Submanifolds of the nearly Kähler SL(2,R) x SL(2,R)

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Sous-variétés du nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

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Abstract

A strict nearly Kähler manifold is an almost Hermitian manifold for which the covariant derivative of the almost complex structure is non-degenerate and skew symmetric. Strict nearly Kähler manifolds are quite rare. In six dimensions, there are only four homogeneous strict nearly Kähler manifolds : $\mathbb{S}^6$, $\mathbb{S}^3 \times \mathbb{S}^3$, $\mathbb{C}P^3$ and $F(\mathbb{C}^3)$. In the pseudo-Riemannian setting, there are six analogues. We consider $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, the analogue of $\mathbb{S}^3 \times \mathbb{S}^3$.

We first study the strict nearly Kähler structure of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The metric and almost complex structure are both induced from a pseudo-Riemannian submersion $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We give an expression for the full isometry group.

In this space we study different types of submanifolds, such as surfaces, Lagrangian submanifolds and hypersurfaces. We divide Lagrangian submanifolds into four different types, which depend on their behavior with respect to an almost product structure.

We classify all totally geodesic and extrinsically homogeneous Lagrangian submanifolds. Moreover, we study totally geodesic surfaces and hypersurfaces with constant sectional curvature.

Beknopte samenvatting

Een strikte nagenoeg-Kählervariëteit is een bijna-Hermitische variëteit waarvoor de covariante afgeleide van de bijna-complexe structuur niet-ontaard en scheef-symmetrisch is. Dergelijke variëteiten zijn vrij zeldzaam. In dimensie zes zijn er slechts vier homogene strikte nagenoeg-Kählervariëteiten: $\mathbb{S}^6$, $\mathbb{S}^3 \times \mathbb{S}^3$, $\mathbb{C}P^3$ en $F(\mathbb{C}^3)$. In de pseudo-Riemannse setting bestaan er zes analogen. We beschouwen $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, analoog aan $\mathbb{S}^3 \times \mathbb{S}^3$.

We bestuderen eerst de strikte nagenoeg-Kählerstructuur van $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. De metriek en bijna-complexe structuur zijn beide geïnduceerd door een pseudo-Riemannse submersie $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We geven een uitdrukking voor de volledige isometriegroep.

In deze ruimte bestuderen we verschillende soorten deelvariëteiten, zoals oppervlakken, Lagrangiaanse deelvariëteiten en hyperoppervlakken. Lagrangiaanse deelvariëteiten verdelen we in vier verschillende types, afhankelijk van hun gedrag ten opzichte van een bijna-productstructuur.

We classificeren alle totaal geodetische en extrinsiek homogene Lagrangiaanse deelvariëteiten. Bovendien bestuderen we totaal geodetische oppervlakken en hyperoppervlakken met constante sectionele kromming.

Résumé

Le but de cette thèse est d'élargir la compréhension de la variété presque kählerienne $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ et de ses sous-variétés. Les variétés presque kähleriennes sont une relaxation riemannienne de la définition de variété Kähler : une variété presque hermitienne (M, g, J) est presque kählerienne si la dérivée covariante de la structure presque complexe est antisymétrique. De plus, afin d'éviter une éventuelle intersection avec les variétés de Kähler, nous exigeons que ∇J soit non dégénéré, ce que nous appelons une variété stricte presque Kähler.

Les sous-variétés presque kähleriennes sont l'une des seize classes de variétés presque hermitiennes classées par Gray et Hervella dans [27]. Selon Gray, la classe la plus intéressante, les variétés presque kähleriennes, possèdent de nombreuses propriétés intéressantes. Par exemple, d'un point de vue physique, ce sont les seules variétés à six dimensions à admettre un spineur Killing [23, 28]. De plus, les variétés strictes presque kähleriennes sont intéressantes en géométrie multiplectique car elles sont 2-plectiques [37].

Ils possèdent également de nombreuses propriétés de division utiles. À savoir, étant donné une variété presque kählerienne M, il existe une variété presque kählerienne stricte M_1 et une variété de Kähler M_2 telle que $M = M_1 \times M_2$.

Dans les dimensions deux et quatre, variétés presque kähleriennes sont automatiquement variétés de Kähler. Par conséquent, nous trouvons les plus petits exemples intéressants de variétés presque kählerienne en dimension six. En dimension huit, Gray [26] a prouvé qu'une variété presque kählerienne est toujours de Kähler ou le produit d'une variété presque kählerienne stricte à six dimensions avec une surface de Kähler. En dimension dix, Nagy [41] a prouvé qu'une variété presque Kähler est soit Kähler, un produit d'une variété Kähler stricte à six dimensions avec une variété Kähler à quatre dimensions, soit une variété Kähler stricte, qu'il a classée comme espaces de twisteurs au-dessus de variétés Kähler-quaternioniques avec constante d'Einstein positive.

Nagy a également apporté d'importantes contributions sur les propriétés de division des variétés strictes presque kähleriennes.

Theorem 1 (Nagy [41]). *Soit M une variété simplement connexe, complète, stricte, presque kählerienne. Il s'agit alors d'un produit riemannien dont les facteurs appartiennent à l'une des classes suivantes :*

1. *variétés à six dimensions presque kähleriennes,*
2. *variétés homogènes presque kähleriennes (satisfaisant certaines conditions),*
3. *espaces de twisteurs sur espaces Kähler-quaternioniques avec constante d'Einstein positive.*

Nous voyons donc que les variétés presque kähleriennes à six dimensions constituent une base importante pour le cas de dimension supérieure.

Conjecturé depuis longtemps par Gray, Butruille a classé toutes les variétés homogènes à six dimensions proches de Kähler en quatre exemples.

Theorem 2 (Butruille [10]). *Soit M une variété homogène simplement connexe stricte presque kählerienne. Alors il est isométrique à*

— $\mathbb{S}^6 = G_2/\mathrm{SU}(3)$,
— $\mathbb{S}^3 \times \mathbb{S}^3 = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/\Delta\mathrm{SU}(2)$,
— $\mathbb{C}P^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1)\mathrm{U}(1)$,
— $F(\mathbb{C}^3) = \mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$.

Notez que l'hypothèse d'être homogène est importante. Pendant de nombreuses années, on a cru qu'il existait des variétés non homogènes presque kähleriennes, mais il n'existait pas d'exemples explicites de variétés complètes. Podestà et Spiro ont étudié les actions de cohomogénéité-un sur des variétés presque Kähler dans [46] et [47] et ils ont construit des variétés presque kähleriennes inhomogènes mais incomplètes. Enfin, en 2017, Foscolo et Haskins [22] ont construit les premiers exemples de variétés complètes non homogènes à six dimensions presque kähleriennes : $\mathbb{S}^6$ et $\mathbb{S}^3 \times \mathbb{S}^3$ portent des structures presque kähleriennes qui ne sont pas homogènes mais qui ont une cohomogénéité. De plus, ils ont conjecturé que $\mathbb{C}P^3$ ne porte que la structure homogène presque kählerienne.

Dans un contexte pseudo-riemannien, aucun théorème de classification n'est connu. Cependant, en utilisant un type de dualité similaire à celui utilisé pour les espaces symétriques non compacts, Kath [32] et Schäfer [48] ont construit plusieurs analogues pseudo-riemanniens des quatre espaces du théorème 2 (voir Figure 1). Dans [1], les auteurs prétendent avoir un exemple d'une variété

pseudo-riemannienne homogène à six dimensions, presque kählerienne, qui n'est pas un dual d'une variété riemannienne.

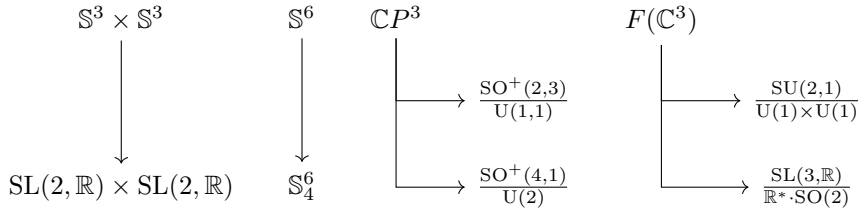


FIGURE 1 – Les duaux pseudo-riemanniens des variétés homogènes à six dimensions presque kählerienne.

Ici, $\mathbb{S}^6_4$ est la pseudo-sphère à l'intérieur de $\mathbb{R}^7_4$, de description homogène $G_{2(2)}/\mathrm{SU}(2,1)$, et $\mathbb{R}^* \cdot \mathrm{SO}(2)$ désigne le groupe $(\mathbb{R}^* \times \mathrm{SO}(2))/\mathbb{Z}_2$. La variété presque kählerienne $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ a une description homogène $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\Delta\mathrm{SL}(2,\mathbb{R})$. Dans cette thèse, nous nous concentrerons sur ce dernier espace.

La théorie des sous-variétés est une généralisation à plusieurs dimensions de l'étude des surfaces dans $\mathbb{R}^3$. Autrement dit, il s'agit de l'étude des immersions isométriques et de la manière dont les invariants intrinsèques et extrinsèques sont liés les uns aux autres. En géométrie presque hermitienne, on s'intéresse également à la façon dont la structure presque complexe de l'espace ambiant agit sur les espaces tangents de la sous-variété. Si la structure presque complexe mappe le fibré tangent dans le fibré normal, nous disons que la sous-variété est totalement réelle. De plus, si la dimension de la sous-variété est maximale, c'est-à-dire la moitié de la dimension de l'espace ambiant, la sous-variété est lagrangienne.

En géométrie symplectique, une sous-variété lagrangienne d'une variété symplectique (M,ω) est une sous-variété telle que ω s'annule partout et sa dimension est égale à $\frac{1}{2}\dim(M)$. Dans la géométrie de Kähler, ω est la forme de Kähler $g(J\cdot,\cdot)$, ce qui fait coïncider les deux définitions de la sous-variété lagrangienne.

Les sous-variétés lagrangiennes sont particulièrement intéressantes en géométrie presque kählerienne : toute sous-variété lagrangienne d'une variété presque Kähler stricte à six dimensions est automatiquement minimale.

Le principal outil pour étudier les sous-variétés dans cette thèse est une structure presque produit P sur le variété presque kählerienne $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Défini explicitement comme l'échange des vecteurs tangents de chaque facteur de

$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, nous pouvons considérer cette structure comme le tenseur qui détermine la relation entre la métrique du produit et la métrique presque Kähler. Analogue à $\mathbb{S}^3 \times \mathbb{S}^3$ [40], il n'existe que trois structures de presque produits différentes sur $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ satisfaisant des équations spécifiques. Nous utilisons cette propriété pour montrer que le groupe $\mathcal{G}$ d'isométries de la variété presque kählerienne $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ qui préservent la structure presque complexe J a 12 composantes connexes différentes.

Theorem 3. *Le groupe d'isométrie de la variété pseudo-presque kählerienne* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *est le produit semi-direct* $\big(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\big) \rtimes \big(\mathbb{Z}_2 \times S_3\big)$*, où* S_3 *est le groupe symétrique d'ordre 6.*

Maintenant, étant donné que nous souhaitons étudier les sous-variétés qui ont une relation particulière avec la structure presque complexe, nous définirons la congruence jusqu'au groupe $\mathcal{G}$. Autrement dit, deux sous-variétés sont congruentes si et seulement s'il existe une isométrie dans $\mathcal{G}$ qui correspond l'une à l'autre.

Les sous-variétés des $\mathbb{S}^3 \times \mathbb{S}^3$ presque kähleriennes ont été largement étudiées par de nombreux auteurs. Une notion importante de fonctions d'angle a été introduite dans [20] et [57]. À savoir, la structure presque produit P induit trois fonctions θ_1, θ_2 et θ_3, qui codent des informations extrinsèques substantielles. Dans [57], les auteurs ont également classé toutes les sous-variétés lagrangiennes totalement géodésiques des $\mathbb{S}^3 \times \mathbb{S}^3$ presque kähleriennes :

Theorem 4 (Dioos, Hu, Vrancken, Wang and Zhang)**.** *Une sous-variété lagrangienne totalement géodésique de* $\mathbb{S}^3 \times \mathbb{S}^3$ *est congrue à éventuellement un sous-ensemble ouvert de l'un des exemples suivants :*

1. $f_1 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u)$,
2. $f_2 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, -\mathbf{i}u\mathbf{i})$,

où $\mathbf{i}$ *est le quaternion unité.*

Les sous-variétés lagrangiennes totalement géodésiques ont des fonctions d'angle constantes et au moins deux d'entre elles sont égales modulo π. Dans [57], les auteurs ont en fait donné six exemples différents, mais en utilisant les isométries de $\mathbb{S}^3 \times \mathbb{S}^3$ décrites dans [40] nous pouvons réduire la liste à seulement deux.

Dans [6] les auteurs ont classé toutes les sous-variétés lagrangiennes avec des fonctions à angle constant. Ce faisant, ils ont également classé toutes les sous-variétés lagrangiennes extrinsèquement homogènes :

Theorem 5 (Bektas, Moruz, Van der Veken and Vrancken)**.** *Une sous-variété lagrangienne extrinsèquement homogène* $H \cdot (1,1)$ *de* $\mathbb{S}^3 \times \mathbb{S}^3$ *avec* H *sous-groupe*

de Lie de $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ *est conforme à un sous-ensemble ouvert de l'un des exemples suivants,*

1. $f_1 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u)$, *avec* $H = \mathrm{SU}(2)$,
2. $f_2 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, -\mathbf{i}u\mathbf{i})$, *avec* $H = \mathrm{SU}(2)$,
3. $f_3 \colon \mathbb{R}P^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (-\mathbf{j}u\mathbf{j}u^{-1}, -\mathbf{i}u\mathbf{i}u^{-1})$, *avec* $H = \mathrm{SU}(2)$,
4. $f_4 \colon \mathbb{T}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (u, v, w) \mapsto (e^{u\mathbf{i}}e^{-w\mathbf{k}}, e^{v\mathbf{j}}e^{-w\mathbf{k}})$, *avec* $H = \mathbb{T}^3$,

où $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *sont les quaternions unitaires, et* $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. *Dans tous les cas, l'isotropie* $H_{(1,1)}$ *est triviale, sauf pour la sous-variété* f_3, *où l'isotropie est* $\mathbb{Z}_2$.

Les cartes f_1 et f_2 sont totalement géodésiques. Toutes les cartes à l'exception de f_2 sont des immersions de formes spatiales où la première est une sphère de courbure sectionnelle constante $\frac{3}{2}$, la troisième est un espace projectif de courbure sectionnelle constante $\frac{3}{8}$ et le dernier est un tore plat.

En relâchant l'hypothèse à un seul angle constant, dans [7] les auteurs ont construit des sous-variétés lagrangiennes dans $\mathbb{S}^3 \times \mathbb{S}^3$ à partir de surfaces minimales dans $\mathbb{S}^3$:

Theorem 6 (Bektas, Moruz, Van der Veken, Vrancken)**.** *Soit* $f = (p, q) \colon M \to \mathbb{S}^3 \times \mathbb{S}^3$ *une immersion lagrangienne dans le variété presque kählerienne* $\mathbb{S}^3 \times \mathbb{S}^3$. *Supposons que dp n'ait nulle part un rang maximal. Alors* $p(M) \subset \mathbb{S}^3$ *est une surface minimale de* $\mathbb{S}^3$.

Inversement, si $p : \Sigma \to \mathbb{S}^3$ *est une surface minimale de* $\mathbb{S}^3$, *il existe une immersion lagrangienne* $f \colon M \to \mathbb{S}^3 \times \mathbb{S}^3$ *satisfaisant* $f = (p, q)$ *pour certaine application* $q \colon M \to \mathbb{S}^3$.

L'objectif principal de cette thèse est l'étude des sous-variétés lagrangiennes. Le but est d'essayer de reproduire les idées trouvées dans [6], [7], [19], [20], [57] en introduisant la notion de fonctions angulaires, qui sont déterminées par l'action du tenseur P sur la sous-variété. La restriction de P sur TM se divise en quatre types différents, également connus dans la littérature sous le nom de types de Petrov. Nous étudions toujours chaque type séparément, et nous montrons qu'il existe des sous-variétés lagrangiennes de chaque type.

Le premier problème que nous abordons est l'étude de sous-variétés lagrangiennes totalement géodésiques. Nous montrons que, à congruence près, il en existe trois différents : un espace anti-de Sitter et deux espaces anti-de Sitter avec des métriques de type Berger, l'un étiré dans une direction spatiale et l'autre dans une direction temporelle. Contrairement à $\mathbb{S}^3 \times \mathbb{S}^3$, nous obtenons une sous-variété supplémentaire, qui découle de la nature pseudo-riemannienne de $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$.

Theorem 7. *Toute sous-variété lagrangienne totalement géodésique de la variété pseudo-presque kählerienne* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *est congrue à l'image de l'une des cartes suivantes, éventuellement limitée à un sous-ensemble ouvert :*

(1) $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (\mathrm{Id}_2, u)$,
(2) $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (u, \boldsymbol{i}u\boldsymbol{i})$,
(3) $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (u, -\boldsymbol{k}u\boldsymbol{k})$,

où Id_2 *est la matrice d'identité, et* $\boldsymbol{i}$, $\boldsymbol{k}$ *sont les coquaternions.*

A l'inverse, les cartes (1), (2) *et* (3) *sont des immersions lagrangiennes totalement géodésiques.*

Nous avons étendu ce résultat à l'étude de sous-variétés lagrangiennes extrinsèquement homogènes. Dans $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ nous avons constaté qu'en plus des cinq analogues (les trois totalement géodésiques, $\mathrm{PSL}(2,\mathbb{R})$ et $\mathbb{R}^2 \times \mathbb{S}^1$), nous obtenons trois exemples exotiques, où l'un est une famille d'immersions.

Theorem 8. *Soit* $f : (M, g) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *une immersion lagrangienne extrinsèquement homogène dans le pseudo-presque kählerienne* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. *Alors* $f(M)$ *est congru à un sous-ensemble ouvert de l'image de l'un des plongements suivants, dont l'image est l'orbite de* $(\mathrm{Id}_2, \mathrm{Id}_2)$ *par* $H \subset \mathrm{Iso}_o(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ *:*

(M, g)	f	H	Remarques
$(\mathrm{SL}(2,\mathbb{R}), \frac{2}{3}\langle,\rangle)$	$u \mapsto (u, u)$	$\mathrm{SL}(2,\mathbb{R})$	tot. géodésique $K = -\frac{3}{2}$
$(\mathrm{SL}(2,\mathbb{R}), g^+_{\kappa,\tau})$	$u \mapsto (u, \boldsymbol{i}u\boldsymbol{i})$	$\mathrm{SL}(2,\mathbb{R})$	tot. géodésique
$(\mathrm{SL}(2,\mathbb{R}), g^-_{\kappa,\tau})$	$u \mapsto (u, -\boldsymbol{k}u\boldsymbol{k})$	$\mathrm{SL}(2,\mathbb{R})$	tot. géodésique
$(\mathrm{PSL}(2,\mathbb{R}), \frac{8}{3}\langle,\rangle)$	$[u] \mapsto (\boldsymbol{i}u\boldsymbol{i}u^{-1}, \boldsymbol{j}u\boldsymbol{j}u^{-1})$	$\mathrm{SL}(2,\mathbb{R})$	$K = -\frac{3}{8}$
$\mathbb{R}^3_1/\mathbb{Z}$	$(u, v, w) \mapsto (e^{v\boldsymbol{i}}e^{-u\boldsymbol{k}}, e^{w\boldsymbol{j}}e^{-u\boldsymbol{k}})$	$\mathbb{R}^2 \times \mathbb{S}^1$	$K = 0$
$(\mathbb{R}^3, \hat{g})$	ι	$\mathbb{R} \ltimes_{\varphi_o} \mathbb{R}^2$	$K = -\frac{3}{2}$
$(\mathbb{R}^3/H_\lambda, g_\lambda)$	f_λ	$\frac{\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2}{H_\lambda}$	$K = -\frac{3}{2}$
$(\mathbb{R}^3, \tilde{g})$	$\jmath$	$\mathbb{R} \ltimes_{\varphi_2} \mathbb{R}^2$	

où H agit sur $f(M)$ toujours avec une isotropie triviale sauf pour $\mathrm{PSL}(2,\mathbb{R})$*, où il agit avec une isotropie $\mathbb{Z}_2$. Ici K est la courbure sectionnelle de $f(M)$ et $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ sont les matrices*

$$\boldsymbol{i}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \boldsymbol{j}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \boldsymbol{k}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

De plus, $\langle,\rangle$ est la métrique donnée dans (2.2)*, $g^{+}_{\kappa,\tau}$ et $g^{-}_{\kappa,\tau}$ sont des métriques de type Berger, étirées respectivement dans une direction spatiale et temporelle ; $\hat{g}$, ι et φ_o sont donnés dans l'exemple 21 ; g_λ, f_λ, H_λ et φ_1 sont donnés dans l'exemple 22 ; $\tilde{g}$, $\jmath$ et φ_2 sont donnés dans l'exemple 23.*

À l'inverse, les cartes répertoriées dans le tableau ci-dessus sont des sous-variétés lagrangiennes extrinsèquement homogènes de $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$*. De plus, toutes les immersions ne sont pas congruentes les unes aux autres, y compris les différentes immersions de la famille f_λ.*

Le théorème 8 montre également qu'il existe des sous-variétés lagrangiennes pour chaque type des types de Petrov sauf pour le type IV. Dans [7] les auteurs montrent une manière de construire une sous-variété lagrangienne à partir d'une sous-variété minimale de $\mathbb{S}^3$. Nous utilisons la même méthode pour montrer qu'il existe une sous-variété lagrangienne de type IV.

Theorem 9. *Soit $p:\Sigma\to\mathrm{SL}(2,\mathbb{R})$ une surface minimale de $(\mathrm{SL}(2,\mathbb{R}),\langle,\rangle)$, où $\langle,\rangle$ est la métrique lorentzienne donnée dans* (2.2)*. Soit ω une solution de l'équation* cosh*-Gordon avec $\omega<\log\frac{1}{\sqrt{3}}$. Supposons que $\{u,v\}$ soient des coordonnées nulles avec $\langle\frac{\partial}{\partial u},\frac{\partial}{\partial v}\rangle=e^{\omega}$ et $\sigma(\frac{\partial}{\partial u},\ tpartderu)=\sigma(\frac{\partial}{\partial v},\frac{\partial}{\partial v})=-1$ où σ est la composante de la deuxième forme fondamentale dans la direction de l'unité normale ξ. Ensuite la distribution sur $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ s'étend sur $(p\alpha_i,q\beta_i)$ avec $dp(\frac{\partial}{\partial u})=p\alpha_1$, $dp(\frac{\partial}{\partial v})=p\alpha_2$, $\alpha_3=0$ et*

$$\beta_1=\frac{1}{2}\alpha_1-\frac{e^{-\omega}}{2}\alpha_2-\frac{1}{4}\omega_u e^{-\omega}\alpha_1\times\alpha_2$$

$$\beta_2=\frac{e^{-\omega}}{2}\alpha_1+\frac{1}{2}\alpha_2+\frac{e^{-\omega}\omega_v}{4}\alpha_1\times\alpha_2$$

$$\beta_3=-\sqrt{\frac{3}{2}}e^{-\omega}\alpha_1\times\alpha_2$$

est intégrable et la variété intégrale est une sous-variété lagrangienne de type IV.

Nous étudions également des surfaces de $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. C'est-à-dire ces immersions pseudo-riemanniennes $f:\Sigma\to\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ où Σ est

une variété (pseudo-)riemannienne bidimensionnelle. En particulier, nous nous intéressons aux surfaces totalement géodésiques, ces surfaces dont les géodésiques sont aussi des géodésiques de $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Dans l'analogue riemannien $\mathbb{S}^3 \times \mathbb{S}^3$ de $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, des surfaces totalement géodésiques ont été étudiées dans [9] et [19]. Dans [9] les auteurs ont classé jusqu'à congruence toutes les surfaces presque complexes totalement géodésiques. Les surfaces presque complexes, également appelées J-holomorphes, sont les surfaces telles que la structure presque complexe J préserve l'espace tangent.

Theorem 10 (Bolton, Dioos, Dillen, Vrancken)**.** *Toute surface presque complexe totalement géodésique de $\mathbb{S}^3 \times \mathbb{S}^3$ est congrue à un sous-ensemble ouvert de l'une des immersions suivantes :*

1. $\mathbb{S}^2(\frac{2}{3}) \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto \frac{1}{2}(1 - \sqrt{3}x, 1 + \sqrt{3}x)$,
2. $\mathbb{T}^2 \to \mathbb{S}^3 \times \mathbb{S}^3 : (s,t) \mapsto (e^{s\mathbf{i}}, e^{t\mathbf{i}})$,

où $\mathbf{i}$ est le quaternion unité. A l'inverse, les immersions listées ci-dessus sont des surfaces totalement géodésiques presque complexes.

Dans [19] l'auteur a classé des surfaces totalement géodésiques totalement réelles de $\mathbb{S}^3 \times \mathbb{S}^3$. C'est-à-dire les surfaces sur lesquelles J mappe l'espace tangent dans l'espace normal.

Theorem 11 (Dioos)**.** *Toute surface totalement géodésique totalement réelle de $\mathbb{S}^3 \times \mathbb{S}^3$ est congruente à un sous-ensemble ouvert de l'immersion suivante :*

— $\mathbb{S}^2(\frac{2}{3}) \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (\iota(u), 1)$

où ι est l'immersion totalement géodésique de $\mathbb{S}^2$ dans $\mathbb{S}^3$. A l'inverse, l'immersion ci-dessus est une surface totalement géodésique totalement réelle.

Ghandour et Vrancken [25] ont classé toutes les surfaces totalement géodésiques presque complexes, nous nous concentrons donc sur les surfaces totalement réelles.

Theorem 12 (Ghandour, Vrancken)**.** *Toute surface totalement géodésique presque complexe de la variété pseudo-presque kählerienne $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ est congrue à un sous-ensemble ouvert de l'image de l'une des immersions suivantes :*

1. $H^2(-\frac{3}{2}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : x \mapsto \frac{1}{2}(\mathrm{Id}_2 - \sqrt{3}x, \mathrm{Id}_2 + \sqrt{3}x)$,
2. $\mathbb{T}^2 \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : (s,t) \mapsto (e^{t\boldsymbol{k}}, e^{s\boldsymbol{k}})$,
3. $\mathbb{R}^2 \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : (s,t) \mapsto (e^{t\boldsymbol{i}}, e^{s\boldsymbol{i}})$,

où $\boldsymbol{i}$ et $\boldsymbol{k}$ sont les coquaternions.

Nous obtenons un résultat de classification pour des surfaces totalement géodésiques totalement réelles de $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Theorem 13. *Une surface totalement géodésique totalement réelle de* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *est congruente à un sous-ensemble ouvert de l'image de l'une des immersions suivantes :*

(1) $f_1 \colon H_0^2(-\frac{3}{2}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : p \to (p, \mathrm{Id}_2)$,

(2) $f_2 \colon H_1^2(-\frac{3}{2}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : p \to (p, \mathrm{Id}_2)$,

A l'inverse, les immersions listées ci-dessus sont des immersions totalement réelles totalement géodésiques.

De plus, nous avons étudié les hypersurfaces de $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, plus spécifiquement les hypersurfaces à courbure sectionnelle constante. Dans le cadre d'un travail en cours destiné à toutes les variétés presque Kähler à six dimensions, nous montrons qu'il n'existe pas d'hypersurfaces de $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ à courbure sectionnelle constante.

Étant donné une variété riemannienne n-dimensionnelle (M, g), on entend par hypersurface de M une sous-variété de dimension $n-1$. Les hypersurfaces de variétés presque kählerienes à six dimensions ont été largement étudiées. Par exemple, Loubeau et Deschamps [18] ont montré qu'il n'y a pas d'hypersurfaces totalement géodésiques ou totalement ombilicales dans $\mathbb{C}P^3$ et $F(\mathbb{C}^3)$. Cela est également vrai pour $\mathbb{S}^3 \times \mathbb{S}^3$ (voir par exemple [31]), donc la seule variété à six dimensions presque kählerienne avec une hypersurface totalement géodésique est la sphère $\mathbb{S}^6$. Cela n'est pas surprenant, compte tenu du résultat de Nikolayevsky [43], qui affirme qu'une variété homogène avec une hypersurface totalement géodésique doit être un produit déformé de $\mathbb{R}^n$ avec une variété homogène, une produit torsadé de $\mathbb{R}$ avec une variété homogène, ou produit riemannien d'une forme spatiale avec un espace homogène. Nous montrons qu'il n'existe pas d'hypersurfaces à courbure sectionnelle constante dans $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ au moyen d'une version raffinée de l'équation de Codazzi.

Theorem 14. *Il n'y a pas d'hypersurfaces dans la variété pseudo-presque kählerienne* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *avec une courbure sectionnelle constante.*

La structure de la thèse est la suivante. Dans le chapitre 1, nous donnons quelques préliminaires nécessaires à la suite de la thèse. Dans le chapitre 2, nous introduisons la structure presque Kähler de $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ et nous fournissons une expression pour le groupe d'isométrie. Au chapitre 3, nous étudions les sous-variétés lagrangiennes. Nous les divisons en quatre types, et nous classons toutes les sous-variétés lagrangiennes totalement géodésiques et extrinsèquement homogènes. De plus, nous montrons que chaque type possède au moins une sous-variété lagrangienne sous-jacente. Dans le chapitre 4,

nous classons toutes les surfaces totalement géodésiques totalement réelles de $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Enfin, au chapitre 5, nous montrons qu'il n'existe pas d'hypersurfaces à courbure sectionnelle constante.

Contents

Introduction (anglais)

The aim of this thesis is to broaden the understanding of the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and its submanifolds. Nearly Kähler manifolds are a Riemannian relaxation of the definition of a Kähler manifold: an almost Hermitian manifold (M, g, J) is nearly Kähler if the covariant derivative of the almost complex structure is skew-symmetric. Moreover, to avoid a possible intersection with Kähler manifolds, we require the condition on M of ∇J being non-degenerate, which we call strict nearly Kähler. The nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is the pseudo-Riemannian dual of $\mathbb{S}^3 \times \mathbb{S}^3$, one of the only four six-dimensional homogeneous strict nearly Kähler manifolds, classified by Butruille [10].

Submanifold theory is a generalization to higher dimensions of the study of surfaces in $\mathbb{R}^3$. That is, it is the study of isometric immersions and how the intrinsic and extrinsic invariants relate to each other. In almost Hermitian geometry, we are also interested in how the almost complex structure of the ambient space acts on the tangent spaces of the submanifold. If the almost complex structure maps the tangent bundle into the normal bundle, we say that the submanifold is totally real. Moreover, if the dimension of the submanifold is maximal, i.e. half of the dimension of the ambient space, the submanifold is Lagrangian. This definition agrees with the notion of Lagrangian submanifold in symplectic geometry, since the Kähler form $g(J\cdot, \cdot)$ vanishes everywhere on the submanifold.

Lagrangian submanifolds arise mostly from Hamiltonian mechanics. By Arnold–Liouville theorem, a level set of a function of an integrable system is a Lagrangian submanifold. Moreover, if the submanifold is compact the is a torus. Further, Weinstein–Darboux theorem states that if a submanifold is Lagrangian with respect to two different symplectic forms, then the manifolds are symplectomorphic in a neighborhood around the submanifold.

Lagrangian submanifolds are particularly interesting in nearly Kähler geometry:

any Lagrangian submanifold of a six-dimensional strict nearly Kähler manifold is automatically minimal. Moreover, Lagrangian submanifolds are special Lagrangian [4]. That is, the real part of the Kähler form coincides with the volume form.

The main tool for studying submanifolds in this thesis is an almost product structure P on the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Defined explicitly by swapping tangent vectors of each factor of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, we can think of this structure as the tensor that determines the relation between the product metric and the nearly Kähler metric of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Analogous to $\mathbb{S}^3 \times \mathbb{S}^3$ [40], there exist only three different almost product structures on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ satisfying some specific equations. We use this property to show that the group $\mathcal{G}$ of isometries of the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ that preserve the almost complex structure J has 12 different connected components. Now, given that we are interested in studying submanifolds that have a special relation with the almost complex structure we will define congruency up to the group $\mathcal{G}$. That is, two submanifolds are congruent if and only if there exists an isometry in $\mathcal{G}$ that maps one into the other.

The main focus is on the study of Lagrangian submanifolds. Following the steps for $\mathbb{S}^3 \times \mathbb{S}^3$ in [57] and [20], we introduce the notion of angle functions, which are determined by the action of the tensor P on the submanifold. The restriction of P onto TM divides into four different types, also known in the literature as Petrov types. We always study each type separately, and we show that there exist Lagrangian submanifolds of each type.

The first problem we address is the study of totally geodesic Lagrangian submanifolds. We show that, up to congruence, there are three different ones: an anti-de Sitter space and two anti-de Sitter spaces with Berger-like metrics, one stretched in a spacelike direction and the other in a time-like direction. In contrast with $\mathbb{S}^3 \times \mathbb{S}^3$, we get an extra submanifold, which arises from the pseudo-Riemannian nature of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We extended this result to the study of extrinsically homogeneous Lagrangian submanifolds. For $\mathbb{S}^3 \times \mathbb{S}^3$ (see [6]) there are four extrinsically homogeneous Lagrangian submanifolds: the two totally geodesic ones, an $\mathbb{R}P^3$ with constant sectional curvature, and a flat three-torus. In $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we found that besides the five analogues (the three totally geodesic ones, $\mathrm{PSL}(2,\mathbb{R})$ and $\mathbb{R}^2 \times \mathbb{S}^1$), we get three exotic examples, where one is a family of immersions.

We also study totally geodesic surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Ghandour and Vrancken [25] classified all almost complex totally geodesic surfaces, therefore we focus on totally real surfaces. We provide a classification result, where we show that there are only two examples: $H^2(-\frac{3}{2})$ and $H^2_1(-\frac{3}{2})$.

Further, we study hypersurfaces of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, more specifically constant sectional curvature hypersurfaces. As part of an ongoing work intended for all six-dimensional nearly Kähler manifolds, we show that there exist no hypersurfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ with constant sectional curvature.

The layout of the thesis is as follows. In Chapter 1, we give some preliminaries necessary for the rest of the thesis. In Chapter 2, we introduce the nearly Kähler structure of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and we provide an expression for the isometry group. In Chapter 3, we study Lagrangian submanifolds. We divide them into four types, and we classify all totally geodesic and extrinsically homogeneous Lagrangian submanifolds. Moreover, we show that each type has at least one underlying Lagrangian submanifold. In Chapter 4, we classify all totally real totally geodesic surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Finally, in Chapter 5, we show that there are no hypersurfaces with constant sectional curvature.

Chapter 1

Preliminaries

In this chapter we present the basic concepts of differential, Riemannian and Hermitian geometry. Further, we introduce the reader to nearly Kähler geometry and we provide some homogeneous examples.

1.1 Smooth manifolds

We can think of a differential manifold as a continuous set of parameters with some relations between them.

Definition 1. A topological space M is said to be **locally Euclidean** of dimension n if for each point $p \in M$ there exists an open neighborhood U_p around p and a homeomorphism $\varphi_p \colon U_p \to \varphi_p(U_p) \subset \mathbb{R}^n$ such that $\varphi_p(U_p)$ is an open subset of $\mathbb{R}^n$. The pairs (U_p, φ_p) are called **coordinate charts**.

Definition 2. A $\mathcal{C}^\infty$**-atlas** of a locally Euclidean space M is a collection of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that:

1. $\mathcal{A}$ is an open covering of M: $M = \cup_{\alpha \in I} U_\alpha$.
2. The charts are compatible: For all $\alpha, \beta \in I$ such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{U_{\alpha\beta}} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$ is $\mathcal{C}^\infty$.

Definition 3. A n**-dimensional differentiable manifold** is a Hausdorff, second countable locally Euclidean space M of dimension n with a maximal $\mathcal{C}^\infty$-atlas $\mathcal{A}$, in the sense that if $\mathcal{A}'$ is another $\mathcal{C}^\infty$-atlas such that $\mathcal{A} \subset \mathcal{A}'$ then $\mathcal{A} = \mathcal{A}'$.

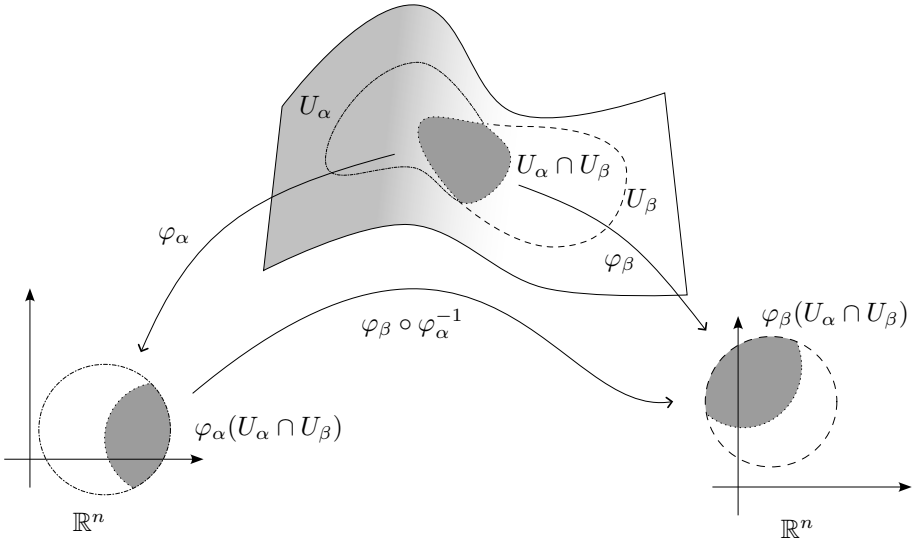


Figure 1.1 – The compatibility of charts

Remark 1. We will often refer to an n-dimensional differentiable manifold as a n-manifold, a n-smooth manifold, a manifold M^n, etc.

Example 1. Let $\mathbb{S}^n$ be the n-dimensional sphere. That is, the subset of $\mathbb{R}^{n+1}$

$$\mathbb{S}^n = \{x \in \mathbb{R}^n : \|x\| = 1\},$$

where $\|\cdot\|$ is the standard Euclidean norm. We denote by p_1 and p_2 the north and south poles, respectively. The maps $\varphi_i \colon U_i \to \mathbb{R}^n$, where $U_i = \mathbb{S}^n \setminus p_i$ and $i = 1, 2$ given by

$$\varphi_i(x) = \left(\tfrac{x_1}{1+(-1)^i x_0}, \ldots, \tfrac{x_n}{1+(-1)^i x_0}\right)$$

define a $\mathcal{C}^\infty$-atlas. Taking the maximal atlas that contains it we get a differentiable structure on $\mathbb{S}^n$.

Definition 4. Let $(M, \mathcal{A})$ be a differentiable manifold. A function $f \colon M \to \mathbb{R}$ is said to be **differentiable** at $p \in M$ there exist a coordinate chart (U, φ) in $\mathcal{A}$ with $p \in U$ such that $f \circ \varphi^{-1} \colon \varphi(U) \subset \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^\infty$ at p. A function $f \colon M \to \mathbb{R}$ is said to be **differentiable** or **smooth** if it is differentiable at every point $p \in M$.

Definition 5. Given two smooth manifolds M^m and N^n and a map $f \colon M \to N$ we said that f is **differentiable** at $p \in M$ if there exist coordinate charts

(U, φ) and (V, ψ) around p and $f(p)$ respectively, such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} \colon \varphi(U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$ is differentiable. As before, if f is differentiable for all $p \in M$ then it is said to be **differentiable**, or **smooth**.

Definition 6. Let M^n be a smooth manifold and let $p \in M$. We define $\boldsymbol{\mathcal{C}^\infty(p)}$ as the set of all smooth functions defined on open neighborhoods around p.

Given two smooth functions f and g on $\mathcal{C}^\infty(p)$ defined in U and V respectively, we may define the sum $f + g \colon U \cap V \to \mathbb{R}$ as $(f + g)(p) = f(p) + g(p)$. In the same way we define fg and λf, where $\lambda \in \mathbb{R}$.

Definition 7. A **tangent vector** to M at p is a map $v \colon \mathcal{C}^\infty(p) \to \mathbb{R}$ such that

1. $v(f + g) = v(f) + v(g)$ and $v(\lambda f) = \lambda v(f)$, where $\lambda \in \mathbb{R}$,
2. $v(fg) = v(f)g(p) + f(p)v(g)$.

Remark 2. The set T_pM of all tangent vectors to an n-manifold M at a point p forms a n-dimensional vector space, called the **tangent space** at p.

Given a coordinate chart $(U, \varphi = (x_1, x_2, \ldots, x_n))$ on a manifold M^n, a point $p \in U$ we define the tangent vectors $\left.\frac{\partial}{\partial x_i}\right|_p$ by

$$\left.\frac{\partial}{\partial x_i}\right|_p (f) = \left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} f \circ \varphi^{-1}.$$

These tangent vectors are linearly independent and form a basis of T_pM. Moreover, if $v = \sum_i a_i \left.\frac{\partial}{\partial x_i}\right|_p$ then $a_i = v(x_i)$.

If $\gamma \colon I \to M$, $I \subset \mathbb{R}$, is a curve in a manifold M, we define its velocity at $t_o \in I$ as

$$\gamma'(t_o) = \sum_{i=1}^{n} r_i'(t_o) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t_o)},$$

where $r_i = x_i \circ \gamma$. Moreover, for any tangent vector $v \in T_pM$ there exists a curve $\gamma \colon I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Example 2. Let V be a n-dimensional real vector space. As it is homeomorphic to $\mathbb{R}^n$ it is clearly a differentiable manifold. We can identify the tangent space at a point $p \in V$ with V itself, via the isomorphism

$$v \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} (p + tv).$$

Definition 8. Given smooth manifolds M^m and N^n and a smooth map $F \colon M \to N$ the **differential** of F at p is the linear map $dF_p \colon T_pM \to T_{F(p)}N$ defined by $dF_p(v)(f) = v(f \circ F)$, where $f \in \mathcal{C}^\infty(F(p))$.

Example 3. Given a smooth manifold M^n, the set $TM = \cup_{p\in M} T_pM$ possesses a canonical smooth structure, of dimension $2n$. We call TM the tangent bundle of M.

Definition 9. Let M^m and N^n be smooth manifolds. A map $f\colon M \to N$ is called a(n):

- **Immersion** if df_p is injective for all $p \in M$. If moreover, $M \subset N$ and the inclusion is an immersion we say that M is an **immersed submanifold** of N.
- **Embedding** if f is an immersion such that M is homeomorphic to $f(M)$, where $f(M)$ carries the induced topology.
- **Submersion** if df_p is surjective for all $p \in M$.

Definition 10. Let M^n be a smooth manifold and let U be an open subset of M. A **smooth vector field** on U is a smooth map $X\colon U \to TM$ such that $X_p = X(p) \in T_pM$ for all $p \in U$. Given a function $f\colon U \to \mathbb{R}$ we define the function $X(f)\colon U \to \mathbb{R}$ as $X(f)(q) = X_q(f)$.

Remark 3. Given a smooth vector field X on a manifold M, we can write it locally as $X_p = \sum_i a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p$, where $p \in U$ and $(U, (x_1, \ldots, x_n))$ is a coordinate chart, and $a_i\colon U \to \mathbb{R}$ are smooth functions.

The set of all smooth vector fields on a manifold M is denoted by $\mathfrak{X}(M)$.

Definition 11. Let X be a smooth vector field on a manifold M. A curve $\gamma\colon I \subset \mathbb{R} \to M$ is said to be an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}$ for all t.

Example 4. Let $\mathbb{H}$ be the quaternion algebra. That is, $\mathbb{R}^4$ with the non-commutative product given by the following table:

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

where 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the canonical basis of $\mathbb{R}^4$.

As mentioned in Example 1, the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ is a smooth manifold. We define vector fields on $\mathbb{S}^3$ by

$$X_1(p) = p\mathbf{i}, \quad X_2(p) = p\mathbf{j}, \quad X_3(p) = p\mathbf{k}.$$

It can be easily proven that these vector fields are smooth, as the multiplication by a fixed quaternion is smooth.

The integral curves for X_1, X_2 and X_3 through a point p are respectively given by

$$pe^{\mathbf{i}t} := p(1\cos t + \mathbf{i}\sin t),$$

$$pe^{\mathbf{j}t} := p(1\cos t + \mathbf{j}\sin t),$$

$$pe^{\mathbf{k}t} := p(1\cos t + \mathbf{k}\sin t).$$

Definition 12. Given $X, Y \in \mathfrak{X}(M)$, the set of all smooth vector fields in M, we define their **Lie bracket** as the smooth vector field $[X, Y]$ given by $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$, with $f \in \mathcal{C}^\infty(p)$.

Proposition 1. *Let $(U, (x_1, \ldots, x_n))$ be a coordinate chart of a manifold M. Let $X, Y, Z \in \mathfrak{X}(M)$ and let f be a smooth function on M. Then,*

1. $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ *for all i and j.*
2. $[X, Y] = -[Y, X]$,
3. $[X, Y + Z] = [X, Y] + [Y, Z]$,
4. $[X, fY] = X(f)Y + f[X, Y]$,
5. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The last property is known as the Jacobi identity.

Definition 13. A set of vector fields on a manifold M is called a **frame** if at each point p it is a basis of the tangent space T_pM.

Proposition 2. *Given a frame $\{X_1, \ldots, X_n\}$ on a manifold M that satisfies $[X_i, X_j] = 0$ for all $i, j = 1, \ldots, n$, there exists a coordinate chart $(U, (x_1, \ldots, x_n))$ such that $X_i = \frac{\partial}{\partial x_i}$ for all $i = 1, \ldots, n$.*

1.2 Riemannian manifolds

Definition 14. A **Riemannian manifold** is a pair (M, g) where M is a smooth manifold and g assigns to each point p an inner product g_p in T_pM, which varies smoothly from point to point. The map g is know as the **Riemannian metric** of M, or simply as the metric. Alternatively, if g_p is indefinite we say that the pair (M, g) is a **pseudo-Riemannian manifold**.

Definition 15. A map $\phi\colon (M^n, g_1) \to (N^n, g_2)$ between Riemannian manifolds is said to be an **isometry** if it is a diffeomorphism and preserves the metric:

$g_2(d\phi X, d\phi Y) = g_1(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. If there exists such a map, then we say that (M, g_1) and (N, g_2) are **isometric**.

Proposition 3. *Let $\varphi, \psi \colon M \to N$ be isometries, where M and N are connected (pseudo-)Riemannian manifolds. Suppose that there exists a point $p \in M$ such that $\varphi(p) = \psi(p)$ and $d\varphi_p$ coincides with $d\psi_p$ in T_pM. Then $\varphi = \psi$ everywhere.*

Example 5. The canonical Riemannian metric on the n-dimensional Euclidean space $\mathbb{R}^n$ is defined by identifying $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ and adding the usual inner product $\langle , \rangle$ on $\mathbb{R}^n$.

Example 6. In Example 1 we defined the n-dimensional sphere embedded in $\mathbb{R}^n$. The tangent space at a point p is given by $\{p\}^\perp \subset \mathbb{R}^n$. The restriction of the inner product of $\mathbb{R}^n$ to $\{p\}^\perp$ induces a Riemannian metric on $\mathbb{S}^n$. This metric is know as the round metric.

Example 7. Consider the half space $\{(x_1, \ldots, x_n) : x_1 > 0\} \subset \mathbb{R}^n$ with the metric

$$g\left(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\right) = \delta_{ij} \frac{1}{x_1^2}.$$

This Riemannian manifold is called hyperbolic space of dimension n.

Remark 4. Given a curve $\gamma \colon [a, b] \to M$ in a Riemannian manifold (M, g) we may define the length of γ by

$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

Hence, a Riemannian metric defines a distance on M:

$$d(p, q) = \inf\{L(\gamma) : \gamma \text{ joins } p \text{ and } q\}$$

The topology induced by d coincides with the original topology on M.

Definition 16. A linear connection is a map $\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \mapsto \nabla_X Y$ that satisfies the following properties:

1. It is $\mathcal{C}^\infty$-linear in the first factor: $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$.
2. It is $\mathbb{R}$-linear on the second factor: $\nabla_X(\lambda Y + Z) = \lambda \nabla_X Y + \nabla_X Z$.
3. It satisfies the Leibniz rule on the second factor: $\nabla_X fY = X(f)Y + f\nabla_X Y$.

This definition can be extended to tensors, that is $\mathcal{C}^\infty$ multilinear maps defined for all T_pM such vary smoothly from point to point. Namely, given a (p, q)-tensor

$$T \colon \underbrace{\mathfrak{X}(M) \otimes \ldots \otimes \mathfrak{X}(M)}_{p \text{ times}} \to \underbrace{\mathfrak{X}(M) \otimes \ldots \otimes \mathfrak{X}(M)}_{q \text{ times}}$$

we define the covariant derivative of T with respect to $Y \in \mathfrak{X}(M)$ as the (p,q)-tensor given by

$$(\nabla_Y T)(X_1,\dots,X_p) = \nabla_Y(T(X_1,\dots,X_p)) - \sum_{i=1}^{p} T(X_1,\dots,\nabla_Y X_i,\dots,X_p).$$

Given a Riemannian manifold (M,g) there exists a unique linear connection which is:

1. symmetric, i.e. $\nabla_X Y - \nabla_Y X = [X,Y]$,
2. metric compatible, i.e. $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Definition 17. The unique connection defined above is called the **Levi-Civita connection** associated to g.

Although there is no general explicit formula for this connection, we may compute it by means of the so-called Koszul formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])\Big),$$

where $X, Y, Z \in \mathfrak{X}(M)$.

Definition 18. A vector field X along a curve γ in a (pseudo-)Riemannian manifold is said to be **parallel** along γ if $\nabla_{\gamma'} X = 0$, where ∇ is the Levi-Civita connection.

Note that to define a parallel vector field we do not need a metric, therefore we may define parallelity for any linear connection. If the connection is compatible with the metric then parallel vector fields have constant length.

Definition 19. A curve γ is called a **geodesic** if its velocity is a parallel vector field along γ: $\nabla_{\gamma'}\gamma' = 0$.

Lemma 1. *Let M be a smooth manifold with a linear connection ∇ (not necessarily torsion-free or metric compatible). Given $p \in M$ and $X \in T_pM$ there exist a unique maximal geodesic γ_X such that $\gamma_X(0) = p$ and $\gamma_X'(0) = X$.*

Here maximality refers to the domain of the geodesic.

Definition 20. Given a point p in a smooth manifold with a linear connection, we define the **exponential map** $\exp_p \colon U \subset T_pM \to M$ by

$$\exp(X) = \gamma_X(1),$$

where U is the subset of T_pM where $\gamma_X(1)$ is defined.

Lemma 2. *Let M be a smooth manifold with a linear connection. Given $p \in M$, there exist neighborhoods U and V of p and $0 \in T_pM$, respectively, such that $\exp_p \colon V \to U$ is a diffeomorphism. Such neighborhood U of p is called a* ***normal neighborhood****.*

An important consequence for a manifold M to carry a Riemannian metric g is that the metric induces a distance on M. To be more precise, given a curve $\gamma \colon [a, b] \to M$ we can measure its length by

$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

Consequently, we can define a distance d between two points $p, q \in M$:

$$d(p, q) = \inf_{\gamma} \{L(\gamma) : \gamma \text{ joins } p \text{ and } q\}.$$

This distance induces a topology on M, which coincides with the topology of smooth manifold. This is not true for pseudo-Riemannian, wince we cannot deduce a natural distance from it.

Theorem 15 (Hopf-Rinow)**.** *Let M be a Riemannian manifold. Then the following are equivalent:*

1. *(M, d) is complete as topological metric space. That is, all the Cauchy sequences converge.*
2. *M is geodesically complete, that is geodesics are defined for all $t \in \mathbb{R}$.*
3. *The exponential map is defined in all T_pM for all p.*
4. *Closed and bounded subsets of M are compact.*

Definition 21. Given a (pseudo-)Riemannian manifold (M, g) **the curvature tensor** R is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The curvature tensor is sometimes written as $R(X, Y, Z, W) = g(R(X, Y)Z, W)$. We will only use it in the next proposition.

Proposition 4. *The curvature tensor R of a Riemannian manifold satisfies:*

1. *$R(X, Y, Z, W) + R(Y, X, Z, W) = 0$,*
2. *$R(X, Y, Z, W) + R(X, Y, W, Z) = 0$,*
3. *$R(X, Y, Z, W) - R(Z, W, X, Y) = 0$,*
4. *$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$. (First Bianchi identity)*

Definition 22. Given a (pseudo)-Riemannian manifold (M, g), we define the **sectional curvature** K of a non-degenerate plane $\Pi \subset T_pM$ by

$$K_p(\Pi) = \frac{g_p(R_p(X,Y)Y, X)}{g_p(X,X)g_p(Y,Y) - g_p(X,Y)^2}$$

where X_p and Y_p form a basis of Π.

Theorem 16 (Schur's Lemma)**.** *Let M be a Riemannian manifold such that $K_p(\Pi) = \kappa(p)$ for all planes $\Pi \subset T_pM$ and a smooth function κ. Then κ is constant.*

Complete Riemannian manifolds with constant sectional curvature are known as space forms. In Example 5, Example 6 and Example 7 we defined the three different space forms, up to a rescaling. Next, we define the pseudo-Riemannian space forms.

Example 8 (Pseudo-Riemannian space forms)**.** The pseudo-Euclidean space $\mathbb{R}^n_k$ is defined as the manifold $\mathbb{R}^n$ with the pseudo-Riemannian metric $\langle x, y\rangle_k = -x_1y_1 - \ldots - x_ky_k + x_{x+1}y_{x+1} + \ldots + x_ny_n$. The manifold $\mathbb{R}^n_k$ has constant sectional curvature equal to zero, and when $k = 1$ it is known as Minkowski space. If $k = 0$, we have the usual Euclidean space.

Now consider the pseudo-Riemannian sphere $\mathbb{S}^{n-1}_k(c)$ of curvature $c > 0$ as the pseudo-Riemannian submanifold of $\mathbb{R}^n_k$ given by

$$\mathbb{S}^{n-1}_k(c) = \left\{x \in \mathbb{R}^n_k : \langle x, x\rangle_k = \tfrac{1}{c}\right\}.$$

It has positive constant sectional curvature c. If $k = 0$ it is the usual Riemannian sphere and if $k = 1$ is called the de Sitter space.

The pseudo-hyperbolic space $H^{n-1}_{k-1}(c)$ of curvature $c < 0$ is the pseudo-Riemannian submanifold of $\mathbb{R}^n_k$ given by

$$H^{n-1}_{k-1}(c) = \left\{x \in \mathbb{R}^n_k : \langle x, x\rangle_k = \tfrac{1}{c}\right\}.$$

If $k = 1$ the it is the usual hyperbolic space and if $k = 2$ it is called the anti-de Sitter space.

We denote by $\tilde{\mathbb{S}}^{n-1}_k$ and $\tilde{H}^{n-1}_{k-1}$ their respective universal covers.

Theorem 17 (Killing-Hopf)**.** *Let M^n be a complete, simply connected Riemannian manifold with constant sectional curvature c. Then it is isometric to either $H^n(c)$, $\mathbb{R}^n$ or $\mathbb{S}^n(c)$.*

Theorem 18 (O'Neill)**.** *Let M^n be a complete, simply connected Lorentzian manifold with constant sectional curvature c. Then it is isometric to either $\tilde{H}^n_1(c)$, $\mathbb{R}^n_1$ or $\tilde{\mathbb{S}}^n_1(c)$.*

Remark 5. Contrary to the Riemannian case, pseudo-Riemannian metrics do not induce distances on the underlying manifold. Hence, there is no canonical choice of distance to define completeness. Thus, in Theorem 18 complete means geodesically complete, i.e. geodesics are defined for all $t \in \mathbb{R}$.

We finish this section with a result for linear connections that can be found in [53].

Let M and M' smooth manifolds of dimension n with linear connections and let $p \in M$ and $p' \in M'$. Choose a linear isomorphism $\varphi\colon T_pM \to T_{p'}M'$. Choose open convex neighborhoods V and $V' = \varphi(V)$ of $0 \in T_pM$ and $0 \in T_{p'}M'$, respectively. Let $U = \exp_p(V)$ and $U' = \exp_{p'}(V')$. There exists a diffeomorphism $f\colon U \to U'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V' \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_{p'}} \\ U & \xrightarrow{f} & U' \end{array} \tag{1.1}$$

commutes. Moreover, for every $q \in U$, we have linear isomorphisms $\varphi_q\colon T_qM \to T_{f(q)}M'$ such that the diagram

$$\begin{array}{ccc} T_pM & \xrightarrow{\varphi} & T_{p'}M' \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau'} \\ T_qM & \xrightarrow{\varphi_q} & T_{f(q)}M' \end{array}$$

commutes, where τ and τ' are the parallel transports along geodesics passing through p and p', respectively.

Theorem 19 (Local version of Cartan-Ambrose-Hicks theorem)**.** *Let M and M' be smooth manifolds of dimension n, equipped with linear connections. Let R, R' and T, T' their respective curvature and torsion tensors. Let $p \in M$, $p' \in M'$ and let U and U' be normal neighborhoods around p and p', respectively. Suppose that for every $q \in U$ the linear isomorphism φ_q maps R_q into $R'_{f(q)}$ and T_q into $T'_{f(q)}$, i.e.*

$$\varphi_q R(X,Y)Z = R'(\varphi_q X, \varphi_q Y)\varphi_q Z, \qquad \varphi_q T(X,Y) = T'(\varphi_q X, \varphi_q Y).$$

Then $f\colon U \to U'$ is an affine diffeomorphism (i.e. connection-preserving) and $df_q\colon T_qM \to T_{f(q)}M'$ coincides with φ_q.

Corollary 1. *Let (M, g) and (M', g') be (pseudo-)Riemannian manifolds and let R and R' be their respective curvature tensors, associated to the Levi-Civita*

connection. Let $p \in M$, $p' \in M'$, U, U' normal neighborhoods around p and p' respectively, and $\varphi\colon T_pM \to T_{p'}M'$ be a linear isometry.

Suppose that for every $q \in U$, and for every $u_1, \dots, u_4 \in T_qM$ it holds

$$g(R(u_1,u_2)u_3,u_4)_q = g'(R'(u_1',u_2')u_3',u_4')_{f(q)},$$

where f is the diffeomorphism given in (1.1), $u_i' = \tau'(\varphi(\tau^{-1}(u_i)))$, τ and τ' are the parallel transports along the geodesics that join p with q and p' with $f(q)$, respectively. Then f is a local isometry between M and M'.

1.3 Lie groups

Definition 23. A **Lie group** is a group G with a smooth structure, such that the multiplication $m\colon G \times G \to G, (g,h) \mapsto gh$ and inverse $i\colon G \to G, g \mapsto g^{-1}$ are smooth.

Definition 24. A **Lie algebra** is a vector space $\mathfrak{g}$ together with a bilinear operation $[,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the **Lie bracket**. Such operation must satisfy the following properties:

1. $[X,Y] = -[Y,X]$,
2. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

for all $X, Y, Z \in \mathfrak{g}$.

Definition 25. An **isomorphism of Lie groups** G and H is a group isomorphism $\Phi\colon G \to H$ such that it is differentiable. On the other hand, an **isomorphism of Lie algebras** $\mathfrak{g}$ and $\mathfrak{h}$ is a vector space isomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ such that $\varphi[X,Y]_{\mathfrak{g}} = [\varphi X, \varphi Y]_{\mathfrak{h}}$, where $X, Y \in \mathfrak{g}$.

Although the definitions of Lie group and Lie algebra seem to be unrelated, any Lie group has an associated Lie algebra and any Lie algebra has a simply connected associated Lie group.

Definition 26. Let G be a Lie group and denote by $L_g\colon G \to G$ and $R_g\colon G \to G$ the left and right multiplications by g, respectively. A **left-invariant vector field** on G is a vector X such that $(dL_g)_h X_h = X_{gh}$. Similarly we can define right-invariant vector fields.

We denote by $\mathrm{Lie}(G)$ the set of all left-invariant vector fields on G. With the Lie bracket $[,]$ of smooth vector fields, $\mathrm{Lie}(G)$ and $[,]$ form a Lie algebra. Denote the identity of G by e. Then the map that sends $X \mapsto X_e$ is a vector

space isomorphism between $\mathrm{Lie}(G)$ and T_eG. Hence, we can copy a Lie algebra structure to T_eG.

Associating a Lie group to a given Lie algebra is a more difficult task than the other way around. Many Lie groups can have the same associated Lie algebra, as for instance $\mathrm{SO}(n)$, $\mathrm{O}(n)$ and $\mathrm{Spin}(n)$, or $\mathbb{R}$ and $\mathbb{S}^1$. Thus, adding the condition of being simply connected, we obtain what is known as Lie's third theorem.

Theorem 20 (Lie's third theorem)**.** *Given a Lie algebra $\mathfrak{g}$ there exists a unique (up to isomorphism) simply connected Lie group G such that* $\mathrm{Lie}(G) = \mathfrak{g}$.

The sum of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ denoted by $\mathfrak{g} \oplus \mathfrak{h}$ is the Lie algebra with Lie bracket given by $[X_1 + Y_1, X_2 + Y_2] = [X_1, X_2]_{\mathfrak{g}} + [Y_1, Y_2]_{\mathfrak{h}}$. Because of the overlap in notation, we will clarify whether we are talking of sum of Lie algebras or sum of vector spaces.

Definition 27. A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a vector subspace invariant by the Lie bracket. An **ideal** of $\mathfrak{g}$ is a vector subspace $\mathfrak{a}$ such that $[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$.

As before, Lie subgroups and Lie subalgebras are tightly linked.

Proposition 5. *Given a Lie group G with Lie algebra $\mathfrak{g}$, for each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there exists a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.*

Definition 28. A Lie algebra $\mathfrak{g}$ is **abelian** if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. A Lie algebra is said to be **simple** if does not contain any proper ideals. If it does not contain any proper abelian ideals is known as **semisimple** Lie algebra.

Proposition 6. *Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:*

1. *$\mathfrak{g}$ is semisimple,*
2. *$\mathfrak{g}$ is a direct sum of simple Lie algebras,*
3. *The Killing form of $\mathfrak{g}$ is non-degenerate,*

where the Killing form is the quadratic form given by

$$B(X, Y) = \mathrm{tr}(Z \mapsto [X, [Y, Z]]). \tag{1.2}$$

Definition 29. A **reductive** Lie algebra is a Lie algebra $\mathfrak{g}$ such that for every ideal $\mathfrak{a}$ there exists another ideal $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, as sum of Lie algebras.

We denote by $\mathfrak{z}(\mathfrak{g})$ the center of a Lie algebra $\mathfrak{g}$, i.e. $\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0, \text{ for all } Y \in \mathfrak{g}\}$. The center of a Lie algebra is an abelian ideal.

The lower central series of a Lie algebra $\mathfrak{g}$ is given by $\mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}]$. If there exists a $k \in \mathbb{N}$ such that $\mathfrak{g}_k = 0$ then we say that $\mathfrak{g}$ is **nilpotent**. The upper central series of $\mathfrak{g}$ is given by $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$. If there exists a $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$ then we say that $\mathfrak{g}$ is **solvable**.

Proposition 7. *A Lie algebra $\mathfrak{g}$ is reductive if and only if $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ as a sum of Lie algebras, with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.*

Therefore, we can interpret reductive Lie algebras as a generalization of semisimple Lie algebras, since any reductive Lie algebra is the sum of an abelian Lie algebra and a collection of simple Lie algebras.

The exponential map

Given a morphism of Lie groups, its differential at the identity element is a morphism of Lie algebras. Moreover, the converse is true if we require simply connectedness.

Theorem 21 (Lie's second theorem)**.** *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and let H be a Lie group with Lie algebra $\mathfrak{h}$. Suppose that there exists a morphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{h}$. Then there exists a morphism of Lie groups $\Phi\colon G \to H$ such that $d_e\Phi$ coincides with φ, where e is the identity element of G.*

A **monoparametric subgroup** of a Lie group G is a morphism $\gamma\colon (\mathbb{R}, +) \to G$. Let $X \in \mathfrak{g}$ be an element of the Lie algebra of G. The Lie subalgebra $\mathrm{Span}\{X\}$ is an abelian subalgebra, therefore by Lie's second theorem we obtain that there exists an associated monoparametric subgroup γ_X of G such that $\gamma_X'(0) = X$.

We define the **exponential map** $\exp\colon \mathfrak{g} \to G$ by

$$\exp(X) = \gamma_X(1).$$

In general the exponential map is not surjective. For instance, $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective, as the element $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not on its image. This is also related to the fact that $\mathrm{SL}(2, \mathbb{R})$ with the Killing form as Lorentzian metric is not geodesically connected.

Actions of Lie groups

Definition 30. Let G be a Lie group and M a smooth manifold. An **action** of G on M is a smooth map $\phi\colon G \times M \to M : (g, p) \mapsto g \cdot p$ such that

1. $e \cdot p = p$,
2. $g_2 \cdot (g_1 \cdot p) = (g_2 g_1) \cdot p$,

where e is the identity element of G.

The action is **transitive** if for any arbitrary pair of points p, q in M there exists an element $g \in G$ such that $g \cdot p = q$.

The **orbit** of an element $p \in M$ by an action of a group G on M is the set

$$G \cdot p = \{g \cdot p \in M : g \in G\}.$$

The **isotropy subgroup** of $p \in M$ (also known as the stabilizer) is the subgroup of G given by

$$G_p = \{g \in G : g \cdot p = p\}.$$

Remark 6. Given an action of a Lie group on a smooth manifold M, the isotropy group of a point is a closed subgroup (and therefore a Lie subgroup of G) and the orbits are immersed submanifolds of M. The orbit $G \cdot p$ of a point $p \in M$ is diffeomorphic to G/G_p the set of all cosets.

Remark 7. Given a Riemannian manifold, the set of all its isometries is a Lie group with the composition as group law.

Definition 31. A **homogeneous Riemannian manifold** is a Riemannian manifold M whose isometry group acts transitively on M.

Example 9. The orthogonal group $\mathrm{O}(n)$ acts by isometries on $\mathbb{R}^n$. The orbit of any unit-length element of $\mathbb{R}^n$ is the unit sphere $\mathbb{S}^{n-1}$. The stabilizer of each point in the sphere is isomorphic to $\mathrm{O}(n-1)$ therefore, $\mathbb{S}^{n-1} \cong \mathrm{O}(n)/\mathrm{O}(n-1)$.

Similarly, n-hyperbolic spaces are orbits of the action of $\mathrm{SO}(n-1,1)$ on $\mathbb{R}^n_1$. Thus $H^{n-1} = \mathrm{SO}(n-1,1)/\mathrm{O}(n-1)$.

Bianchi classification

In the late 19th century, Bianchi managed to classify all three-dimensional real Lie algebras in 11 classes, up to isomorphism. Most of them are a semidirect products $\mathbb{R} \ltimes \mathbb{R}^2$. Namely, their simply connected associated Lie group is $\mathbb{R} \times \mathbb{R}^2$ with the group law $(t,u) \cdot (s,v) = (t+s, \varphi(s)u + v)$, where $\varphi \colon \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$. All such groups can also be expressed as a group of 3×3 matrices, where the elements are of the form

$$\left(\begin{array}{c|c} \varphi(s) & v \\ \hline 0 & 1 \end{array}\right).$$

We give the classification below, and e_1, e_2, e_3 denotes a basis of each Lie algebra.

— Type I. The Lie algebra is abelian. That is $[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0$. The associated Lie group is $\mathbb{R}^3$.

— Type II. Known as the Heisenberg Lie algebra, this is the nilpotent Lie algebra with the bracket given by $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$. The associated Lie group is the Heisenberg group, often denoted by Nil_3. It is the the semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$ with

$$\varphi(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

— Type III. The Lie algebra is given by the relations $[e_1, e_2] = [e_1, e_3] = 0$ and $[e_2, e_3] = e_2$. The Lie group is $\mathbb{R} \ltimes \mathbb{R}^2$ with

$$\varphi(s) = \begin{pmatrix} e^{as} & 0 \\ 0 & 1 \end{pmatrix}, \quad (a \in \mathbb{R} \setminus \{0\})$$

— Type IV. The Lie algebra is given by the relations $[e_1, e_2] = 0$, $[e_3, e_1] = e_1$ and $[e_3, e_2] = e_1 + e_2$. The Lie group is $\mathbb{R} \ltimes \mathbb{R}^2$ with

$$\varphi(s) = \begin{pmatrix} e^{as} & e^{as}s \\ 0 & e^{as} \end{pmatrix}, \quad (a \in \mathbb{R} \setminus \{0\}).$$

— Type V. The Lie algebra is defined by the relations $[e_1, e_2] = 0$, $[e_3, e_1] = e_1$ and $[e_3, e_2] = e_2$. The Lie group is $\mathbb{R} \ltimes \mathbb{R}^2$ with

$$\varphi(s) = \begin{pmatrix} e^{as} & 0 \\ 0 & e^{as} \end{pmatrix}, \quad (a \in \mathbb{R} \setminus \{0\}).$$

— Type VI. The Lie algebra is defined by the relations $[e_1, e_2] = 0$, $[e_3, e_1] = e_1$ and $[e_3, e_2] = ce_2$, $c \in \mathbb{R} \setminus \{0\}$. The Lie group is $\mathbb{R} \ltimes \mathbb{R}^2$ with

$$\varphi(s) = \begin{pmatrix} e^{as} & 0 \\ 0 & e^{acs} \end{pmatrix}, \quad (a \in \mathbb{R} \setminus \{0\}).$$

— Type VI_0. The Lie algebra is defined by the relations $[e_1, e_2] = 0$, $[e_3, e_1] = e_1$ and $[e_2, e_3] = e_2$. The Lie group is $\mathbb{R} \ltimes \mathbb{R}^2$ with

$$\varphi(s) = \begin{pmatrix} e^{as} & 0 \\ 0 & e^{-as} \end{pmatrix}, \quad (a \in \mathbb{R} \setminus \{0\}).$$

— Type VII. This is an infinite family. They are the Lie algebra of the semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$ where $\varphi'(0)$ has complex non-real non-imaginary eigenvalues.

— Type VII_0. The Lie algebra is defined by the relations $[e_1, e_2] = 0$, $[e_1, e_3] = e_2$ and $[e_3, e_2] = e_1$. The Lie group is the connected group of isometries of $\mathbb{R}^2$, given by $\mathbb{R} \ltimes \mathbb{R}^2$ with

$$\varphi(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}.$$

— Type VIII. The Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$. It has the relations $[e_1, e_2] = 2e_3$, $[e_1, e_3] = 2e_2$ and $[e_3, e_2] = 2e_1$. The Lie group is the universal cover of $\mathrm{SL}(2, \mathbb{R})$, the 2 by 2 matrices of determinant 1.

— Type IX. The Lie algebra $\mathfrak{su}(2)$. It has the relations $[e_1, e_2] = 2e_3$, $[e_3, e_1] = 2e_2$ and $[e_2, e_3] = 2e_1$. The Lie group is $\mathrm{SU}(2)$, the group of all 2 by 2 complex unitary matrices of determinant 1. It is also isomorphic to the 3-sphere, seen as the set of all unit quaternions.

Note that all Lie algebras from type II to type VII_0 are solvable. Type VIII and IX are the only ones that are simple. The exponential map of all Lie algebras is surjective except for type VIII.

1.4 Submanifold theory

Although the study of submanifolds is not exclusive to Riemannian geometry, we think of submanifold theory as the study of Riemannian immersions. That is, an immersion $f\colon (M, g) \to (N, \tilde{g})$ such that $\tilde{g}((df)X, (df)Y) = g(X, Y)$, for all $X, Y \in \mathfrak{X}(M)$.

A Riemannian immersion $f\colon (M, g) \to (N, \tilde{g})$ determines an orthogonal splitting of the tangent bundle of the target space: $TN = (df)(TM) \oplus TM^{\perp}$, which we will call the tangent part and the normal part.

Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and N associated to g and $\tilde{g}$, respectively. Given two vector fields X, Y on M, we can split $\tilde{\nabla}_{(df)X}(df)Y$ into a tangent and a normal part:

$$\tilde{\nabla}_{(df)X}(df)Y = (df)\nabla_X Y + h(X, Y).$$

This equation is known as the **Gauss formula**, and h is a symmetric, normal tensor called the **second fundamental form** of M.

If instead we take a normal vector field $\xi \in \Gamma(TM^{\perp})$ and a tangent vector field $X \in \mathfrak{X}(M)$ we get

$$\tilde{\nabla}_{(df)X}\xi = -S_{\xi}(df)X + \nabla^{\perp}_X \xi.$$

This is know as the **Weingarten formula** and $\nabla^{\perp}$ is a connection $\nabla^{\perp}\colon \Gamma(TM) \times \Gamma(TM^{\perp}) \to \Gamma(TM^{\perp})$ called the normal connection. On the

other hand, S_ξ is linear operator symmetric with respect to $\tilde{g}$. Moreover, the shape operator associated to ξ, is linear on ξ and satisfies

$$\tilde{g}(S_\xi(df)X, (df)Y) = \tilde{g}(h(X,Y),\xi).$$

Remark 8. As the immersion f preserves g, we will not make a difference between g and $\tilde{g}$. Also, we will omit writing f onwards.

Denote by $R^\perp$ the normal curvature tensor, that is the tensor given by

$$R^\perp(X,Y)\xi = \nabla^\perp_X\nabla^\perp_Y\xi - \nabla^\perp_Y\nabla^\perp_X\xi - \nabla^\perp_{[X,Y]}\xi.$$

We also write $(\overline{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

Let $\tilde{R}$ and R be the curvature tensors of N and M respectively. Then $\tilde{R}$ splits into a tangent part and a normal part, which yields the so0called Gauss, Codazzi and Ricci equations.

1. Gauss equation: $(\tilde{R}(X,Y)Z)^\top = R(X,Y)Z - S_{h(Y,Z)}X + S_{h(X,Z)}Y$.
2. Codazzi equation: $(\tilde{R}(X,Y)Z)^\perp = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z)$.
3. Ricci equation: $g(\tilde{R}(X,Y)\xi,\eta) = g(R^\perp(X,Y)\xi,\eta) - g([S_\xi, S_\eta]X, Y)$.

Here, X, Y and Z are vector fields on M and ξ and η are normal vector fields.

These equations set the basis of submanifold theory. Namely, since any Riemannian immersion must satisfy them, they are the basic necessary conditions for a existence-and-uniqueness-of-submanifolds type of result.

The most classical result in this area, is the so-called Bonnet theorem [21]. It states that given h and g in $\mathbb{R}^3$ satisfying the Gauss and Codazzi equations, there exists a unique (up to congruence) surface of $\mathbb{R}^3$ with metric g and second fundamental form h. Note that, since the codimension is only one, the Ricci equation becomes trivial.

A similar result [15] can be obtained when the ambient space is a simply connected space form, even for the pseudo-Riemannian case. Note that for submanifolds of space forms, the Gauss and Codazzi equations can be written "intrinsically". That is, since the left-hand side only depends on the metric of the submanifold, the equations can be written only in terms of tensors defined on the submanifold. For this reason, when the ambient space becomes less symmetric the fundamental equations are not sufficient anymore to guarantee the existence (and uniqueness) of a submanifold.

Nonetheless, in [16] Daniel provided sufficient and necessary conditions for the existence and uniqueness of a hypersurface in $\mathbb{S}^n \times \mathbb{R}$ or $H^n \times \mathbb{R}$. Separately, Kowalczyk [33] and Lira, Tojeiro and Vitório [36] generalized this result for

submanifolds of a product of two space forms. Also, Piccione and Tausk [45] worked on this problem for homogeneous manifolds.

There does not exist (to the knowledge of the author) a similar result in nearly Kähler theory. Hence, we study submanifolds with strong properties in order to compensate for the lack of the sufficient conditions for the existence of submanifolds.

Definition 32. A **totally geodesic** submanifold of a (pseudo-)Riemannian manifold M is a submanifold such that its geodesics are also geodesics of M. Equivalently, a submanifold is totally geodesic if the second fundamental form vanishes everywhere.

Definition 33. The **mean curvature field** H on a (pseudo-)Riemannian submanifold M is defined as the trace of the second fundamental form. Namely,

$$H = \frac{1}{n}\sum_{i=1}^{n} \eta_i h(E_i, E_i),$$

where $\{E_i\}_i$ is an orthogonal frame on M^n and $\eta_i = g(E_i, E_i) = \pm 1$.

If H vanishes everywhere we say that M is a **minimal submanifold**.

Definition 34. A submanifold M of a (pseudo-)Riemannian manifold is said to be **totally umbilical** if $h(X, Y) = g(X, Y)H$ for all $X, Y \in \mathfrak{X}(M)$.

Definition 35. Let G be a connected Lie group acting on a (pseudo-) Riemannian manifold M by isometries. Orbits of points $p \in M$ by G are called **extrinsically homogeneous** submanifolds (also known as equivariant submanifolds).

1.5 Almost Hermitian manifolds

In the same way as we defined a smooth manifold, we could define a "holomorphic manifold". That is, a topological space with coordinate charts mapping into $\mathbb{C}^n$ and with holomorphic transitions.

Definition 36. Let M be a smooth manifold. A **holomorphic atlas** is a set $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of open subsets $U_\alpha \subseteq M$ and charts $\varphi_\alpha \colon U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$ such that the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are holomorphic. Such coordinate charts are called holomorphic charts.

Definition 37. A **complex manifold** M of dimension n is a smooth manifold of dimension $2n$ equipped with a maximal holomorphic atlas.

Definition 38. Given a complex manifold M, a holomorphic function is a function $f\colon M \to \mathbb{C}$ such that $f \circ \varphi_\alpha^{-1}\colon \varphi(U_\alpha) \to \mathbb{C}$ is holomorphic, for all charts $(U_\alpha, \varphi_\alpha)$.

Analogous to the definition of smooth functions and smooth maps, we may define holomorphic functions and holomorphic maps. Two complex manifolds are called biholomorphic if there exist a holomorphic homeomorphism whose inverse is also holomorphic.

Example 10 (Complex torus)**.** A complex torus is defined as the quotient $\mathbb{C}^n/\Gamma$ where $\Gamma = \oplus_{j=1}^{2n}\mathbb{Z}v_j$, a discrete subgroup of $\mathbb{C}^n$. Since $\mathbb{C}^n/\Gamma$ and $\mathbb{C}^n$ are locally homeomorphic we can easily define charts into $\mathbb{C}^n$. Two complex tori $\mathbb{C}^n/\Gamma_1$ and $\mathbb{C}^n/\Gamma_2$ are biholomorphic if and only if Γ_1 and Γ_2 are isomorphic.

In general, even-dimensional spheres are not complex manifolds. However, any complex projective space can be seen as a complex manifold.

Example 11 (Complex projective space (for more information see [39]))**.** We define the complex projective space as

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where $x \sim y$ if there exists $\lambda \in \mathbb{C}$ such that $x = \lambda y$.

We define the open subsets $\tilde{U}_i$ in $\mathbb{C}P^n$ as

$$\tilde{U}_i = \{[x] \in \mathbb{C}P^n : x_i \neq 0\}.$$

Then we define the holomorphic charts $\varphi_i\colon \tilde{U}_i \to \mathbb{C}^n$ by

$$\varphi_i([x]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

Now we generalize the notion of the complex unit $i \in \mathbb{C}$ to a tensor on a complex manifold. Take a holomorphic chart $(U, \varphi = (x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n))$ on a complex manifold M and define $J_U\colon TU \to TU\colon \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}$. We see that $(J_U)_p$ is independent of U, thus we can define a tensor $J\colon TM \to TM$. This tensor satisfies $J^2 = -\operatorname{Id}$ and is known as the **complex structure** on M.

We can generalize the concept of complex manifold to an even-dimensional smooth manifold that admits such a tensor J. These manifolds are called almost complex manifolds. In general, an almost complex manifold is not necessarily a complex manifold. Such is the case of the sphere $\mathbb{S}^6$, which we see below.

Example 12. We define the octonion algebra $\mathbb{O}$ as $\mathbb{R}^8$ with the non-associative operation given by the following table:

$\bullet$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

The set of all imaginary octonions $\mathrm{Im}(\mathbb{O})$ carries a natural cross product defined as $\alpha \times \beta = \frac{1}{2}(\alpha\beta - \beta\alpha)$.

Consider $\mathbb{S}^6$ immersed in $\mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$ and define the almost complex structure J on $\mathbb{S}^6$ by

$$J_p(X) = p \times X.$$

The sphere $\mathbb{S}^6$ with J is not a complex manifold and this can be proven using Theorem 22. It is still an open problem to determine wether $\mathbb{S}^6$ can be endowed with a complex structure or not.

Example 13. Every even-dimensional Lie group G^{2n} is parallelizable. That is, the tangent bundle is trivial:

$$TG = G \times \mathbb{R}^{2n}.$$

Hence we can define an almost complex structure J by $J_p v = (p, Iv)$ where

$$I = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Moreover, any endomorphism J of the Lie algebra $\mathrm{Lie}(G)$ that satisfies $J^2 = -\mathrm{Id}$ induces an almost complex structure on G.

Definition 39. An almost complex structure on a smooth manifold M^{2n} is said to be **integrable** if M admits the structure of a complex manifold and the associated complex structure coincides with J.

Remark 9. By applying the determinant to the equation $J^2 = -\mathrm{Id}$ we can easily see that any smooth manifold admitting an almost complex structure is necessarily even-dimensional.

Knowing wether a manifold admits a holomorphic atlas or not may be a hard task, but given an almost complex structure there is an easy criterion to determine if it is integrable or not.

Theorem 22 (Newlander-Nirenberg)**.** *Let (M, J) be an almost complex manifold. Then J is integrable if and only if $N_J \equiv 0$, where N_J is the Nijenhuis tensor associated to J defined by*

$$N_J(X,Y) = [JX, JY] - [X,Y] - J([JX,Y] + [X,JY]).$$

Definition 40. Let M^{2n} be a smooth manifold with an almost complex structure J and a Riemannian metric g. Then M is an **almost Hermitian manifold** if $g(JX, JY) = g(X,Y)$ for all $X, Y \in \mathfrak{X}(M)$. In that case, the differential two-form given by $\omega(X,Y) = g(JX,Y)$ is known as the **Kähler form**.

Definition 41. An almost Hermitian manifold (M, g, J) is said to be **Kähler** if J is integrable and ω is closed.

From this definition we can see that Kähler geometry, the study of Kähler manifolds, is the intersection of three important areas of differential geometry: Riemannian geometry, complex geometry, and symplectic geometry. We understand by symplectic manifolds those manifolds that admit a closed, non-degenerate two-form which is globally defined.

Theorem 23. *An almost Hermitian manifold (M, g, J) is Kähler if and only if $\nabla J \equiv 0$.*

The covariant derivative of the Kähler form satisfies $(\nabla_X \omega)(Y,Z) = g((\nabla_X J)Y, Z)$. Notice that from the previous theorem it follows that a manifold is Kähler if and only if ω is parallel.

Hervella and Gray studied almost Hermitian manifolds to a great extent. In [27], they divided almost Hermitian manifolds into 16 different classes. These came from combining four different properties that a tensor $\alpha \in TM^* \otimes TM^* \otimes TM^*$ may satisfy:

1. $\alpha(X,X,Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$,
2. $\alpha(X,Y,Z) + \alpha(Y,Z,X) + \alpha(Z,X,Y) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$,
3. $\alpha(X,Y,Z) - \alpha(JX, JY, Z) = \bar{\alpha}(Z) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$,
4. $$\alpha(X,Y,Z) = -\frac{1}{2(n-1)}\big(g(X,Y)\bar{\alpha}(Z) - g(X,Z)\bar{\alpha}(Y) - g(X,JY)\bar{\alpha}(JZ) + g(X,JZ)\bar{\alpha}(JZ)\big)$$

   for all $X, Y, Z \in \mathfrak{X}(M)$,

where $\bar{\alpha} = \sum_i \alpha(E_i, E_i, Z)$ and $\{E_i\}$ is an arbitrary orthonormal frame. Notice that

$$\overline{\nabla\omega}(Z) = \sum_{i=1}^{n} \nabla\omega(E_i, E_i, Z) = \sum_{i=1}^{n} g((\nabla_{E_i} J)E_i, Z) = \delta\omega(Z)$$

where δ is the codifferential operator.

Denote by W_i the space of all $(3,0)$ tensors that satisfy the property i above, for $i = 1, 2, 3, 4$. Then the covariant derivative of the Kähler form of any almost Hermitian manifold lies in $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$.

Now we define classes of manifolds which are associated to subspaces of W. We denote by $\mathcal{U}$ the class of almost Hermitian manifolds for which the covariant derivative of ω lies in a subspace $U \subset W$. The class associated to the subspace $\{0\} \subset W$ consists of those manifolds for which the covariant derivative of the Kähler form is zero. As we have seen before, these are the Kähler manifolds.

The class $\mathcal{W}_1$ consists of those almost Hermitian manifolds M such that

$$0 = \nabla_X \omega(X, Y) = g((\nabla_X J)X, Y) \text{ for all } X, Y \in \mathfrak{X}(M),$$

which is equivalent to $(\nabla_X J)X = 0$ for all $X \in \mathfrak{X}(M)$. These manifolds are known as **nearly Kähler** manifolds.

The class $\mathcal{W}_2$ consists of those almost Hermitian manifolds that satisfy

$$0 = \mathop{\mathfrak{S}}_{X,Y,Z} (\nabla_X \omega)(Y, Z) = d\omega(X, Y, Z),$$

which is equivalent to have a closed Kähler form. These are know as **almost Kähler** manifolds.

The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ is known as the class of **quasi-Kähler** manifolds and it is equal to the class of those manifolds that satisfy

$$(\nabla_X \omega)(Y, Z) + (\nabla_{JX} \omega)(JY, Z) = 0.$$

The class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ is the class of **semi-Kähler** (or balanced) manifolds, and it is equal to the class of those manifolds that satisfy $\delta\omega = 0$.

It is immediate that nearly Kähler and almost Kähler manifolds are quasi-Kähler, and quasi-Kähler manifolds are semi-Kähler. Moreover, a manifold that is nearly Kähler and almost Kähler, is Kähler.

Examples

Example 14 (Almost Kähler)**.** Finding an example of an almost Kähler manifold which is not Kähler was a big challenge. In [51] it is shown that the

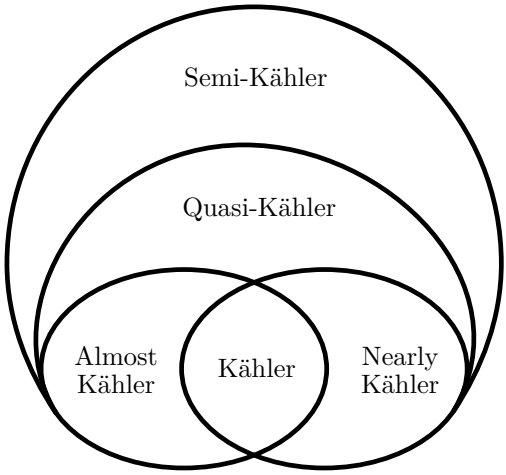


Figure 1.2 – The inclusion of some classes of almost Hermitian manifolds.

tangent bundle of any Riemannian manifold admits an almost Kähler structure. It is Kähler if and only if the base manifold is flat.

Example 15 (Hypersurfaces of $\mathbb{R}^7$ [27])**.** Similarly to the almost complex structure defined on $\mathbb{S}^6$ we may define an almost complex structure on a hypersurface of $\mathbb{R}^7$.

Let M^6 be a submanifold of $\mathbb{R}^7$ and let ξ be the unit normal vector field, that is $\langle \xi, \xi \rangle = 1$ and $\langle X, \xi \rangle = 0$ for all $X \in \mathfrak{X}(M)$. Then we define the almost complex structure J on M as

$$J(X) = \xi \times X,$$

where $\times$ is the cross product in $\mathbb{R}^7$ defined in Example 12.

Note that if $M = \mathbb{S}^6$ then $\xi = p$. Moreover, $\mathbb{S}^6$ is the only hypersurface of $\mathbb{R}^7$ that is nearly Kähler but not Kähler.

Let V be a three-dimensional vector subspace of $\mathbb{R}^7$ that is closed under the cross product and let $\Sigma \subset V$ be a surface. Then $M = \Sigma \times \mathbb{R}^4$ is semi-Kähler. Moreover, if Σ is a minimal surface then J is integrable. Thus, if Σ is flat then M is Kähler. If instead Σ is isometric to a sphere then M is quasi-Kähler but not almost Kähler nor nearly Kähler.

1.6 Nearly Kähler manifolds

First known as almost Tachibana spaces, the first notion of nearly Kähler manifold was introduced by Tachibana in [50]. Later on, Gray coined the term nearly Kähler as we know it today.

Definition 42. A **nearly Kähler manifold** is an almost Hermitian manifold (M, g, J) such that ∇J is skew symmetric, where ∇ is the Levi-Civita connection associated to g.

Although the nearly Kähler condition is point-wise, it has topological implications, for instance having even first Betti number [54].

Note that the Riemannian products of nearly Kähler and Kähler manifolds are nearly Kähler manifolds as well. Thus, the notion of strict nearly Kähler excludes the cases when you can extract a Kähler factor from a nearly Kähler manifold.

Definition 43. A nearly Kähler manifold M is said to be **strict** if $\nabla_X J \not\equiv 0$ for all $X \in \mathfrak{X}(M)$.

Gray showed that nearly Kähler manifolds split into strict nearly Kähler and Kähler manifolds.

Proposition 8 (Gray [26])**.** *Given a nearly Kähler manifold M there exists a strict nearly Kähler manifold M_1 and a Kähler manifold M_2 such that $M \cong M_1 \times M_2$.*

Next, we present some general properties of the covariant derivative of the almost complex structure.

Proposition 9. *Let (M, g, J) be a nearly Kähler manifold. Then the tensor $G = \nabla J$ satisfies the following:*

1. $G(X, JY) + JG(X, Y) = G(JX, Y) + JG(X, Y) = 0$,
2. $G(JX, JY) + G(X, Y) = 0$,
3. $g(G(X, Y), Z) + g(G(X, Z), Y) = 0$,
4. $g(G(X, Y), JZ) + g(G(X, Z), JY) = 0$.

By checking with a basis, we can easily see that two- and four-dimensional nearly Kähler manifolds are automatically Kähler. Furthermore, eight-dimensional nearly Kähler manifolds are either Kähler or the product of a strict nearly Kähler six-manifold and a Kähler surface.

Proposition 10 (Gray [26])**.** *There do not exist strict nearly Kähler manifolds of dimension 8.*

It is worth to note that there is a confusion in the literature regarding the definition of strict nearly Kähler manifold. In some sources strict nearly Kähler manifolds are defined as being non-Kähler. This ambiguity originates from the fact that this is true in six dimensions:

Proposition 11. *A nearly Kähler manifold of dimension 6 is strict nearly Kähler if and only if it is not Kähler.*

In principle this result is a direct corollary of Proposition 8, but we provide a proof here where we can see the obstruction of strictness more clearly.

Proof. Let M be a nearly Kähler manifold of dimension 6. It is immediate that if M is strict it is not Kähler.

Suppose now that M is not Kähler but there exists a smooth vector field Z on M such that $\nabla_Z J \equiv 0$. Since M is not Kähler, there exist $X, Y \in \mathfrak{X}(M)$ such that $(\nabla_X J)Y \neq 0$. Moreover, the nearly Kähler condition allows us to assume that $\{X, JX, Y, JY, Z, JZ\}$ is an orthonormal frame on M. We have

$$\begin{aligned}(\nabla_X J)Y = {} & g((\nabla_X J)Y, X)X + g((\nabla_X J)Y, JX)JX \\ & + g((\nabla_X J)Y, Y)Y + g((\nabla_X J)Y, JY)JY \\ & + g((\nabla_X J)Y, Z)Z + g((\nabla_X J)Y, JZ)JZ.\end{aligned}$$

From the properties in Proposition 9 it follows that

$$\begin{aligned}(\nabla_X J)Y &= g((\nabla_X J)Y, Z)Z + g((\nabla_X J)Y, JZ)JZ \\ &= -g((\nabla_X J)Z, Y)Z - g((\nabla_X J)Z, JY)JZ \\ &= g((\nabla_Z J)X, Y)Z + g((\nabla_Z J)X, JY)JZ \\ &= 0\end{aligned}$$

which is a contradiction. □

Six-dimensional strict nearly Kähler manifolds are always Einstein, that is, the Ricci tensor is a multiple of the Riemannian metric. These manifolds are interesting from a physics point of view since, for instance, they are the only ones to admit a Killing spinor [23, 28]. Moreover, nearly Kähler manifolds

are exactly those six-dimensional manifolds for which their seven-dimensional cone has holonomy G_2 [5]. Furthermore, strict nearly Kähler manifolds are interesting in multiplectic geometry as they are 2-plectic [37].

Nearly Kähler manifolds are also related to the so-called quaternionic Kähler manifolds. A quaternionic Kähler manifold is a Riemannian manifold of dimension $4n$ such that its holonomy is a subgroup of $(\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2$. This definition might seem arbitrary, but it is quite natural after learning Berger's holonomy theorem.

Theorem 24 (Berger [8])**.** *Let M be a simply connected irreducible and non-locally symmetric Riemannian manifold. Then the holonomy group of M is either $SO(n)$ or one of the following exceptional holonomies:*

1. $\mathrm{U}(n)$,
2. $\mathrm{SU}(n)$,
3. $\mathrm{Sp}(n)$,
4. $(\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2$,
5. $\mathrm{Spin}(7)$,
6. G_2.

Recall that a symmetric space is Riemannian manifold such that for every point p there exists a local isometry φ_p such that $\varphi_p(p) = p$ and $(d\varphi_p)_p = -\mathrm{Id}$.

Quaternionic Kähler manifolds are a generalization of hyperkähler manifolds with non-flat Ricci curvature. Hence, quaternionic Kähler manifolds admit a fiber bundle where the fibers are two-spheres, arising from the possible quaternionic structures we can find on them. These $\mathbb{S}^2$-bundles are known as twistor spaces, and they have a close link with nearly Kähler geometry.

Quaternionic Kähler manifolds are automatically Einstein, and when the Einstein constant is positive and they are geodesically complete, they are called **positive quaternionic Kähler**.

Theorem 25 (Nagy [42])**.** *Let M be a 10-dimensional complete nearly Kähler manifold. Then it is isometric to either a product of a strict nearly Kähler six-manifold with a Kähler surface or a twistor space over an eight-dimensional positive quaternionic Kähler manifold.*

There exist only 3 eight-dimensional positive quaternionic spaces, the symmetric spaces $\mathbb{H}P^2$, $\mathbb{G}r_2(\mathbb{C}^4)$ and $G_2/\mathrm{SO}(4)$. Their twistor spaces are listed in [34], thus all the 10-dimensional strict nearly Kähler manifolds are classified.

A quaternionic Kähler manifold which is also a symmetric space is called a Wolf space. Salamon and LeBrun conjectured that every positive quaternionic

Kähler manifold is a symmetric space, but so far it has only been proven for dimensions 4 and 8.

Theorem 26 (Nagy [41])**.** *Let M be a simply connected, complete strict nearly Kähler manifold. Then it is a Riemannian product whose factors are in one of the following classes:*

1. *six-dimensional nearly Kähler manifolds,*
2. *homogeneous nearly Kähler manifolds (satisfying certain conditions),*
3. *twistor spaces over positive quaternionic Kähler spaces.*

The conditions that homogeneous nearly Kähler manifolds satisfy in Theorem 26 are quite involved, and they are not relevant for this thesis. In dimension six, it was believed for many years that there were only four homogeneous examples, but the proof did not come until 2005.

Theorem 27 (Butruille [10])**.** *Let M be a homogeneous simply connected strict nearly Kähler manifold. Then it is isometric to either $\mathbb{S}^6$, the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, the nearly Kähler $\mathbb{C}P^3$ or the nearly Kähler $F(\mathbb{C}^3)$.*

Note that the hypothesis of being homogeneous is important. The existence of non-homogeneous nearly Kähler manifolds was conjectured long ago. Podestà and Spiro studied cohomogeneity-one actions on nearly Kähler manifolds in [46] and [47] and they constructed inhomogeneous but non-complete nearly Kähler manifolds. Finally in 2017, Foscolo and Haskins [22] constructed the first examples of complete non-homogeneous six-dimensional nearly Kähler manifolds: $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ carry nearly Kähler structures which are not homogenous but have cohomogeneity one. Moreover they conjectured that $\mathbb{C}P^3$ only carries the homogeneous nearly Kähler structure.

The nearly Kähler $\mathbb{S}^6$

For more information about $\mathbb{S}^6$ we refer the reader to [24].

The nearly Kähler structure of $\mathbb{S}^6$ was defined in Example 12. In Example 9 we saw that $\mathbb{S}^6 = \mathrm{O}(7)/\mathrm{O}(6)$. However, certain isometries in $\mathrm{O}(7)$ do not preserve the almost complex structure. Namely, elements $\varphi \in \mathrm{O}(7)$ do not necessarily satisfy $d\varphi \circ J = \pm J \circ d\varphi$. From the definition of the cross product in $\mathbb{R}^7$ we can see that the elements from $\mathrm{O}(7)$ that preserve J are those who preserve the octonion product. The set of automorphisms of $\mathbb{O}$ is the smallest exceptional Lie group and it is denoted by G_2. This group acts transitively on $\mathbb{S}^6$ with isotropy $\mathrm{SU}(3)$. Hence,

$$\mathbb{S}^6 = \frac{G_2}{\mathrm{SU}(3)}.$$

The nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

For more information about $\mathbb{S}^3 \times \mathbb{S}^3$ we refer the reader to [40].

Take $\mathbb{S}^3 \subset \mathbb{H} \cong \mathbb{R}^4$ the set of all unit-length quaternions. The tangent space at a point $p \in \mathbb{S}^3$ is given by $T_p\mathbb{S}^3 = \{p\alpha : \alpha \in \mathrm{Im}(\mathbb{H})\}$. We define a Riemannian metric on $\mathbb{S}^3 \times \mathbb{S}^3$ by

$$g\left((p\alpha, q\beta), (p\gamma, q\delta)\right) = \frac{4}{3}\left(\langle p\alpha, p\gamma\rangle + \langle q\beta, q\delta\rangle\right) - \frac{2}{3}\left(\langle p\alpha, p\delta\rangle + \langle q\beta, q\gamma\rangle\right).$$

We also define an almost complex structure J on $\mathbb{S}^3 \times \mathbb{S}^3$ compatible with g:

$$J(p\alpha, q\beta) = \frac{1}{\sqrt{3}}(p(2\beta - \alpha), q(\beta - 2\alpha)).$$

One can check that $\left(\mathbb{S}^3 \times \mathbb{S}^3, g, J\right)$ is a nearly Kähler manifold.

The group $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ acts transitively by isometries on $\mathbb{S}^3 \times \mathbb{S}^3$ as follows:

$$(a, b, c) \cdot (p, q) = (apc^{-1}, bqc^{-1}).$$

Hence $\mathbb{S}^3 \times \mathbb{S}^3$ is a homogeneous manifold described as

$$\mathbb{S}^3 \times \mathbb{S}^3 = \frac{\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3}{\Delta\mathbb{S}^3},$$

where $\Delta\mathbb{S}^3 = \{(a, a, a) : a \in \mathbb{S}^3\}$.

The nearly Kähler $\mathbb{C}P^3$

For more information about $\mathbb{C}P^3$ we refer the reader to [35].

We denote the Kähler metric and complex structure by g_o and J_o, respectively.

Now we define the so-called Hopf fibration. Let $\pi\colon \mathbb{S}^7 \to \mathbb{C}P^3$ be the submersion given by $\pi(p) = [p]$. The vertical distribution associated to π is the distribution on $\mathbb{S}^7$ given by $\mathcal{V} = \{v \in \mathfrak{X}(\mathbb{S}^7) : d\pi(v) = 0\}$. This gives rise to the horizontal distribution, that is $\mathcal{H} = \mathcal{V}^\perp$.

Thinking of $\mathbb{S}^7$ as the immersed sphere in $\mathbb{H} \times \mathbb{H} \cong \mathbb{R}^8$, we can define three distinguished vector fields:

$$Y_1(p, q) = (p\mathbf{i}, q\mathbf{i}), \quad Y_2(p, q) = (p\mathbf{j}, q\mathbf{j}), \quad Y_3(p, q) = (p\mathbf{k}, q\mathbf{k}),$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit quaternions. The vertical distribution $\mathcal{V}$ on $\mathbb{S}^7$ is spanned by Y_1. We split $\mathcal{H}$ as

$$\mathcal{H} = \tilde{\mathcal{D}}_1 \oplus \tilde{\mathcal{D}}_2,$$

where $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$ are two- and four-dimensional distributions respectively on $\mathbb{S}^7$ defined by $\tilde{\mathcal{D}}_1 = \mathrm{Span}\{Y_2, Y_3\}$ and $\tilde{\mathcal{D}}_2 = (\mathcal{V} \oplus \tilde{\mathcal{D}}_1)^\perp$.

Denoting $\mathcal{D}_i = d\pi(\tilde{\mathcal{D}}_i)$ we obtain

$$T\mathbb{C}P^3 = d\pi(\mathcal{H}) = \mathcal{D}_1 \oplus \mathcal{D}_2.$$

We define the nearly Kähler structure $(\mathbb{C}P^3, g, J)$ by

$$g = \begin{cases} g_o & \text{on } \mathcal{D}_2, \\ 2g_o & \text{on } \mathcal{D}_4, \end{cases} \qquad \text{and} \qquad J = \begin{cases} -J_o & \text{on } \mathcal{D}_2, \\ J_o & \text{on } \mathcal{D}_4. \end{cases}$$

The Lie group $\mathrm{Sp}(2)$ acts on $\mathbb{S}^7 \subset \mathbb{H}^2$. This action descends to the nearly Kähler $\mathbb{C}P^3$, where it acts transitively with isotropy group $\mathrm{SU}(2) \times \mathrm{U}(1)$. Therefore

$$\mathbb{C}P^3 = \frac{\mathrm{Sp}(2)}{\mathrm{SU}(2) \times \mathrm{U}(1)}.$$

The nearly Kähler $F(\mathbb{C}^3)$

For more information about the flag manifold we refer the reader to [18].

The manifold of full flags in $\mathbb{C}^3$ is defined as the set of chains (V_1, V_2) of complex vector subspaces $V_1 \subset V_2 \subset \mathbb{C}^3$, where $\dim_{\mathbb{C}}(V_1) = 1$ and $\dim_{\mathbb{C}}(V_2) = 2$.

It is a well known fact that complex flag manifolds admit a Kähler structure (see for instance [29]), which we denote by $(F(\mathbb{C}^3), g_o, J_o)$. There exists a submersion onto $\mathbb{C}P^2$ given by

$$\begin{array}{rccc} \pi\colon & F(\mathbb{C}^3) & \longrightarrow & \mathbb{C}P^2 \\ & (\mathrm{Span}\{x\}, V_2) & \mapsto & [x]. \end{array}$$

Defining $\mathcal{V} = \{v \in \mathfrak{X}(F(\mathbb{C}^3)) : d\pi(v) = 0\}$ and $\mathcal{H} = \mathcal{V}^\perp$ with respect to g_o, we obtain a nearly Kähler structure $(F(\mathbb{C}^3), g, J)$ given by

$$g = \begin{cases} \frac{1}{2}g_o & \text{on } \mathcal{V}, \\ g_o & \text{on } \mathcal{H}, \end{cases} \qquad \text{and} \qquad J = \begin{cases} -J_o & \text{on } \mathcal{V}, \\ J_o & \text{on } \mathcal{H}. \end{cases}$$

The action of $\mathrm{SU}(3)$ on $\mathbb{C}^3$ extends to a transitive action by isometries on $F(\mathbb{C}^3)$ with isotropy group $\mathrm{U}(1) \times \mathrm{U}(1)$. Therefore,

$$F(\mathbb{C}^3) = \frac{\mathrm{SU}(3)}{\mathrm{U}(1) \times \mathrm{U}(1)}.$$

Pseudo-Riemannian nearly Kähler manifolds and T-duality

Next we define the notion of T-duality, which allows us to obtain analogues of homogeneous spaces, preserving important structures.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{m}$ be an $\mathrm{ad}(\mathfrak{h})$-invariant complement of $\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{ad}(X)(Y) = [X, Y] \in \mathfrak{m}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$. Let T be a Lie automorphism such that $T^2 = \mathrm{Id}$ and $T(\mathfrak{h}) = \mathfrak{h}$, $T(\mathfrak{m}) = \mathfrak{m}$. Denote by $\mathfrak{g}_+$ and $\mathfrak{g}_-$ the eigenspaces of T associated to 1 and -1, respectively. Then we have $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, sum of vector spaces. Moreover, $\mathfrak{g}_+$ is a Lie subalgebra of $\mathfrak{g}$, $[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-$, and $[\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+$. The automorphism T preserving $\mathfrak{h}$ and $\mathfrak{m}$ is equivalent to $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{g}_+) \oplus (\mathfrak{h} \cap \mathfrak{g}_-)$ and $\mathfrak{m} = (\mathfrak{m} \cap \mathfrak{g}_+) \oplus (\mathfrak{m} \cap \mathfrak{g}_-)$.

Define the Lie algebra $\mathfrak{g}' = \mathfrak{g}_+ \oplus i\mathfrak{g}_-$. That is, the vector space $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ with Lie bracket given by

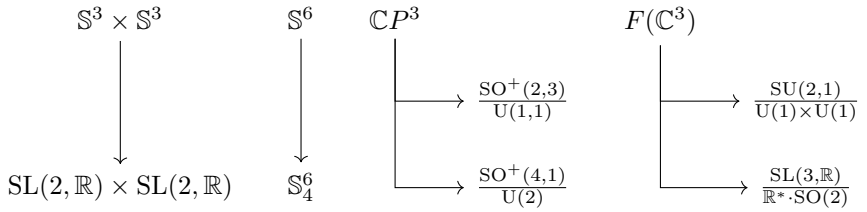
$$[(X, Y), (Z, W)] = ([X, Z] - [Y, W], [Y, Z] + [X, W]). \tag{1.3}$$

Similarly, define $\mathfrak{h}'$ and $\mathfrak{m}'$ by

$$\mathfrak{h}' = (\mathfrak{h} \cap \mathfrak{g}_+) \oplus i(\mathfrak{h} \cap \mathfrak{g}_-), \quad \mathfrak{m}' = (\mathfrak{m} \cap \mathfrak{g}_+) \oplus i(\mathfrak{m} \cap \mathfrak{g}_-).$$

The Lie algebra $\mathfrak{g}'$ is a real form of $\mathfrak{g}_{\mathbb{C}}$, i.e. $\mathfrak{g}' \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$.

Let G, H, G' and H' be Lie groups associated to the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{g}'$ and $\mathfrak{h}'$. We say that the homogeneous spaces G/H and G'/H' are T-duals. This way, Kath [32] and Schäfer [48] constructed T-duals of each one of the six-dimensional homogeneous nearly Kähler spaces, which are pseudo-Riemannian nearly Kähler manifolds:



Here, $\mathbb{S}^6_4$ is the pseudo-sphere inside $\mathbb{R}^7_4$, with homogeneous description $G_{2(2)}/\mathrm{SU}(2,1)$, and $\mathbb{R}^* \cdot \mathrm{SO}(2)$ denotes the group $(\mathbb{R}^* \times \mathrm{SO}(2))/\mathbb{Z}_2$. The nearly Kähler manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ has homogeneous description $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\Delta\mathrm{SL}(2,\mathbb{R})$. We will focus on this last space in the rest of the chapters.

Chapter 2

The nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

In this chapter we present the nearly Kähler structure of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Moreover, we provide an expression for its isometry group (see Theorem 28). The contents of this chapter can also be found in [2], [3] and [25].

2.1 The group $\mathrm{SL}(2,\mathbb{R})$

Let $\mathrm{SL}(2,\mathbb{R})$ be the Lie group defined as $\mathrm{SL}(2,\mathbb{R}) = \{a \in M(2,\mathbb{R}) : \det(a) = 1\}$. The associated Lie algebra is given by

$$\mathfrak{sl}(2,\mathbb{R}) = \{X \in M(2,\mathbb{R}) : \mathrm{Tr}(X) = 0\}$$

This is a three-dimensional non-compact simple Lie algebra which is spanned by the so-called split-quaternions

$$\boldsymbol{i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{k} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.1}$$

with Lie brackets

$$[\boldsymbol{i},\boldsymbol{j}] = 2\boldsymbol{k}, \quad [\boldsymbol{i},\boldsymbol{k}] = 2\boldsymbol{j}, \quad [\boldsymbol{j},\boldsymbol{k}] = -2\boldsymbol{i}.$$

Moreover, they satisfy $\boldsymbol{i}^2 = \boldsymbol{j}^2 = -\boldsymbol{k}^2 = -\boldsymbol{i}\boldsymbol{j}\boldsymbol{k} = \mathrm{Id}_2$.

Given a matrix a in $M(2,\mathbb{R})$, we denote by $\mathrm{adj}(a)$ the adjugate matrix, i.e.

$$\mathrm{adj}\begin{pmatrix} x & y \\ w & z \end{pmatrix} = \begin{pmatrix} z & -y \\ -w & x \end{pmatrix}.$$

Define in $M(2,\mathbb{R})$ the inner product given by

$$\langle x, y\rangle = -\frac{1}{2}\operatorname{tr}(\mathrm{adj}(x)y). \tag{2.2}$$

Thus, we see that $\mathrm{SL}(2,\mathbb{R}) = \{a \in M(2,\mathbb{R}) : \langle a, a\rangle = -1\}$. Moreover, we deduce that

$$T_a\mathrm{SL}(2,\mathbb{R}) = a^{\perp} = \{a\alpha : \alpha \in \mathfrak{sl}(2,\mathbb{R})\}.$$

The inner product $\langle , \rangle$ defines a pseudo-Riemannian metric on $M(2,\mathbb{R})$ of signature $(2,2)$. The submanifold $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow M(2,\mathbb{R})$ inherits a pseudo-Riemannian metric of signature $(1,2)$. More explicitly, the frame $\{X_1, X_2, X_3\}$ given by

$$X_1(a) = a\boldsymbol{i}, \quad X_2(a) = a\boldsymbol{j}, \quad X_3(a) = a\boldsymbol{k}, \tag{2.3}$$

satisfies

$$\langle X_1, X_1\rangle = \langle X_2, X_2\rangle = -\langle X_3, X_3\rangle = 1$$

and $\langle X_i, X_j\rangle = 0$ for $i \neq j$.

Remark 10. Since $M(2,\mathbb{R})$ with this metric is isometric to $\mathbb{R}^4_2$, the manifold $\mathrm{SL}(2,\mathbb{R})$ is isometric to the anti-de Sitter space $H^3_1(-1)$. Namely, the map $\mathfrak{F} : H^3_1(-1) \to \mathrm{SL}(2,\mathbb{R})$ given by

$$\mathfrak{F}(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_2 - x_0 & x_1 - x_3 \\ x1 + x_3 & x_0 + x_2 \end{pmatrix}, \tag{2.4}$$

is an isometry.

The adjugate matrix has the same role as the conjugate of a quaternion for the structure of $\mathbb{S}^3$, as we can see in the following result.

Proposition 12. *Let $a, b \in \mathrm{SL}(2,\mathbb{R})$ and $\alpha, \beta \in \mathfrak{sl}(2,\mathbb{R})$. Then,*

1. $\mathrm{adj}(a) = a^{-1}$,
2. $\mathrm{adj}(ab) = \mathrm{adj}(b)\,\mathrm{adj}(a)$,
3. $\mathrm{adj}(\alpha) = -\alpha$,
4. $\langle a, a\rangle = -\det(a)$,
5. $\langle a\alpha, a\beta\rangle = \langle \alpha a, \beta a\rangle = \langle \alpha, \beta\rangle$,

Remark 11. Although we always write the inverse matrix for elements of $\mathrm{SL}(2,\mathbb{R})$, in computations either by hand or using software it is always recommendable to use the adjugate matrix, as it reduces considerably the computational cost.

There is a natural bilinear operation $\times$ in $\mathfrak{sl}(2,\mathbb{R})$, defined by

$$\alpha \times \beta = \frac{1}{2}(\alpha\beta - \beta\alpha).$$

This operation is a Lorentzian cross-product, that is, a bilinear operation that satisfies

(1) $\alpha \times \beta = -\beta \times \alpha$,
(2) $\langle \alpha \times \beta, \gamma\rangle = \langle \beta \times \gamma, \alpha\rangle = \langle \gamma \times \alpha, \beta\rangle$,
(3) $\langle \alpha \times \beta, \alpha\rangle = \langle \alpha \times \beta, \beta\rangle = 0$,
(4) $\langle \alpha \times \beta, \alpha \times \beta\rangle = -\langle \alpha, \alpha\rangle\langle \beta, \beta\rangle + \langle \alpha, \beta\rangle^2$.

Note that the third property actually follows from the first and second one. This cross product also satisfies

$$\alpha\beta = \alpha \times \beta + \langle \alpha, \beta\rangle \,\mathrm{Id}_2\,. \tag{2.5}$$

The metric $\langle , \rangle$ on $\mathrm{SL}(2,\mathbb{R})$ is a multiple of the Killing form, the quadratic form for Lie groups given in 1.2, hence is a bi-invariant metric. As a consequence, the Riemannian exponential map coincides with the Lie group exponential map.

Proposition 13. *The geodesics of* $\mathrm{SL}(2,\mathbb{R})$ *are given by* $t \mapsto pe^{t\alpha}$*, where* $\alpha \in \mathfrak{sl}(2,\mathbb{R})$ *and* $p \in \mathrm{SL}(2,\mathbb{R})$.

2.2 The nearly Kähler structure of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

In the previous chapter we mentioned that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is constructed by means of T-duality. In this section we will make that construction more explicit.

Recall that the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ is the homogeneous space $G/H = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta\mathrm{SU}(2)}$. Then, $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as a sum of Lie algebras, the Lie subalgebra $\mathfrak{h} = \Delta\mathfrak{su}(2) = \{(X, X, X) : X \in \mathfrak{su}(2)\}$ and the vector space $\mathfrak{m} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \{0\}$. Take the involutive Lie automorphism T of $\mathfrak{g}$ given by

$$T(X, Y, Z) = (-X^t, -Y^t, -Z^t).$$

The eigenspaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are

$$\mathfrak{g}_+ = \mathrm{Span}\{(\mathbf{k}, 0, 0), (0, \mathbf{k}, 0), (0, 0, \mathbf{k})\},$$

$$\mathfrak{g}_- = \mathrm{Span}\{(\mathbf{i}, 0, 0), (\mathbf{j}, 0, 0), (0, \mathbf{i}, 0), (0, \mathbf{j}, 0), (0, 0, \mathbf{i}), (0, 0, \mathbf{j})\},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the matrices in $\mathfrak{su}(2)$ given by

$$\mathbf{i} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.6}$$

It is easy to see that T preserves $\mathfrak{h}$ and $\mathfrak{m}$. The Lie algebra $\mathfrak{g}' = \mathfrak{g}_+ \oplus i\mathfrak{g}_-$ with the Lie bracket in (1.3) is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, as a sum of Lie algebras. Therefore $\mathbb{S}^3 \times \mathbb{S}^3$ and $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ are T-duals.

The tangent space of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ at a point (p,q) is

$$T_{(p,q)}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})) = \{(p\alpha, q\beta) : \alpha, \beta \in \mathfrak{sl}(2,\mathbb{R})\}.$$

Let $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be the pseudo-Riemannian product with the product metric that arises from the one in (2.2). Now take $\pi\colon \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ the submersion given by

$$\pi(a,b,c) = (ac^{-1}, bc^{-1}). \tag{2.7}$$

Let g be the metric on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ such that π is a pseudo-Riemannian submersion. More explicitly,

$$g((p\alpha, q\beta), (p\gamma, q\delta)) = \frac{2}{3}\langle (p\alpha, q\delta), (p\gamma, q\beta)\rangle - \frac{1}{3}\langle (p\beta, q\alpha), (p\gamma, q\delta)\rangle, \tag{2.8}$$

where $\langle , \rangle$ is the product metric. It follows that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ acts transitively by isometries on $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), g)$, therefore

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) = \frac{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})}{\Delta \mathrm{SL}(2,\mathbb{R})}.$$

We may also define the metric g from a Hermitian point of view. Take $\theta \in \mathrm{Aut}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ the automorphism of order three given by

$$\theta(a,b,c) = (b,c,a).$$

Note that

$$\begin{aligned} 0 &= \theta_*^3 - \mathrm{Id} \\ &= (\theta_* - \mathrm{Id})(\theta_*^2 + \theta_* + \mathrm{Id}) \end{aligned}$$

and that $(\theta_* - \mathrm{Id})$ only vanishes for vertical vector fields of the submersion π given in (2.7). Hence, if we evaluate (2.2) with a horizontal vector field we obtain that on the set of horizontal vector fields $\mathcal{H}$,

$$\begin{aligned} 0 &= \theta_*^2 + \theta_* + \mathrm{Id} \\ &= (\theta_* + \frac{1}{2}\,\mathrm{Id})^2 + \frac{3}{4}\,\mathrm{Id}\,. \end{aligned} \tag{2.9}$$

Then, the tensor $\frac{2}{\sqrt{3}}\theta_* + \frac{1}{\sqrt{3}}\,\mathrm{Id}$ is an almost complex structure on $\mathcal{H}$, which we can identify with $T(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$.

Explicitly, we take the almost complex structure J on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$J(p\alpha, q\beta) = \frac{1}{\sqrt{3}}(p(\alpha - 2\beta), q(2\alpha - \beta)). \tag{2.10}$$

The product metric $\langle , \rangle$ on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is not compatible with J, i.e. $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), \langle , \rangle)$ is not an almost Hermitian manifold. However, we may construct a new metric from $\langle , \rangle$ that is compatible with J:

$$g'(X,Y) = \frac{1}{4}\langle X, Y\rangle + \frac{1}{4}\langle JX, JY\rangle.$$

Proposition 14. *The manifold* $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), g', J)$ *is an almost Hermitian manifold. Moreover,* g *and* g' *coincide.*

We define the $\{U_1, U_2, U_3, V_1, V_2, V_3\}$ as the frame on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ by

$$U_i(p,q) = (X_i(p), 0), \quad V_i(p,q) = (0, X_i(q)),$$

where the vector fields X_i are given in (2.3). The components of the metric in this frame are

$$g(U_i, U_j) = g(V_i, V_j) = \begin{cases} \frac{2}{3}\delta_{ij} & \text{for } i = 1,2, \\ -\frac{2}{3}\delta_{ij} & \text{for } i = 3, \end{cases}$$

and

$$g(U_i, V_j) = \begin{cases} -\frac{1}{3}\delta_{ij} & \text{for } i = 1,2, \\ \frac{1}{3}\delta_{ij} & \text{for } i = 3. \end{cases}$$

The Lie brackets of vector fields of the frame are

$$[U_i, U_j] = -2\varepsilon_{ijk}\eta_k U_k, \quad [U_i, V_j] = -2\varepsilon_{ijk}\eta_k V_k, \quad [V_i, V_k] = 0,$$

where $\eta_i = 1$ if $i = 1,2$, $\eta_3 = -1$ and ε_{ijk} is the Levi-Civita symbol:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of (1,2,3),} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of (1,2,3),} \\ 0 & \text{all other cases.} \end{cases}$$

Lemma 3. *The Levi-Civita connection* $\tilde{\nabla}$ *of* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *associated to* g *is given by*

$$\tilde{\nabla}_{U_i} U_j = -\varepsilon_{ijk}\eta_k U_k,$$

$$\tilde{\nabla}_{V_i} V_j = -\varepsilon_{ijk}\eta_k V_k,$$

$$\tilde{\nabla}_{U_i} V_j = \varepsilon_{ijk}\frac{\eta_k}{3}(U_k - V_k),$$

$$\tilde{\nabla}_{V_i} U_j = \varepsilon_{ijk}\frac{\eta_k}{3}(V_k - U_k).$$

Moreover, the covariant derivative of J is given by

$$(\tilde{\nabla}_{U_i} J)U_j = \frac{2}{3\sqrt{3}}\varepsilon_{ijk}\eta_k(U_k + 2V_k),$$

$$(\tilde{\nabla}_{V_i} J)V_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}\eta_k(2U_k + V_k),$$

$$(\tilde{\nabla}_{U_i} J)V_j = (\tilde{\nabla}_{V_i} J)U_j = \frac{2}{3\sqrt{3}}\varepsilon_{ijk}\eta_k(U_k - V_k).$$

From the previous lemma we can deduce that $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), g, J)$ is a nearly Kähler manifold. As such, it satisfies all properties given in Proposition 9. In what follows, we denote by G the covariant derivative of J.

Lemma 4 (Ghandour, Vrancken [25])**.** *The tensor* G *satisfies*

1. $(\nabla_X G)(Y,Z) = -\frac{2}{3}\Big(g(X,Z)JY - g(X,Y)JZ - g(JY,Z)X\Big)$,
2. $g(G(X,Y),G(Z,W)) = -\frac{2}{3}\Big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)$
$\qquad + g(JX,Z)g(JW,Y) - g(JX,W)g(JZ,Y)\Big)$,
3. $G(X,G(Z,W)) = \frac{2}{3}\Big(g(X,Z)W - g(X,W)Z + g(JX,Z)JW$
$\qquad - g(JX,W)JZ\Big)$.

An explicit expression for the tensor G is given in [25]:

$$G((a\alpha,b\beta),(a\gamma,b\delta)) = \frac{2}{3\sqrt{3}}\Big(a(-\alpha\times\gamma - \alpha\times\delta + \gamma\times\beta + 2\beta\times\delta), \\ b(-2\alpha\times\gamma + \alpha\times\delta - \gamma\times\beta + \beta\times\delta)\Big), \tag{2.11}$$

where $a, b \in \mathrm{SL}(2,\mathbb{R})$ and $\alpha, \beta, \gamma, \delta \in \mathfrak{sl}(2,\mathbb{R})$. We introduce the tensor P given by

$$P(p\alpha, q\beta) = (p\beta, q\alpha), \tag{2.12}$$

where $(p\alpha, q\beta) \in T_{(p,q)}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$. This tensor plays an important role in the structure of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, as we will see later.

Lemma 5. *The tensor P is an almost product structure. Namely, P satisfies*

$$P^2 = \mathrm{Id}, \quad g(PX, PY) = g(X, Y).$$

Moreover, it satisfies the following conditions:

1. $g(PX, Y) = g(X, PY)$,
2. $PJ = -JP$,
3. $G(PX, PY) + PG(X, Y) = 0$,
4. $(\tilde{\nabla}_X P)Y = \frac{1}{2}\Big(JG(X, PY) + JPG(X, Y)\Big)$.

Proof. That P is a almost product structure as well as 1 and 2 follow from a straightforward computation. Property 3 follows From the expression of G in (2.11). By checking with a basis, we can prove 4. □

We denote by $\tilde{R}$ the curvature tensor of the nearly Kähler connection $\tilde{\nabla}$. It is explicitly given by the following formula

$$\begin{aligned}
\tilde{R}(X,Y)Z = &-\tfrac{5}{6}\Big(g(Y,Z)X - g(X,Z)Y\Big) \\
&- \tfrac{1}{6}\Big(g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ\Big) \\
&- \tfrac{2}{3}\Big(g(PY,Z)PX - g(PX,Z)PY \\
&\qquad + g(JPY,Z)JPX - g(JPX,Z)JPY\Big).
\end{aligned} \tag{2.13}$$

The following proposition is stated in [40] for $\mathbb{S}^3 \times \mathbb{S}^3$, but the proof also works for $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Proposition 15. *Let P' be an almost product structure that satisfies 1, 2 and 3 of Lemma 5 and Equation* (2.13)*, then*

$$P' = \cos\tau P + \sin\tau JP$$

for $\tau \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$. Conversely, such tensors are almost product structures and satisfy 1, 2 and 3 of Lemma 5 and Equation (2.13)*. Moreover, Property 4 of Lemma 5 is trivially satisfied by P'.*

2.2.1 The relation between the product structure and the nearly Kähler structure

Any (pseudo-)Riemannian product carries a canonical product structure:

$$Q(X,Y) = (-X,Y). \tag{2.14}$$

Moreover, this product structure is parallel with respect to the Levi-Civita connection associated to the product metric.

In our setting, there is an explicit relation between the nearly Kähler metric g and the product metric $\langle , \rangle$:

$$\langle X,Y\rangle = 2g(X,Y) + g(X,PY). \tag{2.15}$$

The almost product structures Q and P on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ are related by

$$QX = -\frac{1}{\sqrt{3}}(2PJX - JX). \tag{2.16}$$

The Levi-Civita connection ∇^E associated to the product metric $\langle , \rangle$ on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is related to $\tilde{\nabla}$ by

$$\nabla^E_X Y = \tilde{\nabla}_X Y + \frac{1}{2}\Big(JG(X,PY) + JG(Y,PX)\Big). \tag{2.17}$$

The product manifold $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), \langle , \rangle)$ is isometrically immersed in $M(2,\mathbb{R}) \times M(2,\mathbb{R}) \cong \mathbb{R}^8_4$, thus we can use the Gauss formula to find a relation between ∇^E and the Euclidean connection D of $\mathbb{R}^8_4$. That is, at any point $(a,b) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we have

$$D_X Y = \nabla^E_X Y + h^E(X,Y),$$

where h^E is the second fundamental form. Since the normal space at (a,b) of the immersion is spanned by (a,b) and $(-a,b)$ we have that

$$h^E(X,Y) = \frac{\langle D_X Y, (a,b)\rangle}{\langle (a,b),(a,b)\rangle}(a,b) + \frac{\langle D_X Y, (-a,b)\rangle}{\langle (-a,b),(-a,b)\rangle}(-a,b).$$

Finally, using that $D_X(a,b) = X$, $D_X(-a,b) = QX$ and the compatibility of D with $\langle , \rangle$, we obtain that

$$D_X Y = \nabla^E_X Y + \frac{1}{2}\langle X,Y\rangle(a,b) + \frac{1}{2}\langle QX,Y\rangle(-a,b). \tag{2.18}$$

2.2.2 The isometry group

The connected component of the identity of the isometry group of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is

$$\mathrm{Iso}_o(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})) = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}),$$

where an element $\phi_{(a,b,c)}$ acts on a point (p,q) by $\phi_{(a,b,c)}(p,q) = (apc^{-1}, bqc^{-1})$.

The isometries $\phi_{(a,b,c)}$ preserve P and J, in the sense that $d\phi_{(a,b,c)} \circ J = J \circ d\phi_{(a,b,c)}$ and $d\phi_{(a,b,c)} \circ P = P \circ d\phi_{(a,b,c)}$. These isometries are not the only ones that satisfy these properties. Given three matrices a, b and c with determinant -1, the map $(p,q) \mapsto (apc^{-1}, bqc^{-1})$ is also an isometry that preserves J and P.

Denote by $\mathrm{SL}^{\pm}(2,\mathbb{R})$ the group of all matrices in $M(2,\mathbb{R})$ with determinant ± 1. We can write any matrix of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ as $\boldsymbol{i}^k a$, where $\boldsymbol{i}$ is the matrix given in (2.1), $k \in \{0,1\}$ and $a \in \mathrm{SL}(2,\mathbb{R})$. Thus, we have

$$(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{Z}_2 \subset \mathrm{Iso}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})).$$

Permutations of elements of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ also give rise to isometries of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$:

$$\begin{aligned}
&\Psi_{0,0}(p,q) = (p,q), && \Psi_{1,0}(p,q) = (q,p),\\
&\Psi_{0,2\pi/3}(p,q) = (pq^{-1}, q^{-1}), && \Psi_{1,2\pi/3}(p,q) = (q^{-1}, pq^{-1}),\\
&\Psi_{0,4\pi/3}(p,q) = (qp^{-1}, p^{-1}), && \Psi_{1,4\pi/3}(p,q) = (p^{-1}, qp^{-1}).
\end{aligned} \tag{2.19}$$

Except for $\Psi_{0,0}$, these isometries are not included in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Moreover, each one of these is in a different connected component of $\mathrm{Iso}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ and satisfies

$$J \circ d\Psi_{\kappa,\tau} = (-1)^{\kappa} d\Psi_{\kappa,\tau} \circ J, \qquad P \circ d\Psi_{\kappa,\tau} = d\Psi_{\kappa,\tau} \circ (\cos\tau P + \sin\tau JP).$$

Later on, we prove that these are *all* the isometries of the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

A key result in the classification of Riemannian homogeneous nearly Kähler manifolds by Butruille [10] is the existence of a unique nearly Kähler structure on $\mathbb{S}^3 \times \mathbb{S}^3$. Consequently, the almost complex structure on $\mathbb{S}^3 \times \mathbb{S}^3$ is unique up to sign.

However, in [48] it is shown that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ has a unique *left-invariant* nearly Kähler structure, which does not necessarily imply that J is unique up

to sign. Therefore, by isometry group of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we mean the set of all diffeomorphisms preserving the almost Hermitian structure. That is, those isometries $\mathcal{F}$ of $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), g)$ that preserve J, i.e. $\mathcal{F}_* J = \pm J \mathcal{F}_*$.

The following lemma is a well known result.

Lemma 6. *Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ be bases of $\mathfrak{sl}(2,\mathbb{R})$. If $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle$ for all i, $j \in \{1,2,3\}$, then there exists a matrix c in $\mathrm{SL}^{\pm}(2,\mathbb{R})$ such that $c\alpha_i c^{-1} = \beta_i$. In other words, $\mathrm{SL}^{\pm}(2,\mathbb{R})/\mathbb{Z}_2$ is isomorphic to $\mathrm{SO}(2,1)$.*

With this lemma and Proposition 15 we prove the following statement.

Theorem 28. *The isometry group of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is the semi-direct product $\big(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\big) \rtimes \big(\mathbb{Z}_2 \times S_3\big)$, where S_3 is the symmetric group of order 6 generated by $\{\Psi_{1,0}, \Psi_{1,4\pi/3}\}$.*

Proof. We already know that the given group is included in $\mathrm{Iso}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$. Here we show the opposite inclusion.

Let $\mathcal{F}$ be an isometry of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. That means, there exists a $\kappa_0 \in \{0,1\}$ satisfying

$$\mathcal{F}_* J = (-1)^{\kappa_0} J \mathcal{F}_*.$$

As $\mathcal{F}_* P (\mathcal{F}^{-1})_*$ is an almost product structure satisfying 1, 2 and 3 in Lemma 5 and Equation (2.13), Proposition 15 implies that

$$\mathcal{F}_* P (\mathcal{F}^{-1})_* = \cos\tau_0 P + \sin\tau_0 JP,$$

for some $\tau_0 \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$. By taking the composition $\mathcal{F} \circ \Psi_{\kappa_0,(-1)^{\kappa_0}\tau_0}$ we may assume that $\mathcal{F}$ preserves P and J. Let $(p_o, q_o) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ such that $\mathcal{F}(\mathrm{Id}_2, \mathrm{Id}_2) = (p_o, q_o)$. Then by taking the composition $\mathcal{F} \circ \phi_{(p_o^{-1}, q_o^{-1}, \mathrm{Id}_2)}$ we may also assume that $\mathcal{F}(\mathrm{Id}_2, \mathrm{Id}_2) = (\mathrm{Id}_2, \mathrm{Id}_2)$.

Let $\alpha \in \mathfrak{sl}(2,\mathbb{R})$. Then we write $\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(\alpha,0) = (\beta,\gamma)$. Since $\mathcal{F}$ preserves P we know that $\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(0,\alpha) = (\gamma,\beta)$. We compute

$$\begin{aligned}\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}J(\alpha,0) &= \frac{1}{\sqrt{3}}\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(\alpha,2\alpha)\\ &= \frac{1}{\sqrt{3}}\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(\alpha,0) + \frac{1}{\sqrt{3}}\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(0,2\alpha)\\ &= \frac{1}{\sqrt{3}}(\beta,\gamma) + \frac{2}{\sqrt{3}}(\gamma,\beta)\\ &= \frac{1}{\sqrt{3}}(\beta+2\gamma, 2\beta+\gamma).\end{aligned}$$

On the other hand, as $\mathcal{F}$ preserves J, we have that $\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}J(\alpha,0)$ equals

$$\begin{aligned}J\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(\alpha,0) &= J(\beta,\gamma)\\ &= \frac{1}{\sqrt{3}}(\beta-2\gamma, 2\beta-\gamma).\end{aligned}$$

Therefore, we obtain that $\gamma = 0$. Moreover, since $\mathcal{F}$ is an isometry, we deduce that $\mathcal{F}_*$ maps a set $\{(\alpha_1,0),(\alpha_2,0),(\alpha_3,0)\}$ to a set $\{(\beta_1,0),(\beta_2,0),(\beta_3,0)\}$ such that $\langle\alpha_i,\alpha_j\rangle = \langle\beta_i,\beta_j\rangle$.

Now, using Lemma 6, we may compose $\mathcal{F}$ with an isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \rtimes \mathbb{Z}_2$ to assume that $\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(\alpha,0) = (\alpha,0)$ for all $\alpha \in \mathfrak{sl}(2,\mathbb{R})$. Since $\mathcal{F}$ preserves P, we have

$$\begin{aligned}\mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(\alpha,\beta) &= \mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(\alpha,0) + \mathcal{F}_{*(\mathrm{Id}_2,\mathrm{Id}_2)}(0,\beta)\\ &= (\alpha,0) + P(\beta,0)\\ &= (\alpha,\beta).\end{aligned}$$

Since by Proposition 3 isometries are determined by a point and the differential at that point, the argument above shows that $\mathcal{F}^{-1}$ is in $\big(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\big) \rtimes \big(\mathbb{Z}_2 \times S_3\big)$, hence $\mathcal{F}$ also belongs to this group. □

An element (a,b,c,Ψ,k) acts on a point (p,q) by

$$(a,b,c,k,\Psi)\cdot(p,q) = \Psi \circ \phi_{\boldsymbol{i}^k(a,b,c)}(p,q). \tag{2.20}$$

The isometry group is a semidirect product since for the action in (2.20) to be a Lie group action, the group law has to be

$$\begin{aligned}(a_1, b_1, c_1, k_1, \Psi_1) \circ (a_2, b_2, c_2, k_2, \Psi_2) \\ = (\sigma_2(a_1, b_1, c_1) \cdot (a_2, b_2, c_2), \Psi_1 \circ \Psi_2, k_1 + k_2)\end{aligned}$$

where σ_2 is the permutation of $(1, 2, 3)$ in S_3 associated to Ψ_2.

Note that in the literature there is no similar expression of the isometry group for the Riemannian analogue. The same proof can be done for $\mathbb{S}^3 \times \mathbb{S}^3$, obtaining

$$\mathrm{Iso}(\mathbb{S}^3 \times \mathbb{S}^3) = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes S_3.$$

Chapter 3

Lagrangian submanifolds of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

This chapter is based on a collaboration with Joeri Van der Veken [3], the preprint by the author [2], and an unpublished work by the author.

Introduction

Given an almost Hermitian manifold (M, g, J) there are two distinguished types of submanifolds: Those for which the almost complex structure J preserves the tangent space and those for which the almost complex structure maps the tangent space into the normal space. The former kind are called **almost complex** submanifolds, and the latter are known as **totally real** submanifolds. If the dimension of a totally real submanifold is maximal, we say that it is a Lagrangian submanifold. We can see that the dimension of a **Lagrangian submanifold** is always half of the dimension of the ambient space.

In symplectic geometry, a Lagrangian submanifold of a symplectic manifold (M, ω) is a submanifold such that ω vanishes everywhere and its dimension is equal to $\frac{1}{2}\dim(M)$. In Kähler geometry ω is the Kähler form $g(J\cdot, \cdot)$, making both definitions of Lagrangian submanifold coincide.

The submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ have been widely studied by many authors. An important notion of angle functions was introduced in [20] and [57]. Namely, the almost product structure P induces three functions θ_1,

θ_2 and θ_3, which encode substantial extrinsic information. In [57] the authors also classified all totally geodesic Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$:

Theorem 29 (Dioos, Hu, Vrancken, Wang and Zhang)**.** *A totally geodesic Lagrangian submanifold of $\mathbb{S}^3 \times \mathbb{S}^3$ is congruent to an open subset of one of the following examples:*

1. $f_1 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u)$,
2. $f_2 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, -\mathbf{i}u\mathbf{i})$,

where $\mathbf{i}$ *is the unit quaternion given in* (2.6).

Totally geodesic Lagrangian submanifolds have constant angle functions and at least two of them are equal modulo π. In [57] the authors actually gave six different examples, but using the isometries of $\mathbb{S}^3 \times \mathbb{S}^3$ described in [40] we can reduce the list to just two.

In [6] the authors classified all Lagrangian submanifolds with constant angle functions. By doing so, they also classified all extrinsically homogeneous Lagrangian submanifolds:

Theorem 30 (Bektas, Moruz, Van der Veken and Vrancken)**.** *An extrinsically homogeneous Lagrangian submanifold $H \cdot (1,1)$ of $\mathbb{S}^3 \times \mathbb{S}^3$ is congruent to an open subset of one of the following examples,*

1. $f_1 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u)$, *with* $H = \mathrm{SU}(2)$,
2. $f_2 \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, -\mathbf{i}u\mathbf{i})$, *with* $H = \mathrm{SU}(2)$,
3. $f_3 \colon \mathbb{R}P^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (-\mathbf{j}u\mathbf{j}u^{-1}, -\mathbf{i}u\mathbf{i}u^{-1})$, *with* $H = \mathrm{SU}(2)$,
4. $f_4 \colon \mathbb{T}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (u, v, w) \mapsto (e^{u\mathbf{i}}e^{-w\mathbf{k}}, e^{v\mathbf{j}}e^{-w\mathbf{k}})$, *with* $H = \mathbb{T}^3$,

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *are the unit quaternions given in* (2.6)*, and* $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$*. In all cases the isotropy $H_{(1,1)}$ is trivial, except for the submanifold f_3, where the isotropy is $\mathbb{Z}_2$.*

From Theorem 29 we know that the maps f_1 and f_2 are totally geodesic. All maps except for f_2 are immersions of space forms where the first is a sphere of constant sectional curvature $\frac{3}{2}$, the third is a projective space of constant sectional curvature $\frac{3}{8}$ and the last one is a flat torus.

By relaxing the hypothesis to just one angle being constant, in [7] the authors constructed Lagrangian submanifolds in $\mathbb{S}^3 \times \mathbb{S}^3$ from minimal surfaces in $\mathbb{S}^3$:

Theorem 31 (Bektas, Moruz, Van der Veken, Vrancken)**.** *Let $f = (p, q) \colon M \to \mathbb{S}^3 \times \mathbb{S}^3$ be a Lagrangian immersion into the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. Suppose that dp has nowhere maximal rank. Then $p(M) \subset \mathbb{S}^3$ is a minimal surface of $\mathbb{S}^3$.*

Conversely, if $p : \Sigma \to \mathbb{S}^3$ is a minimal surface of $\mathbb{S}^3$, there exists a Lagrangian immersion $f\colon M \to \mathbb{S}^3 \times \mathbb{S}^3$ satisfying $f = (p, q)$ for some map $q\colon M \to \mathbb{S}^3$.

In this chapter we obtain similar results for the nearly Kähler $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. In Section 3.1 we divide Lagrangian submanifolds into four types, which depend on their behavior with respect to the almost product structure P. In Section 3.2 we classify all totally geodesic Lagrangian submanifolds of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. In Section 3.3 we classify all extrinsically homogeneous Lagrangian submanifolds of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Finally, in Section 3.4 we construct an example of a Lagrangian submanifold of type IV of Lemma 11.

We begin by stating some facts about Lagrangian submanifolds of nearly Kähler manifolds, which can be found in [48].

Proposition 16. *Let M be a Lagrangian submanifold of a (pseudo-)nearly Kähler manifold N, and let h and S be the second fundamental form and shape operator, respectively. Then, M satisfies the following properties:*

1. *$g(G(X,Y),Z) = 0$, i.e. $G(X,Y)$ is normal for all $X, Y \in \mathfrak{X}(M)$,*
2. *$g(h(X,Y),JZ) = g(h(X,Z),JY)$, i.e. the tensor $g(h(\cdot,\cdot),J\cdot)$ is totally symmetric,*
3. *$G(X,Y) = \nabla^{\perp}_X JY - J\nabla_X Y$,*
4. *$S_{JX}Y = -Jh(X,Y)$,*

where $G = \nabla J$. Moreover, if N is six-dimensional then M is orientable and minimal.

3.1 The angle functions

Determining the curvature tensor of submanifolds is key while studying the intrinsic information of the immersion. However, as we can see in the expression for $\tilde{R}$ in Equation (2.13), we first need to know how the almost product structure P behaves in the tangent space of the submanifold.

Given a Lagrangian submanifold M of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, the restriction of P to TM can be written as $P|_{TM} = A + JB$, where $A, B\colon TM \to TM$ are the tangent and normal parts of P.

Lemma 7. *The tensors A and B on a Lagrangian submanifold M of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ are symmetric with respect to g, they commute with each other and satisfy $A^2 + B^2 = \mathrm{Id}$.*

Proof. Let X and Y be tangent vectors fields on M. Using that P is symmetric with respect to g and that it anticommutes with J, we compute

$$g(PX, Y) = g(AX + JBX, Y) = g(AX, Y),$$

and on the other hand

$$g(X, PY) = g(X, AY + JBY) = g(X, AY),$$

thus A is symmetric. In a similar way, with the equation $g(PX, JY) = g(X, PJY) = -g(X, JPY)$, we deduce that B is symmetric as well.

Now, using that $P^2 = \mathrm{Id}$ we compute

$$\begin{aligned} X = P^2X = P(AX + JBX) = PAX - JPBX \\ = A^2X + JBAX - J(ABX + JB^2X) \\ = (A^2 + B^2)X + J(BA - AB)X. \end{aligned}$$

□

Given a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, the Gauss and Codazzi equations in Section 1.4 give us expressions for the tangent and normal parts of the curvature tensor $\tilde{R}$, in terms of the curvature tensor R and second fundamental form H of M. In this setting these equations become:

$$\begin{aligned} R(X,Y)Z = &-\tfrac{5}{6}\big(g(Y,Z)X - g(X,Z)Y\big) \\ &-\tfrac{2}{3}\big(g(AY,Z)AX - g(AX,Z)AY + g(BY,Z)BX \\ &- g(BX,Z)BY\big) - S_{h(X,Z)}Y + S_{h(Y,Z)}X, \end{aligned} \tag{3.1}$$

$$\begin{aligned} (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = &-\tfrac{2}{3}\big(g(AY,Z)JBX - g(AX,Z)JBY \\ &- g(BY,Z)JAX + g(BX,Z)JAY\big). \end{aligned} \tag{3.2}$$

In a Riemannian setting, Lemma 7 would imply that A and B are both simultaneously diagonalizable, but this is no longer the case in a pseudo-Riemannian setting. Therefore, we state the following lemma, which holds for operators A and B on vector spaces.

A basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3_1$ is said to be Δ_i-orthonormal if the matrix of inner products is given by Δ_i, where

$$\Delta_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A positive oriented frame on a Lorentzian manifold M is said to be a Δ_i-orthonormal frame if it is a Δ_i-orthonormal basis at each point.

A part of the following lemma can be found in [38].

Lemma 8. *Let A and B be two commuting symmetric operators with respect to a Lorentzian metric on a three-dimensional vector space V. Then A and B must take one of the following forms, with respect to a Δ_i-orthonormal basis.*

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad B = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \qquad \text{with } \Delta_i = \Delta_1, \tag{3.3}$$

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} \mu_1 & \mu_2 & 0 \\ -\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \text{with } \Delta_i = \Delta_1, \quad \mu_2 \neq 0, \tag{3.4}$$

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad B = \begin{pmatrix} \mu_1 & \varepsilon & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}, \qquad \text{with } \Delta_i = \Delta_2, \tag{3.5}$$

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad B = \begin{pmatrix} \mu & 0 & 1 \\ 0 & \mu & 0 \\ 0 & 1 & \mu \end{pmatrix}, \qquad \text{with } \Delta_i = \Delta_2, \tag{3.6}$$

$$A = \begin{pmatrix} \lambda_1 & \varepsilon & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} \mu_1 & b & c \\ 0 & \mu_1 & 0 \\ 0 & c & \mu_2 \end{pmatrix}, \qquad \text{with } \Delta_i = \Delta_2, \quad c\lambda_1 = c\lambda_2, \tag{3.7}$$

$$A = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \qquad B = \begin{pmatrix} \mu & b & c \\ 0 & \mu & 0 \\ 0 & c & \mu \end{pmatrix}, \qquad \text{with } \Delta_i = \Delta_2, \tag{3.8}$$

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \gamma & \delta & 0 \\ -\delta & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \text{with } \Delta_i = \Delta_3, \quad \beta \neq 0, \tag{3.9}$$

with $\varepsilon = \pm 1$.

From [44] we know that the matrix A has four possible forms. The proof of Lemma 8 follows from considering the four forms of A and an easy computation,

by imposing to A and B the conditions of being symmetric with respect to the metric, and applying a linear isometry to the basis if necessary.

Lemma 9. *Let A and B be two symmetric operators with respect to a Lorentzian metric on a three-dimensional vector space V. Assume that they commute and that $A^2 + B^2 = \mathrm{Id}$. Then A and B must take one of the following forms, with respect to a Δ_i-orthonormal basis.*

1. $$A = \begin{pmatrix} \cos 2\theta_1 & 0 & 0 \\ 0 & \cos 2\theta_2 & 0 \\ 0 & 0 & \cos 2\theta_3 \end{pmatrix}, \quad B = \begin{pmatrix} \sin 2\theta_1 & 0 & 0 \\ 0 & \sin 2\theta_2 & 0 \\ 0 & 0 & \sin 2\theta_3 \end{pmatrix},$$

with $\Delta_i = \Delta_1$ and $\theta_1, \theta_2, \theta_3 \in [0, \pi)$.

2. $$A = \begin{pmatrix} \cosh\lambda & 0 & 0 \\ 0 & \cosh\lambda & 0 \\ 0 & 0 & \cos 2\theta \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \sinh\lambda & 0 \\ -\sinh\lambda & 0 & 0 \\ 0 & 0 & \sin 2\theta \end{pmatrix},$$

with $\Delta_i = \Delta_1$, $\lambda \in \mathbb{R}$ and $\theta \in [0, \pi)$.

3. $$A = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $\Delta_i = \Delta_2$ and $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.

4. $$A = \begin{pmatrix} \cos 2\theta_1 & \varepsilon & 0 \\ 0 & \cos 2\theta_1 & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix},$$

$$B = \begin{pmatrix} \sin 2\theta_1 & \frac{-(c^2 + 2\varepsilon \cos 2\theta_1)}{2 \sin 2\theta_1} & c \\ 0 & \sin 2\theta_1 & 0 \\ 0 & c & \sin 2\theta_2 \end{pmatrix},$$

with $\Delta_i = \Delta_2$, $\theta_1, \theta_2 \in [0, \pi)$, $\theta_1 \neq 0, \pi/2$, $\varepsilon = \pm 1$ and $c \in \mathbb{R}$. If $c \neq 0$, then $\cos 2\theta_1 = \cos 2\theta_2$ and $\sin 2\theta_1 = -\sin 2\theta_2$.

5. $A = \begin{pmatrix} -\varepsilon & \varepsilon & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}$, $B = \begin{pmatrix} 0 & t & \sqrt{2} \\ 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$,

with $\Delta_i = \Delta_2$, $\varepsilon = \pm 1$ and $t \in \mathbb{R}$.

6. $A = \begin{pmatrix} \cos 2\theta & 0 & 1 \\ 0 & \cos 2\theta & 0 \\ 0 & 1 & \cos 2\theta \end{pmatrix}$,

$B = \begin{pmatrix} \sin 2\theta & -(\csc 2\theta)^3/2 & -\cot 2\theta \\ 0 & \sin 2\theta & 0 \\ 0 & -\cot 2\theta & \sin 2\theta \end{pmatrix}$,

with $\Delta_i = \Delta_2$, and $\theta \neq 0, \pi/2$.

7. $A = \begin{pmatrix} s\cos 2\theta_1 & x & 0 \\ -x & s\cos 2\theta_1 & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix}$,

$B = \begin{pmatrix} s\sin 2\theta_1 & y & 0 \\ -y & s\sin 2\theta_1 & 0 \\ 0 & 0 & \sin 2\theta_2 \end{pmatrix}$,

with $\Delta_i = \Delta_3$, $s = \sqrt{1 + x^2 + y^2}$, $x \neq 0$, $y \sin 2\theta_1 = -x \cos 2\theta_1$ and $\theta_1, \theta_2 \in [0, \pi)$.

Proof. Lemma 8 provides us with seven different forms A and B can take. We analyze the equation $A^2 + B^2 = \mathrm{Id}$ on each type of matrix separately.

Type 1 (3.3): Computing $A^2 + B^2 = \mathrm{Id}$ in (3.3) immediately yields Case *1* of Lemma 9.

Type 2 (3.4): It follows from $A^2 + B^2 = \mathrm{Id}$ that $\mu_1 = 0$, $\lambda_1^2 - \mu_2^2 = 1$ and $\lambda_2^2 + \mu_3^2 = 1$. Hence, we get Case *2* of Lemma 9.

Type 3 (3.5): Case *3* is immediate from computing $A^2 + B^2 = \mathrm{Id}$, as we obtain that $\mu_1 = 0$. It follows that $\lambda = \pm 1$ and that $\mu_2 = 0$. This is Case *3* of Lemma 9.

Type 4 (3.6): We easily see that under no conditions $A^2 + B^2$ can be equal to the identity in this case.

Type 5 (3.7): Computing $A^2 + B^2 = \mathrm{Id}$ in (3.7) yields the equations

$$\lambda_1^2 + \mu_1^2 = 1, \tag{3.10}$$

$$\lambda_2^2 + \mu_2^2 = 1, \tag{3.11}$$

$$2\varepsilon\lambda_1 + 2b\mu_1 + c^2 = 0, \tag{3.12}$$

$$c(\mu_1 + \mu_2) = 0. \tag{3.13}$$

Suppose $c \neq 0$. Then because of Equation (3.13) and $c\lambda_1 = c\lambda_2$ we have that $\lambda_1 = \lambda_2$ and $\mu_1 = -\mu_2$. If $\mu_1 = 0$ then (3.10) and (3.12) imply that $\lambda_1 = \lambda_2 = -\varepsilon$ and $c = \sqrt{2}$, which is Case *5* of Lemma 9. If instead $\mu_1 \neq 0$, by Equation (3.10) we can write $\lambda_1 = \cos 2\theta_1$ and $\mu_1 = \sin 2\theta_1$, then by (3.12) we have $b = -(c^2 + 2\varepsilon\cos 2\theta_1)/(2\sin 2\theta_1)$, which leave us with Case *4* of Lemma 9.

Now suppose that $c = 0$. If $\mu_1 = 0$ then we get a contradiction from equations (3.10) and (3.12). Therefore μ_1 must be different from zero, and from (3.10), (3.11) and (3.12) we get that $\lambda_1 = \cos 2\theta_1$, $\lambda_2 = \cos 2\theta_2$, $\mu_1 = \sin 2\theta_1$, $\mu_2 = \sin 2\theta_2$ and $b = -\varepsilon\cot 2\theta_1$, which is again Case *4* of Lemma 9.

Type 6 (3.8): Computing $A^2 + B^2 = \mathrm{Id}$ yields

$$\lambda^2 + \mu^2 = 1, \tag{3.14}$$

$$\lambda + c\mu = 0, \tag{3.15}$$

$$2b\mu + c^2 + 1 = 0. \tag{3.16}$$

From (3.14) and (3.15) we easily see that $\mu \neq 0$, thus $c = -\lambda/\mu$. By replacing this in (3.16) we get that $b = -1/(2\mu^3)$. Finally by (3.14) we have that $\lambda = \cos 2\theta$, $\mu = \sin 2\theta$, $c = -\cot 2\theta$ and $b = -\csc(2\theta)^3/2$, which is Case *6* in Lemma 9.

Type 7 (3.9): Again, we compute $A^2 + B^2 = \mathrm{Id}$ in (3.9), from which follows

$$\alpha^2 - \beta^2 + \gamma^2 - \delta^2 = 1, \tag{3.17}$$

$$\alpha\beta + \gamma\delta = 0, \tag{3.18}$$

$$\lambda^2 + \gamma^2 = 1. \tag{3.19}$$

We can transform (3.17) into

$$\left(\frac{\alpha}{\sqrt{1+\beta^2+\delta^2}}\right)^2 + \left(\frac{\gamma}{\sqrt{1+\beta^2+\delta^2}}\right)^2 = 1,$$

thus we obtain that $\alpha = \sqrt{1+\beta^2+\delta^2}\cos\theta$ and $\gamma = \sqrt{1+\beta^2+\delta^2}\sin\theta$ for some θ. Equation (3.18) becomes $\sqrt{1+\beta^2+\delta^2}(\beta\cos\theta + \delta\sin\theta) = 0$, hence we can conclude that $\delta = -\beta\cot\theta$ since if $\sin\theta = 0$ then $\beta = 0$, which contradicts the last condition of (3.9). Renaming $x = \beta$, $y = \delta$ and $s = \sqrt{1+x^2+y^2}$ we obtain Case *7* of Lemma 9. □

Remark 12. In [38], only the case $\varepsilon = 1$ is treated in equations (3.5) and (3.7). However, the case $\varepsilon = -1$ is essentially different, as there is no change of basis preserving the metric that can take one case into the other.

Now, fixing a point $p \in M$ we know that there exists a basis of T_pM as in Lemma 9. There exists an open dense subset of M such that at each point we can extend this basis locally to a Δ_i-orthonormal smooth frame $\{E_1, E_2, E_3\}$ on M.

To see this, take p in M such that A and B take one of the forms of Lemma 9 and the eigenvalues have maximal multiplicity. The coefficients of the characteristic polynomial of A and B are smooth since both are smooth tensors. On the other hand, the roots of a polynomial of grade 3 can always be written in a smooth combination of the coefficients. Hence, the roots are always smooth and therefore most of the functions involved in Lemma 9. The functions x, y and s in type *7* of Lemma 9 can be written in terms of the eigenvalues, therefore they are smooth as well. It follows that θ_1 is smooth too.

We can apply the same argument as in Lemma 1.2 in [49] and extend the basis to a frame on an open subset where the multiplicities of the eigenvalues are constant. Moreover, we assume that outside of subset the multiplicities are different. Now take the complement of the closure of this open subset. We obtain an open subset where we can apply this process again. After iterating we obtain an open dense subset where we can always extend the frame locally.

Finally, by using that the expression for A is smooth and therefore the frame is smooth, we conclude that the functions c and t in types *4* and *5* are smooth since the frame is smooth and the tensor B is smooth.

Recall from the first equation of Proposition 16 it follows that $G(X,Y)$ is a normal vector to M for any tangent vectors X, Y on M. Let $\{E_1, E_2, E_3\}$ be a Δ_i-orthonormal frame of the tangent space of M. By Proposition 9 and Lemma 4 we get $JG(E_j, E_k) = \varepsilon\sqrt{\frac{2}{3}}E_l$ with $\varepsilon = \pm 1$ depending on Δ_i, as the following table shows.

	Δ_1	Δ_2	Δ_3
$JG(E_1, E_2)$	$\sqrt{\frac{2}{3}}E_3$	$\sqrt{\frac{2}{3}}E_3$	$\sqrt{\frac{2}{3}}E_3$
$JG(E_1, E_3)$	$-\sqrt{\frac{2}{3}}E_2$	$-\sqrt{\frac{2}{3}}E_1$	$\sqrt{\frac{2}{3}}E_2$
$JG(E_2, E_3)$	$-\sqrt{\frac{2}{3}}E_1$	$\sqrt{\frac{2}{3}}E_2$	$\sqrt{\frac{2}{3}}E_1$

(3.20)

Lemma 10. *Let M be a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and P the almost product structure given in (2.12). Assume that A and B are such that $P|_M = A + JB$. Let $\mathcal{F}$ be an isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Then $P|_{\mathcal{F}(M)} = \tilde{A} + J\tilde{B}$ with*

$$\tilde{A} = \mathcal{F}_*(\cos\tau A + (-1)^\kappa \sin\tau B)\mathcal{F}_*^{-1}$$

$$\tilde{B} = \mathcal{F}_*(-\sin\tau A + (-1)^\kappa \cos\tau B)\mathcal{F}_*^{-1}$$

where τ and κ are such that $d\mathcal{F} \circ P = (\cos\tau P + \sin\tau JP) \circ d\mathcal{F}$ and $d\mathcal{F} \circ J = (-1)^\kappa \circ d\mathcal{F}$.

Proof. We know from Section 2.2.2 that there exists such numbers κ and τ as in the last statement of the proposition. Let $\mathcal{F}_*X \in \mathfrak{X}(\mathcal{F}(M))$. On the one hand we have that

$$\begin{aligned}\mathcal{F}_*PX &= \mathcal{F}_*(AX) + \mathcal{F}_*(JBX)\\ &= \mathcal{F}_*(AX) + (-1)^\kappa J\mathcal{F}_*(BX),\end{aligned}$$

and on the other hand

$$\begin{aligned}\mathcal{F}_*PX &= (\cos\tau P + \sin\tau JP)\mathcal{F}_*X\\ &= \cos\tau(\tilde{A}\mathcal{F}_*X + J\tilde{B}\mathcal{F}_*X) + \sin\tau J(\tilde{A}\mathcal{F}_*X + J\tilde{B}\mathcal{F}_*X).\end{aligned}$$

Since $\mathcal{F}$ commutes with J (up to sign), the Lagrangian condition is preserved. Therefore, after comparing tangent and normal parts; we obtain

$$\mathcal{F}_*AX = (\cos\tau\tilde{A} - \sin\tau\tilde{B})\mathcal{F}_*X,$$

$$(-1)^\kappa\mathcal{F}_*BX = (\sin\tau\tilde{A} + \cos\tau\tilde{B})\mathcal{F}_*X.$$

Equivalently;

$$\tilde{A}\mathcal{F}_*X = \mathcal{F}_*(\cos\tau AX + (-1)^\kappa \sin\tau BX),$$

$$\tilde{B}\mathcal{F}_*X = \mathcal{F}_*(-\sin\tau AX + (-1)^\kappa \cos\tau BX).$$

□

With this information we can state a stronger version of Lemma 9.

Lemma 11. *Let M be a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ and P the almost product structure given in* (2.12)*. Then there exists a Lagrangian submanifold N congruent to M such that the restriction of P to N can be written as $P|_N = A + JB$, where $A, B : TN \to TN$ must have one of the following forms, with respect to a Δ_i-orthonormal frame $\{E_1, E_2, E_3\}$.*

I. $$A = \begin{pmatrix} \cos 2\theta_1 & 0 & 0 \\ 0 & \cos 2\theta_2 & 0 \\ 0 & 0 & \cos 2\theta_3 \end{pmatrix}, \quad B = \begin{pmatrix} \sin 2\theta_1 & 0 & 0 \\ 0 & \sin 2\theta_2 & 0 \\ 0 & 0 & \sin 2\theta_3 \end{pmatrix},$$

with $\Delta_i = \Delta_1$ and $\theta_1 + \theta_2 + \theta_3 = 0$ modulo π.

II. $$A = \begin{pmatrix} \cos 2\theta_1 & 1 & 0 \\ 0 & \cos 2\theta_1 & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix}, \quad B = \begin{pmatrix} \sin 2\theta_1 & -\cot 2\theta_1 & 0 \\ 0 & \sin 2\theta_1 & 0 \\ 0 & 0 & \sin 2\theta_2 \end{pmatrix},$$

with $\Delta_i = \Delta_2$, $2\theta_1 + \theta_2 = 0$ modulo π and $\theta_1 \neq 0, \pi/2$.

III. $$A = \begin{pmatrix} -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}, \qquad B = \pm \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-4}{3\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

with $\Delta_i = \Delta_2$.

IV. $$A = \begin{pmatrix} \cosh\psi\cos 2\theta_1 & \sinh\psi\sin\theta_2 & 0 \\ -\sinh\psi\sin\theta_2 & \cosh\psi\cos 2\theta_1 & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix},$$

$$B = \begin{pmatrix} \cosh\psi\sin 2\theta_1 & \sinh\psi\cos\theta_2 & 0 \\ -\sinh\psi\cos\theta_2 & \cosh\psi\sin 2\theta_1 & 0 \\ 0 & 0 & \sin 2\theta_2 \end{pmatrix},$$

with $\Delta_i = \Delta_3$, $2\theta_1 + \theta_2 = 0$ modulo π, $\theta_2 \neq 0, \pi$ and $\psi \neq 0$.

The functions θ_i and ψ are called the **angle functions**. Since the type is invariant under congruent transformations, we say that a Lagrangian submanifold M is of type I (or sometimes diagonalizable type), II, III or IV. Likewise, if M is of type i we say that A and B take type i form on M.

Proof. The next equation follows from applying P to (3.20), using Lemma 5:

$$PE_i = -\alpha JPG(E_j, E_k) = \alpha JG(PE_j, PE_k), \tag{3.21}$$

where α can be either $\sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}}$, depending on which case of (3.20) we are in. For Case *1*, Case *2* and Case *7* of Lemma 9 the triple (i, j, k) is a permutation of $(1, 2, 3)$. For cases *3* to *6* of Lemma 9 this is not true anymore, and i is equal either to j or to k. We consider the seven cases of Lemma 9 applied to Equation (3.21), one by one.

Case 1. Using a similar procedure as in [20], we get $\theta_1 + \theta_2 + \theta_3 = 0$ modulo π.

Case 2. Taking $i = 3$, $j = 1$ and $k = 2$ in (3.21) and looking at the component of JE_3 we conclude that $\theta = 0$. Now we apply to M the isometry $\Psi_{4\pi/3,1}$ given in (2.19). The tensor P restricted to $\Psi_{4\pi/3,1}(M)$ can be written as $\tilde{A} + J\tilde{B}$, where

$$\tilde{A} = -\frac{1}{2}A + \frac{\sqrt{3}}{2}B, \qquad \tilde{B} = \frac{\sqrt{3}}{2}A + \frac{1}{2}B.$$

This is Case IV of Lemma 11 with $\theta_1 = \theta_2 = \frac{\pi}{3}$.

Case 3. With $i = 1$, $j = 1$ and $k = 3$ in (3.21) we obtain $\varepsilon_1 = 1$. As there is no possible change of basis that can transform ε_2 into 1, we apply either $\Psi_{2\pi/3,1}$ or $\Psi_{4\pi/3,1}$ to M, depending on its value. Proceeding in the same way as in the previous case, we restrict P to the image of the isometry. After a change of basis, we get Case II of Lemma 11, with $\theta_1 = \theta_2 = \frac{\pi}{3}$ if $\varepsilon_2 = 1$ and $\theta_1 = \theta_2 = \frac{2\pi}{3}$ if $\varepsilon_2 = -1$.

Case 4. Taking $i = 1$, $j = 1$ and $k = 1$ in (3.21) yields

$$\cos 2\theta_1 = \cos 2(\theta_1 + \theta_2), \qquad \sin 2\theta_1 = -\sin 2(\theta_1 + \theta_2).$$

Thus,

$$\begin{aligned}\cos 2(2\theta_1 + \theta_2) &= \cos 2\theta_1 \cos 2(\theta_1 + \theta_2) - \sin 2\theta_1 \sin 2(\theta_1 + \theta_2)\\ &= \cos^2 2(\theta_1 + \theta_2) + \sin^2 2(\theta_1 + \theta_2)\\ &= 1.\end{aligned}$$

Then $2\theta_1 + \theta_2 = 0 \bmod \pi$. Looking at the component of E_3 in (3.21) when taking $i = 2$, $j = 2$ and $k = 3$ we obtain $c \sin 2\theta_1 = 0$, therefore $c = 0$ as $\sin 2\theta_1 \neq 0$.

If $\varepsilon = 1$, we have Case II of Lemma 11.

If $\varepsilon = -1$, we would like to transform it into 1. We apply either $\Psi_{2\pi/3,0}$ or $\Psi_{2\pi/3,1}$ to M and restrict P to the image. The component $\tilde{A}_{12}$ is $1 + \sqrt{3}\cot 2\theta_1$ or $1 - \sqrt{3}\cot 2\theta_1$. These two values cannot be negative at the same time, thus we can always choose one to be positive. After a change of basis, we get Case II of Lemma 11.

Case 5. On the left hand side of (3.21) we have $PE_1 = -E_1$ and on the right hand side we have

$$-\sqrt{\frac{3}{2}}JG(PE_1, PE_3) = -\sqrt{\frac{3}{2}}JG(-E_1, -E_3) = E_1,$$

which is a contradiction. Therefore, there is no Lagrangian submanifold with such a frame.

Case 6. Taking the component of E_3 on both sides of (3.21) with $i = 2$, $j = 2$ and $k = 3$ yields $-2\cos 2\theta = 1$. Hence $\theta = \frac{\pi}{3}$ or $\theta = \frac{2}{3}\pi$. That is

$$A = \begin{pmatrix} -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}, \quad B = \pm \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{4}{3\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Case 7. Taking $i = 1$, $j = 2$ and $k = 3$ in (3.21) gives us

$$\cos 2\theta_1 = \cos 2(\theta_1 + \theta_2),$$

$$\sin 2\theta_1 = -\sin 2(\theta_1 + \theta_2).$$

Therefore as in the previous cases we have $\cos 2(2\theta_1 + \theta_2) = 1$. We take $i = 2$, $j = 1$ and $k = 3$ in (3.21) and we look at the components of E_1, JE_1:

$$\begin{aligned} x &= -x\cos 2\theta_2 + y\sin 2\theta_2, \\ y &= x\sin 2\theta_2 + y\cos 2\theta_2. \end{aligned} \tag{3.22}$$

Note that the equations in (3.22) can be written as the equation $Rv = v$ where $v = (x, y)^t$ and

$$R = \begin{pmatrix} \cos(\pi - 2\theta_2) & \sin(\pi - 2\theta_2) \\ \sin(\pi - 2\theta_2) & -\cos(\pi - 2\theta_2) \end{pmatrix}.$$

Hence, v is an eigenvector of the matrix R associated to the eigenvalue 1. Now, R is a reflection in the plane with respect to the straight line $s \mapsto s(\cos\left(\frac{\pi}{2} - \theta_2\right), \sin\left(\frac{\pi}{2} - \theta_2\right))^t = s(\sin\theta_2, \cos\theta_2)^t$, therefore v lies in that subspace. So $x = \sinh\psi\sin\theta_2$, $y = \sinh\psi\cos\theta_2$ for some $\psi \in \mathbb{R}$ different from zero. Finally, replacing x and y in the matrices of Case 7 of Lemma 9 yields what we desired. □

3.1.1 The tensor ∇P

Let M be a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and let $\{E_1, E_2, E_3\}$ be a frame given by Lemma 11. As the normal space is spanned by

$\{JE_1, JE_2, JE_3\}$, we may define functions ω_{ij}^k and h_{ij}^k by $\nabla_{E_i} E_j = \sum_k \omega_{ij}^k E_k$ and $h(E_i, E_j) = \sum_k h_{ij}^k JE_k$. From Property 2 of Proposition 16 and the compatibility of the connection with the metric, we obtain the following symmetries.

First, for frames $\{E_i\}$ associated to Δ_1 and Δ_3 we have

$$\delta_k \omega_{ij}^k = -\delta_j \omega_{ik}^j, \qquad h_{ij}^k = h_{ji}^k = \delta_j \delta_k h_{ik}^j,$$

where

$$\delta_i = g(E_i, E_i).$$

This implies that $\omega_{ij}^j = 0$ for all $i, j = 1, 2, 3$.

If the frame is associated to Δ_2 we get

$$\omega_{ij}^k = -\omega_{i\widehat{k}}^{\widehat{j}}, \qquad h_{ij}^k = h_{ji}^k = h_{i\widehat{k}}^{\widehat{j}}$$

where $\widehat{2} = 1$, $\widehat{1} = 2$ and $\widehat{3} = 3$. As before, we have that $\omega_{i3}^3 = 0$. Also, if $j = 1$, $k = 2$ or $j = 2$, $k = 1$ then $\omega_{ij}^k = 0$.

We recall the expression for the covariant derivative of the tensor P, previously given in Lemma 5:

$$(\tilde{\nabla}_X P)Y = \frac{1}{2}\Big(JG(X, PY) + JPG(X, Y)\Big). \tag{3.23}$$

This equation is useful since it gives conditions on the components h_{ij}^k and ω_{ij}^k. As it depends on P, we are forced to divide between the four cases of Lemma 11.

Lagrangian submanifolds of diagonalizable type

Lemma 12. *Let M be a Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type I form in Lemma 11 with respect to a Δ_1-orthonormal frame $\{E_1, E_2, E_3\}$. Except for h_{12}^3, all the components of the second fundamental form are given by the derivatives of the angle functions θ_1, θ_2 and θ_3:*

$$E_i(\theta_j) = -\delta_i \delta_j h_{jj}^i, \tag{3.24}$$

where $\delta_i = g(E_i, E_i)$. Also

$$h_{ij}^k \cos(\theta_j - \theta_k) = (\tfrac{1}{\sqrt{6}} \delta_k \varepsilon_{ij}^k - \omega_{ij}^k) \sin(\theta_j - \theta_k), \tag{3.25}$$

for $j \neq k$.

Proof. Taking $X = E_1$ and $Y = E_2$ in (3.23) and comparing the components in $E_1, E_2, E_3, JE_1, JE_2$ and JE_3 yields the following six equations

$$\begin{aligned}
(h^2_{11}\cos(\theta_1-\theta_2)+\omega^2_{11}\sin(\theta_1-\theta_2))\sin(\theta_1+\theta_2)&=0,\\
(h^2_{11}\cos(\theta_1-\theta_2)+\omega^2_{11}\sin(\theta_1-\theta_2))\cos(\theta_1+\theta_2)&=0,\\
(h^1_{22}-E_1(\theta_2))\sin(2\theta_2)&=0,\\
(h^1_{22}-E_1(\theta_2))\cos(2\theta_2)&=0,\\
(h^3_{12}\cos(\theta_2-\theta_3)+(-\tfrac{1}{\sqrt{6}}+\omega^3_{12})\sin(\theta_2-\theta_3))\sin(\theta_2+\theta_3)&=0,\\
(h^3_{12}\cos(\theta_2-\theta_3)+(-\tfrac{1}{\sqrt{6}}+\omega^3_{12})\sin(\theta_2-\theta_3))\cos(\theta_2+\theta_3)&=0.
\end{aligned}$$

Since sine and cosine never vanish at the same time (3.24) and (3.25) hold for $i = 1$ and $j = 2$. The other equations follow in a similar way, by choosing different X and Y in (3.23). □

Lagrangian submanifolds of type II

The covariant derivative of P in Equation (3.23) yields the following lemma for type II Lagrangian submanifolds.

Lemma 13. *Let M be a Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. Suppose that A and B are of type II in Lemma 11 with respect to a Δ_2-orthonormal frame $\{E_1, E_2, E_3\}$. Then $h(E_1, E_1) = 0$. Moreover, the derivatives of the angles are given by*

$$\begin{aligned}
&E_1(\theta_1) = -h^1_{11} = 0, && E_2(\theta_1) = -h^2_{22}, && E_3(\theta_1) = -h^3_{12}\\
&E_1(\theta_2) = -h^2_{33}, && E_2(\theta_2) = -h^1_{33}, && E_3(\theta_2) = -h^3_{33}.
\end{aligned} \tag{3.26}$$

Furthermore,

$$h^1_{33} = -2h^2_{22}, \qquad h^2_{33} = -2h^1_{11} = 0, \qquad h^3_{33} = -2h^3_{12}. \tag{3.27}$$

Proof. Computing Equation (3.23) with $X = E_2$, $Y = E_1$ and looking at the components of E_2 and JE_2 we obtain

$$h^1_{11}\sin 2\theta_1 = 0, \qquad h^1_{11}\cos 2\theta_1 = 0.$$

Since sine and cosine never vanish at the same time we get that $h^1_{11} = 0$. Same can be done computing Equation (3.23) with $X = E_1$, $Y = E_1$ and $X = E_3$,

$Y = E_1$ and looking in the directions of E_2 and JE_2 , obtaining $h^2_{11} = h^3_{11} = 0$. Looking in the direction of E_1 and JE_1 of the same equations, we obtain the derivatives of the function θ_1. We derive $E_i(\theta_2)$ by computing Equation (3.23) with $X = E_i$ and $Y = E_3$.

We obtain the last statement either from Proposition 16, or from (3.26) and the fact that $2\theta_1 + \theta_2 = 0$. □

Lagrangian submanifolds of type III

For Lagrangian submanifolds of type III, Equation (3.23) gives expressions for all functions ω^k_{ij}, given in the following lemma.

Lemma 14. *Let M be a Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type III form in Lemma 11 with respect to a Δ_2-orthonormal frame $\{E_1, E_2, E_3\}$. Then we have $h(E_1, E_1) = 0$ and*

$$
\begin{aligned}
h^3_{12} &= \omega^1_{11} = \omega^3_{11} = \omega^2_{33} = 0,\\
\omega^3_{12} &= \frac{\sqrt{2} + (-1)^{k+1} 3h^2_{22}}{2\sqrt{3}},\\
\omega^1_{31} &= \frac{\sqrt{2} + (-1)^{k+1} 12h^2_{22}}{2\sqrt{3}},\\
\omega^3_{21} &= -\frac{\sqrt{2} + (-1)^{k} 6h^2_{22}}{2\sqrt{3}},\\
\omega^2_{22} &= \frac{(-1)^k (h^2_{22} - 3h^3_{22})}{\sqrt{3}},\\
\omega^1_{33} &= \frac{(-1)^{k+1} (4h^2_{22} - 3h^3_{22})}{2\sqrt{3}},\\
\omega^3_{22} &= \frac{(-1)^{k+1} (9h^1_{22} - 8h^2_{22} + 6h^3_{22})}{6\sqrt{3}},
\end{aligned}
\tag{3.28}
$$

where $(-1)^k$ with $k \in \{0, 1\}$ the sign of B in Lemma 11.

Proof. From computing Equation (3.23) with $X = E_i$, $Y = E_1$, $i = 1, 2, 3$ and looking at the components in the direction of JE_2 it follows that $h(E_1, E_1) = 0$. Now, if we compute Equation (3.23) with $X = E_i$, $Y = E_1$, $X = E_2$ $Y = E_3$ in

the direction of E_1 and JE_2 respectively we get $h^3_{12} = \omega^3_{11} = 0$. The rest of the equations are obtained in the same way. □

Lagrangian submanifolds of type IV

Let M be a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of type IV in Lemma 11. From this Lemma we know that $2\theta_1 + \theta_2 = 0$ modulo π. Thus, we write $\theta_2 = k\pi - 2\theta_1$ where $k = 0, 1$. Suppose that $k = 1$. The tensors A, B on the image of M by the isometry $\Psi_{0,1}$ given in (2.19), corresponds to the case with angle functions θ_1, θ_2, $-\psi$ and $k = 0$.

Contrary to type II Lagrangian submanifolds, here it is necessary to distinguish between $k = 0$ and $k = 1$, since both θ_2 and $2\theta_2$ appear in the expressions for A and B.

From Proposition 16 we may assume that

$$h^i_{33} = h^i_{22} - h^i_{11}$$

for $i = 1, 2, 3$. Now, from Equation (3.23) we obtain the next lemma.

Lemma 15. *Let M be a Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type IV form in Lemma 11 with respect to a Δ_3-orthonormal frame $\{E_1, E_2, E_3\}$. Then the derivatives of the angle functions are given by:*

$$\begin{aligned} &E_1(\theta_1) = \frac{h^1_{22} - h^1_{11}}{2}, \quad E_2(\theta_1) = \frac{h^2_{11} - h^2_{22}}{2}, \quad E_3(\theta_1) = \frac{h^3_{22} - h^3_{11}}{2}, \\ &E_1(\psi) = 2h^2_{11}, \quad E_2(\psi) = -2h^1_{22}, \quad E_3(\psi) = -2h^3_{12}. \end{aligned} \tag{3.29}$$

Moreover, we have the following expressions for the functions ω^k_{ij}:

$$\begin{aligned} \omega^2_{11} &= \frac{1}{2}(h^1_{11} + h^1_{22}) \coth\psi, \\ \omega^1_{22} &= -\frac{1}{2}(h^2_{11} + h^2_{22}) \coth\psi, \\ \omega^1_{32} &= \frac{1}{2}(h^3_{11} + h^3_{22}) \coth\psi - \frac{1}{\sqrt{6}}. \end{aligned} \tag{3.30}$$

Also,

$$\begin{aligned}
\omega^3_{11} &= \frac{h^3_{11}\sin 6\theta_1 + h^3_{12}\sinh\psi}{\cos 6\theta_1 - \cosh\psi},\\
\omega^3_{12} &= \frac{h^3_{12}\sin 6\theta_1 - h^3_{11}\sinh\psi}{\cos(6\theta_1) - \cosh\psi} + \frac{1}{\sqrt{6}},\\
\omega^3_{22} &= \frac{h^3_{22}\sin 6\theta_1 - h^3_{12}\sinh\psi}{\cos 6\theta_1 - \cosh\psi},\\
\omega^3_{21} &= \frac{h^3_{12}\sin 6\theta_1 + h^3_{22}\sinh\psi}{\cos 6\theta_1 - \cosh\psi} - \frac{1}{\sqrt{6}},\\
\omega^1_{33} &= \frac{(h^1_{11} - h^1_{22})\sin 6\theta_1 + (h^2_{22} - h^1_{11})\sinh\psi}{\cos 6\theta_1 - \cosh\psi},\\
\omega^2_{33} &= \frac{(h^2_{11} - h^2_{22})\sin 6\theta_1 - (h^1_{22} - h^1_{11})\sinh\psi}{\cos 6\theta_1 - \cosh\psi}.
\end{aligned} \tag{3.31}$$

Proof. From computing $(\nabla_{E_1}P)E_1$ and looking at the components in the direction of E_2 and JE_2 we get the equation

$$\begin{pmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ -\sin 2\theta_1 & \cos 2\theta_1 \end{pmatrix} \begin{pmatrix} \cosh\psi\left(E_1(\psi) - 2h^2_{11}\right) \\ -\sinh\psi\left(2E_1(\theta_1) + h^1_{11} - h^1_{22}\right) \end{pmatrix} = 0.$$

From $(\nabla_{E_2}P)E_2$ and $(\nabla_{E_3}P)E_3$ we derive the rest of the equations in Equation (3.29) in a similar way.

Having Equation (3.29) we look at the rest of the components of $(\nabla_{E_i}P)E_i$ and we obtain the expressions in (3.30).

By computing $(\nabla_{E_i}P)E_j$ with $i \neq j$ we come to linear equations which yield (3.31).

□

3.2 Totally geodesic Lagrangian submanifolds

In this section we classify totally geodesic Lagrangian submanifolds of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The classification can also be found in [3].

Theorem 32. *Any totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *is congruent to the image of one of the following maps, possibly restricted to an open subset:*

(1) $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (\mathrm{Id}_2, u)$,
(2) $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (u, \boldsymbol{i}u\boldsymbol{i})$,
(3) $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (u, -\boldsymbol{k}u\boldsymbol{k})$,

where Id_2 *is the identity matrix, and* $\boldsymbol{i}$, $\boldsymbol{k}$ *are the matrices given in* (2.1).

Conversely, the maps (1), (2) *and* (3) *are totally geodesic Lagrangian immersions.*

The layout of this section is as follows. In Subsection 3.2.1 we present some examples of totally geodesic Lagrangian submanifolds of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Subsections 3.2.2 and 3.2.3 address the existence of totally geodesic Lagrangian submanifolds of diagonalizable type and of the rest of the types in Lemma 11, respectively. In Subsection 3.2.4 we prove Theorem 32.

In [57], the authors stated that any totally geodesic Lagrangian submanifold of $\mathbb{S}^3 \times \mathbb{S}^3$ is congruent to an immersion of a list of 6 examples. This classification can be simplified, using isometries equivalent to the ones given in (2.19). This way, the list is reduced to just two examples, similar to immersions (1) and (2) of Theorem 32. Hence, immersion (3) is a new example, which arises from the pseudo-Riemannian nature of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

We can understand the geometry of the immersions of Theorem 32 via the identification of $\mathrm{SL}(2,\mathbb{R})$ with the anti-de Sitter space $H_1^3(-1)$; namely via the map

$$H_1^3(-1) \subset \mathbb{R}_2^4 \to \mathrm{SL}(2,\mathbb{R}) : (x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 - x_2 & x_3 - x_1 \\ x_3 + x_1 & x_0 + x_2 \end{pmatrix},$$

which is an isometry between $H_1^3(-1)$ and $\mathrm{SL}(2,\mathbb{R})$ with the metric introduced in Equation (2.2). Here, $\mathbb{R}_2^4$ denotes $\mathbb{R}^4$ equipped with the indefinite inner product

$$\langle x, y\rangle = -x_0y_0 - x_1y_1 + x_2y_2 + x_3y_3,$$

and the three-dimensional anti-de Sitter space with constant sectional curvature $c < 0$ is defined as $H_1^3(c) = \{x \in \mathbb{R}^4 : \langle x, x\rangle = 1/c\}$.

Note that the three immersions of Theorem 32 induce essentially different Riemannian structures on $\mathrm{SL}(2,\mathbb{R})$. The first immersion induces a metric with constant sectional curvature, that is homothetic to the standard metric. The second immersion turns $\mathrm{SL}(2,\mathbb{R})$ into an anti-de Sitter space with a Berger-like metric stretched in a spacelike direction, and the third immersion turns it again into an anti-de Sitter space with a Berger-like metric, but now stretched in a timelike direction. These metrics have been studied more generally in [11] and [12].

3.2.1 Examples of totally geodesic Lagrangian submanifolds

We proceed by giving three examples of totally geodesic Lagrangian submanifolds of the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Recall the frame $\{X_1, X_2, X_3\}$ on $\mathrm{SL}(2,\mathbb{R})$ given in (2.3), that is

$$X_1(a) = a\boldsymbol{i}, \qquad X_2(a) = a\boldsymbol{j}, \qquad X_3(a) = a\boldsymbol{k}, \tag{3.32}$$

for which

$$\langle X_1, X_1\rangle = \langle X_2, X_2\rangle = -\langle X_3, X_3\rangle = 1,$$

where $\langle , \rangle$ is the metric given in (2.2). Then $\{X_3, X_2, X_1\}$ is a Δ_1-orthonormal frame on $\mathrm{SL}(2,\mathbb{R})$.

Example 16. Consider the immersion of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (\mathrm{Id}_2, u).$$

We have

$$df(X_1(u)) = (0, u\boldsymbol{i})_{(\mathrm{Id}_2,u)},$$

$$df(X_2(u)) = (0, u\boldsymbol{j})_{(\mathrm{Id}_2,u)},$$

$$df(X_3(u)) = (0, u\boldsymbol{k})_{(\mathrm{Id}_2,u)}.$$

Then we compute

$$Pdf(X_1(u)) = (\boldsymbol{i}, 0)_{(\mathrm{Id}_2,u)},$$

$$Pdf(X_2(u)) = (\boldsymbol{j}, 0)_{(\mathrm{Id}_2,u)},$$

$$Pdf(X_3(u)) = (\boldsymbol{k}, 0)_{(\mathrm{Id}_2,u)},$$

and

$$Jdf(X_1(u)) = -\frac{1}{\sqrt{3}}(2\boldsymbol{i}, u\boldsymbol{i})_{(\mathrm{Id}_2,u)},$$

$$Jdf(X_2(u)) = -\frac{1}{\sqrt{3}}(2\boldsymbol{j}, u\boldsymbol{j})_{(\mathrm{Id}_2,u)},$$

$$Jdf(X_3(u)) = -\frac{1}{\sqrt{3}}(2\boldsymbol{k}, u\boldsymbol{k})_{(\mathrm{Id}_2,u)}.$$

From this, it immediately follows that $g(df(X_i(u)), Jdf(X_j(u))) = 0$ for all $i, j = 1, 2, 3$. Therefore f is a Lagrangian immersion. Moreover, we notice that

$$Pdf(X_i(u)) = -\tfrac{1}{2}df(X_i(u)) - \tfrac{\sqrt{3}}{2}Jdf(X_i(u)),$$

hence f is of diagonalizable type and the angle functions are constant and all are equal to $\frac{2}{3}\pi$.

In Lemma 3 we compute $\tilde{\nabla}_{X_i}X_j$ for $i, j = 1, 2, 3$, and we see that it is always tangent. Hence, the submanifold is totally geodesic.

Example 17. Consider the immersion of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$f : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \colon u \mapsto (u, \boldsymbol{iui}).$$

Recall that the split quaternions $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ anticommute. Then we may compute

$$df(X_1(u)) = (u\boldsymbol{i}, \boldsymbol{iuii})_{(u,\boldsymbol{iui})} = (u\boldsymbol{i}, \boldsymbol{iuii})_{(u,\boldsymbol{iui})},$$

$$df(X_2(u)) = (u\boldsymbol{j}, \boldsymbol{iuji})_{(u,\boldsymbol{iui})} = (u\boldsymbol{j}, \boldsymbol{iui}(-\boldsymbol{j}))_{(u,\boldsymbol{iui})},$$

$$df(X_3(u)) = (u\boldsymbol{k}, \boldsymbol{iuki})_{(u,\boldsymbol{iui})} = (u\boldsymbol{k}, \boldsymbol{iui}(-\boldsymbol{k}))_{(u,\boldsymbol{iui})}.$$

By the definition of J in (2.10) we get

$$Jdf(X_1(u)) = \frac{1}{\sqrt{3}}(-u\boldsymbol{i}, \boldsymbol{iuii})_{(u,\boldsymbol{iui})},$$

$$Jdf(X_2(u)) = \sqrt{3}(u\boldsymbol{j}, \boldsymbol{iuij})_{(u,\boldsymbol{iui})},$$

$$Jdf(X_3(u)) = \sqrt{3}(u\boldsymbol{k}, \boldsymbol{iuik})_{(u,\boldsymbol{iui})}.$$

Hence, we can easily compute $g(df(X_i(u)), Jdf(X_j(u))) = 0$ for $i, j = 1, 2, 3$, therefore f is a Lagrangian immersion. Moreover, we have

$$Pdf(X_1(u)) = (u\boldsymbol{i}, \boldsymbol{iuii})_{(u,\boldsymbol{iui})} = df(X_1(u)),$$

$$Pdf(X_2(u)) = (-u\boldsymbol{j}, \boldsymbol{iuij})_{(u,\boldsymbol{iui})} = -df(X_2(u)),$$

$$Pdf(X_3(u)) = (-u\boldsymbol{k}, \boldsymbol{iuik})_{(u,\boldsymbol{iui})} = -df(X_3(u)).$$

To obtain a Δ_1-orthonormal frame we have to reorder the vectors X_i to $\{X_3, X_2, X_1\}$. Consequently, the immersion f is of diagonalizable type with constant angle functions $(\theta_1, \theta_2, \theta_3) = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$.

In order to prove that f is totally geodesic, we compute $\tilde{\nabla}_X Y$ for X, Y tangent vector fields and we see that the result is tangent to the submanifold.

We only compute $\tilde{\nabla}_{X_1} X_2$, since the rest of the cases are similar. From Equations (2.11), (2.17) and (2.18) we obtain

$$\begin{aligned}
\tilde{\nabla}_{X_1} X_2 &= D_{X_1} D_{X_2} f - \frac{1}{2}\langle X_1, X_2\rangle f - \frac{1}{2}\langle QX_1, X_2\rangle Qf \\
&\quad - \frac{1}{2} J \circ (\mathrm{Id} + P) G(X_1, PX_2) \\
&= (u\boldsymbol{ij}, -\boldsymbol{i}u\boldsymbol{ij}) - \frac{1}{2}\langle (u\boldsymbol{i}, \boldsymbol{i}u\boldsymbol{ii}), (u\boldsymbol{j}, -\boldsymbol{i}u\boldsymbol{ij})\rangle (u, \boldsymbol{i}u\boldsymbol{i}) \\
&\quad - \frac{1}{2}\langle (-u\boldsymbol{i}, \boldsymbol{i}u\boldsymbol{ii}), (u\boldsymbol{j}, -\boldsymbol{i}u\boldsymbol{ij})\rangle (-u, \boldsymbol{i}u\boldsymbol{i}) \\
&\quad - \frac{1}{2} J \circ (\mathrm{Id} + P) G((u\boldsymbol{i}, \boldsymbol{i}u\boldsymbol{ii}), (-u\boldsymbol{j}, \boldsymbol{i}u\boldsymbol{ij})) \\
&= df(X_3) - \frac{1}{2} J \circ (\mathrm{Id} + P) G((u\boldsymbol{i}, \boldsymbol{i}u\boldsymbol{ii}), (-u\boldsymbol{j}, \boldsymbol{i}u\boldsymbol{ij})) \\
&= df(X_3) - \frac{12}{3\sqrt{3}} J(u\boldsymbol{k}, \boldsymbol{i}u\boldsymbol{ik}) \\
&= df(X_3) + \frac{4}{3} df(X_3).
\end{aligned}$$

Example 18. Consider the immersion of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (u, -\boldsymbol{k}u\boldsymbol{k}).$$

The minus sign is added such that $f(\mathrm{Id}_2) = (\mathrm{Id}_2, \mathrm{Id}_2)$.

We have

$$\begin{aligned}
df(X_1(u)) &= (u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{ik})_{(u,-\boldsymbol{k}u\boldsymbol{k})} = (u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{i}))_{(u,-\boldsymbol{k}u\boldsymbol{k})}, \\
df(X_2(u)) &= (u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{jk})_{(u,-\boldsymbol{k}u\boldsymbol{k})} = (u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{j}))_{(u,-\boldsymbol{k}u\boldsymbol{k})}, \\
df(X_3(u)) &= (u\boldsymbol{k}, -\boldsymbol{k}u\boldsymbol{kk})_{(u,-\boldsymbol{k}u\boldsymbol{k})}.
\end{aligned}$$

By definition of J in (2.10) we obtain

$$\begin{aligned}
Jdf(X_1(u)) &= \sqrt{3}(u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{ki})_{(u,-\boldsymbol{k}u\boldsymbol{k})}, \\
Jdf(X_2(u)) &= \sqrt{3}(u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{kj})_{(u,-\boldsymbol{k}u\boldsymbol{k})}, \\
Jdf(X_3(u)) &= \frac{1}{\sqrt{3}}(-u\boldsymbol{k}, -\boldsymbol{k}u\boldsymbol{kk})_{(u,-\boldsymbol{k}u\boldsymbol{k})}.
\end{aligned}$$

Hence we can easily compute $g(df(X_i(u)), Jdf(X_j(u))) = 0$ for $i, j = 1, 2, 3$, therefore f is a Lagrangian immersion. Moreover we have that

$$Pdf(X_1(u)) = (-u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{k}\boldsymbol{i})_{(u,-\boldsymbol{k}u\boldsymbol{k})} = -df(X_1(u)),$$

$$Pdf(X_2(u)) = (-u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{k}\boldsymbol{j})_{(u,-\boldsymbol{k}u\boldsymbol{k})} = -df(X_2(u)),$$

$$Pdf(X_3(u)) = (u\boldsymbol{k}, -\boldsymbol{k}u\boldsymbol{k}\boldsymbol{k})_{(u,-\boldsymbol{k}u\boldsymbol{k})} = df(X_3(u)).$$

Again, since the Δ_1-orthonormal frame is $\{X_3, X_2, X_1\}$, the immersion is of diagonalizable type and the angle functions are constant and equal to $(0, \frac{\pi}{2}, \frac{\pi}{2})$.

In order to prove that f is totally geodesic, we compute $\tilde{\nabla}_X Y$ for X, Y tangent vector fields and we see that the result is tangent to the submanifold.

We only compute $\tilde{\nabla}_{X_1} X_2$, since the rest of the cases are similar. From Equations (2.11), (2.17) and (2.18) we obtain

$$\begin{aligned}
\tilde{\nabla}_{X_1} X_2 &= D_{X_1} D_{X_2} f - \tfrac{1}{2}\langle X_1, X_2\rangle f - \frac{1}{2}\langle QX_1, X_2\rangle Qf \\
&\quad - \frac{1}{2} J \circ (\mathrm{Id} + P) G(X_1, PX_2) \\
&= (u\boldsymbol{i}\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{j})) - \frac{1}{2}\langle (u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{i})), (u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{j}))\rangle (u, -\boldsymbol{k}u\boldsymbol{k}) \\
&\quad - \frac{1}{2}\langle (-u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{i})), (u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{j}))\rangle (-u, -\boldsymbol{k}u\boldsymbol{k}) \\
&\quad - \frac{1}{2} J \circ (\mathrm{Id} + P) G((u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{i})), (-u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{k}\boldsymbol{j})) \\
&= df(X_3) - \frac{1}{2} J \circ (\mathrm{Id} + P) G((u\boldsymbol{i}, -\boldsymbol{k}u\boldsymbol{k}(-\boldsymbol{i})), (-u\boldsymbol{j}, -\boldsymbol{k}u\boldsymbol{k}\boldsymbol{j})) \\
&= df(X_3).
\end{aligned}$$

Remark 13. The immersion $f : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ given by $f : u \mapsto (u, \boldsymbol{j}u\boldsymbol{j})$ is a Lagrangian immersion congruent to Example 17. Indeed, take the isometry of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ given by $\phi : (p, q) \mapsto (cpc, -cqc)$, where c is the matrix in $\mathrm{SL}(2, \mathbb{R})$ given by $e^{\frac{\pi}{4}\boldsymbol{k}}$. This matrix satisfies that $c\boldsymbol{j} = \boldsymbol{i}c$ and $\boldsymbol{j}c = -c\boldsymbol{i}$. Hence

$$\phi \circ f(u) = (cuc, -c\boldsymbol{j}u\boldsymbol{j}c) = (v, \boldsymbol{i}v\boldsymbol{i}),$$

where $v = cuc$.

3.2.2 Lagrangian submanifolds of diagonalizable type

Proposition 17. *Let M be a Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of diagonalizable type. If the angle functions θ_i are constant and $h^3_{12} = 0$, then M is totally geodesic. Conversely, if the submanifold is totally geodesic then the angles are constant.*

Proof. By Lemma 12, if the angle functions are constant then $h^j_{ii} = 0$ for all $i, j = 1, 2, 3$. Using the symmetries of h^k_{ij} we conclude that the submanifold is totally geodesic. The converse is immediate by Lemma 12. □

Now notice that

$$\begin{aligned} -\delta_k E_k(h^i_{jj}) + \delta_i E_i(h^k_{jj}) &= \delta_k\delta_i\delta_j E_k(E_i(\theta_j)) - \delta_i\delta_k\delta_j E_i(E_k(\theta_j)) \\ &= \delta_k\delta_i\delta_j [E_k, E_i](\theta_j) \\ &= \delta_k\delta_i\delta_j (\nabla_{E_k} E_i - \nabla_{E_i} E_k)(\theta_j) \\ &= \delta_k\delta_i\delta_j \sum_l (\omega^l_{ki} - \omega^l_{ik}) E_l(\theta_j) \\ &= \delta_k\delta_i \sum_l \delta_l (\omega^l_{ik} - \omega^l_{ki}) h^l_{jj}. \end{aligned} \tag{3.33}$$

Proposition 18. *Let M be a Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of diagonalizable type. If two angles are equal modulo π, then M is totally geodesic.*

Proof. We assume that $\theta_1 = \theta_2 \bmod \pi$. From (3.24) we get $-h^i_{22} = h^i_{11}$ and $h^2_{i1} = 0$, for all i. Thus $h^1_{11} = h^2_{11} = h^1_{22} = h^2_{22} = h^3_{12} = 0$. By Proposition 16 the submanifold M is minimal, then $-h^i_{11} + h^i_{22} + h^i_{33} = 0$ for $i = 1, 2, 3$. Hence, h^1_{33} and h^2_{33} also vanish. The remaining components are related by $h^3_{11} = -h^3_{22}$ and $h^3_{33} = 2h^3_{11}$. Taking $i = 2, j = 1, k = 1$ in (3.33) we obtain $0 = (\omega^3_{21} - \omega^3_{12})h^3_{11}$. Computing both sides of the Codazzi equation in (3.2) with $X = E_1, Y = E_2, Z = E_2$ yields

$$0 = h^3_{11}(\sqrt{\tfrac{2}{3}} - 3\omega^3_{12} + \omega^3_{21}).$$

Suppose $h^3_{11} \neq 0$. Then $\omega^3_{12} = \omega^3_{21} = \frac{1}{\sqrt{6}}$ and hence by (3.25) we have $0 = (-\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}})\sin(\theta_1 - \theta_3)$, so $\theta_1 = \theta_3 \bmod \pi$. By (3.24) we have that $h^3_{11} = -h^3_{33} = 0$, which is a contradiction. Thus, h^3_{11} must be zero and therefore the

submanifold is totally geodesic. The proof is similar (up to signs) for a different choice of the pair of angles. □

Lemma 16. *Let $f : M \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be a Lagrangian immersion into the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, and assume that M is of diagonalizable type with Δ_1-orthonormal frame $\{E_1, E_2, E_3\}$. Let $\theta_1, \theta_2, \theta_3$ be their respective angle functions. Then the Lagrangian immersions $\Psi_{1,0} \circ f$, $\Psi_{1,4\pi/3} \circ f$, where $\Psi_{\kappa,\tau}$ are the isometries given in (2.19), are also of diagonalizable type and their respective angle functions are*

$$\theta_i^{(1)} = \pi - \theta_i, \qquad \theta_i^{(2)} = \frac{\pi}{3} - \theta_i.$$

Proof. The proofs of Theorem 3 and Theorem 4 of [6] can be replicated for this case, taking the adjugate matrix instead of the conjugate of a quaternion. □

Lemma 17. *Consider a totally geodesic Lagrangian submanifold of diagonalizable type of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. After a possible permutation of the angles, we have one of the following:*

1. $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})$,
2. $(2\theta_1, 2\theta_2, 2\theta_3) = (0, \pi, \pi)$,
3. $(2\theta_1, 2\theta_2, 2\theta_3) = (\pi, \pi, 0)$.

Proof. Taking $X = E_i$, $Y = E_j$ and $Z = E_j$ in the Codazzi equation yields $\sin(2(\theta_i - \theta_j)) = 0$, thus all pairs of angles $2\theta_i$ and $2\theta_j$ are equal up to a multiple of π. This, together with the fact that the sum of the angles is equal to zero modulo 2π, implies that $6\theta_i = 0$ modulo π for all i. Therefore, replacing E_2 with E_3 and E_3 with $-E_2$ if necessary, we obtain the following possibilities.

(1) $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})$,
(2) $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$,
(3) $(2\theta_1, 2\theta_2, 2\theta_3) = (0, 0, 0)$,
(4) $(2\theta_1, 2\theta_2, 2\theta_3) = (0, \pi, \pi)$,
(5) $(2\theta_1, 2\theta_2, 2\theta_3) = (\pi, \pi, 0)$,
(6) $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3})$,
(7) $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$,
(8) $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{2\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{3})$,
(9) $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{5\pi}{3}, \frac{5\pi}{3}, \frac{2\pi}{3})$.

Suppose that M is a Lagrangian submanifold of diagonalizable type with angles as in (3). Then by Lemma 16, applying the isometry $\Psi_{1,4\pi/3}$ in (2.19) produces a congruent Lagrangian submanifold with angles as in (1). We can apply a similar argument to see that cases (2), (3), (6), (7), (8) and (9) are congruent to one of the cases (1), (4) or (5). Hence we may restrict our attention to the latter cases. Notice that in cases (4) and (5) we cannot do a permutation with

E_1 since, unlike E_2 and E_3, this vector is timelike. Later on we will prove that these cases are not isometric. □

In Section 3.2.1 we proved that the immersions in Theorem 32 are Lagrangian and totally geodesic. To complete the proof of this theorem, it remains to be shown that any totally geodesic Lagrangian immersion is locally congruent to one of the Examples 16-18, depending on the possible angle functions in Lemma 17. This is what we prove in the upcoming propositions.

Proposition 19. *Let M be a totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of diagonalizable type. Assume that $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})$. Then M is locally congruent to the submanifold $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \colon u \mapsto (\mathrm{Id}_2, u)$.*

Proof. As M is of type I in Lemma 11, we have that $A = -\frac{1}{2}\,\mathrm{Id}$ and $B = -\frac{\sqrt{3}}{2}$ are multiples of the identity we have that $PX = -\frac{1}{2}X - \frac{\sqrt{3}}{2}JX$ for any vector field X tangent to M. It follows immediately that $QX = X$ where Q is the almost product structure given in (2.16). Using the Gauss equation we can compute the curvature tensor of M and also the sectional curvature, which is equal to $-\frac{3}{2}$. Then M is locally isometric to $\mathrm{SL}(2,\mathbb{R})$ equipped with the metric $\frac{2}{3}g_0$, where g_0 is the metric defined in (2.2). Now write $f = (p, q)$. By the definition of Q, we have that

$$(dp(v), 0) = \tfrac{1}{2}(df(v) - Qdf(v)) = 0, \quad (0, dq(v)) = \tfrac{1}{2}(df(v) + Qdf(v)) = df(v),$$

with $v \in T\mathrm{SL}(2,\mathbb{R})$. Hence p should be a constant matrix in $\mathrm{SL}(2,\mathbb{R})$. The previous equation also implies that dq is a linear isomorphism, then q is a local diffeomorphism. Therefore, we may assume that $q(u)$ is actually equal to u. Applying an isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we may assume that p is equal to Id_2. □

The immersion in Example 17 is the immersion of $\mathrm{SL}(2,\mathbb{R})$ with a Berger-like metric, stretched in the spacelike direction. We can construct such a metric by taking on $\mathrm{SL}(2,\mathbb{R})$ the frame $\{X_1, X_2, X_3\}$ given in (3.32), and the metric $g^+_{\kappa,\tau}$ given by

$$g^+_{\kappa,\tau}(X,Y) = \tfrac{4}{\kappa}\left(\langle X, Y\rangle + (\tfrac{4\tau^2}{\kappa} - 1)\langle X, X_1\rangle\langle Y, X_1\rangle\right),$$

where $\kappa > 0$ and τ are constants and $\langle , \rangle$ is the metric on $\mathrm{SL}(2,\mathbb{R})$ given in (2.2). It follows from a straightforward computation that

$$[X_1, X_2] = 2X_3, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = -2X_1.$$

We take the vector fields

$$\tilde{E}_1 = \tfrac{\sqrt{\kappa}}{2} X_3,\ \tilde{E}_2 = \tfrac{\sqrt{\kappa}}{2} X_2 \text{ and } \tilde{E}_3 = \tfrac{\kappa}{4\tau} X_1, \tag{3.34}$$

which form a Δ_1-orthonormal frame on $\mathrm{SL}(2,\mathbb{R})$ with respect to the metric g_+. We denote the Levi-Civita connection associated to $g^+_{\kappa,\tau}$ by ∇^+. It follows from the Koszul formula that $\nabla^+_{\tilde{E}_i} \tilde{E}_i = 0$ and that

$$\begin{aligned}
&\nabla^+_{\tilde{E}_1} \tilde{E}_2 = -\tau \tilde{E}_3, && \nabla^+_{\tilde{E}_2} \tilde{E}_3 = \tau \tilde{E}_1,\\
&\nabla^+_{\tilde{E}_1} \tilde{E}_3 = \tau \tilde{E}_2, && \nabla^+_{\tilde{E}_3} \tilde{E}_1 = (\tau - \frac{\kappa}{2\tau}) \tilde{E}_2,\\
&\nabla^+_{\tilde{E}_2} \tilde{E}_1 = \tau \tilde{E}_3, && \nabla^+_{\tilde{E}_3} \tilde{E}_2 = (\tau - \frac{\kappa}{2\tau}) \tilde{E}_1.
\end{aligned}$$

The following result is a corollary of Theorem 19.

Proposition 20. *Let (M^n, g) and $(N^n, \widehat{g})$ be pseudo-Riemannian manifolds with Levi-Civita connections ∇ and $\widehat{\nabla}$, respectively. Suppose that there exist constants c^k_{ij}, $i,j,k \in 1,\dots,n$ such that for all $p \in M$ and $p' \in N$ there exist pseudo-orthonormal frames $\{E_1,\dots,E_n\}$ around p, $\{F_1,\dots,F_n\}$ around p' with the same signatures such that $\nabla_{E_i} E_j = \sum_{i=1}^n c^k_{ij} E_k$, $\widehat{\nabla}_{F_i} F_j = \sum_{i=1}^n c^k_{ij} F_k$. Then there exists a local isometry that maps a neighborhood of p into a neighborhood of $\tilde{p}$ and maps $\{E_i\}$ to $\{F_i\}$.*

Proof. Let $p \in N$ and $p' \in \tilde{N}$, and let $\varphi\colon T_pN \to T_{p'}\tilde{N}$ be the linear isomorphism given by $\varphi((E_i)_p) = (F_i)_{p'}$. Let U be a normal neighborhood around p and let f be the diffeomorphism given in (1.1). Let $q \in U$ and denote by $\tilde{E}_i(t)$ and $\tilde{F}_i(t)$ the parallel transports of $(E_i)_p$ and $(F_i)_{p'}$ along the geodesics γ and $\widehat{\gamma}$ that join p with q and p' with $f(q)$, respectively. Note that $\widehat{\gamma}'(0) = \varphi(\gamma'(0))$.

We write

$$\begin{aligned}
\gamma'(t) &= \sum_j a_j(t) E_j, && \widehat{\gamma}'(t) = \sum_j b_j(t) F_j,\\
\tilde{E}_i(t) &= \sum_j x^i_j(t) E_j, && \tilde{F}_i(t) = \sum_j y^i_j(t) F_j,
\end{aligned}$$

with a_j, b_j, x_j^i and y_j^i smooth functions that satisfy $a_j(0) = b_j(0)$, $x_j^i(0) = y_j^i(0) = \delta_{ij}$, for all $i, j = 1, \ldots, n$. We compute

$$0 = \nabla_{\frac{\partial}{\partial t}} \gamma'(t) = \sum_j \tfrac{\partial}{\partial t}(a_j(t))E_j + a_j(t)\nabla_{\sum_k a_k(t)E_k} E_j$$

$$= \sum_{j,k,l} \tfrac{\partial}{\partial t}(a_j(t))E_j + a_j(t)a_k(t)c_{kj}^l E_l.$$

This is a first order system of differential equations, with initial conditions $a_i(0)$. By doing the same computation for $\widehat{\gamma}'$ we show that the functions b_j are solutions of the same system, and therefore $a_j(t) = b_j(t)$. We apply the same reasoning for $\tilde{E}_i$ and $\tilde{F}_i$, obtaining $x_j^i(t) = y_j^i(t)$.

Now we compare R with $\widehat{R}$, the curvature tensors of M and N respectively.

$$g(R(\tilde{E}_i, \tilde{E}_j)\tilde{E}_k, \tilde{E}_l) = \sum_{\alpha,\beta,\kappa,\delta} x_\alpha^i x_\beta^j x_\kappa^k x_\delta^l g(R(E_\alpha, E_\beta)E_\kappa, E_\delta)$$

$$= \sum_{\alpha,\beta,\kappa,\delta,s} \eta_\delta x_\alpha^i x_\beta^j x_\kappa^k x_\delta^l (c_{\beta\kappa}^s c_{\alpha s}^\delta - c_{\alpha\kappa}^s c_{\beta s}^\delta - c_{s\kappa}^\delta (c_{\alpha\beta}^s - c_{\beta\alpha}^s))$$

where $\eta_\delta = g(E_\delta, E_\delta)$. On the other hand computing $\widehat{g}(\widehat{R}(\tilde{F}_i, \tilde{F}_j)\tilde{F}_k, \tilde{F}_l)$ yields the same result. Therefore, by Corollary 1, f is a local isometry.

Finally, we write $E_i = \sum_j \bar{x}_j^i \tilde{E}_j$, where $\bar{x}_j^i$ are the entries of the inverse of the matrix $(x_j^i)_{ij}$. We can also see that $F_i = \sum_j \bar{x}_j^i \tilde{F}_j$. Then, by Theorem 19 we have

$$df_q(E_i) = \sum_j \bar{x}_j^i \tau' \circ \varphi \circ \tau^{-1}(\tilde{E}_i) = \sum_j \bar{x}_j^i \tilde{F}_j = F_i.$$

□

We will use this proposition to show that cases (2) and (3) of Lemma 17 are locally $\mathrm{SL}(2,\mathbb{R})$ with Berger-like metrics.

Proposition 21. *Let M be a totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, of diagonalizable type. Assume that $(2\theta_1, 2\theta_2, 2\theta_3) = (\pi, \pi, 0)$. Then M is congruent to possibly an open subset of the submanifold $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\colon u \mapsto (u, \boldsymbol{iui})$.*

Proof. Let $\{E_1, E_2, E_3\}$ be a Δ_1-orthonormal frame such that $JG(E_1, E_2) = \sqrt{\frac{2}{3}} E_3$ and A and B take the form of type I in Lemma 11 with $(2\theta_1, 2\theta_2, 2\theta_3) =$

$(\pi, \pi, 0)$. Thus

$$PE_1 = -E_1, \quad PE_2 = -E_2, \quad PE_3 = E_3. \tag{3.35}$$

From Lemma 12 we obtain that $\omega^j_{i3} = 0$ for $i, j = 1, 2, 3$ with the exceptions of $i = 1$, $j = 2$ and $i = 2$, $j = 1$. We also get

$$\omega^3_{12} = \frac{1}{\sqrt{6}}, \quad \omega^3_{21} = -\frac{1}{\sqrt{6}}.$$

From the equation of Gauss in (3.1) we obtain the following equations:

$$\begin{aligned} E_1(\omega^1_{32}) - E_3(\omega^2_{11}) + \left(\omega^1_{32} + \frac{1}{\sqrt{6}}\right)\omega^1_{22} = 0, \\ E_2(\omega^1_{32}) - E_3(\omega^2_{11}) + \left(\omega^1_{32} + \frac{1}{\sqrt{6}}\right)\omega^2_{11} = 0, \\ E_1(\omega^1_{22}) - E_2(\omega^2_{11}) - \sqrt{\frac{2}{3}}\omega^1_{32} - (\omega^2_{11})^2 + (\omega^1_{22})^2 + \frac{5}{3} = 0. \end{aligned} \tag{3.36}$$

Define the 1-form ω by

$$\omega(E_1) = -\omega^2_{11}, \quad \omega(E_2) = -\omega^1_{22}, \quad \omega(E_3) = \omega^1_{32} + \frac{5}{\sqrt{6}}.$$

Using (3.36) we can prove that ω is closed. Hence there exists a local function φ such that $d\varphi = \omega$. Now define the new frame

$$F_1 = \cosh\varphi E_1 + \sinh\varphi, \quad F_2 = \sinh\varphi E_2 + \cosh\varphi E_3, \quad F_3 = -E_3.$$

This new frame is still Δ_1-orthonormal which satisfies (3.35) and $JG(F_1, F_2) = \sqrt{\frac{2}{3}}F_3$. We have

$$\begin{aligned} &\nabla_{F_1}F_1 = 0 && \nabla_{F_1}F_2 = -\frac{1}{\sqrt{6}}F_3, && \nabla_{F_1}F_3 = \frac{1}{\sqrt{6}}F_2, \\ &\nabla_{F_2}F_1 = \frac{1}{\sqrt{6}}F_3, && \nabla_{F_2}F_2 = 0, && \nabla_{F_2}F_3 = \frac{1}{\sqrt{6}}F_1, \\ &\nabla_{F_3}F_1 = -\frac{5}{\sqrt{6}}F_2, && \nabla_{F_3}F_2 = -\frac{5}{\sqrt{6}}F_1, && \nabla_{F_3}F_3 = 0. \end{aligned}$$

By Proposition 20 we have that M is locally isometric to $\mathrm{SL}(2, \mathbb{R})$ with a Berger-like metric stretched in the spacelike direction with $\tau = \frac{1}{\sqrt{6}}$ and $\kappa = 2$.

Now using (2.17) we may write

$$\begin{aligned}
&\nabla^E_{F_1} F_1 = 0, && \nabla^E_{F_1} F_2 = -\frac{1}{\sqrt{6}} F_3, && \nabla^E_{F_1} F_3 = \sqrt{\frac{3}{2}} F_2,\\
&\nabla^E_{F_2} F_1 = \frac{1}{\sqrt{6}} F_3, && \nabla^E_{F_2} F_2 = 0, && \nabla^E_{F_2} F_3 = \sqrt{\frac{3}{2}} F_1,\\
&\nabla^E_{F_3} F_1 = -\sqrt{\frac{3}{2}} F_2, && \nabla^E_{F_3} F_2 = -\sqrt{\frac{3}{2}} F_1, && \nabla^E_{F_3} F_3 = 0.
\end{aligned} \tag{3.37}$$

where ∇^E is the Levi-Civita connection associated to the product metric. We can identify the frame $\{F_i\}_i$ on $\mathrm{SL}(2,\mathbb{R})$ with the frame given in (3.34), i.e.,

$$F_1 = \frac{1}{\sqrt{2}} X_3, \quad F_2 = \frac{1}{\sqrt{2}} X_2, \quad F_3 = \sqrt{\frac{3}{2}} X_1. \tag{3.38}$$

Now writing the immersion $f = (p, q)$ and $df(F_i) = D_{F_i} f = (p\alpha_i, q\beta_i)$, where α_i, β_i are matrices in $\mathfrak{sl}(2,\mathbb{R})$, we obtain

$$\beta_1 = -\alpha_1, \qquad \beta_2 = -\alpha_2, \qquad \beta_3 = \alpha_3. \tag{3.39}$$

because of Equation (3.35).

It follows from (2.15) that α_i are mutually orthogonal and their lengths are given by

$$-\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = \frac{1}{2}, \quad \langle \alpha_3, \alpha_3 \rangle = \frac{3}{2}.$$

From the properties of the Lorentzian cross product we deduce that

$$\alpha_1 \times \alpha_2 = \varepsilon \frac{1}{\sqrt{6}} \alpha_3, \quad \alpha_2 \times \alpha_3 = -\varepsilon \sqrt{\frac{3}{2}} \alpha_1, \qquad \alpha_3 \times \alpha_1 = \varepsilon \sqrt{\frac{3}{2}} \alpha_2,$$

where $\varepsilon = \pm 1$. We compute

$$D_{F_i} D_{F_j} f = (p\alpha_i \times \alpha_j + \langle \alpha_i, \alpha_j \rangle p + p F_i(\alpha_j),\ q\beta_i \times \beta_j + \langle \beta_i, \beta_j \rangle q + q F_i(\beta_j)),$$

where

$$F(\alpha) = F(\langle \alpha, \boldsymbol{i} \rangle)\boldsymbol{i} + F(\langle \alpha, \boldsymbol{j} \rangle)\boldsymbol{j} - F(\langle \alpha, \boldsymbol{k} \rangle)\boldsymbol{k}.$$

Applying (2.18) it follows that

$$\nabla^E_{F_i} F_j = (p\alpha_i \times \alpha_j + p F_i(\alpha_j),\ q\beta_i \times \beta_j + q F_i(\beta_j)).$$

Comparing the above equation with (3.37) and recalling that $F_i(p,q) = (p\alpha_i, q\beta_i)$ we obtain

$$F_1(\alpha_1) = 0, \qquad F_2(\alpha_1) = \tfrac{1}{\sqrt{6}}(1+\varepsilon)\alpha_3, \qquad F_3(\alpha_1) = -\sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_2,$$

$$F_1(\alpha_2) = -\tfrac{1}{\sqrt{6}}(1+\varepsilon)\alpha_3, \quad F_2(\alpha_2) = 0, \qquad F_3(\alpha_2) = -\sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_1,$$

$$F_1(\alpha_3) = \sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_2, \qquad F_2(\alpha_3) = \sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_1, \quad F_3(\alpha_3) = 0.$$

Making use of (3.38) yields

$$X_3(\alpha_1) = 0, \qquad X_2(\alpha_1) = \tfrac{1}{\sqrt{3}}(1+\varepsilon)\alpha_3, \quad X_1(\alpha_1) = -\sqrt{3}(1+\varepsilon)\alpha_2,$$

$$X_3(\alpha_2) = -\tfrac{1}{\sqrt{3}}(1+\varepsilon)\alpha_3, \quad X_2(\alpha_2) = 0, \qquad X_1(\alpha_2) = -(1+\varepsilon)\alpha_1,$$

$$X_3(\alpha_3) = \sqrt{3}(1+\varepsilon)\alpha_2, \qquad X_2(\alpha_3) = \sqrt{3}(1+\varepsilon)\alpha_1, \quad X_1(\alpha_3) = 0.$$

We can write the same equations for β_i. Taking into account (3.39), we conclude ε must be equal to -1. Therefore α_i is constant for all i. Hence, there exists an isometry on $\mathrm{SL}(2,\mathbb{R})$, namely conjugation by a matrix c in $\mathrm{SL}^{\pm}(2,\mathbb{R})$, such that

$$\alpha_1 = \frac{1}{\sqrt{2}} c\boldsymbol{k}c^{-1}, \qquad \alpha_2 = \frac{1}{\sqrt{2}} c\boldsymbol{j}c^{-1}, \qquad \alpha_3 = \sqrt{\frac{3}{2}} c\boldsymbol{i}c^{-1}.$$

We have that all totally geodesic Lagrangian submanifolds with angle functions $(\pi, \pi, 0)$ satisfy this differential equation. Suppose that $f(\mathrm{Id}) = (p_o, q_o)$, choosing an isometry $(p,q) \mapsto (p_o^{-1}p, q_o^{-1}q)$ we may assume that $f(\mathrm{Id}_2) = (\mathrm{Id}_2, \mathrm{Id}_2)$. We obtain that the unique solution of the system $D_{F_i} f = (p\alpha_i, q\beta_i)$ is $p(u) = cuc^{-1}$, $q(u) = c\boldsymbol{i}u\boldsymbol{i}c^{-1}$. Taking an isometry on the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, given in Theorem 28, we see that f is congruent to $(u, \boldsymbol{i}u\boldsymbol{i})$. □

The third example of a totally geodesic immersion is locally isometric to $\mathrm{SL}(2,\mathbb{R})$ with a Berger-like metric, which is stretched in the direction of a timelike component. We can construct such a metric by taking on $\mathrm{SL}(2,\mathbb{R})$ the frame $\{X_1, X_2, X_3\}$ given in (3.32), and the metric $g^-_{\kappa,\tau}$ given by

$$g^-_{\kappa,\tau}(X,Y) = \tfrac{4}{\kappa}\left(\langle X, Y\rangle + (1 - \tfrac{4\tau^2}{\kappa})\langle X, X_3\rangle\langle Y, X_3\rangle\right),$$

where κ, τ are constants and $\langle , \rangle$ is the metric on $\mathrm{SL}(2,\mathbb{R})$ given in (2.2). It follows from a straightforward computation that

$$[X_1, X_2] = 2X_3, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = -2X_1.$$

We take the vector fields

$$\tilde{E}_1 = \tfrac{\kappa}{4\tau} X_3, \tilde{E}_2 = \tfrac{\sqrt{\kappa}}{2} X_2 \tag{3.40}$$

and $\tilde{E}_3 = \frac{\sqrt{\kappa}}{2} X_1$, which form a pseudo-orthonormal frame on $\mathrm{SL}(2,\mathbb{R})$ with respect to the metric $\tilde{g}$. We denote ∇^- as the Levi-Civita connection associated to $\tilde{g}$. It follows from the Koszul formula that $\nabla^-_{\tilde{E}_i} \tilde{E}_i = 0$ and that

$$\begin{aligned}
&\nabla^-_{\tilde{E}_1} \tilde{E}_2 = (\tau - \frac{\kappa}{2\tau}) \tilde{E}_3, && \nabla^-_{\tilde{E}_2} \tilde{E}_3 = \tau \tilde{E}_1, \\
&\nabla^-_{\tilde{E}_1} \tilde{E}_3 = (\frac{\kappa}{2\tau} - \tau) \tilde{E}_2, && \nabla^-_{\tilde{E}_3} \tilde{E}_1 = -\tau \tilde{E}_2, \\
&\nabla^-_{\tilde{E}_2} \tilde{E}_1 = \tau \tilde{E}_3, && \nabla^-_{\tilde{E}_3} \tilde{E}_2 = -\tau \tilde{E}_1.
\end{aligned}$$

Proposition 22. *Let M be a totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Assume that $(2\theta_1, 2\theta_2, 2\theta_3) = (\pi, \pi, 0)$. Then M is locally isometric to the submanifold $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$: $u \mapsto (u, -\boldsymbol{k}u\boldsymbol{k})$.*

Proof. Let $\{E_1, E_2, E_3\}$ be a Δ_1-orthonormal frame such that $JG(E_1, E_2) = \sqrt{\frac{2}{3}} E_3$ and A and B take the form of type I in Lemma 11 with $(2\theta_1, 2\theta_2, 2\theta_3) = (0, \pi, \pi)$. Thus

$$PE_1 = E_1, \quad PE_2 = -E_2, \quad PE_3 = -E_3. \tag{3.41}$$

From Lemma 12 we obtain that $\omega^j_{i1} = 0$ for $i, j = 1, 2, 3$. We also get

$$\omega^3_{21} = -\frac{1}{\sqrt{6}}, \quad \omega^1_{32} = \frac{1}{\sqrt{6}}.$$

From the Gauss Equation (3.1) we obtain the following equations:

$$\begin{aligned}
E_1(\omega^3_{22}) - E_2(\omega^3_{12}) + \left(\omega^3_{12} + \frac{1}{\sqrt{6}}\right) \omega^2_{33} = 0, \\
-E_1(\omega^2_{33}) - E_3(\omega^3_{12}) + \left(\omega^3_{12} + \frac{1}{\sqrt{6}}\right) \omega^3_{22} = 0, \\
E_2(\omega^2_{33}) + E_3(\omega^3_{22}) - \sqrt{\frac{2}{3}} \omega^3_{12} - (\omega^3_{22})^2 - (\omega^2_{33})^2 + \frac{5}{3} = 0.
\end{aligned} \tag{3.42}$$

Define the 1-form ω by

$$\omega(E_1) = -\omega^3_{12} + \frac{5}{\sqrt{6}}, \quad \omega(E_2) = -\omega^3_{22}, \quad \omega(E_3) = \omega^2_{33}.$$

Using (3.42) we can prove that ω is closed. Hence there exists a local function φ such that $d\varphi = \omega$. Now define the new frame

$$F_1 = -E_1, \qquad F_2 = -\cos\varphi E_2 - \sin\varphi E_3, \qquad F_3 = \sin\varphi E_2 - \cos\varphi E_3.$$

This new frame is still Δ_1-orthonormal which satisfies (3.41) and $JG(F_1, F_2) = \sqrt{\frac{2}{3}}F_3$. We have

$$\begin{aligned}
&\nabla_{F_1}F_1 = 0 && \nabla_{F_1}F_2 = -\frac{5}{\sqrt{6}}F_3, && \nabla_{F_1}F_3 = \frac{5}{\sqrt{6}}F_2,\\
&\nabla_{F_2}F_1 = \frac{1}{\sqrt{6}}F_3, && \nabla_{F_2}F_2 = 0, && \nabla_{F_2}F_3 = \frac{1}{\sqrt{6}}F_1,\\
&\nabla_{F_3}F_1 = -\frac{1}{\sqrt{6}}F_2, && \nabla_{F_3}F_2 = -\frac{1}{\sqrt{6}}F_1, && \nabla_{F_3}F_3 = 0.
\end{aligned}$$

By Proposition 20 we have that M is locally isometric to $\mathrm{SL}(2,\mathbb{R})$ with a Berger-like metric stretched in the timelike direction with $\tau = \frac{1}{\sqrt{6}}$ and $\kappa = 2$. Now using (2.17) we may write

$$\begin{aligned}
&\nabla^E_{F_1}F_1 = 0, && \nabla^E_{F_1}F_2 = -\sqrt{\frac{3}{2}}F_3, && \nabla^E_{F_1}F_3 = \sqrt{\frac{3}{2}}F_2,\\
&\nabla^E_{F_2}F_1 = \sqrt{\frac{3}{2}}F_3, && \nabla^E_{F_2}F_2 = 0, && \nabla^E_{F_2}F_3 = \frac{1}{\sqrt{6}}F_1,\\
&\nabla^E_{F_3}F_1 = -\sqrt{\frac{3}{2}}F_2, && \nabla^E_{F_3}F_2 = -\frac{1}{\sqrt{6}}F_1, && \nabla^E_{F_3}F_3 = 0.
\end{aligned} \tag{3.43}$$

where ∇^E is the Levi-Civita connection associated to the product metric. We can identify the frame $\{F_i\}_i$ on $\mathrm{SL}(2,\mathbb{R})$ with the frame given in (3.40), i.e.,

$$F_1 = \sqrt{\frac{3}{2}}X_3, \qquad F_2 = \frac{1}{\sqrt{2}}X_2, \qquad F_3 = \frac{1}{\sqrt{2}}X_1. \tag{3.44}$$

Now writing the immersion $f = (p, q)$ and $df(F_i) = D_{F_i}f = (p\alpha_i, q\beta_i)$, where α_i, β_i are matrices in $\mathfrak{sl}(2,\mathbb{R})$, we obtain

$$\beta_1 = \alpha_1, \qquad \beta_2 = -\alpha_2, \qquad \beta_3 = -\alpha_3. \tag{3.45}$$

because of Equation (3.41).

It follows from (2.15) that α_i are mutually orthogonal and their lengths are given by

$$\langle\alpha_1, \alpha_1\rangle = -\frac{3}{2}, \qquad \langle\alpha_2, \alpha_2\rangle = \langle\alpha_3, \alpha_3\rangle = \frac{1}{2}.$$

Thus we can write the Lorentzian cross products as

$$\alpha_1\times\alpha_2=\varepsilon\sqrt{\frac{3}{2}}\alpha_3,\quad \alpha_2\times\alpha_3=-\varepsilon\frac{1}{\sqrt{6}}\alpha_1,\qquad \alpha_3\times\alpha_1=\varepsilon\sqrt{\frac{3}{2}}\alpha_2,$$

where $\varepsilon=\pm1$. We compute

$$D_{F_i}D_{F_j}f=(p\alpha_i\times\alpha_j+\langle\alpha_i,\alpha_j\rangle p+pF_i(\alpha_j),\ q\beta_i\times\beta_j+\langle\beta_i,\beta_j\rangle q+qF_i(\beta_j)),$$

therefore applying (2.18) it follows that

$$\nabla^E_{F_i}F_j=(p\alpha_i\times\alpha_j+pF_i(\alpha_j),\ q\beta_i\times\beta_j+qF_i(\beta_j)).$$

Comparing the above equation with (3.43) we obtain

$$F_1(\alpha_1)=0,\qquad F_2(\alpha_1)=\sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_3,\quad F_3(\alpha_1)=-\sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_3,$$

$$F_1(\alpha_2)=-\sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_3,\quad F_2(\alpha_2)=0,\qquad F_3(\alpha_2)=-\tfrac{1}{\sqrt{6}}(1+\varepsilon)\alpha_1,$$

$$F_1(\alpha_3)=\sqrt{\tfrac{3}{2}}(1+\varepsilon)\alpha_2,\qquad F_2(\alpha_3)=\tfrac{1}{\sqrt{6}}(1+\varepsilon)\alpha_1,\quad F_3(\alpha_3)=0.$$

Making use of (3.44) yields

$$X_3(\alpha_1)=0,\qquad X_2(\alpha_1)=\sqrt{3}(1+\varepsilon)\alpha_3,\quad X_1(\alpha_1)=-\sqrt{3}(1+\varepsilon)\alpha_3,$$

$$X_3(\alpha_2)=(1+\varepsilon)\alpha_3,\qquad X_2(\alpha_2)=0,\qquad X_1(\alpha_2)=-\tfrac{1}{\sqrt{3}}(1+\varepsilon)\alpha_1,$$

$$X_3(\alpha_3)=-(1+\varepsilon)\alpha_2,\quad X_2(\alpha_3)=\tfrac{1}{\sqrt{3}}(1+\varepsilon)\alpha_1,\quad X_1(\alpha_3)=0.$$

We can write the same equations for β_i. Taking into account (3.45), we conclude ε must be equal to -1. Therefore α_i is constant for all i. Hence, there exist an isometry of $\mathrm{SL}(2,\mathbb{R})$, namely conjugation by a matrix c in $\mathrm{SL}^{\pm}(2,\mathbb{R})$, such that

$$\alpha_1=\sqrt{\frac{3}{2}}c\boldsymbol{k}c^{-1},\qquad \alpha_2=\frac{1}{\sqrt{2}}c\boldsymbol{j}c^{-1},\qquad \alpha_3=\frac{1}{\sqrt{2}}c\boldsymbol{i}c^{-1}.$$

By applying an isometry of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ we may assume that $f(\mathrm{Id}_2)=(\mathrm{Id}_2,\mathrm{Id}_2)$. We obtain that the unique solution of the system $D_{F_i}f=(p\alpha_i,q\beta_i)$ is $p(u)=cuc^{-1}$, $q(u)=-c\boldsymbol{k}u\boldsymbol{k}c^{-1}$. As the map $(p,q)\mapsto(cpc^{-1},cqc^{-1})$ is an isometry for all $c\in\mathrm{SL}^{\pm}(2,\mathbb{R})$ we obtain that f is congruent to $(p(u),q(u))$. □

3.2.3 Lagrangian submanifolds of the non-diagonalizable types

In this subsection we show that there do not exist totally geodesic Lagrangian submanifolds in types II, III and IV of Lemma 11.

Lemma 18. *There are no totally geodesic Lagrangian immersions into the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *corresponding to type* II *of Lemma* 11.

Proof. Suppose that M is a totally geodesic Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ associated to type II of Lemma 11. The left hand side of the Codazzi equation in (3.2) for a totally geodesic submanifold is always zero. Computing the right hand side for $X = E_1$, $Y = E_2$ and $Z = E_2$ yields

$$-\frac{4}{3}(\sin 2\theta_1 + \cos 2\theta_1 \cot 2\theta_1) JE_1$$

which cannot be zero, therefore a contradiction. □

Lemma 19. *There are no totally geodesic Lagrangian immersions into the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *corresponding to type* III *of Lemma* 11.

Proof. Suppose that M is a totally geodesic Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ associated to type III of Lemma 11. As in the previous lemma, the left hand side of the Codazzi equation is zero. The component in the direction of JE_1 of the right hand side of the Codazzi equation with $X = E_1, Y = E_2, Z = E_2$ is $\pm\frac{8}{9\sqrt{3}}$, which is a contradiction. □

Lemma 20. *There are no totally geodesic Lagrangian immersions into the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *corresponding to type* IV *of Lemma* 11.

Proof. Suppose that M is a totally geodesic Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ associated to type IV of Lemma 11. The left hand side of the Codazzi equation is always zero for totally geodesic submanifolds and the right hand side of the Codazzi equation for $X = E_1, Y = E_2, Z = E_2$ is

$$\frac{2}{3}\sinh(2\psi)\cos(2\theta_1 + \theta_2) JE_2$$

which is not zero since ψ must be different from zero and $2\theta_1 + \theta_2$ is equal to zero modulo π. □

3.2.4 Proof of the Theorem 32

Proof of Theorem 32. In Examples 16, 17 and 18 we showed that the three immersions of Theorem 32 are totally geodesic and Lagrangian.

Let M be a totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. By Lemma 11, there are four cases to consider. In Lemmas 18, 19 and 20 we proved that there are no totally geodesic Lagrangian submanifolds in types II, III and IV of Lemma 11.

In Lemma 17 we have seen that any totally geodesic Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of diagonalizable type is congruent to a submanifold with angle functions $(\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})$, $(\pi, \pi, 0)$ or $(0, \pi, \pi)$. In Proposition 19, we showed that all totally geodesic Lagrangian submanifolds with angle functions $(\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})$ are locally congruent. The totally geodesic Lagrangian submanifolds with angle functions $(0, \pi, \pi)$ or $(\pi, \pi, 0)$ are classified up to congruence by Propositions 21 and 22. □

3.3 Extrinsically homogeneous Lagrangian submanifolds

In this section we study and provide a classification result of extrinsically homogeneous Lagrangian submanifolds of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Recall that a submanifold M is said to be extrinsically homogeneous if there exists a Lie subgroup H of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ such that $M = H \cdot (p,q)$. Note that we can choose (p,q) to be $(\mathrm{Id}_2, \mathrm{Id}_2)$ since conjugated subgroups have congruent orbits through different points. Namely, if g is an isometry in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ that maps (p,q) to $(\mathrm{Id}_2, \mathrm{Id}_2)$, we have

$$g(H \cdot (p,q)) = gHg^{-1} \cdot (g \cdot (p,q)) = gHg^{-1} \cdot (\mathrm{Id}_2, \mathrm{Id}_2).$$

Theorem 33. *Let $f : (M,g) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be an extrinsically homogeneous Lagrangian immersion into the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Then $f(M)$ is congruent to an open subset of the image of one of the following embeddings, whose image is the orbit of $(\mathrm{Id}_2, \mathrm{Id}_2)$ by $H \subset \mathrm{Iso}_o(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$:*

(M,g)	f	H	Remarks
$(\mathrm{SL}(2,\mathbb{R}), \frac{2}{3}\langle,\rangle)$	$u \mapsto (u,u)$	$\mathrm{SL}(2,\mathbb{R})$	Totally geodesic $K = -\frac{3}{2}$
$(\mathrm{SL}(2,\mathbb{R}), g^+_{\kappa,\tau})$	$u \mapsto (u, \boldsymbol{i}u\boldsymbol{i})$	$\mathrm{SL}(2,\mathbb{R})$	Totally geodesic
$(\mathrm{SL}(2,\mathbb{R}), g^-_{\kappa,\tau})$	$u \mapsto (u, -\boldsymbol{k}u\boldsymbol{k})$	$\mathrm{SL}(2,\mathbb{R})$	Totally geodesic
$(\mathrm{PSL}(2,\mathbb{R}), \frac{8}{3}\langle,\rangle)$	$[u] \mapsto (\boldsymbol{i}u\boldsymbol{i}u^{-1}, \boldsymbol{j}u\boldsymbol{j}u^{-1})$	$\mathrm{SL}(2,\mathbb{R})$	$K = -\frac{3}{8}$
$\mathbb{R}^3_1/\mathbb{Z}$	$(u,v,w) \mapsto$ $(e^{v\boldsymbol{i}}e^{-u\boldsymbol{k}}, e^{w\boldsymbol{j}}e^{-u\boldsymbol{k}})$	$\mathbb{R}^2 \times \mathbb{S}^1$	$K = 0$
$(\mathbb{R}^3, \hat{g})$	ι	$\mathbb{R} \ltimes_{\varphi_o} \mathbb{R}^2$	$K = -\frac{3}{2}$
$(\mathbb{R}^3/H_\lambda, g_\lambda)$	f_λ	$\frac{\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2}{H_\lambda}$	$K = -\frac{3}{2}$
$(\mathbb{R}^3, \tilde{g})$	$\jmath$	$\mathbb{R} \ltimes_{\varphi_2} \mathbb{R}^2$	

where H acts on $f(M)$ always with trivial isotropy except for $\mathrm{PSL}(2,\mathbb{R})$, where it acts with isotropy $\mathbb{Z}_2$. Here K is the sectional curvature of $f(M)$ and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the matrices

$$\boldsymbol{i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{k} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore, $\langle , \rangle$ is the metric given in (2.2), $g^+_{\kappa,\tau}$ and $g^-_{\kappa,\tau}$ are Berger-like metrics stretched in a spacelike and timelike direction, respectively; $\hat{g}$, ι and φ_o are given in Example 21; g_λ, f_λ, H_λ and φ_1 are given in Example 22; $\tilde{g}$, $\jmath$ and φ_2 are given in Example 23.

Conversely, the maps listed in the table above are extrinsically homogeneous Lagrangian submanifolds of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. *Moreover, all immersions are not congruent to each other, including the different immersions of the family f_λ.*

In Subsection 3.3.1 we study the frame $\{E_i\}_{i=1,\dots,3}$ introduced in Lemma 11 for extrinsically homogeneous submanifolds. In Subsections 3.3.2, 3.3.3 and 3.3.4 we classify the extrinsically homogeneous Lagrangian submanifolds of types I, II and III respectively. In Subsection 3.3.5 we show that there are no extrinsically homogeneous Lagrangian submanifolds of type IV. Finally, in Subsection 3.3.6 we prove Theorem 33.

3.3.1 The uniqueness of the frames

In this subsection we first prove that for each case of Lemma 11 there is a unique frame $\{E_i\}_i$ with respect to which P takes that particular shape on an extrinsically homogeneous Lagrangian submanifold. Consequently, the associated angle functions, h^k_{ij} and ω^k_{ij} are constant. Afterwards, we describe the examples given in Theorem 33 and provide a classification for each type of Lagrangian submanifold.

We consider each case of Lemma 11 separately.

Lagrangian submanifolds of type I

It is straightforward to check that for type I Lagrangian submanifolds the frame $\{E_1, E_2, E_3\}$ is unique if and only if the functions θ_i are all different modulo π. Later on, we will see that if two of them are equal, the submanifold is totally geodesic. Hence, now we focus on the case where all angles are different modulo π.

Proposition 23. *Let M be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. *Suppose that $\{E_1, E_2, E_3\}$ is the unique Δ_1-orthonormal frame such that A and B take type I form in Lemma 11. Then the functions θ_i, h^k_{ij} and ω^k_{ij} are constant.*

Proof. We have to show that $\theta_{ip} = \theta_{iq}$ for any two points p and q in M. By hypothesis there is a Lie subgroup H of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ such

that H acts transitively on M. Therefore, there exists an isometry $\phi \in H$ such that $\phi(p) = q$. We have that

$$P_p E_{ip} = \cos 2\theta_{ip} E_{ip} + \sin 2\theta_{ip} J_p E_{ip}$$

In Subsection 2.2.2 we saw that isometries in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ preserve P and J. Thus, we apply ϕ to both sides:

$$\begin{aligned} P_q \phi_* E_{ip} = \phi_* P_p E_{ip} &= \phi_*(\cos 2\theta_{ip} E_{ip} + \sin 2\theta_{ip} J_p E_{ip}) \\ &= \cos 2\theta_{ip} \phi_* E_{ip} + \sin 2\theta_{ip} \phi_* J_p E_{ip} \\ &= \cos 2\theta_{ip} \phi_* E_{ip} + \sin 2\theta_{ip} J_q \phi_* E_{ip} \end{aligned}$$

Since $\{E_i\}_i$ is the unique frame with respect to which A and B are diagonal, we have $\phi_* E_{ip} = E_{iq}$ and $\theta_{iq} = \theta_{ip}$.

It follows from $\phi(M) = M$ that ϕ preserves ∇ and h. Thus using a similar argument we get that ω_{ij}^k and h_{ij}^k are constant. □

Lagrangian submanifolds of type II

Proposition 24. *Let M be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take the type II form in Lemma 11 with respect to a Δ_2-orthonormal frame $\{E_1, E_2, E_3\}$. If $\theta_1 \neq \theta_2$ modulo π then the frame is unique up to signs. If instead $\theta_1 = \theta_2$ there is a unique frame, up to signs, such that $h_{22}^1 = g(h(E_2, E_2), JE_2) = 0$. In both cases, the functions θ_i, h_{ij}^k and ω_{ij}^k are constant.*

Proof. The last statement follows from the uniqueness (even if it is up to sign) of the frame as in the proof of Proposition 23.

Suppose that $\theta_1 \neq \theta_2$ and that $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ is a frame on M such that

$$P\tilde{E}_1 = \cos 2\tilde{\theta}_1 \tilde{E}_1 + \sin 2\tilde{\theta}_1 J\tilde{E}_1,$$

$$P\tilde{E}_2 = \tilde{E}_1 + \cos 2\tilde{\theta}_1 \tilde{E}_2 - \cot 2\tilde{\theta}_1 J\tilde{E}_1 + \sin 2\tilde{\theta}_1 J\tilde{E}_2$$

$$P\tilde{E}_3 = \cos 2\tilde{\theta}_2 \tilde{E}_3 + \sin 2\tilde{\theta}_2 J\tilde{E}_3.$$

Hence, at any point of M the eigenvalues of A and B are $\{\cos 2\theta_1, \cos 2\theta_2\}$ and $\{\sin 2\theta_1, \sin 2\theta_2\}$, respectively. Moreover, the associated eigenvectors are the same. The eigenspace associated to $\cos 2\theta_1$ and $\sin 2\theta_1$ is lightlike and the one

associated to $\cos 2\theta_2$ and $\sin 2\theta_2$ is spacelike, therefore $\tilde{\theta}_1 = \theta_1$ and $\tilde{\theta}_2 = \theta_2$ modulo π. As the eigenvectors E_3 and $\tilde{E}_3$ are both unit length, we derive that $\tilde{E}_3 = \pm E_3$. Similarly, we get $\tilde{E}_1 = cE_1$ for $c \in \mathbb{R}$ and therefore $\tilde{E}_2 = c^{-1}E_2$. Computing $A\tilde{E}_2$ we produce

$$A\tilde{E}_2 = \frac{1}{c}(E_1 + \cos 2\theta_1 E_2) = \frac{1}{c^2}\tilde{E}_1 + \cos 2\theta_1 \tilde{E}_2$$

thus $c^2 = 1$. Since $\{\tilde{E}_i\}$ also has to satisfy the relations in (3.20) we obtain $\tilde{E}_3 = E_3$.

Suppose now that $\theta_2 = \theta_1$ modulo π. This means that the eigenspace associated to $\cos 2\theta_1$ is two-dimensional. Therefore any linear isometry that preserves the eigenspace preserves the form of A and B. Let T be the linear isometry defined by $TE_i = \tilde{E}_i$. After some computations we obtain that T has the form

$$T = \begin{pmatrix} \varepsilon & -\varepsilon\frac{t^2}{2} & t \\ 0 & \varepsilon & 0 \\ 0 & -\varepsilon t & 1 \end{pmatrix},$$

for $\varepsilon = \pm 1$ and some $t \in \mathbb{R}$.

Computing $h(\tilde{E}_2, \tilde{E}_2)$ and using Lemma 13 yields $\tilde{h}^1_{22} = h^1_{22} - 2th^3_{22}$ and $\tilde{h}^3_{22} = h^3_{22}$. Suppose that $h^3_{22} = 0$, then the Gauss equation (3.1) with $X = E_2, Y = E_3, Z = E_3$ implies $1/6 = 3/2$, a contradiction. Then we can choose $t = h^1_{22}/(2h^3_{22})$, thus $\tilde{h}^1_{22} = 0$. In the same way, we can obtain that it is the unique (up to sign) frame with this condition. As before, we conclude that ω^k_{ij} and h^k_{ij} are constant for this frame. □

Lagrangian submanifolds of type III

Proposition 25. *Let M be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type III form in Lemma 11 with respect to a Δ_2-orthogonal frame $\{E_1, E_2, E_3\}$. Then the frame is unique and the functions θ_i, h^k_{ij} and ω^k_{ij} are constant.*

Proof. Suppose that $\{\tilde{E}_i\}_i$ is another Δ_2-orthonormal frame with respect to which A and B take type III form in Lemma 11. We denote by T the linear isometry given by $TE_i = \tilde{E}_i$. We write

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

First notice that E_1 spans the unique eigenspace of A and B, thus $t_{21} = t_{31} = 0$. Computing $g(TE_1, TE_2)$ we furnish $t_{22}t_{11} = 1$. In the same way, computing $g(TE_1, TE_3)$ and $g(TE_3, TE_3)$ we come to $t_{23} = 0$ and $t_{33} = \varepsilon = \pm 1$. Computing $ATE_2 = -\frac{1}{2}TE_2 + TE_3$ we get that $t_{13} = t_{32}$ and $t_{11} = \varepsilon$. Computing $g(E_2, E_3)$ and $g(E_2, E_2)$ we obtain $t_{13} = 0$ and $t_{12} = 0$. By asking $JG(TE_1, TE_2) = \sqrt{\frac{2}{3}}TE_3$ we see that $\varepsilon = 1$.

As in the previous propositions, the last statement follows from the uniqueness of the frame. □

Lagrangian submanifolds of type IV

Proposition 26. *Let M be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type IV form in Lemma 11 with respect to Δ_3-orthonormal frame $\{E_1, E_2, E_3\}$. Then the frame is unique and the functions θ_i, ψ, h^k_{ij} and ω^k_{ij} are constant.*

Proof. In order to simplify the proof we write A and B as

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix}, \quad B = \begin{pmatrix} \gamma & \delta & 0 \\ -\delta & \gamma & 0 \\ 0 & 0 & \sin 2\theta_2 \end{pmatrix},$$

with respect to $\{E_i\}_i$.

Suppose there exists a Δ_3-orthonormal frame $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ and functions $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ and $\tilde{\theta}_2$ such that A and B take the form

$$A = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} & 0 \\ -\tilde{\beta} & \tilde{\alpha} & 0 \\ 0 & 0 & \cos 2\tilde{\theta}_2 \end{pmatrix}, \quad B = \begin{pmatrix} \tilde{\gamma} & \tilde{\delta} & 0 \\ -\tilde{\delta} & \tilde{\gamma} & 0 \\ 0 & 0 & \sin 2\tilde{\theta}_2 \end{pmatrix},$$

with respect to $\{\tilde{E}_i\}_i$.

Since $\cos 2\theta_2$ and $\sin 2\theta_2$ are the only eigenvalues of A and B, $\tilde{\theta}_2 = \theta_2$ modulo π and $\tilde{E}_3 = \varepsilon E_3$ with $\varepsilon = \pm 1$. We denote by T the linear isometry defined by $TE_i = \tilde{E}_i$. Given that $\{\tilde{E}_i\}$ is an Δ_3-orthonormal frame, we may assume that $\tilde{E}_1$ and $\tilde{E}_2$ do not have components in the direction of E_3. Hence, we can write T as

$$T = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & \varepsilon \end{pmatrix},$$

with $t \in \mathbb{R}$. Requiring $ATE_1 = (\tilde{\alpha}TE_1 - \tilde{\beta}TE_2)$ and $ATE_2 = (\tilde{\beta}TE_1 + \tilde{\alpha}TE_2)$ we obtain

$$\alpha \cosh t + \beta \sinh t = \tilde{\alpha} \cosh t - \tilde{\beta} \sinh t,$$

$$-\beta \cosh t + \alpha \sinh t = -\tilde{\beta} \cosh t + \tilde{\alpha} \sinh t,$$

$$\beta \cosh t + \alpha \sinh t = \tilde{\beta} \cosh t + \tilde{\alpha} \sinh t,$$

$$\alpha \cosh t - \beta \sinh t = \tilde{\alpha} \cosh t + \tilde{\beta} \sinh t.$$

Combining these equations we get

$$(\alpha - \tilde{\alpha}) \cosh t = 0,$$

$$(\beta - \tilde{\beta}) \cosh t = 0,$$

therefore $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \beta$. We may use the same argument to deduce that $\tilde{\delta} = \delta$ and $\tilde{\gamma} = \gamma$. We compute again $ATE_1 = (\tilde{\alpha}TE_1 - \tilde{\beta}TE_2)$ and we derive $t = 0$. From $JG(\tilde{E}_1, \tilde{E}_2) = \sqrt{\frac{2}{3}}\tilde{E}_3$ it follows that $\varepsilon = 1$.

In a similar way as in the proofs of propositions 23-25 we obtain that α, β, γ and δ are constant. Computing $\frac{\alpha}{\delta}$ and $\frac{\alpha}{\beta}$ we obtain that ψ, θ_1 and θ_2 are constant as well. Finally using the uniqueness of the frame we get that h^k_{ij} and ω^k_{ij} are constant. □

3.3.2 Extrinsically homogeneous Lagrangian submanifolds of type I

In the previous section we saw a characterization of totally geodesic Lagrangian submanifolds of type I:

Proposition 18. *Let M be a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of type I in Lemma 11. If two angles are equal modulo π, then M is totally geodesic.*

Moreover, in Theorem 32 all the totally geodesic Lagrangian submanifolds are classified up to congruence, which we rewrite to fit better in this section.

Theorem 32. *Let $f : (M, g) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be a totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Then $f(M)$ is congruent to an open subset of the following extrinsically homogeneous*

Lagrangian embeddings:

(M,g)	f	H
$(\mathrm{SL}(2,\mathbb{R}), \frac{2}{3}\langle,\rangle)$	$u \mapsto (u,u)$	$\{(u,u,\mathrm{Id}_2) : u \in \mathrm{SL}(2,\mathbb{R})\}$
$(\mathrm{SL}(2,\mathbb{R}), g^+_{\kappa,\tau})$	$u \mapsto (u, \boldsymbol{i}u\boldsymbol{i})$	$\{(u, \boldsymbol{i}u\boldsymbol{i}, \mathrm{Id}_2) : u \in \mathrm{SL}(2,\mathbb{R})\}$
$(\mathrm{SL}(2,\mathbb{R}), g^-_{\kappa,\tau})$	$u \mapsto (u, -\boldsymbol{k}u\boldsymbol{k})$	$\{(u, -\boldsymbol{k}u\boldsymbol{k}, \mathrm{Id}_2) : u \in \mathrm{SL}(2,\mathbb{R})\}$

where H acts transitively on M with null isotropy, and $g^+_{\kappa,\tau}$, $g^-_{\kappa,\tau}$ are Berger-like metrics on $\mathrm{SL}(2,\mathbb{R})$ stretched in spacelike and timelike directions, respectively.

This theorem implies that any Lagrangian submanifold of type I with two equal angle functions modulo π is extrinsically homogeneous. Therefore, to complete the classification of extrinsically homogeneous Lagrangian submanifolds we assume that the submanifold is not totally geodesic and that all angle functions are different modulo π.

Proposition 27. *Let M be a non-totally geodesic extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ of type I. Let θ_i, $i = 1,2,3$ be the angle functions associated to the Δ_1-orthonormal frame with respect ot which A and B are diagonal. Then $(\theta_1,\theta_2,\theta_3)$ is a permutation of $(0,\pi/3,2\pi/3)$ and the manifold M has constant sectional curvature. Moreover, the sectional curvature is either equal to 0 or to $-\frac{3}{8}$.*

Proof. Lagrangian submanifolds of type I are essentially an analogue of Lagrangian submanifolds of $\mathbb{S}^3\times\mathbb{S}^3$. In [6] the authors proved for $\mathbb{S}^3\times\mathbb{S}^3$ that the angle functions of non-totally geodesic Lagrangian submanifolds are constant and a permutation of $(0, \frac{\pi}{3}, \frac{2\pi}{3})$. The same argument works for $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

By Lemma 10 we may assume that $(\theta_1,\theta_2,\theta_3) = (0,\pi/3,2\pi/3)$. From Lemma 12 we know that all the functions h^k_{ij} are equal to zero, except for h^3_{12}, which from Proposition 23 we know is constant. Then the Codazzi equation (3.2) with $X = E_1$, $Y = E_2$, $Z = E_2$ yields that h^3_{12} is either equal to $\frac{1}{2\sqrt{2}}$ or to $-\frac{1}{\sqrt{2}}$. Both cases imply that the sectional curvature is constant. In the former case the sectional curvature is equal to $-\frac{3}{8}$ and in the latter case the sectional curvature is equal to 0. □

Example 19. Let $f : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ be the isometric immersion given by $u \mapsto (\boldsymbol{i}u\boldsymbol{i}u^{-1}, \boldsymbol{j}u\boldsymbol{j}u^{-1})$ and let $\{X_1,X_2,X_3\}$ be the frame on $\mathrm{SL}(2,\mathbb{R})$

given in (2.3). We may compute

$$df(X_1) = (0, \boldsymbol{j}u\boldsymbol{j}u^{-1}(-2u\boldsymbol{i}u^{-1})),$$

$$df(X_2) = (\boldsymbol{i}u\boldsymbol{i}u^{-1}(-2u\boldsymbol{j}u^{-1}), 0),$$

$$df(X_3) = (\boldsymbol{i}u\boldsymbol{i}u^{-1}(-2u\boldsymbol{k}u^{-1}), \boldsymbol{j}u\boldsymbol{j}u^{-1}(-2u\boldsymbol{k}u^{-1})).$$

It follows from the definition of J in (2.10) that

$$Jdf(X_1) = \frac{1}{\sqrt{3}}(\boldsymbol{i}u\boldsymbol{i}u^{-1}(4u\boldsymbol{i}u^{-1}), \boldsymbol{j}u\boldsymbol{j}u^{-1}(2u\boldsymbol{i}u^{-1})),$$

$$Jdf(X_2) = \frac{1}{\sqrt{3}}(\boldsymbol{i}u\boldsymbol{i}u^{-1}(-2u\boldsymbol{j}u^{-1}), \boldsymbol{j}u\boldsymbol{j}u^{-1}(-4u\boldsymbol{j}u^{-1})),$$

$$Jdf(X_3) = \frac{1}{\sqrt{3}}(\boldsymbol{i}u\boldsymbol{i}u^{-1}(2u\boldsymbol{k}u^{-1}), \boldsymbol{j}u\boldsymbol{j}u^{-1}(-2u\boldsymbol{k}u^{-1})).$$

We can easily check that f is a Lagrangian immersion by computing $g(Jdf(X_i), df(X_j)) = 0$ for $i, j = 1, 2, 3$. Moreover, we have

$$Pdf(X_1) = (\boldsymbol{i}u\boldsymbol{i}u^{-1}(-2u\boldsymbol{i}u^{-1}), 0) = -\frac{1}{2}df(X_1) - \frac{\sqrt{3}}{2}Jdf(X_1),$$

$$Pdf(X_2) = (0, \boldsymbol{j}u\boldsymbol{j}u^{-1}(-2u\boldsymbol{j}u^{-1})) = -\frac{1}{2}df(X_2) + \frac{\sqrt{3}}{2}Jdf(X_2),$$

$$Pdf(X_3) = (\boldsymbol{i}u\boldsymbol{i}u^{-1}(-2u\boldsymbol{k}u^{-1}), \boldsymbol{j}u\boldsymbol{j}u^{-1}(-2u\boldsymbol{k}u^{-1})) = df(X_3).$$

Let H be the subgroup of $\mathrm{Iso}_o(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ given by $\{(\boldsymbol{i}u\boldsymbol{i}, \boldsymbol{j}u\boldsymbol{j}, u) : u \in \mathrm{SL}(2,\mathbb{R})\} \cong \mathrm{SL}(2,\mathbb{R})$. Then $f(\mathrm{SL}(2,\mathbb{R})) = H \cdot (\mathrm{Id}_2, \mathrm{Id}_2)$. Notice that H acts on $f(\mathrm{SL}(2,\mathbb{R}))$ with isotropy $\mathbb{Z}_2$. Hence, the embedding $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : [u] \mapsto (\boldsymbol{i}u\boldsymbol{i}u^{-1}, \boldsymbol{j}u\boldsymbol{j}u^{-1})$ is congruent to f.

Proposition 28. *Any extrinsically homogeneous non-totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *of type I with* $h^3_{12} = \frac{1}{2\sqrt{2}}$ *is congruent to an open subset of the image of* $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : [u] \mapsto (\boldsymbol{i}u\boldsymbol{i}u^{-1}, \boldsymbol{j}u\boldsymbol{j}u^{-1})$.

Proof. Let $f\colon M \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be a non-totally geodesic extrinsically homogeneous Lagrangian immersion of type I. Let $\{E_1, E_2, E_3\}$ be the frame on M such that $JG(E_1, E_2) = \sqrt{\frac{2}{3}}E_3$ with angle functions given by $(\theta_1, \theta_2, \theta_3) = (0, \pi/3, 2\pi/3)$. Moreover, assume that $h^3_{12} = g(h(E_1, E_2), E_3) = \frac{1}{2\sqrt{2}}$. We have

$$PE_1 = E_1, \quad PE_2 = -\tfrac{1}{2}E_2 + \tfrac{\sqrt{3}}{2}JE_2, \quad PE_3 = -\tfrac{1}{2}E_3 - \tfrac{\sqrt{3}}{2}JE_3.$$

Hence, according to Equation (2.16), we have

$$QE_1 = \sqrt{3}JE_1, \quad QE_2 = -E_2, \quad QE_3 = E_3. \tag{3.46}$$

From Proposition 27 it follows that M has constant sectional curvature $-\frac{3}{8}$. Thus M is locally isometric to $(\mathrm{SL}(2,\mathbb{R}), \frac{8}{3}\langle,\rangle)$ (see [44]), where $\langle,\rangle$ is the metric given in (2.2). Then we may identify

$$E_1 = \sqrt{\tfrac{3}{8}}X_3, \quad E_2 = \sqrt{\tfrac{3}{8}}X_2, \quad E_3 = \sqrt{\tfrac{3}{8}}X_1,$$

where $\{X_1, X_2, X_3\}$ is the frame on $\mathrm{SL}(2,\mathbb{R})$ given in (2.3). Now we write the immersion $f(u) = (p(u), q(u))$ and $df(E_i)_u = (D_{E_i}f)_u = (p(u)\alpha_i(u), q(u)\beta_i(u))$ where $\alpha_i(u), \beta_i(u) \in \mathfrak{sl}(2,\mathbb{R})$. By Equation (3.46) we have $\alpha_1 = \beta_1$, $\beta_2 = 0$ and $\alpha_3 = 0$. We know from (2.17) that $\nabla^E_{E_1}E_1 = \nabla^E_{E_2}E_2 = \nabla^E_{E_3}E_3 = 0$ and

$$\begin{aligned}
&\nabla^E_{E_1}E_2 = \nabla^E_{E_1}E_3 = 0,\\
&\nabla^E_{E_2}E_1 = -\sqrt{\tfrac{3}{2}}E_3 = -\sqrt{\tfrac{3}{2}}(0, q\beta_3),\\
&\nabla^E_{E_2}E_3 = -\tfrac{1}{2}\sqrt{\tfrac{3}{2}}(E_1 + QE_1) = -\sqrt{\tfrac{3}{2}}(0, q\alpha_1),\\
&\nabla^E_{E_3}E_1 = \sqrt{\tfrac{3}{2}}E_2 = \sqrt{\tfrac{3}{2}}(p\alpha_2, 0),\\
&\nabla^E_{E_3}E_2 = \tfrac{1}{2}\sqrt{\tfrac{3}{2}}(E_1 - QE_1) = \sqrt{\tfrac{3}{2}}(p\alpha_1, 0).
\end{aligned} \tag{3.47}$$

Throughout this proof, we will denote by $\langle,\rangle_\times$ the product metric associated to the metric $\langle,\rangle$ on $\mathrm{SL}(2,\mathbb{R})$ given in (2.2). By Equation 2.15 E_1, E_2, E_3 are orthogonal with respect to the product metric $\langle,\rangle_\times$ and their lengths are

$$\langle E_1, E_1\rangle_\times = -3, \quad \langle E_2, E_2\rangle_\times = \langle E_3, E_3\rangle_\times = \tfrac{3}{2}.$$

This implies that

$$\langle\alpha_1, \alpha_1\rangle = -\tfrac{3}{2}, \quad \langle\alpha_2, \alpha_2\rangle = \langle\beta_3, \beta_3\rangle = \tfrac{3}{\cdot}2 \tag{3.48}$$

On the one hand, Equation (2.18) yields

$$D_{E_i}D_{E_j}f = \nabla^E_{E_i}E_j + \tfrac{1}{2}\langle E_i, E_j\rangle(p, q) + \tfrac{1}{2}\langle E_i, QE_j\rangle(-p, q),$$

and on the other hand, by Equation (2.5), we obtain

$$\begin{aligned}
D_{E_i}D_{E_j}f &= (p\alpha_i\alpha_j + pE_i(\alpha_j), q\beta_i\beta_j + qE_i(\beta_j))\\
&= (p(\alpha_i \times \alpha_j) + \langle\alpha_i, \alpha_j\rangle p + pE_i(\alpha_j),\\
&\qquad q(\beta_i \times \beta_j) + \langle\beta_i, \beta_j\rangle q + qE_i(\beta_j)).
\end{aligned}$$

Therefore

$$\nabla^E_{E_i} E_j = (p\alpha_i \times \alpha_j + pE_i(\alpha_j), q\beta_i \times \beta_j + qE_i(\beta_j)),$$

where $E_i(\alpha) = d\alpha(E_i)$ thinking of α as a map from $\mathrm{SL}(2,\mathbb{R})$ into $\mathfrak{sl}(2,\mathbb{R})$. Hence, using (3.47) we obtain

$$\alpha_1 \times \alpha_2 = -\sqrt{\tfrac{3}{2}}\beta_3,$$

and also

$$\begin{aligned}
&E_1(\alpha_1) = 0, && E_2(\alpha_1) = -\sqrt{\tfrac{3}{2}}\beta_3, && E_3(\alpha_1) = \sqrt{\tfrac{3}{2}}\alpha_2,\\
&E_1(\alpha_2) = \sqrt{\tfrac{3}{2}}\beta_3, && E_2(\alpha_2) = 0, && E_3(\alpha_2) = \sqrt{\tfrac{3}{2}}\alpha_1,\\
&E_1(\beta_3) = -\sqrt{\tfrac{3}{2}}\alpha_2, && E_2(\beta_3) = -\sqrt{\tfrac{3}{2}}\alpha_1, && E_3(\beta_3) = 0.
\end{aligned}$$

In terms of the vector fields X_i this translates into the following differential equations:

$$\begin{aligned}
&X_1(\alpha_1) = 2\alpha_2, && X_2(\alpha_1) = -2\beta_3, && X_3(\alpha_1) = 0,\\
&X_1(\alpha_2) = 2\alpha_1, && X_2(\alpha_2) = 0, && X_3(\alpha_2) = 2\beta_3,\\
&X_1(\beta_3) = 0, && X_2(\beta_3) = -2\alpha_1, && X_3(\beta_3) = -2\alpha_2.
\end{aligned} \tag{3.49}$$

From (3.48) and Lemma 6 we know that there exists $c \in \mathrm{SL}^{\pm}(2,\mathbb{R})$ such that

$$\alpha_1(\mathrm{Id}_2) = -\sqrt{\tfrac{3}{2}}c\boldsymbol{k}c^{-1}, \quad \alpha_2(\mathrm{Id}_2) = -\sqrt{\tfrac{3}{2}}c\boldsymbol{j}c^{-1}, \quad \beta_3(\mathrm{Id}_2) = -\sqrt{\tfrac{3}{2}}c\boldsymbol{i}c^{-1}. \tag{3.50}$$

Therefore, as the solution of the system (3.49) with initial conditions (3.50) is unique, we have that

$$\alpha_1(u) = -\sqrt{\tfrac{3}{2}}cu\boldsymbol{k}u^{-1}c^{-1}, \;\; \alpha_2(u) = -\sqrt{\tfrac{3}{2}}u\boldsymbol{j}u^{-1}c^{-1}, \;\; \beta_3(u) = -\sqrt{\tfrac{3}{2}}cu\boldsymbol{i}u^{-1}c^{-1}.$$

We can check easily that they satisfy the equations in (3.49). By the homogeneity of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we can take initial conditions $f(\mathrm{Id}_2) = (\mathrm{Id}_2, \mathrm{Id}_2)$. Applying the isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by $(p,q) \mapsto (cpc^{-1}, cqc^{-1})$ we may assume that

$$\alpha_1(u) = -\sqrt{\tfrac{3}{2}}u\boldsymbol{k}u^{-1}, \quad \alpha_2(u) = -\sqrt{\tfrac{3}{2}}u\boldsymbol{j}u^{-1}, \quad \beta_3(u) = -\sqrt{\tfrac{3}{2}}cu\boldsymbol{i}u^{-1}.$$

Then, the immersion $f = (p,q)$ given by $p = \boldsymbol{i}u\boldsymbol{i}u^{-1}$, $q = \boldsymbol{j}u\boldsymbol{j}u^{-1}$ is the unique solution of the differential equation $D_{E_i}p = p\alpha_i$, $D_{E_i}q = q\beta_i$, with $i = 1,2,3$. □

Example 20. Let H be the Lie subgroup of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$H = \{(e^{v\boldsymbol{i}}, e^{w\boldsymbol{j}}, e^{u\boldsymbol{k}}) : v, w \in \mathbb{R},\ u \in [0, 2\pi)\} \cong \mathbb{R}^2 \times \mathbb{S}^1.$$

The subgroup H acts transitively on the submanifold $f : \mathbb{R}^3_1/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$f(u, v, w) = (e^{v\boldsymbol{i}} e^{-u\boldsymbol{k}}, e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}}).$$

Moreover, the isotropy of H is trivial.

The derivatives of f are given by

$$f_u = (-e^{v\boldsymbol{i}} e^{-u\boldsymbol{k}}\boldsymbol{k}, -e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}}\boldsymbol{k}),$$

$$f_v = (e^{v\boldsymbol{i}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{i} e^{-u\boldsymbol{k}}, 0),$$

$$f_w = (0, e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{j} e^{-u\boldsymbol{k}}).$$

Applying the almost complex structure yields the following expressions:

$$Jf_u = \frac{1}{\sqrt{3}}(e^{v\boldsymbol{i}} e^{-u\boldsymbol{k}}\boldsymbol{k}, -e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}}\boldsymbol{k}),$$

$$Jf_v = \frac{1}{\sqrt{3}}(e^{v\boldsymbol{i}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{i} e^{-u\boldsymbol{k}}, 2e^{v\boldsymbol{i}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{i} e^{-u\boldsymbol{k}}),$$

$$Jf_w = \frac{1}{\sqrt{3}}(-2e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{j} e^{-u\boldsymbol{k}}, -e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{j} e^{-u\boldsymbol{k}}).$$

We can easily check that

$$g(Jf_u, f_v) = g(Jf_u, f_w) = g(Jf_v, f_w) = 0,$$

which shows that this submanifold is Lagrangian. After applying the tensor P we obtain

$$Pf_u = (-e^{v\boldsymbol{i}} e^{-u\boldsymbol{k}}\boldsymbol{k}, -e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}}\boldsymbol{k}) = f_u,$$

$$Pf_v = (0, e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{i} e^{-u\boldsymbol{k}}) = -\frac{1}{2} f_v + \frac{\sqrt{3}}{2} Jf_v,$$

$$Pf_w = (e^{w\boldsymbol{j}} e^{-u\boldsymbol{k}} e^{u\boldsymbol{k}}\boldsymbol{j} e^{-u\boldsymbol{k}}, 0) = -\frac{1}{2} f_w - \frac{\sqrt{3}}{2} Jf_w.$$

Thus, f is a flat, extrinsically homogeneous Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of type I with constant angles $(\theta_1, \theta_2, \theta_3) = (0, \frac{\pi}{3}, \frac{2\pi}{3})$.

Proposition 29. *Any extrinsically homogeneous non-totally geodesic Lagrangian submanifold of type I of the nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *with* $h^3_{12} = -\frac{1}{\sqrt{2}}$ *is congruent to an open subset of the image of the Lagrangian embedding* $f : \mathbb{R}^3_1/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *given by*

$$f(u,v,w) = \left(e^{v\boldsymbol{i}}e^{-u\boldsymbol{k}}, e^{w\boldsymbol{j}}e^{-u\boldsymbol{k}}\right).$$

Proof. Notice first that all the coefficients of the connection and second fundamental form vanish except for $h^3_{12} = g(h(E_1,E_2),JE_3) = -\frac{1}{\sqrt{2}}$. We also know that the angle functions are given by $(\theta_1,\theta_2,\theta_3) = (0,\frac{\pi}{3},\frac{2\pi}{3})$. So we can find a local frame such that $JG(E_1,E_3) = \sqrt{\frac{2}{3}}E_3$ and

$$PE_1 = E_1, \quad PE_2 = -\tfrac{1}{2}E_2 + \tfrac{\sqrt{3}}{2}JE_2, \quad PE_3 = -\tfrac{1}{2}E_3 - \tfrac{\sqrt{3}}{2}JE_3.$$

From the relation between Q and P in (2.16) it follows that

$$QE_1 = \sqrt{3}JE_1, \quad QE_2 = -E_2, \quad QE_3 = E_3. \tag{3.51}$$

Using that $h^j_{ii} = 0$, $h^3_{12} = -\frac{1}{\sqrt{2}}$ and Equation (3.25), we deduce that $[E_i,E_j] = 0$ for $i,j = 1,2,3$. Then we write $E_1 = f_u$, $E_2 = f_v$, $E_3 = f_w$ for u,v,w local coordinates. Thus, Equation (3.51) implies that

$$p_w = 0, \qquad q_v = 0, \qquad q_u = p_u. \tag{3.52}$$

Moreover, we have $\nabla^E_{f_u} f_u = \nabla^E_{f_v} f_v = \nabla^E_{f_w} f_w = 0$ and

$$\nabla^E_{f_u} f_v = \nabla^E_{f_v} f_u = -\tfrac{1}{2}\sqrt{\tfrac{3}{2}}f_w - \tfrac{3}{2\sqrt{2}}Jf_w,$$

$$\nabla^E_{f_u} f_w = \nabla^E_{f_w} f_u = \tfrac{1}{2}\sqrt{\tfrac{3}{2}}f_v - \tfrac{3}{2\sqrt{2}}Jf_v,$$

$$\nabla^E_{f_v} f_w = \nabla^E_{f_w} f_v = 0.$$

From the relation between the Euclidean metric with the nearly Kähler metric in (2.15) we know that f_u, f_v, f_w are also orthogonal with respect to the induced Euclidean product metric. Furthermore, their inner products are given by

$$\langle f_u, f_u\rangle = -3, \quad \langle f_v, f_v\rangle = \langle f_w, f_w\rangle = \tfrac{3}{2}.$$

Furthermore,

$$\langle f_u, Qf_u\rangle = 0, \quad \langle f_u, Qf_v\rangle = 0, \quad \langle f_u, Qf_w\rangle = 0,$$

$$\langle f_v, Qf_w\rangle = 0, \quad \langle f_v, Qf_v\rangle = -\tfrac{3}{2}, \quad \langle f_w, Qf_w\rangle = \tfrac{3}{2}.$$

Thus, using (3.52) we get that p_u, p_v and q_w are orthogonal and

$$\langle p_u, p_u\rangle = -\tfrac{3}{2}, \quad \langle p_v, p_v\rangle = \langle p_w, p_w\rangle = \tfrac{3}{2}. \tag{3.53}$$

From the expression for the Euclidean connection D of $\mathbb{R}^8_4$ given in Equation (2.18) we obtain

$$\begin{aligned} f_{uu} &= -\tfrac{3}{2}f, & f_{uv} &= -\tfrac{1}{2}\sqrt{\tfrac{3}{2}}f_w - \tfrac{3}{2\sqrt{2}}Jf_w,\\ f_{uw} &= \tfrac{1}{2}\sqrt{\tfrac{3}{2}}f_v - \tfrac{3}{2\sqrt{2}}Jf_v, & f_{vv} &= \tfrac{3}{4}f - \tfrac{3}{4}Qf,\\ f_{vw} &= 0, & f_{ww} &= \tfrac{3}{4}f + \tfrac{3}{4}Qf. \end{aligned}$$

Hence we produce differential equations for p and q:

$$\begin{aligned} &p_{uu} = -\tfrac{3}{2}p, \quad p_{vv} = \tfrac{3}{2}p, \quad p_{uv} = \sqrt{\tfrac{3}{2}}pq^{-1}q_w,\\ &q_{uu} = -\tfrac{3}{2}q, \quad q_{ww} = \tfrac{3}{2}q, \quad q_{uw} = -\sqrt{\tfrac{3}{2}}qp^{-1}p_v. \end{aligned} \tag{3.54}$$

By applying an isometry of the type $(p, q) \mapsto (ap, bq)$ we may assume that $p(0) = \mathrm{Id}_2$, $q(0) = \mathrm{Id}_2$. Now, because of Equation (3.53), there exists a matrix $c \in \mathrm{SL}^{\pm}(2, \mathbb{R})$ such that

$$p_u(0) = \sqrt{\tfrac{3}{2}}c\boldsymbol{k}c^{-1}, \;\; p_v(0) = \sqrt{\tfrac{3}{2}}c\boldsymbol{i}c^{-1}, \;\; q_u(0) = \sqrt{\tfrac{3}{2}}c\boldsymbol{k}c^{-1}, \;\; q_w(0) = \sqrt{\tfrac{3}{2}}c\boldsymbol{j}c^{-1}.$$

Applying the isometry $(p, q) \mapsto (cpc^{-1}, cqc^{-1})$ we obtain that any solution of the system (3.54) is congruent to the map

$$f(u, v, w) = \left(e^{\sqrt{\frac{3}{2}}v\boldsymbol{i}} e^{-\sqrt{\frac{3}{2}}u\boldsymbol{k}}, e^{\sqrt{\frac{3}{2}}w\boldsymbol{j}} e^{-\sqrt{\frac{3}{2}}u\boldsymbol{k}} \right)$$

Then, changing the coordinates by $u \to \sqrt{\frac{2}{3}}u$, $v \to \sqrt{\frac{2}{3}}v$ and $w \to \sqrt{\frac{2}{3}}w$, we get the map $(u, v, w) \mapsto (e^{v\boldsymbol{i}}e^{-u\boldsymbol{k}}, e^{w\boldsymbol{j}}e^{-u\boldsymbol{k}})$. □

3.3.3 Extrinsically homogeneous Lagrangian submanifolds of type II

Proposition 30. *Let M be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler* $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. *Suppose that A and*

B take type II form in Lemma 11 with respect to a Δ_2-orthonormal frame $\{E_1, E_3, E_3\}$. Then $\theta_1 = \theta_2 = \pi/3$ modulo π. Also, all the components of the second fundamental form and of the connection are equal to zero except for h^3_{22}, ω^3_{12}, ω^3_{21}, ω^1_{31} and ω^3_{22}. There are only two possibilities for the value of these constants:

1. $h^3_{22} = -\frac{\sqrt{2}}{3}$, $\quad \omega^3_{12} = -\sqrt{\frac{3}{2}}$, $\quad \omega^3_{21} = -\sqrt{\frac{3}{2}}$, $\quad \omega^1_{31} = 0$, $\quad \omega^2_{33} = 0$,
2. $h^3_{22} = \frac{2\sqrt{2}}{3}$, $\quad \omega^3_{12} = \sqrt{\frac{3}{2}}$, $\quad \omega^3_{21} = \sqrt{\frac{3}{2}}$, $\quad \omega^1_{31} = \sqrt{\frac{3}{2}}$.

Moreover, in both cases the sectional curvature is constant and equal to $-\frac{3}{2}$.

Proof. As indicated in Proposition 24, we have to distinguish between two cases: when $\theta_1 = \theta_2$ and when $\theta_1 \neq \theta_2$.

Suppose first that $\theta_1 \neq \theta_2$. By Proposition 24 both angles and the functions h^k_{ij}, ω^k_{ij} are constant. It follows from computing the Codazzi equation (3.2) with $X = E_3$, $Y = E_2$ and $Z = E_3$ that $\sin(2(\theta_1 - \theta_2)) = 0$, hence θ_1 and θ_2 are equal modulo $\pi/2$. Recall that for type II submanifolds $\sin 2\theta_1$ is different from zero, $2\theta_1 + \theta_2 = 0$ modulo π and the angles are different modulo π. Therefore $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{2\pi}{3}$ or $\theta_1 = \frac{5\pi}{6}$, $\theta_2 = \frac{\pi}{3}$. From Lemma 10 we know that these two cases are congruent via the isometry $\Psi_{0,1}$ given in (2.19). It follows from Equation (3.23) that

$$\omega^1_{11} = \omega^3_{11} = \omega^2_{33} = 0, \quad \omega^3_{21} = -\frac{1}{\sqrt{6}}, \quad \omega^3_{12} = \frac{1}{\sqrt{6}}, \quad \omega^1_{31} = \frac{\sqrt{3}}{2} h^3_{22} + \frac{1}{\sqrt{6}}.$$

Computing the Codazzi equation (3.2) with $X = E_3$, $Y = E_2$, $Z = E_3$ and $X = E_1$, $Y = E_2$, $Z = E_2$ we obtain

$$\frac{4 - 3h^3_{22}\left(3h^3_{22} + \sqrt{2}\right)}{3\sqrt{3}} = 0, \quad 8\sqrt{3} - 3\sqrt{6}h^3_{22} = 0,$$

which is a contradiction.

Suppose now that $\theta_1 = \theta_2$ modulo π. Using that $2\theta_1 + \theta_2 = 0$ modulo π we deduce $\theta_1 = \theta_2 = \pi/3$ or $2\pi/3$. By Lemma 10 we know that these cases are congruent via the isometry $\Psi_{0,1}$ given in (2.19). Thus, we only consider the cases where $\theta_1 = \theta_2 = \pi/3$.

By Lemma 13 we have $h^2_{22} = h^3_{12} = 0$. Moreover, by Proposition 24 we may assume that $h^1_{22} = 0$ and that all the functions h^k_{ij}, ω^k_{ij} are constant. Hence, from Equation (3.23) we obtain

$$\omega^1_{11} = \omega^3_{11} = \omega^2_{22} = \omega^2_{33} = 0.$$

We also get

$$\omega_{21}^3 = \sqrt{3}h_{22}^3 - \frac{1}{\sqrt{6}}, \qquad \omega_{31}^1 = \frac{\sqrt{3}}{2}h_{22}^3 + \frac{1}{\sqrt{6}}.$$

Thus computing the Codazzi equation with $X = E_3$, $Y = E_2$ and $Z = E_2$ yields $\omega_{33}^1 = 0$. Moreover, we obtain

$$-9(h_{22}^3)^2 + 3\sqrt{2}h_{22}^3 + 4 = 0.$$

Hence $h_{22}^3 = -\frac{\sqrt{2}}{3}$ or $h_{22}^3 = \frac{2\sqrt{2}}{3}$.

If $h_{22}^3 = -\frac{\sqrt{2}}{3}$ then the Codazzi equation with $X = E_1$, $Y = E_2$ and $Z = E_2$ implies $\omega_{12}^3 = -\sqrt{\frac{3}{2}}$. Also we obtain $\omega_{22}^3 = 0$.

If instead $h_{22}^3 = \frac{2\sqrt{2}}{3}$ we obtain $\omega_{12}^3 = \sqrt{\frac{3}{2}}$. A straightforward computation shows that both cases have constant sectional curvature equal to $-\frac{3}{2}$. □

Now we exhibit two examples of extrinsically homogeneous Lagrangian submanifolds of type II in Lemma 11.

Example 21. Let $\mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2$ be the Bianchi group of type V with group law given by

$$(t, u) \cdot (s, v) = (t + s, \varphi_0(s)u + v)$$

where $\varphi_0 \colon \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is given by

$$\varphi_0(s) = \begin{pmatrix} e^{-2s} & 0 \\ 0 & e^{-2s} \end{pmatrix}.$$

Let $\hat{g}$ be the right invariant metric such that its components with respect to the frame of right invariant vector fields $\left\{ \frac{\partial}{\partial t}, e^{-2t}\frac{\partial}{\partial u_1}, e^{-2t}\frac{\partial}{\partial u_2} \right\}$ are given by

$$\begin{pmatrix} \frac{8}{3} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

In fact, this Lorentzian manifold is simply connected, geodesically complete and it has constant sectional curvature equal to $-\frac{3}{2}$. Hence, by Theorem 18 it is isometric to $\tilde{H}_1^3(-\frac{3}{2})$, the universal cover of the anti-de Sitter space.

Now let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ spanned by $\{e_1, e_2, e_3\}$ where

$$e_1 = \Big(\boldsymbol{i}, -\boldsymbol{i}, -\boldsymbol{i}\Big),$$

$$e_2 = \left(\frac{9}{4}(\boldsymbol{j} - \boldsymbol{k}), \frac{1}{2}(\boldsymbol{j} + \boldsymbol{k}), 0\right),$$

$$e_3 = \left(\frac{9}{4}(-\boldsymbol{j} + \boldsymbol{k}), 0, \frac{1}{2}(\boldsymbol{j} + \boldsymbol{k})\right),$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are given in (2.1). The Lie algebra $\mathfrak{h}$ is a Bianchi Lie algebra of type V with brackets

$$[e_1, e_2] = -2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = 0.$$

Note that the exponential map of a Bianchi Lie algebra of type V is surjective, then the Lie subgroup of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ associated with $\mathfrak{h}$ is equal to $\exp(\mathfrak{h})$. Therefore $\mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2$ is the universal cover of $\exp(\mathfrak{h})$. Moreover, the map

$$(w, u, v) \mapsto \exp(we_1 + ue^w w \operatorname{csch}(w) e_2 + ve^w w \operatorname{csch}(w) e_3)$$

defined at $w = 0$ as $\exp(ue_2 + ve_3)$, is a group isomorphism, thus $\exp \mathfrak{h} \cong \mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2$. One can check that the immersion $\iota : \mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2 \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$\iota(w, u, v) = \exp(we_1 + ue^w w \operatorname{csch}(w) e_2 + ve^w w \operatorname{csch}(w) e_3) \cdot (\mathrm{Id}_2, \mathrm{Id}_2) \quad (3.55)$$

is a Lagrangian immersion, whose image is extrinsically homogeneous. The frame given by

$$E_1 = -e^{-2w} \iota_u - e^{-2w} \iota_v, \qquad E_2 = \tfrac{1}{3} e^{-2w} \iota_u - \tfrac{1}{3} e^{-2w} \iota_v, \qquad E_3 = \sqrt{\tfrac{3}{8}} \iota_w,$$

is a Δ_2-orthonormal frame with respect to which A and B take type II form in Lemma 11, with angle functions $\theta_1 = \theta_2 = \frac{\pi}{3}$.

Example 22. Let $\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2$ be the Bianchi group of type III with group law

$$(t, u) \cdot (s, v) = (t + s, \varphi_1(s)u + v)$$

where $\varphi_1 \colon \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is given by

$$\varphi_1(s) = \begin{pmatrix} e^{2s} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let g_λ be the right invariant metric on $\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2$ such that its components with respect to the frame of right invariant vector fields $\left\{ \frac{\partial}{\partial t}, e^{2t}\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right\}$ are given by

$$\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{2(\lambda-1)}{3} \end{pmatrix},$$

where λ is an arbitrary real number. As in Example 21, this Lorentzian manifold is simply connected, geodesically complete and it has constant sectional curvature equal to $-\frac{3}{2}$. Therefore it is isometric to $\tilde{H}^3_1(-\frac{3}{2})$.

Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ spanned by $\{e_1, e_2, e_3\}$ where

$$e_1 = (\boldsymbol{i}, 0, 0),$$

$$e_2 = (\tfrac{1}{2}(\boldsymbol{j} + \boldsymbol{k}), 0, 0),$$

$$e_3 = (0, -\tfrac{\lambda+7}{6}\boldsymbol{j} + \tfrac{11-\lambda}{6}\boldsymbol{k}, -\tfrac{\lambda+9}{6}\boldsymbol{j} + \tfrac{9-\lambda}{6}\boldsymbol{k}).$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are given in (2.1). The Lie algebra $\mathfrak{h}$ is a Bianchi Lie algebra of type III with brackets

$$[e_1, e_2] = 2e_2, \qquad [e_1, e_3] = 0, \qquad [e_2, e_3] = 0.$$

Note that the exponential map of a Bianchi Lie algebra of type III is surjective, then the Lie subgroup of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ associated with $\mathfrak{h}$ is equal to $\exp(\mathfrak{h})$. Therefore $\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2$ is the universal cover of $\exp(\mathfrak{h})$. The map $\phi_\lambda \colon \mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2 \to \exp(\mathfrak{h})$ given by

$$\phi_\lambda(u, v, w) = \exp\left(we_1 + \frac{uwe^{-w}}{\sinh w} e_2 + ve_3 \right)$$

with $\frac{we^{-w}}{\sinh w}$ extended to 1 when $w = 0$, is a surjective homomorphism with

$$H_\lambda = \ker(\phi_\lambda) \cong \begin{cases} \mathbb{Z} & \text{when } \lambda = \frac{2n^2}{n^2-m^2},\ m > n > 0 \text{ integers,} \\ \{0\} & \text{otherwise.} \end{cases}$$

One can check that the map $f_\lambda \colon (\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)/H_\lambda \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$f_\lambda(u, v, w) = \phi_\lambda(u, v, w) \cdot (\mathrm{Id}_2, \mathrm{Id}_2),$$

is a Lagrangian immersion, whose image is extrinsically homogeneous. The frame $\{E_1, E_2, E_3\}$ given by

$$E_1 = e^{2w}(f_\lambda)_u, \qquad E_2 = \frac{1-\lambda}{3} e^{2w}(f_\lambda)_u + (f_\lambda)_v, \qquad E_3 = \sqrt{\frac{3}{2}}(f_\lambda)_w,$$

is a Δ_2-orthonormal frame with respect to which A and B take type II form in Lemma 11 with $\theta_1 = \theta_2 = \frac{\pi}{3}$.

Remark 14. For any pair λ_1, λ_2 the subgroups $\phi_{\lambda_i}(\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ are non-conjugate. That is, there does not exist an automorphism of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ preserving the isotropy subgroup $\Delta\mathrm{SL}(2,\mathbb{R})$ that maps $\phi_{\lambda_1}(\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)$ into $\phi_{\lambda_2}(\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)$. This can be easily seen since conjugations by elements of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ preserve the indefinite inner product of $\mathfrak{sl}(2,\mathbb{R})$ given in (2.2).

Proposition 31. *Let $f\colon M \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type II form in Lemma 11 with respect to a Δ_2-orthonormal frame $\{E_1, E_2, E_3\}$. Then M is congruent to an open subset of either the image of the immersion in Example 21, or the image of the immersion in Example 22.*

Proof. Because of Proposition 30 we may assume that $\theta_1 = \theta_2 = \frac{2}{3}\pi$. Moreover, we divide in two cases.

Suppose that ω_{ij}^k and h_{ij}^k satisfy (1) in Proposition 30. We take the frame $\{\rho E_1, \rho E_2, E_3\}$ where ρ is a non-vanishing smooth function and solution of

$$E_1(\rho) = E_2(\rho) = 0, \qquad E_3(\rho) = \sqrt{\tfrac{3}{2}}\rho. \tag{3.56}$$

It is easy to check that ρ indeed exists and that

$$[\rho E_1, \rho E_2] = [\rho E_1, E_3] = [\rho E_2, E_3] = 0.$$

Hence, there exist local coordinates u, v, w such that $\rho E_1 = f_u$, $\rho E_2 = f_v$ and $E_3 = f_w$ and hence $\rho(w) = e^{\sqrt{\frac{3}{2}}w}$. It follows from the relation between Q and P given in (2.16) that

$$Qf_u = -f_u, \quad Qf_v = -\frac{2}{3}f_u - f_v + \frac{2}{\sqrt{3}}Jf_u, \quad Qf_w = -f_w.$$

Writing $f = (p, q)$ yields

$$q_u = q_w = 0, \qquad q_v = \tfrac{2}{3}qp^{-1}p_u. \tag{3.57}$$

Using the relation between the nearly Kähler connection $\tilde{\nabla}$ and the product connection ∇^E given in (2.17) we obtain

$$\begin{aligned}
&\nabla^E_{f_u} f_u = 0, && \nabla^E_{f_u} f_v = -\sqrt{\tfrac{3}{2}}\rho^2 f_w,\\
&\nabla^E_{f_u} f_w = \sqrt{\tfrac{3}{2}} f_u, && \nabla^E_{f_v} f_u = -\sqrt{\tfrac{3}{2}}\rho^2 f_w,\\
&\nabla^E_{f_v} f_v = -\sqrt{\tfrac{2}{3}}\rho^2 f_w, && \nabla^E_{f_v} f_w = \tfrac{1}{\sqrt{6}} f_u + \sqrt{\tfrac{3}{2}} f_v - \tfrac{1}{\sqrt{2}} J f_u,\\
&\nabla^E_{f_w} f_u = \sqrt{\tfrac{3}{2}} f_u, && \nabla^E_{f_w} f_w = 0,\\
&\nabla^E_{f_w} f_v = \tfrac{1}{\sqrt{6}} f_u + \sqrt{\tfrac{3}{2}} f_v - \tfrac{1}{\sqrt{2}} J f_u.
\end{aligned} \tag{3.58}$$

Now we compute

$$\begin{aligned}
&\langle f_u, f_u\rangle = 0, && \langle f_v, f_v\rangle = \rho^2, && \langle f_w, f_w\rangle = \tfrac{3}{2},\\
&\langle f_u, f_v\rangle = \tfrac{3}{2}\rho^2, && \langle f_u, f_w\rangle = 0, && \langle f_v, f_w\rangle = 0,
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
&\langle f_u, Qf_u\rangle = 0, && \langle f_u, Qf_w\rangle = 0, && \langle f_v, Qf_w\rangle = 0,\\
&\langle f_u, Qf_v\rangle = -\tfrac{3}{2}\rho^2, && \langle f_v, Qf_v\rangle = -\rho^2, && \langle f_w, Qf_w\rangle = -\tfrac{3}{2},
\end{aligned} \tag{3.60}$$

where $\langle , \rangle$ is the product metric associated to the metric on $\mathrm{SL}(2,\mathbb{R})$ given in (2.2). In particular, we have

$$\begin{aligned}
&\langle p_u, p_u\rangle = 0, && \langle p_v, p_v\rangle = \rho^2, && \langle p_w, p_w\rangle = \tfrac{3}{2}\\
&\langle p_u, p_v\rangle = \tfrac{3}{2}\rho^2, && \langle p_u, p_w\rangle = 0, && \langle p_v, p_w\rangle = 0.
\end{aligned} \tag{3.61}$$

Here, $\langle , \rangle$ is the metric on $\mathrm{SL}(2,\mathbb{R})$ given in (2.2).

To compute the second derivatives of f, we use the expression for the Euclidean connection of $\mathbb{R}^8_4$ in (2.18). Plugging (3.58), (3.59) and (3.60) into (2.18) we obtain

$$\begin{aligned}
&f_{uu} = 0, && f_{uv} = -\sqrt{\tfrac{3}{2}}\rho^2 f_w + \tfrac{3}{4}\rho^2 f - \tfrac{3}{4}\rho^2 Qf,\\
&f_{uw} = \sqrt{\tfrac{3}{2}} f_u, && f_{vv} = -\sqrt{\tfrac{2}{3}}\rho^2 f_w + \tfrac{1}{2}\rho^2 f - \tfrac{1}{2}\rho^2 Qf,\\
&f_{vw} = \tfrac{1}{\sqrt{6}} f_u + \sqrt{\tfrac{3}{2}} f_v - \tfrac{1}{\sqrt{2}} J f_u, && f_{ww} = \tfrac{3}{4} f - \tfrac{3}{4} Qf.
\end{aligned}$$

Looking at each component of f we obtain differential equations for p and q:

$$p_{uu} = 0, \qquad p_{uv} = -\sqrt{\tfrac{3}{2}}\rho^2 p_w + \tfrac{3}{2}\rho^2 p, \qquad p_{uw} = \sqrt{\tfrac{3}{2}} p_u,$$
$$p_{vw} = \sqrt{\tfrac{3}{2}} p_v, \qquad p_{vv} = -\sqrt{\tfrac{2}{3}}\rho^2 p_w + \rho^2 p, \qquad p_{ww} = \tfrac{3}{2} p, \qquad q_{vv} = 0. \tag{3.62}$$

The other derivatives of q are zero because of (3.57).

Applying an isometry of the type $(p,q) \mapsto (ap, bq)$, we may assume initial conditions $(p(0), q(0)) = (\mathrm{Id}_2, \mathrm{Id}_2)$. Then from (3.61) and (3.57) it follows that there exists $c \in \mathrm{SL}^{\pm}(2,\mathbb{R})$ such that

$$p_u(0) = c \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} c^{-1}, \qquad p_v(0) = c \begin{pmatrix} 0 & \frac{1}{3} \\ 3 & 0 \end{pmatrix} c^{-1},$$
$$p_w(0) = \sqrt{\frac{3}{2}} c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c^{-1}, \quad q_v(0) = c \begin{pmatrix} 0 & \frac{2}{3} \\ 0 & 0 \end{pmatrix} c^{-1}.$$

Applying the isometry $(p,q) \mapsto (cpc^{-1}, cqc^{-1})$ we obtain that any solution of (3.62) is congruent to an open subset of the immersion $f = (p,q)$ where

$$p(u,v,w) = \begin{pmatrix} e^{\sqrt{\frac{3}{2}}w} & e^{\sqrt{\frac{3}{2}}w}\left(u + \frac{v}{3}\right) \\ 3e^{\sqrt{\frac{3}{2}}w} v & e^{\sqrt{\frac{3}{2}}w}\left(v^2 + 3uv\right) + e^{-\sqrt{\frac{3}{2}}w} \end{pmatrix}, \quad q(v) = \begin{pmatrix} 1 & \frac{2v}{3} \\ 0 & 1 \end{pmatrix}.$$

Finally, taking the change of coordinates $w \to 2\sqrt{\frac{2}{3}}w$, $u \to -\frac{1}{2}(u+v)$ and $v \to \frac{3}{2}(u-v)$ we get the immersion in (3.55).

Now suppose that ω^k_{ij} and h^k_{ij} satisfy (2) in Proposition 30. In this case, ω^3_{22} is constant. We define the constant λ as $\omega^3_{22} = \sqrt{\frac{2}{3}}(1-\lambda)$.

Take the frame $\left\{\rho E_1, -\frac{1}{\sqrt{6}}\omega^3_{22}E_1 + E_2, E_3\right\}$, where ρ is a non-vanishing smooth function and solution of

$$E_1(\rho) = E_2(\rho) = 0, \qquad E_3(\rho) = -\sqrt{6}E_3.$$

Using Proposition 30 we can easily check that this is a coordinate frame. We call this frame $\{f_u, f_v, f_w\}$. First we notice that $\rho = e^{-\sqrt{6}w}$. We obtain from Equation (2.16) that

$$Qf_u = -f_u, \quad Qf_v = -\tfrac{2}{3\rho}f_u - f_v + \tfrac{2}{\sqrt{3}\rho}Jf_u, \quad Qf_w = -f_w.$$

We deduce

$$q_u = q_w = 0, \qquad q_v = \tfrac{2}{3\rho} q p^{-1} p_u. \tag{3.63}$$

Equation (2.17) gives us the following expressions

$$\nabla^E_{f_u} f_u = 0, \qquad \nabla^E_{f_u} f_v = \sqrt{\frac{3}{2}}\rho f_w,$$

$$\nabla^E_{f_v} f_v = -\sqrt{\frac{2}{3}} f_w + \sqrt{2} J f_w, \quad \nabla^E_{f_u} f_w = -\sqrt{\frac{3}{2}} f_u,$$

$$\nabla^E_{f_w} f_w = 0, \qquad \nabla^E_{f_v} f_w = \tfrac{2\lambda-1}{\sqrt{6}\rho} f_u - \sqrt{\tfrac{3}{2}} f_v + \tfrac{1}{\sqrt{2}\rho} J f_u.$$

From Equation (2.15) it follows

$$\langle f_u, f_u\rangle = 0, \qquad \langle f_u, f_v\rangle = \tfrac{3\rho}{2}, \qquad \langle f_u, f_w\rangle = 0,$$

$$\langle f_v, f_v\rangle = \lambda, \qquad \langle f_v, f_w\rangle = 0, \qquad \langle f_w, f_w\rangle = \tfrac{3}{2}.$$

and

$$\langle f_u, Qf_u\rangle = 0, \qquad \langle f_u, Qf_v\rangle = -\tfrac{3\rho}{2}, \qquad \langle f_u, Qf_w\rangle = 0,$$

$$\langle f_v, Qf_v\rangle = -\lambda, \qquad \langle f_v, Qf_w\rangle = 0, \qquad \langle f_w, Qf_w\rangle = -\tfrac{3}{2}.$$

In particular, we have

$$\begin{aligned} &\langle p_u, p_u\rangle = 0, \quad \langle p_v, p_v\rangle = \lambda, \quad \langle p_w, p_w\rangle = \tfrac{3}{2}, \\ &\langle p_u, p_v\rangle = \tfrac{3}{2}\rho, \quad \langle p_u, p_w\rangle = 0, \quad \langle p_v, p_w\rangle = 0. \end{aligned} \tag{3.64}$$

We may use Equation (2.18) to compute

$$f_{uu} = 0, \qquad f_{vv} = -\sqrt{\frac{2}{3}} f_w + \sqrt{2} J f_w + \frac{1}{2}\lambda f - \frac{1}{2}\lambda Q f,$$

$$f_{ww} = \tfrac{3}{4} f - \tfrac{3}{4} Q f, \qquad f_{uw} = -\sqrt{\frac{3}{2}} f_u,$$

$$f_{vw} = \tfrac{2\lambda-1}{\sqrt{6}\rho} f_u - \sqrt{\tfrac{3}{2}} f_v + \tfrac{1}{\sqrt{2}\rho} J f_u, \quad f_{uv} = \sqrt{\frac{3}{2}}\rho f_w + \frac{3\rho}{4} f - \frac{3\rho}{4} Q f.$$

Hence we obtain

$$\begin{aligned} &p_{uu} = 0, \qquad p_{uv} = \sqrt{\tfrac{3}{2}}\rho p_w + \tfrac{3}{2}\rho p, \qquad p_{uw} = -\sqrt{\tfrac{3}{2}} p_u, \\ &p_{vv} = \lambda p, \qquad p_{vw} = \tfrac{1}{\rho}\sqrt{\tfrac{2}{3}}\lambda p_u - \sqrt{\tfrac{3}{2}} p_v, \qquad p_{ww} = \tfrac{3}{2} p, \\ &q_{vv} = 2\sqrt{\tfrac{2}{3}} q p^{-1} p_w. \end{aligned} \tag{3.65}$$

Applying an isometry of the type $(p, q) \mapsto (ap, bq)$ we may assume that $p(0) = \mathrm{Id}_2$ and $q(0) = \mathrm{Id}_2$. From (3.64) and (3.63) it follows that there exists a matrix $c \in \mathrm{SL}^{\pm}(2\mathbb{R})$ such that

$$p_u(0) = c \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} c^{-1}, \qquad p_v(0) = c \begin{pmatrix} 0 & \frac{\lambda}{3} \\ 3 & 0 \end{pmatrix} c^{-1},$$

$$p_w(0) = \sqrt{\frac{3}{2}} c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c^{-1}, \quad q_v(0) = c \begin{pmatrix} 0 & \frac{2}{3} \\ 0 & 0 \end{pmatrix} c^{-1}.$$

Applying the isometry $(p, q) \mapsto (cpc^{-1}, cqc^{-1})$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we obtain that any solution of (3.65) is congruent to the solution with $c = \mathrm{Id}_2$. After the change of coordinates $w \to \sqrt{\frac{3}{2}} w$, $u \to u e^{-\sqrt{\frac{3}{2}} w}$, we obtain that such solution is the immersion f_λ in Example 22. □

3.3.4 Extrinsically homogeneous Lagrangian submanifolds of type III

Example 23. Let $\mathbb{R} \ltimes_{\varphi_2} \mathbb{R}^2$ be the Bianchi group of type VI with the group law

$$(t, u) \cdot (s, v) = (t + s, \varphi_2(s)u + v)$$

where $\varphi_2 \colon \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is given by

$$\varphi_2(s) = \begin{pmatrix} e^{-2s} & 0 \\ 0 & e^{s} \end{pmatrix}.$$

Let $\tilde{g}$ be the right invariant metric such that its components with respect to the frame of right invariant vector fields $\left\{ \frac{\partial}{\partial t}, e^{-2t} \frac{\partial}{\partial u_1}, e^{t} \frac{\partial}{\partial u_2} \right\}$ are given by

$$\begin{pmatrix} 0 & 2 & \frac{7}{3}\sqrt{\frac{2}{3}} \\ 2 & 0 & 8\sqrt{\frac{2}{3}} \\ \frac{7}{3}\sqrt{\frac{2}{3}} & 8\sqrt{\frac{2}{3}} & \frac{128}{9} \end{pmatrix}.$$

Now let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ spanned by $\{e_1, e_2, e_3\}$ where

$$\begin{aligned}
e_1 = &\left(\tfrac{1}{18}\left(27 + 2\sqrt{6}\right)\boldsymbol{i} - \left(2\sqrt{\tfrac{2}{3}} + \tfrac{3}{4}\right)\boldsymbol{j} + \left(\tfrac{8}{3}\sqrt{\tfrac{2}{3}} + \tfrac{3}{4}\right)\boldsymbol{k}, \right. \\
&- \left(1 + \tfrac{17}{12\sqrt{6}}\right)\boldsymbol{i} + \tfrac{1}{96}\left(48 - 17\sqrt{6}\right)\boldsymbol{j} - \left(\tfrac{1}{2} + \tfrac{85}{48\sqrt{6}}\right)\boldsymbol{k}, \\
&\left. - \tfrac{1}{2}\boldsymbol{i} + \tfrac{1}{4}\left(1 - 3\sqrt{6}\right)\boldsymbol{j} + \tfrac{1}{4}\left(3\sqrt{6} - 1\right)\boldsymbol{k} \right), \\
e_2 = &\left(0, \sqrt{\tfrac{2}{3}}\boldsymbol{i} + \tfrac{1}{2}\sqrt{\tfrac{3}{2}}\boldsymbol{j} + \tfrac{5}{2\sqrt{6}}\boldsymbol{k}, 0\right), \\
e_3 = &\left(\tfrac{8}{9}\left(2 + 3\sqrt{6}\right)\boldsymbol{i} - \tfrac{2}{3}\left(7 + 2\sqrt{6}\right)\boldsymbol{j} + \tfrac{2}{9}\left(37 + 6\sqrt{6}\right)\boldsymbol{k}, 0, -6\boldsymbol{j} + 6\boldsymbol{k}\right).
\end{aligned}$$

The Lie algebra $\mathfrak{h}$ is a Bianchi Lie algebra of type VI with brackets

$$[e_1, e_2] = -2e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = 0.$$

Note that the exponential map of a Bianchi Lie algebra of type VI is surjective, then the Lie subgroup of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ associated with $\mathfrak{h}$ is equal to $\exp(\mathfrak{h})$. Therefore $\mathbb{R} \ltimes_{\varphi_2} \mathbb{R}^2$ is the universal cover of $\exp(\mathfrak{h})$. Moreover, the map defined as

$$(v, u, w) \mapsto \exp\left(ve_1 + \frac{2ue^{2v}v}{e^{2v} - 1}e_2 + \frac{vw}{e^v - 1}e_3\right)$$

and as $\exp(2ue_2 + we_3)$ when $v = 0$, is a group isomorphism. Thus $\exp(\mathfrak{h}) \cong \mathbb{R} \ltimes_{\varphi_2} \mathbb{R}^2$. One can check that the immersion $\jmath : \mathbb{R} \ltimes_{\varphi_2} \mathbb{R}^2 \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given by

$$\jmath(u, v, w) = \exp\left(ve_1 + \frac{2ue^{2v}v}{e^{2v} - 1}e_2 + \frac{vw}{e^v - 1}e_3\right) \cdot (\mathrm{Id}_2, \mathrm{Id}_2)$$

is a Lagrangian immersion, whose image is extrinsically homogeneous. The frame given by

$$E_1 = -\sqrt{\frac{3}{2}}e^{-2v}\jmath_u, \quad E_2 = -\frac{1}{\sqrt{6}}\jmath_v, \quad E_3 = \frac{7e^{-2v}}{4\sqrt{6}}\jmath_u + \sqrt{6}\jmath_v - \frac{3}{4}e^v\jmath_w.$$

is a Δ_2-orthonormal frame with respect to which A and B take type III form from Lemma 11.

Proposition 32. *Let M be an extrinsically homogeneous Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type III form from Lemma 11 with respect to a Δ_2-orthonormal frame $\{E_1, E_2, E_3\}$. Then M is congruent to an open subset of the submanifold given in Example 23.*

Proof. By Lemma 10 we may apply the isometry $\Psi_{0,1}$ in (2.19) and assume that the sign of B is -1. Proposition 25 implies that the components of the second fundamental form and of the connection associated to E_1, E_2 and E_3 are all constant. From the Codazzi equation (3.2) with $X = E_1$, $Y = E_2$ and $Z = E_2$ we have $h^2_{22} = \frac{2\sqrt{2}}{3}$. Then, computing the Codazzi equation with $X = E_3$, $Y = E_2$ and $Z = E_2$ we obtain

$$h^1_{22} = -\frac{13}{18\sqrt{2}}, \qquad h^3_{22} = \frac{5\sqrt{2}}{9}.$$

We define the frame $\left\{\rho E_1, E_2, \rho^{-1/2}(\frac{7}{12\sqrt{6}}E_1 + \sqrt{6}E_2 + \frac{1}{\sqrt{6}}E_3)\right\}$ where ρ is a non-vanishing smooth function that satisfies

$$E_1(\rho) = 0, \quad E_2(\rho) = \sqrt{\tfrac{2}{3}}\rho, \quad E_3(\rho) = -2\sqrt{6}\rho.$$

From (3.28) we can check that ρ indeed exists and that the defined frame is a coordinate frame. We write said frame as $\{f_u, f_v, f_w\}$. Hence $\rho = e^{\sqrt{\frac{2}{3}}v}$. Equation (2.16) yields

$$Qf_u = f_u,$$

$$Qf_v = -\frac{23}{18\rho}f_u - 3f_v + 2\sqrt{\frac{2\rho}{3}}f_w - \frac{7}{6\sqrt{3}\rho}Jf_u - 4\sqrt{3}Jf_v + 2\sqrt{2\rho}Jf_w,$$

$$Qf_w = -\frac{7\rho^{-3/2}}{\sqrt{6}}f_u - 4\sqrt{\frac{6}{\rho}}f_v + 5f_w - \frac{5\rho^{-3/2}}{3\sqrt{2}}Jf_u - 12\sqrt{\frac{2}{\rho}}Jf_v + 4\sqrt{3}Jf_w.$$

If we denote $f = (p, q)$ then the first equation implies that $p_u = 0$. From the second and third equations we obtain

$$p_v = pq^{-1}\left(-\frac{5q_u}{6\rho} - 4q_v + 2\sqrt{\frac{2}{3}}\sqrt{\rho}q_w\right), \quad p_w = pq^{-1}\left(4q_w - \frac{13q_u + 72\rho q_v}{3\sqrt{6}\rho^{3/2}}\right). \tag{3.66}$$

The relation between the connection ∇^E associated to the product metric and the nearly Kähler connection $\tilde{\nabla}$ given in Equation (2.17) yields

$$\begin{aligned}
\nabla^E_{f_u} f_u &= 0,\\
\nabla^E_{f_u} f_v &= -\tfrac{11}{4\sqrt{6}} f_u - 3\sqrt{6}\rho f_v + 3\rho^{3/2} f_w + \tfrac{1}{\sqrt{2}} J f_u,\\
\nabla^E_{f_u} f_w &= -\tfrac{13}{4\sqrt{\rho}} f_u - 18\sqrt{\rho} f_v + 3\sqrt{6}\rho f_w + \sqrt{\tfrac{3}{\rho}} J f_u,\\
\nabla^E_{f_v} f_v &= \tfrac{49}{48\sqrt{6}\rho} f_u + \tfrac{29}{2\sqrt{6}} f_v - \tfrac{7\sqrt{\rho}}{4} f_w - \tfrac{17}{9\sqrt{2}\rho} J f_u - 5\sqrt{2} J f_v + 2\sqrt{3\rho} J f_w,\\
\nabla^E_{f_v} f_w &= \tfrac{91}{96\rho^{3/2}} f_u + \tfrac{53}{4\sqrt{\rho}} f_v - \tfrac{43}{4\sqrt{6}} f_w - \tfrac{25}{6\sqrt{3}\rho^{3/2}} J f_u - 7\sqrt{\tfrac{3}{\rho}} J f_v + \tfrac{9}{\sqrt{2}} J f_w,\\
\nabla^E_{f_w} f_w &= \tfrac{19}{3\sqrt{6}\rho^2} f_u + \tfrac{12\sqrt{6}}{\rho} f_v - \tfrac{10}{\sqrt{\rho}} f_w - \tfrac{17}{3\sqrt{2}\rho^2} J f_u - \tfrac{12\sqrt{2}}{\rho} J f_v + 6\sqrt{\tfrac{3}{\rho}} J f_w.
\end{aligned}$$

From the relation between the product metric and the nearly Kähler metric in Equation (2.15) it follows

$$\begin{aligned}
&\langle f_u, f_u\rangle = 0, \qquad \langle f_v, f_v\rangle = 0, \qquad \langle f_w, f_w\rangle = \tfrac{4}{\rho},\\
&\langle f_u, f_v\rangle = \tfrac{3}{2}\rho, \qquad \langle f_u, f_w\rangle = 3\sqrt{\tfrac{3\rho}{2}}, \qquad \langle f_v, f_w\rangle = \tfrac{5}{8}\sqrt{\tfrac{3}{2\rho}},
\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}
&\langle f_u, Qf_u\rangle = 0, \qquad \langle f_v, Qf_v\rangle = -\tfrac{4}{3}, \qquad \langle f_w, Qf_w\rangle = -\tfrac{4}{\rho},\\
&\langle f_u, Qf_v\rangle = \tfrac{3}{2}\rho, \qquad \langle f_u, Qf_w\rangle = 3\sqrt{\tfrac{3\rho}{2}}, \qquad \langle f_v, Qf_w\rangle = -\tfrac{49}{8\sqrt{6\rho}}.
\end{aligned} \tag{3.68}$$

In particular we have

$$\begin{aligned}
&\langle q_u, q_u\rangle = 0, \qquad \langle q_v, q_v\rangle = -\tfrac{2}{3}, \qquad \langle q_w, q_w\rangle = 0,\\
&\langle q_u, q_v\rangle = \tfrac{3}{2}\rho, \qquad \langle q_u, q_w\rangle = 3\sqrt{\tfrac{3\rho}{2}}, \qquad \langle q_v, q_w\rangle = -\tfrac{17}{8\sqrt{6\rho}}.
\end{aligned} \tag{3.69}$$

Now we compute

$$
\begin{aligned}
f_{uu} &= 0,\\
f_{uv} &= -\tfrac{11}{4\sqrt{6}} f_u - 3\sqrt{6}\rho f_v + 3\rho^{3/2} f_w + \tfrac{1}{\sqrt{2}} J f_u + \tfrac{3\rho}{4} f + \tfrac{3\rho}{4} Q f,\\
f_{uw} &= -\tfrac{13}{4\sqrt{\rho}} f_u - 18\sqrt{\rho} f_v + 3\sqrt{6}\rho f_w + \sqrt{\tfrac{3}{\rho}} J f_u + \tfrac{3}{2}\sqrt{\tfrac{3}{2}}\sqrt{\rho} f + \tfrac{3}{2}\sqrt{\tfrac{3\rho}{2}} Q f,\\
f_{vv} &= \tfrac{49}{48\sqrt{6}\rho} f_u + \tfrac{29}{2\sqrt{6}} f_v - \tfrac{7\sqrt{\rho}}{4} f_w - \tfrac{17}{9\sqrt{2}\rho} J f_u - 5\sqrt{2} J f_v + 2\sqrt{3\rho} J f_w - \tfrac{2}{3} Q f,\\
f_{vw} &= \tfrac{91}{96\rho^{3/2}} f_u + \tfrac{53}{4\sqrt{\rho}} f_v - \tfrac{43}{4\sqrt{6}} f_w - \tfrac{25}{6\sqrt{3}\rho^{3/2}} J f_u - 7\sqrt{\tfrac{3}{\rho}} J f_v + \tfrac{9}{\sqrt{2}} J f_w\\
&\quad + \tfrac{5}{16}\sqrt{\tfrac{3}{2\rho}} f - \tfrac{49}{16\sqrt{6\rho}} Q f,\\
f_{ww} &= \tfrac{19}{3\sqrt{6}\rho^2} f_u + \tfrac{12\sqrt{6}}{\rho} f_v - \tfrac{10}{\sqrt{\rho}} f_w - \tfrac{17}{3\sqrt{2}\rho^2} J f_u - \tfrac{12\sqrt{2}}{\rho} J f_v + 6\sqrt{\tfrac{3}{\rho}} J f_w\\
&\quad + \tfrac{2}{\rho} f - \tfrac{2}{\rho} Q f.
\end{aligned}
$$

Hence q satisfies

$$
\begin{aligned}
q_{uu} &= 0,\\
q_{ww} &= 0,\\
q_{uv} &= \frac{1}{8}\left(12\rho\left(q - 2\sqrt{6} q_v + 2\sqrt{\rho} q_w\right) - 5\sqrt{6} q_u\right),\\
q_{uw} &= \frac{1}{4\sqrt{\rho}}\left(6\rho\left(\sqrt{6} q - 12 q_v + 2\sqrt{6\rho} q_w\right) - 17 q_u\right),\\
q_{vv} &= \frac{1}{864\rho}\left(323\sqrt{6} q_u - 72\rho\left(8q - 17\sqrt{6} q_v + 13\sqrt{\rho} q_w\right)\right),\\
q_{vw} &= \frac{1}{288\rho^{3/2}}\left(289 q_u - 6\rho\left(17\sqrt{6} q - 204 q_v + 30\sqrt{6\rho} q_w\right)\right),
\end{aligned}
\tag{3.70}
$$

and p satisfies

$$\begin{aligned} p_{vv} &= \frac{2}{3}p - \frac{1}{18\sqrt{6}\rho}\left(19pq^{-1}q_u + 72\rho pq^{-1}q_v\right), \\ p_{vw} &= \frac{1}{6\rho^{3/2}}\left(4\rho\left(\sqrt{6}p - 6pq^{-1}q_v\right) - 7pq^{-1}q_u\right), \\ p_{ww} &= \frac{1}{9\rho^2}\left(36\rho\left(p - 2\sqrt{6}pq^{-1}q_v + \sqrt{\rho}pq^{-1}q_w\right) - 17\sqrt{6}pq^{-1}q_u\right). \end{aligned} \tag{3.71}$$

After applying an isometry of the type $(p, q) \mapsto (ap, bq)$, we may assume that $p(0) = \mathrm{Id}_2$ and $q(0) = \mathrm{Id}_2$. From (3.66) and (3.69) it follows that there exists a matrix $c \in \mathrm{SL}^{\pm}(2, \mathbb{R})$ such that

$$q_u(0) = c\begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{pmatrix}c^{-1},$$

$$q_w(0) = c\begin{pmatrix} 0 & 0 \\ 3\sqrt{\frac{3}{2}} & 0 \end{pmatrix}c^{-1},$$

$$q_v(0) = c\begin{pmatrix} \frac{1}{72}\left(-17 + 6\sqrt{6}\right) & -\frac{17}{36} \\ \frac{1}{144}\left(233 - 12\sqrt{6}\right) & \frac{1}{72}\left(17 - 6\sqrt{6}\right) \end{pmatrix}c^{-1},$$

$$p_v(0) = c\begin{pmatrix} \frac{1}{9} - \sqrt{\frac{2}{3}} & \frac{2}{9} \\ \sqrt{\frac{2}{3}} - \frac{1}{18} & \sqrt{\frac{2}{3}} - \frac{1}{9} \end{pmatrix}c^{-1},$$

$$p_w(0) = c\begin{pmatrix} -2 + \frac{2}{9}\sqrt{6} & \frac{4}{3}\sqrt{\frac{2}{3}} \\ 2 - \frac{1}{3}\sqrt{\frac{2}{3}} & 2 - \frac{2}{9}\sqrt{6} \end{pmatrix}c^{-1}.$$

By applying the isometry $(p, q) \mapsto (cpc^{-1}, cqc^{-1})$ we obtain that any solution of the system of differential equations given in (3.70) and (3.71) is congruent to the solution with $c = \mathrm{Id}_2$. This solution is the map given in Example 23 after the change of coordinates $v \to \frac{v}{\sqrt{6}}$, $u \to \sqrt{\frac{3}{2}}u$, $w \to \frac{1}{4}\sqrt{\frac{3}{2}}w$. □

3.3.5 Extrinsically homogeneous Lagrangian submanifolds of type IV

Proposition 33. *There are no extrinsically homogeneous Lagrangian submanifolds of the pseudo-nearly Kähler* $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ *of type IV in Lemma 11.*

Proof. Let M be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Suppose that A and B take type IV form with respect to a Δ_3-orthonormal frame $\{E_1, E_2, E_3\}$. Then by Proposition 26 the functions ψ, θ_1, θ_2, h^k_{ij} and ω^k_{ij} are constant. Thus by Lemma 15 the functions h^k_{ij} are all zero except for h^3_{11}. Recall that we can write $\theta_2 = -2\theta_1$. Computing the Codazzi equation (3.2) with $X = E_3$, $Y = E_1$, $Z = E_1$ and $X = E_1$, $Y = E_2$, $Z = E_2$ yields

$$(h^3_{11})^2 = \frac{1}{3}(\cosh\psi - \cos 6\theta_1)\cosh\psi, \tag{3.72}$$

and

$$\sqrt{\frac{2}{3}}h^3_{11} = \frac{4\sinh(\psi)\left(-\cos 6\theta_1 \cosh\psi + 3(h^3_{11})^2 + \cosh^2(\psi)\right)}{3(\cos 6\theta_1 - \cosh\psi)}. \tag{3.73}$$

Plugging (3.72) into (3.73) gives

$$h^3_{11} = -2\sqrt{\frac{2}{3}}\sinh 2\psi.$$

Comparing both values of $(h^3_{11})^2$ we derive $\cos 6\theta_1 = 9\cosh\psi - 8\cosh 3\psi$. Then writing the Codazzi equation (3.2) with $X = E_1$, $Y = E_3$ and $Z = E_2$ we obtain $\sinh 2\psi = 0$. This is a contradiction since by Lemma 11, ψ is different from zero. □

3.3.6 Proof of Theorem 33

Proof of Theorem 33. By Lemma 11 we separate the argument into four cases.

In Section 3.2 we showed that any totally geodesic submanifold is of type I and is congruent to one of the first three examples given in Theorem 33. By Proposition 27, Proposition 28 and Proposition 29, any non-totally geodesic extrinsically homogeneous Lagrangian submanifold of type I is congruent to either Example 20 or to Example 19.

Proposition 31 implies that any extrinsically homogeneous Lagrangian submanifold of type II is congruent to an open subset of either the image of the immersion in Example 21 or the image of the immersion in Example 22.

Proposition 32 states that any extrinsically homogeneous Lagrangian submanifold of type III is congruent to the one given in Example 23.

Proposition 33 shows that there are no extrinsically homogeneous Lagrangian submanifolds of type IV.

Except for the first, sixth and seventh examples in Theorem 33, all the submanifolds are not isometric and therefore not congruent. Hence, it only remains to distinguish between the aforementioned cases.

First, the first submanifold in Theorem 33 is the first example of Theorem 32, therefore the only one of these three that is totally geodesic. The sixth submanifold is the immersion $\imath$ given in Example 21 and the seventh one is the family of immersions f_λ given in Example 22.

Suppose that ι is congruent to f_λ for some λ. That means, there exists a isometry $\mathcal{F}$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ that maps one into the other. Suppose that $\mathcal{F} \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \rtimes \mathbb{Z}_2$. These isometries preserve P and J, hence A and B have the same shape with respect to $\{E_i\}_i$ and with respect to $\{\mathcal{F}_* E_i\}_i$. Hence,

$$\begin{aligned} -\frac{\sqrt{2}}{3} &= g(h(E_2,E_2), JE_3) \\ &= g(\mathcal{F}_* h(E_2,E_2), \mathcal{F}_* JE_3) \\ &= g(h(\mathcal{F}_* E_2, \mathcal{F}_* E_2), J\mathcal{F}_* E_3) \\ &= \frac{2\sqrt{2}}{3}, \end{aligned}$$

which is a contradiction.

In Theorem 28 we showed that the isometry group of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is a semidirect product of S_3 with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \rtimes \mathbb{Z}_2$. Therefore, to complete the proof we can assume that $\mathcal{F} \in S_3$, i.e. $\mathcal{F} = \Psi_{\kappa,\tau}$ for some $\kappa \in \{0,1\}$, $\tau \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$. Moreover, we can assume that $\tau \neq 0$, since otherwise $\mathcal{F}$ preserves P, and therefore we may use the same argument as before, up to sign. From Lemma 10 it follows that P restricted to $\mathcal{F}(M)$ takes the shape $\tilde{A} + J\tilde{B}$, where

$$\tilde{A} = \cos\tau A + (-1)^\kappa \sin\tau B,$$

$$\tilde{B} = -\sin\tau A + (-1)^\kappa \cos\tau B.$$

For $\tau \neq 0$ these matrices have a different form than A and B, as it can be seen in Lemma 9. Therefore, there does not exist such a Δ_2-orthonormal frame $\{E_i\}_i$ such that A and B take type II form in Lemma 11 on $\mathcal{F}(M)$, which is a contradiction.

Similar arguments can be used to distinguish between f_{λ_1} and f_{λ_2} for $\lambda_1 \neq \lambda_2$, by considering the function $\omega^2_{33} = \sqrt{\frac{2}{3}}(1-\lambda)$ instead of h^3_{22}. □

3.4 An example of a type IV Lagrangian submanifold

In Lemma 11 we defined four types of Lagrangian submanifolds. So far we have only seen examples of type I, II and III, so it is natural to ask if there exist Lagrangian submanifolds of type IV at all.

In this section, following the construction for $\mathbb{S}^3 \times \mathbb{S}^3$ given in [7], we construct an example of a type IV Lagrangian submanifold.

Theorem 34. *Let $p : \Sigma \to \mathrm{SL}(2,\mathbb{R})$ be a minimal surface of $(\mathrm{SL}(2,\mathbb{R}), \langle,\rangle)$, where $\langle,\rangle$ is the Lorentzian metric given in (2.2). Let ω be a solution of the* cosh*-Gordon equation with $\omega < \log \frac{1}{\sqrt{3}}$. Suppose that $\{u, v\}$ are null coordinates with $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\rangle = e^{\omega}$ and $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = \sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = -1$ where σ is the component of the second fundamental form in the direction of the unit normal ξ. Then the distribution on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ spanned by $(p\alpha_i, q\beta_i)$ with $dp(\frac{\partial}{\partial u}) = p\alpha_1$, $dp(\frac{\partial}{\partial v}) = p\alpha_2$, $\alpha_3 = 0$ and*

$$\beta_1 = \frac{1}{2}\alpha_1 - \frac{e^{-\omega}}{2}\alpha_2 - \frac{1}{4}\omega_u e^{-\omega}\alpha_1 \times \alpha_2$$

$$\beta_2 = \frac{e^{-\omega}}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{e^{-\omega}\omega_v}{4}\alpha_1 \times \alpha_2$$

$$\beta_3 = -\sqrt{\frac{3}{2}}e^{-\omega}\alpha_1 \times \alpha_2$$

is integrable and the integral manifold is a Lagrangian submanifold of type IV in Lemma 11 with $\tanh\left(\frac{\psi}{2}\right) = e^{\omega}$ and $\theta_1 = \theta_2 = 2\pi/3$.

In order to prove this, we need first to introduce some theory about minimal submanifolds of $\mathrm{SL}(2,\mathbb{R}) = H^3_1(-1)$.

3.4.1 Time-like minimal surfaces of $\mathrm{SL}(2,\mathbb{R})$

Given a two-dimensional pseudo-Riemannian manifold Σ (which we will call a timelike surface), a null coordinate frame is a set of coordinates $\{u, v\}$ on Σ such that $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\rangle = e^{\omega}$ and $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u}\rangle = \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v}\rangle = 0$, where ω is a function on Σ.

For such coordinates, the Levi-Civita connection ∇ on Σ becomes

$$\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u} = \omega_u \frac{\partial}{\partial u}, \quad \nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v} = \nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial u} = 0, \quad \nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v} = \omega_v \frac{\partial}{\partial v}. \tag{3.74}$$

For the existence of null coordinates on surfaces see [52]. Here we give a stronger version of the existence result for minimal surfaces in $H^3_1(-1)$.

Lemma 21. *Given a minimal surface Σ of the three-dimensional anti-de Sitter space $H^3_1(-1)$, there exist local null coordinates $\{u, v\}$ on Σ that satisfy $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = 0$ and one of the following:*

1. $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = \pm 1$, $\sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = \pm 1$,
2. $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = 0$, $\sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = \pm 1$,
3. $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = \sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = 0$,

where σ is the component of the second fundamental form in the direction of the unit normal.

Proof. Let $\{e_1, e_2\}$ be a local null frame on Σ such that $\langle e_1, e_2\rangle = 1$. Let ∇ be the Levi-Civita connection on Σ associated to $\langle , \rangle$. Then

$$\nabla_{e_1} e_1 = a e_1, \quad \nabla_{e_1} e_2 = -a e_2, \quad \nabla_{e_2} e_1 = -b e_1, \quad \nabla_{e_2} e_2 = b e_2,$$

for some real functions a, b. To find null coordinates $\{u, v\}$ on Σ we take arbitrary non-zero functions λ_1, λ_2 on Σ and we compute

$$[\lambda_1 e_1, \lambda_2 e_2] = \lambda_1 e_1(\lambda_2) e_2 - \lambda_2 e_2(\lambda_1) e_1 + \lambda_1 \lambda_2 b e_1 - \lambda_1 \lambda_2 a e_2.$$

Hence $\{\lambda_1 e_1, \lambda_2 e_2\}$ is a null coordinate frame if and only if

$$e_1(\lambda_2) = a\lambda_2, \qquad e_2(\lambda_1) = b\lambda_1. \tag{3.75}$$

We show that (3.75) always has a solution by constructing it explicitly.

Let ξ be the unit normal vector field on Σ and let σ be the component of the second fundamental form h of the surface in the direction of ξ. Since Σ is minimal we have $\sigma(e_1, e_2) = 0$. Call $\alpha = \sigma(e_1, e_1)$ and $\beta = \sigma(e_2, e_2)$. We divide in three different cases: when both α and β are different from zero, exactly one of them is zero, and both are zero.

Case 1: α and β are both different from zero.

We can take a change of basis such that $\alpha = \pm\beta$. Namely, take the new frame $\{\tilde{e}_1, \tilde{e}_2\}$ on Σ given by $\tilde{e}_1 = \mu e_1$, $\tilde{e}_2 = \frac{1}{\mu} e_2$ where $\mu^2 \alpha = \pm \frac{1}{\mu^2} \beta$ and μ is a non-vanishing function. Then $\sigma(\tilde{e}_1, \tilde{e}_1) = \pm\sigma(\tilde{e}_2, \tilde{e}_2)$.

As the ambient manifold is a space form, the Codazzi equation (see Section 1.4) becomes $\nabla h(e_2, e_1, e_1) = \nabla h(e_1, e_2, e_1)$ and $\nabla h(e_1, e_2, e_2) = \nabla h(e_2, e_1, e_2)$. It follows

$$e_1(\alpha) = -2a\alpha, \qquad e_2(\alpha) = -2b\alpha. \tag{3.76}$$

Since we are working in a neighborhood where $\alpha \neq 0$, the functions $\lambda_1 = \lambda_2 = |\alpha|^{-\frac{3}{2}}$ are smooth. From (3.76) it follows that λ_1 and λ_2 are solutions of (3.75).

We call $\frac{\partial}{\partial u} = \lambda_1 e_1$, $\frac{\partial}{\partial v} = \lambda_2 e_2$. Since we assume that $\alpha = \pm\beta$ we have $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = \pm\sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v})$. Using (3.74) we compute the Codazzi equation and we obtain that $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})$ is constant. Finally after rescaling the coordinates we get 1 in Lemma 21.

Case 2. $\alpha = 0$, $\beta \neq 0$.

Again, by changing the basis $\{e_1, e_2\}$ we may assume that $\alpha = 0, \beta = \pm 1$. From the Codazzi equation $\nabla h(e_1, e_2, e_2) = \nabla h(e_2, e_1, e_2)$ it follows that $0 = e_1(\beta) = -2a\beta$. Hence $a = 0$. By computing $[e_1, e_2](f)$ we know that there exists a solution of the system of differential equations

$$\begin{cases} e_1(f) = f, \\ e_2(f) = bf. \end{cases} \tag{3.77}$$

Taking λ_2 a constant and λ_1 a solution of (3.77) we get a solution of (3.75).

Calling $\frac{\partial}{\partial u} = \lambda_1 e_1$ and $\frac{\partial}{\partial v} = \lambda_2 e_2$ it follows that $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = 0$, $\sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = \pm\lambda_2^2$. After a rescaling we see that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ satisfy 2 in Lemma 21.

Case 3: $\alpha = \beta = 0$.

If $\alpha = \beta = 0$ then Σ is a time-like totally geodesic surface of $H_1^3(-1)$. Hence Σ is isometric to the two-dimensional anti-de Sitter space $H_1^2(-1)$, which admits null coordinates. This is 3 in Lemma 21. □

Proposition 34. *Let $p\colon \Sigma \to H_1^3(-1)$ be a minimal time-like surface of $H_1^3(-1)$. Let $\{u, v\}$ be null coordinates on Σ and let $\alpha = \sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})$, $\beta = \sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v})$ where σ is the component of the second fundamental form of Σ in the direction of the unit normal ξ. Then the function ω defined by $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle = e^{\omega}$ satisfies one of the following:*

1. $\omega_{uv} = 2\cosh\omega$ *when* $\alpha = \beta = -1$, *(*cosh*-Gordon equation)*
2. $\omega_{uv} = 2\sinh\omega$ *when* $-\alpha = \beta = -1$, *(*sinh*-Gordon equation)*
3. $\omega_{uv} = e^{\omega}$ *when* $\alpha = 0$ *or* $\beta = 0$. *(Liouville equation)*

Proof. Writing the Gauss and Weingarten formulas for the immersion $\Sigma \to H_1^3(-1) \to \mathbb{R}_2^4$ yields

$$\begin{aligned} p_{uu} &= \omega_u p_u + \alpha\xi, \\ p_{vu} &= p_{uv} = e^{\omega} p, \\ p_{vv} &= \omega_v p_v + \beta\xi, \\ \xi_u &= -\alpha e^{-\omega} p_v, \\ \xi_v &= -\beta e^{-\omega} p_u. \end{aligned} \tag{3.78}$$

Therefore

$$p_{uuv} = (\omega_{uv} - \alpha\beta e^{-\omega})p_u + \alpha_v \xi + \omega_u e^{\omega} p, \qquad p_{uvu} = \omega_u e^{\omega} p + e^{\omega} p_u$$

Hence ω satisfies

$$\omega_{uv} = e^{\omega} + \alpha\beta e^{-\omega}. \tag{3.79}$$

From Lemma 21 and taking $-\xi$ as the unit normal if necessary, we obtain either $\alpha = \beta = -1$, $\alpha = -\beta = -1$, $\alpha = 0$, $\beta = -1$ or $\alpha = \beta = 0$. □

3.4.2 The projection of a Lagrangian submanifold

Proposition 35. *Let $f\colon M \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be a Lagrangian submanifold of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of type IV in Lemma 11. Suppose that $f = (p, q)$ and that dp has nowhere maximal rank. Then $p(M) \subset \mathrm{SL}(2,\mathbb{R})$ is a time-like minimal surface, where $\mathrm{SL}(2,\mathbb{R})$ carries the metric given in* (2.2).

Proof. Let M be a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. By hypothesis A and B are of IV in Lemma 11 with respect to a Δ_3-orthonormal frame $\{E_1, E_2, E_3\}$. Let X be a non-zero vector field on M such that $dp(X) = 0$. Then

$$df(X) = (0, dq(X)) = \frac{1}{2}(Qdf(X) + df(X)), \tag{3.80}$$

where Q is the tensor defined in (2.14). From (2.16) it follows that equation (3.80) becomes

$$PX = -\frac{1}{2}X - \frac{\sqrt{3}}{2}JX.$$

By assuming X to be a unit vector we conclude that $X = \pm E_3$ and that $\theta_1 = \theta_2 = \frac{2\pi}{3}$. Hence the tangent space of $p(M)$ is spanned by $v_1 = dp(E_1)$ and $v_2 = dp(E_2)$. Writing $dp(E_i) = \frac{1}{2}(E_i - QE_i)$ yields the non-zero vector fields

$$v_1 = -\sinh^2\left(\frac{\psi}{2}\right) E_1 + \frac{\sinh(\psi)}{2\sqrt{3}} E_2 + \frac{\sinh^2\left(\frac{\psi}{2}\right)}{\sqrt{3}} JE_1 + \frac{\sinh(\psi)}{2} JE_2,$$

$$v_2 = -\frac{\sinh(\psi)}{2\sqrt{3}} E_1 - \sinh^2\left(\frac{\psi}{2}\right) E_2 - \frac{\sinh(\psi)}{2} JE_1 + \frac{\sinh^2\left(\frac{\psi}{2}\right)}{\sqrt{3}} JE_2.$$

It follows from a straightforward computation that

$$\langle v_1, v_1 \rangle = 1 - \cosh(\psi),$$

$$\langle v_1, v_2 \rangle = 0,$$

$$\langle v_2, v_2 \rangle = \cosh(\psi) - 1.$$

By Lemma 11, the function ψ is different from zero. Therefore $p(M)$ is a non-degenerate time-like surface of $\mathrm{SL}(2,\mathbb{R})$.

It is easy to check that $\xi = \frac{1}{\sqrt{6}} E_3 + \frac{1}{\sqrt{2}} JE_3$ lies in the first factor of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and that it is a unit normal vector field to $p(M)$.

Denote by σ the component of the second fundamental form of $p(M)$ in the direction of ξ. Then

$$\sigma(v_1, v_1) = \sqrt{2} h^3_{11} - \frac{\sinh(\psi)}{\sqrt{3}},$$

$$\sigma(v_1, v_2) = \sqrt{2} h^3_{12},$$

$$\sigma(v_2, v_2) = \sqrt{2} h^3_{11} - \frac{\sinh(\psi)}{\sqrt{3}}.$$

Since the mean curvature vector field H is given by

$$H = \big(-\sigma(v_1, v_1) + \sigma(v_2, v_2) \big) \xi,$$

the surface is minimal. □

Note that by taking the frame $\{E_1, -E_2, E_3\}$ in Lemma 11 we may assume that the function ψ is positive.

Consider the map $\tilde{\mathcal{P}}\colon p(M) \to \mathrm{SO}(2,2)$ given by

$$\tilde{\mathcal{P}}(x) = \begin{pmatrix} p(x) & \frac{v_1}{\sqrt{\cosh(\psi)-1}} & \frac{v_2}{\sqrt{\cosh(\psi)-1}} & \xi \end{pmatrix}.$$

The differential of $\tilde{\mathcal{P}}$ is given by $d\tilde{\mathcal{P}} = \tilde{\mathcal{P}}\tilde{\Omega}$ where $\tilde{\Omega}$ is given by

$$\begin{pmatrix} 0 & -\sqrt{\cosh(\psi)-1}\,\eta_1 & \sqrt{\cosh(\psi)-1}\,\eta_2 & 0 \\ \sqrt{\cosh(\psi)-1}\,\eta_1 & 0 & -\operatorname{csch}(\psi)h^1_{11}\eta_1 + \operatorname{csch}(\psi)h^2_{11}\eta_2 & -(\sqrt{\tfrac{2}{3}} + \operatorname{csch}(\psi)h^3_{11})\eta_3 \\ \sqrt{\cosh(\psi)-1}\,\eta_2 & -(\sqrt{\tfrac{2}{3}} + \operatorname{csch}(\psi)h^3_{11})\eta_3 & 0 & \frac{\sqrt{2}h^3_{12}\eta_1 + (\sqrt{2}h^3_{11} - \frac{1}{\sqrt{3}}\sinh(\psi))\eta_2}{\sqrt{\operatorname{csch}(\psi)-1}} \\ 0 & \frac{(\sqrt{2}h^3_{11} - \frac{1}{\sqrt{3}}\sinh(\psi))\eta_1 + \sqrt{2}h^3_{12}\eta_2}{\sqrt{\operatorname{csch}(\psi)-1}} & -\frac{\sqrt{2}h^3_{12}\eta_1 + (\sqrt{2}h^3_{11} - \frac{1}{\sqrt{3}}\sinh(\psi))\eta_2}{\sqrt{\operatorname{csch}(\psi)-1}} & 0 \end{pmatrix}$$

where $\eta_i(E_j) = \delta_{ij}$.

In [7] the authors study three different cases: when $p(M)$ is not totally geodesic in $\mathbb{S}^3$ and $\tilde{\mathcal{P}}$ is an immersion, when $p(M)$ is totally geodesic in $\mathbb{S}^3$, and when $p(M)$ is not totally geodesic in $\mathbb{S}^3$ and $\tilde{\mathcal{P}}$ is not an immersion. Due to complications that the first two cases present for type IV Lagrangian submanifolds, in this section we will only consider the last case.

Suppose that $\tilde{\mathcal{P}}$ is not an immersion. Then the component of η_3 in $\tilde{\Omega}$ must vanish. That is

$$h^3_{11} = -\sqrt{\frac{2}{3}}\sinh(\psi).$$

From the Codazzi equation we obtain that $h^3_{12} = 0$. Moreover, we get

$$E_3(h^1_{11}) = E_3(h^2_{11}) = 0,$$

and also

$$\begin{aligned} E_1(h^1_{11}) &= -E_2(h^2_{11}), \\ 0 &= \coth(\psi) - 3(h^1_{11})^2\coth(\psi) + 3(h^2_{11})^2\coth(\psi) + \operatorname{csch}(\psi) \\ &\quad - \cosh(2\psi)\operatorname{csch}(\psi) - \cosh(3\psi)\operatorname{csch}(\psi) - E_1(h^2_{11}) - E_2(h^1_{11}). \end{aligned} \tag{3.81}$$

Note that the Lie brackets of the vector fields E_1, E_2, E_3 are

$$[E_1, E_2] = h^1_{11} \coth(\psi) E_1 + h^2_{11} \coth(\psi) E_2 - \sqrt{\frac{2}{3}}(2\cosh(\psi) + 1) E_3,$$

$$[E_1, E_3] = 0,$$

$$[E_2, E_3] = 0.$$

Then, we take the frame $\{X_1, X_2, X_3\}$ given by

$$\begin{aligned} X_1 &= -\frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}} E_1 + \frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}} E_2 - \frac{h^1_{11} + h^2_{11}}{2\sqrt[4]{3^3}\sinh^{\frac{3}{2}}(\psi)} E_3, \\ X_2 &= \frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}} E_1 + \frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}} E_2 + \frac{h^1_{11} - h^2_{11}}{2\sqrt[4]{3^3}\sinh^{\frac{3}{2}}(\psi)} E_3, \\ X_3 &= E_3. \end{aligned} \tag{3.82}$$

Using (3.81) we may check that $[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0$. Thus, there exist coordinates $\{u, v, t\}$ such that

$$X_1 = \tfrac{\partial}{\partial u}, \quad X_2 = \tfrac{\partial}{\partial v}, \quad X_3 = \tfrac{\partial}{\partial t}.$$

From (3.29) we get expressions for the derivatives of ψ:

$$\begin{aligned} \psi_u &= -\frac{\sqrt{2}(h^1_{11} + h^2_{11})}{\sqrt[4]{3}\sqrt{\sinh(\psi)}}, \\ \psi_v &= -\frac{\sqrt{2}(h^1_{11} - h^2_{11})}{\sqrt[4]{3}\sqrt{\sinh(\psi)}}, \\ \psi_t &= 0. \end{aligned}$$

Hence, we express h^1_{11} and h^2_{11} in terms of ψ and its derivatives:

$$\begin{aligned} h^1_{11} &= -\frac{\sqrt[4]{3}\sqrt{\sinh(\psi)}(\psi_u + \psi_v)}{2\sqrt{2}}, \\ h^2_{11} &= \frac{\sqrt[4]{3}\sqrt{\sinh(\psi)}(\psi_v - \psi_u)}{2\sqrt{2}}. \end{aligned} \tag{3.83}$$

Plugging in these values in (3.82) we obtain

$$\begin{aligned} X_1 &= -\frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}}E_1 + \frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}}E_2 + \frac{\psi_u \operatorname{csch}(\psi)}{2\sqrt{6}}E_3, \\ X_2 &= \frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}}E_1 + \frac{1}{\sqrt{2}\sqrt[4]{3}\sqrt{\sinh(\psi)}}E_2 - \frac{\psi_v \operatorname{csch}(\psi)}{2\sqrt{6}}E_3, \\ X_3 &= E_3. \end{aligned} \tag{3.84}$$

Writing $dp = \frac{1}{2}(\mathrm{Id} - Q)$ we obtain

$$\begin{aligned} dp(\tfrac{\partial}{\partial u}) = &-\frac{\sinh(\psi) - \sqrt{3}\cosh(\psi) + \sqrt{3}}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}E_1 - \frac{\sinh(\psi) + \sqrt{3}\cosh(\psi) - \sqrt{3}}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}E_2 \\ &- \frac{\sqrt{3}\sinh(\psi) + \cosh(\psi) - 1}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}JE_1 + \frac{-\sqrt{3}\sinh(\psi) + \cosh(\psi) - 1}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}JE_2, \\ dp(\tfrac{\partial}{\partial v}) = &-\frac{\sinh(\psi) + \sqrt{3}\cosh(\psi) - \sqrt{3}}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}E_1 + \frac{\sinh(\psi) - \sqrt{3}\cosh(\psi) + \sqrt{3}}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}E_2 \\ &+ \frac{-\sqrt{3}\sinh(\psi) + \cosh(\psi) - 1}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}JE_1 + \frac{\sqrt{3}\sinh(\psi) + \cosh(\psi) - 1}{2\sqrt{2}3^{3/4}\sqrt{\sinh(\psi)}}JE_2. \end{aligned}$$

Comparing the equation we obtained for minimal surfaces in H^3_1, on the one hand we have $\langle dp(X_1), dp(X_2)\rangle = \frac{1}{\sqrt{3}}\tanh\left(\frac{\psi}{2}\right)$ and on the other hand $\langle dp(\frac{\partial}{\partial u}), dp(\frac{\partial}{\partial v})\rangle = e^{\omega}$. Hence

$$e^{\omega} = \frac{1}{\sqrt{3}}\tanh\left(\frac{\psi}{2}\right).$$

Then

$$\omega_u = \frac{\psi_u}{\sinh(\psi)}, \qquad \omega_v = \frac{\psi_v}{\sinh(\psi)}.$$

With a straightforward computation we can check that

$$\begin{aligned} \nabla^E_{\partial u} dp(\partial u) &= \omega_u dp(\partial u) - \xi, \\ \nabla^E_{\partial v} dp(\partial v) &= \omega_v dp(\partial v) - \xi. \end{aligned}$$

Now we write the frame $\{E_1, E_2, E_3\}$ in terms of the coordinate frame:

$$\begin{aligned} E_1 &= \frac{\sqrt[4]{3}\sqrt{\sinh(\psi)}}{\sqrt{2}}\left(-\tfrac{\partial}{\partial u} + \tfrac{\partial}{\partial v} + \frac{\operatorname{csch}(\psi)(\psi_u + \psi_v)}{2\sqrt{6}}\,\tfrac{\partial}{\partial t}\right), \\ E_2 &= \frac{\sqrt[4]{3}\sqrt{\sinh(\psi)}}{\sqrt{2}}\left(\tfrac{\partial}{\partial u} + \tfrac{\partial}{\partial v} + \frac{\operatorname{csch}(\psi)(-\psi_u + \psi_v)}{2\sqrt{6}}\,\tfrac{\partial}{\partial t}\right), \\ E_3 &= \tfrac{\partial}{\partial t}. \end{aligned} \tag{3.85}$$

3.4.3 The reverse construction

In this subsection we construct a Lagrangian submanifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of type IV in Lemma 11 starting from a minimal surface in $\mathrm{SL}(2,\mathbb{R})$. We begin by stating and proving a standard result in differential equations, adapted to our setting.

Lemma 22. *Let β_1, β_2 and β_3 be matrices in $\mathfrak{sl}(2,\mathbb{R})$ that depend on variables u, v and t. If*

$$\begin{aligned} \frac{\partial \beta_1}{\partial v} - \frac{\partial \beta_2}{\partial u} &= 2\beta_1 \times \beta_2, \\ \frac{\partial \beta_1}{\partial t} - \frac{\partial \beta_3}{\partial u} &= 2\beta_1 \times \beta_3, \\ \frac{\partial \beta_2}{\partial t} - \frac{\partial \beta_3}{\partial v} &= 2\beta_2 \times \beta_3, \end{aligned} \tag{3.86}$$

then the system of partial differential equations

$$\frac{\partial q}{\partial u} = q\beta_1, \qquad \frac{\partial q}{\partial v} = q\beta_2, \qquad \frac{\partial q}{\partial t} = q\beta_3, \tag{3.87}$$

has a solution in $\mathrm{SL}(2,\mathbb{R})$ with initial condition $q(0,0,0) = \mathrm{Id}_2$.

Proof. Define the matrix $B_i \in M(4,\mathbb{R})$ by

$$B_i = \begin{pmatrix} \beta_i^{\intercal} & 0 \\ 0 & \beta_i^{\intercal} \end{pmatrix}.$$

Thinking of q as a vector in $\mathbb{R}^4$, we write $\frac{\partial q}{\partial u} = q\beta_1$ as the linear ordinary differential equation $\frac{\partial q}{\partial u} = B_1 q$. This differential equation always has a solution, and the set of all solutions forms a four-dimensional vector space. Hence, we

write $q = Cy$ where the columns of C form a basis of said vector space. The matrix C depends on u, v, t and y depends only on v and t. We differentiate q with respect to v and we get

$$B_2 Cy = \frac{\partial q}{\partial v} = \frac{\partial C}{\partial v} y + C \frac{\partial y}{\partial v}.$$

It follows that

$$\frac{\partial y}{\partial v} = C^{-1} \left(B_2 C - \frac{\partial C}{\partial v} \right) y.$$

Then, if $C^{-1} \left(B_2 C - \frac{\partial C}{\partial v} \right)$ does not depend on u, we can repeat the process for v and t, and therefore find a solution. Note that $\frac{\partial C}{\partial u} = B_1 C$ and that $\frac{\partial}{\partial u} C^{-1} = -C^{-1} (\frac{\partial}{\partial u} C) C^{-1}$. We compute

$$\begin{aligned}
\frac{\partial}{\partial u} \left(C^{-1} B_2 C - C^{-1} \frac{\partial C}{\partial v} \right) &= \frac{\partial C}{\partial u}^{-1} B_2 C + C^{-1} \frac{\partial B_2}{\partial u} C + C^{-1} B_2 \frac{\partial C}{\partial u} \\
&\quad - \frac{\partial C}{\partial u}^{-1} \frac{\partial C}{\partial v} - C^{-1} \frac{\partial^2 C}{\partial u \partial v} \\
&= -C^{-1} B_1 C C^{-1} B_2 C + C^{-1} \frac{\partial B_2}{\partial u} C \\
&\quad + C^{-1} B_2 B_1 C + C^{-1} B_1 C C^{-1} \frac{\partial C}{\partial v} \\
&\quad - C^{-1} \left(\frac{\partial B_1}{\partial v} C + B_1 \frac{\partial C}{\partial v} \right) \\
&= C^{-1} \left(-B_1 B_2 + \frac{\partial B_2}{\partial u} + B_2 B_1 - \frac{\partial B_1}{\partial v} \right) C.
\end{aligned}$$

Using the first equation of (3.86) we obtain that the last term of the equation above equals to zero. Repeating the process we see that (3.87) has a solution. Moreover, by requiring $q(0,0,0) = \mathrm{Id}_2$, we see that $\langle q, q \rangle$ is constant and equal to -1. □

Proof of Theorem 34. Let $p \colon \Sigma \to \mathrm{SL}(2,\mathbb{R})$ be a minimal surface of $\mathrm{SL}(2,\mathbb{R})$ on which we take coordinates u and v such that $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle = e^{\omega}$, $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle = \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle = 0$. Also we take $\sigma(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = \sigma(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = -1$ where σ is the component of the second fundamental form in the direction of a unit normal ξ. Let ω be the solution of $\omega_{uv} = 2\cosh(\omega)$ with $\omega < \ln\left(\frac{1}{\sqrt{3}}\right)$. Denote $\frac{\partial}{\partial u} p = p\alpha_1$,

$\frac{\partial}{\partial v} p = p\alpha_2$ and let β_1, β_2, and β_3 be the matrices in $\mathfrak{sl}(2,\mathbb{R})$ given by

$$\beta_1 = \frac{1}{2}\alpha_1 - \frac{e^{-\omega}}{2}\alpha_2 - \frac{1}{4}\omega_u e^{-\omega}\alpha_1 \times \alpha_2,$$

$$\beta_2 = \frac{e^{-\omega}}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{e^{-\omega}\omega_v}{4}\alpha_1 \times \alpha_2,$$

$$\beta_3 = -\sqrt{\frac{3}{2}} e^{-\omega}\alpha_1 \times \alpha_2.$$

We use the identities

$$\begin{aligned}
\alpha_1 \times (\alpha_1 \times \alpha_2) &= -e^{\omega}\alpha_1, \\
(\alpha_1 \times \alpha_2) \times \alpha_2 &= -e^{\omega}\alpha_2, \\
\xi &= e^{-\omega} p\alpha_1 \times \alpha_2, \\
\alpha_{1u} &= \omega_u \alpha_1 - e^{-\omega}\alpha_1 \times \alpha_2, \\
\alpha_{2u} &= -\alpha_1 \times \alpha_2, \\
(\alpha_1 \times \alpha_2)_u &= e^{\omega}\alpha_1 + \alpha_2 + \omega_u \alpha_1 \times \alpha_2, \\
\alpha_{1v} &= \alpha_1 \times \alpha_2, \\
\alpha_{2v} &= \omega_v \alpha_2 - e^{-\omega}\alpha_1 \times \alpha_2, \\
(\alpha_1 \times \alpha_2)_v &= \alpha_1 - e^{\omega}\alpha_2 + \omega_v \alpha_1 \times \alpha_2,
\end{aligned}$$

to check that β_1, β_2 and β_3 satisfy (3.86). Therefore, there exists a submanifold $f\colon M \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ with $f = (p, q)$.

Using the definition of the pseudo-nearly Kähler metric g in (2.8) and of the almost complex structure (2.10), it follows from a straightforward computation that f is a Lagrangian immersion.

Take the Δ_3-orthonormal frame on $\{E_i\}$ on M defined as

$$\begin{aligned}
E_1(q) &= q\tilde{\beta}_1, \quad E_1(p) = p\tilde{\alpha}_1, \\
E_2(q) &= q\tilde{\beta}_2, \quad E_1(p) = p\tilde{\alpha}_2, \\
E_3(q) &= q\tilde{\beta}_3, \quad E_1(p) = p\tilde{\alpha}_3,
\end{aligned} \tag{3.88}$$

where $\tilde{\alpha}_i, \tilde{\beta}_i$ are the matrices in $\mathfrak{sl}(2,\mathbb{R})$ given by

$$\tilde{\alpha}_1 = -\frac{\sqrt{3}e^{\omega/2}}{\sqrt{1-3e^{2\omega}}}\alpha_1 + \frac{\sqrt{3}e^{\omega/2}}{\sqrt{1-3e^{2\omega}}}\alpha_2,$$

$$\tilde{\alpha}_2 = \frac{\sqrt{3}e^{\omega/2}}{\sqrt{1-3e^{2\omega}}}\alpha_1 + \frac{\sqrt{3}e^{\omega/2}}{\sqrt{1-3e^{2\omega}}}\alpha_2.$$

and

$$\tilde{\beta}_1 = -\frac{1}{2}\sqrt{3}e^{-\frac{\varpi}{2}}\left(e^{\omega}-1\right)\frac{1}{\sqrt{1-3e^{2\omega}}}\alpha_1 + \frac{1}{2}\sqrt{3}e^{-\frac{\varpi}{2}}\left(e^{\omega}+1\right)\frac{1}{\sqrt{1-3e^{2\omega}}}\alpha_2,$$

$$\tilde{\beta}_2 = \frac{1}{2}\sqrt{3}e^{-\frac{\varpi}{2}}\left(e^{\omega}+1\right)\frac{1}{\sqrt{1-3e^{2\omega}}}\alpha_1 + \frac{1}{2}\sqrt{3}e^{-\frac{\varpi}{2}}\left(e^{\omega}-1\right)\frac{1}{\sqrt{1-3e^{2\omega}}}\alpha_2,$$

$$\tilde{\beta}_3 = -\sqrt{\frac{3}{2}}e^{-\omega}\alpha_1 \times \alpha_2.$$

We write the restriction of the almost product structure P to M as $A + JB$, where $A, B\colon TM \to TM$. Then A and B are of type IV in Lemma 11 with respect to $\{E_1, E_2, E_3\}$, with $\theta_1 = \theta_2 = \frac{2\pi}{3}$ and ψ a positive function which satisfies

$$e^{\omega} = \frac{1}{\sqrt{3}}\tanh\left(\frac{\psi}{2}\right).$$

□

Chapter 4

Totally geodesic surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

This chapter is based on unpublished joint work with K. Dekimpe. As such, it can also be found in his PhD thesis [17].

Introduction

In this chapter we study pseudo-Riemannian surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. That is, those pseudo-Riemannian immersions $f : \Sigma \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ where Σ is a 2-dimensional (pseudo-)Riemannian manifold. In particular, we are interested in totally geodesic surfaces, those surfaces whose geodesics are also geodesics of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

In the Riemannian analogue $\mathbb{S}^3 \times \mathbb{S}^3$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, totally geodesic surfaces were studied in [9] and [19]. In [9] the authors classified up to congruence all totally geodesic almost complex surfaces. Almost complex surfaces, also known as J-holomorphic, are those surfaces such that the almost complex structure J preserves the tangent space.

Theorem 35 (Bolton, Dillen, Dioos, Vrancken)**.** *Any totally geodesic almost complex surface of $\mathbb{S}^3 \times \mathbb{S}^3$ is congruent to an open subset of one of the following immersions:*

1. $\mathbb{S}^2(\frac{2}{3}) \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto \frac{1}{2}(1 - \sqrt{3}x, 1 + \sqrt{3}x)$,
2. $\mathbb{T}^2 \to \mathbb{S}^3 \times \mathbb{S}^3 : (s,t) \mapsto (e^{s\mathbf{i}}, e^{t\mathbf{i}})$,

where $\mathbf{i}$*, is the unit quaternion. Conversely, the immersions listed above are totally geodesic almost complex surfaces.*

In [19] the author classified totally geodesic totally real surfaces of $\mathbb{S}^3 \times \mathbb{S}^3$. That is, those surfaces where J maps the tangent space into the normal space.

Theorem 36 (Dioos)**.** *Any totally geodesic totally real surface of* $\mathbb{S}^3 \times \mathbb{S}^3$ *is congruent to an open subset of the following immersion:*

$$\mathbb{S}^2(\tfrac{2}{3}) \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (\iota(u), 1)$$

where ι *is the totally geodesic immersion of* $\mathbb{S}^2$ *into* $\mathbb{S}^3$*. Conversely, the immersion above is a totally geodesic totally real surface.*

On the other hand, surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ have just started to be studied. Ghandour and Vrancken [25] classified all almost complex surfaces totally geodesic of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Theorem 37 (Ghandour, Vrancken)**.** *Any almost complex totally geodesic surface of the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *is congruent to an open subset of the image of one of the following immersions:*

1. $H^2(-\frac{3}{2}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : x \mapsto \frac{1}{2}(\mathrm{Id}_2 - \sqrt{3}x, \mathrm{Id}_2 + \sqrt{3}x)$,
2. $\mathbb{T}^2 \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : (s,t) \mapsto (e^{t\boldsymbol{k}}, e^{s\boldsymbol{k}})$,
3. $\mathbb{R}^2 \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : (s,t) \mapsto (e^{t\boldsymbol{i}}, e^{s\boldsymbol{i}})$,

where $\boldsymbol{i}$ *is the matrix given in* (2.1)*.*

We focus on the study of totally real surfaces in the nearly Kähler manifold $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), J, g)$. These surfaces are characterized by the property that the almost complex structure J maps tangent spaces of the surface into normal spaces. In this section, we present some examples of totally real surfaces in the nearly Kähler manifold by investigating semi-Riemannian surfaces within Lagrangian submanifolds of this ambient space. Subsequently, in Section 4.1, our attention is directed towards the analysis of totally geodesic totally real surfaces. We demonstrate that the examples presented in this section are essentially the only possible surfaces of this kind.

We recall Theorem 32 given in Chapter 3.

Theorem 32. *Let* M *be an totally geodesic Lagrangian immersion in the nearly Kähler* $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), g)$*. Then it is locally congruent to one of*

the following immersions:

$$F_1 : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : p \mapsto (p, p);$$

$$F_2 : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : p \mapsto (p, -\boldsymbol{k}p\boldsymbol{k});$$

$$F_3 : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : p \mapsto (p, \boldsymbol{i}p\boldsymbol{i});$$

where $\boldsymbol{i}$ *and* $\boldsymbol{k}$ *are part of the split quaternions, defined in (2.1).*

The immersion F_1 induces a metric tensor g_1 on $\mathrm{SL}(2,\mathbb{R})$, defined as $g_1 = \frac{2}{3}\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ represents the standard semi-Euclidean metric on $\mathrm{SL}(2,\mathbb{R})$. The isometry $\mathfrak{F}$, as defined in Equation (2.4), establishes that $(\mathrm{SL}(2,\mathbb{R}), g_1)$ is locally isometric to the semi-Riemannian space form $H_1^3\left(-\frac{3}{2}\right)$.

We now utilize these immersions to present examples of totally geodesic and totally real surfaces in the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

From its definition we see that the totally geodesic property is transitive. Namely, given two totally geodesic immersions $f_1 \colon L \to M$ and $f_2 \colon M \to N$ the immersion $f_2 \circ f_1 \colon L \to N$ is totally geodesic. Then, totally geodesic surfaces of totally geodesic Lagrangian submanifolds of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ are totally real.

The subsequent proposition provides a classification for the totally geodesic submanifolds of the semi-Riemannian space form $H_s^n(c)$.

Proposition 35 ([13])**.** *Every totally geodesic semi-Riemannian submanifold of dimension* k *in a semi-Riemannian hyperbolic space* $H_s^n(c)$ *is congruent to an open part of a semi-hyperbolic* k*-subspace* $H_t^k(c)$ *of* $H_s^n(c)$*, defined as*

$$H_t^k(c) = \{(x_1, \ldots, x_{t+1}, 0, \ldots, x_{s+1}, \ldots, x_{k+1}, 0, \ldots, 0) \in H_s^n(c)\},$$

with $k < n$ *and* $0 \leq t \leq s$*.*

This proposition demonstrates that every spacelike totally geodesic surface in $H_1^3(c)$ is congruent to an open part of the semi-hyperbolic two-space $H_0^2(c)$. Similarly, every Lorentzian totally geodesic surface in $H_1^3(c)$ is congruent to an open part of the semi-hyperbolic two-space $H_1^2(c)$. Utilizing the isometry $\mathfrak{F}$ allows us to establish that every spacelike or Lorentzian totally geodesic surface in $(\mathrm{SL}(2,\mathbb{R}), \frac{2}{3}\langle,\rangle)$ is locally congruent to an open portion of $\mathfrak{F}(H_0^2(-\frac{3}{2}))$ or $\mathfrak{F}(H_1^2(-\frac{3}{2}))$. Consequently, we obtain the following examples of totally geodesic totally real surfaces in the nearly Kähler manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, when considering the classification of totally geodesic Lagrangian submanifolds of this space, as presented in Theorem 32.

Example 24. The following immersions are totally geodesic totally real surfaces in the nearly Kähler $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), g)$:

1. $f_1 \colon \mathfrak{F}(H_0^2(-\frac{3}{2})) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : p \to (p, \mathrm{Id}_2)$,
2. $f_2 \colon \mathfrak{F}(H_1^2(-\frac{3}{2})) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) : p \to (p, \mathrm{Id}_2)$,

Finally, we obtain the following classification result, which is direct consequence of Theorem 40 and Theorem 41.

Theorem 39. *Let Σ be a (non-degenerate) totally geodesic totally real surface of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Then it is congruent to an open subset of the image of either the immersion 1 or the immersion 2 in Example 24. Conversely, the immersions in Example 24 are totally real totally geodesic surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.*

The structure of this chapter is as follows: In Section 4.1 we provide some general theory for totally real surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. In Section 4.2 we study spacelike surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and we provide a classification of totally geodesic totally real spacelike surfaces in Theorem 40. In Section 4.3 we study Lorentzian surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and we provide a classification of totally geodesic totally real Lorentzian surfaces in Theorem 41.

4.1 Totally real surfaces

In this section, our aim is to provide some general theory of totally real surfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ that we will use to conclude in a classification result in the following sections. A significant consequence of a surface being totally real is that it can only be either spacelike or Lorentzian, since it cannot be negative definite. This fact becomes evident later in this section as we proceed with constructing a local frame on the surface.

Let Σ denote a totally real surface in the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We begin our analysis by defining the distribution $\mathcal{D}$ given by

$$\mathcal{D} = T\Sigma + JT\Sigma + G(T\Sigma \times T\Sigma) + JG(T\Sigma \times T\Sigma). \tag{4.1}$$

In the case of totally real surfaces, it is important to note that the dimension of $\mathcal{D}$ is at least four. Consequently, considering a local semi-orthonormal frame $\{U, V\}$ on the surface Σ, it immediately follows that the set $\{U, V, JU, JV\}$ consists of linearly independent vector fields since Σ is totally real. Additionally, from the properties of the tensor G as outlined in Proposition 9, we can deduce that the vector fields $G(U,V)$ and $JG(U,V)$ are both linearly independent from the aforementioned set. Thus, the dimension of the distribution $\mathcal{D}$ is at least six, and as $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is a six-dimensional manifold, its dimension is precisely six.

This means that the set of vector fields

$$\mathcal{G} = \{U, V, JU, JV, G(U,V), JG(U,V)\}, \tag{4.2}$$

forms a local frame for $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ when restricted to the surface Σ. This implies that the almost product structure P, defined in Equation (2.12), restricted to the surface Σ, can be described in terms of the frame $\mathcal{G}$ as follows:

$$PU = a_1U + a_2V + a_3JU + a_4JV + a_5G(U,V) + a_6JG(U,V), \tag{4.3}$$

$$PV = b_1U + b_2V + b_3JU + b_4JV + b_5G(U,V) + b_6JG(U,V), \tag{4.4}$$

where $a_1, \ldots, a_6, b_1, \ldots, b_6$ are smooth functions on the surface Σ.

Now, we aim to define the almost product structure P on the frame $\mathcal{G}$. By using the fact that J and P anti-commute, we can straightforwardly find expressions for PJU and PJV. However, the expressions for $PG(U,V)$ and $PJG(U,V)$ are more intricate. Applying Lemma 4 and Lemma 5, we observe that these expressions depend on the restriction of the nearly Kähler metric g to the surface Σ.

As the nearly Kähler manifold $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), g)$ is a six-dimensional semi-Riemannian manifold of index 2, the surface Σ can be spacelike, Lorentzian, or negative definite. However, the case where the totally real surface Σ is negative definite can be excluded because the almost complex structure J is compatible with the nearly Kähler metric g. Indeed, suppose that the surface Σ is negative definite with a semi-orthonormal frame $\{U, V\}$. The compatibility condition then implies that

$$g(U,U) = g(V,V) = g(JU,JU) = g(JV,JV) = -1,$$

which is not possible for four linearly independent vector fields on a six-dimensional manifold with index 2. Thus the surface Σ is either spacelike or Lorentzian and these cases are treated separately in Sections 4.2 and 4.3.

Remark 15. We define a new symmetric operator denoted as P^Σ on the surface Σ, which is the restriction of the almost product structure operator P and projection to the tangent bundle $T\Sigma$. Notably, this restriction ensures that the operator P^Σ itself remains symmetric and it is a tensor on the surface. Consider a semi-orthonormal frame $\{E_1, E_2\}$ defined on Σ, where the following conditions hold:

$$g(E_1,E_1) = 1, \quad g(E_2,E_2) = -1, \quad g(E_1,E_2) = 0.$$

We can express the operator P^Σ relative to this frame as follows:

$$P^\Sigma : T\Sigma \to T\Sigma : X \mapsto g(PX,E_1)E_1 - g(PX,E_2)E_2.$$

It is worth remarking that this description of the operator is applicable to any frame, including lightlike frames. To achieve this, one can simply rephrase the new frame using the frame $\{E_1, E_2\}$. The operator P^Σ plays a crucial role in the analysis of totally geodesic totally real surfaces in the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, as shown in the subsequent sections.

4.2 Spacelike surfaces

We now focus on the case of spacelike totally real surfaces in the nearly Kähler manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, which are totally geodesic, and present a complete classification in Theorem 40. Before proving this theorem, we establish some important results concerning the nearly Kähler connection ∇, Lemma 23, and the sectional curvature of the surface Σ. These results provide the necessary background for the subsequent proof of the classification theorem.

Let us thus assume that the surface Σ is spacelike. In this case, the tangent frame $\{U, V\}$ on Σ satisfies

$$g(U,U) = g(V,V) = 1.$$

The compatibility of the almost complex structure J and the properties of the tensor G on the nearly Kähler manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, as depicted in Lemma 4, allow us to express the metric g with respect to the frame $\mathcal{G}$ as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix}.$$

Remark 16. It is worth noting that as the operator P^Σ is symmetric it is always diagonalizable on the spacelike surface Σ. This property allows us to choose the spacelike frame $\{U, V\}$ in such a way that

$$P^\Sigma(U) = a_1 U, \quad P^\Sigma(V) = b_2 V,$$

for smooth functions a_1, b_2 on Σ. Consequently, Equations (4.3)-(4.4) imply that the functions a_2 and b_1 can be set to zero, and we have

$$g(PU, V) = g(U, PV) = 0.$$

The following lemma demonstrates how the nearly Kähler connection ∇ on the totally geodesic surface Σ can be constructed with respect to the frame $\mathcal{G}$.

Lemma 23. *The nearly Kähler connection ∇ on a spacelike, totally geodesic totally real surface Σ, with respect to the frame $\mathcal{G}$ defined in Equation (4.2), is constructed as follows:*

$$\begin{aligned}
&\nabla_U U = AV, && \nabla_V U = -BV,\\
&\nabla_U V = -AU, && \nabla_V V = BV,\\
&\nabla_U JU = AJV, && \nabla_V JU = -BJV - G(U,V),\\
&\nabla_U JV = -AJU + G(U,V), && \nabla_V JV = BJU,\\
&\nabla_U G(U,V) = \frac{2}{3}JV, && \nabla_V G(U,V) = -\frac{2}{3}JU,\\
&\nabla_U JG(U,V) = 0, && \nabla_V JG(U,V) = 0,
\end{aligned}$$

where A and B are smooth functions on Σ.

Proof. Recall that the tensor field G was defined as $G(X,Y) = (\nabla_X J)Y$ for all vector fields $X, Y \in \mathfrak{X}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$. Using this expression together with Lemma 4 then immediately proves the expression for the nearly Kähler connection ∇ with respect to the frame $\mathcal{G}$. □

Note that these expressions depend on the causal character of the surface Σ, as the tensor field ∇G is expressed in terms of the nearly Kähler metric g. As Σ is a totally geodesic surface, the Gauss and Codazzi equations in Section 1.4 show that

$$\begin{aligned}
&(\tilde{R}(U,V)U)^T = R(U,V)U && (\tilde{R}(U,V)V)^T = R(U,V)V,\\
&(\tilde{R}(U,V)U)^\perp = 0 && (\tilde{R}(U,V)V)^\perp = 0.
\end{aligned}$$

where R is the Riemann curvature tensor of Σ and $\tilde{R}$ the Riemann curvature tensor of the ambient space $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. On the other hand we have that one can relate the Riemann tensor of Σ with the sectional curvature K at each point $p \in \Sigma$ through Definition 22:

$$\begin{aligned}
R(U,V)U &= K(p)(g(V,U)U - g(U,U)V) = -K(p)V,\\
R(U,V)V &= K(p)(g(V,V)U - g(U,V)V) = K(p)U.
\end{aligned}$$

Combining the Gauss and Codazzi equations with the previous formulas thus yields

$$\tilde{R}(U,V)U = -K(p)V, \quad \tilde{R}(U,V)V = K(p)U.$$

Equation (2.13) gives an explicit formula for the Riemann tensor of the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, thus one has the following relation for the sectional curvature K at each point $p \in \Sigma$:

$$\begin{aligned} -K(p)V = \frac{5}{6}V - \frac{1}{2}g(JV,U)JU - \frac{2}{3}(g(PV,U)PU - g(PU,U)PV) \\ - \frac{2}{3}(g(JPV,U)JPU - g(JPU,U)JPV), \\ K(p)U = -\frac{5}{6}U + \frac{1}{2}g(JU,V)JV - \frac{2}{3}(g(PV,V)PU - g(PU,V)PV) \\ - \frac{2}{3}(g(JPV,V)JPU - g(JPU,V)JPV). \end{aligned}$$

These formulas are for general spacelike totally geodesic surfaces, so after using that Σ is totally real and that the frame $\{U,V\}$ was chosen in such a way that $g(U,PV) = 0$, the previous formulas reduce to

$$-K(p)V = \frac{5}{6}V + \frac{2}{3}(g(PU,U)PV - g(JPV,U)JPU + g(JPU,U)JPV), \tag{4.5}$$

$$K(p)U = -\frac{5}{6}U - \frac{2}{3}(g(PV,V)PU + g(JPV,V)JPU - g(JPU,V)JPV). \tag{4.6}$$

We can now classify all spacelike totally geodesic totally real surfaces in the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Theorem 40. *Let Σ be a spacelike totally geodesic totally real surface in the nearly Kähler $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), J, g)$. Then it is locally congruent to immersion f_1 of Example 24.*

Proof. By assuming that Σ is spacelike, we can define the almost product structure P on the entire frame $\mathcal{G}$ by utilizing the anti-commutation between P and J, as well as the relationship between P and the tensor G as presented in

Lemma 5. This yields the following expressions:

$$PU = a_1U + a_3JU + a_4JV + a_5G(U,V) + a_6JG(U,V),$$

$$PV = b_2V + b_3JU + b_4JV + b_5G(U,V) + b_6JG(U,V),$$

$$PJU = a_3U + a_4V - a_1JU + a_6G(U,V) - a_5JG(U,V),$$

$$PJV = b_3U + b_4V - b_2JV + b_6G(U,V) - b_5JG(U,V),$$

$$\begin{aligned} PG(U,V) = &-\frac{2}{3}(a_5b_2 - a_6b_4 + a_4b_6)U - \frac{2}{3}(a_6b_3 + a_1b_5 - a_3b_6)V \\ &+ \frac{2}{3}(a_6b_2 + a_5b_4 - a_4b_5)JU + \frac{2}{3}(-a_5b_3 + a_3b_5 + a_1b_6)JV \\ &- (a_1b_2 + a_4b_3 - a_3b_4)G(U,V) + (a_3b_2 + a_1b_4)JG(U,V), \end{aligned}$$

$$\begin{aligned} PJG(U,V) = &\frac{2}{3}(a_6b_2 + a_5b_4 - a_4b_5)U + \frac{2}{3}(-a_5b_3 + a_3b_5 + a_1b_6)V \\ &+ \frac{2}{3}(a_5b_2 - a_6b_4 + a_4b_6)JU + \frac{2}{3}(a_6b_3 + a_1b_5 - a_3b_6)JV \\ &+ (a_3b_2 + a_1b_4)G(U,V) + (a_1b_2 + a_4b_3 - a_3b_4)JG(U,V), \end{aligned}$$

where once again we used that the vector field PU is orthogonal to V. We now use the involutive, compatibility and symmetric properties of the almost product structure P, together with Equations (4.5)-(4.6) to determine conditions for the functions $a_1, \ldots, a_6, b_2 \ldots, b_6$. Lemma 5 shows an expression for the tensor field ∇P in terms of the almost complex structure J and the tensor G:

$$P\nabla_X Y = \nabla_X PY - \frac{1}{2}\left(JG(X,PY) + JPG(X,Y)\right), \tag{4.7}$$

for all vector fields $X, Y \in \mathfrak{X}(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$. This equation provides a refinement, together with Lemma 23, of the conditions on the functions $a_1, \ldots, a_6, b_1 \ldots b_6$.

The symmetry condition of the almost product structure P immediately yields

$$a_4 = b_3.$$

Using this equality in Equations (4.5)-(4.6) then shows that the following conditions must hold:

$$a_1a_4 = 0, \quad b_2a_4 = 0, \tag{4.8}$$

where one can distinguish two cases, i.e. when a_4 is not equal to zero or when a_4 is equal to zero and we treat these cases separately in the following sections.

The function $\mathbf{a_4}$ is not equal to zero.

As the function a_4 is not equal to zero, one immediately has from Equation (4.8) that the functions a_1 and b_2 are both zero. Equations (4.5)-(4.6) then show that

$$a_4a_6 = a_3b_6, \quad a_4a_5 = a_3b_5, \quad a_6b_4 = a_4b_6, \quad a_4b_5 = a_5b_4.$$

Computing $g(PU, PJV) = g(U, JV)$ and plugging in $\{U, G(U,V)\}$, $\{V, G(U,V)\}$, $\{V, G(U,V)\}$ and $\{V, JG(U,V)\}$ in 1 in Lemma 5, complements the above conditions with the following relations:

$$a_6b_5 = a_5b_6, \qquad b_5 = a_4a_6 - a_3b_6, \quad a_5 = a_6b_4 - a_4b_6,$$

$$a_6 = a_4b_5 - a_5b_4, \quad b_6 = a_4a_5 - a_3b_5.$$

Note that a direct consequence of P being involutive is that not all functions $a_1, \ldots, a_6$ nor all functions $b_1, \ldots, b_6$ can be simultaneously zero. Keeping this remark in mind, Table 4.1 then shows that there is only one possible solution for this system of equations.

Table 4.1 – Possible solutions when $a_4 = b_3 \neq 0$ and Σ is spacelike.

Case 1	$b_6 = b_5 = b_2 = a_6 = a_5 = a_1 = 0$

Let us further analyze this case by examining the compatibility condition in Lemma 5. Plugging in $\{U, V\}$ in the equation, allows us to obtain

$$a_3 = -b_4,$$

while computing $P^2U = U$ yields $a_4^2 + b_4^2 = 1$. Therefore, there exists a smooth function ϕ on the surface Σ such that

$$a_4 = \cos\phi, \quad b_4 = \sin\phi.$$

These conditions satisfy the symmetry, compatibility, and involutive conditions of the almost product structure P.

Now, according to Lemma 23 and substituting $\{X, Y\}$ with $\{U, V\}$ in equation (4.7), we find that the previous equation simplifies to

$$A \sin\phi JU - A\cos\phi JV = -(A + U(\phi)) \sin\phi JU + (A + U(\phi)) \cos\phi JV + \frac{\sin\phi}{2} G(U,V) + \frac{1}{2} JG(U,V),$$

which leads to a contradiction as the term involving $JG(U, V)$ is never zero. Hence, we can conclude that there are no spacelike totally geodesic totally real surfaces for which the function a_4 is nonzero.

The function $\mathbf{a_4}$ is equal to zero.

Let us now consider the case when the function a_4 is equal to zero. From Equations (4.5)-(4.6), we can immediately deduce the following conditions:

$$a_3 b_2 = a_1 b_4, \qquad a_1 b_5 = -a_3 b_6, \qquad a_3 b_5 = a_1 b_6,$$

$$a_5 b_2 = -a_6 b_4, \qquad a_6 b_2 = a_5 b_4.$$

Computing $g(PU, PV) - g(U, V)$ and plugging in $\{U, G(U, V)\}$, $\{U, JG(U, V)\}$, $\{V, G(U, V)\}$ and $\{V, JG(U, V)\}$ in *1* in Lemma 5, yields additional conditions:

$$a_5 b_5 = -a_6 b_6, \qquad a_6 b_5 = a_5 b_6, \qquad (b_2 - 1) a_5 = a_6 b_4,$$

$$(b_2 + 1) a_6 = -a_5 b_4, \qquad (a_1 - 1) b_5 = a_3 b_6, \qquad (a_1 + 1) b_6 = -a_3 b_5.$$

The solutions to these equations, considering the conditions derived from Equations (4.5)-(4.6), can be obtained by excluding cases where all functions $a_1, \ldots, a_6$ or $b_1, \ldots, b_6$ are zero. This leads to the two cases as presented in Table 4.2.

Table 4.2 – Possible solutions when $a_4 = b_3 = 0$ and Σ is spacelike.

Case 2	$b_6 = b_5 = a_6 = a_5 = 0,\ b_4 \neq 0$ and $a_1 = \frac{a_3 b_2}{b_4}$
Case 3	$b_6 = b_5 = b_4 = a_6 = a_5 = a_3 = 0,\ b_2 \neq 0$

We now distinguish between these two cases.

*__Case 2.__ By applying the conditions obtained in this case, a straightforward calculation reveals that $P^2 V = V$ yields the equation $b_2^2 + b_4^2 = 1$. Thus, we can consider a smooth function ϕ on the surface Σ such that

$$b_2 = \cos\phi, \quad b_4 = \sin\phi.$$

Computing $g(PU, PU) - g(U, U) = 0$ yields $a_3^2 = \sin^2\phi$, leading us to consider two different cases:

$$a_3 = \sin\phi,\ a_1 = \cos\phi \quad \text{and} \quad a_3 = -\sin\phi,\ a_1 = -\cos\phi.$$

It is worth noting that in both cases, the compatibility, symmetry, and involutive conditions of the almost product structure P are satisfied. Equations (4.5)-(4.6) now show that the sectional curvature K of Σ has a constant value of $-\frac{3}{2}$ if $a_3 = \sin\phi$ and a constant value of $-\frac{1}{6}$ if $a_3 = -\sin\phi$.

* *The function a_3 is equal to* $\sin\phi$. According to Lemma 23 and plugging in $\{X, Y\}$ with $\{U, V\}$ in Equation (4.7), we have

$$U(\phi)\sin\phi V - U(\phi)\cos\phi JV = \frac{1}{2}(\sin 2\phi - \sin\phi)G(U, V) + \frac{1}{2}(\cos 2\phi - \cos\phi)JG(U, V),$$

which holds only when the angle ϕ is a constant, taking values from the set $\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ (up to an integer multiple of 2π). Since we assumed in this case that b_4 is not zero, we find that the angle ϕ must be either $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. A straightforward calculation demonstrates that Equation (4.7) is always satisfied on the surface Σ for these constant values of ϕ. Consequently, our almost product structure P is well-defined on Σ. Furthermore, Equations (4.5)-(4.6) show that the sectional curvature K of Σ has a constant value of $-\frac{3}{2}$ for both values of the constant angle function ϕ.

* *The function a_3 is equal to* $-\sin\phi$. In this case combining Lemma 23 with Equation (4.7), where we once again substitute $\{X, Y\}$ with $\{U, V\}$, yields

$$2A\cos\phi U + 2A\sin\phi JU = U(\phi)\sin\phi V - U(\phi)\cos\phi JV - \frac{1}{2}(\sin\phi - \sin 2\phi)G(U, V) + \frac{1}{2}(\cos\phi + \cos 2\phi)JG(U, V).$$

Note that this expression can only be valid when the function A is zero and angle function ϕ has the constant value π, which would imply that the function b_4 is equal to zero. This is thus a contradiction.

***Case 3.** In this case one finds, after a substitution of $\{X, Y\}$ with $\{U, U\}$ and $\{V, V\}$ in the compatibility condition in Lemma 5, that the functions a_1 and b_2 have to satisfy

$$a_1^2 = 1, \quad b_2^2 = 1,$$

which results in four distinct cases. Equations (4.5)-(4.6) then reveal that the sectional curvature of Σ has a constant value of $-\frac{3}{2}$ if $a_1 = b_2 = 1$ or if $a_1 = b_2 = -1$, while it has a constant value of $-\frac{1}{6}$ in the other two cases. We can analyze these four cases simultaneously by combining Lemma 23 together with Equation (4.7). Substituting the vector fields $\{X, Y\}$ with $\{U, U\}$,$\{U, V\}$,$\{V, U\}$ and $\{V, V\}$ gives rise to the following equations:

$$Ab_2V = Aa_1V, \qquad Aa_1U = Ab_2U - \frac{1}{2}(a_1 - 1)b_2JG(U, V),$$

$$Ba_1U = Bb_2U, \qquad Bb_2V = Ba_1V + \frac{1}{2}(b_2 - 1)a_1JG(U, V),$$

which indicates that the almost product structure P is only well defined on the surface Σ when both functions a_1 and b_1 are equal to the constant value 1.

The table below now presents all the possibilities for the almost product structure P with respect to the frame $\mathcal{G}$ on the spacelike surface Σ.

Table 4.3 – Almost product structure on the spacelike surface Σ.

	Almost product structure P	Sectional curvature K
Case A	$PU = U$ $PV = V$	$-\frac{3}{2}$
Case B	$PU = -\frac{1}{2}U + \frac{\sqrt{3}}{2}JU$ $PV = -\frac{1}{2}V + \frac{\sqrt{3}}{2}JV$	$-\frac{3}{2}$
Case C	$PU = -\frac{1}{2}U - \frac{\sqrt{3}}{2}JU$ $PV = -\frac{1}{2}V - \frac{\sqrt{3}}{2}JV$	$-\frac{3}{2}$

Once again, we treat the cases separately and conduct a more detailed analysis of the vector fields U and V on the surface Σ. Specifically, at the identity element (Id, Id) of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, the vector fields U and V can be expressed as (α, β) and (γ, δ), respectively. Here, $\alpha, \beta, \gamma, \delta$ are elements of the Lie algebra $\mathfrak{sl}_2\mathbb{R}$, and the expression for the almost product structure P imposes certain conditions on these matrices.

***Case A.** In this case the vector fields U and V are both eigenvector fields with eigenvalue 1 of the almost product structure P. The expression for P, as defined in Equation (2.12), now immediately shows that one should have that $\alpha = \beta$ and $\gamma = \delta$, thus at the identity one has $U = (\alpha, \alpha)$ and $V = (\gamma, \gamma)$. We now determine the length of the matrices α and γ, with respect to the

semi-Euclidean metric $\langle\ ,\ \rangle$, as Equation (2.15) shows that

$$\langle \alpha, \alpha \rangle = \frac{1}{2} \langle U, U \rangle = \frac{1}{2}(2g(U,U) + g(U, PU)) = \frac{3}{2},$$

$$\langle \gamma, \gamma \rangle = \frac{1}{2} \langle V, V \rangle = \frac{1}{2}(2g(V,V) + g(V, PV)) = \frac{3}{2},$$

where U and V are spacelike unit normal vector fields with respect to the nearly Kähler metric g. After applying an isometry of the ambient space one can take

$$\alpha = \sqrt{\frac{3}{2}}\, \boldsymbol{i}, \quad \gamma = \sqrt{\frac{3}{2}}\, \boldsymbol{j},$$

with $\boldsymbol{i}$ and $\boldsymbol{j}$ the split-quaternions defined in Equation (2.1). Since the surface Σ is assumed to be totally geodesic, we can deduce from the expressions of the vector fields U and V at the identity that it must lie in the first Lagrangian immersion of Theorem 32. This implies that Σ is locally congruent to the spacelike totally geodesic totally real surface f_1 in Example 24.

*__Case B.__ From the definition of the almost complex structure J, defined in Equation (2.10), one has that the almost product structure P in this case yields

$$(\beta, \alpha) = (-\beta, \alpha - \beta), \quad (\delta, \gamma) = (-\delta, \gamma - \delta)$$

which shows that both β and δ are the zero matrix and that at the identity one has $U = (\alpha, 0)$ and $V = (\gamma, 0)$. An analogous calculation as in the previous case shows that $\langle \alpha, \alpha \rangle = \langle \gamma, \gamma \rangle = 3$. Thus after applying an isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ one can fix α and γ as

$$\alpha = \sqrt{3}\, \boldsymbol{i}, \quad \gamma = \sqrt{3}\, \boldsymbol{j}.$$

Applying the isometry $\Psi_{1,4\pi/3}$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given in (2.19), we obtain that Σ is locally congruent to the immersion in **Case A**.

*__Case C.__ From the definition of the almost complex structure J, defined in Equation (2.10), one has that the almost product structure P in this case yields

$$(\beta, \alpha) = (\beta - \alpha, -\alpha), \quad (\delta, \gamma) = (\delta - \gamma, -\gamma)$$

which shows that both α and γ are the zero matrix. An analogous calculation as in the previous case shows that $\langle \beta, \beta \rangle = \langle \delta, \delta \rangle = 3$. Thus after applying an isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ one can fix β and δ as

$$\beta = \sqrt{3}\, \boldsymbol{i}, \quad \delta = \sqrt{3}\, \boldsymbol{j}.$$

Applying the isometry $\Psi_{0,2\pi/3}$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ given in (2.19), we obtain that Σ is locally congruent to the immersion in **Case A**.

□

4.3 Lorentzian surfaces

We now consider the case where the totally geodesic totally real surface Σ is Lorentzian. One can then take a local semi-orthonormal frame $\{U, V\}$ on Σ that satisfies

$$g(U,U) = 1, \quad g(V,V) = -1. \tag{4.9}$$

The compatibility of the almost complex structure J and the properties of the tensor G on the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, as shown in Lemma 4, allow us to express the nearly Kähler metric g with respect to the frame $\mathcal{G}$ as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}.$$

Note that the symmetric operator P^{Σ}, defined in Remark 15, is not always diagonalizable on the Lorentzian surface Σ. Therefore, we state a result similar to 9, which can be found in [44].

Lemma 24. *Let V be a two-dimensional Lorentzian vector space and A be a symmetric linear endomorphism. Then there exists a basis $\{e_1, e_2\}$ of V with inner products given by the matrix Δ_i, with respect to which A takes the one of the following forms:*

$$\begin{aligned} &\textit{Type I:} \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &\textit{Type II:} \quad A = \begin{pmatrix} \lambda_1 & 0 \\ \varepsilon & \lambda_1 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ &\textit{Type III:} \quad A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{4.10}$$

with $\lambda_i, \alpha, \beta \in \mathbb{R}$ and $\varepsilon = \pm 1$.

It is worth remarking that there exists a third possibility for symmetric operators of Lorentz spaces of dimension greater than or equal to three. In Proposition 36, we demonstrate that if Σ is a Lorentzian totally geodesic totally real surface, then P^{Σ} must be of type I. Thus P^{Σ} is always diagonalizable, which is similar to the spacelike case in Section 4.2.

We first consider the following lemma, which is the Lorentzian analogue of Lemma 23. Its proof follows a similar line of reasoning and can therefore be omitted.

Lemma 25. *The nearly Kähler connection ∇ on a Lorentzian, totally geodesic totally real surface Σ, with respect to the frame $\mathcal{G}$ defined in Equation (4.2), is constructed as follows:*

$$\begin{aligned}
&\nabla_U U = AV, && \nabla_V U = BV,\\
&\nabla_U V = AU, && \nabla_V V = BV,\\
&\nabla_U JU = AJV, && \nabla_V JU = BJV - G(U,V),\\
&\nabla_U JV = AJU + G(U,V), && \nabla_V JV = BJU,\\
&\nabla_U G(U,V) = \frac{2}{3}JV, && \nabla_V G(U,V) = \frac{2}{3}JU,\\
&\nabla_U JG(U,V) = 0, && \nabla_V JG(U,V) = 0,
\end{aligned}$$

where A and B are smooth functions on Σ.

We again focus on the form of the almost product structure P when restricted to the surface and obtain conditions for the functions $a_1, \ldots, a_6, b_1, \ldots, b_6$, defined in Equations (4.3)-(4.4). Before classifying all the totally geodesic totally real surfaces in the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we first show that the operator P^Σ on such a surface has to be of type I.

Proposition 36. *Let Σ be a Lorentzian totally geodesic totally real surface in the nearly Kähler $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), J, g)$. The symmetric operator P^Σ, defined in Remark 15 is then of type I in Lemma 24.*

Remark 17. We prove this theorem by demonstrating that the operator P^Σ cannot be of type II or type III. These two cases are analyzed separately and it is shown that they both lead to contradictions.

The operator P^Σ is of type II

Lemma 24 then shows that one can choose *null* vector fields $\tilde{U}, \tilde{V}$ on the surface Σ satisfying

$$g(\tilde{U},\tilde{U}) = g(\tilde{V},\tilde{V}) = 0, \quad g(\tilde{U},\tilde{V}) = 1,$$

such that the operator P^Σ satisfies

$$P^\Sigma(\tilde{U}) = a_1\tilde{U} + \varepsilon\tilde{V}, \quad P^\Sigma(\tilde{V}) = a_1\tilde{V},$$

for a smooth function a_1 on Σ. The almost product structure P can then, using Equations (4.3)-(4.4), be expressed with respect to the frame $\{\tilde{U}, \tilde{V}\}$ as follows:

$$P\tilde{U} = a_1\tilde{U} + \varepsilon\tilde{V} + a_3 J\tilde{U} + a_4 J\tilde{V} + a_5 G(\tilde{U},\tilde{V}) + a_6 JG(\tilde{U},\tilde{V}),$$

$$P\tilde{V} = a_1\tilde{V} + b_3 J\tilde{U} + b_4 J\tilde{V} + b_5 G(\tilde{U},\tilde{V}) + b_6 JG(\tilde{U},\tilde{V}).$$

We have thus imposed the conditions $a_1 = b_2$, $a_2 = \varepsilon$, and $b_1 = 0$. With these conditions, we can now define the almost product structure P on the frame $\tilde{\mathcal{G}}$, which is similar to the frame $\mathcal{G}$ defined in Equation (4.2), but now using the null vector fields $\tilde{U}$ and $\tilde{V}$. It is important to note that the expressions for P, with respect to the frame $\tilde{\mathcal{G}}$, are not identical to the expressions found when the surface Σ is spacelike, as they depend on the induced metric on the surface. In this proof, we omit writing out these expressions as the construction is completely analogous. As Σ is a totally geodesic totally real surface, one can again use the Gauss and Codazzi equations in Section 1.4 to find a Lorentzian analogue of Equations (4.5)-(4.6), yielding in this case

$$\begin{aligned} K(p)\tilde{U} = &-\frac{5}{6}\tilde{U} - \frac{2}{3}(g(P\tilde{V},\tilde{U})P\tilde{U} - g(P\tilde{U},\tilde{U})P\tilde{V}) \\ &- \frac{2}{3}(g(JP\tilde{V},\tilde{U})JP\tilde{U} - g(JP\tilde{U},\tilde{U})JP\tilde{V}), \end{aligned} \tag{4.11}$$

$$\begin{aligned} -K(p)\tilde{V} = &\frac{5}{6}\tilde{V} - \frac{2}{3}(g(P\tilde{V},\tilde{V})P\tilde{U} - g(P\tilde{U},\tilde{V})P\tilde{V}) \\ &- \frac{2}{3}(g(JP\tilde{V},\tilde{V})JP\tilde{U} - g(JP\tilde{U},\tilde{V})JP\tilde{V}). \end{aligned} \tag{4.12}$$

We now follow a similar approach as in the proof of Theorem 40, where we classified all spacelike totally real surfaces, which are totally geodesic. Plugging the vector fields $\{\tilde{U}, J\tilde{V}\}$ in the symmetry condition of P immediately yields the condition $a_3 = b_4$. Equation (4.11) then implies that the function b_3 vanishes everywhere, thus $b_3 = 0$. Replacing $\{X, Y\}$ with $\{\tilde{U}, J\tilde{V}\}$ in the compatibility condition of P leads to the following equations:

$$a_1^2 + b_4^2 + \frac{2}{3}(a_5 b_5 + a_6 b_6) = 1, \quad b_5^2 + b_6^2 = 0.$$

From these equations, we can deduce that $b_5 = b_6 = 0$. Furthermore, there exists a smooth function ϕ on the surface Σ such that

$$a_1 = \cos\phi, \quad b_4 = \sin\phi.$$

Applying these conditions in Equation (4.11) then yields

$$a_4 \cos\phi = \varepsilon \sin\phi, \quad a_5 \cos\phi = -a_6 \sin\phi, \quad a_6 \cos\phi = a_5 \sin\phi.$$

The first equation shows that $\cos\phi$ is never zero, so we can define the function a_4 as $a_4 = \varepsilon \tan\phi$. The last two equations imply that the only possibility for these equations to hold is when $a_5 = a_6 = 0$. We can conclude this part of the proof by considering the compatibility condition of P with g, where we plug in $\{\tilde{U}, J\tilde{V}\}$, resulting in the condition

$$\sec\phi = 0,$$

which can clearly never be satisfied. Thus the operator P^Σ can never be of type II.

The operator P^Σ is of type III

In this case Lemma 24 shows that one can choose the semi-orthonormal frame $\{U, V\}$, Equation 4.9, in such a way that

$$P^\Sigma(U) = a_1 U + a_2 V, \quad P^\Sigma(V) = -a_2 U + a_1 V,$$

with a_1, a_2 smooth functions on the surface Σ and a_2 nowhere vanishing.

The almost product structure P can now, using Equations (4.3)-(4.4), be described with respect to the frame $\{U, V\}$ as follows:

$$PU = a_1 U + a_2 V + a_3 JU + a_5 G(U,V) + a_6 JG(U,V),$$

$$PV = -a_2 U + a_1 V + b_4 JV + b_5 G(U,V) + b_6 JG(U,V).$$

We have thus imposed the conditions $a_1 = b_2$, and $a_2 = -b_2$. With these conditions, we can now define the almost product structure P on the frame $\mathcal{G}$, defined in Equation (4.2). Once again, the expressions for P with respect to the frame $\mathcal{G}$ is not identical to the expressions found when the surface Σ is spacelike, as they depend on the induced metric on the surface. We do not explicitly state these expressions as the construction in this case is completely similar.

As Σ is a totally geodesic totally real surface, one can again use the Gauss and Codazzi equations given in Section 1.4 to find a Lorentzian analogue of

Equations (4.5)-(4.6), yielding in this case

$$-K(p)V = \frac{5}{6}V - \frac{2}{3}(g(PV,U) - g(PU,U)PV) \tag{4.13}$$
$$- \frac{2}{3}(g(JPV,U)JPU - g(JPU,U)JPV),$$
$$-K(p)U = \frac{5}{6}U - \frac{2}{3}(g(PV,V)PU - g(PU,V)PV) \tag{4.14}$$
$$- \frac{2}{3}(g(JPV,V)JPU - g(JPU,V)JPV).$$

Plugging in $\{U, V\}$ in the symmetry condition of P with the vector fields shows that $a_4 = -b_3$. Equations (4.13)-(4.14) then yield the following conditions:

$$a_2a_3 = a_1a_4 = 0, \quad a_1(a_3 - b_4) = 0, \quad a_1a_4 = a_2b_4.$$

This system of equations has its potential solutions listed in the table below, as the function a_2 was assumed to never be zero. A distinction is made between these three cases.

Table 4.4 – Possible solutions when P^Σ is of type III.

Case (a)	$a_3 = b_4,\ a_4 \neq 0$ and $a_1 = \frac{a_2b_4}{a_4}$
Case (b)	$a_1 = a_3 = b_4 = 0$
Case (c)	$a_3 = a_4 = b_4 = 0$

*__Case (a).__ In this scenario, it is possible to consider the existence of a continuous function α on Σ such that the following relationships hold: $a_1 = \alpha a_2$ and $a_3 = \alpha a_4$. Plugging in $\{U, JV\}$ in the compatibility condition of P with the metric g yields the relation $a_6b_5 = a_5b_6$. Thus there exists a function β such that $b_5 = \beta a_5$ and $b_6 = \beta a_6$. Equations (4.13)-(4.14) then imply the following conditions:

$$(a_2a_5 + a_4a_6)(1 + \alpha\beta) = 0, \quad (a_2a_5 + a_4a_6)(\alpha - \beta), \tag{4.15}$$

$$(a_4a_5 - a_2a_6)(1 + \alpha\beta) = 0, \quad (a_4a_5 - a_2a_6)(\alpha - \beta). \tag{4.16}$$

The only way in which this system of equations can be satisfied is when $a_5 = a_6 = 0$, as the function a_2 is never zero. Applying the involutive condition of P on the vector field U then results in

$$(a_2^2 + a_4^2)\alpha = 0, \quad (a_2^2 + a_4^2)(-1 + \alpha^2) = 1,$$

which is clearly a contradiction.

*__Case (b).__ Equations (4.13)-(4.14) in this case give rise to the following equations:

$$a_2a_5 = -a_4a_6, \quad a_2b_6 = a_4b_5, \quad a_2b_5 = -a_4b_5, \quad a_2a_6 = a_4a_5.$$

The only possible solution, when taking into account the non-vanishing of the function a_2, is when the functions a_5, a_6, b_5 and b_6 are all zero. The involutive condition of P on the vector field U then yields

$$a_2^2 + a_4^2 = -1,$$

again resulting in a contradiction.

*__Case (c).__ Equations (4.13)-(4.14) in this case imply the following conditions:

$$a_2a_5 = -a_1b_5, \quad a_2b_5 = a_1a_5, \quad a_2a_6 = -a_1b_6, \quad a_2b_6 = a_1a_6.$$

Once again the only possible solution is when the functions a_5, a_6, b_5 and b_6 are all zero. Finally applying the involutive condition of P on the vector field U results in

$$a_2^2 = a_1^2 - 1, \quad a_1a_2 = 0,$$

which is once again a contradiction, as the function a_2 is assumed to be non-vanishing.

Thus the operator P^Σ cannot be of type III, thereby concluding the proof of the proposition.

Type I surfaces

We can now classify all Lorentzian totally geodesic totally real surfaces in the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Theorem 41. *Let Σ be a Lorentzian totally geodesic totally real surface in the nearly Kähler $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), J, g)$. Then it is locally congruent to immersion f_2 of Example 24.*

Proof. Proposition 36 immediately shows that the operator P^Σ in this case has to be of type I in Lemma 24. This lemma shows that one can choose the semi-orthonormal frame $\{U, V\}$, defined in Equation (4.9), in such a way that

$$P^\Sigma(U) = a_1U, \quad P^\Sigma(V) = b_2V,$$

for smooth functions a_1, b_2 on the surface Σ, implying that P^Σ is diagonalizable. In this case one can describe the almost product structure P, using Equations (4.3)-(4.4), as follows:

$$PU = a_1 U + a_3 JU + a_5 G(U,V) + a_6 JG(U,V),$$

$$PV = b_2 V + b_4 JV + b_5 G(U,V) + b_6 JG(U,V).$$

We have thus imposed the conditions $a_2 = b_1 = 0$, which are exactly the same as in Section 4.2, where Σ was assumed to be spacelike. Now, with these conditions in place, we can proceed to define the almost product structure P on the frame $\mathcal{G}$ as defined in Equation (4.2). However, once again the expressions for P with respect to the frame $\mathcal{G}$ are not identical to those found when the surface Σ is spacelike. Although we do not explicitly state these expressions, the construction follows a similar approach as before. The Lorentzian analogue of Equations (4.5)-(4.6), when P^Σ is of type I, can straightforwardly be calculated to yield:

$$-K(p)V = \frac{5}{6}V + \frac{2}{3}(g(PU,U)PV - g(JPV,U)JPU + g(JPU,U)JPV), \tag{4.17}$$

$$-K(p)U = \frac{5}{6}U - \frac{2}{3}(g(PV,V)PU + g(JPV,V)JPU - g(JPU,V)JPV). \tag{4.18}$$

We now follow a similar approach as in Section 4.2 to derive conditions for the functions $a_1, \ldots, a_6, b_1, \ldots, b_6$. By plugging in $\{U, V\}$ in the symmetry condition of P for the almost product structure P, we obtain the immediate condition:

$$a_4 = -b_3.$$

Next, combining this condition with Equations (4.17)-(4.18) leads to the following results:

$$a_1 a_4 = 0, \quad b_2 a_4 = 0.$$

This implies that we can once again distinguish between the case where the function a_4 is zero and the case where it does not vanish.

The function $\mathbf{a_4}$ is not equal to zero.

When the function a_4 is not zero, one immediately has from the previous equation that the functions a_1 and b_2 are both zero. Equations (4.17)-(4.18) in

this case straightforwardly show that

$$a_4a_6 = a_3b_6, \quad a_4a_5 = -a_3b_5, \quad a_6b_4 = a_4b_6, \quad a_5b_4 = a_4b_5.$$

We can plug in $\{U, JV\}$ in the compatibility condition of P with the metric g and $\{U, G(U,V)\}$,$\{U, JG(U,V)\}$,$\{V, G(U,V)\}$ and $\{V, JG(U,V)\}$ in the symmetry condition of P to obtain

$$a_6b_5 = a_5b_6, \qquad a_5 = a_4b_6 - a_6b_4, \quad a_6 = a_4b_5 - a_5b_4,$$

$$b_5, = -a_3b_6 - a_4b_6, \quad b_6 = -a_3b_5 - a_4a_5.$$

Table 4.5 shows that there is only one possible solution for these equations, if one uses the extra condition that not all functions $a_1, \ldots, a_6$ nor all functions $b_1, \ldots, b_6$ can vanish.

Table 4.5 – Possible solutions when $a_4 = b_3 \neq 0$ and Σ is Lorentzian.

Case 1'	$b_6 = b_5 = a_6 = a_5 = 0$

Notice that these conditions are exactly the same as those in the spacelike case, as shown in Table 4.1. Now, we further analyze this solution by determining the possible values of the functions a_3, a_4, b_3, b_4. Starting with the compatibility condition of P with g, we find that $a_3 = -b_4$. Additionally, plugging in $\{U, U\}$ in this equation yields $b_4^2 - a_4^2 = 1$. Consequently, we can define a smooth function ϕ on the surface Σ to express the functions as follows:

$$b_4 = \cosh\phi, \quad a_4 = \sinh\phi.$$

These conditions then ensure that the compatibility, symmetry and involutive conditions of the almost product structure P are always satisfied on the surface Σ. Equations (4.17)-(4.18) also show that Σ has constant sectional curvature with value $-\frac{1}{6}$. The nearly Kähler connection, restricted to the surface Σ, is described by Lemma 25 and shows, together with substituting $\{X, Y\}$ with $\{U, V\}$ in Equation (4.7), that

$$U(\phi)\cosh\phi JU + 2\sinh\phi(A + U(\phi))JV = -\frac{3\cosh\phi}{2}G(U,V) + \frac{1}{2}JG(U,V)$$

This is clearly a contradiction, as the term containing $JG(U,V)$ is never zero. As a result, there exist no Lorentzian, totally geodesic totally real surfaces in the nearly Kähler manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ where the function a_4 does not vanish.

The function a_4 is equal to zero.

In this case the function a_4 is exactly zero, which results in Equations (4.17)-(4.18) yielding the following conditions:

$$a_1 b_4 = a_3 b_2, \quad a_1 b_5 = -a_3 b_6,$$

$$a_1 b_6 = a_3 b_5, \quad a_5 b_2 = -a_6 b_4, \quad a_5 b_4 = a_6 b_2.$$

Plugging in $\{U, V\}$ and $\{U, JV\}$ in the compatibility condition of P with the metric g shows that

$$a_5 b_5 = -a_6 b_6 \quad a_5 b_6 = a_6 b_5,$$

while plugging in $\{U, G(U,V)\}$,$\{U, JG(U,V)\}$,$\{V, G(U,V)\}$ and $\{V, JG(U,V)\}$ in the symmetry condition of P gives rise to the additional equations:

$$(b_2 - 1)a_5 = a_6 b_4, \quad (b_2 + 1)a_6 = -a_5 b_4,$$

$$(a_1 - 1)b_5 = a_3 b_6 \quad (a_1 + 1)b_6 = -a_3 b_5.$$

Solving these equations, while again considering the additional condition that the functions $a_1, \ldots, a_6$ or $b_2, \ldots b_6$ cannot all be simultaneously be zero, results in the following table.

Table 4.6 – Possible solutions when $a_4 = b_3 = 0$ and Σ is Lorentzian.

Case 2'	$b_6 = b_5 = a_6 = a_5 = 0, b_4 \neq 0$ and $a_1 = \frac{a_3 b_2}{b_4}$
Case 3'	$b_6 = b_5 = b_4 = a_6 = a_5 = a_3 = 0$ and $b_2 \neq 0$

We separate our analysis of these two cases.

***Case 2'.** Replacing $\{V, V\}$ in the compatibility condition of P in this case, shows that $b_2^2 + b_4^2 = 1$. This implies that one can consider a smooth function ϕ on the surface Σ such that

$$b_2 = \cos\phi, \quad b_4 = \sin\phi.$$

Applying the involutive condition of P on the vector field U shows that $a_3^2 = \sin^2\phi$, thus one gets two distinct cases for the functions a_1 and a_3, given by

$$a_1 = \cos\phi,\ a_3 = \sin\phi, \quad \text{and} \quad a_1 = -\cos\phi,\ a_3 = -\sin\phi.$$

The compatibility, symmetry and involutive conditions of the almost product structure P on the surface Σ are now again satisfied in both cases, while the sectional curvature K of the surface Σ has a constant value of $-\frac{3}{2}$ if $a_3 = \sin\phi$ and a constant value of $-\frac{1}{6}$ if $a_3 = -\sin\phi$. A distinction is made between these two cases.

* *The function a_3 is equal to $\sin\phi$.* Lemma 25 and Equation (4.7) now show that

$$-U(\phi)\sin\phi V + U(\phi)\cos\phi JV = \frac{1}{2}(\sin\phi + \sin 2\phi)G(U,V) + \frac{1}{2}(\cos 2\phi - \cos\phi)JG(U,V),$$

Similar to Section 4.2, when Σ was spacelike, we can make a corresponding argument for the case where a_4 is zero. In this scenario, the angle function ϕ should be a constant function, and its possible values are in the set $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$, with angles again determined up to an integer multiple of 2π. Additionally, Equation (4.7) is always satisfied on the surface Σ for these values of ϕ. As a result, for these particular values of ϕ, the almost product structure P is well-defined on the nearly Kähler manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ when restricted to the totally real, totally geodesic surface Σ.

* *The function a_3 is equal to $-\sin\phi$.* Substituting once again $\{X, Y\}$ with $\{U, V\}$ in Equation (4.7) yields

$$A\cos\phi U + A\sin\phi JU = U(\phi)\sin\phi V - U(\phi)\cos\phi JV + \frac{1}{4}(\sin 2\phi - \sin\phi)G(U,V) + \frac{1}{2}(\cos\phi + \cos 2\phi)JG(U,V).$$

One can immediately see that this can only be satisfied when the function A is zero and the angle function ϕ is a constant function with value an integer multiple of π, yielding a contradiction as this would make the function b_4 vanish.

***Case 3'.** Plugging in $\{U, U\}$ and $\{V, V\}$ in the compatibility condition of P with g then straightforwardly yields

$$a_1 = \pm 1, \quad b_2 = \pm 1,$$

trivially satisfying the compatibility, symmetry and involutive conditions of the almost product structure P on the surface Σ. Equations (4.17)-(4.18) then shows that the sectional curvature K of Σ has a constant value of $-\frac{3}{2}$ if $a_1 = b_2 = \pm 1$

and a constant value of $-\frac{1}{6}$ in the other two cases. It is possible to analyze these four cases simultaneously, by combining Lemma 25 together with Equation (4.7), as substituting the vector fields $\{X, Y\}$ with $\{U, U\}$,$\{U, V\}$,$\{V, U\}$ and $\{V, V\}$ gives rise to the following equations:

$$Ab_2V = Aa_1V, \qquad Aa_1U = Ab_2U - \frac{1}{2}(a_1 - 1)b_2JG(U, V),$$

$$Ba_1U = Bb_2U, \qquad Bb_2V = Ba_1V + \frac{1}{2}(1 - b_2)a_1JG(U, V).$$

These equations are only satisfied if $a_1 = b_2 = 1$, thus only then is the almost product structure P well-defined on the totally geodesic, totally real surface Σ. These conditions make it a surface with constant sectional curvature $K = -\frac{3}{2}$.

The following table now finally yields all the possibilities for the almost product structure P with respect to the frame $\mathcal{G}$ on the Lorentzian surface Σ, which is identical to Table 4.3, where Σ was assumed to be spacelike.

Table 4.7 – Almost product structure on the Lorentzian surface Σ.

	Almost product structure P	Sectional curvature K
Case A	$PU = U$ $PV = V$	$-\frac{3}{2}$
Case B	$PU = -\frac{1}{2}U + \frac{\sqrt{3}}{2}JU$ $PV = -\frac{1}{2}V + \frac{\sqrt{3}}{2}JV$	$-\frac{3}{2}$
Case C	$PU = -\frac{1}{2}U - \frac{\sqrt{3}}{2}JU$ $PV = -\frac{1}{2}V - \frac{\sqrt{3}}{2}JV$	$-\frac{3}{2}$

We once again analyze the vector fields U and V at the identity element (Id, Id) of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and write $U = (\alpha, \beta)$ and $V = (\gamma, \delta)$ at this point, with $\alpha, \beta, \gamma, \delta \in \mathfrak{sl}_2\mathbb{R}$.

***Case A'.** One can immediately see, as in Section 4.2 , that the matrices α, β, γ and δ satisfy $\alpha = \beta$ and $\gamma = \delta$, which means that at the identity we have $U = (\alpha, \alpha)$ and $V = (\gamma, \gamma)$. The lengths of the matrices with respect to the semi-Euclidean metric $\langle\ ,\ \rangle$ are then determined as

$$\langle \alpha, \alpha \rangle = \frac{3}{2}, \quad \langle \gamma, \gamma \rangle = -\frac{3}{2}.$$

Thus after applying an isometry of the nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ one can fix α and γ as

$$\alpha = \sqrt{\frac{3}{2}}\boldsymbol{i}, \quad \gamma = \sqrt{\frac{3}{2}}\boldsymbol{k},$$

with $\boldsymbol{i}$ and $\boldsymbol{j}$ the split-quaternions defined in Equation (2.1). Because the surface Σ is assumed to be totally geodesic, one can deduce from the expressions of the vector fields U and V at the identity that it must lie in the first Lagrangian immersion of Theorem 32. Thus the surface Σ is locally congruent to the Lorentzian totally geodesic totally real immersion f_2 in Example 24.

***Case B'.** In this case, as in Section 4.2, one has that $\beta = \delta = 0$ and the expressions $U = (\alpha, 0)$ and $V = (\gamma, 0)$ hold at the identity $(\mathrm{Id}_2, \mathrm{Id}_2)$. The lengths of the matrices with respect to the semi-Euclidean metric $\langle\ ,\ \rangle$ are then determined as

$$\langle \alpha, \alpha \rangle = 3, \quad \langle \gamma, \gamma \rangle = -3,$$

which means that applying an isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ allows us to fix α and γ as

$$\alpha = \sqrt{3}\boldsymbol{i}, \quad \gamma = \sqrt{3}\boldsymbol{k}.$$

Applying the isometry $\Psi_{1,4\pi/3}$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we see that this surface is mapped into the surface of ***Case A'**.

***Case C'.** Analogous to the spacelike surfaces in Section 4.2, one immediately has that $\alpha = \gamma = 0$ and the expressions $U = (0, \beta)$ and $V = (0, \delta)$ hold at the identity $(\mathrm{Id}_2, \mathrm{Id}_2)$. The lengths of the matrices with respect to the semi-Euclidean metric $\langle\ ,\ \rangle$ are then determined as

$$\langle \beta, \beta \rangle = 3, \quad \langle \delta, \delta \rangle = -3,$$

which means that applying an isometry of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ allows us to fix β and δ as

$$\beta = \sqrt{3}\boldsymbol{i}, \quad \delta = \sqrt{3}\boldsymbol{k}.$$

Using a similar reasoning as in the previous case, one can deduce that the surface Σ in this case also has to be locally congruent to the Lorentzian totally geodesic totally real immersion f_2 given in Example 24.

Therefore we can conclude the proof of the theorem. □

Chapter 5

Hypersurfaces of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

This work is based on an unpublished work with my colleagues K. Dekimpe and M. D'haene.

5.1 Introduction

Given a n-dimensional Riemannian manifold (M, g), we understand by a hypersurface of M a submanifold of dimension $n-1$.

Hypersurfaces of six-dimensional nearly Kähler manifolds have been widely studied. For instance, Loubeau and Deschamps [18] showed that there are no totally geodesic or totally umbilical hypersurfaces in $\mathbb{C}P^3$ and $F(\mathbb{C}^3)$. This is also true for $\mathbb{S}^3 \times \mathbb{S}^3$ (see for instance [31]), so the only six-dimensional nearly Kähler manifold with a totally geodesic hypersurface is the sphere $\mathbb{S}^6$. This does not come as a surprise, taking into account the result by Nikolayevsky [43], who states that a homogeneous manifold with a totally geodesic hypersurface must be a warped product of $\mathbb{R}^n$ with a homogeneous manifold, a twisted product of $\mathbb{R}$ with a homogeneous manifold, or the Riemannian product of a space form with a homogeneous space.

In this chapter, we show that there are no constant sectional curvature hypersurfaces in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ through means of a refined version of the Gauss equation. First, in Section 5.2, we show a general result for hypersurfaces.

Then, we distinguish between two cases, when the vector field $P\xi$, with ξ the unit normal vector field, depends linearly on ξ and $J\xi$, and when it does not. We use the result of Section 5.2 to prove that in both cases the shape operator has a special form. Then in Section 5.5 we show the following result.

Theorem 42. *There are no hypersurfaces in the pseudo-nearly Kähler* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *with constant sectional curvature.*

5.2 The main equation

Lemma 26. *Let M be a hypersurface of a (pseudo)-Riemannian manifold N of dimension at least three, and let R, $\tilde{R}$ be their respective curvature tensors. Then*

$$\mathop{\mathfrak{S}}_{X,Y,W} g(\tilde{R}(X,Y)Z, SW) = \mathop{\mathfrak{S}}_{X,Y,W} g(R(X,Y)Z, SW). \tag{5.1}$$

where S is the shape operator associated to the immersion of M into N, and $\mathop{\mathfrak{S}}_{X,Y,W}$ is the cyclic sum over X, Y and W. Moreover, if M has constant sectional curvature the right hand side vanishes.

Proof. By taking the cyclic sum over the Gauss equation in Section 1.4 we obtain 5.1. Moreover, if M has constant sectional curvature then $R(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y)$. Plugging in this expression in the right hand side of 5.1 we see that it vanishes everywhere.

□

Analogously to previous chapters, we approach the study of constant sectional curvature hypersurfaces by dividing them depending on the behavior of the almost product structure P on the tangent space of the hypersurface.

Let ξ be the ε-unit normal vector field to the hypersurface M, i.e. $g(\xi,\xi) = \varepsilon = \pm 1$. The vector field $P\xi$ might or might not be contained in the span of ξ and $J\xi$. We treat these cases separately.

5.3 The vector field $P\xi$ is independent of ξ and $J\xi$

Define the vector field X on M by

$$X = P\xi - \varepsilon g(P\xi,\xi)\xi - \varepsilon g(P\xi, J\xi)J\xi.$$

Suppose that $P\xi$, $J\xi$ and ξ are linearly independent. Then

$$F = \{J\xi, X, JX, G(\xi, X), JG(\xi, X)\}$$

is a frame on M. We define the functions θ_1 and θ_2 on M by

$$\theta_1 = g(P\xi, \xi), \qquad \theta_2 = g(P\xi, J\xi). \tag{5.2}$$

Then by lemmas 4 and 5, F is an orthogonal frame with

$$g(J\xi, J\xi) = \varepsilon, \quad g(X, X) = \varepsilon(1 - \theta_1^2 - \theta_2^2),$$

$$g(G(\xi, X), G(\xi, X)) = -\frac{2}{3}(1 - \theta_1^2 - \theta_2^2).$$

Note that the linear independency of $P\xi$, ξ, and $J\xi$ is equivalent to $\theta_1^2 + \theta_2^2 \neq 1$. It follows from lemmas 4 and 5 that P takes the shape

$$\begin{pmatrix} \varepsilon\theta_1 & \varepsilon\theta_2 & (1 - \theta_1^2 - \theta_2^2) & 0 & 0 & 0 \\ \varepsilon\theta_2 & -\varepsilon\theta_1 & 0 & -(1 - \theta_1^2 - \theta_2^2) & 0 & 0 \\ 1 & 0 & -\varepsilon\theta_1 & \varepsilon\theta_2 & 0 & 0 \\ 0 & -1 & \varepsilon\theta_2 & \varepsilon\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

with respect to the frame $\{\xi, F\}$.

To simplify notation, we denote

$$E_2 = J\xi, \quad E_3 = X, \quad E_4 = JX \quad E_5 = G(\xi, X), \quad E_6 = JG(\xi, X). \tag{5.3}$$

Then, we write

$$h(E_i, E_j) = h_{ij}\xi,$$

where h is the second fundamental form.

Proposition 37. *Let M be a hypersurface of the pseud-nearly Kähler $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ with constant sectional curvature. Suppose that $\theta_1^2 + \theta_2^2 \neq 1$, where θ_1, θ_2 are the functions given in* (5.2)*. Then M is totally umbilical.*

Proof. By Lemma 26 the right hand side of (5.1) vanishes. Then, by taking $X = E_2$, $Y = E_3$, $Z = E_5, E_6$ and $W = E_4, E_5$ in (26) we obtain

$$h_{25} = h_{26} = 0, \quad h_{34} = -\varepsilon\theta_2 h_{24}.$$

Similarly, taking $X = E_2, E_5$, $Y = E_6$, $Z = E_2, E_4, E_5$ and $W = E_2, E_3, E_4$ yields

$$h_{34} = h_{23} = h_{56} = 0, \quad h_{66} = h_{55}.$$

Plugging in these values in (5.1) with $X = E_2$, $Y = E_3, E_4$, $Z = E_3$ and $W = E_5, E_6$ we produce

$$h_{36} = -2\varepsilon\theta_1 h_{45}, \quad h_{35} = -2\varepsilon\theta_1 h_{46}, \quad h_{46} = 2\varepsilon\theta_2 h_{45}.$$

From computing (5.1) with $X = E_2, E_3$, $Y = E_5$, $Z = E_3, E_4$ and $W = E_6$ it follows

$$h_{24} = 0, \quad h_{55} = -\tfrac{2}{3}\varepsilon h_{33}.$$

Moreover, taking $X = E_2$, $Y = E_6$, $Z = E_5, E_6$ and $W = E_3, E_4$ we get

$$h_{33} = h_{44} = h_{22}(1 - \theta_1^2 - \theta_2^2) = h_{22}(1 - \theta_1^2 - \theta_2^2).$$

Finally, by plugging in all these values in (5.1) with $X = E_2$, $Y = E_4$, $Z = E_2, E_3, E_4$, $W = E_5, E_6$ we obtain

$$h_{45}(4\theta_2^2 - 1) = 0, \quad h_{45}(1 - \theta_1\left(8\theta_1\theta_2^2 + \varepsilon\right)) = 0, \quad h_{45}(1 + 8\varepsilon\theta_1\theta_2^2) = 0,$$

which imply $h_{45} = 0$.

Therefore, the second fundamental form satisfies $h(X,Y) = \varepsilon h_{22} g(X,Y)\xi$. □

5.4 The vector field $P\xi$ is not independent of ξ and $J\xi$

Suppose now that $P\xi \in \mathrm{Span}\{\xi, J\xi\} := \mathcal{D}$. The tensor P preserves $\mathcal{D}$ and its orthogonal complement. Let U and V be eigenvectors of $P|_{\mathcal{D}}$ associated to 1, with $g(U,U) = -\varepsilon$ and $g(V,V) = 1$. Then JU and JV are eigenvectors associated to -1. By computing $g(P\xi, P\xi)$ we obtain $\theta_1^2 + \theta_2^2 = 1$. Hence we can write $\theta_1 = \cos\theta$, $\theta_2 = \sin\theta$.

Again, to simplify notation, we write

$$E_2 = J\xi, \quad E_3 = U, \quad E_4 = V, \quad E_5 = JU, \quad E_6 = JV, \tag{5.4}$$

and $h(E_i, E_j) = h_{ij}\xi$.

Proposition 38. *Let M be a hypersurface of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ with constant sectional curvature, and let ξ be the normal vector field with $g(\xi,\xi) = \varepsilon = \pm 1$. Suppose that $\theta_1^2 + \theta_2^2 = 1$ where θ_1 and θ_2 are given in* (5.2)*. Then the shape operator S satisfies*

$$S = \begin{pmatrix} h_{22} & 0 & 0 & 0 & 0 \\ 0 & h_{33} & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon h_{33} & 0 & 0 \\ 0 & 0 & 0 & h_{33} & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon h_{33} \end{pmatrix},$$

with respect to the frame $\{E_i\}_i$ *on* M *given in* (5.4).

Proof. By Lemma 26 the shape operator S satisfies

$$\mathop{\mathfrak{S}}_{X,Y,W} g(\tilde{R}(X,Y)Z, SW) = 0, \tag{5.5}$$

where $\tilde{R}$ is the curvature tensor of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Plugging in $X = E_2$, $Y = E_3, E_6$, $Z = E_4, E_5, E_6$ and $W = E_4, E_5$ in (5.5) we obtain $h_{23} = h_{24} = h_{25} = h_{26} = 0$.

By taking $X = E_5$, $Y = E_3$, $Z = E_3$, $W = E_4$ and $X = E_3$, $Y = E_6$, $Z = E_5$, $W = E_5$ in (5.5) we obtain $h_{36} = 3h_{45}$ and $h_{36} = \frac{1}{3}h_{45}$, hence $h_{36} = h_{45} = 0$. In a similar way we obtain $h_{34} = h_{56} = 0$, $h_{35} = h_{46} = 0$.

Now plug in $\{X, Y, Z, W\}$ in (5.5) as all permutations of $\{E_3, E_4, E_5, E_6\}$. Then $h_{44} = -\varepsilon h_{33}$, $h_{55} = h_{33}$ and $h_{66} = -\varepsilon h_{33}$. □

5.5 Proof of Theorem 42

Proof of Theorem 42. Let M be a hypersurface of the pseudo-nearly Kähler $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ with constant sectional curvature c. Let ξ the normal vector field with $g(\xi,\xi) = \varepsilon = \pm 1$.

We divide in two cases, when $P\xi$ depends linearly on ξ and $J\xi$ and when it does not. Recall that this is equivalent to $\theta_1^2 + \theta_2^2$ being equal to or different from 1, where $\theta_1 = g(P\xi,\xi)$ and $\theta_2 = g(P\xi, J\xi)$.

Suppose that $\theta_1^2 + \theta_2^2 \neq 1$. Then by Proposition 37, M is a totally umbilical hypersurface. The equation of Gauss given in Chapter 1, Section 1.4 can be written as

$$\begin{aligned} c\,(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) = {} & g((\tilde{R}(X,Y)Z), W) \\ & + g(h(Y,Z), h(X,W)) \\ & - g(h(X,Z), h(Y,W)). \end{aligned} \tag{5.6}$$

Plugging in $X = E_6$, $Y = E_5$, $Z = E_3$ and $W = E_4$ in (5.6) where $\{E_i\}_i$ is the frame given in (5.3) yields $\theta_1^2 + \theta_2^2 = 1$, which is a contradiction.

Now suppose that $\theta_1^2 + \theta_2^2 = 1$. By taking $X = E_5$, $Y = E_4$, $Z = E_3$ and $W = E_6$ in (5.6) where $\{E_i\}_i$ is the frame given in (5.4) gives $\varepsilon/2 = 0$, also a contradiction. □

Conclusion

Overview

We started by presenting the pseudo-Riemannian nearly Kähler structure of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ as the T-dual of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. Then, we provided an expression of the isometry group, as a tool for classification of submanifolds up to congruency. Later, we studied Lagrangian submanifolds and we divided them into four types. We considered each type separately and we classified all totally geodesic and extrinsically homogeneous Lagrangian submanifolds.

We found that there are three examples of totally geodesics Lagrangian submanifolds, which corresponds to the Riemannian case, but with one extra example. On the contrary, there were exotic examples of extrinsically homogeneous Lagrangian submanifolds. Besides the analogues of the submanifolds in the Riemannian case, we found three extra examples, where one is actually a family of immersions.

We classified all totally real and totally geodesic surfaces, finding a hyperbolic plane and anti-de Sitter plane, immersed in one of the Lagrangian submanifolds. Also, we showed that there are no hypersurfaces with constant sectional curvature.

Results that were not included

Besides the work included in this thesis, I also have some results I decided to exclude for various reasons.

The construction in Section 3.4 for a type IV Lagrangian submanifold, was done for all types. During the first year of my PhD, looking for explicit examples of

all the types in Lemma 11, I followed the construction given in [7] for $\mathbb{S}^3 \times \mathbb{S}^3$.

For type I Lagrangian submanifolds, the results are practically the same than for $\mathbb{S}^3 \times \mathbb{S}^3$, with the exception that we have to consider different cases depending on the index of the surface in $\mathrm{SL}(2,\mathbb{R})$. I did not include this result since it did not add any value to the thesis.

For type II Lagrangian submanifolds, there are two cases to consider, since the projection to the first factor is either a minimal Lorentzian surface in $\mathrm{SL}(2,\mathbb{R})$ or a null curve in $\mathrm{SL}(2,\mathbb{R})$ with special properties. I did not add this result since some cases lead to a dead end.

For type III Lagrangian submanifolds, the projection to one of the factors is always a degenerate surface. Hence, I did not include it since the computations were cumbersome and did not produce result.

Besides $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, I also studied submanifolds of other spaces. Chapter 5 is part of an ongoing work with K. Dekimpe and M. D'haene. We saw that besides $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, there are no constant sectional curvature hypersurfaces in $\mathbb{C}P^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$. The aim is to prove this for all six-dimensional homogeneous nearly Kähler manifolds (besides $\mathbb{S}^6$). During my stay in China in May 2024, I worked with M. D'haene, X. Chen, Z. Hu and L. Vrancken on totally geodesic almost complex surfaces of the analogue of the flag manifold $\mathrm{SL}(3,\mathbb{R})/\mathbb{R}^* \cdot \mathrm{SO}(2)$. The result is not included due to time constraints.

Future research

The obvious next step is to prove that a totally geodesic surface of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is either almost complex or totally real. This poses an extra difficulty, since we have to consider an extra function: the angle that the almost complex structure J forms with the tangent space.

As mentioned before, finishing the classification of hypersurfaces with constant sectional curvature of nearly Kähler manifolds comes in the near future.

There are many results in $\mathbb{S}^3 \times \mathbb{S}^3$ that can be translated to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. For instance, Z. Yao, Z. Hu and X. Zhang studied hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ to a great extent (see [30, 31, 55, 56]). Particularly interested in Hopf hypersurfaces, they found a family of immersions of $\mathbb{S}^2 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$. It is clear that in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we will find at least two families: $H^2 \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $H^2_1 \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. It would be interesting to know if there exist more examples.

Also, there is close to no research done in the pseudo-Riemannian nearly Kähler six-manifolds. Vrancken and Cwiklinski studied totally geodesic surfaces of one of the analogues of the flag manifold in [14]. As mentioned before, totally geodesic surfaces in the other analogue were considered by the author, X. Chen, M. D'haene, Z. Hu and L. Vrancken. In $\mathbb{S}^6_4$ and the analogues of $\mathbb{C}P^3$ there are no results yet.

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