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Feuilletages singuliers en géométrie différentielle

Hadi Nahari

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Hadi Nahari. Feuilletages singuliers en géométrie différentielle. Mathématiques [math]. Université Claude Bernard - Lyon I, 2024. Français. NNT : 2024LYO10174 . tel-04790592

HAL Id: tel-04790592

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THESE de DOCTORAT DE L'UNIVERSITE CLAUDE BERNARD LYON 1

Ecole Doctorale N° 512
École doctorale InfoMaths (ED 512)

Discipline : Mathématiques

Soutenue publiquement 01/10/2024, par :
Hadi NAHARI

Feuilletages singuliers en géométrie différentielle

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Singular foliations in differential geometry



Hadi Nahari
Thèse de doctorat

Université Claude Bernard Lyon 1
École doctorale InfoMaths (ED 512)
Spécialité : Mathématiques
n°. d'ordre :

Feuilletages singuliers en géométrie différentielle

Thèse présentée en vue d'obtenir le diplôme de
Doctorat de l'Université de Lyon

soutenue publiquement le octobre 2024 par
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*“Mathematics is not about numbers, equations, computations,
or algorithms: it is about understanding.”
- William Paul Thurston*

Acknowledgements

This thesis marks the culmination of a long journey that began in my high school years. From a curious student to these final days of my PhD studies, this path would not have been possible without the support and guidance of many individuals who have helped me along the way. I feel incredibly fortunate to have been able to pursue my passions, and I deeply appreciate all the valuable assistance and advice that have made this achievement possible.

I would like to thank my supervisor, Thomas Strobl, for his invaluable insights throughout the completion of this thesis. His support extended beyond the scientific aspects of my PhD studies; he has also been a wonderful friend whom I will continue to rely on in the future. I am grateful for his influence on my mathematical journey and even more appreciative of the broader perspectives that have changed the way I view the world.

I owe the foundation of my mathematical knowledge to the Mathematics Department of Sharif University of Technology and its esteemed professors. In particular, I would like to express my deepest gratitude to Siavash Shahshahani for his significant influence on my academic career. On a broader scale, I am profoundly appreciative of his contributions and legacy as one of the pioneering Iranian mathematicians who dedicated many years to advancing mathematics education in Iran.

I am grateful to the Faculty of Mathematics at *École Normale Supérieure de Lyon*, where I experienced one of the best years of my life as a master's student. The courses I took there were invaluable during my PhD studies, and the initial steps of the research projects presented in this manuscript began there in my master thesis.

Finally, I would like to express my heartfelt thanks to my parents, Mehdi and Elham, and my sister, Fatemeh. Their unwavering belief in me and constant support, even from thousands of kilometers away, have been invaluable. The trust and freedom they gave me to pursue my passions have been the foundation of my achievements and will continue to inspire my future endeavors. This thesis is dedicated to them.

Hadi Nahari
July 2024
Paris, France

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The first purpose of this thesis is to introduce and study a notion of singular Riemannian foliations which is adapted to the module definition of a singular foliation. Finding a commonly accepted definition of singular foliations was a longstanding debate since the 1960s [L18]. This debate ended with the definition proposed by Androulidakis and Skandalis [AS09]. They defined singular foliations as follows: ¹

Definition 1.1. *A singular foliation on M is defined as a $C^\infty(M)$ -submodule \mathcal{F} of the module of compactly supported vector fields on M , which is locally finitely generated and closed with respect to the Lie bracket of vector fields.*

This definition induces a decomposition of M into injectively immersed submanifolds called leaves [H62], thus yielding singular foliations in the more traditional sense (see, e.g., [L18]). But the association is not one-to-one: several singular foliations give rise to the same leaf decomposition. However, in the case where all the leaves have the same dimension, the relation is one-to-one and Definition 1.1 becomes equivalent to the usual notion of a regular foliation. Examples of singular foliations are induced on the underlying manifold by, e.g., Poisson manifolds, Lie algebroids, and Lie ∞ -algebroids.

Now let us add a Riemannian structure g to the above setting. Inspired by [KS16, KS19], but stripping off unnecessary data from the definitions given there, we propose:

Definition 1.2. *A singular Riemannian foliation on a Riemannian manifold (M, g) is defined as a singular foliation \mathcal{F} on (M, g) such that for every vector field $X \in \mathcal{F}$ we have:*

$$\mathcal{L}_X g \in \Omega^1(M) \odot g_b(\mathcal{F}), \tag{1}$$

where $g_b: \mathfrak{X}(M) \rightarrow \Omega^1(M)$, $X \mapsto g(X, \cdot)$ is the standard musical isomorphism and \odot stands for the symmetric tensor product.

With this definition, every geodesic perpendicular to one leaf turns out to stay perpendicular to all the leaves it meets, thus yielding singular Riemannian foliations in the

1. A singular foliation can be equivalently defined as an involutive and locally finitely generated subsheaf of the sheaf of smooth vector fields on M closed under multiplication by $C^\infty(M)$ [LGLS20] (see also [GZ19]). This has the advantage that one can replace $C^\infty(M)$ by an arbitrary sheaf of rings \mathcal{O} on M .

more traditional sense [M98]. The converse is not always true:² A singular Riemannian foliation in the sense of Molino is not always a singular Riemannian foliation. For a regular foliation, Definition 1.2 becomes equivalent to the usual notion of a (regular) Riemannian foliation ([H58], [R59]). Examples of singular Riemannian foliations are given by isometric Lie group actions on Riemannian manifolds and, more generally, orbit decompositions induced by Riemannian groupoids [dHF18].

Our notion of singular Riemannian foliations behaves well under the *pullback operation* of [AS09]. This permits us to provide a definition of *Morita equivalence* between singular Riemannian foliations. It implies *Hausdorff Morita equivalence* for the underlying singular foliations, as defined in [GZ19]. In the fore-cited work it is shown that the leaf spaces of Hausdorff Morita equivalent singular foliations are homeomorphic. Here we will establish:

Theorem A. *Let $(N_1, g_1, \mathcal{F}_1)$ and $(N_2, g_2, \mathcal{F}_2)$ be Morita equivalent singular Riemannian foliations. Then their leaf spaces are isometric as pseudo-metric spaces.*

A second purpose of this thesis is to introduce the *category of \mathcal{I} -Poisson manifolds \mathbf{IPois}* . For its objects, the intention is to generalize coisotropic submanifolds (see, e.g., [MR86]) to the singular setting. For simplicity of the presentation, in the Introduction we provide the definition of objects for the subcategory of *semi-strict \mathcal{I} -Poisson manifolds $\mathbf{ssIPois}$* of \mathbf{IPois} , which are constructed simply out of Poisson manifolds:³

Definition 1.3. *A semi-strict \mathcal{I} -Poisson manifold is a triple $(P, \{\cdot, \cdot\}, \mathcal{I})$ where \mathcal{I} is a subsheaf of smooth functions on a Poisson manifold $(P, \{\cdot, \cdot\})$ which is closed under multiplication by smooth functions, locally finitely generated, and for every open subset $U \subset P$, $\mathcal{I}(U) \subset C^\infty(U)$ is a Poisson subalgebra, i.e.*

$$\{\mathcal{I}(U), \mathcal{I}(U)\} \subset \mathcal{I}(U).$$

To describe dynamics, one needs a compatible Hamiltonian, i.e. a function $H \in N(\mathcal{I})$ where

$$N(\mathcal{I}) := \{f \in C^\infty(P) : \{f|_U, \mathcal{I}(U)\} \subset \mathcal{I}(U) \text{ for every open subset } U\}.$$

We then call $(P, \{\cdot, \cdot\}, \mathcal{I}, H)$ a (semi-strict) dynamical \mathcal{I} -Poisson manifold and the corresponding category **(ss)dynIPois**.

The property that a singular foliation is locally finitely generated is crucial for the existence of the induced leaf decomposition. Similarly, the condition “locally finitely generated” in Definition 1.3 is essential for showing that the flow of any $H \in N(\mathcal{I})$, if complete, preserves the sheaf \mathcal{I} (see Proposition 4.28 for the precise statement).

Definition 1.4. *A smooth map $\varphi: P_1 \rightarrow P_2$ between $(P_1, \{\cdot, \cdot\}_1, \mathcal{I}_1)$ and $(P_2, \{\cdot, \cdot\}_2, \mathcal{I}_2)$ is a morphism of (semi-strict) \mathcal{I} -Poisson manifolds, iff the two obvious conditions $\varphi^*(\mathcal{I}_2(P_2)) \subset \mathcal{I}_1(P_1)$ and $\varphi^*N(\mathcal{I}_2) \subset N(\mathcal{I}_1)$ are complemented by*

$$\{\varphi^*f, \varphi^*g\}_1 - \varphi^*\{f, g\}_2 \in \mathcal{I}_1(P_1) \quad \forall f, g \in N(\mathcal{I}_2). \quad (2)$$

*For dynamical \mathcal{I} -Poisson manifolds we add the condition $\varphi^*H_2 - H_1 \in \mathcal{I}_1$.*

2. In the real analytic setting, we will provide such a counterexample in the third chapter of this thesis (cf. also **Question** below).

3. For the complete version see Definitions 4.13, 4.30, and 4.32 below. The more general notion permits to cover also examples such as Hamiltonian quasi-Poisson manifolds [AKSM02], see Example 4.17.

These are also the morphisms of the general category, when “semi-strict” in the parenthesis is dropped. With this notion of morphisms, the category **Pois** of Poisson manifolds is a full subcategory of **(ss)IPois** for the choice of the zero ideal. In general, however, the morphisms between (semi-strict) \mathcal{I} -Poisson manifolds are not necessarily Poisson maps between the underlying Poisson manifolds—an important feature in several applications.

The condition (2) is optimal to ensure that φ^* descends to a Poisson morphism on the level of reductions: In fact, every (semi-strict) \mathcal{I} -Poisson manifold $(P, \{\cdot, \cdot\}, \mathcal{I})$ induces a Poisson algebra structure on $N(\mathcal{I})/\mathcal{I}(P)$. In the case of coisotropic reductions [MR86], this algebra coincides with the algebra of smooth functions on the reduced Poisson manifold. The algebraic formulation here is, however, also applicable in the general context of \mathcal{I} -Poisson manifolds, where, e.g., the vanishing set of the ideal $\mathcal{I}(P)$ does not need to be a submanifold anymore. The conditions in Definition 1.4 ensure that there is a canonical contravariant functor F from **(ss)IPois** to **PoisAlg**, the category of Poisson algebras.

The third purpose of this manuscript is to bring the two aforementioned subjects together and, in particular, to use \mathcal{I} -Poisson geometry so as to learn more about singular (Riemannian) foliations.

Starting from a singular foliation (M, \mathcal{F}) and viewing every vector field in \mathcal{F} as a smooth function on T^*M , we construct a semi-strict \mathcal{I} -Poisson manifold $(T^*M, \{\cdot, \cdot\}_{T^*M}, \mathcal{I}_{\mathcal{F}})$. Moreover, every metric g on M defines a compatible Hamiltonian (making the semi-strict \mathcal{I} -Poisson manifold dynamical) if and only if the metric satisfies condition (1). Thus there is a canonical map from singular (Riemannian) foliations to the objects of **ssIPois** and **ssdynIPois**, respectively. As we will see, this construction is not only conceptually illuminating, it also has technical advantages: we will use it to find elegant proofs of several properties of singular (Riemannian) foliations, like to show, e.g., that Definition 1.2 automatically induces a singular Riemannian foliation in the sense of [M98].

To complete the above map on objects to a functor, one would need a proper definition of the categories **SF** and **SRF** of singular (Riemannian) foliations. Surprisingly, already for singular foliations, in the literature there is not yet any satisfactory proposal for what a morphism between general singular foliations should be. However, the situation changes if one restricts to submersions and Riemannian submersions in the case of singular foliations and singular Riemannian foliations, respectively, because in these cases the previously mentioned pullback operations are defined. For example, a Riemannian submersion $\pi: (N, h) \rightarrow (M, g)$ between two singular Riemannian foliations (N, h, \mathcal{F}_N) and (M, g, \mathcal{F}_M) which satisfies $\pi^{-1}\mathcal{F}_M = \mathcal{F}_N$ should definitely be considered as a morphism. Let us call **SF**₀ and **SRF**₀ the two (sub)categories with such restricted morphisms. In this paper we show in particular:

Theorem B. *There are canonical functors $\Psi: \mathbf{SF}_0 \rightarrow \mathbf{IPois}$ and $\Phi: \mathbf{SRF}_0 \rightarrow \mathbf{dynIPois}$.*

As a side result, we will find that for $\mathcal{F}_M = 0$, $\Phi(\pi)$ becomes an ordinary Poisson map if and only if the horizontal distribution $(\ker d\pi)^\perp$ is integrable—correcting [BWY21], where this map has been considered as well, but claimed to always be Poisson.

Composing the functor Ψ , evaluated on a singular foliation (M, \mathcal{F}) , with the functor $F: \mathbf{IPois} \rightarrow \mathbf{PoisAlg}$, we obtain the (reduced) Poisson algebra $A(\mathcal{F}) := N(\mathcal{I}_{\mathcal{F}})/\mathcal{I}_{\mathcal{F}}(T^*M)$. This algebra provides an invariant of Hausdorff Morita equivalence, since we will prove:

Theorem C. *Let (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) be Hausdorff Morita equivalent singular foliations. Then the reduced Poisson algebras $A(\mathcal{F}_1)$ and $A(\mathcal{F}_2)$ are isomorphic.*

The final purpose of this thesis is to study the exceptional example of the singular octonionic Hopf foliation, constructed upon the octonionic Hopf fibration on $S^{15} \subset \mathbb{O}^2 \cong \mathbb{R}^{16}$. Hopf fibrations associated to the complex numbers and quaternions are well-known examples in differential geometry and mathematical physics [H31, U03]. Due to the difficulties arising by the lack of associativity in the algebra of octonions \mathbb{O} , the octonionic Hopf fibration is less studied in the literature. However, it has remarkable properties due to the nature of the octonions [OPPV13, BC21]. In the context of this thesis, the study of this example was motivated by the following question:

Question: Let (M, g, \mathcal{F}) be a triple inducing a leaf decomposition which is a singular Riemannian foliation in the sense of Molino [M98]. Is it possible to find a singular Riemannian foliation (M, g, \mathcal{F}') in the sense of Definition 1.2 which has the same leaf decomposition as (M, g, \mathcal{F}) ?

The *singular octonionic Hopf foliation* introduced in Section 8 will be seen to provide a counterexample to this in the real analytic setting. For the smooth setting, this is still an open problem. Let us remark in parenthesis that the likewise question if every leaf decomposition called a singular foliation in the traditional sense can be obtained as the leaf decomposition of a singular foliation in the sense of Definition 1.1 is still an open problem as well.

The *singular octonionic Hopf leaf decomposition* \mathcal{L}_{OH} is defined using the octonionic lines in $\mathbb{O}^2 \cong \mathbb{R}^{16}$ defined as [GWZ86, OPPV13]

$$l_m := \{(x, m \cdot x) \in \mathbb{O}^2 : x \in \mathbb{O}\}. \quad (3)$$

for every $m \in \mathbb{O}$, called the slope of the line, together with the octonionic line of slope ∞

$$l_\infty := \{(0, x) \in \mathbb{O}^2 : x \in \mathbb{O}\}.$$

Definition 1.5. *The singular Hopf leaf decomposition of \mathbb{O}^2 is defined as the family of the leaves*

$$\begin{aligned} L_{m,r} &:= l_m \cap S(r) \quad \forall m \in \mathbb{O}, \\ L_{\infty,r} &:= l_\infty \cap S(r), \end{aligned} \quad (4)$$

together with the origin in \mathbb{O}^2 . Here, $S(r) := \{(x, y) \in \mathbb{D}^2 : \|x\|^2 + \|y\|^2 = r^2\}$ is the sphere of radius $r > 0$ in \mathbb{D}^2 .

The leaf decomposition \mathcal{L}_{OH} is known to be locally *non-homogeneous*, i.e. there is no isometric Lie group action, even locally, generating the leaves of \mathcal{L}_{OH} as its orbits [MR19, GWZ86, L93].

A central starting point of our discussion is the following lemma, characterizing the vector fields tangent to the leaves in \mathcal{L}_{OH} : A vector field $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{X}(\mathbb{O}^2)$ with $u, v \in C^\infty(\mathbb{O}^2, \mathbb{O})$ is tangent to the leaves of \mathcal{L}_{OH} , if and only if

$$u \cdot \bar{y} + x \cdot \bar{v} = 0 \quad \& \quad (5)$$

$$\langle x, u \rangle = \langle y, v \rangle = 0 \quad (6)$$

for all $(x, y) \in \mathbb{O}^2 \cong \mathbb{R}^{16}$. This characterization is an essential tool in the study of \mathcal{L}_{OH} , and provides the ingredient for the computational part of the thesis, done using Eisenbaud's Macaulay2⁴. In Section 8, we use this lemma to provide an alternative proof for the non-homogeneity of \mathcal{L}_{OH} . Moreover, we improve the classic result of non-homogeneity for \mathcal{L}_{OH} as follows:

Theorem D. *Let \mathcal{F}_0 be any singular foliation on \mathbb{O}^2 having \mathcal{L}_{OH} as its leaf decomposition. Then $(\mathbb{O}^2, \mathcal{F}_0)$ is not Hausdorff Morita equivalent to any singular foliation \mathcal{F} on some Riemannian manifold (M, g) , whose leaf decomposition is locally given by orbits of some isometric Lie group action.*

The non-homogeneity of \mathcal{L}_{OH} gives the non-existence of an isometric Lie group action around the origin in $(\mathbb{R}^{16}, g_{st})$ which induces \mathcal{L}_{OH} . Moreover, there is, more generally, also no known (unconstrained) Lie group action around the origin generating these leaves.

In Section 7 we provide a Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$. Its construction is based on considering the *rescaling function* $\lambda: \mathbb{O}^2 \times \mathbb{O}^2 \rightarrow \mathbb{R}$ defined by the formula

$$\lambda(F, G, x, y) = \sqrt{1 + 2(\langle x, F \rangle + \langle y, G \rangle + \langle x \cdot \bar{y}, F \cdot \bar{G} \rangle) + \|x\|^2 \|F\|^2 + \|y\|^2 \|G\|^2}, \quad (7)$$

for all (F, G) and $(x, y) \in \mathbb{O}^2$.

Definition 1.6. *The Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$ is given by the following data:*

- *The manifold of arrows \mathcal{G} is an open subset of $\mathbb{O}^2 \times \mathbb{O}^2$ given by*

$$\mathcal{G} := \mathbb{O}^2 \times \mathbb{O}^2 \setminus \mathcal{C}$$

where $\mathcal{C} = \{(F, G, x, y) \in \mathbb{O}^2 \times \mathbb{O}^2 : \lambda(F, G, x, y) = 0\}$.

- *For every arrow $g = (F, G, x, y) \in \mathcal{G}$, the source and the target maps are given by:*

$$\begin{aligned} s(g) &= (x, y), \\ t(g) &= \frac{1}{\lambda(g)} (x + \|x\|^2 F + (x \cdot \bar{y}) \cdot G, y + \|y\|^2 G + (y \cdot \bar{x}) \cdot F). \end{aligned}$$

- *The product $m(g', g) \equiv g' \cdot g$ for a pair of composable arrows $g' = (F', G', x', y')$ and $g = (F, G, x, y)$ is given by*

$$g' \cdot g := (F + \lambda(g) \cdot F', G + \lambda(g) \cdot G', x, y).$$

- *The unit map $u: \mathbb{O}^2 \rightarrow \mathcal{G}$ is given by associating the arrow $1_{(x,y)} := (0, 0, x, y)$ to every object $(x, y) \in \mathbb{O}^2$.*
- *The inverse $i: \mathcal{G} \rightarrow \mathcal{G}$ applied to an arrow $g = (F, G, x, y)$ gives $g^{-1} \equiv i(g)$ by means of*

$$g^{-1} = (-F/\lambda(g), -G/\lambda(g), t(g)).$$

Theorem E. *The orbits of the Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$ are precisely the leaves in \mathcal{L}_{OH} .*

4. macaulay2.com

In the same section, we then differentiate this Lie groupoid to a Lie algebroid. It consists of the trivial vector bundle $E_0 = \underline{\mathbb{O}^2}$ over \mathbb{O}^2 and the anchor map $\rho: E_0 \rightarrow T\mathbb{O}^2 \cong \underline{\mathbb{O}^2}$ given by

$$\rho \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \|x\|^2 u + (x \cdot \bar{y}) \cdot v - (\langle x, u \rangle + \langle y, v \rangle) x \\ \|y\|^2 v + (y \cdot \bar{x}) \cdot u - (\langle x, u \rangle + \langle y, v \rangle) y \end{pmatrix}. \quad (8)$$

The Lie bracket evaluated on constant global sections $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in \Gamma(E_0)$ is defined by

$$\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right] = (\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} u' \\ v' \end{pmatrix} - (\langle x, u' \rangle + \langle y, v' \rangle) \begin{pmatrix} u \\ v \end{pmatrix}. \quad (9)$$

This Lie algebroid realizes \mathcal{L}_{OH} as leaf decomposition induced by a singular foliation \mathcal{F}_{OH} in the sense of Definition 1.1. This singular foliation turns out to be maximal among all singular foliations with the same leaf decomposition. More precisely, utilizing Equations (5) and (6), together with computations done using Macaulay2 in Appendix 2, prove the following result: The singular octonionic Hopf foliation \mathcal{F}_{OH} is generated by all vector fields tangent to the leaves of \mathcal{L}_{OH} .

As mentioned before, the singular foliation \mathcal{F}_{OH} gives an answer to our question, in the real analytic setting. More precisely, since the octonionic Hopf fibration is known to be a regular Riemannian foliation [GWZ86], using Molino's homothetic transformation lemma [M98] we observe that the triple $(\mathbb{O}^2, ds^2, \mathcal{F}_{OH})$ induces a singular Riemannian foliation in the sense of Molino. However, in the real analytic setting we prove the following result:

Theorem F. *Let \mathcal{F} be any singular foliation on the real analytic Riemannian manifold (\mathbb{O}^2, ds^2) having \mathcal{L}_{OH} as its leaf decomposition. Then the triple $(\mathbb{O}^2, ds^2, \mathcal{F})$ induces a singular Riemannian foliation in the sense of Molino[M98], but not in the sense of Definition 1.2.*

Finally, we complete the study of \mathcal{F}_{OH} by extending the Lie algebroid $(E_0, [\cdot, \cdot], \rho)$ to a universal Lie 3-algebroid. The *universal Lie ∞ -algebroid* of a singular foliation is introduced in [LGLS20]. Lie ∞ -algebroids, first appeared in [V10] as higher analogues of Lie algebroids, are defined as a positively graded vector bundle $E = \bigoplus_{i \geq 0} E_{-i}$ over a manifold M , together with a family of graded skew-symmetric and multilinear maps $l_k: \wedge^k \Gamma(E) \rightarrow \Gamma(E)$ of degree $2 - k$, called k -brackets and an anchor $\rho: E_0 \rightarrow TM$. The brackets and the anchor are required to satisfy some compatibility conditions. In [LGLS20], it is proven that for a foliated manifold (M, \mathcal{F}) , upon the existence of a *geometric resolution*, one can associate a Lie ∞ -algebroid to the singular foliation, called a universal Lie ∞ -algebroid of \mathcal{F} . This association turns out to be unique up to homotopy, and leads to invariants of singular foliations. However, this association is not constructive, and different singular foliations may need different techniques to compute the universal Lie ∞ -algebroid. For example, for linear foliations obtained by the actions of some subgroups of the general linear group, the construction of the associated universal Lie ∞ -algebroids is explained in [S23].

To construct a universal Lie 3-algebroid of \mathcal{F}_{OH} , in the first step, we use an exact sequence found via Macaulay2 in Appendix 2. This leads to finding a geometric resolution for the foliated manifold $(\mathbb{O}^2, \mathcal{F}_{OH})$. There, we have $E_0 := \underline{\mathbb{O}^2}, E_{-1} := \underline{\mathbb{R} \oplus \mathbb{O} \oplus \mathbb{R}}$, and

$E_{-2} := \underline{\mathbb{R}}$. We have $E_i = 0$ for $i \geq 3$. The anchor map $\rho: E_0 \rightarrow T\mathbb{O}^2 \cong \underline{\mathbb{O}^2}$ and the 2-bracket restricted to sections of degree zero, coincide with the anchor and the Lie bracket of the Lie algebroid $(E_0, [\cdot, \cdot], \rho)$. The 1-bracket on global sections $\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \in \Gamma(E_{-1})$ and $t \in \Gamma(E_{-2})$ is given by

$$\begin{aligned} d^{(1)} \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} &:= \begin{pmatrix} \mu x + a \cdot y \\ \nu y + \bar{a} \cdot x \end{pmatrix}, \\ d^{(2)}(t) &:= \begin{pmatrix} -\|y\|^2 t \\ (x \cdot \bar{y}) t \\ -\|x\|^2 t \end{pmatrix}, \end{aligned}$$

and the 2-bracket on other sections is defined as

$$\begin{aligned} \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] &:= \begin{pmatrix} -2\langle y, \bar{a} \cdot u \rangle + 2\langle y, v \rangle \mu \\ x \cdot (\bar{u} \cdot a) + (a \cdot v) \cdot \bar{y} - \mu(x \cdot \bar{v}) - \nu(u \cdot \bar{y}) \\ -2\langle x, a \cdot v \rangle + 2\langle x, u \rangle \nu \end{pmatrix}, \\ \left[\begin{pmatrix} u \\ v \end{pmatrix}, t \right] &:= 2(\langle x, u \rangle + \langle y, v \rangle)t, \\ \left[\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right] &:= 4\langle a, a' \rangle - 2\mu\nu' - 2\mu'\nu. \end{aligned} \tag{10}$$

The other brackets are set to be zero.

In Proposition 10.10 we prove that these data define a Lie 3-algebroid, which is in addition minimal at the origin, i.e. all the 1-brackets vanish at the origin. Using the results of [LGLS20], we prove:

Theorem G. *The Lie algebroid $(E_0, [\cdot, \cdot], \rho)$ and similarly the Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{O}^2$ have the minimal dimension among Lie algebroids and Lie groupoids over \mathcal{O}^2 which induce the singular foliation \mathcal{F}_{OH} .*

The structure of this thesis is as follows:

Section 2 contains a short review of the definitions and main properties of singular foliations related to the goal of this manuscript, in particular the notion of Hausdorff Morita equivalence of singular foliations.

In *Section 3* we introduce singular Riemannian foliations and study some of their properties. We show (in Theorem 3.6 below) that every finitely generated singular Riemannian foliation admits an almost Lie algebroid structure with connection to turn the singular Riemannian foliation into an almost Killing Lie algebroid [KS19]. We define Morita equivalence of singular Riemannian foliations, show that it defines an equivalence relation, and prove Theorem A.

Section 4 introduces the category **IPois**, the reduction functor F to **PoisAlg**, and provides several examples and properties of \mathcal{I} -Poisson manifolds.

In *Section 5* we show how singular foliations and singular Riemannian foliations give rise to particular \mathcal{I} -Poisson and dynamical \mathcal{I} -Poisson manifolds, respectively.

In *Section 5.4* we prove Theorems [B](#) and [C](#).

In *Section 6*, we provide a self-contained background on the *normed division algebras* and their properties. In particular, for the case of octonions, we mention many identities which turn out to be useful in calculations, in the absence of the associativity. We conclude the section by introducing the singular Hopf leaf decomposition associated to each of the four normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} .

In *Section 7*, we explain the construction of the Lie groupoid which induces the singular octonionic Hopf leaf decomposition as its orbits. We then differentiate it to a Lie algebroid of rank 16, realizing \mathcal{L}_{OH} as the leaf decomposition of a singular foliation.

Section 8 is devoted to the study of \mathcal{F}_{OH} . Non-homogeneity and maximality of \mathcal{F}_{OH} are explained as well.

In *Section 9*, we study \mathcal{F}_{OH} as a singular Riemannian foliation in the sense of Molino [\[M98\]](#) and in our sense. We explain how this example distinguishes the two definitions in the real analytic setting.

In *Section 10*, finally after recalling the Universal Lie ∞ -algebroid of singular foliations, we construct the Universal Lie 3-algebroid of $(\mathbb{O}^2, \mathcal{F}_{OH})$, and prove the minimality of the rank for the Lie algebroid $(E_0, [\cdot, \cdot], \rho)$, and correspondingly for the Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$.

The definition of almost Killing Lie algebroids as well as part of the proof of [Theorem 3.6](#) (in the form of [Proposition 1.5](#)) are deferred to *Appendix 1*. The computations done via Eisenbaud's Macaulay2 are explained in *Appendix 2*.

Singular Riemannian foliations

As a well-studied branch of differential geometry, *regular foliations* are defined as smooth decompositions of manifolds into connected and injectively immersed submanifolds of a same dimension k called *leaves* [MM10]. Using the Frobenius theorem [F77], they can be equivalently defined as involutive k -dimensional smooth distributions on manifolds. In the case of regular foliations, *Riemannian foliations* have been studied in differential geometry since the 1950s (see for example [H58] and [R59]). They are defined as regular foliations on Riemannian manifolds with the property that every geodesic perpendicular to one leaf stays perpendicular to all the leaves it meets.

If we let the dimension of the leaves to vary, a smooth singular leaf decomposition could not be equivalently characterized with as certain involutive submodules of vector fields. However, the second definition [AS09], contains more information than a leaf decomposition. For singular foliations as singular leaf decompositions, definition of singular Riemannian foliations in the traditional setting was introduced by Molino [M98], by requiring the above property of geodesics for the now possibly singular leaf decomposition. Examples of such foliations are given by isometric Lie group actions on Riemannian manifolds and, more generally, orbit decompositions induced by Riemannian groupoids [dHF18]. But, for the more recent definition of singular foliations by [AS09], using the same property for the induced leaf decomposition seems inadequate due to the additional information carried by singular foliations.

In this chapter, we present a new definition of singular Riemannian foliations which also depends on the choice of the underlying singular foliation. We use a compatibility condition which first appeared in the study of gauge theory for standard sigma models [KS16] and was used in the definition of Killing Cartan Lie algebroids. Every singular Riemannian foliation in this sense turns out to have a leaf decomposition that is a singular Riemannian foliation in the sense of Molino [M98]. The converse is not always true. For regular foliations, on the other hand, the two notions coincide.

2 Background on singular foliations

In what follows, M is assumed to be a smooth manifold and $\mathfrak{X}_c(M)$ denotes the $C^\infty(M)$ -module of compactly supported vector fields on M . For more details and examples of singular foliations see [AS09, LGLS20] or [LGLR22].

2.1 Basic definitions and examples

Definition 2.1. A $C^\infty(M)$ -submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$ is said to be locally finitely generated if, for every point $q \in M$, there exists an open neighborhood $U \subset M$ containing q such that the submodule $\iota_U^{-1}(\mathcal{F}) := \{X|_U : X \in \mathcal{F}, \text{supp}(X) \subset U\}$ of $\mathfrak{X}_c(U)$ is finitely generated. This means that there exist finitely many vector fields $X_1, \dots, X_N \subset \mathfrak{X}(U)$ such that

$$\iota_U^{-1}(\mathcal{F}) = \langle X_1, \dots, X_N \rangle_{C^\infty(U)}. \quad (11)$$

Remark 2.2. Note that the generators of $\iota_U^{-1}(\mathcal{F})$ in Definition 2.1 are not required to be compactly supported. This allows for more flexibility in constructing singular foliations on M .

Example 2.3. Let $M = \mathbb{R}$. Then the $C^\infty(\mathbb{R})$ -module $\mathfrak{X}_c(\mathbb{R})$ is globally generated by the single vector field $\frac{d}{dx}$. On the other hand, the $C^\infty(\mathbb{R})$ -submodule of compactly supported vector fields which vanish on \mathbb{R}_- is not locally finitely generated around 0.

Definition 2.4. A singular foliation on M is defined as $C^\infty(M)$ -submodule \mathcal{F} of $\mathfrak{X}_c(M)$ which is locally finitely generated and closed with respect to the Lie bracket of vector fields. The pair (M, \mathcal{F}) is then called a foliated manifold.

For every foliated manifold (M, \mathcal{F}) the leaf L_q passing through $q \in M$ is given by the set of points which can be joined to q by the flows of finitely many vector fields in \mathcal{F} .

Theorem 2.5 ([H62]). Let (M, \mathcal{F}) be a foliated manifold. Then for every point $q \in M$, the subset L_q is an injectively immersed submanifold of M .

The space of leaves of (M, \mathcal{F}) , i.e. the quotient space obtained from the equivalence relation of belonging to the same leaf, is denoted by M/\mathcal{F} .

Let L_q be the leaf passing through the point $q \in M$ in a foliated manifold (M, \mathcal{F}) . Then, by definition of the leaves, $T_q L_q$ can be identified with $\{X|_q : X \in \mathcal{F}\} \subset T_q M$, which motivates:

Definition 2.6. For every point $q \in M$ in a foliated manifold (M, \mathcal{F}) , the tangent of \mathcal{F} at q is defined as

$$F_q := \{X|_q : X \in \mathcal{F}\} \subset T_q M.$$

If $q \mapsto \dim(F_q)$ is constant on M , we obtain *regular foliations* as particular singular ones. In this case, by the Frobenius theorem [F77], there is a one-to-one correspondence between the leaf decomposition of the foliation and the module of vector fields generating it. This is no more the case if the singular foliation is non-regular; there *always* exist different modules generating the same leaf decomposition then (for an example, see Example 2.9 below). Note also that in the non-regular case all the vector fields tangent

to the leaves of a given singular foliation do not necessarily define a singular foliation anymore: the module of Example 2.3, despite not being a singular foliation since not finitely generated, induces a leaf decomposition, which can be obtained also by an honest singular foliation with the single generator $\chi \frac{d}{dx}$. Here $\chi \in C^\infty(\mathbb{R})$ can be chosen, e.g., as the function

$$\chi(x) = \begin{cases} \exp(\frac{-1}{x^2}) & x > 0 \\ 0 & x \leq 0 \end{cases}. \quad (12)$$

Remark 2.7. *The function $M \rightarrow \mathbb{N}$ given by $q \mapsto \dim(F_q)$ is lower semi-continuous. As a result, the subset $U \subset M$ of the continuity set of $\dim(F_q)$ is open and dense, and $\iota_U^{-1}(\mathcal{F})$ induces a regular foliation over each connected component of U [AS09].*

The following example shows the importance of being locally finitely generated:

Example 2.8. *On $M = \mathbb{R}^2$, consider the module \mathcal{G} generated by the vector fields ∂_x and $\mathcal{X}(x)\partial_y$, where \mathcal{X} is the function defined in (12), together with all their multiple commutator Lie brackets. Then, by construction, \mathcal{G} is closed under the Lie bracket. However, it is not locally finitely generated as a $C^\infty(M)$ -module since with each derivative on \mathcal{X} we obtain a new, independent coefficient in front of ∂_y . As a consequence, we loose the well-behavedness of a leaf-decomposition: although every two points in \mathbb{R}^2 can be connected by a sequence of flows of vector fields in \mathcal{G} (so that, in this sense, there would be only one leaf that is \mathbb{R}^2 itself), the tangent of \mathcal{G} at every point in the left half-plane is only one-dimensional.*

As mentioned above, Definition 2.4 contains more information than a well-behaved decomposition of M into leaves:

Example 2.9. *Consider $M = \mathbb{R}$ and fix some $k \in \mathbb{N}$. Then the vector fields vanishing at least to order k at the origin form a singular foliation \mathcal{F}_k . While the leaf decomposition is the same for all k , one has $\mathcal{F}_{k+1} \subsetneq \mathcal{F}_k$.*

To capture some of this additional information contained in the definition of a singular foliation, we extract some more data from the module \mathcal{F} by the following definition of [AS09].

Definition 2.10. *Let (M, \mathcal{F}) be a foliated manifold. For every point $q \in M$, the fiber of \mathcal{F} at q is defined as:*

$$\mathcal{F}_q := \mathcal{F}/I_q \cdot \mathcal{F}$$

where $I_q := \{f \in C^\infty(M) : f(q) = 0\}$ is the vanishing ideal of q in $C^\infty(M)$.

Remark 2.11. *The function $M \rightarrow \mathbb{N}$ sending $q \rightarrow \dim(\mathcal{F}_q)$ is upper semi-continuous, and $\dim(\mathcal{F}_q)$ gives the minimal number of vector fields locally generating \mathcal{F} around $q \in M$ [AS09].*

Note that for every point $q \in M$ the evaluation map $ev_q: \mathcal{F}_q \rightarrow F_q, [X] \rightarrow X|_q$ is a homomorphism of vector spaces and induces the following short exact sequence:

$$0 \rightarrow \ker(ev_q) \hookrightarrow \mathcal{F}_q \xrightarrow{ev_q} F_q \rightarrow 0. \quad (13)$$

here $[X]$ denotes the equivalence class of the vector field $X \in \mathcal{F}$.

It is not difficult to see that the Lie bracket on \mathcal{F} induces a Lie bracket on the finite-dimensional vector space $\ker(ev_q) \subset \mathcal{F}_q$.

Definition 2.12. *The vector space $\mathfrak{g}_q^{\mathcal{F}} := \ker(ev_q)$ together with the bracket inherited by \mathcal{F}_q defines the isotropy Lie algebra of \mathcal{F} at q .*

In the case of regular foliations, the map $ev_q: \mathcal{F}_q \rightarrow F_q$ is a vector space isomorphism and $\mathfrak{g}_q^{\mathcal{F}} = 0$. So one can say that the isotropy Lie algebra $\mathfrak{g}_q^{\mathcal{F}}$ characterizes in part the singularity of \mathcal{F} at $q \in M$.

In Example 2.9 all fibers and isotropy Lie algebras at the origin are isomorphic. This changes, if we increase the dimension of M :

Example 2.13. *Let $M = \mathbb{R}^n$, $n \geq 2$, and let \mathcal{F} be the singular foliation generated by vector fields vanishing at the origin at least of order $k \in \mathbb{N}$. There are always only two leaves $M \setminus \{0\}$ and $\{0\}$, but the fiber at the origin has different dimensions for different choices of k , $\dim \mathcal{F}_0 = \binom{k+n-1}{n-1}$.*

2.2 Hausdorff Morita equivalence

Definition 2.14 ([AS09, GZ19]). *Let $f: N \rightarrow M$ be a smooth map and let \mathcal{F} be a singular foliation on M . Then f is said to be transverse to \mathcal{F} , if for every $q \in M$ one has*

$$d_q f(T_q N) + F_{f(q)} = T_{f(q)} M.$$

Example 2.15. *If $S \subset M$ is transverse to the leaves of a foliated manifold (M, \mathcal{F}) , then the inclusion map $\iota_S: S \hookrightarrow M$ is transverse to \mathcal{F} .*

Example 2.16. *A submersion $\pi: N \rightarrow M$ is transverse to every singular foliation on M .*

Proposition 2.17 ([AS09]). *Let (M, \mathcal{F}) be a foliated manifold and let $f: N \rightarrow M$ be a smooth map transverse to \mathcal{F} . Then the $C^\infty(N)$ -module $f^{-1}\mathcal{F}$ generated by vector fields on N projectable to \mathcal{F} is a singular foliation on N .*

Here, a vector field $V \in \mathfrak{X}(N)$ is called *projectable to \mathcal{F}* , if there exists a vector field $X \in \mathcal{F}$ such that for every point $q \in N$ we have:

$$d_q f(V|_q) = X|_{f(q)}.$$

Definition 2.18 ([GZ19]). *Two foliated manifolds (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) are Hausdorff Morita equivalent if there exists a smooth manifold N and surjective submersions with connected fibers $\pi_i: N \rightarrow M_i$, $i = 1, 2$ such that:*

$$\pi_1^{-1}\mathcal{F}_1 = \pi_2^{-1}\mathcal{F}_2.$$

In this case we write $(M_1, \mathcal{F}_1) \sim_{ME} (M_2, \mathcal{F}_2)$.

It is shown in [GZ19] that the singular foliations underlying Morita equivalent Lie algebroids [G01] or Morita equivalent Poisson manifolds [X91] are Hausdorff Morita equivalent. Also the Morita equivalence of regular foliations [M98] is a special case. Hausdorff Morita equivalence defines an equivalence relation on foliated manifolds—something that holds true for Poisson manifolds only upon restriction to those integrating to a symplectic groupoid. The main fact about Hausdorff Morita equivalent foliated manifolds is that they have Morita equivalent holonomy groupoids (as open topological groupoids) defined in [AS09].

Theorem 2.19 ([GZ19]). *Let (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) be foliated manifolds which are Hausdorff Morita equivalent by means of (N, π_1, π_2) . Then:*

(i) *The map sending the leaf passing through $q \in M_1$ to the leaf of \mathcal{F}_2 containing $\pi_2(\pi_1^{-1}(q))$ is a homeomorphism between the leaf spaces. It preserves the codimension of leaves and the property of being an embedded leaf.*

(ii) *Let $q_1 \in N_1$ and $q_2 \in N_2$ be points in corresponding leaves. Choose transversal slices S_{q_1} at q_1 and S_{q_2} at q_2 . Then the foliated manifolds $(S_{q_1}, \iota_{S_{q_1}}^{-1} \mathcal{F}_1)$ and $(S_{q_2}, \iota_{S_{q_2}}^{-1} \mathcal{F}_2)$ are diffeomorphic and the isotropy Lie algebras $\mathfrak{g}_{q_1}^{\mathcal{F}_1}$ and $\mathfrak{g}_{q_2}^{\mathcal{F}_2}$ are isomorphic.*

Example 2.20. *For smooth, connected manifolds M and N , $(M, \mathfrak{X}_c(M))$ and $(N, \mathfrak{X}_c(N))$ are always Hausdorff Morita equivalent. On the other hand, $(M, 0)$ and $(N, 0)$ are Hausdorff Morita equivalent only if M and N are diffeomorphic.*

Example 2.21. *Let (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) be isomorphic foliated manifolds, i.e. there exists a diffeomorphism $\Phi: M_1 \rightarrow M_2$ satisfying $\Phi_*(\mathcal{F}_1) = \mathcal{F}_2$. Then we have $(M_1, \mathcal{F}_1) \sim_{ME} (M_2, \mathcal{F}_2)$ by choosing $N = M_1$, $\pi_1 = \text{Id}_{M_1}$ and $\pi_2 = \Phi$ in Definition 2.18.*

3 Module singular Riemannian foliations

In what follows, (M, g) denotes a Riemannian manifold. We begin by recalling the first approach to defining singular Riemannian foliations, following [M98]. Then, we introduce a second approach that considers additional information carried by a singular foliation. We examine its properties and also explore Morita equivalence for both approaches.

3.1 Approaches for defining singular Riemannian foliations

The first approach to define singular Riemannian foliation, due to Molino [M98], requires compatibility between the Riemannian metric and the leaf decomposition of the singular foliation. To distinguish it from the second approach, we add the suffix 'geometric' before singular Riemannian foliation.

Definition 3.1. *Let \mathcal{F} be a singular foliation on (M, g) . We call the triple (M, g, \mathcal{F}) a geometric singular Riemannian foliation, if every geodesic orthogonal to a leaf at one point is orthogonal to all the leaves it meets.*

In this text, we focus mainly on the following definition of singular Riemannian foliations, streamlining the one given in [KS19]⁵:

Definition 3.2. *Let \mathcal{F} be a singular foliation on (M, g) . We call the triple (M, g, \mathcal{F}) a module singular Riemannian foliation, if for every vector field $X \in \mathcal{F}$ we have:*

$$\mathcal{L}_X g \in \Omega^1(M) \odot g_b(\mathcal{F}). \quad (14)$$

Here \odot stands for the symmetric tensor product and g_b is the map on sections induced by the musical isomorphism $g_b: TM \rightarrow T^*M$, $(q, v) \mapsto g_q(v, \cdot)$. Let $(g_b)^{-1}: \Omega^1(M) \rightarrow \mathcal{X}(M)$ denote the corresponding inverse map and $g^{-1} \in \Gamma(S^2TM)$ the 2-tensor inducing it. Then, by means of $\mathcal{L}_X(g_b)^{-1} = -(g_b)^{-1} \circ (\mathcal{L}_X g_b) \circ (g_b)^{-1}$, we can express the defining property of a module singular Riemannian foliation also in the following form

Lemma 3.3. *The triple (M, g, \mathcal{F}) is a module singular Riemannian foliation if and only if*

$$\mathcal{L}_X g^{-1} \in \mathfrak{X}(M) \odot \mathcal{F}$$

for every vector field $X \in \mathcal{F}$.

As a consequence of the following lemma and proposition, it is enough to check Equation (14) locally for a family of generators:

Lemma 3.4. *Let (M, \mathcal{F}) be a foliated manifold such that $\mathcal{F} = \langle X_1, \dots, X_N \rangle_{C^\infty(M)}$ for some positive integer N . Then the triple (M, g, \mathcal{F}) is a module singular Riemannian foliation if and only if there exist $\omega_a^b \in \Omega^1(M)$ for $a, b = 1, \dots, N$ such that*

$$\mathcal{L}_{X_a} g = \sum_{b=1}^N \omega_a^b \odot g_b(X_b).$$

5. For the relation of module singular Riemannian foliations with the notion defined in [KS19] see Appendix 1 as well as Theorem 3.6 below.

Proof. First assume that (M, g, \mathcal{F}) is a module singular Riemannian foliation. Choose a partition of unity $\{\rho_i\}_{i=1}^\infty$ subordinate to a locally finite cover $\{U_i\}_{i=1}^\infty$ of M . For every $a = 1, \dots, N$ we have

$$\begin{aligned} \mathcal{L}_{X_a} g &= \sum_{i=1}^\infty \rho_i \mathcal{L}_{X_a} g = \sum_{i=1}^\infty (\mathcal{L}_{\rho_i X_a} g - (d\rho_i) \odot g_b(X_a)) \\ &= \sum_{i=1}^\infty \left(\sum_{b=1}^N \eta_{i,a}^b \odot g_b(X_b) - (d\rho_i) \odot g_b(X_a) \right) \\ &= \sum_{b=1}^N \omega_a^b \odot g_b(X_b), \end{aligned}$$

for some 1-forms $\eta_{i,a}^b$ on M and $\omega_a^b := \sum_{i=1}^\infty \eta_{i,a}^b - \delta_a^b d\rho_i$. For the converse, let X be a vector field in \mathcal{F} . By assumption, there exist $f^1, \dots, f^N \in C_c^\infty(M)$ such that $X = \sum_{a=1}^N f^a X_a$. It follows that

$$\begin{aligned} \mathcal{L}_X g &= \sum_{a=1}^N \mathcal{L}_{f^a X_a} g = \sum_{a=1}^N f^a \mathcal{L}_{X_a} g + (df^a) \odot g_b(X_a) \\ &= \sum_{a=1}^N \omega_a^b \odot g_b(f^a X_b) + (df^a) \odot g_b(X_a) \in \Omega^1(M) \odot g_b(\mathcal{F}), \end{aligned}$$

■

An important property of the definition of a geometric singular Riemannian foliation is that the defining condition is local. This is less trivial in the case of module singular Riemannian foliations.

Proposition 3.5. *The triple (M, g, \mathcal{F}) is a module singular Riemannian foliation if and only if for every point $q \in M$ there exist an open neighborhood $U \subset M$ around q such that $(U, g_U, \iota_U^{-1} \mathcal{F})$ is a module singular Riemannian foliation, where g_U is the restriction to U of g .*

Proof. If (M, g, \mathcal{F}) is a module singular Riemannian foliation, then restricting both sides of Equation (14) to any open subset $U \in M$ implies that $(U, g_U, \iota_U^{-1} \mathcal{F})$ is a module singular Riemannian foliation. It remains to prove the converse. Choose a partition of unity $\{\rho_i\}_{i=1}^\infty$ subordinate to a locally finite cover $\{U_i\}_{i=1}^\infty$ of M , with open subsets U_i small enough such that $\iota_{U_i}^{-1} \mathcal{F} = \langle X_{i,1}, \dots, X_{i,N_i} \rangle_{C_c^\infty(U_i)}$ for some positive integer N_i and vector fields $X_{i,1}, \dots, X_{i,N_i} \in \mathfrak{X}(U_i)$. Then for every vector field $X \in \mathcal{F}$,

$$X = \sum_{i=1}^\infty \rho_i X.$$

Moreover, for every positive integer i , there exist functions $f^{i,1}, \dots, f^{i,N_i} \in C_c^\infty(U_i)$ such that

$$\rho_i X = \sum_{a=1}^{N_i} f^{i,a} X_{i,a},$$

and consequently

$$X = \sum_{i=1}^{\infty} \sum_{a=1}^{N_i} f^{i,a} X_{i,a}.$$

This together with Lemma 3.4 now permit us to prove that (M, g, \mathcal{F}) is a module singular Riemannian foliation:

$$\mathcal{L}_X g = \sum_{i=1}^{\infty} \sum_{a=1}^{N_i} f^{i,a} \mathcal{L}_{X_{i,a}} g_{U_i} + (df^{i,a}) \odot (g)_b(X_{i,a}),$$

which proves $\mathcal{L}_X g \in \Omega^1(M) \odot g_b(\mathcal{F})$ since X is compactly supported and only finitely many $f^{i,a}$ are nonzero on $\text{supp}(X)$. \blacksquare

Every finitely generated singular foliation is image of the anchor map of an almost Lie algebroid [LGLS20] (see Appendix 1). For module singular Riemannian foliations, one has furthermore:

Theorem 3.6.

- For every module singular Riemannian foliation (M, g, \mathcal{F}) with \mathcal{F} finitely generated, there exists an almost Lie algebroid $(A, \rho, [\cdot, \cdot]_A)$ over M equipped with a connection $\nabla: \Gamma(A) \rightarrow \Gamma(T^*M \otimes A)$ such that $\mathcal{F} := \rho(\Gamma_c(A))$ and

$${}^A\nabla g = 0, \tag{15}$$

where ${}^A\nabla$ is the A -connection induced by ∇ , see Equation (100) in Appendix 1.

- Let $(A, \rho, [\cdot, \cdot]_A)$ be an almost Lie algebroid over a Riemannian manifold (M, g) , such that the triple $(M, g, \mathcal{F} := \rho(\Gamma_c(A)))$ is a module singular Riemannian foliation. Then there exists a connection ∇ on A such that (15) holds true.

Proof. The proof of the first part of the Theorem can be performed by a straightforward adaptation of the proof of Proposition 1.5 in the Appendix. In particular, the almost Lie algebroid A then can be chosen to be trivial, $A = M \times \mathbb{R}^r$, where r is the number of generators of \mathcal{F} .

We prove the second part of the Theorem, where now one is given a particular, not necessarily trivial almost Lie algebroid A inducing \mathcal{F} , as follows: There exists a vector bundle $V \rightarrow M$ such that $(\tilde{A} := A \oplus V) \rightarrow M$ is a trivial vector bundle of rank N . Consequently there exist sections $e_1, \dots, e_N \in \Gamma(A)$ and $v_1, \dots, v_N \in \Gamma(V)$ such that $e_1 + v_1, \dots, e_N + v_N$ is a global frame for \tilde{A} . Now we define the almost Lie algebroid $(\tilde{A}, \tilde{\rho}, [\cdot, \cdot]_{\tilde{A}})$, where the bracket and the anchor map are the trivial prolongation of $[\cdot, \cdot]_A$ and ρ to \tilde{A} (since in an almost Lie algebroid one does not need to satisfy the Jacobi identity for the bracket, this extension does not pose any problems here). By assumption $\tilde{\rho}(\Gamma_c(\tilde{A})) = \rho(\Gamma_c(A))$ defines a module singular Riemannian foliation on (M, g) . According to Lemma 3.4, this is equivalent to the existence of 1-forms $\omega_a^b \in \Omega^1(M)$ such that

$$\mathcal{L}_{X_a} g = \sum_{b=1}^N \omega_a^b \odot \iota_{X_b} g \quad \forall a = 1, \dots, N. \tag{16}$$

Here $X_a := \tilde{\rho}(e_a + v_a) = \rho(e_a)$. Now define a connection $\tilde{\nabla}$ on \tilde{A} by

$$\tilde{\nabla}(e_a + v_a) = \sum_{b=1}^N \omega_a^b \otimes (e_b + v_b),$$

which induces a connection on A as follows: Let $s \in \Gamma(A) \subset \Gamma(\tilde{A})$, then

$$\nabla_X s := \text{Pr}_A \circ (\tilde{\nabla}_X s) \quad \forall X \in \mathfrak{X}(M),$$

where $\text{Pr}_A: \tilde{A} \rightarrow A$ is the projection to the first component. In particular, for every e_a , there exist unique functions $f_a^b \in C^\infty(M)$ for $b = 1, \dots, N$ such that $e_a = \sum_{b=1}^N f_a^b(e_b + v_b)$ and we have

$$\nabla_X e_a = \text{Pr}_A \circ (\tilde{\nabla}_X (\sum_{b=1}^N f_a^b(e_b + v_b))) = \sum_{b=1}^N X(f_a^b) e_b + \sum_{b,c=1}^N (f_a^b \iota_X \omega_b^c) e_c.$$

Now for every vector field $X \in \mathfrak{X}(M)$, we have:

$$\begin{aligned} 2g(\rho(\nabla_X e_a), X) &= 2g\left(\rho\left(\sum_{b=1}^N X(f_a^b) e_b + \sum_{b,c=1}^N (f_a^b \iota_X \omega_b^c) e_c\right), X\right) \\ &= 2 \sum_{b=1}^N f_a^b \left(\sum_{c=1}^N (\iota_X \omega_b^c) g(X_c, X) \right) + 2 \sum_{b=1}^N X(f_a^b) g(X_b, X) \\ &= \sum_{b=1}^N f_a^b \left(\sum_{b=1}^N \omega_b^c \odot \iota_{X_c} g \right) (X, X) + \sum_{b=1}^N (df_a^b \odot \iota_{X_b} g) (X, X) \\ &= \sum_{b=1}^N (f_a^b \mathcal{L}_{X_b} g + df_a^b \odot \iota_{X_b} g) (X, X) \\ &= (\mathcal{L}_{X_a} g) (X, X), \end{aligned}$$

and, by Lemma 1.4 in the Appendix below, the statement then follows. \blacksquare

So locally one can define singular foliations also as an equivalence class of almost Lie algebroids and module singular Riemannian foliations as an equivalence class of almost Lie algebroids over a Riemannian base with an appropriately compatible connection. (For some related cohomology see also [HS22]).

3.2 Comparison between the two approaches

Using the language of almost Lie algebroids, the following proposition is Theorem 7 in [KS19]. It will be proven in an alternative, more direct way in the present manuscript, using the techniques of \mathcal{I} -Poisson geometry:

Proposition 3.7. *Every module singular Riemannian foliation is a geometric singular Riemannian foliation.*

Note that the converse is not true, at least not for every choice of the module \mathcal{F} :

Example 3.8. *Consider $\mathcal{F} = \langle (x^2 + y^2)(x\partial_y - y\partial_x) \rangle_{C_c^\infty(\mathbb{R}^2)}$ on $M = \mathbb{R}^2$ equipped with the standard metric g_{st} . The leaves are circles centered at the origin, which is a geometric singular Riemannian foliation, but it does not satisfy Equation (14). More precisely, for $V := (x^2 + y^2)(x\partial_y - y\partial_x)$, a simple calculation implies that*

$$\mathcal{L}_V g_{st} = 4 \left[\frac{x dx + y dy}{x^2 + y^2} \right] \odot (g_{st})_b(V),$$

on $\mathbb{R}^2 \setminus (0,0)$. Evidently, the 1-form $\frac{x dx + y dy}{x^2 + y^2}$ fails to have a smooth extension to the origin.

Despite the fact that the singular foliation introduced in Example 3.8 does not define a module singular Riemannian foliation on \mathbb{R}^2 , it is possible to find a module singular Riemannian foliation with the same leaf decomposition. In fact, the singular foliation \mathcal{F}_0 generated by the vector field $V_0 = x\partial_y - y\partial_x$ is a Killing vector field for the standard metric g_{st} , and Equation (19) is obviously satisfied.

Question: Let (M, g, \mathcal{F}) be a geometric singular Riemannian foliation. Is it possible to find a module singular Riemannian foliation (M, g, \mathcal{F}') , having the same leaf decomposition as (M, g, \mathcal{F}) ?

For the smooth setting, this is still an open problem. A counterexample for the polynomial or analytic setting is the singular octonionic Hopf foliation, introduced in Section 8.

For singular foliations there is a pullback under maps transverse to the singular foliation, see Definition 2.17 and the text following it. To adapt this to the context of singular Riemannian foliations, we consider the following:

Definition 3.9. Let $\pi: (N, h) \rightarrow (M, g)$ be a smooth submersion between Riemannian manifolds. It is called a Riemannian submersion if, for every $q \in N$, the restriction $d_q\pi: \mathcal{H}_q \rightarrow T_{\pi(q)}M$ of $d_q\pi$ to $\mathcal{H}_q = (\ker d_q\pi)^\perp \subset T_qN$ is an isometry. The smooth distribution $\mathcal{H} = (\mathcal{H}_q)_{q \in N}$ of rank $\dim(M)$ is called the horizontal distribution of π .

Lemma 3.10. Let $\pi: (N, h) \rightarrow (M, g)$ be a Riemannian submersion and (M, \mathcal{F}) a singular foliation. Then the pullback singular foliation can be generated as follows

$$\pi^{-1}\mathcal{F} = \langle \mathcal{F}^{\mathcal{H}} + \Gamma(\ker d\pi) \rangle_{C_c^\infty(N)}, \quad (17)$$

where $\mathcal{F}^{\mathcal{H}}$ is the horizontal lift of \mathcal{F} .

Proof. By definition 2.17 the inclusion $\langle \mathcal{F}^{\mathcal{H}} + \Gamma(\ker d\pi) \rangle_{C_c^\infty(N)} \subset \pi^{-1}\mathcal{F}$ is evident. Now let W be a projectable vector field on N projecting to \mathcal{F} , i.e. there exists a vector field $X \in \mathcal{F}$ such that $d_q\pi(W|_q) = X|_{\pi(q)}$. On the other hand, if we decompose W into its horizontal part W_H and its vertical part W_V , we have $d_q\pi(W_H|_q) = X|_{\pi(q)}$, which gives $X_H = V^{\mathcal{H}}$. This means that generators of $\pi^{-1}\mathcal{F}$ belongs to $\mathcal{F}^{\mathcal{H}} + \Gamma(\ker d\pi)$, consequently $\pi^{-1}\mathcal{F} = \langle \mathcal{F}^{\mathcal{H}} + \Gamma(\ker d\pi) \rangle_{C_c^\infty(N)}$. \blacksquare

Proposition 3.11. Let $\pi: (N, h) \rightarrow (M, g)$ be a Riemannian submersion and let (M, g, \mathcal{F}) be a module singular Riemannian foliation. Then $(N, h, \pi^{-1}\mathcal{F})$ is a module singular Riemannian foliation as well. The same statement holds true for geometric singular Riemannian foliations.

Proposition 3.11 will be proven in Section 5.4 below. As a consequence, and by the fact that (regular) Riemannian foliations are locally modeled on Riemannian submersions [M98], we obtain:

Proposition 3.12. Let (M, \mathcal{F}) be a regular foliation on a Riemannian manifold (M, g) . Then (M, g, \mathcal{F}) is a geometric singular Riemannian foliation if and only if it is a module singular Riemannian foliation.

Example 3.13. Let G be a Lie group acting by isometries on (M, g) . Then after Lemma 3.4 the $C^\infty(M)$ -submodule $\mathcal{F} \subset \mathfrak{X}_c(M)$ generated by fundamental vector fields is a module singular Riemannian foliation on (M, g) , since every fundamental vector field X is a Killing vector field: $\mathcal{L}_X g = 0$.

Example 3.14. *The proof of Theorem 1 in [KS19] shows that the geometric singular Riemannian foliation induced on the manifold of objects of a Riemannian groupoid—*as defined in [dHF18]*—is a module singular Riemannian foliation.*

3.3 Morita equivalence of singular Riemannian foliations

Definition 3.15. *Two module singular Riemannian foliations $(M_1, g_1, \mathcal{F}_1)$ and $(M_2, g_2, \mathcal{F}_2)$ are Morita equivalent if there exists a Riemannian manifold (N, h) together with two surjective Riemannian submersions with connected fibers $\pi_i: (N, h) \rightarrow (M_i, g_i)$ for $i = 1, 2$ such that*

$$\pi_1^{-1}\mathcal{F}_1 = \pi_2^{-1}\mathcal{F}_2$$

and we write $(N_1, g_1, \mathcal{F}_1) \sim_{ME} (N_2, g_2, \mathcal{F}_2)$.

Remark 3.16. *This notion of Morita equivalence can be defined for geometric singular Riemannian foliations as well as for module ones. Consequently, if two module singular Riemannian foliations are Morita equivalent then they are also Morita equivalent as geometric singular Riemannian foliations. Moreover, if we forget about Riemannian metrics, we obtain Hausdorff Morita equivalent foliated manifolds.*

While for Hausdorff Morita equivalence of singular foliations transitivity of the equivalence relation is relatively easy to show, this is more involved in case of the additional Riemannian structure due to the presence of the metric.

Proposition 3.17. *The Morita equivalence of module singular Riemannian foliations defines an equivalence relation.*

Proof. Reflexivity is evident from the definition and for the self-equivalence the identity map defines a Morita equivalence between a module singular Riemannian foliation and itself. Now we prove the transitivity as follows: Assume that $(M_1, g_1, \mathcal{F}_1) \sim_{ME} (M_2, g_2, \mathcal{F}_2)$ given by $\pi_i: (U, g_U) \rightarrow (M_i, g_i)$ for $i = 1, 2$ and $(M_2, g_2, \mathcal{F}_2) \sim_{ME} (M_3, g_3, \mathcal{F}_3)$ given by $\eta_i: (W, g_W) \rightarrow (M_i, g_i)$ for $i = 2, 3$. Now consider the smooth manifold $U \times_{\pi_2 \times \eta_2} W$ defined as

$$U \times_{\pi_2 \times \eta_2} W := \{(u, w) \in U \times W \mid \pi_2(u) = \eta_2(w) \in M_2\}$$

with canonical projections $p_U: U \times_{\pi_2 \times \eta_2} W \rightarrow U$ and $p_W: U \times_{\pi_2 \times \eta_2} W \rightarrow W$. Note that the tangent space at $(u, w) \in U \times_{\pi_2 \times \eta_2} W$ is given by

$$T_{(u,w)}(U \times_{\pi_2 \times \eta_2} W) = \{(X, Y) \in T_u U \times T_w W \mid d_u \pi_2(X) = d_w \eta_2(Y)\}$$

since every smooth curve on $U \times_{\pi_2 \times \eta_2} W$ can be expressed as (γ_U, γ_W) where γ_U and γ_W are smooth curves on U and W , respectively, such that $\pi_2(\gamma_U) = \eta_2(\gamma_W)$. We now define a Riemannian metric g on $U \times_{\pi_2 \times \eta_2} W$ as follows:⁶

$$g((X_1, Y_1), (X_2, Y_2)) := g_U(X_1, X_2) + g_W(Y_1, Y_2) - g_2(d_u \pi_2(X_1), d_u \pi_2(X_2)) \quad (18)$$

where $(X_i, Y_i) \in T_{(u,w)}(U \times_{\pi_2 \times \eta_2} W)$ for $i = 1, 2$, and note that $d_u \pi_U(X_1) = d_w \eta_W(Y_i)$ for $i = 1, 2$. It is clearly smooth and symmetric. In addition we have

$$g((X, Y), (X, Y)) = \|X\|^2 + \|Y\|^2 - \|d_u \pi_2(X)\|^2 = \|X\|^2 + \|Y\|^2 - \|d_w \eta_2(Y)\|^2 \geq 0$$

6. We were informed that this idea has been used already in [W83] and [dHF18].

for every $(X, Y) \in T_{(u,w)}(U \times_{\pi_2} W)$ since π_U and η_W are Riemannian submersions, and it is zero if and only if both X and Y are zero vectors. Hence $(U \times_{\pi_2} W, g)$ defines a Riemannian manifold. Now we claim that the projections p_U and p_W are Riemannian submersions. We have

$$\ker(d_{(u,w)}p_U) = \{(0, Y) \in T_u U \times T_w W \mid d_w \eta_2(Y) = 0\},$$

so its orthogonal complement is given by

$$H_{(u,w)} = \{(X, Y) \in T_{(u,w)}(U \times_{\pi_2} W) \mid g_W(Y, Z) = 0 \quad \forall Z \in \ker(d_w \eta_2)\}.$$

Using the fact that η_W is a Riemannian submersion, for every two vectors (X_1, Y_1) and (X_2, Y_2) in $H_{(u,w)}$ we have:

$$\begin{aligned} g((X_1, Y_1), (X_2, Y_2)) &= g_U(X_1, X_2) + g_W(d_w \eta_2(Y_1), d_w \eta_2(Y_2)) - g_2(d_w \eta_2(Y_1), d_w \eta_2(Y_2)) \\ &= g_U(X_1, X_2) = g_U(d_{(u,w)}p_U(X_1, Y_1), d_{(u,w)}p_U(X_2, Y_2)) \end{aligned}$$

which proves that p_U is a Riemannian submersion. It has connected fibers since for every $u \in U$, we have $p_U^{-1}(u) = \{u\} \times \eta_2^{-1}(\pi_2(u))$, which is connected. Similarly it is shown that p_W is a Riemannian submersion with connected fibers. These two Riemannian submersions are surjective by construction. So the Riemannian manifold $(U \times_{\pi_2} W, g)$ and the surjective Riemannian submersions with connected fibers $\pi_1 \circ p_U$ and $\pi_2 \circ p_W$ define a Morita equivalence between $(N_1, g_1, \mathcal{F}_1)$ and $(N_3, g_3, \mathcal{F}_3)$. This completes the proof. \blacksquare

Although the leaf space of a singular Riemannian foliation may not be topologically well-behaved, it inherits a pseudo-metric space structure from the Riemannian metric. Following [PPT10], for every two leaves L_1 and L_2 of a SRF (M, g, \mathcal{F}) , the distance between them is defined by

$$d_{M/\mathcal{F}}(L_1, L_2) := \inf \left\{ \sum_{i=1}^N L_g(\gamma_i) \right\}.$$

Here the infimum is taken over all *discrete paths* $(\gamma_1, \dots, \gamma_N)$ joining L_1 and L_2 , i.e. a family of piecewise smooth curves $\gamma_1, \dots, \gamma_N: [0, 1] \rightarrow M$ for some positive integer N , such that $\gamma_1(0) \in L_1$, $\gamma_N(1) \in L_2$ and $\gamma_i(1)$ and $\gamma_{i+1}(0)$ belong to the same leaf for each $i = 1, \dots, n-1$.

As a corollary of Remark 3.16 and Theorem 2.19, there exists a homeomorphism between the leaf spaces of Morita equivalent module singular Riemannian foliations. The following theorem is the Riemannian counterpart of part (i) of Theorem 2.19:

Theorem 3.18. *Let $(N_1, g_1, \mathcal{F}_1)$ and $(N_2, g_2, \mathcal{F}_2)$ be Morita equivalent module singular Riemannian foliations. Then the homeomorphism between the leaf spaces given in Theorem 2.19 is distance preserving.*

Proof. Assume that $(M_1, g_1, \mathcal{F}_1) \sim_{ME} (M_2, g_2, \mathcal{F}_2)$ is given by $\pi_i: (N, h) \rightarrow (M_i, g_i)$ for $i = 1, 2$. Let L_1 and L'_1 be two leaves in $(M_1, g_1, \mathcal{F}_1)$ and let L_2 and L'_2 be their corresponding leaves in $(M_2, g_2, \mathcal{F}_2)$. Consider a discrete path $(\gamma_1, \dots, \gamma_n)$ joining L_1 and L'_1 . By lifting each γ_i into finitely many piecewise smooth horizontal paths, one

obtains a discrete path $(\eta_1, \dots, \eta_{n'})$ for some $n' \geq n$ on n joining $\pi_1^{-1}(L_1)$ and $\pi_1^{-1}(L'_1)$ with the same length as $(\gamma_1, \dots, \gamma_n)$ —since the lifts are horizontal with respect to the Riemannian submersion π_1 . Since π_2 is a Riemannian submersion, $(\pi_2(\eta_1), \dots, \pi_2(\eta_{n'}))$ is a discrete path joining L_2 and L'_2 with a length which is smaller than or equal to the length of $(\gamma_1, \dots, \gamma_n)$ —since the lifts are not necessarily horizontal with respect to π_2 . Consequently

$$d_{M_1/\mathcal{F}_1}(L_1, L'_1) \geq d_{M_2/\mathcal{F}_2}(L_2, L'_2).$$

Similarly $d_{M_2/\mathcal{F}_2}(L_2, L'_2) \geq d_{M_1/\mathcal{F}_1}(L_1, L'_1)$, which implies $d_{M_1/\mathcal{F}_1}(L_1, L'_1) = d_{M_1/\mathcal{F}_2}(L_2, L'_2)$. This proves the statement. \blacksquare

To define a category **SRF** of module singular Riemannian foliations one needs to specify their morphisms. We are not going to do this in the present manuscript. But since any good notion of such morphisms should include Riemannian submersions which satisfy that the pullback of the singular foliation on the base agrees with the singular foliation on the total space, we define the following full subcategory **SRF**₀:

Definition 3.19. *The category **SRF**₀ has module singular Riemannian foliations as its objects and Riemannian submersions $\pi: (N, h, \mathcal{F}_N) \rightarrow (M, g, \mathcal{F}_M)$ satisfying $\pi^{-1}\mathcal{F}_M = \mathcal{F}_N$ as its morphisms.*

3.4 Singular foliations as sheaves

One can equivalently define singular foliations as an involutive and locally finitely generated subsheaf of the sheaf of vector fields \mathfrak{X} . This definition is equivalent to Definition 2.4 in the smooth setting, but it offers advantages when working with sheaves of polynomial, real analytic, or holomorphic functions (See [LGLS20] or [GZ19]). Notably, since these sheaves of rings are Noetherian, the condition of being locally finitely generated is automatically satisfied and can therefore be omitted. We observe that the definition of module singular Riemannian foliations can be directly adapted to the definition of singular foliations as sheaves.

In what follows, M is a smooth or real analytic manifold, or an affine variety. For every open subset $U \subset M$, let $\mathcal{O}: U \mapsto \mathcal{O}(U)$ be the sheaf of smooth, real analytic or polynomial functions. We denote by $\mathfrak{X}(U)$ the $\mathcal{O}(U)$ -module of vector fields on U and the sheaf of vector fields by $\mathfrak{X}: U \mapsto \mathfrak{X}(U)$.

Definition 3.20. *A sheaf $\mathcal{F}: U \mapsto \mathcal{F}(U)$ of \mathcal{O} -modules on M is called locally finitely generated, if for every point $q \in M$, there exists a neighborhood U containing q and finitely many sections $X_1, \dots, X_N \in \mathcal{F}(U)$, such that for every open subset $V \subset U$ we have*

$$\mathcal{F}(V) = \langle X_1|_V, \dots, X_N|_V \rangle_{\mathcal{O}(V)}$$

Definition 3.21. *A singular foliation on an \mathcal{O} -manifold M is a subsheaf $\mathcal{F} \subset \mathfrak{X}$ of the sheaf of vector fields, which is locally finitely generated and involutive, i.e. the $\mathcal{O}(U)$ -module of vector fields $\mathcal{F}(U)$ is closed under the Lie bracket of vector fields. The pair (M, \mathcal{F}) is referred to as a foliated manifold.*

Theorem 2.5 stays valid and implies the partition of the foliated manifold (M, \mathcal{F}) into leaves. Moreover, except the pullback operation, all definitions and properties mentioned in Section 2 can be similarly defined and verified for Definition 3.21 of singular foliations.

Remark 3.22. *The two definitions coincide in the smooth setting. More precisely, if \mathcal{O} is the sheaf of smooth functions on the smooth manifolds M , there is a one-to-one correspondence between locally finitely generated subsheaves of the sheaf of vector fields and the $C^\infty(M)$ -submodules of the compactly supported vector fields [GZ19].*

Definition 3.2 of module singular Riemannian foliation, can be directly adapted to Definition 3.21 of singular foliations, as follows:

Definition 3.23. *Let \mathcal{F} be a singular foliation on the Riemannian manifold (M, g) . The triple (M, g, \mathcal{F}) defines a module singular Riemannian foliation, if for every open subset $U \subset M$ and $X \in \mathcal{F}(U)$ we have*

$$\mathcal{L}_X g \in \Omega^1(M) \odot g_b(\mathcal{F}(U)). \quad (19)$$

Here, \odot stands for the symmetric inner product of 1-forms and $g_b : TM \rightarrow T^*M$ is the musical isomorphism given by $(q, v) \mapsto g_q(v, \cdot)$ for all $(q, v) \in TM$.

In particular, for an open subset U with finitely many generators $X_1, \dots, X_N \in \mathcal{F}(U)$, Equation (19) is satisfied if and only if there exist 1-forms $\omega_a^b \in \Omega^1(U)$ such that

$$\mathcal{L}_{X_a} g = \omega_a^b \odot g(X_b, \cdot). \quad (20)$$

Remark 3.24. *As a consequence of Proposition 3.5, Definition 3.23 is equivalent to Definition 3.2 in the smooth setting.*

The idea of defining \mathcal{I} -Poisson manifolds arises from an algebraic perspective on the coisotropic reduction of Poisson manifolds [MR86]. Generalizing the well-known example of the symplectic reduction [MW74], coisotropic reduction is defined for a Poisson manifold together with a closed submanifold whose annihilating functions form an ideal closed with respect to the Poisson bracket. This ensures that the Hamiltonian flows of functions in this ideal decompose the coisotropic submanifold into leaves. Under certain regularity conditions, the leaf space inherits a Poisson structure.

In differential geometry and mathematical physics, the regularity conditions for applying the coisotropic reduction are not always satisfied, as shown in examples of this chapter. To study examples with singularities and emphasize the algebraic aspects of the reduction, useful ideas have been developed in [SW83, AGJ90] and more recently in [DEW19].

In this chapter, we introduce \mathcal{I} -Poisson manifolds in their most general sense. We will clarify the relation and the differences between this definition and the aforementioned notions. In addition, we define the category **IPois** bply introducing the morphisms of \mathcal{I} -Poisson manifolds as a natural relaxation of Poisson maps. At the end of this chapter, we utilize this framework to study singular (Riemannian) foliations and prove some results.

4 \mathcal{I} -Poisson manifolds

4.1 Background on Poisson manifolds

We start by recalling the necessary definitions and examples from Poisson geometry. See [dSW99, LGPV13, CFM21] for a more detailed introduction to Poisson geometry

Definition 4.1. A Poisson manifold $(P, \{\cdot, \cdot\})$ consists of a smooth manifold P and an \mathbb{R} -bilinear Lie bracket $\{\cdot, \cdot\} : C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P)$ satisfying the Leibniz rule

$$\{f, gh\} = g\{f, h\} + \{f, h\}h$$

for every $f, g, h \in C^\infty(P)$.

The Leibniz rule implies that for every function $f \in C^\infty(P)$, $\{f, \cdot\}$ and $\{\cdot, f\}$ are derivations of $C^\infty(P)$, or equivalently vector fields in $\mathfrak{X}(P)$. This implies that there exists a bivector field $\Pi \in \mathfrak{X}^2(P) = \Gamma(\Lambda^2(TM))$ such that for every two functions $f, g \in C^\infty(P)$ we have:

$$\{f, g\} = \Pi(df, dg).$$

Remark 4.2. The Jacobi identity for $\{\cdot, \cdot\}$ is equivalent to $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket on multi-vector fields \mathfrak{X}^\bullet . The vector field $X_f = \{f, \cdot\} = \Pi(df, \cdot)$ is called the Hamiltonian vector field of the function f . Functions with vanishing Hamiltonian vector fields are called Casimir functions.

Remark 4.3. A Poisson algebra is an associative algebra over some field k equipped with a Lie bracket $\{\cdot, \cdot\}$, where the Leibniz rule gives the compatibility condition between these two products on A . The triple $(C^\infty(P), \cdot, \{\cdot, \cdot\})$ where \cdot stands for the pointwise product of functions is an example of Poisson algebras.

Example 4.4. Let $M = \mathbb{R}^{2n}$ for some positive integer $n \in \mathbb{N}$ and choose local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$. Then the bracket $\{\cdot, \cdot\}$ defined by

$$\{f, g\} := \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) \quad (21)$$

for every $f, g \in C^\infty(M)$ defines the standard Poisson bracket on M .

Example 4.5. More generally, every symplectic manifold (M, ω) defines a Poisson manifold with the bracket $\{\cdot, \cdot\}$ defined by

$$\{f, g\} := -\omega(X_f, X_g)$$

for every $f, g \in C^\infty(M)$, where X_f and X_g are Hamiltonian vector fields of f and g satisfying $\iota_{X_f}\omega = df$ and $\iota_{X_g}\omega = dg$. In particular the cotangent bundle of a smooth manifold is a Poisson manifold, with the Poisson bracket given by Equation 21 in Darboux's coordinates.

One can show that the assignment $f \mapsto X_f$ defines a morphism of Lie algebras. This proves the following definition:

Proposition 4.6. *Let $\mathcal{F}_\Pi \subset \mathfrak{X}(P)$ be the $C^\infty(P)$ -submodule of the vector fields generated by all Hamiltonian vector fields. Then (P, \mathcal{F}_Π) defines a singular foliation, called the symplectic foliation.*

Example 4.7. *Let \mathfrak{g} be a finite dimensional Lie algebra with Lie bracket $[\cdot, \cdot]$. For every smooth function $f \in C^\infty(\mathfrak{g}^*)$ and every element $\xi \in \mathfrak{g}^*$, the linear map $d_\xi f: T_\xi \mathfrak{g}^* \cong \mathfrak{g}^* \rightarrow \mathbb{R}$ is naturally identified with an element $\underline{d_\xi f} \in \mathfrak{g}$. Then \mathfrak{g}^* is a Poisson manifold with the Lie-Poisson bracket*

$$\{f, g\}(\xi) := \langle \xi, [\underline{d_\xi f}, \underline{d_\xi g}] \rangle$$

for every two functions $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. The Jacobi identity holds by the Jacobi identity of the Lie bracket $[\cdot, \cdot]$, and the Leibniz rule is a consequence of the Leibniz rule for the differential d . In this example, the symplectic foliation has the coadjoint orbits in \mathfrak{g}^* as its leaves.

Definition 4.8. *A smooth map $\pi: (P_1, \{\cdot, \cdot\}_1) \rightarrow (P_2, \{\cdot, \cdot\}_2)$ is called a Poisson map if for every two functions $f, g \in C^\infty(P_2)$ we have:*

$$\{\pi^* f, \pi^* g\}_1 = \pi^* \{f, g\}_2. \quad (22)$$

The following example in [GZ19] reveals the relation of Morita equivalence of Poisson manifolds and Hausdorff Morita equivalence of singular foliations.

Example 4.9 ([GZ19]). *Poisson manifolds (N_1, Π_1) and (N_2, Π_2) are Morita equivalent [X91] if there exist a symplectic manifold (U, ω) and surjective submersions $\varphi_i: U \rightarrow N_i$ for $i = 1, 2$, which are Poisson and anti-Poisson maps respectively, and such that $\ker(d\varphi_1)_u$ and $\ker(d\varphi_2)_u$ are symplectic orthogonal subspaces of $T_u U$ for all $u \in U$. Let \mathcal{F}_1 and \mathcal{F}_2 be the symplectic foliations generated by Hamiltonian vector fields on N_1 and N_2 respectively. Assume that (N_1, Π_1) and (N_2, Π_2) are Morita equivalent, then we have the following descriptions of $\Gamma(\ker(d\varphi_1))$ and $\Gamma(\ker(d\varphi_2))$:*

$$\begin{aligned} \Gamma(\ker(d\varphi_1)) &= \langle \{X_{\varphi_2^* g}: g \in C^\infty(N_2)\} \rangle_{C^\infty(U)} \\ \Gamma(\ker(d\varphi_2)) &= \langle \{X_{\varphi_1^* g}: g \in C^\infty(N_1)\} \rangle_{C^\infty(U)} \end{aligned}$$

where $X_f := \Pi^\#(df)$ stands for the Hamiltonian vector field of the function f . Consequently

$$\varphi_i^{-1}(\mathcal{F}_i) = \langle \{X_{\varphi_i^* g}: g \in C^\infty(N_i)\} + \Gamma(\ker(d\varphi_i)) \rangle_{C^\infty(U)}$$

for $i = 1, 2$. It follows that

$$\varphi_1^{-1}(\mathcal{F}_1) = \langle \{X_{\varphi_1^* g}: g \in C^\infty(N_1)\} + \{X_{\varphi_2^* g}: g \in C^\infty(N_2)\} \rangle_{C^\infty(U)} = \varphi_2^{-1}(\mathcal{F}_2)$$

which proves Corollary 2.24 in [GZ19]: If two Poisson manifolds are Morita equivalent, then their symplectic foliations are Hausdorff Morita equivalent.

Remark 4.10. *The above notion of Morita equivalence for Poisson manifolds defines an equivalence relation on the family of integrable Poisson manifolds [X91], i.e. Poisson manifolds having a complete symplectic realization or, equivalently, such that their Lie algebroids defined over their cotangent bundles integrate to Lie groupoids. If one drops the integrability condition, one has problems already with the self-equivalence.*

4.2 Definitions and examples

Definition 4.11. *A subsheaf \mathcal{I} of a sheaf of rings \mathcal{O} on a manifold P is called locally finitely generated if for every $q \in P$ there exist an open neighborhood $U \subset P$ containing q and finitely many sections $g_1, \dots, g_N \in \mathcal{O}(U)$ such that $\mathcal{I}(V) = \langle g_1|_V, \dots, g_N|_V \rangle_{\mathcal{O}(V)}$ for every open subset $V \subset U$.*

Remark 4.12. *In this text, we mostly work with \mathcal{O} being the sheaf of smooth functions, but one may equally consider sheaves of polynomial, real analytic or holomorphic functions for the appropriate choice of P .*

Definition 4.13. *An \mathcal{I} -Poisson manifold is a triple $(P, \{\cdot, \cdot\}, \mathcal{I})$ where \mathcal{I} is a locally finitely generated subsheaf of smooth functions on P , such that for every open subset $U \subset P$ we have*

1. $\mathcal{I}(U)$ is a $C^\infty(U)$ -module,
2. $\mathcal{I}(U)$ is closed under the bracket,
3. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} \in \mathcal{I}(P), \quad \forall f, g, h \in N(\mathcal{I}),$

where $N(\mathcal{I}) := \{f \in C^\infty(P) : \{f|_U, \mathcal{I}(U)\} \subset \mathcal{I}(U) \text{ for every open subset } U\}$.

We call $\{\cdot, \cdot\}$ the \mathcal{I} -Poisson bracket and $N(\mathcal{I})$ the \mathcal{I} -Poisson normalizer.

Example 4.14. *Every Poisson manifold $(P, \{\cdot, \cdot\})$ is canonically an \mathcal{I} -Poisson manifold for \mathcal{I} generated by the zero function.*

Example 4.15. *Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold and $I = \langle f_1, \dots, f_N \rangle_{C^\infty(P)}$ be a finitely generated ideal of $C^\infty(P)$ which is a Poisson subalgebra. Then the sheaf \mathcal{I} defined by*

$$U \mapsto \mathcal{I}(U) := \langle f_1|_U, \dots, f_N|_U \rangle_{C^\infty(U)}$$

defines an \mathcal{I} -Poisson manifold.

Example 4.16. *Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold and $C \subset P$ an embedded coisotropic submanifold. Then the triple $(P, \{\cdot, \cdot\}, \mathcal{I}_C)$ where $\mathcal{I}_C(U) := \{f \in C^\infty(U) : f|_{C \cap U} \equiv 0\}$ for every open subset $U \subset P$ defines an \mathcal{I} -Poisson manifold. Note that in this example, \mathcal{I}_C is in general not finitely generated, only locally so.*

Example 4.17. *Let $(P, \{\cdot, \cdot\}, \Phi)$ be a Hamiltonian quasi-Poisson manifold [AKSM02]: P is a G -manifold for a compact Lie group G , $\{\cdot, \cdot\} : \wedge^2 C^\infty(P) \rightarrow C^\infty(P)$ is an \mathbb{R} -bilinear bracket satisfying the Leibniz rule, such that*

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = \phi_P(df, dg, dh) \quad \forall f, g, h \in C^\infty(P) \quad (23)$$

where $\phi_P \in \mathfrak{X}^3(P)$ is a 3-vector field induced by the cartan 3-tensor $\phi \in \wedge^3 \mathfrak{g}$, and $\Phi : P \rightarrow G$ a G -equivariant map satisfying the moment map condition

$$\{\Phi^* f, \cdot\} = \frac{1}{2} \Phi^*((e_a^L + e_a^R) \cdot f)(e_a)_P \quad \forall f \in C^\infty(G), \quad (24)$$

where (e_a) is a basis for \mathfrak{g} , $e_a^L, e_a^R \in \mathfrak{X}(G)$ are the left-invariant and right-invariant vector fields associated to e_a respectively, and $(e_a)_P \in \mathfrak{X}(P)$ is the fundamental vector field induced by e_a .

Let P_* be the open subset of P on which G acts freely. Fix a conjugacy class $C \subset G$ and let $\mathcal{I} \subset C^\infty(P)$ to be defined as the vanishing ideal of $C_* := \Phi^{-1}(C) \cap P_*$. Now the triple $(P_*, \{\cdot, \cdot\}, \mathcal{I})$ defines an \mathcal{I} -Poisson manifold: In Definition 4.13, Condition 1 is clear and Condition 2 is a consequence of the moment map condition and G -equivariance of Φ . It remains to show that Condition 3 is satisfied: The moment map condition gives

$$\mathcal{N}(\mathcal{I}) = \{f \in C^\infty(P) : f|_{C_*} \in C^\infty(C_*)^G\},$$

which implies that for $f, g, h \in \mathcal{N}(\mathcal{I})$, the function $\phi_P(df, dg, dh)$ vanishes on C_* since $\phi_P|_{C_*} \in \mathfrak{X}^3(C_*)$.

Under some conditions, a reduction process applied to Examples 4.16 and 4.17 results in reduced Poisson manifolds:

Example 4.18. In Example 4.16, the Hamiltonian vector fields of functions in \mathcal{I}_C are tangent to C and they are closed under the Lie bracket, hence defining a singular foliation on C . If this singular foliation is regular and the quotient map $\pi: C \rightarrow C_{red}$ to the leaf space C_{red} is a smooth submersion, then C_{red} inherits a Poisson bracket $\{\cdot, \cdot\}_{red}$ such that $\pi^* \{f, g\}_{red} = \{F, G\}|_C$, where F and G are smooth functions on P satisfying $F|_C = \pi^* f$ and $G|_C = \pi^* g$. This process is called the coisotropic reduction [MR86].

Example 4.19. In Example 4.17, Theorem 6.1 in [AKSM02] implies that the quotient $C_{red} := C_*/G$ inherits a Poisson bracket $\{\cdot, \cdot\}_{red}$.

Remark 4.20. The notion of an \mathcal{I} -Poisson manifold is motivated by generalizing Examples 4.16 and 4.17 and their reductions to a potentially singular setting, where the quotient C_{red} does not need to exist as a manifold and the reduction is performed algebraically.

As a consequence of Conditions 2 and 3 of Definition 4, the quotient $\mathcal{N}(\mathcal{I})/\mathcal{I}(P)$ forms a Poisson algebra. This motivates the following definition:

Definition 4.21. The reduced Poisson algebra of the \mathcal{I} -Poisson manifold $(P, \{\cdot, \cdot\}, \mathcal{I})$ is defined to be the Poisson algebra $\mathcal{R}(\mathcal{I}) := \mathcal{N}(\mathcal{I})/\mathcal{I}(P)$.

Remark 4.22. This is a straightforward generalization of the set of Dirac observables [D50]. The algebra $\mathcal{R}(\mathcal{I})$ also appears in [SW83] as an algebraic method of reducing Hamiltonian G -spaces with singular moment maps.

Example 4.23. If C_{red} in Example 4.18 is smooth, then $\mathcal{R}(\mathcal{I}_C)$ is isomorphic to the Poisson algebra $C^\infty(C_{red})$. Similarly, in Example 4.19, the Poisson algebra of functions on C_{red} is isomorphic to the Poisson algebra $\mathcal{R}(\mathcal{I})$.

Example 4.24. Let G be a connected Lie group acting on a Poisson manifold $(P, \{\cdot, \cdot\})$ by Poisson diffeomorphisms with a G -equivariant moment $\mu: P \rightarrow \mathfrak{g}^*$. Following [SW83], the subsheaf $\mathcal{I} \subset C^\infty$ generated by smooth functions $\langle \mu, \mathfrak{g} \rangle$ is a Poisson subalgebra and one has

$$\mathcal{R}(\mathcal{I}) \cong (C^\infty(P)/\mathcal{I})^G.$$

Moreover, if G is compact, then Proposition 5.12 in [AGJ90] states that

$$\mathcal{R}(\mathcal{I}) \cong C^\infty(P)^G/\mathcal{I}^G.$$

Example 4.25. Let $P = T^*\mathbb{R}^n$, $n > 1$, with coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ and $\mathcal{I} \subset C^\infty$ the subsheaf generated by the $n(n-1)/2$ functions $q^i p_j - q^j p_i$ for $1 \leq i < j \leq n$. This is a special case of Example 4.24 for the diagonal action of $G = SO(n)$ on $T^*\mathbb{R}^n$. We have

$$\mathcal{R}(\mathcal{I}) \cong W^\infty(D),$$

where $D \subset \mathbb{R}^3$ is defined by

$$D := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = x_3^2 \text{ and } x_3 \geq 0\}$$

and $W^\infty(D)$ stands for the smooth functions on D in the sense of Whitney, i.e. the restriction of $C^\infty(\mathbb{R}^3)$ to D . For more details and proofs see Theorem 5.6 and Example 5.11(a) of [AGJ90].

The Poisson bracket on $W^\infty(D)$ can be understood as follows: Identify \mathbb{R}^3 with the Poisson manifold $\mathfrak{so}(2,1)^*$ and, simultaneously, with 2+1 dimensional Minkowski space. The symplectic leaves of $\mathfrak{so}(2,1)^*$ then consist of spacelike vectors of a fixed Minkowski norm (one-sheeted hyperboloids), null vectors decompose into the origin, the forward light cone, and the backward light cone as three distinct leaves, and finally timelike vectors of a fixed norm yield two leaves each (two-sheeted hyperboloids). Then restriction to D corresponds precisely to restricting to the forward lightcone and the origin in this Minkowski space. This bracket does not depend on the extension of a function on D to the ambient space since D is the collection of (two) symplectic leaves.

Remark 4.26. If in the previous example one restricts to the polynomial functions, such that $\mathcal{I} \subset \mathbb{R}[q^1, \dots, q^n, p_1, \dots, p_n]$, one finds

$$\mathcal{R}(\mathcal{I}) \cong S^\bullet(\mathfrak{so}(2,1)) / \langle x_1^2 + x_2^2 - x_3^2 \rangle,$$

i.e. the polynomial functions on $\mathfrak{so}(2,1)^*$ modulo the ideal generated by the quadratic Casimir. So one loses the restriction $x_3 \geq 0$ that one finds in the smooth setting.

4.3 Dynamical \mathcal{I} -Poisson manifolds

Definition 4.27. A dynamical \mathcal{I} -Poisson manifold denoted by $(P, \{\cdot, \cdot\}, \mathcal{I}, H)$ consists of an \mathcal{I} -Poisson manifold $(P, \{\cdot, \cdot\}, \mathcal{I})$ and a Hamiltonian function $H \in N(\mathcal{I})$. Its reduction is defined to be the pair $(\mathcal{R}(\mathcal{I}), [H])$ where $[H] \in \mathcal{R}(\mathcal{I})$ is the equivalence class of H .

The following proposition reveals one of the main properties of dynamical \mathcal{I} -Poisson manifolds.

Proposition 4.28. Let $(P, \{\cdot, \cdot\}, \mathcal{I}, H)$ be a dynamical \mathcal{I} -Poisson manifold. Then the Hamiltonian flow of H locally preserves \mathcal{I} , i.e. for every $q_0 \in P$ there exists an open neighborhood $U \subset P$ around q_0 such that $\Phi_H^t|_U$ is defined for $t \in (-\epsilon, \epsilon)$ and

$$(\Phi_H^t)^* \mathcal{I}(\Phi_H^t(U)) = \mathcal{I}(U). \quad (25)$$

In the case that the Hamiltonian vector field X_H is complete, this implies that, for all $t \in \mathbb{R}$, one has $(\Phi_H^t)^* \mathcal{I} \circ \Phi_H^t = \mathcal{I}$ and, in particular, that the ideal $\mathcal{I}(P)$ is preserved,

$$(\Phi_H^t)^* \mathcal{I}(P) = \mathcal{I}(P).$$

Proof. Choose an open neighborhood $W \subset P$ around q_0 where $\mathcal{I}(W)$ is generated by finitely many functions g_1, \dots, g_N for some positive integer N . Then by the existence and uniqueness theorem for ODEs there exist an open subset $U \subset W$ containing q_0 and an interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, such that $\Phi_H^t|_U$ is defined for $t \in (-\epsilon, \epsilon)$. By the definition of the \mathcal{I} -Poisson normalizer in Definition 4.13, there exist functions $\lambda_a^b \in C^\infty(U)$, $a, b = 1, \dots, N$, such that:

$$\{H, g_a\} = \sum_{b=1}^N \lambda_a^b g_b.$$

Using this equation, we obtain:

$$\frac{d}{dt}((\Phi_H^t)^* g_a) = (\Phi_H^t)^* \{H, g_a\} = \sum_{b=1}^N ((\Phi_H^t)^* \lambda_a^b) ((\Phi_H^t)^* g_b). \quad (26)$$

Now, let $X_p(t) \in \mathbb{R}^N$ be a column vector with a -th component equal to $g_a \circ \Phi_H^t(p)$ for $a = 1, \dots, N$ and let $A_p(t)$ be the N by N matrix $(\lambda_a^b \circ \Phi_H^t(p))_{a,b=1}^N$. Equation (26) then transforms into the following family of non-autonomous linear ODEs

$$\frac{d}{dt} X_p(t) = A_p(t) X_p(t). \quad (27)$$

This equation and its initial conditions depend smoothly on $p \in U$. It is standard knowledge that solutions to (27) take the form:

$$X_p(t) = \Psi_p(t) X_p(0). \quad (28)$$

Here $\Psi_p(t) = (\psi_a^b(t, p))_{a,b=1}^N$ is the fundamental matrix of the ODE, satisfying $\Psi_p(0) = I_N$ and

$$\frac{d}{dt} \Psi_p(t) = A_p(t) \Psi_p(t).$$

$\Psi_p(t)$ is sometimes also called the (time-) ordered exponential of $A_p(t)$.

Since $A_p(t)$ and $X_p(0)$ depend smoothly on p , the components of the fundamental matrix, $\psi_a^b(t, p)$, depend smoothly on p as well. Now, Equation (28) can be written as

$$(\Phi_H^t)^* g_a(p) = \sum_{b=1}^N \psi_a^b(t, p) g_b(p),$$

which implies the inclusion $(\Phi_H^t)^* \mathcal{I}(\Phi_H^t(U)) \subset \mathcal{I}(U)$.

To prove equality, we first observe that the inclusion yields also $(\Phi_H^{-t})^* \mathcal{I}(U) \subset \mathcal{I}(\Phi_H^t(U))$ for every $t \in (-\epsilon, \epsilon)$. Thus, for every $f \in \mathcal{I}(U)$, one has $(\Phi_H^{-t})^* f \in \mathcal{I}(\Phi_H^t(U))$. But on the other hand, we have the obvious identity

$$f = (\Phi_H^t)^* ((\Phi_H^{-t})^* f),$$

and therefore $f \in (\Phi_H^t)^* \mathcal{I}(\Phi_H^t(U))$. ■

The following example shows that the condition of being locally finitely generated in the definition of \mathcal{I} -Poisson manifolds is crucial for Proposition 4.28 to hold true:

Example 4.29. Consider the Poisson manifold $M = T^*\mathbb{R} \cong \mathbb{R}^2$ with coordinates (q, p) and standard Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p}.$$

Let \mathcal{I} be the subsheaf of C^∞ vanishing on $\{q < 0\} \subset M$, which is not locally finitely generated around every point on the p -axis, but still closed under the Poisson bracket. Then the coordinate function p is an element of $N(\mathcal{I})$ since $X_p = \frac{\partial}{\partial q}$ preserves \mathcal{I} . But the Hamiltonian flow of X_p is given by $\Phi_{X_p}^t(q, p) = (q + t, p)$, which evidently does not preserve \mathcal{I} if $t > 0$.

4.4 The categories IPois and dynIPois

In order to define the category of \mathcal{I} -Poisson manifolds, we introduce a notion of morphisms and show that they can be composed:

Definition 4.30. Let $\varphi: (P_1, \{\cdot, \cdot\}_1, \mathcal{I}_1) \rightarrow (P_2, \{\cdot, \cdot\}_2, \mathcal{I}_2)$ be a smooth map between two \mathcal{I} -Poisson manifolds. We call it an \mathcal{I} -Poisson map if the following three conditions are satisfied:

$$\varphi^*(\mathcal{I}_2(P_2)) \subset \mathcal{I}_1(P_1), \quad (29)$$

$$\varphi^*N(\mathcal{I}_2) \subset N(\mathcal{I}_1), \quad (30)$$

$$\{\varphi^*f, \varphi^*g\}_1 - \varphi^*\{f, g\}_2 \in \mathcal{I}_1(P_1) \quad \forall f, g \in N(\mathcal{I}_2). \quad (31)$$

For dynamical \mathcal{I} -Poisson manifolds we add the condition $\varphi^*H_2 - H_1 \in \mathcal{I}_1$.

Proposition 4.31. The composition of two \mathcal{I} -Poisson maps is an \mathcal{I} -Poisson map.

Proof. Consider the following \mathcal{I} -Poisson maps:

$$\begin{aligned} \varphi: (P_1, \{\cdot, \cdot\}_1, \mathcal{I}_1) &\rightarrow (P_2, \{\cdot, \cdot\}_2, \mathcal{I}_2) \\ \psi: (P_2, \{\cdot, \cdot\}_2, \mathcal{I}_2) &\rightarrow (P_3, \{\cdot, \cdot\}_3, \mathcal{I}_3). \end{aligned}$$

Equations (29) and (30) for $\psi \circ \varphi$ follow directly from those equations for ψ and φ . It is thus enough to verify Equation (31) for the composition. For all $f, g \in N(\mathcal{I}_3)$ we have:

$$\begin{aligned} &\{f \circ \psi \circ \varphi, g \circ \psi \circ \varphi\}_1 - \{f, g\}_3 \circ \psi \circ \varphi \\ &= \{(f \circ \psi) \circ \varphi, (g \circ \psi) \circ \varphi\}_1 - \{f \circ \psi, g \circ \psi\}_2 \circ \varphi \\ &+ (\{f \circ \psi, g \circ \psi\}_2 - \{f, g\}_3 \circ \psi) \circ \varphi \\ &\in \mathcal{I}_1(P_1) + \varphi^*\mathcal{I}_2(P_2) \subset \mathcal{I}_1(P_1), \end{aligned}$$

where we used Equations (29) and (31) for φ and Equations (30) and (31) for ψ in the last line of the proof. A similar computation shows that morphisms of dynamical \mathcal{I} -Poisson manifolds can be composed as well. \blacksquare

Definition 4.32. The category **IPois** and **dynIPois** consist of \mathcal{I} -Poisson manifolds together with \mathcal{I} -Poisson maps and dynamical \mathcal{I} -Poisson manifolds together with dynamical \mathcal{I} -Poisson maps, respectively. By requiring the \mathcal{I} -Poisson bracket to be a Poisson bracket, we obtain a subcategory which we call (dynamical) semi-strict \mathcal{I} -Poisson manifolds **ssIPois** (**ssdynIPois**). Similarly, the category **sIPois** (**sdynIPois**) of strict (dynamical) \mathcal{I} -Poisson manifolds is defined by requiring that the \mathcal{I} -Poisson bracket is a Poisson bracket and that the morphisms are Poisson maps.

Remark 4.33. While $ssIPois$ is a full subcategory of $IPois$, $sIPois$ is not.

Remark 4.34. The three conditions in Definition 4.30 are the minimal conditions for the map φ^* to induce a morphism of Poisson algebras $\tilde{\varphi}: \mathcal{R}(\mathcal{I}_2) \rightarrow \mathcal{R}(\mathcal{I}_1)$. In particular, we obtain a functor F from $IPois^{op}$ to $PoisAlg$, the category of Poisson algebras. We call F the reduction functor.

Remark 4.35. Viewing Poisson manifolds $(P, \{\cdot, \cdot\})$ as \mathcal{I} -Poisson manifolds $(P, \{\cdot, \cdot\}, 0)$, \mathcal{I} -Poisson maps are precisely Poisson maps. This identifies the category of Poisson manifolds $Pois$ with a full subcategory of $sIPois$.

Remark 4.36. There is a functor from $IPois$ to C_3Alg , the category of coisotropic triples of algebras as introduced in [DEW19]. On the level of objects, one associates the triple $(C^\infty(P), N(\mathcal{I}), \mathcal{I}(P))$ to every \mathcal{I} -Poisson manifold $(P, \{\cdot, \cdot\}, \mathcal{I})$, while a morphism φ in our sense gives rise to a morphism φ^* in C_3Alg due to the first two defining conditions (29) and (30).

5 Singular (Riemannian) foliations through \mathcal{I} -Poisson manifolds

5.1 Vector fields as functions on the cotangent bundle

Let M be a smooth manifold. We denote by $C_k^\infty(T^*M) \subset C^\infty(T^*M)$ the algebra of homogeneous polynomials of degree k in the fiber coordinates of T^*M with coefficients in $C^\infty(M)$. Every vector field $X \in \mathfrak{X}(M)$ defines an element $\overline{X} \in C_1^\infty(T^*M)$ on the cotangent bundle of M , defined as

$$\overline{X}(q, p) = \langle p, X|_q \rangle$$

for every $(q, p) \in T^*M$, where $q \in M$ and $p \in T_q^*M$ and $\langle \cdot, \cdot \rangle$ denotes the canonical pairing. This construction can be naturally extended to the sections of $\Gamma(S^k(TM))$ to obtain elements in $C_k(T^*M)$.

Lemma 5.1. *Let $X, Y \in \mathfrak{X}(M)$ be two vector fields on M . Then*

$$\{\overline{X}, \overline{Y}\}_{T^*M} = \overline{[X, Y]}, \quad (32)$$

where $\{\cdot, \cdot\}_{T^*M}$ is the canonical Poisson bracket on T^*M and $[\cdot, \cdot]$ is the Lie bracket of vector fields on M .

Proof. Let (q^1, \dots, q^n) be a local coordinate system on M , and $(q^1, \dots, q^n, p_1, \dots, p_n)$ the corresponding canonical local coordinates on T^*M . In this coordinate system \overline{X} and \overline{Y} can be written as $\overline{X} = \sum_{i=1}^n X^i p_i$ and $\overline{Y} = \sum_{i=1}^n Y^i p_i$, where X^i, Y^i are the components of V, W in the above coordinate system. The following calculation proves the lemma:

$$\{\overline{X}, \overline{Y}\}_{T^*M} = \sum_{i=1}^n \left(\sum_{j=1}^n X^j \frac{\partial Y^i}{\partial q^j} - Y^j \frac{\partial X^i}{\partial q^j} \right) p_i = \sum_{i=1}^n [X, Y]^i p_i = \overline{[X, Y]}.$$

■

Lemma 5.1 and the Leibniz rule for the Lie derivative of tensor fields imply:

Corollary 5.2. *Let S be an element of $\Gamma(S^k(TM))$ for some $k \geq 0$, and \overline{S} be its corresponding element in $C_k^\infty(T^*M)$. Then for every vector field $X \in \mathfrak{X}(M)$ we have:*

$$\{\overline{X}, \overline{S}\}_{T^*M} = \overline{\mathcal{L}_X S}. \quad (33)$$

5.2 Singular foliations and \mathcal{I} -Poisson manifolds

Now let (M, \mathcal{F}) be a foliated manifold. Define a $C^\infty(T^*M)$ -submodule $\mathcal{J}_{\mathcal{F}} \subset C_c^\infty(T^*M)$ by

$$\mathcal{J}_{\mathcal{F}} := \langle \overline{X} : X \in \mathcal{F} \rangle_{C_c^\infty(T^*M)}.$$

Note that the generators of $\mathcal{J}_{\mathcal{F}}$ are not required to be compactly supported on T^*M .

Then we define the sub-presheaf $\mathcal{I}_{\mathcal{F}}$ of the sheaf of smooth functions on T^*M by

$$\mathcal{I}_{\mathcal{F}}(U) := \{f \in C^\infty(U) : \rho f \in \mathcal{J}_{\mathcal{F}} \quad \forall \rho \in C_c^\infty(U)\} \quad (34)$$

for every open subset $U \subset T^*M$.

Proposition 5.3. *The presheaf $\mathcal{I}_{\mathcal{F}}$ defined in Equation (34) is a subsheaf of the sheaf of smooth functions on T^*M .*

Proof. The locality of $\mathcal{I}_{\mathcal{F}}$ is evident, since $\mathcal{I}_{\mathcal{F}}$ is a sub-presheaf of the sheaf of smooth functions on T^*M . To verify the gluing property, let $\{U_i\}_{i=1}^{\infty}$ be an arbitrary open cover of T^*M and let $f \in C^{\infty}(T^*M)$ be such that $f|_{U_i} \in \mathcal{I}_{\mathcal{F}}(U_i)$ for every positive integer i . We prove that $f \in \mathcal{I}_{\mathcal{F}}(T^*M)$ as follows: it is enough to show that for every $\rho \in C_c^{\infty}(T^*M)$, we have $\rho f \in \mathcal{J}_{\mathcal{F}}$. Since $\text{supp}(\rho)$ is compact, it can be covered by finitely many open subsets U_{i_1}, \dots, U_{i_N} in $\{U_i\}_{i=1}^{\infty}$. Choose a partition of unity $\sigma_0, \sigma_{i_1}, \dots, \sigma_{i_N}$ subordinate to the open cover $\{U_0 := T^*M \setminus \text{supp}(\rho), U_{i_1}, \dots, U_{i_N}\}$ of T^*M and write

$$\rho f = \sum_{k=1}^N \rho \sigma_{i_k} f|_{U_{i_k}}.$$

The latter implies that $\rho f \in \mathcal{J}_{\mathcal{F}}$, since by definition of $\mathcal{I}_{\mathcal{F}}(U_{i_k})$, for each $k = 1, \dots, N$ we have $\rho \sigma_{i_k} f|_{U_{i_k}} \in \mathcal{J}_{\mathcal{F}}$. \blacksquare

We prove that the sheaf $\mathcal{I}_{\mathcal{F}}$ satisfies the properties of Definition 4.13, in the following lemmas:

Lemma 5.4. *For every open subset $U \subset T^*M$ we have*

$$\{\mathcal{I}_{\mathcal{F}}(U), \mathcal{I}_{\mathcal{F}}(U)\}_{T^*M} \subset \mathcal{I}_{\mathcal{F}}(U)$$

Proof. Let $f, g \in \mathcal{I}_{\mathcal{F}}(U)$. It is enough to show that for every $\rho \in C_c^{\infty}(U)$ we have $\rho\{f, g\} \subset \mathcal{J}_{\mathcal{F}}$. Choose a compactly supported function $\sigma \in C_c^{\infty}(U)$ such that $\sigma|_{\text{supp}(\rho)} \equiv 1$. One obtains

$$\rho\{f, g\}_{T^*M} = \{\sigma f, \rho g\}_{T^*M} - \{\sigma f, \rho\}_{T^*M} g - \rho f \{\sigma, g\}_{T^*M} \in \mathcal{J}_{\mathcal{F}},$$

since the first term belongs to $\mathcal{J}_{\mathcal{F}}$ by Lemma 5.1, the second term is inside $\mathcal{J}_{\mathcal{F}}$ by Definition of $\mathcal{I}_{\mathcal{F}}(U)$, and the last term vanishes identically. Consequently $\{f, g\}_{T^*M} \in \mathcal{I}_{\mathcal{F}}(U)$. \blacksquare

Lemma 5.5. *Let $U \subset M$ be an open subset such that $\iota_U^{-1}\mathcal{F} = \langle X_1, \dots, X_N \rangle_{C_c^{\infty}(U)}$ for finitely many vector fields $X_1, \dots, X_N \in \mathfrak{X}(U)$. Then*

$$\mathcal{I}_{\mathcal{F}}(V) = \langle \overline{X_1}|_V, \dots, \overline{X_N}|_V \rangle_{C^{\infty}(V)},$$

for every open subset $V \subset T^*U$.

Proof. We first prove that $\langle \overline{X_1}|_V, \dots, \overline{X_N}|_V \rangle_{C^{\infty}(V)} \subset \mathcal{I}_{\mathcal{F}}(V)$. Let $\sum_{a=1}^N \lambda^a \overline{X_a}|_V$ be an element of $\langle \overline{X_1}|_V, \dots, \overline{X_N}|_V \rangle_{C^{\infty}(V)}$ and take an arbitrary $\rho \in C_c^{\infty}(V)$. By choosing a compactly supported function $h \in C_c^{\infty}(U)$ such that $h|_{\text{supp}(\rho)} \equiv 1$ (when viewing h as an element of $C_0^{\infty}(T^*U)$), we have

$$\rho \sum_{a=1}^N \lambda^a \overline{X_a}|_V = \sum_{a=1}^N \rho \lambda^a h \overline{X_a} \in \mathcal{J}_{\mathcal{F}},$$

since $\rho \lambda^a \in C_c^{\infty}(V)$ and $h X_a \in \mathcal{F}$ for all $a = 1, \dots, N$. To prove equality, let $f \in \mathcal{I}_{\mathcal{F}}(V)$. Choose a partition of unity $\{\rho_i\}_{i=1}^{\infty}$ subordinate to a locally finite cover $\{V_i\}_{i=1}^{\infty}$ of V . Since

for every i we have $\rho_i f \in \mathcal{J}_{\mathcal{F}}$ and $V \subset T^*U$, there exist functions $\lambda_i^1, \dots, \lambda_i^N \in C_c^\infty(T^*U)$ such that

$$\rho_i f = \sum_{a=1}^N \lambda_i^a |_{V} \overline{X_a} |_{V}.$$

This implies

$$\begin{aligned} f &= \sum_{i=1}^{\infty} \rho_i f = \sum_{i=1}^{\infty} \sum_{a=1}^N \lambda_i^a |_{V} \overline{X_a} |_{V} \\ &= \sum_{a=1}^N \left(\sum_{i=1}^{\infty} \lambda_i^a |_{V} \right) \overline{X_a} |_{V}, \end{aligned}$$

which completes the proof. ■

Corollary 5.6. *Let (M, \mathcal{F}) be a foliated manifold. Then the triple $(T^*M, \{\cdot, \cdot\}_{T^*M}, \mathcal{I}_{\mathcal{F}})$ is an \mathcal{I} -Poisson manifold.*

5.3 Singular Riemannian foliations and dynamical \mathcal{I} -Poisson manifolds

For every Riemannian manifold (M, g) , its cotangent bundle T^*M carries a natural Hamiltonian function H_g :

$$H_g(q, p) = \frac{1}{2} \langle p, g_b^{-1}(p) \rangle$$

for every $(q, p) \in T^*M$, where $g_b: T_q M \rightarrow T_q^* M$ is the musical isomorphism $v \mapsto g(v, \cdot)$. In local Darboux coordinates this becomes $H_g(q^1, \dots, q^n, p_1, \dots, p_n) = \frac{1}{2} \sum_{i,j=1}^n g^{ij} p_i p_j$ where the matrix (g^{ij}) is the inverse to the matrix of the Riemannian metric (g_{ij}) in the coordinates (q^1, \dots, q^n) . Equivalently, we can define also H_g using the isomorphism $\Gamma(S^2(TM)) \cong C_2^\infty(T^*M)$, under which it becomes identified with $g^{-1} := \sum_{i,j=1}^n g^{ij} \partial_i \odot \partial_j$, i.e. $H_g = \frac{1}{2} \overline{g^{-1}}$.

The following fact about H_g is standard knowledge, which we still prove for completeness.

Proposition 5.7. *The Hamiltonian flow of H_g is the image of the geodesic flow under the musical isomorphism, i.e. for every geodesic $\gamma: (-\epsilon, \epsilon) \rightarrow M$ and every $t \in (-\epsilon, \epsilon)$, we have*

$$\Phi_{H_g}^t(\gamma(0), g(\dot{\gamma}(0), \cdot)) = (\gamma(t), g(\dot{\gamma}(t), \cdot)).$$

Proof. Assume that (q^1, \dots, q^n) is a normal coordinate system centered at $q \in M$, i.e. $g_{ij}(q) = \delta_{ij}$ and $\partial_k g_{ij}(q) = 0$ for $i, j, k = 1, \dots, n$. For every $p \in T_q^* M$ we have

$$X_{H_g}(q, p) = \sum_{i=1}^n p_i \frac{\partial}{\partial q^i} \Big|_q.$$

Let $\gamma: (-\epsilon, \epsilon) \rightarrow M$ be a geodesic passing through q at $t = 0$; in particular, $\ddot{q}^i(0) = 0$. Then $((\gamma(t), g(\dot{\gamma}(t), \cdot)))$ is a curve on T^*M passing through $(q, p) = ((\gamma(0), g(\dot{\gamma}(0), \cdot)))$ at

$t = 0$; in local coordinates, $((\gamma(t), g(\dot{\gamma}(t)), \cdot) = (q^1(t), \dots, q^n(t), p_1(t), \dots, p_n(t))$ where $p_i(t) = \sum_{j=1}^n g_{ij}(q(t))\dot{q}^j(t)$. Then, since $\dot{p}_i(0) = 0$, we have

$$\frac{d}{dt}\Big|_{t=0}((\gamma(t), g(\dot{\gamma}(t)), \cdot)) = \sum_{i=1}^n \dot{q}^i(0) \frac{\partial}{\partial q^i}\Big|_q.$$

On the other hand, $\dot{q}^i(0) = g(\dot{\gamma}(0), \frac{\partial}{\partial q^i}\Big|_q) = p_i(0)$, which indeed gives

$$X_{H_g}(q, p) = \frac{d}{dt}\Big|_{t=0}((\gamma(t), g(\dot{\gamma}(t)), \cdot)).$$

■

Lemma 5.8. *Let (M, \mathcal{F}) be a foliated manifold. We have*

$$N(\mathcal{I}_{\mathcal{F}}) = \{f \in C^\infty(T^*M) : \{f, \mathcal{J}_{\mathcal{F}}\}_{T^*M} \subset \mathcal{J}_{\mathcal{F}}\}.$$

Proof. The inclusion $N(\mathcal{I}_{\mathcal{F}}) \subset \{f \in C^\infty(T^*M) : \{f, \mathcal{J}_{\mathcal{F}}\}_{T^*M} \subset \mathcal{J}_{\mathcal{F}}\}$ is satisfied by Definition 4.13 and the fact that $\mathcal{J}_{\mathcal{F}}$ is equal to the set of compactly supported elements in $\mathcal{I}_{\mathcal{F}}(T^*M)$. Now let $f \in C^\infty(T^*M)$ be such that $\{f, \mathcal{J}_{\mathcal{F}}\}_{T^*M} \subset \mathcal{J}_{\mathcal{F}}$. Let $U \subset T^*M$ be an open subset and $g \in \mathcal{I}_{\mathcal{F}}(U)$. For every $\rho \in C_c^\infty(U)$ we have

$$\rho \{f|_U, g\}_{T^*M} = \{f, \rho g\}_{T^*M} - \{f, \rho\}_{T^*M} g \in \mathcal{J}_{\mathcal{F}},$$

since $\rho g \in \mathcal{J}_{\mathcal{F}}$ and $\{f, \rho\}_{T^*M}$ is compactly supported in U . The definition of $\mathcal{I}_{\mathcal{F}}(U)$ then implies that $\{f|_U, \mathcal{I}_{\mathcal{F}}(U)\}_{T^*M} \subset \mathcal{I}_{\mathcal{F}}(U)$. Since U is arbitrary, we obtain $f \in N(\mathcal{I}_{\mathcal{F}})$. ■

Now we can state an equivalent definition of module singular Riemannian foliations through \mathcal{I} -Poisson geometry.

Proposition 5.9. *A singular foliation \mathcal{F} on a Riemannian manifold (M, g) defines a module singular Riemannian foliation, if and only if*

$$H_g \in N(\mathcal{I}_{\mathcal{F}}). \tag{35}$$

Proof. Assume that (M, g, \mathcal{F}) is a module singular Riemannian foliation. By Lemma 3.3, for every $X \in \mathcal{F}$ we have

$$\mathcal{L}_X g^{-1} \in \mathfrak{X}(M) \odot \mathcal{F}.$$

Using the isomorphism $\Gamma(S^2(TM)) \cong C_2^\infty(T^*M)$ and Corollary 5.2, we obtain

$$\{\overline{X}, H_g\}_{T^*M} = \frac{1}{2} \overline{\mathcal{L}_X g^{-1}} \in \overline{\mathfrak{X}(M) \odot \mathcal{F}},$$

which together with the Leibniz rule imply $\{\mathcal{J}_{\mathcal{F}}, H_g\}_{T^*M} \subset \mathcal{J}_{\mathcal{F}}$. Lemma 5.8 then implies that $H_g \in N(\mathcal{I}_{\mathcal{F}})$. Conversely assume that $H_g \in N(\mathcal{I}_{\mathcal{F}})$. After Proposition 3.5 we can assume that $\mathcal{F} = \langle X_1, \dots, X_N \rangle_{C_c^\infty(M)}$. Using Lemma 5.5, $H_g \in N(\mathcal{I}_{\mathcal{F}})$ implies that for every $a = 1, \dots, N$ there exist functions $f_a^1, \dots, f_a^N \in C^\infty(T^*M)$ such that

$$\frac{1}{2} \overline{\mathcal{L}_{X_a} g^{-1}} = \{\overline{X_a}, H_g\} = \sum_{b=1}^N f_a^b \overline{X_b}, \tag{36}$$

where we used Corollary 5.2 for the first equality. Locally, for each a and b , consider the first-order Taylor approximation of $f_a^b(q, p)$ with respect to the fiber coordinates $\{p_i\}$ around $(q, 0)$

$$f_a^b(q, p) = f_a^b(q, 0) + \lambda_a^b(q, p) + o(\|p\|) \quad (37)$$

where $\lambda_a^b(q, p)$ is linear in fiber coordinates. Equations (36) and (37) then imply that

$$\{\overline{X}_a, H_g\} - \sum_{b=1}^N \lambda_a^b \overline{X}_b = \sum_{b=1}^N [f_a^b(q, 0) + o(\|p\|)] \overline{X}_b.$$

The left-hand side of the last equation is quadratic in fiber-coordinates, while the right-hand side is not. This implies that both sides are identically zero. Consequently

$$\{\overline{X}_a, H_g\} = \sum_{b=1}^N \lambda_a^b \overline{X}_b,$$

for some $\lambda_a^b \in C_1^\infty(T^*M)$. Lemma 3.4 then implies that $\mathcal{L}_X g^{-1} \in \mathfrak{X}(M) \odot \mathcal{F}$. \blacksquare

Now we are able to present the proof of Proposition 3.7.

Proof. [Proposition 3.7] Let (M, g, \mathcal{F}) be a module singular Riemannian foliation. As the statement is local, we can assume that \mathcal{F} is finitely generated, i.e. there exist vector fields $X_1, \dots, X_N \in \mathfrak{X}(M)$ for some positive integer N , such that $\mathcal{F} = \langle X_1, \dots, X_N \rangle_{C_c^\infty(M)}$. By Lemma 5.5, $\mathcal{I}_{\mathcal{F}}$ is generated by functions $\overline{X}_1, \dots, \overline{X}_N$. By Proposition 5.9, for every $a = 1, \dots, N$ there exist functions $\lambda_a^1, \dots, \lambda_a^N \in C_1^\infty(T^*M)$ such that

$$\{H_g, \overline{X}_a\} = \sum_{b=1}^N \lambda_a^b \overline{X}_b.$$

Assume that $\gamma: (-\epsilon, \epsilon) \rightarrow M$ is a geodesic such that $\dot{\gamma}(0) \perp F_{\gamma(0)}$, i.e. the geodesic is orthogonal to the leaf at $t = 0$. Then the ideal $\mathcal{I}_{\mathcal{F}}(T^*M)$ vanishes at $(q_0, p_0) = (\gamma(0), g(\dot{\gamma}(0), \cdot)) \in T^*M$. Since $\Phi_{H_g}^t(q_0)$ is defined for $-\epsilon < t < \epsilon$, for every $r < \epsilon$ there exists an open neighborhood $U \subset M$ of q_0 such that $\Phi_{H_g}^t$ is defined for $t \in (-r, r)$ on U . According to Proposition 5.7,

$$\begin{aligned} g(\dot{\gamma}(t), X_a(\gamma(t))) &= \overline{X}_a(\gamma(t), g_{\gamma(t)}(\dot{\gamma}(t), \cdot)) \\ &= \overline{X}_a \circ \Phi_{H_g}^t|_U(\gamma(0), g_{\gamma(0)}(\dot{\gamma}(0), \cdot)) \\ &= \Phi_{H_g}^t|_U^* \overline{X}_a(q_0, p_0) \end{aligned}$$

for $a = 1, \dots, N$. But now, according to Proposition 4.28, the function $(\Phi_{H_g}^t|_U)^* \left(\overline{X}_a|_{\Phi_{H_g}^t(U)} \right)$ is an element in $\mathcal{I}_{\mathcal{F}}(U)$ for $t \in (-r, r)$. This means that for t in this interval, $\dot{\gamma}(t) \perp F_{\gamma(t)}$. As $r < \epsilon$ is arbitrary, the proof is complete. \blacksquare

5.4 The functor Φ and reduction

At the end of Section 3 we introduced the category \mathbf{SRF}_0 and in Section 4 we introduced the category of \mathcal{I} -Poisson manifolds \mathbf{IPois} . In this section we will provide a functor from the first to the second category, by sending a module singular Riemannian foliation (M, g, \mathcal{F}) to the \mathcal{I} -Poisson manifold $(T^*M, \{\cdot, \cdot\}, \mathcal{I}_{\mathcal{F}})$ and every surjective

Riemannian submersion $\pi: (M_1, g_1) \rightarrow (M_2, g_2)$ to the bundle map $\varphi_\pi := (g_2)_b \circ d\pi \circ (g_1)_b^{-1}: (T^*M_1, \{\cdot, \cdot\}_1) \rightarrow (T^*M_2, \{\cdot, \cdot\}_2)$, see Theorem 5.18 below. φ_π is precisely the map making the following diagram commute:

$$\begin{array}{ccc} TM_1 & \xrightarrow{(g_1)_b} & T^*M_1 \\ d\pi \downarrow & & \downarrow \varphi_\pi \\ TM_2 & \xrightarrow{(g_2)_b} & T^*M_2 \end{array}$$

The map φ_π is not a Poisson map in general:⁷

Example 5.10. Let $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the projection $(x, y, z) \mapsto (x, y)$ in the canonical coordinates. Equipping \mathbb{R}^3 and \mathbb{R}^2 with the following metric tensors

$$\begin{aligned} g_3 &:= dx \otimes dx + (1 + x^2)dy \otimes dy - xdy \otimes dz - xdz \otimes dy + dz \otimes dz \quad \text{and} \\ g_2 &:= d\underline{x} \otimes d\underline{x} + d\underline{y} \otimes d\underline{y}, \end{aligned}$$

respectively, turns π into a Riemannian submersion. Here $(\underline{x}, \underline{y})$ denote the coordinates on \mathbb{R}^2 . In the induced coordinates $(\underline{x}, \underline{y}, p_{\underline{x}}, p_{\underline{y}})$ and (x, y, z, p_x, p_y, p_z) on $T^*\mathbb{R}^2$ and $T^*\mathbb{R}^3$, respectively, the map φ_π is given by

$$\varphi_\pi(x, y, z, p_x, p_y, p_z) = (x, y, p_x, p_y + xp_z).$$

This is not a Poisson map, since $\{\varphi_\pi^*p_{\underline{x}}, \varphi_\pi^*p_{\underline{y}}\} = \{p_x, p_y + xp_z\} = p_z \neq 0$.

In the last example the obstruction for φ_π to be a Poisson map is that the horizontal distribution of the Riemannian submersion π , which is generated by vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, is not integrable; the corresponding connection has curvature.

The map φ_π still preserves the Poisson bracket up to some ideal of functions $\mathcal{I}_{\ker d\pi}$:

Definition 5.11. Let $\pi: M_1 \rightarrow M_2$ be a submersion. The subsheaf of smooth functions $\mathcal{I}_{\ker d\pi}$ on T^*M_1 is defined as the corresponding sheaf $\mathcal{I}_{\mathcal{F}}$ for the regular foliation $\mathcal{F} := \Gamma_c(\ker d\pi)$.

When there is no ambiguity, for simplicity, we denote the ideal $\mathcal{I}_{\ker d\pi}(T^*M_1)$ by $\mathcal{I}_{\ker d\pi}$.

Remark 5.12. It is not difficult to see that for every open subset $U \subset T^*M$, The ideal $\mathcal{I}_{\ker d\pi}(U)$ is the vanishing ideal of the submanifold $\text{Ann}(\ker d\pi) \cap U \subset U$. Here $\text{Ann}(\ker d\pi)$ stands for the annihilator of the subbundle $\ker d\pi \subset TM_1$. Moreover, since $\text{Ann}(\ker d\pi)$ is an embedded submanifold, we have:

$$\mathcal{C} := \{(q, p) \in T^*M_1: f(q, p) = 0 \quad \forall f \in \mathcal{I}_{\ker d\pi}\} \equiv \text{Ann}(\ker d\pi). \quad (38)$$

Lemma 5.13. Let $\pi: (M_1, g_1) \rightarrow (M_2, g_2)$ be a Riemannian submersion. Then for every $f, g \in C^\infty(T^*M_2)$:

$$\{f \circ \varphi_\pi, g \circ \varphi_\pi\}_1 - \{f, g\}_2 \circ \varphi_\pi \in \mathcal{I}_{\ker d\pi} \quad (39)$$

$$\{f \circ \varphi_\pi, \mathcal{I}_{\ker d\pi}\}_1 \subset \mathcal{I}_{\ker d\pi}. \quad (40)$$

7. In contrast to what is claimed in [BWY21].

Proof. Choose local Darboux coordinates (q_2^i, p_i^2) on T^*M_2 and $(q_1^i, q_1^\alpha, p_i^1, p_\alpha^1)$ on T^*M_1 , such that $q_2^i \circ \pi = q_1^i$ (this is possible since π is assumed to be a submersion). In particular, $\mathcal{I}_{\ker d\pi}$ is generated by the momenta p_α^1 . Now note that at every point $q \in M_1$,

$$d_q\pi\left(\frac{\partial}{\partial q_1^i}\Big|_q\right) = \frac{\partial}{\partial q_2^i}\Big|_{\pi(q)},$$

since for every function $f \in C^\infty(M_2)$

$$d_q\pi\left(\frac{\partial}{\partial q_1^i}\Big|_q\right) \cdot f = \frac{\partial(f \circ \pi)}{\partial q_1^i}(q) = \frac{\partial f}{\partial q_2^i}(\pi(q)) \frac{\partial(q_2^i \circ \pi)}{\partial q_1^i}(q) = \frac{\partial f}{\partial q_2^i}(\pi(q)).$$

In particular, since φ_π is a bundle map, we have

$$q_1^i = q_2^i \circ \varphi_\pi. \quad (41)$$

Next we prove that upon restriction to the vanishing submanifold \mathcal{C} of $\mathcal{I}_{\ker d\pi}$,

$$\mathcal{C} = \{(q, p) \in T^*M_1 : f(q, p) = 0 \quad \forall f \in \mathcal{I}_{\ker d\pi}\} \equiv \text{Ann}(\ker d\pi), \quad (42)$$

one has $p_i^1 = p_i^2 \circ \varphi_\pi$. Indeed, let (q, p) be a point in T^*M_1 and $X = (g_1)_b^{-1}(p)$. Then

$$p_i^1(q, p) = p\left(\frac{\partial}{\partial q_1^i}\Big|_q\right) = g_1(X, \frac{\partial}{\partial q_1^i}\Big|_q) = g_1(X, (\frac{\partial}{\partial q_1^i}\Big|_q)^H) + g_1(X, (\frac{\partial}{\partial q_1^i}\Big|_q)^V)$$

where $(\frac{\partial}{\partial q_1^i}\Big|_q)^H$ and $(\frac{\partial}{\partial q_1^i}\Big|_q)^V$ are the horizontal and vertical parts of the vector $\frac{\partial}{\partial q_1^i}\Big|_q$ with respect to g_1 , respectively. Using that π is a Riemannian submersion and that there exist functions A_α such that $(\frac{\partial}{\partial q_1^i}\Big|_q)^V = \sum_\alpha A_\alpha(q) \frac{\partial}{\partial q_1^\alpha}\Big|_q$, this implies:

$$p_i^1(q, p) = g_2(d_q\pi(X), \frac{\partial}{\partial q_2^i}\Big|_{\pi(q)}) + \sum_\alpha g_1(X, A_\alpha(q) \frac{\partial}{\partial q_1^\alpha}\Big|_q).$$

Consequently, by definition of φ_π ,

$$p_i^1(q, p) = p_i^2 \circ \varphi_\pi(q, p) + \sum_\alpha A_\alpha(q) p_\alpha^1(q, p)$$

and thus

$$p_i^1|_{\mathcal{C}} = (p_i^2 \circ \varphi_\pi)|_{\mathcal{C}}. \quad (43)$$

Now for every $f \in C^\infty(T^*M_2)$, we have:

$$\frac{\partial(f \circ \varphi_\pi)}{\partial q_1^i}(q, p) = \frac{\partial f}{\partial q_2^i}(\varphi_\pi(q, p)) \frac{\partial(q_2^i \circ \varphi_\pi)}{\partial q_1^i}(q, p) + \frac{\partial f}{\partial p_j^2}(\varphi_\pi(q, p)) \frac{\partial(p_j^2 \circ \varphi_\pi)}{\partial q_1^i}(q, p). \quad (44)$$

Since $\frac{\partial}{\partial q_1^i}\Big|_{\mathcal{C}}$ is tangent to \mathcal{C} , for every point $(q, p) \in \mathcal{C}$, we may use Equation (43) to transform Equation (44) into:

$$\begin{aligned} \frac{\partial(f \circ \varphi_\pi)}{\partial q_1^i}(q, p) &= \frac{\partial f}{\partial q_2^i}(\varphi_\pi(q, p)) \frac{\partial q_1^i}{\partial q_1^i}(q, p) + \frac{\partial f}{\partial p_j^2}(\varphi_\pi(q, p)) \frac{\partial p_j^1}{\partial q_1^i}(q, p) \\ &= \frac{\partial f}{\partial q_2^i}(\varphi_\pi(q, p)). \end{aligned} \quad (45)$$

In a similar way, using the chain rule and that $\frac{\partial}{\partial p_i^1}\Big|_{\mathcal{C}}$ and $\frac{\partial}{\partial q_1^\alpha}\Big|_{\mathcal{C}}$ are tangent to \mathcal{C} , for every function $f \in C^\infty(T^*M_2)$ and every $(q, p) \in \mathcal{C}$, one finds

$$\frac{\partial(f \circ \varphi_\pi)}{\partial p_i^1}(q, p) = \frac{\partial f}{\partial p_i^2}(\varphi_\pi(q, p)), \quad (46)$$

$$\frac{\partial(f \circ \varphi_\pi)}{\partial q_1^\alpha}(q, p) = 0. \quad (47)$$

For every two functions $f, g \in C^\infty(T^*M_2)$, upon restriction to $|_{\mathcal{C}}$ we have:

$$\begin{aligned}
\{f \circ \varphi_\pi, g \circ \varphi_\pi\}_1 |_{\mathcal{C}} &= \sum_i \left(\frac{\partial(f \circ \varphi_\pi)}{\partial p_i^1} \frac{\partial(g \circ \varphi_\pi)}{\partial q_1^i} - \frac{\partial(g \circ \varphi_\pi)}{\partial p_i^1} \frac{\partial(f \circ \varphi_\pi)}{\partial q_1^i} \right) |_{\mathcal{C}} \\
&+ \sum_\alpha \left(\frac{\partial(f \circ \varphi_\pi)}{\partial p_\alpha^1} \frac{\partial(g \circ \varphi_\pi)}{\partial q_1^\alpha} - \frac{\partial(g \circ \varphi_\pi)}{\partial p_\alpha^1} \frac{\partial(f \circ \varphi_\pi)}{\partial q_1^\alpha} \right) |_{\mathcal{C}} \\
&= \sum_i \left(\frac{\partial(f \circ \varphi_\pi)}{\partial p_i^1} \frac{\partial(g \circ \varphi_\pi)}{\partial q_1^i} - \frac{\partial(g \circ \varphi_\pi)}{\partial p_i^1} \frac{\partial(f \circ \varphi_\pi)}{\partial q_1^i} \right) |_{\mathcal{C}} \\
&= \sum_i \left(\frac{\partial f}{\partial p_i^2} \circ \varphi_\pi \right) |_{\mathcal{C}} \left(\frac{\partial g}{\partial q_2^i} \circ \varphi_\pi \right) |_{\mathcal{C}} - \left(\frac{\partial g}{\partial p_i^2} \circ \varphi_\pi \right) |_{\mathcal{C}} \left(\frac{\partial f}{\partial q_2^i} \circ \varphi_\pi \right) |_{\mathcal{C}} \\
&= \{f, g\}_2 \circ \varphi_\pi |_{\mathcal{C}}
\end{aligned}$$

Here in the first equality we used just the definition of the Poisson bracket, in the second one we used Equation (47), thereafter Equations (45) and (46), and finally again the definition of the bracket. Note that every function on T^*M_1 vanishing on \mathcal{C} is an element of $\mathcal{I}_{\ker d\pi}$, which proves Equation (39).

Equation (47) implies Equation (40) as well, since $\mathcal{I}_{\ker d\pi}$ is locally generated by coordinate functions p_α for $\alpha = 1, \dots, k$, and we have

$$\{p_\alpha, f \circ \varphi_\pi\}_1 |_{\mathcal{C}} = \frac{\partial(f \circ \varphi_\pi)}{\partial q_1^\alpha} |_{\mathcal{C}} = 0.$$

which gives $\{f \circ \varphi_\pi, p_\alpha\}_1 \in \mathcal{I}_{\ker d\pi}$. ■

Corollary 5.14. *The restriction $\varphi_\pi|_{\mathcal{C}}: \mathcal{C} \rightarrow T^*M_2$ is a surjective submersion. It coincides with the projection to the leaf space for the coisotropic reduction of $\mathcal{C} \subset T^*M_1$.*

Proof. Choosing the same local coordinates as in the proof of Lemma 5.13, $(q_1^i, q_1^\alpha, p_i^1)$ give local coordinates for \mathcal{C} and Equations (41) and (43) ensure that $\varphi_\pi(q_1^i, q_1^\alpha, p_i^1) = (q_1^i, p_i^1)$. ■

To study the obstruction for φ_π to be a Poisson map, we first prove the following lemma which describes the horizontal distribution in terms of the map φ_π :

Lemma 5.15. *Let $\pi: (N, h) \rightarrow (M, g)$ be a Riemannian submersion and let X be a vector field on M . Then the horizontal lift of X is given by a vector field V on N satisfying*

$$\bar{V} = (\varphi_\pi)^* \bar{X}, \tag{48}$$

which is an element in $C_1^\infty(T^*N)$.

Proof. Define $V \in \mathfrak{X}(N)$ by $\bar{V} := (\varphi_\pi)^*(\bar{X}) \in C_1^\infty(T^*M)$. Using Corollary 5.2 and Lemma 5.13 for every function $f \in C^\infty(M)$, we have

$$\begin{aligned}
V \cdot \pi^* f &= \{\bar{V}, \overline{\pi^* f}\} = \{(\varphi_\pi)^* \bar{X}, \pi^* f\} \\
&= \{(\varphi_\pi)^* \bar{X}, (\varphi_\pi)^* f\} = (\varphi_\pi)^* \{\bar{X}, f\} \\
&= \pi^*(X \cdot f),
\end{aligned}$$

which means that V is projectable and projects to X . In addition, for every vertical vector $v \in \ker d_q \pi$ and $i = 1, \dots, n$ we have

$$\begin{aligned}
h(v, V|_q) &= \bar{V}(h_b(v)) = (\varphi_\pi)^* \bar{X}(h_b(v)) \\
&= \bar{X}(\varphi_\pi \circ h_b(v)) = \bar{X}(g_b \circ d_q \pi(v)) \\
&= 0,
\end{aligned}$$

showing that V is the horizontal lift of X . ■

The following identifies the obstruction for φ_π to be a Poisson map:

Proposition 5.16. *Let $\pi: (N, h) \rightarrow (M, g)$ be a Riemannian submersion. Then the map $\varphi_\pi := g_b \circ d\pi \circ h_b^{-1}$ is a Poisson map if and only if the horizontal distribution $\mathcal{H} \subset TM$ of π is integrable.*

Proof. Let $\{f_i\}_{i=1}^n$ be a local orthonormal frame around a point $q \in M$ and $\{e_i\}_{i=1}^n$ their horizontal lifts. By Lemma 5.15 we have $\bar{e}_i = (\varphi_\pi)^* \bar{f}_i$ for $i = 1, \dots, n$. If φ_π is a Poisson map, the family of functions $(\varphi_\pi)^*(\bar{f}_i) \in C_1^\infty(T^*N)$ is closed under the Poisson bracket, and consequently the horizontal distribution locally generated by vector fields e_i is integrable. This proves the if part of the proposition.

Conversely assume that \mathcal{H} is integrable. It is enough to check the condition of being a Poisson map on smooth functions in $C_0^\infty(T^*M) \oplus C_1^\infty(T^*M)$ only. First, for every $f, g \in C_0^\infty(T^*M)$ we have $\{f \circ \varphi_\pi, g \circ \varphi_\pi\}_{T^*N} = \{f, g\}_{T^*M} = 0$. Second, for every $\bar{X} \in C_1^\infty(T^*M)$ and $f \in C_0^\infty(T^*M)$ we have

$$\begin{aligned} \{\bar{X} \circ \varphi_\pi, f \circ \varphi_\pi\}_{T^*N} &= X^H \cdot (f \circ \varphi_\pi) \\ &= (X \cdot f) \circ \varphi_\pi \\ &= \{\bar{X}, f\}_{T^*M} \circ \varphi_\pi. \end{aligned}$$

Finally, by Lemma 5.15 and integrability of \mathcal{H} , for every $\bar{X}, \bar{Y} \in C_1^\infty(T^*M)$ one obtains

$$\begin{aligned} \{\bar{X} \circ \varphi_\pi, \bar{Y} \circ \varphi_\pi\}_{T^*N} &= \overline{[X^H, Y^H]} \\ &= \overline{[X, Y]^H} \\ &= \overline{[X, Y]} \circ \varphi_\pi \\ &= \{\bar{X}, \bar{Y}\}_{T^*M} \circ \varphi_\pi. \end{aligned}$$

■

Lemma 5.17. *Let $\pi: (N, h) \rightarrow (M, g)$ be a Riemannian submersion. Then*

$$H_h - H_g \circ \varphi_\pi \in \mathcal{I}_{\ker d\pi}.$$

Proof. It is enough to show that the left-hand side vanishes on \mathcal{C} , defined in Equation (38). For every $(q, p) \in \mathcal{C}$, we have

$$\begin{aligned} H_h(q, p) &= \frac{1}{2} \langle p, (h_b)^{-1}(p) \rangle = \frac{1}{2} \langle p, [(h_b)^{-1}(p)]^H \rangle \\ &= \frac{1}{2} h((h_b)^{-1}(p), [(h_b)^{-1}(p)]^H) \\ &= \frac{1}{2} h([(h_b)^{-1}(p)]^H, [(h_b)^{-1}(p)]^H) \\ &= \frac{1}{2} g(d_q \pi \circ (h_b)^{-1}(p), d_q \pi \circ (h_b)^{-1}(p)) \\ &= \frac{1}{2} \langle \varphi_\pi(p), (g_b)^{-1}(\varphi_\pi(p)) \rangle \\ &= H_g \circ \varphi_\pi(q, p). \end{aligned}$$

■

Now we are able to prove the well-behavedness of module singular Riemannian foliations under Riemannian submersions.

Proof. [**Proposition 3.11**] Let (M, g, \mathcal{F}) be a module singular Riemannian foliation and $\pi: (N, h) \rightarrow (M, g)$ a Riemannian submersion. By Lemma 3.10

$$\mathcal{F}_N = \langle \mathcal{F}_M^{\mathcal{H}} + \Gamma(\ker d\pi) \rangle_{C^\infty(N)} \quad (49)$$

where $\mathcal{F}_M^{\mathcal{H}}$ consists of horizontal lifts of vector fields in \mathcal{F}_M . By Lemma 5.15 we have

$$\mathcal{J}_{\mathcal{F}_N} = \langle (\varphi_\pi)^* \overline{\mathcal{F}_M} + \mathcal{I}_{\ker d\pi} \rangle_{C^\infty(T^*N)},$$

where $\overline{\mathcal{F}_M} := \{\overline{X} : X \in \mathcal{F}_M\}$. Finally, it remains to check $H_g \in N(\mathcal{I}_{\pi^{-1}\mathcal{F}})$. By Lemma 5.8 it is enough to verify the following:

$$\begin{aligned} \{H_g, (\varphi_\pi)^* \overline{\mathcal{F}_M} + \mathcal{I}_{\ker d\pi}\} &= \{H_g - H_h \circ \varphi_\pi, (\varphi_\pi)^* \overline{\mathcal{F}_M}\} + \{H_h \circ \varphi_\pi, (\varphi_\pi)^* \overline{\mathcal{F}_M}\} \\ &+ \{H_g - H_h \circ \varphi_\pi, \mathcal{I}_{\ker d\pi}\} + \{H_h \circ \varphi_\pi, \mathcal{I}_{\ker d\pi}\} \\ &\subset (\varphi_\pi)^* \overline{\mathcal{F}_M} + \mathcal{I}_{\ker d\pi}. \end{aligned}$$

Here we used Lemmas 5.13 and 5.17 to prove the inclusion. ■

The following theorem is the main result of this section:

Theorem 5.18. *The map sending every module singular Riemannian foliation (M, g, \mathcal{F}) to the corresponding dynamical \mathcal{I} -Poisson manifold $(T^*M, \{\cdot, \cdot\}_{T^*M}, \mathcal{I}_{\mathcal{F}}, H_g)$ and every morphism π of singular Riemannian foliations within \mathbf{SRF}_0 to the map φ_π defines a functor $\Phi: \mathbf{SRF}_0 \rightarrow \mathbf{dynIPois}$.*

Proof. It is enough to show that Φ preserves the morphisms. A morphism π within \mathbf{SRF}_0 is a Riemannian submersion $\pi: (N, h, \mathcal{F}_N) \rightarrow (M, g, \mathcal{F}_M)$ such that $\mathcal{F}_N = \pi^{-1}(\mathcal{F}_M)$. Similar to the previous proof we have

$$\mathcal{J}_{\mathcal{F}_N} = \langle (\varphi_\pi)^* \overline{\mathcal{F}_M} + \mathcal{I}_{\ker d\pi} \rangle_{C^\infty(T^*N)}$$

and therefore the pullback $(\varphi_\pi)^* \mathcal{I}_{\mathcal{F}_M}(T^*M)$ and $\mathcal{I}_{\ker d\pi}$ lie inside $\mathcal{I}_{\mathcal{F}_N}(T^*N)$. By Lemma 5.13, for every $f \in N(\mathcal{I}_{\mathcal{F}_M})$ we have

$$\begin{aligned} \{f \circ \varphi_\pi, (\varphi_\pi)^* \overline{\mathcal{F}_M} + \mathcal{I}_{\ker d\pi}\}_{T^*N} &\subset \{f, \overline{\mathcal{F}_M}\}_{T^*M} \circ \varphi_\pi + \{f \circ \varphi_\pi, \mathcal{I}_{\ker d\pi}\}_{T^*N} + \mathcal{I}_{\ker d\pi} \\ &\subset (\varphi_\pi)^* \overline{\mathcal{F}_M} + \mathcal{I}_{\ker d\pi} \end{aligned}$$

which implies that $\{f \circ \varphi_\pi, \mathcal{J}_{\mathcal{F}_N}\}_{T^*N} \subset \mathcal{J}_{\mathcal{F}_N}$, and consequently $(\varphi_\pi)^* N(\mathcal{I}_{\mathcal{F}_M})$ lies inside $N(\mathcal{I}_{\mathcal{F}_N})$. Using Lemma 5.13 again, for every $f, g \in N(\mathcal{I}_{\mathcal{F}_M})$

$$\{f \circ \varphi_\pi, g \circ \varphi_\pi\}_{T^*N} - \{f, g\}_{T^*M} \circ \varphi_\pi \in \mathcal{I}_{\ker d\pi} \subset \mathcal{I}_{\mathcal{F}_N}(T^*N).$$

These together with Lemma 5.17 complete the proof. ■

Theorem 5.19. *Let (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) be Hausdorff Morita equivalent singular foliations. Then the Poisson algebras $\mathcal{R}(\mathcal{I}_{\mathcal{F}_1})$ and $\mathcal{R}(\mathcal{I}_{\mathcal{F}_2})$ are isomorphic.*

Here $\mathcal{R}(\mathcal{I}_{\mathcal{F}_i}) \equiv N(\mathcal{I}_{\mathcal{F}_i})/\mathcal{I}_{\mathcal{F}_i}(T^*M_i)$, $i = 1, 2$, see Definition 4.21.

The proof of this theorem will be a consequence of the following two lemmas.

Lemma 5.20. *Let $\pi: (N, h) \rightarrow (M, g)$ be a surjective Riemannian submersion with connected fibers and \mathcal{F} be a singular foliation on M . If $f \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$ for some $f \in C^\infty(T^*M)$, then $f \in \mathcal{I}_{\mathcal{F}}(T^*M)$.*

Proof. We first demonstrate that the result holds true for finitely generated singular foliations. Let $\mathcal{F} = \langle X_1, \dots, X_N \rangle_{C_c^\infty(M)}$ and let $Y_1, \dots, Y_K \in \mathfrak{X}(N)$ be generators of the regular foliation $\Gamma_c(\ker d\pi)$ for some positive integers N and K . Lemma 3.10 implies that $\pi^{-1}\mathcal{F} = \langle X_1^H, \dots, X_N^H, Y_1, \dots, Y_K \rangle_{C_c^\infty(N)}$. Consequently, for every open subset $V \subset T^*N$, we obtain

$$\mathcal{I}_{\pi^{-1}\mathcal{F}}(V) = \langle \overline{X_1} \circ \varphi_\pi|_V, \dots, \overline{X_N} \circ \varphi_\pi|_V, \overline{Y_1}|_V, \dots, \overline{Y_K}|_V \rangle_{C^\infty(V)}, \quad (50)$$

where we used Lemmas 5.15 and 5.5.

Let us assume for a moment that there exists a global section $s: T^*M \rightarrow \mathcal{C}$ for the surjective submersion $\varphi_\pi|_{\mathcal{C}}: \mathcal{C} \rightarrow T^*M$ (see Corollary 5.14). Since $f \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$, Equation (50) implies that there exist smooth functions $\lambda^1, \dots, \lambda^N, \eta^1, \dots, \eta^K \in C^\infty(T^*N)$ such that

$$f \circ \varphi_\pi = \sum_{a=1}^N \lambda^a \cdot (\overline{X_a} \circ \varphi_\pi) + \sum_{b=1}^K \eta^b \cdot \overline{Y_b}. \quad (51)$$

Since $\varphi_\pi \circ s \circ \varphi_\pi = \varphi_\pi$ and $\overline{Y_b}|_{\mathcal{C}} = 0$, composing both sides of Equation (51) by $s \circ \varphi_\pi$ gives

$$\begin{aligned} f \circ \varphi_\pi &= f \circ \varphi_\pi \circ s \circ \varphi_\pi = \sum_{a=1}^N (\lambda^a \circ s \circ \varphi_\pi) \cdot (\overline{X_a} \circ \varphi_\pi \circ s \circ \varphi_\pi) \\ &= \left(\sum_{a=1}^N (\lambda^a \circ s) \cdot \overline{X_a} \right) \circ \varphi_\pi. \end{aligned}$$

This implies that $f = \sum_{a=1}^N (\lambda^a \circ s) \cdot \overline{X_a} \in \mathcal{I}_{\mathcal{F}}(T^*M)$, since φ_π is surjective.

If a global section does not exist, we can choose an open covering $\{U_i\}_{i=1}^\infty$ of T^*M such that for every positive integer i there exists a local section $s_i: U_i \rightarrow \mathcal{C}$. Using the same argument as for the global case, we may show that $f|_{U_i} \in \mathcal{I}_{\mathcal{F}}(U_i)$ for each i . Since $\mathcal{I}_{\mathcal{F}}$ is a sheaf on T^*M , we have $f \in \mathcal{I}_{\mathcal{F}}(T^*M)$.

For the general case, choose an open covering $\{U_i\}_{i=1}^\infty$ of M such that for every positive integer i the pullback $\iota_{U_i}^{-1}\mathcal{F}$ is finitely generated. The finitely generated case discussed before then implies that $f|_{T^*U_i} \in \mathcal{I}_{\mathcal{F}}(T^*U_i)$ for every i , and since $\mathcal{I}_{\mathcal{F}}$ is a sheaf, we obtain $f \in \mathcal{I}_{\mathcal{F}}(T^*M)$. ■

Lemma 5.21. *Let $\pi: (N, h) \rightarrow (M, g)$ be a surjective Riemannian submersion with connected fibers and \mathcal{F} be a finitely generated singular foliation on M . Then for every $F \in N(\mathcal{I}_{\pi^{-1}\mathcal{F}})$, there exists some $f \in C^\infty(T^*M)$ such that $F - f \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$.*

Proof. We proceed as in the beginning of the proof of Lemma 5.20, establishing Equation (50) and assuming first again that there is a global section $s: T^*M \rightarrow \mathcal{C} = \text{Ann}(\ker d\pi)$ for the surjection $\varphi_\pi|_{\mathcal{C}}: \mathcal{C} \rightarrow T^*M$. Define, in addition, $f := F \circ s \in C^\infty(T^*M)$.

We now will prove that for every $x \in T^*M$, there exists an open neighborhood $V_x \subset T^*N$ such that $(F - f \circ \varphi_\pi)|_{V_x} \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(V_x)$. Since $\mathcal{I}_{\pi^{-1}\mathcal{F}}$ is a sheaf, this implies the desired $F - f \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$. The proof is divided into the following three cases:

Case 1. $[x \notin \mathcal{C}]$: We choose an open subset $V_x \subset T^*N$ such that $\overline{V_x} \cap \mathcal{C} = \emptyset$. Let $\rho \in C^\infty(T^*N)$ with $\text{supp}(\rho) \subset T^*N \setminus \mathcal{C}$ and $\rho|_{\overline{V_x}} \equiv 1$. Then since $\rho \cdot (F - f \circ \varphi_\pi)$ vanishes on \mathcal{C} , we obtain

$$(F - f \circ \varphi_\pi)|_{V_x} = \rho \cdot (F - f \circ \varphi_\pi)|_{V_x} \in \mathcal{I}_{\ker d\pi}(V_x) \subset \mathcal{I}_{\pi^{-1}\mathcal{F}}(V_x).$$

Case 2. $[x \in s(T^*M) \subset \mathcal{C}]$: Choose local coordinates (q^i, q^α) centered at the base-point of x and (q_M^i) on N and M , respectively, where $i \in \{1, \dots, m := \dim M\}$ and $\alpha \in \{m+1, \dots, n := \dim N\}$, which are compatible with the submersion π , i.e. $\pi(q^i, q^\alpha) = (q^i)$. Let $(q^i, q^\alpha, p_i, p_\alpha)$ be the corresponding Darboux coordinates on some open neighborhood $V_x \subset T^*N$ centered at x . As a consequence, in particular $\varphi_\pi(q^i, q^\alpha, p_i, 0) = (q^i, p_i)$ (see Corollary 5.14) and $\mathcal{I}_{\ker d\pi}(V_x) = \langle p_\alpha \rangle_{C^\infty(V_x)}$. For simplicity also assume that, in these local coordinates, $s \circ \varphi_\pi(q^i, q^\alpha, p_i, 0) = (q^i, 0, p_i, 0)$. Then, for every arbitrary point $(q_0^i, q_0^\alpha, p_i^0, 0)$ in $V_x \cap \mathcal{C}$, we have

$$\begin{aligned} (F - f \circ \varphi_\pi)(q_0^i, q_0^\alpha, p_i^0, 0) &= F(q_0^i, q_0^\alpha, p_i^0, 0) - F(q_0^i, 0, p_i^0, 0) \\ &= \int_0^1 \frac{d}{dt} F(q_0^i, tq_0^\alpha, p_i^0, 0) dt \\ &= \int_0^1 \left(\sum_\beta \left\{ q_0^\beta p_\beta, F \right\}_{T^*N} \right) (q_0^i, tq_0^\alpha, p_i^0, 0) dt \\ &= \int_0^1 \left(\sum_\beta q_0^\beta \{p_\beta, F\}_{T^*N} \right) (q_0^i, tq_0^\alpha, p_i^0, 0) dt. \end{aligned} \quad (52)$$

Since $F \in N(\mathcal{I}_{\pi^{-1}\mathcal{F}})$, for every β there exist smooth functions $\lambda_\beta^1, \dots, \lambda_\beta^N, \eta_\beta^1, \dots, \eta_\beta^K \in C^\infty(V_x)$ such that $\{p_\beta, F\}_{T^*N} = \sum_a \lambda_\beta^a \cdot (\overline{X_a} \circ \varphi_\pi|_{V_x}) + \sum_b \eta_\beta^b \cdot (\overline{Y_b}|_{V_x})$. Implementing this into Equation (52) gives

$$\begin{aligned} &(F - f \circ \varphi_\pi)(q_0^i, q_0^\alpha, p_i^0, 0) \\ &= \int_0^1 \left(\sum_\beta q_0^\beta \left(\sum_a \lambda_\beta^a \cdot (\overline{X_a} \circ \varphi_\pi|_{V_x}) \right) \right) (q_0^i, tq_0^\alpha, p_i^0, 0) dt \\ &= \sum_a \left(\Lambda^a \cdot (\overline{X_a} \circ \varphi_\pi|_{V_x}) \right) (q_0^i, q_0^\alpha, p_i^0, 0) \end{aligned} \quad (53)$$

where $\Lambda^a \in C^\infty(V_x)$ is defined as

$$\Lambda^a(q^i, q^\alpha, p_i, p_\alpha) := \int_0^1 \left(\sum_\beta q^\beta \lambda_\beta^a \right) (q^i, tq^\alpha, p_i, p_\alpha) dt.$$

Equation (53) implies that $(F - f \circ \varphi_\pi) - \sum_a \Lambda^a \cdot (\overline{X_a} \circ \varphi_\pi|_{V_x})$ vanishes on $V_x \cap \mathcal{C}$ and consequently this difference is an element of $\mathcal{I}_{\ker d\pi}(V_x)$. Since $\sum_a \Lambda^a \cdot (\overline{X_a} \circ \varphi_\pi|_{V_x}) \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(V_x)$ we obtain

$$(F - f \circ \varphi_\pi)|_{V_x} \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(V_x).$$

Case 3. $[x \in \mathcal{C} \setminus s(T^*M)]$: Define $x_0 = s \circ \varphi_\pi(x)$. Since x and x_0 belong to the same fiber of $\varphi_\pi|_{\mathcal{C}}$, there exist compactly supported functions $h_1, \dots, h_l \in \mathcal{I}_{\ker d\pi}$ for some positive integer l such that their Hamiltonian flows connect x_0 to x , i.e.

$$x = \Phi_{h_l}^1 \circ \dots \circ \Phi_{h_1}^1(x_0).$$

Then the global section $s' := \Phi_{h_l}^1 \circ \dots \circ \Phi_{h_1}^1 \circ s$ passes through the point x . After **Case 2** for the function $f' := F \circ s'$, there exists an open neighborhood V_x around x such that $(F - f' \circ \varphi_\pi)|_{V_x} \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(V_x)$. It remains to show that $(f' - f) \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$. For arbitrary $y \in \mathcal{C}$, defining $\Phi_{h_0}^t := \text{Id}_{T^*N}$ and $y_0 = s \circ \varphi_\pi(y)$ gives

$$\begin{aligned} (f' - f) \circ \varphi_\pi(y) &= (F \circ s' - F \circ s) \circ \varphi_\pi(y) \\ &= \sum_{i=1}^l (F \circ \Phi_{h_i}^1 \circ \dots \circ \Phi_{h_0}^1 - F \circ \Phi_{h_{i-1}}^1 \circ \dots \circ \Phi_{h_0}^1)(y_0) \\ &= \sum_{i=1}^l \int_0^1 \frac{d}{dt} F \circ \Phi_{h_i}^t \circ \Phi_{h_{i-1}}^1 \circ \dots \circ \Phi_{h_0}^1(y_0) dt \\ &= \sum_{i=1}^l \int_0^1 \{h_i, F\}_{T^*N} \circ \Phi_{h_i}^t \circ \Phi_{h_{i-1}}^1 \circ \dots \circ \Phi_{h_0}^1(y_0) dt. \end{aligned} \quad (54)$$

Since $F \in N(\mathcal{I}_{\pi^{-1}\mathcal{F}})$, for every i there exist smooth functions $\lambda_i^1, \dots, \lambda_i^N, \eta_i^1, \dots, \eta_i^K \in C^\infty(T^*N)$ such that $\{h_i, F\}_{T^*N} = \sum_a \lambda_i^a \cdot (\overline{X}_a \circ \varphi_\pi) + \sum_b \eta_i^b \cdot (\overline{Y}_b)$. Implementing this into Equation (54), making use of the fact that the flows of the h_i s preserve \mathcal{C} , and noting that the \overline{Y}_b s vanish on \mathcal{C} , this gives

$$\begin{aligned} &(f' - f) \circ \varphi_\pi(y) \\ &= \sum_{i=1}^l \int_0^1 \left(\sum_a \lambda_i^a \cdot (\overline{X}_a \circ \varphi_\pi) \right) \circ \Phi_{h_i}^t \circ \Phi_{h_{i-1}}^1 \circ \dots \circ \Phi_{h_1}^1(y_0) dt \\ &= \sum_a \Lambda^a(y) \cdot (\overline{X}_a \circ \varphi_\pi(y)). \end{aligned} \quad (55)$$

Here we defined $\Lambda^a \in C^\infty(T^*N)$ by

$$\Lambda^a(z) := \sum_{i=1}^l \int_0^1 \lambda_i^a \circ \Phi_{h_i}^t \circ \Phi_{h_{i-1}}^1 \circ \dots \circ \Phi_{h_1}^1 \circ s \circ \varphi_\pi(z) dt \quad \forall z \in T^*N.$$

Equation (55) implies that $(f' - f) \circ \varphi_\pi - \sum_a \Lambda^a(y) \cdot (\overline{X}_a \circ \varphi_\pi(y))$ vanishes on \mathcal{C} and, equivalently, it thus belongs to $\mathcal{I}_{\ker d\pi}$ and since $\sum_a \Lambda^a \cdot (\overline{X}_a \circ \varphi_\pi) \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$. This gives $(f' - f) \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$, which completes the proof in **Case 3**.

If a global section does not exist, we can choose a locally finite open covering $\{U_i\}_{i=1}^\infty$ of T^*M with a partition of unity $\{\rho_i\}_{i=1}^\infty$ subordinate to it, such that for every positive integer i there exists a local section $s_i: U_i \rightarrow \mathcal{C}$. Similar to the global case, we can show that for $f_i := F \circ s_i \in C^\infty(U_i)$, we have $F|_{\varphi_\pi^{-1}(U_i)} - f \circ \varphi_\pi|_{\varphi_\pi^{-1}(U_i)} \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(\varphi_\pi^{-1}(U_i))$. Defining $f := \sum_{i=1}^\infty \rho_i f_i$, we claim that $F - f \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$. This is equivalent to showing that for every $\sigma \in C_c^\infty(T^*N)$ we have $\sigma \cdot (F - f \circ \varphi_\pi) \in \mathcal{J}_{\pi^{-1}\mathcal{F}}$. Since $\text{supp}(\sigma)$ is compact, it can be covered by finitely many open subsets $\varphi_\pi^{-1}(U_{i_1}), \dots, \varphi_\pi^{-1}(U_{i_n})$ of the

covering $\{\varphi_\pi^{-1}(U_i)\}_{i=1}^\infty$. This gives

$$\begin{aligned}
\sigma \cdot (F - f \circ \varphi_\pi) &= \sum_{i=1}^{\infty} \sigma \cdot (\rho_i \circ \varphi_\pi) \cdot (F - f \circ \varphi_\pi) \\
&= \sum_{a=1}^n \sigma \cdot (\rho_{i_a} \circ \varphi_\pi) \cdot (F|_{\varphi_\pi^{-1}(U_{i_a})} - f_{i_a} \circ \varphi_\pi|_{\varphi_\pi^{-1}(U_{i_a})}) \\
&\in \mathcal{J}_{\pi^{-1}\mathcal{F}},
\end{aligned} \tag{56}$$

since $\sigma \cdot (\rho_{i_a} \circ \varphi_\pi) \in C_c^\infty(\varphi_\pi^{-1}(U_{i_a}))$. This completes the proof. \blacksquare

Proof. [Theorem 5.19] It is enough to show that for every surjective submersion $\pi: N \rightarrow M$ with connected fibers over a foliated manifold (M, \mathcal{F}) , the Poisson algebras $\mathcal{R}(\mathcal{I}_\mathcal{F})$ and $\mathcal{R}(\mathcal{I}_{\pi^{-1}\mathcal{F}})$ are isomorphic.

To do so, we first choose Riemannian metrics g_M and g_N such that π becomes a Riemannian submersion. This can be done as follows: choose a Riemannian metric g_M on M , a fiber metric g^\perp on $\ker d\pi \subset TN$, and a subbundle $\mathcal{H} \subset TN$ complementary to $\ker d\pi$; one then declares these two subbundles to be orthogonal to one another and defines $g_N = (\pi^*g_M)|_{\mathcal{H}} + g^\perp$.

Injectivity of $\tilde{\varphi}_\pi$ is a direct consequence of Lemma (5.20). It remains to prove that $\tilde{\varphi}_\pi$ is surjective. It follows from showing that, for every $F \in N(\mathcal{I}_{\pi^{-1}\mathcal{F}})$, there exists $f \in N(\mathcal{I}_\mathcal{F})$ such that $F - f \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$. To do so we choose an open covering $\{U_i\}_{i=1}^\infty$ of M such that, for every positive integer i , the pullback $\iota_{U_i}^{-1}\mathcal{F}$ is finitely generated. Let $\{V_a\}_{a=1}^\infty$ be a locally finite refinement of the covering $\{T^*U_i\}_{i=1}^\infty$ of T^*M and let $\{\rho_a\}_{a=1}^\infty$ be a partition of unity subordinate to $\{V_a\}_{a=1}^\infty$. Lemma 5.21 then implies that for every a there exists $f_a \in C^\infty(V_a)$ such that $F|_{\varphi_\pi^{-1}(V_a)} - f_a \circ \varphi_\pi|_{\varphi_\pi^{-1}(V_a)} \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(\varphi_\pi^{-1}(V_a))$. Using the same argument as in the proof of Lemma 5.21 (see Equation (56)), for $f := \sum_{a=1}^\infty \rho_a f_a$ we have $F - f \circ \varphi_\pi \in \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$. To complete the proof, we show that $f \in N(\mathcal{I}_\mathcal{F})$ as follows: Since $f \circ \varphi_\pi \in N(\mathcal{I}_{\pi^{-1}\mathcal{F}})$, Equation (39) of Lemma 5.13 implies that $\{f, \mathcal{J}_\mathcal{F}\}_{T^*M} \circ \varphi_\pi \subset \varphi_\pi^* \mathcal{J}_\mathcal{F} \subset \mathcal{I}_{\pi^{-1}\mathcal{F}}(T^*N)$. As a consequence of Lemma 5.20 we have $\{f, \mathcal{J}_\mathcal{F}\}_{T^*M} \subset \mathcal{J}_\mathcal{F}$, which together with Lemma 5.8 gives $f \in N(\mathcal{I}_\mathcal{F})$. \blacksquare

The octonionic Hopf singular foliation

The singular leaf decomposition \mathcal{L}_{OH} of $\mathbb{R}^{16} \cong \mathbb{O}^2$ induced by the Hopf construction for octonions \mathbb{O} [MR19] is not locally induced by any isometric Lie group action, has no known Lie group action generating it. In this chapter we construct a *Lie groupoid* $\mathcal{G} \Rightarrow \mathbb{O}^2$ whose orbits coincide with \mathcal{L}_{OH} . Its Lie algebroid $E_0 = \text{Lie}(\mathcal{G})$ is of the form $\mathbb{O}^4 \rightarrow \mathbb{O}^2$ with polynomial structure functions. Its sheaf of sections induces a singular foliation $\mathcal{F}_{OH} := \rho(\Gamma(E_0))$ on \mathbb{O}^2 , which we call the *singular octonionic Hopf foliation*. \mathcal{F}_{OH} is shown to be maximal among all singular foliations \mathcal{F} generating \mathcal{L}_{OH} —in the polynomial, the real analytic, as well as in the smooth setting.

We extend E_0 to a Lie 3-algebroid, which is a minimal length representative of the universal Lie ∞ -algebroid of the singular octonionic Hopf foliation. This permits to prove that E_0 is the minimal rank Lie algebroid and that \mathcal{G} the lowest dimensional Lie groupoid which generate \mathcal{F}_{OH} .

The leaf decomposition \mathcal{L}_{OH} is one of the few known examples of a singular Riemannian foliation in the sense of Molino which cannot be generated by local isometries (local non-homogeneity). We improve this result by showing that any smooth singular foliation \mathcal{F} inducing \mathcal{L}_{OH} cannot be even Hausdorff Morita equivalent to a singular foliation \mathcal{F}_M on a Riemannian manifold (M, g) generated by local isometries. Furthermore, we show that there is no real analytic singular foliation \mathcal{F} generating \mathcal{L}_{OH} which turns $(\mathbb{R}^{16}, g_{st}, \mathcal{F})$ into a module singular Riemannian foliation.

6 Normed division algebras and Hopf fibrations

Throughout this section, we cover the basic definitions and properties of normed division algebras, particularly focusing on the non-associative case of octonions. We recall the Hopf fibrations associated with these algebras and construct the corresponding singular Hopf leaf decompositions. For a detailed introduction to the algebra of octonions, we refer to [B02].

6.1 Basic properties of normed division algebras

Definition 6.1. *A normed division algebra is a (finite-dimensional) Euclidean vector space $(\mathbb{D}, \langle \cdot, \cdot \rangle)$ equipped with the structure of a unital \mathbb{R} -algebra satisfying*

$$\|a \cdot b\| = \|a\| \cdot \|b\| \quad (57)$$

for all $a, b \in \mathbb{D}$, where the norm is the one induced by the inner product.

Since (57) implies $a \cdot b = 0$ can be satisfied only if $a = 0$ or $b = 0$, this is a division algebra, as suggested by the name. The existence of a unit element 1 implies the embedding $\iota: \mathbb{R} \rightarrow \mathbb{D}, \alpha \mapsto \alpha 1$. The orthogonal projections of an element $a \in \mathbb{D}$ to $\text{im}(\iota)$ and $\text{im}(\iota)^\perp$ permit us to define the *real part* and the *imaginary part* of a , respectively: $\text{Re}(a) = \langle a, 1 \rangle 1$ and $\text{Im}(a) = a - \text{Re}(a)$ and to generalize complex conjugation by means of the involution:

$$\bar{a} := \text{Re}(a) - \text{Im}(a). \quad (58)$$

Theorem 6.2 ([H98]). *Every normed division algebra is isomorphic to one of the following four: The real numbers \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} , the octonions \mathbb{O} .*

They can be obtained successively by the Cayley-Dickson construction starting from the real numbers, doubling the dimension in each step. In the process one loses in the first step that all elements are real, in the second step commutativity, and in the last step associativity. Continuing Cayley-Dickson further then violates Equation (57). While non-associative, the octonions are still an alternative algebra, i.e. the associator $[a, b, c] = a \cdot (b \cdot c) - (a \cdot b) \cdot c$ is skew-symmetric for all $a, b, c \in \mathbb{O}$. This implies in particular

$$a \cdot (b \cdot a) = (a \cdot b) \cdot a, \quad (59)$$

which, for this reason, we will simply write as $a \cdot b \cdot a$ henceforth.

In addition, the real components of the arguments of the associator do not contribute. Thus one has, for example, $[a, b, c] = -[\bar{b}, \bar{a}, c]$, which, when written out, gives

$$a \cdot (b \cdot c) + \bar{b} \cdot (\bar{a} \cdot c) = (a \cdot b) \cdot c + (\bar{b} \cdot \bar{a}) \cdot c. \quad (60)$$

This equation yields several ones that we will use in this chapter and which we will derive from it now.

As a direct consequence of the Cayley-Dickson construction, we have $\overline{a \cdot b} = \bar{b} \cdot \bar{a}$ for all $a, b \in \mathbb{O}$ as well as

$$\bar{a} \cdot a = a \cdot \bar{a} = \|a\|^2, \quad (61)$$

where the embedding ι is understood on the right-hand side. Replacing a by $a + b$ in Equation (61) then recovers the standard inner product by the formula.

$$\langle a, b \rangle = \frac{1}{2}(a \cdot \bar{b} + b \cdot \bar{a}) = \operatorname{Re}(a \cdot \bar{b}). \quad (62)$$

This in particular implies that Equation (60) can be rewritten as

$$a \cdot (b \cdot c) + \bar{b} \cdot (\bar{a} \cdot c) = 2\langle a, \bar{b} \rangle \cdot c. \quad (63)$$

Equation (61) implies that every $a \neq 0$ has an inverse $a^{-1} = \bar{a}/\|a\|^2$. If we replace b and c in Equation (63) by \bar{a} and b , respectively, we obtain $a \cdot (\bar{a} \cdot b) = \|a\|^2 \cdot b$, which, for $a \neq 0$, implies

$$a \cdot (a^{-1} \cdot b) = b, \quad (64)$$

$$(b \cdot a^{-1}) \cdot a = b, \quad (65)$$

where the second equation follows from the first one by conjugation.

Polarization of Equation (57) leads to

$$\langle a \cdot b, a \cdot c \rangle = \|a\|^2 \langle b, c \rangle = \langle b \cdot a, c \cdot a \rangle, \quad (66)$$

which yields the frequently used equations (for $a \neq 0$, replace c by $a^{-1} \cdot c$ and by $c \cdot a^{-1}$, respectively):

$$\langle a \cdot b, c \rangle = \langle b, \bar{a} \cdot c \rangle, \quad (67)$$

$$\langle b \cdot a, c \rangle = \langle b, c \cdot \bar{a} \rangle. \quad (68)$$

Replacing c by a in Equation (63), we obtain the conjugation formula

$$a \cdot b \cdot a = 2\langle a, \bar{b} \rangle a - \|a\|^2 \bar{b}. \quad (69)$$

Finally, these equations imply the well-known Moufang identities: for every $a, b, c \in \mathbb{O}$ we have

$$(a \cdot b) \cdot (c \cdot a) = a \cdot (b \cdot c) \cdot a, \quad (70)$$

$$a \cdot (b \cdot (a \cdot c)) = (a \cdot b \cdot a) \cdot c, \quad (71)$$

$$((a \cdot b) \cdot c) \cdot b = a \cdot (b \cdot c \cdot b). \quad (72)$$

Let us illustrate this for Equation (70): Using Equations (63) and (65), we have

$$\begin{aligned} (a \cdot b) \cdot (c \cdot a) &= 2\langle a \cdot b, \bar{c} \rangle \cdot a - \bar{c} \cdot ((\bar{b} \cdot \bar{a}) \cdot a) \\ &= 2\langle a, (\bar{b} \cdot \bar{c}) \rangle \cdot a - \|a\|^2 \cdot (\bar{b} \cdot \bar{c}) \\ &= a \cdot (b \cdot c) \cdot a, \end{aligned}$$

where we used Equation (63) once more to obtain the last equality.

6.2 Hopf fibrations

The Hopf fibration can be constructed for each of the four normed division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} . [GWZ86] As explained in this subsection, there is a significant difference between the first three and the octonions, due to the lack of associativity for the latter.

We start by describing the construction for the case of associative normed division algebras. Here \mathbb{A} stands for \mathbb{R} , \mathbb{C} or \mathbb{H} . Associativity implies that the set of units $U = \{u \in \mathbb{A} : \|u\|^2 = 1\}$ in \mathbb{A} forms a Lie group. As multiplication by elements of norm one preserves the norm of every element in \mathbb{A} , we can consider the right Lie group action of U on the unit sphere $S \subset \mathbb{A}^2$, defined by

$$(a, b) \cdot u := (a \cdot u, b \cdot u) \quad (73)$$

for every $(a, b) \in \mathbb{A}^2$ and $u \in U$. This action is free. It is proper as well since both U and S are compact manifolds. Consequently, the right Lie group action $S \times U \rightarrow S$ defines a principal U -bundle. Table 1 below provides an overview of the three corresponding bundles.

\mathbb{A}	Structure group	Total space	Base manifold
\mathbb{R}	$\mathbb{Z}/2\mathbb{Z} \cong S^0$	S^1	S^1
\mathbb{C}	$U(1) \cong S^1$	S^3	S^2
\mathbb{H}	$SU(2) \cong S^3$	S^7	S^4

Table 1 –

Remark 6.3. *If we replace the right action in the construction of Hopf fibrations by a left action, we obtain the same fibrations for the cases of $\mathbb{A} = \mathbb{R}, \mathbb{C}$. However, due to non-commutativity, the case of $\mathbb{A} = \mathbb{H}$ will have a different fibration. Although we again obtain a fibration of S^7 into 3-spheres, these fibers are not identically the same as the fibers in the construction by the right action described above. To confirm this, note that in our description for any point (a', b') on the fiber passing through $(a, b) \in S^7 \subset \mathbb{H}^2$, we have $b' \cdot \bar{a}' = b \cdot \bar{a}$, which is not true in the other description.*

Due to the lack of associativity, the set of unit octonions $S^7 \subset \mathbb{O}$ does not form a Lie group. Consequently, right multiplication by unit octonions is not a Lie group action and may fail to construct the octonionic Hopf fibration. To see the disadvantages of such a construction, it is useful to introduce the following more explicit description of octonions:

Definition 6.4. *The algebra \mathbb{O} of octonions is generated by an orthonormal basis $e_0 \equiv 1$ and $\{e_i\}_{i=1}^7$, where e_0 is chosen to be the unit and the multiplication among the remaining basis elements can be defined by*

$$e_i \cdot e_j = -\delta_{ij} 1 + \epsilon_{ijk} e_k, \quad i, j = 1, \dots, 7. \quad (74)$$

Here δ_{ij} is the Kronecker delta, and ϵ_{ijk} is a completely anti-symmetric tensor with value 1 when $ijk \in \{123, 145, 176, 246, 257, 347, 365\}$ and 0 for all other triples.

In particular, Definition 6.4 identifies the underlying vector space of \mathbb{O} with \mathbb{R}^8 , where the identification is given by $a = \sum_{i=0}^7 a_i e_i \mapsto (a_0, a_1, \dots, a_7)$. When this identification is understood, by abuse of notation, we write $a = (a_0, a_1, \dots, a_7) = (a^i)_{i=0}^7$.

Now for every $(x, y) \in S^{15} \subset \mathbb{O}^2$ consider the 7-sphere defined as

$$S_{(x,y)}^7 := \{(x \cdot u, y \cdot u) : u \in \mathbb{O}, \|u\| = 1\} \subset S^{15} \subset \mathbb{O}^2.$$

If right multiplication by octonions of norm 1 induces an honest fibration, for every point $(x', y') \in S_{(x,y)}^7$, we should have $S_{(x',y')}^7 = S_{(x,y)}^7$. Equivalently, for every $(x, y) \in S^{15} \subset \mathbb{O}^2$ and every two unit octonions u_1 and u_2 , there should exist a unique unit octonion u_3 such that

$$(x \cdot u_1) \cdot u_2 = x \cdot u_3, \quad (75)$$

$$(y \cdot u_1) \cdot u_2 = y \cdot u_3. \quad (76)$$

But it is not difficult to find an example violating this necessary condition: If we choose $(x, y) = (e_1/\sqrt{2}, e_2/\sqrt{2})$, $u_1 = e_5$ and $u_2 = e_4$, in Equation (75) we find the unique solution $u_3 = -e_1$, but the unique solution for Equation (76) turns out to be $u_3 = e_1$. Consequently, right multiplication by unit octonions does not fibrate S^{15} into 7-spheres. In particular, it does not induce the octonionic Hopf fibration.

6.3 Singular Hopf leaf decomposition

Since the usual construction fails for the octonionic Hopf fibration, we describe the construction of Hopf fibrations using \mathbb{D} -lines, which can also be used in the case of octonions.

Definition 6.5. *Let \mathbb{D} be a normed division algebra. For every $m \in \mathbb{D}$, the \mathbb{D} -line in \mathbb{D}^2 with slope m is defined as*

$$l_m := \{(x, m \cdot x) \in \mathbb{D}^2 : x \in \mathbb{D}\}. \quad (77)$$

One also defines \mathbb{D} -line with slope ∞ :

$$l_\infty := \{(0, x) \in \mathbb{D}^2 : x \in \mathbb{D}\}.$$

Definition 6.6. *Let \mathbb{D} be a normed division algebra. The singular Hopf leaf decomposition of \mathbb{D}^2 is defined as*

$$\begin{aligned} L_{m,r} &:= l_m \cap S(r) \quad \forall m \in \mathbb{D}, \\ L_{\infty,r} &:= l_\infty \cap S(r), \end{aligned} \quad (78)$$

together with the origin in \mathbb{D}^2 . Here, $S(r) := \{(x, y) \in \mathbb{D}^2 : \|x\|^2 + \|y\|^2 = r^2\}$ is the sphere of radius $r > 0$ in \mathbb{D}^2 .

The Hopf fibration associated to any of the normed division algebras $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and \mathbb{O} can be defined as the partition of the unit sphere $S(1) \subset \mathbb{D}^2$ into the leaves $L_{m,1}$, for all $m \in \mathbb{D}$, and the leaf $L_{\infty,1}$.

Remark 6.7. *It can be easily verified that the definition of Hopf fibrations using \mathbb{A} -lines coincides with the fibrations described in Table 1. Note that in the case of quaternions, if one prefers to work with left action, the definition of \mathbb{H} -lines should be modified as*

$$l_m := \{(x, x \cdot m) \in \mathbb{H}^2 : x \in \mathbb{H}\} \quad (79)$$

for the two fibrations to coincide.

7 The Lie groupoid and its induced Lie algebroid

In this section, we describe the construction of a Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$ which has \mathcal{L}_{OH} as its orbits. The corresponding Lie algebroid is computed using the right-invariant vector fields.

7.1 The Lie groupoid

A prominent role in its construction of the Lie groupoid will be played by the following function:

Definition 7.1. *The (octonionic) rescaling function $\lambda: \mathbb{O}^2 \times \mathbb{O}^2 \rightarrow \mathbb{R}$ is defined by the following formula:*

$$\lambda(F, G, x, y) = \sqrt{1 + 2(\langle x, F \rangle + \langle y, G \rangle + \langle x \cdot \bar{y}, F \cdot \bar{G} \rangle) + \|x\|^2 \|F\|^2 + \|y\|^2 \|G\|^2}. \quad (80)$$

We refer to $F = (F^i)_{i=0}^7, G = (G^i)_{i=0}^7$ as *arrow coordinates* and to $x = (x^i)_{i=0}^7, y = (y^i)_{i=0}^7$ as *object coordinates*.

The following lemma ensures that the radicand in Definition 7.1 is non-negative. Consequently, the rescaling function λ is defined throughout $\mathbb{O}^2 \times \mathbb{O}^2$.

Lemma 7.2. *For all $(F, G, x, y) \in \mathbb{O}^2 \times \mathbb{O}^2$, one has*

$$\begin{aligned} & \|x\|^2 [1 + 2(\langle x, F \rangle + \langle y, G \rangle + \langle x \cdot \bar{y}, F \cdot \bar{G} \rangle) + \|x\|^2 \|F\|^2 + \|y\|^2 \|G\|^2] \\ &= \|x + \|x\|^2 F + (x \cdot \bar{y}) \cdot G\|^2 \end{aligned}$$

Proof. We first expand the right-hand side of the identity:

$$\begin{aligned} & \|x + \|x\|^2 F + (x \cdot \bar{y}) \cdot G\|^2 = \langle x + \|x\|^2 F + (x \cdot \bar{y}) \cdot G, x + \|x\|^2 F + (x \cdot \bar{y}) \cdot G \rangle = \\ &= \|x\|^2 + 2\|x\|^2 (\langle x, F \rangle + \langle x, (x \cdot \bar{y}) \cdot G \rangle) + \langle (x \cdot \bar{y}) \cdot G, F \rangle + \|x\|^4 \|F\|^2 + \|x\|^2 \|y\|^2 \|G\|^2. \end{aligned}$$

The result then follows by noting that $\langle (x \cdot \bar{y}) \cdot G, F \rangle = \langle x \cdot \bar{y}, F \cdot \bar{G} \rangle$ due to Equation (68) and that $\langle x, (x \cdot \bar{y}) \cdot G \rangle = \langle y, G \rangle$ as a consequence of Equations (67) and (65). ■

Theorem 7.3. *The following data define a Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$, whose orbits coincide with the singular octonionic Hopf leaf decomposition \mathcal{L}_{OH} .*

— *The manifold of arrows \mathcal{G} is an open subset of $\mathbb{O}^2 \times \mathbb{O}^2$ given by*

$$\mathcal{G} := \mathbb{O}^2 \times \mathbb{O}^2 \setminus \mathcal{C}$$

where $\mathcal{C} = \{(F, G, x, y) \in \mathbb{O}^2 \times \mathbb{O}^2 : \lambda(F, G, x, y) = 0\}$.

— *For every arrow $g = (F, G, x, y) \in \mathcal{G}$, the source map $s: \mathcal{G} \rightarrow \mathbb{O}^2$ and the target map $t: \mathcal{G} \rightarrow \mathbb{O}^2$ are given by:*

$$\begin{aligned} s(g) &= (x, y), \\ t(g) &= \frac{1}{\lambda(g)} (x + \|x\|^2 F + (x \cdot \bar{y}) \cdot G, y + \|y\|^2 G + (y \cdot \bar{x}) \cdot F). \end{aligned}$$

- The multiplication map $m: \mathcal{G}^{(2)} := \{(g', g) \in \mathcal{G} \times \mathcal{G} : s(g') = t(g)\} \rightarrow \mathcal{G}$, is defined as follows: The product $m(g', g) \equiv g' \cdot g$ of composable arrows $g' = (F', G', x', y')$ and $g = (F, G, x, y)$ is given by

$$g' \cdot g := (F + \lambda(g) \cdot F', G + \lambda(g) \cdot G', x, y).$$

- The unit map $u: \mathbb{O}^2 \rightarrow \mathcal{G}$ is given by associating the arrow $1_{(x,y)} := (0, 0, x, y)$ to every object $(x, y) \in \mathbb{O}^2$.
- The inverse $i: \mathcal{G} \rightarrow \mathcal{G}$ applied to an arrow $g = (F, G, x, y)$ gives $g^{-1} \equiv i(g)$ by means of

$$g^{-1} = (-F/\lambda(g), -G/\lambda(g), t(g)).$$

We first establish some identities for the structure maps of \mathcal{G} in the following lemmas, which will be used in the proof of the theorem.

Lemma 7.4. *For every arrow $g \in \mathcal{G}$, both $s(g)$ and $t(g)$ belong to the same octonionic line.*

Proof. If $x = 0$, both $s(g)$ and $t(g)$ belong to the octonionic line l_∞ . If x is non-zero, put $m = y \cdot x^{-1}$. Then evidently $s(g) = (x, y) \in l_m$. To show that also $t(g)$ belongs to l_m , we first show that $(\|x\|^2 F, (y \cdot \bar{x}) \cdot F)$ and $((x \cdot y) \cdot G, \|y\|^2 G)$ belong to the octonionic line l_m :

$$((y \cdot \bar{x}) \cdot F) \cdot (\|x\|^2 F)^{-1} = ((y \cdot \bar{x}) \cdot F) \cdot \frac{\overline{(\|x\|^2 F)}}{\|x\|^4 \cdot \|F\|^2} = (y \cdot \frac{x}{\|x\|^2}) \cdot \frac{\|F\|^2}{\|F\|^2} = y \cdot x^{-1}$$

and

$$(\|y\|^2 G) \cdot ((x \cdot \bar{y}) \cdot G)^{-1} = (\|y\|^2 G) \cdot \frac{\overline{G} \cdot (y \cdot \bar{x})}{\|x\|^2 \|y\|^2 \|G\|^2} = \frac{\|G\|^2}{\|G\|^2} \cdot (y \cdot \frac{\bar{y}}{\|x\|^2}) = y \cdot x^{-1}.$$

As l_m is a vector space, we obtain that also $t(g) \in l_m$. ■

Corollary 7.5. *For every arrow $g = (F, G, x, y) \in \mathcal{G}$ with $t(g) = (x', y')$, we have*

$$y \cdot \bar{x} = y' \cdot \bar{x}'.$$

Lemma 7.6. *For composable arrows $(g', g) \in \mathcal{G}^{(2)}$ the map λ is multiplicative, i.e. it satisfies $\lambda(g' \cdot g) = \lambda(g') \cdot \lambda(g)$.*

Proof. Using Lemma 7.2 and Corollary 7.5, we have

$$\begin{aligned} \lambda(g') \cdot \lambda(g) &= \frac{\|x' + \|x'\|^2 F' + (x' \cdot \bar{y}') \cdot G'\|}{\|x'\|} \cdot \lambda(g) \\ &= \frac{\|\lambda(g) \cdot x' + \|x\|^2 \lambda(g) \cdot F' + (x \cdot \bar{y}) \cdot \lambda(g) \cdot G'\|}{\|x\|} \\ &= \frac{\|(x + \|x\|^2 F + (x \cdot \bar{y}) \cdot G) + \|x\|^2 \lambda(g) \cdot F' + (x \cdot \bar{y}) \cdot \lambda(g) \cdot G'\|}{\|x\|} \\ &= \frac{\|x + \|x\|^2 \cdot (F + \lambda(g) \cdot F') + (x \cdot \bar{y}) \cdot (G + \lambda(g) \cdot G')\|}{\|x\|} \\ &= \lambda(F + \lambda(g) \cdot F', G + \lambda(g) \cdot G, x, y) = \lambda(g' \cdot g) \end{aligned}$$

■

Lemma 7.7. *The multiplication map $m: \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ is associative.*

Proof. Consider the arrows $g'', g', g \in \mathcal{G}$, such that $s(g'') = t(g')$ and $s(g') = t(g)$. Using Lemma 7.6, we have

$$\begin{aligned} g'' \cdot (g' \cdot g) &= g'' \cdot (F + \lambda(g) \cdot F', G + \lambda(g) \cdot G', x, y) \\ &= (F + \lambda(g) \cdot F' + \lambda(g' \cdot g) \cdot F'', G + \lambda(g) \cdot G' + \lambda(g' \cdot g) \cdot G'', x, y) \\ &= (F + \lambda(g) \cdot F' + \lambda(g) \cdot \lambda(g') \cdot F'', G + \lambda(g) \cdot G' + \lambda(g) \cdot \lambda(g') \cdot G'', x, y) \\ &= (F' + \lambda(g') \cdot F'', G' + \lambda(g') \cdot G'', x', y') \cdot g \\ &= (g'' \cdot g') \cdot g. \end{aligned}$$

■

Lemma 7.8. *For composable arrows $(g', g) \in \mathcal{G}^{(2)}$, we have $t(g' \cdot g) = t(g')$ and $s(g' \cdot g) = s(g)$.*

Proof. Let $g' = (F', g', x', y')$ and $g = (F, G, x, y)$ with $(x', y') = t(F, G, x, y)$. Lemma 7.6 together with Corollary 7.5 and $\|x'\| = \|x\|$ and $\|y'\| = \|y\|$ imply that

$$\begin{aligned} t(g' \cdot g) &= t(F + \lambda(g) \cdot F', G + \lambda(g) \cdot G, x, y) \\ &= \frac{1}{\lambda(g') \cdot \lambda(g)} (x + \|x\|^2 (F + \lambda(g) \cdot F') + (x \cdot \bar{y}) \cdot (G + \lambda(g) \cdot G'), y + \|y\|^2 G + (y \cdot \bar{x}) \cdot F) \\ &= \frac{1}{\lambda(g')} (x' + \|x'\|^2 F' + (x' \cdot \bar{y}') \cdot G', y' + \|y'\|^2 G' + (y \cdot \bar{x}) \cdot F') = t(g'). \end{aligned}$$

The second identity is obvious from the definition. ■

Lemma 7.9. *For every arrow $g \in \mathcal{G}$, one has $g^{-1} \cdot g = 1_{s(g)}$. In addition, $t \circ i = s$.*

Proof. For every arrow $g = (F, G, x, y)$ $g^{-1} \cdot g = (0, 0, x, y) = 1_{(x, y)}$ is evident by the definition. Lemma 7.8 implies, moreover, that $t \circ i(g) = t(g^{-1} \cdot g) = t(1_{(x, y)}) = s(g)$ ■

Corollary 7.10. *For every arrow $g = (F, G, x, y)$, the composition $g \cdot g^{-1}$ is defined and equal to $1_{t(g)}$.*

Proof. The compositability of two arrows is a direct consequence of Lemma 7.9. For $g \cdot g^{-1} = (0, 0, t(g)) = 1_{t(g)}$ use that $\lambda(g^{-1}) = \frac{1}{\lambda(g)}$ by Lemma 7.6. ■

Proof of Theorem 7.3. It is clear that \mathcal{G} is a smooth manifold and the maps s, t, i and u are smooth. In addition, Lemmas 7.7, 7.8 and 7.9 imply that the structure maps satisfy the compatibility conditions in the definition of a groupoid.

The source map is obviously a surjective submersion. The target map is a surjection as well, cf. Lemma 7.9. In order to prove that the target map t is also a submersion, it is sufficient to show that for every arrow $g \in \mathcal{G}$, there is a local section passing through it. Let $g = (F_0, G_0, x_0, y_0) \in \mathcal{G}$ and consider its inverse arrow $g^{-1} = (F, G, x_1, y_1)$. There exists an open neighborhood $U \subset \mathbb{O}^2$ with $(F, G, x, y) \in \mathcal{G}$ for all $(x, y) \in U$. We now define a smooth map $\sigma: U \rightarrow \mathcal{G}$ by means of $\sigma(x, y) := i(F, G, x, y)$. We have $\sigma(x_1, y_1) = g$ and

$$t \circ \sigma(x, y) = t \circ i(F, G, x, y) = s(F, G, x, y) = (x, y), \quad (81)$$

where we used Lemma 7.9 to obtain the last equality. This implies that σ is a local section passing through g , and consequently, t is a submersion. As a result, $\mathcal{G}^{(2)}$ is a smooth manifold and the multiplication map m is smooth. This completes the proof that $\mathcal{G} \rightrightarrows \mathbb{O}^2$ is a Lie groupoid.

It remains to prove that the orbits of $\mathcal{G} \rightrightarrows \mathbb{O}^2$ coincide with the leaves in \mathcal{L}_{OH} . According to Lemma 7.2 one has $\|t(g)\| = \|s(g)\|$ for all $g \in \mathcal{G}$. Together with Lemma 7.4 this implies that the orbits of \mathcal{G} are included in the leaves of \mathcal{L}_{OH} . In addition, for every $(x, m \cdot x) \in L_{m,r}$, we consider the arrow

$$g_{m,r} = \left(\frac{x-1}{\|x\|}, 0, \|x\|, m \cdot \|x\| \right) \in \mathcal{G}$$

which satisfies $s(g_{m,r}) = (\|x\|, m \cdot \|x\|)$ and

$$\begin{aligned} t(g_{m,r}) &= \frac{1}{\lambda(g_{m,r})} (\|x\| + \|x\| \cdot (x-1), m \cdot \|x\| + m \cdot \|x\| \cdot (x-1)) \\ &= \frac{\|x\|}{\lambda(g_{m,r})} \cdot (x, m \cdot x) = (x, m \cdot x). \end{aligned}$$

The last equality holds since

$$\begin{aligned} \lambda(g_{m,r}) &= \sqrt{1 + 2 \langle \|x\|, \frac{x-1}{\|x\|} \rangle + \|x\|^2 \frac{\|x-1\|^2}{\|x\|^2}} \\ &= \sqrt{1 + 2 \langle 1, x-1 \rangle + \|x-1\|^2} \\ &= \sqrt{\|1 + (x-1)\|^2} = \|x\|. \end{aligned}$$

Similarly for $(0, y) \in L_{\infty,r}$: The source of $g = (0, \frac{y-1}{\|y\|, 0, \|y\|})$ is $(0, \|y\|)$ while its target is $(0, y)$. Consequently, all points in a leaf of \mathcal{L}_{OH} can be joined to a single point of the same leaf by an arrow in $\mathcal{G} \rightrightarrows \mathbb{O}^2$. \blacksquare

The same construction can be applied for any of the normed division algebras \mathbb{D} , i.e. also for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. In the latter cases it then yields a Lie groupoid $\mathcal{G}_{\mathbb{A}} \rightrightarrows \mathbb{A}^2$ for which the orbits give precisely the leaves of the singular Hopf foliation associated to the respective associative division algebra \mathbb{A} . The dimension of $\mathcal{G}_{\mathbb{A}}$ equals to $4 \dim(\mathbb{A})$, i.e. 4, 8, and 16, respectively. This is significantly bigger than the ones found for the corresponding action Lie groupoids $\mathbb{A}^2 \rtimes U_{\mathbb{A}} \rightrightarrows \mathbb{A}^2$, where $U_{\mathbb{A}} = \{u \in \mathbb{A} : \|u\| = 1\}$ denotes the group of unitary elements found in the first column of Table 1; these Lie groupoids have dimensions 2, 5, and 11, respectively. This is in contrast to $\mathcal{G} = \mathcal{G}_{\mathbb{O}}$ whose dimension 32 turns out to be minimal (see Theorem 10.11 below). The relation of the Lie groupoids $\mathcal{G}_{\mathbb{A}}$ with the corresponding action groupoids is clarified in the following proposition.

Proposition 7.11. *Let $\mathbb{A}^2 \rtimes U_{\mathbb{A}} \rightrightarrows \mathbb{A}^2$ be the action groupoid for the diagonal right action of unitary elements $U_{\mathbb{A}}$ on \mathbb{A}^2 . Then the smooth map $\varphi: \mathcal{G}_{\mathbb{A}} \rightarrow \mathbb{A}^2 \rtimes U_{\mathbb{A}}$ given by*

$$\varphi(F, G, x, y) := \left((x, y), \frac{1 + \bar{x} \cdot F + \bar{y} \cdot G}{\|1 + \bar{x} \cdot F + \bar{y} \cdot G\|} \right),$$

defines a Lie groupoid morphism covering the identity on \mathbb{A}^2 .

Proof. First we observe that the rescaling function $\lambda: \mathbb{A}^2 \times \mathbb{A}^2 \rightarrow \mathbb{R}$, defined by Equation (80), simplifies drastically in the case when the division algebra is associative: For every $g = (F, G, x, y) \in \mathcal{G}_{\mathbb{A}}$ we have

$$\lambda(g) = \frac{\|x + \|x\|^2 F + x \cdot \bar{y} \cdot G\|}{\|x\|} = \frac{\|x \cdot (1 + \bar{x} \cdot F + \bar{y} \cdot G)\|}{\|x\|} = \|1 + \bar{x} \cdot F + \bar{y} \cdot G\|.$$

Similarly, for the target map one now finds:

$$\begin{aligned} t(g) &= \frac{1}{\lambda(g)} (x + \|x\|^2 F + x \cdot \bar{y} \cdot G, y + \|y\|^2 G + y \cdot \bar{y} \cdot F) \\ &= \frac{1}{\lambda(g)} (x \cdot (1 + \bar{x} \cdot F + \bar{y} \cdot G), y \cdot (1 + \bar{x} \cdot F + \bar{y} \cdot G)) = (x, y) \cdot \frac{1 + \bar{x} \cdot F + \bar{y} \cdot G}{\|1 + \bar{x} \cdot F + \bar{y} \cdot G\|}, \end{aligned}$$

which is evidently equal to the target of $\varphi(g)$ in the action groupoid.

To prove that φ is a morphism of Lie groupoids, it remains to show that it also preserves the multiplication: Let $g' = (F', G', x', y')$ and $g = (F, G, x, y)$ be arrows such that $(x', y') = t(g)$. On the one hand, we have

$$\begin{aligned} \varphi(g') \cdot \varphi(g) &= \left((x', y'), \frac{1 + \bar{x}' \cdot F' + \bar{y}' \cdot G'}{\lambda(g')} \right) \cdot \left((x, y), \frac{1 + \bar{x} \cdot F + \bar{y} \cdot G}{\lambda(g)} \right) \\ &= \left((x, y), \frac{(1 + \bar{x} \cdot F + \bar{y} \cdot G) \cdot (1 + \bar{x}' \cdot F' + \bar{y}' \cdot G')}{\lambda(g) \cdot \lambda(g')} \right) \end{aligned}$$

and, on the other hand,

$$\varphi(g' \cdot g) = \left((x, y), \frac{1 + \bar{x} \cdot (F + \lambda(g) \cdot F') + \bar{y} \cdot (G + \lambda(g) \cdot G')}{\lambda(g' \cdot g)} \right).$$

Using Lemma 7.6, to establish $\varphi(g' \cdot g) = \varphi(g') \cdot \varphi(g)$ it suffices to show that

$$(1 + \bar{x} \cdot F + \bar{y} \cdot G) \cdot (1 + \bar{x}' \cdot F' + \bar{y}' \cdot G') = 1 + \bar{x} \cdot (F + \lambda(g) \cdot F') + \bar{y} \cdot (G + \lambda(g) \cdot G')$$

or, equivalently, that

$$(1 + \bar{x} \cdot F + \bar{y} \cdot G) \cdot (\bar{x}' \cdot F' + \bar{y}' \cdot G') = \lambda(g) \cdot (\bar{x} \cdot F' + \bar{y} \cdot G').$$

But as $(1 + \bar{x} \cdot F + \bar{y} \cdot G)^{-1} = \overline{(1 + \bar{x} \cdot F + \bar{y} \cdot G) / \lambda(g)^2}$, the latter equation becomes equivalent to

$$(\bar{x}' \cdot F' + \bar{y}' \cdot G') = \overline{\left(\frac{1 + \bar{x} \cdot F + \bar{y} \cdot G}{\lambda(g)} \right)} \cdot (\bar{x} \cdot F' + \bar{y} \cdot G'),$$

which holds true since

$$(x', y') = \left(x \cdot \left(\frac{1 + \bar{x} \cdot F + \bar{y} \cdot G}{\lambda(g)} \right), y \cdot \left(\frac{1 + \bar{x} \cdot F + \bar{y} \cdot G}{\lambda(g)} \right) \right).$$

■

7.2 The induced Lie algebroid

We differentiate the Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$ introduced in the previous subsection so as to obtain a corresponding Lie algebroid. We follow the conventions of [CFM21]. In what follows, for every vector space V and smooth manifold M , we denote the corresponding trivial vector bundle by \underline{V} .

Proposition 7.12. *The Lie algebroid associated to the Lie groupoid $\mathcal{G} \rightrightarrows \mathbb{O}^2$ is given by*

- The trivial vector bundle $E_0 = \underline{\mathbb{O}^2}$ over \mathbb{O}^2 .
- The anchor map $\rho: E_0 \rightarrow T\mathbb{O}^2 \cong \underline{\mathbb{O}^2}$:

$$\rho \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \|x\|^2 u + (x \cdot \bar{y}) \cdot v - (\langle x, u \rangle + \langle y, v \rangle) x \\ \|y\|^2 v + (y \cdot \bar{x}) \cdot u - (\langle x, u \rangle + \langle y, v \rangle) y \end{pmatrix}, \quad (82)$$

for every $\begin{pmatrix} u \\ v \end{pmatrix} \in E_0$ based at $(x, y) \in \mathbb{O}^2$.

- The Lie bracket evaluated on constant global sections $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in \Gamma(E_0)$:

$$\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right] = (\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} u' \\ v' \end{pmatrix} - (\langle x, u' \rangle + \langle y, v' \rangle) \begin{pmatrix} u \\ v \end{pmatrix}. \quad (83)$$

Proof. The vector bundle E_0 can be identified with $\ker ds|_{u(\mathbb{O}^2)} \subset \mathcal{G}|_{u(\mathbb{O}^2)}$. As $s: \mathcal{G} \rightarrow \mathbb{O}^2$ is projection to the second component of $\mathcal{G} \subset \mathbb{O}^2 \times \mathbb{O}^2$, E_0 is the trivial, rank 16 vector bundle $\underline{\mathbb{O}^2} \cong \mathbb{O}^2 \times \mathbb{O}^2$. Assuming that the trivialization is given by the constant global frame $\frac{\partial}{\partial F^i}$ and $\frac{\partial}{\partial G^i}$ for $i = 0, 1, \dots, 7$, we may identify every vector field

$$\langle u, \partial_F \rangle + \langle v, \partial_G \rangle := \sum_{i=0}^7 (u^i \frac{\partial}{\partial F^i} + v^i \frac{\partial}{\partial G^i}) \in \Gamma(E_0) \quad (84)$$

with $\begin{pmatrix} u \\ v \end{pmatrix} \in \Gamma(\underline{\mathbb{O}^2})$, where u and v can be thought of as \mathbb{O} -valued functions on \mathbb{O}^2 .

To evaluate the anchor map $\rho: E_0 \rightarrow T\mathbb{O}^2 \cong \underline{\mathbb{O}^2}$, it suffices to compute $\rho \begin{pmatrix} e_i \\ 0 \end{pmatrix}$ and $\rho \begin{pmatrix} 0 \\ e_i \end{pmatrix}$: Let us represent the vector $\frac{\partial}{\partial F^i} \Big|_{1(x,y)}$ as the velocity vector of the smooth curve $g(\tau): \tau \rightarrow (\tau \cdot e_i, 0, x, y) \in s^{-1}(x, y)$, for which we have $g(0) = 1_{(x,y)}$ and $\dot{g}(0) = \frac{\partial}{\partial F^i} \Big|_{1(x,y)}$. On this curve, the rescaling function is given by

$$\lambda(g(\tau)) = \sqrt{1 + 2\tau \langle x, e_i \rangle + \tau^2 \|x\|^2}$$

which implies that

$$\frac{d}{d\tau} \left(\lambda(g(\tau)) \right) \Big|_{\tau=0} = x^i. \quad (85)$$

For every $i = 0, \dots, 7$ we can compute $\rho \begin{pmatrix} e_i \\ 0 \end{pmatrix}$ as

$$\begin{aligned} \rho \begin{pmatrix} e_i \\ 0 \end{pmatrix} &= \text{dt} \left(\frac{\partial}{\partial F^i} \right) = \frac{d}{d\tau} \Big|_{\tau=0} \text{t}(g(\tau)) \\ &= \frac{d}{d\tau} \Big|_{\tau=0} \left[\frac{1}{\lambda(g(\tau))} \begin{pmatrix} x + \tau \|x\|^2 e_i \\ y + \tau (y \cdot \bar{x}) \cdot e_i \end{pmatrix} \right] \\ &= -x^i \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \|x\|^2 e_i \\ (y \cdot \bar{x}) \cdot e_i \end{pmatrix} \\ &= \begin{pmatrix} \|x\|^2 e_i - x_i x \\ (y \cdot \bar{x}) \cdot e_i - x_i y \end{pmatrix}. \end{aligned}$$

$\rho \begin{pmatrix} 0 \\ e_i \end{pmatrix}$ can be computed in a similar way. Equation (82) then follows from the $C^\infty(\mathbb{O}^2)$ -linearity of ρ .

Given the anchor map, it is sufficient to specify the Lie algebroid bracket $[\cdot, \cdot]: \Gamma(E_0) \wedge \Gamma(E_0) \rightarrow \Gamma(E_0)$ on constant sections: We first compute the Lie bracket of the right-invariant vector fields induced by $\frac{\partial}{\partial F^i} \Big|_{u(\mathbb{O}^2)}$ and $\frac{\partial}{\partial G^i} \Big|_{u(\mathbb{O}^2)}$ on \mathcal{G} . Let $g = (F, G, x, y) \in \mathcal{G}$ be an arrow with $\text{t}(g) = (x', y')$. Consider the curve $g(\tau): \tau \mapsto (\tau e_i, 0, x', y')$ which satisfies $g(0) = 1_{(x', y')}$ and $\dot{g}(0) = \frac{\partial}{\partial F^i} \Big|_{1_{(x', y')}} \cdot$.

The right-invariant vector field $\left(\frac{\partial}{\partial F^i} \right)^R \in \mathfrak{X}(\mathcal{G})$ induced by $\frac{\partial}{\partial F^i} \Big|_{u(\mathbb{O}^2)}$ is given by

$$\begin{aligned} \frac{\partial}{\partial F^i}{}^R(g) &= \frac{d}{d\tau} \Big|_{\tau=0} [g(\tau) \cdot g] = \frac{d}{d\tau} \Big|_{\tau=0} (F + \tau \lambda(g) \cdot e_i, G, x, y) \\ &= \lambda(g) \frac{\partial}{\partial F^i} \Big|_g \end{aligned}$$

and similarly

$$\frac{\partial}{\partial G^i}{}^R(g) = \lambda(g) \frac{\partial}{\partial G^i} \Big|_g.$$

To calculate commutators of the above vector fields, we need to compute for example

$$\left(\left(\frac{\partial}{\partial F^i} \right)^R \cdot \lambda \right) (1_{(x, y)}),$$

a calculation we performed already in Equation (85). This implies in particular

$$\left[\frac{\partial}{\partial F^i}{}^R, \frac{\partial}{\partial F^j}{}^R \right] \Big|_{1_{(x, y)}} = x^i \frac{\partial}{\partial F^j}{}^R \Big|_{1_{(x, y)}} - x^j \frac{\partial}{\partial F^i}{}^R \Big|_{1_{(x, y)}} = x^i \frac{\partial}{\partial F^j} \Big|_{1_{(x, y)}} - x^j \frac{\partial}{\partial F^i} \Big|_{1_{(x, y)}}.$$

In a similar fashion, one obtains

$$\begin{aligned} \left[\frac{\partial}{\partial F^i}{}^R, \frac{\partial}{\partial G^j}{}^R \right] \Big|_{1_{(x, y)}} &= x^i \frac{\partial}{\partial G^j} \Big|_{1_{(x, y)}} - y^j \frac{\partial}{\partial F^i} \Big|_{1_{(x, y)}} \\ \left[\frac{\partial}{\partial G^i}{}^R, \frac{\partial}{\partial G^j}{}^R \right] \Big|_{1_{(x, y)}} &= y^i \frac{\partial}{\partial G^j} \Big|_{1_{(x, y)}} - y^j \frac{\partial}{\partial G^i} \Big|_{1_{(x, y)}}. \end{aligned}$$

Equation (83) now follows by \mathbb{R} -linearity of the bracket. ■

8 The singular octonionic Hopf foliation

The Lie algebroid introduced in the previous section realizes \mathcal{L}_{OH} as the leaf decomposition of a *singular foliation* on $\mathbb{O}^2 \cong \mathbb{R}^{16}$. It provides the setting in which we will prove our result on the non-homogeneity of the singular octonionic Hopf foliation, see Theorem 8.9 below.

8.1 The singular foliation and non-homogeneity

Definition 8.1. *The $C^\infty(\mathbb{O}^2)$ -module \mathcal{F}_{OH} generated by elements of $\rho(E_0)$ for the Lie algebroid given in Proposition 7.12 defines a singular foliation on \mathbb{O}^2 , referred to as the singular octonionic Hopf foliation.*

We can characterize vector fields tangent to the leaves in \mathcal{F}_{OH} as solutions of a system of functional equations. Throughout this section, every vector field

$$\langle u, \frac{\partial}{\partial x} \rangle + \langle v, \frac{\partial}{\partial y} \rangle = \sum_{i=1}^7 (u^i \partial x^i + v^i \partial y^i)$$

is identified with $\begin{pmatrix} u \\ v \end{pmatrix} \in \Gamma(\underline{\mathbb{O}^2})$, where u and v can be thought of as \mathbb{O} -valued functions on \mathbb{O}^2 .

Lemma 8.2. *A vector field $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{X}(\mathbb{O}^2)$ is tangent to the leaves of \mathcal{L}_{OH} , if and only if*

$$u \cdot \bar{y} + x \cdot \bar{v} = 0, \tag{86}$$

$$\langle x, u \rangle = \langle y, v \rangle = 0. \tag{87}$$

for all $(x, y) \in \mathbb{O}^2 \cong \mathbb{R}^{16}$.

Proof. Assume that $\begin{pmatrix} u \\ v \end{pmatrix}$ is tangent to all the leaves in \mathcal{L}_{OH} . For $x = 0$, the vector field $\begin{pmatrix} u \\ v \end{pmatrix}$ being tangent to the octonionic line l_∞ gives $u = 0$, implying Equation (86). As the vector field is also tangent to the spheres $S(r)$ for every $r \geq 0$ and $\langle x, u \rangle = 0$, one obtains Equation (87).

For $(x, y) = (x_0, y_0)$ with $x_0 \neq 0$, consider the integral curve $\gamma: \mathbb{R} \rightarrow \mathbb{O}^2$ of the vector field $\begin{pmatrix} u \\ v \end{pmatrix}$ satisfying $\gamma(0) = (x_0, y_0)$. Writing $\gamma(t) = (x(t), y(t))$, one has $(x(0), y(0)) = (x_0, y_0)$, $\dot{x}(t) = u(x(t), y(t))$, and $\dot{y}(t) = v(x(t), y(t))$. The curve γ is contained in the octonionic line l_m with $m = y_0 \cdot x_0^{-1}$. This implies that $m = y(t) \cdot x(t)^{-1}$ for all t . Consequently, $y(t) = m \cdot x(t)$, and differentiation with respect to t gives $v(x(t), y(t)) = m \cdot u(x(t), y(t))$. Evaluation at $t = 0$ yields

$$(y \cdot x^{-1}) \cdot u = v. \tag{88}$$

Taking the inner product of both sides with y and using the identities (67) and (64), we have

$$\langle y, v \rangle = \langle y, (y \cdot x^{-1}) \cdot u \rangle = \frac{1}{\|x\|^2} \langle (x \cdot \bar{y}) \cdot y, u \rangle = \frac{\|y\|^2}{\|x\|^2} \langle x, u \rangle \tag{89}$$

Since, in addition, $\langle x, u \rangle + \langle y, v \rangle (1 + \frac{\|y\|^2}{\|x\|^2}) \langle x, u \rangle = 0$ by $\begin{pmatrix} u \\ v \end{pmatrix}$ being tangent to the spheres $S^{15}(r)$, Equation (87) follows.

Multiplying both sides of Eq. (88) by x^{-1} from the right and using the third Moufang identity(72), we obtain

$$y \cdot (x^{-1} \cdot u \cdot x^{-1}) = v \cdot x^{-1}.$$

Multiplying both sides by $\|x\|^4$ and using Equation (69) imply that

$$\|x\|^2 v \cdot \bar{x} = y \cdot (\bar{x} \cdot u \cdot \bar{x}) = y \cdot (2\langle \bar{x}, \bar{u} \rangle \bar{x} - \|x\|^2 \bar{u}) = -\|x\|^2 y \cdot \bar{u},$$

which gives $v \cdot \bar{x} = -y \cdot \bar{u}$ or, by conjugation, to Eq. (86).

Conversely, assume that $\begin{pmatrix} u \\ v \end{pmatrix}$ satisfies Equations (86) and (87). These equation together with Equation (69) and the third Moufang identity (72) imply Equation (88), by following the above calculation from back to front. Now, consider some integral curve $\gamma(t) = (x(t), y(t))$. Since $\dot{x}(t) = u(x(t), y(t))$ and $\dot{y}(t) = v(x(t), y(t))$, Equation (87) implies that $\|x(t)\|^2$ and $\|y(t)\|^2$ are constant in t . As a first consequence, if γ passes through the origin it must be a constant curve, and if it intersects $L_{\infty, r}$, it stays in $L_{\infty, r}$. If $x(t_0) \neq 0$ for some t_0 , then one has $x(t) \neq 0$ for all t . Differentiation with respect to t of the equation $(y(t) \cdot x(t)^{-1}) \cdot x(t) = y(t)$ gives

$$0 = (y(t) \cdot x(t)^{-1}) \cdot \dot{x}(t) - \dot{y}(t) + \left[\frac{d}{dt} (y(t) \cdot x(t)^{-1}) \right] \cdot x(t) = \left[\frac{d}{dt} (y(t) \cdot x(t)^{-1}) \right] \cdot x(t), \quad (90)$$

where we used Equation (88) in the last equality. Since $x(t) \neq 0$, this implies that $y(t) \cdot x(t)^{-1}$ equals to a constant $m \in \mathbb{O}$ and in particular $\gamma(t) = (x(t), m \cdot x(t))$ lies in the octonionic line l_m for all t . In addition, Equation (87) ensures that it lies in some leaf of \mathcal{L}_{OH} .

We have shown that, if the integral curve γ intersects a leaf in \mathcal{L}_{OH} , it stays in that leaf. As a result, the vector field $\begin{pmatrix} u \\ v \end{pmatrix}$ is tangent to the leaves of \mathcal{L}_{OH} . \blacksquare

Proposition 8.3. *The singular octonionic Hopf foliation \mathcal{F}_{OH} is generated by all vector fields tangent to the leaves of \mathcal{L}_{OH} .*

Proof. Using the exact sequence described in Appendix 2 for the case of $C^\infty(M)$ -modules, kernel of the map $J: \Gamma(\underline{\mathbb{O}^2}) \rightarrow \Gamma(\underline{\mathbb{R} \oplus \mathbb{O} \oplus \mathbb{R}})$ given by

$$J \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \langle x, u \rangle \\ u \cdot \bar{y} + x \cdot \bar{v} \\ \langle y, v \rangle \end{pmatrix}$$

is generated by the image of the anchor $\rho: \Gamma(E_0) \rightarrow \mathfrak{X}(\underline{\mathbb{O}^2}) \cong \Gamma(\underline{\mathbb{O}^2})$. Lemma 8.2 implies the result. \blacksquare

This characterization of the vector fields tangent to the leaves in \mathcal{L}_{OH} , together with the non-associativity of octonions, gives rise to the following important lemma:

Lemma 8.4. *Let $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{X}(\underline{\mathbb{O}^2})$ be a vector field with $u, v \in C^\infty(\underline{\mathbb{O}^2}, \mathbb{O})$ linear in coordinates x^i and y^i . If $\begin{pmatrix} u \\ v \end{pmatrix}$ is tangent to \mathcal{L}_{OH} , then $u = v \equiv 0$.*

Proof. We prove the statement by contradiction. Assume that X is a non-zero linear vector field on $\mathbb{O}^2 \cong \mathbb{R}^{16}$, tangent to the leaves of \mathcal{L}_{OH} . Equation (86) then implies that u is forced to be independent of y and v be independent of x . Consequently, there are \mathbb{R} -linear maps $A, B: \mathbb{O} \rightarrow \mathbb{O}$ such that $u(x, y) = Ax$ and $v(x, y) = By$. Now, Equation (86) and (87) can be rewritten as

$$(Ax) \cdot \bar{y} + x \cdot \overline{(By)} = 0, \quad (91)$$

$$\langle Ax, x \rangle = 0. \quad (92)$$

Choosing $y = 1$ in Equation (91) gives

$$Ax = -x \cdot \overline{(B1)},$$

which together with Equation (92) imply that $B1$ is an imaginary element of \mathbb{O} . Putting $x = 1$ in the last equation we obtain $A1 = B1$. Similarly, with $x = 1$ in Equation (91), we obtain $By = -y \cdot \overline{(A1)}$. Denoting $-\overline{(A1)} = -\overline{(B1)} \in \text{Im}(\mathbb{O})$ by c , we have shown

$$Ax = x \cdot c, \quad By = y \cdot c.$$

This turns Equation (91) into

$$(x \cdot c) \cdot \bar{y} = x \cdot (c \cdot \bar{y}),$$

which must hold true for all x and y . But this is possible if and only if $c \in \text{Re}(\mathbb{O})$. Since we already showed that c is purely imaginary, this implies that $c = 0$, and consequently that $u = v = 0$, which is in contradiction with our assumption. \blacksquare

Definition 8.5. Let \mathcal{F} be a singular foliation on a Riemannian manifold.

- \mathcal{F} is called *homogeneous* if there exists a Lie group of isometries $G \subset \text{Isom}(M, g)$, such that the leaves of \mathcal{F} are generated as the orbits of the action.
- \mathcal{F} is called *locally homogeneous* if for every point $q \in M$, there exists an open subset U such that $\iota_U^{-1} \mathcal{F}$ is homogeneous on (U, g_U) .

Otherwise, \mathcal{F} is called (locally) *non-homogeneous*.

Since the fundamental vector fields of isometries of (\mathbb{R}^n, g_{st}) which leave the origin fixed are always linear, we have the following immediate corollary of Lemma 8.4:

Corollary 8.6. The singular foliation \mathcal{F}_{OH} on $\mathbb{O}^2 \cong \mathbb{R}^{16}$, equipped with the standard Riemannian metric, is both non-homogeneous and locally non-homogeneous at the origin.

Remark 8.7. Corollary 8.6 can be used also as an alternative proof of the classical result that the octonionic Hopf fibration of S^{15} is non-homogeneous, which was shown in [L93] by examining all isometric Lie group actions on \mathbb{R}^{16} , and (implicitly) [GWZ86] by different methods.

Remark 8.8. The local non-homogeneity of the singular leaf decomposition \mathcal{L}_{OH} around the origin has been discussed in [MR19].

Finally, we improve the previous classical results, and show that local non-homogeneity does not hold even up to Hausdorff Morita equivalence.

Theorem 8.9. *Let \mathcal{F}_0 be any singular foliation on \mathbb{O}^2 having \mathcal{L}_{OH} as its leaf decomposition. Then $(\mathbb{O}^2, \mathcal{F}_0)$ is not Hausdorff Morita equivalent to any locally homogeneous singular foliation \mathcal{F} on some Riemannian manifold (M, g) .*

Proof. Assume that $(\mathbb{O}^2, \mathcal{F}_0) \sim_{ME} (M, \mathcal{F})$ for some locally homogeneous singular foliation \mathcal{F} on a Riemannian manifold (M, g) . Using the first part of Theorem 2.19, the origin in \mathbb{O}^2 , as the zero-dimensional leaf of \mathcal{F}_0 , corresponds to a leaf $L_q^{\mathcal{F}} \subset M$ of \mathcal{F} for some $q \in M$. Denote the orthogonal complement of $T_q L_q^{\mathcal{F}}$ in $T_q M$ by $\nu_q := (T_q L_q^{\mathcal{F}})^\perp$. We define the slice $S_q \subset M$ at $q \in L_q^{\mathcal{F}}$ as the image of the exponential map $\exp_q: \nu_q^\epsilon \rightarrow M$, where ν_q^ϵ stands for the vectors of length smaller than some $\epsilon > 0$, chosen small enough such that ν_q^ϵ lies inside the domain of definition of \exp_q .

The singular foliation $\iota_{S_q}^{-1} \mathcal{F}$ (see Example 2.15 and Proposition 2.17) is generated by vector fields in \mathcal{F} tangent to the slice S_q . The second part of Theorem 2.19 then implies that the foliated manifold $(S_q, \iota_{S_q}^{-1} \mathcal{F})$ is isomorphic to a neighborhood of the origin in the foliated manifold $(\mathbb{O}^2, \mathcal{F}_0)$. In particular, $(S_q, \iota_{S_q}^{-1} \mathcal{F})$ has a single leaf of dimension 0 and all the other leaves are diffeomorphic to 7-spheres.

On the other hand, by assumption there exists an open subset $U \subset M$ containing q , such that $\iota_{S_q}^{-1} \mathcal{F}$ is induced by a Lie group $G \subset \text{Isom}(U, g_U)$, acting on U by isometries. Assuming that ϵ is chosen small enough to have $S_q = \exp_q(\nu_q^\epsilon) \subset U$, we claim that the stabilizer H_q of G at q acts orthogonally on S_q and determines a singular foliation \mathcal{F}' . More precisely, since H_q acts on S_q by isometries and for every $h \in H_q \subset G$, we have

$$h \cdot \exp_q(v) = \exp_q(d_q L_h(v)),$$

we can define the left Lie group action $H_q \times \nu_q^\epsilon \rightarrow \nu_q^\epsilon$ by the mapping $(h, v) \rightarrow d_q L_h(v)$. Note that H_q acts linearly and since

$$g(d_q L_h(v), d_q L_h(v)) = L_h^* g(v, v) = g(v, v),$$

it preserves the norm. Consequently, it defines an orthogonal left Lie group action on \mathbb{O}^2 . But according to [L93], an orthogonal Lie group action on $(\mathbb{R}^{16}, g_{st})$ cannot induce 7-dimensional leaves. This contradicts what we found above. \blacksquare

9 \mathcal{L}_{OH} and singular Riemannian foliations

9.1 \mathcal{F}_{OH} , a counterexample

Question: Let (M, g, \mathcal{F}) be a geometric singular Riemannian foliation. Is it possible to find a module singular Riemannian foliation (M, g, \mathcal{F}') , having the same leaf decomposition as (M, g, \mathcal{F}) ?

Here, we claim that for \mathcal{O} being the sheaf of real analytic functions on a real analytic manifold, the singular octonionic Hopf foliation provides a counterexample.

Consider \mathbb{O}^2 as a real analytic manifold, equipped with the standard Riemannian metric g_{st} . The restriction of \mathcal{F}_{OH} to $S^{15} \subset \mathbb{O}^2$ induces the octonionic Hopf fibration, which is known to be a regular Riemannian foliation. M98's homothetic transformation lemma [M98] then implies that $(\mathbb{O}^2, g_{st}, \mathcal{F}_{OH})$ itself defines a geometric singular Riemannian foliation.

Theorem 9.1. *Let \mathcal{F} be any singular foliation on the real analytic Riemannian manifold (\mathbb{O}^2, g_{st}) , having \mathcal{L}_{OH} as its leaf decomposition. Then the geometric singular Riemannian foliation $(\mathbb{O}^2, g_{st}, \mathcal{F})$ is not a module singular Riemannian foliation.*

Proof. Let $(\mathbb{O}^2, g_{st}, \mathcal{F})$ be a module singular Riemannian foliation on the real analytic manifold \mathbb{O}^2 , with \mathcal{L}_{OH} as its leaf decomposition, and consider the vector field $X \in \mathcal{F}$. The Taylor expansion around the origin gives homogeneous polynomials $P_k(x, y)$ and $Q_k(x, y)$ of degree k , such that

$$X(x, y) = \langle P_k(x, y), \frac{\partial}{\partial x} \rangle + \langle Q_k(x, y), \frac{\partial}{\partial y} \rangle.$$

Denote the homogeneous part of degree k in X by X_k . Since X is tangent to the leaves in \mathcal{L}_{OH} , we have $X(0, 0) = 0$ which translates into

$$P_0(0, 0) = Q_0(x, y) \equiv 0.$$

In addition, Equation (86) implies that

$$\sum_{k=1}^{\infty} P_k(x, y) \cdot \bar{y} + x \cdot \overline{Q_k(x, y)} = 0$$

and in particular

$$\langle P_1(x, y), \frac{\partial}{\partial x} \rangle + \langle Q_1(x, y), \frac{\partial}{\partial y} \rangle = 0.$$

Similarly, we can show that the linear vector field X_1 satisfies Equation (87), and Lemma 8.2 implies that it is tangent to the leaves of \mathcal{L}_{OH} . But according to Lemma 8.4, there is no non-zero linear vector field tangent to the leaves in \mathcal{L}_{OH} , showing that $X_0 = X_1 = 0$. Consequently we have $X = \sum_{k=2}^{\infty} X_k$

As a direct consequence of the result above, every element in the space $\Omega^1(\mathbb{O}^2) \odot (g_{st})_b(\mathcal{F})$ vanishes at least quadratically at the origin.

In the other hand, we have

$$\mathcal{L}_X g_{st} = \sum_{k=2}^{\infty} \mathcal{L}_{X_k} g_{st}.$$

For $k = 2$, the homogeneous part $\mathcal{L}_{X_2} g_{st}$ is a symmetric 2-tensor of degree 1 in x and y . This term cannot be an element of $\Omega^1(\mathbb{O}^2) \odot (g_{st})_b(\mathcal{F})$, and X_2 is not a Killing vector field, consequently we obtain $X_2 = 0$.

Recursively, we can prove that $X_k = 0$ for all k , which implies $X = 0$. This completes the proof. ■

10 Universal Lie ∞ -algebroid of \mathcal{F}_{OH}

In [LGLS20], it is proven that for every singular foliation which admits a *geometric resolution*, one can associate a *Lie ∞ -algebroid* inducing it. (The necessary notions will be recalled in Section 10.1). This association turns out to be unique up to homotopy and leads to invariants of the singular foliation. In Section 10.2, we construct a Lie 3-algebroid that represents the universal Lie ∞ -algebroid of \mathcal{F}_{OH} . This Lie 3-algebroid will be used then in to prove that $\mathcal{G} \Rightarrow \mathbb{O}^2$ has the minimal dimension among all Lie groupoids having \mathcal{L}_{OH} as their orbits.

10.1 Lie ∞ -algebroids of singular foliations

In what follows, M is a smooth or real analytic manifold, or an affine variety over $K = \mathbb{R}$ or \mathbb{C} , whose sheaf of functions is denoted by \mathcal{O} . For every vector bundle $F \rightarrow M$, the space of sections $\Gamma(F)$ is viewed as a sheaf of \mathcal{O} -modules.

Definition 10.1. *A (split) Lie ∞ -algebroid $(E, (l_k)_{k \geq 1}, \rho)$ consists of a positively-graded vector bundle $E = \bigoplus_{i \geq 0} E_{-i}$ over a manifold M , equipped with:*

1. *A family of graded skew-symmetric and multilinear maps $l_k: \wedge^k \Gamma(E) \rightarrow \Gamma(E)$ of degree $2 - k$, called k -brackets, for all integers $k \geq 1$,*
2. *A vector bundle morphism $\rho: E_0 \rightarrow TM$, called the anchor.*

These must satisfy the following conditions:

- *For $k \neq 2$, the k -brackets l_k are \mathcal{O} -linear,*
- *The 2-bracket l_2 is \mathcal{O} -linear, except when at least one of the entries is of degree zero: for sections $x \in \Gamma(E_0)$, $y \in \Gamma(E)$, and $f \in \mathcal{O}$, the 2-bracket satisfies*

$$l_2(x, fy) = f l_2(x, y) + (\rho(x) \cdot f) y;$$

- *For every $x \in \Gamma(E_{-1})$, one has $\rho(l_1(x)) = 0$;*
- *The k -brackets satisfy the higher Jacobi identities: for every positive integer n and sections $x_1, \dots, x_n \in \Gamma(E)$,*

$$\sum_{i=1}^n (-1)^{i(n-i)} \sum_{\sigma \in \text{Un}(i, n-i)} \epsilon(\sigma) l_{n-i+1}(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0,$$

where $\text{Un}(i, n-i)$ stands for the set of $(i, n-i)$ -unshuffles, and $\epsilon(\sigma)$ is the signature of the permutation σ given by

$$x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(n)} = \epsilon(\sigma) x_1 \wedge \dots \wedge x_n.$$

A Lie n -algebroid is a Lie ∞ -algebroid with $E_{-i} = 0$ for $i \geq n$.

The \mathcal{O} -linearity of l_k with $k \neq 2$ implies that for these k we have $l_k: \wedge^k E \rightarrow E$. (We do not distinguish between the vector bundle morphisms and the induced map on sections).

Example 10.2. A Lie ∞ -algebroid $(E, (l_k)_{k \geq 1}, \rho)$ with $l_k = 0$ for all $k \geq 3$ is called a dg-Lie algebroid. Denoting the 1-bracket by d and the 2-bracket by $[\cdot, \cdot]$, the higher Jacobi identities become:

$$d \circ d = 0, \quad (93)$$

$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy], \quad (94)$$

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0, \quad (95)$$

for all homogeneous sections x, y , and z . Here, $|x|$ denotes the degree of the homogeneous section x .

In a Lie ∞ -algebroid, writing $d^{(i)} := l_1|_{E_{-i}}$, the higher Jacobi identities implies that the sequence

$$\cdots \longrightarrow E_{-3} \xrightarrow{d^{(3)}} E_{-2} \xrightarrow{d^{(2)}} E_{-1} \xrightarrow{d^{(1)}} E_0 \xrightarrow{\rho} TM$$

is a chain complex, called the *linear part* of the Lie ∞ -algebroid.

Definition 10.3. Let \mathcal{F} be a singular foliation on M . A geometric resolution of \mathcal{F} is a chain complex

$$\cdots \longrightarrow E_{-3} \xrightarrow{d^{(3)}} E_{-2} \xrightarrow{d^{(2)}} E_{-1} \xrightarrow{d^{(1)}} E_0 \xrightarrow{\rho} TM$$

with $\rho(\Gamma(E_0)) = \mathcal{F}$, such that for every open subset $U \subset M$ the chain complex

$$\cdots \longrightarrow \Gamma_U(E_{-3}) \xrightarrow{d^{(3)}} \Gamma_U(E_{-2}) \xrightarrow{d^{(2)}} \Gamma_U(E_{-1}) \xrightarrow{d^{(1)}} \Gamma_U(E_0) \xrightarrow{\rho} \mathcal{F}(U) \longrightarrow 0$$

is an exact sequence of $\mathcal{O}(U)$ -modules. The geometric resolution (E, d, ρ) is said to be minimal at $q \in M$, if $d_q^{(i)}: E_{-i}|_q \rightarrow E_{(i-1)}|_q$ vanishes for all $i \geq 1$.

Lemma 10.4 ([LGLS20]). Let \mathcal{F} be a singular foliation on an \mathcal{O} -manifold M .

For \mathcal{O} being the sheaf of smooth functions on M , geometric resolutions \mathcal{F} are in one-to-one correspondence with resolutions of the \mathcal{O} -module \mathcal{F} by locally finitely generated projective \mathcal{O} -modules.

For \mathcal{O} being the sheaf of polynomial or real analytic functions on M , geometric resolutions are in one-to-one correspondence with resolutions of \mathcal{F} by finitely generated free \mathcal{O} -modules.

The flatness theorems of Malgrange [T68] imply the following proposition on the transition between different choices of \mathcal{O} on \mathbb{R}^n in the study of geometric resolutions.

Proposition 10.5 ([LGLR22]). Let \mathcal{F} be a singular foliation on the affine variety \mathbb{R}^n . A geometric resolution of \mathcal{F} for \mathcal{O} being the sheaf of polynomial functions, is also a geometric resolution for \mathcal{O} being the sheaf of smooth functions.

Definition 10.6. Let \mathcal{F} be a singular foliation on M . A universal Lie ∞ -algebroid of \mathcal{F} is a Lie ∞ -algebroid $(E, (l_k)_{k \geq 1}, \rho)$, such that its linear part is a geometric resolution of \mathcal{F} . If $E_{-i} = 0$ for $i \geq n$, it is called a universal Lie n -algebroid.

As it is clear from Definition 10.6, for a singular foliation to be induced by a universal Lie ∞ -algebroid, it is required to admit a geometric resolution. There are examples of singular foliations which do not admit a geometric resolution, see Example 3.38 in [LGLS20]. However, if \mathcal{O} is the sheaf of polynomial or real analytic functions, Hilbert's syzygy theorem ensures the existence of a geometric resolution in a neighborhood of every point.

Theorem 10.7 ([LGLS20]). *Let \mathcal{F} be a singular foliation on M , admitting a geometric resolution (E, d, ρ) . There exist a universal Lie ∞ -algebroid, having (E, d, ρ) as its linear part.*

Such a universal Lie ∞ -algebroid is unique up to some precise notion of homotopies; for details on this and about why it is universal, see [LGLS20].

Let us recall that for a foliated manifold (M, \mathcal{F}) the minimal number of locally generating vector fields for \mathcal{F} around $q \in M$ is equal to the dimension of the fiber \mathcal{F}_q .

Proposition 10.8 ([LGLS20]). *Let $(E, (l_k)_{k \geq 1}, \rho)$ be a universal Lie ∞ -algebroid of a singular foliation \mathcal{F} on M . If the linear part (E, l_1, ρ) is minimal at $q \in M$, then $\text{rank}(E_0) = \dim(\mathcal{F}_q) =: r$. In particular, the rank of every Lie algebroid inducing \mathcal{F} in a neighborhood of q is at least r .*

10.2 The universal Lie 3-algebroid of \mathcal{F}_{OH}

We start the construction of the Lie 3-algebroid by choosing a geometric resolution for \mathcal{F}_{OH} . Let us first view \mathcal{F}_{OH} as a sheaf of \mathcal{O} -modules over the affine variety $\mathbb{O}^2 \cong \mathbb{R}^{16}$, where \mathcal{O} is the sheaf of polynomial functions. This is possible since \mathcal{F}_{OH} is generated by polynomial vector fields. In this setting, according to Lemma 10.4, it suffices to find a free resolution of the module \mathcal{F}_{OH} . This can be done using Eisenbaud's Macaulay2, as explained in Appendix 2. The result is as follows:

The graded vector bundle of the geometric resolution is given by the trivial vector bundles E_0, E_{-1} and E_{-2} over \mathbb{O}^2 , where

$$E_0 := \underline{\mathbb{O}^2}, \quad E_{-1} := \underline{\mathbb{R} \oplus \mathbb{O} \oplus \mathbb{R}}, \quad E_{-2} := \underline{\mathbb{R}}.$$

The vector bundle morphisms $\rho, d^{(1)}$ and $d^{(2)}$ in the sequence

$$0 \longrightarrow E_{-2} \xrightarrow{d^{(2)}} E_{-1} \xrightarrow{d^{(1)}} E_0 \xrightarrow{\rho} T\mathbb{O}^2 \cong \underline{\mathbb{O}^2} \quad (96)$$

are given by the following evaluations on the constant global sections $\begin{pmatrix} u \\ v \end{pmatrix} \in \Gamma(E_0)$,

$\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \in \Gamma(E_{-1})$ and $t \in \Gamma(E_{-2})$:

$$\begin{aligned} \rho \begin{pmatrix} u \\ v \end{pmatrix} &:= \begin{pmatrix} \|x\|^2 u + (x \cdot \bar{y}) \cdot v - (\langle x, u \rangle + \langle y, v \rangle) x \\ \|y\|^2 v + (y \cdot \bar{x}) \cdot u - (\langle x, u \rangle + \langle y, v \rangle) y \end{pmatrix}, \\ d^{(1)} \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} &:= \begin{pmatrix} \mu x + a \cdot y \\ \nu y + \bar{a} \cdot x \end{pmatrix}, \\ d^{(2)}(t) &:= \begin{pmatrix} -\|y\|^2 t \\ (x \cdot \bar{y}) t \\ -\|x\|^2 t \end{pmatrix}. \end{aligned} \tag{97}$$

Lemma 10.9. *The triple $(E = \bigoplus_{i=0}^2 E_{-i}, d, \rho)$ is a geometric resolution of \mathcal{F}_{OH} on the smooth manifold $\mathbb{O}^2 \cong \mathbb{R}^{16}$. Moreover, this geometric resolution is minimal at the origin.*

Proof. Since $(E = \bigoplus_{i=0}^2 E_{-i}, d, \rho)$ is a geometric resolution of \mathcal{F}_{OH} as a sheaf of modules over the ring of polynomials, Proposition 10.5 implies that it is also a geometric resolution for \mathcal{F}_{OH} when \mathcal{O} is the sheaf of smooth functions on \mathbb{R}^n . Clearly, both $d^{(1)}$ and $d^{(2)}$ vanish at the origin, showing that the geometric resolution is minimal at this point. \blacksquare

We construct a degree 2-bracket $[\cdot, \cdot]: \Gamma(E) \wedge \Gamma(E) \rightarrow \Gamma(E)$ of degree 0 as follows:

$$\begin{aligned} \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right] &:= (\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} u' \\ v' \end{pmatrix} - (\langle x, u' \rangle + \langle y, v' \rangle) \begin{pmatrix} u \\ v \end{pmatrix}, \\ \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] &:= \begin{pmatrix} -2\langle y, \bar{a} \cdot u \rangle + 2\langle y, v \rangle \mu \\ x \cdot (\bar{u} \cdot a) + (a \cdot v) \cdot \bar{y} - \mu(x \cdot \bar{v}) - \nu(u \cdot \bar{y}) \\ -2\langle x, a \cdot v \rangle + 2\langle x, u \rangle \nu \end{pmatrix}, \\ \left[\begin{pmatrix} u \\ v \end{pmatrix}, t \right] &:= 2(\langle x, u \rangle + \langle y, v \rangle) t, \\ \left[\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right] &:= 4\langle a, a' \rangle - 2\mu\nu' - 2\mu'\nu, \end{aligned} \tag{98}$$

and extend it to all sections of E using Leibniz rule. We define $l_k = 0$ for all $k \geq 3$.

Proposition 10.10. *The triple $(E, (l_k)_{k \geq 1}, \rho)$ defines a Lie 3-algebroid. It is a universal Lie 3-algebroid of \mathcal{F}_{OH} , whose linear part is a geometric resolution minimal at the origin.*

Proof. Since $l_k = 0$ for $k \geq 3$, we have to show that $(E, (l_k)_{k \geq 1}, \rho)$ is a dg-Lie algebroid. Note that as (E, d, ρ) is a geometric resolution, we have $\rho \circ d = 0$ and $d \circ d = 0$. It suffices to verify Equations (94) and (95) in Example (10.2).

Step 1. Verifying Equation (94): For constant global sections $\begin{pmatrix} u \\ v \end{pmatrix} \in \Gamma(E_0)$ and

$\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \in \Gamma(E_{-1})$, we have

$$\begin{aligned}
d^{(1)}\left(\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}\right]\right) &= d^{(1)}\left(\begin{pmatrix} -2\langle y, \bar{a}\cdot u \rangle + 2\langle y, v \rangle \mu \\ x\cdot(\bar{u}\cdot a) + (a\cdot v)\cdot\bar{y} - \mu(x\cdot\bar{v}) - \nu(u\cdot\bar{y}) \\ -2\langle x, a\cdot v \rangle + 2\langle x, u \rangle \nu \end{pmatrix}\right) \\
&= \begin{pmatrix} -2\langle y, \bar{a}\cdot u \rangle x + 2\langle y, v \rangle \mu x + (x\cdot(\bar{u}\cdot a))\cdot y + \|y\|^2 a\cdot v - \mu(x\cdot\bar{v})\cdot y - \nu\|y\|^2 u \\ -2\langle x, a\cdot v \rangle y + 2\langle x, u \rangle \nu y + (y\cdot(\bar{v}\cdot\bar{a}))\cdot x + \|x\|^2 \bar{a}\cdot u - \nu(y\cdot\bar{u})\cdot x - \mu\|x\|^2 v \end{pmatrix} \\
&= \begin{pmatrix} -(x\cdot\bar{y})\cdot(\bar{a}\cdot u) + \|y\|^2 a\cdot v + \mu(x\cdot\bar{y})\cdot v - \nu\|y\|^2 u \\ -(y\cdot\bar{x})\cdot(a\cdot v) + \|x\|^2 \bar{a}\cdot u + \nu(y\cdot\bar{x})\cdot u - \mu\|x\|^2 v \end{pmatrix},
\end{aligned}$$

where we used Equation (63) to obtain the second equality, and

$$\begin{aligned}
\left[\begin{pmatrix} u \\ v \end{pmatrix}, d^{(1)}\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}\right] &= \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu x + a\cdot y \\ \nu y + \bar{a}\cdot x \end{pmatrix}\right] \\
&= \begin{pmatrix} \mu(\|x\|^2 u + (x\cdot\bar{y})\cdot v - (\langle x, u \rangle + \langle y, v \rangle)x) + a\cdot(\|y\|^2 v + (y\cdot\bar{x})\cdot u - (\langle x, u \rangle + \langle y, v \rangle)y) \\ \nu(\|y\|^2 v + (y\cdot\bar{x})\cdot u - (\langle x, u \rangle + \langle y, v \rangle)y) + \bar{a}\cdot(\|x\|^2 u + (x\cdot\bar{y})\cdot v - (\langle x, u \rangle + \langle y, v \rangle)x) \end{pmatrix} \\
&\quad + (\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} \mu x + a\cdot y \\ \nu y + \bar{a}\cdot x \end{pmatrix} - (\langle x, \mu x + a\cdot y \rangle + \langle y, \nu y + \bar{a}\cdot x \rangle) \begin{pmatrix} u \\ v \end{pmatrix}.
\end{aligned}$$

After cancellations and using Equations (67) and (68) in the last term, it becomes equal to

$$= \begin{pmatrix} a\cdot((y\cdot\bar{x})\cdot u) + \|y\|^2 a\cdot v + \mu(x\cdot\bar{y})\cdot v - \nu\|y\|^2 u \\ -\bar{a}\cdot((x\cdot\bar{y})\cdot v) + \|x\|^2 \bar{a}\cdot u + \nu(y\cdot\bar{x})\cdot u - \mu\|x\|^2 v \end{pmatrix} - 2\langle a, x\cdot\bar{y} \rangle \begin{pmatrix} u \\ v \end{pmatrix},$$

which together with Equation (63) gives

$$d^{(1)}\left(\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}\right]\right) = \left[\begin{pmatrix} u \\ v \end{pmatrix}, d^{(1)}\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}\right].$$

For global constant sections $\begin{pmatrix} u \\ v \end{pmatrix} \in \Gamma(E_0)$ and $t \in \Gamma(E_{-2})$ one has

$$d^{(2)}\left(\left[\begin{pmatrix} u \\ v \end{pmatrix}, t\right]\right) = 2(\langle x, u \rangle + \langle y, v \rangle)d^{(t)}(t) = 2(\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} -\|y\|^2 t \\ (x\cdot\bar{y})t \\ -\|x\|^2 t \end{pmatrix}.$$

On the other hand, since by Lemma 7.4 and Corollary 7.5 the functions $\|x\|^2$, $\|y\|^2$ and $x\cdot\bar{y}$ are constant along orbits of $\mathcal{G} \Rightarrow \mathbb{O}^2$, we have

$$\rho \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} -\|y\|^2 t \\ (x\cdot\bar{y})t \\ -\|x\|^2 t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives

$$\begin{aligned}
\left[\begin{pmatrix} u \\ v \end{pmatrix}, d^{(2)}t\right] &= \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} -\|y\|^2 t \\ (x\cdot\bar{y})t \\ -\|x\|^2 t \end{pmatrix}\right] \\
&= \begin{pmatrix} -2\langle y, (y\cdot\bar{x})\cdot u \rangle t - 2\|y\|^2 \langle y, v \rangle t \\ x\cdot(\bar{u}\cdot(x\cdot\bar{y}))t + ((x\cdot\bar{y})\cdot v)\cdot\bar{y}t + \|y\|^2 x\cdot\bar{v}t + \|x\|^2 u\cdot\bar{y}t \\ -2\langle x, (x\cdot\bar{y})\cdot v \rangle t - 2\|x\|^2 \langle x, u \rangle t \end{pmatrix}
\end{aligned}$$

Using Equation (67), the first and the third components become $2(\langle x, u \rangle + \langle y, v \rangle)(-\|y\|^2 t)$ and $2(\langle x, u \rangle + \langle y, v \rangle)(-\|x\|^2 t)$, respectively. After Moufang identities (71) and (72), and using Equation (63), the second component equals $2(\langle x, u \rangle + \langle y, v \rangle)((x \cdot \bar{y})t)$. These together give

$$d^{(2)}\left(\left[\begin{pmatrix} u \\ v \end{pmatrix}, t\right]\right) = \left[\begin{pmatrix} u \\ v \end{pmatrix}, d^{(2)}t\right].$$

Finally, for constant global sections $\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \in \Gamma(E_{-1})$ one has

$$d^{(2)}\left(\left[\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}\right]\right) = d^{(2)}(4\langle a, a' \rangle - \mu\nu' - \mu'\nu) = \begin{pmatrix} -2\|y\|^2(2\langle a, a' \rangle - \mu\nu' - \mu'\nu) \\ 2(x \cdot \bar{y})(2\langle a, a' \rangle - \mu\nu' - \mu'\nu) \\ -2\|x\|^2(2\langle a, a' \rangle - \mu\nu' - \mu'\nu) \end{pmatrix}.$$

On the other hand, one has

$$\begin{aligned} & [d^{(1)}\left(\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}\right)] - [d^{(1)}\left(\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, d^{(1)}\left(\begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}\right)\right)] \\ &= [d^{(1)}\left(\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}\right)] + \left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\} = \left[\begin{pmatrix} \mu x + a \cdot y \\ \nu y + \bar{a} \cdot x \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right] + \left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\} \\ &= \begin{pmatrix} -2\langle y, \bar{a}' \cdot (\mu x + a \cdot y) \rangle + 2\langle y, \nu y + \bar{a} \cdot x \rangle \mu' \\ x \cdot ((\mu \bar{x} + \bar{y} \cdot \bar{a}) \cdot a') + (a' \cdot (\nu y + \bar{a} \cdot x)) \cdot \bar{y} - \mu'(x \cdot (\nu \bar{y} + \bar{x} \cdot a)) - \nu'((\mu x + a \cdot y) \cdot \bar{y}) \\ -2\langle x, a' \cdot (\nu y + \bar{a} \cdot x) \rangle + 2\langle x, \mu x + a \cdot y \rangle \nu' \end{pmatrix} \\ &+ \left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\} \\ &= \begin{pmatrix} -2\|y\|^2(\langle a, a' \rangle - \mu'\nu) + 2\langle a, x \cdot \bar{y} \rangle \mu' - 2\langle a', x \cdot \bar{y} \rangle \mu \\ (\mu\|x\|^2 + \nu\|y\|^2)a' - (\mu'\|x\|^2 + \nu'\|y\|^2)a + x \cdot ((\bar{y} \cdot \bar{a}) \cdot a') + (a' \cdot (\bar{a} \cdot x)) \cdot \bar{y} - (x \cdot \bar{y})(\mu\nu' + \mu'\nu) \\ -2\|x\|^2(\langle a, a' \rangle - \mu\nu') + 2\langle a, x \cdot \bar{y} \rangle \nu' - 2\langle a', x \cdot \bar{y} \rangle \mu' \end{pmatrix} \\ &+ \left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\} \\ &= \begin{pmatrix} -2\|y\|^2(2\langle a, a' \rangle - \mu\nu' - \mu'\nu) \\ x \cdot ((\bar{y} \cdot \bar{a}) \cdot a' + (\bar{y} \cdot \bar{a}') \cdot a) + (a \cdot (\bar{a}' \cdot x) + a' \cdot (\bar{a} \cdot x)) \cdot \bar{y} - (x \cdot \bar{y})(\mu\nu' + \mu'\nu) \\ -2\|x\|^2(\langle a, a' \rangle - \mu\nu' - \mu'\nu) \end{pmatrix}, \end{aligned}$$

where $\left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\}$ stands for the terms obtained by switching of the sections.

Equation (63) then implies

$$d^{(2)}\left(\left[\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}\right]\right) = [d^{(1)}\left(\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}\right)] - [d^{(1)}\left(\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, d^{(1)}\left(\begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}\right)\right)].$$

The other cases are obvious for degree reasons.

Step 2. Verifying Equation (95): Similar to approach of the first step, we proceed by verifying Equation (95) for some particular choices of constant global sections. For

all the other choices, the Jacobiator vanishes for degree reasons.

Since restriction to E_0 coincides with the Lie algebroid of $\mathcal{G} \Rightarrow \mathbb{O}^2$, for three sections of degree 0, the Jacobi identity is already satisfied.

For constant global sections $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in \Gamma(E_0)$ and $\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \in \Gamma(E_{-1})$, we claim that

$$\left[\begin{pmatrix} u \\ v \end{pmatrix}, \left[\begin{pmatrix} u' \\ v' \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] \right] - \left[\begin{pmatrix} u' \\ v' \end{pmatrix}, \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] \right] - \left[\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right], \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (99)$$

We proceed by showing that each component of the result vanishes. Note that

$$\left[\begin{pmatrix} u \\ v \end{pmatrix}, \left[\begin{pmatrix} u' \\ v' \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] \right] = \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} x \cdot (\bar{u}' \cdot a) - 2\langle y, \bar{a} \cdot u' \rangle + 2\langle y, v' \rangle \mu \\ (a \cdot v') \cdot \bar{y} - \mu(x \cdot \bar{v}') - \nu(u' \cdot \bar{y}) \\ -2\langle x, a \cdot v' \rangle + 2\langle x, u' \rangle \nu \end{pmatrix} \right],$$

and

$$\begin{aligned} \left[\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right], \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] &= [(\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} u' \\ v' \end{pmatrix} - (\langle x, u' \rangle + \langle y, v' \rangle) \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}] \\ &= (\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} x \cdot (\bar{u}' \cdot a) - 2\langle y, \bar{a} \cdot u' \rangle + 2\langle y, v' \rangle \mu \\ (a \cdot v') \cdot \bar{y} - \mu(x \cdot \bar{v}') - \nu(u' \cdot \bar{y}) \\ -2\langle x, a \cdot v' \rangle + 2\langle x, u' \rangle \nu \end{pmatrix} \\ &\quad - \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\}. \end{aligned}$$

The first component of the left-hand side of Equation (99) then decomposes into

$$\begin{aligned} &= -2\langle \|y\|^2 v + (y \cdot \bar{x}) \cdot u - (\langle x, u \rangle + \langle y, v \rangle) y, \bar{a} \cdot u' \rangle + 2\langle \|y\|^2 v + (y \cdot \bar{x}) \cdot u - (\langle x, u \rangle + \langle y, v \rangle) y, v' \rangle \mu \\ &\quad - 2\langle y, ((\bar{a} \cdot u') \cdot \bar{x} + y \cdot (\bar{v}' \cdot \bar{a}) - \mu(v' \cdot \bar{x}) - \nu(y \cdot \bar{u}')) \cdot u \rangle + 2\langle y, v \rangle (-2\langle y, \bar{a} \cdot u' \rangle + 2\langle y, v' \rangle \mu) \\ &\quad - (\langle x, u \rangle + \langle y, v \rangle) (-2\langle y, \bar{a} \cdot u' \rangle + 2\langle y, v' \rangle \mu) \\ &\quad - \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\}. \end{aligned}$$

Straightforward cancellations then turn it into

$$\begin{aligned} &= -2\langle \|y\|^2 v + (y \cdot \bar{x}) \cdot u, \bar{a} \cdot u' \rangle + 2\langle (y \cdot \bar{x}) \cdot u, v' \rangle \mu \\ &\quad - 2\langle y, ((\bar{a} \cdot u') \cdot \bar{x} + y \cdot (\bar{v}' \cdot \bar{a}) - \mu(v' \cdot \bar{x}) - \nu(y \cdot \bar{u}')) \cdot u \rangle \\ &\quad + 4\langle x, u \rangle (\langle y, \bar{a} \cdot u' \rangle - \langle y, v' \rangle \mu) \\ &\quad - \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\}, \end{aligned}$$

which vanishes after the following consequences of Equations (63), (67) and (68):

$$\begin{aligned} \langle (y \cdot \bar{x}) \cdot u, v' \rangle &= \langle -(y \cdot \bar{u}) \cdot x + 2\langle x, u \rangle y, v' \rangle = -\langle y, (v' \cdot x) \cdot u \rangle + 2\langle x, u \rangle \langle y, v' \rangle, \\ \langle y, ((\bar{a} \cdot u') \cdot \bar{x}) \cdot u \rangle &= \langle (y \cdot \bar{u}) \cdot x, \bar{a} \cdot u' \rangle = \langle -(y \cdot \bar{x}) \cdot u + 2\langle x, u \rangle y, \bar{a} \cdot u' \rangle, \\ \langle y, (y \cdot (\bar{v}' \cdot \bar{a})) \cdot u \rangle &= \langle \|y\|^2 v', \bar{a} \cdot u \rangle, \\ \langle y, (y \cdot \bar{u}) \cdot u' \rangle &= \|y\|^2 \langle u, u' \rangle = \langle y, (y \cdot \bar{u}') \cdot u \rangle. \end{aligned}$$

The third component vanishes in a similar way. Finally, the second component equals to

$$\begin{aligned}
&= (\|x\|^2 u + (x \cdot \bar{y}) \cdot v - (\langle x, u \rangle + \langle y, v \rangle) x) \cdot (\bar{u}' \cdot a) + (a \cdot v') \cdot (\|y\|^2 \bar{v} + \bar{u} \cdot (x \cdot \bar{y}) - (\langle x, u \rangle + \langle y, v \rangle) \bar{y}) \\
&- \mu((\|x\|^2 u + (x \cdot \bar{y}) \cdot v - (\langle x, u \rangle + \langle y, v \rangle) x) \cdot \bar{v}') - \nu(u' \cdot (\|y\|^2 \bar{v} + \bar{u} \cdot (x \cdot \bar{y}) - (\langle x, u \rangle + \langle y, v \rangle) \bar{y})) \\
&+ x \cdot (\bar{u} \cdot (x \cdot (\bar{u}' \cdot a) + (a \cdot v') \cdot \bar{y} - \mu(x \cdot \bar{v}') - \nu(u' \cdot \bar{y}))) + ((x \cdot (\bar{u}' \cdot a) + (a \cdot v') \cdot \bar{y} - \mu(x \cdot \bar{v}') - \nu(u' \cdot \bar{y}))) \cdot v \cdot \bar{y} \\
&- (-2\langle y, \bar{a} \cdot u' \rangle + 2\langle y, v' \rangle \mu) \cdot (x \cdot \bar{v}) - (-2\langle x, a \cdot v' \rangle + 2\langle x, u' \rangle \nu) \cdot (u \cdot \bar{y}) \\
&- (\langle x, u \rangle + \langle y, v \rangle) \cdot (x \cdot (\bar{u}' \cdot a) + (a \cdot v') \cdot \bar{y} - \mu(x \cdot \bar{v}') - \nu(u' \cdot \bar{y})) \\
&- \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&= \mu(-\|x\|^2 u \cdot \bar{v}' - ((x \cdot \bar{y}) \cdot v) \cdot \bar{v}' - (x \cdot \bar{u} \cdot x) \cdot \bar{v}' - ((x \cdot \bar{v}') \cdot v) \cdot \bar{y} - 2\langle y, v' \rangle x \cdot \bar{v} + 2\langle x, u \rangle x \cdot v' + 2\langle y, v \rangle x \cdot \bar{v}') \\
&+ \nu(-\|y\|^2 u' \cdot \bar{v} - u' \cdot (\bar{u} \cdot (x \cdot \bar{y})) - x \cdot (\bar{u} \cdot (u' \cdot \bar{y})) - u' \cdot (\bar{y} \cdot v \cdot \bar{y}) - 2\langle x, u' \rangle u \cdot \bar{y} + 2\langle x, u \rangle u' \cdot \bar{y} + 2\langle y, v \rangle u' \cdot \bar{y}) \\
&+ (\|x\|^2 u + x \cdot \bar{u} \cdot x) \cdot (u' \cdot a) + (a \cdot v') \cdot (\|y\|^2 \bar{v} + \bar{y} \cdot v \cdot \bar{y}) + ((x \cdot \bar{y}) \cdot v) \cdot (\bar{u}' \cdot a) + (a \cdot v') \cdot (\bar{u} \cdot (x \cdot \bar{y})) \\
&+ x \cdot (\bar{u} \cdot ((a \cdot v') \cdot \bar{y})) + ((x \cdot (\bar{u}' \cdot a)) \cdot v) \cdot \bar{y} + 2\langle y, \bar{a} \cdot u' \rangle x \cdot \bar{v} + 2\langle x, a \cdot v' \rangle u \cdot \bar{y} \\
&- 2(\langle x, u \rangle + \langle y, v \rangle) x \cdot (\bar{u}' \cdot a) - 2(\langle x, u \rangle + \langle y, v \rangle) (a \cdot v') \cdot \bar{y} \\
&- \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\}.
\end{aligned}$$

Here, the coefficient of μ vanishes as a result of the following identities, concluded from Equation (63):

$$\begin{aligned}
\|x\|^2 u \cdot \bar{v}' + (x \cdot \bar{u} \cdot x) \cdot \bar{v}' &= 2\langle x, u \rangle x \cdot \bar{v}', \\
((x \cdot \bar{y}) \cdot v) \cdot \bar{v}' &= 2\langle y, v \rangle x \cdot \bar{v}' - 2\langle y, v' \rangle x \cdot \bar{v} + ((x \cdot \bar{v}) \cdot v') \cdot \bar{y}.
\end{aligned}$$

The coefficient of ν vanishes in a similar way. Using the identities $\|x\|^2 u + x \cdot \bar{u} \cdot x = 2\langle x, u \rangle x$ and $\|y\|^2 \bar{v} + \bar{y} \cdot v \cdot \bar{y} = 2\langle y, v \rangle \bar{y}$, the remaining terms simplify into

$$\begin{aligned}
&= ((x \cdot \bar{y}) \cdot v) \cdot (\bar{u}' \cdot a) + (a \cdot v') \cdot (\bar{u} \cdot (x \cdot \bar{y})) \\
&+ x \cdot (\bar{u} \cdot ((a \cdot v') \cdot \bar{y})) + ((x \cdot (\bar{u}' \cdot a)) \cdot v) \cdot \bar{y} + 2\langle y, \bar{a} \cdot u' \rangle x \cdot \bar{v} + 2\langle x, a \cdot v' \rangle u \cdot \bar{y} \\
&- 2\langle y, v \rangle x \cdot (\bar{u}' \cdot a) - 2\langle x, u \rangle (a \cdot v') \cdot \bar{y} \\
&- \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\}.
\end{aligned}$$

Again, iterative use of Equation (63) gives

$$\begin{aligned}
((x \cdot \bar{y}) \cdot v) \cdot (\bar{u}' \cdot a) &= 2\langle y, v \rangle x \cdot (\bar{u}' \cdot a) - 2\langle y, \bar{a} \cdot u' \rangle x \cdot \bar{v} + 2\langle u', a \cdot v \rangle x \cdot \bar{y} - ((x \cdot (\bar{u}' \cdot a)) \cdot v) \cdot \bar{y}, \\
(a \cdot v') \cdot (\bar{u} \cdot (x \cdot \bar{y})) &= 2\langle x, u \rangle (a \cdot v') \cdot \bar{y} - 2\langle x, a \cdot v' \rangle u \cdot \bar{y} + 2\langle u, a \cdot v' \rangle x \cdot \bar{y} - x \cdot (\bar{u} \cdot ((a \cdot v') \cdot \bar{y})),
\end{aligned}$$

which implies that the second component vanishes as well, and Equation (99) is verified.

For global constant sections $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in \Gamma(E_0)$ and $t \in \Gamma(E_{-2})$, we have

$$\begin{aligned}
& \left[\begin{pmatrix} u \\ v \end{pmatrix}, \left[\begin{pmatrix} u' \\ v' \end{pmatrix}, t \right] \right] - \left[\begin{pmatrix} u' \\ v' \end{pmatrix}, \left[\begin{pmatrix} u \\ v \end{pmatrix}, t \right] \right] - \left[\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right], t \right] \\
&= \left[\begin{pmatrix} u \\ v \end{pmatrix}, 2(\langle x, u' \rangle + \langle y, v' \rangle)t \right] - [(\langle x, u \rangle + \langle y, v \rangle) \begin{pmatrix} u' \\ v' \end{pmatrix}, t] - \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\} \\
&= 2(\langle \|x\|^2 u + (x \cdot \bar{y}) \cdot v - (\langle x, u \rangle + \langle y, v \rangle)x, u' \rangle + \langle \|y\|^2 v + (y \cdot \bar{x}) \cdot u - (\langle x, u \rangle + \langle y, v \rangle)y, v' \rangle)t \\
&+ 4(\langle x, u \rangle + \langle y, v \rangle)(\langle x, u' \rangle + \langle y, v' \rangle)t - 2(\langle x, u \rangle + \langle y, v \rangle)(\langle x, u' \rangle + \langle y, v' \rangle)t \\
&- \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} \right\} = 0
\end{aligned}$$

which vanishes due to the following consequences of Equation (67):

$$\begin{aligned}
\langle (x \cdot \bar{y}) \cdot v, u' \rangle &= \langle (y \cdot \bar{x}) \cdot u', v \rangle, \\
\langle (x \cdot \bar{y}) \cdot v', u \rangle &= \langle (y \cdot \bar{x}) \cdot u, v' \rangle.
\end{aligned}$$

Finally, for the constant global sections $\begin{pmatrix} u \\ v \end{pmatrix} \in \Gamma(E_0)$ and $\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \in \Gamma(E_{-1})$ we have

$$\begin{aligned}
& \left[\begin{pmatrix} u \\ v \end{pmatrix}, \left[\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right] \right] - \left[\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right] \right] - \left[\begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix}, \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \right] \right] \\
&= \left[\begin{pmatrix} u \\ v \end{pmatrix}, \mu\nu' - \langle a, a' \rangle \right] - \left[\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix}, \begin{pmatrix} -2\langle y, \bar{a}' \cdot u \rangle + 2\langle y, v \rangle \mu' \\ x \cdot (\bar{u} \cdot a') + (a' \cdot v) \cdot \bar{y} - \mu(x \cdot \bar{v}) - \nu'(u \cdot \bar{y}) \\ -2\langle x, a' \cdot v \rangle + 2\langle x, u \rangle \nu' \end{pmatrix} \right] \\
&+ \left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\} \\
&= 2(\langle x, u \rangle + \langle y, v \rangle)(\mu\nu' - \langle a, a' \rangle) - 2\mu(-\langle x, a' \cdot v \rangle + \langle x, u \rangle \nu') \\
&- 2\nu(-\langle y, \bar{a}' \cdot u \rangle + \langle y, v \rangle \mu') + 2\langle a, x \cdot (\bar{u} \cdot a') + (a' \cdot v) \cdot \bar{y} - \mu'(x \cdot \bar{v}) - \nu'(u \cdot \bar{y}) \rangle \\
&+ \left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\},
\end{aligned}$$

which simplifies into

$$\begin{aligned}
&= -2(\langle x, u \rangle + \langle y, v \rangle)\langle a, a' \rangle + 2\langle y, v \rangle(\mu\nu' - \mu'\nu) + 2\langle a, x \cdot (\bar{u} \cdot a') + (a' \cdot v) \cdot \bar{y} \rangle \\
&+ 2(\mu\langle x, a' \cdot v \rangle + \nu\langle y, \bar{a}' \cdot u \rangle - \mu'\langle x, a \cdot v \rangle - \nu'\langle y, \bar{a} \cdot u \rangle) \\
&+ \left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\}.
\end{aligned}$$

By anti-symmetry, the second and the fourth terms of the expression cancel with the corresponding terms in $\left\{ \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \leftrightarrow \begin{pmatrix} \mu' \\ a' \\ \nu' \end{pmatrix} \right\}$. The remaining terms can be written as

$$= 2\langle a, x \cdot (\bar{u} \cdot a') + u \cdot (\bar{x} \cdot a') - 2\langle x, u \rangle a' \rangle + 2\langle a, (a' \cdot y) \cdot \bar{v} + (a' \cdot v) \cdot \bar{y} - 2\langle y, v \rangle a' \rangle = 0$$

which vanishes as a result of Equation (63). ■

Theorem 10.11. *The Lie algebroid $(E_0, [\cdot, \cdot], \rho)$ and correspondingly the Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{O}^2$ have the minimal dimension among Lie algebroids and Lie groupoids over \mathcal{O}^2 , with \mathcal{F}_{OH} as their orbits.*

Proof. Since the universal Lie 3-algebroid of \mathcal{F}_{OH} introduced in Proposition 10.10 is minimal at the origin, and $\text{rank}(E_0) = 16$, Proposition 10.8 implies that every such Lie algebroid is of dimension at least 16. As a consequence, since \mathcal{O}^2 is a 16-dimensional manifold, every Lie groupoid inducing \mathcal{F}_{OH} has a manifold of arrows with dimension at least 32. ■

1 Almost Killing Lie algebroids

In the appendix we recall the notion of almost Killing Lie algebroids as defined previously in [KS19] and provide their relation to module singular Riemannian foliations defined in this paper. (See, in particular, Proposition 1.5 below, but also Theorem 3.6 in the main text).

Definition 1.1. *A vector bundle $A \rightarrow M$ equipped with a vector bundle morphism $\rho: A \rightarrow TM$ covering the identity is called an anchored bundle. Let (A, ρ) be an anchored bundle equipped with a skew-symmetric bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$. The triple $(A, \rho, [\cdot, \cdot]_A)$ is called an almost Lie algebroid if the induced map $\rho: \Gamma(A) \rightarrow \mathfrak{X}(M)$ preserves the brackets, and the Leibniz rule is satisfied:*

$$[s, fs']_A = (\rho(s) \cdot f) s' + f[s, s']_A.$$

Definition 1.2. *Let (A, ρ) be an anchored bundle over M and $E \rightarrow M$ a vector bundle over the same base. An A -connection on E is a $C^\infty(M)$ -linear map ${}^A\nabla$ from $\Gamma(A)$ to $\text{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(E))$ satisfying*

$${}^A\nabla_s(fe) = (\rho(s) \cdot f) e + f{}^A\nabla_s e,$$

for every $f \in C^\infty(M)$, $e \in \Gamma(E)$ and $s \in \Gamma(A)$.

An anchored bundle (A, ρ) together with an ordinary connection on A , $\nabla: \Gamma(A) \rightarrow \Gamma(T^*M \otimes A)$, defines an A -connection ${}^A\nabla$ on TM by:

$${}^A\nabla_s X := \mathcal{L}_{\rho(s)} X + \rho(\nabla_X s), \quad (100)$$

valid for every $s \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$. Note that by assuming the Leibniz rule and the commutativity of ${}^A\nabla_s$ with contractions, these derivations can be extended to arbitrary tensor powers of TM and T^*M .

Definition 1.3. *Let $(A, \rho, [\cdot, \cdot]_A)$ be an almost Lie algebroid over a Riemannian manifold (M, g) and $\nabla: \Gamma(A) \rightarrow \Gamma(T^*M \otimes A)$ a connection on A . Then (A, ∇) and (M, g) are called compatible if*

$${}^A\nabla g = 0,$$

where the A -connection ${}^A\nabla$ is defined by Equation (100). The triple (A, ∇, g) is called a Killing almost Lie algebroid over M .

Lemma 1.4. *Let (A, ρ) be an anchored vector bundle over a Riemannian manifold (M, g) , and let ∇ be an ordinary connection on A . The triple (A, ∇, g) satisfies ${}^A\nabla g = 0$ if and only if for every $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(A)$ we have*

$$(\mathcal{L}_{\rho(s)}g)(X, Y) = g(\rho(\nabla_X s), Y) + g(X, \rho(\nabla_Y s)).$$

Proof. By Equation (100), for every vector field $X \in \mathfrak{X}(M)$

$$\begin{aligned} ({}^A\nabla_s g)(X, X) &= {}^A\nabla_s(g(X, X)) - 2g({}^A\nabla_s X, X) \\ &= (\mathcal{L}_{\rho(s)}g)(X, X) - 2g(\rho(\nabla_X s), X). \end{aligned}$$

Consequently, ${}^A\nabla g = 0$ if and only if

$$(\mathcal{L}_{\rho(s)}g)(X, X) = 2g(\rho(\nabla_X s), X).$$

■

Proposition 1.5. *Let (M, \mathcal{F}) be a singular foliation on a Riemannian manifold (M, g) . Then the triple (M, g, \mathcal{F}) is a module singular Riemannian foliation if and only if it is locally generated by Killing almost Lie algebroids, i.e. $\forall q \in M$, there exist an open neighborhood $U \in M$ containing q and a Killing almost Lie algebroid (A_U, ∇, g_U) over (U, g_U) such that $\rho(\Gamma_c(A_U)) = \iota_U^{-1}\mathcal{F}$.*

Proof. Assume that (M, g, \mathcal{F}) is a module singular Riemannian foliation and $q \in M$. Then there exists an open neighborhood $U \in M$ containing q such that $\iota_U^{-1}\mathcal{F}$ is generated by finitely many vector fields $V_1, \dots, V_N \in \mathfrak{X}(U)$ for some positive integer N . By involutivity of $\iota_U^{-1}\mathcal{F}$, the trivial vector bundle A_U of rank N with a frame $e_1, \dots, e_N \in \Gamma(A_U)$ together with the anchor map $\rho: A_U \rightarrow TM$, $e_a \mapsto V_a$ for $a = 1, \dots, N$, can be equipped with an almost Lie algebroid structure. By Lemma 3.4 there exist 1-forms $\omega_a^b \in \Omega^1(U)$ such that

$$\mathcal{L}_{V_a}g = \sum_{b=1}^N \omega_a^b \odot \iota_{V_b}g \quad \forall a, b = 1, \dots, N.$$

Now if we define $\nabla e_a = \sum_{b=1}^N \omega_a^b \otimes e_b$, for every $X, Y \in \mathfrak{X}(U)$, we have

$$\begin{aligned} (\mathcal{L}_{\rho(e_a)}g)(X, Y) &= (\mathcal{L}_{V_a}g)(X, Y) \\ &= \sum_{b=1}^N ((\iota_X \omega_a^b)g(V_b, Y) + (\iota_Y \omega_a^b)g(X, V_b)) \\ &= g\left(\rho\left(\sum_{b=1}^N \iota_X \omega_a^b e_b\right), Y\right) + g\left(X, \rho\left(\sum_{b=1}^N \iota_Y \omega_a^b e_b\right)\right) \\ &= g(\rho(\nabla_X e_a), Y) + g(X, \rho(\nabla_Y e_a)). \end{aligned}$$

Consequently, by Lemma 1.4, (A_U, ∇_U, g_U) is a Killing almost Lie algebroid and we have $\rho(\Gamma_c(A_U)) = \iota_U^{-1}\mathcal{F}$. Conversely, Assume that (M, \mathcal{F}) is locally generated by Killing almost Lie algebroids. Let $q \in M$, and take a neighborhood $U \in M$ containing q with a Killing almost Lie algebroid (A_U, ∇, g_U) over (U, g_U) such that $\rho(\Gamma_c(A_U)) = \iota_U^{-1}\mathcal{F}$. By choosing U small enough, we can assume that A_U is trivial and there is a global frame $e_1, \dots, e_N \in \Gamma(A_U)$. Then there exist 1-forms $\omega_a^b \in \Omega^1(U)$ such that

$$\nabla e_a = \sum_{b=1}^N \omega_a^b \otimes e_b \quad \forall a, b = 1, \dots, N.$$

With $V_a := \rho(e_a)$ for $a, b = 1, \dots, N$, by Lemma 1.4, for every $X, Y \in \mathfrak{X}(U)$ one has

$$\begin{aligned} (\mathcal{L}_{V_a}g)(X, Y) &= \sum_{b=1}^N ((\iota_X \omega_a^b)g(V_b, Y) + (\iota_Y \omega_a^b)g(X, V_b)) \\ &= \left(\sum_{b=1}^N \omega_a^b \odot \iota_{V_b}g \right) (X, Y). \end{aligned}$$

This implies, using Lemma 3.4 and Proposition 3.5, that (M, g, \mathcal{F}) is a module singular Riemannian foliation. ■

2 Computations with Macaulay2

In this appendix, we use Eisenbud's Macaulay2 to construct a short exact sequence of \mathcal{O} -modules, where \mathcal{O} stands for the sheaf of smooth, polynomial, or real analytic functions on $\mathbb{O}^2 \cong \mathbb{R}^{16}$. This exact sequence will be used in Section 8.1 to prove the maximality of \mathcal{F}_{OH} , and in Section 10.2 to construct a universal Lie 3-algebroid for \mathcal{F}_{OH} . This can be done as follows: In [Macaulay2Web](#), first specify the ring of polynomial functions on \mathbb{R}^{16} by entering the following code in the first line:

```
R = QQ[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7,
      y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7]
```

Then, considering the characterizing equations of Lemma 8.2 for vector fields tangent to the leaves in \mathcal{L}_{OH} , we construct the morphism of \mathcal{O} -modules $J: \Gamma(\underline{\mathbb{O}^2}) \rightarrow \Gamma(\underline{\mathbb{R} \oplus \mathbb{O} \oplus \mathbb{R}})$ given by

$$J \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \langle x, u \rangle \\ u \cdot \bar{y} + x \cdot \bar{v} \\ \langle y, v \rangle \end{pmatrix}$$

whose kernel gives the desired \mathcal{O} -module of vector fields tangent to \mathcal{L}_{OH} . In Macaulay2 one can use the following code to specify this morphism as a matrix, which is a result of considering the octonion multiplication as a product on \mathbb{R}^8 :

```
F = image matrix {{ x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7,
                    0, 0, 0, 0, 0, 0, 0, 0 },
                  { y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7,
                    x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7 },
                  { -y_1, y_0, -y_3, y_2, -y_5, y_4, y_7, -y_6,
                    x_1, -x_0, x_3, -x_2, x_5, -x_4, -x_7, x_6 },
                  { -y_2, y_3, y_0, -y_1, -y_6, -y_7, y_4, y_5,
                    x_2, -x_3, -x_0, x_1, x_6, x_7, -x_4, -x_5 },
                  { -y_3, -y_2, y_1, y_0, -y_7, y_6, -y_5, y_4,
                    x_3, x_2, -x_1, -x_0, x_7, -x_6, x_5, -x_4 },
                  { -y_4, y_5, y_6, y_7, y_0, -y_1, -y_2, -y_3,
                    x_4, -x_5, -x_6, -x_7, -x_0, x_1, x_2, x_3 },
                  { -y_5, -y_4, y_7, -y_6, y_1, y_0, y_3, -y_2,
                    x_5, x_4, -x_7, x_6, -x_1, -x_0, -x_3, x_2 },
                  { -y_6, -y_7, -y_4, y_5, y_2, -y_3, y_0, y_1,
                    x_6, x_7, x_4, -x_5, -x_2, x_3, -x_0, -x_1 },
                  { -y_7, y_6, -y_5, -y_4, y_3, y_2, -y_1, y_0,
                    x_7, -x_6, x_5, x_4, -x_3, -x_2, x_1, -x_0 },
                  { 0, 0, 0, 0, 0, 0, 0, 0,
                    y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7 }}
```

This map can be completed to an exact sequence, using the code

```
resolution minimalPresentation F
```

Finally, to view the matrices associated with the differential, use the codes

o3.dd_1

,

o3.dd_2

and

o3.dd_2

This exact sequence can be translated back to the framework of octonions as follows

$$0 \longrightarrow \Gamma(E_{-2}) \xrightarrow{d^{(2)}} \Gamma(E_{-1}) \xrightarrow{d^{(1)}} \Gamma(E_0) \xrightarrow{\rho} \Gamma(\underline{\mathbb{O}^2}) \xrightarrow{J} \Gamma(\underline{\mathbb{R} \oplus \mathbb{O} \oplus \mathbb{R}}) \quad (101)$$

where

$$E_0 := \underline{\mathbb{O}^2}, \quad E_{-1} := \underline{\mathbb{R} \oplus \mathbb{O} \oplus \mathbb{R}}, \quad E_{-2} := \underline{\mathbb{R}}.$$

and for global sections $\begin{pmatrix} u \\ v \end{pmatrix} \in \Gamma(E_0)$, $\begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} \in \Gamma(E_{-1})$ and $t \in \Gamma(E_{-2})$:

$$\begin{aligned} \rho \begin{pmatrix} u \\ v \end{pmatrix} &:= \begin{pmatrix} \|x\|^2 u + (x \cdot \bar{y}) \cdot v - (\langle x, u \rangle + \langle y, v \rangle) x \\ \|y\|^2 v + (y \cdot \bar{x}) \cdot u - (\langle x, u \rangle + \langle y, v \rangle) y \end{pmatrix}, \\ d^{(1)} \begin{pmatrix} \mu \\ a \\ \nu \end{pmatrix} &:= \begin{pmatrix} \mu x + a \cdot y \\ \nu y + \bar{a} \cdot x \end{pmatrix}, \\ d^{(2)}(t) &:= \begin{pmatrix} -\|y\|^2 t \\ (x \cdot \bar{y}) t \\ -\|x\|^2 t \end{pmatrix}. \end{aligned} \quad (102)$$

Finally, using Malgrange's flatness theorem, this remains exact as a sequence of modules over real analytic or smooth functions.

Bibliography

- [AKSM02] A. Alekseev, Y. Kosmann-Schwarzbach, E. Meinrenken, *Quasi-Poisson manifolds*, *Canad. J. Math.* 54(1), (2002), 3-29.
- [AS09] I. Androulidakis and G. Skandalis, *The holonomy groupoid of a singular foliation*, *J. Reine Angew. Math.* 626, (2009), 1–37.
- [AGJ90] J. M. Arms, M. J. Gotay and G. F. Jennings *Geometric and algebraic reduction for singular momentum maps*, *Adv. Math.* 79, (1990), 43-103.
- [B02] J. Baez, *The octonions*, *Bull. Amer. Math. Soc.* 39, (2002), 145-205.
- [BWY21] J. C. Baez, D. Weisbart and A. M. Yassine, *Open systems in classical mechanics*, *J. Math. Phys.* 62, 042902, (2021).
- [BC21] F. Baudoin and G. Cho, *The Subelliptic Heat Kernel of the Octonionic Hopf Fibration*, *Potential Anal* 55, (2021), 211–228.
- [dSW99] A. Cannas da Silva and A. J. Weinstein, *Geometric models for noncommutative algebras*, *Berkeley Mathematics Lecture Notes*, 10. American Mathematical Society, Providence, RI, (1999).
- [CFM21] M. Crainic, R. L. Fernandes and I. Marcut, *Lectures on Poisson geometry*, *Grad. Stud. in Math.* (2021).
- [dHF18] M. del Hoyo and R. L. Fernandes, *Riemannian metrics on Lie groupoids*, *J. Reine Angew. Math.* 735, (2018), 143-173.
- [D50] P. Dirac, *Generalized Hamiltonian Dynamics*, *Canad. J. Math.* 2, (1950), 129-148.
- [DEW19] M. Dippell, C. Esposito and S. Waldmann *Coisotropic Triples, Reduction and Classical Limit*, *Documenta Math.* 24, (2019), 1811–1853.
- [F77] G. Frobenius, *Über das Pfaffsche Problem*, *J. Reine Angew. Math.* (82), (1877), 230–315.
- [GZ19] A. Garmendia and M. Zambon, *Hausdorff morita equivalence of singular foliations*, *Ann. Glob. Anal. Geom.* 55, (2019), 99-132.

- [GWZ86] H. Gluck, F. Warner, and W. Ziller *The Geometry of the Hopf fibrations*. Enseign. Math. 32 (1986), 173-198.
- [G01] V. L. Ginzburg, *Grothendieck groups of Poisson vector bundles*, J. Symplectic Geom. 1, (2001), 121–169.
- [HS22] A. Hancharuk and T. Strobl, *BFV extensions for mechanical systems with Lie-2 symmetry*, Phys. Rev. D 107, 025014, (2023).
- [H58] A. Haefliger, *Structures feuilletées et cohomologie à valeur dans un faisceau de groupoides*, Comment. Math. Helv., 32, (1958), 248–329.
- [H62] R. Hermann, *The differential geometry of foliations, II*, J. Appl. Math. Mech. 11, (1962), 303–315.
- [H31] H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. 104, (1931), 637–665.
- [H98] A. Hurwitz, *Über die Composition der quadratischen Formen von beliebig vielen Variablen*, Nachr. Ges. Wiss. Göttingen, (1898), 309–316.
- [L18] S. Lavau, *A short guide through integration theorems of generalized distributions*, Differ. Geom. Appl. 61, (2018), 42–58.
- [L93] D. Lu, *Homogeneous foliations of spheres*, Trans. Amer. Math. Soc. 340 (1993), 95-102.
- [KS16] A. Kotov and T. Strobl, *Gauging without initial symmetry*, J. Geom. Phys. 99, (2016), 184-189.
- [KS19] A. Kotov and T. Strobl, *Lie algebroids, gauge theories, and compatible geometrical structures*, Rev. Math. Phys. 31, (2019).
- [LGLR22] C. Laurent-Gengoux, R. Louis and L. Ryvkin *Geometry of singular foliations: a draft of an introduction*, CRM Barcelona, Poisson geometry summer school, (2022).
- [LGLS20] C. Laurent-Gengoux, S. Lavau and T. Strobl *The universal Lie ∞ -algebroid of a singular foliation*, Documenta Math. 25, (2020), 1571-1652.
- [LGPV13] C. Laurent-Gengoux, A. Pichereau and P. Vanhaecke, *Poisson structures*, Grundlehren Math. Wiss., vol. 347, Heidelberg: Springer, (2013).
- [M98] P. Molino, *Singular Riemannian foliations*, Progress in Mathematics, vol 73. Birkhäuser Boston. (1998).
- [MR86] J. E. Marsden and T. Ratiu, *Reduction of Poisson manifolds*, Lett. Math. Phys. 11.2, (1986), 161–169.
- [MM10] I. Moerdijk and J. Mrcun, *Introduction to foliations and Lie groupoids*, Cambridge University Press, (2010).
- [MW74] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. 5, (1974), 121–130.

- [MR19] R. A. E. Mendes and M. Radeschi, *A slice theorem for singular Riemannian foliations, with applications*, Trans. Amer. Math. Soc. 371 (2019), 4931-4949.
- [OPPV13] L. Ornea, M. Parton, P. Piccinni and V. Vuletescu, *Spin(9) geometry of the octonionic Hopf fibration*, Transform. Groups 18, (2013), 845–864.
- [PPT10] M. Pflaum, H. Posthuma and X. Tang and T. Ratiu, *Geometry of orbit spaces of proper Lie groupoids*, J. Reine Angew. Math. 694, (2014) 49–84.
- [R59] B. L. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. Math. (2), 69, (1959), 119-132.
- [S23] K. Singh, *On the universal L_∞ algebroid of linear foliations*, J. Lie Theory 33, (2023), 925–952.
- [SW83] J. Sniatycki and A. Weinstein, *Reduction and quantization for singular momentum mappings*, Lett. Math. Phys. 7.2 (1983), 155–161.
- [T68] J. C. Tougeron, *Idéaux des fonctions différentiables*, I. Ann. Inst. Fourier (Grenoble) 18, (1968), 177–240.
- [U03] H. K. Urbantke, *The Hopf fibration—seven times in physics*, J. Geom. Phys. 46, (2003), 125-150.
- [W83] H. E. Winkelkemper, *The graph of a foliation*, Ann. Glob. Anal. Geom. 1, (1983), 51-75.
- [V10] T. Voronov, *Q-manifolds and higher analogs of Lie algebroids*, XXIX workshop on geometric methods in physics, Selected papers, (2010), 191–202.
- [X91] P. Xu, *Morita equivalence of Poisson manifolds*, Commun. Math. Phys. 142, (1991), 493–509.

List of Notations

$1_{(x,y)}$	Unit arrow for the object (x, y)
\mathcal{F}_q	Fiber of \mathcal{F} at $q \in M$
\mathcal{I}^G	Set of G -invariant elements in \mathcal{I}
\mathcal{O}	Sheaf of rings of polynomial, real analytic or smooth functions
$d_q f$	Differential of f at $q \in M$
ι_S	Inclusion map of $S \subset M$
$\mathcal{R}(\mathcal{I})$	Reduced Poisson algebra of an \mathcal{I} -Poisson manifold
$\mathfrak{g}_q^{\mathcal{F}}$	Isotropy Lie algebra of \mathcal{F} at $q \in M$
\mathfrak{X}_c	Module of compactly supported vector fields
\odot	Symmetric tensor product
\overline{X}	A vector field X viewed as a function on the cotangent bundle
Φ_H^t	Hamiltonian flow of a function H
\underline{V}	Trivial vector bundle with a vector space V as fibers
ev_q	Evaluation at $q \in M$
F_q	Tangent of the singular foliation \mathcal{F} at $q \in M$
g_b	Musical isomorphism $(q, v) \mapsto g_q(v, \cdot)$
g_U	Restriction of the Riemannian metric g to $U \subset M$
L_q	Leaf passing through $q \in M$
$S(r)$	Sphere of radius r
V^{\perp_g}	orthogonal complement of V with respect to g
W^∞	Set of smooth functions in the sense of Whitney
$X _q$	Evaluation of the vector field X at a point $q \in M$
$X^{\mathcal{H}}$	Horizontal lift of the vector field X
${}^A\nabla$	A -connection induced by an ordinary connection on A

- A -connection, 67
- G -equivariance, 19
- G -manifold, 18
- \mathbb{D} -line, 43
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- \mathcal{I} -Poisson manifold, 18
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 symplectic manifold, 16

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 transverse to \mathcal{F} , 4

 universal Lie ∞ -algebroid of \mathcal{F} , 58
 upper semi-continuous, 3

Feuilletages singuliers en géométrie différentielle

Résumé. Les feuilletages singuliers sont définis comme certains sous-modules de champs vectoriels, dont les flots décomposent la variété en feuilles injectivement immergées de dimensions potentiellement différentes. En présence d'une métrique riemannienne, la recherche de compatibilité entre les feuilles et la structure riemannienne conduit à la notion de feuilletages riemanniens singuliers géométriques, tels que décrits par Molino. Ici, nous adoptons une approche différente pour adapter la notion de feuilletages riemanniens singuliers afin de capturer les riches propriétés algébriques du feuilletage singulier. Notre concept interagit naturellement avec la géométrie de Poisson et motive la définition des variétés I-Poisson comme une relaxation de la catégorie des variétés de Poisson. Nous comparons les feuilletages riemanniens singuliers dans notre sens et celui de Molino, montrant que notre notion est plus restrictive et se distingue par l'exemple du feuilletage singulier octonionique de Hopf. Construit sur la fibration de Hopf octonionique, le feuilletage singulier octonionique de Hopf illustre un feuilletage singulier localement non homogène, c'est-à-dire qu'il ne peut pas être généré par des isométries locales. Cependant, nous construisons un groupoïde de Lie induisant ce feuilletage singulier et un algébroïde de Lie universel, démontrant la minimalité de la dimension pour le groupoïde de Lie.

Mots-clés : Feuilletages singuliers, Géométrie différentielle, Théorie de Lie, Géométrie de Poisson

Singular foliations in differential geometry

Abstract. Singular foliations are defined as certain submodules of vector fields, whose flows decompose the manifold into injectively immersed leaves of possibly different dimensions. In the presence of a Riemannian metric, seeking compatibility between the leaves and the Riemannian structure leads to the notion of geometric singular Riemannian foliations, as described by Molino. Here, we adopt a different approach to adapt the notion of singular Riemannian foliations to capture the rich algebraic properties of the singular foliation. Our notion naturally interacts with Poisson geometry and motivates the definition of I-Poisson manifolds as a relaxation of the category of Poisson manifolds. We compare singular Riemannian foliations in our sense and Molino's, showing that our notion is more restrictive and distinguishable by the example of the singular octonionic Hopf foliation. Constructed upon the octonionic Hopf fibration, the singular octonionic Hopf foliation exemplifies a locally non-homogeneous singular foliation, i.e., it cannot be generated by local isometries. However, we construct a Lie groupoid inducing this singular foliation and a universal Lie infinity algebroid, demonstrating the minimality of the dimension for the Lie groupoid.

Keywords: Singular foliations, Differential geometry, Lie theory, Poisson geometry

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