham, Hryniewicz, and Salaomão [ABHSa18]. They constructed dynamically convex domains in \mathbb{R}^4 that do not satisfy a weak version of Viterbo conjecture. This led to the following alternative in dimension 4: either Question B has a negative answer, or Viterbo conjecture is false. In December 2020, Chaidez and Edtmair [CE22] definitively settled Question B by constructing the first examples of dynamically convex domains in \mathbb{R}^4 that are not symplectically convex. This result relies on a "quantitative criterion" satisfied by every symplectically convex domain. More precisely, this criterion is based on the product of the *Ruelle invariant* and the *systolic ratio*¹²:

Theorem ([CE22]). *Let U be a symplectically convex domain in \mathbb{R}^4 , then*

$$c \leq \text{ru}(U) \cdot \text{sys}(U)^{\frac{1}{2}} \leq C \tag{4}$$

where c and C are positive constants independent of the domain U .

¹⁰i.e. a bounded convex domain $U = \{x \in \mathbb{R}^{2n} \mid H(x) \leq 1\}$ with H some smooth function such that there exists a constant $a > 0$ for which $H^n(x) \geq aId$ for any $x \in \mathbb{R}^{2n}$.

¹¹See Appendix 4.4.

¹²See Section 1.3.

Main Results of this Thesis

In 2020, Gutt, Hutchings, and Ramos [GHGBR22] showed that dynamically convex domains in \mathbb{R}^4 , which have the additional property of being *toric*, satisfy the strong version of Viterbo conjecture¹³. A *toric* domain $X \subset \mathbb{C}^2 \simeq \mathbb{R}^4$ is a domain invariant under the action of the torus \mathbb{T}^2 i.e. if $(z_1, z_2) \in X$, $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2$ then

$$(e^{i\theta_1}, e^{i\theta_2}).(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in X.$$

This symmetry allows us to associate to each toric domain a unique region in the positive quadrant of \mathbb{R}^2 called the *moment image*. A particularity of these domains is that their symplectic capacities, dynamical convexity, and Ruelle invariant can be more easily studied on their moment image.

Similar to Question B, the result of [GHGBR22] suggests the following question.

Question B-bis. *Is any **toric** dynamically convex domain symplectomorphic to a convex one?*

The first result of this thesis provides a negative answer to this question. We produce a systematic way to create new examples of dynamically convex domains in \mathbb{R}^4 that are not symplectically convex. In particular, this method applies to toric domains.

Theorem 1 ([DGZ24]). *Let X_Ω be a dynamically convex toric domain in \mathbb{R}^4 , there exists a small perturbation in terms of volume, such that the resulting domain $X_{\hat{\Omega}}$ is still dynamically convex but is not symplectically convex.*

Furthermore, we estimate the constants in the quantitative criterion of Chaidez and Edtmair using *strictly monotone* toric domains¹⁴.

Theorem 2 ([DGZ24]). *Let \mathcal{SM}_4 denotes the set of strictly monotone toric domains in \mathbb{R}^4 and \mathcal{CT}_4 the set of geometrically convex toric domains in \mathbb{R}^4 . We have*

$$\inf_{X_\Omega \in \mathcal{SM}_4} \left(\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \right) = \frac{1}{2}.$$

and

$$\sup_{X_\Omega \in \mathcal{CT}_4} \left(\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \right) = 3.$$

In particular, the optimal constant c and C in the criterion of Chaidez and Edtmair satisfy $c \leq \frac{1}{2}$ and $C \geq 3$.

¹³See Conjecture 1.1.7.

¹⁴According to [GHGBR22], this is equivalent to being dynamically convex for a toric domain.

This construction naturally extends to higher dimensions. Thus, by generalizing their quantitative criterion for symplectic convexity, Chaidez and Edtmair answered Question B in any dimension using our construction. This joint work with Jean Gutt and Jun Zhang led to the following article, **Symplectic non-convexity of toric domains**, published in *Communications in Contemporary Mathematics* (Vol. 26, No. 04, 2350010 (2024)).

Motivated by the Banach-Mazur distance on convex sets in \mathbb{R}^d , Ostrover and Polterovich suggested a nonlinear symplectic analogue called the coarse symplectic distance d_c . Let U and V be domains in \mathbb{R}^4 :

$$d_c(U, V) := \inf \left\{ \log \lambda \geq 0 \mid \frac{1}{\lambda}U \xrightarrow{\text{Symp}} V \xrightarrow{\text{Symp}} \lambda U \right\}$$

Given the examples previously found by Chaidez-Edtmair and D.-Gutt-Zhang, one may ask:

Question C. *How far away dynamically convex domains can be from symplectically convex domains with respect to the coarse symplectic distance?*

The second part of this thesis aimed to answer this question. Based on a classic result by John, every convex domain in \mathbb{R}^4 is sandwiched between an ellipsoid and its dilation by a constant depending only on the dimension. From the perspective of coarse symplectic distance, this means that any symplectically convex domain is at distance at most $\log 2$ from ellipsoids. This provides a second criterion for symplectic convexity.

Proposition ([DGRZ23]). *Let U be a symplectically convex domain in \mathbb{R}^4 , then*

$$d_c(U, \mathcal{E}_4) := \inf_{E \in \mathcal{E}_4} d_c(U, E) \leq \log 2$$

where \mathcal{E}_4 denotes the set of ellipsoids in \mathbb{R}^4 .

Relying on ECH capacities¹⁵, we found a family of toric dynamically convex domains $(X_{\Omega_p})_{p \in]0,1]}$ such that $d_c(X_{\Omega_p}, \mathcal{E}_4)$ tends to infinity as p goes to 0. In particular, by triangle inequality, as p goes to 0, the distance from X_{Ω_p} to the set of symplectically convex domains tends to infinity. This answers Question C: dynamically convex domains can be arbitrarily far away from symplectically convex domains with respect to the coarse symplectic distance. More precisely, we have the following theorem:

Theorem 3 ([DGRZ23]). *For X_{Ω_p} such that $p < \frac{1}{5}$, we have:*

$$d_c(X_{\Omega_p}, \mathcal{E}_4) \geq \frac{1}{8} \log \left(\frac{g(p)}{1 + \log 4 + \log g(p)} \right) \quad (5)$$

¹⁵A family of symplectic capacities only defined in dimension 4 (see Subsection 2.3.2).

where $g(p) = 2^{\frac{2}{p}-2} \text{Vol}_{\mathbb{R}^4}(X_{\Omega_p})$ tends to $+\infty$ as $p \rightarrow 0$. In particular, when p satisfies the following condition:

$$\frac{g(p)}{1 + \log 4 + \log g(p)} > 2^8, \quad (6)$$

then X_{Ω_p} is dynamically convex but not symplectically convex.

This joint work with Jean Gutt, Vinicius G.B. Ramos, and Jun Zhang led to the following submitted preprint, **Coarse distance from dynamically convex to convex** (arXiv:2308.06604).

Manuscript Structure

The structure of the manuscript is as follows. In the **Preliminary Chapter**, we review the basic definitions of symplectic and contact geometry as well as some connections between the two.

In **Chapter 1**, we motivate symplectic convexity by introducing the Viterbo conjecture and then present dynamic convexity, along with some of its applications and the result of Chaidez and Edtmair in dimension 4.

In **Chapter 2**, we define toric domains, study the dynamics on their boundary, and establish a formula for their Ruelle invariant. Then, we calculate their symplectic capacities in many cases.

Chapter 3 is partly based on the article "Symplectic non-convexity of toric domains" published in collaboration with Jean Gutt and Jun Zhang. It deals with the construction of dynamically convex toric domains that are not symplectically convex.

Chapter 4 is essentially composed of the preprint "Coarse distance from dynamically convex to convex" in collaboration with Jean Gutt, Vinicius G.B. Ramos, and Jun Zhang. In this chapter, we show that dynamically convex domains are arbitrarily far away from symplectically convex domains in terms of coarse symplectic distance. Additionally, we answer Question B independently of the work of Chaidez and Edtmair.

Finally, **Chapter Perspectives** addresses some research directions motivated by the results of this thesis.

Chapter 0

Preliminary

In this Chapter, we cover the basic material of symplectic and contact geometry that will be needed to the study of this thesis. In Section 0.1, we give the formal definition of a symplectic manifold and give some examples such as interior of star-shaped domains. Then we introduce the notion of symplectomorphism and state the Darboux theorem. In Section 0.2, we define contact manifolds, give some examples and review basic notions of the Reeb dynamic of a contact form. In Section 0.3 and Section 0.4, we show how to construct contact manifolds from symplectic manifolds and vice-versa. Finally, in Section 0.5, we review some relations between the symplectic interior of a domain with its contact boundary.

0.1 Symplectic geometry

Definition 0.1.1. *Let X be a smooth manifold, the couple (X, ω) is called a **symplectic manifold** if ω is a non-degenerate closed 2-form. In this case, ω is called a **symplectic form**.*

The non-degeneracy condition implies that $\dim(X) = 2n$ is even. Thus $\omega^{\wedge n}$ is a volume form on X by non-degeneracy of ω and therefore any symplectic manifold is orientable.

Example 0.1.2. *The following smooth manifolds are symplectic:*

- \mathbb{R}^{2n} with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ and symplectic form

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$$

*which is usually referred as the **standard** symplectic form.*

- \mathbb{C}^n with complex coordinates (z_1, \dots, z_n) and symplectic form

$$\omega = \frac{1}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$$

- S^2 with its volume form.

Non-example 0.1.3. If X is compact, then by Stokes' theorem the cohomology class $[\omega^{\wedge n}]$ is non-zero and so is $[\omega] \in H^2(X, \mathbb{R})$. Since $H^2(S^{2n}, \mathbb{R})$ are trivial for $n > 2$, there is no symplectic structure on the spheres S^{2n} for $n > 2$.

Example 0.1.4. An historically important example is T^*M , the cotangent bundle of a smooth manifold M . Indeed, the cotangent bundle can be seen as the phase space of a mechanical system where M is the space of positions. It was one of the first motivation that lead to the study of symplectic geometry.

In this thesis, we will mainly be interested by a large class of symplectic manifolds which are called *star-shaped domains*.

Definition 0.1.5. A **domain** U in \mathbb{R}^{2n} is the closure of a connected open set. It is called **star-shaped** if for any point $p \in U$, the straight line between p and 0 is contained in U . Whenever a star-shaped domain is bounded and has a smooth boundary which is transverse to the radial vector field, we call it **nice**.

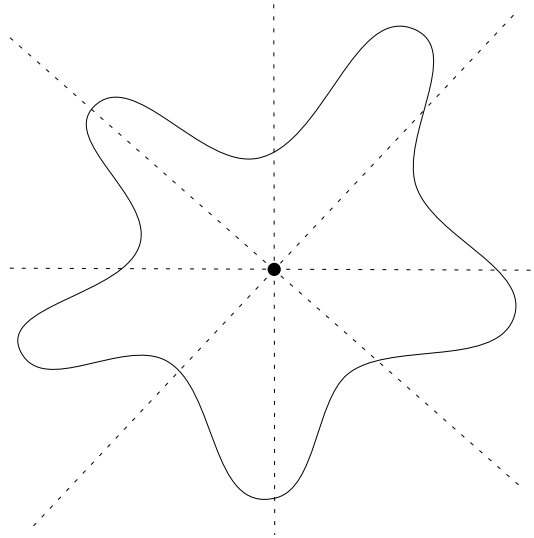


Figure 1: A nice star-shaped domain in \mathbb{R}^2 .

In the rest of this thesis, any domain will be supposed to be bounded unless otherwise stated. Remark that any star-shaped domains is homologically trivial.

Proposition 0.1.6. *Let U be a domain in \mathbb{R}^{2n} , then $(\text{int}(U), \omega_0|_{\text{int}(U)})$ is a symplectic manifold.*

From now on, we will sometimes denote by U the associated symplectic manifold $(\text{int}(U), \omega_0|_{\text{int}(U)})$.

As in other geometries, we are interested in diffeomorphisms that preserve the structure (Erlangen program). Historically, it was noticed that the time flow in the phase space of a mechanical system, or any other coordinates change which preserve the Hamiltonian, were such diffeomorphisms.

Definition 0.1.7. *Let (X_1, ω_1) and (X_2, ω_2) be two symplectic manifolds, a diffeomorphism $\varphi: X_1 \rightarrow X_2$ such that $\varphi^*\omega_2 = \omega_1$ is called a **symplectomorphism**. In this case, (X_1, ω_1) and (X_2, ω_2) are said to be **symplectomorphic**.*

We will denote by $\text{Symp}(X, \omega)$ the set of symplectomorphisms from (X, ω) to itself and $\text{Sp}(2n)$ the set of linear symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$. In terms of symplectic geometry, symplectomorphic manifolds are considered as the same. A goal of the field is to classify symplectic manifolds up to symplectomorphism. It appears that symplectic geometry presents surprising flexibility phenomenon as the following theorem by Darboux shows.

Theorem 0.1.8 (Darboux). *Let (X, ω) be a $2n$ -dimensional symplectic manifold. For any $p \in X$ there is a (symplectic) coordinate chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ centered on p such that on \mathcal{U} we have:*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i =: \omega_0$$

In particular, a consequence from the above theorem is that symplectic manifolds are locally indistinguishable from one another. This is in contrast with Riemannian geometry where local curvature is a local invariant. This leads to the study of global invariants such as the ones defined in Section 1.1.

0.2 Contact geometry

Contact geometry is often seen as the odd-dimensional counterpart of symplectic geometry. In our study, it will arise as the geometry of the smooth boundary of some symplectic manifolds.

Definition 0.2.1. *Let Y be a $(2n+1)$ -manifold, the couple (Y, ξ) is called a **contact manifold** if $\xi = \ker(\alpha) \subset TY$ is a maximally non-integrable hyperplane field (i.e.*

$\forall \alpha$ such that $\xi = \ker(\alpha)$ then $\alpha \wedge (d\alpha)^n \neq 0$). Such an hyperplane field ξ is called a **contact structure** and any such 1-form is called a **contact form**. Whenever the 1-form is defined globally, the structure is called **coorientable**.

From now on, we will only consider coorientable contact structure. The condition on the contact structure implies that $(d\alpha)|_{\xi}$ is non-degenerate and so each hyperplane $(\xi_x, d\alpha|_{\xi_x})$ is a symplectic vector space¹ for $x \in Y$. Moreover, $\alpha \wedge (d\alpha)^n$ is a volume form and so Y is orientable.

Remark 0.2.2. If a 1-form α defines a contact structure ξ then any 1-form defining ξ can be written as $f\alpha$ with $f : Y \rightarrow \mathbb{R}$ a non-vanishing function.

Example 0.2.3. On \mathbb{R}^{2n+1} with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$,

$$\alpha_{std} = dz - \sum_{i=1}^n y_i dx_i$$

is a contact form defining the so-called standard contact structure $\xi_{std} = \ker(\alpha_{std})$. This contact structure is invariant under translation along the x_i and z axis.

Example 0.2.4. On \mathbb{R}^{2n+1} with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$,

$$\alpha_{rad} = dz + \sum_{i=1}^n x_i dy_i - y_i dx_i$$

is a contact form defining a contact structure $\xi_{rad} = \ker(\alpha_{rad})$.

As in the symplectic case, let us define the corresponding geometry preserving transformations.

Definition 0.2.5. Let (Y_1, ξ_1) and (Y_2, ξ_2) be two contact manifolds, a diffeomorphism $\varphi : Y_1 \rightarrow Y_2$ such that $\varphi_*\xi_1 = \xi_2$ is called a **contactomorphism**. In other words, if $\xi_i = \ker(\alpha_i)$ then $\varphi^*\alpha_2 = f\alpha_1$ for some non-vanishing function f (cf Remark 0.2.2). Whenever f is constant equal to 1, φ is called a **strict contactomorphism**. In this case, (Y_1, ξ_1) and (Y_2, ξ_2) are said to be (strict) **contactomorphic**.

Proposition 0.2.6. $(\mathbb{R}^{2n+1}, \xi_{std})$ and $(\mathbb{R}^{2n+1}, \xi_{rad})$ are strictly contactomorphic by the following diffeomorphism:

$$\begin{aligned} \mathbb{R}^{2n+1} &\longrightarrow \mathbb{R}^{2n+1} \\ (x_1, \dots, x_n, y_1, \dots, y_n, z) &\longmapsto \left(\frac{\mathbf{x} + \mathbf{y}}{2}, \frac{\mathbf{y} - \mathbf{x}}{2}, z + \frac{1}{2} \sum_{i=1}^n x_i y_i \right). \end{aligned}$$

¹A symplectic vector space (V, ω) is a finite dimensional vector space V equipped with a bilinear, anti-symmetric and non-degenerate application ω .

Let us mention a Darboux's analogue in contact geometry.

Theorem 0.2.7 (Contact Darboux). *Let (Y, ξ) be a $(2n+1)$ -dimensional contact manifold, for any $p \in Y$ there is a coordinate chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n, z)$ centered on p such that on \mathcal{U} we have:*

$$\alpha = dz - \sum_{i=1}^n y_i dx_i =: \alpha_{std}$$

0.2.1 Reeb dynamic

To any contact manifold with a prescribed contact form, one can naturally associate a unique vector field. One might expect that the geometry of the manifold constraints the dynamic of this vector field, and vice versa. Therefore, we shall study its dynamic to recover geometric information of the manifold.

Definition 0.2.8. *Let (Y, α) be a contact manifold, there is a unique vector field R_α associated to α which satisfies:*

- $d\alpha(R_\alpha, \cdot) = 0$
- $\alpha(R_\alpha) = 1$

R_α is called the **Reeb vector field** of α .

Remark 0.2.9. *Let $(Y, \xi = \ker(\alpha))$ be a contact manifold, the Reeb vector field induces a splitting of the tangent bundle:*

$$TY = \mathbb{R}R_\alpha \oplus \xi$$

The flow φ_t associated to the Reeb vector field is a strict contactomorphism:

$$(\varphi_t)^* \alpha = \alpha$$

It induces a linear transformation $d\varphi_t$ which leaves invariant this splitting, in particular for any $x \in Y$

$$(d\varphi_t)_x : \xi_x \rightarrow \xi_{\varphi_t(x)}$$

is a linear symplectomorphism.

Example 0.2.10. *The Reeb vector field of the standard contact form α_{std} on \mathbb{R}^{2n+1} is given by*

$$R_{\alpha_{std}} = \partial_z$$

The second condition in Definition 0.2.8 implies that the Reeb vector field cannot vanish, in particular its flow doesn't admit any fixed point. Therefore, we are interested in the periodic orbits.

Definition 0.2.11. Let (Y, α) be a contact manifold, a periodic orbit γ of the Reeb vector field R_α is called a **closed** (or **periodic**) **Reeb orbit**.

- Its **action** is the real number defined by

$$\mathcal{A}(\gamma) := \int_\gamma \alpha$$

which corresponds to the period of this orbit.

- We call it **non-degenerate** if the linearized Poincaré return map $P_\gamma : \xi_{\gamma(0)} \rightarrow \xi_{\gamma(0)}$ doesn't admit 1 as eigenvalue. The contact form α is called non-degenerate if all closed Reeb orbit are non-degenerate.
- The **minimal action** of (Y, α) is defined as the smallest period of its periodic orbits

$$T_{\min}(Y, \alpha) := \inf \{ \mathcal{A}(\gamma) \mid \gamma \text{ is a closed Reeb orbit} \}$$

Proposition 0.2.12. Let (Y_1, α_1) and (Y_2, α_2) be two contact manifolds such that $\varphi : (Y_1, \alpha_1) \rightarrow (Y_2, \alpha_2)$ is a strict contactomorphism, then

$$\varphi_* R_{\alpha_1} = R_{\alpha_2}$$

This implies that strictly contactomorphic manifolds have conjugated Reeb flows. In particular they share the number of periodic orbits, the minimal action, etc. Following this remark, let us consider the following example of contactomorphic manifolds which have different dynamics.

Example 0.2.13. The family of contact forms on S^3 indexed by the real parameter $t \geq 0$:

$$\alpha_t = (x_1 dy_1 - y_1 dx_1) + (1 + t)(x_2 dy_2 - y_2 dx_2)$$

induces contactomorphic contact structures because of Gray stability theorem (see Theorem 0.2.14 below). However, they are not strictly contactomorphic because of Proposition 0.2.12 and the fact that they yield different Reeb dynamics.

In fact, the Reeb vector field associated to α_0 is

$$R_{\alpha_0} = (x_1 \partial_{y_1} - y_1 \partial_{x_1}) + (x_2 \partial_{y_2} - y_2 \partial_{x_2})$$

and one can verify that the Reeb orbits are the fibers of the Hopf fibration of S^3 . In particular, there exists infinitely many closed Reeb orbits. However, the Reeb vector

field of α_t is

$$R_{\alpha_t} = (x_1\partial_{y_1} - y_1\partial_{x_1}) + \frac{1}{1+t}(x_2\partial_{y_2} - y_2\partial_{x_2})$$

which can be seen as the Reeb vector field of the standard radial contact form on the boundary of a symplectic ellipsoid of radii 1 and $1+t$ (see Equation 2.1). In particular, whenever $t \in \mathbb{R}_{\geq 0} \setminus \mathbb{Q}$, the Reeb vector field has only two simple closed Reeb orbits.

Theorem 0.2.14 (Gray stability theorem). *Let ξ_t be a smooth family of contact structures on a closed manifold Y , then there exists an isotopy $(\varphi_t)_{t \in [0,1]}$ of Y such that $d\varphi_t(\xi_0) = \xi_t$.*

Proof. The proof relies on the Moser trick which is also used to prove Darboux's theorem. The idea of Moser trick consists in defining the isotopy as the flow of a time-dependent vector field V_t such that $\varphi_0 = id$. We can write the equation of the theorem as:

$$\varphi_t^* \alpha_t = \lambda_t \alpha_0$$

where $\lambda_t : Y \rightarrow \mathbb{R}_+$ is a family smooth function. We differentiate with respect to the parameter t and use again the previous equation:

$$\varphi_t^*(\dot{\alpha}_t + \mathcal{L}_{X_t}\alpha_t) = \dot{\lambda}_t \alpha_0 = \frac{\dot{\lambda}_t}{\lambda_t} \varphi_t^* \alpha_t$$

We introduce the function $g_t = \frac{d}{dt}(\log \lambda_t) \circ \varphi_t^{-1}$ and use Cartan formula:

$$\varphi_t^*(\dot{\alpha}_t + d(\alpha_t(V_t)) + \iota_{V_t}d\alpha_t) = \varphi_t^*(g_t\alpha_t)$$

We will search V_t in ξ_t and so we get

$$\dot{\alpha}_t + \iota_{V_t}d\alpha_t = g_t\alpha_t$$

We evaluate on the Reeb vector field R_{α_t}

$$\dot{\alpha}_t(R_{\alpha_t}) = g_t$$

This defines uniquely g_t . Moreover, $g_t\alpha_t - \dot{\alpha}_t$ is a 1-form on ξ_t since $R_{\alpha_t} \in \ker(g_t\alpha_t - \dot{\alpha}_t)$ and $TY = \mathbb{R}R_{\alpha_t} \oplus \xi_t$. Hence, by non-degeneracy of $(d\alpha_t)|_{\xi_t}$, V_t is uniquely defined. \square

0.3 From symplectic geometry to contact geometry

In this Section, we give the definition of a Liouville vector field and describe how it allows us to define a contact structure on any well-behaved hypersurface contained in a given symplectic manifold. In particular, when the symplectic manifold is (the interior of) a nice star-shaped domains in \mathbb{R}^{2n} , this allows us to define a contact structure on its boundary.

Definition 0.3.1. Let (X, ω) be a symplectic manifold, a vector field $V \in \Gamma(TX)$ satisfying

$$\mathcal{L}_V \omega = \omega$$

is called a **Liouville** vector field.

Remark 0.3.2. If there exists a Liouville vector field V on (X, ω) , then by Cartan formula, ω is exact:

$$\omega = \mathcal{L}_V \omega = d(\iota_V \omega)$$

Proposition 0.3.3. Let Y be a smooth hypersurface contained in a symplectic manifold (X, ω) such that a Liouville vector field V is transverse to Y , then $(Y, (\iota_V \omega)|_Y)$ is a contact manifold.

Definition 0.3.4. A smooth hypersurface of \mathbb{R}^{2n} is **star-shaped** if it bounds a nice star-shaped domain (see Definition 0.1.5).

Example 0.3.5. Consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0 = \sum_i dx_i \wedge dy_i)$, then the radial vector field

$$V = \frac{1}{2} \sum_{i=1}^n (x_i \partial_{x_i} + y_i \partial_{y_i})$$

is a Liouville vector field. Hence

$$\lambda_0 := \iota_V \omega_0 = \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$$

is a contact form on any star-shaped hypersurface of \mathbb{R}^{2n} , it satisfies $\omega_0 = d\lambda_0$.

Example 0.3.6. In particular, $S^{2n-1} = \partial B^{2n}$ is a star-shaped hypersurface in \mathbb{R}^{2n} bounding the unit ball. Therefore, it is a contact manifold with contact form λ_0 . Its Reeb vector field is given by

$$R = 2 \sum_{i=1}^n (x_i \partial_{y_i} - y_i \partial_{x_i})$$

In particular, there exists infinitely many periodic orbits.

Whenever two nice star-shaped domains are symplectomorphic, one might ask what is the relation between their contact boundaries. In general, not so much is known, but for the case of two globally² symplectomorphic domains, we have the following proposition.

Proposition 0.3.7. *Let U and V be two nice star-shaped domains of \mathbb{R}^{2n} . If there exists a symplectomorphism $\Phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $\Phi(U) = V$, then their contact boundary ∂U and ∂V are strict contactomorphic.*

Remark 0.3.8. *In particular, for nice star-shaped domain U , any contact invariant defined on ∂U is preserved under global symplectomorphisms.*

Proof. Assume that there is some symplectomorphism $\Phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $\Phi(U) = V$. By definition $\Phi^*\omega_0 = \omega_0$ with $\omega_0 = d\lambda_0$ being the standard symplectic form on \mathbb{R}^{2n} . Therefore, since $d(\Phi^*\lambda_0 - \lambda_0) = 0$ and $H^1(U, \mathbb{R}) = H^1(V, \mathbb{R}) = 0$, there exists some function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that

$$\Phi^*\lambda_0 = \lambda_0 + df$$

By the following lemma cited in [CE22], Φ is isotopic to some strict contactomorphism between ∂U and ∂V . \square

Lemma 0.3.9 ([CE22] Lemma 3.5). *Let $\phi : (Y, \lambda) \rightarrow (Y', \lambda')$ a diffeomorphism between two compact contact manifolds such that $\phi^*\lambda' = \lambda + df$ for some function $f : Y \rightarrow \mathbb{R}$, then ϕ is isotopic to a strict contactomorphism between (Y, λ) and (Y', λ') .*

Proof. We use Moser's trick: we want to construct an isotopy φ_t as the flow of some vector field X_t such that $\varphi_t^*\lambda_t = \lambda$ where $\lambda_t = (1-t)\lambda + t(\lambda + df)$. Then for $t = 1$, we will have $\varphi_1^*(\lambda + df) = \lambda$ and so ϕ is isotopic to the strict contactomorphism

$$\phi \circ \varphi_1 : (Y, \lambda) \rightarrow (Y', \lambda')$$

By differentiating $\varphi_t^*\lambda_t = \lambda$, we get:

$$0 = \frac{d}{dt}(\varphi_t^*\lambda_t) = \varphi_t^* \left(\frac{d}{dt}\lambda_t + \mathcal{L}_{X_t}\lambda_t \right)$$

therefore by Cartan formula for Lie derivative:

$$df + d\iota_{X_t}\lambda_t + \iota_{X_t}d\lambda_t = 0$$

²i.e. there exists a global symplectomorphism that sends one to the other (see Section 0.5).

if we look for X_t in $\ker(\lambda_t)$ one gets

$$\iota_{X_t} d\lambda_t = -df$$

which by non-degeneracy of $d\lambda_t$ on $\ker(\lambda_t)$ defines uniquely the vector field $X_t \in \ker(\lambda_t)$. In fact, λ_t is a contact form on $\ker(\lambda_t)$. Set $v_t = \lambda_t \wedge d\lambda_t = \lambda_t \wedge (d\lambda_t)^{n-1}$ for any $t \in [0, 1]$. For $t = 0$ and $t = 1$, v_t is a volume form since λ_0 and λ_1 are contact. And on the critical set of f , v_0 and v_1 coincide, therefore since the space of top-dimensional forms is one-dimensional, we deduce that this convex combination of volume form is a volume form. Finally, the flow induced by X_t is complete since Y is compact. \square

0.4 From contact geometry to symplectic geometry

In this Section, we go in the opposite direction by associating a symplectic manifold to any contact manifold: the symplectization. Then we show that for any nice star-shaped domains, a strict contactomorphism between their boundary induces a global symplectomorphism between the corresponding domains.

Definition 0.4.1. *Let (Y, α) be a contact $(2n - 1)$ -manifold, then the (extrinsic) **symplectization** of (Y, α) is the following symplectic $2n$ -manifold*

$$(\mathbb{R}_t \times Y, d(e^t \alpha))$$

Example 0.4.2. *The symplectization of $(S^{2n-1}, \lambda_0|_{S^{2n-1}})$ is symplectomorphic to $(\mathbb{R}^{2n} \setminus \{0\}, \omega_0)$ by*

$$\begin{aligned} \mathbb{R} \times S^{2n-1} &\rightarrow \mathbb{R}^{2n} \setminus \{0\} \\ (t, x_1, \dots, x_n, y_1, \dots, y_n) &\rightarrow (e^{t/2} x_1, \dots, e^{t/2} x_n, e^{t/2} y_1, \dots, e^{t/2} y_n) \end{aligned}$$

Now, we go into the opposite direction of Proposition 0.3.7.

Proposition 0.4.3. *Let U and V be two nice star-shaped domains of \mathbb{R}^4 . If ∂U and ∂V are strict contactomorphic, then there exists a symplectomorphism $\Phi : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that $\Phi(U) = V$.*

The proof is based on an argument from Ramos in [Ram15] (see also Remark 4.5 in [ABHSA18]). Note that, in some cases, it can be extended in higher dimensions (see Section 6 in [ABE23]).

Proof. Assume that there is some strict contactomorphism φ between ∂U and ∂V , this induces a symplectomorphism between their symplectization: $(]-\delta, +\infty[\times \partial U, d(e^t \lambda_0))$ and $(]-\delta, +\infty[\times \partial V, d(e^t \lambda_0))$ with $\delta > 0$.

Therefore, we have a symplectomorphism $\tilde{\varphi}$ between open neighborhoods of $\mathbb{R}^4 \setminus U$ and $\mathbb{R}^4 \setminus V$. Since there is no non trivial closed embedded submanifold in the exact symplectic manifold \mathbb{R}^4 , this extends to a symplectomorphism of \mathbb{R}^4 which sends U to V by the following theorem. □

Theorem 0.4.4 ([MS12], Theorem 9.4.2). *Let (M, ω) be a connected symplectic 4-manifold and $K \subset M$ be a compact subset such that the following holds.*

- (i) *There is no symplectically embedded 2-sphere $S \subset M$ with self-intersection number $S \cdot S = -1$.*
- (ii) *There exists a symplectomorphism $\psi : \mathbb{R}^4 \setminus V \rightarrow M \setminus K$, where $V \subset \mathbb{R}^4$ is a star-shaped compact set.*

Then (M, ω) is symplectomorphic to (\mathbb{R}^4, ω_0) . Moreover, for every open neighbourhood $U \subset M$ of K , the symplectomorphism can be chosen equal to ψ^{-1} on $M \setminus U$.

Remark 0.4.5. *Under the assumptions from Proposition 0.4.3, one has directly that*

$$\text{Vol}_{\omega_0^2}(U) = \text{Vol}_{\omega_0^2}(V)$$

indeed:

$$\begin{aligned} \text{Vol}_{\omega_0^2}(U) &= \int_U d\lambda_0 \wedge d\lambda_0 = \int_U d(\lambda_0 \wedge d\lambda_0) = \int_{\partial U} \lambda_0 \wedge d\lambda_0 \\ &= \int_{\partial V} \lambda_0 \wedge d\lambda_0 = \int_V d\lambda_0 \wedge d\lambda_0 \\ &= \text{Vol}_{\omega_0^2}(V) \end{aligned}$$

where we used Stokes' theorem twice.

0.5 Relations between the symplectic interior and the contact boundary

For any nice star-shaped domain U in \mathbb{R}^{2n} , its interior $(\text{int}(U), \omega_0|_{\text{int}(U)})$ is a symplectic manifold and its boundary $(\partial U, \lambda_0|_{\partial U})$ is a contact manifold. When there is no ambiguity, we will denote by U the associated symplectic manifold and by ∂U the associated contact manifold. In the two previous sections, we have seen that under some strong hypothesis we have

Proposition 0.5.1. *Let U and V be two nice star-shaped domains of \mathbb{R}^{2n} . Then ∂U and ∂V are strict contactomorphic if and only if there exists a symplectomorphism $\Phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $\Phi(U) = V$.*

In other words, the contact boundary constraints the global symplectic nature of the domain. However, it doesn't answer the following harder question which has been the subject of various studies

Question (inspired by Eliashberg and Hofer). *What part of the information about the contact boundary ∂U can be seen from the interior of U ?*

Without further assumptions, Eliashberg and Hofer showed in [EH96] that there exists convex domains with smooth boundaries such that their interior are symplectomorphic but their boundary are not strict contactomorphic. Recently, the authors in [ABE23] observed that the convex domains constructed by Katok in [Kat73] were other examples of such domains. In [HE92], Eliashberg and Hofer showed that if two nice star-shaped domains have symplectomorphic interior, then their action spectrum is equal. A stronger result has been established in [CFHW96], under the assumption that the contact form on the boundary is non-degenerate. Therefore, whenever mentioning symplectomorphisms between domains, one has to be careful. We shall use the following convention for the rest of this thesis.

Convention. *Given two compact star-shaped domains U and V in \mathbb{R}^{2n} , they are said*

- *(globally) symplectomorphic if there exists $\Phi \in \text{Symp}(\mathbb{R}^{2n}, \omega_0)$ such that*

$$\Phi(U) = V$$

- *intrinsically symplectomorphic if there exists a symplectomorphism*

$$\Psi : \text{int}(U) \rightarrow \text{int}(V)$$

Evidently, two (globally) symplectomorphic star-shaped domains are *intrinsically* symplectomorphic by taking $\Psi = \Phi|_{\text{int}(U)}$. Remark that by [EH96], the converse is not true. In what follows, any symplectomorphism will be supposed to be global unless otherwise stated.

Chapter 1

Symplectic convexity

Convex domains in \mathbb{R}^{2n} have very strong symplectic rigidity properties such as dynamical convexity and also conjectural properties such as Viterbo conjecture. General star-shaped domains do not satisfy the two previous properties (see [Her]). However, convexity is not necessarily preserved under symplectic transformations. For instance, a symplectomorphism of \mathbb{R}^2 is an area and orientation preserving diffeomorphism, which does not preserve convexity in general.

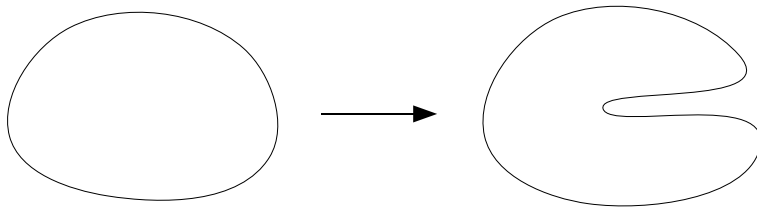


Figure 1.1: Two symplectomorphic domains in \mathbb{R}^2 .

This lead symplectic geometers to ask whether we could define a symplectic analogue of convexity. Naively, we have the following definition:

Definition 1.0.1. *A domain U in \mathbb{R}^{2n} is symplectically convex if it is symplectomorphic to a convex domain C in \mathbb{R}^{2n} .*

Verifying that a given domain U is symplectically convex is difficult since it requires to provide an explicit symplectomorphism φ and to confirm that the image $\varphi(U)$ is indeed convex. In the opposite direction, it is somewhat easier to show that a domain is not symplectically convex. First, find a symplectic property satisfied by convex domains, then, construct a domain for which this property doesn't hold. One of the most well-known example of such property is the *dynamical convexity* (defined in Section 1.2).

Whether dynamically convex domains are all symplectically convex or not is a 20 years old question which was recently answered negatively in dimension 4 by

the ground breaking work of Chaidez and Edtmair [CE22]. They first established a symplectic convexity criterion (see Subsection 1.3.2) based on the Ruelle invariant (see Definition 1.3.1), then constructed dynamically convex domains which do not satisfy this criterion.

In an other direction, symplectic convexity can be studied through the scope of symplectic embeddings' problems. This refers to the well known Viterbo conjecture which relates symplectic capacities and the volume in an isoperimetric type inequality.

1.1 Viterbo conjecture

In this Section, we first discuss about symplectic embeddings problems and introduce symplectic capacities of domains. Then, we state several versions of the Viterbo conjecture and some of its consequences.

1.1.1 Symplectic embeddings

A central problem in modern symplectic geometry is to study embeddings between symplectic manifolds of same dimension (see [Sch05, Sch17] for a survey).

Definition 1.1.1. *Let (X_1, ω_1) and (X_2, ω_2) be two symplectic manifolds, a smooth embedding $\varphi: X_1 \hookrightarrow X_2$ is called a symplectic embedding if $\varphi^*\omega_2 = \omega_1$.*

In this case, we will shortly denote by

$$(X_1, \omega_1) \xrightarrow{Symp} (X_2, \omega_2)$$

for "there exists a symplectic embedding of (X_1, ω_1) into (X_2, ω_2) ". Whether one can symplectically embed a domain of \mathbb{R}^{2n} into another is still a widely unsolved problem even for simplest forms such as balls, ellipsoids and polydisks. In fact, complete solutions have been described only in few cases, for instance, for ellipsoids in dimension four by McDuff [McD11]. Recall that an ellipsoid $E(a, b)$ of symplectic radii $a > 0$ and $b > 0$ is defined by:

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}$$

Theorem 1.1.2 ([McD11], Theorem 1.1). *Let $a, b, c, d > 0$,*

$$\text{int}(E(a, b)) \xrightarrow{Symp} E(c, d) \text{ if and only if } N_k(a, b) \leq N_k(c, d)$$

where $(N_k(a, b))_{k \geq 0}$ is the sequence of non-negative integer linear combinations of a and b arranged in increasing order.

Note that determining if $N_k(a, b) \leq N_k(c, d)$ for any integer k can be a highly non-trivial problem. More generally, one can remark that any symplectic embedding preserves the volume form associated to the symplectic structure. Therefore, a first naive criterion to symplectically embed a domain into another is the following volume constraint.

$$\text{int}(U) \xrightarrow{\text{Symp}} V \implies \text{Vol}(U) \leq \text{Vol}(V)$$

However, symplectic geometry is more rigid than volume-preserving geometry. Let us define the $2n$ -ball $B^{2n}(r)$ of symplectic radius r by

$$B^{2n}(r) := \left\{ z \in \mathbb{C}^n \mid \pi \|z\|^2 \leq r \right\}$$

and the $2n$ -cylinder $Z^{2n}(R)$ of symplectic radius R by

$$Z^{2n}(R) := \left\{ z \in \mathbb{C}^n \mid \pi |z_1|^2 \leq R \right\}.$$

Then, $\psi : (z_1, \dots, z_n) \mapsto \left(\sqrt{\frac{R}{r}} z_1, \dots, \sqrt{\frac{r}{R}} z_n \right)$ is a volume preserving diffeomorphism such that $\psi(B^{2n}(r)) \subset Z^{2n}(R)$ for any $r > R$. But due to Gromov [Gro85] we have

Theorem 1.1.3 ([Gro85], Gromov).

$$\text{int} \left(B^{2n}(r) \right) \xrightarrow{\text{Symp}} Z^{2n}(R) \iff r \leq R$$

1.1.2 Symplectic capacities

The previous result lead Gromov to introduce in [Gro85] the first following quantity which he referred to as the *radius* of a symplectic manifold. For any domain U in \mathbb{R}^{2n} let

$$c_B(U) := \sup \left\{ r > 0 \mid B^{2n}(r) \xrightarrow{\text{Symp}} U \right\} \text{ and } c_Z(U) := \inf \left\{ R > 0 \mid U \xrightarrow{\text{Symp}} Z^{2n}(R) \right\}$$

These two positive numbers are usually called the **ball capacity** of U (or **Gromov width**¹ according to [EH89]) and the **cylindrical capacity** of U . These define applications satisfying the following properties:

- (1) **Monotonicity:** If $U \xrightarrow{\text{Symp}} V$ then $c(U) \leq c(V)$

¹This name, usually used in the literature, might be misleading since, historically, Gromov himself defined the *symplectic width* in another way (see Section 4 in [Gro87]).

(2) **Conformality:** If r is a positive real number then $c(rU) = r^2c(U)$

(3) **Ball normalization:** $c(B^{2n}(1)) = c(Z^{2n}(1)) = 1$

In general, one can define a **symplectic capacity** as a map

$$c : \{\text{domains of } \mathbb{R}^{2n}\} \rightarrow [0, +\infty]$$

satisfying (1) and (2). Moreover, if it satisfies (3), we call it *ball-normalized*. It gives rise to a whole class of symplectic invariants which measure the obstruction to the existence of a symplectic embedding between two domains ². One has directly the following fact:

Remark 1.1.4. *If U is a domain in \mathbb{R}^{2n} then for every ball-normalized symplectic capacity c , we have*

$$c_B(U) \leq c(U) \leq c_Z(U)$$

Since their introduction by Gromov, Ekeland and Hofer, many symplectic capacities have been defined. For example, the Hofer-Zehnder capacity c_{HZ} and the Viterbo capacity c_{SH} [Vit92], but also families of capacities indexed by integers: the Gutt-Hutchings capacities c_k^{GH} (see subsection 2.4.1), the ECH capacities c_k^{ECH} (see subsection 2.3.2) and the Ekeland-Hofer capacities c_k^{EH} (see [CHL⁺07] for a survey on the subject). Moreover, many of them have been shown to be equal on convex domains to the minimal action on the boundary (see proof of Theorem 1.12 in [GHGBR22] for details):

Theorem 1.1.5 (Ekeland, Hofer, Zehnder, Abbondandolo-Kang, Irie). *If U is a bounded convex domain in \mathbb{R}^{2n} with smooth boundary then:*

$$c_1^{EH}(U) = c_{HZ}(U) = c_{SH}(U) = c_1^{GH}(U) = T_{min}(\partial U)$$

In particular, the minimal action is a ball-normalized symplectic capacity for convex domains with smooth boundaries.

1.1.3 Viterbo conjecture

In [Vit00], trying to "relate the symplectic way of measuring size, using so-called capacities, to the classical Riemannian approach, using the volume"³, Viterbo stated the following, now famous, conjecture about symplectic capacities:

²In fact, symplectic capacities can be defined for larger classes of symplectic manifolds but we will restrict ourselves to domains in \mathbb{R}^{2n} .

³See Introduction in [Vit00].

Conjecture 1.1.6 ([Vit00], Section 5, Remark 1). *If U is a convex domain in \mathbb{R}^{2n} and c is a ball-normalized symplectic capacity, then:*

$$c(U) \leq (n! \text{Vol}(U))^{1/n}$$

where $\text{Vol}(U)$ is the euclidean volume of U .

Moreover, equality case holds if and only if the domain is symplectomorphic to the ball.

In other words, among convex domains a symplectic capacity takes its maximal value at the unit ball.

At first, the convexity assumption seems a bit surprising since the inequality is symplectically invariant but convexity is not. There are mainly two reasons for that. In fact, using John's ellipsoid theorem⁴, Viterbo showed in [Vit00] that there exists some constant $\gamma_n \geq 1$ such that

$$c(U) \leq \gamma_n (n! \text{Vol}(U))^{1/n}$$

for any convex domain U in \mathbb{R}^{2n} . Moreover, due to a result by Hermann in [Her], there exists a nice star-shaped domain with arbitrarily small volume but with cylindrical capacity greater than one⁵. Therefore, the conjecture above doesn't hold for the class of nice star-shaped domains.

Then, following Theorem 1.1.5, a strong version of Viterbo conjecture can be considered.⁶

Conjecture 1.1.7 (Strong Viterbo conjecture). *All ball-normalized symplectic capacities agree on convex domains of \mathbb{R}^{2n} .*

By the volume constraint, the Viterbo conjecture is true for c_B , therefore it follows that the strong version of the conjecture implies the inequality in Conjecture 1.1.6.

Since then, Viterbo conjecture has drawn attention even outside the realm of symplectic geometry as it implies the following long-standing conjecture thanks to a result by Artstein-Avidan, Karasev and Ostrover [AKO13, Theorem 1.6].

Conjecture 1.1.8 (Mahler 1939). *Let X be a n -dimensional⁷ normed space,*

$$\text{Vol}(B \times B^o) \geq \frac{4^n}{n!}$$

⁴See Section 4.1.

⁵Note that the example constructed by Hermann is *toric* (see Definition 2.1.1), and other examples of such toric domains were constructed in [GHGBR22, Section 5].

⁶In fact, the strong Viterbo conjecture seems to have been stated in the 90's, even before the Viterbo conjecture.

⁷Strong Viterbo conjecture in dimension $2n$ implies Mähler conjecture in dimension n .

where B is the unit ball of X , B° the one of its dual X^* and Vol is the natural symplectic volume on $X \times X^*$.

So far, the conjecture has only been proved by Mahler for $n = 2$ [Mah39] and by Iriyeh-Shibata for $n = 3$ [IS20]. Following Theorem 1.1.5, if we replace the capacity in the Viterbo conjecture by the minimal action T_{min} , one get the following weak version of Conjecture 1.1.6

Conjecture 1.1.9 (Weak Viterbo conjecture). *If U is a convex domain in \mathbb{R}^{2n} with smooth boundary, then:*

$$\text{sys}(U) := \frac{T_{min}(\partial U)^n}{n! \text{Vol}(U)} \leq 1$$

where sys is called the systolic ratio and is invariant under scaling of the domain.

Let us finish this section by reviewing some recent *local* results about the Viterbo conjectures. In [ABHSa18], the authors showed that the weak Viterbo conjecture holds in a \mathcal{C}^3 -neighborhood of the ball in dimension 4. Furthermore, in [Edt23], Edtmair showed that even the strong version holds in a \mathcal{C}^3 -neighborhood of the ball. Finally, this was generalized by the previous author, Abbondandolo and Benedetti in [ABE23], where they showed that the strong Viterbo conjecture holds in a \mathcal{C}^2 -neighborhood of the unit ball in any dimension. However, they showed that this does not hold for some nice star-shaped domains which are \mathcal{C}^1 -close to the ball.

The Viterbo conjectures are all symplectically invariant statements about convex domains, but as we pointed out previously, convexity is not necessarily preserved under symplectic transformation. This leads us to the following notion of convexity which is symplectically invariant.

1.2 Dynamical convexity

In [HWZ98], by studying the periodic orbits on the boundary of strongly convex domains, Hofer, Wysocki and Zehnder showed in dimension 4 the following theorem⁸.

Theorem 1.2.1 ([HWZ98], Theorem 3.4). *Let $U \subset \mathbb{R}^{2n}$ be a strongly convex domain with smooth boundary (i.e its boundary has uniformly positive curvature). Then for any closed Reeb orbit γ of ∂U ,*

$$\overline{CZ}(\gamma) \geq n + 1$$

⁸In fact, in [HWZ98], they proved the theorem only for $n = 2$. However, by a Remark at the bottom of page 222 in [HWZ98], the authors explain that the same arguments of their proof yield the theorem in \mathbb{R}^{2n} for $n > 2$.

where $\overline{CZ}(\gamma)$ is the canonical Conley-Zehnder index of γ (see Appendix 4.4).

Assumption 1.2.2. Originally, Hofer, Wysocki and Zehnder used the lower semi-continuous extension of the Conley-Zehnder index. In this thesis, we consider the Robbin-Salamon extension of the Conley-Zehnder index that we call the canonical Conley-Zehnder index. A priori, they are different, but in this thesis we **assume** that the dynamical convexity is independent of the choice of extension CZ of the Conley-Zehnder index. This assumption is mainly needed to prove Proposition 2.2.4 which has been already proven in [GHGBR22, Proposition 1.8] using the lower semi-continuous extension. Moreover, all the computations in the toric case (see Chapter 2) lead us to think that the definition below is, indeed, independent of the choice of extension.

According to Section 0.3, this property is invariant under (global) symplectomorphisms, thus is a natural candidate for the notion of symplectic convexity.

Definition 1.2.3. Let $U \subset \mathbb{R}^{2n}$ be a nice star-shaped domain. If for all closed Reeb orbits γ of ∂U ,

$$\overline{CZ}(\gamma) \geq n + 1$$

then U is called **dynamically convex**.

The theorem above raised the following natural question:

Question 1.2.4. Is every dynamically convex domain in \mathbb{R}^{2n} (globally) symplectomorphic to a convex domain?

Remark 1.2.5. Consider two nice star-shaped domains in \mathbb{R}^{2n} which are intrinsically symplectomorphic. It is not known whether if one is dynamically convex, so is the other.

Historically, the above problem was formulated in terms of contact geometry. In fact, there is a one-to-one correspondence between nice star-shaped domains in \mathbb{R}^{2n} and contact forms on S^{2n-1} for the standard structure. Indeed, if $Y = \partial U$ is the boundary of a nice star-shaped domain $U \subset \mathbb{R}^{2n}$ we have a diffeomorphism

$$\begin{aligned} \psi : S^{2n-1} &\longrightarrow Y \\ z &\longmapsto h(z)z \end{aligned}$$

which consists of projecting along rays coming from the origin. More precisely:

$$h(z) := \inf\{\mu \mid \mu z \in Y\}.$$

The diffeomorphism ψ induces a contact form defining the standard contact structure on S^{2n-1} by the following equation:

$$\alpha := \psi^* \left(\lambda_0 \Big|_Y \right) = h^2 \lambda_0 \Big|_{S^{2n-1}}$$

In particular, the Reeb flow on Y is equivalent to the Reeb flow of α on S^{2n-1} . This motivates the following definitions:

Definition 1.2.6. *A contact form α for the standard contact structure of S^{2n-1} is called (strongly) **convex** if its corresponding hypersurface bounds a (strongly) convex domain in \mathbb{R}^{2n} .*

Definition 1.2.7. *A contact form α for the standard contact structure of S^{2n-1} is called **dynamically convex** if for all closed Reeb orbits γ of ∂U ,*

$$\overline{CZ}(\gamma) \geq n + 1$$

Thus, Theorem 1.2.1 can be restated as

Theorem 1.2.8. *Any strongly convex contact form α on S^{2n-1} is dynamically convex.*

According to Section 0.3, dynamical convexity of contact forms is, indeed, invariant under strict contactomorphisms. In this formalism, the above question can be stated as follows.

Question 1.2.9. *Is every dynamically convex contact form on S^{2n-1} also convex?*

A first step towards an answer to this question was achieved by Abbondandolo, Bramham, Hryniewicz and Salomão in [ABHS18].

Theorem 1.2.10 ([ABHS18], Theorem 1.1). *For every $\varepsilon > 0$, there exists a dynamically convex contact form α on S^3 such that:*

$$2 - \varepsilon < \text{sys}(S^3, \alpha) < 2$$

Now recall that, in terms of contact forms, the weak Viterbo conjecture is equivalent to the fact that if α is a convex contact form on S^3 , then

$$\text{sys}(S^3, \alpha) \leq 1$$

where

$$\text{sys}(S^3, \alpha) = \frac{T_{\min}(S^3, \alpha)^2}{\text{Vol}(S^3, \alpha)}$$

Therefore, the authors in [ABHS18] constructed a dynamically convex contact form that doesn't satisfy the weak Viterbo conjecture.

Corollary 1.2.11. *Either the weak Viterbo conjecture is false in \mathbb{R}^4 or there exists dynamically convex contact forms on S^3 which are not convex.*

Finally, the answer to Question 1.2.9 in dimension 3 was given in [CE22] (see following Section 1.3) and in any dimensions in [CE] by the same authors (see Section 3.4).

Other applications

Dynamical convexity plays a more general role than being a symplectic notion of convexity. First, it plays an important role in the existence of periodic orbits. In [Rab79], Rabinowitz proved the existence of a closed orbit on any star-shaped hypersurfaces in \mathbb{R}^{2n} . More generally, a conjecture attributed to Weinstein [Wei79] states that for any contact manifold in certain classes, there exists a closed orbit. In [Vit87], Viterbo proved the Weinstein conjecture for contact type hypersurface⁹ in \mathbb{R}^{2n} . In [Tau07], Taubes proved the conjecture on any contact 3-manifold. Then, one might ask what is the minimal number of distinct closed orbits. Under the hypothesis of being dynamically convex we have the following results.

Theorem 1.2.12 ([GK16], Theorem 1.1). *For any star-shaped hypersurface in \mathbb{R}^{2n} which is non-degenerate¹⁰ and dynamically convex, there are at least n simple¹¹ closed Reeb orbits.*

Theorem 1.2.13 ([HWZ98]). *Let α be a dynamically convex contact form on S^3 , then there is either 2 or infinitely many simple closed Reeb orbits.*

Recently, the dynamical convexity hypothesis in the previous theorem was removed by Cristofaro-Gardiner, Hryniewicz, Hutchings and Liu in [CGHHL23].

Theorem 1.2.14 ([CGHHL23], Theorem 1.1). *Let (Y, ξ) a closed connected 3-manifold such that $c_1(\xi) \in H^2(Y, \mathbb{Z})$ is torsion and $\xi = \ker(\lambda)$. Then λ has either 2 or infinitely many simple closed Reeb orbits.*

Corollary 1.2.15 ([CGHHL23], Corollary 1.3). *For any star-shaped hypersurface in \mathbb{R}^4 , there is either 2 or infinitely many simple closed Reeb orbits.*

Furthermore, in Symplectic Field Theory, contact invariants such as Cylindrical Contact Homology encounter transversality issues in their constructions. However,

⁹Generalization of star-shaped hypersurfaces (see Definition 3.5.32 in [MS17]).

¹⁰A hypersurface Y in \mathbb{R}^{2n} is called non-degenerate if the standard contact form $\lambda_0|_Y$ is non-degenerate.

¹¹A *simple* closed Reeb orbit is an embedded close Reeb orbit.

in [HN14], the authors showed that Cylindrical Contact Homology is well defined for dynamically convex contact forms in dimension 3.

Finally, results in [Edt23] and [HHR23] imply the following step towards the proof of the strong Viterbo conjecture in dimension 4.

Corollary 1.2.16 ([Edt23], Corollary 1.8). *For any dynamically convex domains $U \subset \mathbb{R}^4$, we have*

$$c_Z(U) = c_1^{ECH}(U)$$

where c_Z is the cylindrical capacity defined in Section 1.1 and c_1^{ECH} is the first ECH capacity (see Subsection 2.3.2).

1.3 Ruelle invariant and Chaidez-Edtmair criterion in dimension 4

In this Section, we review the definition of the Ruelle invariant in dimension 3 and explain its uses by Chaidez and Edtmair in [CE22] to answer Question 1.2.9.

1.3.1 Ruelle invariant

To any closed contact 3-manifold (Y, α) which is a homology 3-sphere we can associate the Ruelle invariant $\text{Ru}(Y, \alpha) \in \mathbb{R}$ in the following way.

Observe that the Reeb flow $\varphi_t^{R_\alpha}$ preserves the contact form α . In particular, it preserves the contact structure ξ . Then, for any time t and any fixed point $y \in Y$, the pushforward, or linearization is the following map

$$(\varphi_t^{R_\alpha})_* : \xi_y \longrightarrow \xi_{\varphi_t^{R_\alpha}(y)}$$

Under a given trivialization τ of the contact structure, this linearization can be regarded as a linear transformation of \mathbb{R}^2 , denoted by $\Phi_{y,t}^\tau$. For any real $T \geq 0$, the path $\Phi = \{\Phi_{y,t}^\tau\}_{t \in [0,T]}$ defines an element of the universal cover of the symplectic group $\widetilde{\text{Sp}}(2)$. Together with the rotation number, $\rho : \widetilde{\text{Sp}}(2) \rightarrow \mathbb{R}$ (see subsection 1.2 in [Hut22] or subsection 2.1 in [CE22]), this yields a real number $\rho(y, T, \tau) := \rho(\{\Phi_{y,t}^\tau\}_{t \in [0,T]})$ and the following limit

$$\text{rot}(y) = \lim_{T \rightarrow +\infty} \frac{\rho(y, T, \tau)}{T} \tag{1.1}$$

is well-defined. In particular, $\text{rot}(y)$ is independent of the trivialization τ , in fact, ξ has a unique trivialization up to homotopy since $[Y, S^1] \simeq H^1(Y, \mathbb{Z})$ and Y has the homology of S^3 . In general, $\text{rot}(y)$ only depends on the homotopy class of a

trivialization. As elaborated by Hutchings in [Hut22], subsection 1.2, right above Definition 1.3 (and which traces back to Ruelle in [Rue85]), we have that $\text{rot}(y)$ is an integrable function. Proposition 2.13 in [CE22] also proves these properties.

Definition 1.3.1. *Suppose the closed contact 3-manifold (Y, α) is a homology 3-sphere, then its Ruelle invariant is defined by*

$$\text{Ru}(Y, \alpha) := \int_Y \text{rot}(y) \alpha \wedge d\alpha.$$

In particular, if U is a nice star-shaped domain in \mathbb{R}^4 , then we define

$$\text{Ru}(U) := \text{Ru} \left(\partial U, \lambda_0 \Big|_{\partial U} \right)$$

Moreover, it is in fact a symplectic invariant:

Proposition 1.3.2. *Suppose the closed contact 3-manifolds (Y_1, α_1) and (Y_2, α_2) are homology 3-spheres such that there exists some strict contactomorphism $f : (Y_1, \alpha_1) \rightarrow (Y_2, \alpha_2)$, then*

$$\text{Ru}(Y_1, \alpha_1) = \text{Ru}(Y_2, \alpha_2)$$

Proof. The existence of a strict contactomorphism implies that the Reeb flows are conjugated:

$$\forall t \in \mathbb{R}, \forall y \in Y_1, \phi_2(t, f(y)) = f(\phi_1(t, y))$$

thus by differentiating we have $d\phi_2(t, f(y))df_y = df_{\phi_1(t, y)}d\phi_1(t, y)$. Moreover, given a trivialization τ_1 of $\xi_1 = \ker(\alpha_1)$, we get a trivialization τ_2 of $\xi_2 = \ker(\alpha_2)$ by the following equation:

$$\forall y \in Y_1, \tau_1(y) = \tau_2(f(y))df_y$$

therefore the linearized Reeb flows are given by: $\Phi_2^{T_2}(T, f(y)) = \Phi_1^{T_1}(T, y)$ and finally $\text{rot}_{\tau_1} = f^*\text{rot}_{\tau_2}$. Hence

$$\begin{aligned} \text{Ru}(Y_2, \alpha_2) &= \int_{Y_2} \text{rot}_{\tau_2} \alpha_2 \wedge d\alpha_2 = \int_{f^{-1}(Y_2)} f^*(\text{rot}_{\tau_2} \alpha_2 \wedge d\alpha_2) \\ &= \int_{Y_1} \text{rot}_{\tau_1} \alpha_1 \wedge d\alpha_1 \\ &= \text{Ru}(Y_1, \alpha_1) \end{aligned}$$

□

This invariant was later involved in the following quantitative criterion of symplectic convexity developed by Chaidez and Edtmair.

1.3.2 Chaidez-Edtmair's criterion

In [CE22], the authors constructed the first dynamically convex contact forms that are not convex by establishing a criterion for convex contact forms using the Ruelle invariant and the systolic ratio. More precisely, the Ruelle ratio is defined as

$$\text{ru}(Y, \alpha) = \frac{\text{Ru}(Y, \alpha)}{\text{Vol}(Y, \alpha)^{\frac{1}{2}}}$$

where

$$\text{Vol}(Y, \alpha) = \int_Y \alpha \wedge d\alpha.$$

We can remark that this ratio is invariant under scaling of the contact form.

Definition 1.3.3. *Let (Y, α) be a contact 3-manifold which is a homology 3-sphere, we define the Chaidez-Edtmair's invariant as:*

$$c_{CE}(Y, \alpha) = \text{ru}(Y, \alpha) \cdot \text{sys}(Y, \alpha)^{\frac{1}{2}}$$

The above quantity is invariant under strict contactomorphisms, in particular Chaidez and Edtmair showed the following proposition

Theorem 1.3.4 ([CE22], Proposition 1.9). *There exists two positive constants c and C such that for any given convex¹² contact form α on S^3 , we have*

$$c \leq c_{CE}(S^3, \alpha) \leq C \tag{1.2}$$

Moreover, based on methods of the authors in [ABHSa18, ABHS18], they constructed contact forms on S^3 which violate these bounds.

Proposition 1.3.5 ([CE22], Proposition 1.12). *For every $\varepsilon > 0$, there exists a dynamically convex contact form α on S^3 satisfying*

$$\text{Vol}(S^3, \alpha) = 1 \quad \text{sys}(S^3, \alpha) \geq 1 - \varepsilon \quad \text{Ru}(S^3, \alpha) \leq \varepsilon$$

and there exists a dynamically convex contact form β on S^3 satisfying

$$\text{Vol}(S^3, \beta) = 1 \quad \text{sys}(S^3, \beta) \geq 1 - \varepsilon \quad \text{Ru}(S^3, \beta) \geq \varepsilon^{-1}$$

Therefore, they answered negatively Question 1.2.9 in dimension 3, thus answering a long standing conjecture which goes back to Hofer, Wysocki and Zehnder (cf Theorem 1.2.1).

¹²See Definition 1.2.6.

Corollary 1.3.6. *There exists dynamically convex contact forms on S^3 which are not convex.*

Coming back to nice star-shaped domains in \mathbb{R}^4 , Theorem 1.3.4 and Corollary 1.3.6 can be rephrased equivalently (see Definition 1.2.6) as follows:

Theorem 1.3.7 ([CE22], Proposition 3.1). *Given a nice star-shaped domain U in \mathbb{R}^4 , if U is symplectically convex, then*

$$c \leq c_{CE}(U) \leq C \tag{1.3}$$

where $c_{CE}(U) := c_{CE}(\partial U, \lambda_0|_{\partial U})$ and c and C are positive constants, independent of the domain U .

Corollary 1.3.8. *There exist dynamically convex domains in \mathbb{R}^4 that are not symplectically convex.*

The previous corollary is the starting point of this thesis. In the rest of this manuscript, we develop new techniques to construct examples of such domains with or without referring to Chaidez-Edtmair's criterion. In particular, all of our examples of such domains have the additional property of being *toric*.

Chapter 2

Toric domains

In this Chapter, we introduce the so-called toric domains of \mathbb{R}^4 which are domains invariant under the \mathbb{T}^2 action. In Section 2.1, we give the definition and examples of large class of such domains. In Section 2.2, we study the Reeb dynamic on their boundary and compute their Ruelle invariant. In Section 2.3, we give several formulas to compute the symplectic capacities of toric domains and recall the proof that they satisfy the strong Viterbo conjecture. Finally, in Section 2.4, we mention the extensions in higher dimensions and how to compute some other symplectic capacities of these domains.

2.1 Definition

Toric domains, also called Reinhardt domains, are often studied as toy models by symplectic geometers to investigate symplectic embeddings' problems [Her, CCGF⁺14, Ram15, CG14, OR19, GHGBR22, ORS23]. They are defined as follows:

Definition 2.1.1. *A domain $X \subset \mathbb{C}^2$ is called **toric** if it is invariant under the action of the 2-torus \mathbb{T}^2 i.e. if $(z_1, z_2) \in X$, $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2$ then*

$$(e^{i\theta_1}, e^{i\theta_2}).(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in X$$

The symmetry of toric domains allow us to describe these four-dimensional object in terms of a two-dimensional figure.

Proposition 2.1.2. *Any toric domain containing the origin can be written as $X_\Omega = \mu^{-1}(\Omega)$ where Ω is an open neighbourhood of the origin in $\mathbb{R}_{\geq 0}^2$ and*

$$\begin{aligned} \mu : \mathbb{C}^2 &\longrightarrow \mathbb{R}_{\geq 0}^2 \\ (z_1, z_2) &\longmapsto (\pi|z_1|^2, \pi|z_2|^2) \end{aligned}$$

μ is called the moment map and Ω is called the moment image of X_Ω . The factor π is chosen to insure that $\text{Vol}_{\mathbb{R}^4}(X_\Omega) = \text{Vol}_{\mathbb{R}^2}(\Omega)$.

Example 2.1.3. Recall that for any $a, b > 0$, the ellipsoid of symplectic radii a and b is defined by

$$E(a, b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}. \quad (2.1)$$

Thus $E(a, b) = X_{T(a, b)}$ where

$$T(a, b) := \left\{ (x, y) \in \mathbb{R}_{\geq 0}^2 \mid \frac{x}{a} + \frac{y}{b} \leq 1 \right\}$$

The polydisk as defined by

$$P(a, b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \leq a, \pi|z_2|^2 \leq b \right\}. \quad (2.2)$$

has moment image showed in Figure 2.1.

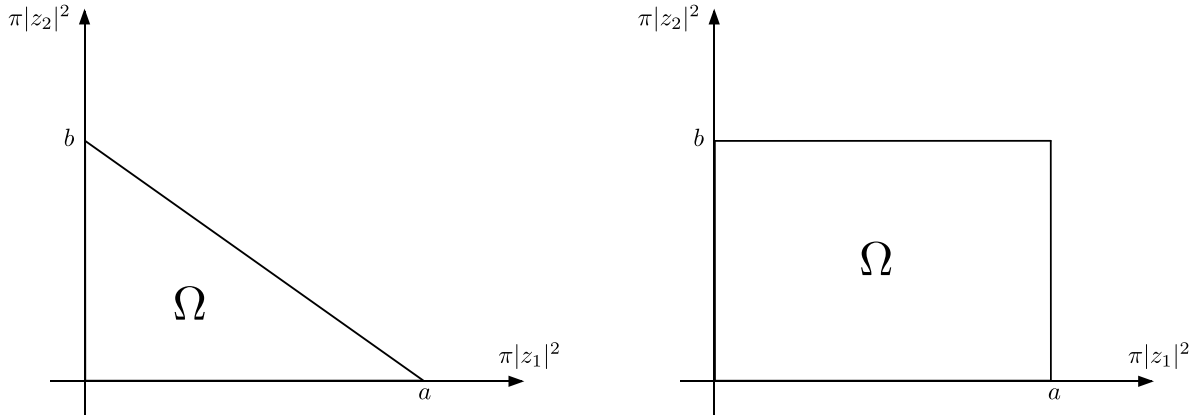
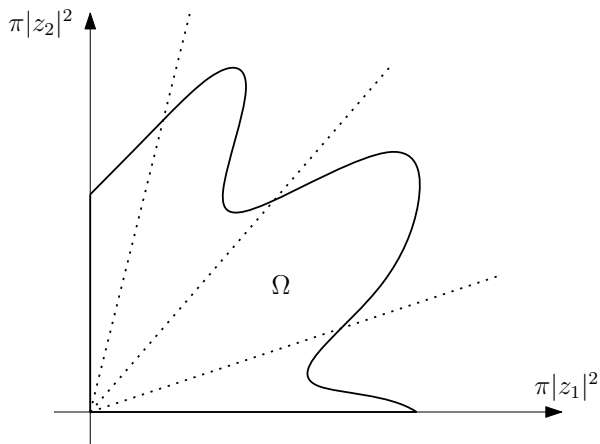


Figure 2.1: An ellipsoid $E(a, b)$ and a polydisk $P(a, b)$

We will see that part of the information of a given toric domain X_Ω can be read off the geometry of its moment image Ω .

Lemma 2.1.4 ([Her], Lemma 2.5). *Let X_Ω be toric domain, then it is a nice star-shaped domain if and only if Ω is star-shaped with respect to the origin in $\mathbb{R}_{\geq 0}^2$ and $\overline{\partial_+ \Omega}$ is a smooth curve, where $\partial_+ \Omega := \partial \Omega \cap \mathbb{R}_{> 0}^2$.*

One should keep in mind that $\mu^{-1}(\overline{\partial_+ \Omega}) = \partial X_\Omega$. Examples of nice star-shaped toric domains are given by the following definitions of (*weakly*) *convex* toric domains and *concave* toric domains.



- (1) Moment image Ω is the region enclosed by bold black curve.
- (2) $X_\Omega = \mu^{-1}(\Omega)$ and it is star-shaped by the transversality from dotted rays.

Figure 2.2: An example of a star-shaped toric domain.

Definition 2.1.5. A *weakly convex* toric domain X_Ω is a nice star-shaped toric domain such that Ω is convex. Moreover, it is a *convex* toric domain if

$$\widehat{\Omega} := \{\mu \in \mathbb{R}^2 \mid (|\mu_1|, |\mu_2|) \in \Omega\}$$

is convex and compact in \mathbb{R}^2 .

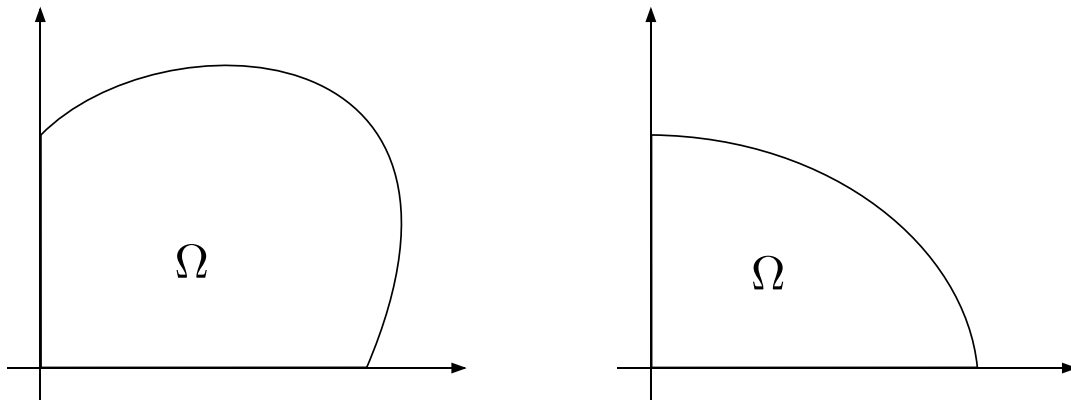


Figure 2.3: A *weakly convex* toric domain and a *convex* toric domain.

Definition 2.1.6. A *concave* toric domain X_Ω is a nice star-shaped toric domain such that $\mathbb{R}_{\geq 0}^2 \setminus \Omega$ is convex.

Remark 2.1.7. All convex toric domains are geometrically convex as domains in \mathbb{R}^4 . Unfortunately, the nomenclature can be confusing since there are concave toric domains which are also geometrically convex.

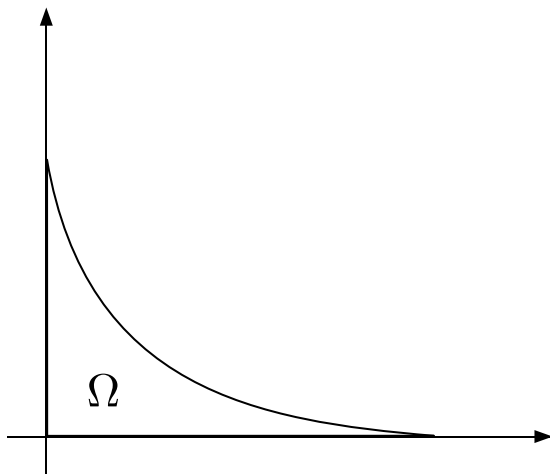


Figure 2.4: A *concave* toric domain.

Going back to symplectic geometry, it follows from the previous lemma that any nice star-shaped toric domain X_Ω is an open symplectic manifold with its boundary ∂X_Ω being a contact manifold as seen in Example 0.3.5.

2.2 Reeb dynamic on the boundary ∂X_Ω

In this section, we study the Reeb dynamic on the boundary ∂X_Ω of any nice star-shaped toric domain. First, we show the existence of closed Reeb orbits and compute their action, then compute the linearized Reeb flow in order to establish a formula for the Ruelle invariant of nice star-shaped toric domains.

2.2.1 Reeb orbits and their action

Let us consider $\mathbb{R}_{\geq 0}^2$ with coordinates (w_1, w_2) , recall that $\partial_+ \Omega := \partial \Omega \cap \mathbb{R}_{> 0}^2$ and define $p_1 := \overline{\partial_+ \Omega} \cap \{w_2 = 0\}$, $p_2 := \overline{\partial_+ \Omega} \cap \{w_1 = 0\}$. We will study the Reeb dynamic in each of these components (see Figure 2.5).

Orbits coming from $\partial_+ \Omega$

For any point $p = (w_1, w_2) \in \partial_+ \Omega$, consider the polar coordinate $(w_1, \theta_1, w_2, \theta_2)$. Then, for any $z \in \mu^{-1}(p)$, one can verify that the Reeb vector field R is

$$R(z) = \frac{2\pi}{\nu_1(p)w_1 + \nu_2(p)w_2} \left(\nu_1(p) \frac{\partial}{\partial \theta_1} + \nu_2(p) \frac{\partial}{\partial \theta_2} \right) \quad (2.3)$$

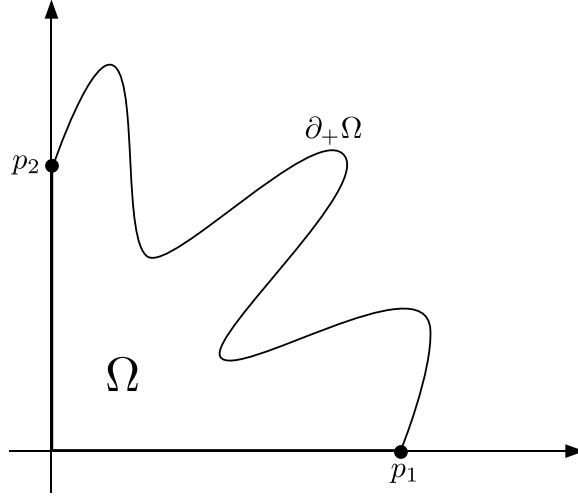


Figure 2.5: The three different components where live the Reeb orbits

where $(\nu_1(p), \nu_2(p))$ is the unit normal vector of $\partial_+\Omega$ at point p , pointing outward of Ω . By (2.3), for a point $p = (w_1, w_2) \in \partial_+\Omega = \partial\Omega \cap \mathbb{R}_{>0}^2$, we know that

$$\text{a Reeb trajectory at } \mu^{-1}(p) \text{ is closed} \quad \text{if and only if} \quad \exists q \in \mathbb{Q}, \nu_2(p) = q\nu_1(p). \quad (2.4)$$

By (2.3), for any $z \in \mu^{-1}(p)$, the Reeb orbits are given by

$$\begin{aligned} \phi_R^t(z) &= \phi_{R_\alpha}^t(w_1, \theta_1, w_2, \theta_2) \\ &= (w_1, \theta_1 + \Theta_1 \cdot t, w_2, \theta_2 + \Theta_2 \cdot t) \end{aligned}$$

where

$$\Theta_1 = \frac{2\pi\nu_1(p)}{\nu_1(p)w_1 + \nu_2(p)w_2} \quad \text{and} \quad \Theta_2 = \frac{2\pi\nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2},$$

Moreover, assume that $\exists q \in \mathbb{Q}, \nu_2(p) = q\nu_1(p)$. Denote by $h_p \in \mathbb{R}_{>0}$ the unique non-zero positive scalar such that

- (i) $(h_p\nu_1(p), h_p\nu_2(p)) \in \mathbb{Z}^2$;
- (ii) $h_p\nu_1(p), h_p\nu_2(p)$ are coprime.

For brevity, denote $(m_p, n_p) := (h_p\nu_1(p), h_p\nu_2(p))$. By (i), the orbits in $\mu^{-1}(p)$ are closed. And by (ii) they have all the same minimal action $\mathcal{A}(\gamma_{(m_p, n_p)})$ where $\gamma_{(m_p, n_p)}$ represents the family of orbits in $\mu^{-1}(p)$. Hence,

$$\mathcal{A}(\gamma_{(m_p, n_p)}) = (\nu_1(p)w_1 + \nu_2(p)w_2) \cdot h_p = m_p w_1 + n_p w_2. \quad (2.5)$$

This can be viewed as the inner product of the (integer-normalized) normal vector

(m_p, n_p) and the position vector (w_1, w_2) (for point p). Due to our hypothesis that X_Ω is star-shaped, the action $\mathcal{A}(\gamma_{(m_p, n_p)})$ is always positive, even though the vector (m_p, n_p) does not have both of its components positive.

Orbits coming from p_1 and p_2 .

For $p_1 = (w_1, 0) \in \overline{\partial_+ \Omega}$, consider the mixed coordinates $(w_1, \theta_1, x_2, y_2)$. Then, for any $z \in \mu^{-1}(p_1)$, one can verify that the Reeb vector field R is

$$R(z) = \frac{2\pi}{w_1} \frac{\partial}{\partial \theta_1} \quad (2.6)$$

Therefore, there is a unique closed Reeb orbit up to reparameterization that we will denote γ_{p_1} :

$$\gamma_{p_1}(t) = \left(w_1, \theta_1 + \frac{2\pi}{w_1} t, 0, 0 \right)$$

And its action is given by:

$$\mathcal{A}(\gamma_{p_1}) = w_1$$

Similarly, there is a unique closed Reeb orbit at $p_2 = (0, w_2)$ with symplectic action $\mathcal{A}(\gamma_{p_2}) = w_2$.

2.2.2 Linearized Reeb flow on $\partial_+ \Omega$

We can now, compute the linearized Reeb flow. As said previously, $\nu_1(p)w_1 + \nu_2(p)w_2 > 0$ for any $p \in \partial_+ \Omega$ due to our hypothesis that X_Ω is star-shaped. Moreover, the contact 2-plane at z is given by,

$$\xi_z = \left\{ a_1 \frac{\partial}{\partial w_1} + b_1 \frac{\partial}{\partial \theta_1} + a_2 \frac{\partial}{\partial w_2} + b_2 \frac{\partial}{\partial \theta_2} \mid \begin{array}{l} \nu_1(p)a_1 + \nu_2(p)a_2 = 0 \\ w_1 b_1 + w_2 b_2 = 0 \end{array} \right\}, \quad (2.7)$$

and one can choose a basis of ξ_z as follows,

$$e_1(p) = -\nu_2(p) \frac{\partial}{\partial w_1} + \nu_1(p) \frac{\partial}{\partial w_2} \quad \text{and} \quad e_2(p) = -w_2 \frac{\partial}{\partial \theta_1} + w_1 \frac{\partial}{\partial \theta_2}. \quad (2.8)$$

Note that $(e_1(p), e_2(p))$ is an *ordered* basis in that $(\omega_0)_z(e_1(p), e_2(p)) > 0$. Using this basis, along any Reeb trajectory $\gamma = (\gamma(t))_{t \in [0, T]}$, one can chose a trivialization $\tau : \gamma^* \xi \rightarrow \gamma \times \mathbb{R}^2$ explicitly defined as follows. For any $(z, v) \in (\gamma^* \xi)_z$ where $z \in \gamma$ and $v \in \xi_z$,

$$\tau(p)((z, v)) = (z, (v_R, v_\theta)) \quad \text{where} \quad v = v_R e_1(p) + v_\theta e_2(p). \quad (2.9)$$

Moreover, under this trivialization, the differentials of the Reeb flow along the trajectory γ form a path in $\text{Sp}(2)$, denoted by Φ . The following lemma is crucial.

Lemma 2.2.1. *With respect to the trivialization given in (2.9), along the Reeb trajectory $\gamma = (\gamma(t))_{t \in [0, T]}$ the resulting path Φ in $\text{Sp}(2)$ from the differentials of the Reeb flow is*

$$\Phi = \left\{ \begin{pmatrix} 1 & 0 \\ f(t) & 1 \end{pmatrix} \middle| t \in [0, T] \right\}$$

where $f(t)$ is a linear function of t depending only on $\gamma(0)$ and $e_1(\mu(\gamma(0)))$ in (2.8). In particular, $\rho(\gamma(0), T, \tau) = 0$.

Proof. Suppose $\gamma(0) \in \mu^{-1}(p)$ for some $p \in \partial_+ \Omega$. For $v \in \xi_{\gamma(0)}$ and any $t \in [0, T]$, to compute the differential $(d\phi_R^t|_{\xi_{\gamma(0)}})(v)$, we need to take a locally defined smooth path $r(s) : (-\epsilon, \epsilon) \rightarrow \partial X_\Omega$ for ϵ sufficiently small such that $r(0) = \gamma(0)$ and $r'(0) = v$. Denote for brevity $r(s) = (w_1(s), \theta_1(s), w_2(s), \theta_2(s))$ where $w_1(0) = w_1$ and $w_2(0) = w_2$. For any $s \in (-\epsilon, \epsilon)$, by (2.8) and (2.9),

$$\begin{aligned} r(s) &= r(0) + sv + o(s) \\ &= (w_1 - s\nu_2(p)v_R, \theta_1 - sw_2v_\theta, w_2 + s\nu_1(p)v_R, \theta_2 + sw_1v_\theta) + o(s). \end{aligned}$$

Note that the approximation term $o(s)$ exist to guarantee that $r(s) \in \partial X_\Omega$. Then, by (2.3), recall we have

$$\begin{aligned} \phi_R^t(r(s)) &= \phi_{R_\alpha}^t(w_1(s), \theta_1(s), w_2(s), \theta_2(s)) \\ &= (w_1(s), \theta_1(s) + \Theta_1(s) \cdot t, w_2(s), \theta_2(s) + \Theta_2(s) \cdot t) \end{aligned}$$

where

$$\Theta_1(s) = \frac{2\pi\nu_1(p(s))}{\nu_1(p(s))w_1(s) + \nu_2(p(s))w_2(s)} \quad \text{and} \quad \Theta_2(s) = \frac{2\pi\nu_2(p(s))}{\nu_1(p(s))w_1(s) + \nu_2(p(s))w_2(s)},$$

Observe that the denominator of $\Theta_1(s)$ and $\Theta_2(s)$ can be simplified as follows,

$$\nu_1(p(s))w_1(s) + \nu_2(p(s))w_2(s) = \nu_1(p(s))w_1 + \nu_2(p(s))w_2 + o(s).$$

In particular, it converges to $\nu_1(p)w_1 + \nu_2(p)w_2$ as $s \rightarrow 0$. Then, by the definition of

a differential and computations above,

$$\begin{aligned} (d\phi_R^t)(v) &= \lim_{s \rightarrow 0} \frac{\phi_R^t(r(s)) - \phi_R^t(r(0))}{s} \\ &= (-\nu_2(p)v_R, -w_2v_\theta, \nu_1(p)v_R, w_1v_\theta) \\ &\quad + \left(0, \lim_{s \rightarrow 0} \frac{\Theta_1(s) - \frac{2\pi\nu_1(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} \cdot t, 0, \lim_{s \rightarrow 0} \frac{\Theta_2(s) - \frac{2\pi\nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} \cdot t \right). \end{aligned}$$

Meanwhile, further simplifications yield

$$\lim_{s \rightarrow 0} \frac{\Theta_1(s) - \frac{2\pi\nu_1(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} = 2\pi \cdot \frac{(\nu_1(p)\nu_2(p(s))'|_{s=0} - \nu_1(p(s))'|_{s=0}\nu_2(p))(-w_2)}{(\nu_1(p)w_1 + \nu_2(p)w_2)^2}, \quad (2.10)$$

and similarly,

$$\lim_{s \rightarrow 0} \frac{\Theta_2(s) - \frac{2\pi\nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} = 2\pi \cdot \frac{(\nu_1(p)\nu_2(p(s))'|_{s=0} - \nu_1(p(s))'|_{s=0}\nu_2(p))w_1}{(\nu_1(p)w_1 + \nu_2(p)w_2)^2}, \quad (2.11)$$

where the $\nu_i(p(s))'|_{s=0}$ denotes the derivative with respect to the variable s and then evaluated at $s = 0$. For brevity, denote by

$$A(p; v) := 2\pi \cdot \frac{\nu_1(p)\nu_2(p(s))'|_{s=0} - \nu_1(p(s))'|_{s=0}\nu_2(p)}{(\nu_1(p)w_1 + \nu_2(p)w_2)^2},$$

the common factor in (2.10) and (2.11). Then

$$(d\phi_{R_\alpha}^t)(v) = (-\nu_2(p)v_R, (A(p; v) + v_\theta)(-w_2)t, \nu_1(p)v_R, (A(p; v) + v_\theta)w_1t). \quad (2.12)$$

In particular,

$$d\phi_R^t(e_1(p)) = e_1(p) + (A(p; e_1(p))t)e_2(p) \quad \text{and} \quad d\phi_R^t(v) = e_2(p).$$

Representing this by a matrix with respect to the basis $(e_1(p), e_2(p))$, one gets that

$$d\phi_R^t|_{\xi_{\gamma(0)}} = \begin{pmatrix} 1 & 0 \\ A(p; e_1(p))t & 1 \end{pmatrix}. \quad (2.13)$$

Thus we prove the first conclusion by setting $f(t) := A(p; e_1(p))t$. Moreover, the second conclusion is straightforward, since each matrix representation of the differential $d\phi_R^t|_{\xi_z}$ as in (2.13) is similar to a shear matrix, which does not contribute any rotations. \square

2.2.3 Ruelle invariant

The Ruelle invariant can be easily computed for nice star-shaped toric domains by the following proposition that generalizes [Hut22, Proposition 1.11]. In fact, here, there is no hypothesis that the profile curve, as the boundary $\partial_+\Omega = \partial\Omega \cap \mathbb{R}_{>0}^2$, has slopes everywhere negative (cf. [Hut22, footnote on page 6]).

Proposition 2.2.2. *Let X_Ω be any 4-dimensional nice star-shaped toric domain. Then its Ruelle invariant is given by*

$$\text{Ru}(X_\Omega) = a(\Omega) + b(\Omega)$$

where $a(\Omega)$ and $b(\Omega)$ are the w_1 -intercept and w_2 -intercept, respectively, of the moment image Ω in $\mathbb{R}_{\geq 0}^2$, in (w_1, w_2) -coordinate.

Proof. Note that the trivialization in (2.9) does not extend to the entire ∂X_Ω (since the polar coordinate is not well-defined for the points where $w_1 = 0$ or $w_2 = 0$). For any globally defined trivialization $\bar{\tau}$, compared with the trivialization via the polar coordinate, the only difference of the rotation number at point $z \in \mu^{-1}(p)$ for $p \in \partial_+\Omega$ comes from how much the function $\theta_1 + \theta_2$ changes along the Reeb flow. Indeed, as explained in the proof of Lemma 2.2 in [Hut22], moving along the circle given by a rotation of either θ_1 or θ_2 results in the desired change of the factor ν_R in (2.9), with respect to the trivialization $\bar{\tau}$. Then Lemma 2.2.1 yields

$$\text{rot}(z) = \lim_{T \rightarrow \infty} \frac{\rho(z, T, \bar{\tau})}{T} = 0 + \frac{(d\theta_1 + d\theta_2)(R_\alpha(z))}{2\pi} = \frac{\nu_1(p) + \nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}$$

where $p = (w_1, w_2)$. Note that $\text{rot}(z)$ is in fact a function of $p \in \partial_+\Omega$. Then by the definition of Ruelle invariant,

$$\begin{aligned} \text{Ru}(X_\Omega) &= \int_{\partial X_\Omega} \text{rot}(z) \alpha \wedge d\alpha \\ &= \int_{\partial_+\Omega} \frac{\nu_1(p) + \nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2} (w_1 dw_2 - w_2 dw_1) \end{aligned}$$

where the second equality comes from a change of variable via the moment map μ defined in (2.1.2) (and restricted to ∂X_Ω). Suppose the profile curve $\overline{\partial_+\Omega}$ is parametrized by $\{(w_1(s), w_2(s))\}_{s \in [0,1]}$ such that

$$w_1(0) = a(\Omega), \quad w_1(1) = 0 \quad \text{and} \quad w_2(0) = 0, \quad w_2(1) = b(\Omega),$$

where $a(\Omega)$ and $b(\Omega)$ are the w_1 -intercept and w_2 -intercept. We may assume after a

change of parametrization that

$$w_1 dw_2 - w_2 dw_1 = (w_1(s)w_2'(s) - w_2(s)w_1'(s))ds.$$

Meanwhile, observe that

$$(\nu_1(p), \nu_2(p)) = (\nu_1(s), \nu_2(s)) = \left(\frac{-w_2'(s)}{\sqrt{|w_1'(s)|^2 + |w_2'(s)|^2}}, \frac{w_1'(s)}{\sqrt{|w_1'(s)|^2 + |w_2'(s)|^2}} \right). \quad (2.14)$$

Therefore, by (2.14),

$$\begin{aligned} \text{Ru}(X_\Omega) &= \int_0^1 \frac{\nu_1(s) + \nu_2(s)}{\nu_1(s)w_1(s) + \nu_2(s)w_2(s)} (w_1(s)w_2'(s) - w_2(s)w_1'(s))ds \\ &= \int_0^1 \frac{-w_2'(s) + w_1'(s)}{-w_2'(s)w_1(s) + w_1'(s)w_2(s)} (w_1(s)w_2'(s) - w_2(s)w_1'(s))ds \\ &= \int_0^1 w_2'(s) - w_1'(s)ds \\ &= (w_2(1) - w_2(0)) - (w_1(1) - w_1(0)) = b(\Omega) + a(\Omega). \end{aligned}$$

□

2.2.4 Monotone toric domains

For any nice star-shaped toric domains, the above study of their dynamic can be summarized as follows: up to reparameterization there is one closed Reeb orbit γ_{p_i} at the w_i -intercept and for any point p on $\partial_+\Omega$ with rational slope, there exists a S^1 -family of periodic orbits $\gamma_{(m_p, n_p)}$. Moreover, one can compute their canonical Conley-Zehnder index (see Appendix 4.4):

Proposition 2.2.3 ([HZ], Theorem 1.7). *Let γ_{p_i} for $i = 1, 2$ and $\gamma_{(m_p, n_p)}$ be the two type of closed Reeb orbits in ∂X_Ω . Moreover, let assume that $\nu_2(p_1)$ and $\nu_1(p_2)$ are never zero, and suppose that p is not an inflection point of $\partial_+\Omega$. Then the following formulas compute their canonical CZ-indices. For any $k \in \mathbb{N}_{>0}$,*

$$\overline{\text{CZ}}(\gamma_{p_1}^k) = 2k + 2 \left\lfloor k \frac{\nu_2(p_1)}{\nu_1(p_1)} \right\rfloor + 1 \quad \text{and} \quad \overline{\text{CZ}}(\gamma_{p_2}^k) = 2k + 2 \left\lfloor k \frac{\nu_1(p_2)}{\nu_2(p_2)} \right\rfloor + 1 \quad (2.15)$$

and

$$\overline{\text{CZ}}(\gamma_{(km_p, kn_p)}) = \begin{cases} 2k(m_p + n_p) + \frac{1}{2} & \text{if } \partial_+\Omega \text{ is convex near } p \\ 2k(m_p + n_p) - \frac{1}{2} & \text{if } \partial_+\Omega \text{ is concave near } p \end{cases} \quad (2.16)$$

where $\partial_+\Omega$ is convex near p if locally at p , Ω lies entirely in one side of $T_p\partial_+\Omega$, and it is concave otherwise.

This induces the following purely geometrical interpretation of dynamical convexity for nice star-shaped toric domains in terms of the moment image (see Figure 2.6).

Proposition 2.2.4 ([GHGBR22], Proposition 1.8.). *Let X_Ω be a nice star-shaped toric domain in \mathbb{R}^4 , then X_Ω is dynamically convex if and only if the outward normal vector at any point of $\overline{\partial_+\Omega}$ has positive components.*

Proof. Assume that X_Ω is dynamically convex, according to Equation 2.15 it follows that the coordinates of the outward normal vector $(\nu_1(p_i), \nu_2(p_i))$ at p_i satisfy

$$\frac{\nu_2(p_1)}{\nu_1(p_1)} \geq 0 \quad \text{and} \quad \frac{\nu_1(p_2)}{\nu_2(p_2)} \geq 0$$

The case where these ratios are zero is a degenerate case for which the formulas of Proposition 2.2.3 do not hold. In fact, one can show that, in this case, the corresponding canonical Conley-Zehnder index is equal to 2 and so the above inequalities are strict. Since X_Ω is star-shaped, they can't be both non-positive. Therefore, they are both positive. Furthermore, Equation 2.16 implies

$$m_p + n_p \geq 2$$

since m_p and n_p are both natural integers we get the desired conclusion. The converse follows from the same observations. \square

Remark 2.2.5. *In dimension 4, the geometric condition in Proposition 2.2.4 can be reformulated as follows: $\overline{\partial_+\Omega}$ is the graph of a function with strictly negative derivative. Even if this formulation is more visual, we will prefer the one from Proposition 2.2.4 as it extends more easily to higher dimensions (see Section 2.4).*

Following [GHGBR22], these toric domains are called *strictly monotone* toric domains.

Definition 2.2.6. *A (strictly) monotone toric domain is a nice star-shaped toric domain such that for every $\mu \in \overline{\partial_+\Omega} = \partial\Omega \cap \mathbb{R}_{\geq 0}^2$ the outward unit normal vector at μ has (strictly) non-negative components.*

Remark that any monotone toric domain can be approximated by strictly monotone toric domains with respect to the Hausdorff distance. In particular, this approximation can be expressed in terms of moment image.

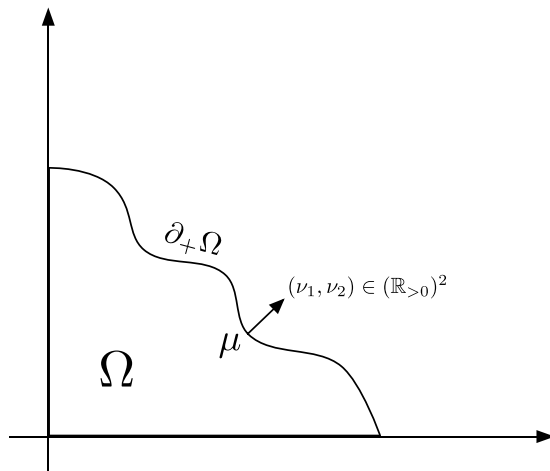


Figure 2.6: A monotone toric domain.

2.3 Symplectic capacities formulas

In many cases, one can compute the symplectic capacities of toric domains, making them an interesting class of domains on which to study symplectic embeddings. In this Section, we first establish a formula for the ball capacity of a monotone toric domain. Then, we introduce the ECH capacities and some of their application to symplectic embeddings. Finally, by using ECH capacities, we give a proof that all ball-normalized symplectic capacities coincide on monotone toric domains. In particular, any ball-normalized symplectic capacity of monotone toric domain can be computed via its ball capacity.

2.3.1 Ball-normalized capacities

Recall that a ball-normalized symplectic capacity c is a symplectic capacity such that $c(B^4(1)) = c(Z^4(1)) = 1$. The following theorem provides a geometric way to compute the ball capacity of any monotone toric domain:

Theorem 2.3.1 ([GHGBR22], Theorem 1.11). *Let X_Ω be a monotone toric domain in \mathbb{R}^4 ,*

$$c_B(X_\Omega) = c_1^{GH}(X_\Omega) = \sup\{c \mid B(c) \subset X_\Omega\}$$

where c_1^{GH} is the first Gutt-Hutchings capacity¹.

This means that the ball capacity can be computed by finding the width of the biggest ball that can be geometrically included into X_Ω . In terms of moment image,

¹Cf subsection 2.4.1.

the right hand side of the equation is the width of the biggest isosceles right triangle, with edges on the axis of $\mathbb{R}_{\geq 0}^2$, that can fit into the moment image Ω .

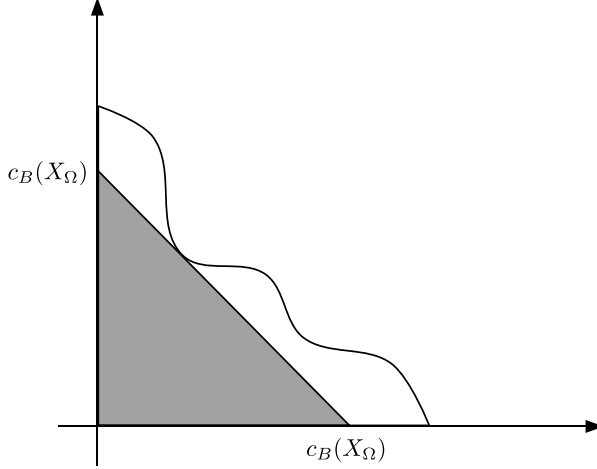


Figure 2.7: The ball capacity of a toric domain X_Ω on its moment image.

We follow the proof of [GHGBR22] which relies on the following lemma. For $a, b > 0$, let us consider the L -shaped domain:

$$L(a, b) = \{(w_1, w_2) \in \mathbb{R}_{\geq 0}^2 \mid w_1 \leq a \text{ or } w_2 \leq b\}$$

Lemma 2.3.2 ([GHGBR22], Lemma 3.1). *Let $a, b > 0$, then*

$$c_1^{GH}(L(a, b)) = a + b$$

Proof of Theorem 2.3.1. Let $\bar{c} = \sup\{c \mid B(c) \subset X_\Omega\}$, so by definition $\bar{c} \leq c_B(X_\Omega)$. Moreover, since X_Ω is monotone, there exists some $a, b > 0$ such that $a + b = \bar{c}$ and $X_\Omega \subset L(a, b)$, by Lemma 2.3.2 this implies $c_1^{GH}(X_\Omega) \leq c_1^{GH}(L(a, b)) = a + b = \bar{c}$. The desired conclusion follows from Remark 1.1.4. \square

Remark 2.3.3. *It follows from the above proof that*

$$c_B(X_\Omega) = \min\{a + b \mid X_\Omega \subset L(a, b)\} = \min\{a + b \mid (a, b) \in \overline{\partial_+ \Omega}\}$$

Moreover, one can show that

Proposition 2.3.4. *Let X_Ω be a strictly monotone toric domain in \mathbb{R}^4 ,*

$$c_B(X_\Omega) = T_{\min}(\partial X_\Omega)$$

Proof. For any closed Reeb orbit γ corresponding to a point $p = (w_1, w_2) \in \partial_+\Omega$, consider (m_p, n_p) the integer-valued normal vector at p , then

$$\mathcal{A}(\gamma_{(m_p, n_p)}) = m_p w_1 + n_p w_2 \geq w_1 + w_2 \geq c_B(X_\Omega)$$

where the first inequality comes from X_Ω being strictly monotone and the second from Remark 2.3.3.

For any closed Reeb orbit γ corresponding to a point p_i with $i \in \{1, 2\}$,

$$\mathcal{A}(\gamma_{p_i}) = w_i \geq \sup\{c \mid B(c) \subset X_\Omega\} = c_B(X_\Omega)$$

Combining both inequalities one gets $T_{min}(\partial X_\Omega) \geq c_B(X_\Omega)$.

Moreover, there exists some closed Reeb orbit γ at the point $p = (w_1, w_2) \in \overline{\partial_+\Omega}$ such that $w_1 + w_2 = c_B(X_\Omega)$. In fact, the slope at p of $\partial_+\Omega$ is $-1 \in \mathbb{Q}$ and so the integer valued normal vector at p is $(1, 1)$. Finally, $\mathcal{A}(\gamma) = w_1 + w_2 = c_B(X_\Omega)$ and $c_B(X_\Omega) \geq T_{min}(\partial X_\Omega)$. \square

In particular, we can compute easily the systolic ratio $\text{sys}(X_\Omega) = \frac{T_{min}(\partial X_\Omega)^2}{2\text{Vol}(X_\Omega)}$ of any strictly monotone toric domains X_Ω and therefore, combined with Proposition 2.2.2, Chaidez-Edtmair's invariant $c_{CE}(X_\Omega)$ (see Definition 1.3.3). Remark that Proposition 2.3.4 doesn't hold for every monotone toric domains.

Proposition 2.3.5. *For any $0 < \varepsilon < 1$, there exists a monotone toric domain X_Ω such that*

$$c_B(X_\Omega) = 1 \quad \text{and} \quad T_{min}(\partial X_\Omega) \leq \varepsilon$$

Proof. Start with the unit ball $B^4(1)$ and add a little bump near the point $(1 - \varepsilon, \varepsilon)$ on the moment image (see Figure 2.8). According to Theorem 2.3.1, we have directly $c_B(X_\Omega) = 1$. Then for some $\delta > 0$, we made the bump so that the normal vector at $(1 - \varepsilon + \delta, \varepsilon)$ is $\nu = (0, 1)$. Thus, there is a closed Reeb orbit γ whose action is equal to $\mathcal{A}(\gamma) = (1 - \varepsilon + \delta) \times 0 + \varepsilon \times 1 = \varepsilon$ and finally $T_{min} \leq \varepsilon$. \square

2.3.2 ECH capacities

In [Hut11], Hutchings associates to any symplectic 4-manifold (X, ω) , a sequence of symplectic capacities indexed by positive integers

$$0 \leq c_1^{ECH}(X, \omega) \leq c_2^{ECH}(X, \omega) \leq \dots \leq \infty$$

called ECH capacities of (X, ω) . In addition to the classical properties verified by symplectic capacities, they satisfy the following ones:

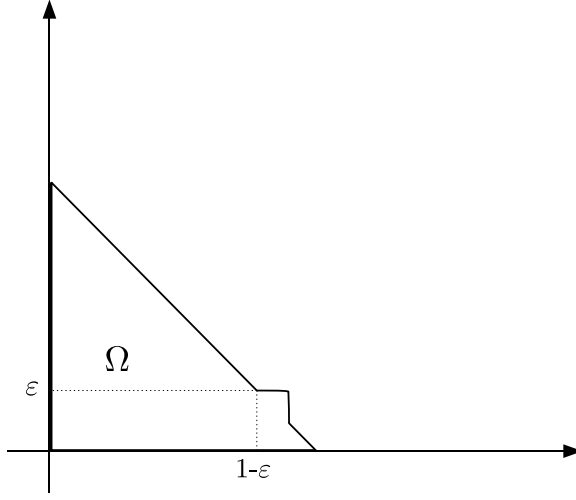


Figure 2.8: A monotone toric domain with arbitrarily small minimal action.

- (Disjoint union) Let $\{(X_i, \omega_i)\}$ be a sequence of four-manifolds, then:

$$c_k^{ECH} \left(\coprod_i (X_i, \omega_i) \right) = \sup_{\sum_i k_i = k} \sum_i c_{k_i}^{ECH}(X_i, \omega_i) \quad (2.17)$$

- (Ball) If $a > 0$, then:

$$c_k^{ECH}(B(a)) = da \quad (2.18)$$

where $d \in \mathbb{N}$ is the unique integer such that

$$d^2 + d \leq 2k \leq d^3 + 3d$$

Moreover, since $c_k^{ECH}(Z(a)) = ka$, only the first ECH capacity is ball-normalized.

In [CG14], Cristofaro-Gardiner showed that these capacities give a sharp obstruction to symplectically embed a concave toric domain into a weakly convex toric domain.

Theorem 2.3.6 ([CG14], Theorem 1.2). *Let X_Ω be a concave toric domain and $X_{\Omega'}$ a weakly convex toric domain, then there exists a symplectic embedding*

$$\text{int}(X_\Omega) \xrightarrow{\text{Symp}} \text{int}(X_{\Omega'})$$

if and only if

$$\forall k \geq 0, \quad c_k^{ECH}(X_\Omega) \leq c_k^{ECH}(X_{\Omega'})$$

The proof relies on the following notion of *weight expansion*.

Weight expansion

In [McD09], McDuff showed that the problem of whether one ellipsoid symplectically embeds into another or not, is equivalent to a ball packing problem. In her proof, she introduced the *weight expansion*² $W(a, b)$ of an ellipsoid $E(a, b)$ which is a (possibly infinite) sequence of positive numbers associated to it. The weight expansion relates to the ellipsoids embedding problem as it gives the corresponding ball packing problem. Then, in [McD11], McDuff completely solved the ellipsoids embedding problem (see Theorem 1.1.2) by using the fact that ECH capacities give sharp obstructions to the ball packing problems. In fact, the ECH capacities of an ellipsoid can be computed in terms of its weight expansion.

This was later generalized by the authors in [CCGF⁺14] for a large class of toric domains for which they defined the weight expansion. In particular, they established the following formula for concave toric domains.

Theorem 2.3.7 ([CCGF⁺14], Theorem 1.4.). *Let X_Ω be a concave toric domain with weight expansion $W(\Omega) = \{a_i\}_{i \in I}$, then for any $k \in \mathbb{N}$ we have*

$$c_k^{ECH}(X_\Omega) = c_k^{ECH} \left(\prod_i B(a_i) \right)$$

Remark that due to properties (2.17) and (2.18) of ECH capacities, it means that we can compute explicitly the ECH capacities of any concave toric domain. In practice, this is a challenging problem. Later on, Cristofaro-Gardiner established the corresponding formula for weakly convex toric domains to show its Theorem 2.3.6.

Theorem 2.3.8 ([CG14], Theorem A.1.). *Let X_Ω be a weakly convex toric domain with weight expansion $W(\Omega) = (b; b_1, b_2, \dots)$, then for any $k \in \mathbb{N}$ we have*

$$c_k^{ECH}(X_\Omega) = \inf_{l \geq 0} \left\{ c_{k+l}^{ECH}(B(b)) - c_l^{ECH} \left(\prod_i B(b_i) \right) \right\}$$

We will now define this weight expansion for concave toric domains and, following [CCGF⁺14], give the proof to one inequality in Theorem 2.3.7.

To define the weight expansion, let us start with a simple example. Let X_Ω be a concave toric domain X_Ω such that $\partial_+ \Omega$ is piece-wise linear with rational slopes and have one non-smooth point (see Figure 2.9). We will define the weight expansion inductively.

First, consider the biggest real r such that the isosceles right triangle of radius r fits into Ω , $r = w_0$ is called the first weight. It decomposes Ω into three triangles

²Also called sometimes the *weight decomposition* or *weight sequence*.

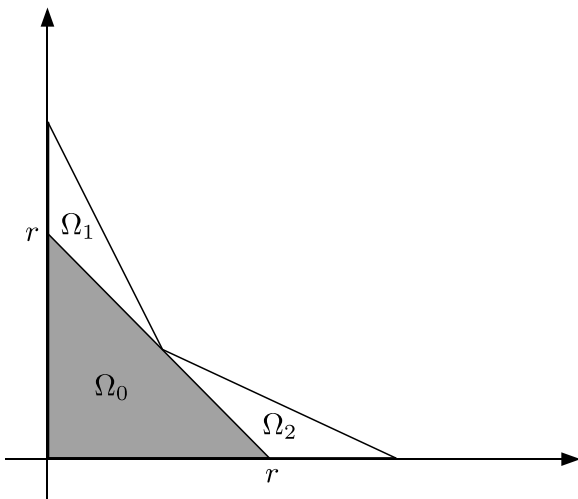


Figure 2.9: The starting procedure of the weight decomposition.

Ω_0 , Ω_1 and Ω_2 equivalent up to translation and action of $\text{GL}(2, \mathbb{Z})$ to right triangles with rational slopes. Then, compute the first weight w_1 , w_2 of each triangle Ω_1 and Ω_2 in the same manner and repeat this method inductively. The weight expansion is given by the list of these weights: $W(\Omega) = \{w_0, w_1, w_2, \dots\}$. Finally, this procedure naturally extends to any concave toric domain.

The weight expansion plays a particular role in ball-packing problems due to the following proposition by Traynor [Tra95]:

Proposition 2.3.9 ([Tra95]). *Let $T \subset \mathbb{R}_{\geq 0}^2$ be an open triangle equivalent up to translation and action of $\text{GL}(2, \mathbb{Z})$ to an isosceles right triangle of radius a , then there is a symplectic embedding*

$$\text{int}(B(a)) \xrightarrow{\text{Symp}} X_T$$

Therefore, given a concave toric domain X_Ω , we have a weight expansion $W(\Omega) = \{a_i\}_{i \in I}$ and associated triangles $\{T_i\}_{i \in I}$ which, by Traynor's proposition, induce symplectic embeddings

$$\text{int}(B(a_i)) \xrightarrow{\text{Symp}} X_{T_i}$$

Then, we have a ball-packing $\coprod_i \text{int}(B(a_i)) \xrightarrow{\text{Symp}} X_\Omega$ and so by monotonicity:

$$c_k^{ECH} \left(\coprod_i B(a_i) \right) \leq c_k^{ECH}(X_\Omega)$$

The reader can refer to [CCGF⁺14] for the reverse inequality.

Volume property

In [CGRH12], the authors showed that ECH capacities possess the important following additional property.

Theorem 2.3.10 ([CGRH12], Theorem 1.1). *For any compact domain X in \mathbb{R}^4 with piece-wise smooth boundary,*

$$\lim_{k \rightarrow \infty} \frac{c_k^{ECH}(X)^2}{k} = 4\text{Vol}(X) \quad (2.19)$$

This so-called *volume property* have many dynamical implications as developed in [Iri15, Iri22, CDPT22]. A direct consequence of this property combined with Theorem 2.3.7 is the following proposition.

Proposition 2.3.11. *Let X_Ω be a concave toric domain with weight decomposition $W(\Omega) = \{a_i\}_{i \in I}$, then*

$$\text{Vol}(X_\Omega) = \text{Vol}\left(\coprod_i B(a_i)\right) = \frac{1}{2} \sum_i a_i^2$$

Proof.

$$\text{Vol}(X_\Omega) = \frac{1}{4} \lim_{k \rightarrow \infty} \frac{c_k^{ECH}(X_\Omega)^2}{k} = \frac{1}{4} \lim_{k \rightarrow \infty} \frac{c_k^{ECH}(\coprod_i B(a_i))^2}{k} = \text{Vol}\left(\coprod_i B(a_i)\right) = \frac{1}{2} \sum_i a_i^2$$

□

Remark 2.3.12. *One can prove the above proposition without relying on Theorem 2.3.10. First, note that by the weight decomposition, $\Omega = \coprod_{i \in I} f_i(T(a_i, a_i))$ where $W(\Omega) = \{a_i\}_{i \in I}$ and $f_i \in \text{GL}(2, \mathbb{Z})$. Since $\det(f_i) \in \{-1, +1\}$, then we have*

$$\text{Vol}\left(\coprod_{i \in I} f_i(T(a_i, a_i))\right) = \text{Vol}\left(\coprod_{i \in I} T(a_i, a_i)\right).$$

For k large enough, Equation 2.19 tells us that ECH capacities asymptotically recover the volume. Therefore, symplectic embedding obstructions coming from ECH capacities lie in the *error term* as defined by

$$e_k(X) = c_k^{ECH}(X) - 2\sqrt{k\text{Vol}(X)} \quad (2.20)$$

For some toric domains, Hutchings showed that this error term asymptotically recovers the Ruelle invariant.

Theorem 2.3.13 ([Hut22], Theorem 1.10). *Let X_Ω be a strictly convex or strictly concave toric domain, then*

$$\lim_{k \rightarrow +\infty} e_k(X_\Omega) = -\frac{1}{2} \text{Ru}(X_\Omega) \quad (2.21)$$

Here, we give the following lemma which is inspired by the proof of Lemma 3.8. in [Hut22]. This will be needed in the proof of the main theorem in Chapter 4.

Lemma 2.3.14. *Let X_Ω be a strictly concave toric domain, then for any $k \in \mathbb{N}$, $e_k(X_\Omega) \leq 0$, equivalently*

$$c_k^{ECH}(X_\Omega) \leq 2\sqrt{k \text{Vol}(X_\Omega)}$$

Proof. By equations (2.17) and (2.18), we get

$$c_k^{ECH}(X_\Omega) = \sup \left\{ \sum_i a_i d_i \mid \sum_i (d_i^2 + d_i) \leq 2k \right\} \quad (2.22)$$

Without loss of generality, suppose that the non-negative integers d_i are zeros for $i > k$. Then the supremum is attained and let $d(k)_i$ be the integers realizing the maximum. Equation 2.18 becomes

$$c_k^{ECH}(X_\Omega) = \sum_i a_i d(k)_i \quad (2.23)$$

$$\sum_i (d(k)_i^2 + d(k)_i) \leq 2k \quad (2.24)$$

Applying Cauchy-Schwarz inequality and using (2.24) we get

$$\begin{aligned} \sum_i a_i \sqrt{d(k)_i^2 + d(k)_i} &\leq \sqrt{\sum_i a_i^2} \sqrt{2k} \\ &\leq \sqrt{2 \text{Vol}(X_\Omega)} \sqrt{2k} \end{aligned}$$

Finally, by (2.23) we have

$$c_k^{ECH}(X_\Omega) - 2\sqrt{k \text{vol}(X_\Omega)} \leq - \sum_i a_i \left(\sqrt{d(k)_i^2 + d(k)_i} - d(k)_i \right) \leq 0$$

□

2.3.3 Strong Viterbo conjecture

In [Her, Proposition 2.4], Hermann showed that all ball-normalized symplectic capacities agree on geometrically convex domains of \mathbb{R}^4 which are toric. This was later

generalized in [GHGBR22], where the authors showed that the strong Viterbo conjecture (see Conjecture 1.1.7) holds for dynamically convex toric domains of \mathbb{R}^4 .

Theorem 2.3.15 ([GHGBR22], Theorem 1.7.). *All ball-normalized symplectic capacities agree on monotone toric domains in \mathbb{R}^4 .*

Here, we follow will the proof in [GHGBR22].

Proof. In order to prove the theorem, the goal is to show that for any monotone toric domain X_Ω , we have $c_B(X_\Omega) = c_Z(X_\Omega)$. In particular, by Remark 1.1.4 we only need $c_B(X_\Omega) \geq c_Z(X_\Omega)$.

Let r be the maximal positive real number such that the isosceles right triangle of radius r fit into Ω , by Theorem 2.3.1 $c_B(X_\Omega) = r$. Without loss of generality we can suppose that X_Ω is strictly monotone (because the capacities are continuous with respect to the Hausdorff distance) and so our moment image Ω is included in the moment image Ω' of a concave toric domain (see Figure 2.10).

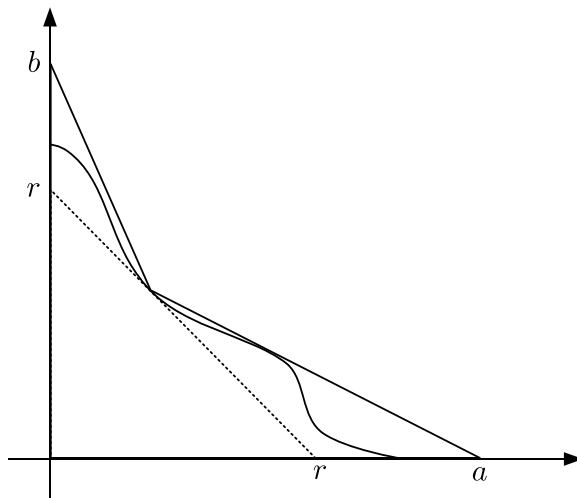


Figure 2.10: The concave toric domain $X_{\Omega'}$.

By applying the following lemma to $X_{\Omega'}$, one gets

$$\text{int}(X_{\Omega'}) \xrightarrow{\text{Symp}} P(r, \max(b, a - r))$$

and since $P(r, \max(b, a - r)) \subset Z^4(r)$ we have $c_Z(X_\Omega) \leq r$ i.e. $c_Z(X_\Omega) \geq c_B(X_\Omega)$, which was the desired conclusion. \square

Lemma 2.3.16 ([GHGBR22], Lemma 4.5.). *Let X_Ω be a concave toric domain such that $\partial_+\Omega$ has endpoints $(0, b)$ and $(a, 0)$ and let $r = c_B(X_\Omega)$, then there exists a symplectic embedding:*

$$\text{int}(X_\Omega) \xrightarrow{\text{Symp}} P(r, \max(b, a - r))$$

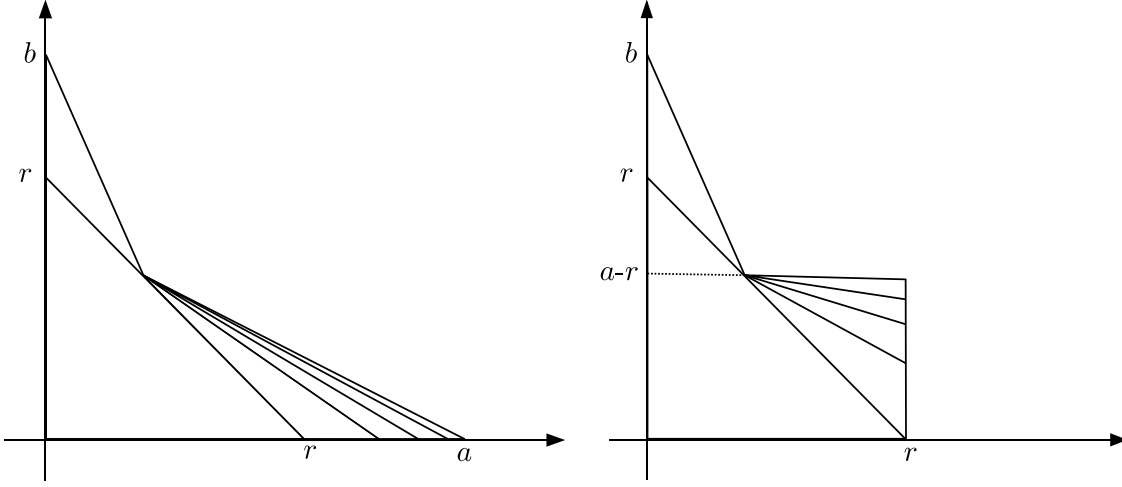


Figure 2.11: Ball packing into the polydisk $P(r, \max(b, a - r))$.

Proof of Lemma 2.3.16. Let X_Ω be a concave toric domain, then there exists a weight sequence $W(\Omega) = \{a_i\}_{i \in I}$ and triangles $\{T_i\}_{i \in I}$ which we can pack into $P(r, \max(r, b - a))$ (see Figure 2.11) so that

$$\bigsqcup_i \text{int}(B(a_i)) \xrightarrow{\text{Symp}} P(r, \max(r, b - a))$$

Therefore by monotonicity, we get

$$c_k^{ECH} \left(\bigsqcup_i \text{int}(B(a_i)) \right) \leq c_k^{ECH}(P(r, \max(r, b - a)))$$

and by Theorem 2.3.7 we have

$$c_k^{ECH}(X_\Omega) \leq c_k^{ECH}(P(r, \max(r, b - a)))$$

Finally, by Theorem 2.3.6 there exists a symplectic embedding:

$$\text{int}(X_\Omega) \xrightarrow{\text{Symp}} P(r, \max(b, a - r))$$

□

2.4 Higher dimensions

Toric domains can still be defined in higher dimensions as domains X in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ which are invariant under the action of the n -dimensional torus \mathbb{T}^n .

Definition 2.4.1. Let Ω be a domain in $\mathbb{R}_{\geq 0}^n$, a toric domain X_Ω is defined as follows

$$X_\Omega = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid (\pi|z_1|^2, \dots, \pi|z_n|^2) \in \Omega \right\}$$

In [CE], Chaidez and Edtmair generalized the Ruelle invariant in \mathbb{R}^{2n} and they showed that it can be easily computed as in Proposition 2.2.2 (see [CE, Proposition 5.6]). Definition 2.2.6 of strictly monotone toric domain extends naturally and they are still dynamically convex (see [CE, Proposition 5.8]). However the reverse as in Proposition 2.2.4 is false, there exists dynamically convex toric domains in higher dimensions which are not monotone. Moreover, Theorem 2.3.1 also holds in \mathbb{R}^{2n} (see [GHGBR22, Theorem 1.11]) and finally Cristofaro-Gardiner and Hind showed that the strong Viterbo conjecture for monotone toric domains (cf Theorem 2.3.15) holds in any dimension (see [CGH, Theorem 1.2]). Let us now review two classes of symplectic capacities which are of particular interest in higher dimensions: the Gutt-Hutchings capacities and the cube-normalized capacities.

2.4.1 Gutt-Hutchings capacities

As we have seen, ECH capacities are powerful tools for the study of symplectic and contact geometry. However, these are only defined in dimension 4. In [GH18], Gutt and Hutchings defined a new sequence of symplectic capacities by imitating the definition of ECH capacities. Using positive S^1 -equivariant symplectic homology, they showed the following theorem.

Theorem 2.4.2 ([GH18], Theorem 1.1 and Remark 1.3). *There exists symplectic capacities c_k^{GH} for star-shaped domains in \mathbb{R}^{2n} satisfying the following axioms:*

- **(Conformality)** *If U is a star-shaped domain in \mathbb{R}^{2n} and $r > 0$, then*

$$c_k^{GH}(rU) = r^2 c_k^{GH}(U)$$

- **(Increasing)** $c_1^{GH}(U) \leq c_2^{GH}(U) \leq \dots < +\infty$
- **(Monotonicity)** *If U and V are star-shaped domains in \mathbb{R}^{2n} such that*

$$U \xrightarrow{\text{Symp}} V, \text{ then } c_k^{GH}(U) \leq c_k^{GH}(V) \text{ for all } k.$$

- **(Reeb orbits)** *If U is a nice star-shaped domain such that $\lambda_0|_{\partial U}$ is non-degenerate, then*

$$c_k^{GH}(U) = \mathcal{A}(\gamma)$$

for some Reeb orbit γ of $\lambda_0|_{\partial U}$ with $\overline{CZ}(\gamma) = 2k + n - 1$.

Sometimes, Gutt-Hutchings capacities are denoted by c_k^{GH} where GH stands for *Contact Homology*. Moreover, they established the following formulas for convex and concave toric domains.

Theorem 2.4.3 ([GH18], Theorem 1.6). *Let X_Ω be a convex toric domain in \mathbb{R}^{2n} . Then*

$$c_k^{GH}(X_\Omega) = \min \left\{ \|v\|_\Omega^* \mid v = (v_1, \dots, v_n) \in \mathbb{N}^n, \sum_{i=1}^n v_i = k \right\} \quad (2.25)$$

where for any $v \in \mathbb{R}_{\geq 0}^n$, $\|v\|_\Omega^* = \max\{\langle v, w \rangle \mid w \in \Omega\}$.

Theorem 2.4.4 ([GH18], Theorem 1.14). *Let X_Ω be a concave toric domain in \mathbb{R}^{2n} . Then*

$$c_k^{GH}(X_\Omega) = \max \left\{ [v]_\Omega \mid v = (v_1, \dots, v_n) \in \mathbb{N}_{>0}^n, \sum_{i=1}^n v_i = k + n - 1 \right\} \quad (2.26)$$

where for any $v \in \mathbb{R}_{\geq 0}^n$, $[v]_\Omega = \min\{\langle v, w \rangle \mid w \in \overline{\partial_+ \Omega}\}$.

In particular, it follows that

$$c_k^{GH}(P(a_1, \dots, a_n)) = k \cdot \min\{a_1, \dots, a_n\} \quad (2.27)$$

and

$$c_k^{GH}(E(a_1, \dots, a_n)) = M_k(a_1, \dots, a_n) \quad (2.28)$$

where $(M_k(a_1, \dots, a_n))_{k \in \mathbb{N}^*}$ denote the sequence of positive integer multiples of a_1, \dots, a_n arranged in non-decreasing order. In particular it coincides with the Ekeland-Hofer capacities which are known to be hard to compute (see [EH90, Proposition 4-5]). This motivated the following conjecture which was recently announced to be proven by Gutt and Ramos:

Conjecture 2.4.5. *Let U be a compact star-shaped domain in \mathbb{R}^{2n} , then*

$$c_k^{GH}(U) = c_k^{EH}(U)$$

for any positive integer k .

In [KL0], Kerman and Liang explored the "blind spots" of these capacities. More precisely, they showed that, in contrast with ECH capacities, Gutt-Hutchings capacities do not recover the volume for nice star-shaped toric domains in \mathbb{R}^4 (see [KL0, Theorem 1.1]). They also answered negatively to the recognition problem in the following sense:

Theorem 2.4.6 ([KL0], Theorem 1.3). *There is a 1-dimensional smooth family of nice star-shaped toric domains in \mathbb{R}^4 all of which have the same Gutt-Hutchings capacities and volume, but no two of which are intrinsically symplectomorphic.*

2.4.2 Cube-normalized symplectic capacities

Consider the two following toric domains:

$$C^{2n}(r) := P(r, \dots, r) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \pi|z_1|^2 \leq r, \dots, \pi|z_n|^2 \leq r\}$$

which is called the cube of radius r . And

$$N^{2n}(R) := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \min_i \{\pi|z_i|^2\} \leq R\}$$

which is called the NDUC³ (non-disjoint union of cylinders) of size R .

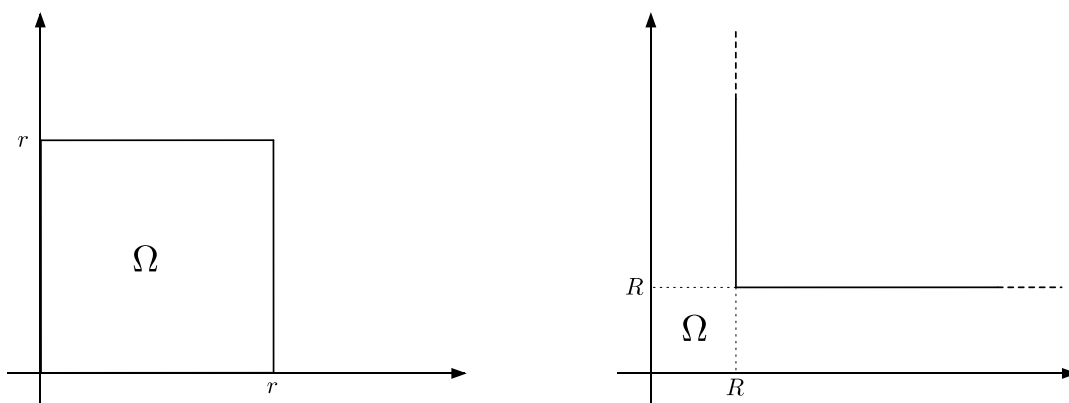


Figure 2.12: The four-dimensional cube and the NDUC.

In [GH18], Gutt and Hutchings showed the following non-squeezing theorem:

Theorem 2.4.7 ([GH18], Proposition 1.20).

$$C^{2n}(r) \xrightarrow{Symplectic} N^{2n}(R) \iff r \leq R$$

As for ball-normalized capacities, this motivated the following definition.

Definition 2.4.8 ([GPR22], Definition 4). A symplectic capacity c is called **cube-normalized** if

$$c(C^{2n}(1)) = c(N^{2n}(1)) = 1$$

A natural candidate for such capacity is the following *cube capacity*:

$$c_P(X) := \sup \left\{ r \mid C^{2n}(r) \xrightarrow{Symplectic} X \right\}$$

In fact, by definition $c_P(C^{2n}(1)) = 1$ and by the theorem above, $c_P(N^{2n}(1)) = 1$.

³Referred to as the L-shaped domain in Subsection 2.3.1.

The other most well-known example of such capacity is the Lagrangian capacity c_{Lag} . Introduced first by Cieliebak-Mohnke in [CM18], Pereira showed in [Per22] that it is cube-normalized (the proof being under assumptions from Remark 2.4.11). For any symplectic manifold (X, ω) , if L is a Lagrangian submanifold of X , then we define

$$A_{\min}(L) := \inf \left\{ \int_{\sigma} \omega \mid \sigma \in \pi_2(X, L), \int_{\sigma} \omega > 0 \right\}$$

which is called the minimal symplectic area of L .

Definition 2.4.9 ([CM18], Section 1.2). *Let (X, ω) be a symplectic manifold, define the Lagrangian capacity of (X, ω) as*

$$c_{\text{Lag}}(X, \omega) := \sup \{ A_{\min}(L) \mid L \subset X \text{ embedded Lagrangian torus} \}$$

By [CM18, Corollary 1.8] and [CM18, p.216], we know that this symplectic capacity is, in fact, not ball-normalized

$$c_{\text{Lag}}(B^{2n}(1)) = \frac{1}{n} \leq 1 = c_{\text{Lag}}(Z^{2n}(1))$$

In terms of its computation, Theorem 6.41 in [Per22] provides a convenient way to read off $c_{\text{Lag}}(X_{\Omega})$ directly from its moment image Ω for convex toric domains. Explicitly, it is just the coordinate value of the diagonal that intersects $\partial\Omega$ that we will denote by

$$\delta_{\Omega} := \sup \{ a \mid (a, \dots, a) \in \Omega \}$$

Then we have the following result in dimension 4

Theorem 2.4.10 ([Per22], Theorem 6.41). *If X_{Ω} is a 4-dimensional convex toric domain then*

$$c_{\text{Lag}}(X_{\Omega}) = \delta_{\Omega}$$

Remark 2.4.11. *Under some assumptions about linearized contact homology (see [Per22, Section 7.1]), Pereira showed that Theorem 2.4.10 extends to any dimension and also holds for concave toric domain (see [Per22, Theorem 7.65]). It was pointed out by J. Gutt that we only need to show the result for the NDUC. Finally, by [GPR22, Remark 8], the result directly extends to monotone toric domains.*

Under the same assumptions, a similar theorem as Theorem 2.3.15 holds in any dimension for cube-normalized symplectic capacities.

Theorem 2.4.12 ([GPR22], Theorem 10). *All cube-normalized symplectic capacities agree on monotone toric domains in \mathbb{R}^{2n} .*

Chapter 3

Non-symplectically convex toric domains

In this Chapter, based on the criterion from Chaidez-Edtmair’s Theorem 1.3.7 via Ruelle invariant and systolic ratio of the boundary of star-shaped domains, we provide elementary operations on domains that can kill the symplectic convexity. These operations only result in small perturbations in terms of domains’ volume. Moreover, one of the operations is a systematic way to produce examples of dynamically convex but not symplectically convex toric domains. Then, we are able to provide concrete bounds for the constants that appear in Chaidez–Edtmair’s criterion (1.3). Finally, we review how Chaidez and Edtmair applied one of these operations to answer Question 1.2.9 in any dimensions.

Theorem 3.0.1. *Given any nice star-shaped toric domain X_Ω in \mathbb{R}^4 , there exist small perturbations of X_Ω , in terms of the volume, such that the resulting domains $X_{\widehat{\Omega}}$ are still star-shaped but the product $\text{ru}(X_{\widehat{\Omega}}) \cdot \text{sys}(X_{\widehat{\Omega}})^{\frac{1}{2}}$ can be arbitrarily small or arbitrarily large. In particular, the resulting domains $X_{\widehat{\Omega}}$ are not symplectically convex.*

The notation $X_{\widehat{\Omega}}$ indicates that the perturbations of X_Ω promised in Theorem 3.0.1 can be carried on directly on the moment image Ω . Indeed, the proof of Theorem 3.0.1 in Section 3.1 and Section 3.2 provides two explicit constructions of such perturbations - strangulation operation and strain operation. A schematic picture below, Figure 3.1, illustrate these two operations on the level of moment images. More explicitly, given a moment image Ω of a toric star-shaped domain X_Ω , the strangulation operation removes a small part (blue shaded region) along a ray from Ω , while the strain operation adds a thin triangle (red shaded region) to Ω . Refined pictures of these operations with precise parameters will show up in Figure 3.3 and Figure 3.4. The resulting new toric domain is denoted by $X_{\widehat{\Omega}}$ with its moment image $\widehat{\Omega}$.

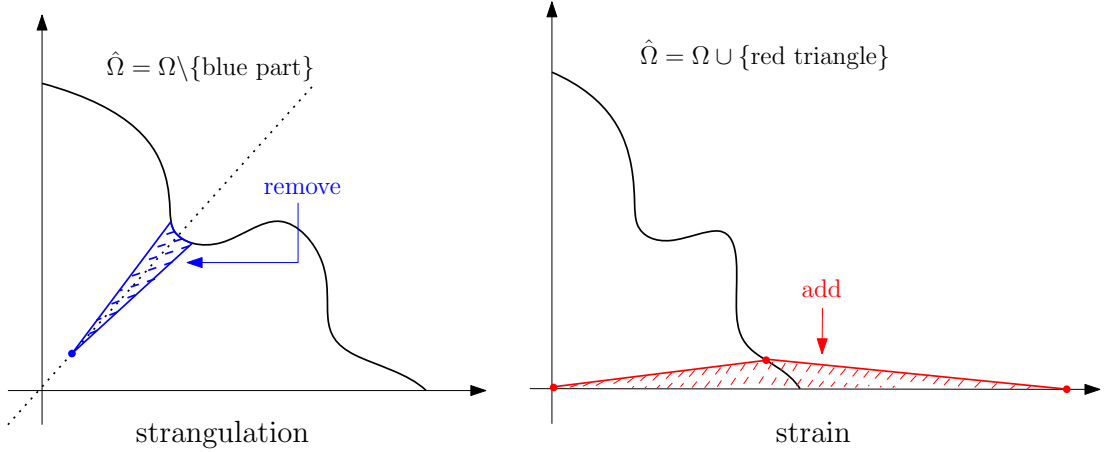


Figure 3.1: Strangulation operation and strain operation.

We will apply Theorem 1.3.7 to deduce that $X_{\hat{\Omega}}$ is not symplectically convex by deforming Ω until passing below the lower bound c or over the upper bound C in criterion (1.3), even though we don't know explicitly how big c and C are in general. As expected, there will be non-trivial estimations of the ratios

$$\text{ru}(X_{\hat{\Omega}}) \quad \text{and} \quad \text{sys}(X_{\hat{\Omega}}).$$

Remark that if one applies Theorem 1.2.1 from [HWZ98], symplectic convexity can be killed by breaking the dynamical convexity. In fact, this can be achieved as well by a C^0 -close perturbation.

Proposition 3.0.2. *Given any strictly monotone toric domain X_{Ω} in \mathbb{R}^4 there exist C^0 -close perturbations of X_{Ω} such that the resulting domains $X_{\tilde{\Omega}}$ are still star-shaped but no longer dynamically convex. In particular, the resulting domains $X_{\tilde{\Omega}}$ are not symplectically convex.*

Proof. Given a strictly monotone toric domain $X_{\Omega} \subset \mathbb{R}^4$, let us modify the profile curve of Ω in $\mathbb{R}_{\geq 0}^2$ near the intersection point p_1 of $\overline{\partial_+ \Omega}$ with the w_1 -axis so that in a neighborhood of p_1 , $\overline{\partial_+ \Omega}$ is a straight line with slope $n \in \mathbb{N}$ (see Figure 3.2). Let denote by $\widetilde{\gamma}_{p_1}$ the new closed Reeb orbit at p_1 . Then, up to some constant, the normal vector at p_1 has coordinates $(1, -\frac{1}{n})$ and according to Proposition 2.2.3

$$\overline{CZ}(\widetilde{\gamma}_{p_1}) = 2 + 2[-n] + 1 = 3 - 2n$$

which is always negative for $n \geq 2$. Given such n , one can choose a small enough neighborhood such that the corresponding perturbation is small in terms of volume. \square

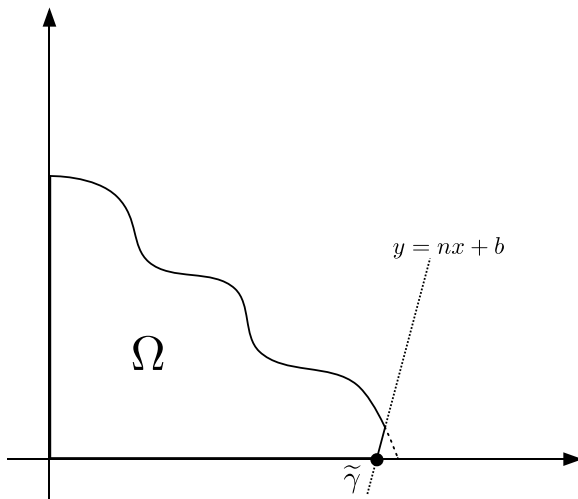


Figure 3.2: A small perturbation near the x-axis which kills symplectic convexity.

Here, we emphasize that our operations in Theorem 3.0.1 are fundamentally different. In particular, method one - strangulation in section 3.1 - can be distinguished with the operation elaborated above via symplectic capacities (for instance, the minimal action), even though it always goes beyond the category of dynamically convex toric domains due to Proposition 2.2.4. Method two - strain in section 3.2 - can be carried out even within the category of dynamically convex domains. In particular, we have the following useful result.

Corollary 3.0.3. *For any dynamically convex toric domain X_Ω in \mathbb{R}^4 , there exists a small perturbation in terms of the volume such that the resulting domain $X_{\hat{\Omega}}$ is still dynamically convex but not symplectically convex.*

Proof. This directly comes from the construction of strain operation in section 3.2, Corollary 3.2.2, and Proposition 2.2.4. \square

Note that Corollary 3.0.3 provides a variety of examples as in Corollary 1.3.8. In sharp contrast to the example produced in subsection 1.5 in [CE22] (which is closely related to the one invented in [ABHSA18]), Corollary 3.0.3 above is to our best knowledge the first systematic way to produce *toric* such examples and it is more direct and much simpler than the one in [CE22].

3.1 Method one: strangulation

Since X_Ω is a star-shaped domain, we can assume, without loss of generality, that the diagonal of $\mathbb{R}_{>0}^2$ intersects $\partial\Omega$ at point (w_*, w_*) such that a neighborhood within $\mathbb{R}_{>0}^2$ of the subset $\{(w, w) \in \mathbb{R}_{>0}^2 \mid 0 \leq w < w_*\}$ lies in the interior of Ω . In general,

there always exists some ray in $\mathbb{R}_{\geq 0}^2$ satisfying this condition. By our assumption, since $\overline{\partial_+ \Omega}$ is smooth, for any given $\epsilon > 0$, there exists some angle $\theta(\epsilon)$ such that the unbounded sector with vertex (ϵ, ϵ) , divided in half by the diagonal, and angle equal to $2\theta(\epsilon)$, intersects Ω in a closed region $\mathcal{S}(\epsilon)$ with points $(w_1, w_2) \in \partial\Omega \cap \mathcal{S}$ satisfying

$$|w_1 - w_*| \leq \epsilon \quad \text{and} \quad |w_2 - w_*| \leq \epsilon.$$

Now, carry on the following *strangulation* operation on Ω , that is, define

$$\widehat{\Omega} := \Omega \setminus (\text{int}(\mathcal{S}(\epsilon)) \cup \text{int}(\partial\Omega \cap \mathcal{S}(\epsilon))).$$

For a picture of this operation, see Figure 3.3.

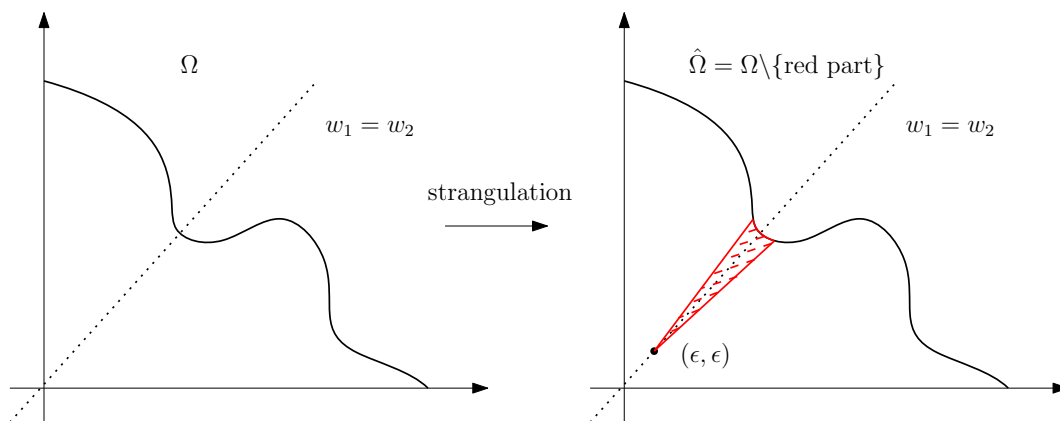


Figure 3.3: Strangulation operation.

After smoothing all singularities of $\widehat{\Omega}$, we have that the resulting domain, still denoted by $\widehat{\Omega}$, is again a closed domain in $\mathbb{R}_{\geq 0}^2$ with its pre-image under the moment map $\mu^{-1}(\widehat{\Omega}) =: X_{\widehat{\Omega}}$ being a star-shaped domain. Moreover, $X_{\widehat{\Omega}}$ satisfies the following quantitative properties.

Lemma 3.1.1. *The strangulation operation on X_{Ω} results in a star-shaped domain $X_{\widehat{\Omega}}$ which satisfies*

- (1) $\text{Vol}(X_{\widehat{\Omega}}) = \text{Vol}(X_{\Omega}) - O(\epsilon)$.
- (2) $\text{sys}(X_{\widehat{\Omega}}) \leq O(\epsilon)$.
- (3) $\text{Ru}(X_{\widehat{\Omega}}) = \text{Ru}(X_{\Omega})$.

Here, $O(\epsilon)$ represents a constant, proportional to ϵ in Figure 3.3, that can be arbitrarily small.

Assuming Lemma 3.1.1, we continue the proof of Theorem 3.0.1 via the strangu-lation operation. First, we have the following computations on volumes from Stokes' theorem,

$$\begin{aligned}\text{Vol}(\partial X_{\widehat{\Omega}}, \lambda) &= \int_{\partial X_{\widehat{\Omega}}} \lambda \wedge d\lambda = \int_{X_{\widehat{\Omega}}} d(\lambda \wedge d\lambda) \\ &= \int_{X_{\widehat{\Omega}}} d\lambda \wedge d\lambda = 2\text{Vol}(X_{\widehat{\Omega}}).\end{aligned}$$

Second, suppose ϵ is sufficiently small so that $O(\epsilon) < \frac{\text{Vol}(X_{\Omega})}{2}$. Then (1) and (2) in Lemma 3.1.1 imply that

$$\text{ru}(X_{\widehat{\Omega}})^2 = \frac{\text{Ru}(X_{\widehat{\Omega}})^2}{2\text{Vol}(X_{\widehat{\Omega}})} = \frac{\text{Ru}(X_{\Omega})^2}{2\text{Vol}(X_{\Omega}) - 2O(\epsilon)} \leq 2 \cdot \text{ru}(X_{\Omega})^2,$$

where, in particular, the upper bound $2 \cdot \text{ru}(X_{\Omega})^2$ is finite. Third, (3) in Lemma 3.1.1 implies that

$$\text{ru}(X_{\widehat{\Omega}}) \cdot \text{sys}(X_{\widehat{\Omega}})^{\frac{1}{2}} \leq \sqrt{2} \cdot \text{ru}(X_{\Omega}) \cdot \sqrt{O(\epsilon)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Therefore, the product $\text{ru}(X_{\widehat{\Omega}}) \cdot \text{sys}(X_{\widehat{\Omega}})^{\frac{1}{2}}$ will be lower than the constant c appearing in criterion (1.3), whenever ϵ is sufficiently small. In conclusion, the domain $X_{\widehat{\Omega}}$ is not symplectically convex.

Proof of Lemma 3.1.1. When $\epsilon < w_*$, the 4-dimensional volume of X_{Ω} and $X_{\widehat{\Omega}}$ (with respect to the standard symplectic structure on \mathbb{R}^4) satisfy

$$|\text{Vol}(X_{\widehat{\Omega}}) - \text{Vol}(X_{\Omega})| \leq \pi \cdot 2(w_* + \epsilon)^2 \cdot \frac{\theta(\epsilon)}{\pi} = 8w_*^2 \cdot \theta(\epsilon) \quad (3.1)$$

which goes to 0 as ϵ goes to 0 (since $\theta(\epsilon)$ goes to 0). Therefore, $X_{\widehat{\Omega}}$ is indeed a small perturbation in terms of the volume of X_{Ω} . This proves (1).

By the discussion above on the closed Reeb orbits in (2.4), applied to the new domain $\widehat{\Omega}$, there exists a closed Reeb orbit at $p = (\epsilon, \epsilon)$ corresponding to the normal vector $(1, 1)$. In particular, by (2.5) its action is $1 \cdot \epsilon + 1 \cdot \epsilon = 2\epsilon$. Denoting by \widehat{T}_{\min} the minimal period of a closed Reeb orbit of $\partial X_{\widehat{\Omega}}$, we have

$$\widehat{T}_{\min} \leq 2\epsilon$$

Therefore,

$$\text{sys}(X_{\hat{\Omega}}) \leq \frac{4\epsilon^2}{\text{Vol}(\partial X_{\hat{\Omega}}, \lambda)} = \frac{4\epsilon^2}{2\text{Vol}(X_{\hat{\Omega}})} \leq \frac{4\epsilon^2}{2\text{Vol}(X_{\Omega}) - 16w_*^2 \cdot \theta(\epsilon)}$$

where the second inequality comes from (3.1). Hence, when ϵ is sufficiently small so that $w_*^2 \cdot \theta(\epsilon) < \frac{\text{Vol}(X_{\Omega})}{16}$, we have

$$\text{sys}(X_{\hat{\Omega}}) \leq \frac{4\epsilon^2}{\text{Vol}(X_{\Omega})} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (3.2)$$

This proves (2).

Finally, since the strangulation operation does not change the w_1 or w_2 -intercepts of the original domain Ω , due to Proposition 2.2.2, the Ruelle invariant does not change, that is, $\text{Ru}(X_{\hat{\Omega}}) = \text{Ru}(X_{\Omega})$. This proves (3). \square

Remark 3.1.2. *It is not necessary to carry out the strangulation operation along the diagonal, as we did above. In general, most rays starting from the origin work in a similar way. An extreme case is to carry out such an operation along the w_1 -axis or w_2 -axis. The only difference is that the Ruelle invariant will change but gets smaller so we still obtain the result that the product of ratios $\text{ru} \cdot \text{sys}^{\frac{1}{2}}$ will be eventually smaller than the constant c in the criterion (1.3). In fact, such an operation has been investigated in [Ush22] on ellipsoids, called truncated ellipsoid (see Example One in Section 4.3).*

3.2 Method two: strain

Given a star-shaped domain X_{Ω} , suppose that the w_1 -intercept of $\overline{\partial_+ \Omega}$ is $a > 0$. Consider a generic small perturbation of Ω near $(a, 0)$ but with the w_1 -intercept a fixed, which also results in a small perturbation of X_{Ω} in terms of the volume, such that in a neighborhood N of $(a, 0)$, the boundary $\overline{\partial_+ \Omega}$ has a constant slope k , either positive or negative (but not equal to $\pm\infty$). This can be achieved due to our hypothesis that $\overline{\partial_+ \Omega}$ is smooth, and we can consider N sufficiently small so that the minimal period of the Reeb orbit of ∂X_{Ω} changes in an arbitrarily small way. For brevity, we still denote the domain after this perturbation by Ω .

Next, for any $\epsilon > 0$, sufficiently small so that the (unique) point $(w_*(\epsilon), \epsilon) \in \partial_+ \Omega$ for some $w_* > 0$ lies in the neighborhood N above, we have $\frac{\epsilon - 0}{w_*(\epsilon) - a} = k$, that is,

$$w_*(\epsilon) = \frac{\epsilon}{k} + a. \quad (3.3)$$

Consider the following triangle

$$\mathcal{T}(\epsilon) := \text{the triangle determined by vertices } (0, 0), (w_*(\epsilon), \epsilon), \text{ and } \left(\frac{1}{\sqrt{\epsilon}}, 0\right)$$

where ϵ is sufficiently small so that

$$\frac{-\epsilon}{\frac{1}{\sqrt{\epsilon}} - w_*(\epsilon)} > k \text{ if } k < 0. \quad (3.4)$$

This can be achieved since (3.4) is equal to $k(a - \frac{1}{\sqrt{\epsilon}}) > 0$, so when $\epsilon \rightarrow 0$, we have $a - \frac{1}{\sqrt{\epsilon}} < 0$ (since $k < 0$). Then consider the following *strain* operation on Ω , that is,

$$\hat{\Omega} := \Omega \cup \mathcal{T}(\epsilon).$$

For a picture of this operation, see Figure 3.4.

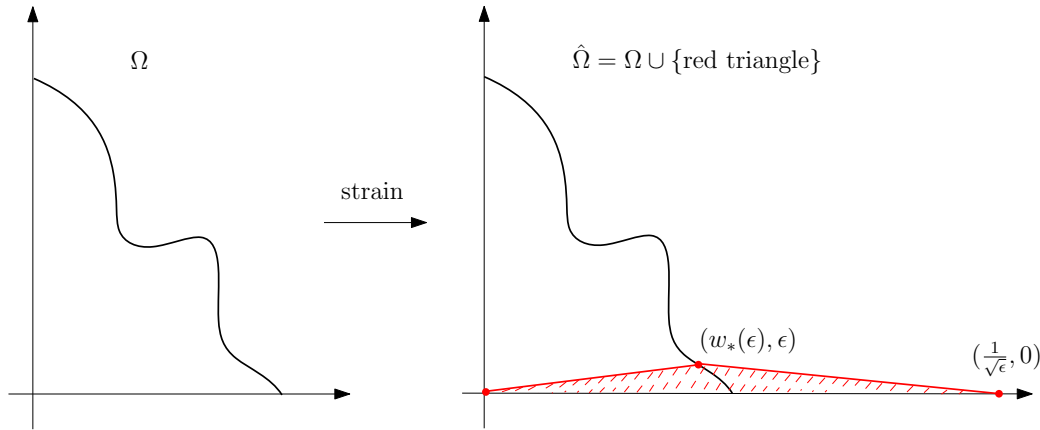


Figure 3.4: Strain operation.

Observe that condition (3.4) together with the hypothesis that X_Ω is star-shaped, implies that $\Omega \subset \hat{\Omega}$ and the pre-image $X_{\hat{\Omega}} = \mu^{-1}(\hat{\Omega})$ is again star-shaped. In particular, X_Ω being star-shaped is used to deal with the case when $k > 0$. Similarly to Lemma 3.1.1, we have the following quantitative result for $X_{\hat{\Omega}}$.

Lemma 3.2.1. *The strain operation on X_Ω results in a star-shaped domain $X_{\hat{\Omega}}$ that satisfies*

- (1) $\text{Vol}(X_{\hat{\Omega}}) = \text{Vol}(X_\Omega) + O(\epsilon)$.
- (2) $\text{sys}(X_{\hat{\Omega}}) \geq A$, where A is a constant independent of ϵ .
- (3) $\text{Ru}(X_{\hat{\Omega}}) \geq \frac{1}{O(\epsilon)}$.

Here, $O(\epsilon)$ represents a constant, proportional to ϵ in Figure 3.3, that can be chosen arbitrarily small.

Assuming Lemma 3.2.1, we continue the proof of Theorem 3.0.1 via the strain operation. The conclusions (1) and (3) in Lemma 3.2.1 imply that, if $O(\epsilon) < \text{Vol}(X_\Omega)$, we have

$$\text{ru}(X_{\widehat{\Omega}})^2 = \frac{\text{Ru}(X_{\widehat{\Omega}})^2}{2\text{Vol}(X_{\widehat{\Omega}})} = \frac{\text{Ru}(X_{\widehat{\Omega}})^2}{2\text{Vol}(X_\Omega) + 2O(\epsilon)} \geq \frac{1}{O(\epsilon)^2} \cdot \frac{1}{4\text{Vol}(X_\Omega)}.$$

Then we have

$$\text{ru}(X_{\widehat{\Omega}}) \cdot \text{sys}(X_{\widehat{\Omega}})^{\frac{1}{2}} \geq \frac{1}{O(\epsilon)^2} \cdot \frac{A}{4\text{Vol}(X_\Omega)} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0. \quad (3.5)$$

Hence, the product of the ratios will be larger than the constant C appearing in criterion (1.3). Therefore, the domain $X_{\widehat{\Omega}}$ is not symplectically convex.

Proof of Lemma 3.2.1. Comparing the difference of the volume in \mathbb{R}^4 , we have

$$|\text{Vol}(X_{\widehat{\Omega}}) - \text{Vol}(X_\Omega)| \leq \frac{\epsilon \cdot \frac{1}{\sqrt{\epsilon}}}{2} = \frac{\sqrt{\epsilon}}{2} \quad (3.6)$$

which goes to 0 as ϵ goes to 0. This proves (1).

This operation possibly introduces various new closed Reeb orbits. Besides the one corresponding to the w_1 -intercept point $(\frac{1}{\sqrt{\epsilon}}, 0)$ with large action, others will concentrate only near the point $p = (w_*(\epsilon), \epsilon)$, after smoothing $\widehat{\Omega}$ at p . By (2.4), these closed Reeb orbits correspond to the pairs of integers,

$$(m_p, n_p) \in \mathbb{Z}_{>0} \times \mathbb{Z} \text{ with } \min\left\{-\frac{1}{k}, 0\right\} \leq \frac{n_p}{m_p} \leq \frac{\sqrt{\epsilon} - w_*(\epsilon)}{\epsilon}.$$

Concerning their action, we have by (2.5),

$$\begin{aligned} \mathcal{A}(\gamma_{(m_p, n_p)}) &= m_p w_*(\epsilon) + n_p \epsilon \\ &= m_p \left(w_*(\epsilon) + \frac{n_p}{m_p} \epsilon \right) \\ &\geq m_p \left(\frac{\epsilon}{k} + a + \min\left\{-\frac{\epsilon}{k}, 0\right\} \right) \geq \frac{a}{2}, \end{aligned}$$

when ϵ is sufficiently small. We denote as above, T_{\min} the minimal period of a closed Reeb orbit on ∂X_Ω and \widehat{T}_{\min} the minimal period of a closed Reeb orbit on $\partial X_{\widehat{\Omega}}$. If $T_{\min} < \frac{a}{2}$, then obviously $\widehat{T}_{\min} = T_{\min}$. If, on the other hand, $T_{\min} \geq \frac{a}{2}$, then $\widehat{T}_{\min} \geq \frac{a}{2}$.

Meanwhile, by assumption, $T_{\min} \leq a$. Therefore, in either case, we have

$$\hat{T}_{\min} \geq \frac{T_{\min}}{2}.$$

In particular, the lower bound $\frac{T_{\min}}{2}$ is independent of the parameter ϵ .

Now, for the ratios in discussion, by (3.6),

$$\text{sys}(X_{\hat{\Omega}}) \geq \frac{T_{\min}^2}{4\text{Vol}(\partial X_{\hat{\Omega}}, \lambda)} = \frac{T_{\min}^2}{8\text{Vol}(X_{\hat{\Omega}})} \geq \frac{T_{\min}^2}{8\text{Vol}(X_{\Omega}) + 4\sqrt{\epsilon}}$$

When ϵ is sufficiently small, say $\sqrt{\epsilon} < \text{Vol}(X_{\Omega})$, we have

$$\text{sys}(X_{\hat{\Omega}}) \geq \frac{1}{12} \cdot \frac{T_{\min}^2}{\text{Vol}(X_{\Omega})} (:= A) \quad \text{which is independent of } \epsilon. \quad (3.7)$$

This proves (2).

Finally, the strain operation results in an essential change of the Ruelle invariant. By Proposition 2.2.2,

$$\text{Ru}(X_{\hat{\Omega}}) = (w_2\text{-intercept of } \overline{\partial_+ \Omega}) + \frac{1}{\sqrt{\epsilon}} \geq \frac{1}{\sqrt{\epsilon}}.$$

This proves (3). □

As an immediate consequence from the strain operation, we have the following.

Corollary 3.2.2. *The product of ratios $\text{ru} \cdot \text{sys}^{\frac{1}{2}}$ is unbounded on the category of monotone toric domains.*

Proof. Note that the strain operation is closed within the category of strictly monotone toric domains. Hence, it is closed within the category of dynamically convex toric domains by Proposition 2.2.4, since by the definition of a monotone toric domain, near the w_1 -intercept the corresponding slope k is always negative. Then the desired conclusion follows from the computation (3.5). □

3.3 Constants in Chaidez-Edtmair's criterion

As the proof of Theorem 3.0.1 is essentially fighting against the optimal constant

$$c = \inf_{X \in \mathcal{C}_4} \text{ru}(X) \cdot \text{sys}(X)^{\frac{1}{2}} \quad \text{and} \quad C = \sup_{X \in \mathcal{C}_4} \text{ru}(X) \cdot \text{sys}(X)^{\frac{1}{2}}$$

appearing in (1.3) (where \mathcal{C}_4 denotes the set of symplectically convex domains in \mathbb{R}^4), one may be curious about how small or large these constants are. In general, due

to the complexity of the proof of (1.3) in [CE22], it seems difficult to read off any bounds for c and C directly. However, for strictly monotone toric domains, we are able to estimate the product $\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}}$ by the following result, which yields concrete bounds for c and C in (1.3) (cf [CE22, Remark 1.11]).

Theorem 3.3.1. *Let \mathcal{SM}_4 denotes the set of strictly monotone toric domains in \mathbb{R}^4 and \mathcal{CT}_4 the set of geometrically convex toric domains in \mathbb{R}^4 . We have*

$$\inf_{X_\Omega \in \mathcal{SM}_4} \left(\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \right) = \frac{1}{2}.$$

and

$$\sup_{X_\Omega \in \mathcal{CT}_4} \left(\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \right) = 3.$$

In particular, the optimal constant c and C in the criterion (1.3) satisfy $c \leq \frac{1}{2}$ and $C \geq 3$, respectively.

Remark 3.3.2. *In the proof of Theorem 3.0.1, strain operation in section 3.2, shows that*

$$\sup_{X_\Omega \in \mathcal{SM}_4} \left(\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \right) = +\infty$$

see Corollary 3.2.2. For the conclusion of Theorem 3.3.1, the upper bound is realised by the toric domain whose moment image is given by the blue curve of Figure 3.6. Moreover, the lower bound $\frac{1}{2}$ can be arbitrarily approximated by the family of polydisks $P(a, b)$ (its definition given by (2.2)) with $b \rightarrow \infty$, we have

$$\text{ru}(P(a, b)) \cdot \text{sys}(P(a, b))^{\frac{1}{2}} = \frac{(a+b)}{\sqrt{2ab}} \cdot \frac{a}{\sqrt{2ab}} = \frac{(a+b)}{2b} \xrightarrow{b \rightarrow +\infty} \frac{1}{2}$$

In contrast with Corollary 3.2.2, in this section, we give a proof of Theorem 3.3.1, which provides a uniform bounds of the product of ratios $\text{ru} \cdot \text{sys}^{\frac{1}{2}}$ for such domains (when they are geometrically convex in \mathbb{R}^4). Let us start from the following useful result. Recall that $c_B(X_\Omega)$ the ball capacity of a toric domain X_Ω , measures the largest $B^4(a)$ that can be symplectically embedded into X_Ω .

Lemma 3.3.3. *Let X_Ω be a monotone toric domain where the w_1 -intercept and w_2 -intercept of $\overline{\partial_+ \Omega}$ are $(a, 0)$ and $(0, b)$ respectively. Suppose that $b \geq a$, then*

$$\text{Vol}(X_\Omega) \leq b \cdot c_B(X_\Omega).$$

Proof. By Proposition 2.3.1, we know that $c_B(X_\Omega)$ is equal to the largest $L > 0$ such that the right triangle with vertices $(0, L)$, $(L, 0)$ and $(0, 0)$ is contained in Ω . Denote by (s, t) one of these intersection points. Then observe that X_Ω being monotone

implies that $X_\Omega \subset P(s, b) \cup P(a, t)$. Therefore, we have

$$\begin{aligned} \text{Vol}(X_\Omega) &\leq sb + at = sb + a(-s + c_B(X_\Omega)) \\ &= s(b - a) + ac_B(X_\Omega) \leq b \cdot c_B(X_\Omega) \end{aligned}$$

where the last inequality comes from $s \leq c_B(X_\Omega)$. \square

Now, we are ready to give the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. For a strictly monotone toric domain X_Ω , Proposition 2.3.4 shows that the minimal period of a Reeb orbit is equal to $c_B(X_\Omega)$. Without loss of generality, assume $b \geq a$. Then by Proposition 2.2.2 and Lemma 3.3.3, we have

$$\begin{aligned} \text{ru}(X_\Omega)^2 \cdot \text{sys}(X_\Omega) &= \frac{(a+b)^2}{4} \cdot \frac{c_B(X_\Omega)^2}{\text{Vol}(X_\Omega)^2} \\ &\geq \frac{(a+b)^2}{4} \frac{c_B(X_\Omega)^2}{(b \cdot c_B(X_\Omega))^2} \\ &\geq \frac{(a+b)^2}{4b^2} = \frac{1}{4} \left(1 + \frac{a}{b}\right)^2 \geq \frac{1}{4}. \end{aligned}$$

Thus, we complete the proof of the first conclusion.

Now, suppose furthermore that X_Ω is geometrically convex in \mathbb{R}^4 . Up to a rescaling, assume the w_1 -intercept of Ω is 1 while the w_2 -intercept of Ω is still b . Up to a reflection between w_1 and w_2 , we can assume that $b \geq 1$. Therefore, we have $\text{Ru}(X_\Omega) = 1 + b$ for any such domain and so

$$\text{ru}(X_\Omega)^2 \cdot \text{sys}(X_\Omega) = \frac{(1+b)^2 c_B(X_\Omega)^2}{4 \text{Vol}(X_\Omega)^2}$$

depends only on $\frac{c_B(X_\Omega)}{\text{Vol}(X_\Omega)}$. We thus aim to bound above this quantity among strictly monotone toric domains which are geometrically convex.

Proposition 2.3 in [GHGBR22] states that the following subset

$$\tilde{\Omega} := \{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid (\pi|\mu_1|^2, \pi|\mu_2|^2) \in \Omega\}$$

is a convex subset in \mathbb{R}^2 if and only if X_Ω is geometrically convex. In particular, when restricted to $\mathbb{R}_{\geq 0}^2$, the boundary $\partial\tilde{\Omega}$ is the graph of a decreasing concave function $\mu_2 = g(\mu_1)$. Since g is concave, we have $g(\mu_1) \geq \sqrt{b}(1 - \mu_1)$ for all $\mu_1 \in [0, 1]$. Meanwhile, if we fix $c = c_B(X_\Omega)$, we also have $g(\mu_1) \geq \sqrt{c - \mu_1^2}$ for all $\mu_1 \in [0, 1]$ (since $B^4(c) \subset X_\Omega$). Therefore, g is above the broken curve consisting of the two previous curves. Hence, among these g , the one whose domain maximizes $\frac{c_B}{\text{Vol}}$ is the

one minimizing the volume i.e. the convex hull of this broken curve, see Figure 3.5. It has the following boundary:

$$g_c(\mu_1) = \begin{cases} \sqrt{b} - \sqrt{\frac{b-c}{c}}\mu_1 & \text{if } 0 \leq \mu_1 \leq \sqrt{\frac{c}{b}}(b-c) \\ \sqrt{c - \mu_1^2} & \text{if } \sqrt{\frac{c}{b}}(b-c) \leq \mu_1 \leq c \\ \sqrt{\frac{c}{1-c}}(1 - \mu_1) & \text{if } c \leq \mu_1 \leq 1 \end{cases}$$

By a change of variables $w_i = \mu_i^2$, we know that the boundary $\overline{\partial_+ \Omega}$ (minus the

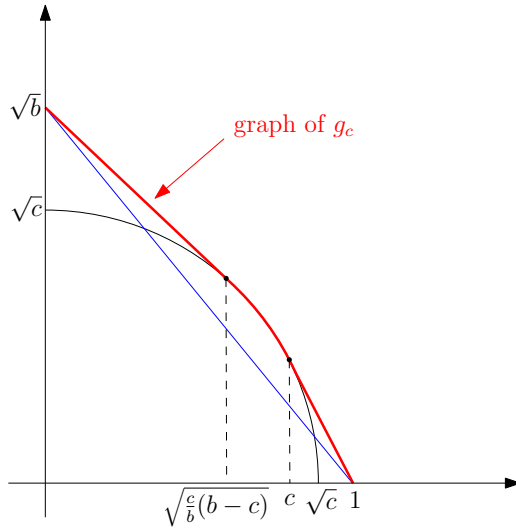


Figure 3.5: Graph of g_c in red.

components on w_1 -axis and w_2 -axis) is a function $w_2 = f_c(w_1) := g_c(\sqrt{w_1})^2$ given by

$$f_c(w_1) = \begin{cases} \left(\sqrt{b} - \sqrt{\frac{b-c}{c}}\sqrt{w_1}\right)^2 & \text{if } 0 \leq w_1 \leq \frac{c}{b}(b-c) \\ c - w_1 & \text{if } \frac{c}{b}(b-c) \leq w_1 \leq c^2 \\ \frac{c}{1-c}(1 - \sqrt{w_1})^2 & \text{if } c^2 \leq w_1 \leq 1 \end{cases}$$

also see Figure 3.6.

Denote by X_{f_c} the strictly monotone toric domain such that f_c is the boundary of its moment map minus the w_1 and w_2 -axis. Then we have

$$\frac{c_B(X_\Omega)}{\text{Vol}(X_\Omega)} \leq \frac{c_B(X_{f_c})}{\text{Vol}(X_{f_c})}.$$

Meanwhile, by integrating along the graph $f_c(w_1)$, we get that

$$\text{Vol}(X_{f_c}) = \frac{c^2}{2} + \frac{(b-c)^2 c}{6b} + \frac{c(1-c)^2}{6},$$

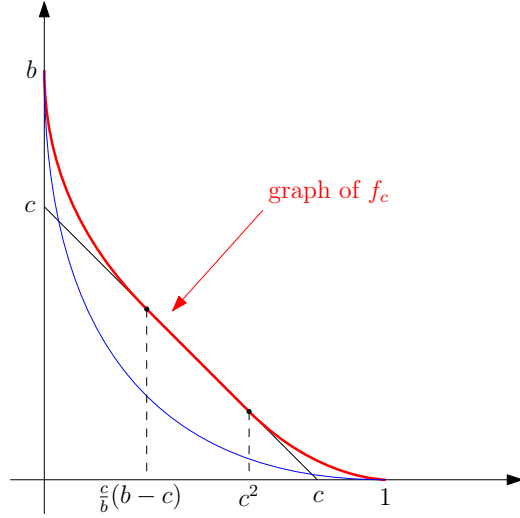


Figure 3.6: Graph of f_c in red.

therefore,

$$\frac{c_B(X_{f_c})}{\text{Vol}(X_{f_c})} = \frac{c}{\text{Vol}(X_{f_c})} = \frac{6}{3c + \frac{(b-c)^2}{b} + (1-c)^2}.$$

Moreover, since X_Ω is geometrically convex with the w_1 -intercept and w_2 -intercept being $a(\Omega) = 1$ and $b(\Omega) = b$, respectively, we have $\frac{b}{1+b} \leq c_B(X_\Omega) \leq 1$, since the ball capacity of the domain with boundary $w_2 = b(1 - \sqrt{w_1})^2$ is $\frac{b}{1+b}$. Therefore, we get

$$\max_{c \in [\frac{b}{1+b}, 1]} \left\{ \frac{c_B(X_{f_c})}{\text{Vol}(X_{f_c})} \right\} = \frac{6}{1+b}$$

where the maximum is obtained for $c = \frac{b}{1+b}$ i.e. for the domain whose boundary is given by $f(w_1) = b(1 - \sqrt{w_1})^2$. Hence,

$$\text{ru}(X_\Omega)^2 \cdot \text{sys}(X_\Omega) = \frac{(1+b)^2 c_B(X_\Omega)^2}{4\text{Vol}(X_\Omega)^2} \leq 9.$$

Thus we completed the proof of the second conclusion. \square

3.4 Higher dimensions

Let us conclude this chapter by briefly explaining how the strain operation was applied by Chaidez and Edtmair to answer Question 1.2.9 in higher dimensions.

In [CE], Chaidez and Edtmair generalized the upper bound of their criterion 1.3 to higher dimensions.

Theorem 3.4.1 ([CE], Theorem 1.12). *Given a star-shaped domain X in \mathbb{R}^{2n} , if X is symplectically convex, then*

$$c_{CE}(X) \leq C(n)$$

where $C(n)$ is a positive constant, independent of the domain X .

Then, since the strain operation of Section 3.2 naturally extends to any dimension, they constructed *strained* concave toric domains which are not symplectically convex.

Proposition 3.4.2 ([CE], Proposition 5.16). *Given any concave toric domain X_Ω in \mathbb{R}^{2n} , there exist small perturbations, in terms of the volume, such that the resulting domains $X_{\hat{\Omega}}$ are still concave toric domains but $c_{CE}(X_{\hat{\Omega}})$ can be arbitrarily large. In particular, the resulting domains $X_{\hat{\Omega}}$ are not symplectically convex.*

This exhibited the first family of examples in \mathbb{R}^{2n} that are dynamically convex but not symplectically convex (for $n > 2$). Thus answering Question 1.2.9 in any dimensions:

Corollary 3.4.3 ([CE], Corollary 1.19). *There exists dynamically convex star-shaped domains in \mathbb{R}^{2n} that are not symplectically convex.*

Chapter 4

A metric story

According to Corollary 1.3.8 and Corollary 3.0.3, it was showed that symplectic convexity and dynamical convexity are two distinct notions. Here, we study this difference with a metric-geometrical approach. In particular, we address the following question:

Question 4.0.1. *How far can dynamically convex domains be away from symplectically convex domains?*

Let us introduce the following notations, let \mathcal{C}_4 be the set of symplectically convex domains in \mathbb{R}^4 , and let \mathcal{D}_4 denote the set of dynamically convex domains (see Definition 1.2.3). From Theorem 1.2.1, it follows that $\mathcal{C}_4 \subset \mathcal{D}_4$, however in [CE22], Chaidez and Edtmair showed that $\mathcal{C}_4 \subsetneq \mathcal{D}_4$. This result was later supported in [DGZ24] with Corollary 3.0.3 and extended in any dimensions by Chaidez-Edtmair in [CE] (see Corollary 3.4.3). Moreover, let \mathcal{T}_4 and \mathcal{M}_4 denote the sets consisting of the star-shaped toric domains and of the **strictly** monotone toric domains, respectively. In [GHGBR22], it was shown that, $\mathcal{M}_4 = \mathcal{T}_4 \cap \mathcal{D}_4$ (see Proposition 2.2.4). The following Figure 4.1 illustrates the schematic relations between \mathcal{T}_4 , \mathcal{C}_4 , \mathcal{M}_4 , and \mathcal{D}_4 , where \mathcal{M}_4 is the shaded region.

In this chapter, we answer Question 4.0.1 by showing that there exists dynamically convex domains in \mathbb{R}^4 which lie arbitrarily far away from symplectically convex domains with respect to the coarse symplectic Banach-Mazur distance (defined in Section 4.1). In particular, this exhibits the first dynamically convex domains which are shown to be not symplectically convex without referring to Chaidez-Edtmair's criterion (1.3). Moreover, in Section 4.3, we study the difference between these two symplectic convexity's criterion. Finally, in Section 4.4, we investigate the large-scale geometry of the pseudo-metric space of star-shaped domains equipped with the coarse distance.

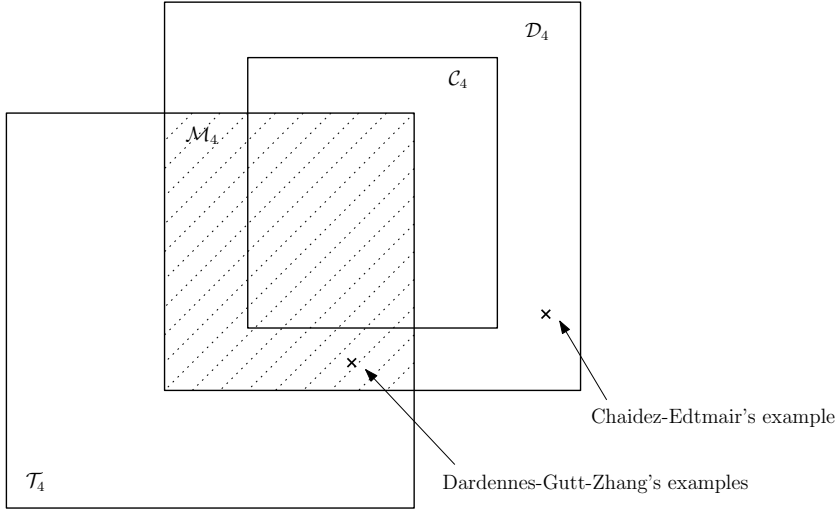


Figure 4.1: Relations between \mathcal{T}_4 , \mathcal{C}_4 , \mathcal{M}_4 , and \mathcal{D}_4 .

4.1 Symplectic John's ellipsoid theorem

In this Section, we introduce the coarse symplectic Banach-Mazur distance and develop a "new" symplectic convexity criterion relying on a well-known theorem by John.

The classical John's ellipsoid theorem [Joh48] says that convex domains in \mathbb{R}^n are close to ellipsoids. More precisely,

Theorem 4.1.1 (John's ellipsoid, 1948). *: Let U be a convex domain of \mathbb{R}^n , then there exists an ellipsoid $E \subset \mathbb{R}^n$ such that*

$$E \subset U \subset o + n \cdot (E - o)$$

where o is the center of E .

This can be rephrased in the following metric setting. Explicitly, denote by $\text{Conv}(\mathbb{R}^n)$ the convex domains in \mathbb{R}^n . A quantitative comparison between any two convex domains inside $\text{Conv}(\mathbb{R}^n)$ is given by the so-called Banach-Mazur distance [Rud00],

$$d_{\text{BM}}(U, V) := \inf \left\{ \log \lambda \geq 0 \mid \exists A \in \text{GL}(n), u, v \in \mathbb{R}^n \text{ such that } \frac{1}{\lambda}(U + u) \subset A(V + v) \subset \lambda(U + u) \right\} \quad (4.1)$$

where $\cdot + u$ and $\cdot + v$ denote translations while $\frac{1}{\lambda} \cdot$ and $\lambda \cdot$ stand for dilations. It is easily verified that d_{BM} defines a pseudo-metric on $\text{Conv}(\mathbb{R}^n)$ and $d_{\text{BM}}(U, V) = 0$ if and only if U is an affine transformation of V . Since d_{BM} is defined up to affine

transformations in \mathbb{R}^n , consider

$$\text{Ell}(n) := \{\text{ellipsoids in } \mathbb{R}^n\}$$

Then, Theorem 4.1.1 says that for any $U \in \text{Conv}(n)$, we have

$$d_{\text{BM}}(U, \text{Ell}(n)) := \inf_{E \in \text{Ell}(n)} d_{\text{BM}}(U, E) \leq \frac{1}{2} \log n \quad (4.2)$$

in particular, the distance is finite.

A non-linear symplectic analogue of the Banach-Mazur distance (4.1) was first suggested by Ostrover and Polterovich and further developed by [Ush22, SZ21]. It is defined as follows.

Definition 4.1.2 (Coarse symplectic Banach-Mazur distance). *For $U, V \subset \mathbb{R}^4$ star-shaped domains,*

$$d_c(U, V) = \inf \left\{ \log \lambda \geq 0 \mid \frac{1}{\lambda} U \hookrightarrow V \hookrightarrow \lambda U \right\} \quad (4.3)$$

where \hookrightarrow represents a symplectic embedding via some global symplectomorphism ϕ of \mathbb{R}^4

According to [Ush22, Definition 1.3] d_c is named as the coarse symplectic Banach-Mazur distance, we briefly call d_c the *coarse distance* in this chapter.

Remark 4.1.3. *Note that it is equivalent in the above definition to consider intrinsic or global- symplectic embedding. In fact, due to the so-called Extension after restriction principle for star-shaped domains of \mathbb{R}^{2n} (see [Sch01, Proposition 1.7]), the existence of an intrinsic symplectic embedding $\frac{1}{\lambda} U \hookrightarrow V$ implies the existence of a globally symplectic embedding $\frac{1}{\lambda+\varepsilon} U \hookrightarrow V$ for any $\varepsilon > 0$.*

Following what is usually done in any metric space, given \mathcal{A}, \mathcal{B} two collections of subsets of \mathbb{R}^4 , we define the distance from \mathcal{A} to \mathcal{B} as:

$$\sigma_{d_c}(\mathcal{A}, \mathcal{B}) = \sup_{U \in \mathcal{A}} \inf_{V \in \mathcal{B}} d_c(U, V). \quad (4.4)$$

Note that $\sigma_{d_c}(-, -)$ is not symmetric in general. For instance, if $\mathcal{A} \subset \mathcal{B}$, then $\sigma_{d_c}(\mathcal{A}, \mathcal{B}) = 0$ while $\sigma_{d_c}(\mathcal{B}, \mathcal{A})$ could be large. Moreover, $\sigma_{d_c}(-, -)$ satisfies a monotonicity property, namely, $\sigma_{d_c}(\mathcal{A}, \mathcal{B}') \leq \sigma_{d_c}(\mathcal{A}, \mathcal{B})$ if $\mathcal{B} \subset \mathcal{B}'$. In what follows, for simplicity, let us denote σ_{d_c} by σ . Recall also that \mathcal{E}_4 consists of all the symplectic ellipsoids in \mathbb{R}^4 . Our symplectic convexity's criterion can be formulated as follows:

Proposition 4.1.4 (Symplectic John's ellipsoid theorem). $\sigma(\mathcal{C}_4, \mathcal{E}_4) \leq \log 2$.

Proof. Up to symplectomorphism, let $U \in \mathcal{C}_4$ be a convex domain of \mathbb{R}^4 , by the classical John's ellipsoid theorem 4.1.1, there exists an ellipsoid $E \subset \mathbb{R}^4$ such that

$$E \subset U \subset o + 4 \cdot (E - o) \tag{4.5}$$

where o is the center of E . Note that this E is not necessarily a symplectic ellipsoid. However, by Williamson's theorem on standard forms for symplectic ellipsoids (cf [MS17, Lemma 2.4.6]), there exists a linear symplectomorphism $\phi \in \text{Sp}(4)$ such that $\phi(E) \in \mathcal{E}_4$. Without loss of generality, let us assume that $\phi(E) = E(a, b)$ for some $0 < a \leq b$ (see Equation 2.1). Then, since the shifting (by o) is also a symplectomorphism, the relation (4.5) implies that $E(a, b) \hookrightarrow U \hookrightarrow 4E(a, b)$. By a rescaling, one gets

$$\frac{1}{2}E(4a, 4b) \hookrightarrow U \hookrightarrow 2E(4a, 4b).$$

Therefore, by the definition (4.3), we have $d_c(U, \mathcal{E}_4) \leq \log 2$. □

Remark 4.1.5. *Let us clarify the notation of rescaling $\alpha E(a, b)$ for $\alpha \in \mathbb{R}_{>0}$ that appears in the proof of Proposition 4.1.4. Here is the definition,*

$$\alpha E(a, b) := \left\{ (\alpha z_1, \alpha z_2) \in \mathbb{C}^2 \mid \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\}$$

for $\alpha \in \mathbb{R}_{>0}$. In particular, we have an identification that $\alpha E(a, b) = E(\alpha^2 a, \alpha^2 b)$. This implies that $E(a, b) \hookrightarrow U$ if and only if $\frac{1}{\alpha} E(\alpha a, \alpha b) \hookrightarrow U$.

Remark 4.1.6. *Proposition 4.1.4 holds in any $2n$ -dimensional case. More explicitly, we have $d_c(\mathcal{C}_{2n}, \mathcal{E}_{2n}) \leq \frac{1}{2} \log(2n)$.*

4.2 Counterexample

The main result of this section is to answer Question 4.0.1 under the distance d_c . It turns out that the d_c -distance from the set of dynamically convex domains to the set of symplectically convex domains is unbounded.

Theorem 4.2.1. $\sigma(\mathcal{D}_4, \mathcal{C}_4) = +\infty$.

Along with the proof of Theorem 4.2.1, we discover a family of dynamically convex domains X_{Ω_p} (see (4.6)), parametrized by $p \in (0, 1]$, that are not symplectomorphic to convex ones when p is sufficiently small. It is the first family of dynamically convex domains whose symplectic non-convexity can be verified *without* referring to Chaidez-Edtmair's criterion. Moreover, the verification is only based on the classical machinery - ECH capacities (see subsection 2.3.2), where a concrete estimation on

how small p can be so that X_{Ω_p} is not symplectically convex is obtained (see Remark 4.2.4). Note that an estimation on p could also be obtained via Chaidez-Edtmair's criterion (see Remark 4.2.2).

Remark 4.2.2. *Soon after the first version of [DGRZ23] appears to the public, Oliver Edtmair informed us that, by chasing the arguments in [CE22], one was able to obtain an explicit estimate of C in the criterion (1.3), though not presented in [CE22]. As a consequence, based on a careful calculation of $c_{\text{CE}}(X_{\Omega_p})$, this upper bound also helps to estimate, in a relatively precisely manner, the parameter p in X_{Ω_p} for symplectic non-convexity.*

Now consider the following toric domain denoted by $X_{\Omega_p} \in \mathcal{M}_4$, where Figure 4.2 shows the moment image Ω_p . Explicitly, for $p \in (0, 1]$, let

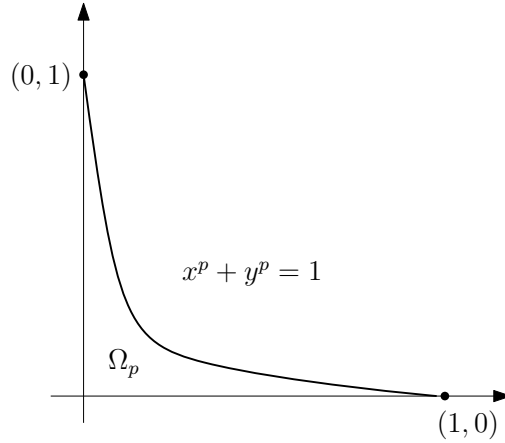


Figure 4.2: X_{Ω_p} for $p \in (0, 1]$.

$$X_{\Omega_p} := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \left(\pi |z_1|^2 \right)^p + \left(\pi |z_2|^2 \right)^p < 1 \right\}. \quad (4.6)$$

Define $d_c(X_{\Omega_p}, \mathcal{E}_4) = \inf_{E \in \mathcal{E}_4} d_c(X_{\Omega_p}, E)$, where \mathcal{E}_4 consists of symplectic ellipsoids in \mathbb{R}^4 defined by (2.1).

Theorem 4.2.3. *For toric domain X_{Ω_p} defined as above, we have the following estimation for $p < \frac{1}{5}$,*

$$d_c(X_{\Omega_p}, \mathcal{E}_4) \geq \frac{1}{8} \log \left(\frac{g(p)}{1 + \log 4 + \log g(p)} \right) \quad (4.7)$$

where $g(p) = 2^{\frac{2}{p}-2} \text{Vol}_{\mathbb{R}^4}(X_{\Omega_p})$ tends to $+\infty$ when $p \rightarrow 0$. In particular, when p

satisfies the condition that

$$\frac{g(p)}{1 + \log 4 + \log g(p)} \geq 256, \quad (4.8)$$

then X_{Ω_p} is dynamically convex but not symplectically convex.

The proof of the second conclusion of Theorem 4.2.3 directly comes from Proposition 4.1.4, while the first conclusion needs a sophisticated argument based on ECH capacities given after.

Remark 4.2.4. From the second conclusion of Theorem 4.2.3, one can estimate $p \in (0, 1]$ so that X_{Ω_p} is symplectically non-convex. Explicitly, from the inductive relation that $g(\frac{1}{k+1}) = \frac{2k+2}{2k+1}g(\frac{1}{k})$ for any positive integer k , one can show that for $p \leq \frac{1}{62460059}$, the condition (4.8) is satisfied and such X_{Ω_p} is not symplectically convex. Here, we emphasize that estimation (4.8) of Theorem 4.2.3 can certainly be improved, where the estimation from Chaidez-Edtmair's criterion (1.3) mentioned in Remark 4.2.2 potentially results in a better estimation on p .

Remark 4.2.5. For two domains $U, V \subset \mathbb{R}^4$, we call them symplectically equivalent if $d_c(U, V) = 0$. Note that if U is intrinsically symplectomorphic to V , then they are symplectically equivalent by the definition of d_c . Potentially, the symplectic equivalence relation is weaker (i.e. more general) than being intrinsically symplectomorphic. Back to X_{Ω_p} , even though the criterion (1.3) applies to verify its symplectic non-convexity, our Theorem 4.2.3 in fact shows that they are not even symplectically equivalent to any convex domains when p is sufficiently small.

Proof. Suppose that $E(a, b) \hookrightarrow X_{\Omega_p} \hookrightarrow \lambda E(a, b)$ for some $E(a, b)$ and $\lambda \geq 1$. We assume without loss of generality that $a \leq b$. Note that X_{Ω_p} is a concave toric domain. For each $k \in \mathbb{N}$, let $x_k(p)$ denote the x -intercept of the line of slope $-\frac{1}{k}$, which is tangent to the curve $x^p + y^p = 1$. It follows from a straight-forward computation that

$$x_k(p) = (1 + k^{\frac{p}{p-1}})^{-\frac{1}{p}} + k(1 + k^{\frac{p}{1-p}})^{-\frac{1}{p}}.$$

Consider a certain truncation of Ω_p , defined by

$$X_k(p) = \{(x, y) \in [0, \infty)^2 \mid x^p + y^p \leq 1 \text{ and } \max(x, y) \leq x_k(p)\}$$

which is shown in Figure 4.3. Let (w_1, w_2, \dots) be the weight decomposition of Ω_p (see the corresponding definition in Subsection 2.3.2). Let $k = \lfloor b/a \rfloor$ and denote by c_k the k -th ECH capacity.

$$ka = c_k(E(a, b)) \leq c_k(X_{\Omega_p}) = c_k\left(\bigsqcup_{i=1}^k B(w_i)\right). \quad (4.9)$$

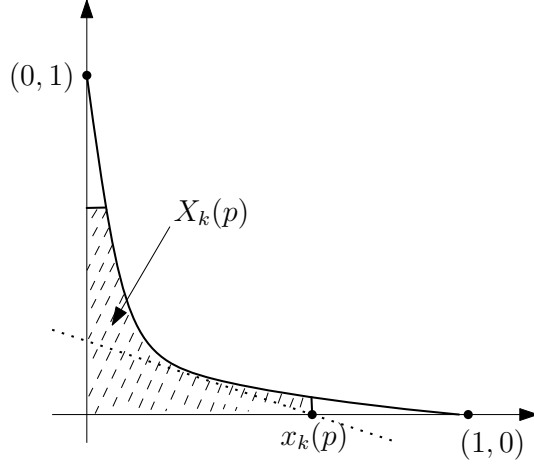


Figure 4.3: $X_k(p)$ for $p \in (0, 1]$.

From Lemma 2.3.14, we obtain

$$c_k \left(\bigsqcup_{i=1}^k B(w_i) \right) \leq 2 \sqrt{k \cdot \text{Vol} \left(\bigsqcup_{i=1}^k B(w_i) \right)}. \quad (4.10)$$

It follows from the definition of the weight decomposition that $\bigsqcup_{i=1}^k B(w_i) \subset X_k(p)$. So

$$\begin{aligned} \text{Vol} \left(\bigsqcup_{i=1}^k B(w_i) \right) &\leq \text{Vol}(X_k(p)) = 2^{-\frac{2}{p}} + 2 \int_{2^{-\frac{1}{p}}}^{x_k(p)} (1 - x^p)^{\frac{1}{p}} dx \\ &= 2^{-\frac{2}{p}} \left(1 + 2 \int_1^{2^{\frac{1}{p}} x_k(p)} (2 - u^p)^{\frac{1}{p}} du \right). \end{aligned} \quad (4.11)$$

We note that the integrand in the last integral above increases as $p \rightarrow 0$ and it converges pointwise to $\frac{1}{u}$. Moreover it is readily verified that $2^{\frac{1}{p}} x_k(p)$ also increases as $p \rightarrow 0$ and it converges to $2\sqrt{k}$. So

$$2 \int_1^{2^{\frac{1}{p}} x_k(p)} (2 - u^p)^{\frac{1}{p}} du \leq 2 \int_1^{2\sqrt{k}} \frac{1}{u} du = 2 \log(2\sqrt{k}) = \log 4 + \log k. \quad (4.12)$$

Combining (4.9), (4.10), (4.11) and (4.12), we obtain $ka \leq 2 \cdot 2^{-\frac{1}{p}} \sqrt{k(1 + \log 4 + \log k)}$. So

$$a \leq 2 \cdot 2^{-\frac{1}{p}} \sqrt{\frac{1 + \log 4 + \log k}{k}}. \quad (4.13)$$

Since $X_{\Omega_p} \hookrightarrow \lambda E(a, b)$, we have

$$2 \cdot 2^{-\frac{1}{p}} = c_1^{\text{ECH}}(X_{\Omega_p}) \leq \lambda^2 c_1^{\text{ECH}}(E(a, b)) = \lambda^2 a. \quad (4.14)$$

It follows from (4.13) and (4.14) that $a \leq \lambda^2 a \sqrt{\frac{1+\log 4+\log k}{k}}$. So

$$\lambda^4 \geq \frac{k}{1 + \log 4 + \log k}. \quad (4.15)$$

Note that this estimation is not enough to deduce the requested large d_c -distance conclusion since when p changes, the constant k (so that the corresponding ellipsoid $E(a, b)$ with $k = \lfloor b/a \rfloor$ that embeds into X_{Ω_p}) may change.

Now, from (4.13), we also obtain the following estimations,

$$\begin{aligned} a &\leq \sqrt{\frac{(1 + \log 4 + \log k) \text{Vol}(X_{\Omega_p})}{k \cdot g(p)}} \\ &\leq \sqrt{\frac{(1 + \log 4 + \log k) \text{Vol}(\lambda E(a, b))}{k \cdot g(p)}} \\ &= \sqrt{\frac{(1 + \log 4 + \log k) \lambda^4 ab}{2k \cdot g(p)}} \\ &\leq a \sqrt{\frac{(1 + \log 4 + \log k) \lambda^4 (k + 1)}{2k \cdot g(p)}} \\ &\leq a \sqrt{\frac{(1 + \log 4 + \log k) \lambda^4}{g(p)}} \end{aligned}$$

where the final step comes from the estimation $\frac{k+1}{2k} \leq 1$ for all $k \in \mathbb{N}$. So, we get

$$\lambda^4 \geq \frac{g(p)}{1 + \log 4 + \log k}. \quad (4.16)$$

From (4.15) and (4.16) it follows that

$$\lambda^4 \geq \frac{\max(g(p), k)}{1 + \log 4 + \log k}. \quad (4.17)$$

Now, let us study $p \mapsto g(p)$. From a change of variables we obtain

$$\begin{aligned} g(p) &= 2^{\frac{2}{p}-2} \text{Vol}(X_{\Omega_p}) = 2^{\frac{2}{p}-2} \int_0^1 (1 - x^p)^{\frac{1}{p}} dx \\ &= \frac{1}{4} \int_0^{2^{\frac{1}{p}}} (2 - u^p)^{\frac{1}{p}} du. \end{aligned}$$

One observes that $\lim_{p \rightarrow 0} (2 - u^p)^{\frac{1}{p}} = \frac{1}{u}$ and $\lim_{p \rightarrow 0} 2^{\frac{1}{p}} = +\infty$. Therefore $\lim_{p \rightarrow 0} g(p) =$

$+\infty$. From the same equation, we see that g is decreasing on $(0, 1]$ since $p \mapsto (2 - u^p)^{\frac{1}{p}}$ is decreasing for any $u \in [0, 2^{1/p}]$. Moreover we have

$$\begin{aligned} \int_0^1 (1 - x^p)^{\frac{1}{p}} dx &= \frac{1}{p} \int_0^1 (1 - t)^{\frac{1}{p}} t^{\frac{1}{p}-1} dt = \frac{1}{p} B\left(\frac{1}{p}, \frac{1}{p} + 1\right) = \frac{1}{p} \frac{\Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{1}{p} + 1\right)}{\Gamma\left(\frac{2}{p} + 1\right)} \\ &= \frac{1}{2p} \frac{\Gamma\left(\frac{1}{p}\right)^2}{\Gamma\left(\frac{2}{p}\right)}. \end{aligned}$$

Then, by the Legendre duplication formula we get

$$g(p) = \frac{\sqrt{\pi}}{4p} \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p} + \frac{1}{2}\right)}$$

And we recover the asymptotic of $g(p)$ when p goes to 0 by Stirling's formula

$$g(p) \underset{p \rightarrow 0}{\sim} \sqrt{\frac{\pi}{16p}}.$$

Finally, remark that

$$g\left(\frac{1}{5}\right) = 2^{2 \cdot 5 - 2} \cdot \frac{5}{2} \cdot \frac{\Gamma(5)^2}{\Gamma(2 \cdot 5)} = 2^8 \cdot \frac{5}{2} \cdot \frac{4!^2}{9!} = \frac{64}{63} > 1.$$

So for $p < \frac{1}{5}$, it follows that $\lfloor g(p) \rfloor \geq 1$. In this case,

$$\begin{aligned} \min_k \frac{\max(g(p), k)}{1 + \log 4 + \log k} &= \min \left(\frac{g(p)}{1 + \log 4 + \log \lfloor g(p) \rfloor}, \frac{\lfloor g(p) \rfloor + 1}{1 + \log 4 + \log(\lfloor g(p) \rfloor + 1)} \right) \\ &\geq \frac{g(p)}{1 + \log 4 + \log g(p)}. \end{aligned}$$

Taking the infimum of over all triples (λ, a, b) , we obtain

$$f(p)^4 \geq \frac{g(p)}{1 + \log 4 + \log g(p)}$$

where $f(p) = \inf\{\lambda \geq 1 \mid E(a, b) \hookrightarrow X_{\Omega_p} \hookrightarrow \lambda E(a, b) \text{ for some } a, b > 0\}$. Then one can show that $f(p)$ is linked to the coarse distance d_c by

$$d_c(X_{\Omega_p}, \mathcal{E}_4) = \frac{1}{2} \log f(p)$$

thus we obtain (4.7). Then the desired conclusion results from the following triangular

inequality,

$$d_c(X_{\Omega_p}, \mathcal{E}_4) \leq d_c(X_{\Omega_p}, \mathcal{C}_4) + \sigma(\mathcal{C}_4, \mathcal{E}_4)$$

and Proposition 4.1.4. □

Here, we point out that if we “linearize” the profile curve in Ω_p of X_{Ω_p} considered in the proof of Theorem 4.2.3 above, denoted by Ω_p^{lin} and shown in Figure 4.4, then following conclusion holds.

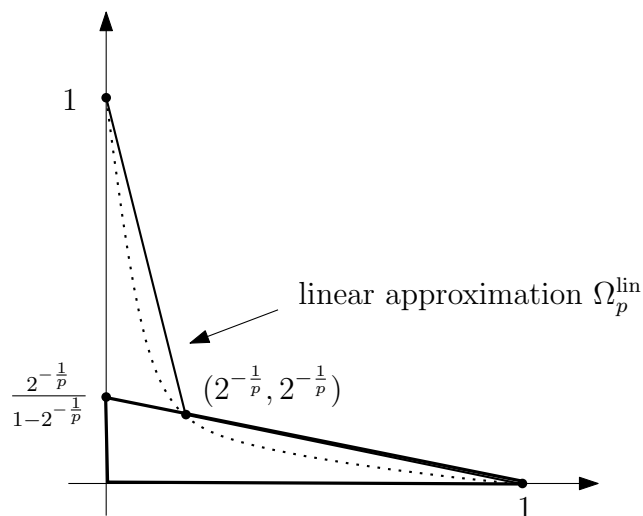


Figure 4.4: A linear approximation of X_{Ω_p} from Ω_p^{lin} .

Proposition 4.2.6. *For any $p \in (0, 1]$, we have $d_c(X_{\Omega_p^{\text{lin}}}, \mathcal{C}_4) \leq \frac{1}{2} \log 3$.*

Proof. On the one hand, by inclusion, the ellipsoid $E(1, \frac{2^{-1/p}}{1-2^{-1/p}})$ shown by the bold edges in Figure 4.4 embeds inside $X_{\Omega_p^{\text{lin}}}$. On the other hand, by Lemma 2.3.16, $X_{\Omega_p^{\text{lin}}} \hookrightarrow P(1, 2^{-1/p+1})$. Since $P(1, 2^{-1/p+1}) \subset (3 - 2^{-1/p+1})E(1, \frac{2^{-1/p}}{1-2^{-1/p}})$ trivially, one gets the following relation,

$$E\left(1, \frac{2^{-1/p}}{1-2^{-1/p}}\right) \hookrightarrow X_{\Omega_p^{\text{lin}}} \hookrightarrow 3E\left(1, \frac{2^{-1/p}}{1-2^{-1/p}}\right)$$

This implies that

$$d_c(X_{\Omega_p^{\text{lin}}}, \mathcal{E}_4) \leq \frac{1}{2} \log 3$$

which completes the proof. □

Remark 4.2.7. *Here is a way to explain the essential difference between Theorem*

4.2.1 and Proposition 4.2.6, by simply comparing the respective volumes, where

$$\frac{\text{Vol}_{\mathbb{R}^4}(X_{\Omega_p^{\text{lin}}})}{\text{Vol}_{\mathbb{R}^4}(X_{\Omega_p})} \rightarrow +\infty$$

as $p \rightarrow 0$.

4.3 Differences in two criteria

As mentioned above, before [DGRZ23], the only technique that was known for proving that dynamically convex domains are not symplectically convex was Chaidez-Edtmair's criterion based on the Ruelle invariant. It turns out that this criterion and the metric-geometrical approach used in this chapter are quite different in essence as we now explain.

The following result shows that the Ruelle invariant based criterion (1.3) and d_c provide two different perspectives to study symplectic non-convexity.

Theorem 4.3.1. *There exist sequences of star-shaped toric domains $\{X_{\Omega_1^k}\}$ and $\{X_{\Omega_2^k}\}$ in \mathbb{R}^4 such that*

$$\lim_{k \rightarrow \infty} \frac{c_{\text{CE}}(X_{\Omega_1^k})}{c_{\text{CE}}(X_{\Omega_1^1})} = 0 \quad \text{and} \quad d_c(X_{\Omega_1^k}, X_{\Omega_1^1}) < A$$

$$\lim_{k \rightarrow \infty} \frac{c_{\text{CE}}(X_{\Omega_2^k})}{c_{\text{CE}}(X_{\Omega_2^1})} = +\infty \quad \text{and} \quad d_c(X_{\Omega_2^k}, X_{\Omega_2^1}) < B$$

where A and B are positive constants independent of the integer k .

Remark 4.3.2. *Since d_c reflects the changes of symplectic capacities, Theorem 4.3.1 also implies that c_{CE} and symplectic capacities are independent to each other in general. It would be interesting to explore other numerical characterizations of toric domains (cf. recent work from Hutchings [Hut24]).*

Remark 4.3.3 (Informed by Oliver Edtmair). *The non-toric examples constructed in Proposition 1.3.5 with small and large c_{CE} in fact lie in arbitrarily small neighborhoods of the round ball with respect to the coarse distance d_c . This shows another evidence of the independence between c_{CE} and d_c .*

Proof. The proof is given by two different examples.

Example One: Let us consider the strangulation operation on $B^4(1)$ or for brevity $B(1)$. By definition, it cuts off a part of $B^4(1)$, which is, on the level of the moment image, a thin triangle symmetric to the diagonal $y = x$. See Figure 4.5, where $\epsilon(\delta)$ is

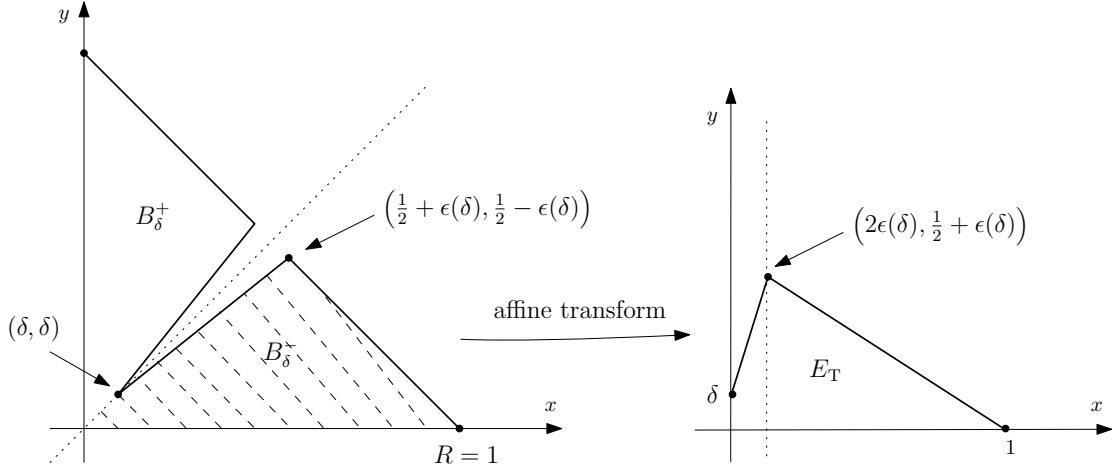


Figure 4.5: A strangulation on $B(1)$, resulting a toric domain B_δ .

proportional to δ , Denote the resulting toric domain by B_δ . Then obviously we have $B_\delta \hookrightarrow B(1)$ simply by inclusion. Now, cut through the diagonal and taking closures, we obtain two toric domains B_δ^- and B_δ^+ (which are in fact symplectomorphic to each other by symmetry). Apply the affine transform

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

then the moment image of B_δ^- transfers to a region E_T where the corresponding toric domain (still denoted by) E_T is an example of so-called *truncated ellipsoid* defined in [Ush22]. More explicitly, by an elementary calculation, it is a truncated ellipsoid from $E\left(1, \frac{1+2\epsilon(\delta)}{2-4\epsilon(\delta)}\right)$, with truncated parameters δ and $\frac{\frac{1}{2}+\epsilon(\delta)-\delta}{2\epsilon(\delta)}$. These parameters are denoted by ϵ and β respectively in [Ush22]. For our purpose, let us denote

$$\beta(\delta) := \frac{\frac{1}{2} + \epsilon(\delta) - \delta}{2\epsilon(\delta)}. \quad (4.18)$$

Since affine transformations on moment images induce symplectomorphisms on the corresponding toric domains, we know that

$$(E_T)^- \sqcup (E_T)^+ \simeq B_\delta^- \sqcup B_\delta^+ \hookrightarrow B_\delta \quad (4.19)$$

where $(E_T)^-$ and $(E_T)^+$ stand for two copies of the truncated ellipsoid above. One of the more striking properties of truncated ellipsoids is Corollary 5.3 in [Ush22], which

proves that

$$d_c \left(E_{\mathbb{T}}^-, E \left(1, \frac{1 + 2\epsilon(\delta)}{2 - 4\epsilon(\delta)} \right) \right) \leq 2 \log \left(\frac{1 + \beta(\delta)}{\beta(\delta)} \right) := \log C(\delta). \quad (4.20)$$

Hence, with all the notations above, we have

$$\begin{aligned} \frac{1}{C(\delta)} B \left(\frac{1 + 2\epsilon(\delta)}{2 - 4\epsilon(\delta)} \right) &\hookrightarrow \frac{1}{C(\delta)} B \left(\frac{1 + 2\epsilon(\delta)}{2 - 4\epsilon(\delta)} \right) \sqcup \frac{1}{C(\delta)} B \left(\frac{1 + 2\epsilon(\delta)}{2 - 4\epsilon(\delta)} \right) \\ &\hookrightarrow \frac{1}{C(\delta)} E \left(1, \frac{1 + 2\epsilon(\delta)}{2 - 4\epsilon(\delta)} \right) \sqcup \frac{1}{C(\delta)} E \left(1, \frac{1 + 2\epsilon(\delta)}{2 - 4\epsilon(\delta)} \right) \\ &\hookrightarrow (E_{\mathbb{T}})^- \sqcup (E_{\mathbb{T}})^+ \hookrightarrow B_{\delta} \end{aligned}$$

where the first embedding is just the inclusion; the second embedding comes from the ball capacity of an ellipsoid; the third embedding comes from (4.20); the last embedding is just (4.19). To sum up, we obtain the following embedding relations,

$$B \left(\frac{1}{C(\delta)^2} \cdot \frac{1 + 2\epsilon(\delta)}{2 - 4\epsilon(\delta)} \right) \hookrightarrow B_{\delta} \hookrightarrow B(1).$$

Thus,

$$d_c(B(1), B_{\delta}) \leq \frac{1}{4} \log \left(C(\delta)^2 \cdot \frac{2 - 4\epsilon(\delta)}{1 + 2\epsilon(\delta)} \right)$$

Then when $\delta \rightarrow 0$, we have

$$\epsilon(\delta) \rightarrow 0, \text{ so } \frac{2 - 4\epsilon(\delta)}{1 + 2\epsilon(\delta)} \rightarrow 2; \quad \beta(\delta) \rightarrow +\infty, \text{ so } C(\delta) \rightarrow 1.$$

Therefore, $d_c(B(1), B_{\delta}) \leq \frac{1}{4} \log 2$ for δ sufficiently small. In particular, $B(1)$ and B_{δ} do not differ much in terms of any symplectic capacity. However, by Lemma 3.1.1, we know $c_{\text{CE}}(B_{\delta}) \rightarrow 0$ as $\delta \rightarrow 0$.

Example Two: Let us consider the strain operation on $B^4(99)$ or for brevity $B(99)$. By definition, we add two small ‘‘tail’’ triangles along each axis on the moment image of $B(99)$. See Figure 4.6, where small triangles are shaded, they are symmetric with respect to $y = x$, and the horizontal one has height 1 and base 99. Denote the resulting domain by X . Note that

$$\text{Vol}_{\mathbb{R}^4}(X) = \frac{99^2}{2} + 2 \cdot \frac{99}{2} < \frac{100^2}{2} = \text{Vol}_{\mathbb{R}^4}(B(100)).$$

Then obviously we have $B(99) \hookrightarrow X$. Now, we claim that $X \hookrightarrow B(100)$. Since X is

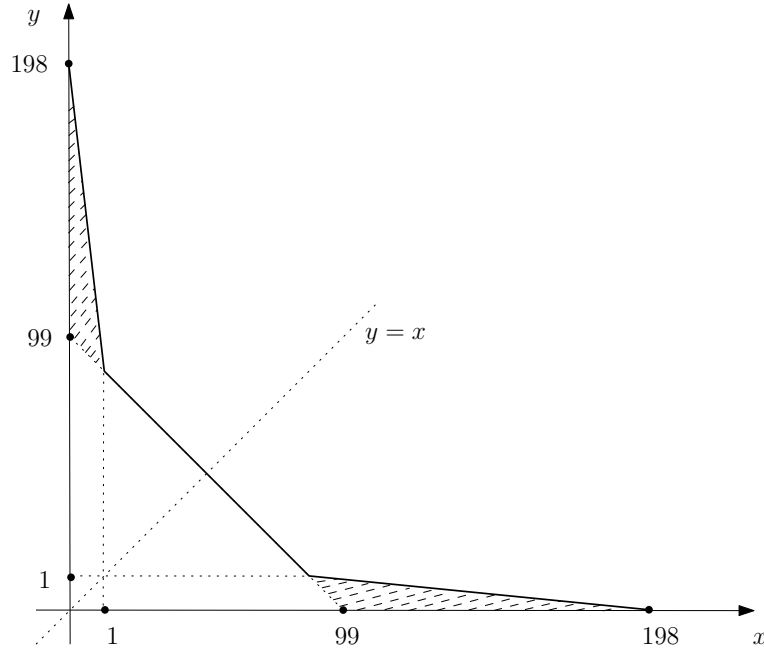


Figure 4.6: A strain on $B(99)$, resulting a toric domain X .

concave toric, $B(100)$ is convex toric, and the weight sequence of X is

$$(w_1, w_2, \dots) = (99, 1, \underbrace{\dots, 1}_{198\text{-many}})$$

such required embedding can be transferred to a ball-packing problem by [CG14, Theorem 2.1]. In fact, we claim that

$$B(99) \cup \underbrace{B(1) \cup \dots \cup B(1)}_{199\text{-many}} \hookrightarrow B(100).$$

Indeed, the shaded tail triangles in X (for instance, the horizontal one) can be divided into small triangles where each is equivalent to the triangle with vertices $(98, 0), (99, 0), (98, 1)$, up to affine transformation. For an illustration, see the left picture in Figure 4.7. Then by further affine transformations, we can move the “start” triangle into $B(100) \setminus B(99)$, on the level of their moment image denoted by $\Delta(100) \setminus \Delta(99)$. The right picture in Figure 4.7 shows how the small triangle, shaded one with vertices $(100, 0), (101, 0), (98, 1)$, is placed inside $\Delta(100) \setminus \Delta(99)$, precisely located at the shaded small triangle with vertices $(99, 1), (100, 1), (99, 2)$. Since

$$\text{Area}_{\mathbb{R}^2}(\Delta(100) \setminus \Delta(99)) = \frac{100^2}{2} - \frac{99^2}{2} = \frac{199}{2} = 199 \cdot \text{Area}_{\mathbb{R}^2}(\text{small } \Delta),$$

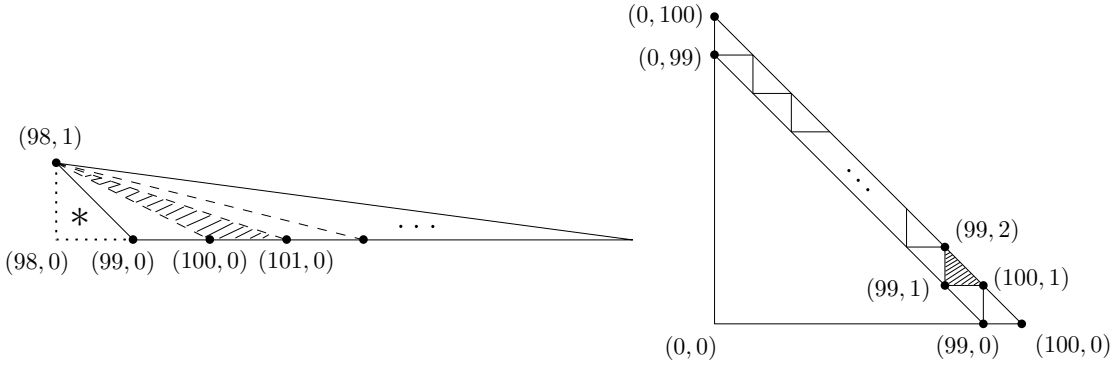


Figure 4.7: Packing the tail triangles of X into $B(100)$.

such procedure can be inductively carried out and pack 199-many small Δ into $\Delta(100) \setminus \Delta(99)$. This corresponds to a packing of 199-many $B(1)$ into $B(100) \setminus B(99)$. Thus we obtain the desired claim.

Note that the same argument works identically if we replace the tail triangle in X by one with the furthest vertex being $(k, 0)$ (but with volume fixed). Denote the resulting toric domain by X_k , then we complete the proof by Lemma 3.2.1 where $c_{\text{CE}}(X_k) \rightarrow \infty$ when $k \rightarrow \infty$. \square

4.4 Large-scale geometry

Consider the following pseudo-metric space

$$(\mathcal{S}_4, d_c) = (\{\text{star-shaped domains in } \mathbb{R}^4\}, d_c).$$

From a perspective of the coarse geometry that focuses on large-scale geometrical phenomena, one could ask how many linearly independent directions in (\mathcal{S}_4, d_c) go to infinity, usually called the rank of a quasi-flat in (\mathcal{S}_4, d_c) . More precisely,

Definition 4.4.1. *Let (X_1, d_1) and (X_2, d_2) be two metric spaces, a map $\psi : (X_1, d_1) \rightarrow (X_2, d_2)$ is called a quasi-isometric embedding if there exists constants $A, B > 0$ such that for any $x, y \in X_1$*

$$\frac{1}{A}d_1(x, y) - B \leq d_2(\psi(x), \psi(y)) \leq Ad_1(x, y) + B$$

Definition 4.4.2. *Let (X, d) be a metric space, a rank n quasi-flat in (X, d) is a quasi-isometric embedding $(\mathbb{R}^n, \|\cdot\|_\infty) \rightarrow (X, d)$. The quasi-flat rank of (X, d) is the supremum over such existing n .*

The higher the rank is or the more directions there are, the richer star-shaped

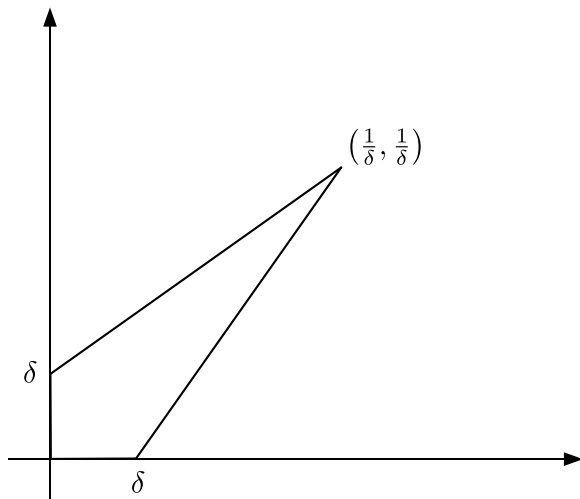


Figure 4.8: The 1-finger domain from Hermann.

domains in \mathbb{R}^4 will be in terms of d_c . Immediately from the relation $\mathcal{E}_4 \subset \mathcal{S}_4$ together with Proposition 4.1.4, one concludes that there are at least 2 such directions, given by \mathcal{C}_4 consisting of all the symplectically convex domains.

Moreover, the classical example, 1-finger shape from Hermann in [Her], shows that there exists a third direction. Let us provide a new proof based on the Lagrangian capacity (cf Subsection 2.4.2).

Proposition 4.4.3. *There exists a family of toric domains in \mathbb{R}^4 , denoted by X_δ , such that $d_c(X_\delta, \mathcal{E}_4) \geq -\frac{1}{2}\log(\delta)$. In particular, $d_c(X_\delta, \mathcal{E}_4) \rightarrow +\infty$ when $\delta \rightarrow 0$.*

Proof. Consider the Hermann's finger-type toric domain, denoted by X_δ , with the tip point being $(\frac{1}{\delta}, \frac{1}{\delta})$ (see Figure 4.8). Choose the w_1 -intercept and the w_2 -intercept so that $\text{Vol}(X_\delta) = 1$ for any δ . Then for any $E(a, b)$ such that $X_\delta \hookrightarrow E(a, b)$, monotonicity of the Lagrangian capacity c_{Lag} implies

$$\frac{1}{\delta} = c_{\text{Lag}}(X_\delta) \leq c_{\text{Lag}}(E(a, b)) = \frac{ab}{a+b}.$$

Where the two equalities come from Theorem 2.4.10. Since $a+b \geq 2\sqrt{ab}$, we know that $\frac{1}{\delta} \leq \frac{\sqrt{ab}}{2}$. This give the following constraint

$$\frac{2}{\delta^2} \leq \frac{ab}{2} = \text{Vol}(E(a, b)). \quad (4.21)$$

For this $E(a, b)$, if there exists an embedding $\frac{1}{\alpha}E(a, b) \hookrightarrow X_\delta$, then the volume con-

straint implies that $\frac{1}{\alpha^2} \text{Vol}(E(a, b)) \leq \text{Vol}(X_\delta) = 1$. Together with (4.21), we have

$$\frac{2}{\delta^2} \leq \alpha^2 \quad \text{which implies that} \quad \frac{1}{\delta} \leq \frac{\sqrt{2}}{\delta} \leq \alpha.$$

In other words, we have shown that for any $E(a, b)$ such that $\frac{1}{\alpha}E(a, b) \hookrightarrow X_\delta \hookrightarrow E(a, b)$, we have $\alpha \geq \frac{1}{\delta}$. By definition, this implies $d_c(X_\delta, \mathcal{E}_4) \geq -\frac{1}{2}\log(\delta)$ which is the desired conclusion. \square

Finally, by Theorem 4.2.3 and a quick observation via the comparison of Lagrangian capacity c_{Lag} (see Subsection 2.4.2) of X_{Ω_p} and Hermann's example, the family X_{Ω_p} for $p \in (0, 1]$ indicates the existence of a new direction going to infinity in (\mathcal{S}_4, d_c) .

Therefore, the previous examples suggest that the quasi-flat rank of (\mathcal{S}_4, d_c) is greater than 4. In fact, Cristofaro-Gardiner and Hind showed in [CGH] the following theorem.

Theorem 4.4.4 ([CGH], Theorem 1.2). *The space of star-shaped domains in \mathbb{R}^4 , with respect to the coarse distance d_c , has infinite quasi-flat rank.*

Perspectives

Let us conclude this thesis with the following perspectives that could lead to future research. There are two main directions.

L^p -balls and the Viterbo conjecture

The toric domains $B_p(\mathbb{C}) := X_{\Omega_p}$ considered in Section 4.2 are often referred as *complex* L^p -balls. Indeed, it is known (by upcoming work of Aydin and Schlenk) that, up to some dilatation factor, they are intrinsically¹ symplectomorphic to the *real* L^{2p} -balls for any $p > 0$:

$$B_{2p}(\mathbb{R}) := \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \mid |x_1|^{2p} + |y_1|^{2p} + |x_2|^{2p} + |y_2|^{2p} \leq 1 \right\}$$

One can easily check that $B_{2p}(\mathbb{R})$ is geometrically convex if and only if $p \geq \frac{1}{2}$. This suggests that condition (4.8) on p from Theorem 4.2.3, so that $B_p(\mathbb{C})$ is not symplectically convex, could be improved. Moreover, the limit case for the upper constant in Theorem 3.3.1 is attained by the real L^1 -ball, that is exactly for $p = \frac{1}{2}$. This lead to the following conjecture

Conjecture 4.4.5. $B_{2p}(\mathbb{R})$ is symplectically convex if and only if $p \geq \frac{1}{2}$

It means that L^p -balls might be no longer symplectically convex as soon as they are no more geometrically convex. This would enlighten some new rigidity phenomenon in symplectic geometry. Let $\mathbf{p} := \sup \{p > 0 \mid B_{2p}(\mathbb{R}) \text{ is not symplectically convex}\}$, we have directly $\mathbf{p} < \frac{1}{2}$ and by Remark 4.2.4,

$$\mathbf{p} > \frac{1}{62460059}$$

One way to significantly improve this constant would be to compute the d_c distance from symplectically convex domains to ellipsoids. In fact, by Proposition 4.1.4, $\sigma(\mathcal{C}_4, \mathcal{E}_4) \leq \log 2$ and, in the standard Banach-Mazur setting, this upper bound is

¹See Section 0.5.

sharp exactly for the standard simplex. However, due to Aydin and Schlenk, this standard simplex is symplectomorphic to the $B_{\frac{1}{2}}(\mathbb{C})$ and so its coarse distance to the set of ellipsoids is strictly less than $\log 2$. This suggests that $\sigma(\mathcal{C}_4, \mathcal{E}_4) < \log 2$.

Question 4.4.6. *What is the value of $\sigma(\mathcal{C}_4, \mathcal{E}_4)$?*

On the other hand, since $\mathcal{E}_4 \subset \mathcal{M}_4$, the monotonicity of $\sigma(-, -)$ implies that

$$\sigma(\mathcal{C}_4, \mathcal{M}_4) \leq \log 2$$

However, for any symplectically convex domain $U \in \mathcal{C}_4$, it is not clear how to approximate U via non-ellipsoids domains $V \in \mathcal{M}_4$. Nevertheless, since \mathcal{M}_4 contains substantially more elements than \mathcal{E}_4 , we expect that the upper bound of $\sigma(\mathcal{C}_4, \mathcal{M}_4)$ is much lower than $\log 2$. In fact, there are several examples of domains in [Ram15],[OR19] and [ORS23] such that the distance is zero. This suggests the following conjecture

Conjecture 4.4.7. $\sigma(\mathcal{C}_4, \mathcal{M}_4) = 0$

Since the strong Viterbo conjecture is true for elements in \mathcal{M}_4 (see Theorem 2.3.15), this would lead to a proof of Conjecture 1.1.7 in dimension 4. Note that due to [CGH, Theorem 1.2], a generalization of the above conjecture would lead to the same conclusion in higher dimensions.

Symplectic proximity

Let us emphasize that this approach is sensitive to the notion of *symplectic proximity* chosen. Indeed, if two domains are intrinsically symplectomorphic, then the coarse distance between them is equal to zero. Let us consider the following stronger version of the above conjecture:

Conjecture 4.4.8. *Let U be a symplectically convex domain of \mathbb{R}^4 , then there exists a monotone toric domain X_Ω such that $\text{int}(U)$ is symplectomorphic to $\text{int}(X_\Omega)$.*

However, due to recent results from Hutchings [Hut24], this is known to be false. This suggests the following hierarchy between different notions of symplectic proximity:

$$\text{globally symplectomorphic} > \text{intrinsically symplectomorphic} \geq d_c = 0$$

By results from Eliashberg and Hofer [EH96] or Abbondandolo, Benedetti and Edtmair [ABE23] (see Section 0.5), we know that there exists intrinsically symplectomorphic domains which are not globally symplectomorphic. On the other hand, the following question is still open:

Question 4.4.9. *Let U and V be two star-shaped domains in \mathbb{R}^4 such that $d_c(U, V) = 0$. Are they intrinsically symplectomorphic?*

A priori, we cannot use dynamical tools to distinguish these two notions because of the previous results cited above. Answering this question might lead to discover new intrinsic symplectic invariants.

Appendix

The Conley-Zehnder index

In this Appendix, we recall the properties of the Conley-Zehnder index following [Gut14], then we introduce one of its extension: the Robbin-Salamon index. Finally we describe how one can associate such index to any periodic Reeb orbit for some contact manifolds.

First introduced in [CZ84], Conley-Zehnder index associates an integer to any continuous path of symplectic matrices starting at the identity and such that ending matrix doesn't admit 1 as eigenvalue. In [RS93], Robbin and Salamon extended this index on any continuous paths of symplectic matrix starting at the identity, as expected this so-called Robbin-Salamon index agrees with the Conley-Zehnder index whenever the ending matrix doesn't possess 1 as eigenvalue. However, in the degenerate case Robbin-Salamon's index might not be an integer. In the contact setting, one can associate a CZ-index to any non-degenerate closed Reeb orbit and this extends to degenerate closed Reeb orbits via the RS-index.

Let $SP(2n) := \left\{ \Psi : [0, 1] \rightarrow Sp(2n) \mid \Psi(0) = \text{Id} \text{ and } \det(\Psi(1) - \text{Id}) \neq 0 \right\}$ where the elements of $SP(2n)$ are supposed to be continuous.

Theorem 4.4.10 ([Sal99], Section 2.4). *The Conley Zehnder index is a map*

$$CZ : SP(2n) \rightarrow \mathbb{Z}$$

satisfying the following properties

- (Naturality) *For any continuous path $\Phi : [0, 1] \rightarrow Sp(2n)$,*

$$CZ(\Phi\Psi\Phi^{-1}) = CZ(\Psi)$$

- (Homotopy) *The Conley-Zehnder index is constant on the components of $SP(2n)$*

- (Zero) If $\Psi(s)$ has no eigenvalue on the unit circle for $s > 0$ then $CZ(\Psi) = 0$
- (Product) $CZ(\Psi' \oplus \Psi'') = CZ(\Psi') + CZ(\Psi'')$
- (Loop) If $\Psi \in \text{SP}(2n)$ is a loop, then $CZ(\Phi\Psi) = CZ(\Phi) + 2\mu(\Psi)$ where μ is the Maslov index
- (Normalization) if $S \in \mathbb{R}^{2n \times 2n}$ is a symmetric non-degenerate matrix with all eigenvalues of absolute value $< 2\pi$ and if $\Psi(t) = \exp(J_0St)$ for $t \in [0, 1]$ and $J_0 = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$, then $RS(\Psi) = \frac{1}{2}\text{Sign}(S)$ where $\text{Sign}(S)$ is the signature of S .
- (Determinant) $(-1)^{n-CZ(\Psi)} = \text{Sign} \det(\text{Id} - \Psi(1))$
- (Inverse) $CZ(\Psi^{-1}) = CZ(\Psi^T) = -CZ(\Psi)$

This map is characterized by the homotopy, loop and signature properties (see [Gut14, Proposition 37]). For paths of symplectic matrices whose ending matrix has 1 as eigenvalue, Robbin and Salamon introduced the following extension of the Conley-Zehnder index.

Theorem 4.4.11 ([Gut14], Theorem 1.1). *The Robbin-Salamon index for a continuous path of symplectic matrices is characterized by the following properties:*

- (Homotopy) *it is invariant under homotopies with fixed endpoints*
- (Catenation) *it is additive under catenation of paths*
- (Zero) *it vanishes on any path $\psi : [a, b] \rightarrow \text{Sp}(2n)$ of matrices such that $\dim \text{Ker}(\psi(t) - \text{Id}) = k$ is constant on $[a, b]$.*
- (Normalization) *if $S \in \mathbb{R}^{2n \times 2n}$ is a symmetric matrix with all eigenvalues of absolute value $< 2\pi$ and if $\psi(t) = \exp(J_0St)$ for $t \in [0, 1]$ and $J_0 = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$, then $RS(\psi) = \frac{1}{2}\text{Sign}(S)$ where $\text{Sign}(S)$ is the signature of S .*

In particular, one have the following property which is crucial in Proposition 2.2.3.

Proposition 4.4.12 ([Gut14], Proposition 4.9). *The Robbin-Salamon index of a symplectic shear $\psi_t = \begin{pmatrix} \text{Id} & B(t) \\ 0 & \text{Id} \end{pmatrix}$ with $B(t)$ symmetric, is equal to*

$$RS(\psi) = \frac{1}{2}\text{Sign}B(0) - \frac{1}{2}\text{Sign}B(1)$$

There are other extensions of the Conley-Zehnder index which are not all equivalent (see Section 2.2 in [CE22] for the four-dimensional case and Section 2.3 in [CE] for any dimension).

Index of a closed Reeb orbit

In the manner of Subsection 1.3.1, one can associate a path of symplectic matrices to any closed Reeb orbit. Let (S^{2n-1}, ξ) be the $(2n-1)$ -sphere with standard contact structure $\xi = \ker(\alpha)$. For any closed Reeb orbit γ of α , consider a trivialization of the contact structure along γ :

$$\tau : \xi_\gamma \simeq \mathbb{R}^{2n-2}$$

For any $t > 0$, the Reeb flow φ_t induces a path of symplectic maps

$$d\varphi_t(\gamma(0)) : \xi_{\gamma(0)} \rightarrow \xi_{\gamma(t)}$$

which can be identified as a symplectic matrix via the trivialization τ

$$\mathbb{R}^{2n-2} \xrightarrow{\tau(\gamma(0))^{-1}} \xi_{\gamma(0)} \xrightarrow{d\varphi_t(\gamma(0))} \xi_{\gamma(t)} \xrightarrow{\tau(\gamma(t))} \mathbb{R}^{2n-2}$$

Define

$$\Phi_\gamma^\tau(t) := \tau(\gamma(t)) \circ d\varphi_t(\gamma(0)) \circ \tau(\gamma(0))^{-1}$$

the composition of the above maps. For any $T > 0$, this gives us a path of symplectic matrices:

$$\begin{aligned} \Phi_{\gamma,T}^\tau : [0, T] &\longrightarrow \mathrm{Sp}(2n-2) \\ t &\longmapsto \Phi_\gamma^\tau(t) \end{aligned}$$

which starts at the identity since $d\varphi_0 = \mathrm{Id}$.

Definition 4.4.13. *For any closed Reeb orbit γ with action $T := \mathcal{A}(\gamma)$, we define its Conley-Zehnder index relative to the trivialization τ as*

$$CZ^\tau(\gamma) := RS\left(\Phi_{\gamma,T}^\tau\right)$$

In particular, if γ is non-degenerate, we have

$$CZ^\tau(\gamma) = CZ\left(\Phi_{\gamma,T}^\tau\right)$$

Canonical Conley-Zehnder index

First, remark that the previous index depends on the trivialization. In order to introduce a definition which is independent of the choice of trivialization, we need the following proposition that holds in dimension 4. Note that similar ideas work in higher dimensions.

Proposition 4.4.14 ([HZ]). *Let U be a nice star-shaped domain in \mathbb{R}^4 and denote by ξ the standard contact distribution on ∂U . For any closed Reeb orbit γ on ∂U and two trivializations τ, τ' of ξ along γ , we have*

$$2c_1^\tau(\xi_\gamma) + CZ^\tau(\gamma) = 2c_1^{\tau'}(\xi_\gamma) + CZ^{\tau'}(\gamma)$$

where c_1^τ is the relative first Chern class.

In particular, the following quantity is well defined

Definition 4.4.15. *Let U be a nice star-shaped domain in \mathbb{R}^4 , let γ be a periodic Reeb orbit on ∂U , its canonical Conley-Zehnder index is defined as follows:*

$$\overline{CZ}(\gamma) = 2c_1^\tau(\xi_\gamma) + CZ^\tau(\gamma).$$

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Titre : Non-convexité symplectique des domaines toriques

Mots clés : géométrie symplectique, géométrie de contact, convexité, domaines toriques, capacités symplectiques, conjecture de Viterbo

Résumé : La convexité joue un rôle particulier en géométrie symplectique, pourtant ce n'est pas une notion invariante par symplectomorphisme. Dans un article fondateur, Hofer, Wysocki et Zehnder ont montré que tout domaine fortement convexe est dynamiquement convexe, une notion, qui elle, est invariante par symplectomorphisme. Depuis plus de vingt ans, l'existence ou non de domaines dynamiquement convexes qui ne sont pas symplectomorphes à un convexe est restée une question ouverte. Récemment, Chaidez et Edtmair ont répondu à cette question en dimension 4. Ils ont établi un critère "quantitatif" de convexité symplectique puis ont construit des domaines dynamiquement convexes qui ne vérifient pas ce critère. Dans cette thèse, nous utilisons ce critère pour construire de nouveaux exemples de tels domaines en dimension 4, qui ont la propriété supplémentaire d'être torique. De plus, nous estimons les constantes intervenant dans ce critère. Ce travail en collaboration avec Jean Gutt et Jun Zhang a été ensuite utilisé par Chaidez et Edtmair pour résoudre la question initiale en toute dimension. Dans un second temps, en collaboration avec Jean Gutt, Vinicius G.B.Ramos et Jun Zhang, nous étudions la distance des domaines dynamiquement convexes aux domaines symplectiquement convexes. Nous montrons qu'en dimension 4, celle-ci est arbitrairement grande aux yeux d'un analogue symplectique de la distance de Banach-Mazur. Au passage, nous reproveons de manière indépendante l'existence de domaines dynamiquement convexes non symplectiquement convexes en dimension 4.

Title: Symplectic non-convexity of toric domains

Key words: symplectic geometry, contact geometry, convexity, toric domains, symplectic capacities, Viterbo conjecture

Abstract: Convexity plays a special role in symplectic geometry, but it is not a notion that is invariant by symplectomorphism. In a seminal work, Hofer, Wysocki and Zehnder showed that any strongly convex domain is dynamically convex, a notion that is invariant by symplectomorphism. For more than twenty years, the existence or not of dynamically convex domains that are not symplectomorphic to a convex domain has remained an open question. Recently, Chaidez and Edtmair answered this question in dimension 4. They established a "quantitative" criterion of symplectic convexity and constructed dynamically convex domains that do not satisfy this criterion. In this thesis, we use this criterion to construct new examples of such domains in dimension 4, which have the additional property of being toric. Moreover, we estimate the constants involved in this criterion. This work in collaboration with Jean Gutt and Jun Zhang was later used by Chaidez and Edtmair to solve the initial question in all dimensions. Furthermore, in collaboration with Jean Gutt, Vinicius G.B.Ramos and Jun Zhang, we study the distance from dynamically convex domains to symplectically convex domains. We show that in dimension 4, this distance is arbitrarily large with respect to a symplectic analogue of the Banach-Mazur distance. Additionally, we independently reprove the existence of dynamically convex domains that are not symplectically convex in dimension 4.